Classical Measurements in Curved Space-Times

FERNANDO DE FELICE
AND DONATO BINI

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CLASSICAL MEASUREMENTS IN CURVED SPACE-TIMES

The theory of relativity describes the laws of physics in a given space-time. However, a physical theory must provide observational predictions expressed in terms of measurements, which are the outcome of practical experiments and observations.

Ideal for researchers with a mathematical background and a basic knowledge of relativity, this book will help in the understanding of the physics behind the mathematical formalism of the theory of relativity. It explores the informative power of the theory of relativity, and shows how it can be used in space physics, astrophysics, and cosmology. Readers are given the tools to pick out from the mathematical formalism the quantities which have physical meaning, which can therefore be the result of a measurement. The book considers the complications that arise through the interpretation of a measurement which is dependent on the observer who performs it. Specific examples of this are given to highlight the awkwardness of the problem.

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Preface

A physical measurement is meaningful only if one identifies in a non-ambiguous way who is the observer and what is being observed. The same observable can be the target of more than one observer so we need a suitable algorithm to compare their measurements. This is the task of the theory of measurement which we develop here in the framework of general relativity.

Before tackling the formal aspects of the theory, we shall define what we mean by observer and measurement and illustrate in more detail the concept which most affected, at the beginning of the twentieth century, our common way of thinking, namely the relativity of time.

We then continue on our task with a review of the entire mathematical machinery of the theory of relativity. Indeed, the richness and complexity of that machinery are essential to define a measurement consistently with the geometrical and physical environment of the system under consideration.

Most of the material contained in this book is spread throughout the literature and the topic is so vast that we had to consider only a minor part of it, concentrating on the general method rather than single applications. These have been extensively analyzed in Clifford Will's book (Will, 1981), which remains an essential milestone in the field of experimental gravity. Nevertheless we apologize for all the references that would have been pertinent but were overlooked.

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Notation

$\Re$: The real line.

$\Re^4$: The space of the quadruplets of real numbers.

$\{x^\alpha\}|_{\alpha=0,1,2,3}$: A quadruplet of local coordinates.

$\{e_\alpha\}$: A field of bases (frames) for the tangent space.

$\{\omega^\alpha\}$: A field of dual bases (dual frames) $\omega^\alpha(e_\beta) = \delta^\alpha{}_\beta$.

$\mathrm{g} = g_{\alpha\beta}\omega^\alpha \otimes \omega^\beta$: The metric tensor.

g^{-1}: Inverse metric.

g: Determinant of the metric.

$X^\#$: A tangent vector field (with contravariant components).

$X^\flat$: The 1-form (with covariant components) g-isomorphic to X.

$\lrcorner$ (left contraction): Contraction of the rightmost contravariant index of the first tensor with the leftmost covariant index of the second tensor, that is $[S \lrcorner T]^{\cdots}{}_{\ldots} = S^{\cdots\alpha}T_{\alpha\ldots}$.

$\llcorner$ (right contraction): Contraction of the rightmost covariant index of the first tensor with the leftmost contravariant index of the second tensor, that is $[S \llcorner T]^{\cdots}{}_{\ldots} = S_{\ldots\alpha}T^{\alpha\cdots}$.

$\overset{p}{\lrcorner}$ (left p-contraction): Contraction of the rightmost p contravariant indices of the first tensor with the leftmost p covariant indices of the second tensor, i.e. $S \overset{p}{\lrcorner} T \equiv S^{\alpha\ldots\beta_1\ldots\beta_p}T_{\beta_1\ldots\beta_p\ldots}$.

$\overset{p}{\llcorner}$ (right p-contraction): Contraction of the rightmost p covariant indices of the first tensor with the leftmost p contravariant indices of the second tensor, i.e. $S \overset{p}{\llcorner} T \equiv S^{\alpha\cdots}{}_{\beta_1\ldots\beta_p}T^{\beta_1\ldots\beta_p\cdots}$.

$\binom{r}{s}$-tensor: A tensor r-times contravariant and s-times covariant.

$[\alpha_1\ldots\alpha_p]$: Antisymmetrization of the p indices.

$(\alpha_1\ldots\alpha_p)$: Symmetrization of the p indices.

$[\mathrm{ALT}\, S]_{\alpha_1\ldots\alpha_p} = S_{[\alpha_1\ldots\alpha_p]}$.

$[\mathrm{SYM}\, S]_{\alpha_1\ldots\alpha_p} = S_{(\alpha_1\ldots\alpha_p)}$.

$e_\gamma(\cdot)$: γ-component of a frame derivative.

∇: Covariant derivative.

∇_{e_α}: α-component of the covariant derivative relative to the frame $\{e_\sigma\}$.

$\epsilon_{\alpha_1\ldots\alpha_4} = \epsilon_{[\alpha_1\ldots\alpha_4]}$: Levi-Civita alternating symbol.

$\eta_{\alpha_1\ldots\alpha_4} = g^{1/2}\epsilon_{\alpha_1\ldots\alpha_4}$; $\eta^{\alpha_1\ldots\alpha_4} = -g^{-1/2}\epsilon^{\alpha_1\ldots\alpha_4}$: The unit volume 4-form.

$\delta^{\alpha_1\ldots\alpha_4}{}_{\beta_1\ldots\beta_4} = \epsilon^{\alpha_1\ldots\alpha_4}\epsilon_{\beta_1\ldots\beta_4} = -\eta^{\alpha_1\ldots\alpha_4}\eta_{\beta_1\ldots\beta_4}$: Generalized Kronecker delta.

$[{}^*S]_{\alpha_{p+1}\ldots\alpha_4} = \frac{1}{p!}\, S_{\alpha_1\ldots\alpha_p}\, \eta^{\alpha_1\ldots\alpha_p}{}_{\alpha_{p+1}\ldots\alpha_4}$: Hodge dual of $S_{\alpha_1\ldots\alpha_p}$.

"$\cdot$": Scalar g-product, i.e. $u\cdot v = g(u,v) = g_{\alpha\beta}u^\alpha v^\beta$ for any pair of vectors $(u,\, v)$.

$\wedge$: The exterior or wedge product, i.e. $u\wedge v = u\otimes v - v\otimes u$ for any pair $(u,\, v)$ of vectors or 1-forms.

$\{e_{\hat\alpha}\}$: An orthonormal frame (tetrad).

$\frac{D}{ds}$: Absolute derivative along a curve γ with parameter s, i.e. $D/ds = \nabla_{\dot\gamma}$.

$a(u)$: Acceleration vector of the world line with tangent vector field u, i.e. $a(u) = \nabla_u u$.

$\frac{D_{(\mathrm{fw},u)}}{ds}$: The Fermi-Walker derivative along the curve with parameter s. For any vector field X: $\frac{D_{(\mathrm{fw},u)}\, X}{ds} = \frac{DX}{ds} \pm [a(u)(u\cdot X) - u(a(u)\cdot X)]$.

$\mathcal{C}_X$: The congruence of curves with tangent field X.

$\omega(X)$: The vorticity tensor of the congruence $\mathcal{C}_X$ (the same symbol also denotes the vorticity vector).

$\theta(X)$: The expansion tensor of the congruence $\mathcal{C}_X$.

$\Theta(X) = \mathrm{Tr}\,\theta(X)$: The trace of the expansion tensor of the congruence $\mathcal{C}_X$.

$\pounds_X$: Lie derivative along the congruence $\mathcal{C}_X$.

$C^\gamma{}_{\alpha\beta}$: Structure functions of a given frame.

$\omega^{\alpha_1\ldots\alpha_p} = p!\omega^{[\alpha_1}\otimes\cdots\otimes\omega^{\alpha_p]}$: The dual basis tensor of a space of p-forms.

$\delta T = {}^*d[{}^*T]$: Divergence of a p-form T.

$\Delta_{(\mathrm{dR})} = \delta d + d\delta$: de Rham operator.

LRS_u: Local rest space of u.

$P(u)$: u-spatial projector operator which generates LRS_u.

$T(u)$: u-temporal projector operator which generates the time axis of u.

$[P(u)S] \equiv S(u)$: Total u-spatial projection of a tensor S such that

$$[S(u)]^{\alpha_1 \cdots}{}_{\beta_1 \ldots} = P(u)^{\alpha_1}{}_{\sigma_1} \cdots P(u)^{\rho_1}{}_{\beta_1} \cdots S^{\sigma_1 \cdots}{}_{\rho_1 \ldots}.$$

$[\pounds(u)_X]$: u-spatially projected Lie derivative. For any tensor T it is

$$[\pounds(u)_X T]^{\alpha \ldots}{}_{\beta \ldots} = P(u)^{\alpha}_{\sigma} \cdots P(u)^{\rho}{}_{\beta} \cdots [\pounds_X T]^{\sigma} \cdots{}_{\rho \ldots}.$$

$\nabla(u)_{\rm lie} = \pounds(u)_u$: u-spatial Lie temporal derivative.

$\nabla(u) = P(u)\nabla$: u-spatially projected covariant derivative.

$P(u)\frac{D_{({\rm fw},X)}}{ds}$: u-spatially projected Fermi-Walker derivative along a curve with unit tangent vector X.

$d(u) = P(u)d$: u-spatially projected exterior derivative.

"$\cdot_u$": u-spatial inner product, i.e. $X \cdot_u Y = P(u)_{\alpha\beta} X^\alpha Y^\beta$.

"$\times_u$" : u-spatial cross product, i.e. $[X \times_u Y]^\alpha = \eta(u)^\alpha{}_{\rho\sigma} X^\rho Y^\sigma$.

$\eta(u)^\alpha{}_{\rho\sigma} = u_\beta \eta^{\beta\alpha}{}_{\rho\sigma}$: u-spatial 4-volume.

${\rm grad}_u = \nabla(u)$: u-spatial gradient.

${\rm curl}_u = \nabla(u)\times_u$: u-spatial curl.

${\rm div}_u = \nabla(u)\cdot_u$: u-spatial divergence.

${\rm Scurl}_u$: Symmetrized ${\rm curl}_u$, i.e. $[{\rm Scurl}_u\, A]^{\alpha\beta} = \eta(u)^{\gamma\delta(\alpha} \nabla(u)_\gamma A^{\beta)}{}_\delta$.

$C_{({\rm fw})ab}$: Fermi-Walker rotation coefficients, i.e. $C_{({\rm fw})ab} = e_b \cdot \nabla_u e_a$.

$C_{({\rm lie})}{}^b{}_a$: Lie rotation coefficients, i.e. $C_{({\rm lie})}{}^b{}_a = \omega^b(\pounds(u)_u e_a)$.

$\nabla(u)_{({\rm fw})} = P(u)\nabla_u$: u-spatial Fermi-Walker temporal derivative.

$\nabla(u)_{({\rm tem})}. \equiv \nabla(u)_{({\rm fw})}$ or $\nabla(u)_{({\rm lie})}$

$\nu(U,u)$: Relative spatial velocity of U with respect to u.

$\nu(u,U)$: Relative spatial velocity of u with respect to U.

$\gamma(U,u) = \gamma(u,U) = \gamma$: Lorentz factor of the two observers u and U.

$\hat\nu(u,U)$: Unitary relative velocity vector of u with respect to U.

ζ: Angular velocity.

$\omega(k,u)$: Frequency of the light ray k with respect to the observer u.

$||\nu(U,u)|| = ||\nu(u,U)|| = \nu$: Magnitude of the relative velocity of the two observers u and U.

$B(U,u)$: Relative boost from u to U.

$P(U,u) = P(U)P(u)$: Mixed projector operator from LRS_u into LRS_U.

$B_{(\mathrm{lrs})}(U,u) = P(U)B(U,u)P(u)$: Boost from LRS_u into LRS_U.

$B_{(\mathrm{lrs})}(U,u)^{-1} = B_{(\mathrm{lrs})}(u,U)$: Inverse boost from LRS_U to LRS_u.

$B_{(\mathrm{lrs})_u}(U,u) = P(U,u)^{-1} \llcorner B_{(\mathrm{lrs})}(U,u)$.

$\frac{D_{(\mathrm{lie},U)}X}{d\tau_U} = [U,\, X]$: Lie derivative of X along $\mathcal{C}_U$.

$\tau_{(U,\,u)}$: Relative standard time parameter, i.e. $d\tau_{(U,\,u)} = \gamma(U,\,u)d\tau_U$.

$\ell_{(U,\,u)}$: Relative standard length parameter, i.e. $d\ell_{(U,\,u)} = \gamma(U,\,u)||\nu(U,\,u)||d\tau_U$.

$\frac{D_{(\mathrm{fw},U,u)}}{d\tau_{(U,\,u)}} = P(u)\frac{D}{d\tau_{(U,\,u)}}$: Projected absolute covariant derivative along U.

$a_{(\mathrm{fw},U,u)} = P(u)\frac{D\nu(U,u)}{d\tau_{(U,u)}}$: Relative acceleration of U with respect to u.

$(\nabla X)_{\alpha\beta} \equiv \nabla_\beta X_\alpha$.

Physical dimensions

We are using geometrized units with $G = 1 = c$, G and c being Newton's gravitational constant and the speed of light in vacuum, respectively. Symbols are as they appear in the text; the reader is advised that more than one symbol may be used for the same item and conversely the same symbol may refer to different items. Reference is made to the observers $(U,\, u)$.

Time	t	$\rightarrow$	$[L]^1$
Space	r, x, y, z	$\rightarrow$	$[L]^1$
Mass	$\mathcal{M}, m, \mu_0$	$\rightarrow$	$[L]^1$
Angular velocity	ζ	$\rightarrow$	$[L]^{-1}$
Energy	E	$\rightarrow$	$[L]^1$
Specific energy			
(in units of mc^2)	E, γ	$\rightarrow$	$[L]^0$
Specific angular momentum			
(in units of mc)	$L, \lambda, \Lambda, \ell$	$\rightarrow$	$[L]^1$
of a rotating source (Kerr)	a	$\rightarrow$	$[L]^1$
Spin	S	$\rightarrow$	$[L]^2$
Specific spin	S/m	$\rightarrow$	$[L]^1$
4-velocity	U, u	$\rightarrow$	$[L]^0$
Relative velocity	$\nu(U,u)$	$\rightarrow$	$[L]^0$
Force	F	$\rightarrow$	$[L]^0$
Acceleration	$a(U)$	$\rightarrow$	$[L]^{-1}$
Expansion	$\Theta(U)$	$\rightarrow$	$[L]^{-1}$

Vorticity	$(\omega(U)_{\alpha\beta}\,\omega(U)^{\alpha\beta})^{1/2}$	$\rightarrow$	$[L]^{-1}$
Electric charge	Q, e	$\rightarrow$	$[L]^{1}$
Space-time curvature	$(R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta})^{1/2}$	$\rightarrow$	$[L]^{-2}$
Electric field	$E(U)$	$\rightarrow$	$[L]^{-1}$
Magnetic field	$B(U)$	$\rightarrow$	$[L]^{-1}$
Frequency	$\omega(k,u)$	$\rightarrow$	$[L]^{-1}$
Gravitational wave amplitude	$h_{+,\times}$	$\rightarrow$	$[L]^{0}$
Strain	$S(U)$	$\rightarrow$	$[L]^{-2}$.

Conversion factors

We list here conversion factors from conventional CGS to geometrized units. For convenience we denote the quantities in CGS units with a tilde ($\sim$).

Name	CGS units	Geometrized units
Mass	$\tilde{M}$	$\mathcal{M} = \frac{G\tilde{M}}{c^2}$
Electric charge	$\tilde{Q}$	$Q = \tilde{Q}\sqrt{\frac{G}{4\pi\epsilon_0 c^4}}$
Velocity	$\tilde{v}$	$\nu = \frac{\tilde{v}}{c}$
Acceleration	$\tilde{a}$	$a = \frac{\tilde{a}}{c^2}$
Force	$\tilde{F}$	$F = \frac{G\tilde{F}}{c^4}$
Electric field	$\tilde{E}$	$E = \tilde{E}\sqrt{\frac{4\pi\epsilon_0 G}{c^4}}$
Magnetic field	$\tilde{H}$	$H = \tilde{H}\sqrt{\frac{4\pi\epsilon_0 G}{c^2}}$
Energy	$\tilde{\mathcal{E}}$	$\mathcal{E} = \frac{G\tilde{\mathcal{E}}}{c^4}$
Specific energy	$\frac{\tilde{\mathcal{E}}}{\tilde{M}c^2}$	$\frac{\mathcal{E}}{\mathcal{M}} \equiv \gamma = \frac{\tilde{\mathcal{E}}}{\tilde{M}c^2}$
Angular momentum	$\tilde{L}$	$L = \frac{G\tilde{L}}{c^3}$
Angular momentum in units of $\tilde{M}c$	$\frac{\tilde{L}}{\tilde{M}c}$	$\lambda = \frac{L}{\mathcal{M}} = \frac{\tilde{L}}{\tilde{M}c}$

1
Introduction

A physical measurement requires a collection of devices such as a clock, a theodolite, a counter, a light gun, and so on. The operational control of this instrumentation is exercised by the observer, who decides what to measure, how to perform a measurement, and how to interpret the results. The observer's laboratory covers a finite spatial volume and the measurements last for a finite interval of time so we can define as the *measurement's domain* the space-time region in which a process of measurement takes place. If the background curvature can be neglected, then the measurements will not suffer from curvature effects and will then be termed *local*. On the contrary, if the curvature is strong enough that it cannot be neglected over the measurement's domain, the response of the instruments will depend on the position therein and therefore they require a careful calibration to correct for curvature perturbations. In this case the measurements carrying a signature of the curvature will be termed *non-local*.

1.1 Observers and physical measurements

A laboratory is mathematically modeled by a family of non-intersecting time-like curves having u as tangent vector field and denoted by $\mathcal{C}_u$; this family is also termed the *congruence*. Each curve of the congruence represents the history of a point in the laboratory. We choose the parameter τ on the curves of $\mathcal{C}_u$ so as to make the tangent vector field u unitary; this choice is always possible for non-null curves. Let Σ be a space-like three-dimensional section of C_u spanned by the curves which cross a selected curve γ_* of the congruence orthogonally. The concepts of unitarity and orthogonality are relative to the assumed background metric. The curve γ_* will be termed the *fiducial* curve of the congruence and referred to as the *world line* of the observer. Let the point of intersection of Σ with γ_* be $\gamma_*(\tau)$; as τ varies continuously over γ_*, the section Σ spans a four-dimensional volume which represents the space-time history of the observer's laboratory. Whenever we limit the extension of Σ to a range much smaller than the average radius of its induced curvature, we can identify $\mathcal{C}_u$ with the single curve γ_* and Σ with the point $\gamma_*(\tau)$. Any time-like curve γ with tangent vector u can then be identified as the world line of an *observer*, which will be referred to as "the observer u." If the parameter τ on γ is such as to make the tangent vector unitary, then its physical meaning is that of the *proper time* of the observer u, i.e. the time read on his clock in units of the speed of light in vacuum.

This concept of observer, however, needs to be specialized further, defining a *reference frame* adapted to him. A reference frame is defined by a clock which marks the time as a parameter on γ, as already noted, and by a *spatial* frame made of three space-like directions identified at each point on γ by space-like curves stemming orthogonally from it. While the time direction is uniquely fixed by the vector field u, the spatial directions are defined up to spatial rotations, i.e. transformations which do not change u; obviously there are infinitely many such spatial perspectives.

The result of a physical measurement is mathematically described by a scalar, a quantity which is invariant under general coordinate transformations. A scalar quantity, however, is not necessarily a physical measurement. The latter, in fact, needs to be defined with respect to an observer and in particular to one of the infinitely many spatial frames adapted to him. The aim of the relativistic theory of measurement is to enable one to devise, out of the tensorial representation of a physical system and with respect to a given frame, those scalars which describe specific properties of the system.

The measurements are in general observer-dependent so, as stated, a criterion should also be given for comparing measurements made by different observers. A basic role in this procedure of comparison is played by the Lorentz group of transformations. A measurement which is observer-independent is termed *Lorentz invariant*. Lorentz invariant measurements are of key importance in physics.

1.2 Interpretation of physical measurements

The description of a physical system depends both on the observer and on the chosen frame of reference. In most cases the result of a measurement is affected by contributions from the background curvature and from the peculiarity of the reference frame. As long as it is not possible to discriminate among them, a measurement remains plagued by an intrinsic ambiguity. We shall present a few examples where this situation arises and discuss possible ways out. The most important among the observer-dependent measurements is that of time intervals. Basic to Einstein's theory of relativity is the relativity of time. Hence we shall illustrate this concept first, dealing with inertial frames for the sake of clarity.

1.3 Clock synchronization and relativity of time

The theory of special relativity, formally issued in 1905 (Einstein, 1905), presupposes that inertial observers are fully equivalent in describing physical laws. This requirement, known as the *principle of relativity*, implies that one has to abandon the concepts of absolute space and absolute time. This step is essential in order to envisage a model of reality which is consistent with observations and in particular with the behavior of light. As is well known, the speed of light c, whose value in vacuum is $2.997\,924\,58 \times 10^5$ km s^{-1}, is independent of the observer who measures it, and therefore is an absolute quantity.

Since time plays the role of a coordinate with the same prerogatives as the spatial ones, one needs a criterion for assigning a value of that coordinate, let us say t, to each space-time point. The criterion of time labeling, also termed *clock synchronization*, should be the same in all frames if we want the principle of relativity to make sense, and this is assured by the universality of the velocity of light. In fact, one uses a light ray stemming from a fiducial point with spatial coordinates x_0, for example, and time coordinate equal to zero, then assigns to each point of spatial coordinates $x_0 + \Delta x$ crossed by the light ray the time $t = \Delta x/c$. In this way, assuming the connectivity of space-time, we can label each of its points with a value of t. Clearly one must be able to fix for each of them the spatial separation Δx from the given fiducial point, but that is a non-trivial procedure which will be discussed later in the book.

The relativity of time is usually stated by saying that if an observer u compares the time t read on his own clock with that read on the clock of an observer u' moving uniformly with respect to u and instantaneously coincident with it, then u finds that t' differs from t by some factor K, as $t' = Kt$.[1] On the other hand, if the comparison is made by the observer u', because of the equivalence of the inertial observers he will find that the time t marked by the clock of u differs from the time t' read on his own clock when they instantaneously coincide, by the *same* factor, as $t = Kt'$. The factor K, which we denote as the *relativity factor*, is at this stage unknown except for the obvious facts that it should be positive, it should depend only on the magnitude of the relative velocity for consistency with the principle of relativity, and finally that it should reduce to one when the relative velocity is equal to zero. Our aim is to find the factor K and explain why it differs in general from one. A similar analysis can be found in Bondi's K-calculus (Bondi, 1980; see also de Felice, 2006). In what follows we shall not require knowledge of the Lorentz transformations nor of any concept of relativity.

Let us consider an inertial frame S with coordinates (x, y, z) and time t. The time axes in S form a congruence of curves each representing the history of a static observer at the corresponding spatial point. Denote by u the fiducial observer of this family, located at the spatial origin of S. At each point of S there exists a clock which marks the time t of that particular event and which would be read by the static observer spatially fixed at that point. All static observers

[1] The choice of a linear relation is justified a posteriori since it leads to the correct theory of relativity.

in S are equivalent to each other since we require that their time runs with the same rate. A clock which is attached to each point of S will be termed an S-clock.

Let S' be another inertial frame with spatial coordinates (x', y', z') and time t'. We require that S' moves uniformly along the x-axis of S with velocity ν. The x-axis of S will be considered spatially coincident with the x'-axis of S', with the further requirement that the origins of x and x' coincide at the time $t = t' = 0$. In this case the relative motion is that of a recession. In the frame S' the totality of time axes forms a congruence of curves each representing the history of a static observer. At each point of S' there is a clock, termed an S'-clock, which marks the time of that particular event and is read by the static observer fixed at the corresponding spatial position. The S'-clocks mark the time t' with the same rate; hence the static observers in S' are equivalent to each other. Finally we denote by u' the fiducial observer of the above congruence of time axes, fixed at the spatial origin of S'.

Let the systems S and S' be represented by the 2-planes (ct, x) and (ct', x') respectively;[2] we then assume that from the spatial origin of S and at time t_u, a light signal is emitted along the x-axis and towards the observer u'. The light signal reaches the observer u' at the time marked by the local S-clock, given by

$$t_{u'} = \frac{t_u}{1 - \nu/c}. \tag{1.1}$$

At this event, the observer u' can read two clocks which are momentarily coincident, namely the S-clock which marks the time $t_{u'}$ as in (1.1) and his own clock which marks a time $t'_{u'}$. In general the time beating on a given clock is driven by a sequence of events; in our case the time read on the clock of the observer u, at the spatial origin of S, follows the emission of the light signals. If these are emitted with continuity[3] then the time marked by the clock of the observer u will be a continuous function on S which we still denote by t_u. The time read on the S-clocks which are *crossed* by the observer u' along his path marks the instants of recording by u' of the light signals emitted by u. The events of reception by u', however, do not belong to the history of one observer only, but each of them, having a different spatial position in S, belongs to the history of the static observer located at the corresponding spatial point.

Let us now consider the same process as seen in the frame S'. The space-time of S' is carpeted by S'-clocks each marking the time t' read by the static observers fixed at each spatial point of S'. The observer u', at the spatial origin of S', receives at time $t'_{u'}$ the light signal emitted by the observer u who is seen receding along the negative direction of the x'-axis. The emission of the light signals by

[2] This is possible without loss of generality because of the homogeneity and isotropy of space.
[3] By continuity here we mean that the time interval between any two events of emission (or of reception) goes to zero.

the observer u occurs at times t'_u read on the S'-clocks *crossed* by u along his path and given by

$$t'_u = \frac{t'_{u'}}{1 + \nu/c}. \tag{1.2}$$

The time on the clock of the observer u', denoted by $t'_{u'}$, runs continuously with the recording of the light signals emitted by u. Meanwhile the observer u can read two clocks, momentarily coincident, namely his own clock which marks a time t_u and the S'-clock which is crossed by u during his motion which marks a time t'_u. Also in this case we have to remember that t'_u is *not* the time read on the clock of one single observer but is the time read at each instant on an S'-clock belonging to the static observer fixed at the corresponding spatial position in S'.

To summarize, the time read on the S-clocks set along the path of u' in S is $t_{u'}$ while the time marked by the clock carried by u' is $t'_{u'}$. Analogously the time read on the S'-clocks set along the path of u in S' is t'_u while the time read by u on his own clock is given by t_u. Our aim is to find the relation between $t'_{u'}$ and $t_{u'}$ in the frame S and that between t_u and t'_u in the frame S'. In both cases we are comparing times read on clocks which are in relative motion but instantaneously coincident.

The observers u and u' located at the spatial origins of S and S' respectively cannot *read* each other's clocks because they will be far apart after the initial time $t = t' = 0$ when they are assumed to coincide. In order to find the relation between t_u and $t'_{u'}$ one has to go through the intermediate steps where

(i) the observer u at the spatial origin of S correlates the time t_u, read on his own clock at the emission of the light signals, to the time $t_{u'}$, marked by the S-clocks when they are reached by the light signals and simultaneously crossed by the observer u' along his path in S;
(ii) the observer u' at the spatial origin of S' correlates the time $t'_{u'}$, read on his own clock, to the time t'_u marked by the S'-clocks when a light signal was emitted and simultaneously crossed by u along his path in S'.

The two points of view are not symmetric; in fact, the light signals are emitted by u and received by u' in both cases. These *intermediate steps* allow us to establish the relativity of time.

The principle of relativity ensures the complete equivalence of the inertial observers in the sense that they will always draw the same conclusions from an equal set of observations. In our case, comparing the points of view of the two observers, we deduce that the ratio between the time $t_{u'}$ that u' reads on each S-clock when he crosses it, and the time $t'_{u'}$ that he reads on his own clock at the same instant, is the same as the ratio between the time t'_u that u reads on each S'-clock which he crosses during his motion in S', and the time t_u that he reads on his own clock at the same instant, namely:

$$\frac{t_{u'}}{t'_{u'}} = \frac{t'_u}{t_u}. \tag{1.3}$$

Taking into account (1.1), relation (1.3) becomes

$$t'_{u'} = t_{u'}\frac{t_u}{t'_u} = \frac{t^2_{u'}}{t'_u}\left(1 - \frac{\nu}{c}\right). \tag{1.4}$$

Then, from (1.2),

$$t'_{u'} = \frac{t^2_{u'}}{t'_{u'}}\left(1 - \frac{\nu^2}{c^2}\right). \tag{1.5}$$

Along the path of u' in S, we have

$$t'_{u'} = \sqrt{1 - \frac{\nu^2}{c^2}}\, t_{u'}. \tag{1.6}$$

Similarly, from (1.3) and (1.2) we have

$$t_u = t'_{u'}\frac{t'_u}{t_{u'}} = \frac{t'^2_u}{t_{u'}}\left(1 + \frac{\nu}{c}\right). \tag{1.7}$$

Hence, from (1.1),

$$t_u = \frac{t'^2_u}{t_u}\left(1 - \frac{\nu^2}{c^2}\right). \tag{1.8}$$

Along the path of u in S' we finally have

$$t_u = \sqrt{1 - \frac{\nu^2}{c^2}}\, t'_u. \tag{1.9}$$

Thus the factor K turns out to be equal to $\sqrt{1 - (\nu/c)^2}$.

The above considerations have been made under the assumption that the observers u and u' are receding from each other. However the above result should still hold if the observers u and u' are approaching instead. We shall prove that this is actually the case.

Indeed the time rates of their clocks depend on the *sense* of the relative motion. In fact, if the two observers move away from each other the light signals emitted by one of them will be seen by the other with a delay, hence at a slower rate, because each signal has to cover a longer path than the previous one. If the observers instead approach each other then the signal emitted by one will be seen by the other with an anticipation due to the relative approaching motion, and so at a faster rate. This is what actually occurs to the time rates of the clocks carried by the observers u and u', namely t_u and $t'_{u'}$. In fact, from (1.6) and (1.1) we deduce, from the point of view of the observer u, that

$$t'_{u'} = \sqrt{\frac{1+\nu/c}{1-\nu/c}}\, t_u. \tag{1.10}$$

Hence, if $\nu > 0$ (u' recedes from u) then $t'_{u'} > t_u$, i.e. u judges the clock of u' to be ticking at a slower rate with respect to his own; if $\nu < 0$ (u' approaching u) then $t'_{u'} < t_u$, that is u now judges the clock of u' to be ticking at a faster rate with respect to his own. Despite this, the difference marked by the clocks of the two frames when they are instantaneously coincident must be independent of the sense of the relative motion. This will be shown in what follows.

Let us consider two frames S and S' approaching each other with velocity ν along the respective coordinate axes x and x'. Let u be the observer at rest at the spatial origin of S and u' the one at rest at the spatial origin of S'. From the point of view of S, the observer u' approaches u along a straight line of equation:

$$x = -\nu t + x_0 \tag{1.11}$$

where x_0 is the spatial position of u' at the initial time $t = 0$. The observer u emits light signals along the x-axis at times t_u. These signals move towards the observer u' and meet him at the events of observation at times $t_{u'}$ read by u' on the S-clocks that he crosses along his path. The equation of motion of the light signals will be in general

$$x = c(t - t_u) \tag{1.12}$$

and so the instant of observation by u' is given by the intersection of the line (1.12), which describes the motion of the light ray, and the line (1.11) which describes that of the observer u', namely

$$c\,(t_{u'} - t_u) = -\nu t_{u'} + x_0 \tag{1.13}$$

which leads to

$$t_{u'} = \frac{t_u + x_0/c}{1+\nu/c}. \tag{1.14}$$

Let us stress what was said before: while times t_u are read on the clock of u at rest in the spatial origin of S, the instants $t_{u'}$ are marked by the S-clocks spatially coincident with the position of the observer u' when he detects the light signals.

Let us now illustrate how the same process is seen in the frame S'. In this case the observer u approaches u' along the axis x' with relative velocity ν and therefore along a straight line of equation

$$x' = \nu t' - x'_0. \tag{1.15}$$

The position of u at the initial time $t' = 0$ is given by some value of the coordinate x' which we set equal to $-x'_0$, with x'_0 positive. This value is related to x_0, which appears in (1.11), by an explicit relation that we here ignore.[4] The observer u sends light signals at times t'_u. These reach the observer u' set in the spatial origin of S' at the instants $t'_{u'}$ read on his own clock. The motion of these signals is described by a straight line whose equation is given in general by

$$x' = c\,(t' - t'_{u'}). \tag{1.16}$$

The time of emission by the observer u is fixed by the intersection of the line (1.16) which describes the motion of the light signal with the line (1.15) which describes the motion of u, namely

$$t'_u = \frac{t'_{u'} - x'_0/c}{1 - \nu/c}. \tag{1.17}$$

Let us recall again here that while $t'_{u'}$ is the time read by u' on his own clock set stably at the spatial origin of S', the time t'_u is marked by the S'-clocks which are instantaneously coincident with the moving observer u. Here we exploit the equivalence between inertial frames regarding the reading of the clocks which lead to (1.3). After some elementary mathematical steps we obtain

$$(t'_{u'})^2 = \left(1 - \frac{\nu^2}{c^2}\right)(t_{u'})^2 - \left(1 - \frac{\nu}{c}\right)\frac{x_0}{c}t_{u'} + \frac{x'_0}{c}t'_{u'}. \tag{1.18}$$

This relation, deduced in the case of approaching observers, does not coincide with the analogous relation (1.6) deduced in the case of observers receding from each other. Although we do not know what the relation between x_0 and x'_0 is, we can prove the symmetry between this case and the previously discussed one.

Let us consider the extension of the light trajectories stemming from u to u' in the frames S and S', until they intersect the world line of static observers located at x_0 and x'_0 respectively. Let us denote these observers as u_0 and u'_0. In the frame S, the light signals intercept the observer u_0 at times, read on the clock of u_0, given by

$$t_{u_0} = t_u + x_0/c. \tag{1.19}$$

Then Eq. (1.14) can also be written as

$$t_{u'} = \frac{t_{u_0}}{1 + \nu/c}. \tag{1.20}$$

The time marked by the clock of u_0 at the arrival of the light signals runs with the same rate as that of the time marked by the clock of u at the emission of the

[4] The quantities x_0 and x'_0 are related by a Lorentz transformation but here we cannot introduce it because the latter presupposes that one knows the relation between the time rates of the spatially coincident clocks of the frames S and S', which instead we want to deduce.

same signals, since u and u_0 have zero relative velocity and therefore are to be considered as the *same* observer located at different spatial positions. Then we can still denote t_{u_0} as t_u and write relation (1.20) as

$$t_{u'} = \frac{t_u}{1 + \nu/c}. \tag{1.21}$$

A similar argument can be repeated in the frame S'. The time read on the clock of u'_0, at the intersections of the light rays with the history of the observer u'_0, is equal to

$$t'_{u'_0} = t'_{u'} - x'_0/c. \tag{1.22}$$

Relation (1.17) can be written as

$$t'_u = \frac{t'_{u'_0}}{1 - \nu/c}. \tag{1.23}$$

The time read on the clock of u'_0 runs at the same rate as that of u' since u'_0 and u' are to be considered as the same observer but located at different spatial positions. From this it follows that (1.23) can also be written as

$$t'_u = \frac{t'_{u'}}{1 - \nu/c}. \tag{1.24}$$

We clearly see that the relative motion of approach of u to u' is equivalent to a relative motion of recession between the observers u' and u_0 in S and between u and u'_0 in S'. The result of this comparison is the same as that shown in the relations (1.6) and (1.9), which are then independent of the sense of the relative motion, as expected. Moreover, this conclusion implies for consistency that, setting in (1.18)

$$t'_{u'} = \sqrt{1 - \frac{\nu^2}{c^2}}\, t_{u'}, \tag{1.25}$$

it follows that

$$-\left(1 - \frac{\nu}{c}\right) \frac{x_0}{c} t_{u'} + \frac{x'_0}{c} t'_{u'} = 0. \tag{1.26}$$

From this and (1.25) we further deduce that

$$x'_0 = \frac{x_0}{\sqrt{1 - \nu^2/c^2}} (1 - \nu/c). \tag{1.27}$$

One should notice here that (1.26) must be solved with respect to x'_0 and *not* with respect x_0 because the corresponding relation (1.25) between times, which implies (1.26), is relative to the situation where the observer u is the one who observes the moving frame S'; hence we have to express all quantities of S' in terms of the coordinates of S. Equations (1.6) and (1.9) are the starting point for the arguments which lead to the Lorentz transformations. From the latter, one deduces a posteriori that Eq. (1.27) is just the Lorentz transform of the spatial coordinate of the point of S with coordinates $(x_0,\, t_u = x_0/c)$; the corresponding point of S' will have coordinates $(x'_0,\, t'_{u'} = \sqrt{1-\nu^2/c^2}\, t_{u'})$.

The above analysis shows the important fact that the relativity of time is the result of the conspiracy of three basic facts, namely the finite velocity of light, the equivalence of the inertial observers, and the uniqueness of the clock synchronization procedure.

2

The theory of relativity: a mathematical overview

In the theory of relativity space and time loose their individuality and become indistinguishable in a continuous network termed *space-time*. The latter provides the unique environment where all phenomena occur and all observers and observables live undisclosed until they are forced to be distinguished according to their role. Unlike other interactions, gravity is not generated by a field of force but is just the manifestation of a varied background geometry. A variation of the background geometry may be induced by a choice of coordinates or by the presence of matter and energy distributions. In the former case the geometry variations give rise to inertial forces which act in a way similar but not fully equivalent to gravity; in the latter case they generate gravity, whose effects however are never completely disentangled from those generated by inertial forces.

2.1 The space-time

A space-time is described by a four-dimensional differentiable manifold M endowed with a pseudo-Riemannian metric g. Any open set $U \in M$ is homeomorphic to $\Re^4$ meaning that it can be described in terms of local coordinates x^α, for example, with $\alpha = 0, 1, 2, 3$. These coordinates induce a coordinate basis $\{\partial/\partial x^\alpha \equiv \partial_\alpha\}$ for the tangent space TM over U with dual $\{dx^\alpha\}$. It is often convenient to work with non-coordinate bases $\{e_\alpha\}$ with dual $\{\omega^\alpha\}$ satisfying the duality condition

$$\omega^\alpha(e_\beta) = \delta^\alpha{}_\beta. \tag{2.1}$$

If one expresses these vector fields in terms of coordinate components they are given by

$$e_\alpha = e^\beta{}_\alpha\, \partial_\beta, \qquad \omega^\alpha = \omega^\alpha{}_\beta\, dx^\beta, \tag{2.2}$$

where identical indices set diagonal to one another indicate that sums are to be taken only over the range of identical values. In (2.2) the matrices $\{e^{\beta}{}_{\alpha}\}$ and $\{\omega^{\alpha}{}_{\beta}\}$ are inverse to each other, i.e.

$$e^{\alpha}{}_{\beta}\,\omega^{\beta}{}_{\mu} = \delta^{\alpha}{}_{\mu}, \quad e^{\alpha}{}_{\beta}\,\omega^{\mu}{}_{\alpha} = \delta^{\mu}{}_{\beta}. \tag{2.3}$$

From the above relations it follows

$$\partial_{\alpha} = \omega^{\beta}{}_{\alpha}\,e_{\beta}, \qquad dx^{\alpha} = e^{\alpha}{}_{\beta}\,\omega^{\beta}. \tag{2.4}$$

A field of bases $\{e_\alpha\}$ is said to form a *frame*. The structure functions of $\{e_\alpha\}$ are the Lie brackets of the basis vectors

$$[e_\alpha, e_\beta] = C^{\gamma}{}_{\alpha\beta}\,e_{\gamma}, \tag{2.5}$$

and are defined from (2.2) as

$$C^{\gamma}{}_{\alpha\beta} = -2e_{[\alpha}\left(\omega^{\gamma}{}_{|\sigma|}\right)e^{\sigma}{}_{\beta]} \tag{2.6}$$

where $e_\gamma(\,\cdot\,)$ denotes a frame derivative. Here square brackets mean antisymmetrization (to be defined shortly) of the enclosed indices, and indices between vertical bars are ignored by this operation. A general tensor field can be expressed in terms of frame components as

$$S = S^{\alpha\cdots}{}_{\beta\ldots}e_{\alpha}\otimes\cdots\otimes\omega^{\beta}\otimes\cdots\,, \qquad S^{\alpha\cdots}{}_{\beta\ldots} = S(\omega^{\alpha},\ldots,e_{\beta},\ldots). \tag{2.7}$$

It is convenient to adopt the notation

$$\omega^{\alpha_1\ldots\alpha_p} = p!\omega^{[\alpha_1}\otimes\cdots\otimes\omega^{\alpha_p]}. \tag{2.8}$$

Hence a p-form K – a totally antisymmetric p-times covariant tensor field, also referred to as a $\binom{0}{p}$-tensor – can be written as

$$K = \frac{1}{p!}K_{\alpha_1\ldots\alpha_p}\omega^{\alpha_1\ldots\alpha_p}. \tag{2.9}$$

A similar notation $e_{\alpha_1\ldots\alpha_p}$ can be used to express p-vector fields, namely the totally antisymmetric p-times contravariant $\binom{p}{0}$-tensor fields.

A tensor field whose components are antisymmetric in a subset of p covariant indices is termed a *tensor-valued p-form*

$$\begin{aligned} S &= S^{\alpha\cdots}{}_{\beta\ldots[\gamma_1\ldots\gamma_p]}e_{\alpha}\otimes\cdots\otimes\omega^{\beta}\otimes\cdots\otimes\omega^{\gamma_1}\otimes\cdots\otimes\omega^{\gamma_p} \\ &= \frac{1}{p!}S^{\alpha\cdots}{}_{\beta\ldots\gamma_1\ldots\gamma_p}e_{\alpha}\otimes\cdots\otimes\omega^{\beta}\otimes\cdots\otimes\omega^{\gamma_1\ldots\gamma_p}. \end{aligned} \tag{2.10}$$

Right and left contractions

Tensor products are defined by a suitable contraction of the tensorial indices. Contraction of one index of each tensor is represented by the symbol $\llcorner$ (right contraction) or $\lrcorner$ (left contraction). If S and T are two arbitrary tensor fields, the left contraction $S \lrcorner T$ denotes a contraction between the rightmost contravariant index of S and the leftmost covariant index of T (i.e. $S^{\cdots\alpha}_{\cdots}T^{\cdots}_{\alpha\ldots}$), and the right contraction $S \llcorner T$ denotes a contraction between the rightmost covariant index of S and the leftmost contravariant index of T (i.e. $S^{\cdots}_{\ldots\alpha}T^{\alpha\cdots}_{\cdots}$), assuming in each case that such indices exist. If B is a $\binom{1}{1}$-tensor field, namely

$$B = B^{\alpha}{}_{\beta}\, e_{\alpha} \otimes \omega^{\beta}, \tag{2.11}$$

and it acts on a vector field X by right contraction, we have

$$B \llcorner X = [B \llcorner X]^{\alpha} e_{\alpha} = B^{\alpha}{}_{\beta} X^{\beta} e_{\alpha}. \tag{2.12}$$

Similarly, if A and B are two $\binom{1}{1}$-tensor fields, their right contraction is given by

$$A \llcorner B = [A \llcorner B]^{\alpha}{}_{\beta}\, e_{\alpha} \otimes \omega^{\beta} = [A^{\alpha}{}_{\gamma} B^{\gamma}{}_{\beta}]\, e_{\alpha} \otimes \omega^{\beta}\,. \tag{2.13}$$

The identity transformation is represented by the unit tensor or Kronecker delta tensor δ:

$$\delta = \delta^{\alpha}{}_{\beta}\, e_{\alpha} \otimes \omega^{\beta}. \tag{2.14}$$

The trace of a $\binom{1}{1}$-tensor is defined as

$$\mathrm{Tr}\, A = A^{\alpha}{}_{\alpha}. \tag{2.15}$$

Any $\binom{1}{1}$-tensor can be decomposed into a pure-trace and a trace-free (TF) part

$$A = A^{(\mathrm{TF})} + \frac{1}{4}[\mathrm{Tr}\, A]\delta. \tag{2.16}$$

Contraction over p indices of two general tensors is symbolically described as (right p-contraction)

$$S \overset{p}{\llcorner} T \equiv S^{\alpha\cdots}{}_{\beta_1\ldots\beta_p} T^{\beta_1\ldots\beta_p\cdots} \tag{2.17}$$

or (left p-contraction)

$$S \overset{p}{\lrcorner} T \equiv S^{\alpha\ldots\,\beta_1\ldots\beta_p} T_{\beta_1\ldots\beta_p\ldots}. \tag{2.18}$$

Change of frame

Given a non-degenerate $\binom{1}{1}$-tensor field A, i.e. such that the determinant $\det(A^{\alpha}{}_{\beta})$ is everywhere non-vanishing, one can prove that the elements of the inverse matrix are the components of a tensor A^{-1} which is termed the *inverse tensor* of A:

$$A^{-1} \llcorner A = \delta, \qquad \left(A^{-1}\right)^{\alpha}{}_{\gamma}\, A^{\gamma}{}_{\beta} = \delta^{\alpha}{}_{\beta}. \tag{2.19}$$

Non-degenerate $\binom{1}{1}$-tensor fields induce frame transformations as

$$\bar{e}_\alpha = A^{-1} \llcorner e_\alpha = \left(A^{-1}\right)^\beta{}_\alpha\, e_\beta, \qquad \bar{\omega}^\alpha = \omega^\alpha \llcorner A = A^\alpha{}_\beta\, \omega^\beta. \tag{2.20}$$

The space-time metric

Let $(g_{\alpha\beta})$ be a symmetric non-degenerate matrix of signature $+2$ and denote by $(g^{\alpha\beta})$ its inverse. Then

$$\mathrm{g} = g_{\alpha\beta}\, \omega^\alpha \otimes \omega^\beta, \qquad \mathrm{g}^{-1} = g^{\alpha\beta} e_\alpha \otimes e_\beta \tag{2.21}$$

locally define a pseudo-Riemannian metric tensor (g) and its inverse (g^{-1}) satisfying

$$\mathrm{g}^{-1} \lrcorner \mathrm{g} = \delta, \qquad g^{\alpha\gamma} g_{\gamma\beta} = \delta^\alpha{}_\beta. \tag{2.22}$$

Using the notation $g = |\det(g_{\alpha\beta})|$ one finds $\det(g_{\alpha\beta}) = -g$ and

$$d \ln g = g^{\alpha\beta} dg_{\alpha\beta} = \mathrm{Tr}\,[\mathrm{g}^{-1} \lrcorner d\mathrm{g}]. \tag{2.23}$$

Let us choose an orientation on M, namely an everywhere non-zero 4-form $\tilde{O}$; a frame $\{e_\alpha\}$ is termed *oriented* if $\tilde{O}(e_0, e_1, e_2, e_3) > 0$.

Since the metric is non-degenerate, it determines an isomorphism between the tangent and cotangent spaces at each point of the manifold which in index-notation corresponds to "raising" and "lowering" indices. For a vector field X and a 1-form θ one has

$$\begin{aligned} X^\flat &= \mathrm{g} \llcorner X, & X_\alpha &= g_{\alpha\beta} X^\beta, \\ \theta^\sharp &= \mathrm{g}^{-1} \lrcorner \theta, & \theta^\alpha &= g^{\alpha\beta}\theta_\beta, \end{aligned} \tag{2.24}$$

where, as is customary, we use the sharp ($\sharp$) and flat ($\flat$) notation for an arbitrary tensor to mean the tensor obtained by raising or lowering, respectively, *all* of the indices which are not already of the appropriate type.

Unit volume 4-form

The Levi-Civita permutation symbols $\epsilon_{\alpha_1 \ldots \alpha_4}$ and $\epsilon^{\alpha_1 \cdots \alpha_4}$ are totally antisymmetric with

$$\epsilon_{0123} = 1 = \epsilon^{0123}, \tag{2.25}$$

so that they vanish unless $\alpha_1 \ldots \alpha_4$ is a permutation of $0 \ldots 3$, in which case their value is the sign of the permutation. The components of the unit volume 4-form are related to the Levi-Civita symbols by

$$\eta_{\alpha_1\alpha_2\alpha_3\alpha_4} = g^{1/2} \epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4}, \qquad \eta^{\alpha_1\alpha_2\alpha_3\alpha_4} = -g^{-1/2} \epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4}. \tag{2.26}$$

Generalized Kronecker deltas

The following identities define the generalized Kronecker deltas and relate them to the Levi-Civita alternating symbols and to the unit volume 4-form η:

$$
\begin{aligned}
\delta^{\alpha_1\ldots\alpha_4}{}_{\beta_1\ldots\beta_4} &= \epsilon^{\alpha_1\ldots\alpha_4}\epsilon_{\beta_1\ldots\beta_4} = -\eta^{\alpha_1\ldots\alpha_4}\eta_{\beta_1\ldots\beta_4},\\
\delta^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_p} &= \frac{1}{(4-p)!}\delta^{\alpha_1\ldots\alpha_p\gamma_{p+1}\ldots\gamma_4}{}_{\beta_1\ldots\beta_p\gamma_{p+1}\ldots\gamma_4}\\
&= -\frac{1}{(4-p)!}\eta^{\alpha_1\ldots\alpha_p\gamma_{p+1}\ldots\gamma_4}\eta_{\beta_1\ldots\beta_p\gamma_{p+1}\ldots\gamma_4}.
\end{aligned}
\tag{2.27}
$$

A familiar alternative definition is given by

$$
\delta^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_p} = \det\begin{pmatrix} \delta^{\alpha_1}{}_{\beta_1} & \cdots & \delta^{\alpha_1}{}_{\beta_p}\\ \vdots & & \vdots\\ \delta^{\alpha_p}{}_{\beta_1} & \cdots & \delta^{\alpha_p}{}_{\beta_p}\end{pmatrix} = p!\,\delta^{\alpha_1}{}_{[\beta_1}\cdots\delta^{\alpha_p}{}_{\beta_p]}. \tag{2.28}
$$

Note that the second of the relations (2.27) holds in a four-dimensional Riemannian manifold with signature +2; in a three-dimensional Euclidean manifold one has instead

$$
\delta^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_p} = \frac{1}{(3-p)!}\eta^{\alpha_1\ldots\alpha_p\gamma_{p+1}\ldots\gamma_3}\eta_{\beta_1\ldots\beta_p\gamma_{p+1}\ldots\gamma_3}. \tag{2.29}
$$

Symmetrization and antisymmetrization

Given an object with only covariant or contravariant indices, or a subset of only covariant or contravariant indices from those of a mixed object, one can always project out the purely symmetric and purely antisymmetric parts. For example for a $\binom{0}{p}$-tensor field one has

$$
\begin{aligned}
[\mathrm{ALT}\,S]_{\alpha_1\ldots\alpha_p} &= S_{[\alpha_1\ldots\alpha_p]} = \frac{1}{p!}\delta^{\beta_1\ldots\beta_p}{}_{\alpha_1\ldots\alpha_p}S_{\beta_1\ldots\beta_p},\\
[\mathrm{SYM}\,S]_{\alpha_1\ldots\alpha_p} &= S_{(\alpha_1\ldots\alpha_p)} = \frac{1}{p!}\sum_\sigma S_{\sigma(\alpha_1\ldots\alpha_p)},
\end{aligned}
\tag{2.30}
$$

where the sum is over all the permutations σ of $\{\alpha_1\ldots\alpha_p\}$.

If T is a $\binom{0}{2}$-tensor, its symmetric and antisymmetric parts are given by

$$
T_{(\mu\nu)} = \frac{1}{2}\left(T_{\mu\nu} + T_{\nu\mu}\right) \tag{2.31}
$$

$$
T_{[\mu\nu]} = \frac{1}{2}\left(T_{\mu\nu} - T_{\nu\mu}\right). \tag{2.32}
$$

Exterior product

The exterior or wedge product of p 1-forms ω^{α_i} $(i = 1, \dots p)$ is defined as

$$\begin{aligned}\omega^{\alpha_1} \wedge \dots \wedge \omega^{\alpha_p} &\equiv p!\, \omega^{[\alpha_1} \otimes \dots \otimes \omega^{\alpha_p]} \\ &= \delta^{\alpha_1 \dots \alpha_p}{}_{\beta_1 \dots \beta_p} \omega^{\beta_1} \otimes \dots \otimes \omega^{\beta_p} \\ &= \omega^{\alpha_1 \dots \alpha_p},\end{aligned} \tag{2.33}$$

where notation (2.8) has been used.

From Eq. (2.33), the wedge product of a p-form S with a q-form T has the following expression

$$[S \wedge T]_{\alpha_1 \dots \alpha_{p+q}} = \frac{(p+q)!}{p!q!} S_{[\alpha_1 \dots \alpha_p} T_{\alpha_{p+1} \dots \alpha_{p+q}]}, \tag{2.34}$$

so that the relation

$$S \wedge T = (-1)^{pq}\, T \wedge S \tag{2.35}$$

holds identically.

Hodge duality operation

The Hodge duality operation associates with a p-form S the $(4-p)$-form ${}^*S$ with components

$$[{}^*S]_{\alpha_{p+1} \dots \alpha_4} = \frac{1}{p!}\, S_{\alpha_1 \dots \alpha_p}\, \eta^{\alpha_1 \dots \alpha_p}{}_{\alpha_{p+1} \dots \alpha_4}, \tag{2.36}$$

i.e. with η at the right-hand side of the p-form S and with a proper contraction of the first indices of η (*right duality convention*). From the above definition one finds that a p-form S satisfies the identity

$$^{**}S \equiv (-1)^{p-1} S. \tag{2.37}$$

The duality operation applied to the wedge product of a p-form S and a q-form T (with $q \geq p$) implies the following identity:

$${}^*(S \wedge {}^*T) = (-1)^{1+(4-q)(q-p)}\, \frac{1}{p!} S^\sharp \,\underset{\lrcorner}{p}\, T. \tag{2.38}$$

With $p = 0$ and $q = 1$ this reduces to (2.37), while when $p = q$ one has

$${}^*(S \wedge {}^*T) = (-1)\, \frac{1}{p!} S^\sharp \,\underset{\lrcorner}{p}\, T = {}^*\eta\, \frac{1}{p!} S^\sharp \,\underset{\lrcorner}{p}\, T, \tag{2.39}$$

since ${}^*\eta = (-1)$. Removing the duality operation from each side leads to

$$S \wedge {}^*T = \frac{1}{p!} (S^\sharp \,\underset{\lrcorner}{p}\, T)\, \eta. \tag{2.40}$$

2.2 Derivatives on a manifold

Differentiation on a manifold is essential to derive physical laws and define physical measurements. It stems from appropriate criteria for comparison between the algebraic structures at any two points of the manifold. Clearly differentiation allows one to define new quantities, as we shall see next.

Covariant derivative and connection

The covariant derivative $\nabla_{e_\gamma} S$ of an arbitrary $\binom{p}{q}$-tensor S along the e_γ frame direction is a $\binom{p}{q+1}$-tensor with components

$$\begin{aligned}[\nabla_{e_\gamma} S]^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_q} &\equiv \nabla_\gamma S^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_q} \\ &= e_\gamma\left(S^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_q}\right) + \Gamma^{\alpha_1}{}_{\delta\gamma}\, S^{\delta\ldots}{}_{\beta_1\ldots} + \cdots - \Gamma^{\delta}{}_{\beta_1\gamma}\, S^{\alpha_1\ldots}{}_{\delta\ldots} - \cdots.\end{aligned} \tag{2.41}$$

Here the coefficients $\Gamma^{\alpha}{}_{\gamma\delta}$ are the components of a linear connection on M defined as

$$\nabla_{e_\alpha} e_\beta = \Gamma^{\gamma}{}_{\beta\alpha}\, e_\gamma \quad \leftrightarrow \quad \nabla_{e_\alpha}\, \omega^\beta = -\Gamma^{\beta}{}_{\gamma\alpha}\, \omega^\gamma. \tag{2.42}$$

The covariant derivative satisfies the following rules. For any pair of vector fields X and Y and of real functions f and h, we have

$$\nabla_{fX+hY} = f\nabla_X + h\nabla_Y. \tag{2.43}$$

From this it follows that

$$\nabla_X S = X^\alpha \nabla_{e_\alpha} S \equiv X^\alpha \nabla_\alpha S. \tag{2.44}$$

A connection is said to be compatible with the *metric* if the latter is constant under covariant differentiation, that is

$$\begin{aligned}0 = \nabla_\gamma g_{\alpha\beta} &= e_\gamma(g_{\alpha\beta}) - g_{\delta\beta}\Gamma^{\delta}{}_{\alpha\gamma} - g_{\alpha\delta}\Gamma^{\delta}{}_{\beta\gamma} \\ &= e_\gamma(g_{\alpha\beta}) - \Gamma_{\beta\alpha\gamma} - \Gamma_{\alpha\beta\gamma},\end{aligned} \tag{2.45}$$

where we set

$$\Gamma_{\alpha\beta\gamma} \equiv g_{\alpha\sigma}\Gamma^{\sigma}{}_{\beta\gamma}. \tag{2.46}$$

Given two vector fields X and Y, consider the covariant derivative of Y in the direction of X as

$$\nabla_X Y = \left[X^\alpha e_\alpha(Y^\delta) + \Gamma^{\delta}{}_{\beta\alpha} X^\alpha Y^\beta\right] e_\delta. \tag{2.47}$$

From (2.47) and (2.5) we deduce the following relation

$$\nabla_X Y - \nabla_Y X = [X, Y] - [2\Gamma^{\delta}{}_{[\beta\alpha]} - C^{\delta}{}_{\alpha\beta}] X^\alpha Y^\beta e_\delta. \tag{2.48}$$

The quantity

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \tag{2.49}$$

is a $\binom{1}{3}$-tensor termed *torsion*. In the theory of relativity the torsion is assumed to be identically zero. Hence, X and Y being arbitrary, the following relation holds

$$\Gamma^{\delta}{}_{[\beta\alpha]} = \frac{1}{2} C^{\delta}{}_{\alpha\beta}. \tag{2.50}$$

From this, Eq. (2.45) can be inverted, yielding

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left[e_{\beta}(g_{\delta\gamma}) + e_{\gamma}(g_{\beta\delta}) - e_{\delta}(g_{\gamma\beta}) + C_{\delta\gamma\beta} + C_{\beta\delta\gamma} - C_{\gamma\beta\delta} \right], \tag{2.51}$$

where we set $C_{\alpha\beta\gamma} = g_{\alpha\sigma} C^{\sigma}{}_{\beta\gamma}$. In the event that the structure functions $C_{\alpha\beta\gamma}$ are all zero – in which case the bases e_{α} are termed *holonomous* – the components of the linear connection are known as *Christoffel symbols*.

Curvature

The curvature or Riemann tensor is defined as

$$R(e_{\gamma}, e_{\delta}) e_{\beta} = \left([\nabla_{e_{\gamma}}, \nabla_{e_{\delta}}] - \nabla_{[e_{\gamma}, e_{\delta}]} \right) e_{\beta} = R^{\alpha}{}_{\beta\gamma\delta}\, e_{\alpha}. \tag{2.52}$$

Hence the components in a given frame are

$$R^{\alpha}{}_{\beta\gamma\delta} = e_{\gamma} \left(\Gamma^{\alpha}{}_{\beta\delta} \right) - e_{\delta} \left(\Gamma^{\alpha}{}_{\beta\gamma} \right) - C^{\epsilon}{}_{\gamma\delta} \Gamma^{\alpha}{}_{\beta\epsilon} + \Gamma^{\alpha}{}_{\epsilon\gamma} \Gamma^{\epsilon}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\epsilon\delta} \Gamma^{\epsilon}{}_{\beta\gamma}. \tag{2.53}$$

This tensor is explicitly antisymmetric in its last pair of indices and so defines a $\binom{1}{1}$-tensor-valued 2-form Ω:

$$\Omega = e_{\alpha} \otimes \omega^{\beta} \otimes \Omega^{\alpha}{}_{\beta} = e_{\alpha} \otimes \omega^{\beta} \otimes \frac{1}{2} R^{\alpha}{}_{\beta\gamma\delta}\, \omega^{\gamma\delta}. \tag{2.54}$$

Covariant derivative and curvature

A remarkable property of the covariant derivative is that it does not commute with itself. In fact, given a vector X and using coordinate components ($C^{\gamma}{}_{\alpha\beta} = 0$) we have

$$\nabla_{[\alpha} \nabla_{\beta]} X^{\gamma} = \frac{1}{2} R^{\gamma}{}_{\sigma\alpha\beta} X^{\sigma}, \tag{2.55}$$

while given a 1-form ω we have

$$\nabla_{[\alpha} \nabla_{\beta]} \omega_{\gamma} = \frac{1}{2} \omega_{\sigma} R^{\sigma}{}_{\gamma\beta\alpha}. \tag{2.56}$$

Relations (2.55) and (2.56) can be generalized to arbitrary tensors; for a $\binom{1}{1}$-tensor $T_{\rho}{}^{\sigma}$ we have

$$\nabla_{[\alpha} \nabla_{\beta]} T_{\rho}{}^{\sigma} = \frac{1}{2} T_{\mu}{}^{\sigma} R^{\mu}{}_{\rho\beta\alpha} + \frac{1}{2} R^{\sigma}{}_{\mu\alpha\beta} T_{\rho}{}^{\mu}. \tag{2.57}$$

Applying the above relation to the metric tensor we have

$$\nabla_{[\alpha}\nabla_{\beta]}g_{\rho\sigma} = \frac{1}{2}g_{\mu\sigma}R^{\mu}{}_{\rho\beta\alpha} + \frac{1}{2}g_{\rho\mu}R^{\mu}{}_{\sigma\beta\alpha} \equiv 0. \tag{2.58}$$

Hence

$$R_{\sigma\rho\beta\alpha} + R_{\rho\sigma\beta\alpha} = 0, \tag{2.59}$$

implying that the totally covariant (or contravariant) Riemann tensor is antisymmetric with respect to the first pair of indices as well.

Ricci and Bianchi identities

The Ricci identities are given by

$$3R^{\alpha}{}_{[\beta\gamma\delta]} = R^{\alpha}{}_{\beta\gamma\delta} + R^{\alpha}{}_{\gamma\delta\beta} + R^{\alpha}{}_{\delta\beta\gamma} = 0, \tag{2.60}$$

whereas the Bianchi identities are given by

$$3\nabla_{[\epsilon}R_{\gamma\delta]\beta}{}^{\alpha} = \nabla_{\epsilon}R_{\gamma\delta\beta}{}^{\alpha} + \nabla_{\delta}R_{\epsilon\gamma\beta}{}^{\alpha} + \nabla_{\gamma}R_{\delta\epsilon\beta}{}^{\alpha} = 0. \tag{2.61}$$

Finally, if we make the algebraic sum of the Ricci identities (2.60) for each permutation of the indices of a totally covariant Riemann tensor,

$$R_{\alpha[\beta\gamma\delta]} + R_{\delta[\alpha\beta\gamma]} - R_{\gamma[\delta\alpha\beta]} - R_{\beta[\gamma\delta\alpha]} = 0\,, \tag{2.62}$$

it follows that

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}, \tag{2.63}$$

i.e. the Riemann tensor is symmetric under exchange of the two pairs of indices.

Ricci tensor and curvature scalar

From the symmetries of the curvature tensor its *first* contraction generates the Ricci curvature tensor

$$R_{\alpha\beta} = R^{\gamma}{}_{\alpha\gamma\beta}, \tag{2.64}$$

which is symmetric due to the Ricci identities. Its trace defines the curvature scalar

$$R \equiv R^{\alpha}{}_{\alpha} = R^{\gamma\alpha}{}_{\gamma\alpha}. \tag{2.65}$$

Weyl and Einstein tensors

The Weyl tensor is a suitable combination of the curvature tensor, the Ricci tensor, and the curvature scalar, as follows:

$$\begin{aligned} C^{\alpha\beta}{}_{\gamma\delta} &= R^{\alpha\beta}{}_{\gamma\delta} - 2R^{[\alpha}{}_{[\gamma}\delta^{\beta]}{}_{\delta]} + \frac{R}{6}\delta^{\alpha\beta}_{\gamma\delta} \\ &= R^{\alpha\beta}{}_{\gamma\delta} - 2\delta^{[\alpha}{}_{[\gamma}S^{\beta]}{}_{\delta]}, \end{aligned} \tag{2.66}$$

where

$$S^{\beta}{}_{\alpha} = R^{\beta}{}_{\alpha} - \frac{1}{6}R\delta^{\beta}{}_{\alpha}. \tag{2.67}$$

The Weyl tensor is trace-free, i.e. $C^{\alpha}{}_{\beta\alpha\delta} = 0$, and is invariant under conformal transformations of the metric.

The one-fold contraction of the Bianchi identities (using the trace-free property of the Weyl tensor (2.66)) reduces to

$$\begin{aligned} 0 &= \nabla_{[\epsilon}R_{\gamma\delta]}{}^{\alpha\beta}\delta^{\gamma}{}_{\alpha} \\ &= \nabla_{\alpha}R_{\delta\epsilon}{}^{\alpha\beta} + 2\nabla_{[\epsilon}R_{\delta]}{}^{\beta} \\ &= \nabla_{\alpha}C_{\delta\epsilon}{}^{\alpha\beta} + \nabla_{[\epsilon}\left(R^{\beta}{}_{\delta]} - \frac{R}{6}\delta^{\beta}{}_{\delta]}\right). \end{aligned} \tag{2.68}$$

The two-fold contraction of the Bianchi identities leads to

$$0 = \nabla_{[\epsilon}R^{\alpha\beta}{}_{\gamma\delta]}\delta^{\gamma}{}_{\alpha}\delta^{\delta}{}_{\beta}. \tag{2.69}$$

Defining the Einstein tensor as

$$G^{\delta}{}_{\epsilon} \equiv R^{\delta}{}_{\epsilon} - \frac{1}{2}\delta^{\delta}{}_{\epsilon}R, \tag{2.70}$$

relation (2.69) implies identically that

$$\nabla_{\delta}G^{\delta}{}_{\epsilon} = 0. \tag{2.71}$$

Cotton tensor

The Cotton tensor (Cotton, 1899; Eisenhart, 1997) is defined as

$$R^{\beta}{}_{\delta\epsilon} \equiv 2\nabla_{[\epsilon}\left(R^{\beta}{}_{\delta]} - \frac{R}{6}\delta^{\beta}{}_{\delta]}\right) = 2\nabla_{[\epsilon}S^{\beta}{}_{\delta]}. \tag{2.72}$$

Hence the divergence of the Weyl tensor can be written in a more compact form as

$$\nabla_{\alpha}C_{\delta\epsilon}{}^{\alpha\beta} = -\frac{1}{2}R^{\beta}{}_{\delta\epsilon}. \tag{2.73}$$

The property of the Einstein tensor of being divergence-free leads to the trace-free property of the Cotton tensor:

$$R^{\beta}{}_{\alpha\beta} = \nabla_\beta G^{\beta}{}_{\alpha} = 0. \tag{2.74}$$

The totally antisymmetric part of the fully covariant (or contravariant) Cotton tensor is zero due to the symmetry of the Ricci and metric tensors, that is

$$3R_{[\alpha\beta\gamma]} = R_{\alpha\beta\gamma} + R_{\beta\gamma\alpha} + R_{\gamma\alpha\beta} = 0. \tag{2.75}$$

Finally from (2.71) it follows that the divergence of the Cotton tensor is also zero:

$$\nabla_\beta R^{\beta}{}_{\delta\epsilon} = 0. \tag{2.76}$$

Absolute derivative along a world line

Given a curve γ parameterized by λ and denoting by $\dot{\gamma}$ the corresponding tangent vector field, the absolute derivative of any tensor field $S(\lambda)$ defined along the curve is

$$\frac{DS}{d\lambda} = \nabla_{\dot{\gamma}} S. \tag{2.77}$$

Given a local coordinate system $\{x^\alpha\}$, the curve is described by the functions $\gamma^\alpha(\lambda) = x^\alpha(\gamma(\lambda)) \equiv x^\alpha(\lambda)$, so that

$$[\dot{\gamma}(\lambda)]^\alpha = \frac{dx^\alpha}{d\lambda}(\lambda).$$

For a vector field X defined on γ, its absolute derivative takes the form

$$\frac{DX}{d\lambda} = \left(\frac{dX^\alpha(\lambda)}{d\lambda} + \Gamma^{\alpha}{}_{\gamma\beta}(\lambda)\dot{\gamma}^\beta(\lambda)X^\gamma(\lambda)\right) e_\alpha. \tag{2.78}$$

Parallel transport and geodesics

A tensor field S defined on a curve γ with parameter λ is said to be parallel transported along the curve if its absolute derivative along γ is zero:

$$\frac{DS}{d\lambda} = 0. \tag{2.79}$$

A curve γ whose tangent vector $\dot{\gamma}$ satisfies the equation

$$\frac{D\dot{\gamma}}{d\lambda}(\lambda) = f(\lambda)\dot{\gamma}(\lambda), \tag{2.80}$$

where $f(\lambda)$ is a function defined on γ, is termed *geodesic*. It is always possible to re-parameterize the curve with $\sigma(\lambda)$ such that Eq. (2.80) becomes

$$\frac{D\dot{\gamma}'}{d\sigma}(\sigma) = 0, \tag{2.81}$$

where

$$\dot{\gamma}'^{\alpha} = \frac{dx^\alpha}{d\sigma}.$$

In this case the parameter σ is said to be *affine* and it is defined up to linear transformations. From Eq. (2.81) we see that a characteristic feature of an affine geodesic is that the tangent vector is parallel transported along it.

Fermi-Walker derivative and transport

Let us introduce a new parameter s along a non-null curve γ such that the tangent vector of the curve $u^\alpha = dx^\alpha/ds$ is unitary, that is, $u \cdot u = \pm 1$. If the curve is time-like then $u \cdot u = -1$ and the parameter s is termed *proper time*, as already noted. The curvature vector (or acceleration) of the curve is defined as

$$a(u) = \frac{Du}{ds}. \tag{2.82}$$

Consider a vector field $X(s)$ on γ. We define the Fermi-Walker derivative of X along γ as the vector field $D_{(\mathrm{fw},u)}X/ds$ having components

$$\begin{aligned}\left(\frac{D_{(\mathrm{fw},u)}}{ds}X\right)^\alpha &\equiv \left(\frac{DX}{ds}\right)^\alpha \pm [a(u)^\alpha(u\cdot X) - u^\alpha(a(u)\cdot X)] \\ &= \left(\frac{DX}{ds}\right)^\alpha \pm [a(u)\wedge u]^\alpha{}_\gamma X^\gamma,\end{aligned} \tag{2.83}$$

where the signs $\pm$ should be chosen according to the causal character of the curve: $+$ for a time-like curve and $-$ for a space-like one.

The generalization to a tensor field $T \in \binom{1}{1}$, for example, is the following:

$$\begin{aligned}\left(\frac{D_{(\mathrm{fw},u)}}{ds}T\right)^\alpha{}_\beta \equiv \left(\frac{DT}{ds}\right)^\alpha{}_\beta &\pm ([a(u)\wedge u]^\alpha{}_\gamma T^\gamma \beta \\ &- [a(u)\wedge u]^\gamma{}_\beta T^\alpha{}_\gamma)\,.\end{aligned} \tag{2.84}$$

The tensor field T is said to be Fermi-Walker transported along the curve if its Fermi-Walker derivative is identically zero.

Lie derivative

The Lie derivative along a congruence $\mathcal{C}_X$ of curves with tangent vector field X is denoted $\pounds_X$ and defined as follows:

(i) For a scalar field f,
$$\pounds_X f = X(f). \tag{2.85}$$

(ii) For a vector field Y,
$$\pounds_X Y = [X, Y]. \tag{2.86}$$

(iii) For a 1-form ω,
$$[\pounds_X \omega]_\beta = X^\gamma e_\gamma(\omega_\beta) + \omega_\alpha e_\beta(X^\alpha). \tag{2.87}$$

(iv) For a general tensor $S^{\alpha\dots}{}_{\beta\dots}$,
$$\begin{aligned}[\pounds_X S]^{\alpha\dots}{}_{\beta\dots} = X^\gamma e_\gamma(S^{\alpha\dots}{}_{\beta\dots}) - S^{\mu\dots}{}_{\beta\dots} e_\mu(X^\alpha) + \dots \\ + S^{\alpha\dots}{}_{\mu\dots} e_\beta(X^\mu) + \dots .\end{aligned} \tag{2.88}$$

From (2.86), the Lie operator applied to the vectors of a frame leads to (2.5), that is
$$\pounds_{e_\alpha} e_\beta = [e_\alpha, e_\beta] = C^\gamma{}_{\alpha\beta}\, e_\gamma. \tag{2.89}$$

A general tensor field S is said to be Lie transported along a congruence $\mathcal{C}_X$ if its Lie derivative with respect to X is identically zero.

Exterior derivative

The exterior derivative is a differential operator d which associates a p-form to a $(p+1)$-form according to the following properties:

(i) d is additive, i.e. $d(S+T) = dS + dT$ for arbitrary forms S, T.
(ii) If f is a 0-form (i.e. a scalar function), df is the ordinary differential of the function f. If $\{x^\alpha\}$ denotes a local coordinate system then $df = \partial_\alpha f dx^\alpha$. In a non-coordinate frame $\{e_\alpha\}$ with dual $\{\omega^\alpha\}$ one has instead $df = e_\alpha(f)\omega^\alpha$.
(iii) If S is a p-form and T a q-form then the following relation holds:
$$d(S \wedge T) = dS \wedge T + (-1)^p S \wedge dT. \tag{2.90}$$
(iv) $d^2 S = d(dS) = 0$ for all forms S.

As a first application let us consider the exterior derivatives of the dual frame 1-forms ω^α. We have, recalling (2.6),
$$\begin{aligned} d\omega^\alpha &= d(\omega^\alpha{}_\beta\, dx^\beta) = \partial_\mu(\omega^\alpha{}_\beta) dx^\mu \wedge dx^\beta \\ &= \partial_\mu(\omega^\alpha{}_\beta) e^\mu{}_\sigma\, e^\beta{}_\rho\, \omega^\sigma \wedge \omega^\rho \\ &= e_{[\sigma}(\omega^\alpha{}_{|\beta|}) e^\beta{}_{\rho]}\, \omega^\sigma \wedge \omega^\rho \\ &= -\frac{1}{2} C^\alpha{}_{\sigma\rho}\, \omega^\sigma \wedge \omega^\rho. \end{aligned} \tag{2.91}$$

One can use this result to evaluate the exterior derivative of $\omega^{\alpha\beta} = \omega^\alpha \wedge \omega^\beta$; from (2.90) and (2.33) we have

$$\begin{aligned} d\omega^{\alpha\beta} &= d\omega^\alpha \wedge \omega^\beta - \omega^\alpha \wedge d\omega^\beta \\ &= -\frac{1}{2}\left[C^\alpha{}_{\mu\nu}\,\omega^\mu \wedge \omega^\nu \wedge \omega^\beta + (-1)C^\beta{}_{\mu\nu}\,\omega^\alpha \wedge \omega^\mu \wedge \omega^\nu\right] \\ &= -\frac{1}{2}\left[C^\alpha{}_{\mu\nu}\,\omega^{\mu\nu\beta} + (-1)C^\beta{}_{\mu\nu}\,\omega^{\alpha\mu\nu}\right] . \end{aligned} \tag{2.92}$$

From this it follows that the exterior derivative of a 2-form S,

$$S = \frac{1}{2}S_{\alpha\beta}\,\omega^\alpha \wedge \omega^\beta \equiv \frac{1}{2}S_{\alpha\beta}\,\omega^{\alpha\beta}, \tag{2.93}$$

is given by

$$\begin{aligned} dS &= \frac{1}{3!}[dS]_{\beta\mu\nu}\,\omega^{\beta\mu\nu} \\ &= d\left(\frac{1}{2}S_{\alpha\beta}\,\omega^\alpha \wedge \omega^\beta\right) \\ &= \frac{1}{2}e_\gamma(S_{\alpha\beta})\,\omega^{\gamma\alpha\beta} + \frac{1}{2}S_{\alpha\beta}\left(-\frac{1}{2}C^\alpha{}_{\mu\nu}\,\omega^{\mu\nu}\right) \wedge \omega^\beta \\ &\quad - \frac{1}{2}S_{\alpha\beta}\,\omega^\alpha \wedge \left(-\frac{1}{2}C^\beta{}_{\mu\nu}\,\omega^{\mu\nu}\right) \\ &= \frac{1}{2}e_\gamma(S_{\alpha\beta})\omega^{\gamma\alpha\beta} - \frac{1}{4}S_{\alpha\beta}C^\alpha{}_{\mu\nu}\,\omega^{\mu\nu\beta} \\ &\quad + \frac{1}{4}S_{\alpha\beta}C^\beta{}_{\mu\nu}\,\omega^{\alpha\mu\nu} \\ &= \frac{1}{2}e_{[\beta}(S_{\mu\nu]})\omega^{\beta\mu\nu} - \frac{1}{2}S_{\alpha\beta}C^\alpha{}_{\mu\nu}\,\omega^{\beta\mu\nu} \\ &= \frac{1}{2}\left(e_{[\beta}(S_{\mu\nu]}) - \frac{1}{2}S_{\alpha\beta}C^\alpha{}_{\mu\nu}\right)\omega^{\beta\mu\nu} ; \end{aligned} \tag{2.94}$$

hence

$$[dS]_{\beta\mu\nu} = 3\left(e_{[\beta}(S_{\mu\nu]}) - \frac{1}{2}S_{\alpha\beta}C^\alpha{}_{\mu\nu}\right) . \tag{2.95}$$

From the above analysis a general formula for the exterior derivative of the basis p-forms $\omega^{\alpha_1\ldots\alpha_p}$ follows as

$$\begin{aligned} d\omega^{\alpha_1\ldots\alpha_p} &= -\frac{1}{2}C^{\alpha_1}{}_{\rho\sigma}\,\omega^{\rho\sigma\alpha_2\ldots\alpha_p} + \cdots - \frac{1}{2}(-1)^{p-1}C^{\alpha_p}{}_{\rho\sigma}\,\omega^{\alpha_1\ldots\alpha_{p-1}\rho\sigma} \\ &= -\frac{1}{2}\sum_{i=1}^{p}(-1)^{i-1}C^{\alpha_i}{}_{\beta\gamma}\,\omega^{\alpha_1\ldots\alpha_{i-1}\beta\gamma\alpha_{i+1}\ldots\alpha_p}. \end{aligned} \tag{2.96}$$

As a consequence, the exterior derivative of a p-form S,

$$S = \frac{1}{p!}S_{\alpha_1\ldots\alpha_p}\omega^{\alpha_1\ldots\alpha_p}, \tag{2.97}$$

is a $(p+1)$-form defined as

$$dS = \frac{1}{(p+1)!}[dS]_{\alpha_1\ldots\alpha_{p+1}}\omega^{\alpha_1\ldots\alpha_{p+1}}, \tag{2.98}$$

where the components $[dS]_{\alpha_1\ldots\alpha_{p+1}}$ are given by

$$\begin{aligned}[dS]_{\alpha_1\ldots\alpha_{p+1}} &= (p+1)(e_{[\alpha_1}(S_{\alpha_2\ldots\alpha_{p+1}]}) \\ &\qquad - \frac{1}{2}p\, C^{\beta}{}_{[\alpha_1\alpha_2}S_{|\beta|\alpha_3\ldots\alpha_{i-1}\alpha_{i+1}\ldots\alpha_{p+1}]}).\end{aligned} \tag{2.99}$$

In a coordinate frame the second term of (2.99) vanishes. Furthermore, this result may be re-expressed in terms of a covariant derivative as

$$[dS]_{\alpha_1\ldots\alpha_{p+1}} = (p+1)\nabla_{[\alpha_1}S_{\alpha_2\ldots\alpha_{p+1}]}. \tag{2.100}$$

In fact, applying condition (2.90) to (2.98) we obtain

$$dS = \frac{1}{p!}dS_{\alpha_1\ldots\alpha_p}\wedge\omega^{\alpha_1\ldots\alpha_p} + \frac{1}{p!}S_{\alpha_1\ldots\alpha_p}d\omega^{\alpha_1\ldots\alpha_p}. \tag{2.101}$$

By combining this relation with (2.96) we finally obtain (2.100).

The divergence operator

The divergence operator δ of a p-form is defined as

$$\delta T = {}^*[d^*T]. \tag{2.102}$$

If $T = T_\alpha\omega^\alpha$ is a 1-form we have

$$[{}^*T]_{\alpha\beta\gamma} = \eta^{\delta}{}_{\alpha\beta\gamma}T_\delta \tag{2.103}$$

and

$$[d^*T]_{\mu\alpha\beta\gamma} = 4\nabla_{[\mu}(\eta^{\delta}{}_{\alpha\beta\gamma]}T_\delta), \tag{2.104}$$

so that

$$\begin{aligned}\delta T = {}^*[d^*T] &= \frac{1}{4!}\eta^{\mu\alpha\beta\gamma}[d^*T]_{\mu\alpha\beta\gamma} \\ &= \frac{1}{3!}\eta^{\mu\alpha\beta\gamma}\eta^{\delta}{}_{\alpha\beta\gamma}\nabla_\mu T_\delta \\ &= -\frac{1}{3!}\delta^{\mu\alpha\beta\gamma}_{\delta\alpha\beta\gamma}\nabla_\mu T^\delta \\ &= -\delta^{\mu}_{\delta}\nabla_\mu T^\delta = -\nabla_\delta T^\delta.\end{aligned} \tag{2.105}$$

If $T = \frac{1}{2}T_{\alpha\beta}\omega^\alpha\wedge\omega^\beta$ is a 2-form we have

$$[{}^*T]_{\mu\nu} = \frac{1}{2}\eta_{\mu\nu}{}^{\alpha\beta}T_{\alpha\beta} \tag{2.106}$$

and

$$[d^*T]_{\lambda\mu\nu} = \frac{3}{2}\nabla_{[\lambda}(\eta_{\mu\nu]}{}^{\alpha\beta}\, T_{\alpha\beta}), \tag{2.107}$$

so that

$$\begin{aligned}[\delta T]^\sigma = [{}^*[d^*T]]^\sigma &= \frac{1}{3}\eta^{\lambda\mu\nu\sigma}\nabla_\lambda\left(\frac{3}{2}\eta_{\mu\nu}{}^{\alpha\beta}\, T_{\alpha\beta}\right) \\ &= \frac{1}{4}2!\delta^{\sigma\lambda}_{\alpha\beta}\nabla_\lambda T^{\alpha\beta} = -\nabla_\beta T^{\beta\sigma}.\end{aligned} \tag{2.108}$$

Finally, if T is a p-form we have

$$[\delta T]_{\alpha_2\ldots\alpha_p} = -\nabla^\alpha T_{\alpha\alpha_2\ldots\alpha_p}. \tag{2.109}$$

We introduce also the notation

$$\operatorname{div} T = -\delta T \tag{2.110}$$

and extend this operation to any tensor field $T = T^{\alpha_1\ldots\alpha_n} e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_n}$ as

$$\operatorname{div} T = [\nabla_\alpha T^{\alpha\alpha_2\ldots\alpha_n}]\, e_{\alpha_2}\otimes\cdots\otimes e_{\alpha_n}. \tag{2.111}$$

De Rham Laplacian

The de Rham Laplacian operator for p-forms is defined by

$$\Delta_{(\mathrm{dR})} = \delta d + d\delta. \tag{2.112}$$

It differs from the Laplacian Δ, namely

$$[\Delta S]_{\alpha_1\ldots\alpha_p} = -\nabla_\alpha\nabla^\alpha S_{\alpha_1\ldots\alpha_p}, \tag{2.113}$$

by curvature terms

$$\begin{aligned}[\Delta_{(\mathrm{dR})}S]_{\alpha_1\ldots\alpha_p} = [\Delta S]_{\alpha_1\ldots\alpha_p} &+ \sum_{i=1}^{p} R^\beta{}_{\alpha_i} S_{\alpha_1\ldots\alpha_{i-1}\beta\,\alpha_{i+1}\ldots\alpha_p} \\ &- \sum_{i\neq j=1}^{p} R^\beta{}_{\alpha_i}{}^\gamma{}_{\alpha_j} S_{\alpha_1\ldots\alpha_{i-1}\beta\,\alpha_{i+1}\ldots\alpha_{j-1}\gamma\,\alpha_{j+1}\ldots\alpha_p}.\end{aligned}$$

As an example let us consider the electromagnetic 1-form A with $F = dA$, and $dF = 0$, $\delta F = -4\pi J$. Then one finds

$$\Delta_{(\mathrm{dR})}A - d(\delta A) = -4\pi J, \qquad \Delta_{(\mathrm{dR})}F = -4\pi(dJ), \tag{2.114}$$

showing that $[\Delta_{(\mathrm{dR})}, d] = 0$.

Let us write explicitly the de Rham Laplacian of the Faraday 2-form F. We have

$$\begin{aligned}[\Delta_{(\mathrm{dR})}F]_{\alpha\beta} = [\Delta F]_{\alpha\beta} &- R^\mu{}_\alpha F_{\mu\beta} + R^\mu{}_\beta F_{\mu\alpha} \\ &- R^\mu{}_\alpha{}^\gamma{}_\beta F_{\mu\gamma} + R^\mu{}_\beta{}^\gamma{}_\alpha F_{\mu\gamma}.\end{aligned} \tag{2.115}$$

One can re-derive the de Rham Laplacian of F in the form (2.114) starting from the homogeous Maxwell's equations $dF = 0$ written in coordinate components:

$$\nabla_\gamma F_{\alpha\beta} + \nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} = 0. \tag{2.116}$$

Covariant differentiation ∇^γ of both sides gives

$$-\Delta F_{\alpha\beta} + 2\nabla^\gamma \nabla_{[\alpha} F_{\beta]\gamma} = 0. \tag{2.117}$$

Recalling the non-commutativity of the covariant derivatives acting in this case on $F_{\mu\nu}$, namely

$$[\nabla_\gamma, \nabla_\alpha] F_\beta{}^\gamma = R_\beta{}^{\mu\gamma}{}_\alpha F_{\mu\gamma} + R_\gamma{}^{\mu\gamma}{}_\alpha F_{\beta\mu}, \tag{2.118}$$

replace the second term of Eq. (2.117) and use the non-homogeous Maxwell's equations $\nabla^\gamma F_{\beta\gamma} = 4\pi J_\beta$. Finally we obtain the relation $\Delta_{(\mathrm{dR})} F = -4\pi(dJ)$.

For a 3-form S we have instead

$$\begin{aligned}[\Delta_{(\mathrm{dR})} S]_{\alpha_1\alpha_2\alpha_3} = [\Delta S]_{\alpha_1\alpha_2\alpha_p} &+ R^\beta{}_{\alpha_1} S_{\beta\alpha_2\alpha_3} + R^\beta{}_{\alpha_2} S_{\alpha_1\beta\alpha_3} + R^\beta{}_{\alpha_3} S_{\alpha_1\alpha_2\beta} \\ &- 2R^\beta{}_{[\alpha_2}{}^\gamma{}_{\alpha_3]} S_{\beta\gamma\alpha_1} - 2R^\beta{}_{[\alpha_1}{}^\gamma{}_{\alpha_2]} S_{\beta\gamma\alpha_3} + 2R^\beta{}_{[\alpha_1}{}^\gamma{}_{\alpha_3]} S_{\beta\gamma\alpha_2},\end{aligned}$$

which can also be written as

$$[\Delta_{(\mathrm{dR})} S]_{\alpha_1\alpha_2\alpha_3} = [\Delta S]_{\alpha_1\alpha_2\alpha_p} + 3R^\beta{}_{[\alpha_1} S_{|\beta|\alpha_2\alpha_3]} - 6R^\beta{}_{[\alpha_1}{}^\gamma{}_{\alpha_2} S_{|\beta\gamma|\alpha_3]}.$$

2.3 Killing symmetries

The Lie derivative of a general tensor field along a congruence of curves describes the "intrinsic" behavior of this field along the congruence. If the field is the metric tensor then the vanishing of its Lie derivative implies that the vector field ξ, tangent to the congruence, is an isometry for the metric, and this field is called a *Killing vector field*. From the definitions of the covariant and Lie derivatives, a Killing vector field satisfies the equations

$$\nabla_{(\alpha}\xi_{\beta)} = 0. \tag{2.119}$$

In a remarkable paper, Papapetrou (1966) pointed out that a Killing vector field ξ can be considered as the vector potential generating an electromagnetic field

$$F_{\alpha\beta} = \nabla_\alpha \xi_\beta, \tag{2.120}$$

with current $J^\alpha = R^\alpha{}_\beta \xi^\beta$ (which vanishes in vacuum space-times) satisfying the Lorentz gauge $\nabla_\alpha \xi^\alpha = 0$. Fayos and Sopuerta (1999) introduced the term *Papapetrou field* for such an electromagnetic field. Their result is that Papapetrou fields constitute a link between the Killing symmetries and the algebraic structure of a space-time established by the alignment of the principal null directions (see the following subsection) of the Papapetrou field with those of the Riemann tensor.

A symmetric 2-tensor $K_{\mu\nu}$ is termed a *Killing tensor* if it satisfies the relation

$$\nabla_\lambda K_{\mu\nu} + \nabla_\nu K_{\lambda\mu} + \nabla_\mu K_{\nu\lambda} = 3\nabla_{(\lambda} K_{\mu\nu)} = 0. \tag{2.121}$$

2.4 Petrov classification

The algebraic properties of the space-time curvature and in particular of the Weyl tensor play a central role in Einstein's theory. In the most general space-time there exist four distinct null eigenvectors l of the Weyl tensor, known as *principal null directions* (PNDs), which satisfy the Penrose-Debever equation,

$$l_{[\alpha} C_{\beta]\mu\nu[\gamma} l_{\delta]} l^\mu l^\nu = 0. \tag{2.122}$$

When some of them coincide we have algebraically special cases. The multiplicity of the principal null directions leads to the canonical Petrov classification:

- Type I (four distinct PNDs)
- Type II (one pair of PNDs coincides)
- Type D (two pairs of PNDs coincide)
- Type III (three PNDs coincide)
- Type N (all four PNDs coincide)
- Type 0 (no PNDs).

2.5 Einstein's equations

A physical measurement crucially depends on the space-time which supports it. A space-time is a solution of Einstein's equations, which in the presence of a cosmological constant and for any source term are given by

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta}. \tag{2.123}$$

Here Λ is the cosmological constant; $T_{\alpha\beta}$ is the energy-momentum tensor of the source of the gravitational field, and satisfies the local causality conditions and the energy conditions (Hawking and Ellis, 1973). The most relevant $T_{\alpha\beta}$ are:

(i) $T_{\alpha\beta} = 0$, empty space-time.
(ii) $T_{\alpha\beta} = \frac{1}{4\pi}[\nabla_\alpha\phi\nabla_\beta\phi - \frac{1}{2}g_{\alpha\beta}[\nabla_\mu\phi\nabla^\mu\phi + m^2\phi^2]]$, massive ($m$) scalar field.
(iii) $T_{\alpha\beta} = (\mu + p)u_\alpha u_\beta + p g_{\alpha\beta}$, perfect fluid, where μ is the energy density of the fluid, p is its hydrostatic (isotropic) pressure, and u^α is the 4-velocity of the fluid element.
(iv) $T_{\alpha\beta} = \frac{1}{4\pi}[F_{\alpha\mu}F_\beta{}^\mu - \frac{1}{4}g_{\alpha\beta}F_{\mu\nu}F^{\mu\nu}]$, electromagnetic field. In this case Einstein's equations are coupled to Maxwell's equations through the Faraday 2-form field F, which can be written as

$$dF = 0, \qquad -\delta F = 4\pi J. \tag{2.124}$$

The first of these equations implies the existence of a 4-potential A such that $F = dA$, while the second one leads to the conservation of the electromagnetic current $\delta J = 0$ (or $\nabla_\alpha J^\alpha = 0$). Equivalently, Eqs. (2.124) can be written in terms of covariant derivatives as

$$\nabla_\beta {}^*F^{\alpha\beta} = 0, \qquad \nabla_\beta F^{\alpha\beta} = 4\pi J^\alpha. \tag{2.125}$$

In what follows we shall consider exact solutions of Einstein's equations which are of physical interest.

2.6 Exact solutions

We shall list the main exact solutions of Einstein's equations which provide natural support for most of the physical measurements considered here.

Constant space-time curvature solutions

Let us consider a maximally symmetric space-time which satisfies the relation

$$R^{\alpha\beta}{}_{\gamma\delta} = \frac{1}{12} R \delta^{\alpha\beta}_{\gamma\delta}, \quad \to \quad R_{\alpha\beta} = \frac{1}{4} R g_{\alpha\beta}, \tag{2.126}$$

where the space-time Ricci scalar R is constant. Relation (2.126) is equivalent to the vanishing of the trace-free components of the Ricci tensor, that is,

$$[R^{\mathrm{TF}}]_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{4} R g_{\alpha\beta} = 0. \tag{2.127}$$

In this case it follows that:

(i) if $T_{\alpha\beta} = 0$ and $\Lambda \neq 0$ (empty space-time), Einstein's equations (2.123) are satisfied with

$$\Lambda = \frac{1}{4} R; \tag{2.128}$$

(ii) if $T_{\alpha\beta} \neq 0$ describes a perfect fluid and either $\Lambda = 0$ or $\Lambda \neq 0$, Einstein's equations are satisfied with

$$\mu = \frac{1}{32\pi} R = -p, \quad \to \quad T_{\alpha\beta} = p g_{\alpha\beta}, \tag{2.129}$$

a condition which physically describes the vacuum state.

Among this family of constant space-time curvature solutions we distinguish three relevant cases (Stephani *et al.*, 2003):

(i) $R > 0$, de Sitter (dS):

$$ds^2 = -dt^2 + \alpha^2 \cosh^2\left(\frac{t}{\alpha}\right)[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)], \tag{2.130}$$

where $t \in (-\infty, +\infty)$, $\chi \in [0, \pi]$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$.

(ii) $R = 0$, Minkowski (M).

(iii) $R < 0$, Anti-de Sitter (AdS):

$$ds^2 = -dt^2 + \alpha^2 \cos^2\left(\frac{t}{\alpha}\right)[d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2)], \tag{2.131}$$

where $t \in (-\infty, +\infty)$, $\chi \in [0, \pi]$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$.

Constant spatial curvature solutions

Space-time solutions of Einstein's equations having spatial sections $t =$ constant, with constant curvature are the Friedmann-Robertson-Walker solutions, with metric

$$ds^2 = -dt^2 + a(t)^2[d\chi^2 + f(\chi)^2(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{2.132}$$

Of course dS, AdS, and also M are special cases.

Gödel solution

The Gödel solution (Gödel, 1949) describes a rotating universe with metric

$$ds^2 = -dt^2 + dx^2 - \frac{1}{2}U^2 dy^2 - 2U dt dy + dz^2, \tag{2.133}$$

where $U = e^{\sqrt{2}\omega x}$ and ω is a constant. Metric (2.133) describes an empty non-expanding but rotating universe and ω has the meaning of the angular velocity of a test particle with respect to a nearby fiducial particle.

This is a type D solution with a pressureless perfect fluid as a source. The 4-velocity u of the fluid is a Killing vector, which is not hypersurface-forming.

Kasner solution

This is an empty space-time solution (Kasner, 1925) whose metric is given by

$$ds^2 = -dt^2 + t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2, \quad t > 0, \tag{2.134}$$

where p_1, p_2, p_3 are constant parameters satisfying the conditions $p_1^2+p_2^2+p_3^2=1$ and $p_1+p_2+p_3=1$. It describes a space-time while approaching a curvature singularity at $t=0$, the latter being either of cosmological nature or emerging from gravitational collapse. It is of Petrov type I, except for the case $p_2=p_3=2/3$, $p_1=-1/3$, which corresponds to type D, and the case $p_1=p_2=p_3=0$, which is Minkowski.

Schwarzschild solution

This solution was obtained in 1915 by Schwarzschild (1916a; 1916b) and independently by Droste, a student of Lorentz, in 1916. In asymptotically spherical coordinates the space-time metric is given by

$$ds^2=-\left(1-\frac{2\mathcal{M}}{r}\right)dt^2+\left(1-\frac{2\mathcal{M}}{r}\right)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2), \qquad (2.135)$$

with $t\in(-\infty,+\infty)$, $r\in(2\mathcal{M},\infty)$, $\theta\in[0,\pi]$, $\phi\in[0,2\pi]$, and $\mathcal{M}$ being the total mass of the source. It describes the space-time outside a spherically symmetric, electrically neutral fluid source. $r=0$ is a space-like curvature singularity which is hidden by an event horizon at $r=2\mathcal{M}$.

Reissner-Nordström solution

This is a generalization of the Schwarzschild solution to an electrically charged sphere (Reissner, 1916; Nordström, 1918). In asymptotically spherical coordinates the space-time metric is given by

$$\begin{aligned}ds^2=&-\left(1-\frac{2\mathcal{M}}{r}+\frac{Q^2}{r^2}\right)dt^2+\left(1-\frac{2\mathcal{M}}{r}+\frac{Q^2}{r^2}\right)^{-1}dr^2\\&+r^2(d\theta^2+\sin^2\theta d\phi^2),\end{aligned} \qquad (2.136)$$

where Q and $\mathcal{M}$ are the electric charge and the total mass of the source, respectively. The electromagnetic field F and the 4-vector potential A are given by

$$F=-\frac{Q}{r^2}dt\wedge dr, \qquad A=-\frac{Q}{r}dt. \qquad (2.137)$$

Here $r=0$ is a time-like curvature singularity which may be hidden by an event horizon at $r=r_+=\mathcal{M}+\sqrt{\mathcal{M}^2-Q^2}$ if $\mathcal{M}>Q$.[1] When $\mathcal{M}<Q$ we have no horizons and the singularity at $r=0$ is termed *naked*.

[1] The solution admits also an inner horizon at $r=r_-=\mathcal{M}-\sqrt{\mathcal{M}^2-Q^2}$.

Kerr solution

Written in Boyer-Lindquist coordinates, Kerr space-time (Kerr, 1963; Kerr and Shild, 1967) is given by

$$ds^2 = -dt^2 + \frac{2\mathcal{M}r}{\Sigma}(a\sin^2\theta d\phi - dt)^2 + (r^2 + a^2)\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2, \tag{2.138}$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 + a^2 - 2\mathcal{M}r$. Here $\mathcal{M}$ and a are respectively the total mass and the specific angular momentum of the source. This reduces to the Schwarzschild solution when $a = 0$. It admits an event horizon[2] at $r = r_+ = \mathcal{M} + \sqrt{\mathcal{M}^2 - a^2}$, if $a \leq \mathcal{M}$ and a curvature ring-singularity in the above coordinates is found at $r = 0$ and $\theta = \pi/2$. This singularity is time-like and allows the curvature invariant $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ to be directional depending on how it is approached.

The Kerr solution is most popular in the astrophysical community because, under suitable conditions, it describes the space-time generated by a rotating black hole (Bardeen, 1970; Ruffini, 1973; Rees, Ruffini and Wheeler, 1974; Ruffini, 1978).

Kerr-Newman solution

This solution is a generalization of Kerr's to electrically charged sources (Newman *et al.*, 1965). In Boyer and Lindquist coordinates the space-time metric is given by

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2\mathcal{M}r - Q^2}{\Sigma}\right)dt^2 - \frac{2a\sin\theta(2\mathcal{M}r - Q^2)}{\Sigma}dtd\phi \\ &+ \sin^2\theta\left(r^2 + a^2 + \frac{a^2\sin^2\theta(2\mathcal{M}r - Q^2)}{\Sigma}\right)d\phi^2 \\ &+ \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2, \end{aligned} \tag{2.139}$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 + a^2 - 2\mathcal{M}r + Q^2$. Here $\mathcal{M}$, a, and Q are respectively the total mass, the specific angular momentum, and the charge of the source. The horizon and the curvature singularity have the same properties as in the Kerr metric.

[2] As in the Reissner-Nordström solution, the Kerr solution also admits an inner event horizon at $r = r_- = \mathcal{M} - \sqrt{\mathcal{M}^2 - a^2}$.

Single gravitational plane-wave solution

A plane gravitational wave (Bondi, Pirani, and Robinson, 1959) is a non-trivial solution of the vacuum Einstein equations with a five-parameter group of motion. The space-time metric is given by

$$ds^2 = -dt^2 + dz^2 + L^2(e^{2\beta}dx^2 + e^{-2\beta}dy^2), \tag{2.140}$$

with L, β depending only on the parameter $u = (-t+z)/\sqrt{2}$ and such that $d^2L/du^2 + \beta^2 L = 0$. This solution is of Petrov type N. It has the same symmetry properties as an electromagnetic plane wave in Minkowski space-time; hence we deduce the associated Killing vectors considering the electromagnetic case. In this case the Faraday 2-form is given by

$$F_{\mu\nu} = A(u)\left([k \wedge r]_{\mu\nu} \cos\theta(u) + [k \wedge s]_{\mu\nu} \sin\theta(u)\right), \tag{2.141}$$

where k_μ is the null wave vector, r_μ and s_μ are unitary space-like vectors which are mutually orthogonal and also orthogonal to k:

$$k \cdot r = k \cdot s = s \cdot r = 0, \quad s \cdot s = 1 = r \cdot r. \tag{2.142}$$

The wave amplitude $A(u)$ and its polarization $\theta(u)$ are both functions of $u = k_\mu x^\mu$. The following properties hold: $\partial_\nu F^{\mu\nu} = 0$ and $\partial_{[\rho} F_{\mu\nu]} = 0$. Without loss of generality one can choose the coordinate system so that

$$k = \frac{1}{\sqrt{2}}(\partial_t + \partial_z), \quad r = \partial_x, \quad s = \partial_y, \tag{2.143}$$

and

$$u = k_\mu x^\mu = \frac{-t+z}{\sqrt{2}}, \tag{2.144}$$

as already defined. It is easy to verify that $F^{\mu\nu}$ is invariant under the action of a five-parameter subgroup of the Lorentz group, with generators

(i) $\xi_1 = k$, generating translation on the plane $-t+z =$constant
(ii) $\xi_2 = r$, generating translation along x
(iii) $\xi_3 = s$, generating translation along y
(iv) $\xi_4 = x^\rho[s_\rho k^\mu + k_\rho s^\mu]$, generating null rotations about s
(v) $\xi_5 = x^\rho[r_\rho k^\mu + k_\rho r^\mu]$, generating null rotations about r.

The only non-vanishing structure functions are

$$C^1{}_{24} = C^1{}_{35} = 1. \tag{2.145}$$

3
Space-time splitting

The concept of space-time brings into a unified scenario quantities which, in the pre-relativistic era, carried distinct notions like *time* and *space*, *energy* and *momentum*, *mechanical power* and *force*, *electric* and *magnetic* fields, and so on. In everyday experience, however, our intuition is still compatible with the perception of a three-dimensional space and a one-dimensional time; hence a physical measurement requires a local recovery of the pre-relativistic type of separation between space and time, yet consistent with the principle of relativity. To this end we need a specific algorithm which allows us to perform the required splitting, identifying a "space" and a "time" relative to any given observer. This is accomplished locally by means of a congruence of time-like world lines with a future-pointing unit tangent vector field u, which may be interpreted as the 4-velocity of a family of observers. These world lines are naturally parameterized by the proper time τ_u defined on each of them from some initial value. The splitting of the tangent space at each point of the congruence into a local time direction spanned by vectors parallel to u, and a local rest space spanned by vectors orthogonal to u (hereafter LRS_u), allows one to decompose all space-time tensors and tensor equations into spatial and temporal components. (Choquet-Bruhat, Dillard-Bleick and DeWitt-Morette 1977).

3.1 Orthogonal decompositions

Let g be the four-dimensional space-time metric with signature +2 and components $g_{\alpha\beta}$ ($\alpha, \beta = 0, 1, 2, 3$), ∇ its associated covariant derivative operator, and η the unit volume 4-form which assures space-time orientation (see (2.26)). Assume that the space-time is time-oriented and let u be a future-pointing unit time-like ($u^\alpha u_\alpha = -1$) vector field which identifies an observer. The local splitting of the tangent space into orthogonal subspaces uniquely related to the given observer u is accomplished by a temporal projection operator $T(u)$ which generates vectors parallel to u and a spatial projection operator $P(u)$ which generates LRS_u. These operators, in mixed form, are defined as follows:

$$
\begin{aligned}
T(u) &= -u^\sharp \otimes u^\flat \\
P(u) &= I + u^\sharp \otimes u^\flat ,
\end{aligned}
\tag{3.1}
$$

where $I \equiv \delta^\alpha{}_\beta$ is the identity on the tangent spaces of the manifold M. The definitions (3.1) imply that

$$
P(u) \llcorner u = 0, \quad T(u) \llcorner u = u\,, \tag{3.2}
$$

and

$$
P(u) \llcorner P(u) = P(u), \quad T(u) \llcorner T(u) = T(u), \quad T(u) \llcorner P(u) = 0\,. \tag{3.3}
$$

In terms of components the above relations can be written

$$
\begin{aligned}
&T(u)^\alpha{}_\beta = -u^\alpha u_\beta, \\
&P(u)^\alpha{}_\beta = \delta^\alpha{}_\beta + u^\alpha u_\beta, \\
&P(u)^\alpha{}_\mu u^\mu = 0, \\
&T(u)^\alpha{}_\mu u^\mu = u^\alpha, \\
&P(u)^\alpha{}_\mu P(u)^\mu{}_\beta = P(u)^\alpha{}_\beta, \\
&T(u)^\alpha{}_\mu T(u)^\mu{}_\beta = T(u)^\alpha{}_\beta, \\
&T(u)^\alpha{}_\mu P(u)^\mu{}_\beta \equiv 0.
\end{aligned}
\tag{3.4}
$$

Given a $\binom{r}{s}$-tensor S, let us denote by $[P(u)S]$ and $[T(u)S]$ its fully spatial and temporal projections, obtained by acting with the corresponding operators on *all* of its indices. In terms of components these are given by

$$
[P(u)S]^{\alpha\cdots}{}_{\beta\ldots} = P(u)^\alpha{}_\gamma \cdots P(u)^\delta{}_\beta \cdots S^{\gamma\cdots}{}_{\delta\ldots} \tag{3.5}
$$

$$
[T(u)S]^{\alpha\cdots}{}_{\beta\ldots} = T(u)^\alpha{}_\gamma \cdots T(u)^\delta{}_\beta \cdots S^{\gamma\cdots}{}_{\delta\ldots}\,. \tag{3.6}
$$

The *splitting* of S relative to a given observer is the set of tensors which arise from the spatial and temporal projection of *each* of its indices, as we shall now illustrate.

Splitting of a vector

If S is a vector field, its splitting gives rise to a scalar field and a spatial vector field:

$$
S \quad \leftrightarrow \quad \{u\cdot S, [P(u)S]\}\,. \tag{3.7}
$$

In terms of components these are given by

$$
S^\alpha \quad \leftrightarrow \quad \{u_\gamma S^\gamma, P(u)^\alpha{}_\gamma S^\gamma\}. \tag{3.8}
$$

With respect to the observer u, the vector S then admits the following representation:

$$
\begin{aligned}
S^\alpha &= [T(u)S]^\alpha + [P(u)S]^\alpha \\
&= -(u_\gamma S^\gamma)u^\alpha + P(u)^\alpha{}_\gamma\, S^\gamma ,
\end{aligned}
\tag{3.9}
$$

also termed *1 + 3-splitting.*

Splitting of a $\binom{1}{1}$-tensor

If S is a mixed $\binom{1}{1}$-tensor field, then its splitting consists of a scalar field, a spatial vector field, a spatial 1-form, and a spatial $\binom{1}{1}$-tensor field, namely

$$
S^\alpha{}_\beta \leftrightarrow \{u^\delta u_\gamma S^\gamma{}_\delta,\ P(u)^\alpha{}_\gamma u^\delta S^\gamma{}_\delta,\ P(u)^\delta{}_\alpha u_\gamma S^\gamma{}_\delta,\ P(u)^\alpha{}_\gamma P(u)^\delta{}_\beta S^\gamma{}_\delta\}.
$$

In terms of these fields, the tensor S admits the following representation:

$$
\begin{aligned}
S^\alpha{}_\beta &= [T(u)^\alpha{}_\gamma + P(u)^\alpha{}_\gamma][T(u)^\delta{}_\beta + P(u)^\delta{}_\beta]S^\gamma{}_\delta \\
&= (u^\delta u_\gamma S^\gamma{}_\delta)u^\alpha u_\beta - u^\alpha u_\gamma P(u)^\delta{}_\beta S^\gamma{}_\delta \\
&\quad - u^\delta u_\beta P(u)^\alpha{}_\gamma S^\gamma{}_\delta + [P(u)S]^\alpha{}_\beta ,
\end{aligned}
\tag{3.10}
$$

for any chosen observer u.

The local spatial and temporal projections of a $\binom{p}{q}$-tensor are easily generalized. Let us consider the metric tensor $g_{\alpha\beta}$. From (3.2) and (3.3) one finds that

$$
g_{\alpha\beta} = P(u)_{\alpha\beta} + T(u)_{\alpha\beta}\,;
$$

hence, the spatial metric $[P(u)\mathrm{g}]_{\alpha\beta} = P(u)_{\alpha\beta}$ is the only non-trivial spatial field which arises from the splitting of the space-time metric.

Splitting of p-forms

Given a p-form

$$
S = S_{[\alpha_1\ldots\alpha_p]}\omega^{\alpha_1}\otimes\cdots\otimes\omega^{\alpha_p} \equiv \frac{1}{p!}S_{\alpha_1\ldots\alpha_p}\omega^{\alpha_1}\wedge\ldots\wedge\omega^{\alpha_p},
\tag{3.11}
$$

we define the *electric part* of S relative to the observer u by the quantity

$$
\left[S^{(\mathrm{E})}(u)\right]_{\alpha_1\ldots\alpha_{p-1}} = -u^\sigma S_{\sigma\alpha_1\ldots\alpha_{p-1}}
\tag{3.12}
$$

or, in a more compact form, $S^{(\mathrm{E})}(u) = -u \,\lrcorner\, S$. Similarly we define the *magnetic part* of S by the quantity

$$
\left[S^{(\mathrm{M})}(u)\right]_{\alpha_1\ldots\alpha_p} = P(u)^{\beta_1}{}_{\alpha_1}\ldots P(u)^{\beta_p}{}_{\alpha_p}S_{\beta_1\ldots\beta_p}
\tag{3.13}
$$

or, in a more compact form, $S^{(\mathrm{M})}(u) = P(u)S$. From the above definitions we deduce the representation of S in terms of its spatial and temporal decompositions:

$$S = u^\flat \wedge S^{(\mathrm{E})}(u) + S^{(\mathrm{M})}(u), \tag{3.14}$$

or, in components,

$$S_{\alpha_1\ldots\alpha_p} = p! u_{[\alpha_1}[S^{(\mathrm{E})}(u)]_{\alpha_2\ldots\alpha_p]} + [S^{(\mathrm{M})}(u)]_{\alpha_1\ldots\alpha_p}. \tag{3.15}$$

Notice that the contraction $u \rfloor S$ is automatically spatial due to the antisymmetry of S (since $u \rfloor (u \rfloor S) = 0$ or equivalently $u^\alpha u^\beta S_{\beta\alpha\gamma_3\ldots\gamma_p} = 0$).

Consider the splitting of the unit volume 4-form η. From its properties and having in mind that $[P(u)\eta] \equiv 0$, one deduces the following representation:

$$\eta = -u^\flat \wedge \eta(u). \tag{3.16}$$

To express this in terms of components, let us recall (2.34) and the antisymmetrization rule for four indices. From (2.30), this is given by

$$\mathcal{R}_{[\alpha\beta\gamma\delta]} = \frac{1}{4}\left(\mathcal{R}_{\alpha[\beta\gamma\delta]} - \mathcal{R}_{\beta[\alpha\gamma\delta]} + \mathcal{R}_{\gamma[\alpha\beta\delta]} - \mathcal{R}_{\delta[\alpha\beta\gamma]}\right) \tag{3.17}$$

for any tensor $\mathcal{R}$. Hence we have

$$\begin{aligned}[u^\flat \wedge \eta(u)]_{\alpha\beta\gamma\delta} &= 4u_{[\alpha}\eta(u)_{\beta\gamma\delta]} \\ &= \left[2u_{[\alpha}\eta(u)_{\beta]\gamma\delta} + 2u_{[\gamma}\eta(u)_{\delta]\alpha\beta}\right],\end{aligned} \tag{3.18}$$

where the spatial unit volume 3-form

$$\eta(u)_{\alpha\beta\gamma} = u^\delta \eta_{\delta\alpha\beta\gamma} \tag{3.19}$$

is the only non-trivial spatial field which arises from the splitting of the volume 4-form. From the above it then follows that

$$\eta_{\alpha\beta\gamma\delta} = -2u_{[\alpha}\eta(u)_{\beta]\gamma\delta} - 2u_{[\gamma}\eta(u)_{\delta]\alpha\beta}. \tag{3.20}$$

In (2.36) and (2.37) we saw that, using the space-time (Hodge) duality operation (*), one can associate with any p-form S (with $0 \leq p \leq 4$) a $(4-p)$-form. Similarly a spatial duality operation ($^{*(u)}$) is defined for a spatial p-form S ($u \rfloor S = 0$) replacing η with $\eta(u)$, namely

$$^{*(u)}S_{\alpha_1\ldots\alpha_{3-p}} = \frac{1}{p!} S_{\beta_1\ldots\beta_p}\eta(u)^{\beta_1\ldots\beta_p}{}_{\alpha_1\ldots\alpha_{3-p}}. \tag{3.21}$$

As an example, given a spatial 2-form S, its spatial dual is a vector given by

$$[^{*(u)}S]^\alpha = \frac{1}{2}\eta(u)^{\alpha\beta\gamma}S_{\beta\gamma}. \tag{3.22}$$

This operation satisfies the property

$$^{*(u)*(u)}S = S. \tag{3.23}$$

Let us now consider the splitting of ${}^*S$ where S is given by (3.14). We have

$$
\begin{aligned}
{}^*S &= u^\flat \wedge [{}^*S]^{(\mathrm{E})}(u) + [{}^*S]^{(\mathrm{M})}(u) \\
&= {}^*[u^\flat \wedge S^{(\mathrm{E})}(u) + S^{(\mathrm{M})}(u)] \\
&= {}^*[u^\flat \wedge S^{(\mathrm{E})}(u)] + {}^*[S^{(\mathrm{M})}(u)] \\
&= {}^{*(u)}S^{(\mathrm{E})}(u) + {}^*[{}^{*(u)}[{}^{*(u)}[S^{(\mathrm{M})}(u)]]] \\
&= {}^{*(u)}S^{(\mathrm{E})}(u) + (-1)^{p-1}u^\flat \wedge {}^{*(u)}[S^{(\mathrm{M})}(u)],
\end{aligned}
\tag{3.24}
$$

where, in the third line, we have used (3.23). Comparing the first and the last line we have

$$
[{}^*S]^{(\mathrm{E})}(u) = (-1)^{p-1}{}^{*(u)}[S^{(\mathrm{M})}(u)], \qquad [{}^*S]^{(\mathrm{M})}(u) = {}^{*(u)}S^{(\mathrm{E})}(u). \tag{3.25}
$$

Splitting of differential operators

We saw in Chapter 2 that in general relativity one encounters several space-time tensorial differential operators which act on tensor fields. Let us recall them: if T is a tensor field of any rank, we have:

(i) the Lie derivative of T along the direction of a given vector field X: $[\pounds_X T]$
(ii) the covariant derivative of T: ∇T
(iii) the absolute derivative of T along a curve with unit tangent vector X and parameterized by s: $\nabla_X T \equiv DT/ds$
(iv) the Fermi-Walker derivative of T along a non-null curve with unit tangent vector X and parameterized by s: $D_{(\mathrm{fw},X)}T/ds$, defined in (2.83).

Finally if S is a p-form, one has

(v) the exterior derivative of S: dS.

Application of the spatial projection into the LRS_u of a family of observers u to the space-time derivatives (i) to (v) yields new operators which can be more easily compared with those defined in a three-dimensional Euclidean space. Given a tensor field T of components $T^{\alpha\cdots}{}_{\beta\ldots}$ we have in fact

(i) the spatially projected Lie derivative along a vector field X

$$
[\pounds(u)_X T]^{\alpha\cdots}{}_{\beta\ldots} \equiv P(u)^\alpha{}_\sigma \ldots P(u)^\rho{}_\beta \ldots [\pounds_X T]^{\sigma\cdots}{}_{\rho\ldots}; \tag{3.26}
$$

when $X = u$ we also use the notation

$$
\nabla(u)_{(\mathrm{lie})}T \equiv \pounds(u)_u T, \tag{3.27}
$$

and this operation will be termed the *spatial-Lie temporal derivative*

(ii) the spatially projected covariant derivative along any e_γ frame direction

$$\nabla(u)_\gamma T \equiv P(u)\nabla_\gamma T, \tag{3.28}$$

namely

$$[\nabla(u)_\gamma T]^{\alpha\ldots}{}_{\beta\ldots} = P(u)^\alpha{}_{\alpha_1}\ldots P(u)^{\beta_1}{}_\beta \ldots P(u)^\sigma{}_\gamma \nabla_\sigma T^{\alpha_1\ldots}{}_{\beta_1\ldots} \tag{3.29}$$

(iii) the spatially projected absolute derivative along a curve with unit tangent vector X

$$[P(u)\nabla_X T]^{\alpha\ldots}{}_{\beta\ldots} = P(u)^\alpha{}_{\alpha_1}\ldots P(u)^{\beta_1}{}_\beta \ldots [\nabla_X T]^{\alpha_1\ldots}{}_{\beta_1\ldots} \tag{3.30}$$

(iv) the spatially projected Fermi-Walker derivative along a curve with unit tangent vector X and parameterized by s

$$\left[P(u)\frac{D_{(\mathrm{fw},X)}T}{ds}\right]^{\alpha\ldots}{}_{\beta\ldots} = P(u)^\alpha{}_\sigma \ldots P(u)^\rho{}_{\beta\ldots}\left[\frac{D_{(\mathrm{fw},X)}T}{ds}\right]^{\sigma\ldots}{}_{\rho\ldots} \tag{3.31}$$

(v) the spatially projected exterior derivative of a p-form S

$$d(u)S \equiv P(u)dS, \tag{3.32}$$

namely

$$[d(u)S]_{\alpha_1\ldots\alpha_p\beta} = P(u)^{\beta_1}{}_{\alpha_1}\ldots P(u)^\sigma{}_\beta [dS]_{\beta_1\ldots\sigma}. \tag{3.33}$$

Note that all these spatial differential operators are well defined since they arise from the spatial projection of space-time differential operators. Moreover, there are relations among them which we shall analyze below. From their definitions it is clear that both the Fermi-Walker and the Lie derivatives of the vector field u along itself vanish identically (and so do the projections orthogonal to u of these derivatives). The only derivative of u along itself which is meaningful, i.e. different than zero, is the covariant derivative

$$P(u)\nabla_u u = \nabla_u u = a(u). \tag{3.34}$$

3.2 Three-dimensional notation

Given a family of observers u, a vector X is termed *spatial with respect to* u when it lives in the local rest space of u, i.e. it satisfies the condition $X_\alpha u^\alpha = 0$. It is then convenient to introduce 3-dimensional vector notation for the spatial inner product and the spatial cross product of two spatial vector fields X and Y. The *spatial inner product* is defined as

$$X \cdot_u Y = P(u)_{\alpha\beta}X^\alpha Y^\beta \tag{3.35}$$

while the *spatial cross product* is

$$[X \times_u Y]^\alpha = \eta(u)^\alpha{}_{\beta\gamma}X^\beta Y^\gamma, \tag{3.36}$$

where $\eta(u)^\alpha{}_{\beta\gamma} = u_\sigma \eta^{\sigma\alpha}{}_{\beta\gamma}$ as stated.

In terms of the above definitions we can define spatial gradient, curl, and divergence operators of functions f and spatial vector fields X as

$$\begin{aligned}\mathrm{grad}_u\, f &= \nabla(u) f,\\ \mathrm{curl}_u\, X &= \nabla(u) \times_u X,\\ \mathrm{div}_u\, X &= \nabla(u) \cdot_u X.\end{aligned} \tag{3.37}$$

In terms of components these relations are given by

$$\begin{aligned}[\mathrm{grad}_u\, f]_\alpha &= \nabla(u)_\alpha f = P(u)^{\alpha\beta} e_\beta(f),\\ [\mathrm{curl}_u\, X]^\alpha &= \eta(u)^{\alpha\beta\gamma} \nabla(u)_\beta X_\gamma = u^\sigma \eta_\sigma{}^{\alpha\beta\gamma} \nabla_\beta X_\gamma,\\ [\mathrm{div}_u\, X] &= \nabla(u)_\alpha X^\alpha = P(u)^{\alpha\beta} \nabla_\alpha X_\beta.\end{aligned} \tag{3.38}$$

It is useful to extend the above definitions to

(i) the spatial cross product of a vector X by a symmetric tensor A
(ii) the spatial cross product of two symmetric spatial tensors A and B
(iii) the spatial inner product of two symmetric spatial tensors A and B
(iv) the trace of the above tensor product

as shown below:

$$\begin{aligned}[X \times_u A]^{\alpha\beta} &= \eta(u)^{\gamma\delta(\alpha} X_\gamma A^{\beta)}{}_\delta,\\ [A \times_u B]_\alpha &= \eta(u)_{\alpha\beta\gamma} A^\beta{}_\delta\, B^{\delta\gamma},\\ [A \cdot_u B]_\alpha{}^\beta &= A_{\alpha\gamma} B^{\gamma\beta},\\ \mathrm{Tr}[A \cdot_u B] &= A_{\alpha\beta} B^{\alpha\beta}.\end{aligned} \tag{3.39}$$

One may also introduce a symmetric curl operation or *Scurl*. For a spatial and symmetric $\binom{2}{0}$-tensor A this is defined by

$$[\mathrm{Scurl}_u\, A]^{\alpha\beta} = \eta(u)^{\gamma\delta(\alpha} \nabla(u)_\gamma\, A^{\beta)}{}_\delta. \tag{3.40}$$

The Scurl operation can be extended to a spatial antisymmetric $\binom{2}{0}$-tensor Ω, as follows:

$$\begin{aligned}[\mathrm{Scurl}_u\, \Omega]^{\alpha\beta} &= \eta(u)^{\gamma\delta(\alpha} \nabla(u)_\gamma\, \Omega^{\beta)}{}_\delta\\ &= \nabla(u)_\gamma\, \eta(u)^{\gamma\delta(\alpha} \eta(u)^{\beta)}{}_{\delta\sigma} [{}^{*(u)}\Omega]^\sigma\\ &= -\mathrm{div}_u\, [{}^{*(u)}\Omega] P(u)^{\alpha\beta} + \nabla(u)^{(\alpha} [{}^{*(u)}\Omega]^{\beta)}.\end{aligned} \tag{3.41}$$

Similarly, a spatial divergence of spatial tensors can be defined as in (3.38)$_3$, that is

$$[\mathrm{div}_u\, X]^{\alpha\ldots\beta} = \nabla(u)_\sigma X^{\sigma\alpha\ldots\beta}, \quad X = P(u)X. \tag{3.42}$$

Finally, the above operations performed on spatial fields may be extended to non-spatial ones. For example, if

$$X = X^{\parallel} u + X^{\perp}$$

is a non-spatial field we define

$$[\mathrm{curl}\, X]^\alpha \equiv \eta(u)^{\alpha\beta\gamma}\nabla(u)_\beta X_\gamma = 2X^{\|}\omega(u)^\alpha + [\mathrm{curl}_{\mathrm{u}}\, X]^\alpha.$$

Analogous definitions hold for the other spatial operations considered above.

3.3 Kinematics of the observer's congruence

Let us now consider the splitting of the covariant derivative $\nabla_\beta u^\alpha$. This operation generates two spatial fields, namely the acceleration vector field $a(u)$ and the kinematical tensor field $k(u)$, defined as

$$\begin{aligned} a(u) &= P(u)\nabla_u u, \\ k(u) &= -\nabla(u)u = \omega(u) - \theta(u), \end{aligned} \tag{3.43}$$

where

$$\begin{aligned} \omega(u)_{\alpha\beta} &= P(u)^\mu_\alpha P(u)^\nu_\beta \nabla_{[\mu}u_{\nu]}, \\ \theta(u)_{\alpha\beta} &= P(u)^\mu_\alpha P(u)^\nu_\beta \nabla_{(\mu}u_{\nu)} \\ &= \frac{1}{2}[\pounds(u)_u P(u)]_{\alpha\beta}, \end{aligned} \tag{3.44}$$

are the components of tensor fields $\omega(u)$ and $\theta(u)$ having the meaning respectively of *vorticity* and *expansion*. From the above definitions, the tensor field $\nabla_\beta u^\alpha$ can be written as

$$\nabla_\beta u^\alpha = -a(u)^\alpha u_\beta - k(u)^\alpha{}_\beta. \tag{3.45}$$

The expansion tensor field $\theta(u)$ may itself be decomposed into its trace-free and pure-trace parts:

$$\theta(u) = \sigma(u) + \frac{1}{3}\Theta(u)P(u), \tag{3.46}$$

where the trace-free tensor field $\sigma(u)$ ($\sigma(u)^\alpha{}_\alpha = 0$) is termed *shear* and the scalar

$$\Theta(u) = \nabla_\alpha u^\alpha \tag{3.47}$$

is termed the *volumetric (or isotropic) scalar expansion.*

We can also define the vorticity vector field $\omega(u) = 1/2\ \mathrm{curl}_u\, u$ as the spatial dual of the spatial rotation tensor:

$$\omega(u)^\alpha = \frac{1}{2}\eta(u)^{\alpha\beta\gamma}\omega(u)_{\beta\gamma} = \frac{1}{2}\eta^{\sigma\alpha\beta\gamma}u_\sigma\nabla_\beta u_\gamma. \tag{3.48}$$

Although we use the same symbol for the vorticity tensor and the associated vector they can be easily distinguished by the context.

3.4 Adapted frames

Given a field of observers u, a frame $\{e_\alpha\}$ with $\alpha = 0, 1, 2, 3$ (with dual ω^α) is termed *adapted to* u if $e_0 = u$ and e_a with $a = 1, 2, 3$ are orthogonal to u, that is $u \cdot e_a = 0$. From this it follows that $\omega^0 = -u^\flat$. Obviously the components of u relative to the frame are simply $u^\alpha = \delta^\alpha_0$ and the metric tensor is given by

$$g^\flat = -u^\flat \otimes u^\flat + P(u)_{ab}\omega^a \otimes \omega^b. \tag{3.49}$$

In this section all indices denote components relative to the frame $\{e_\alpha\}$. The evolution of the frame vectors along the world lines of u is governed by the relations $\nabla_u e_\alpha = e_\sigma \Gamma^\sigma{}_{\alpha 0}$. Let us consider separately the cases $\alpha = 0$ and $\alpha = a$:

(i) When $\alpha = 0$ it follows that

$$\nabla_u e_0 = \nabla_u u = a(u) = e_\sigma \Gamma^\sigma{}_{00}. \tag{3.50}$$

Recalling that $u \cdot a(u) = 0$, one finds that $\Gamma^0{}_{00} = 0$ and therefore that

$$\Gamma^b{}_{00} = a(u)^b. \tag{3.51}$$

(ii) When $\alpha = a$ we have

$$\nabla_u e_a = \Gamma^\sigma{}_{a0} e_\sigma = \Gamma^0{}_{a0} e_0 + \Gamma^c{}_{a0} e_c. \tag{3.52}$$

The dot product of these terms with $-u$ gives

$$-u \cdot \nabla_u e_a = \Gamma^0{}_{a0} = a(u)_a. \tag{3.53}$$

We then have two expressions for the acceleration, namely

$$a(u)_a = \Gamma^0{}_{a0}, \quad \text{and} \quad a(u)^a = \Gamma^a{}_{00}. \tag{3.54}$$

Before proceeding, let us define

$$\Gamma_{\alpha\beta\gamma} \equiv e_\alpha \cdot e_\sigma \Gamma^\sigma{}_{\beta\gamma}; \tag{3.55}$$

expanding the sum we have

$$\begin{aligned}\Gamma_{\alpha\beta\gamma} &= (e_\alpha \cdot e_0)\Gamma^0{}_{\beta\gamma} + (e_\alpha \cdot e_b)\Gamma^b{}_{\beta\gamma} \\ &= \delta^0_\alpha g_{00}\Gamma^0{}_{\beta\gamma} + \delta^c_\alpha P(u)_{cb}\Gamma^b{}_{\beta\gamma}.\end{aligned} \tag{3.56}$$

Hence it follows that

$$\Gamma_{0\beta\gamma} = -\Gamma^0{}_{\beta\gamma}, \qquad \Gamma_{c\beta\gamma} = P(u)_{cb}\Gamma^b{}_{\beta\gamma}. \tag{3.57}$$

From the latter, lowering the up-indices in (3.54) we obtain

$$\Gamma_{0a0} + \Gamma_{a00} = 2\Gamma_{(0a)0} = 0\,; \tag{3.58}$$

moreover the transport law for the inner product $(e_a \cdot u)$ along u itself yields

$$\nabla_u(e_a \cdot u) = -\Gamma^0{}_{a0} + \Gamma_{a00} = 0 \tag{3.59}$$

from (3.58). Therefore the antisymmetry of the first pair of indices $(0a)$ ensures that the spatial vectors of the frame remain orthogonal to u while moving along the world line of u itself. From (3.52) and (3.55) we have also

$$\Gamma_{ba0} = e_b \cdot \nabla_u e_a = e_b \cdot [P(u)\nabla_u e_a] \equiv C_{(\mathrm{fw})ba}, \tag{3.60}$$

where $C_{(\mathrm{fw})ba}$ are termed *Fermi-Walker structure functions.*[1] From (3.60) relation (3.52) can be written as

$$\nabla_u e_a = a(u)_a e_0 + C_{(\mathrm{fw})}{}^b{}_a e_b, \tag{3.61}$$

implying that

$$P(u)\nabla_u e_a = C_{(\mathrm{fw})}{}^b{}_a e_b. \tag{3.62}$$

Let us now prove that $C_{(\mathrm{fw})}{}^b{}_a$ can also we written as

$$C_{(\mathrm{fw})}{}^b{}_a = C_{(\mathrm{lie})}{}^b{}_a - k(u)^b{}_a, \tag{3.63}$$

where

$$C_{(\mathrm{lie})}{}^b{}_a \equiv \omega^b(\pounds(u)_u e_a), \tag{3.64}$$

recalling that ω^b is the dual basis; Eq. (3.64) implies that

$$P(u)\pounds_u e_a = \pounds(u)_u e_a = C_{(\mathrm{lie})}{}^b{}_a e_b. \tag{3.65}$$

In terms of the structure functions of the frame, we have by definition

$$\begin{aligned}\pounds(u)_u e_a \equiv P(u)\pounds_u e_a = P(u)[u, e_a] = P(u)\, C^\alpha{}_{0a} e_\alpha \\ = C^b{}_{0a} e_b,\end{aligned} \tag{3.66}$$

where we have used the relations $P(u)e_0 = 0$ and $P(u)e_b = e_b$; therefore

$$C_{(\mathrm{lie})}{}^b{}_a = C^b{}_{0a}. \tag{3.67}$$

Using (3.45) we obtain

$$\pounds_u e_a = \nabla_u e_a - \nabla_{e_a} u = \nabla_u e_a + k(u)^c{}_a e_c, \tag{3.68}$$

so, projecting both sides orthogonally to u yields

$$\pounds(u)_u e_a = P(u)\nabla_u e_a + k(u)^c{}_a e_c, \tag{3.69}$$

which leads to (3.63).

Using (3.45), one obtains

$$\begin{aligned}\nabla_{e_a} e_0 &= \Gamma^\gamma{}_{0a} e_\gamma = -k(u)^b{}_a e_b, \\ \nabla_{e_a} e_b &= \Gamma^\gamma{}_{ba} e_\gamma = -k(u)_{ba} e_0 + \Gamma^c{}_{ba} e_c.\end{aligned} \tag{3.70}$$

Summarizing, we have

$$\begin{aligned}&\Gamma^a{}_{00} = a(u)^a, &&\Gamma^0{}_{a0} = a(u)_a, &&\Gamma^b{}_{a0} = C_{(\mathrm{fw})}{}^b{}_a, \\ &\Gamma^b{}_{0a} = -k(u)^b{}_a, &&\Gamma^0{}_{ba} = -k(u)_{ba}.\end{aligned} \tag{3.71}$$

[1] A more correct notation for the Fermi-Walker structure functions should be $C_{(\mathrm{fw},u,e_c)ba}$, i.e. a symbol which includes explicit dependence on the frame. Clearly when the frame is fixed the simplified notation proves useful.

We can also express the structure functions in terms of kinematical quantities. In fact, from the definition

$$e_\alpha C^\alpha{}_{\beta\gamma} = [e_\beta, e_\gamma] = \nabla_{e_\beta} e_\gamma - \nabla_{e_\gamma} e_\beta \tag{3.72}$$

we have

$$\begin{aligned} e_\alpha C^\alpha{}_{0b} &= \nabla_u e_b - \nabla_{e_b} u \\ &= a(u)_b u + [C_{(\mathrm{fw})}{}^c{}_b + k(u)^c{}_b] e_c \\ &= a(u)_b u + C_{(\mathrm{lie})}{}^c{}_b e_c, \end{aligned} \tag{3.73}$$

so that

$$C^0{}_{0b} = a(u)_b, \qquad C^c{}_{0b} = C_{(\mathrm{lie})}{}^c{}_b. \tag{3.74}$$

Similarly

$$\begin{aligned} e_\alpha C^\alpha{}_{bc} &= \nabla_{e_b} e_c - \nabla_{e_c} e_b \\ &= 2\omega(u)_{bc} u + 2\Gamma^d{}_{[cb]} e_d, \end{aligned} \tag{3.75}$$

so that

$$C^0{}_{bc} = 2\omega(u)_{bc}, \qquad C^d{}_{bc} = 2\Gamma^d{}_{[cb]}. \tag{3.76}$$

We recall that the structure functions satisfy the Jacobi identities

$$[[e_\alpha, e_\beta], e_\rho] + [[e_\beta, e_\rho], e_\alpha] + [[e_\rho, e_\alpha], e_\beta] = 0, \tag{3.77}$$

which implies the following differential relations:

$$e_{[\rho}(C^\sigma{}_{\alpha\beta]}) - C^\sigma{}_{\nu[\rho} C^\nu{}_{\alpha\beta]} = 0. \tag{3.78}$$

Splitting such relations using an observer-adapted frame gives

$$\begin{aligned} 0 &= -\nabla(u)_{(\mathrm{lie})} \omega(u)_{ab} + \nabla(u)_{[a} a(u)_{b]}, \\ 0 &= -\nabla(u)_{(\mathrm{lie})} C^d{}_{ab} + 2e_{[a}\left(C_{(\mathrm{lie})}{}^f{}_{b]}\right) + 2a(u)_{[a} C_{(\mathrm{lie})}{}^f{}_{b]}, \\ 0 &= -\nabla(u)_{[r} \omega(u)_{ba]} + a(u)_{[r} \omega(u)_{ba]}, \\ 0 &= -e_{[r} C^c{}_{ab]} + C^c{}_{s[r} C^s{}_{ab]} + 2C_{(\mathrm{lie})}{}^c{}_{[r} \omega(u)_{ab]}. \end{aligned} \tag{3.79}$$

The first of these relations can also be written as

$$[\nabla(u)_{(\mathrm{lie})} + \Theta(u)] \omega(u)^s - \frac{1}{2} [\mathrm{curl}_u\, a(u)]^s = 0, \tag{3.80}$$

where we have used the property

$$\nabla(u)_{(\mathrm{lie})} \eta(u)_{abc} = \Theta(u) \eta(u)_{abc}. \tag{3.81}$$

The third equation becomes

$$-\operatorname{div}_u \omega(u) + a(u) \cdot \omega(u) = 0 \tag{3.82}$$

after contraction of both sides with $\eta(u)^{rba}$.

Spatial-Fermi-Walker and spatial-Lie temporal derivatives

In the previous subsection we introduced the Fermi-Walker structure functions $C_{(\mathrm{fw})}{}^b{}_a$, as follows:

$$P(u)\nabla_u e_a = C_{(\mathrm{fw})}{}^b{}_a e_b. \tag{3.83}$$

A similar relation was defined for the Lie derivative projected along u, which we also termed a *spatial-Lie temporal derivative* (see (3.27)):

$$£(u)_u e_a = P(u)£_u e_a = C_{(\mathrm{lie})}{}^b{}_a e_b = C^b{}_{0a} e_b. \tag{3.84}$$

It is useful to handle both these operations with a unified notation[2]

$$\nabla(u)_{(\mathrm{tem})} e_a = C_{(\mathrm{tem})}{}^b{}_a e_b, \tag{3.85}$$

where “tem” refers either to “fw” or to “lie”, with the following definitions:

$$\nabla(u)_{(\mathrm{fw})} e_a \equiv P(u)\nabla_u e_a, \qquad \nabla(u)_{(\mathrm{lie})} e_a \equiv P(u)£_u e_a = £(u)_u e_a. \tag{3.86}$$

Therefore if X is a vector field orthogonal to u, i.e. $X \cdot u = 0$, we have

$$\begin{aligned}
\nabla(u)_{(\mathrm{tem})} X = \nabla(u)_{(\mathrm{tem})}(X^a e_a) &= \frac{dX^a}{d\tau_u} e_a + X^a C_{(\mathrm{tem})}{}^b{}_a e_b \\
&= \left(\frac{dX^b}{d\tau_u} + X^a C_{(\mathrm{tem})}{}^b{}_a \right) e_b \\
&= (\nabla(u)_{(\mathrm{tem})} X^b) e_b.
\end{aligned} \tag{3.87}$$

The operation $\nabla(u)_{(\mathrm{fw})} = P(u)\nabla_u$ is termed a *spatial-Fermi-Walker temporal derivative*. It can be extended to non-spatial fields. If we apply this operation to the vector field u itself we have

$$\nabla(u)_{(\mathrm{fw})} u = P(u)\nabla_u u = a(u), \tag{3.88}$$

and if $X = fu$ we have

$$\begin{aligned}
\nabla(u)_{(\mathrm{fw})} X &= \nabla(u)_{(\mathrm{fw})}(fu) = f a(u), \\
\nabla(u)_{(\mathrm{lie})} X &= \nabla(u)_{(\mathrm{lie})}(fu) = 0.
\end{aligned} \tag{3.89}$$

[2] This notation has been introduced by R. T. Jantzen, who also considered two more spatial temporal derivatives, corotating Fermi-Walker and Lie$^\flat$. Details can be found in Jantzen, Carini, and Bini (1992).

Hence the temporal derivatives so defined through their action on purely spatial and purely temporal fields can now act on any space-time field.

Frame components of the Riemann tensor

From the definition

$$e_\alpha R^\alpha{}_{\beta\gamma\delta} = [\nabla_{e_\gamma}, \nabla_{e_\delta}]e_\beta - C^\sigma{}_{\gamma\delta}\nabla_{e_\sigma}e_\beta, \tag{3.90}$$

we have

$$\begin{aligned} e_\alpha R^\alpha{}_{0b0} &= [\nabla_{e_b}, \nabla_u]u - C^\sigma{}_{b0}\nabla_{e_\sigma}u \\ &= \nabla_{e_b}(a(u)^c e_c) - \nabla_u(\nabla_{e_b}u) - C^0{}_{b0}a(u)^c e_c - C^c{}_{b0}\nabla_{e_c}u \\ &= \left\{[\nabla(u)_b + a(u)_b]a(u)^c + \nabla(u)_{(\mathrm{fw})}k(u)^c{}_b - [k(u)^2]^c{}_b\right\} e_c, \end{aligned} \tag{3.91}$$

where

$$[k(u)^2]^c{}_b \equiv k(u)^c{}_s k(u)^s{}_b,$$

so that

$$R^c{}_{0b0} = [\nabla(u)_b + a(u)_b]a(u)^c + \nabla(u)_{(\mathrm{fw})}k(u)^c{}_b - [k(u)^2]^c{}_b, \tag{3.92}$$

where

$$\nabla(u)_{(\mathrm{fw})}k(u)^c{}_b = \nabla_u k(u)^c{}_b + C_{(\mathrm{fw})}{}^c{}_f k(u)^f{}_b - C_{(\mathrm{fw})}{}^f{}_b k(u)^c{}_f. \tag{3.93}$$

Similarly

$$\begin{aligned} e_\alpha R^\alpha{}_{bcd} &= [\nabla_{e_c}, \nabla_{e_d}]e_b - C^0{}_{cd}\nabla_u e_b - C^f{}_{cd}\nabla_{e_f}e_b \end{aligned} \tag{3.94}$$

$$\begin{aligned} &= \nabla_{e_c}(\nabla_{e_d}e_b) - \nabla_{e_d}(\nabla_{e_c}e_b) + 2\omega(u)_{cd}\nabla_u e_b \\ &\quad - C^f{}_{cd}\nabla_{e_f}e_b \\ &= \nabla_{e_c}(-k(u)_{bd}u + \Gamma^f{}_{bd}e_f) - \nabla_{e_d}(-k(u)_{bc}u + \Gamma^f{}_{bc}e_f) \\ &\quad + 2\omega(u)_{cd}(a(u)_b u + C_{(\mathrm{fw})}{}^f{}_b e_f) \\ &\quad - C^f{}_{cd}(-k(u)_{bf}u + \Gamma^r{}_{bf}e_r), \end{aligned} \tag{3.95}$$

that is,

$$\begin{aligned} e_\alpha R^\alpha{}_{bcd} &= [-e_c(k(u)_{bd}) - \Gamma^s{}_{bd}k(u)_{sc} + e_d(k(u)_{bc}) + \Gamma^s{}_{bc}k(u)_{sd} \\ &\quad - 2\omega(u)_{cd}a(u)_b + C^s{}_{cd}k(u)_{bs}]u \\ &\quad + [2e_{[c}(\Gamma^f{}_{|b|d]}) + 2\Gamma^f{}_{s[c}\Gamma^s{}_{|b|d]} - 2k(u)_{b[c}k(u)^f{}_{d]} \\ &\quad - 2\omega_{cd}C_{(\mathrm{fw})}{}^f{}_b - C^s{}_{cd}\Gamma^f{}_{bs}]e_f. \end{aligned} \tag{3.96}$$

Analyzing the time and space components of the Riemann tensor relative to the basis, we have

$$R^0{}_{bcd} = -2[\nabla(u)_{[c}k(u)_{|b|d]} + \omega(u)_{cd}a(u)_b] \tag{3.97}$$

and

$$\begin{aligned} R^f{}_{bcd} &= R_{(\mathrm{fw})}{}^f{}_{bcd} - 2k(u)_{b[c}k(u)^f{}_{d]} \\ &= R_{(\mathrm{fw})}{}^f{}_{bcd} + 2k(u)^f{}_{[c}k(u)_{|b|d]}, \end{aligned} \tag{3.98}$$

where

$$\begin{aligned} R_{(\mathrm{fw})}{}^f{}_{bcd} &= 2e_{[c}\left(\Gamma^f{}_{|b|d]}\right) + 2\Gamma^s{}_{b[c}\Gamma^f{}_{|s|d]} - C^s{}_{cd}\Gamma^f{}_{bs} \\ &\quad - 2\omega(u)_{cd}C_{(\mathrm{fw})}{}^f{}_b. \end{aligned} \tag{3.99}$$

This tensor is termed a *Fermi-Walker spatial Riemann tensor*;[3] it can be written in invariant form as follows

$$\begin{aligned} R_{(\mathrm{fw})}(u)(X,Y)Z &= \{[\nabla(u)_X, \nabla(u)_Y]Z - \nabla(u)_{[X,Y]}\}Z \\ &\quad - 2\omega(u)(X,Y)\nabla(u)_{(\mathrm{fw})}Z, \end{aligned} \tag{3.100}$$

where X, Y, and Z are spatial fields with respect to u and we note that

$$[X,Y] = P(u)[X,Y] - 2\omega^\flat(X,Y)\,u. \tag{3.101}$$

The Fermi-Walker spatial Riemann tensor does not have all the symmetries of a three-dimensional Riemann tensor. For instance it does not satisfy the Ricci identities. In fact we have

$$0 = R^f{}_{[bcd]} = R_{(\mathrm{fw})}{}^f{}_{[bcd]} - 2k(u)_{[bc}k(u)^f{}_{d]}. \tag{3.102}$$

Recalling that the antisymmetric part of a $\binom{0}{3}$-tensor X_{bcd} can be written as

$$\begin{aligned} X_{[bcd]} &= \frac{1}{3}\left(X_{[bc]d} + X_{[cd]b} + X_{[db]c}\right) \\ &= \frac{1}{3}\left(X_{b[cd]} + X_{c[db]} + X_{d[bc]}\right), \end{aligned} \tag{3.103}$$

it follows that (3.102) can be written more conveniently as

$$R_{(\mathrm{fw})}{}^f{}_{[bcd]} = 2k(u)^f{}_{[b}\omega(u)_{cd]}. \tag{3.104}$$

From the latter one can construct a new Riemann tensor with all the necessary symmetries. Ferrarese (1965) has shown that the symmetry-obeying Riemann tensor, denoted by $R_{(\mathrm{sym})}{}^{ab}{}_{cd}$, is related to the Fermi-Walker Riemann tensor (3.100) by

$$\begin{aligned} R_{(\mathrm{sym})}{}^{ab}{}_{cd} &= R_{(\mathrm{fw})}{}^{ab}{}_{cd} - 2\omega(u)^{ab}\omega(u)_{cd} - 4\theta(u)^{[a}{}_{[c}\omega(u)^{b]}{}_{d]} \\ &= R^{ab}{}_{cd} + 2k(u)^b{}_{[c}k(u)^a{}_{d]} - 2\omega(u)^{ab}\omega(u)_{cd} \\ &\quad - 4\theta(u)^{[a}{}_{[c}\omega(u)^{b]}{}_{d]}, \end{aligned} \tag{3.105}$$

[3] A Lie spatial Riemann tensor can be defined similarly, replacing the Fermi-Walker structure functions $C_{(\mathrm{fw})}{}^f{}_b$ with the corresponding Lie structure functions $C_{(\mathrm{lie})}{}^f{}_b$ according to (3.63).

or

$$
\begin{aligned}
R^{ab}{}_{cd} = R_{(\mathrm{sym})}{}^{ab}{}_{cd} + 2\theta(u)^a{}_{[c}\theta(u)^b{}_{d]} + 2\omega^{ab}\omega_{cd} \\
+ 2\omega(u)^a{}_{[c}\omega(u)^b{}_{d]},
\end{aligned} \tag{3.106}
$$

where (3.98) has been used.

In the special case of a Born-rigid congruence of observers u, i.e. $\theta(u) = 0$, we find that (3.105) can be written as

$$
\begin{aligned}
R_{(\mathrm{sym})}{}^{ab}{}_{cd} &= R_{(\mathrm{fw})}{}^{ab}{}_{cd} - 2\omega(u)^{ab}\omega(u)_{cd} \\
&= R^{ab}{}_{cd} + 2\omega(u)^b{}_{[c}\omega(u)^a{}_{d]} - 2\omega(u)^{ab}\omega(u)_{cd},
\end{aligned} \tag{3.107}
$$

whereas in the special case of a vorticity-free congruence, i.e. $\omega(u) = 0$, we have

$$
\begin{aligned}
R_{(\mathrm{sym})}{}^{ab}{}_{cd} &= R_{(\mathrm{fw})}{}^{ab}{}_{cd} \\
&= R^{ab}{}_{cd} + 2\theta(u)^b{}_{[c}\theta(u)^a{}_{d]}.
\end{aligned} \tag{3.108}
$$

Together with the spatial symmetric Riemann tensor $R_{(\mathrm{sym})}{}^{ab}{}_{cd}$ we also introduce the spatial symmetric Ricci tensor, $R_{(\mathrm{sym})}{}^a{}_b = R_{(\mathrm{sym})}{}^{ca}{}_{cb}$, as well as the associated scalar $R_{(\mathrm{sym})} = R_{(\mathrm{sym})}{}^a{}_a$.

3.5 Comparing families of observers

Let u and U be two unitary time-like vector fields. Define the relative spatial velocity of U with respect to u as

$$
\begin{aligned}
\nu(U,u)^\alpha &= -(u^\sigma U_\sigma)^{-1} P(u)^\alpha{}_\beta U^\beta \\
&= -(u^\sigma U_\sigma)^{-1} \left[U^\alpha + (u^\rho U_\rho) u^\alpha\right].
\end{aligned} \tag{3.109}
$$

The space-like vector field $\nu(U,u)^\alpha$, orthogonal to u, can also be written as

$$
\nu(U,u)^\alpha = ||\nu(U,u)||\hat{\nu}(U,u)^\alpha, \tag{3.110}
$$

where $\hat{\nu}(U,u)$ is the unitary vector giving the direction of $\nu(U,u)$ in the rest frame of u. Similar formulas hold for the relative velocity of u with respect to U; hence the 4-velocities of the two observers admit the following decomposition:

$$
\begin{aligned}
U &= \gamma(U,u)[u + \nu(U,u)] \\
&= \gamma(U,u)[u + ||\nu(U,u)||\hat{\nu}(U,u)],
\end{aligned} \tag{3.111}
$$

and

$$
\begin{aligned}
u &= \gamma(u,U)[U + \nu(u,U)] \\
&= \gamma(u,U)[U + ||\nu(u,U)||\hat{\nu}(u,U)].
\end{aligned} \tag{3.112}
$$

Both the spatial relative velocity vectors have the same magnitude,

$$
||\nu(U,u)|| = [\nu(U,u)_\alpha \nu(U,u)^\alpha]^{1/2} = [\nu(u,U)_\alpha \nu(u,U)^\alpha]^{1/2}.
$$

The common gamma factor is related to that magnitude by

$$\gamma(U,u) = \gamma(u,U) = [1 - ||\nu(U,u)||^2]^{-1/2} = -U_\alpha u^\alpha \,; \tag{3.113}$$

hence we recognize it as the relative Lorentz factor. It is convenient to abbreviate $\gamma(U,u)$ by γ and $||\nu(U,u)||$ by ν when their meaning is clear from the context and there are no more than two observers involved.

Let us notice here that by substituting (3.111) into (3.112) we obtain the following relation:

$$-\,\hat{\nu}(u,U) = \gamma[\hat{\nu}(U,u) + \nu u], \tag{3.114}$$

which together with

$$U = \gamma[u + \nu\hat{\nu}(U,u)] \tag{3.115}$$

yields a unique *relative boost* $B(U,u)$ from u to U, namely

$$\begin{aligned} B(U,u)u &= U \\ &= \gamma[u + \nu\hat{\nu}(U,u)] \\ B(U,u)\hat{\nu}(U,u) &= -\hat{\nu}(u,U) \\ &= \gamma[\hat{\nu}(U,u) + \nu u]. \end{aligned} \tag{3.116}$$

The inverse relations hold by interchanging U with u. The boost acts as the identity on the intersection of their local rest spaces $LRS_u \cap LRS_U$.

Maps between LRSs

The spatial measurements of two observers in relative motion can be compared by relating their respective LRSs. Let U and u be two such observers and LRS_U and LRS_u their LRSs. There exist several maps between these LRSs, as we now demonstrate.

Combining the projection operators $P(U)$ and $P(u)$ one can form the following "mixed projection" maps:

(i) $P(U,u)$ from LRS_u into LRS_U, defined as

$$P(U,u) = P(U)P(u) : LRS_u \to LRS_U, \tag{3.117}$$

with its inverse:

$$P(U,u)^{-1} : LRS_U \to LRS_u \tag{3.118}$$

(ii) $P(u,U)$ from LRS_U into LRS_u, defined as

$$P(u,U) = P(u)P(U) : LRS_U \to LRS_u, \tag{3.119}$$

with its inverse:

$$P(u,U)^{-1} : LRS_u \to LRS_U. \tag{3.120}$$

Note that $P(U,u) \neq P(u,U)^{-1}$ shown as, from their representations,

$$\begin{aligned}
P(U,u) &= P(u) + \gamma\nu U \otimes \hat{\nu}(U,u), \\
P(U,u)^{-1} &= P(U) + \nu U \otimes \hat{\nu}(u,U), \\
P(u,U) &= P(U) + \gamma\nu u \otimes \hat{\nu}(u,U), \\
P(u,U)^{-1} &= P(u) + \nu u \otimes \hat{\nu}(U,u).
\end{aligned} \tag{3.121}$$

Let us deduce each of the above relations.

(a) Relation $(3.121)_1$ is easily verified. Consider a vector $X \in LRS_u$; then, by definition,

$$P(U,u)X = P(U)P(u)X = P(U)X = X + U(U \cdot X) \; ; \tag{3.122}$$

but

$$U \cdot X = \gamma(u \cdot X + \nu\hat{\nu}(U,u) \cdot X) = \gamma\nu\hat{\nu}(U,u) \cdot X.$$

Hence

$$P(U,u)X = X + \gamma\nu U\hat{\nu}(U,u) \cdot X = [P(u) + \gamma\nu U \otimes \hat{\nu}(U,u)] \llcorner X, \tag{3.123}$$

which completes the proof.

(b) Let us now verify $(3.121)_2$. Consider a vector $X = P(U,u)Y \in LRS_U$, with $Y \in LRS_u$; then, by using the relation $u = \gamma(U + \nu\hat{\nu}(u,U))$ and the property $Y \cdot u = 0$, we have

$$Y \cdot U = -\nu Y \cdot \hat{\nu}(u,U).$$

Hence

$$\begin{aligned}
Y &= P(U)Y - U(U \cdot Y) = P(U,u)Y + \nu U[Y \cdot \hat{\nu}(u,U)] \\
&= P(U,u)Y + \nu U[P(U,u)Y \cdot \hat{\nu}(u,U)].
\end{aligned} \tag{3.124}$$

Substituting $Y = P(U,u)^{-1}X$ gives

$$\begin{aligned}
P(U,u)^{-1}X &= P(U)X + \nu U[X \cdot \hat{\nu}(u,U)] \\
&= [P(U) + \nu U \otimes \hat{\nu}(u,U)] \llcorner X,
\end{aligned} \tag{3.125}$$

which completes the proof.

(c) Relations $(3.121)_3$ and $(3.121)_4$ are straightforwardly verified by exchanging U with u in $(3.121)_1$ and $(3.121)_2$ respectively.

One can then show that

$$\begin{aligned}
P(U,u)\hat{\nu}(U,u) &= -\gamma\hat{\nu}(u,U), \\
P(u,U)^{-1}\hat{\nu}(U,u) &= -\frac{1}{\gamma}\hat{\nu}(u,U).
\end{aligned} \tag{3.126}$$

Note that, from the property (3.3) of the projection operator $P(U)$, we have

$$P(U,u) = P(U) \llcorner P(u) = P(U) \llcorner P(U,u) = P(U,u) \llcorner P(u). \tag{3.127}$$

This is a mixed tensor which is spatial with respect to u in its covariant index and with respect to U in its contravariant index.[4] Moreover the following relations hold:

$$\begin{aligned} P(U) &= P(U,u) \llcorner P(U,u)^{-1}, \\ P(u) &= P(U,u)^{-1} \llcorner P(U,u). \end{aligned} \tag{3.128}$$

Boost maps

Similarly to what we have done in combining projection maps, the boost $B(U,u)$ induces an invertible map between the local rest spaces of the given observers defined as

$$B_{(\mathrm{lrs})}(U,u) \equiv P(U)B(U,u)P(u) : LRS_u \to LRS_U. \tag{3.129}$$

It also acts as the identity on the intersection of their subspaces $LRS_U \cap LRS_u$. Because the boost is an isometry, exchanging the role of U and u in (3.129) leads to the inverse boost:

$$B_{(\mathrm{lrs})}(U,u)^{-1} \equiv B_{(\mathrm{lrs})}(u,U) : LRS_U \to LRS_u. \tag{3.130}$$

Before giving explicit representations of $B_{(\mathrm{lrs})}(U,u)$ and its inverse $B_{(\mathrm{lrs})}(U,u)^{-1}$ we note that any map between the local rest spaces of two observers may be expressed only in terms of quantities which are spatial with respect to those observers.

The representations of the boost and its inverse can be given in terms of the associated tensors

$$B_{(\mathrm{lrs})\,u}(U,u), \quad B_{(\mathrm{lrs})\,U}(U,u), \quad B_{(\mathrm{lrs})\,u}(u,U), \quad B_{(\mathrm{lrs})\,U}(u,U), \tag{3.131}$$

defined by

$$\begin{aligned} B_{(\mathrm{lrs})\,u}(U,u) &= P(U,u)^{-1} \llcorner B_{(\mathrm{lrs})}(U,u), \\ B_{(\mathrm{lrs})\,U}(U,u) &= B_{(\mathrm{lrs})}(U,u) \llcorner P(U,u)^{-1}, \end{aligned} \tag{3.132}$$

with the corresponding expressions for the inverse boost obtained simply by exchanging the roles of U and u, and with

$$B_{(\mathrm{lrs})}(U,u) = B_{(\mathrm{lrs})\,U}(U,u) \llcorner P(U,u) = P(U,u) \llcorner B_{(\mathrm{lrs})\,u}(U,u). \tag{3.133}$$

The explicit expression for $B_{(\mathrm{lrs})\,u}(U,u)$, for example, is given by

$$B_{(\mathrm{lrs})\,u}(U,u) = P(u) + \frac{1-\gamma}{\gamma} \hat{\nu}(U,u) \otimes \hat{\nu}(U,u). \tag{3.134}$$

[4] This is a kind of two-point tensor that Schouten (1954) termed a *connecting tensor*.

This can be shown as follows. Let $X \in LRS_u$; then

$$\begin{aligned} B_{(\mathrm{lrs})u}(U,u)X &= P(U,u)^{-1}[B_{(\mathrm{lrs})}(U,u)X] \\ &= P(U,u)^{-1}[B_{(\mathrm{lrs})}(U,u)X^{||}\hat{\nu}(U,u) + X^{\perp}], \end{aligned} \tag{3.135}$$

where $X^{||} = X \cdot \hat{\nu}(U,u)$ and $X^{\perp} = X - X^{||}\hat{\nu}(U,u)$, that is

$$X = X^{||}\hat{\nu}(U,u) + X^{\perp}. \tag{3.136}$$

We have also used the fact that the boost reduces to the identity for vectors not belonging to the boost plane, as $X^{\perp}$. In this case the boost plane is spanned by the vectors u and $\hat{\nu}(U,u)$. Taking into account (3.116), that is,

$$B_{(\mathrm{lrs})}(U,u)\hat{\nu}(U,u) = -\hat{\nu}(u,U), \tag{3.137}$$

as well as the linearity of the boost map, we have

$$\begin{aligned} B_{(\mathrm{lrs})u}(U,u)X &= P(U,u)^{-1}[X - X^{||}\hat{\nu}(u,U) - X^{||}\hat{\nu}(U,u)] \\ &= P(U,u)^{-1}X - X^{||}P(U,u)^{-1}[\hat{\nu}(u,U) + \hat{\nu}(U,u)] \\ &= X + \frac{1-\gamma}{\gamma}X^{||}\hat{\nu}(U,u), \end{aligned} \tag{3.138}$$

where $P(U,u)^{-1}X = X$ because $X \in LRS_u$:

$$\begin{aligned} P(U,u)^{-1}X &= P(U,u)^{-1}P(u)X = P(U,u)^{-1}P(U)P(u)X \\ &= P(U,u)^{-1}P(U,u)X = P(u)X = X. \end{aligned} \tag{3.139}$$

Hence $P(U,u)^{-1}\hat{\nu}(U,u) = \hat{\nu}(U,u)$, because $\nu(U,u)$ belongs to LRS_u. Moreover, from (3.126), by exchanging the roles of U and u, we find

$$\begin{aligned} P(U,u)^{-1}\hat{\nu}(u,U) &= -\frac{1}{\gamma}\hat{\nu}(U,u) = \frac{1}{\gamma}B(u,U)\hat{\nu}(u,U) \\ &= \frac{1}{\gamma}B(U,u)^{-1}\hat{\nu}(u,U). \end{aligned} \tag{3.140}$$

Therefore

$$\begin{aligned} B_{(\mathrm{lrs})u}(U,u)X &= X - X^{||}\left(-\frac{1}{\gamma}+1\right)\hat{\nu}(U,u) \\ &= X - (X\cdot\hat{\nu}(U,u))\left(-\frac{1}{\gamma}+1\right)\hat{\nu}(U,u) \\ &= \left[P(u) - \frac{\gamma-1}{\gamma}\hat{\nu}(U,u)\otimes\hat{\nu}(U,u)^{\flat}\right]\llcorner X, \end{aligned} \tag{3.141}$$

which is equivalent to (3.134).

Similarly, for the inverse boost $B_{(\mathrm{lrs})}(u,U)$, one has

$$
\begin{aligned}
B_{(\mathrm{lrs})\,u}(u,U) &= P(u) - \frac{\gamma}{\gamma+1}\nu(U,u)\otimes\nu(U,u)^\flat \\
&= P(u) - \frac{\gamma-1}{\gamma}\hat\nu(U,u)\otimes\hat\nu(U,u)^\flat, \\
B_{(\mathrm{lrs})\,U}(u,U) &= P(U) - \frac{\gamma}{\gamma+1}\nu(u,U)\otimes\nu(u,U)^\flat \\
&= P(U) - \frac{\gamma-1}{\gamma}\hat\nu(u,U)\otimes\hat\nu(u,U)^\flat.
\end{aligned}
\tag{3.142}
$$

Thus, if S is a vector field such that $S\in LRS_U$, then its inverse boost is the vector belonging to LRS_u:

$$
B_{(\mathrm{lrs})}(u,U)S = [P(u) - \gamma(\gamma+1)^{-1}\nu(U,u)\otimes\nu(U,u)^\flat]\llcorner P(u,U)S.
\tag{3.143}
$$

In components, the latter relation becomes

$$
B_{(\mathrm{lrs})}(u,U)^\alpha{}_\beta S^\beta = [P(u)^\alpha{}_\mu - \gamma(\gamma+1)^{-1}\nu(U,u)^\alpha\nu(U,u)_\mu]P(u,U)^\mu{}_\beta S^\beta.
\tag{3.144}
$$

3.6 Splitting of derivatives along a time-like curve

Consider a congruence of curves $\mathcal{C}_U$ with tangent vector field U and proper time τ_U as parameter. We know, at this stage, that the evolution along $\mathcal{C}_U$ of any tensor field can be specified by one of the following space-time derivatives:

(i) the absolute derivative along $\mathcal{C}_U$: $D/d\tau_U = \nabla_U$
(ii) the Fermi-Walker derivative along $\mathcal{C}_U$: $D_{(\mathrm{fw},U)}/d\tau_U$
(iii) the space-time Lie derivative along $\mathcal{C}_U$: $\pounds_U$, for which we use also the notation $D_{(\mathrm{lie},U)}/d\tau_U = \pounds_U$.

The actions of the Fermi-Walker and Lie derivatives on a vector field X are related to the absolute derivative as follows:

$$
\begin{aligned}
\frac{D_{(\mathrm{fw},U)}X}{d\tau_U} &= \nabla_U X + a(U)(U\cdot X) - U(a(U)\cdot X) \\
&= P(U)\nabla_U X - U\nabla_U(X\cdot U) + a(U)(X\cdot U), \\
\frac{D_{(\mathrm{lie},U)}X}{d\tau_U} &= [U,X] = \nabla_U X + a(U)(U\cdot X) + k(U)\llcorner X,
\end{aligned}
\tag{3.145}
$$

where $k(U)=\omega(U)-\theta(U)$ is the kinematical tensor of the congruence $\mathcal{C}_U$, defined in (3.43). For $X=U$ we have

$$
\frac{D_{(\mathrm{fw},U)}U}{d\tau_U} = 0, \qquad \frac{D_{(\mathrm{lie},U)}U}{d\tau_U} = 0,
\tag{3.146}
$$

whereas

$$\frac{DU}{d\tau_U} = a(U). \tag{3.147}$$

If X is spatial with respect to U, i.e. $X \cdot U = 0$, we have instead

$$\begin{aligned}
\frac{D_{(\mathrm{fw},U)}X}{d\tau_U} &= P(U)\nabla_U X, \\
\frac{D_{(\mathrm{lie},U)}X}{d\tau_U} &= \nabla_U X - k(U) \llcorner X \\
&= P(U)\nabla_U X + U(a(U)\cdot X) + k(U) \llcorner X.
\end{aligned} \tag{3.148}$$

The projection of $D_{(\mathrm{lie},U)}X/d\tau_U$ orthogonal to U gives, as in (3.148),

$$\begin{aligned}
P(U)\frac{D_{(\mathrm{lie},U)}X}{d\tau_U} &= P(U)\nabla_U X + k(U) \llcorner X \\
&= \frac{D_{(\mathrm{fw},U)}}{d\tau_U} X + k(U) \llcorner X.
\end{aligned} \tag{3.149}$$

Let u be another family of observers whose world lines have as parameter the proper time τ_u. One can introduce, on the congruence $\mathcal{C}_U$ whose unit tangent vector field can be written as

$$U = \gamma(U,u)[u + \nu(U,u)], \tag{3.150}$$

two new parameterizations $\tau_{(U,u)}$ and $\ell_{(U,u)}$ as follows:

$$\frac{d\tau_{(U,u)}}{d\tau_U} = \gamma(U,u), \qquad \frac{d\ell_{(U,u)}}{d\tau_U} = \gamma(U,u)||\nu(U,u)||, \tag{3.151}$$

where $\tau_{(U,u)}$ corresponds to the proper times of the observers u when their curves are crossed by a given curve of $\mathcal{C}_U$, and $\ell_{(U,u)}$ corresponds to the proper length on $\mathcal{C}_U$.

The projection orthogonal to u of the absolute derivative along U is expressed as

$$\begin{aligned}
P(u)\frac{D}{d\tau_U} = P(u)\nabla_U &= \gamma[P(u)\nabla_u + P(u)\nabla_{\nu(U,u)}] \\
&= \gamma[P(u)\nabla_u + \nabla(u)_{\nu(U,u)}].
\end{aligned} \tag{3.152}$$

We note that in the above equation the derivative operation $P(u)\nabla_u$ is just what we have termed the spatial-Fermi-Walker temporal derivative, i.e. $\nabla(u)_{(\mathrm{fw})}$, in (3.86). For a vector field X we can then write

$$
\begin{aligned}
P(u)\frac{D_{(\mathrm{fw},U)}X}{d\tau_U} &\equiv \frac{D_{(\mathrm{fw},U,u)}X}{d\tau_U} \\
&= P(u)\frac{DX}{d\tau_U} + [P(u,U)a(U)](U\cdot X) \\
&\quad - \gamma\nu(U,u)(a(U)\cdot X),
\end{aligned}
\tag{3.153}
$$

$$
\begin{aligned}
P(u)\frac{D_{(\mathrm{lie},U)}X}{d\tau_U} &\equiv \frac{D_{(\mathrm{lie},U,u)}X}{d\tau_U} \\
&= P(u)\frac{DX}{d\tau_U} + P(u,U)a(U)(U\cdot X) \\
&\quad + P(u)[k(U)\llcorner X].
\end{aligned}
\tag{3.154}
$$

We shall now examine these three projected absolute derivatives in detail.

Projected absolute derivative

Consider the absolute derivative of u along U, namely $\nabla_U u$. Since u is unitary, then

$$
u\cdot\frac{Du}{d\tau_U} = 0
$$

and we can write, from (3.43),

$$
\begin{aligned}
\frac{Du}{d\tau_U} &= P(u)\frac{Du}{d\tau_U} = \gamma[P(u)\nabla_u u + P(u)\nabla_{\nu(U,u)}u] \\
&= \gamma[\nabla(u)_{(\mathrm{fw})}u + P(u)\nabla_{\nu(U,u)}u] \\
&= \gamma[a(u) - k(u)\llcorner\nu(U,u)] \\
&= \gamma[a(u) + \omega(u)\times_u \nu(U,u) + \theta(u)\llcorner\nu(U,u)].
\end{aligned}
\tag{3.155}
$$

Let us denote the above quantity as a (minus) Fermi-Walker gravitational force, that is

$$
\begin{aligned}
F^{(G)}_{(\mathrm{fw},U,u)} &= -\frac{Du}{d\tau_U} \\
&= -\gamma[a(u) + \omega(u)\times_u \nu(U,u) + \theta(u)\llcorner\nu(U,u)].
\end{aligned}
\tag{3.156}
$$

It should be stressed here that, although $F^{(G)}_{(\mathrm{fw},U,u)}$ is referred to as a gravitational force, it contains contributions by true gravity and by inertial forces.

Consider now the case of X orthogonal to u, i.e. $X\cdot u = 0$. The projection onto LRS_u of the absolute derivative of X along U gives

$$
\begin{aligned}
P(u)\frac{DX}{d\tau_U} &= \gamma[P(u)\nabla_u X + \nabla(u)_{\nu(U,u)}X] \\
&\equiv \frac{D_{(\mathrm{fw},U,u)}X}{d\tau_U}.
\end{aligned}
\tag{3.157}
$$

This differential operator plays an important role since both Fermi-Walker and Lie derivatives along U can be expressed in terms of it. It is therefore worthwhile to write down the above expression in terms of (adapted) frame components:

$$\begin{aligned} P(u)\frac{DX}{d\tau_U} &= \frac{dX^a}{d\tau_U}e_a + \gamma\left[X^a P(u)\nabla_u e_a + \nu(U,u)^b X^a \nabla(u)_b e_a\right] \\ &= \left\{\frac{dX^b}{d\tau_U} + \gamma\left[X^a\left(C_{(\mathrm{fw})}{}^b{}_a + \nu(U,u)^c\,\Gamma^b{}_{ac}\right)\right]\right\} e_b, \end{aligned} \tag{3.158}$$

where we set $X = X^a e_a$. Introducing the relative standard time parameterization $\tau(U,u)$ defined in (3.151), we have

$$P(u)\frac{DX}{d\tau_{(U,u)}} = \frac{D_{(\mathrm{fw},U,u)}X}{d\tau_{(U,u)}} = \left(\frac{D_{(\mathrm{fw},U,u)}X}{d\tau_{(U,u)}}\right)^a e_a \tag{3.159}$$

or, in components,

$$\left(\frac{D_{(\mathrm{fw},U,u)}X}{d\tau_{(U,u)}}\right)^b = \frac{dX^b}{d\tau_{(U,u)}} + X^a\left(C_{(\mathrm{fw})}{}^b{}_a + \nu(U,u)^c\,\Gamma^b{}_{ac}\right). \tag{3.160}$$

A particular vector field which is orthogonal to u and is defined all along $\mathcal{C}_U$ is the field of relative velocities, $\nu(U,u)$. We introduce the acceleration of U relative to u as

$$a_{(\mathrm{fw},U,u)} = \frac{D_{(\mathrm{fw},U,u)}}{d\tau_{(U,u)}}\nu(U,u) = \gamma^{-1}P(u)\frac{D}{d\tau_U}\nu(U,u). \tag{3.161}$$

Considering instead the unit vector $\hat\nu(U,u)$, this quantity can be written as

$$a_{(\mathrm{fw},U,u)} = P(u)\frac{D}{d\tau_{(U,u)}}[\nu\hat\nu(U,u)], \tag{3.162}$$

where $\nu = ||\nu(U,u)||$. Finally we have

$$a_{(\mathrm{fw},U,u)} = \hat\nu(U,u)\frac{d\nu}{d\tau_{(U,u)}} + \nu P(u)\frac{D}{d\tau_{(U,u)}}\hat\nu(U,u). \tag{3.163}$$

It is therefore quite natural to denote the first term as a tangential Fermi-Walker acceleration $a^{(T)}_{(\mathrm{fw},U,u)}$ of U relative to u, and the second as a centripetal Fermi-Walker acceleration $a^{(C)}_{(\mathrm{fw},U,u)}$ of U relative to u:

$$a_{(\mathrm{fw},U,u)} = a^{(T)}_{(\mathrm{fw},U,u)} + a^{(C)}_{(\mathrm{fw},U,u)}, \tag{3.164}$$

where

$$\begin{aligned} a^{(T)}_{(\mathrm{fw},U,u)} &= \hat\nu(U,u)\,\frac{d\nu}{d\tau_{(U,u)}}, \\ a^{(C)}_{(\mathrm{fw},U,u)} &= \nu\,P(u)\frac{D}{d\tau_{(U,u)}}\hat\nu(U,u) \\ &= \nu\frac{D_{(\mathrm{fw},U,u)}}{d\tau_{(U,u)}}\hat\nu(U,u). \end{aligned} \tag{3.165}$$

To generalize the classical mechanics notion of centripetal acceleration we need to convert the relative standard time parameterization into an analogous relative standard length parameterization:[5]

$$d\ell_{(U,u)} = \nu d\tau_{(U,u)}. \tag{3.166}$$

With this parameterization we have

$$\begin{aligned} a^{(C)}_{(\mathrm{fw},U,u)} &= \nu^2 P(u) \frac{D}{d\ell_{(U,u)}} \hat{\nu}(U,u) \\ &= \nu \frac{D_{(\mathrm{fw},U,u)}}{d\tau_{(U,u)}} \hat{\nu}(U,u) \\ &= \frac{\nu^2}{\mathcal{R}_{(\mathrm{fw},U,u)}} \hat{\eta}_{(\mathrm{fw},U,u)} \\ &= \nu^2 k_{(\mathrm{fw},U,u)} \hat{\eta}_{(\mathrm{fw},U,u)}, \end{aligned} \tag{3.167}$$

where $\hat{\eta}_{(\mathrm{fw},U,u)}$ is a unit space-like vector orthogonal to $\hat{\nu}(U,u)$, $k_{(\mathrm{fw},U,u)}$ is the Fermi-Walker relative curvature, and $\mathcal{R}_{(\mathrm{fw},U,u)}$ is the curvature radius of the curve such that

$$k_{(\mathrm{fw},U,u)} \hat{\eta}_{(\mathrm{fw},U,u)} = \frac{\hat{\eta}_{(\mathrm{fw},U,u)}}{\mathcal{R}_{(\mathrm{fw},U,u)}} = P(u) \frac{D}{d\ell_{(U,u)}} \hat{\nu}(U,u). \tag{3.168}$$

Clearly, whether geometrically or physically motivated, one can replace the spatial-Fermi-Walker temporal derivative with the spatial-Lie temporal derivative defining the corresponding quantities. Doing this, one really understands the power of the notation used. For example (and for later use) one can define the Lie relative curvature of a curve and the associated curvature radius:

$$k_{(\mathrm{lie},U,u)} \hat{\eta}_{(\mathrm{lie},U,u)} = \frac{\hat{\eta}_{(\mathrm{lie},U,u)}}{\mathcal{R}_{(\mathrm{lie},U,u)}} = \frac{D_{(\mathrm{lie},U,u)}}{d\ell_{(U,u)}} \hat{\nu}(U,u). \tag{3.169}$$

Projected Fermi-Walker derivative

The projected Fermi-Walker derivative for a general vector field X is given by (3.153). In the case $X \cdot u = 0$, and using the standard decomposition of $U = \gamma(u + \nu(U,u))$, we have

$$\begin{aligned} P(u) \frac{D_{(\mathrm{fw},U)} X}{d\tau_U} \equiv \frac{D_{(\mathrm{fw},U,u)} X}{d\tau_U} &\\ = P(u) \frac{DX}{d\tau_U} &+ \gamma P(u,U) a(U) (\nu(U,u) \cdot X) \\ &- \gamma \nu(U,u) (P(u,U) a(U) \cdot X). \end{aligned} \tag{3.170}$$

[5] In fact the Euclidean space definition involves spatial orbits parameterized by the (spatial) curvilinear abscissa.

Setting

$$P(u,U)a(U)=\gamma F(U,u), \tag{3.171}$$

we find that

$$\begin{aligned} P(u)\frac{D_{(\mathrm{fw},U)}X}{d\tau_U} &= P(u)\frac{DX}{d\tau_U}+\gamma^2 F(U,u)(\nu(U,u)\cdot X)\\ &\quad -\gamma^2\nu(U,u)(F(U,u)\cdot X)\\ &= P(u)\frac{DX}{d\tau_U}\\ &\quad +\gamma^2 X\times_u (F(U,u)\times_u \nu(U,u)). \end{aligned} \tag{3.172}$$

Projected Lie derivative

Finally, for the projected Lie derivative we have, from (3.154),

$$\begin{aligned} P(u)\frac{D_{(\mathrm{lie},U)}X}{d\tau_U} &\equiv \frac{D_{(\mathrm{lie},U,u)}X}{d\tau_U}\\ &= P(u)\frac{DX}{d\tau_U}+\gamma^2 F(U,u)(\nu(U,u)\cdot X)\\ &\quad + P(u)[k(U)\llcorner X]. \end{aligned} \tag{3.173}$$

4
Special frames

The definition and interpretation of a physical measurement are better achieved if reference is made to a system of Cartesian axes which matches an instantaneous inertial frame. With respect to such a frame, a measurement is expressed in terms of the projection on its axes of the tensors or tensor equations characterizing the phenomenon under investigation. Two different Cartesian frames are connected by a general Lorentz transformation; hence the properties of the Lorentz group play a key role in the theory of measurements. In Newtonian mechanics the local rest spaces of all observers coincide as a result of the absoluteness of time but in relativistic mechanics the local rest spaces do not coincide, because of relativity of time; hence a comparison between any two quantities which belong to the local rest spaces of different observers requires a non-trivial mapping among them. This is what we are going to discuss now.

4.1 Orthonormal frames

Let u be the 4-velocity of an observer; at any point of his world line let us choose in LRS_u three mutually orthogonal unit space-like vectors $e(u)_{\hat a}$ with $\hat a = 1, 2, 3$. In what follows these vectors will be simply denoted by $e_{\hat a}$ whenever it is not necessary to specify the time-like vector u which is orthogonal to them. They satisfy the relation

$$e_{\hat a} \cdot e_{\hat b} = \delta_{\hat a \hat b}. \tag{4.1}$$

It is clear that the triad $\{e_{\hat a}\}_{\hat a = 1,\,2,\,3}$ is an orthonormal basis for LRS_u.

If we write $u \equiv e_{\hat 0}$, then the set $\{e_{\hat\alpha}\}$ with $\hat\alpha = 0, 1, 2, 3$ itself forms an orthonormal basis for the local tangent space and is termed a *tetrad* (Pirani, 1956b). The vectors $e_{\hat\alpha}$ satisfy the condition

$$e_{\hat\alpha} \cdot e_{\hat\beta} = \eta_{\hat\alpha\hat\beta} \equiv \mathrm{diag}\,[-1, 1, 1, 1], \tag{4.2}$$

which ensures that $\{e_{\hat{\alpha}}\}$ is an instantaneous inertial frame, as stated. Hereafter the orthonormal frame vectors as well as all tetrad components will be denoted by hatted indices. From (4.2), hatted indices are raised and lowered by the Minkowski metric $\eta_{\hat{\alpha}\hat{\beta}}$.

Let $e_{\hat{\alpha}}$, $\bar{e}_{\hat{\alpha}}$ be two orthonormal frames defined at a certain point of the spacetime manifold. Then

$$e_{\hat{\alpha}} \cdot e_{\hat{\beta}} = \bar{e}_{\hat{\alpha}} \cdot \bar{e}_{\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}}, \tag{4.3}$$

where dot defines the inner product induced by the background metric, that is $e_{\hat{\alpha}} \cdot e_{\hat{\beta}} \equiv g_{\mu\nu} e_{\hat{\alpha}}{}^{\mu} e_{\hat{\beta}}{}^{\nu}$. If L is a matrix which generates a change of frame, we have

$$\bar{e}_{\hat{\alpha}} = e_{\hat{\beta}} L^{\hat{\beta}}{}_{\hat{\alpha}}. \tag{4.4}$$

The orthonormality of both frames places restrictions on the components of the matrix L, i.e.

$$\eta_{\hat{\alpha}\hat{\beta}} = L^{\hat{\gamma}}{}_{\hat{\alpha}} L^{\hat{\delta}}{}_{\hat{\beta}} \eta_{\hat{\gamma}\hat{\delta}}. \tag{4.5}$$

The set of matrices L which satisfy (4.5) forms a group known as the Lorentz group of transformations. We shall impose the conditions

$$L^{\hat{0}}{}_{\hat{0}} > 0, \qquad \det(L) = 1, \tag{4.6}$$

which imply that future-pointing time-like vectors remain future-pointing. Lorentz transformations which satisfy conditions (4.6) are termed *orthochronous* and *proper*.

If u is a vector field whose integral curves form a congruence $\mathcal{C}_u$, and $\{e_{\hat{\alpha}}\}$ is a field of orthonormal bases over the above congruence, then the transport law for any vector of the tetrad along any direction of the frame itself is given by

$$\nabla_{e_{\hat{\alpha}}} e_{\hat{\beta}} = \Gamma^{\hat{\gamma}}{}_{\hat{\beta}\hat{\alpha}} e_{\hat{\gamma}}. \tag{4.7}$$

Here $\Gamma^{\hat{\gamma}}{}_{\hat{\beta}\hat{\alpha}}$ are the frame components of the connection coefficients, also termed *Ricci rotation coefficients*. The requirement that the frame remains orthonormal over the congruence is fulfilled by the condition

$$\Gamma_{(\hat{\gamma}\hat{\beta})\hat{\alpha}} = 0, \tag{4.8}$$

where we set

$$\Gamma_{\hat{\gamma}\hat{\beta}\hat{\alpha}} \equiv \eta_{\hat{\gamma}\hat{\sigma}} \Gamma^{\hat{\sigma}}{}_{\hat{\beta}\hat{\alpha}}\ . \tag{4.9}$$

Since the frame components of the metric are constant, from (2.51) it follows that the connection coefficients have the form

$$\Gamma^{\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}} = \frac{1}{2} \eta^{\hat{\alpha}\hat{\delta}} \left[C_{\hat{\delta}\hat{\gamma}\hat{\beta}} + C_{\hat{\beta}\hat{\delta}\hat{\gamma}} - C_{\hat{\gamma}\hat{\beta}\hat{\delta}} \right], \tag{4.10}$$

where $C^{\hat\gamma}{}_{\hat\beta\hat\delta}$ are the structure functions of the given frame. Using (3.45) and (3.60), and following the reasoning of Section 3.2, we have

$$\begin{aligned}
\nabla_{e_{\hat 0}} e_{\hat 0} &= \Gamma^{\hat a}{}_{\hat 0\hat 0} e_{\hat a} = a(u)^{\hat a} e_{\hat a},\\
\nabla_{e_{\hat 0}} e_{\hat a} &= \Gamma^{\hat\gamma}{}_{\hat a\hat 0} e_{\hat\gamma} = a(u)_{\hat a} e_{\hat 0} + C_{(\mathrm{fw})}{}^{\hat b}{}_{\hat a} e_{\hat b},\\
\nabla_{e_{\hat a}} e_{\hat 0} &= \Gamma^{\hat\gamma}{}_{\hat 0\hat a} e_{\hat\gamma} = -k(u)^{\hat b}{}_{\hat a} e_{\hat b},\\
\nabla_{e_{\hat a}} e_{\hat b} &= \Gamma^{\hat\gamma}{}_{\hat b\hat a} e_{\hat\gamma} = -k(u)_{\hat b\hat a} e_{\hat 0} + \Gamma^{\hat c}{}_{\hat b\hat a} e_{\hat c},
\end{aligned} \tag{4.11}$$

where $C_{(\mathrm{fw})}{}^{\hat b}{}_{\hat a}$ are related to $C_{(\mathrm{lie})}{}^{\hat b}{}_{\hat a}$ by (3.63) with hatted indices. Summarizing, we have

$$\begin{aligned}
&\Gamma^{\hat a}{}_{\hat 0\hat 0} = a(u)^{\hat a}, \qquad &&\Gamma^{\hat 0}{}_{\hat a\hat 0} = a(u)_{\hat a}, \qquad &&\Gamma^{\hat b}{}_{\hat a\hat 0} = C_{(\mathrm{fw})}{}^{\hat b}{}_{\hat a},\\
&\Gamma^{\hat b}{}_{\hat 0\hat a} = -k(u)^{\hat b}{}_{\hat a}, \qquad &&\Gamma^{\hat 0}{}_{\hat b\hat a} = -k(u)_{\hat b\hat a},
\end{aligned} \tag{4.12}$$

where each component has a precise physical meaning.

Let U be the unit time-like vector tangent to the world line γ of a particle and let it be analyzed by a family of observers u represented by a congruence of curves $\mathcal{C}_u$. Let $\{e_{\hat\alpha}\}$ be a field of tetrads which are adapted to u throughout the congruence $\mathcal{C}_u$. Clearly γ intersects at each of its points a curve of the congruence $\mathcal{C}_u$. We denote by $u|_\gamma$ and $\{e_{\hat a}\}_\gamma$ the restrictions on γ of the vector fields $u = e_{\hat 0}$ and $\{e_{\hat a}\}$.

With respect to the observer u, the vector field U admits the following representation:

$$U = \gamma(U,u)[u + \nu(U,u)^{\hat a} e_{\hat a}] \equiv \gamma[u + \nu\hat\nu(U,u)^{\hat a} e_{\hat a}]. \tag{4.13}$$

It is physically relevant to calculate the transport law for the vectors of $\{e_{\hat\alpha}\}_\gamma$ along γ as judged by the observer u himself. From Eqs. (4.13) and (4.11) we have

$$\begin{aligned}
\nabla_U e_{\hat a} &= \gamma \nabla_u e_{\hat a} + \gamma \nu^{\hat c} \nabla_{e_{\hat c}} e_{\hat a}\\
&= \gamma \left[a(u)_{\hat a} - k(u)_{\hat a\hat c}\nu^{\hat c}\right] u + \gamma \left[C_{(\mathrm{fw})}{}^{\hat d}{}_{\hat a} + \nu^{\hat c}\Gamma^{\hat d}{}_{\hat a\hat c}\right] e_{\hat d},\\
\nabla_U u &= \gamma \nabla_u u + \gamma \nu^{\hat c} \nabla_{e_{\hat c}} u\\
&= \gamma \left[a(u)^{\hat b} - k(u)^{\hat b}{}_{\hat c}\nu^{\hat c}\right] e_{\hat b},
\end{aligned} \tag{4.14}$$

where we have set $\nu(U,u)^{\hat a} = \nu^{\hat a}$ to simplify notation. Projecting onto LRS_u and recalling (3.156), we obtain

$$\begin{aligned}
P(u)\nabla_U e_{\hat a} &\equiv \frac{D_{(\mathrm{fw},U,u)}}{d\tau_U} e_{\hat a} = \gamma \left[C_{(\mathrm{fw})}{}^{\hat d}{}_{\hat a} + \nu^{\hat c}\Gamma^{\hat d}{}_{\hat a\hat c}\right] e_{\hat d},\\
P(u)\nabla_U u &= \nabla_U u = -F^{(G)}_{(\mathrm{fw},U,u)}{}^{\hat b} e_{\hat b}.
\end{aligned} \tag{4.15}$$

By using as a parameter the relative standard time $\tau_{(U,u)}$, the first line of (4.15) becomes

$$\frac{D_{(\mathrm{fw},U,u)}}{d\tau_{(U,u)}} e_{\hat a} = \left[C_{(\mathrm{fw})}{}^{\hat d}{}_{\hat a} + \nu^{\hat c}\Gamma^{\hat d}{}_{\hat a\hat c}\right] e_{\hat d}. \tag{4.16}$$

This equation represents the transport law along γ of the vectors of the triad $\{e_{\hat a}\}_\gamma$ as judged by u. Clearly the vectors $\{e_{\hat a}\}_\gamma$ undergo a rotation described by the tensor

$$\mathcal{R}^{\hat d}{}_{\hat a} = C_{(\mathrm{fw})}{}^{\hat d}{}_{\hat a} + \nu^{\hat c}\Gamma^{\hat d}{}_{\hat a\hat c}. \tag{4.17}$$

The antisymmetry of the Fermi-Walker structure functions $C_{(\mathrm{fw})\,\hat d\hat a}$ allows us to define a *Fermi-Walker angular velocity* vector $\zeta_{(\mathrm{fw})}$ by

$$C_{(\mathrm{fw})\,\hat d\hat a} = -\epsilon_{\hat d\hat a\hat f}\zeta^{\hat f}_{(\mathrm{fw})}, \tag{4.18}$$

which depends entirely on the properties of the tetrad $\{e_{\hat\alpha}\}$. Here we set $\epsilon_{\hat d\hat a\hat f} = \eta(u)_{\hat d\hat a\hat f}$ from (3.19). It is easy to verify that

$$C_{(\mathrm{fw})}{}^{\hat d}{}_{\hat a} e_{\hat d} = \zeta_{(\mathrm{fw})} \times_u e_{\hat a}. \tag{4.19}$$

In addition, from the antisymmetry of the first two indices of the Ricci rotation coefficients, the second term on the right-hand side of (4.16) can be written as

$$\nu^{\hat c}\Gamma^{\hat d}{}_{\hat a\hat c} = -\epsilon^{\hat d}{}_{\hat a\hat f}\zeta^{\hat f}_{(\mathrm{sc})}, \tag{4.20}$$

where $\zeta_{(\mathrm{sc})}$ is an instantaneous angular velocity vector, termed the *spatial curvature angular velocity*, which illustrates the behavior of the vector field $e_{\hat a}|_\gamma$ along $\nu(U,u)$ in LRS_u. It is easy to verify that

$$\nu^{\hat c}\Gamma^{\hat d}{}_{\hat a\hat c} e_{\hat d} = \zeta_{(\mathrm{sc})} \times_u e_{\hat a}. \tag{4.21}$$

In terms of (4.18) and (4.20), Eq. (4.16) can be written as

$$\frac{D_{(\mathrm{fw},U,u)}}{d\tau_{(U,u)}} e_{\hat a} = [\zeta_{(\mathrm{fw})} + \zeta_{(\mathrm{sc})}] \times_u e_{\hat a}. \tag{4.22}$$

Clearly, (4.22) allows us to describe the behavior along γ of any tensor defined in LRS_u. Consider as an example the unitary relative velocity direction $\hat\nu(U,u) = \hat\nu(U,u)^{\hat a} e_{\hat a} \equiv \hat\nu$. From (4.16) it follows that

$$\begin{aligned}\frac{D_{(\mathrm{fw},U,u)}\hat\nu}{d\tau_{(U,u)}} &= \left(\frac{D_{(\mathrm{fw},U,u)}\hat\nu}{d\tau_{(U,u)}}\right)^{\hat d} e_{\hat d}\\ &= \left[\frac{d\,\hat\nu^{\hat d}}{d\tau_{(U,u)}} + \hat\nu^{\hat a}\left(C_{(\mathrm{fw})}{}^{\hat d}{}_{\hat a} + \nu^{\hat c}\Gamma^{\hat d}{}_{\hat a\hat c}\right)\right] e_{\hat d}.\end{aligned} \tag{4.23}$$

Recalling the definition of the centripetal part of the Fermi-Walker acceleration of the particle U relative to u given in (3.165), namely

$$\nu \frac{D_{(\mathrm{fw},U,u)}\hat\nu}{d\tau_{(U,u)}} = a^{(C)}_{(\mathrm{fw},U,u)}, \tag{4.24}$$

we have

$$a^{(C)}_{(\mathrm{fw},U,u)} = \nu \left(\frac{d\,\hat{\nu}^{\hat{d}}}{d\tau_{(U,u)}} \right) e_{\hat{d}} + \left(\zeta_{(\mathrm{fw})} + \zeta_{(\mathrm{sc})}\right) \times_u \nu. \tag{4.25}$$

Tetrads $\{u = e_{\hat{0}}, e_{\hat{a}}\}$ may be further specified by particular transport laws for the triad $\{e_{\hat{a}}\}$ along the world line of u itself, as we will show next.

Fermi-Walker frames

A vector field X is said to be Fermi-Walker transported along the congruence $\mathcal{C}_u$ when its Fermi-Walker derivative along $\mathcal{C}_u$ is equal to zero, i.e., from (2.83)

$$\frac{D_{(\mathrm{fw},u)} X}{d\tau_u} = \nabla_u X + [a(u) \wedge u] \llcorner X = 0. \tag{4.26}$$

If $X = u$, then the above relation holds identically. If $X = e_{\hat{a}}$ we have instead

$$\begin{aligned} \frac{D_{(\mathrm{fw},u)} e_{\hat{a}}}{d\tau_u} &= \nabla_u e_{\hat{a}} - a(u)_{\hat{a}} u \\ &= a(u)_{\hat{a}} u + C_{(\mathrm{fw})}{}^{\hat{b}}{}_{\hat{a}} e_{\hat{b}} - a(u)_{\hat{a}} u \\ &= C_{(\mathrm{fw})}{}^{\hat{b}}{}_{\hat{a}} e_{\hat{b}} = 0. \end{aligned} \tag{4.27}$$

Since $\{e_{\hat{b}}\}$ is a basis in LRS_u, then $C_{(\mathrm{fw})}{}^{\hat{b}}{}_{\hat{a}} = 0$. In this case the tetrad frame is termed Fermi-Walker.

Absolute Frenet-Serret frames

A Frenet-Serret frame $\{E_{\hat{\alpha}}\}$ ($\alpha = 0, 1, 2, 3$) along a time-like curve with unit tangent vector U is defined as a solution of the following equations:

$$\begin{aligned} \frac{DE_{\hat{0}}}{d\tau_U} &= \kappa E_{\hat{1}}, & \frac{DE_{\hat{1}}}{d\tau_U} &= \kappa E_{\hat{0}} + \tau_1 E_{\hat{2}}, \\ \frac{DE_{\hat{2}}}{d\tau_U} &= -\tau_1 E_{\hat{1}} + \tau_2 E_{\hat{3}}, & \frac{DE_{\hat{3}}}{d\tau_U} &= -\tau_2 E_{\hat{2}}, \end{aligned} \tag{4.28}$$

where $E_{\hat{0}} = U$ and

$$\kappa = \Gamma^{\hat{1}}{}_{\hat{0}\hat{0}}, \qquad \tau_1 = C_{(\mathrm{fw})}{}^{\hat{2}}{}_{\hat{1}}, \qquad \tau_2 = C_{(\mathrm{fw})}{}^{\hat{3}}{}_{\hat{2}} \tag{4.29}$$

are respectively the magnitude of the acceleration and the first and second torsions of the world line. τ_1 and τ_2 are the components of the Frenet-Serret angular velocity $\omega_{(\mathrm{FS})} = \tau_1 E_{\hat{3}} + \tau_2 E_{\hat{1}}$ of the spatial triad $\{E_{\hat{a}}\}$ with respect to a Fermi-Walker frame defined along U. The frame $\{E_{\hat{\alpha}}\}$ (with dual $W^{\hat{\alpha}}$) is termed an *absolute* Frenet-Serret frame.

In compact form we have

$$\frac{DE_{\hat\alpha}}{d\tau_U} = E_{\hat\beta} C^{\hat\beta}{}_{\hat\alpha}, \tag{4.30}$$

with

$$C = C_{\mathcal{K}} + C_{\mathcal{T}}. \tag{4.31}$$

Here

$$C_{\mathcal{K}} = \kappa(E_{\hat 1} \otimes W^{\hat 0} + E_{\hat 0} \otimes W^{\hat 1}), \tag{4.32}$$

and

$$C_{\mathcal{T}} = \tau_{\hat 1}(E_{\hat 1} \otimes W^{\hat 2} + E_{\hat 2} \otimes W^{\hat 1}) + \tau_2(E_{\hat 2} \otimes W^{\hat 3} - E_{\hat 3} \otimes W^{\hat 2}), \tag{4.33}$$

$W^{\hat\alpha}$ being the dual frame of $E_{\hat\alpha}$. Therefore

$$C_{\mathcal{K}}^{\sharp} = -\kappa[W^{\hat 0} \wedge W^{\hat 1}] = -[a(U) \wedge U]^{\sharp}, \tag{4.34}$$

and we have the following relation for the Fermi-Walker derivative along U of a generic vector X:

$$\frac{D_{(\mathrm{fw},U)}X}{d\tau_U} = \frac{DX}{d\tau_U} - C_{\mathcal{K}} \,\llcorner\, X. \tag{4.35}$$

This form of the Fermi-Walker derivative along a time-like world line will be used to generalize to the case of a null world line (see, for instance, Bini *et al.*, 2006).

Relative Frenet-Serret frames

Let us consider again a curve γ with tangent vector field U and a family of observers u represented by the congruence $\mathcal{C}_u$ of its integral curves. These cross the world line of U and at each of its points one defines a vector representing the (unique) 3-velocity of the particle U, as measured locally by the observer u, namely

$$\nu(U,u) = \nu\hat\nu(U,u), \tag{4.36}$$

recalling that $U = \gamma[u + \nu\hat\nu(U,u)]$; clearly $\nu(U,u) \in LRS_u$. We shall now construct, along the world line of U, a frame in LRS_u following a Frenet-Serret procedure.

The first vector of this frame is chosen as the unit space-like direction $\hat\nu(U,u)$. The remaining two vectors are the relative normal $\hat\eta_{(\mathrm{fw},U,u)}$ which, apart from the sign, is defined as the normalized $P(u)$-projected covariant derivative of $\hat\nu(U,u)$ along U and its cross product with $\hat\nu(U,u)$, namely the relative binormal $\hat\beta_{(\mathrm{fw},U,u)} = \hat\nu(U,u) \times_u \hat\eta_{(\mathrm{fw},U,u)}$. Let $\{E_{(\mathrm{fw},U,u)\hat a}\} = \{\hat\nu(U,u), \hat\eta_{(\mathrm{fw},U,u)}, \hat\beta_{(\mathrm{fw},U,u)}\}$ be this frame.

Projecting the covariant derivatives along U of these vectors onto LRS_u and re-parameterizing the derivatives with respect to the spatial arc length

$$d\ell_{(U,u)} = \gamma(U,u)\nu(U,u)d\tau_U, \tag{4.37}$$

namely

$$\frac{D_{(\mathrm{fw},U,u)}}{d\tau_U} E_{(\mathrm{fw},U,u)\hat{a}} = \gamma\nu \frac{D_{(\mathrm{fw},U,u)}}{d\ell_{(U,u)}} E_{(\mathrm{fw},U,u)\hat{a}}, \tag{4.38}$$

leads to the *relative Frenet-Serret equations*:

$$\frac{D_{(\mathrm{fw},U,u)}}{d\ell_{(U,u)}} \hat{\nu}(U,u) = k_{(\mathrm{fw},U,u)} \hat{\eta}_{(\mathrm{fw},U,u)}, \tag{4.39}$$

$$\frac{D_{(\mathrm{fw},U,u)}}{d\ell_{(U,u)}} \hat{\eta}_{(\mathrm{fw},U,u)} = -k_{(\mathrm{fw},U,u)} \hat{\nu}(U,u) + \tau_{(\mathrm{fw},U,u)} \hat{\beta}_{(\mathrm{fw},U,u)}, \tag{4.40}$$

$$\frac{D_{(\mathrm{fw},U,u)}}{d\ell_{(U,u)}} \hat{\beta}_{(\mathrm{fw},U,u)} = -\tau_{(\mathrm{fw},U,u)} \hat{\eta}_{(\mathrm{fw},U,u)}. \tag{4.41}$$

Here the coefficients $k_{(\mathrm{fw},U,u)}$ and $\tau_{(\mathrm{fw},U,u)}$ are respectively the relative Fermi-Walker curvature and torsion of the world line of U as measured by u. The frame $E_{(\mathrm{fw},U,u)\hat{a}}$ is termed a *relative Frenet-Serret frame*.

Equation (4.39) does not allow one to fix the signs of the curvature $k_{(\mathrm{fw},U,u)}$ and of the frame vector $\hat{\eta}_{(\mathrm{fw},U,u)}$ individually. Once these signs are fixed, although arbitrarily, the third vector $\hat{\beta}_{(\mathrm{fw},U,u)} = \hat{\nu}_{(U,u)} \times_u \hat{\eta}_{(\mathrm{fw},U,u)}$ is chosen so as to make the frame right-handed, and this fixes the sign of the torsion.

The relative Frenet-Serret equations can be written in a more compact form as

$$\frac{D_{(\mathrm{fw},U,u)}}{d\ell_{(U,u)}} E_{(\mathrm{fw},U,u)\hat{a}} = \omega_{(\mathrm{fw},U,u)} \times_u E_{(\mathrm{fw},U,u)\hat{a}}, \tag{4.42}$$

where

$$\omega_{(\mathrm{fw},U,u)} = \tau_{(\mathrm{fw},U,u)} \hat{\nu}(U,u) + k_{(\mathrm{fw},U,u)} \hat{\beta}_{(\mathrm{fw},U,u)} \tag{4.43}$$

defines the relative Frenet-Serret angular velocity of the spatial frame.

The world line with 4-velocity U is said to be *u-relatively straight* if the Fermi-Walker relative curvature vanishes, $k_{(\mathrm{fw},U,u)} = 0$; and *$u$-relatively flat* if the Fermi-Walker relative torsion vanishes, $\tau_{(\mathrm{fw},U,u)} = 0$. When the relative curvature vanishes identically the relative Frenet-Serret frame can still be defined by adopting an appropriate limiting procedure. On the other hand, when it vanishes only at an isolated point, one must allow it to have either sign when it is non-zero in order to extend the relative normal continuously through this point.

Comoving relative Frenet-Serret frame

Consider again a test particle with 4-velocity U being the target of an observer u. The centripetal acceleration of the particle is measured by the observer u in his own LRS_u, while the centrifugal acceleration is an "inertial acceleration" measured by the particle U in *its* own LRS_U. In Newtonian mechanics these rest frames coincide, the time being absolute; therefore one can simply consider the centripetal acceleration as opposite to the centrifugal one.

In general relativity the observer and the test particle in relative motion have different rest frames, so a comparison between the two accelerations is only possible with a suitable geometrical representation of the centrifugal acceleration of the test particle in the rest frame of the observer.

If u is a vector field and $\mathcal{C}_u$ is the congruence of its integrable curves, at each point where the particle U crosses a curve of $\mathcal{C}_u$ the vectors u and U define a boost of $(u, \nu(U, u))$ into $(U, -\nu(u, U))$ and leaves the orthogonal 2-space $LRS_u \cap LRS_U$ invariant, as discussed in Section 3.3. Let

$$\hat{\mathcal{V}}(u, U) = -\hat{\nu}(u, U) = \gamma(\nu u + \hat{\nu}(U, u)) \tag{4.44}$$

be the negative of the relative velocity of u with respect to U as in (3.114).

The projection into the rest space of U of the covariant derivative of any vector field X defined on the world line of U and lying in LRS_U leads to the usual Fermi-Walker derivative along U:

$$\frac{D_{(\mathrm{fw},U)}X}{d\tau_U} = P(U)\frac{DX}{d\tau_U} = \gamma\nu\frac{D_{(\mathrm{fw},U)}X}{d\ell_{(U,u)}}, \qquad X \in LRS_U. \tag{4.45}$$

One can construct a Frenet-Serret-like frame starting from $\hat{\mathcal{V}}(u, U)$ by adding two new vectors, both lying in LRS_U, $\hat{\mathcal{N}}_{(\mathrm{fw},u,U)}$ being the normal and $\hat{\mathcal{B}}_{(\mathrm{fw},u,U)}$ the binormal of the world line of U. They are solutions of the Frenet-Serret relations

$$\begin{aligned}
\frac{D_{(\mathrm{fw},U)}}{d\ell_{(U,u)}}\hat{\mathcal{V}}(u, U) &= \mathcal{K}_{(\mathrm{fw},u,U)}\hat{\mathcal{N}}_{(\mathrm{fw},u,U)},\\
\frac{D_{(\mathrm{fw},U)}}{d\ell_{(U,u)}}\hat{\mathcal{N}}_{(\mathrm{fw},u,U)} &= -\mathcal{K}_{(\mathrm{fw},u,U)}\hat{\mathcal{V}}(u, U) + \mathcal{T}_{(\mathrm{fw},u,U)}\hat{\mathcal{B}}_{(\mathrm{fw},u,U)},\\
\frac{D_{(\mathrm{fw},U)}}{d\ell_{(U,u)}}\hat{\mathcal{B}}_{(\mathrm{fw},u,U)} &= -\mathcal{T}_{(\mathrm{fw},u,U)}\hat{\mathcal{N}}_{(\mathrm{fw},u,U)}.
\end{aligned} \tag{4.46}$$

Let us now prove that the following relation holds:

$$\begin{aligned}
\mathcal{K}_{(\mathrm{fw},u,U)}\hat{\mathcal{N}}_{(\mathrm{fw},u,U)} &= \gamma k_{(\mathrm{fw},U,u)}\hat{\eta}_{(\mathrm{fw},U,u)}\\
&\quad + \hat{\nu}(U, u) \times_u [\hat{\nu}(U, u) \times_u F^{(G)}_{(\mathrm{fw},U,u)}].
\end{aligned} \tag{4.47}$$

First recall that the Fermi-Walker derivative along U of a vector orthogonal to U reduces to the projection on LRS_U of its covariant derivative along U itself. Hence, differentiating (4.44) with respect to the parameter $\ell(U, u)$, we obtain

$$
\begin{aligned}
\frac{D_{(\mathrm{fw},U)}\hat{\mathcal{V}}(u,U)}{d\ell_{(U,u)}} &= \frac{1}{\gamma\nu}P(U)\frac{D}{d\tau_U}[\gamma(\nu u+\hat{\nu}(U,u))]\\
&= \frac{1}{\gamma\nu}P(U)\left[\frac{d(\gamma\nu)}{d\tau_U}u+\gamma\nu\frac{Du}{d\tau_U}+\hat{\nu}(U,u)\frac{d\gamma}{d\tau_U}+\gamma\frac{D\hat{\nu}(U,u)}{d\tau_U}\right]\\
&= \frac{1}{\gamma\nu}P(U)\left[\frac{d\gamma}{\nu d\tau_U}u-\gamma\nu F^{(G)}_{(\mathrm{fw},U,u)}+\hat{\nu}(U,u)\frac{d\gamma}{d\tau_U}+\gamma\frac{D\hat{\nu}(U,u)}{d\tau_U}\right]\\
&= \frac{1}{\gamma\nu}P(U)\left[\frac{d\gamma}{d\tau_U}\left(\frac{u}{\nu}+\hat{\nu}(U,u)\right)-\gamma\nu F^{(G)}_{(\mathrm{fw},U,u)}+\gamma\frac{D\hat{\nu}(U,u)}{d\tau_U}\right],
\end{aligned}
\tag{4.48}
$$

where we have used (3.156) and the relation

$$
\frac{d\gamma}{d\tau_U}=\gamma^3\nu\frac{d\nu}{d\tau_U}. \tag{4.49}
$$

The first term in square brackets in the last line of (4.48) is along U and therefore vanishes after contraction with $P(U)$; hence we have

$$
\frac{D_{(\mathrm{fw},U)}\hat{\mathcal{V}}(u,U)}{d\ell_{(U,u)}}=\frac{1}{\gamma\nu}P(U)\left[-\gamma\nu F^{(G)}_{(\mathrm{fw},U,u)}+\gamma\frac{D\hat{\nu}(U,u)}{d\tau_U}\right]. \tag{4.50}
$$

The last term on the right-hand side of (4.48) can be written as

$$
\begin{aligned}
\frac{D\hat{\nu}(U,u)}{d\tau_U} &= P(u)\frac{D\hat{\nu}(U,u)}{d\tau_U}-uu\cdot\frac{D\hat{\nu}(U,u)}{d\tau_U}\\
&= P(u)\frac{D\hat{\nu}(U,u)}{d\tau_U}+u\hat{\nu}(U,u)\cdot\frac{Du}{d\tau_U}\\
&= P(u)\frac{D\hat{\nu}(U,u)}{d\tau_U}-u\hat{\nu}(U,u)\cdot F^{(G)}_{(\mathrm{fw},U,u)}\\
&= \gamma\nu k_{(\mathrm{fw},U,u)}\hat{\eta}_{(\mathrm{fw},U,u)}-u\hat{\nu}(U,u)\cdot F^{(G)}_{(\mathrm{fw},U,u)}.
\end{aligned}
\tag{4.51}
$$

The projection orthogonal to U of this quantity is

$$
P(U)\frac{D\hat{\nu}(U,u)}{d\tau_U}=\gamma\nu k_{(\mathrm{fw},U,u)}\hat{\eta}_{(\mathrm{fw},U,u)}-\hat{\nu}(U,u)\cdot F^{(G)}_{(\mathrm{fw},U,u)}[P(U)u], \tag{4.52}
$$

noting that

$$
P(U)\hat{\eta}_{(\mathrm{fw},U,u)}=\hat{\eta}_{(\mathrm{fw},U,u)}+U\big(U\cdot\hat{\eta}_{(\mathrm{fw},U,u)}\big)=\hat{\eta}_{(\mathrm{fw},U,u)}, \tag{4.53}
$$

since $U = \gamma[u + \nu\hat{\nu}(U,u)]$ and $\hat{\nu}(U,u)\cdot\hat{\eta}_{(\mathrm{fw},U,u)} = 0$. From the above it then follows that

$$\begin{aligned}
\frac{D_{(\mathrm{fw},U)}\hat{\mathcal{V}}(u,U)}{d\ell_{(U,u)}} &= -P(U)F^{(G)}_{(\mathrm{fw},U,u)} + \gamma k_{(\mathrm{fw},U,u)}\hat{\eta}_{(\mathrm{fw},U,u)} \\
&\quad - \frac{\hat{\nu}(U,u)\cdot F^{(G)}_{(\mathrm{fw},U,u)}}{\nu}(u-\gamma U) \\
&= \gamma k_{(\mathrm{fw},U,u)}\hat{\eta}_{(\mathrm{fw},U,u)} - F^{(G)}_{(\mathrm{fw},U,u)} - U\gamma\nu\hat{\nu}(U,u)\cdot F^{(G)}_{(\mathrm{fw},U,u)} \\
&\quad - \frac{\hat{\nu}(U,u)\cdot F^{(G)}_{(\mathrm{fw},U,u)}}{\nu}(u-\gamma U) \\
&= \gamma k_{(\mathrm{fw},U,u)}\hat{\eta}_{(\mathrm{fw},U,u)} - F^{(G)}_{(\mathrm{fw},U,u)} \\
&\quad - \hat{\nu}(U,u)\cdot F^{(G)}_{(\mathrm{fw},U,u)}\left(\gamma\nu U + \frac{u-\gamma U}{\nu}\right) \\
&= \gamma k_{(\mathrm{fw},U,u)}\hat{\eta}_{(\mathrm{fw},U,u)} - F^{(G)}_{(\mathrm{fw},U,u)} + \hat{\nu}(U,u)\cdot F^{(G)}_{(\mathrm{fw},U,u)}\hat{\nu}(U,u).
\end{aligned}$$

Finally,

$$\begin{aligned}
\frac{D_{(\mathrm{fw},U)}\hat{\mathcal{V}}(u,U)}{d\ell_{(U,u)}} &= \gamma k_{(\mathrm{fw},U,u)}\hat{\eta}_{(\mathrm{fw},U,u)} \\
&\quad + \hat{\nu}(U,u)\times_u\left[\hat{\nu}(U,u)\times_u F^{(G)}_{(\mathrm{fw},U,u)}\right],
\end{aligned} \tag{4.54}$$

which completes the proof.

The cross product of (4.47) with $\hat{\mathcal{V}}(u,U)$ yields

$$\begin{aligned}
\mathcal{K}_{(\mathrm{fw},u,U)}\hat{\mathcal{V}}(u,U)\times_U\hat{\mathcal{N}}_{(\mathrm{fw},u,U)} &= \gamma k_{(\mathrm{fw},U,u)}\hat{\mathcal{V}}(u,U)\times_U\hat{\eta}_{(\mathrm{fw},U,u)} \\
&\quad + (\hat{\nu}(U,u)\cdot F^{(G)}_{(\mathrm{fw},U,u)})\hat{\mathcal{V}}(u,U)\times_U\hat{\nu}(U,u) \\
&\quad - \hat{\mathcal{V}}(u,U)\times_U F^{(G)}_{(\mathrm{fw},U,u)},
\end{aligned} \tag{4.55}$$

that is

$$\begin{aligned}
\mathcal{K}_{(\mathrm{fw},u,U)}\hat{\mathcal{B}}_{(\mathrm{fw},u,U)} &= \gamma k_{(\mathrm{fw},U,u)}\hat{\mathcal{V}}(u,U)\times_U\hat{\eta}_{(\mathrm{fw},U,u)} \\
&\quad + (\hat{\nu}(U,u)\cdot F^{(G)}_{(\mathrm{fw},U,u)})\hat{\mathcal{V}}(u,U)\times_U\hat{\nu}(U,u) \\
&\quad - \hat{\mathcal{V}}(u,U)\times_U F^{(G)}_{(\mathrm{fw},U,u)}.
\end{aligned} \tag{4.56}$$

To proceed further it is necessary to evaluate the cross product in LRS_U of a vector $X(U)$, for example, which belongs to LRS_U, with a vector $Y(u)$, for example, which belongs to LRS_u. The result is the following:

$$\begin{aligned}
X(U)\times_U Y(u) = \gamma\,\{&[P(u,U)X(U)\times_u Y(u)] \\
&- (\nu\cdot P(u,U)X(U))(\nu\times_u Y) \\
&- u[P(u,U)X(U)\cdot(\nu\times_u Y(u))]\}\,.
\end{aligned} \tag{4.57}$$

In our case $X(U) = \hat{\mathcal{V}}(u,U)$ and

$$P(u,U)\hat{\mathcal{V}}(u,U) = \gamma\hat{\nu}(U,u)\,; \tag{4.58}$$

therefore we have

$$\begin{aligned}\hat{\mathcal{V}}(u,U) \times_U Y(u) &= \gamma^2[\hat{\nu}(U,u) \times_u Y(u)] - \gamma^2\nu^2[\hat{\nu}(U,u) \times_u Y(u)]\\ &= \hat{\nu}(U,u) \times_u Y(u).\end{aligned} \tag{4.59}$$

From (4.56) it then follows that

$$\begin{aligned}\mathcal{K}_{(\mathrm{fw},u,U)}\hat{\mathcal{B}}_{(\mathrm{fw},u,U)} &= \gamma k_{(\mathrm{fw},U,u)}\hat{\nu}(U,u) \times_u \hat{\eta}_{(\mathrm{fw},U,u)}\\ &\quad - \hat{\nu}(U,u) \times_u F^{(G)}_{(\mathrm{fw},U,u)}\\ &= \gamma k_{(\mathrm{fw},U,u)}\hat{\beta}_{(\mathrm{fw},U,u)} - \hat{\nu}(U,u) \times_u F^{(G)}_{(\mathrm{fw},U,u)}.\end{aligned} \tag{4.60}$$

The spatial triad

$$\{\mathcal{E}_{(\mathrm{fw},u,U)\hat{a}}\} = \{\hat{\mathcal{V}}(u,U), \hat{\mathcal{N}}_{(\mathrm{fw},u,U)}, \hat{\mathcal{B}}_{(\mathrm{fw},u,U)} = \hat{\mathcal{V}}(u,U) \times_U \hat{\mathcal{N}}_{(\mathrm{fw},u,U)}\} \tag{4.61}$$

is termed a *comoving relative Frenet-Serret frame.*

In compact form the comoving relative Frenet-Serret relations become

$$\frac{D_{(\mathrm{fw},U)}}{d\ell_{(U,u)}}\mathcal{E}_{(\mathrm{fw},u,U)\hat{a}} = \Omega_{(\mathrm{fw},u,U)} \times_U \mathcal{E}_{(\mathrm{fw},u,U)\hat{a}}, \tag{4.62}$$

where $\Omega_{(\mathrm{fw},u,U)} = \mathcal{T}_{(\mathrm{fw},u,U)}\hat{\mathcal{V}}(u,U) + \mathcal{K}_{(\mathrm{fw},u,U)}\hat{\mathcal{B}}_{(\mathrm{fw},u,U)}$ defines the comoving relative angular velocity of this new frame with respect to a Fermi-Walker spatial frame. Again no assumption is made about the choice of signs for the curvature and torsion in this general discussion.

4.2 Null frames

Null frames include null vectors, and are particularly convenient for treating null fields such as electromagnetic and gravitational radiation. These frames can be complex or real.

Complex (Newman-Penrose) null frames

A (complex, null) Newman-Penrose frame $\{l, n, m, \bar{m}\}$ (Newman and Penrose, 1962) can be associated with an orthonormal frame $\{e_{\hat{\alpha}}\}$ in the following way:

$$\begin{aligned}l &= \frac{1}{\sqrt{2}}[e_{\hat{0}} + e_{\hat{1}}], \qquad & n &= \frac{1}{\sqrt{2}}[e_{\hat{0}} - e_{\hat{1}}],\\ m &= \frac{1}{\sqrt{2}}[e_{\hat{2}} + ie_{\hat{3}}], \qquad & \bar{m} &= \frac{1}{\sqrt{2}}[e_{\hat{2}} - ie_{\hat{3}}],\end{aligned} \tag{4.63}$$

with $l\cdot n = -1$ and $m\cdot\bar{m} = 1$. The four vectors $l, n, m, \bar{m}$ are null vectors ($l\cdot l = 0$, $n\cdot n = 0$, $m\cdot m = 0$, $\bar{m}\cdot\bar{m} = 0$); two of them are real (l, n) and the other two are complex conjugate $(m, \bar{m})$. The frame components of the metric are then equal to

$$(g_{ab}) = (g^{ab}) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{4.64}$$

Real null frames

Instead of a complex null frame it is useful to introduce a real null frame made up of two null vectors and two space-like vectors. Choosing the spatial vectors orthogonal to each other and normalized to unity, one constructs a quasi-orthogonal real null frame. Denoting such a frame as $\{E_\alpha\}$ $(\alpha = 1, 2, 3, 4)$, the standard relation with an orthonormal frame $\{e_{\hat{\alpha}}\}$ is the following:

$$\begin{aligned} E_1 &= \frac{1}{\sqrt{2}}(e_{\hat{0}} + e_{\hat{2}}), \\ E_2 &= e_{\hat{1}}, \\ E_3 &= \frac{1}{\sqrt{2}}(e_{\hat{0}} - e_{\hat{2}}), \\ E_4 &= e_{\hat{3}}. \end{aligned} \tag{4.65}$$

The frame components of the metric are given by

$$(g_{ab}) = (E_a \cdot E_b) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (g^{ab}). \tag{4.66}$$

Let us denote by Π and Π' the vector spaces generated by (E_1, E_3) and (E_2, E_4), respectively. The null vectors E_1 and E_3 are defined up to a multiplicative factor, while the spatial vectors E_2 and E_4 can be arbitrarily rotated in the 2-plane Π'. Therefore there arises a set of equivalent frames, each being a suitable Frenet-Serret frame along a null curve.

This frame satisfies the following set of Frenet-Serret evolution equations:

$$\frac{DE_1}{d\lambda} = -\mathcal{K}E_2, \qquad \frac{DE_2}{d\lambda} = \mathcal{T}_1 E_1 - \mathcal{K}E_3,$$

$$\frac{DE_3}{d\lambda} = \mathcal{T}_1 E_2 + \mathcal{T}_2 E_4, \qquad \frac{DE_4}{d\lambda} = \mathcal{T}_2 E_1, \tag{4.67}$$

where λ is an arbitrary parameter along E_1 and the quantities $\mathcal{K}$, $\mathcal{T}_1$, and $\mathcal{T}_2$ play roles analogous to the Frenet-Serret curvature and torsions in the time-like case. The connection matrix

$$\frac{DE_a}{d\lambda} = E_b C^b{}_a \tag{4.68}$$

has components

$$(C^a{}_b) = \begin{pmatrix} 0 & \mathcal{T}_1 & 0 & \mathcal{T}_2 \\ -\mathcal{K} & 0 & \mathcal{T}_1 & 0 \\ 0 & -\mathcal{K} & 0 & 0 \\ 0 & 0 & \mathcal{T}_2 & 0 \end{pmatrix}, \quad (C^{ab}) = \begin{pmatrix} 0 & \mathcal{T}_1 & 0 & \mathcal{T}_2 \\ -\mathcal{T}_1 & 0 & \mathcal{K} & 0 \\ 0 & -\mathcal{K} & 0 & 0 \\ -\mathcal{T}_2 & 0 & 0 & 0 \end{pmatrix}. \tag{4.69}$$

The invariants of the matrix are given by

$$I_1 = \frac{1}{2} C_{ab} C^{ab} = 2\mathcal{K}\mathcal{T}_1, \qquad I_2 = \frac{1}{2} C_{ab}{}^* C^{ab} = 2\mathcal{K}\mathcal{T}_2, \tag{4.70}$$

where

$$({}^*C^{ab}) = \left(\frac{1}{2}\eta^{abcd} C_{cd}\right) = \begin{pmatrix} 0 & \mathcal{T}_2 & 0 & -\mathcal{T}_1 \\ -\mathcal{T}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathcal{K} \\ \mathcal{T}_1 & 0 & \mathcal{K} & 0 \end{pmatrix}. \tag{4.71}$$

Finally, one can evaluate the four complex eigenvalues of the matrix $(C^a{}_b)$,

$$\begin{aligned} \lambda_{1,2} &= \pm\omega \equiv [-\mathcal{K}\mathcal{T}_1 + \mathcal{K}(\mathcal{T}_1^2 + \mathcal{T}_2^2)^{1/2}]^{1/2}, \\ \lambda_{3,4} &= \pm i\chi \equiv i[\mathcal{K}\mathcal{T}_1 + \mathcal{K}(\mathcal{T}_1^2 + \mathcal{T}_2^2)^{1/2}]^{1/2}, \end{aligned} \tag{4.72}$$

which in turn define two non-negative quantities ω and χ.

Let us consider the decompositions

$$\begin{aligned} C &= C_{\mathcal{K}} + C_{\mathcal{T}}, \\ C^\sharp &= C^\sharp_{\mathcal{K}} + C^\sharp_{\mathcal{T}} \end{aligned} \tag{4.73}$$

with

$$\begin{aligned} C_{\mathcal{K}} &= -\mathcal{K}(E_2 \otimes W^1 + E_3 \otimes W^2), \\ C^\sharp_{\mathcal{K}} &= \mathcal{K}(E_2 \wedge E_3), \end{aligned} \tag{4.74}$$

and

$$\begin{aligned} C_{\mathcal{T}} &= [\mathcal{T}_1(E_1 \otimes W^2 + E_2 \otimes W^3) + \mathcal{T}_2(E_1 \otimes W^4 + E_4 \otimes W^3)], \\ C^\sharp_{\mathcal{T}} &= \sqrt{\mathcal{T}_1^2 + \mathcal{T}_2^2}\left(E_1 \wedge \frac{\mathcal{T}_1 E_2 + \mathcal{T}_2 E_4}{\sqrt{\mathcal{T}_1^2 + \mathcal{T}_2^2}}\right), \end{aligned} \tag{4.75}$$

W^a being the dual frame of E_a. The curvature part $C_{\mathcal{K}}$ of the connection matrix generates a null rotation by an angle $|\mathcal{K}|$ in the time-like hyperplane of E_1, E_2, E_3, which leaves the null vector E_3 fixed ($C_{\mathcal{K}} \llcorner E_3 = 0$). The torsion part $C_{\mathcal{T}}$ of the connection matrix generates a null rotation by an angle $\sqrt{\mathcal{T}_1^2 + \mathcal{T}_2^2}$ in the time-like hyperplane of E_1, E_3, $(\mathcal{T}_1^2 + \mathcal{T}_2^2)^{-1/2}(\mathcal{T}_1 E_2 + \mathcal{T}_2 E_4)$, which leaves the null vector E_1 fixed ($C_{\mathcal{T}} \llcorner E_1 = 0$).

Along a Killing trajectory the Frenet-Serret scalars are all constant, leading to considerable simplification of the above formulas. This is the situation for null circular orbits in stationary axisymmetric space-times, for example.

There still remains the problem that the whole Frenet-Serret frame machinery is still subject to re-parameterization, under which only the relative ratios of the curvature and torsions are invariant. The curvature is just the norm of the second derivative,

$$\mathcal{K} = \left(\frac{D^2x}{d\lambda^2} \cdot \frac{D^2x}{d\lambda^2}\right)^{1/2} \geq 0, \tag{4.76}$$

and it transforms as the square of the related rate of change of the parameters $\lambda \to \hat{\lambda}$, that is

$$\mathcal{K} = \hat{\mathcal{K}} \left(\frac{d\hat{\lambda}}{d\lambda}\right)^2 . \tag{4.77}$$

Thus $d\Lambda = \mathcal{K}^{1/2} d\lambda = \hat{\mathcal{K}}^{1/2} d\hat{\lambda}$ is invariant (when non-zero), like the arc length in the non-null case, and can be used to introduce a preferred unit curvature parameterization when the curvature is everywhere non-vanishing, defined up to an additive constant like the arc length in the null case. This was already introduced by Vessiot for the Riemannian case in 1905 (Vessiot, 1905), who called Λ the *pseudo-arc length*.

The full transformation of all the Frenet-Serret quantities is easily calculated. Letting $\Lambda' = d\Lambda/d\lambda$ and using a hatted notation for the new quantities, one finds that the frame vectors undergo a boost and a null rotation, with the invariance of the last vector confirming the invariant nature of the osculating hyperplane for which it is the unit normal:

$$\begin{aligned} &\hat{E}_1 = (\Lambda')^{-1} E_1, \quad \hat{E}_2 = E_2 + \zeta E_1, \\ &\hat{E}_3 = \Lambda' \left[E_3 + \zeta\left(\frac{\zeta}{2}E_1 + E_2\right)\right], \quad \hat{E}_4 = E_4, \end{aligned} \tag{4.78}$$

where $\zeta = \Lambda''/(\mathcal{K}\Lambda')$, and the new Frenet-Serret quantities are given by

$$\hat{\mathcal{K}} = (\Lambda')^{-2}\mathcal{K}, \quad \hat{\mathcal{T}}_1 = \mathcal{T}_1 + \zeta' + \frac{\mathcal{K}}{2}\zeta^2, \quad \hat{\mathcal{T}}_2 = \mathcal{T}_2. \tag{4.79}$$

The pseudo-arc length parameterization is obtained by setting $\Lambda' = \mathcal{K}^{1/2}$ (so that $\zeta = \mathcal{K}'/(2\mathcal{K}^2)$), and the new form of the Frenet-Serret equations, once hats are dropped, can be obtained from (4.67) simply by making the substitutions $\mathcal{K} \to 1$ and $\lambda \to \Lambda$. Note that, in the case of constant curvature, $\Lambda'' = 0$ (and $\zeta = 0$); this is just a simple affine parameter transformation and both torsions remain unchanged, while the frame undergoes a simple boost constant rescaling.

Fermi-Walker transport along the null world line

Exactly as in the case of a time-like curve, one may retain only the torsion part of the connection matrix while absorbing the curvature part into the derivative to define a corresponding generalized Fermi-Walker transport along the null world line, as follows:

$$\frac{D_{(\mathrm{fw})}X^{\alpha}}{d\lambda} = \frac{DX^{\alpha}}{d\lambda} - C_{\mathcal{K}}{}^{\alpha}{}_{\beta}X^{\beta} = 0. \tag{4.80}$$

5

The world function

Spatial and temporal intervals between any two points in space-time are physically meaningful only if their measurement is made along a curve which joins them. A curve is a natural bridge which allows one to connect the algebraic structures at different points; therefore it is an essential tool to any measurement procedure. The mathematical quantities which glue together the concepts of points and curves are the two-point functions.[1] The most important of these is the world function, first introduced by Ruse (1931) and then used by Synge (1960). As Synge himself realized, the world function is well defined only locally; however, a global generalization has recently been found by Cardin and Marigonda (2004), opening the way to a better understanding of its potential in the theory of relativity. In de Felice and Clarke (1990) the world function was exploited to define spatial and temporal separations between points, and to find curvature effects in the measurement of angles and of relative velocities. Following their work we shall recall the main properties and applications of a world function, starting from its very definition.

Consider a smooth curve γ, parameterized by $s \in \Re$. Let $\dot\gamma$ be the field of vectors tangent to γ. The quantity

$$L = \int_\gamma |\dot\gamma \cdot \dot\gamma|^{\frac{1}{2}}\, ds \tag{5.1}$$

does not depend on the parameter s and therefore it provides the seed for a measurement of length on the curve. Given two points P_0 and P_1 in the manifold, there exist an infinite number of smooth curves γ joining them. Hence, setting $\mathrm{P}_0 = \gamma(s_0)$ and $\mathrm{P}_1 = \gamma(s_1)$, the quantity

$$L(s_0, s_1; \gamma) = \int_{s_0}^{s_1} |\dot\gamma \cdot \dot\gamma|^{\frac{1}{2}}\, ds \tag{5.2}$$

[1] One should say *many-point* functions in general but we shall only consider two- and three-point functions.

is a path-dependent function of the two points; clearly it vanishes if the curve is null. We use the term *world function* for the quantity

$$\Omega(s_0, s_1; \gamma) = \frac{1}{2}L^2, \tag{5.3}$$

which is a real two-point function.

The world function is not a map between tensor fields at different points; therefore meaningful applications require a transport law along any given curve.

5.1 The connector

Transport laws underlie the differential operations on the manifold; examples of these are the Lie and absolute derivatives. For the purpose of our analysis we need to discuss in some detail the transport law which leads to the absolute derivative.

Given a curve γ with parameter s and two points on it, $\mathrm{P}_0 = \gamma(s_0)$ and $\mathrm{P}_1 = \gamma(s_1)$, we use the term *connector* on γ for a map

$$\Gamma(s_0, s_1; \gamma) : T_{\mathrm{P}_0}(M) \to T_{\mathrm{P}_1}(M) \tag{5.4}$$

which carries vectors from P_0 to P_1. The effect of this map is described as

$$\Gamma(s_0, s_1; \gamma)u_{(0)} = \check{u}_{(1)} \tag{5.5}$$

for any $u_0 \in T_{\mathrm{P}_0}(M)$ and with $\check{u}_1 \in T_{\mathrm{P}_1}(M)$. Following de Felice and Clarke (1990), we summarize the main properties of this map. The connector is subject to the following conditions:

(i) *Linearity.* We require Γ to be a linear map, i.e. for any set of real numbers c^{A} and vectors $u_{(\mathrm{A})}$ at P_0, we have

$$\Gamma(s_0, s_1; \gamma)(c^{\mathrm{A}}u_{(\mathrm{A})}) = c^{\mathrm{A}}\Gamma(s_0, s_1; \gamma)u_{(\mathrm{A})}. \tag{5.6}$$

(ii) *Consistency.* If γ joins $\mathrm{P}_0 = \gamma(s_0)$ to $\mathrm{P}_1 = \gamma(s_1)$ and γ' joins $\mathrm{P}_1 = \gamma'(s_1)$ to $\mathrm{P}_2 = \gamma'(s_2)$ then

$$\Gamma(s_1, s_2; \gamma')\Gamma(s_0, s_1; \gamma) = \Gamma(s_0, s_2; \gamma \circ \gamma'), \tag{5.7}$$

where the symbol $\circ$ means the concatenation of the curves. Hence the effect of carrying a vector from P_0 to P_1 along γ and then from P_1 to P_2 along γ' is equivalent to carrying that vector from P_0 to P_2 along the curve $\gamma \circ \gamma'$.

(iii) *Parameterization independence.* The result of the action of Γ along a given curve does not depend on its parameterization.

(iv) *Differentiability.* We require that the result of the application of $\Gamma(s_0, s_1; \gamma)$ varies smoothly if we vary the points P_0 and P_1 and deform the whole path between them.

The consequences of the above conditions allow one to describe the connector as a tensor-like two-point function. Let $\{e_\alpha\}$ be a field of bases on some open set of the manifold containing the pair of points and the curve connecting them. From linearity we have

$$\begin{aligned}\Gamma(s_0, s_1; \gamma)u_{(0)} &= u^{\alpha_0}\Gamma(s_0, s_1; \gamma)e_{\alpha_0} = u^{\alpha_0}\Gamma(s_0, s_1; \gamma)_{\alpha_0}{}^{\beta_1}\, e_{\beta_1} \\ &= \check{u}^{\beta_1} e_{\beta_1},\end{aligned} \tag{5.8}$$

where the indices with subscripts 0 and 1 refer to quantities defined at the points P_0 and P_1 respectively. From the property of a basis we have

$$\check{u}^{\beta_1} = u^{\alpha_0}\Gamma(s_0, s_1; \gamma)_{\alpha_0}{}^{\beta_1}, \tag{5.9}$$

where the coefficients $\Gamma(s_0, s_1; \gamma)_{\alpha_0}{}^{\beta_1}$ are the components of the connector. We shall adopt the convention that the first index refers to the tangent basis at the first end-point of Γ and the second index to the tangent basis at the second end-point. From the above relation it follows that

$$\Gamma(s_0, s_0; \gamma)_{\alpha_0}{}^{\beta_0} = \delta^{\beta_0}_{\alpha_0}. \tag{5.10}$$

The requirement of consistency by concatenation implies that

$$\Gamma(s_0, s_1; \gamma)_{\sigma_0}{}^{\mu_1}\Gamma(s_1, s_2; \gamma)_{\mu_1}{}^{\nu_2} = \Gamma(s_0, s_2; \gamma)_{\sigma_0}{}^{\nu_2}. \tag{5.11}$$

Finally the requirement of differentiability, coupled with all the other properties of the connector, leads to the following law of differentiation (see de Felice and Clarke, 1990, for details):

$$\left.\frac{d}{ds}\Gamma(s_0, s; \gamma)_{\alpha_0}{}^{\rho}\right|_s = -\Gamma(s_0, s; \gamma)_{\alpha_0}{}^{\mu}\Gamma^{\rho}{}_{\mu\nu}(s)\dot{\gamma}^{\nu}(s), \tag{5.12}$$

with the initial condition given by (5.10). Here $\Gamma^{\rho}{}_{\mu\nu}$ are the *connection coefficients* on the manifold; they only depend on the point and not on the curves crossing it. It is well established that the connection coefficients are not the components of a $\binom{1}{2}$-tensor; nevertheless they behave as the components of such a tensor under linear coordinate transformations. This follows from the transformation properties of the components of the connector $\Gamma(s_0, s_1; \gamma)_{\alpha_0}{}^{\beta_1}$; the latter behave as a co-vector at P_0 and as a vector at P_1. From (5.9) and with respect to a general coordinate transformation $x'(x)$ we have

$$\Gamma'(s_0, s_1; \gamma)_{\alpha_0}{}^{\beta_1} = \frac{\partial x^{\mu_0}}{\partial x'^{\alpha_0}}\Gamma(s_0, s_1; \gamma)_{\mu_0}{}^{\nu_1}\frac{\partial x'^{\beta_1}}{\partial x^{\nu_1}}. \tag{5.13}$$

Hence from (5.12) it follows that

$$\Gamma'^{\lambda}{}_{\mu\nu}(x') = \frac{\partial x'^{\lambda}}{\partial x^{\alpha}}\frac{\partial x^{\beta}}{\partial x'^{\mu}}\frac{\partial x^{\gamma}}{\partial x'^{\nu}}\Gamma^{\alpha}{}_{\beta\gamma}(x) - \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial^2 x^{\rho}}{\partial x'^{\mu}\partial x'^{\nu}}. \tag{5.14}$$

A basic property of space-time geometry is the compatibility of the connection with the metric, expressed by the identity

$$\nabla_{\alpha} g_{\mu\nu} \equiv 0. \tag{5.15}$$

This relation stems from the requirement that the connector preserves the scalar product. Along any curve connecting points P_0 and P_1 we have

$$(u \cdot v)_0 = (\breve{u} \cdot \breve{v})_1, \tag{5.16}$$

with $u, v \in T_{\mathrm{P}_0}(M)$ and $\breve{u}, \breve{v} \in T_{\mathrm{P}_1}(M)$.

Let us conclude this brief summary by recalling the concept of geodesic on the manifold. A curve γ connecting P_0 to P_1 is geodesic if

$$\Gamma(s_0, s_1; \gamma)\dot{\gamma}_{(0)} = f(s_1)\dot{\gamma}_{(1)}, \tag{5.17}$$

where $f(s)$ is a differentiable function on γ. As stated, a geodesic can always be reparameterized with $s'(s)$ so that $f(s'(s)) = 1$; in this case the parameter s' is termed *affine*. An affine parameter is defined up to linear transformations.

5.2 Mathematical properties of the world function

The world function behaves as a scalar at each of its end-points and can be differentiated at each of them separately; in this case it generates new functions which may behave as a scalar at one point and a vector or 1-form at the other, or as a 1-form at the first point and as a vector at the other, and so on. To define the derivatives of a world function we shall consider only those whose end-points are connected by a geodesic. If we further restrict our analysis to normal neighborhoods then the geodesic connecting any two points is unique. The world function can be written as

$$\Omega(s_0, s_1; \Upsilon) = \frac{1}{2}(s_1 - s_0)^2\, X \cdot X, \tag{5.18}$$

where $X = \dot{\Upsilon}$ denotes the tangent vector to the unique geodesic Υ joining P_0 to P_1 and parameterized by s. We shall now deduce the derivatives of Ω with respect to variations of its two end-points. Let P_0 and P_1 belong to smooth curves $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$, parameterized by t, and let $\Upsilon_t \equiv \Upsilon_{\tilde{\gamma}_0(t) \to \tilde{\gamma}_1(t)}$ be the geodesic connecting points of $\tilde{\gamma}_0$ to points of $\tilde{\gamma}_1$. We then require that the curve connecting the points P_0 and P_1 as they vary independently on the curves $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ is the unique geodesic $\Upsilon_{\mathrm{P}_0 \to \mathrm{P}_1}$. In this case the points we are considering belong also to the geodesic Υ_t so they can be referred to as $\mathrm{P}_0 = \Upsilon_t(s_0)$ and $\mathrm{P}_1 = \Upsilon_t(s_1)$. This situation is depicted in Fig. (5.1). We then have a one-parameter family of geodesics C_X with connecting vector $Y = \dot{\tilde{\gamma}}$, namely $\pounds_X Y = 0$. By definition we have

$$\frac{DX}{ds} = 0, \qquad \frac{DX}{dt} = \frac{DY}{ds}. \tag{5.19}$$

To pursue our task let us write

$$\begin{aligned} \frac{d}{dt}\Omega(\Upsilon_t(s_0), \Upsilon_t(s_1); \Upsilon_t) &= \frac{\partial \Omega}{\partial x^{\alpha_0}} Y^{\alpha_0} + \frac{\partial \Omega}{\partial x^{\alpha_1}} Y^{\alpha_1} \\ &= (s_1 - s_0)^2 \left(\frac{DX}{dt} \cdot X \right), \end{aligned} \tag{5.20}$$

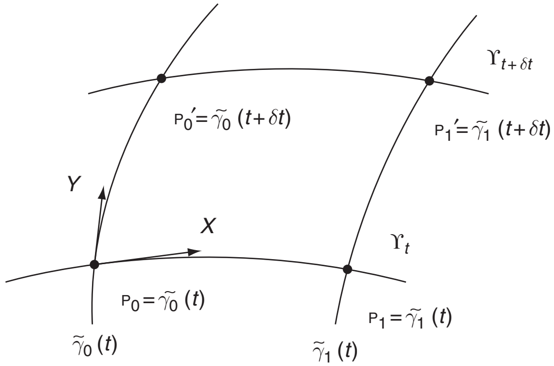

Fig. 5.1. The points $\mathrm{P}_0 = \tilde{\gamma}_0(t)$ and $\mathrm{P}_1 = \tilde{\gamma}_1(t)$ vary on the curves $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ and remain connected by the unique geodesic $\Upsilon_{\tilde{\gamma}_0(t)\to\tilde{\gamma}_1(t)}$. The parameter on the geodesic Υ_t is s with tangent vector X while that on the curves $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ is t with tangent vector Y.

where $\{x^{\alpha_0} = x^\alpha(s_0)\}$ and $\{x^{\alpha_1} = x^\alpha(s_1)\}$ are the local coordinates for $\Upsilon_t(s_0)$ and $\Upsilon_t(s_1)$ respectively. From the equation for geodesic deviation,

$$\frac{D^2Y}{ds^2} = R(X,Y)X, \tag{5.21}$$

we deduce identically that

$$\frac{D^2Y}{ds^2} \cdot X \equiv 0. \tag{5.22}$$

From $(5.19)_1$ we then have

$$\frac{D}{ds}\left(X \cdot \frac{DY}{ds}\right) = \frac{d}{ds}\left(\frac{d}{ds}(X \cdot Y)\right) = 0, \tag{5.23}$$

which implies that

$$\frac{d}{ds}(X \cdot Y) = \kappa, \tag{5.24}$$

where κ is a constant along $\Upsilon_t(s)$. Integration along $\Upsilon_t(s)$ from s_0 to s_1 yields

$$\kappa(s_1 - s_0) = [X \cdot Y]_{s_0}^{s_1}\ ; \tag{5.25}$$

hence we can write (5.24) more conveniently as

$$\frac{d}{ds}(X \cdot Y) = (s_1 - s_0)^{-1}\,[X \cdot Y]_{s_0}^{s_1}\,. \tag{5.26}$$

Setting

$$\Omega_{\alpha_0} = \frac{\partial\Omega}{\partial x^{\alpha_0}}, \qquad \Omega_{\alpha_1} = \frac{\partial\Omega}{\partial x^{\alpha_1}}, \tag{5.27}$$

and recalling (5.19)$_2$, Eq. (5.20) becomes

$$\begin{aligned}\frac{d\Omega}{dt} &= \Omega_{\alpha_0} Y^{\alpha_0} + \Omega_{\alpha_1} Y^{\alpha_1} = (s_1 - s_0)^2 \left(\frac{DY}{ds}\cdot X\right) \\ &= (s_1 - s_0)\left[X_{\alpha_1} Y^{\alpha_1} - X_{\alpha_0} Y^{\alpha_0}\right]; \end{aligned} \tag{5.28}$$

thus

$$\Omega_{\alpha_0} = -(s_1 - s_0) X_{\alpha_0}, \qquad \Omega_{\alpha_1} = (s_1 - s_0) X_{\alpha_1}. \tag{5.29}$$

From the latter we have

$$\Omega_{\alpha_0}\Omega^{\alpha_0} = \Omega_{\alpha_1}\Omega^{\alpha_1} = (s_1 - s_0)^2\, X\cdot X = 2\Omega, \tag{5.30}$$

or equivalently

$$g^{\alpha_0\beta_0}\Omega_{\alpha_0}\Omega_{\beta_0} = g^{\alpha_1\beta_1}\Omega_{\alpha_1}\Omega_{\beta_1} = 2\Omega. \tag{5.31}$$

The quantities in (5.29) are the components of 1-forms defined respectively at $\Upsilon_t(s_0)$ and $\Upsilon_t(s_1)$. Hence differentiating one of them, say $\Omega^\flat(s_0)$, with respect to t, we obtain

$$\left(\frac{D\Omega^\flat(0)}{dt}\right)_{\alpha_0} = \Omega_{\alpha_0\beta_0} Y^{\beta_0} + \Omega_{\alpha_0\beta_1} Y^{\beta_1} = -(s_1 - s_0)\left(\frac{DX}{dt}\right)_{\alpha_0}, \tag{5.32}$$

where $\Omega_{\alpha_0\beta_0} = \nabla_\beta\Omega_\alpha|_{(s_0)}$ etc.

From (5.19)$_2$ we have

$$\left(\frac{DX}{dt}\right)_{s_0} = \left(\frac{DY}{ds}\right)_{s_0}, \tag{5.33}$$

but now we need to know the derivative of the connecting vector field Y. This can be obtained from the general solution of the equation for geodesic deviation (de Felice and Clarke, 1990); we shall omit here the detailed derivation, which can be found in the cited literature, and provide only the results. To first order in the curvature,

$$\begin{aligned}\left(\frac{DY}{ds}\right)^{\alpha_0} &= -\alpha Y^{\alpha_0} + \alpha \Gamma_{\beta_1}{}^{\alpha_0} Y^{\beta_1} \\ &\quad - \alpha^2 Y^{\rho_0}\int_{s_0}^{s_1} (s_1 - s)^2 K^\beta{}_\tau \Gamma_{\rho_0}{}^\tau \Gamma_\beta{}^{\alpha_0} ds \\ &\quad - \alpha^2 Y^{\rho_1}\int_{s_0}^{s_1} (s_1 - s)(s - s_0) K^\sigma{}_\tau \Gamma_{\rho_1}{}^\tau \Gamma_\sigma{}^{\alpha_0} ds \\ &\quad + O(|\mathrm{Riem}|^2), \end{aligned} \tag{5.34}$$

where $O(|\mathrm{Riem}|^2)$ means terms of order 2 or larger in the curvature, $\alpha \equiv (s_1 - s_0)^{-1}$, and

$$K^\alpha{}_\rho = R^\alpha{}_{\mu\nu\rho} X^\mu X^\nu. \tag{5.35}$$

To simplify notation, we also set $\Gamma_{\alpha_0}{}^{\beta} \equiv \Gamma(s_0, s; \Upsilon_t)_{\alpha_0}{}^{\beta}$, these being the components of the connector defined on the curve Υ_t with parameter s. From $(5.19)_1$ and (5.32) we finally have

$$\begin{aligned}\Omega_{\alpha_0\beta_0} &= g_{\alpha_0\beta_0} + \alpha\, g_{\alpha_0\gamma_0} \int_{s_0}^{s_1} (s_1 - s)^2 K^{\rho}{}_{\tau} \Gamma_{\rho}{}^{\gamma_0} \Gamma_{\beta_0}{}^{\tau} ds \\ &\quad + O(|\mathrm{Riem}|^2),\end{aligned} \tag{5.36}$$

$$\begin{aligned}\Omega_{\alpha_0\beta_1} &= -g_{\alpha_0\gamma_0} \Gamma_{\beta_1}{}^{\gamma_0} + \alpha\, g_{\alpha_0\gamma_0} \int_{s_0}^{s_1} (s_1 - s)(s - s_0) K^{\rho}{}_{\tau} \Gamma_{\beta_1}{}^{\tau} \Gamma_{\rho}{}^{\gamma_0} ds \\ &\quad + O(|\mathrm{Riem}|^2).\end{aligned} \tag{5.37}$$

The values of the world function and its derivatives in the limit of coincident end-points are easily obtained as

$$\begin{aligned}\lim_{s_1 \to s_0} \Omega = \lim_{s_1 \to s_0} \Omega_{\alpha} &= 0 \\ \lim_{s_1 \to s_0} \Omega_{\alpha\beta} &= g_{\alpha\beta}(s_0).\end{aligned} \tag{5.38}$$

It is worth pointing out here that these limiting values do not depend on the path along which the end-points have been made to coincide.

When the space-time admits a set of n Killing vectors, say $\xi_{(a)}$ with $a = 1 \ldots n$, a world function connecting points on a geodesic satisfies the following property:

$$\xi_{(a)}^{\alpha_0} \Omega_{\alpha_0} + \xi_{(a)}^{\alpha_1} \Omega_{\alpha_1} = 0. \tag{5.39}$$

Using (5.39) and (5.31), one obtains the explicit form of the world function in special situations.

The simplest example is found in Minkowski space-time. Since in this case the geodesics are straight lines, the world function is just

$$\Omega^{\mathrm{flat}}(x_0, x_1) = \frac{1}{2} \eta_{\alpha\beta} (x_0^{\alpha} - x_1^{\alpha})(x_0^{\beta} - x_1^{\beta}), \tag{5.40}$$

whatever geodesic connects P_0 to P_1.

Recent applications of the world function to detect the time delay and frequency shift of light signals are due to Teyssandier, Le Poncin-Lafitte, and Linet (2008). Because of the potential of the world function approach to measurements, we judge it useful to derive its analytical form in various types of space-time metrics.

5.3 The world function in Fermi coordinates

Consider a general space-time metric and introduce a Fermi coordinate system (T, X, Y, Z) in some neighborhood of an accelerated world line γ with (constant) acceleration $\mathcal{A}$; the spatial coordinates X, Y, Z span the axes of a triad which is Fermi-Walker transported along γ while T measures proper time at the origin of the spatial coordinates. Up to terms linear in the spatial coordinates, one has (see (6.18) of Misner, Thorne, and Wheeler, 1973)

$$\begin{aligned} ds^2 &= (\eta_{\alpha\beta} + 2\mathcal{A}X\delta^0_\alpha\delta^0_\beta)dX^\alpha dX^\beta \\ &= -(1-2\mathcal{A}X)dT^2 + dX^2 + dY^2 + dZ^2 + O(2), \end{aligned} \tag{5.41}$$

a form which is valid within a world tube of radius $1/\mathcal{A}$ so that $|\mathcal{A}X| \ll 1$ is the condition for this approximation to be correct. In what follows we shall give general expressions for both time-like and null geodesics of the Fermi metric (5.41), as well as the expression for the world function (Bini *et al.*, 2008).

To first order in the acceleration parameter $\mathcal{A}$, the time-like geodesics can be written explicitly in terms of an affine parameter λ as

$$\begin{aligned} T(\lambda) &= T(0) + C\lambda + \mathcal{A}C\lambda(C^X\lambda + X(0)), \\ X(\lambda) &= X(0) + C^X\lambda + \frac{1}{2}\mathcal{A}C^2\lambda^2, \\ Y(\lambda) &= Y(0) + C^Y\lambda, \\ Z(\lambda) &= Z(0) + C^Z\lambda, \end{aligned} \tag{5.42}$$

where C, C^X, C^Y, C^Z are integration constants.

Let X^α_A and X^α_B be the Fermi coordinates of two general space-time points A and B connected by a geodesic. By using the explicit expressions (5.42) of the geodesics, and from the definition of the world function, we have

$$\Omega(X_A, X_B) = \frac{1}{2}[-C^2 + (C^X)^2 + (C^Y)^2 + (C^Z)^2], \tag{5.43}$$

where the orbital parameters have to be replaced by the coordinates of the initial and final points. Choosing the affine parameter in such a way that $\lambda = 0$ corresponds to X_A and $\lambda = 1$ to X_B we get the conditions

$$X^0_A = T(0), \quad X^1_A = X(0), \quad X^2_A = Y(0), \quad X^3_A = Z(0), \tag{5.44}$$

and

$$\begin{aligned} X^0_B &= X^0_A + C + \mathcal{A}C(C^X + X^1_A), \\ X^1_B &= X^1_A + C^X + \frac{1}{2}\mathcal{A}C^2, \\ X^2_B &= X^2_A + C^Y, \\ X^3_B &= X^3_A + C^Z. \end{aligned} \tag{5.45}$$

Next, solving the latter equations for C, C^X, C^Y, C^Z yields

$$\begin{aligned} C \ &\simeq (X^0_B - X^0_A)(1 - \mathcal{A}X^1_B), \\ C^X &\simeq (X^1_B - X^1_A) - \frac{1}{2}\mathcal{A}(X^0_B - X^0_A)^2, \\ C^Y &= X^2_B - X^2_A, \\ C^Z &= X^3_B - X^3_A \end{aligned} \tag{5.46}$$

to first order in $\mathcal{A}$. Substituting then in Eq. (5.43) gives the following final expression for the world function:

$$\Omega(X_A, X_B) \simeq \frac{1}{2}\left[\eta_{\alpha\beta} + \mathcal{A}(X_A^1 + X_B^1)\delta^0_\alpha\delta^0_\beta\right](X_A^\alpha - X_B^\alpha)(X_A^\beta - X_B^\beta)$$
$$= \Omega_{\rm flat}(X_A, X_B) + \frac{1}{2}\mathcal{A}(X_A^1 + X_B^1)(X_A^0 - X_B^0)^2, \tag{5.47}$$

to first order in the acceleration parameter $\mathcal{A}$.

5.4 The world function in de Sitter space-time

In isotropic and homogeneous coordinates, the de Sitter metric is given by (Stephani *et al.*, 2003):

$$ds^2 = -dt^2 + e^{2H_0 t}(dx^2 + dy^2 + dz^2), \tag{5.48}$$

where H_0 is the Hubble constant. Metric (5.48) satisfies Einstein's vacuum field equations with non-vanishing cosmological constant $\Lambda = 3H_0^2$ and Weyl flat spatial sections. It describes an empty expanding and non-rotating universe. Let us denote by $\{t, x^i\}$ ($\{x^i\} = \{x, y, z\}$, $i = 1, 2, 3$) the coordinates of an arbitrary event P of this space-time. A first integration of the geodesic equations is easily obtained using the Killing symmetries of the space-time. In terms of an affine parameter s, it gives

$$\frac{dx^i}{ds} = C^i e^{-2H_0 t}, \qquad \left(\frac{dt}{ds}\right)^2 = -\epsilon + C^2 e^{-2H_0 t}, \tag{5.49}$$

where $\epsilon = 0, -1, +1$ correspond to null, time-like, and space-like geodesics respectively, and $\{C^i\}$ are constants with $C^2 = \delta_{ij}C^iC^j$. Introduce a family of fiducial observers

$$n^\flat = -dt, \qquad n = \partial_t, \tag{5.50}$$

with the adapted orthonormal frame

$$e_{\hat t} = \partial_t, \quad e_{\hat i} = e^{-H_0 t}\partial_i. \tag{5.51}$$

The frame components $u^{\hat\alpha}_{(\epsilon)}$ of the vector tangent to the geodesics are given by

$$u_{(\epsilon)} = u^{\hat\alpha}_{(\epsilon)} e_{\hat\alpha} = \sqrt{-\epsilon + C^2 e^{-2H_0 t}}\left[n + \frac{C^i e^{-H_0 t}}{\sqrt{-\epsilon + C^2 e^{-2H_0 t}}}\, e_{\hat i}\right]. \tag{5.52}$$

Equations (5.49) can be easily integrated for the different values of ϵ (i.e. for any causal character) and the results can be summarized as follows:

$$e^{H_0 t} = \frac{C}{\sqrt{\epsilon}}\sin[\sqrt{\epsilon}(\sigma H_0 s + \alpha_0)],$$
$$x^i - x^i_0 = -\frac{C^i}{C^2\sigma H_0}\sqrt{\epsilon}\cot[\sqrt{\epsilon}(\sigma H_0 s + \alpha_0)], \tag{5.53}$$

where x_0^i are integration constants and σ is a sign indicator, corresponding to future-pointing ($\sigma = 1$) or past-pointing ($\sigma = -1$) geodesics, and the null case $\epsilon = 0$ is intended in the sense of the limit.

Let us require now that $s = 0$ corresponds to the space-time point A and $s = 1$ to B. This implies

$$\begin{aligned} e^{H_0 t_A} &= \frac{C}{\sqrt{\epsilon}} \sin[\sqrt{\epsilon}\alpha_0], \quad e^{H_0 t_B} = \frac{C}{\sqrt{\epsilon}} \sin[\sqrt{\epsilon}(\sigma H_0 + \alpha_0)], \\ x_A^i - x_0^i &= -\frac{C^i}{C^2 \sigma H_0} \sqrt{\epsilon} \cot[\sqrt{\epsilon}\alpha_0], \\ x_B^i - x_0^i &= -\frac{C^i}{C^2 \sigma H_0} \sqrt{\epsilon} \cot[\sqrt{\epsilon}(\sigma H_0 + \alpha_0)]. \end{aligned} \tag{5.54}$$

Moreover, from (5.2)

$$L = \int_0^1 \sqrt{|\epsilon|} ds = \sqrt{|\epsilon|} \tag{5.55}$$

and, from (5.3),

$$\Omega(X_A, X_B) = \frac{1}{2}|\epsilon|. \tag{5.56}$$

Using relations (5.54) one can then obtain ϵ as a function of the coordinates of A and B, and hence Ω. To this end it is convenient to write down an equivalent set of equations in place of (5.54) namely

$$\begin{aligned} \sqrt{\epsilon}\alpha_0 &= \arcsin\left(\frac{\sqrt{\epsilon}}{C} e^{H_0 t_A}\right), \\ \sqrt{\epsilon}\sigma H_0 &= \arcsin\left(\frac{\sqrt{\epsilon}}{C} e^{H_0 t_B}\right) - \arcsin\left(\frac{\sqrt{\epsilon}}{C} e^{H_0 t_A}\right), \\ \xi^2 &\equiv \delta_{ij}(x_B^i - x_A^i)(x_B^j - x_A^j) \\ &= \frac{\epsilon}{C^2 H_0^2} [\cot[\sqrt{\epsilon}(\sigma H_0 + \alpha_0)] - \cot[\sqrt{\epsilon}\alpha_0]]^2, \end{aligned} \tag{5.57}$$

in the three unknowns α_0, C^2, and ϵ. Using the the first two of these equations in the third one gives

$$H_0^2 \xi^2 = \frac{\epsilon}{C^2} \left\{ \cot\left[\arcsin\left(\frac{\sqrt{\epsilon}}{C} e^{H_0 t_B}\right)\right] - \cot\left[\arcsin\left(\frac{\sqrt{\epsilon}}{C} e^{H_0 t_A}\right)\right]\right\}^2, \tag{5.58}$$

which can be formally inverted to obtain the quantity $\sqrt{\epsilon}/C$ in terms of t_A, t_B, x_A^i, x_B^i. To see this in detail let us use the notation

$$w = \sqrt{\epsilon}/C, \quad a = e^{H_0 t_A}, \quad b = e^{H_0 t_B}; \tag{5.59}$$

we then have

$$H_0^2 \xi^2 = w^2 \left[\sqrt{\frac{1}{b^2 w^2} - 1} - \sqrt{\frac{1}{a^2 w^2} - 1}\right]^2, \tag{5.60}$$

from which

$$w^2 = \frac{[(a+b)^2 - H_0^2\xi^2 a^2 b^2][H_0^2\xi^2 a^2 b^2 - (a-b)^2]}{4\xi^2 a^4 b^4 H_0^2}. \tag{5.61}$$

Back-substituting this expression for $w = \sqrt{\epsilon}/C$ in the second of Eqs. (5.57) then gives ϵ,

$$\sqrt{\epsilon} = \frac{1}{\sigma H_0}[\arcsin(bw) - \arcsin(aw)], \tag{5.62}$$

and hence Ω, from (5.56).

5.5 The world function in Gödel space-time

In Cartesian-like coordinates $x^\alpha = (t, x, y, z)$, Gödel's metric takes the form (2.133) that we recall here (Gödel, 1949):

$$ds^2 = -dt^2 + dx^2 - \frac{1}{2}U^2 dy^2 - 2U dt dy + dz^2, \tag{5.63}$$

where $U = e^{\sqrt{2}\omega x}$ and ω is a constant. The symmetries of this metric are summarized by five Killing vector fields,

$$\begin{aligned}
\xi_t^\mu &= \partial_t, \quad \xi_y^\mu = \partial_y, \quad \xi_z^\mu = \partial_z, \quad \xi_4^\mu = \partial_x - \sqrt{2}\omega y \partial_y, \\
\xi_5^\mu &= -2e^{-\sqrt{2}\omega x}\partial_t + \sqrt{2}\omega y \partial_x + (e^{-2\sqrt{2}\omega x} - \omega^2 y^2)\partial_y.
\end{aligned} \tag{5.64}$$

Let us consider a geodesic Υ parameterized by an affine parameter λ with tangent vector $P^\mu = (\dot{t}, \dot{x}, \dot{y}, \dot{z})$, where dot means differentiation with respect to λ. The geodesic equations are

$$\begin{aligned}
&\ddot{t} + 2\sqrt{2}\omega \dot{t}\dot{x} + \sqrt{2}\omega U \dot{x}\dot{y} = 0, \\
&\ddot{x} + \sqrt{2}\omega U \dot{t}\dot{y} + \frac{1}{2}U^2\sqrt{2}\omega \dot{y}^2 = 0, \\
&\ddot{y} - 2\sqrt{2}\omega U^{-1}\dot{t}\dot{x} = 0, \\
&\ddot{z} = 0.
\end{aligned} \tag{5.65}$$

Using the Killing symmetries, this system of equations can be fully integrated. First we recall that

$$\begin{aligned}
p_t &= g_{t\alpha}\frac{dx^\alpha}{d\lambda} \equiv -E = \text{constant}, \\
p_y &= g_{y\alpha}\frac{dx^\alpha}{d\lambda} \equiv p_y = \text{constant}, \\
p_z &= \frac{dz}{d\lambda} \equiv p_z = \text{constant}
\end{aligned} \tag{5.66}$$

are conserved quantities. Then, from the above and the metric form (5.63), it follows that

$$\dot{t} = -E - \frac{2p_y}{U}, \qquad \dot{y} = \frac{2E}{U} + \frac{2p_y}{U^2}, \tag{5.67}$$

and

$$z(\lambda) = z_0 + p_z \lambda. \tag{5.68}$$

Using these relations in (5.65) we obtain

$$\ddot{x} - \frac{2\sqrt{2}\omega E p_y}{U} - \frac{2\sqrt{2}\omega p_y^2}{U^2} = 0. \tag{5.69}$$

Multiplying both sides of (5.69) by $2\dot{x}$ we can write it as

$$\frac{d}{d\lambda}\left(\dot{x}^2 + 4Ep_y e^{-\sqrt{2}\omega x} + 2p_y^2 e^{-2\sqrt{2}\omega x}\right) = 0, \tag{5.70}$$

which implies that

$$\dot{x}^2 + 4Ep_y e^{-\sqrt{2}\omega x} + 2p_y^2 e^{-2\sqrt{2}\omega x} = C_1, \tag{5.71}$$

where C_1 is a constant. The normalization condition $p \cdot p = -\epsilon$, with $\epsilon = (1, 0, -1)$ for time-like, null, and space-like orbits respectively, allows one to fix the constant C_1 as

$$C_1 = -\epsilon^2 - E^2 - p_z^2 \ ; \tag{5.72}$$

then Eq. (5.71) can be integrated by standard methods. Use of this solution in both Eqs. (5.67) leads us to the full integration of the geodesic equations. Clearly this allows us to write the exact world function along any non-null geodesic.

In a remarkable paper, Warner and Buchdahl (1980) were able to derive the exact form of the world function for Gödel's space-time, following an alternative approach. Their aim was to find the world function as a solution of a set of differential equations. Following their arguments, let us first recall from (5.39) that, given a set of Killing vectors $\xi_{(a)}$ and a pair of events with coordinates $\{x^{\mu'}\}$ and $\{x^\mu\}$, the world function satisfies the relation

$$\xi^{\mu'}_{(a)} \partial_{\mu'} \Omega + \xi^{\mu}_{(a)} \partial_\mu \Omega = 0. \tag{5.73}$$

We then have

$$\begin{aligned} \partial_{t'}\Omega + \partial_t \Omega &= 0, \\ \partial_{y'}\Omega + \partial_y \Omega &= 0, \\ \partial_{z'}\Omega + \partial_z \Omega &= 0, \end{aligned} \tag{5.74}$$

which imply for Ω a dependence of the type

$$\Omega(t' - t, x, x', y' - y, z' - z; \Upsilon) = \Omega_1(\tau, x, x', \xi; \Upsilon) + \frac{1}{2}(z' - z)^2, \tag{5.75}$$

where Ω_1 is a new function to be determined and

$$\tau = t' - t, \quad \xi = y' - y. \tag{5.76}$$

Using the remaining Killing vectors $\xi_{(4)}$ and $\xi_{(5)}$ we have from (5.64) the additional equations

$$0 = \partial_x \Omega - \sqrt{2}\omega y \partial_y \Omega + \partial_{x'} \Omega - \sqrt{2}\omega y' \partial_{y'} \Omega, \tag{5.77}$$

$$\begin{aligned} 0 &= \sqrt{2}\omega y \partial_x \Omega + (e^{-2\sqrt{2}\omega x} - \omega^2 y^2)\partial_y \Omega - 2e^{-\sqrt{2}\omega x}\partial_t \Omega \\ &\quad + \sqrt{2}\omega y' \partial_{x'} \Omega + (e^{-2\sqrt{2}\omega x'} - \omega^2 y'^2)\partial_{y'} \Omega \\ &\quad - 2e^{-\sqrt{2}\omega x'} \partial_t' \Omega. \end{aligned} \tag{5.78}$$

Introducing the new variables

$$v = e^{-\sqrt{2}\omega x}, \qquad v' = e^{-\sqrt{2}\omega x'}, \tag{5.79}$$

and recalling (5.75), Eq. (5.77) gives

$$v \partial_v \Omega_1 + v' \partial_{v'} \Omega_1 + \xi \partial_\xi \Omega_1 = 0, \tag{5.80}$$

whose solution is

$$\Omega_1 = \Omega_1 \left(\tau, \frac{v}{\xi}, \frac{v'}{\xi}\right). \tag{5.81}$$

Using this form of Ω_1 in (5.78) and denoting

$$\alpha = \frac{v}{\xi}, \qquad \beta = \frac{v'}{\xi}, \tag{5.82}$$

we obtain

$$\alpha(\omega^2 - \beta^2 + \alpha^2)\partial_\alpha \Omega_1 - \beta(\omega^2 - \alpha^2 - \beta^2)\partial_\beta \Omega_1 + 2\,(\alpha - \beta)\partial_\tau \Omega_1 = 0. \tag{5.83}$$

This equation can be considerably simplified with the further change of variables

$$s + r = \frac{\alpha}{\omega}, \qquad s - r = \frac{\beta}{\omega}, \qquad \tau_1 = \omega\tau, \tag{5.84}$$

leading to

$$s(4r^2 + 1)\partial_r \Omega_1 + r(4s^2 + 1)\partial_s \Omega_1 + 4r\partial_{\tau_1} \Omega_1 = 0. \tag{5.85}$$

This equation can be solved with the method of separation of variables. Setting $\Omega_1 = \Omega_1(r) + \Omega_1(s) + \Omega_1(\tau_1)$, we obtain

$$\Omega_1 = \frac{1}{8}\mathcal{K} \ln\left(\frac{4r^2 + 1}{4s^2 + 1}\right) + \bar{\mathcal{K}}\left(\frac{1}{2}\tan^{-1}(2s) - \tau_1\right), \tag{5.86}$$

where $\mathcal{K}$ and $\bar{\mathcal{K}}$ are separation constants. Hence the full world function (5.75) follows.

5.6 The world function of a weak gravitational wave

Consider the metric of a weak gravitational plane wave propagating along the x direction of a coordinate frame with "+, ×" polarization states, written in the form

$$ds^2 = -dt^2 + dx^2 + (1 - h_+)dy^2 + (1 + h_+)dz^2 - 2h_\times dxdz, \tag{5.87}$$

where the wave amplitudes $h_{+/\times} = h_{+/\times}(t-x)$ are functions of $(t-x)$. Let them be given by

$$h_+ = A_+ \sin\omega(t - x), \qquad h_\times = A_\times \cos\omega(t - x), \tag{5.88}$$

where linear polarization is characterized by $A_+ = 0$ or $A_\times = 0$, whereas circular polarization is assured by the condition $A_+ = \pm A_\times$. It is also useful to introduce the *polarization angle*, $\psi = \tan^{-1}(A_\times/A_+)$.

The geodesics of this metric are given by (de Felice, 1979)

$$\begin{aligned} U_{(\mathrm{geo})} = &\frac{1}{2E}[(\epsilon^2 + f + E^2)\partial_t + (\epsilon^2 + f - E^2)\partial_x] \\ &+ [\alpha(1 + h_+) + \beta h_\times]\partial_y + [\beta(1 - h_+) + \alpha h_\times]\partial_z, \end{aligned} \tag{5.89}$$

where α, β, and E are conserved Killing quantities, $\epsilon^2 = 1, 0, -1$ correspond to time-like, null, and space-like geodesics respectively, and to first order in the wave amplitudes

$$f \simeq \alpha^2(1 + h_+) + \beta^2(1 - h_+) + 2\alpha\beta h_\times. \tag{5.90}$$

The parametric equations of the geodesics are then easily obtained:

$$\begin{aligned} t(\lambda) &= E\lambda + t_0 + x(\lambda) - x_0, \\ x(\lambda) &= (\epsilon^2 + \alpha^2 + \beta^2 - E^2)\frac{\lambda}{2E} - \frac{1}{2\omega E^2}[(\alpha^2 - \beta^2)A_+ \cos\omega(E\lambda + t_0 - x_0) \\ &\quad - 2\alpha\beta A_\times \sin\omega(E\lambda + t_0 - x_0)] + x_0, \\ y(\lambda) &= \alpha\lambda + y_0 \\ &\quad - \frac{1}{\omega E}[\alpha A_+ \cos\omega(E\lambda + t_0 - x_0) - \beta A_\times \sin\omega(E\lambda + t_0 - x_0)], \\ z(\lambda) &= \beta\lambda + z_0 \\ &\quad + \frac{1}{\omega E}[\beta A_+ \cos\omega(E\lambda + t_0 - x_0) + \alpha A_\times \sin\omega(E\lambda + t_0 - x_0)], \end{aligned} \tag{5.91}$$

where λ is an affine parameter and x_0, y_0, z_0 are integration constants.

We can then evaluate the world function for two general points P_0 and P_1 connected by a geodesic Υ. A direct calculation gives

$$\begin{aligned} \Omega(s_0, s_1; \Upsilon) = &\ \Omega(s_0, s_1)^{\mathrm{flat}} \\ &+ \frac{A_+}{2\omega}[(y_1 - y_0)^2 - (z_1 - z_0)^2]\frac{\cos\omega(t_1 - x_1) - \cos\omega(t_0 - x_0)}{t_1 - x_1 - (t_0 - x_0)} \\ &- \frac{A_\times}{\omega}(y_1 - y_0)(z_1 - z_0)\frac{\sin\omega(t_1 - x_1) - \sin\omega(t_0 - x_0)}{t_1 - x_1 - (t_0 - x_0)}, \end{aligned} \tag{5.92}$$

where

$$\Omega(s_0, s_1)^{\rm flat} = \frac{1}{2}\eta_{\alpha\beta}(x_0 - x_1)^\alpha (x_0 - x_1)^\beta \tag{5.93}$$

is the world function in Minkowski space-time (see Bini *et al.*, 2009).

5.7 Applications of the world function: GPS or emission coordinates

Let us briefly review the standard construction of GPS coordinates in a flat space-time (Rovelli, 2002). Consider Minkowski space-time in standard Cartesian coordinates (t, x, y, z),

$$ds^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta = -dt^2 + dx^2 + dy^2 + dz^2, \tag{5.94}$$

and four satellites, represented by test particles in geodesic motion. With the above choice of coordinates, time-like geodesics are straight lines:

$$x^\alpha_{\rm A}(\tau^{\rm A}) = U^\alpha_{\rm A}\tau^{\rm A} + x^\alpha_{\rm A}(0), \qquad {\rm A} = 1, \ldots, 4, \tag{5.95}$$

where

$$U_{\rm A} = \gamma_{\rm A}[\partial_t + \nu_{\rm A} n^i_{\rm A}\partial_i] = \cosh\alpha_{\rm A}\partial_t + \sinh\alpha_{\rm A} n^i_{\rm A}\partial_i \tag{5.96}$$

are their (constant) 4-velocities and $\tau^{\rm A}$ is the proper time parameterization along each world line. In (5.96), $\gamma_{\rm A}$ is the Lorentz factor and the linear velocities $\nu_{\rm A}$ are relative to any of the four particles chosen as a fiducial observer; they are related to the rapidity parameters $\alpha_{\rm A}$ by $\nu_{\rm A} = \tanh\alpha_{\rm A}$; $n_{\rm A}$ denotes the space-like unit vectors along the spatial directions of motion. Without any loss of generality, we assume that the satellites all start moving from the origin of the coordinate system O, so hereafter we set $x^\alpha_{\rm A}(0) = 0$, and hence

$$x^\alpha_{\rm A}(\tau) = U^\alpha_{\rm A}\tau^{\rm A}. \tag{5.97}$$

Let us now consider a general space-time point $\bar{P}$ with coordinates $\bar{W}^\alpha$ and a point $P_{\rm A}$ with coordinates $x^\alpha_{\rm A}$ on the world line of the Ath satellite corresponding to an elapsed amount of proper time $\tau^{\rm A}$. A photon emitted at $P_{\rm A}$ follows a null geodesic, i.e. the straight line

$$x^\alpha(\lambda) = K^\alpha\lambda + x^\alpha_{\rm A}, \tag{5.98}$$

where λ is an affine parameter. Such a photon will reach $\bar{P}$ at a certain value $\bar{\lambda}$ of the parameter according to

$$\bar{W}^\alpha = K^\alpha\bar{\lambda} + x^\alpha_{\rm A}, \tag{5.99}$$

implying that

$$U^\alpha_{\rm A}\tau^{\rm A} - \bar{W}^\alpha = -K^\alpha\bar{\lambda}. \tag{5.100}$$

Taking the norm of both sides, we obtain

$$-(\tau^{\mathrm{A}})^2 + ||\bar{W}||^2 - 2\tau^{\mathrm{A}}(U_{\mathrm{A}} \cdot \bar{W}) = 0, \tag{5.101}$$

since K is a null vector. Solving for τ^{A} and selecting the solution corresponding to the past light cone leads to

$$\tau^{\mathrm{A}} = -(U_{\mathrm{A}} \cdot \bar{W}) - \sqrt{(U_{\mathrm{A}} \cdot \bar{W})^2 + ||\bar{W}||^2}. \tag{5.102}$$

These equations give the four proper times τ^{A} (i.e. the GPS or emission coordinates) associated with each satellite in terms of the Cartesian coordinates of the general point $\bar{P}$ in the space-time, i.e. $\tau^{\mathrm{A}} = \tau^{\mathrm{A}}(\bar{W}^0, \ldots, \bar{W}^3)$.

Using Eq. (5.102), one can evaluate the inverse of the transformed metric

$$g^{\mathrm{AB}} = \eta^{\alpha\beta} \frac{\partial \tau^{\mathrm{A}}}{\partial \bar{W}^\alpha} \frac{\partial \tau^{\mathrm{B}}}{\partial \bar{W}^\beta} \equiv \eta^{\alpha\beta} (d\tau^{\mathrm{A}})_\alpha (d\tau^{\mathrm{B}})_\beta = d\tau^{\mathrm{A}} \cdot d\tau^{\mathrm{B}}, \tag{5.103}$$

where the dual frame $(d\tau^{\mathrm{A}})_\alpha = \partial \tau^{\mathrm{A}} / \partial \bar{W}^\alpha$ also satisfies the properties

$$(d\tau^{\mathrm{A}})_\alpha \bar{W}^\alpha = \tau^{\mathrm{A}}, \qquad (d\tau^{\mathrm{A}})_\alpha U_{\mathrm{A}}^\alpha = 1. \tag{5.104}$$

Similarly one can introduce the frame vectors

$$\left(\frac{\partial}{\partial \tau^{\mathrm{A}}}\right)^\alpha = \frac{\partial \bar{W}^\alpha}{\partial \tau^{\mathrm{A}}}, \qquad (d\tau^{\mathrm{A}})_\alpha \left(\frac{\partial}{\partial \tau^{\mathrm{B}}}\right)^\alpha = \delta^{\mathrm{A}}_{\mathrm{B}}. \tag{5.105}$$

It is then easy to show that the condition $g^{\mathrm{AA}} = d\tau^{\mathrm{A}} \cdot d\tau^{\mathrm{A}} = 0$ is fulfilled. In fact, by differentiating both sides of Eq. (5.101) with respect to $\bar{W}^\alpha$ one obtains

$$(d\tau^{\mathrm{A}})_\alpha = \frac{\bar{W}_\alpha - \tau^{\mathrm{A}} U_{\mathrm{A}\alpha}}{\tau^{\mathrm{A}} + (U_{\mathrm{A}} \cdot \bar{W})}, \tag{5.106}$$

which implies that

$$g^{\mathrm{AA}} = (d\tau^{\mathrm{A}})^\alpha (d\tau^{\mathrm{A}})_\alpha = \frac{-(\tau^{\mathrm{A}})^2 + ||\bar{W}||^2 - 2\tau^{\mathrm{A}}(U_{\mathrm{A}} \cdot \bar{W})}{[\tau^{\mathrm{A}} + (U_{\mathrm{A}} \cdot \bar{W})]^2} = 0. \tag{5.107}$$

The metric coefficients $g_{\mathrm{AB}} = (\partial/\partial\tau^{\mathrm{A}}) \cdot (\partial/\partial\tau^{\mathrm{B}}) = \eta^{\alpha\beta}(\partial/\partial\tau^{\mathrm{A}})_\alpha (\partial/\partial\tau^{\mathrm{B}})_\beta$ can be easily obtained as well by expressing the Cartesian coordinates of $\bar{P}$ in terms of the emission coordinates τ^{A}, i.e. $\bar{W}^\alpha = \bar{W}^\alpha(\tau^1, \ldots, \tau^4)$.

To accomplish this, it is enough to invert the transformation (5.102). However, in order to outline a general procedure, we start by considering the equation for the past light cone of the general space-time point $\bar{P}$ with coordinates $\bar{W}^\alpha$ given in terms of the world function, which in the case of flat space-time is simply given by

$$\Omega_{\mathrm{flat}}(x_{\mathrm{A}}, x_{\mathrm{B}}) = \frac{1}{2} \eta_{\alpha\beta} (x_{\mathrm{A}}^\alpha - x_{\mathrm{B}}^\alpha)(x_{\mathrm{A}}^\beta - x_{\mathrm{B}}^\beta). \tag{5.108}$$

The condition ensuring that the past light cone of $\bar{P}$ cuts the emitter world lines is given by

$$\Omega_{\mathrm{flat}}(x_{\mathrm{A}}, \bar{W}) = 0, \qquad x_{\mathrm{A}}^0 < \bar{W}^0, \tag{5.109}$$

for each satellite labeled by the index A. This gives rise to a system of four quadratic equations in the four unknown coordinates $\bar{W}^\alpha$ of the event $\bar{P}$ of the form (5.101) for each $\text{A} = 1, \dots, 4$. To solve this system, start for example by subtracting the last equation from the first three equations to obtain the following system:

$$\begin{aligned}
&\Omega_{\text{flat}}(x_i, \bar{W}) - \Omega_{\text{flat}}(x_4, \bar{W}) = 0 = -2\bar{W} \cdot (x_i - x_4) - (\tau^i)^2 + (\tau^4)^2, \\
&\Omega_{\text{flat}}(x_4, \bar{W}) = 0 = ||\bar{W}||^2 - 2\bar{W} \cdot x_4 - (\tau^4)^2,
\end{aligned} \tag{5.110}$$

with $i = 1, 2, 3$, consisting of three linear equations and only one quadratic equation. Thus we can first solve the linear equations for the coordinates $\bar{W}^1, \bar{W}^2, \bar{W}^3$ in terms of $\bar{W}^0$, which then can be determined from the last quadratic equation. As a result, the coordinates of the event $\bar{P}$ are fully determined in terms of the satellite proper times τ^{A} and the known parameters characterizing their world lines.

An explicit result can easily be obtained for a fixed kinematical configuration of satellites. As an example one can take one of the satellites at rest at the origin O and the other three in motion along each of the three spatial directions:

$$\begin{aligned}
U_1 &= \cosh \alpha_1 \partial_t + \sinh \alpha_1 \partial_x, \\
U_2 &= \cosh \alpha_2 \partial_t + \sinh \alpha_2 \partial_y, \\
U_3 &= \cosh \alpha_3 \partial_t + \sinh \alpha_3 \partial_z, \\
U_4 &= \partial_t,
\end{aligned} \tag{5.111}$$

where α_i, $i = 1, 2, 3$, are the rapidities.

This choice is the one adopted in Bini *et al.* (2008) to construct emission coordinates in curved space-time (but for the metric associated with Fermi coordinates around the world line of an accelerated observer) covering a space-time region around the Earth.

6
Local measurements

The mathematical tools introduced in the previous chapters are essential for dealing with measurements in curved space-times. Here we shall confine our attention to local measurements only, i.e. to those whose measurement domain does not involve space-time curvature explicitly. Given an observer u, the tensorial projection operators $P(u)$ and $T(u)$ allow one to define the observer's rest-space and time dimension in neighborhoods of any of his points which are sufficiently small to allow one to approximate the measurement domain as a point. The above operators, in fact, arise naturally as the infinitesimal limit of the non-local procedure for the measurements of space distances and time intervals, as we will show.

6.1 Measurements of time intervals and space distances

Let γ be the world line of a physical observer; the parameter s on it is taken to be the proper time, so the tangent vector field $\dot{\gamma}$ is normalized as

$$\dot{\gamma} \cdot \dot{\gamma} = -1. \tag{6.1}$$

Here we shall analyze the concepts of *spatial* and *temporal* distances between two events relative to a given observer, referring closely to the analog in Euclidean geometry.

Consider an event P not belonging to γ but sufficiently close to it that a normal neighborhood exists which contains the intersections A_1 and A_2 of γ with the generators of the light cone in P. Referring to Fig. 6.1 we see that all points on γ between A_1 and A_2 can be connected to P by a unique non-time-like geodesic which we shall denote by ζ_s, with parameter σ. Let $\mathrm{A}_1 = \gamma(s_{\mathrm{A}_1})$ and $\mathrm{A}_2 = \gamma(s_{\mathrm{A}_2})$. Then from (5.18) the world function $\Omega(\sigma_{\mathrm{A}}, \sigma_{\mathrm{P}}; \zeta_s)$ along the geodesic ζ_s connecting a general point $\mathrm{A} = \gamma(s)$ with P is given by

$$\Omega(\sigma_{\mathrm{A}}, \sigma_{\mathrm{P}}; \zeta_s) = \frac{1}{2}(\sigma_{\mathrm{P}} - \sigma_{\mathrm{A}})^2 \left(\dot{\zeta}_s \cdot \dot{\zeta}_s\right)_{\mathrm{A}}; \tag{6.2}$$

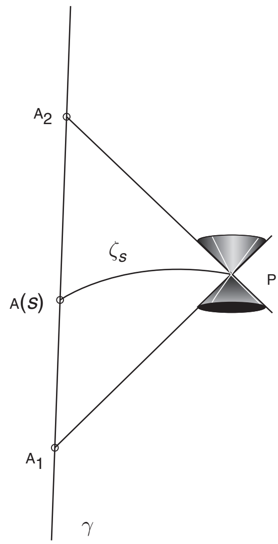

Fig. 6.1. A curve γ is connected to the point P by a light ray emitted at A_1 towards P and recorded at A_2 after being reflected at P. The entire process takes place in a normal neighborhood U_N of the space-time.

the geodesics ζ_s are assumed affine parameterized so $(\dot{\zeta}_s \cdot \dot{\zeta}_s)$ is constant on them. Because P is kept fixed and A is any point on $\gamma : s_{\mathrm{A}_1} \leq s \leq s_{\mathrm{A}_2}$, the world function in (6.2) is only a function of s; hence we write it in general as

$$\Omega(\sigma(s), \sigma_{\mathrm{P}}; \zeta_s) \equiv \Omega(s), \tag{6.3}$$

with the constraints

$$\Omega(s_{\mathrm{A}_1}) = \Omega(s_{\mathrm{A}_2}) = 0. \tag{6.4}$$

Since γ is a smooth curve, there exists a value s_{A_0}: $s_{\mathrm{A}_1} < s_{\mathrm{A}_0} < s_{\mathrm{A}_2}$ at which $\Omega(s)$ has an extreme value, namely at $\mathrm{A}_0 = \gamma(s_{\mathrm{A}_0})$:

$$\left.\frac{d\Omega}{ds}\right|_{\mathrm{A}_0} = 0. \tag{6.5}$$

If P is sufficiently close to γ then the point A_0 is unique. At A_0, (6.5) is equivalent to

$$\Omega_{\alpha_0} \dot{\gamma}^{\alpha_0} = 0, \tag{6.6}$$

where $\dot{\gamma}^{\alpha_0} \equiv \dot{\gamma}^{\alpha}(s_{\mathrm{A}_0})$ and Ω_{α_0} is the derivative at A_0 of the world function along the geodesic $\zeta_s|_{\mathrm{A}_0}$ joining A_0 to P and given from (5.29) as

$$\Omega_{\alpha_0} = -(\sigma_{\mathrm{P}} - \sigma_{\mathrm{A}_0})\xi_{\alpha_0}, \qquad \xi \equiv \dot{\zeta}_s|_{\mathrm{A}_0}, \tag{6.7}$$

where $\xi_{\alpha_0} = \xi_{\alpha}(\sigma_{\mathrm{A}_0})$. From (6.4) and (6.6) we deduce that

$$\dot{\gamma}_{\alpha_0} \xi^{\alpha_0} = 0. \tag{6.8}$$

We define as a measurement of the spatial distance between the observer on γ and the event P the length of the space-like geodesic segment on $\zeta_s|_{\mathrm{A}_0}$ which strikes γ orthogonally at A_0; that is (Synge, 1960),

$$L(\sigma_{\mathrm{A}_0}, \sigma_{\mathrm{P}}; \zeta_s|_{\mathrm{A}_0}) \equiv L(\mathrm{P}, \gamma) = \left(2\Omega(s_{\mathrm{A}_0})\right)^{\frac{1}{2}} = |\sigma_{\mathrm{P}} - \sigma_{\mathrm{A}_0}| \, (\xi \cdot \xi)^{1/2}_{\mathrm{A}_0}. \tag{6.9}$$

The event A_0 on γ is termed *simultaneous* to P with respect to the observer on γ. We now need to express L in terms of directly measurable quantities. Let s_{A_1} and s_{A_2} be the parameters of the events A_1 and A_2 on γ in which a light signal is emitted towards P and received after reflection at P, respectively. The quantity

$$\delta T_\gamma \equiv (s_{\mathrm{A}_2} - s_{\mathrm{A}_1}) \tag{6.10}$$

is a physical time directly readable on the observer's clock. The world function $\Omega(s)$ can be expanded in a power series about s_{A_0} as

$$\Omega(s) = \Omega(s_{\mathrm{A}_0}) + \sum_{n=1}^{\infty} \frac{1}{n!} \left. \frac{d^n\Omega}{ds^n} \right|_{s_{\mathrm{A}_0}} (s - s_{\mathrm{A}_0})^n. \tag{6.11}$$

If we require that the observer's world line be a geodesic ($\ddot{\gamma}^\alpha \equiv a(\dot{\gamma})^\alpha = 0$) then the derivatives of the world function can be written in general as

$$\frac{d^n\Omega}{ds^n} = \Omega_{\alpha_1 \ldots \alpha_n} \dot{\gamma}^{\alpha_1} \ldots \dot{\gamma}^{\alpha_n}. \tag{6.12}$$

Limiting ourselves to second derivatives only, we have that the coefficients $\Omega_{\alpha_0\beta_0}$ are given by (5.36). Assuming that the points A_0 and P are close enough, we can consider the above coefficients to first order in the curvature only. Therefore

$$\Omega_{\alpha_0\beta_0} \approx g_{\alpha_0\beta_0} + \frac{1}{2} \left[S_{\alpha\beta\gamma\delta} \, \xi^\gamma \xi^\delta \right]_{\mathrm{A}_0} (\sigma_{\mathrm{P}} - \sigma_{\mathrm{A}_0})^2, \tag{6.13}$$

where

$$S_{\alpha\beta\gamma\delta} = -\frac{2}{3} R_{\alpha(\gamma|\beta|\delta)}. \tag{6.14}$$

Noticing that along γ we have

$$\left. \frac{d\Omega}{ds} \right|_{\mathrm{A}_0} = 0, \tag{6.15}$$

Eq. (6.11) can be written as

$$\begin{aligned} \Omega(s) = {} & \Omega(s_{\mathrm{A}_0}) - \frac{1}{2}(s - s_{\mathrm{A}_0})^2 \\ & \times \left[1 + (\sigma_{\mathrm{P}} - \sigma_{\mathrm{A}_0}) \xi_\alpha a^\alpha - \frac{1}{2} S_{\alpha\beta\gamma\delta} \, \dot{\gamma}^\alpha \dot{\gamma}^\beta \xi^\gamma \xi^\delta (\sigma_{\mathrm{P}} - \sigma_{\mathrm{A}_0})^2 \right]_{\mathrm{A}_0} \\ & + O(|\mathrm{Riem}|^2). \end{aligned} \tag{6.16}$$

Imposing conditions (6.4) and requiring $a(\dot{\gamma})=0$, we can deduce from (6.16) the values s_{A_1} and s_{A_2} corresponding to the events A_1 and A_2. We then obtain

$$(s_{\mathrm{A}_1/\mathrm{A}_2}-s_{\mathrm{A}_0})^2\left[1-\frac{1}{2}S_{\alpha\beta\gamma\delta}\,u^\alpha u^\beta \xi^\gamma \xi^\delta (\sigma_{\mathrm{P}}-\sigma_{\mathrm{A}_0})^2\right]\approx 2\Omega(s_{\mathrm{A}_0}), \tag{6.17}$$

where we set $u\equiv\dot{\gamma}$ to stress the role of the *observer who makes the measurements* played by the vector field tangent to γ. Equations (6.17) admit the solutions

$$\begin{aligned} s_{\mathrm{A}_1} &\approx s_{\mathrm{A}_0}-[2\Omega(s_{\mathrm{A}_0})]^{\frac{1}{2}}\left[1+\frac{1}{4}S_{\alpha\beta\gamma\delta}\,u^\alpha u^\beta \xi^\gamma \xi^\delta (\sigma_{\mathrm{P}}-\sigma_{\mathrm{A}_0})^2\right]_{\mathrm{A}_0}, \\ s_{\mathrm{A}_2} &\approx s_{\mathrm{A}_0}+[2\Omega(s_{\mathrm{A}_0})]^{\frac{1}{2}}\left[1+\frac{1}{4}S_{\alpha\beta\gamma\delta}\,u^\alpha u^\beta \xi^\gamma \xi^\delta (\sigma_{\mathrm{P}}-\sigma_{\mathrm{A}_0})^2\right]_{\mathrm{A}_0}. \end{aligned} \tag{6.18}$$

From the above formulas and definitions we finally have

$$\delta T_\gamma \approx 2L(\mathrm{P},\gamma)+\frac{1}{2}\left(S_{\alpha\beta\gamma\delta}\,u^\alpha u^\beta \xi^\gamma \xi^\delta\right)_{\mathrm{A}_0} L^3(\mathrm{P},\gamma), \tag{6.19}$$

where we have re-parameterized $\zeta_{s_{\mathrm{A}_0}}$ so that $\xi\cdot\xi=1$. Equation (6.19) gives the first-order curvature contribution to the relationship between the round trip time of a bouncing signal and the geometrical distance between γ and P. Formal solutions of (6.19) are given by

$$L(\mathrm{P},\gamma)_1=\frac{1}{6}\chi^{1/3}\mathcal{A}^{-1}-4\chi^{-1/3}, \tag{6.20}$$

$$L(\mathrm{P},\gamma)_{2,3}=-\frac{1}{2}L(\mathrm{P},\gamma)_1\pm i\sqrt{3}\left[\frac{1}{12}\chi^{1/3}\mathcal{A}^{-1}+2\chi^{-1/3}\right], \tag{6.21}$$

where

$$\mathcal{A}=-\frac{1}{3}\left[R_{\alpha\gamma\beta\delta}\,u^\alpha u^\beta \xi^\gamma \xi^\delta\right]_{\mathrm{A}_0}, \tag{6.22}$$

$$\chi=12\mathcal{A}^2\left[9(\delta T_\gamma)+\sqrt{3}\sqrt{\frac{32}{\mathcal{A}}+27(\delta T_\gamma)^2}\right]. \tag{6.23}$$

Clearly, δT_γ can be read directly on the observer's clock and the quantities $\mathcal{A}$ and χ can be deduced by a suitable space-time modeling once the observer u is fixed. Solutions (6.20), with (6.22) and (6.23), may find application to the Global Positioning System (GPS), with general relativistic corrections to first order in the curvature.

Let us now uncover the role that the projection operators $P(u)$ and $T(u)$ have in defining the measurements of a time interval and a spatial distance, respectively, between two infinitesimally close events and relative to a given observer. Working in the infinitesimal domain we can neglect the curvature in (6.16) and limit ourselves to terms of the second order in $\Delta\sigma$ and Δs. Choose a point A on γ very close to A_0 (here γ is again a general time-like curve); Eq. (6.16) gives

$$2\Omega(s_{\mathrm{A}_0})=2\Omega(s_{\mathrm{A}})+(s_{\mathrm{A}_0}-s_{\mathrm{A}})^2+O(|\Delta\sigma+\Delta s|^3). \tag{6.24}$$

From (6.2) we have

$$2\Omega(s_{\mathrm{A}}) = (\sigma_{\mathrm{P}} - \sigma_{\mathrm{A}})^2 (\xi^\alpha \xi_\alpha)_{\mathrm{A}}, \tag{6.25}$$

so Eqs. (6.1) and (6.5) imply that

$$\begin{aligned} -(\sigma_{\mathrm{P}} - \sigma_{\mathrm{A}})(u^\alpha \xi_\alpha)_{\mathrm{A}} = \left.\frac{d\Omega}{ds}\right|_{\mathrm{A}} &= \left(\frac{d^2\Omega}{ds^2}\right)_{\mathrm{A}_0} (s_{\mathrm{A}} - s_{\mathrm{A}_0}) + O(\Delta s^2) \\ &= -(s_{\mathrm{A}} - s_{\mathrm{A}_0}) + O(|\Delta\sigma + \Delta s|^2). \end{aligned} \tag{6.26}$$

Hence (6.24) becomes

$$\begin{aligned} 2\Omega(s_{\mathrm{A}_0}) &= (\sigma_{\mathrm{P}} - \sigma_{\mathrm{A}})^2 g_{\alpha\beta}(s_{\mathrm{A}})\xi^\alpha(\sigma_{\mathrm{A}})\xi^\beta(\sigma_{\mathrm{A}}) \\ &\quad + (\sigma_{\mathrm{P}} - \sigma_{\mathrm{A}})^2 (u^\alpha \xi_\alpha)^2_{\mathrm{A}} + O(|\Delta\sigma + \Delta s|^3) \\ &= (\sigma_{\mathrm{P}} - \sigma_{\mathrm{A}})^2 \left[(g_{\alpha\beta} + u_\alpha u_\beta)\xi^\alpha \xi^\beta\right]_{\mathrm{A}} \\ &\quad + O(|\Delta\sigma + \Delta s|^3). \end{aligned} \tag{6.27}$$

Recalling that

$$\xi^\alpha = \lim_{\delta\sigma \to 0} \frac{\delta x^\alpha}{\delta\sigma},$$

we deduce, at a point P sufficiently close to γ, and from (6.9), that

$$\delta L(\mathrm{P}, \gamma) = \left[P(u)_{\alpha\beta}\delta x^\alpha \delta x^\beta\right]^{\frac{1}{2}} + O(\delta x^2), \tag{6.28}$$

where $P(u)_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$ and δx^α denotes the coordinate difference between A and P.

From (6.26) we interpret the time interval between the event A and the event A_0 on γ which is simultaneous to P as the temporal separation between the events A and P relative to the observer on γ; it is given by

$$\delta T(\mathrm{A}_0, \mathrm{A}) = -(u_\alpha \delta x^\alpha)_{\mathrm{A}} + O(\delta x^2). \tag{6.29}$$

Comparing (6.28) and (6.29) with the definition of the projection operators $P(u)$ and $T(u)$ one justifies their interpretation.

Relations (6.28) and (6.29) are very useful since they allow one to express the invariant measurements of the spatial distance and the time interval between any two events sufficiently close in terms of coordinates and vector components.

6.2 Measurements of angles

Angles can be measured with great accuracy; therefore their measurements enter in many problems as observables in terms of which one can fix boundary conditions. Clearly an angle is a spatial quantity; hence its measurement must be carried out in the rest space of the observer. Let the observer be represented by his world line γ and denote this by $u \equiv \dot{\gamma}$. Let us evaluate the angle between any two null directions stemming from a given point on γ.

At the space-time point where the measurement takes place, a null vector k admits the following decomposition:

$$k = -(k \cdot u)u + k_\perp = -(k \cdot u)[u + \hat{\nu}(k, u)], \quad \hat{\nu}(k, u) \cdot \hat{\nu}(k, u) = 1, \tag{6.30}$$

where $k_\perp = P(u)k = ||k_\perp||\hat{\nu}(k, u)$ and $\hat{\nu}(k, u)$ is the unitary spatial vector tangent to the local line of sight towards P. If a signal is sent from A to another point, say Q, and with a photon gun similar to the previous one, we identify this direction with a vector at A: $k'_\perp = P(u)k'$, where k' is the vector tangent to the null ray from A to Q. We then define the angle $\Theta_{(k,k')}$ between these two directions at A by

$$\cos \Theta_{(k,k')} = \frac{k'_\perp \cdot k_\perp}{||k_\perp||\,||k'_\perp||} = \hat{\nu}(k', u) \cdot \hat{\nu}(k, u). \tag{6.31}$$

Of course the above formula can be applied to time-like and space-like directions as well.

6.3 Measurements of spatial velocities

The instantaneous spatial velocity of a test particle with 4-velocity U relative to a given observer u has been introduced in (3.109) as the magnitude of the space-like 4-vector,

$$\nu(U, u)^\alpha = -(U^\sigma u_\sigma)^{-1} U^\alpha - u^\alpha. \tag{6.32}$$

Let us now show why this is interpreted as the instantaneous spatial velocity of the particle U with respect to the observer u. Since we are confining our attention to the infinitesimal domain, we shall deal with local measurements only. Let γ' be the world line of the particle with tangent field $U \equiv \dot{\gamma}'$ and assume that the curve γ' strikes the world line γ of u at a point A$'$. At this point the particle and the observer coincide so we can fix their proper times to coincide as well. Consider then a later moment when the particle is at a point P on its world line, still very close to A$'$. Once the particle has reached the point P, the observer u on γ will judge that it has covered a spatial distance

$$\delta L(\mathrm{P}, \gamma) = \left(P(u)_{\alpha\beta}\delta x^\alpha \delta x^\beta\right)^{1/2} \tag{6.33}$$

equal to the length of the (unique) space-like geodesic segment connecting P to the point A$_0$ on γ which is simultaneous with P with respect to u.

A correct way to measure the instantaneous velocity of recession (or approach) of the particle U with respect to the observer u is to track the particle with a light ray (see de Felice and Clarke, 1990, for details). Let A$_1$ be the point of γ which could be connected to P by a light ray; clearly $s_{\mathrm{A}'} < s_{\mathrm{A}_1} < s_{\mathrm{A}_0}$. From the previous discussion it follows that the quantities δx^α in (6.33) are the coordinate differences between P and any point A on γ between A$_1$ and A$_0$. Clearly the 4-vector δx^α is defined at the point A on γ. The approximation of confining ourselves to the infinitesimal domain allows one to identify A and A$_1$ with A$'$.

Hence, once the particle has reached the point P, the observer u will judge that the particle traveling from A$'$ to P took a time, as read on his own clock, equal to

$$\delta T(\mathrm{A}_0, \mathrm{A}') \approx -(u_\alpha \delta x^\alpha). \tag{6.34}$$

This interval of time actually measures the time interval on γ between A$'$, considered as a point of γ, and A$_0$. By definition, the instantaneous spatial velocity of U relative to u is the quantity

$$||\nu(U,u)|| \equiv \lim_{\delta x \to 0} \frac{\delta L(\mathrm{P}, \gamma)}{\delta T(\mathrm{A}_0, \mathrm{A}')} = -\frac{(P(u)_{\alpha\beta} dx^\alpha dx^\beta)^{1/2}}{(dx^\rho u_\rho)}. \tag{6.35}$$

The spatial instantaneous velocity 4-vector can be written as

$$\nu(U,u) = [\nu(U,u)^\alpha \nu(U,u)_\alpha]^{1/2} \hat{\nu}(U,u) \equiv ||\nu(U,u)|| \hat{\nu}(U,u), \tag{6.36}$$

with $\nu(U,u)^\alpha$ given by (6.32) and $\hat{\nu}(U,u)$ being the unitary vector introduced in (3.110). Clearly the 4-vector $\nu(U,u)$ belongs to LRS_u at the point A$'$ of γ. Owing to the symmetry of the projection operators, we have

$$||\nu(U,u)|| = ||\nu(u,U)|| \equiv \nu\,; \tag{6.37}$$

however, vector $\nu(u,U)$ belongs to LRS_U still at the point A$'$ but considered as a point of γ'. From (6.37) it follows as an obvious consequence that the Lorentz factor is

$$\gamma(U,u) = \gamma(u,U) = -(U^\alpha u_\alpha). \tag{6.38}$$

6.4 Composition of velocities

In the theory of relativity velocities add up according to a law which prevents them from becoming larger than the velocity of light, whatever observer one refers to.

Let U denote the 4-velocity of a test particle and u and $\bar{u}$ those of two different observers in relative motion. Confining our attention to the infinitesimal domain, we have

$$\begin{aligned} U &= \gamma(U,u)[u + \nu(U,u)] \\ &= \gamma(U,\bar{u})[\bar{u} + \nu(U,\bar{u})] \end{aligned} \tag{6.39}$$

and

$$\bar{u} = \gamma(\bar{u},u)[u + \nu(\bar{u},u)]. \tag{6.40}$$

From (6.39) it follows that

$$\frac{\gamma(U,\bar{u})}{\gamma(U,u)}[\bar{u} + \nu(U,\bar{u})] = [u + \nu(U,u)]. \tag{6.41}$$

Hence, contracting with $\bar{u}$ and recalling that

$$\bar{u}\cdot\bar{u}=-1=u\cdot u,\qquad \nu(U,\bar{u})\cdot\bar{u}=0,\qquad \nu(\bar{u},u)\cdot u=0,$$

we obtain identically

$$\gamma(\bar{u},u)[1-\nu(U,u)\cdot\nu(\bar{u},u)]=\frac{\gamma(U,\bar{u})}{\gamma(U,u)}. \tag{6.42}$$

Moreover from (6.40) and the first relation in (6.39) we obtain

$$\begin{aligned}\frac{U}{\gamma(U,u)}-u&=\nu(U,u),\\ \frac{\bar{u}}{\gamma(\bar{u},u)}-u&=\nu(\bar{u},u).\end{aligned} \tag{6.43}$$

Subtracting Eqs. (6.43) from one another we find

$$\frac{U}{\gamma(U,u)}-\frac{\bar{u}}{\gamma(\bar{u},u)}=\nu(U,u)-\nu(\bar{u},u), \tag{6.44}$$

and projecting orthogonally to $\bar{u}$,

$$P(\bar{u})\frac{U}{\gamma(U,u)}=\frac{\gamma(U,\bar{u})}{\gamma(U,u)}\nu(U,\bar{u})=P(\bar{u})[\nu(U,u)-\nu(\bar{u},u)]. \tag{6.45}$$

Using the identity (6.42), we can now write

$$\gamma(\bar{u},u)\nu(U,\bar{u})=P(\bar{u})\left[\frac{\nu(U,u)-\nu(\bar{u},u)}{1-\nu(U,u)\cdot\nu(\bar{u},u)}\right]. \tag{6.46}$$

The quantity in the square brackets *lives* in LRS_u; hence it can also be written as

$$P(u)\left[\frac{\nu(U,u)-\nu(\bar{u},u)}{1-\nu(U,u)\cdot\nu(\bar{u},u)}\right]. \tag{6.47}$$

Therefore relation (6.46) can be written as

$$\gamma(\bar{u},u)\nu(U,\bar{u})=P(\bar{u},u)\left[\frac{\nu(U,u)-\nu(\bar{u},u)}{1-\nu(U,u)\cdot\nu(\bar{u},u)}\right], \tag{6.48}$$

where $P(\bar{u},u)=P(\bar{u})P(u):\ LRS_u\to LRS_{\bar{u}}$, or as

$$P(\bar{u},u)^{-1}\gamma(\bar{u},u)\nu(U,\bar{u})=\frac{\nu(U,u)-\nu(\bar{u},u)}{1-\nu(U,u)\cdot\nu(\bar{u},u)}, \tag{6.49}$$

where $P(\bar{u},u)^{-1}:\ LRS_{\bar{u}}\to LRS_u$; this is the velocity composition law.

6.5 Measurements of energy and momentum

Consider the 4-momentum $P = \mu_0 U$ of a point-like test particle with mass μ_0; let us see what information an observer u reads out of its spatial and temporal splittings. In terms of these, the 4-momentum can be written as

$$P_\alpha = P(u)_\alpha{}^\beta P_\beta + T(u)_\alpha{}^\beta P_\beta = p(U,u)_\alpha + u_\alpha E(U,u), \tag{6.50}$$

where $p(U,u)$ is the spatial 4-momentum and $E(U,u)$ its total energy,[1] both relative to u. Let us now justify this interpretation. From (6.39) we have, using a coordinate-free notation,

$$p(U,u) = P(u)P = P + u(u \cdot P) = \mu_0[U + (u \cdot U)u]. \tag{6.51}$$

Recalling that $u \cdot U = -\gamma(U,u)$ and using (6.39), we have

$$p(U,u) = -\mu_0 \nu(U,u)\,(u \cdot U) = \mu_0 \gamma \nu(U,u) = \mu_0 \gamma \nu \hat{\nu}(U,u). \tag{6.52}$$

From (6.36), the magnitude of $p(U,u)$ is given by

$$||p(U,u)|| = \mu_0 \gamma \nu. \tag{6.53}$$

Similarly we have

$$E(U,u) = -(u \cdot P) = -\mu_0\,(u \cdot U) = \gamma \mu_0\,; \tag{6.54}$$

hence, the last two relations justify the physical interpretation of $p(U,u)$ and $E(U,u)$.

6.6 Measurements of frequencies

Consider a photon with 4-momentum k. With respect to an observer u, the 4-vector k admits the decomposition

$$\begin{aligned} k &= T(u)k + P(u)k \\ &= \omega(k,u)u + k_\perp \\ &= \omega(k,u)[u + \hat{\nu}(k,u)], \end{aligned} \tag{6.55}$$

where $\omega(k,u) = -u \cdot k$ represents the frequency of the photon as measured by the observer u; $k_\perp = P(u)k$ is the relative (spatial) momentum and $\hat{\nu}(k,u) = \omega(k,u)^{-1}\, k_\perp$ is a unitary space-like vector which identifies the local line of sight of the observer u. Whenever an observer absorbs a light signal, he measures its frequency and polarization. This operation takes place within a sufficiently small measurement's domain that we can limit our considerations to the local

[1] Here $E(U,u)$ is in units of the velocity of light c to ensure the dimensions of a momentum.

inertial frame where special relativity holds. The surface of discontinuity of an electromagnetic field is described in general by an equation

$$\Phi(x) = 0, \tag{6.56}$$

where $\Phi = \Phi(x)$ is a differentiable function of the coordinates known as the *eikonal* of the wave; it satisfies the equation

$$g^{\alpha\beta} k_\alpha k_\beta = 0 \tag{6.57}$$

(the eikonal equation), where $k_\alpha = \partial_\alpha \Phi$. In the observer's rest frame, reinterpreting Φ as the phase function in the geometrical optics approximation, the instantaneous frequency of the wave is given by

$$\omega(k, u) = -\frac{d\Phi}{d\tau_u}, \tag{6.58}$$

where τ_u is the proper time of the observer u. From the properties of Φ, (6.58) is more conveniently written as

$$\omega(k, u) = -(\partial_\alpha \Phi)\frac{dx^\alpha}{d\tau_u} = -k_\alpha u^\alpha. \tag{6.59}$$

This is the invariant characterization of the frequency of a light signal as measured by the observer u.

Let us now consider two observers with 4-velocities u and u', tangent to the curves γ and γ' respectively. Let the observers be far apart from each other and exchange a light signal. At the event of emission "e" on γ, the observer u measures a frequency $\omega(k, u)_\mathrm{e} = -(u_\alpha k^\alpha)_\mathrm{e}$; the *same* signal will be detected at the event "o" on γ', with frequency $\omega'(k, u')_\mathrm{o} = -(u'_\beta k^\beta)_\mathrm{o}$. The frequency $\omega(k, u)_\mathrm{e}$ at the point of emission should be *evaluated* at the point of observation on γ'. This evaluation is made possible by the properties of the parallel transport along a null geodesic. Along the null geodesic Υ joining "e" to "o" and having tangent vector k, the following relations hold:

$$(u^\alpha k_\alpha)_\mathrm{e} = (\check{u}^\alpha \check{k}_\alpha)_\mathrm{o} = (\check{u}^\alpha k_\alpha)_\mathrm{o}, \tag{6.60}$$

where $\check{u}$ and $\check{k}$ are the parallel propagated vectors along Υ with the further property $k_\alpha = \check{k}_\alpha$. The ratio between the emitted and observed frequencies at the point of observation is called the *frequency shift* and is denoted by

$$(1 + z)_\mathrm{o} = \left(\frac{\omega(\check{k}, \check{u})}{\omega'(k, u')}\right)_\mathrm{o} = \frac{(\check{u}_\alpha k^\alpha)_\mathrm{o}}{(u'_\alpha k^\alpha)_\mathrm{o}}. \tag{6.61}$$

When $(1 + z)_\mathrm{o} < 1$, we have a blue-shift, while when $(1 + z)_\mathrm{o} > 1$ we have a red-shift, both referred to u'.

If the frequency shift of the exchanged signal is due to relative motion, then the shift is known as a Doppler shift. This allows an indirect measurement of the relative velocity, termed the *Doppler velocity*. The variation of the frequency is directly observable and is given, in the observer's rest frame at any event on γ', by

$$\left(\frac{\omega(\check{k},\check{u})}{\omega'(k,u')}\right)_{\rm o} = \left(1 - ||\nu(\check{u},u')||^2\right)^{-1/2}\left[1 - ||\nu(\check{u},u')||\cos\Theta_{(k,\check{u})}\right]. \qquad (6.62)$$

Here $||\nu(\check{u},u')||$ is the magnitude of the instantaneous velocity of $\check{u}$ with respect to u' at the point of observation. Similarly, $\Theta_{(k,\check{u})}$ is the angle between the spatial direction of the light signal and that of the observer who emits it, but referred to the rest frame of the observer u' at o. Obviously the spatial direction of motion of the emitter evaluated in the rest frame of the observer u' is provided by the parallel transported vector $\check{u}$ at the point of observation on γ'. Recalling (6.60), Eq. (6.61) becomes, at "o" on γ',

$$\begin{aligned}\left(\frac{\omega(\check{k},\check{u})}{\omega'(k,u')}\right)_{\rm o} &= \left(\frac{k_\alpha \check{u}^\alpha}{k_\beta u'^\beta}\right)_{\rm o} \\ &= \frac{P(u')_{\alpha\beta}k^\alpha\check{u}^\beta - (u'_\alpha k^\alpha)(u'_\beta\check{u}^\beta)}{(u'_\rho k^\rho)} \\ &= -\frac{(P(u')_{\alpha\beta}k^\alpha\check{u}^\beta)(P(u')_{\rho\sigma}\check{u}^\rho\check{u}^\sigma)^{1/2}}{(P(u')_{\mu\nu}k^\mu k^\nu)^{1/2}(P(u')_{\pi\tau}\check{u}^\pi\check{u}^\tau)^{1/2}} - (u'_\beta\check{u}^\beta), \qquad (6.63)\end{aligned}$$

recalling that $(u'_\beta k^\beta)_{\gamma(s)} < 0$. From (6.31), however, this becomes

$$\begin{aligned}\left(\frac{\omega(\check{k},\check{u})}{\omega'(k,u')}\right)_{\rm o} &= -\cos\Theta_{(k,\check{u})}\left(P(u')_{\rho\sigma}\check{u}^\rho\check{u}^\sigma\right)^{1/2} - (u'^\beta\check{u}_\beta) \\ &= -(u'^\beta\check{u}_\beta)\left[1 + \cos\Theta_{(k,\check{u})}\frac{(P(u')_{\rho\sigma}\check{u}^\rho\check{u}^\sigma)^{1/2}}{(u'^\beta\check{u}_\beta)}\right]. \qquad (6.64)\end{aligned}$$

If we define the spatial velocity of u with respect to u' at "o" as the quantity

$$||\nu(\check{u},u')|| = -\left.\frac{(P(u')_{\rho\sigma}\check{u}^\rho\check{u}^\sigma)^{1/2}}{(u'_\beta\check{u}^\beta)}\right|_{\rm o}, \qquad (6.65)$$

we obtain

$$\left(1 - ||\nu(\check{u},u')||^2\right)^{-1/2} = -(u'_\beta\check{u}^\beta), \qquad (6.66)$$

so (6.62) is recovered.

Let us now better specify the measurement of frequencies and their comparison. In order to characterize the photon's frequency as the *frequency-at-the-emission*, one has to consider the atom as an observer with 4-velocity u who reads the frequency as $\omega_{\rm e} = -(u^\alpha k_\alpha)_{\rm e}$. In this expression the effect of the background geometry enters through the definition of the observer, the null geodesic k, and the scalar product itself. Let the emitted photon be observed by a distant observer with the 4-velocity u'. The *frequency-at-the-observation* is defined as $\omega' \equiv -(u'^\alpha k_\alpha)_{\rm o}$, where k is the same null geodesic which describes the emitted photon. At the point of observation one has in general a different space-time geometry, and the proper time of the atoms will run differently than that of the atoms at the point of emission. In order to make a comparison between the emitted and the observed frequencies, how does the observer at the observation point know about the frequency-at-the-emission? This information is carried by the emitted photon and *deposited* in a spectral line of the electromagnetic spectrum once it reaches the observer u' at the observation point. In this way the emitted frequency is directly *observed* at the observation point. Formally this information transfer is assured by parallel transport of the frequency-at-the-emission along the null geodesic, giving rise to the quantity $(\breve{\omega}_{\rm e})_{\rm o} = -(\breve{u}^\alpha \check{k}_\alpha)_{\rm o}$. Clearly, being the scalar product invariant under parallel transport, we have $(\breve{\omega}_{\rm e})_{\rm o} = \omega_{\rm e}$. This frequency may be compared with the frequency the photon would have had if the *same* atomic transition which occurred at the emission point did occur at the observation point with the local background geometry and with respect to the observer u'. The latter frequency is given by $\omega'_{\rm o}$.

6.7 Measurements of acceleration

Consider the world line of a test particle with tangent vector field U, and let $a(U) = \nabla_U U$ be its 4-acceleration due to the presence of an external non-gravitational force per unit of mass $f(U)$, so that

$$a(U) = f(U). \tag{6.67}$$

If a family of observers u is defined all along the world line of the particle then both its 4-acceleration and the external force can be expressed in terms of measurements made by u.

The local space and time splitting of $a(U)$ relative to u is given by[2]

$$\begin{aligned} a(U) &= P(u)a(U) + T(u)a(U) \\ &= P(u)a(U) - u(u \cdot a(U)). \end{aligned} \tag{6.68}$$

From the property $a(U) \cdot U = 0$ we deduce that $a(U) = P(U)a(U)$; hence (6.68) can be written as

$$\begin{aligned} a(U) &= P(u)P(U)a(U) - u(u \cdot a(U)) \\ &= P(u,U)a(U) - u(u \cdot a(U)). \end{aligned} \tag{6.69}$$

Using again the orthogonality between $a(U)$ and U and the splitting $U = \gamma(u + \nu(U,u))$, we have $u \cdot a(U) = -\nu(U,u) \cdot a(U)$; hence

$$a(U) = P(u,U)a(U) - u(\nu(U,u) \cdot P(u,U)a(U)). \tag{6.70}$$

[2] To simplify notation we often use juxtaposition of tensors to mean inner product. For example we write $P(u,U)a(U)$ instead of $P(u,U) \llcorner a(U)$.

Therefore

$$\begin{aligned} P(u,U)a(U) &= P(u)\nabla_U U \\ &= P(u)\left\{\gamma\nabla_U[u+\nu(U,u)] + [u+\nu(U,u)]\nabla_U\gamma\right\} \\ &= \gamma P(u)\nabla_U[u+\nu(U,u)] + \nu(U,u)\nabla_U\gamma, \end{aligned} \tag{6.71}$$

that is

$$\begin{aligned} P(u,U)a(U) &= \gamma P(u)\frac{Du}{d\tau_U} + P(u)\frac{D(\gamma\nu(U,u))}{d\tau_U} \\ &= \gamma\frac{D_{(\mathrm{fw},U,u)}u}{d\tau_U} + \frac{D_{(\mathrm{fw},U,u)}p(U,u)}{d\tau_U}, \end{aligned} \tag{6.72}$$

where $p(U,u) = \gamma\nu(U,u)$ is the relative linear momentum of the particle per unit mass, with respect to u. We have already defined in (3.156) the gravitational force (per unit mass) as

$$P(u)\frac{Du}{d\tau_U} = -F^{(G)}_{(\mathrm{fw},U,u)}; \tag{6.73}$$

hence we have in the local rest frame of u:

$$P(u,U)a(U) = -\gamma F^{(G)}_{(\mathrm{fw},U,u)} + \frac{D_{(\mathrm{fw},U,u)}p(U,u)}{d\tau_U}. \tag{6.74}$$

Let us define

$$P(u,U)f(U) \equiv \gamma F(U,u), \tag{6.75}$$

then recalling (6.67) we have

$$-\gamma F^{(G)}_{(\mathrm{fw},U,u)} + \frac{D_{(\mathrm{fw},U,u)}p(U,u)}{d\tau_U} = \gamma F(U,u), \tag{6.76}$$

so that, from (3.151), we obtain

$$\frac{D_{(\mathrm{fw},U,u)}p(U,u)}{d\tau_{(U,u)}} = F(U,u) + F^{(G)}_{(\mathrm{fw},U,u)}. \tag{6.77}$$

This is the *force equation* in a Newtonian-like form. Finally, scalar multiplication of both sides of (6.77) by $\nu(U,u)$ gives

$$\nu(U,u)\cdot\frac{D_{(\mathrm{fw},U,u)}p(U,u)}{d\tau_{(U,u)}} = \nu(U,u)\cdot[F(U,u) + F^{(G)}_{(\mathrm{fw},U,u)}]. \tag{6.78}$$

Recalling that $p(U,u) = \gamma\nu(U,u)$, we have

$$\begin{aligned} \nu(U,u) \cdot & \frac{D_{(\mathrm{fw},U,u)}(\gamma\nu(U,u))}{d\tau_{(U,u)}} \\ &= \frac{d\gamma}{d\tau_{(U,u)}}\nu^2 + \gamma\left(\nu(U,u)\cdot\frac{D_{(\mathrm{fw},U,u)}\nu(U,u)}{d\tau_{(U,u)}}\right) \\ &= \frac{d\gamma}{d\tau_{(U,u)}}\left(\nu^2 + \frac{1}{\gamma^2}\right) = \frac{d\gamma}{d\tau_{(U,u)}}, \end{aligned} \tag{6.79}$$

where we have used the relation

$$\frac{d\gamma}{d\tau_{(U,u)}} = \gamma^3 \left(\nu(U,u) \cdot \frac{D_{(\mathrm{fw},U,u)}\nu(U,u)}{d\tau_{(U,u)}} \right). \tag{6.80}$$

Therefore, we have

$$\frac{dE(U,u)}{d\tau_{(U,u)}} = \nu(U,u) \cdot [F(U,u) + F^{(G)}_{(\mathrm{fw},U,u)}], \tag{6.81}$$

where $E(U,u) = \gamma(U,u) \equiv \gamma$ is the energy of the particle per unit of mass. Equation (6.81) is the *power equation* in a Newtonian-like form.

Longitudinal-tranverse splitting of the force equation

The measurement of the acceleration of a test particle in the presence of an external force permitted a Newtonian-like representation of the equations of motion, as shown by (6.77) and (6.81). One can complete this analysis by considering a further splitting of these equations along directions parallel (longitudinal) and orthogonal (transverse) to that of the velocity of the particle relative to the observer u. A key tool for this study is the use of the relative Frenet-Serret frames introduced in Chapter 4.

The transverse splitting of the relative acceleration defines the relative centripetal acceleration. This splitting is accomplished by projecting (6.77) onto the relative Frenet-Serret frame $\{\hat{\nu}(U,u), \hat{\eta}_{(\mathrm{fw},U,u)}, \hat{\beta}_{(\mathrm{fw},U,u)}\}$, as discussed in Eqs. (4.39), (4.40), and (4.41).

Let $||p(U,u)|| = \gamma||\nu(U,u)||$ be the magnitude of the specific (i.e. per unit mass) spatial momentum of U as seen by u, so that

$$p(U,u) = ||p(U,u)||\hat{\nu}(U,u), \tag{6.82}$$

while the gamma factor is the corresponding specific energy $E(U,u) = \gamma$. They satisfy the identity

$$E(U,u)^2 - ||p(U,u)||^2 = 1. \tag{6.83}$$

Then (6.77) takes the form

$$\begin{aligned} F(U,u) + F^{(G)}_{(\mathrm{fw},U,u)} &= \frac{d||p(U,u)||}{d\tau_{(U,u)}}\hat{\nu}(U,u) + ||p(U,u)||\frac{D_{(\mathrm{fw},U,u)}\hat{\nu}(U,u)}{\tau_{(U,u)}} \\ &= \frac{d||p(U,u)||}{d\tau_{(U,u)}}\hat{\nu}(U,u) \\ &\quad + \gamma\nu(U,u)^2 k_{(\mathrm{fw},U,u)}\hat{\eta}_{(\mathrm{fw},U,u)}, \end{aligned} \tag{6.84}$$

where (4.39) has been used.

The relative Frenet-Serret components of this equation, i.e. the projections on the axes of the triad $\{\hat{\nu}(U,u), \hat{\eta}_{(\mathrm{fw},U,u)}, \hat{\beta}_{(\mathrm{fw},U,u)}\}$, are given by

$$\begin{aligned}
\frac{dE(U,u)}{d\ell_{(U,u)}} &= [F(U,u) + F^{(G)}_{(\mathrm{fw},U,u)}] \cdot \hat{\nu}(U,u), \\
\gamma\nu(U,u)^2 k_{(\mathrm{fw},U,u)} &= [F(U,u) + F^{(G)}_{(\mathrm{fw},U,u)}] \cdot \hat{\eta}_{(\mathrm{fw},U,u)}, \\
0 &= [F(U,u) + F^{(G)}_{(\mathrm{fw},U,u)}] \cdot \hat{\beta}_{(\mathrm{fw},U,u)},
\end{aligned} \tag{6.85}$$

where $d\ell_{(U,u)} = \nu\tau_{(U,u)}$ and

$$\frac{d||p(U,u)||}{d\tau_{(U,u)}} = \frac{E(U,u)}{||p(U,u)||}\frac{dE(U,u)}{d\tau_{(U,u)}} = \frac{1}{\nu}\frac{dE(U,u)}{d\tau_{(U,u)}} = \frac{dE(U,u)}{d\ell_{(U,u)}}. \tag{6.86}$$

This equality follows from the identity (6.83), and has been used to express the longitudinal acceleration term in (6.84) as shown by $(6.85)_1$.

Note that if U is tangent to a geodesic, $F(U,u) = 0$ and Eqs. (6.85) imply

$$\begin{aligned}
\frac{dE(U,u)}{d\ell_{(U,u)}} &= F^{(G)}_{(\mathrm{fw},U,u)} \cdot \hat{\nu}(U,u), \\
\gamma\nu(U,u)^2 k_{(\mathrm{fw},U,u)} &= F^{(G)}_{(\mathrm{fw},U,u)} \cdot \hat{\eta}_{(\mathrm{fw},U,u)}, \\
0 &= F^{(G)}_{(\mathrm{fw},U,u)} \cdot \hat{\beta}_{(\mathrm{fw},U,u)}.
\end{aligned} \tag{6.87}$$

On the other hand, the force equation may also be considered from the point of view of the comoving Frenet-Serret frame $\{\hat{\mathcal{V}}(u,U), \hat{\mathcal{N}}_{(\mathrm{fw},u,U)}, \hat{\mathcal{B}}_{(\mathrm{fw},u,U)}\}$ introduced in Chapter 4, Eqs. (4.46).

By applying the derivative $D/d\tau_U$ to the representation of u,

$$u = \gamma(U + \nu(u,U)) = \gamma(U - \nu\hat{\mathcal{V}}(u,U)) \tag{6.88}$$

and solving for $a(U) = f(U)$, one obtains

$$\begin{aligned}
a(U) = \frac{DU}{d\tau_U} &= P(U)\frac{DU}{d\tau_U} \\
&= P(U)\frac{D}{d\tau_U}\left(\gamma^{-1}u - \nu(u,U)\right) \\
&= P(U)\frac{D(\gamma^{-1}u)}{d\tau_U} + P(U)\frac{D}{d\tau_U}\left(\nu\hat{\mathcal{V}}(u,U)\right) \\
&= -\gamma^{-2}\frac{D\gamma}{d\tau_U}P(U)u + \gamma^{-1}P(U)\frac{Du}{d\tau_U} \\
&\quad + \frac{D\nu}{d\tau_U}\hat{\mathcal{V}}(u,U) + \nu P(U)\frac{D\hat{\mathcal{V}}(u,U)}{d\tau_U}.
\end{aligned} \tag{6.89}$$

Using (6.88), we now find that

$$P(U)u = -\gamma\nu\hat{\mathcal{V}}(u,U), \tag{6.90}$$

so that, recalling (3.156) for the definition of the gravitational force, we have for the previous relation

$$
\begin{aligned}
f(U) &= -\gamma\nu \frac{d\nu}{d\tau_U}(-\gamma\nu\hat{\mathcal{V}}(u,U)) - \gamma^{-1}P(U,u)F^{(G)}_{(\mathrm{fw},U,u)} \\
&\quad + \frac{d\nu}{d\tau_U}\hat{\mathcal{V}}(u,U) + \nu P(U)\frac{D\hat{\mathcal{V}}(u,U)}{d\tau_U} \\
&= (1+\gamma^2\nu^2)\frac{d\nu}{d\tau_U}\hat{\mathcal{V}}(u,U) - \gamma^{-1}P(U,u)F^{(G)}_{(\mathrm{fw},U,u)} + \nu P(U)\frac{D\hat{\mathcal{V}}(u,U)}{d\tau_U} \\
&= \gamma^2\frac{d\nu}{d\tau_U}\hat{\mathcal{V}}(u,U) - \gamma^{-1}P(U,u)F^{(G)}_{(\mathrm{fw},U,u)} + \nu P(U)\frac{D\hat{\mathcal{V}}(u,U)}{d\tau_U} \\
&= \gamma^{-1}\frac{d(\gamma\nu)}{d\tau_U}\hat{\mathcal{V}}(u,U) - \gamma^{-1}P(U,u)F^{(G)}_{(\mathrm{fw},U,u)} + \nu P(U)\frac{D\hat{\mathcal{V}}(u,U)}{d\tau_U} \\
&= \gamma^{-1}\frac{d||p(u,U)||}{d\tau_U}\hat{\mathcal{V}}(u,U) + \gamma\nu^2\mathcal{K}_{(\mathrm{fw},u,U)}\hat{\mathcal{N}}_{(\mathrm{fw},u,U)} \\
&\quad - \gamma^{-1}P(U,u)F^{(G)}_{(\mathrm{fw},U,u)}.
\end{aligned} \tag{6.91}
$$

Rearranging terms in this equation leads to the suggestive form

$$
\begin{aligned}
\frac{d||p(u,U)||}{d\tau_{(U,u)}}\hat{\mathcal{V}}(u,U) &= f(U) + \mathcal{F}^{(G)}_{(\mathrm{fw},u,U)} \\
&\quad - \gamma\nu^2\mathcal{K}_{(\mathrm{fw},u,U)}\hat{\mathcal{N}}_{(\mathrm{fw},u,U)},
\end{aligned} \tag{6.92}
$$

where

$$
\mathcal{F}^{(G)}_{(\mathrm{fw},u,U)} = \gamma^{-1}P(U,u)F^{(G)}_{(\mathrm{fw},U,u)} = -\gamma^{-1}P(U)\frac{Du}{d\tau_U} = -P(U)\frac{Du}{d\tau_{(U,u)}}. \tag{6.93}
$$

The last term on the right-hand side of (6.92), being proportional to the derivative of the unit spatial velocity $\hat{\mathcal{V}}(u,U)$ of u with respect to U, is called the *generalized centrifugal force*; it is measured by the particle U itself. Its non-relativistic limit in flat space-time leads to the familiar centrifugal force relative to the usual family of inertial observers. The comoving relative Frenet-Serret decomposition of (6.92) is

$$
\begin{aligned}
\frac{dE(U,u)}{d\ell_{(U,u)}} &= [f(U) + \mathcal{F}^{(G)}_{(\mathrm{fw},u,U)}]\cdot\hat{\mathcal{V}}(u,U), \\
\gamma\nu^2\mathcal{K}_{(\mathrm{fw},u,U)} &= [f(U) + \mathcal{F}^{(G)}_{(\mathrm{fw},u,U)}]\cdot\hat{\mathcal{N}}_{(\mathrm{fw},u,U)}, \\
0 &= [f(U) + \mathcal{F}^{(G)}_{(\mathrm{fw},u,U)}]\cdot\hat{\mathcal{B}}_{(\mathrm{fw},u,U)}.
\end{aligned} \tag{6.94}
$$

These equations are valid as long as ν belongs to the open interval $(0,1)$ and with a slight modification when $\nu = 1$. When $\nu = 0$, the spatial arc length parameterization $d\ell_{(U,u)} = \gamma\nu d\tau_U$ is singular because the particle path in LRS_U collapses to a point. However, since Eqs. (6.94) are just projections along the comoving Frenet-Serret triad of the space-time force equation, they continue to hold in the limit $\nu \to 0$. One may use the terminology *comoving relatively straight* and *comoving relatively flat* for those world lines for which the comoving relative curvature and torsion respectively vanish. By solving Eq. (6.84) for the gravitational force

$$F^{(G)}_{(\mathrm{fw},U,u)} = \gamma\nu^2\hat{\eta}_{(\mathrm{fw},U,u)} - F(U,u) + \frac{d||p(U,u)||}{d\tau_{(U,u)}}\hat{\nu}(U,u) \tag{6.95}$$

and inserting the result into Eq. (4.47), that is

$$\begin{aligned}\mathcal{K}_{(\mathrm{fw},u,U)}\hat{\mathcal{N}}_{(\mathrm{fw},u,U)} &= \gamma k_{(\mathrm{fw},U,u)}\hat{\eta}_{(\mathrm{fw},U,u)} \\ &\quad + \hat{\nu}(U,u)\times_u[\hat{\nu}(U,u)\times_u F^{(G)}_{(\mathrm{fw},U,u)}],\end{aligned} \tag{6.96}$$

one finds the relation

$$\begin{aligned}\mathcal{K}_{(\mathrm{fw},u,U)}\hat{\mathcal{N}}_{(\mathrm{fw},u,U)} &= \gamma^{-1} k_{(\mathrm{fw},U,u)}\hat{\eta}_{(\mathrm{fw},U,u)} \\ &\quad - \hat{\nu}_{(U,u)}\times_u[\hat{\nu}_{(U,u)}\times_u F(U,u)].\end{aligned} \tag{6.97}$$

When the curve is a geodesic, that is when $F(U,u) = 0$, the two curvatures $\mathcal{K}_{(\mathrm{fw},u,U)}$ and $k_{(\mathrm{fw},U,u)}$ only differ by a gamma factor and the two directions $\hat{\mathcal{N}}_{(\mathrm{fw},u,U)}$ and $\hat{\eta}_{(\mathrm{fw},U,u)}$ coincide. In fact, from (6.97) we find

$$\mathcal{K}_{(\mathrm{fw},u,U)}\hat{\mathcal{N}}_{(\mathrm{fw},u,U)} = \gamma^{-1} k_{(\mathrm{fw},U,u)}\hat{\eta}_{(\mathrm{fw},U,u)}, \tag{6.98}$$

so that

$$\mathcal{K}_{(\mathrm{fw},u,U)} = \gamma^{-1} k_{(\mathrm{fw},U,u)}. \tag{6.99}$$

Thus, for a geodesic, the notions of *relative Fermi-Walker straight* (as stated following (4.42)) and *comoving relatively straight* world lines agree with each other.

A relationship between the relative and comoving torsion may be established by differentiating Eq. (4.47) along U and using the last comoving relative Frenet-Serret relation (4.46). After some algebra one finds

$$\begin{aligned}\gamma\nu\mathcal{K}^2_{(\mathrm{fw},u,U)}\mathcal{T}_{(\mathrm{fw},u,U)} &= \gamma^2\nu k_{(\mathrm{fw},U,u)}\tau_{(\mathrm{fw},U,u)}[k_{(\mathrm{fw},U,u)} - F^{(G)}_{(\mathrm{fw},U,u)}\cdot\hat{\eta}_{(\mathrm{fw},U,u)}] \\ &\quad + \frac{d(\gamma k_{(\mathrm{fw},U,u)})}{d\tau_U}[F^{(G)}_{(\mathrm{fw},U,u)}\cdot\hat{\beta}_{(\mathrm{fw},U,u)}] \\ &\quad + \mathcal{K}_{(\mathrm{fw},u,U)}\hat{\mathcal{N}}_{(\mathrm{fw},u,U)}\cdot\nabla_U[\hat{\nu}(U,u)\times_u F^{(G)}_{(\mathrm{fw},U,u)}].\end{aligned} \tag{6.100}$$

In the special case of geodesics, this reduces to

$$\mathcal{T}_{(\mathrm{fw},u,U)} = \tau_{(\mathrm{fw},U,u)}, \tag{6.101}$$

so the notions of relative Fermi-Walker flatness and comoving relative Fermi-Walker flatness also agree.

6.8 Acceleration change under observer transformations

Consider two accelerated time-like world lines with unit tangent vectors U and u. We shall express the acceleration of U in terms of that of u. By definition we have

$$\begin{aligned} a(U) &= P(U)\nabla_U U = \gamma P(U)[\nabla_U u + \nabla_U \nu(U,u)] \\ &= \gamma P(U)[\nabla_U u + P(u)\nabla_U \nu(U,u) - \ u(u\cdot\nabla_U\nu(U,u))] \\ &= \gamma P(U)[\nabla_U u + P(u)\nabla_U \nu(U,u) + \ u(\nu(U,u)\cdot\nabla_U u)]. \end{aligned} \tag{6.102}$$

Recalling (3.156) and (3.161), namely

$$\frac{Du}{d\tau_U} = -F^{(G)}_{(\mathrm{fw},U,u)}, \qquad P(u)\frac{D}{d\tau_U}\nu(U,u) = \gamma a_{(\mathrm{fw},U,u)}, \tag{6.103}$$

relation (6.102) becomes

$$\begin{aligned} a(U) &= \gamma P(U)\left[-F^{(G)}_{(\mathrm{fw},U,u)} + \gamma a_{(\mathrm{fw},U,u)} - u(\nu(U,u)\cdot F^{(G)}_{(\mathrm{fw},U,u)})\right] \\ &= -\gamma P(U)[P(u) + u\otimes\nu(U,u)]\, F^{(G)}_{(\mathrm{fw},U,u)} + \gamma^2 P(U,u) a_{(\mathrm{fw},U,u)}. \end{aligned} \tag{6.104}$$

But, from $(3.121)_4$ it follows that

$$P(u) + u\otimes\nu(U,u) = P(u,U)^{-1}, \tag{6.105}$$

and hence the final result from (3.156) is given by

$$\begin{aligned} a(U) &= -\gamma P(u,U)^{-1} F^{(G)}_{(\mathrm{fw},U,u)} + \gamma^2 P(U,u) a_{(\mathrm{fw},U,u)} \\ &= \gamma^2 \left\{ P(u,U)^{-1}[a(u) + \omega(u)\times_u \nu(U,u) + \theta(u)\llcorner\nu(U,u)] \right. \\ &\quad \left. + P(U,u) a_{(\mathrm{fw},U,u)} \right\}. \end{aligned} \tag{6.106}$$

6.9 Kinematical tensor change under observer transformations

We now consider two time-like congruences of curves, $\mathcal{C}_U$ and $\mathcal{C}_u$; our aim is to express the kinematical tensor of one congruence in terms of the other. From the definition, the kinematical tensor of $\mathcal{C}_U$, $k(U) = -[P(U)\nabla U]$, can be written as

$$\begin{aligned} k(U)^\alpha{}_\beta &= -P(U)^\alpha{}_\sigma P(U)^\delta{}_\beta [\nabla U]^\sigma{}_\delta \\ &= -P(U)^\alpha{}_\sigma P(U)^\delta{}_\beta \nabla_\delta U^\sigma \\ &= -\gamma P(U)^\alpha{}_\sigma P(U)^\delta{}_\beta \nabla_\delta u^\sigma \\ &\quad - \gamma[\nabla(U)\nu(U,u)]^\alpha{}_\beta. \end{aligned} \tag{6.107}$$

We evaluate the term $P(U)\nabla u$ as follows:

$$\begin{aligned}[P(U)\nabla u]^\alpha{}_\beta &= P(U)^\alpha{}_\sigma P(U)^\delta{}_\beta \nabla_\delta u^\sigma \\ &= P(U)^\alpha{}_\sigma P(U)^\delta{}_\beta [-a(u)^\sigma u_\delta - k(u)^\sigma{}_\delta].\end{aligned} \tag{6.108}$$

Taking into account that

$$0 = P(U)^\alpha{}_\beta U^\beta = \gamma P(U)^\alpha{}_\beta [u^\beta + \nu(U,u)^\beta], \tag{6.109}$$

we have

$$P(U)^\alpha{}_\beta u^\beta = -P(U)^\alpha{}_\beta \nu(U,u)^\beta. \tag{6.110}$$

Therefore

$$\begin{aligned}[P(U)\nabla u]^\alpha{}_\beta &= P(U)^\alpha{}_\sigma P(U)^\delta{}_\beta \left[a(u)^\sigma \nu(U,u)_\delta - k(u)^\sigma{}_\delta\right] \\ &= P(U,u)^\alpha{}_\sigma P(U,u)^\delta{}_\beta \left[a(u)^\sigma \nu(U,u)_\delta - k(u)^\sigma{}_\delta\right],\end{aligned} \tag{6.111}$$

or in index-free notation

$$P(U)\nabla u = -P(U,u)[k(u) - a(u)\otimes \nu(U,u)]. \tag{6.112}$$

We can then complete our task, obtaining

$$\begin{aligned}k(U) = {}& \gamma P(U,u)[k(u) - a(u)\otimes \nu(U,u)] \\ & - \gamma \nabla(U)\nu(U,u).\end{aligned} \tag{6.113}$$

Since $\omega(U)^\flat = \mathrm{ALT}\,[k(U)^\flat]$, we have the transformation law for the vorticity 2-form,

$$\begin{aligned}\omega(U)^\flat &= \gamma \mathrm{ALT}\left\{P(U,u)[k(u)^\flat - a(u)^\flat \otimes \nu(U,u)^\flat]\right\} \\ &\quad + \frac{1}{2}\gamma d(U)\nu(U,u)^\flat \\ &= \gamma P(U,u)[\omega(u)^\flat - \frac{1}{2}a(u)\wedge \nu(U,u)] \\ &\quad + \frac{1}{2}\gamma d(U)\nu(U,u)^\flat.\end{aligned} \tag{6.114}$$

We shall now derive the analogous expression for the vorticity vector, namely

$$\begin{aligned}\omega(U) = {}& \gamma^2 P(u,U)^{-1}[\omega(u) + \frac{1}{2}\nu(U,u)\times_u a(u)] \\ & + \frac{1}{2}\gamma\, \mathrm{curl}_U \nu(U,u).\end{aligned} \tag{6.115}$$

Let us write (6.114) in the form

$$\omega(U)^\flat - \frac{1}{2}\gamma d(U)\nu(U,u)^\flat \equiv P(U,u)X, \tag{6.116}$$

where

$$X^\flat = \gamma\left[\omega(u) - \frac{1}{2}a(u)\wedge \nu(U,u)\right]^\flat. \tag{6.117}$$

Next, contract both sides of (6.116) with $(1/2)\eta(U)$. The left-hand side becomes

$$\frac{1}{2}\eta(U)^{\alpha\beta\gamma}\left[\omega(U)_{\alpha\beta}-\frac{\gamma}{2}\,2\nabla(U)_{[\alpha}\nu(U,u)_{\beta]}\right], \tag{6.118}$$

that is

$$\omega(U)^{\gamma}-\frac{\gamma}{2}[\mathrm{curl}_U\nu(U,u)]^{\gamma}. \tag{6.119}$$

To deduce the effect of contracting the right-hand side of (6.116) by $(1/2)\eta(U)$, namely

$$\frac{1}{2}\eta(U)P(U,u)X,$$

let us first recall, from (3.20), that

$$\begin{aligned}\eta(U)_{\beta\gamma\delta}&=U^{\alpha}\eta_{\alpha\beta\gamma\delta}=-2U^{\alpha}\left(u_{[\alpha}\eta(u)_{\beta]\gamma\delta}+u_{[\gamma}\eta(u)_{\delta]\alpha\beta}\right)\\&=\gamma\left[\eta(u)_{\beta\gamma\delta}+\nu^{\alpha}(u_{\beta}\eta(u)_{\alpha\gamma\delta}\right.\\&\qquad\left.+u_{\gamma}\eta(u)_{\alpha\delta\beta}-u_{\delta}\eta(u)_{\alpha\gamma\beta})\right].\end{aligned} \tag{6.120}$$

Then let us recall that

$$P(U,u)_{\alpha}{}^{\mu}=P(u)_{\alpha}{}^{\mu}+\gamma U_{\alpha}\nu^{\mu} \tag{6.121}$$

implies, for any antisymmetric 2-tensor $X\in LRS_u\otimes LRS_u$, the following relation:

$$\begin{aligned}[P(U,u)X]_{\alpha\beta}&=P(U,u)_{\alpha}{}^{\mu}P(U,u)_{\beta}{}^{\nu}X_{\mu\nu}\\&=X_{\alpha\beta}+\gamma U_{\beta}X_{\alpha\sigma}\nu^{\sigma}-\gamma U_{\alpha}X_{\beta\sigma}\nu^{\sigma}\\&=[X+\gamma U\wedge(X\llcorner\nu)]_{\alpha\beta}.\end{aligned} \tag{6.122}$$

From the latter equation we deduce that

$$\eta(U)^{\gamma\alpha\beta}[P(U,u)X]_{\alpha\beta}=\eta(U)^{\gamma\alpha\beta}X_{\alpha\beta}, \tag{6.123}$$

because terms along U vanish after contraction with $\eta(U)$. Using (6.120) we now find that

$$\eta(U)^{\gamma\alpha\beta}[P(U,u)X]_{\alpha\beta}=\gamma[\eta(u)^{\gamma\alpha\beta}-\nu_{\mu}u^{\gamma}\eta(u)^{\mu\beta\alpha}]X_{\alpha\beta}, \tag{6.124}$$

that is

$$\begin{aligned}{}^{*(U)}[P(U,u)X]&=\gamma[{}^{*(u)}X+u\otimes(\nu\llcorner{}^{*(u)}X)]\\&=\gamma[P(u)+u\otimes\nu]\llcorner{}^{*(u)}X\\&=\gamma P(u,U)^{-1}\llcorner{}^{*(u)}X.\end{aligned} \tag{6.125}$$

Since we have

$${}^{*(u)}X=\gamma\left[\omega(u)-\frac{1}{2}a(u)\times_u\nu(U,u)\right], \tag{6.126}$$

Eq. (6.115) follows.

Similarly one can obtain the expansion tensor of $\mathcal{C}_U$ in terms of the kinematical properties of $\mathcal{C}_u$. Since $\theta(U)^\flat = -\mathrm{SYM}\,[k(U)^\flat]$ we have

$$\begin{aligned} -\theta(U)^\flat &= \gamma\mathrm{SYM}\left\{P(U,u)[k(u)^\flat - a(u)^\flat \otimes \nu(U,u)^\flat]\right\} \\ &\quad - \gamma\mathrm{SYM}[\nabla(U)\nu(U,u)^\flat] \\ &= -\gamma P(U,u)\left[\theta(u)^\flat + \frac{1}{2}[a(u)\otimes\nu(U,u) + \nu(U,u)\otimes a(u)]\right] \\ &\quad - \gamma\mathrm{SYM}[\nabla(U)\nu(U,u)^\flat]. \end{aligned} \tag{6.127}$$

6.10 Measurements of electric and magnetic fields

Assume that an observer u moves through an electromagnetic field described by the Faraday 2-form $F_{\alpha\beta}$. The observer learns of this electromagnetic field by studying the behavior of a charged particle. Let U be the 4-velocity of the particle; its trajectory is not a geodesic because of the Lorentz force, hence it has a 4-acceleration

$$a(U)^\alpha = \frac{e}{\mu_0} F^\alpha{}_\beta U^\beta, \tag{6.128}$$

where e is the particle's electric charge and μ_0 its mass. Our aim is to show how the measured force on the charge is defined in terms of the charge and the four-dimensional quantities which characterize the observer and the electromagnetic field. Because $F_{\alpha\beta}$ is a 2-form, we can apply the general splitting formula (3.14), valid for a p-form. The result is:

$$F_{\alpha\beta} = 2u_{[\alpha}E(u)_{\beta]} + [{}^{*(u)}B(u)]_{\alpha\beta}, \tag{6.129}$$

where we have introduced the electric part of F,

$$E(u)^\beta \equiv F^\beta{}_\rho u^\rho, \tag{6.130}$$

and the magnetic part,

$$B(u)^\alpha \equiv {}^*F_\rho{}^\alpha u^\rho = \frac{1}{2}\eta^{\rho\alpha\mu\nu}F_{\mu\nu}u_\rho. \tag{6.131}$$

In coordinate-free notation the definition of such fields is

$$E(u) = F \llcorner u, \qquad B(u) \equiv {}^{*(u)}[P(u)F]. \tag{6.132}$$

Let us decompose the force term $f(U) = (e/\mu_0)F\llcorner U$ into a transverse and a parallel component relative to u, namely

$$f(U) = P(u)\llcorner f(U) + T(u)\llcorner f(U). \tag{6.133}$$

The *transverse* force can be written in coordinate components as

$$\begin{aligned} P(u)^{\alpha}{}_{\beta} f(U)^{\beta} &= \frac{e}{\mu_0} P(u)^{\alpha\beta} F_{\beta\mu} U^{\mu} \\ &= \frac{e}{\mu_0} P(u)^{\alpha\beta} \left[u_{\beta} E(u)_{\mu} - u_{\mu} E(u)_{\beta} \right. \\ &\quad \left. + \eta(u)_{\beta\mu}{}^{\sigma} B(u)_{\sigma} \right] \gamma (u^{\mu} + \nu(U,u)^{\mu}) \\ &= -\frac{e}{\mu_0} \left[\eta_{\sigma\pi\lambda}{}^{\alpha} u^{\sigma} U^{\lambda} B(u)^{\pi} + E(u)^{\alpha} (u_{\rho} U^{\rho}) \right]. \end{aligned} \tag{6.134}$$

As a result the transverse force term becomes

$$P(u) f(U) = \frac{e}{\mu_0} \gamma \left[E(u) + \nu(U,u) \times_u B(u) \right]. \tag{6.135}$$

The *parallel* force component $T(u)f(U)$ can be written more conveniently as

$$T(u)^{\alpha}{}_{\beta} f(U)^{\beta} = -u^{\alpha} \left[u_{\beta} f(U)^{\beta} \right] = -\frac{e}{\mu_0} u^{\alpha} u_{\beta} F^{\beta}{}_{\sigma} U^{\sigma}. \tag{6.136}$$

Recalling that

$$U^{\sigma} = \gamma [u^{\sigma} + \nu(U,u)^{\sigma}], \tag{6.137}$$

we can write (6.136) as

$$T(u)^{\alpha}{}_{\beta} f(U)^{\beta} = u^{\alpha} \nu(U,u)_{\beta} E(U)^{\beta}. \tag{6.138}$$

The magnitude of this quantity measures the power of the electromagnetic action on the particle as seen by u. From the representation of F we have, relative to both the observers u and U,

$$\begin{aligned} F_{\alpha\beta} &= 2u_{[\alpha} E(u)_{\beta]} + [{}^{*(u)} B(u)]_{\alpha\beta} \\ &= 2U_{[\alpha} E(U)_{\beta]} + [{}^{*(U)} B(U)]_{\alpha\beta}\,; \end{aligned} \tag{6.139}$$

hence, contracting with U and using the definitions so far introduced, we obtain

$$\begin{aligned} E(U) &= \gamma P(u,U)^{-1} \left[E(u) + \nu(U,u) \times_u B(u) \right], \\ B(U) &= \gamma P(u,U)^{-1} \left[B(u) - \nu(U,u) \times_u E(u) \right]. \end{aligned} \tag{6.140}$$

If we split the fields along directions parallel and transverse to that of the relative velocity $\hat{\nu}(U,u)$ we obtain

$$E(u) = E^{\|}(u) \hat{\nu}(U,u) + E^{\perp}(u), \tag{6.141}$$

and similarly for $B(u)$. Hence (6.140) take the more familiar form

$$\begin{aligned} [B_{(\mathrm{lrs})}(u,U)\, E(U)]^{\|} &= E^{\|}(u), \\ [B_{(\mathrm{lrs})}(u,U)\, E(U)]^{\perp} &= \gamma [E^{\perp}(u) + \nu(U,u) \times_u B^{\perp}(u)], \end{aligned} \tag{6.142}$$

and

$$\begin{aligned} [B_{(\mathrm{lrs})}(u,U)\, B(U)]^{\|} &= B^{\|}(u), \\ [B_{(\mathrm{lrs})}(u,U)\, B(U)]^{\perp} &= \gamma [B^{\perp}(u) - \nu(U,u) \times_u E^{\perp}(u)]. \end{aligned} \tag{6.143}$$

Recalling that the physical meaning of a 4-vector is encoded in its magnitude, we can deduce that the magnitude of $P(u)f(U)$ is just the magnitude of the Lorentz force, and so the moduli of the electric and the magnetic parts describe the electric field intensity and the magnetic induction.

By introducing the complex vector field $Z(u) = E(u) - iB(u)$ (Landau and Lifshitz, 1975), Eqs. (6.142) and (6.143) can be written together as

$$\begin{aligned}
[B_{(\mathrm{lrs})}(u,U)\,Z(U)]^{\parallel} &= Z^{\parallel}(u),\\
[B_{(\mathrm{lrs})}(u,U)\,Z(U)]^{\perp} &= \cosh\alpha\, Z^{\perp}(u)\\
&\qquad + i\sinh\alpha\,[\hat\nu(U,u)\times_u Z^{\perp}(u)],
\end{aligned} \tag{6.144}$$

where $||\nu(U,u)|| = \tanh\alpha$. In terms of components with respect to an observer-adapted frame $\{e_\alpha\}$ ($e_0 = u$, with e_3 along $\hat\nu(U,u)$), this complex vector representation reflects the isomorphism between the group of proper orthochronous Lorentz transformations and $SO(3,C)$, the group of proper orthogonal transformations. In the 2-plane orthogonal to e_3 within LRS_u one has

$$\begin{bmatrix} [B_{(\mathrm{lrs})}(u,U)\,Z(U)]^{\perp}{}_1 \\ [B_{(\mathrm{lrs})}(u,U)\,Z(U)]^{\perp}{}_2 \end{bmatrix} = \begin{bmatrix} \cosh\alpha & -i\sinh\alpha \\ i\sinh\alpha & \cosh\alpha \end{bmatrix} \begin{bmatrix} Z^{\perp}(u)_1 \\ Z^{\perp}(u)_2 \end{bmatrix}. \tag{6.145}$$

6.11 Local properties of an electromagnetic field

An interesting application is the splitting of the energy-momentum tensor of an electromagnetic field. We have

$$T_{\alpha\beta} = \frac{1}{4\pi}\left(F_{\alpha\rho}F_\beta{}^\rho - \frac{1}{4}g_{\alpha\beta}F_{\rho\sigma}F^{\rho\sigma}\right) = \frac{1}{4\pi}[F^2]^{\mathrm{TF}}{}_{\alpha\beta}. \tag{6.146}$$

An arbitrary observer u will measure:

(i) an energy density

$$\mathcal{E}(u) = T_{\alpha\beta}u^\alpha u^\beta = \frac{1}{8\pi}(E(u)^2 + B(u)^2); \tag{6.147}$$

(ii) a momentum density (Poynting vector)

$$\mathcal{P}(u)^\alpha = -P(u)^\alpha{}_\beta T^\beta{}_\rho u^\rho = \frac{1}{4\pi}[E(u)\times_u B(u)]^\alpha\,; \tag{6.148}$$

(iii) a uniform pressure

$$p(u) = \frac{1}{3}\mathrm{Tr}\,T(u) = \frac{1}{3}\mathcal{E}(u). \tag{6.149}$$

Observers $U_{(\mathrm{em})}$ who measure a vanishing Poynting vector

Given an electromagnetic field and a general observer u, the Poynting vector and the energy density are given by

$$4\pi\mathcal{P}(u) = E(u) \times_u B(u) = \frac{i}{2}[\bar{Z}(u) \times_u Z(u)], \tag{6.150}$$

$$4\pi\mathcal{E}(u) = \frac{1}{2}[E(u)^2 + B(u)^2] = \frac{1}{2}|Z(u)|^2. \tag{6.151}$$

It is well known (see exercise 20.6 of Misner, Thorne, and Wheeler, 1973) that observers exist who see a vanishing Poynting vector or, equivalently, electric and magnetic fields parallel to each other. Let U be an observer in relative motion with respect to u with

$$U = \gamma[u + \nu\hat{\nu}(U,u)] = u\cosh\alpha + \hat{\nu}(U,u)\sinh\alpha\,. \tag{6.152}$$

With respect to U, the magnitude of the Poynting vector is given by

$$\begin{aligned} 4\pi||\mathcal{P}(U)|| &= E(U) \times_U B(U) \\ &= \frac{i}{2}\,\left|\left[\left(\hat{\nu}(U,u)\cdot(\bar{Z}(u)\times_u Z(u))\right)\cosh 2\alpha + i\,|Z(u)|^2\sinh 2\alpha\right]\right| \\ &= \frac{1}{2}\,|Z(u)|^2|\cosh 2\alpha\tanh 2\alpha_{(\mathrm{em})} - \sinh 2\alpha| \\ &= 4\pi\mathcal{E}(u)\,\frac{|\sinh 2(\alpha - \alpha_{(\mathrm{em})})|}{\cosh 2\alpha_{(\mathrm{em})}}, \end{aligned} \tag{6.153}$$

where we have defined

$$\begin{aligned} \tanh 2\alpha_{(\mathrm{em})} &= i\frac{\hat{\nu}(U,u)\cdot(\bar{Z}(u)\times Z(u))}{|Z(u)|^2} \\ &= 2\frac{\hat{\nu}(U,u)\cdot(E(u)\times_u B(u))}{E(u)^2 + B(u)^2}. \end{aligned} \tag{6.154}$$

When $\alpha = \alpha_{(\mathrm{em})}$, i.e. selecting U as a particular observer $U_{(\mathrm{em})}$, we have $||\mathcal{P}(U_{(\mathrm{em})})|| = 0$. With respect to $U_{(\mathrm{em})}$, the electromagnetic energy density takes a minimum value. In fact, the electromagnetic energy density measured by U is in general

$$\begin{aligned} 4\pi\mathcal{E}(U) &= \frac{1}{2}|Z(u)|^2[\cosh 2\alpha - \sinh 2\alpha\tanh 2\alpha_{(\mathrm{em})}] \\ &= 4\pi\mathcal{E}(u)\,\frac{\cosh 2(\alpha - \alpha_{(\mathrm{em})})}{\cosh 2\alpha_{(\mathrm{em})}}. \end{aligned} \tag{6.155}$$

Therefore, when $\alpha = \alpha_{(\mathrm{em})}$, $\mathcal{E}(U_{(\mathrm{em})})$ takes a minimum value equal to

$$\mathcal{E}(U_{(\mathrm{em})}) = \frac{\mathcal{E}(u)}{\cosh 2\alpha_{(\mathrm{em})}}. \tag{6.156}$$

6.12 Time-plus-space $(1+3)$ form of Maxwell's equations

The electromagnetic or Faraday 2-form F is the exterior derivative of a 4-potential 1-form

$$F^\flat = dA = u^\flat \wedge E(u)^\flat + {}^{*(u)}B(u)^\flat. \tag{6.157}$$

The splitting of Maxwell's equations

$$d^2 A^\flat = 0, \quad {}^*d^*F = 4\pi J \tag{6.158}$$

leads to

$$\operatorname{div}_u B(u) + 2\omega(u) \cdot_u E(u) = 0, \tag{6.159}$$
$$\operatorname{curl}_u E(u) + a(u) \times_u E(u) = -[\pounds(u)_u + \Theta(u)]B(u), \tag{6.160}$$
$$\operatorname{div}_u E(u) - 2\omega(u) \cdot_u B(u) = 4\pi\rho(u), \tag{6.161}$$
$$\operatorname{curl}_u B(u) + a(u) \times_u B(u) - [\pounds(u)_u + \Theta(u)]E(u) = 4\pi J(u), \tag{6.162}$$

where $J = \rho(u)u + J(u)$ is the splitting of the 4-current. This shows that Maxwell's equations in their traditional form hold only in an inertial frame where $\omega(u)$, $a(u)$, and $\theta(u)$ vanish.

6.13 Gravitoelectromagnetism

Consider a unit mass charged particle in motion in a given space-time. The equation of motion (6.77),

$$\frac{D_{(\mathrm{fw},U,u)}p(U,u)}{d\tau_{(U,u)}} = F(U,u) + F^{(G)}_{(\mathrm{fw},U,u)}, \tag{6.163}$$

becomes

$$\begin{aligned}\frac{D_{(\mathrm{fw},U,u)}p(U,u)}{d\tau_{(U,u)}} &= e[E(u) + \nu(U,u) \times_u B(u)] + \\ &\quad - \gamma[a(u) + \omega(u) \times_u \nu(U,u) \\ &\quad + \theta(u) \llcorner \nu(U,u)],\end{aligned} \tag{6.164}$$

where $F(U,u)$ has been replaced by the Lorentz force (6.135) (rescaled by a γ factor because on the left-hand side the temporal derivative involves the relative standard time and not the proper time as a parameter along the particle's world line), and the gravitational force $F^{(G)}_{(\mathrm{fw},U,u)}$ is given by (3.156), namely

$$F^{(G)}_{(\mathrm{fw},U,u)} = -\gamma[a(u) + \omega(u) \times_u \nu(U,u) + \theta(u) \llcorner \nu(U,u)]. \tag{6.165}$$

Comparing these two forces leads to the identification of a *gravitoelectric field*

$$E_g(u) = -a(u) \tag{6.166}$$

and a *gravitomagnetic field*

$$B_g(u) = \omega(u), \tag{6.167}$$

so that

$$F^{(G)}_{(\mathrm{fw},U,u)} = \gamma[E_g(u) + \nu(U,u) \times_u B_g(u) + \theta(u) \llcorner \nu(U,u)]. \tag{6.168}$$

Therefore, apart from the gamma factor in the gravitational force and the contribution of the expansion tensor $\theta(u)$, the Lorentz force is closely similar to the gravitational force. This analogy is what we call *gravitoelectromagnetism*, even though the gravitational and electromagnetic interactions remain distinct.

It is worth noting that (1) no linear approximation of the gravitational field has been made here in order to introduce gravitoelectromagnetism (Jantzen, Carini, and Bini, 1992); (2) the analogy between gravity and electromagnetism, developed here starting from the analysis of test particle motion, could be pursued similarly studying fluid or field motions.

6.14 Physical properties of fluids

A physical observer who studies the behavior of a relativistic fluid which is in general away from thermodynamic equilibrium must specify in his own rest frame the parameters which invariantly characterize the fluid and determine its evolution. For this purpose one would need a full thermodynamic treatment (Israel, 1963), which is well beyond the scope of this book. We shall instead outline some of the results in the case of a simple fluid, i.e. when the deviations from local equilibrium are small and the self-gravity of the fluid is neglected.

If T is the energy-momentum tensor of the fluid and u is the vector field tangent to a congruence of observers, the following decomposition holds for each fluid element (Ellis, 1971; Ellis and van Elst, 1998):

$$T = \mathcal{T}(u) + u \otimes q(u) + q(u) \otimes u + \rho(u) u \otimes u, \tag{6.169}$$

where

$$\mathcal{T}(u)_{\alpha\beta} \equiv P(u)_\alpha{}^\rho P(u)_\beta{}^\sigma T_{\rho\sigma}, \tag{6.170}$$

$$q(u)_\alpha \equiv -P(u)_\alpha{}^\rho T_{\rho\sigma} u^\sigma, \tag{6.171}$$

$$\rho(u) \equiv T_{\rho\sigma} u^\rho u^\sigma. \tag{6.172}$$

An insight into the physical interpretation of these quantities arises from the definition of the 4-momentum of an extended body. If we choose a space-like hyperspace Σ in such a way that its unit normal is parallel to u, then the quantities

$$\hat{P}^\alpha = -T^\alpha{}_\beta u^\beta \tag{6.173}$$

turn out to be the components of the 4-momentum density of the fluid. With respect to the observer u this 4-vector can then be decomposed as

$$\begin{aligned}\hat{P}^\alpha &= P(u)^\alpha{}_\beta \hat{P}^\beta + T(u)^\alpha{}_\beta \hat{P}^\beta \\ &= -P(u)^\alpha{}_\beta T^\beta{}_\mu u^\mu + u^\alpha u_\beta T^\beta{}_\mu u^\mu \\ &= q(u)^\alpha + \rho(u) u^\alpha. \end{aligned} \tag{6.174}$$

The first term $q(u)^\alpha$ represents the three-dimensional energy flux density and describes not only the linear momentum of the fluid elements but thermo-conduction processes such as convection, radiation and heat transfer (Landau and Lifshitz, 1959; Novikov and Thorne, 1973). The second term $\rho(u)u^\alpha$ describes an energy density current, so its magnitude $\rho(u)$ is just the energy density of the fluid relative to u. The remaining transverse quantity $\mathcal{T}(u)$ in (6.170), being a symmetric tensor in the three-dimensional LRS_u, can be written as the sum of a trace-free tensor and a trace, as follows:

$$\mathcal{T}(u)_{\alpha\beta} = [\mathcal{T}(u)]^{\mathrm{TF}}{}_{\alpha\beta} + \frac{1}{3}\left[\mathrm{Tr}\, \mathcal{T}(u)\right] P(u)_{\alpha\beta}, \tag{6.175}$$

where $[\mathcal{T}(u)]^{\mathrm{TF}}$ is the trace-free part of $\mathcal{T}(u)$ and $\mathrm{Tr}\,\mathcal{T}(u)$ is its trace. The quantity $[\mathcal{T}(u)]^{\mathrm{TF}}$ is termed the *viscous stress tensor* and describes non-isotropic dissipative processes; to the assumed accuracy (namely small deviations from local equilibrium) the viscous stresses enter the energy-momentum tensor as linear perturbations to the equilibrium configuration and are given in terms of the fluid shear, with a coefficient called the *shear viscosity*. The trace $\mathrm{Tr}\,\mathcal{T}(u)$ is only related to the uniform properties of the fluid; it can be written as

$$\mathrm{Tr}\,\mathcal{T}(u) = 3\left[p(u) + \tilde{\phi}(u)\right], \tag{6.176}$$

where $p(u)$ is the hydrostatic pressure and $\tilde{\phi}(u)$ is a contribution to the pressure arising from the appearance of a volume (or bulk) viscosity. Finally, we define as *comoving with the fluid* that observer U with respect to whom the quantities $q(U)^\alpha$ describe *only* processes of thermo-conduction, the fluid elements having zero momentum (Landau and Lifshitz, 1959). This observer's four-velocity will be hereafter be identified as the 4-velocity of the fluid; the other physical quantities, density $\rho(U)$, pressure $p(U)$, and internal mechanical stresses $\mathcal{T}(U)$, will be denoted by ρ_0, p_0, and $\mathcal{T}_0$, respectively.

In the case of a *perfect fluid*, namely when

$$[\mathcal{T}(U)]^{\mathrm{TF}} = 0, \qquad \tilde{\phi} = 0, \qquad q(U) = 0,$$

the energy-momentum tensor becomes

$$T_{\alpha\beta} = [p(U) + \rho(U)] U_\alpha U_\beta + g_{\alpha\beta}\, p(U). \tag{6.177}$$

A perfect fluid is termed *dust* if, in the comoving frame, $p(U) = 0$; hence it is described by the energy-momentum tensor

$$T_{\alpha\beta} = \rho(U) U_\alpha U_\beta. \tag{6.178}$$

In what follows we will discuss separately the cases of ordinary fluids (i.e. those without thermal stresses) and fluids with thermal stress, discussing both absolute and relative dynamics with respect to a general observer.

Ordinary fluids: absolute dynamics

A fluid is termed *ordinary* when its stresses in the comoving frame, hereafter referred to as *proper stresses*, are purely mechanical. In this case its energy-momentum tensor can be written as

$$T = \rho_0 U \otimes U + \mathcal{T}_0, \qquad \mathcal{T}_0 \llcorner U = 0, \qquad \rho_0 = \mu_0 \hat{\epsilon}_0, \tag{6.179}$$

where U is the 4-velocity field of the fluid elements, ρ_0 includes the matter energy (μ_0) and internal energy ($\hat{\epsilon}_0$), and $\mathcal{T}_0 \equiv \mathcal{T}(U)$ is the proper mechanical stress tensor. In the presence of matter and eventually other external fields the evolution equations of the fluid are

$$\operatorname{div} T = \mu_0 f, \tag{6.180}$$

where the space-time divergence operation div is defined in (2.111) and f represents the action of any external field. Let us consider the splitting of (6.180) in the comoving frame of the fluid. First of all we have

$$\nabla_\alpha T^{\alpha\beta} = \nabla_\alpha(\rho_0 U^\beta) U^\alpha + \rho_0 U^\beta \nabla_\alpha U^\alpha + [\operatorname{div} \mathcal{T}_0]^\beta = \mu_0 f^\beta, \tag{6.181}$$

that is

$$\nabla_U(\rho_0 U) + \rho_0 U \Theta(U) = \mu_0 f_m, \tag{6.182}$$

where $\Theta(U) = \nabla_\alpha U^\alpha$ and

$$f_m \equiv f - \frac{1}{\mu_0} \operatorname{div} \mathcal{T}_0 \tag{6.183}$$

is the total mechanical force, including both internal and external actions given respectively by $(1/\mu_0)\operatorname{div} \mathcal{T}_0$ and f.

Equation (6.182) can then be written in the form

$$\rho_0 a(U) + U \operatorname{div}(\rho_0 U) = \mu_0 f_m. \tag{6.184}$$

Projecting (6.184) orthogonally to U then gives

$$\rho_0 a(U) = \mu_0 P(U) f_m \equiv \mu_0 (f_0^{(e)} + f_0^{(i)}), \tag{6.185}$$

where the contributions from external and internal forces have been made explicit according to the following definitions:

$$f_0^{(e)} = P(U)f \tag{6.186}$$

and

$$\begin{aligned} f_0^{(i)} &\equiv -\frac{1}{\mu_0} P(U)[\mathrm{div}\, \mathcal{T}_0] \\ &= -\frac{1}{\mu_0} \left[\mathrm{div}_U\, \mathcal{T}_0 + a(U) \llcorner \mathcal{T}_0\right], \end{aligned} \tag{6.187}$$

where we have introduced the spatial divergence operator (see (3.37))

$$[\mathrm{div}_U\, \mathcal{T}_0]^\alpha = \nabla(U)_\mu \mathcal{T}_0^{\mu\alpha}. \tag{6.188}$$

The following notation will prove useful:

$$\begin{aligned} &w_0 = -f \cdot U && \text{Proper power of external forces} \\ & && \text{per unit of mass;} \\ &w_0^{(i)} = -U \cdot \mathrm{div}\, \mathcal{T}_0 = \mathrm{Tr}[\mathcal{T}_0 \llcorner \theta(U)] && \text{Power of internal forces} \\ & && \text{per unit proper volume;} \\ &W_0 = \mu_0 w_0 - w_0^{(i)} = -\mu_0 (f_m \cdot U) && \text{Total power per unit} \\ & && \text{of proper volume.} \end{aligned} \tag{6.189}$$

Multiplying Eq. (6.184) by U, one finds the *energy equation*

$$\mathrm{div}\, (\rho_0 U) = W_0. \tag{6.190}$$

Excluding processes of matter creation or annihilation, one must add to this equation the conservation of the proper mass of the fluid,

$$0 = \mathrm{div}\, (\mu_0 U) = \nabla_U \mu_0 + \mu_0 \Theta(U). \tag{6.191}$$

In this case, using the relation $\rho_0 = \mu_0 \hat{\epsilon}_0$, the energy equation (6.190) becomes

$$\nabla_U \hat{\epsilon}_0 = \frac{1}{\mu_0} W_0\,; \tag{6.192}$$

therefore, recalling (6.189), we find

$$\frac{D\hat{\epsilon}_0}{d\tau_U} = w_0 - \frac{1}{\mu_0} w_0^{(i)}, \tag{6.193}$$

which represents the *First Law of Thermodynamics* in the rest frame of the fluid. Summarizing, the fundamental equations in the rest frame of the fluid are given by

$$\begin{aligned} &\rho_0 a(U) = \mu_0 (f_0^{(i)} + f_0^{(e)}) && \text{Equation of motion;} \\ &\frac{D\hat{\epsilon}_0}{d\tau_U} = w_0 - \frac{1}{\mu_0} w_0^{(i)} && \text{Energy equation.} \end{aligned} \tag{6.194}$$

The equation of motion can also be cast in the following form. According to (6.185),

$$\mu_0 f_0^{(i)} = -\mathrm{div}_U\, \mathcal{T}_0 - a(U) \llcorner \mathcal{T}_0, \tag{6.195}$$

that is, $f_0^{(i)}$ contains the fluid acceleration $a(U)$. One can collect terms involving acceleration, rewriting the equation of motion in the form

$$[\rho_0 P(U) - \mathcal{T}_0] \llcorner a(U) = \mu_0 f_0^{(e)} - \mathrm{div}_U\, \mathcal{T}_0, \tag{6.196}$$

similar to the second law of test particle dynamics.

Ordinary fluids: relative dynamics

We now study fluid dynamics as seen by an observer u not comoving with the fluid. Splitting the 4-velocity field of the fluid in the standard way,

$$U = \gamma(U, u)[u + \nu(U, u)], \tag{6.197}$$

the evolution equations with respect to u are obtained by the spatial and temporal projections of (6.194) with respect to u. Projecting (6.185) first orthogonally to U and then orthogonally to u gives

$$\rho_0 P(u, U) a(U) = \mu_0 P(u, U) f_m. \tag{6.198}$$

Using relation (6.106), namely

$$P(u, U) a(U) = -\gamma F^{(G)}_{(\mathrm{fw}, U, u)} + \gamma^2 P(u, U, u) a_{(\mathrm{fw}, U, u)}, \tag{6.199}$$

with

$$P(u, U, u) \equiv P(u, U) \llcorner P(U, u), \tag{6.200}$$

one finds

$$\rho_0 \gamma^2 P(u, U, u) a_{(\mathrm{fw}, U, u)} = \rho_0 \gamma F^{(G)}_{(\mathrm{fw}, U, u)} + \mu_0 P(u, U) f_m. \tag{6.201}$$

Let us now define, following Ferrarese and Bini (2007), the relative energy and mass densities

$$\hat{\mu} = \rho_0 \gamma^2, \qquad \mu = \mu_0 \gamma^2, \tag{6.202}$$

where the presence of the square of the gamma factor has a simple explanation in terms of the Lorentz transformation from the comoving frame U to the observer's frame u: one γ comes from the energy transformation and the other from the volume transformation. Equation (6.201) then becomes

$$\begin{aligned} \hat{\mu} a_{(\mathrm{fw}, U, u)} &= P(u, U, u)^{-1} [\rho_0 \gamma F^{(G)}_{(\mathrm{fw}, U, u)} + \mu_0 P(u, U) f_m] \\ &= P(u, U, u)^{-1} [\rho_0 \gamma F^{(G)}_{(\mathrm{fw}, U, u)} + \mu_0 P(u) f_m - \gamma \nu W_0]. \end{aligned} \tag{6.203}$$

Let us now introduce the total relative force

$$\mu\mathcal{F}^{(\mathrm{tot})}_{(\mathrm{fw},U,u)} = \mu_0 P(u) f_m + \rho_0\gamma F^{(G)}_{(\mathrm{fw},U,u)}, \tag{6.204}$$

which contains the contributions of external, internal, and gravitational forces. The equation of motion becomes

$$\begin{aligned}\hat{\mu} a_{(\mathrm{fw},U,u)} &= \mu P(u,U,u)^{-1}\mathcal{F}^{(\mathrm{tot})}_{(\mathrm{fw},U,u)} - \frac{W_0}{\gamma}\nu \\ &= \mu P(u,U,u)^{-1}\mathcal{F}^{(\mathrm{tot})}_{(\mathrm{fw},U,u)} - W\nu,\end{aligned} \tag{6.205}$$

where $W_0/\gamma \equiv W$ represents the total relative power, namely

$$W = \frac{\mu_0 w_0 - w_0^{(i)}}{\gamma} = \mu\tilde{w} - w^{(i)},$$

where we have introduced, from (6.202), the quantities

$$\tilde{w} \equiv \frac{w_0}{\gamma^3}, \qquad w^{(i)} \equiv \frac{w_0^{(i)}}{\gamma}, \tag{6.206}$$

which represent the relative power of the external forces per unit of mass ($\tilde{w}$) and of the internal forces ($w^{(i)}$), respectively. Therefore

$$\hat{\epsilon}_0 a_{(\mathrm{fw},U,u)} = \left[\mathcal{F}^{(\mathrm{tot})}_{(\mathrm{fw},U,u)} - \nu(\nu\cdot\mathcal{F}^{(\mathrm{tot})}_{(\mathrm{fw},U,u)})\right] - \nu\frac{W}{\mu}, \tag{6.207}$$

$$\frac{d\hat{\epsilon}}{d\tau_{(U,u)}} = \gamma^2\frac{W}{\mu}. \tag{6.208}$$

Here $d\tau_{(U,u)} = \gamma d\tau_U$ is the relative standard time, $\hat{\epsilon} = \hat{\epsilon}_0$, $\hat{\mu} = \hat{\epsilon}\mu$, and we have used the relation

$$P(u,U,u)^{-1} = P(U,u)^{-1}P(u,U)^{-1} = P(u) - \nu(U,u)\otimes\nu(U,u). \tag{6.209}$$

Transformation law for proper mechanical stresses

The transformation of the spatial (with respect to U) and symmetric tensor $\mathcal{T}_0 = \mathcal{T}(U)$ along u and onto LRS_u proceeds in a standard way. Let us denote

$$\mathcal{T}(u,U) = P(u,U)\mathcal{T}(U), \tag{6.210}$$

where the projection is understood to be on each index. Acting on (6.210) with the operator $P(u,U)^{-1}$ and recalling that

$$P(u,U)^{-1}P(u,U) = P(U), \tag{6.211}$$

we obtain

$$\mathcal{T}(U) = P(u,U)^{-1}\mathcal{T}(u,U). \tag{6.212}$$

If u' denotes another family of observers, then

$$\mathcal{T}(u',U) = P(u',U)\mathcal{T}(U); \tag{6.213}$$

hence the transformation law for mechanical stresses follows:

$$\mathcal{T}(u',U) = [P(u',U)P(u,U)^{-1}]\mathcal{T}(u,U). \tag{6.214}$$

As an example let us evaluate the relative power of internal forces $w^{(i)}$. From its definition (see (6.189)) we have

$$w^{(i)} = \frac{w_0^{(i)}}{\gamma} = \frac{1}{\gamma}[\mathcal{T}(U)^{\alpha\beta}\nabla_\beta U_\alpha]. \tag{6.215}$$

Recalling that $U_\alpha = \gamma\,[u_\alpha + \nu(U,u)_\alpha]$, it follows that

$$w^{(i)} = \mathcal{T}(U)^{\alpha\beta}\nabla_\beta(u_\alpha + \nu_\alpha). \tag{6.216}$$

Using the transformation law for mechanical stresses (6.214) one then finds

$$\begin{aligned}
w^{(i)} &= \mathcal{T}(U)^{\alpha\beta}\nabla_\beta(u_\alpha + \nu_\alpha)\\
&= \mathcal{T}(u,U)^{\mu\nu}[P(u,U)^{-1}]^\alpha{}_\mu[P(u,U)^{-1}]^\beta{}_\nu[-u_\beta a(u)_\alpha - k(u)_{\alpha\beta} + \nabla_\beta\nu_\alpha]\\
&= \mathcal{T}(u,U)^{\mu\sigma}[\nabla_\sigma\nu_\mu + \nu_\sigma\nabla_u\nu_\mu] + \mathrm{Tr}\,[\mathcal{T}(u,U)\llcorner\theta(u)]\\
&\quad + [\nu\llcorner\mathcal{T}(U,u)]\cdot[-\gamma^{-1}F^{(G)}_{(\mathrm{fw},U,u)} - 2\nu\llcorner\theta(u) - \nu(\nu\cdot a(u))],
\end{aligned} \tag{6.217}$$

where the representation

$$P(u,U)^{-1} = P(u) + u\otimes\nu(U,u) \tag{6.218}$$

of the mixed projector has been used.

Example: the perfect fluid

In this case $\mathcal{T}_0 = p_0P(U)$, where $P(U)$ is the projection operator orthogonal to U. Using mass conservation we have

$$w_0^{(i)} = p_0\Theta(U). \tag{6.219}$$

According to the notation previously introduced (see Eqs. (6.189)), we also have

$$\mathrm{div}\,\mathcal{T}_0 = \nabla(U)p_0 + p_0U\Theta(U) + p_0a(U), \tag{6.220}$$

$$\mu_0f_m = \mu_0f - \nabla(U)p_0 - p_0U\Theta(U) - p_0a(U), \tag{6.221}$$

$$\mu_0P(U)f_m = \mu_0[f - Uw_0] - [\nabla(U)p_0 + p_0a(U)], \tag{6.222}$$

and the evolution equations reduce to

$$\begin{aligned}
(\rho_0 + p_0)a(U) &= -\nabla(U)p_0 + \mu_0[f - Uw_0],\\
\frac{d\rho_0}{d\tau_U} &= -(\rho_0 + p_0)\Theta(U) + \mu_0w_0.
\end{aligned} \tag{6.223}$$

In terms of the internal energy, the First Law of Thermodynamics becomes

$$\frac{D\hat{\epsilon}_0}{d\tau_U} = w_0 - \frac{p_0}{\mu_0}\Theta(U).$$

One may also introduce the proper entropy s_0 per unit of proper mass,

$$T_K \frac{ds_0}{d\tau_U} = \mu_0 w_0\,, \tag{6.224}$$

where T_K is the temperature.

In the absence of external forces ($f=0$ and hence $w_0=0$) one has the conservation of the entropy density s_0 along the flow lines of U,

$$\frac{Ds_0}{d\tau_U} = 0,$$

and the acceleration of the fluid lines reduces to

$$a(U) = -\frac{1}{\rho_0 + p_0}\nabla(U)p_0.$$

The relative point of view can be obtained directly from (6.205):

$$\begin{aligned} a_{(\mathrm{fw},U,u)} &= -\frac{1}{\rho_0 + p_0}[P(U,u)\nabla p_0 - \nu(\nu \cdot P(U,u)\nabla p_0)] \\ &\quad + \gamma^{-1}[F^{(G)}_{(\mathrm{fw},U,u)} - \nu(\nu \cdot F^{(G)}_{(\mathrm{fw},U,u)})]. \end{aligned} \tag{6.225}$$

Hydrodynamics with thermal flux: absolute formulation

We can generalize our previous results to the case of a fluid with thermal flux. We shall here illustrate only the general procedure, omitting unnecessary detail. The energy-momentum tensor can be written as in (6.179) but having in addition thermal stresses $Q_0 \equiv Q(U)$:

$$\begin{aligned} T &= \rho_0 U \otimes U + S_0, \\ S_0 &= \underbrace{\mathcal{T}_0}_{\text{mechanical}} + \underbrace{Q_0}_{\text{thermal}}, \end{aligned}$$

where

$$Q_0 = U \otimes q_0 + q_0 \otimes U,$$

with $q_0 = q_0(U)$ and

$$\mathrm{ALT}\, S_0 = 0, \quad \mathrm{ALT}\, X_0 = 0, \quad X_0 \llcorner U = 0, \quad q_0 \llcorner U = 0.$$

Let us first consider the splitting of the equations of motion $\operatorname{div} T = \mu_0 f$ in the comoving frame of the fluid. Projection along U and onto LRS_U gives

$$\rho_0 a(U) = \mu_0 P(U)(f_m + f_{th}) = \mu_0 P(U)(f_0^{(i)} + f_0^{(e)} + f_0^{(th)}),$$

$$\frac{d\hat{\epsilon}_0}{d\tau_U} = \frac{W_0}{\mu_0} + (q_c)_0 \equiv \frac{\mathcal{Q}_0}{\mu_0}, \tag{6.226}$$

where $(q_c)_0$ is the thermal conduction power in the comoving frame. We now have a thermal force and a thermal power,

$$\mu_0 P(U) f_{th} = -P(U) \operatorname{div} Q, \tag{6.227}$$

and

$$\mathcal{Q}_0 = -[\mu_0 f - \operatorname{div} S] \cdot U = (\mu_0 q_0^{tot} - w_0^{(i)}) = W_0 + \mu_0 (q_c)_0, \tag{6.228}$$

with

$$\begin{aligned} q_0^{\rm tot} &= w_0 + (q_c)_0, \\ \mu_0 (q_c)_0 &= U \cdot \operatorname{div} Q \equiv -[\nabla \cdot q_0 + a(U) \cdot q_0], \\ w_0^{(i)} &= -U \cdot \operatorname{div} \mathcal{T}. \end{aligned} \tag{6.229}$$

Hydrodynamics with thermal flux: relative formulation

Let us project the evolution equations of the proper frame along and orthogonally to u. We find

$$\hat{\mu} a_{(\mathrm{fw},U,u)} = \mu P(u,U,u)^{-1} F^{\rm tot}_{(\mathrm{fw},U,u)} - \nu \frac{\mathcal{Q}_0}{\gamma}, \tag{6.230}$$

where

$$\mu F^{\rm tot}_{(\mathrm{fw},U,u)} = \mu_0 P(u)[f_m + f_{th}]. \tag{6.231}$$

From the energy theorem we find instead

$$\frac{d\hat{\epsilon}}{d\tau_{(U,u)}} = \gamma^2 \frac{\mathcal{Q}}{\mu}, \tag{6.232}$$

where $\mathcal{Q} = \mathcal{Q}_0/\gamma$.

Example: the viscous fluid of Landau-Lifshitz

The viscous fluid of Landau-Lifshitz (see Misner, Thorne, and Wheeler, 1973, p. 567) is characterized by the following mechanical stress tensor:

$$\mathcal{T}_0 = [p_0 - \zeta \Theta(U)] P(U) - 2\eta \sigma(U), \tag{6.233}$$

where $\sigma(U) = \theta(U) - \frac{1}{3}\Theta(U)P(U)$ is the trace-free part of the expansion field. We have

$$\mu_0(q_c)_0 = -[\mathrm{div}_U\, q_0 + 2a(U)\cdot q_0], \tag{6.234}$$
$$w_0^{(i)} = p_0\Theta(U) - [\zeta\Theta(U)^2 + 2\eta\mathrm{Tr}(\sigma(U)^2)]. \tag{6.235}$$

Also in this case one introduces the scalar entropy s, related to the temperature by the First Law of Thermodynamics (generalized to include the effects of the thermal action),

$$T_K\frac{Ds}{d\tau_U} = q_0^{\mathrm{tot}} + \frac{p_0\Theta(U) - w_0^{(i)}}{\mu_0} \equiv w_0 + (q_c)_0 + [\zeta\Theta(U)^2 + 2\eta\mathrm{Tr}(\sigma(U)^2)], \tag{6.236}$$

and the entropy 4-vector (see Exercise 22.7 of Misner, Thorne, and Wheeler, 1973)

$$\mathcal{S} = \mu_0 sU + \frac{q_0}{T_K}. \tag{6.237}$$

Taking into account the mass conservation law, $\nabla_U(\mu_0 U) = 0$, one finds

$$T_K\nabla\cdot\mathcal{S} = \mu_0 w_0 + [\zeta\Theta(U)^2 + 2\eta\mathrm{Tr}(\sigma(U)^2)] - q_0\cdot[\nabla\ln T_K + a(U)]. \tag{6.238}$$

In the absence of external forces we have $f = 0$ and $w_0 = 0$; therefore the above expressions are considerably simplified.

7
Non-local measurements

The Principle of Equivalence states that gravitational and inertial accelerations cannot be distinguished from each other if one neglects the curvature of the background geometry. But even if curvature is taken into account, the choice of the observer together with a frame adapted to him/her *pollutes* the measurements with inertial contributions which are entangled with the curvature in a non-separable way. The curvature is responsible for a relative acceleration among freely falling particles. This acceleration induces a field of strains which can be measured with a suitable experimental device; however, a relative acceleration also arises from orbital constraints if the bodies are not in free fall. In this case a relativistically complete and correct description of the relative strains must also take into account the properties of the observer and of his frame. A measurement which carries the signature of the space-time curvature within its measurement domain is termed *non-local*.

Here we shall first identify the components of the space-time curvature relative to a given frame and then discuss various ways to measure them.

7.1 Measurement of the space-time curvature

Let u be a vector field whose integral curves form a congruence $\mathcal{C}_u$ which we assume to be representative of a family of observers. With respect to the latter, then, the spatial and temporal splitting of the Riemann tensor identifies the following three spatial fields:

$$\begin{aligned}
\mathcal{E}(u)_{\alpha\beta} &= R_{\alpha\mu\beta\nu}u^\mu u^\nu, \\
\mathcal{H}(u)_{\alpha\beta} &= -R^*{}_{\alpha\mu\beta\nu}u^\mu u^\nu, \\
\mathcal{F}(u)_{\alpha\beta} &= [{}^*R^*]_{\alpha\mu\beta\nu}u^\mu u^\nu,
\end{aligned} \tag{7.1}$$

where $\mathcal{E}(u)_{[\alpha\beta]} = 0 = \mathcal{F}(u)_{[\alpha\beta]}$ and $\mathcal{H}(u)^\alpha{}_\alpha = 0$. The first two quantities are termed, respectively, the *electric* and *magnetic* parts of the Riemann tensor. The

20 independent components of the Riemann tensor are then summarized by the 6 independent components of the electric part (spatial and symmetric tensor), the 8 independent components of the magnetic part (spatial and trace-free tensor), and the 6 independent components of the mixed part (spatial and symmetric tensor).

Consider a frame $\{e_\alpha\}$ (not necessarily orthonormal) adapted to the observer u so that $u = e_0$ with e_a a basis in LRS_u. Then all the above spatial quantities can be written as

$$\begin{aligned}
\mathcal{E}(u)_{ab} &= R_{a0b0},\\
\mathcal{H}(u)_{ab} &= -R^*_{a0b0} = \frac{1}{2}\eta(u)^{cd}{}_b R_{a0cd},\\
\mathcal{F}(u)_{ab} &= [{}^*R^*]_{a0b0} = \frac{1}{4}\eta(u)_a{}^{cd}\eta(u)_b{}^{ef}R_{cdef},
\end{aligned} \tag{7.2}$$

and can be inverted to give

$$\begin{aligned}
R_{a0}{}^{cd} &= \mathcal{H}(u)_{ab}\eta(u)^{bcd},\\
R^{abcd} &= \eta(u)^{abr}\eta(u)^{cds}\mathcal{F}(u)_{rs}.
\end{aligned} \tag{7.3}$$

Using these relations one has also the frame components of the Ricci tensor $R^\alpha{}_\beta = R^{\mu\alpha}{}_{\mu\beta}$,

$$\begin{aligned}
R^0{}_0 &= -\mathcal{E}(u)^c{}_c,\\
R^0{}_a &= \eta(u)_{abc}\mathcal{H}(u)^{bc},\\
R^a{}_b &= -\mathcal{E}(u)^a{}_b - \mathcal{F}(u)^a{}_b + \delta^a{}_b\mathcal{F}(u)^c{}_c,
\end{aligned} \tag{7.4}$$

so that

$$R = R^0{}_0 + R^a{}_a = -2(\mathcal{E}(u)^c{}_c - \mathcal{F}(u)^c{}_c). \tag{7.5}$$

As shown in (2.66), the Riemann tensor can be written in terms of the Weyl tensor, the Ricci tensor, and the scalar curvature; in four dimensions we have

$$\begin{aligned}
R^{\alpha\beta}{}_{\gamma\delta} &= C^{\alpha\beta}{}_{\gamma\delta} + \left(\delta^\alpha{}_{[\gamma}S_{\delta]}{}^\beta - \delta^\beta{}_{[\gamma}S_{\delta]}{}^\alpha\right)\\
&= C^{\alpha\beta}{}_{\gamma\delta} + 2\delta^{[\alpha}{}_{[\gamma}S_{\delta]}{}^{\beta]},
\end{aligned} \tag{7.6}$$

where

$$S_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{6}R\, g_{\alpha\beta}. \tag{7.7}$$

Clearly, in vacuum ($R_{\alpha\beta} = 0, R = 0$), the Weyl and Riemann tensors coincide.

Similarly to the Riemann tensor, the splitting of the Weyl tensor identifies the following two spatial fields, because of the identity ${}^*C^* = -C$ (or ${}^*C = C^*$):

$$E(u)_{\alpha\beta} = C_{\alpha\mu\beta\nu}u^\mu u^\nu, \qquad H(u)_{\alpha\beta} = -C^*{}_{\alpha\mu\beta\nu}u^\mu u^\nu. \tag{7.8}$$

Tensors $E(u)$ and $H(u)$ are termed, respectively, the *electric* and *magnetic* parts of the Weyl tensor and are related to $\mathcal{E}(u)$, $\mathcal{H}(u)$, and $\mathcal{F}(u)$ by the following relations:

$$
\begin{aligned}
E(u) &= \frac{1}{2}[\mathcal{E}(u) - \mathcal{F}(u)]^{(\mathrm{TF})}, \\
H(u) &= \mathrm{SYM}\, \mathcal{H}(u),
\end{aligned} \tag{7.9}
$$

from (7.6). Written in terms of components with respect to a frame adapted to u, we have

$$
\begin{aligned}
E(u)^a{}_b &= \frac{1}{2}\left[\mathcal{E}(u)^a{}_b - \mathcal{F}(u)^a{}_b - \frac{1}{3}\delta^a{}_b(\mathcal{E}(u)^c{}_c - \mathcal{F}(u)^c{}_c)\right], \\
H(u)^{ab} &= \mathcal{H}(u)^{(ab)}\,.
\end{aligned} \tag{7.10}
$$

In Chapter 3 we evaluated the components of the Riemann tensor in an adapted frame; in fact expressions (3.92), (3.97), and (3.98) are equivalent to

$$
\begin{aligned}
\mathcal{E}(u)^a{}_b &= [\nabla(u)_b + a(u)_b]a(u)^a + \nabla(u)_{(\mathrm{fw})}k(u)^a{}_b - [k(u)^2]^a{}_b, \\
\mathcal{H}(u)^{ab} &= -\left[\nabla(u)_{[c}k(u)^a{}_{d]}\eta(u)^{bcd} + 2a(u)^a\omega(u)^b\right], \\
\mathcal{F}(u)^a{}_b &= [\theta(u)^2 - \Theta(u)\theta(u)]^a{}_b - \frac{1}{2}\delta^a{}_b[\mathrm{Tr}\theta(u)^2 - \Theta(u)^2] \\
&\quad + 3\omega(u)^a\omega(u)_b - G_{(\mathrm{sym})}{}^a{}_b,
\end{aligned} \tag{7.11}
$$

where

$$
G_{(\mathrm{sym})}{}^a{}_b = R_{(\mathrm{sym})}{}^a{}_b - \frac{1}{2}\delta^a{}_b R_{(\mathrm{sym})}, \tag{7.12}
$$

with $R_{(\mathrm{sym})}{}^a{}_b = R_{(\mathrm{sym})}{}^{ca}{}_{cb}$ and $R_{(\mathrm{sym})} = R_{(\mathrm{sym})}{}^a{}_a$ from (3.105). Furthermore,

$$
\begin{aligned}
\mathcal{E}(u)^c{}_c &= [\nabla(u)_c + a(u)_c]a(u)^c - \nabla(u)_{(\mathrm{fw})}\Theta(u) + 2\omega(u)^c\omega(u)_c - \mathrm{Tr}\,\theta(u)^2, \\
\mathcal{H}(u)^c{}_c &= 2[\nabla(u)_c - a(u)_c]\omega(u)^c \equiv 0, \\
\mathcal{F}(u)^c{}_c &= -\frac{1}{2}[\mathrm{Tr}\,\theta(u)^2 - \Theta(u)^2] + 3\omega(u)^c\omega(u)_c + \frac{1}{2}R_{(\mathrm{sym})},
\end{aligned} \tag{7.13}
$$

where the trace of $\mathcal{H}(u)$ vanishes identically from (3.82). From (7.11) and recalling the definition of the symmetric curl (see Eqs. (3.40) and (3.41)), we have

$$
\begin{aligned}
H(u)^{ab} = \mathcal{H}(u)^{(ab)} &= -[\mathrm{Scurl}_u\, k(u)]^{ab} - 2a(u)^{(a}\omega(u)^{b)} \\
&= -[\mathrm{Scurl}_u\, \omega(u)]^{ab} + [\mathrm{Scurl}_u\, \theta(u)]^{ab} - 2a(u)^{(a}\omega(u)^{b)} \\
&= \left\{[\mathrm{Scurl}_u\, \theta(u)]^{ab} - 2a(u)^{(a}\omega(u)^{b)} - \nabla(u)^{(a}\omega(u)^{b)}\right\}^{\mathrm{TF}}
\end{aligned} \tag{7.14}
$$

and

$$
\begin{aligned}
E(u)_{ab} = \frac{1}{2}\,&[\nabla(u)_{(a}a(u)_{b)} + a(u)_a a(u)_b - \nabla(u)_{(\mathrm{fw})}\theta(u)_{ab} \\
&+ 2\omega(u)_a\omega(u)_b - 2[\theta(u)^2]_{ab} + \Theta(u)\theta(u)_{ab} - R_{(\mathrm{sym})ab}]^{\mathrm{TF}}
\end{aligned} \tag{7.15}
$$

If u is expansion-free, the electric and magnetic parts of the Weyl tensor reduce to

$$\begin{aligned} E(u)_{ab} &= \frac{1}{2}\left[\nabla(u)_{(a}a(u)_{b)} + a(u)_a a(u)_b + 2\omega(u)_a\omega(u)_b - R_{(\mathrm{sym})ab}\right]^{\mathrm{TF}} \\ H(u)_{ab} &= -[2a(u)^{(a}\omega(u)^{b)} + \nabla(u)^{(a}\omega(u)^{b)}]^{\mathrm{TF}} \end{aligned} \tag{7.16}$$

7.2 Vacuum Einstein's equations in 1+3 form

Einstein's equations without the cosmological constant are given by

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = 8\pi T_{\alpha\beta}. \tag{7.17}$$

In the absence of matter energy sources ($T_{\alpha\beta} = 0$) they become $R_{\alpha\beta} = 0$. Using (7.4) we can write these equations in terms of the kinematical parameters of the observer congruence $\mathcal{C}_u$ and their derivatives. From $(7.2)_1$ and with respect to a frame adapted to u, we have

$$\begin{array}{ll} \mathcal{E}(u)^c{}_c = 0 & (R^0{}_0 = 0), \\ \mathcal{H}(u)^{[ab]} = 0 & (R^0{}_a = 0), \\ \mathcal{E}(u)^a{}_b + \mathcal{F}(u)^a{}_b = \delta^a{}_b\mathcal{F}(u)^c{}_c & (R^a{}_b = 0). \end{array} \tag{7.18}$$

Using the trace-free property of $\mathcal{E}(u)$ in the last equation we find $\mathcal{F}(u)^c{}_c = 0$ so that from the above equations we deduce that

$$\mathcal{E}(u)^a{}_b = -\mathcal{F}(u)^a{}_b. \tag{7.19}$$

Moreover, from $(7.13)_1$, the condition $R^0{}_0 = 0$ gives

$$[\nabla(u)_c + a(u)_c]a(u)^c - \nabla(u)_{(\mathrm{fw})}\Theta(u) + 2\omega(u)^c\omega(u)_c - \mathrm{Tr}\theta(u)^2 = 0. \tag{7.20}$$

From $(7.4)_2$, the components $R^0{}_a = 0$ can be written in the form

$$\eta_{cab}\mathcal{H}(u)^{ab} = 0, \tag{7.21}$$

which is equivalent to

$$-\nabla(u)_a[\theta(u)^a{}_c - \Theta(u)\delta^a{}_c] - [\mathrm{curl}_u\,\omega(u)]_c - 2[a(u)\times_u\omega(u)]_c = 0. \tag{7.22}$$

Finally, $R^a{}_b = 0$ gives

$$\begin{aligned} &-[\nabla(u)_{(\mathrm{lie})} + \Theta(u)]\theta(u)^a{}_b - [\nabla(u)_{(b} + a(u)_{(b}]a(u)^{a)} + R_{(\mathrm{sym})}{}^a{}_b \\ &\qquad - 2[\omega(u)^a\omega(u)_b - \delta^a{}_b\omega(u)^c\omega(u)_c] = 0. \end{aligned} \tag{7.23}$$

This set of equations takes a simplified form when the observer congruence is Born-rigid, i.e. $\theta(u)=0$. In this case we have

$$[\nabla(u)_c + a(u)_c]a(u)^c + 2\omega(u)^c\omega(u)_c = 0, \tag{7.24}$$

$$[\mathrm{curl}_u\, \omega(u)]_c + 2[a(u)\times_u \omega(u)]_c = 0, \tag{7.25}$$

$$\begin{aligned} R_{(\mathrm{sym})ab} &- [\nabla(u)_{(b} + a(u)_{(b}]a(u)_{a)} - 2[\omega(u)_a\omega(u)_b \\ &- P(u)_{ab}\omega(u)^c\omega(u)_c] = 0. \end{aligned} \tag{7.26}$$

Contracting the indices in (7.26) yields

$$R_{(\mathrm{sym})} - \nabla(u)_b a(u)^b - a(u)_a a(u)^a + 4\omega(u)_a\omega(u)^a = 0 \tag{7.27}$$

which, using (7.24), leads to

$$R_{(\mathrm{sym})} + 6\omega(u)_a\omega(u)^a = 0. \tag{7.28}$$

7.3 Divergence of the Weyl tensor in 1+3 form

The divergence of the Weyl tensor $\nabla_\delta\, C^\delta{}_{\alpha\beta\gamma}$ in an adapted frame (1 + 3 form) is represented by the following independent fields:

$$\nabla_\delta\, C^\delta{}_{0a0}, \quad \nabla_\delta\, C^\delta{}_{(ab)0}, \quad \nabla_\delta {}^*C^\delta{}_{0a0}, \quad \nabla_\delta {}^*C^\delta{}_{(ab)0}. \tag{7.29}$$

Other components give non-independent relations due to the trace-free property of the Weyl tensor and the Ricci identity. It is sufficient to deduce the $1+3$ form of the first pair of the above relations because the remaining two are obtained by the substitution $E(u)\to H(u)$ and $H(u)\to -E(u)$, similar to what is done for the electromagnetic field. We have

$$\begin{aligned} \nabla_\delta\, C^\delta{}_{0a0} = e_\delta\left(C^\delta{}_{0a0}\right) + \Gamma^\delta{}_{\mu\delta}C^\mu{}_{0a0} - \Gamma^\sigma{}_{0\delta}C^\delta{}_{\sigma a0} \\ - \Gamma^\sigma{}_{a\delta}C^\delta{}_{0\sigma 0} - \Gamma^\sigma{}_{0\delta}C^\delta{}_{0a\sigma}. \end{aligned} \tag{7.30}$$

Substituting the values (3.71) of the connection coefficients of an adapted frame and using definitions (7.8), this equation can be cast in the form

$$\nabla_\delta\, C^\delta{}_{0a0} = \left\{\mathrm{div}_u\, E(u) - 2^{*(u)}[\theta(u)\llcorner H(u)] - 2H(u)\llcorner\omega(u)\right\}_a. \tag{7.31}$$

Similarly we have

$$\begin{aligned} \nabla_\delta\, C^\delta{}_{(ab)0} = \nabla(u)_{(\mathrm{fw})}E(u)_{ab} &- [\mathrm{Scurl}_u\, H(u)]_{ab} + 2\Theta(u)E(u)_{ab} \\ &- 2[a(u)\times_u H(u)]_{ab} - [\omega(u)\times_u E(u)]_{ab} \\ &- 3[\mathrm{SYM}(\theta(u)\llcorner\, E(u))]^{\mathrm{TF}}_{ab}. \end{aligned} \tag{7.32}$$

Summarizing, the set of equations representing the divergence of the Weyl tensor is the following

$$\begin{aligned} \nabla_\delta\, C^\delta{}_{0a0} &= \{\mathrm{div}_u\, E(u) - 2^{*(u)}[\theta(u)\llcorner H(u)] - 2H(u)\llcorner\omega(u)\}_a\,, \\ \nabla_\delta{}^*\, C^\delta{}_{0a0} &= \{\mathrm{div}_u\, H(u) + 2^{*(u)}[\theta(u)\llcorner E(u)] + 2E(u)\llcorner\omega(u)\}_a\,, \end{aligned}$$

$$
\begin{aligned}
\nabla_\delta C^\delta{}_{(ab)0} &= \nabla(u)_{(\mathrm{fw})}E(u)_{ab} - [\mathrm{Scurl}_u\, H(u)]_{ab} + 2\Theta(u)E(u)_{ab} \\
&\quad - 2[a(u)\times_u H(u)]_{ab} - [\omega(u)\times_u E(u)]_{ab} \\
&\quad - 3[\mathrm{SYM}(\theta(u)\llcorner E(u))]^{\mathrm{TF}}_{ab}, \\
\nabla_\delta{}^* C^\delta{}_{(ab)0} &= \nabla(u)_{(\mathrm{fw})}H(u)_{ab} + [\mathrm{Scurl}_u\, E(u)]_{ab} + 2\Theta(u)H(u)_{ab} \\
&\quad + 2[a(u)\times_u E(u)]_{ab} - [\omega(u)\times_u H(u)]_{ab} \\
&\quad - 3[\mathrm{SYM}(\theta(u)\llcorner H(u))]^{\mathrm{TF}}_{ab}.
\end{aligned} \tag{7.33}
$$

In vacuum ($T_\alpha{}^\beta = 0$) and with respect to an observer u described by an expansion-free ($\theta(u) = 0$) congruence of integral curves, and from (7.8), we can write Einstein's equations in Maxwell-like form (compare with (6.159)–(6.162)):

$$
\begin{aligned}
0 &= \mathrm{div}_u\, E(u) - 2H(u)\llcorner\omega(u), \\
0 &= \mathrm{div}_u\, H(u) + 2E(u)\llcorner\omega(u), \\
0 &= \nabla(u)_{(\mathrm{fw})}E(u) - [\mathrm{Scurl}_u\, H(u)] - 2[a(u)\times_u H(u)] - [\omega(u)\times_u E(u)], \\
0 &= \nabla(u)_{(\mathrm{fw})}H(u) + [\mathrm{Scurl}_u\, E(u)] + 2[a(u)\times_u E(u)] - [\omega(u)\times_u H(u)].
\end{aligned} \tag{7.34}
$$

7.4 Electric and magnetic parts of the Weyl tensor

Let U and u be two different families of observers related by a boost

$$
U = \gamma(U,u)[u + \nu(U,u)]. \tag{7.35}
$$

Both of these observers can be used to split the Weyl tensor (and similarly the Riemann tensor) in terms of associated electric and magnetic parts,

$$
\begin{aligned}
E(u)_{\alpha\beta} &= C_{\alpha\mu\beta\nu}u^\mu u^\nu, & H(u)_{\alpha\beta} &= -C^*_{\alpha\mu\beta\nu}u^\mu u^\nu, \\
E(U)_{\alpha\beta} &= C_{\alpha\mu\beta\nu}U^\mu U^\nu, & H(U)_{\alpha\beta} &= -C^*_{\alpha\mu\beta\nu}U^\mu U^\nu.
\end{aligned} \tag{7.36}
$$

Using (7.35) we have, for example,

$$
E(U)_{\alpha\beta} = \gamma(U,u)^2 C_{\alpha\mu\beta\nu}[u^\mu + \nu(U,u)^\mu][u^\nu + \nu(U,u)^\nu], \tag{7.37}
$$

that is, abbreviating $\gamma(U,u) = \gamma$ and $\nu(U,u) = \nu$,

$$
E(U)_{\alpha\beta} = \gamma^2[E(u)_{\alpha\beta} + C_{\alpha\mu\beta\nu}u^\mu\nu^\nu + C_{\alpha\mu\beta\nu}\nu^\mu u^\nu + C_{\alpha\mu\beta\nu}\nu^\mu\nu^\nu]. \tag{7.38}
$$

Let e_α denote a frame adapted to u (namely $u = e_0$ and e_a with $a = 1,2,3$ spanning the local rest space of u). We then have

$$
\begin{aligned}
E(U)_{ab} &= \gamma^2[E(u)_{ab} + C_{a0bc}\nu^c + C_{acb0}\nu^c + C_{acbd}\nu^c\nu^d] \\
&= \gamma^2[E(u)_{ab} + H(u)_{af}\eta(u)^f{}_{bc}\nu^c + H(u)_{bf}\eta(u)^f{}_{ac}\nu^c \\
&\quad - E(u)_{fg}\eta(u)^f{}_{ac}\eta(u)^g{}_{bd}\nu^c\nu^d], \\
E(U)_{00} &= \gamma^2[C_{0b0c}\nu^b\nu^c] = \gamma^2[E(u)_{bc}\nu^b\nu^c], \\
E(U)_{0b} &= \gamma^2[C_{0cb0}\nu^c + C_{0cbd}\nu^c\nu^d] \\
&= \gamma^2[-E(u)_{cb}\nu^c - H(u)_{cf}\eta(u)^f{}_{bd}\nu^c\nu^d].
\end{aligned} \tag{7.39}
$$

Therefore, in compact form, we have

$$[P(u,U)E(U)]_{\alpha\beta} = \gamma^2[E(u)_{\alpha\beta} + 2H(u)_{(\alpha|f|}\eta(u)^f{}_{\beta)c}\nu^c \\ - E(u)_{fg}\eta(u)^f{}_{\alpha c}\eta(u)^g{}_{\beta d}\nu^c\nu^d] \tag{7.40}$$

and similarly

$$[P(u,U)H(U)]_{\alpha\beta} = \gamma^2[H(u)_{\alpha\beta} - 2E(u)_{(\alpha|f|}\eta(u)^f{}_{\beta)c}\nu^c \\ - H(u)_{fg}\eta(u)^f{}_{\alpha c}\eta(u)^g{}_{\beta d}\nu^c\nu^d]. \tag{7.41}$$

7.5 The Bel-Robinson tensor

In the theory of general relativity, gravitation is described as the curvature of the background geometry; hence any manifestation of pure gravity is presented in terms of the Riemann tensor. Neglecting the contribution to gravity by the local distribution of matter-energy (that is, setting $R_{\alpha\beta} = 0$), a possible generalization of the energy-momentum tensor to the gravitational field is the Bel-Robinson tensor (Bel, 1958), defined by

$$T_{\alpha\beta}{}^{\gamma\delta} = \frac{1}{2}(C_{\alpha\rho\beta\sigma}C^{\gamma\rho\delta\sigma} + {}^*C_{\alpha\rho\beta\sigma}{}^*C^{\gamma\rho\delta\sigma}). \tag{7.42}$$

In standard terminology, we use *super-energy density* and *super-Poynting vector* (Maartens and Bassett, 1998) in reference to the analogous contractions in the electromagnetic case relative to a general observer u, namely

$$\begin{aligned} \mathcal{E}^{(\mathrm{g})}(u) &= T_{\alpha\beta\gamma\delta}u^\alpha u^\beta u^\gamma u^\delta \\ &= \frac{1}{2}[E(u)^2 + H(u)^2], \\ P^{(\mathrm{g})}(u)_\alpha &= T_{\alpha\beta\gamma\delta}u^\beta u^\gamma u^\delta \\ &= [E(u) \times_u H(u)]_\alpha, \end{aligned} \tag{7.43}$$

where the notation (3.39) has been used. Similarly to what was done for the electromagnetic field in Sections 6.11 and 6.12 with the electric and magnetic parts of the Weyl tensor, we can introduce the complex (symmetric trace-free) spatial tensor field $Z^{(\mathrm{g})}(u) = E(u) - iH(u)$, abbreviated to $Z(u)$ in this section, and evaluate the effect of a boost in a general direction U, different from u, in terms of the orthogonal decompositions with respect to the relative spatial velocity of the observers u and U. $Z(u)$ can be decomposed into a scalar $Z^{\parallel\,\parallel}(u)$, a vector $Z^{\parallel\perp}(u)$, and a tensor $Z^{\perp\perp}(u)$, the latter two being orthogonal to $\hat\nu(U,u)$, i.e.

$$Z(u) = Z^{\parallel\,\parallel}(u)\hat\nu(U,u) \otimes \hat\nu(U,u) + Z^{\parallel\perp}(u) \otimes \hat\nu(U,u) \\ + \hat\nu(U,u) \otimes Z^{\parallel\perp}(u) + Z^{\perp\perp}(u), \tag{7.44}$$

where $Z^{\perp\perp}(u)$ can then be further decomposed into its pure-trace part involving $\mathrm{Tr}\, Z^{\perp\perp}(u) = -Z^{\parallel\,\parallel}(u)$ and its trace-free part $Z^{\perp\perp(\mathrm{TF})}(u)$.

Let U be a family of observers; by definition we have

$$Z(U) = E(U) - iH(U). \tag{7.45}$$

Moreover, Eqs. (7.40) and (7.41) imply that

$$\begin{aligned}[P(u,U)Z(U)]_{\alpha\beta} = \gamma^2[&Z(u)_{\alpha\beta} + 2iZ(u)_{(\alpha|f|}\eta(u)^f{}_{\beta)c}\nu^c \\ &- Z(u)_{fg}\eta(u)^f{}_{\alpha c}\eta(u)^g{}_{\beta d}\nu^c\nu^d].\end{aligned} \tag{7.46}$$

Alternatively, one can boost $Z(U)$ onto the local rest space of u; in this case we have

$$B_{(\mathrm{lrs})}(u,U)Z(U) = B_{(\mathrm{lrs})u}(u,U)P(u,U)Z(U), \tag{7.47}$$

where the map $B_{(\mathrm{lrs})u}(u,U)$ has been introduced in (3.134), conveniently rewritten here as

$$\begin{aligned}\left[B_{(\mathrm{lrs})\,u}(u,U)\right]{}^{\alpha}{}_{\beta} &= P(u)^{\alpha}{}_{\beta} - \left(1-\frac{1}{\gamma}\right)\hat\nu^{\alpha}\hat\nu_{\beta} \\ &= [P(u) - \hat\nu\otimes\hat\nu]^{\alpha}{}_{\beta} + \frac{1}{\gamma}[\hat\nu\otimes\hat\nu]^{\alpha}{}_{\beta},\end{aligned} \tag{7.48}$$

with the first term in the square brackets projecting orthogonal to $\hat\nu$ in the local rest space of u.

In terms of components, (7.47) can be written as

$$\begin{aligned}&[B_{(\mathrm{lrs})}(u,U)Z(U)]^{\alpha\beta} \\ &\quad = B_{(\mathrm{lrs})u}(u,U)^{\alpha}{}_{\mu}B_{(\mathrm{lrs})u}(u,U)^{\beta}{}_{\nu}[P(u,U)Z(U)]^{\mu\nu}.\end{aligned} \tag{7.49}$$

To complete the analogy with the corresponding transformation laws for electromagnetic fields in (6.144), we need to replace the magnitude of ν with the rapidity parameter $\nu = \tanh\alpha$. Considering components parallel and perpendicular to the direction of the velocity, one then finds

$$\begin{aligned}[B_{(\mathrm{lrs})}(u,U)Z(U)]^{\parallel\,\parallel} &= Z^{\parallel\,\parallel}(u), \\ [B_{(\mathrm{lrs})}(u,U)Z(U)]^{\parallel\perp} &= \cosh\alpha\, Z^{\parallel\perp}(u) \\ &\quad + i\sinh\alpha\hat\nu(U,u)\times_u Z^{\parallel\perp}(u), \\ [B_{(\mathrm{lrs})}(u,U)Z(U)]^{\perp\perp(\mathrm{TF})} &= \cosh 2\alpha\, Z^{\perp\perp(\mathrm{TF})}(u) \\ &\quad - i\sinh 2\alpha\,\hat\nu(U,u)\times_u Z^{\perp\perp(\mathrm{TF})}(u).\end{aligned} \tag{7.50}$$

Note that the transformation law for the vector $Z^{\parallel\perp}(u)$ is exactly the same as in $(6.144)_2$ for the corresponding electromagnetic vector $Z^{(\mathrm{em})\perp}(u)$, while the one for the tensor $Z(U)^{\perp\perp}$ is formally the same apart from a sign change and having 2α in place of α. The condition that the transformed value of $Z^{\parallel\perp}(U)$ is zero requires the impossible condition $\tanh\alpha = \pm1$, which means that either $Z^{\parallel\perp}(U)$ is initially zero or no observer U can be found for which it becomes zero.

In terms of Z, the super-energy density and the super-Poynting vector are obtained from the Bel-Robinson tensor as

$$\begin{aligned}\mathcal{E}^{(\mathrm{g})}(u) &= \frac{1}{2}\mathrm{Tr}\,[E(u)\cdot E(u)+H(u)\cdot H(u)] = \frac{1}{2}\mathrm{Tr}\,[\bar{Z}(u)\cdot Z(u)],\\ \mathcal{P}^{(\mathrm{g})}(u)_\alpha &= [E(u)\times_u H(u)]_\alpha = \frac{i}{2}\,[\bar{Z}(u)\times_u Z(u)]_\alpha,\end{aligned} \tag{7.51}$$

and their decomposition is

$$\begin{aligned}\mathcal{E}^{(\mathrm{g})}(u) &= \frac{1}{2}\mathrm{Tr}\left[\bar{Z}^{\perp\perp(\mathrm{TF})}(u)\cdot Z^{\perp\perp(\mathrm{TF})}(u)\right]\\ &\quad + \bar{Z}^{\parallel\,\perp}(u)\cdot Z^{\parallel\,\perp}(u) + \frac{3}{4}|Z^{\parallel\,\parallel}(u)|^2,\\ \mathcal{P}^{(\mathrm{g})}(u) &= \frac{i}{2}\,\left[\bar{Z}^{\perp\perp(\mathrm{TF})}(u)\times_u Z^{\perp\perp(\mathrm{TF})}(u) + \bar{Z}^{\parallel\,\perp}(u)\times_u Z^{\parallel\,\perp}(u)\right]\\ &\quad - \hat{\nu}(u,U)\times_u \Im m\left[\bar{Z}^{\parallel\,\parallel}(u)Z^{\parallel\,\perp}(u)\right.\\ &\quad \left. + \bar{Z}^{\perp\perp(\mathrm{TF})}(u) \llcorner Z^{\parallel\,\perp}(u)\right].\end{aligned} \tag{7.52}$$

There are two complementary cases in which the effective transformation reduces to a single electromagnetic-like transformation which one can use to transform the super-Poynting vector to zero: either (i) $Z^{\parallel\perp}(u)=0$ or (ii) $Z^{\parallel\perp}(u)$ is the only non-vanishing part of $Z(u)$. If one starts from a Weyl principal frame (Stephani *et al.*, 2003) in which $Z(u)$ takes its canonical Petrov-type form, and considers boosts along one of the spatial frame vectors, condition (i) is equivalent to requiring that $Z(u)$ have block diagonal form with respect to the chosen frame, which is possible for all Petrov types except III, while condition (ii) describes exactly type III. In that case, however, $Z^{\parallel\perp}(u)$ corresponds to a null electromagnetic field and so one cannot transform the super-Poynting vector to zero.

In the first case, the super-quantities simplify to

$$\begin{aligned}\mathcal{E}^{(\mathrm{g})}(u) &= \frac{1}{2}\mathrm{Tr}\left[\bar{Z}^{\perp\perp(\mathrm{TF})}(u)\cdot Z^{\perp\perp(\mathrm{TF})}(u)\right] + \frac{3}{4}|Z^{\parallel\,\parallel}(u)|^2,\\ \mathcal{P}^{(\mathrm{g})}(u) &= \frac{i}{2}\,\left[\bar{Z}^{\perp\perp(\mathrm{TF})}(u)\times_u Z^{\perp\perp(\mathrm{TF})}(u)\right].\end{aligned} \tag{7.53}$$

Evaluating the magnitude of the transformed super-momentum tensor leads to

$$\begin{aligned}||P^{(\mathrm{g})}(U)|| = \Bigg| -\frac{i}{2}\Big(&\cosh 4\alpha\,\hat{\nu}(U,u)\cdot\left[\bar{Z}^{\perp\perp(\mathrm{TF})}(u)\times_u Z^{\perp\perp(\mathrm{TF})}(u)\right]\\ &+ i\sinh 4\alpha\,\mathrm{Tr}\left[\bar{Z}^{\perp\perp(\mathrm{TF})}(u)\cdot Z^{\perp\perp(\mathrm{TF})}(u)\right]\Big)\Bigg|.\end{aligned} \tag{7.54}$$

As long as $Z^{\perp\perp(\mathrm{TF})}(u)\neq 0$, one can define

$$\tanh 4\alpha_{(g)} = i\frac{\hat{\nu}(U,u)\cdot\left[\bar{Z}^{\perp\perp(\mathrm{TF})}(u)\times_u Z^{\perp\perp(\mathrm{TF})}(u)\right]}{\mathrm{Tr}\left[\bar{Z}^{\perp\perp(\mathrm{TF})}(u)\cdot Z^{\perp\perp(\mathrm{TF})}(u)\right]}, \tag{7.55}$$

but for α to be real and finite, independent of the particular value of $\hat{\nu}(U,u)$, the inequality

$$\frac{||\bar{Z}^{\perp\perp(\mathrm{TF})}(u) \times_u Z^{\perp\perp(\mathrm{TF})}(u)||}{\mathrm{Tr}\left[\bar{Z}^{\perp\perp(\mathrm{TF})}(u) \cdot Z^{\perp\perp(\mathrm{TF})}(u)\right]} < 1 \tag{7.56}$$

must hold. If it does, then one finds

$$||P^{(\mathrm{g})}(U)|| = \mathcal{E}^{(\mathrm{g})\perp\perp(\mathrm{TF})}(u)\,\frac{|\sinh 4(\alpha - \alpha_{(\mathrm{g})})|}{\cosh 4\alpha_{(\mathrm{g})}}, \tag{7.57}$$

and similarly

$$\mathcal{E}^{(\mathrm{g})\perp\perp(\mathrm{TF})}(U) = \mathcal{E}^{(\mathrm{g})\perp\perp(\mathrm{TF})}(u)\,\frac{\cosh 4(\alpha - \alpha_{(\mathrm{g})})}{\cosh 4\alpha_{(\mathrm{g})}}, \tag{7.58}$$

where $\mathcal{E}^{(\mathrm{g})\perp\perp(\mathrm{TF})}(u)$ is the first term in the expression (7.53) for $\mathcal{E}^{(\mathrm{g})}(u)$, contributed to the super-energy by $Z^{\perp\perp(\mathrm{TF})}(u)$, while the remaining contribution to the super-energy from $Z^{\|\,\|}(u)$ does not change since $Z^{\|\,\|}(u)$ is invariant under this family of boosts. Thus the test observer with rapidity $\alpha = \alpha_{(\mathrm{g})}$ sees a vanishing super-Poynting vector and a minimum value of the total super-energy among this family of boosts. The analogy with the electromagnetic case is then complete. A detailed analysis is contained in Bini, Jantzen, and Miniutti (2002).

7.6 Measurement of the electric part of the Riemann tensor

In a series of papers de Felice and coworkers (de Felice and Usseglio-Tomasset, 1991; 1992; 1993; 1996; Semerák and de Felice, 1997) have defined and studied in an observer-dependent way the relative strains among a set of comoving test particles in black hole space-times. Starting from that analysis, we consider here, in full generality, how the definition of relative accelerations and strains is affected, neglecting the background curvature, by the geometric properties of the frame adapted to the fiducial observer.

Old ideas and modern approaches

Our analysis moves from the concept of gravitational compass introduced by Szekeres (1965) (see also Audretsch and Lämmerzahl, 1983; Ciufolini and Demianski, 1986; 1987; Pirani, 1956a; 1956b) and the related discussion about the problem of setting up a preferred frame within which to study the gravitational field. According to Szekeres, a gravitational compass consists of an arrangement of three test particles connected to each other by springs and also to a fiducial observer who is in general accelerated. The behavior of the particles with respect to the observer is then investigated using the geodesic deviation equation, which enables one to deduce the physical significance of the curvature tensor components. When the measurement begins, the apparatus is set free so one can monitor the strains on the springs. The relative acceleration between two nearby particles is completely determined by the electric part of the Riemann tensor, which can be thought of as a symmetric force distribution whose six independent components are the strains on the six springs. When the off-diagonal terms (i.e. the transverse strains) vanish, the springs connecting the test particles to the observer lie along the principal axes of the tidal force matrix, so that the apparatus maps out the local gravitational field, acting just as a compass. In the case of a vacuum space-time, the electric part of the Weyl tensor represents the only curvature contribution to the deviation between any two neighboring trajectories introducing shearing forces, due to its property of being symmetric and trace-free. Actually Szekeres' gravitational compass only describes an idealized situation. For any practical use, in fact, it should be replaced by a "gravity gradiometer," i.e. a device to perform measurements of the local gradient of the tidal gravitational force. The theory of a relativistic gravity gradiometer has been developed by many authors (Mashhoon and Theiss, 1982; Mashhoon, Paik, and Will, 1989) in view of satellite experiments around the Earth in the framework of the post-Newtonian approximation. It should also be noted that a modern observational trend is to use atomic interferometry to build a future generation of highly precise gravity gradiometers (see Matsko, Yu, and Maleki, 2003, and references therein).

More recently Chicone and Mashhoon (2002) have obtained a generalized geodesic deviation equation in Fermi coordinates as well as in arbitrary coordinates as a Taylor expansion in powers of the components of the deviation vector, retaining terms up to first order, but without any restriction on the relative spatial velocities. They then investigated in a number of papers (Chicone and Mashhoon, 2005a; 2005b) the motion of a swarm of free particles (in both non-relativistic and relativistic regimes) relative to a free reference particle which is on a radial escape trajectory away from a collapsed object (a Schwarzschild as well as a Kerr black hole), discussing the astrophysical implications of the related (observer-dependent) tidal acceleration mechanism. The further dependence of the deviation equation on the 4-acceleration of the observer as well as his 3-velocity has been accounted for very recently by Mullari and Tammelo (2006).

In what follows we shall develop the general theory of relativistic strains, generalizing Szekeres' picture as well as the one associated with the relativistic gravity gradiometry in the case when the acceleration strains are present. We also identify which frame is the most convenient for measuring either tidal or inertial forces experienced by an extended body.

The relative acceleration equation

Let us consider a collection of test particles, i.e. a congruence $\mathcal{C}_U$ of time-like world lines, with unit tangent vector U ($U \cdot U = -1$) parameterized by the proper time

τ_U. Let $\gamma_* \in \mathcal{C}_U$ be the world line of the "fiducial observer." In general, the lines of the congruence $\mathcal{C}_U$ as well as of the observer are accelerated, at $a(U) = \nabla_U U$.

The separation between the line γ_* and a generic line of the congruence is represented by a connecting vector Y which satisfies Lie transport along U, that is

$$\pounds_U Y = 0, \tag{7.59}$$

which implies that

$$\nabla_U Y = \nabla_Y U. \tag{7.60}$$

The covariant derivative of U which appears in $\nabla_Y U$ can be written in terms of the kinematical fields of the congruence as in (3.45), that is

$$\nabla_\alpha U^\beta = -a(U)^\beta U_\alpha - k(U)^\beta{}_\alpha, \tag{7.61}$$

where $k(U)^\beta{}_\alpha = \omega(U)^\beta{}_\alpha - \theta(U)^\beta{}_\alpha$ is the kinematical tensor which describes the vorticity of the congruence $\omega(U)_{\alpha\beta} = k(U)_{[\alpha\beta]}$ and its expansion $\theta(U)_{\alpha\beta} = -k(U)_{(\alpha\beta)}$ relative to the observer U. Thus (7.59) becomes

$$\frac{DY}{d\tau_U} = \nabla_Y U = -(Y \cdot U)a(U) - k(U) \llcorner Y. \tag{7.62}$$

The covariant derivative along U of both sides of (7.60) gives rise to the *relative acceleration equation*,

$$\frac{D^2 Y}{d\tau_U^2} = -R(U,Y)U + \nabla_Y a(U), \tag{7.63}$$

or, in components,

$$\frac{D^2 Y^\alpha}{d\tau_U^2} = -R^\alpha{}_{\beta\gamma\delta} U^\beta Y^\gamma U^\delta + Y^\sigma \nabla_\sigma a(U)^\alpha. \tag{7.64}$$

Clearly $R(U,Y)U \equiv R^\alpha{}_{\beta\gamma\delta} U^\beta Y^\gamma U^\delta = \mathcal{E}(U)^\alpha{}_\gamma Y^\gamma$ and $\nabla_Y a(U)$ are respectively the gravitational and the inertial contributions to the relative acceleration. Equation (7.63) can then be conveniently rewritten as follows:

$$\frac{D^2 Y}{d\tau_U^2} = -[\mathcal{E}(U) - \nabla a(U)] \llcorner Y. \tag{7.65}$$

Let us now set up an orthonormal frame $\{E_{\hat\alpha}\} = \{E_{\hat 0} \equiv U, E_{\hat a}\}$ adapted to the congruence U, and write both the Lie transport equation (7.59) and the relative acceleration equation (7.63) with respect to this frame. As before, hatted indices refer to tetrad components. The spatial triad is generic in the sense that it rotates with a certain angular velocity $\zeta_{(\mathrm{fw})}$ with respect to gyro-fixed axes along U:

$$P(U)\nabla_U E_{\hat a} \equiv \nabla(U)_{(\mathrm{fw})} E_{\hat a} = \zeta_{(\mathrm{fw})} \times_U E_{\hat a} \equiv C_{(\mathrm{fw})}{}^{\hat b}{}_{\hat a} E_{\hat b}, \tag{7.66}$$

where, as already defined, $C_{(\mathrm{fw})\hat a \hat b} = -\eta(U)_{\hat a \hat b \hat c}\, \zeta^{\hat c}_{(\mathrm{fw})}$.

Introduce the frame components of Y, i.e. the decomposition $Y = Y^{\hat 0}\, U + Y^{\hat a}\, E_{\hat a}$ (and the notation $\vec Y \equiv P(U)Y = Y^{\hat a}\, E_{\hat a}$). The Lie transport equation (7.59) then becomes

$$\begin{aligned}\nabla_U Y &\equiv \dot Y^{\hat 0}\, U + Y^{\hat 0} a(U) + \dot Y^{\hat a}\, E_{\hat a} + [\vec Y \cdot a(U)]U + \zeta_{(\mathrm{fw})} \times_U \vec Y \\ &= Y^{\hat 0} a(U) - k(U) \llcorner \vec Y,\end{aligned} \tag{7.67}$$

where the relation $Y_{\hat 0} = -Y^{\hat 0}$ has been used and the overdot denotes differentiation with respect to the proper time τ_U ($\dot f = df/d\tau_U$). Combining terms we obtain

$$\left[\dot Y^{\hat 0} + \vec Y \cdot a(U)\right] U + \dot Y^{\hat a}\, E_{\hat a} + \zeta_{(\mathrm{fw})} \times \vec Y = -k(U) \llcorner \vec Y, \tag{7.68}$$

yielding

$$\dot Y^{\hat 0} = -\vec Y \cdot a(U), \tag{7.69}$$

$$\dot Y^{\hat a} + [\zeta_{(\mathrm{fw})} \times \vec Y]^{\hat a} + k(U)^{\hat a}{}_{\hat b} Y^{\hat b} = 0. \tag{7.70}$$

From the definition of $k(U)$ the *relative velocity equation* (7.70) can be written as

$$\dot Y^{\hat a} + [(\zeta_{(\mathrm{fw})} - \omega(U)) \times \vec Y]^{\hat a} - \theta(U)^{\hat a}{}_{\hat b} Y^{\hat b} = 0, \tag{7.71}$$

implying that $\dot Y^{\hat a} = 0$ when the vector $\zeta_{(\mathrm{fw})} = \omega(U)$ and $\theta(U) = 0$. The latter condition is satisfied by a Frenet-Serret frame along a Born-rigid congruence of world lines.

Let us turn now to the relative acceleration equation (7.63). Substituting (7.69) into the first line of (7.67) leads to

$$\nabla_U Y = Y^{\hat 0} a(U) + \dot Y^{\hat a}\, E_{\hat a} + \zeta_{(\mathrm{fw})} \times \vec Y. \tag{7.72}$$

Taking the covariant derivative along U of both sides of Eq. (7.72), we obtain the left-hand side of the relative acceleration equation (7.63):

$$\begin{aligned}\nabla_{UU} Y = {} & [Y^{\hat 0} a(U)^2 - a(U) \cdot (k(U) \llcorner \vec Y)]U - [\vec Y \cdot a(U)]a(U) \\ & + Y^{\hat 0}\left[\dot a(U)^{\hat a} E_{\hat a} + \zeta_{(\mathrm{fw})} \times a(U)\right] + \ddot Y^{\hat a}\, E_{\hat a} \\ & - 2\zeta_{(\mathrm{fw})} \times [k(U) \llcorner \vec Y] - \zeta_{(\mathrm{fw})} \times [\zeta_{(\mathrm{fw})} \times \vec Y] \\ & + \dot\zeta_{(\mathrm{fw})} \times \vec Y,\end{aligned} \tag{7.73}$$

where (7.69) has been taken into account. Let us now evaluate the term $\nabla_Y a(U)$ on the right-hand side of (7.63); a direct calculation shows that

$$\begin{aligned}\nabla_Y a(U) = Y^{\hat{0}}[a(U)^2 U + \dot{a}(U)^{\hat{a}} E_{\hat{a}} + \zeta_{(\mathrm{fw})} \times a(U)] \\ + Y^{\hat{b}} \nabla(U)_{\hat{b}} a(U) - a(U) \cdot (k(U) \llcorner \vec{Y}) U,\end{aligned} \tag{7.74}$$

where $\nabla(U)a(U) = P(U)\nabla a(U) \equiv P(U)^{\mu}_{\alpha} P(U)^{\nu}_{\beta} \nabla_\nu a(U)_\mu$. As a result, from Eqs. (7.73) and (7.74) we obtain

$$\begin{aligned}\nabla_{UU} Y - \nabla_Y a(U) \equiv \ddot{Y}^{\hat{a}}\, E_{\hat{a}} - Y^{\hat{b}} \nabla(U)_{\hat{b}} a(U) - [\vec{Y} \cdot a(U)] a(U) \\ - \zeta_{(\mathrm{fw})} \times [\zeta_{(\mathrm{fw})} \times \vec{Y}] \\ - 2\zeta_{(\mathrm{fw})} \times [k(U) \llcorner \vec{Y}] + \dot{\zeta}_{(\mathrm{fw})} \times \vec{Y} \\ = -\mathcal{E}(U) \llcorner \vec{Y}.\end{aligned} \tag{7.75}$$

In their work, de Felice and coworkers introduced the relative strains as components of the following tensor:

$$S(U) = \nabla(U) a(U) + a(U) \otimes a(U), \tag{7.76}$$

namely $S(U)_{\hat{a}\hat{b}} = \nabla(U)_{\hat{b}} a(U)_{\hat{a}} + a(U)_{\hat{a}} a(U)_{\hat{b}}$. The tensor S will be termed the *Fermi-Walker strain tensor*; it depends on the congruence $\mathcal{C}_U$ and the chosen spatial triad $E_{\hat{a}}$. To make our formulas more compact we also introduce the quantity

$$\begin{aligned}T_{(\mathrm{fw},\,U,\,E)}{}^{\hat{a}}{}_{\hat{b}} = \dot{C}_{(\mathrm{fw})}{}^{\hat{a}}{}_{\hat{b}} - [C^2_{(\mathrm{fw})}]^{\hat{a}}{}_{\hat{b}} - 2C_{(\mathrm{fw})}{}^{\hat{a}}{}_{\hat{c}}\, k(U)^{\hat{c}}{}_{\hat{b}} \\ = \delta^{\hat{a}}_{\hat{b}} \zeta^2_{(\mathrm{fw})} - \zeta^{\hat{a}}_{(\mathrm{fw})} \zeta_{(\mathrm{fw})\hat{b}} - \epsilon^{\hat{a}}{}_{bf} \dot{\zeta}^{\hat{f}}_{(\mathrm{fw})} \\ - 2\epsilon^{\hat{a}}{}_{\hat{f}\hat{c}} \zeta^{\hat{f}}_{(\mathrm{fw})} k(U)^{\hat{c}}{}_{\hat{b}},\end{aligned} \tag{7.77}$$

where $[C^2_{(\mathrm{fw})}]^{\hat{a}}{}_{\hat{b}} = C_{(\mathrm{fw})}{}^{\hat{a}}{}_{\hat{c}}\, C_{(\mathrm{fw})}{}^{\hat{c}}{}_{\hat{b}}$. Clearly $T_{(\mathrm{fw},\,U,\,E)}{}^{\hat{a}}{}_{\hat{b}}$ are the components of a spatial tensor, first derived in Bini, de Felice, and Geralico (2006), which describes how far the chosen frame is from being Fermi-Walker itself; in this case it would vanish. We term this the *twist tensor*. The relative acceleration equation (7.63) (or equivalently (7.65)) then becomes

$$\ddot{Y}^{\hat{a}} + \mathcal{K}_{(U,\,E)}{}^{\hat{a}}{}_{\hat{b}} Y^{\hat{b}} = 0, \tag{7.78}$$

where

$$\mathcal{K}_{(U,\,E)}{}^{\hat{a}}{}_{\hat{b}} = [T_{(\mathrm{fw},\,U,\,E)} - S(U) + \mathcal{E}(U)]^{\hat{a}}{}_{\hat{b}}. \tag{7.79}$$

Equations (7.78) are the basic equations that we shall apply to concrete cases (see Chapter 9). Interesting special cases are listed below:

- *Flat space-time*: $R_{\alpha\beta\gamma\delta}=0$ so that $\mathcal{E}(U)\equiv 0$. In this case we have $\mathcal{K}_{(U,\,E)}=T_{(\mathrm{fw},\,U,\,E)}-S(U)$.
- E_a *spatial triad Fermi-Walker dragged along* U: $\omega_{(\mathrm{fw},\,U,\,E)}=0$. This implies $T_{(\mathrm{fw},\,U,\,E)}=0$, so that $\mathcal{K}_{(U,\,E)}=\mathcal{E}(U)-S(U)$.
- U *geodesic*: $a(U)\equiv 0$. In this case $S(U)=0$ and hence $\mathcal{K}_{(U,\,E)}=T_{(\mathrm{fw},\,U,\,E)}+\mathcal{E}(U)$.
- U *irrotational*: $\omega(U)\equiv 0$, so that $k(U)=-\theta(U)$.
- U *Born-rigid*: $\theta(U)\equiv 0$, so that $k(U)=\omega(U)$.

Clearly we can also consider combinations of the above special cases, for example a congruence of geodesic and irrotational orbits:

- U *geodesic and irrotational*: $a(U)\equiv 0$ and $\omega(U)\equiv 0$ ($k(U)=-\theta(U)$). In this case $S(U)=0$, so that $\mathcal{K}_{(U,\,E)}=T_{(\mathrm{fw},\,U,\,E)}+\mathcal{E}(U)$.

Finally, the case $\mathcal{K}_{(U,\,E)}=0$ corresponds to $\ddot{Y}^a=0$, i.e. the absence of relative accelerations among the particles of the congruence.

Geometrical meaning of the twist tensor

Using the formalism developed in the previous sections we can understand the geometrical meaning of the twist tensor. In fact, for the spatial vector $\vec{Y}=Y^{\hat{a}}E_{\hat{a}}$ the Lie transport equation (7.70) can be written as

$$\nabla_{(\mathrm{lie})}Y=0, \tag{7.80}$$

that is

$$\dot{Y}^{\hat{b}}+C_{(\mathrm{lie})}{}^{\hat{b}}{}_{\hat{a}}Y^{\hat{a}}=0. \tag{7.81}$$

Let us evaluate $\nabla^2_{(\mathrm{fw})}Y$. We have

$$\nabla_{(\mathrm{fw})}Y=(\dot{Y}^{\hat{b}}+C_{(\mathrm{fw})}{}^{\hat{b}}{}_{\hat{a}}Y^{\hat{a}})\,E_{\hat{b}}, \tag{7.82}$$

and then

$$\nabla^2_{(\mathrm{fw})}Y=\left[\ddot{Y}^{\hat{b}}+(\dot{C}_{(\mathrm{fw})}{}^{\hat{b}}{}_{\hat{d}}+[C^2_{(\mathrm{fw})}]^{\hat{b}}{}_{\hat{d}})Y^{\hat{d}}+2C_{(\mathrm{fw})}{}^{\hat{b}}{}_{\hat{d}}\dot{Y}^{\hat{d}}\right]E_{\hat{b}}. \tag{7.83}$$

Replacing $\dot{Y}^{\hat{b}}$ by (7.81) and using (3.63) yields

$$\nabla^2_{(\mathrm{fw})}Y=\left[\ddot{Y}^{\hat{b}}+T_{(\mathrm{fw},\,U,\,E)}{}^{\hat{b}}{}_{\hat{d}}Y^{\hat{d}}\right]E_{\hat{b}}, \tag{7.84}$$

which clarifies the meaning of the twist tensor. Furthermore, Eq. (7.78) becomes

$$\nabla^2_{(\mathrm{fw})}Y^{\hat{a}}+[\mathcal{E}(u)-S(u)]^{\hat{a}}{}_{\hat{b}}Y^{\hat{b}}=0, \tag{7.85}$$

explaining also the name *Fermi-Walker strain tensor* for $S(U)$: a (tem) derivative on the left-hand side would have led instead to a (tem)-strain tensor; as noted after Eq. (3.85), the term "tem" stands for "fw" or "lie".

7.7 Measurement of the magnetic part of the Riemann tensor

The equation for geodesic deviation provides a measurement only of the electric part of the Riemann tensor. Moreover its validity is restricted to the case of infinitesimal variation among neighboring geodesics. Consider two of them, say Υ_1 and Υ_2, with tangent vectors U_1 and U_2 such that

$$\nabla_{U_1} U_1 = 0 = \nabla_{U_2} U_2, \tag{7.86}$$

where in general $U^\alpha = dx^\alpha/ds$, s being the proper time on the corresponding curve. Clearly we require that $x_2(s) = x_1(s) + \delta x(s)$ along a curve connecting the two geodesics. The validity of (7.63) with $a(U) = 0$ requires that the curvature varies slightly over neighboring curves. Denoting by R the magnitude of any curvature component, we have that its variation over nearby curves is small, i.e.

$$\left| \frac{\delta R}{R} \right| \equiv \left| \frac{(\partial_\alpha R)\delta x^\alpha}{R} \right| \ll 1. \tag{7.87}$$

Moreover (7.63) is only valid if the tangent vectors of nearby curves vary along a connecting curve by an infinitesimal amount, that is

$$\frac{|\delta U|}{|U|} \equiv \frac{|U_2(s) - U_1(s)|}{|U_1(s)|} \ll 1\,. \tag{7.88}$$

If this condition is not satisfied, i.e. if

$$\frac{|\delta U|}{|U|} \approx 1, \tag{7.89}$$

then the equation for geodesic deviation takes a different form, termed the *generalized geodesic deviation equation* (Ciufolini, 1986; Ciufolini and Demianski, 1986; 1987). If (7.89) is satisfied, then one has to consider the geodesic equations as

$$\frac{DU_1^\mu}{ds} = \frac{dU_1^\mu}{ds} + \Gamma^\mu{}_{\nu\rho}(x_1) U_1^\nu U_1^\rho = 0, \tag{7.90}$$

$$\begin{aligned}
\frac{DU_2^\mu}{ds} &= \frac{dU_2^\mu}{ds} + \Gamma^\mu{}_{\nu\rho}(x_1 + \delta x) U_2^\nu U_2^\rho \\
&= \frac{d^2}{ds^2}(x_1^\mu + \delta x^\mu) + \Gamma^\mu{}_{\nu\rho}(x_1 + \delta x)\frac{d}{ds}(x_1^\nu + \delta x^\nu)\frac{d}{ds}(x_1^\rho + \delta x^\rho) \\
&= 0.
\end{aligned} \tag{7.91}$$

With a Taylor expansion to first order in δx, the latter equation can be written as

$$\frac{d^2}{ds^2}(x_1^\mu) + \frac{d^2}{ds^2}(\delta x^\mu) + \left[\Gamma^\mu{}_{\nu\rho} + (\partial_\lambda \Gamma^\mu{}_{\nu\rho})\,\delta x^\lambda\right](U_1^\nu + \delta U^\nu)(U_1^\rho + \delta U^\rho) = 0, \tag{7.92}$$

where we put $\delta U^\mu \equiv U_2^\mu(s) - U_1^\mu(s)$ with condition (7.89). If we now subtract Eq. (7.90) from (7.92) we obtain

$$\frac{d^2}{ds^2}(\delta x^\mu) + (\partial_\lambda \Gamma^\mu{}_{\nu\rho})\,\delta x^\lambda U_1^\nu U_1^\rho + 2\Gamma^\mu{}_{\nu\rho} U_1^\nu \delta U^\rho + (\partial_\lambda \Gamma^\mu{}_{\nu\rho})\,\delta x^\lambda \delta U^\nu \delta U^\rho$$
$$+ 2\partial_\lambda \Gamma^\mu{}_{\nu\rho} \delta x^\lambda U_1^\nu \delta U^\rho + \Gamma^\mu{}_{\nu\rho} \delta U^\nu \delta U^\rho = 0. \tag{7.93}$$

An equivalent expression of this equation, using the convenient notation

$$\frac{d^2(\delta x^\mu)}{ds^2} = \delta \ddot{x}^\mu,$$

is

$$\delta \ddot{x}^\mu = -R^\mu{}_{\nu\rho\sigma} U_1^\nu U_1^\sigma \delta x^\rho - (\partial_\lambda \Gamma^\mu{}_{\nu\rho})\,\delta x^\lambda \delta U^\nu \delta U^\rho$$
$$- 2\,(\partial_\lambda \Gamma^\mu{}_{\nu\rho})\,\delta x^\lambda U_1^\nu \delta U^\rho - \Gamma^\mu{}_{\nu\rho} \delta U^\nu \delta U^\rho, \tag{7.94}$$

which to first order in the variation δ reduces to (7.63) with $a(U) = 0$. Let us now express (7.94) in terms of Fermi normal coordinates adapted to the fiducial curve having tangent vector field U_1; in this coordinate system the connection coefficients along the curve vanish but not their derivatives. Therefore we have

$$\delta \ddot{x}^{\hat\alpha} + \partial_{\hat\sigma} \Gamma^{\hat\alpha}{}_{\hat\beta\hat\gamma} \delta x^{\hat\sigma} (U_1^{\hat\beta} U_1^{\hat\gamma} + 2U_1^{\hat\beta} \delta U^{\hat\gamma} + \delta U^{\hat\beta} \delta U^{\hat\gamma}) = 0. \tag{7.95}$$

Let us recall that U_1 is the observer and $\delta \ddot{x}^{\hat\alpha}$ are the measured quantities (observables). The target of the measurement is the components of the Riemann tensor directly related to the derivatives of the connections (see Ciufolini, 1986).

Moreover let us select three particles in such a way that their positions relative to U_1 form an orthonormal Fermi-Walker basis. In this case it would be $U_1^{\hat\alpha} = (1, 0, 0, 0)$. Let the other particles be identified by the following choices:

$$\begin{aligned} \delta x_2^{\hat\alpha} &= (0, 0, 1, 0), \qquad & \delta U_2^{\hat\alpha} &= (0, 1, 0, 0),\\ \delta x_3^{\hat\alpha} &= (0, 0, 0, 1), \qquad & \delta U_3^{\hat\alpha} &= (0, 0, 1, 0),\\ \delta x_4^{\hat\alpha} &= (0, 1, 0, 0), \qquad & \delta U_4^{\hat\alpha} &= (0, 0, 0, 1), \end{aligned}$$

where we set

$$\delta x_i^{\hat\alpha} = x_i^{\hat\alpha} - x_1^{\hat\alpha}, \qquad \delta U_i^{\hat\alpha} = U_i^{\hat\alpha} - U_1^{\hat\alpha} \qquad (i = 2, 3, 4). \tag{7.96}$$

Keeping the dot notation for the derivative with respect to s, we can write Eq. (7.95) for each particle as

$$\begin{aligned} \delta \ddot{x}_2^{\hat\alpha} &= R^{\hat\alpha}{}_{\hat0\hat0\hat2} + 2R^{\hat\alpha}{}_{\hat1\hat0\hat2} + \frac{2}{3} R^{\hat\alpha}{}_{\hat1\hat1\hat2},\\ \delta \ddot{x}_3^{\hat\alpha} &= R^{\hat\alpha}{}_{\hat0\hat0\hat3} + 2R^{\hat\alpha}{}_{\hat2\hat0\hat3} + \frac{2}{3} R^{\hat\alpha}{}_{\hat2\hat2\hat3},\\ \delta \ddot{x}_4^{\hat\alpha} &= R^{\hat\alpha}{}_{\hat0\hat0\hat1} + 2R^{\hat\alpha}{}_{\hat3\hat0\hat1} + \frac{2}{3} R^{\hat\alpha}{}_{\hat3\hat3\hat1}. \end{aligned} \tag{7.97}$$

Assuming that we are in vacuum ($R_{\alpha\beta} = 0$) and rewriting the frame components of the Riemann tensor in terms of electric and magnetic parts, the above system can be written as

$$\delta\ddot{x}_a^{\hat{\alpha}} = A^{\hat{\alpha}}{}_{a\hat{\beta}\hat{\gamma}}\mathcal{E}^{\hat{\beta}\hat{\gamma}} + B^{\hat{\alpha}}{}_{a\hat{\beta}\hat{\gamma}}\mathcal{H}^{\hat{\beta}\hat{\gamma}}, \tag{7.98}$$

with $a = 2, 3, 4$ and $A^{\hat{\alpha}}{}_{a\hat{\beta}\hat{\gamma}}$ and $B^{\hat{\alpha}}{}_{a\hat{\beta}\hat{\gamma}}$ being constant matrices. Inverting this system gives

$$\mathcal{E}_{\hat{\alpha}\hat{\beta}} = \bar{A}_{\hat{\alpha}\hat{\beta}\hat{\gamma}}{}^{d}\delta\ddot{x}_d^{\hat{\gamma}}, \qquad \mathcal{H}_{\hat{\alpha}\hat{\beta}} = \bar{B}_{\hat{\alpha}\hat{\beta}\hat{\gamma}}{}^{d}\delta\ddot{x}_d^{\hat{\gamma}}, \tag{7.99}$$

where again $\bar{A}^{\hat{\alpha}}{}_{a\hat{\beta}\hat{\gamma}}$ and $\bar{B}^{\hat{\alpha}}{}_{a\hat{\beta}\hat{\gamma}}$ are constant matrices, the explicit form of which can be found in Ciufolini (1986) and Ciufolini and Demianski (1986; 1987).

7.8 Curvature contributions to spatial velocity

An observer moving on a curve γ with tangent vector u and proper time s as parameter can only deduce the spatial velocity of a distant particle relative to his own local rest frame by exchanging light signals. At the event A on γ the observer sends a light signal to the particle, which receives it at the event P on γ'. At P the light signal is reflected back to the observer who receives it at the event B on γ. Denote as Υ and Υ' the null geodesics connecting A to P and P to B respectively. Let A_0 be the event on γ, subsequent to A and antecedent to B, which is simultaneous with P with respect to the observer u and such that the space-like geodesic $\zeta_{\mathrm{P}\to\mathrm{A}_0}$ joining P to A_0 is extremal with respect to γ.[1] Repeated reading of the time of emission of light signals at A and of the time of recording of the reflected echo at B allows one to determine the length of the space-like geodesic segment connecting P to A_0, which represents, by definition, the instantaneous spatial distance of the particle at P from the observer on γ; see Fig. 7.1. The relative velocity of the particle with respect to the observer u is then deduced, differentiating the above spatial distance with respect to the observer's proper time.[2] The measurement process involving the events A, P, and B is patently non-local insofar as the measurement domain is finite. Let us recall, however, that a standard determination of the particle velocity is based on the measurement of a frequency shift of the exchanged photons through the application of the Doppler formula. The velocity so determined, however, is an *equivalent* velocity because the frequency shift can also be caused by geometry perturbations which may not be related to the particle's motion at all. As already stated, curvature effects are in general entangled with inertial terms resulting from the choice of the reference frame, so we shall just term as *curvature* any possible combination of them. Our

[1] We assume that the curve γ' lies in a normal neighborhood of γ, so the geodesic $\zeta_{\mathrm{P}\to\mathrm{A}_0}$ is unique.

[2] The relative velocity so determined is along the observer's local line of sight, so it is a *radial* velocity, i.e. a velocity either of recession or of approach.

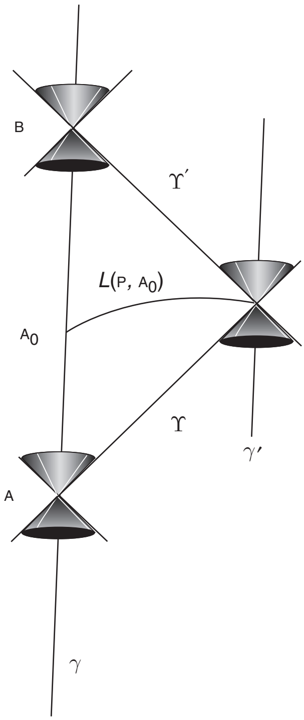

Fig. 7.1. Exchanging light signals with a distant particle, an observer can deduce from time readings on his own clock the spatial distance $L(\mathrm{P}, \mathrm{A}_0)$ covered by the particle. The relative spatial velocity then follows by standard definition.

aim here is to single out the curvature contributions to the frequency shift and, as a consequence, to the relative velocity as well.

The measurement of a relative velocity is the result of a local measurement, which does not contain curvature terms, and a non-local one, which depends explicitly on the curvature. The local (or flat) velocity ν, defined in (6.65), is the magnitude (with sign) of the 4-vector

$$\nu(\check{\ell}', u)^\alpha \equiv -(u_\rho \check{\ell}'^\rho)^{-1}(P(u)^\alpha{}_\sigma \check{\ell}'^\sigma) = -(u_\rho \check{\ell}'^\rho)^{-1}\left[\check{\ell}'^\alpha + u^\alpha(u_\sigma \check{\ell}'^\sigma)\right], \quad (7.100)$$

where ℓ is tangent to the particle's world line γ' with parameter s', for example, and $\check{\ell}'$ is the result of parallel propagating ℓ along the null geodesic Υ' connecting P to B. Clearly relation (7.100) holds at the event of observation B on γ where it is $v(\check{\ell}', u)^\alpha u_\alpha = 0$. As already noted, the local velocity is only a part of the relative velocity $\tilde{\nu}$, which includes both local and non-local contributions. We shall then revisit the general definition of relative velocity (de Felice and Clarke, 1990), making wide use of the connector two-point function. The instantaneous spatial distance L between the particle at P on γ' and the observer as measured by the latter is the length of the space-like segment of $\zeta_{\mathrm{P}\to\mathrm{A}_0}$ connecting P to A_0. The relative velocity $\tilde{\nu}$ is, from (5.3), given by

$$\tilde{\nu} = \frac{dL}{ds} = \frac{d}{ds}(2\Omega)^{1/2} = \frac{1}{L}\left(\frac{d\Omega}{ds}\right). \tag{7.101}$$

Here the differentiation with respect to s is performed varying A_0 on γ and simultaneously varying P on γ' in such a way that the corresponding geodesic segment joining the varied points is kept extremal with respect to γ. It then follows that

$$\tilde{\nu} = \frac{1}{L}\left(\Omega_{\alpha_0}u^{\alpha_0} + \frac{ds'}{ds}\Omega_{\alpha_p}\ell^{\alpha_p}\right) = \frac{1}{L}\frac{ds'}{ds}\Omega_{\alpha_p}\ell^{\alpha_p}, \tag{7.102}$$

with $\Omega_{\alpha_0}u^{\alpha_0} = 0$. Indices α_p and α_0 refer to quantities calculated at P and A_0 respectively. From (5.29) we have, on the space-like geodesic $\zeta_{\mathrm{P}\to\mathrm{A}_0}$,

$$\Omega_{\alpha_p}\ell^{\alpha_p} = -\Omega_{\alpha_0}\check{\ell}^{\alpha_0}, \tag{7.103}$$

where

$$\check{\ell}^{\alpha_0} = \Gamma(\mathrm{P}, \mathrm{A}_0; \zeta_{\mathrm{P}\to\mathrm{A}_0})_{\beta_p}{}^{\alpha_0}\ell^{\beta_p},$$

$\Gamma(\mathrm{P}, \mathrm{A}_0; \zeta_{\mathrm{P}\to\mathrm{A}_0})_{\beta_p}{}^{\alpha_0}$ being the components of the connector relating the points P and A_0 on the geodesic ζ (see de Felice and Clarke, 1990). Hence the quantity (7.102) can be written as

$$\tilde{\nu} = -\frac{1}{L}\left(\frac{ds'}{ds}\right)\Omega_{\alpha_0}\check{\ell}^{\alpha_0}. \tag{7.104}$$

Hereafter we shall omit the subscript $\mathrm{P}\to\mathrm{A}_0$.

In order to express $\tilde{\nu}$ as the magnitude of a 4-vector defined at A_0, let us exploit the property (5.30) of the world function, namely

$$2\Omega = \Omega_{\alpha_0}\Omega^{\alpha_0} = \Omega_{\alpha_p}\Omega^{\alpha_p}. \tag{7.105}$$

Defining

$$\tilde{\nu}_{\alpha_0} = \frac{1}{2\Omega}\left(\frac{ds'}{ds}\right)\Omega_{\alpha_0}\Omega_{\beta_0}\check{\ell}^{\beta_0}, \tag{7.106}$$

Eq. (7.104) can be written as

$$\tilde{\nu} = -(\tilde{\nu}_{\alpha_0}\tilde{\nu}^{\alpha_0})^{1/2}. \tag{7.107}$$

Let us now calculate the ratio ds'/ds. From the properties of the world function, and recalling (5.32) and (5.36), we obtain

$$\begin{aligned}\frac{D\Omega_{\alpha_0}}{ds} &= \Omega_{\alpha_0\beta_0}u^{\beta_0} + \frac{ds'}{ds}\Omega_{\alpha_0\beta_p}\ell^{\beta_p}\\ &= u_{\alpha_0} - \frac{ds'}{ds}\check{\ell}_{\alpha_0} + \chi_{\alpha_0\beta_0}u^{\beta_0} + \frac{ds'}{ds}\zeta_{\alpha_0\beta_p}\ell^{\beta_p},\end{aligned} \tag{7.108}$$

where $\chi_{\alpha_0\beta_0}$ and $\zeta_{\alpha_0\beta_p}$ depend explicitly on the curvature and, to first order in the Riemann tensor, are given by

$$\chi_{\alpha_0\beta_0} = (\sigma_p - \sigma_0)^{-1} g_{\alpha_0\gamma_0} \int_{\sigma_0}^{\sigma_p} (\sigma_p - \sigma)^2 R^\rho{}_{\mu\nu\sigma} \xi^\mu \xi^\nu \Gamma_\rho{}^{\gamma_0} \Gamma_{\beta_0}{}^\sigma \, d\sigma + O(|\mathrm{Riem}|^2), \tag{7.109}$$

$$\zeta_{\alpha_0\beta_p} = (\sigma_p - \sigma_0)^{-1} g_{\alpha_0\sigma_0} \int_{\sigma_0}^{\sigma_p} (\sigma_p - \sigma)(\sigma - \sigma_0) R^\rho{}_{\mu\nu\gamma} \xi^\mu \xi^\nu \Gamma_{\beta_p}{}^\gamma \Gamma_\rho{}^{\sigma_0} \, d\sigma + O(|\mathrm{Riem}|^2). \tag{7.110}$$

Here σ is the parameter on the geodesics ζ whose tangent vector field is ξ. Contracting (7.108) with u^{α_0} and recalling that

$$\frac{D\Omega_{\alpha_0}}{ds} u^{\alpha_0} = -\Omega_{\alpha_0} \dot{u}^{\alpha_0}, \tag{7.111}$$

$\dot{u}$ being the 4-acceleration of γ, leads to

$$-\Omega_{\alpha_0} \dot{u}^{\alpha_0} = -1 - \frac{ds'}{ds} \left(u_{\alpha_0} \check{\ell}^{\alpha_0} \right) + \chi_{\alpha_0\beta_0} u^{\alpha_0} u^{\beta_0} + \frac{ds'}{ds} \zeta_{\alpha_0\beta_p} u^{\alpha_0} \ell^{\beta_p}. \tag{7.112}$$

To first order in the curvature and relative to the observer on γ, we then have

$$\frac{ds'}{ds} = -\left(u_{\alpha_0} \check{\ell}^{\alpha_0} \right)^{-1} (1 - \Omega_{\alpha_0} \dot{u}^{\alpha_0}) \left(1 - \frac{\chi_{\alpha_0\beta_0} u^{\alpha_0} u^{\beta_0}}{1 - \Omega_{\alpha_0} \dot{u}^{\alpha_0}} + \frac{\zeta_{\alpha_0\beta_p} u^{\alpha_0} \ell^{\beta_p}}{u_{\gamma_0} \check{\ell}^{\gamma_0}} \right) + O(|\mathrm{Riem}|^2). \tag{7.113}$$

In the limit of negligible distance between the curves ($\Omega_{\alpha_0} \approx 0$), the above relation reduces to

$$\frac{ds'}{ds} = -\left(u_{\alpha_0} \check{\ell}^{\alpha_0} \right)^{-1} \equiv \bar{\gamma}^{-1}, \tag{7.114}$$

from (6.66). Let us use (7.113) in (7.106) to obtain

$$\begin{aligned} \tilde{\nu}_{\alpha_0} &= -\Omega_{\alpha_0} \frac{\Omega_{\beta_0} \check{\ell}^{\beta_0}}{2\Omega \left(u_{\gamma_0} \check{\ell}^{\gamma_0} \right)} \left(1 - \Omega_{\alpha_0} \dot{u}^{\alpha_0} + \mathcal{R}'_0 + O(\mathrm{Riem}|^2) \right) \\ &= -\xi_{\alpha_0} \frac{\xi_{\beta_0} \check{\ell}^{\beta_0}}{u_{\gamma_0} \check{\ell}^{\gamma_0}} \left(1 - \Omega_{\alpha_0} \dot{u}^{\alpha_0} + \mathcal{R}'_0 + O(|\mathrm{Riem}|^2) \right), \end{aligned} \tag{7.115}$$

from (5.30) with $\xi \cdot \xi = 1$, and

$$\mathcal{R}'_0 = \left(u_{\alpha_0} \check{\ell}^{\alpha_0} \right)^{-1} \zeta_{\alpha_0\beta_p} u^{\alpha_0} \ell^{\beta_p} (1 - \Omega_{\alpha_0} \dot{u}^{\alpha_0}) - \chi_{\alpha_0\beta_0} u^{\alpha_0} u^{\beta_0}. \tag{7.116}$$

Since $\xi_{\alpha_0} u^{\alpha_0} = 0$, Eq. (7.115) can also be written to first order in the curvature:

$$\tilde{\nu}_{\alpha_0} \approx -\xi_{\alpha_0} \xi_{\beta_0} \left[u^{\beta_0} + \check{\ell}^{\beta_0} \left(u^{\gamma_0} \check{\ell}_{\gamma_0} \right)^{-1} \right] \left(1 - \Omega_{\beta_0} \dot{u}^{\beta_0} + \mathcal{R}'_0 \right). \tag{7.117}$$

The quantities

$$\nu'^{\alpha_0} = -\left[u^{\alpha_0} + \check{\ell}^{\alpha_0}\left(u_{\gamma_0}\check{\ell}^{\gamma_0}\right)^{-1}\right] \tag{7.118}$$

closely resemble (7.100), but they differ from those components since these are written in terms of a vector $\check{\ell}$ at A_0 parallel propagated along a space-like geodesic (from P to A_0) and not along a null ray (as in the definition (7.100)). A general component of the vector $\check{\ell}$ at A_0 propagated along the curve ζ from P to A_0 can be written as

$$\check{\ell}_{\alpha_0} = \Gamma(\mathrm{P},\mathrm{A}_0;\zeta)^{\beta_p}{}_{\alpha_0}\ell_{\beta_p}. \tag{7.119}$$

For later convenience let us write (7.119) as follows:

$$\begin{aligned}\check{\ell}_{\alpha_0} &= \Gamma(\mathrm{P},\mathrm{A}_0;\zeta)^{\sigma_p}{}_{\alpha_0}\Gamma(\mathrm{B},\mathrm{P};\Upsilon')^{\rho_b}{}_{\sigma_p}\Gamma(\mathrm{P},\mathrm{B};\Upsilon')^{\beta_p}{}_{\rho_b}\ell_{\beta_p}\\ &= \Gamma(\mathrm{P},\mathrm{A}_0;\zeta)^{\sigma_p}{}_{\alpha_0}\Gamma(\mathrm{B},\mathrm{P};\Upsilon')^{\rho_b}{}_{\sigma_p}\check{\ell}'_{\rho_b},\end{aligned} \tag{7.120}$$

where $\check{\ell}'_{\rho_b}$ are the components of the vector $\check{\ell}'$ which is parallel propagated from P to B along a null geodesic Υ', and the index α_b refers to quantities calculated in B. Let us now parallel propagate $\check{\ell}'_{\rho_b}$ from B to A_0 along the curve γ, obtaining

$$\check{\ell}''_{\alpha_0} = \Gamma(\mathrm{B},\mathrm{A}_0;\gamma)^{\gamma_b}{}_{\alpha_0}\check{\ell}'_{\gamma_b}. \tag{7.121}$$

The difference is

$$\begin{aligned}\Delta\check{\ell}_{\alpha_0} = \check{\ell}_{\alpha_0} - \check{\ell}''_{\alpha_0} = {}&\Gamma(\mathrm{P},\mathrm{A}_0;\zeta)^{\sigma_p}{}_{\alpha_0}\Gamma(\mathrm{B},\mathrm{P};\Upsilon')^{\rho_b}{}_{\sigma_p}\check{\ell}'_{\rho_b}\\ &-\Gamma(\mathrm{B},\mathrm{A}_0;\gamma)^{\gamma_b}{}_{\alpha_0}\check{\ell}'_{\gamma_b}.\end{aligned} \tag{7.122}$$

We can, however, write

$$\begin{aligned}\check{\ell}_{\alpha_0} &= \check{\ell}''_{\alpha_0} + \Delta\check{\ell}_{\alpha_0}\\ &= \Gamma(\mathrm{B},\mathrm{A}_0;\gamma)^{\rho_b}{}_{\alpha_0}\check{\ell}'_{\rho_b}\\ &\quad + \Gamma(\mathrm{P},\mathrm{A}_0;\zeta)^{\sigma_p}{}_{\alpha_0}\Gamma(\mathrm{B},\mathrm{P};\Upsilon')^{\rho_b}{}_{\sigma_p}\check{\ell}'_{\rho_b}\\ &\quad - \Gamma(\mathrm{B},\mathrm{A}_0;\gamma)^{\gamma_b}{}_{\alpha_0}\check{\ell}'_{\gamma_b}\\ &= \Gamma(\mathrm{B},\mathrm{A}_0;\gamma)^{\rho_b}{}_{\alpha_0}\check{\ell}'_{\rho_b} + \mathcal{S}^{\rho_b}{}_{\alpha_0}\check{\ell}'_{\rho_b},\end{aligned} \tag{7.123}$$

where $\check{\ell}'_{\rho_b}$ is, at the point B on γ, the parallel to the vector ℓ in P as the result of a parallel transport along the null geodesic Υ' joining P to B, and $\mathcal{S}^{\rho_b}{}_{\alpha_0}$ is a new quantity which depends on three points and can be expressed as

$$\mathcal{S}^{\gamma_b}{}_{\alpha_0} = \Gamma(\mathrm{P},\mathrm{A}_0;\zeta)^{\sigma_p}{}_{\alpha_0}\Gamma(\mathrm{B},\mathrm{P};\Upsilon')^{\gamma_b}{}_{\sigma_p} - \Gamma(\mathrm{B},\mathrm{A}_0;\gamma)^{\gamma_b}{}_{\alpha_0}. \tag{7.124}$$

This is a three-point tensor which has the property of being zero if any two points coincide or when the curvature is zero (Synge, 1960). Thus, from (7.124), Eq. (7.123) can be written as

$$\check{\ell}_{\alpha_0} = \Gamma(\mathrm{B},\mathrm{A}_0;\gamma)^{\gamma_b}{}_{\alpha_0}\check{\ell}'_{\gamma_b} + \tilde{\mathcal{L}}_{\alpha_0}, \tag{7.125}$$

where $\tilde{\mathcal{L}}_{\alpha_0}$ is the result (at A_0) of acting on $\check{\ell}'_{\gamma_b}$ (at B) with $\tilde{S}^{\gamma_b}{}_{\alpha_0}$. The quantity $\tilde{\mathcal{L}}_{\alpha_0}$ is the result of a comparison at A_0 of the images there of the vector ℓ at P under parallel transport along two different paths, one along the space-like geodesic ζ from P to A_0 and the other along the null geodesic Υ' from P to B and then from B back to A_0 along γ. Hence, since the connection is not integrable, $\tilde{\mathcal{L}}_{\alpha_0}$ is a measure of the integrated curvature over the enclosed area. If we assume, for simplicity, that γ is a geodesic ($\dot{u} = 0$), then

$$u_{\alpha_0} = \check{u}_{\alpha_0} = \Gamma(\mathrm{B}, \mathrm{A}_0; \gamma)^{\gamma_b}{}_{\alpha_0} u_{\gamma_b}. \tag{7.126}$$

Inserting this and (7.125) into ν'_{α_0}, we obtain

$$\begin{aligned}\nu'_{\alpha_0} = -\Big\{ & \check{u}_{\alpha_0} + \left(\Gamma(\mathrm{B}, \mathrm{A}_0; \gamma)^{\gamma_b}{}_{\alpha_0} \check{\ell}'_{\gamma_b} + \tilde{\mathcal{L}}_{\alpha_0}\right) \\ & \times \left[\check{u}^{\beta_0} \left(\Gamma(\mathrm{B}, \mathrm{A}_0; \gamma)^{\gamma_b}{}_{\beta_0} \check{\ell}'_{\gamma_b} + \tilde{\mathcal{L}}_{\beta_0}\right)\right]^{-1} \Big\}. \end{aligned} \tag{7.127}$$

Now since $\check{u}^{\beta_0} \Gamma(\mathrm{B}, \mathrm{A}_0; \gamma)^{\gamma_b}{}_{\beta_0} \check{\ell}'_{\gamma_b} = u^{\gamma_b} \check{\ell}'_{\gamma_b} = -\bar{\gamma}_b$ is the Lorentz factor of the particle ℓ relative to the observer u at B, we can write (7.127) as

$$\nu'_{\alpha_0} = -\left[\check{u}_{\alpha_0} + \left(\Gamma(\mathrm{B}, \mathrm{A}_0; \gamma)^{\gamma_b}{}_{\alpha_0} \check{\ell}'_{\gamma_b} + \tilde{\mathcal{L}}_{\alpha_0}\right) \left(-\bar{\gamma}_b + \tilde{\mathcal{L}}_{\beta_0} \check{u}^{\beta_0}\right)^{-1}\right]. \tag{7.128}$$

Thus to first order in the curvature we have

$$\begin{aligned}\nu'_{\alpha_0} = {} & \check{\nu}_{\alpha_0} + \bar{\gamma}_b^{-1} \tilde{\mathcal{L}}_{\alpha_0} + \bar{\gamma}_b^{-2} \tilde{\mathcal{L}}_{\beta_0} \check{u}^{\beta_0} \Gamma(\mathrm{B}, \mathrm{A}_0; \gamma)^{\gamma_b}{}_{\alpha_0} \check{\ell}'_{\gamma_b} \\ & + O(|\mathrm{Riem}|^2), \end{aligned} \tag{7.129}$$

where

$$\check{\nu}_{\alpha_0} = -\left[\check{u}_{\alpha_0} - \bar{\gamma}^{-1} \Gamma(\mathrm{B}, \mathrm{A}_0; \Upsilon)^{\gamma_b}{}_{\alpha_0} \check{\ell}'_{\gamma_b}\right] \tag{7.130}$$

is the parallel at A_0 of the flat velocity 4-vector defined at B according to (7.100). Then inserting (7.129) in (7.117), with the assumption that $\dot{u} = 0$, we have

$$\begin{aligned}\tilde{\nu}_{\alpha_0} = {} & \xi_{\alpha_0} \xi^{\beta_0} \left[\check{\nu}_{\beta_0} + \tilde{\mathcal{L}}_{\gamma_0} \left(\delta^{\gamma_0}_{\beta_0} \bar{\gamma}_b^{-1} + \bar{\gamma}_b^{-2} u^{\gamma_0} \Gamma(\mathrm{B}, \mathrm{A}_0; \gamma)^{\rho_b}{}_{\beta_0} \check{\ell}'_{\rho_b}\right)\right] \\ & + \xi_{\alpha_0} \xi^{\beta_0} \check{\nu}_{\beta_0} \mathcal{R}'_0 + O(|\mathrm{Riem}|^2). \end{aligned} \tag{7.131}$$

Since all curvature terms are contained in $\tilde{\mathcal{L}}_{\beta_0}$ (Lathrop, 1973), in $\mathcal{R}'_0$ and in the neglected terms, we can express the relative velocity in terms of a flat component $\check{\nu}_{\hat{\xi}} = \check{\nu}_{\beta_0} \xi^{\beta_0}$ and a non-local one $\tilde{\nu}_{\hat{\xi}}$ as follows:

$$\begin{aligned}\tilde{\nu}_{\hat{\xi}} = {} & \check{\nu}_{\hat{\xi}}(1 + \mathcal{R}'_0) + \bar{\gamma}_b^{-1} \tilde{\mathcal{L}}_{\hat{\xi}} + \bar{\gamma}_b^{-2} \tilde{\mathcal{L}}_{\hat{0}} \Gamma(\mathrm{B}, \mathrm{A}_0; \gamma)^{\gamma_b}{}_{\hat{\xi}} \check{\ell}'_{\gamma_b} \\ & + O(|\mathrm{Riem}|^2). \end{aligned} \tag{7.132}$$

Here we have selected a tetrad frame adapted to u on γ so that $\xi = \dot{\zeta}$ coincides at A_0 with one space-like tetrad direction, since $\xi^{\alpha_0} u_{\alpha_0} = 0$ there. Relation (7.132) shows explicitly how the space-time curvature contributes to the measurement of a relative velocity.

7.9 Curvature contributions to the measurements of angles

Modern technology allows one to measure angles with the accuracy needed to detect the curvature of Earth space-time. The aim of this section is to show how one can relate the measurement of angles to the measurement of the background curvature.

Let an observer be moving along a geodesic γ with parameter s and tangent vector field u; moreover he carries a Fermi tetrad $\{E_{\hat{\alpha}}\}$. At the event A he emits a light signal towards a target at P along a spatial direction in his local rest frame, at an angle $\underset{\hat{a}}{\Theta}|_{\mathrm{A}}$ with respect to a given tetrad leg $E_{\hat{a}}$; the angle is given by

$$\cos \underset{\hat{a}}{\Theta}{}_1|_{\mathrm{A}} = \left.\frac{E_{\hat{a}} \cdot k}{(k_\perp \cdot k_\perp)^{\frac{1}{2}}}\right|_{\mathrm{A}}, \tag{7.133}$$

from (6.31); here k is the null vector tangent to the light trajectory. After being reflected at P the signal is recorded by the same observer at the event A_1 some time later on γ. In the local rest frame of u at A_1, the spatial direction of the reflected signal forms a different angle with respect to the *same* tetrad leg $E_{\hat{a}}$, given by

$$\cos \underset{\hat{a}}{\Theta}{}_2|_{\mathrm{A}_1} = \left.\frac{E_{\hat{a}} \cdot k}{(k_\perp \cdot k_\perp)^{\frac{1}{2}}}\right|_{\mathrm{A}_1}. \tag{7.134}$$

We shall see that the difference between these two angles gives a direct measure of the space-time curvature. At a general point $\gamma(s)$ of γ, we have, from (5.29),

$$\Omega_\beta = -(\sigma_{\mathrm{P}} - \sigma_{\gamma(s)})k_\beta, \tag{7.135}$$

where σ is a parameter on the null geodesics from A to P and from P to A_1, so Eqs. (7.133) and (7.134) can be written respectively as

$$\cos \underset{\hat{a}}{\Theta}|_{\mathrm{A}} = \left.\frac{(\Omega_\beta E^\beta_{\hat{a}})}{(\Omega_{\hat{b}}\Omega^{\hat{b}})^{\frac{1}{2}}}\right|_{\mathrm{A}}, \tag{7.136}$$

$$\cos \underset{\hat{a}}{\Theta}{}_2|_{\mathrm{A}_1} = \left.\frac{\Omega_\beta E^\beta_{\hat{a}}}{(\Omega_{\hat{b}}\Omega^{\hat{b}})^{\frac{1}{2}}}\right|_{\mathrm{A}_1}. \tag{7.137}$$

The vector k is null; hence we have

$$(\Omega_{\hat{a}}\Omega^{\hat{a}})^{1/2}_{\mathrm{A}/\mathrm{A}_1} = |\Omega_{\hat{0}}|_{\mathrm{A}/\mathrm{A}_1} = \left|\Omega_\beta u^\beta\right|_{\mathrm{A}/\mathrm{A}_1}, \tag{7.138}$$

so the cosines of the angles can be written as

$$\cos \underset{\hat{a}}{\Theta}{}_{\mathrm{A}/\mathrm{A}_1} = \mp \left.\frac{\Omega_\beta E^\beta_{\hat{a}}}{\Omega_\beta u^\beta}\right|_{\mathrm{A}/\mathrm{A}_1}, \tag{7.139}$$

where the upper sign refers to A and the lower one to A_1. In what follows we shall consider the quantities $\Omega_\beta E^\beta{}_{\hat{a}} \equiv \Omega_{\hat{a}}$ and $\Omega_\beta u^\beta \equiv \Omega_{\hat{0}}$ as smooth functions on γ. Let us then expand $\Omega_{\hat{a}}$ about the point $\mathrm{A}_0 = \gamma(s_0)$, which is the event on γ simultaneous to P:

$$\Omega_{\hat{a}}(s) = \Omega_{\hat{a}}(s_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{d^n}{ds^n} \Omega_{\hat{a}} \right)_{\mathrm{A}_0} (s - s_0)^n. \tag{7.140}$$

From (6.11) and (6.12) and assuming that $\{E_{\hat{a}}\}$ is a Fermi tetrad and the curve γ is a geodesic, we obtain, to first order in the curvature,

$$\begin{aligned}\Omega_{\hat{a}}(s) = \Omega_{\hat{a}}(s_0) &+ (s - s_0)\Omega_{\alpha\beta} u^\beta E^\alpha{}_{\hat{a}} \\ &+ \frac{1}{2}(s - s_0)^2 (\Omega_{\alpha\beta\gamma} u^\beta u^\gamma E^\alpha{}_{\hat{a}}) \\ &+ \frac{1}{6}(s - s_0)^3 (\Omega_{\alpha\beta\gamma\delta} u^\beta u^\gamma u^\delta E^\alpha{}_{\hat{a}}) + O(|\mathrm{Riem}|^2),\end{aligned} \tag{7.141}$$

where

$$\begin{aligned}\Omega_{\alpha\beta\gamma} &\approx -\left(S_{\alpha\beta\gamma\delta}\xi^\delta\right)_{\mathrm{A}_0} (\sigma_{\mathrm{P}} - \sigma_{\mathrm{A}_0}), \\ \Omega_{\alpha\beta\gamma\delta} &\approx \left(S_{\alpha\beta\gamma\delta}\right)_{\mathrm{A}_0},\end{aligned} \tag{7.142}$$

with

$$S_{\alpha\beta\gamma\delta} = -\frac{2}{3} R_{\alpha(\gamma|\beta|\delta)}, \tag{7.143}$$

as in (6.14). Thus from (6.13) we have

$$\begin{aligned}\Omega_{\hat{a}}(s) = \Omega_{\hat{a}}(s_0) &+ \frac{1}{2}(s - s_0)(\sigma_{\mathrm{P}} - \sigma_{\mathrm{A}_0})^2 [S_{\alpha\beta\gamma\delta}\xi^\gamma\xi^\delta u^\beta E^\alpha{}_{\hat{a}}]_{\mathrm{A}_0} \\ &- \frac{1}{2}(s - s_0)^2 (\sigma_{\mathrm{P}} - \sigma_{\mathrm{A}_0}) [S_{\alpha\beta\gamma\delta}\xi^\delta u^\beta u^\gamma E^\alpha{}_{\hat{a}}]_{\mathrm{A}_0} \\ &+ O(|\mathrm{Riem}|^2);\end{aligned} \tag{7.144}$$

here ξ^α are the components of the tangent vector to the geodesic ζ from A_0 to P. The values of the parameter s at the points A and A_1 on γ are given by (6.18); substituting those values in (7.144) for s_1 at A and s_2 at A_1, we have

$$\begin{aligned}\Omega_{\hat{a}}(s_1/s_2) = \Omega_{\hat{a}}(s_0) & \\ &\mp \frac{1}{2}[2\Omega(s_0)]^{\frac{1}{2}} (\sigma_{\mathrm{P}} - \sigma_{\mathrm{A}_0})^2 [S_{\alpha\beta\gamma\delta}\xi^\gamma\xi^\delta u^\beta E^\alpha{}_{\hat{a}}]_{\mathrm{A}_0} \\ &- \Omega(s_0)(\sigma_{\mathrm{P}} - \sigma_{\mathrm{A}_0}) [S_{\alpha\beta\gamma\delta}\xi^\delta u^\beta u^\gamma E^\alpha{}_{\hat{a}}]_{\mathrm{A}_0} + O(|\mathrm{Riem}|^2).\end{aligned} \tag{7.145}$$

With the same argument we deduce that

$$\Omega_{\hat{0}}(s) = (s - s_0) \left[-1 + \frac{1}{2}(\sigma_{\mathrm{P}} - \sigma_{\mathrm{A}_0})^2 (S_{\alpha\beta\gamma\delta} u^\alpha u^\beta \xi^\gamma \xi^\delta)_{\mathrm{A}_0} \right] + O(|\mathrm{Riem}|^2). \tag{7.146}$$

Hence from (6.18) we again have

$$\Omega_{\hat{0}}(s_1/s_2) = \mp[2\Omega(s_0)]^{\frac{1}{2}} \left[-1 + \frac{1}{6}\mathcal{R}_{\mathrm{A}_0}(\sigma_{\mathrm{P}} - \sigma_{\mathrm{A}_0})^2 \right] + O(|\mathrm{Riem}|^2), \tag{7.147}$$

where we set

$$\mathcal{R} = -R_{\alpha\beta\gamma\delta} u^\alpha \xi^\beta u^\gamma \xi^\delta. \tag{7.148}$$

To first order in the curvature, Eq. (7.139) leads to

$$\begin{aligned}\cos \underset{\hat{a}}{\Theta}{}_{\mathrm{A}} \approx -\frac{1}{[2\Omega(s_0)]^{\frac{1}{2}}} \Big\{ &\Omega_{\hat{a}}(s_0) \left[1 + \frac{1}{6}\mathcal{R}_{\mathrm{A}_0}(\sigma_{\mathrm{P}} - \sigma_{\mathrm{A}_0})^2\right] \\ &- \frac{1}{2}[2\Omega(s_0)]^{\frac{1}{2}}(\sigma_{\mathrm{P}} - \sigma_{\mathrm{A}_0})^2 (S_{\alpha\beta\gamma\delta}\xi^\gamma \xi^\delta u^\beta E^\alpha{}_{\hat{a}})_{\mathrm{A}_0} \\ &- \Omega(s_0)(\sigma_{\mathrm{P}} - \sigma_{\mathrm{A}_0})(S_{\alpha\beta\gamma\delta}\xi^\delta u^\beta u^\gamma E^\alpha{}_{\hat{a}})_{\mathrm{A}_0} \Big\}\end{aligned} \tag{7.149}$$

and

$$\begin{aligned}\cos \underset{\hat{a}}{\Theta}{}_{\mathrm{A}_1} \approx -\frac{1}{[2\Omega(s_0)]^{\frac{1}{2}}} \Big\{ &\Omega_{\hat{a}}(s_0) \left[1 + \frac{1}{6}\mathcal{R}_{\mathrm{A}_0}(\sigma_{\mathrm{P}} - \sigma_{\mathrm{A}_0})^2\right] \\ &+ \frac{1}{2}[2\Omega(s_0)]^{\frac{1}{2}}(\sigma_{\mathrm{P}} - \sigma_{\mathrm{A}_0})^2 (S_{\alpha\beta\gamma\delta}\xi^\gamma \xi^\delta u^\beta E^\alpha{}_{\hat{a}})_{\mathrm{A}_0} \\ &- \Omega(s_0)(\sigma_{\mathrm{P}} - \sigma_{\mathrm{A}_0})(S_{\alpha\beta\gamma\delta}\xi^\delta u^\beta u^\gamma E^\alpha{}_{\hat{a}})_{\mathrm{A}_0} \Big\}.\end{aligned} \tag{7.150}$$

The variation of the direction cosine in passing from A to A_1 is given by

$$\Delta(\cos \underset{\hat{a}}{\Theta}) = -(S_{\alpha\beta\gamma\delta}\xi^\gamma \xi^\delta u^\beta E^\alpha{}_{\hat{a}})_{\mathrm{A}_0}(\sigma_{\mathrm{P}} - \sigma_{\mathrm{A}_0})^2. \tag{7.151}$$

This equation gives a measure of how the light gun must be turned relative to a Fermi transported tetrad in order to detect the reflected light signal at A_1.

From (5.1), (6.2), and the definition of $S_{\alpha\beta\gamma\delta}$, Eq. (7.144) can also be written as

$$\Delta(\cos \underset{\hat{a}}{\Theta}) = -\frac{2}{3}(R_{\alpha\beta\gamma\delta}E^\alpha{}_{\hat{a}}\xi^\beta u^\gamma \xi^\delta)_{\mathrm{A}_0} L_u^2(\mathrm{A}_0, \mathrm{P}). \tag{7.152}$$

Here the quantity $\Delta(\cos \underset{\hat{a}}{\Theta})$ is in principle directly measurable; hence, coupling (7.152) with (6.19), one can deduce information about the space-time curvature.

8

Observers in physically relevant space-times

A physical measurement necessarily requires the choice of an observer who makes it. An observer is well defined not only when his state of motion is fixed in the background geometry but also when a frame, adapted to his world line, is chosen as a necessary complement. The most natural observers in a given space-time are the stationary ones whenever this attribute is physically applicable. Of course the significance of a measurement also depends on how *realistic* the space-time which provides the geometrical environment is.

Schwarzschild and Kerr solutions are widely considered in space physics and high-energy astrophysics, since they support models of measurements ready to be compared with observations. Gravitational wave space-time solutions also play a central role. In fact the direct detection of gravitational waves is still a challenge for experimental relativity, so measurements which highlight their properties are of primary importance.

In this chapter we shall analyze the geometrical properties of those space-times and the trajectories that host physically realistic observers.

8.1 Schwarzschild space-time

As outlined in Section 2.6, the Schwarzschild solution describes the vacuum space-time outside a spherical, electrically neutral, and non-rotating source of mass $\mathcal{M}$. As stated in (2.135) its metric is given by

$$ds^2 = -\left(1 - \frac{2\mathcal{M}}{r}\right) dt^2 + \left(1 - \frac{2\mathcal{M}}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (8.1)$$

where $t \in (-\infty, +\infty)$, $r \in (2\mathcal{M}, \infty)$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$ are asymptotically spherical coordinates known as *Schwarzschild coordinates*. Metric (8.1) admits four Killing vector fields, namely

$$\begin{aligned}\xi_0 &= \partial_t,\\ \xi_1 &= \cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi,\\ \xi_2 &= -\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi,\\ \xi_3 &= \partial_\phi,\end{aligned} \tag{8.2}$$

such that

$$[\xi_0, \xi_a] = 0, \quad [\xi_a, \xi_b] = \epsilon_{ab}{}^c \xi_c, \qquad a, b, c = 1, 2, 3, \tag{8.3}$$

where $\epsilon_{ab}{}^c$ is the Euclidean alternating symbol.

The null surface having a space-like section at $r = 2\mathcal{M}$ is both an event horizon and an apparent horizon (Hawking and Ellis, 1973) but it is also a Killing horizon since the time-like Killing vector $\xi_0 = \partial_t$, which manifests the stationarity of the metric, becomes null on it and space-like at $r < 2\mathcal{M}$.

Various coordinate patches

Schwarzschild coordinates best adapt themselves to the spherical symmetry of the solution but they fail to be regular on the horizon at $r = 2\mathcal{M}$, which appears as a *coordinate singularity* for both ingoing and outgoing trajectories. In order to have analytical extensions which enable one to avoid the above coordinate inadequacy and eventually provide a global representation of the Schwarzschild solution, new coordinates are used at the expense of being adapted to the space-time symmetries. Coordinates of this type are the following:

(i) *Eddington-Finkelstein coordinates*

These are $\{u, r, \theta, \phi\}$ or $\{v, r, \theta, \phi\}$, given by

$$u = t - r_*, \qquad v = t + r_*, \tag{8.4}$$

where

$$r_* = r + 2\mathcal{M}\ln\left|\frac{r}{2\mathcal{M}} - 1\right|, \qquad \frac{dr_*}{dr} = \left(1 - \frac{2\mathcal{M}}{r}\right)^{-1}. \tag{8.5}$$

The coordinates u and v are termed *outgoing* and *ingoing*, respectively; the former allows outgoing trajectories to smoothly cross the horizon at $r = 2\mathcal{M}$ while the latter does the same for ingoing trajectories. The forms of the metric in the $\{u, r, \theta, \phi\}$ and $\{v, r, \theta, \phi\}$ coordinate patches are respectively

$$ds^2 = -\left(1 - \frac{2\mathcal{M}}{r}\right)du^2 - 2dudr + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{8.6}$$

and

$$ds^2 = -\left(1 - \frac{2\mathcal{M}}{r}\right)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{8.7}$$

Since they allow an observer to cross the event horizon smoothly, the Eddington-Finkelstein coordinates seem to imply that, by just performing a coordinate transformation, one can obtain two opposite physical situations, namely only ingoing or only outgoing on the horizon. Clearly this is not the case. Both situations are allowed by the Schwarzschild solution, and this can be made explicit by using the following set of coordinates.

(ii) *Kruskal coordinates*

In this coordinate system both outgoing and ingoing coordinates are introduced as follows:

$$\begin{aligned} U_{(\epsilon,\epsilon')} &= \epsilon\sqrt{\epsilon'\left(\frac{r}{2\mathcal{M}}-1\right)}e^{\frac{r}{4\mathcal{M}}}\cosh\left(\frac{t}{4\mathcal{M}}\right), \\ V_{(\epsilon,\epsilon')} &= \epsilon\sqrt{\epsilon'\left(\frac{r}{2\mathcal{M}}-1\right)}e^{\frac{r}{4\mathcal{M}}}\sinh\left(\frac{t}{4\mathcal{M}}\right), \end{aligned} \tag{8.8}$$

where ϵ and ϵ' are sign indicators; according to their values (±1) we build a different coordinate representation of Schwarzschild space-time.

In Kruskal coordinates, the metric becomes

$$ds^2 = \frac{32\mathcal{M}^3}{r}e^{-r/(2\mathcal{M})}(-dV^2+dU^2)+r^2(d\theta^2+\sin^2\theta d\phi^2) \tag{8.9}$$

and it is well defined everywhere $r>0$. In this case the horizon at $r=2\mathcal{M}$ can be smoothly crossed in both senses according to the initial conditions. Metric (8.9) is geodesic complete, that is, every geodesic either reaches the singularity at $r=0$ or can be extended to infinity.

(iii) *Painlevé-Gullstrand coordinates*

The time transformation

$$T = t + \int^r \left(1-\frac{2\mathcal{M}}{r}\right)^{-1}\sqrt{\frac{2\mathcal{M}}{r}}dr \tag{8.10}$$

leads to the following form of the Schwarzschild metric:

$$\begin{aligned} ds^2 &= -\left(1-\frac{2\mathcal{M}}{r}\right)dT^2 + 2\sqrt{\frac{2\mathcal{M}}{r}}dTdr \\ &\quad + dr^2 + r^2(d\theta^2+\sin^2\theta d\phi^2), \end{aligned} \tag{8.11}$$

due to Painlevé and Gullstrand (Painlevé, 1921; Gullstrand, 1922). The peculiarity of these coordinates is that the 3-metric induced on the $T=$ constant hypersurfaces, namely

$$ds^2|_{T=const.} = dr^2 + r^2(d\theta^2+\sin^2\theta d\phi^2), \tag{8.12}$$

is intrinsically flat (i.e. the intrinsic curvature tensor vanishes identically) but not extrinsically flat (the extrinsic curvature is non-zero). In fact, the intrinsic curvature is represented by the Riemann tensor associated with the 3-metric (8.12) and this vanishes identically, while the extrinsic curvature is represented by the tensor

$$K_{\alpha\beta} = -\frac{1}{2}\left[P(\mathcal{N})\pounds_{\mathcal{N}} g\right]_{\alpha\beta}, \tag{8.13}$$

where $\mathcal{N}^\flat = -dT$ is the unit normal to the $T =$ constant hypersurfaces and $P(\mathcal{N})$ projects orthogonally to $\mathcal{N}$.

Curvature invariants

Let us consider the form (8.1) of the Schwarzschild solution. The simplest quadratic curvature invariant is Kretschmann's (Kretschmann 1915a; 1915b):

$$K_1 = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \frac{48\mathcal{M}^2}{r^6}. \tag{8.14}$$

Let us recall here that in this metric the Riemann and Weyl tensors coincide; hence no non-trivial first-order invariants exist. This scalar quantity is regular on the horizon at $r = 2\mathcal{M}$ but diverges at $r = 0$, which persists as the only unavoidable curvature singularity.

Principal null directions, Petrov type and the principal complex null frame

The Schwarzschild solution admits two independent principal null directions, given by

$$k_\pm = \partial_t \pm \left(1 - \frac{2\mathcal{M}}{r}\right)\partial_r; \tag{8.15}$$

therefore it is of Petrov type D. A principal complex null frame has the two real null vectors aligned with the principal null directions (8.15), that is $l \propto k_+$ and $n \propto k_-$. A possible choice is then the following:

$$\begin{aligned} l &= \left(1 - \frac{2\mathcal{M}}{r}\right)^{-1}\partial_t + \partial_r, \\ n &= \frac{1}{2}\left(1 - \frac{2\mathcal{M}}{r}\right)\left[\left(1 - \frac{2\mathcal{M}}{r}\right)^{-1}\partial_t - \partial_r\right], \\ m &= \frac{1}{\sqrt{2}r}\left[\partial_\theta + i\frac{1}{\sin\theta}\partial_\phi\right], \end{aligned} \tag{8.16}$$

which makes l geodesic and such that $l \cdot n = -1$, $m \cdot m = 0$, $\bar{m} \cdot \bar{m} = 0$, and $m \cdot \bar{m} = 1$.

8.2 Special observers in Schwarzschild space-time

The coordinate representations of Schwarzschild space-time help us to identify the observers who best describe the physical situations we might be interested in. We shall consider families of special observers, illustrating their geometrical properties.

Static observers

Static observers are defined to be at rest with respect to the chosen spatial coordinate grid. In Schwarzschild coordinates their 4-velocity is given by

$$m = \frac{1}{\sqrt{-g_{tt}}}\partial_t = \left(1 - \frac{2\mathcal{M}}{r}\right)^{-1/2} \partial_t. \tag{8.17}$$

Consider the following orthonormal frame adapted to m:

$$e_{\hat{0}} = m, \quad e_{\hat{r}} = \left(1 - \frac{2\mathcal{M}}{r}\right)^{1/2} \partial_r, \quad e_{\hat{\theta}} = \frac{1}{r}\partial_\theta, \quad e_{\hat{\phi}} = \frac{1}{r\sin\theta}\partial_\phi. \tag{8.18}$$

The dual of this frame is

$$\omega^{\hat{0}} = -m^\flat = \left(1 - \frac{2\mathcal{M}}{r}\right)^{1/2} dt, \quad \omega^{\hat{r}} = \left(1 - \frac{2\mathcal{M}}{r}\right)^{-1/2} dr,$$
$$\omega^{\hat{\theta}} = r d\theta, \quad \omega^{\hat{\phi}} = r\sin\theta d\phi, \tag{8.19}$$

with $\omega^{\hat{\alpha}}(e_{\hat{\beta}}) = \delta^{\hat{\alpha}}{}_{\hat{\beta}}$.

Static observers form a three-parameter congruence $\mathcal{C}_m$ (the parameters being the spatial coordinates r, θ, and ϕ of these observers) of radially outward accelerated world lines with acceleration described by the 4-vector

$$a(m) \equiv \nabla_m m = \left(1 - \frac{2\mathcal{M}}{r}\right)^{-1/2} \frac{\mathcal{M}}{r^2} e_{\hat{r}}. \tag{8.20}$$

In addition $\mathcal{C}_m$ has vanishing vorticity ($\omega(m) = 0$) and expansion ($\theta(m) = 0$). The identical vanishing of the expansion implies that $\mathcal{C}_m$ is a rigid congruence according to the Born rigidity condition. For later use, we introduce the notation

$$||a(m)|| \equiv \kappa(m) = \left(1 - \frac{2\mathcal{M}}{r}\right)^{-1/2} \frac{\mathcal{M}}{r^2} \tag{8.21}$$

for the magnitude of $a(m)$.

The only non-vanishing frame components of the Riemann tensor with respect to the orthonormal frame $\{e_{\hat{\alpha}}\}$ of (8.18) are

$$R_{\hat{0}\hat{\theta}\hat{0}\hat{\theta}} = R_{\hat{0}\hat{\phi}\hat{0}\hat{\phi}} = -R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = -R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = -\frac{1}{2}R_{\hat{0}\hat{r}\hat{0}\hat{r}} = \frac{1}{2}R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{\mathcal{M}}{r^3}. \tag{8.22}$$

Therefore, with respect to static observers and to the frame (8.18), the Riemann tensor can be expressed in terms of its electric part only:

$$\mathcal{E}(m) = \frac{\mathcal{M}}{r^3}\left(-2e_{\hat{r}} \otimes e_{\hat{r}} + e_{\hat{\theta}} \otimes e_{\hat{\theta}} + e_{\hat{\phi}} \otimes e_{\hat{\phi}}\right). \tag{8.23}$$

The transport properties of the frame (8.18) along the world lines of the congruence $\mathcal{C}_m$ are given by

$$\nabla_m e_{\hat{r}} = \kappa(m)\, m, \quad \nabla_m e_{\hat{\theta}} = 0 = \nabla_m e_{\hat{\phi}}, \tag{8.24}$$

with $\kappa(m)$ given by Eq. (8.21) and, hence, from (3.60), $P(m)\nabla_m e_{\hat{a}} = 0 = e_{\hat{b}} C_{(\mathrm{fw})}{}^{\hat{b}}{}_{\hat{a}}$. It follows that the Fermi-Walker structure functions of the frame vanish identically, i.e. $C_{(\mathrm{fw})}{}^{\hat{a}}{}_{\hat{b}} = 0$, so that each of the spatial directions $e_{\hat{r}}$, $e_{\hat{\theta}}$, and $e_{\hat{\phi}}$ can be aligned with the axis of a gyroscope. Moreover, since the congruence $\mathcal{C}_m$ in this case is vorticity-free and expansion-free, from (3.63) also the Lie structure functions vanish identically, i.e. $C_{(\mathrm{lie})}{}^{\hat{a}}{}_{\hat{b}} = 0$. Finally, from the definition (4.28), we recognize the frame $\{e_{\hat{\alpha}}\}$ in (8.18) as being also a Frenet-Serret frame with curvature and torsions given by

$$\kappa(m) = \left(1 - \frac{2\mathcal{M}}{r}\right)^{-1/2} \frac{\mathcal{M}}{r^2}, \quad \tau_1(m) = 0 = \tau_2(m). \tag{8.25}$$

Observers on spatially circular orbits

Spatially circular orbits are characterized by a unit tangent vector $U^\alpha \equiv dx^\alpha/d\tau_U$ given by

$$U = \Gamma(\partial_t + \zeta\partial_\phi), \tag{8.26}$$

with

$$\Gamma \equiv \frac{dt}{d\tau_U} = \left(1 - \frac{2\mathcal{M}}{r} - \zeta^2 r^2 \sin^2\theta\right)^{-1/2}, \tag{8.27}$$

where $\zeta = d\phi/dt$ is the angular velocity of revolution as it would be measured by a static observer at infinity, where space-time is flat. Indeed the physical interpretation of ζ is only possible if we exploit the asymptotic flatness of the Schwarzschild solution. ζ here is assumed to be constant along the orbit, i.e. satisfying the condition $\pounds_U \zeta = 0$. These orbits form a three-parameter (r, θ, ζ) congruence which has a non-zero acceleration vector

$$a(U) = \Gamma^2\left[\left(1 - \frac{2\mathcal{M}}{r}\right)^{1/2} \frac{\mathcal{M} - r^3\zeta^2\sin^2\theta}{r^2}\, e_{\hat{r}} - r\zeta^2 \sin\theta\cos\theta\, e_{\hat{\theta}}\right], \tag{8.28}$$

a non-zero expansion, whose non-vanishing components are

$$\begin{aligned}\theta(U)_{\hat{r}\hat{\phi}} &= \frac{1}{2}\,[\mathrm{sgn}(\zeta)]\,\Gamma^2 r\sin\theta\left(1-\frac{2\mathcal{M}}{r}\right)\partial_r\zeta,\\ \theta(U)_{\hat{\theta}\hat{\phi}} &= \frac{1}{2}\,[\mathrm{sgn}(\zeta)]\,\Gamma^2\sin\theta\sqrt{1-\frac{2\mathcal{M}}{r}}\,\partial_\theta\zeta,\end{aligned} \tag{8.29}$$

where $\hat{\phi}$ refers to the unit space-like direction $E_{\hat{\phi}}$, orthogonal to U and defined in the 2-plane $(t,\,\phi)$, as

$$E_{\hat{\phi}} = \bar{\Gamma}(\partial_t + \bar{\zeta}\partial_\phi), \quad \bar{\zeta} = -\frac{g_{tt}}{\zeta g_{\phi\phi}}, \quad \bar{\Gamma} = \Gamma|\zeta|\sqrt{\frac{g_{\phi\phi}}{-g_{tt}}}$$

and a non-zero vorticity vector

$$\omega(U) = \omega(U)^{\hat{r}} e_{\hat{r}} + \omega(U)^{\hat{\theta}} e_{\hat{\theta}}, \tag{8.30}$$

where the components can be conveniently written as

$$\begin{aligned}\omega(U)^{\hat{r}} &= \tilde{\omega}^{\hat{r}} + \theta(U)_{\hat{\theta}\hat{\phi}},\\ \omega(U)^{\hat{\theta}} &= \tilde{\omega}^{\hat{\theta}} - \theta(U)_{\hat{r}\hat{\phi}},\end{aligned} \tag{8.31}$$

with

$$\tilde{\omega} = \Gamma^2|\zeta|\left[\left(1-\frac{2\mathcal{M}}{r}\right)^{1/2}\cos\theta\, e_{\hat{r}} - \sin\theta\left(1-\frac{3\mathcal{M}}{r}\right)e_{\hat{\theta}}\right]. \tag{8.32}$$

Let us note here that if ζ is constant over the entire congruence, then $\theta(U) = 0$, i.e. the congruence is Born-rigid and $\omega(U) = \tilde{\omega}$. From (8.28) it follows that circular geodesics exist only on the equatorial plane $\theta = \pi/2$ and are associated with the Keplerian angular velocity

$$\zeta = \pm\zeta_K = \pm\sqrt{\frac{\mathcal{M}}{r^3}}, \tag{8.33}$$

that is

$$U_{K\pm} = \Gamma_K(\partial_t \pm \zeta_K\partial_\phi). \tag{8.34}$$

With respect to a local static observer m, the 4-velocity U of (8.26) can be written as

$$U = \gamma[m + \nu(U,m)^{\hat{\phi}} e_{\hat{\phi}}], \tag{8.35}$$

where

$$\nu(U,m)^{\hat{\phi}} = \left(1-\frac{2\mathcal{M}}{r}\right)^{-1/2} r\zeta\sin\theta \tag{8.36}$$

and

$$\gamma = (1-\nu^2)^{-1/2} = \Gamma\left(1-\frac{2\mathcal{M}}{r}\right)^{1/2}. \tag{8.37}$$

Here γ is the Lorentz factor and

$$\nu = ||\nu(U,m)|| = \left(1 - \frac{2\mathcal{M}}{r}\right)^{-1/2} r|\zeta|\sin\theta \tag{8.38}$$

is the magnitude of the spatial velocity of U relative to m. It can be useful to introduce an effective radius $\mathcal{R}(U,m)$ (abbreviated by $\mathcal{R}$) of the orbit such that the classical relation

$$\nu(U,m)^{\hat\phi} = \mathcal{R}\zeta \tag{8.39}$$

still holds. This implies that

$$\mathcal{R} = r\sin\theta \left(1 - \frac{2\mathcal{M}}{r}\right)^{-1/2} . \tag{8.40}$$

For instance, in terms of $\mathcal{R}$, we have

$$\Gamma^{-2} = \left(1 - \frac{2\mathcal{M}}{r}\right)(1 - \zeta^2\mathcal{R}^2), \tag{8.41}$$

which immediately gives (8.37). At r and θ fixed, this family of spatially circular orbits forms a one-parameter congruence whose parameter is the angular velocity ζ, or equivalently the spatial velocity ν with respect to the static observers (or any other family of observers), or equivalently the rapidity $\alpha = \alpha(U,m)$ defined by

$$\nu = \tanh\alpha. \tag{8.42}$$

Let us specialize to the Schwarzschild case the expression (6.74) of the projection of the acceleration $a(U)$ into the rest space of the observers m. Setting $u = m$ in that formula and recalling that γ and ν are constant along the orbit of U, we have

$$P(m,U)a(U) = -\gamma F^{(G)}_{(\mathrm{fw},U,m)} + \frac{D_{(\mathrm{fw},U,m)}p(U,m)}{d\tau_U}. \tag{8.43}$$

From the definition of $p(U,m) = \gamma\nu(U,m)$, the above expression develops through the following steps:

$$\begin{aligned} P(m,U)a(U) &= -\gamma F^{(G)}_{(\mathrm{fw},U,m)} + \frac{D_{(\mathrm{fw},U,m)}}{d\tau_U}[\gamma\nu(U,m)] \\ &= -\gamma F^{(G)}_{(\mathrm{fw},U,m)} + \gamma\frac{D_{(\mathrm{fw},U,m)}}{d\tau_U}[\nu(U,m)] \\ &= -\gamma F^{(G)}_{(\mathrm{fw},U,m)} + \frac{D_{(\mathrm{fw},U,m)}}{d\tau_{(U,m)}}[\nu(U,m)] \\ &= -\gamma F^{(G)}_{(\mathrm{fw},U,m)} + a_{(\mathrm{fw},U,m)}, \end{aligned} \tag{8.44}$$

where $a_{(\mathrm{fw},U,m)}$ is the relative acceleration of U with respect to m, as in (3.162). We now deduce the explicit expression of $a_{(\mathrm{fw},U,m)}$ for the case under consideration. A direct calculation gives

$$\begin{aligned}
\nabla_U m &= \frac{\Gamma\mathcal{M}}{r^2} e_{\hat r} \equiv -F^{(G)}_{(\mathrm{fw},U,m)},\\
\nabla_U e_{\hat r} &= \Gamma\left[\frac{\mathcal{M}}{r^2} m + \left(1-\frac{2\mathcal{M}}{r}\right)^{1/2} \zeta \sin\theta\, e_{\hat\phi}\right],\\
\nabla_U e_{\hat\theta} &= \Gamma\zeta\cos\theta\, e_{\hat\phi},\\
\nabla_U e_{\hat\phi} &= -\Gamma\zeta\left[\left(1-\frac{2\mathcal{M}}{r}\right)^{1/2} \sin\theta\, e_{\hat r} + \cos\theta\, e_{\hat\theta}\right].
\end{aligned} \tag{8.45}$$

Projecting orthogonally to m, the derivatives (8.45) of the vectors of the spatial frame give

$$\begin{aligned}
P(m)\nabla_U e_{\hat r} &\equiv \frac{D_{(\mathrm{fw},U,m)}}{d\tau_U} e_{\hat r} = \Gamma\left(1-\frac{2\mathcal{M}}{r}\right)^{1/2} \zeta\sin\theta\, e_{\hat\phi},\\
P(m)\nabla_U e_{\hat\theta} &\equiv \frac{D_{(\mathrm{fw},U,m)}}{d\tau_U} e_{\hat\theta} = \Gamma\zeta\cos\theta\, e_{\hat\phi},\\
P(m)\nabla_U e_{\hat\phi} &\equiv \frac{D_{(\mathrm{fw},U,m)}}{d\tau_U} e_{\hat\phi} = -\Gamma\zeta\left[\left(1-\frac{2\mathcal{M}}{r}\right)^{1/2} \sin\theta\, e_{\hat r} + \cos\theta\, e_{\hat\theta}\right].
\end{aligned} \tag{8.46}$$

Let us now recall the definition of Fermi-Walker angular velocity $\zeta_{(\mathrm{fw})}$ and the spatial curvature angular velocity $\zeta_{(\mathrm{sc})}$:

$$\frac{D_{(\mathrm{fw},U,m)}}{d\tau_U} e_{\hat a} = \gamma\left[\zeta_{(\mathrm{fw})} + \zeta_{(\mathrm{sc})}\right] \times_m e_{\hat a}. \tag{8.47}$$

From (8.24) we see that $\zeta_{(\mathrm{fw})} = 0$. Therefore (8.46) allows us to determine $\zeta_{(\mathrm{sc})}$ as

$$\zeta_{(\mathrm{sc})} = \zeta\left[\left(1-\frac{2\mathcal{M}}{r}\right)^{-1/2} \cos\theta\, e_{\hat r} - e_{\hat\theta}\right], \tag{8.48}$$

whose limiting values at infinity and on the equatorial plane are respectively

$$\lim_{r\to\infty} \zeta_{(\mathrm{sc})} = \zeta(\cos\theta\, e_{\hat r} - e_{\hat\theta}), \qquad \lim_{\theta\to\pi/2} \zeta_{(\mathrm{sc})} = -\zeta e_{\hat\theta}. \tag{8.49}$$

Moreover

$$\begin{aligned}
\nabla_U \nu(U,m) &= \nabla_U(\nu(U,m)^{\hat\phi} e_{\hat\phi}) = \nu(U,m)^{\hat\phi}\nabla_U e_{\hat\phi}\\
&= -\Gamma\zeta\nu(U,m)^{\hat\phi}\left[\left(1-\frac{2\mathcal{M}}{r}\right)^{1/2} \sin\theta\, e_{\hat r} + \cos\theta\, e_{\hat\theta}\right]
\end{aligned}$$

and the relation

$$\nabla_U(\nu(U,m)^{\hat\phi}e_{\hat\phi}) = \gamma^{-1}a_{(\mathrm{fw},U,m)} \tag{8.50}$$

lead to the following expression of $a_{(\mathrm{fw},U,m)}$:

$$\begin{aligned} a_{(\mathrm{fw},U,m)} &= -\Gamma^2\zeta\nu(U,m)^{\hat\phi}\left(1-\frac{2\mathcal{M}}{r}\right)^{1/2}\left[\left(1-\frac{2\mathcal{M}}{r}\right)^{1/2}\sin\theta\, e_{\hat r}\right.\\ &\qquad\left. +\cos\theta\, e_{\hat\theta}\right]. \end{aligned} \tag{8.51}$$

Here, the projection orthogonal to m is unnecessary since $\nabla_U[\nu(U,m)^{\hat\phi}e_{\hat\phi}]$ belongs to the 2-plane $(e_{\hat r}, e_{\hat\theta})$. Hence we have as a final result, from (8.37), $(8.45)_1$, and (8.51),

$$\begin{aligned} P(m,U)a(U) &= \Gamma^2\left(1-\frac{2\mathcal{M}}{r}\right)^{1/2}\\ &\quad\times\left\{\frac{\mathcal{M}}{r^2}\,e_{\hat r} - \zeta\nu(U,m)^{\hat\phi}\left[\left(1-\frac{2\mathcal{M}}{r}\right)^{1/2}\sin\theta\, e_{\hat r} + \cos\theta\, e_{\hat\theta}\right]\right\}, \end{aligned} \tag{8.52}$$

as expected from (8.28).

Let us briefly discuss the meaning of the above analysis. The observers who make the measurements are m, while the target of the observation is the particle moving on a spatially circular orbit with 4-velocity U. Because the observers m are static, the particle U crosses one of these observers at each point of his orbit.

The total acceleration of the particle as measured by m, namely $P(m,U)a(U)$, appears to consist of two terms. The first, $-\gamma F^{(G)}_{(\mathrm{fw},U,m)}$, is a term of gravitational type which measures the variation of the 4-velocity m of the observers who cross the particle's orbit U; the second, $a_{(\mathrm{fw},U,m)}$, is a term which measures the variation of the relative velocity vector of U with respect to m, i.e. $\nu(U,m)$, along U itself; all projected into LRS_m; see Fig. 8.1.

It can be useful to identify an orthonormal frame adapted to U, namely $\{E_{\hat 0}\equiv U, E_{\hat a}\}$. Since U is obtained by boosting m in the $e_{\hat\phi}$ direction, we obtain the required frame by also boosting $e_{\hat\phi}$ in the local rest space of U:

$$\begin{aligned} E_{\hat 0} &= \gamma[m+\nu e_{\hat\phi}], &\quad E_{\hat\phi} &= \gamma[\nu m + e_{\hat\phi}],\\ E_{\hat r} &= e_{\hat r}, & E_{\hat\theta} &= e_{\hat\theta}. \end{aligned} \tag{8.53}$$

Note also the following complementary representations for $E_{\hat 0}$ and $E_{\hat\phi}$ in terms of angular velocities:

$$E_{\hat 0} = \Gamma(\partial_t + \zeta\partial_\phi), \qquad E_{\hat\phi} = \bar\Gamma(\partial_t + \bar\zeta\partial_\phi), \tag{8.54}$$

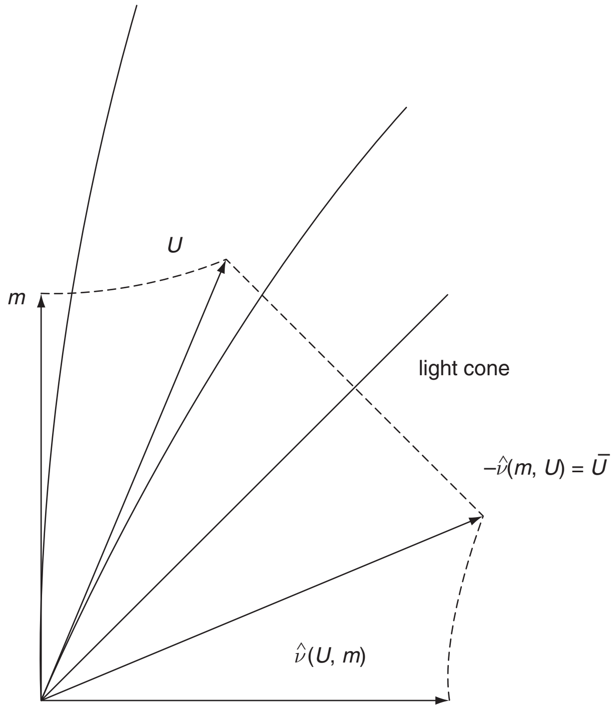

Fig. 8.1. Splitting of the 4-velocity U of a test particle in terms of m (the 4-velocity of a family of test observers) and $\nu(U, m)$ (the associated relative velocity). The transport along U of these terms gives rise to the gravitational force and the 3-acceleration as felt by the observers.

where

$$\bar{\Gamma} = \Gamma\,|\zeta|\,r\sin\theta\left(1 - \frac{2\mathcal{M}}{r}\right)^{-1/2} = \Gamma\mathcal{R}\,\zeta,$$
$$\bar{\zeta} = \frac{1}{r^2\zeta\sin^2\theta}\left(1 - \frac{2\mathcal{M}}{r}\right) = \frac{1}{\mathcal{R}^2\zeta}. \tag{8.55}$$

Introducing the rapidity parameter (8.42), we also have

$$E_{\hat{0}} = \cosh\alpha m + \sinh\alpha e_{\hat{\phi}}, \qquad E_{\hat{\phi}} = \sinh\alpha m + \cosh\alpha e_{\hat{\phi}}. \tag{8.56}$$

The frame (8.53) has been termed *phase-locked* (de Felice, 1991; de Felice and Usseglio-Tomasset, 1991) and plays a special role in exploring the geometrical properties of circular orbits.

Finally let us identify a Frenet-Serret frame along U. Following Iyer and Vishveshwara (1993), we have

$$\begin{aligned} E_{\hat{0}} &= U, \\ E_{\hat{1}} &= \cos\chi e_{\hat{r}} + \sin\chi e_{\hat{\theta}}, \\ E_{\hat{2}} &= E_{\hat{\phi}}, \\ E_{\hat{3}} &= \sin\chi e_{\hat{r}} - \cos\chi e_{\hat{\theta}}, \end{aligned} \tag{8.57}$$

with

$$\tan\chi = -\frac{\zeta^2\sin\theta\cos\theta}{\left(\frac{\mathcal{M}}{r^3} - \zeta^2\sin^2\theta\right)\left(1 - \frac{2\mathcal{M}}{r}\right)^{1/2}} = -\frac{\mathcal{R}}{r}\frac{\zeta^2\cos\theta}{(\zeta_K^2 - \zeta^2\sin^2\theta)}. \tag{8.58}$$

The Frenet-Serret curvature and torsions are then given by

$$\kappa(U)^2 = \Gamma^4 r^2\left[\left(1 - \frac{2\mathcal{M}}{r}\right)\left(\zeta_K^2 - \zeta^2\sin^2\theta\right)^2 + \zeta^4\sin^2\theta\cos^2\theta\right],$$

implying also that

$$\begin{aligned} \tau_1(U)^2 &= \frac{\Gamma^4\zeta^2\sin^2\theta\left(1 - \frac{2\mathcal{M}}{r}\right)\left[\left(\zeta_K^2 - \zeta^2\sin^2\theta\right)\left(1 - \frac{3\mathcal{M}}{r}\right) - \zeta^2\cos^2\theta\right]^2}{\left[\left(1 - \frac{2\mathcal{M}}{r}\right)(\zeta_K^2 - \zeta^2\sin^2\theta)^2 + \zeta^4\sin^2\theta\cos^2\theta\right]}, \\ \tau_2(U)^2 &= \frac{\zeta^2\mathcal{M}^2\cos^2\theta}{r^6\left[\left(1 - \frac{2\mathcal{M}}{r}\right)(\zeta_K^2 - \zeta^2\sin^2\theta)^2 + \zeta^4\sin^2\theta\cos^2\theta\right]}. \end{aligned} \tag{8.59}$$

Observers on equatorial spatially circular orbits

Let us now require that the plane of the orbits in Schwarzschild space-time is fixed at $\theta = \pi/2$, referred to as the *equatorial* plane; we shall still confine our attention to spatially circular orbits. For these orbits the acceleration is only radial, as follows from (8.28), that is

$$\begin{aligned} a(U) &= \left(1 - \frac{2\mathcal{M}}{r} - \zeta^2 r^2\right)^{-1}\left(1 - \frac{2\mathcal{M}}{r}\right)^{1/2} r(\zeta_K^2 - \zeta^2)\, e_{\hat{r}} \\ &= -|\zeta_{\rm c}|\,\frac{\zeta^2 - \zeta_K^2}{\zeta_{\rm c}^2 - \zeta^2}\, e_{\hat{r}}, \end{aligned} \tag{8.60}$$

where

$$|\zeta_{\rm c}| = \frac{1}{r}\left(1 - \frac{2\mathcal{M}}{r}\right)^{1/2} = \frac{1}{\mathcal{R}}. \tag{8.61}$$

This family of orbits is expansion-free and has a vorticity vector given by

$$\omega(U) = -\Gamma^2\zeta r\left(1 - \frac{3\mathcal{M}}{r}\right) e_{\hat{\theta}}. \tag{8.62}$$

In Chapter 4 we discussed *relative Frenet-Serret frames* and introduced in (4.39) the quantity $k_{(\mathrm{fw},U,m)}$ as the Fermi-Walker curvature of the orbit, that is

$$\frac{D_{(\mathrm{fw},U,m)}}{d\ell_{(U,m)}}\hat{\nu}(U,m) = k_{(\mathrm{fw},U,m)}\hat{\eta}_{(\mathrm{fw},U,m)}. \tag{8.63}$$

Similarly, when the Lie-spatial (instead of Fermi-Walker) temporal derivative is involved, one has the Lie-relative curvature $k_{(\mathrm{lie},U,m)}$

$$\frac{D_{(\mathrm{lie},U,m)}}{d\ell_{(U,m)}}\hat{\nu}(U,m) = k_{(\mathrm{lie},U,m)}\hat{\eta}_{(\mathrm{lie},U,m)}. \tag{8.64}$$

In this case, $\mathcal{C}_m$ being vorticity-free and expansion-free, we have, from (3.63),

$$k_{(\mathrm{fw},U,m)} = k_{(\mathrm{lie},U,m)}. \tag{8.65}$$

A direct calculation gives

$$k_{(\mathrm{lie},U,m)} = -\frac{1}{r}\left(1-\frac{2\mathcal{M}}{r}\right)^{1/2} = -\frac{1}{\mathcal{R}}, \tag{8.66}$$

clarifying the geometrical meaning of the effective radius introduced in (8.39); hereafter $k_{(\mathrm{lie},U,m)}$ will be abbreviated simply as $k_{(\mathrm{lie})}$. We then have, from (8.36) and (8.60),

$$a(U) = k_{(\mathrm{lie})}\frac{\zeta^2-\zeta_K^2}{\zeta_{\mathrm{c}}^2-\zeta^2}\, e_{\hat{r}} = k_{(\mathrm{lie})}\gamma^2\left(\nu^2-\nu_K^2\right)e_{\hat{r}}, \tag{8.67}$$

where $\nu_K = \nu(U_{K\pm},m)$ is the Keplerian relative velocity,

$$\nu_K = \left(\frac{\mathcal{M}}{r-2\mathcal{M}}\right)^{1/2}. \tag{8.68}$$

The Frenet-Serret frame (8.53) in this case implies a second torsion identically zero, namely $\tau_2(U)=0$, and

$$\begin{aligned}\kappa(U) &= \Gamma^2\left(1-\frac{2\mathcal{M}}{r}\right)^{1/2}\frac{\mathcal{M}-r^3\zeta^2}{r^2} = \Gamma\left(1-\frac{2\mathcal{M}}{r}\right)\mathcal{R}(\zeta_K^2-\zeta^2),\\ \tau_1(U) &= \zeta\Gamma^2\left(1-\frac{3\mathcal{M}}{r}\right),\end{aligned} \tag{8.69}$$

with

$$\tan\chi = 0, \tag{8.70}$$

and hence $E_{\hat{1}} = e_{\hat{r}}$. We see that, like U, $a(U)$ forms a one-parameter family of vectors, the parameter being the angular velocity ζ (or equivalently the spatial velocity or the rapidity with respect to any family of observers). Expressing $\kappa(U)$ and $\tau_1(U)$ in terms of the rapidity α with respect to the family of static observers gives

$$\begin{aligned}\kappa(U) &= \frac{k_{(\mathrm{lie})}}{\cosh^2 \alpha_K} \sinh(\alpha - \alpha_K)\sinh(\alpha + \alpha_K),\\ \tau_1(U) &= -\frac{1}{2}\partial_\alpha \kappa(U),\end{aligned} \tag{8.71}$$

where $\alpha_K = \alpha(U_K, m)$ and $\tanh \alpha_K = \nu_K$.

One can then consider the *extremely accelerated orbits* as those equatorial spatially circular orbits satisfying the condition

$$\partial_\zeta ||a(U)|| = 0. \tag{8.72}$$

A direct calculation shows that these orbits coincide here with those of the static observers, i.e. with $\zeta = 0$.

8.3 Kerr space-time

The Kerr metric is described by the line element

$$\begin{aligned}ds^2 &= -dt^2 + \frac{2\mathcal{M}r}{\Sigma}(a\sin^2\theta d\phi - dt)^2 + (r^2 + a^2)\sin^2\theta d\phi^2\\ &\quad + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2,\end{aligned} \tag{8.73}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 + a^2 - 2\mathcal{M}r. \tag{8.74}$$

It is useful to introduce the quantity

$$A = (r^2 + a^2)^2 - \Delta a^2 \sin^2\theta \tag{8.75}$$

such that $g_{\phi\phi} = A\sin^2\theta/\Sigma$. The parameters $\mathcal{M}$ and a describe the total mass and specific angular momentum of the source, respectively. The Kerr metric describes the space-time of a rotating black hole when $a < \mathcal{M}$. In this case the coordinates in (8.73) run as $t \in (-\infty, +\infty)$, $r \in (r_+, \infty)$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$, with

$$r_+ = \mathcal{M} + \sqrt{\mathcal{M}^2 - a^2}. \tag{8.76}$$

These are spherical-like coordinates known as *Boyer-Lindquist coordinates*. The inverse Kerr metric is given by

$$\left(\frac{\partial}{\partial s}\right)^2 = \frac{1}{\Sigma}\left\{-\frac{A}{\Delta}\left(\frac{\partial}{\partial t}\right)^2 - \frac{4\mathcal{M}ra}{\Delta}\frac{\partial}{\partial t}\frac{\partial}{\partial \phi} + \frac{\Delta - a^2\sin^2\theta}{\Delta\sin^2\theta}\left(\frac{\partial}{\partial \phi}\right)^2 \right.$$
$$\left. + \Delta\left(\frac{\partial}{\partial r}\right)^2 + \left(\frac{\partial}{\partial \theta}\right)^2\right\}. \tag{8.77}$$

Kerr space-time admits two Killing vectors, which in Boyer-Lindquist coordinates are given by

$$\xi_0 = \partial_t, \qquad \xi_3 = \partial_\phi. \tag{8.78}$$

They express the properties of stationarity and axial symmetry of the solution.

Various coordinate patches

The metric form (8.73) reduces to the Schwarzschild metric if we set $a = 0$. The surface $r = r_+$, where $\Delta = 0$, is a coordinate singularity representing the space-like boundary of an event horizon.[1] It can be removed with a suitable change of coordinates. The same type of coordinates introduced in the Schwarzschild case have been extensively investigated in the literature (see de Felice and Clarke, 1990) and we shall not repeat them here. We shall instead analyze in more detail the following coordinate system, named after Painlevé and Gullstrand.

The Painlevé-Gullstrand coordinates $X^\alpha \equiv (T, R, \Theta, \Phi)$ are related to the Boyer-Lindquist coordinates by the transformation

$$T = t - \int^r f(r)dr\,, \qquad \Phi = \phi - \int^r \frac{a}{r^2+a^2} f(r)dr\,,$$
$$R = r, \qquad \Theta = \theta, \tag{8.79}$$

where

$$f(r) = -\frac{\sqrt{2\mathcal{M}r(r^2+a^2)}}{\Delta}. \tag{8.80}$$

The Kerr metric in Painlevé-Gullstrand coordinates takes the form

$$ds^2 = -\left(1 - \frac{2\mathcal{M}r}{\Sigma}\right)dT^2 + 2\sqrt{\frac{2\mathcal{M}r}{r^2+a^2}}\,dT\,dr$$
$$- \frac{4a\mathcal{M}r}{\Sigma}\sin^2\theta\;dT\;d\Phi + \sin^2\theta\left[(r^2+a^2) + \frac{2a^2\mathcal{M}r}{\Sigma}\sin^2\theta\right]d\Phi^2$$
$$- 2a\sin^2\theta\sqrt{\frac{2\mathcal{M}r}{r^2+a^2}}\,dr\,d\Phi + \frac{\Sigma}{(r^2+a^2)}dr^2 + \Sigma d\theta^2, \tag{8.81}$$

[1] The surface $r = r_+$ is an outer boundary since $\Delta = 0$ admits also the solution $r_- = \mathcal{M} - \sqrt{\mathcal{M}^2 - a^2}$, which is the inner boundary of the event horizon. The inner structure of the Kerr black hole will not be considered here.

which is clearly regular on $r = r_+$. The $a = 0$ limit of (8.81) is (8.11), as expected. The induced metric on $T=$ constant slices is non-diagonal and has the form

$$
\begin{aligned}
ds^2|_{T=const.} = \sin^2\theta \left[(r^2+a^2) + \frac{2\mathcal{M}ra^2}{\Sigma}\sin^2\theta\right] d\Phi^2 \\
- 2a\sin^2\theta\sqrt{\frac{2\mathcal{M}r}{r^2+a^2}}\, dr\, d\Phi \\
+ \frac{\Sigma}{(r^2+a^2)}dr^2 + \Sigma d\theta^2 .
\end{aligned}
\tag{8.82}
$$

The most interesting observers in the Painlevé-Gullstrand coordinate representation are those with world lines orthogonal to the $T =$ constant hypersurfaces. They have 4-velocity

$$
\mathcal{N}^\flat = -dT, \qquad \mathcal{N} = \partial_T - \frac{\sqrt{2\mathcal{M}r(r^2+a^2)}}{\Sigma}\partial_r , \tag{8.83}
$$

and form a geodesic ($a(\mathcal{N}) = 0$) and vorticity-free ($\omega(\mathcal{N}) = 0$) congruence but have a non-vanishing expansion.

Curvature invariants

There are two quadratic curvature Weyl invariants,[2]

$$
K_1 = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}, \quad K_2 = R^{\alpha\beta\gamma\delta}\,{}^*R_{\alpha\beta\gamma\delta}\,; \tag{8.84}
$$

in Boyer-Lindquist coordinates they are given by

$$
\begin{aligned}
K_1 &= \frac{48\mathcal{M}^2}{\Sigma^6}(r^2 - a^2\cos^2\theta)(\Sigma^2 - r^2a^2\cos^2\theta), \\
K_2 &= -\frac{96\mathcal{M}^2 ra\cos\theta}{\Sigma^6}(3r^2 - a^2\cos^2\theta)(r^2 - 3a^2\cos^2\theta)\,.
\end{aligned}
\tag{8.85}
$$

The invariant K_2 is a rotationally induced quantity and its measurement, if operationally feasible, would unambiguously ascertain that we are in a rotating space-time (Ciufolini and Wheeler, 1995). An unavoidable curvature singularity is found where $\Sigma = 0$, i.e. where $r = 0$ and $\theta = \pi/2$.

The forms (8.85) of the curvature invariants show that these quantities change their signs several times if one moves along $\theta =$ constant hypersurfaces; see Fig. 8.2. These invariants can be calculated down to the ring singularity, and one can deduce that they tend to the singularity with different asymptotic values according to how they approach it. This shows that the Kerr singularity has directional properties.

[2] Since the Kerr metric is a vacuum solution, the Weyl and Riemann tensors coincide.

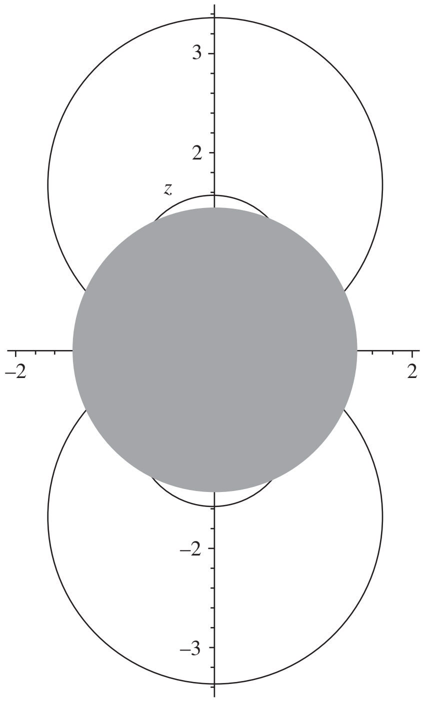

Fig. 8.2. Curvature invariants K_1 and K_2 in Kerr space-time, evaluated for $a/\mathcal{M} = 0.5$ and $\mathcal{M} = 1$. The curves represent the points where the invariants vanish. The closest curve to the horizon (circle in gray) is that of K_2.

Principal null directions, Petrov type and principal complex null frame

The Kerr metric admits two independent principal null directions whose tangent vectors are given by

$$k_{\pm} = \partial_t \pm \frac{\Delta}{(r^2+a^2)}\partial_r + \frac{a}{(r^2+a^2)}\partial_\phi. \tag{8.86}$$

The Kerr metric is of Petrov type D. Associated with the above principal null directions is a complex null frame which can be fixed, leaving the two real null vectors to be aligned with the vectors (8.86) and the two complex conjugate vectors m and $\bar{m}$ as follows (Wald, 1984):

$$\begin{aligned} l &\equiv \frac{r^2+a^2}{\Delta}k_+ = \frac{1}{\Delta}\left[(r^2+a^2)\partial_t + \Delta\partial_r + a\partial_\phi\right], \\ n &\equiv \frac{r^2+a^2}{2\Sigma}k_- = \frac{1}{2\Sigma}\left[(r^2+a^2)\partial_t - \Delta\partial_r + a\partial_\phi\right], \\ m &= \frac{1}{\sqrt{2}(r+ia\cos\theta)}\left(ia\sin\theta\,\partial_t + \partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right). \end{aligned} \tag{8.87}$$

We have $l \cdot n = -1$, $m \cdot m = 0$, $\bar{m} \cdot \bar{m} = 0$, and $m \cdot \bar{m} = 1$. Note that l and k_+ are aligned; the proportionality factor $(r^2 + a^2)/\Delta$ makes l geodesic.

8.4 Special observers in Kerr space-time

The Kerr solution is particularly important in astrophysics since it describes the geometrical environment of a rotating black hole or, in the weak-field limit, that of a rotating spherical source. Because of its prominent role it is essential to describe the orbits which can host physically realistic observers.

Static observers

The 4-velocity of a static observer is given by

$$m = \frac{1}{M}\partial_t, \qquad m^\flat = -M(dt - M_\phi d\phi), \tag{8.88}$$

where M and M_ϕ are respectively the lapse and shift functions defined by

$$M \equiv \sqrt{-g_{tt}} = \sqrt{\frac{\Delta - a^2 \sin^2\theta}{\Sigma}}, \qquad M_\phi = -\frac{g_{t\phi}}{g_{tt}} = -\frac{2\mathcal{M}ra\sin^2\theta}{\Delta - a^2\sin^2\theta}. \tag{8.89}$$

In terms of the lapse and shift functions we have

$$g = -M^2(dt - M_\phi d\phi) \otimes (dt - M_\phi d\phi) + \gamma_{ab} dx^a \otimes dx^b, \tag{8.90}$$

where $\gamma_{ab} = g_{ab} + M^2 M_\phi^2 \delta_a^\phi \delta_b^\phi$.

Static observers only exist outside the ergosphere, defined by the equation $g_{tt} = 0$, that is

$$r = \mathcal{M} \pm \sqrt{\mathcal{M}^2 - a^2\cos^2\theta}. \tag{8.91}$$

A frame adapted to a static observer is the following

$$\begin{aligned}
e(m)_{\hat{t}} &= m, \\
e(m)_{\hat{r}} &= \frac{1}{\sqrt{g_{rr}}}\,\partial_r = \sqrt{\frac{\Delta}{\Sigma}}\,\partial_r, \\
e(m)_{\hat{\theta}} &= \frac{1}{\sqrt{g_{\theta\theta}}}\,\partial_\theta = \sqrt{\frac{1}{\Sigma}}\,\partial_\theta, \qquad (8.92)\\
e(m)_{\hat{\phi}} &= \frac{1}{\sqrt{\gamma_{\phi\phi}}}[\,\partial_\phi + M_\phi\,\partial_t] \\
&= \frac{1}{\sin\theta}\sqrt{\frac{\Delta - a^2\sin^2\theta}{\Delta\Sigma}}\left[\partial_\phi - \frac{2a\mathcal{M}r\sin^2\theta}{\Delta - a^2\sin^2\theta}\,\partial_t\right]. \qquad (8.93)
\end{aligned}$$

The static observers form a congruence of accelerated world lines with acceleration vector

$$a(m) = \frac{\mathcal{M}}{\Sigma^{3/2}(\Delta - a^2\sin^2\theta)}\Big[\sqrt{\Delta}(r^2 - a^2\cos^2\theta)\,e(m)_{\hat r} - 2a^2 r\sin\theta\cos\theta\,e(m)_{\hat\theta}\Big], \tag{8.94}$$

and vorticity vector

$$\omega(m) = \frac{\mathcal{M}}{\Sigma^{3/2}(\Delta - a^2\sin^2\theta)}\Big[2\sqrt{\Delta}ra\cos\theta\,e(m)_{\hat r} + a\sin\theta(r^2 - a^2\cos^2\theta)\,e(m)_{\hat\theta}\Big], \tag{8.95}$$

while the expansion vanishes identically.

One can now evaluate the transport law for the spatial triad $e(m)_{\hat a}$ along the world line of m. We have

$$\begin{aligned} P(m)\nabla_m e(m)_{\hat r} &= -\zeta_{(\mathrm{fw})}{}^{\hat\theta} e(m)_{\hat\phi},\\ P(m)\nabla_m e(m)_{\hat\theta} &= \zeta_{(\mathrm{fw})}{}^{\hat r} e(m)_{\hat\phi},\\ P(m)\nabla_m e(m)_{\hat\phi} &= \zeta_{(\mathrm{fw})}{}^{\hat\theta} e(m)_{\hat r} - \zeta_{(\mathrm{fw})}{}^{\hat r} e(m)_{\hat\theta}, \end{aligned} \tag{8.96}$$

where

$$\begin{aligned} \zeta_{(\mathrm{fw})}{}^{\hat r} &= -\frac{2a\mathcal{M}r\sqrt{\Delta}\cos\theta}{\Sigma^{3/2}(\Sigma - 2\mathcal{M}r)},\\ \zeta_{(\mathrm{fw})}{}^{\hat\theta} &= -\frac{a\mathcal{M}\sin\theta(r^2 - a^2\cos^2\theta)}{\Sigma^{3/2}(\Sigma - 2\mathcal{M}r)}, \end{aligned} \tag{8.97}$$

are the components of the Fermi-Walker angular velocity vector. On the equatorial plane we have $\zeta_{(\mathrm{fw})}{}^{\hat r} = 0$ and

$$\zeta_{(\mathrm{fw})}{}^{\hat\theta} = -\frac{a\mathcal{M}}{r^3(1 - 2\mathcal{M}/r)}. \tag{8.98}$$

Note that here, with an abuse of notation, we have denoted $\zeta_{(\mathrm{fw},m,e(m)_{\hat a})}$ simply by $\zeta_{(\mathrm{fw})}$.

Zero-angular-momentum observers

Observers who have no angular momentum with respect to the flat infinity are termed *ZAMOs*, and move on world lines which are orthogonal to the hypersurfaces $t =$ constant in Boyer-Lindquist coordinates. Their 4-velocity is given by

$$n^\flat = -N dt, \qquad n = \frac{1}{N}(\partial_t - N^\phi \partial_\phi), \tag{8.99}$$

where N and N^ϕ are the corresponding lapse and shift functions

$$N = \sqrt{\frac{\Delta\Sigma}{A}}, \qquad N^\phi = -\frac{2a\mathcal{M}r}{A}. \tag{8.100}$$

In terms of these functions, the space-time metric can be written in the following form:

$$g = -N^2 dt \otimes dt + g_{ab}(dx^a + N^a dt) \otimes (dx^b + N^b dt), \tag{8.101}$$

where $N^a = N^\phi \delta^a_\phi$. ZAMOs exist everywhere outside the outer horizon r_+.

A tetrad frame adapted to ZAMOs is the following:

$$\begin{aligned} e(n)_{\hat{t}} &= n, & e(n)_{\hat{r}} &= \sqrt{\frac{\Delta}{\Sigma}}\partial_r, \\ e(n)_{\hat{\theta}} &= \sqrt{\frac{1}{\Sigma}}\partial_\theta, & e(n)_{\hat{\phi}} &= \frac{1}{\sin\theta}\sqrt{\frac{\Sigma}{A}}\partial_\phi. \end{aligned} \tag{8.102}$$

The world lines of the ZAMO congruence have an acceleration vector

$$a(n) = a(n)^{\hat{r}} e_{\hat{r}} + a(n)^{\hat{\theta}} e_{\hat{\theta}}, \tag{8.103}$$

where

$$\begin{aligned} a(n)^{\hat{r}} &= -\frac{\mathcal{M}}{\sqrt{\Delta}\Sigma^{3/2}A}\left\{a^2\cos^2\theta[(r^2+a^2)^2 - 4\mathcal{M}r^3]\right. \\ &\quad \left. - r^2[(r^2+a^2)^2 - 4a^2\mathcal{M}r]\right\}, \\ a(n)^{\hat{\theta}} &= -\frac{2\sin\theta\cos\theta\mathcal{M}ra^2(r^2+a^2)}{\Sigma^{3/2}A}; \end{aligned} \tag{8.104}$$

they have a non-zero expansion with expansion tensor

$$\begin{aligned} \theta(n) &= \theta(n)^{\hat{r}\hat{\phi}}[e(n)_{\hat{r}} \otimes e(n)_{\hat{\phi}} + e(n)_{\hat{\phi}} \otimes e(n)_{\hat{r}}] \\ &\quad + \theta(n)^{\hat{\theta}\hat{\phi}}[e(n)_{\hat{\theta}} \otimes e(n)_{\hat{\phi}} + e(n)_{\hat{\phi}} \otimes e(n)_{\hat{\theta}}], \end{aligned} \tag{8.105}$$

where

$$\begin{aligned} \theta(n)^{\hat{r}\hat{\phi}} &= \frac{a\mathcal{M}\sin\theta}{\Sigma^{3/2}A}(a^4\cos^2\theta - r^2a^2\cos^2\theta - r^2a^2 - 3r^4), \\ \theta(n)^{\hat{\theta}\hat{\phi}} &= \frac{2ra^3\mathcal{M}\sin^2\theta\cos\theta\sqrt{\Delta}}{\Sigma^{3/2}A}, \end{aligned} \tag{8.106}$$

while the vorticity vanishes identically.

One can now evaluate the transport law for the spatial triad $e(n)_{\hat{a}}$ along the world line of n. We have

$$\begin{aligned} P(n)\nabla_n e(n)_{\hat{r}} &= -\zeta_{(\mathrm{fw})}{}^{\hat{\theta}} e(n)_{\hat{\phi}}, \\ P(n)\nabla_n e(n)_{\hat{\theta}} &= \zeta_{(\mathrm{fw})}{}^{\hat{r}} e(n)_{\hat{\phi}}, \\ P(n)\nabla_n e(n)_{\hat{\phi}} &= \zeta_{(\mathrm{fw})}{}^{\hat{\theta}} e(n)_{\hat{r}} - \zeta_{(\mathrm{fw})}{}^{\hat{r}} e(n)_{\hat{\theta}}, \end{aligned} \tag{8.107}$$

where

$$\begin{aligned}
{\zeta_{(\mathrm{fw})}}^{\hat r} &= -\frac{2a^3\mathcal{M}r\sqrt{\Delta}\sin^2\theta\cos\theta}{\Sigma^{3/2}A} = -\theta(n)^{\hat\theta\hat\phi}, \qquad (8.108)\\
{\zeta_{(\mathrm{fw})}}^{\hat\theta} &= \frac{a\mathcal{M}\sin\theta[a^2\cos^2\theta(a^2-r^2)-r^2(a^2+3r^2)])}{\Sigma^{3/2}A} = \theta(n)^{\hat r\hat\phi}.
\end{aligned}$$

On the equatorial plane we have ${\zeta_{(\mathrm{fw})}}^{\hat r} = 0$ and

$$ {\zeta_{(\mathrm{fw})}}^{\hat\theta} = -\frac{a\mathcal{M}(3r^2+a^2)}{r^2(r^3+a^2r+2a^2\mathcal{M})}. \qquad (8.109)$$

Moreover, the only non-vanishing components of ZAMO kinematical quantities are $a(n)^{\hat r}$ and $\theta(n)^{\hat r\hat\phi}$. Note that here, with an abuse of notation, we have denoted $\zeta_{(\mathrm{fw},n,e(n)_{\hat a})}$ simply by $\zeta_{(\mathrm{fw})}$.

Observers on general spatially circular orbits

Consider now a family of spatially circular orbits with unit tangent vector field

$$U = \Gamma(\partial_t + \zeta\partial_\phi), \qquad (8.110)$$

where $\Gamma > 0$ is defined as

$$\Gamma = \left[1 - \zeta^2\sin^2\theta(r^2+a^2) - \frac{2Mr}{\Sigma}(1-a\zeta\sin^2\theta)^2\right]^{-1/2} \qquad (8.111)$$

and ζ is constant along U, i.e. $\pounds_U\zeta = 0$. This class includes static observers with

$$\zeta_{(\mathrm{static})} = 0 \qquad (8.112)$$

as well as ZAMOs with

$$\zeta_{(\mathrm{ZAMO})} = -N^\phi. \qquad (8.113)$$

One may decompose the vector field U with respect to ZAMOs, obtaining a re-parameterization of the given family of orbits in terms of the relative velocity or the rapidity instead of the angular velocity, that is

$$\begin{aligned}
U &= \gamma[n+\nu(U,n)] = \gamma\left[n + ||\nu(U,n)||e(n)_{\hat\phi}\right]\\
&= \cosh\alpha(U,n)\, n + \sinh\alpha(U,n)e(n)_{\hat\phi}, \qquad (8.114)
\end{aligned}$$

where $\gamma \equiv \gamma(U,n) = \cosh\alpha(U,n)$ is the Lorentz factor and $\nu(U,n) = \nu(U,n)^{\hat\phi}e_{\hat\phi}$. A useful relation between ζ and $\nu(U,n)^{\hat\phi}$ is given by

$$\nu(U,n)^{\hat\phi} = \frac{\sqrt{g_{\phi\phi}}}{N}(\zeta + N^\phi), \qquad (8.115)$$

yielding the following expression for the Lorentz factor:

$$\gamma = \Gamma N. \qquad (8.116)$$

A tetrad adapted to the observers U is given by

$$
\begin{aligned}
e(U)_{\hat t} &= U,\\
e(U)_{\hat r} &= \sqrt{\frac{\Delta}{\Sigma}}\partial_r,\\
e(U)_{\hat \theta} &= \sqrt{\frac{1}{\Sigma}}\partial_\theta,\\
e(U)_{\hat \phi} &= \bar\Gamma[\partial_t + \bar\zeta\partial_\phi],
\end{aligned}
\tag{8.117}
$$

with

$$
\begin{aligned}
\bar\zeta &= -\frac{g_{tt}+\zeta g_{t\phi}}{g_{t\phi}+\zeta g_{\phi\phi}},\\
\bar\Gamma &= (g_{tt}+2\bar\zeta g_{t\phi}+\bar\zeta^2 g_{\phi\phi})^{-1/2} = \Gamma\frac{|g_{t\phi}+\zeta g_{\phi\phi}|}{\sqrt{g_{t\phi}^2-g_{tt}g_{\phi\phi}}}.
\end{aligned}
\tag{8.118}
$$

Explicitly, we have

$$
\bar\zeta = \frac{\Sigma - 2\mathcal{M}r(1-a\zeta\sin^2\theta)}{\sin^2\theta[\zeta(r^2+a^2)\Sigma - 2a\mathcal{M}r(1-a\zeta\sin^2\theta)]}.
\tag{8.119}
$$

Note that, like U in (8.114), $e(U)_{\hat\phi}$ can be written as

$$
\begin{aligned}
e(U)_{\hat\phi} &= \gamma[\,||\nu(U,n)||n + e(n)_{\hat\phi}\,]\\
&= \sinh\alpha(U,n)n + \cos\alpha(U,n)e(n)_{\hat\phi},
\end{aligned}
\tag{8.120}
$$

so that, abbreviating $\alpha(U,n)=\alpha$, one has

$$
e(U)_{\hat\phi} = \frac{dU}{d\alpha}.
\tag{8.121}
$$

The world lines with tangent vectors (8.110) have an acceleration vector

$$
a(U) = a(U)^{\hat r}e(U)_{\hat r} + a(U)^{\hat\theta}e(U)_{\hat\theta},
\tag{8.122}
$$

where

$$
\begin{aligned}
a(U)^{\hat r} &= \frac{\Gamma^2\sqrt{\Delta}}{\sqrt{\Sigma}}\left[\frac{\mathcal{M}(r^2-a^2\cos^2\theta)}{\Sigma^2}(1-a\zeta\sin^2\theta)^2 - r\zeta^2\sin^2\theta\right],\\
a(U)^{\hat\theta} &= -\frac{\Gamma^2\sin\theta\cos\theta}{\sqrt{\Sigma}}\left[\frac{2\mathcal{M}r}{\Sigma^2}[(r^2+a^2)\zeta-a]^2+\Delta\zeta^2\right].
\end{aligned}
\tag{8.123}
$$

Moreover, orbits (8.110) in the special case $\zeta =$constant over the entire congruence (a more restrictive condition than the above mentioned $\pounds_U\zeta=0$) form a Born-rigid congruence of world lines (parameterized by r and θ) with vanishing expansion tensor

$$
\theta(U) = 0,
\tag{8.124}
$$

but with non-zero vorticity vector

$$\omega(U) = \omega(U)^{\hat r} e(U)_{\hat r} + \omega(U)^{\hat\theta} e(U)_{\hat\theta}, \tag{8.125}$$

where

$$\begin{aligned}
\tilde\omega(U)^{\hat r} &= \frac{1}{2}\frac{\Gamma\bar\Gamma}{\sqrt{g_{\theta\theta}}}\left[\partial_\theta\, g_{tt} + (\zeta+\bar\zeta)\partial_\theta\, g_{t\phi} + \zeta\bar\zeta\partial_\theta\, g_{\phi\phi}\right],\\
\tilde\omega(U)^{\hat\theta} &= -\frac{1}{2}\frac{\Gamma\bar\Gamma}{\sqrt{g_{rr}}}\left[\partial_r\, g_{tt} + (\zeta+\bar\zeta)\partial_r\, g_{t\phi} + \zeta\bar\zeta\partial_r\, g_{\phi\phi}\right].
\end{aligned} \tag{8.126}$$

In the general case ($\zeta = \zeta(r,\theta)$ varies over the congruence) the above relations are modified: the congruence is no longer rigid and has

$$\begin{aligned}
\theta(U)_{\hat r\hat\phi} &= \frac{1}{2}\left[\mathrm{sgn}(g_{t\phi}+\zeta g_{\phi\phi})\right]\Gamma^2\sin\theta\frac{\Delta}{\sqrt{\Sigma}}\partial_r\zeta,\\
\theta(U)_{\hat\theta\hat\phi} &= \frac{1}{2}\left[\mathrm{sgn}(g_{t\phi}+\zeta g_{\phi\phi})\right]\Gamma^2\sin\theta\sqrt{\frac{\Delta}{\Sigma}}\partial_\theta\zeta,
\end{aligned} \tag{8.127}$$

where $\hat\phi$ refers to the unit space-like direction $e(U)_{\hat\phi}$ orthogonal to U.

Moreover, the components of the vorticity tensor change and one finds

$$\begin{aligned}
\omega(U)^{\hat r} &= \tilde\omega^{\hat r} + \theta(U)_{\hat\theta\hat\phi},\\
\omega(U)^{\hat\theta} &= \tilde\omega^{\hat\theta} - \theta(U)_{\hat r\hat\phi},
\end{aligned} \tag{8.128}$$

where we have denoted with a tilde the components (8.126) of the vorticity vector corresponding to the case $\zeta =$constant.

For a general circular orbit U, the Fermi-Walker gravitational force as measured by a static observer m is given by

$$\begin{aligned}
F^{(G)}_{(\mathrm{fw},U,m)} &= -\frac{Dm}{d\tau_U} = -\gamma(U,m)[a(m) + \omega(m)\times_m \nu(U,m)]\\
&= -\gamma(U,m)[a(m)^{\hat r} + \omega(m)^{\hat\theta}\nu(U,m)^{\hat\phi}]e(m)_{\hat r}\\
&\quad - \gamma(U,m)[a(m)^{\hat\theta} - \omega(m)^{\hat r}\nu(U,m)^{\hat\phi}]e(m)_{\hat\theta},
\end{aligned} \tag{8.129}$$

while that measured by a ZAMO observer n is given by

$$\begin{aligned}
F^{(G)}_{(\mathrm{fw},U,n)} &= -\frac{Dn}{d\tau_U} = -\gamma(U,n)[a(n) + \theta(n)\llcorner\nu(U,n)]\\
&= -\gamma(U,n)[a(n)^{\hat r} + \theta(n)^{\hat r\hat\phi}\nu(U,n)^{\hat\phi}]e(n)_{\hat r}\\
&\quad - \gamma(U,n)[a(n)^{\hat\theta} + \theta(n)^{\hat\theta\hat\phi}\nu(U,n)^{\hat\phi}]e(n)_{\hat\theta}.
\end{aligned} \tag{8.130}$$

Finally we identify a Frenet-Serret frame along U as the following:

$$\begin{aligned}
E_{\hat{0}} &= U, \\
E_{\hat{1}} &= \cos\chi e_{\hat{r}} + \sin\chi e_{\hat{\theta}}, \\
E_{\hat{2}} &= e(U)_{\hat{\phi}} = \frac{dU}{d\alpha}, \\
E_{\hat{3}} &= \sin\chi e_{\hat{r}} - \cos\chi e_{\hat{\theta}} = -\frac{dE_1}{d\chi},
\end{aligned} \tag{8.131}$$

with

$$\tan\chi = \frac{a(U)^{\hat{\theta}}}{a(U)^{\hat{r}}}, \qquad \kappa(U)^2 = [a(U)^{\hat{r}}]^2 + [a(U)^{\hat{\theta}}]^2, \tag{8.132}$$

where $a(U)^{\hat{r}}$ and $a(U)^{\hat{\theta}}$ are given by (8.123). The general expressions for $\tau(U)_1$ and $\tau(U)_2$ are rather complicated (Iyer and Vishveshwara, 1993). In order to write these relations down explicitly let us introduce the following notation:

$$\begin{aligned}
\mathcal{A} &= g_{tt} + 2\zeta g_{t\phi} + \zeta^2 g_{\phi\phi}, \\
\mathcal{B} &= g_{t\phi} + \zeta g_{\phi\phi},
\end{aligned} \tag{8.133}$$

and

$$\begin{aligned}
\mathcal{A}_{(\hat{a})} &= \partial_{\hat{a}} g_{tt} + 2\zeta\partial_{\hat{a}} g_{t\phi} + \zeta^2\partial_{\hat{a}} g_{\phi\phi}, \\
\mathcal{B}_{(\hat{a})} &= \partial_{\hat{a}} g_{t\phi} + \zeta\partial_{\hat{a}} g_{\phi\phi},
\end{aligned} \tag{8.134}$$

where $a = r, \theta$ and $\partial_{\hat{a}} = (g_{aa})^{-1/2}\partial_a$. Note that, according to our previous notation,

$$\mathcal{A} = -\Gamma^{-2}, \qquad (\sin\chi, \cos\chi) = \frac{1}{2\kappa(U)\mathcal{A}}\left(\mathcal{A}_{(\hat{r})}, \mathcal{A}_{(\hat{\theta})}\right). \tag{8.135}$$

It follows that

$$\mathcal{B} = \frac{1}{2}\partial_\zeta\mathcal{A}, \qquad \mathcal{B}_{(\hat{a})} = \frac{1}{2}\partial_\zeta\mathcal{A}_{(\hat{a})}. \tag{8.136}$$

We then have the following compact expressions:

$$\begin{aligned}
\kappa(U)^2 &= \frac{1}{4}\frac{\left(\mathcal{A}^2_{(\hat{r})} + \mathcal{A}^2_{(\hat{\theta})}\right)}{\mathcal{A}^2}, \\
\tau_1(U)^2 &= \frac{\mathcal{B}^2}{16\mathcal{A}^2 N^2 g_{\phi\phi}\kappa(U)^2}\left[\frac{\mathcal{A}_{(\hat{r})}\mathcal{B}_{(\hat{r})} + \mathcal{A}_{(\hat{\theta})}\mathcal{B}_{(\hat{\theta})}}{\mathcal{B}} - \frac{\mathcal{A}^2_{(\hat{r})} + \mathcal{A}^2_{(\hat{\theta})}}{\mathcal{A}}\right]^2, \\
\tau_2(U)^2 &= \frac{1}{16N^2 g_{\phi\phi}\kappa(U)^2}\left[\frac{\mathcal{A}_{(\hat{r})}\mathcal{B}_{(\hat{\theta})} - \mathcal{A}_{(\hat{\theta})}\mathcal{B}_{(\hat{r})}}{\mathcal{A}}\right]^2.
\end{aligned} \tag{8.137}$$

These relations can be further simplified. Using (8.136) in the expression of $\tau_1(U)^2$ yields

$$\tau_1(U)^2 = \frac{\mathcal{A}^2}{4N^2 g_{\phi\phi}}\left[\partial_\zeta\kappa(U)\right]^2 \tag{8.138}$$

or, from (8.135),

$$\tau_1(U)^2 = \frac{1}{4\Gamma^4 N^2 g_{\phi\phi}} \left[\partial_\zeta \kappa(U)\right]^2 . \tag{8.139}$$

Using relations (8.36) between ζ and $\nu(U,n)^{\hat\phi}$ and (8.116) for the Lorentz factor, one can replace the derivative with respect to ζ with that with respect to ν, obtaining

$$\tau_1(U)^2 = \frac{1}{4\gamma^4} \left[\partial_\nu \kappa(U)\right]^2 . \tag{8.140}$$

Finally, replacing ν with the rapidity parameter α, the above relation becomes

$$\tau_1(U)^2 = \frac{1}{4} \left[\partial_\alpha \kappa(U)\right]^2 . \tag{8.141}$$

A similar procedure, also using (8.135) for $\sin\chi$ and $\cos\chi$, leads to the following expression for $\tau_2(U)$:

$$\tau_2(U)^2 = \frac{1}{4}\kappa(U)^2[\partial_\alpha \chi]^2. \tag{8.142}$$

Using relations (8.137) and their simplifications (8.141) and (8.142), all the intrinsic geometrical properties of time-like spatially circular orbits can be easily studied. It is worth noting that the general expressions for curvature and torsions given here are valid for any stationary and axisymmetric space-time.

Carter's observers

A family of spatially circular orbits with particular properties was found by Carter (1968). Their tangent vector fields are given by

$$\begin{aligned} u_{(\mathrm{car})} &= \frac{r^2+a^2}{\sqrt{\Delta\,\Sigma}}\left[\partial_t + \frac{a}{r^2+a^2}\partial_\phi\right], \\ u^\flat_{(\mathrm{car})} &= -\sqrt{\frac{\Delta}{\Sigma}}\left[dt - a\sin^2\theta d\phi\right]. \end{aligned} \tag{8.143}$$

These belong to the class of spatial circular orbits with

$$\zeta_{(\mathrm{car})} = \frac{a}{r^2+a^2}. \tag{8.144}$$

The main property of these trajectories is that of being the unique time-like world lines belonging to the intersection of the Killing 2-plane $(t,\,\phi)$ with the 2-plane spanned by Kerr principal null directions.

The direction orthogonal to $u_{(\mathrm{car})}$ in the (t,ϕ)-plane is given by

$$\begin{aligned} \bar u_{(\mathrm{car})} &= \frac{a\sin\theta}{\sqrt{\Sigma}}\left[\partial_t + \frac{1}{a\sin^2\theta}\partial_\phi\right], \\ \bar u^\flat_{(\mathrm{car})} &= -\frac{a\sin\theta}{\sqrt{\Sigma}}\left[dt - \frac{r^2+a^2}{a}d\phi\right]. \end{aligned} \tag{8.145}$$

An orthonormal frame adapted to the Carter family of observers is the same as the one in (8.117), with ζ replaced by $\zeta_{(\mathrm{car})}$:

$$\begin{aligned} E(u_{(\mathrm{car})})_{\hat t} &= u_{(\mathrm{car})}, & E(u_{(\mathrm{car})})_{\hat r} &= e_{\hat r}, \\ E(u_{(\mathrm{car})})_{\hat \theta} &= e_{\hat \theta}, & E(u_{(\mathrm{car})})_{\hat \phi} &= \bar u_{(\mathrm{car})}. \end{aligned} \tag{8.146}$$

With respect to this frame, the principal null directions (8.86) of Kerr space-time take the form

$$k_\pm = \frac{\sqrt{\Delta\Sigma}}{r^2+a^2}\left(u_{(\mathrm{car})} \pm e_{\hat r}\right); \tag{8.147}$$

this implies that photons moving along the above directions will appear spatially radial with respect to Carter's observers. If we analyze the vector field $u_{(\mathrm{car})}$ with respect to the static observers in Kerr space-time, we find, for the relative velocity,

$$\nu(u_{(\mathrm{car})}, m)^{\hat\phi} = \sqrt{\gamma_{\phi\phi}}\frac{\zeta_{(\mathrm{car})}}{M(1-M_\phi\zeta_{(\mathrm{car})})} = \frac{a\sin\theta}{\sqrt{\Delta}}, \tag{8.148}$$

where M, M_ϕ, and $\gamma_{\phi\phi}$ are defined by (8.89) and (8.90).

Furthermore, Carter's observers measure the electric and magnetic parts of the Weyl tensor as parallel to each other:

$$\begin{aligned} E(u_{(\mathrm{car})}) &= \frac{\mathcal{M}r(r^2-3a^2\cos^2\theta)}{\Sigma^3}[-2e_{\hat r}\otimes e_{\hat r} + e_{\hat\theta}\otimes e_{\hat\theta} \\ &\quad + \bar u_{(\mathrm{car})}\otimes \bar u_{(\mathrm{car})}], \\ H(u_{(\mathrm{car})}) &= \frac{\mathcal{M}a\cos\theta(3r^2-a^2\cos^2\theta)}{\Sigma^3}[-2e_{\hat r}\otimes e_{\hat r} + e_{\hat\theta}\otimes e_{\hat\theta} \\ &\quad + \bar u_{(\mathrm{car})}\otimes \bar u_{(\mathrm{car})}]. \end{aligned} \tag{8.149}$$

Null spatially circular orbits

Null spatially circular orbits cannot be associated with a physical observer; nonetheless they play an important role as critical conditions for time-like trajectories. Their tangent vector field is given by

$$\ell = \Gamma^{(\mathrm{null})}(\partial_t \pm \zeta_\pm^{(\mathrm{null})}\partial_\phi), \tag{8.150}$$

with

$$\begin{aligned} \zeta_\pm^{(\mathrm{null})} &= \frac{2a\mathcal{M}r}{A} \pm \frac{\Sigma\sqrt{\Delta}}{\sin\theta A} \\ &= \zeta_{(\mathrm{ZAMO})} \pm \frac{\Sigma\sqrt{\Delta}}{\sin\theta A}, \end{aligned} \tag{8.151}$$

where A is given by (8.75) and $\Gamma^{(\mathrm{null})}$ is an arbitrary factor; $\zeta_{(\mathrm{ZAMO})}$ is given by (8.113).

Observers on equatorial circular orbits

There exists a whole collection of special circular orbits in the equatorial plane. They are the corotating (+) and counter-rotating (−) time-like circular geodesics whose angular and linear velocities (with respect to ZAMOs) are respectively[3]

$$\zeta_{K\pm} \equiv \zeta_\pm = \left[a \pm (r^3/\mathcal{M})^{1/2}\right]^{-1}, \tag{8.152}$$

$$\nu_{K\pm} \equiv \nu_\pm = \frac{a^2 \mp 2a\sqrt{\mathcal{M}r} + r^2}{\sqrt{\Delta}(a \pm r\sqrt{r/\mathcal{M}})}. \tag{8.153}$$

These become null when $|\nu_\pm| = 1$, which occurs at

$$r_{K\pm}^{(\mathrm{null})} = 2\mathcal{M}\left\{1 + \cos\left[\frac{2}{3}\arccos\left(\mp\frac{a}{\mathcal{M}}\right)\right]\right\}. \tag{8.154}$$

Obviously co- and counter-rotating time-like circular geodesics exist at $r > r_{K+}^{(\mathrm{null})}$ and $r > r_{K-}^{(\mathrm{null})}$ respectively.

Closely related to these are the *geodesic meeting point* (gmp) orbits defined by the intersection points of the time-like geodesics, with velocity

$$\nu_{(\mathrm{gmp})} = \frac{\nu_+ + \nu_-}{2} = -\frac{a\mathcal{M}(3r^2 + a^2)}{\sqrt{\Delta}(r^3 - a^2\mathcal{M})}. \tag{8.155}$$

Similarly one may consider special orbits having spatial 3-velocity (with respect to ZAMOs) given by

$$\nu_{(\mathrm{pt})} = \frac{2}{\nu_+^{-1} + \nu_-^{-1}} = \frac{(r^2 + a^2)^2 - 4a^2\mathcal{M}r}{a\sqrt{\Delta}(3r^2 + a^2)}. \tag{8.156}$$

Both the above families of orbits play a role in the study of parallel transport (pt) of vectors along circular orbits (Bini, Jantzen, and Mashhoon, 2002).

The linear velocities of the circular geodesics and gmp orbits are related to the ZAMO kinematical quantities

$$\begin{aligned}
a(n)^{\hat{r}} &= \frac{\mathcal{M}\Delta^{-1/2}\left[(r^2 + a^2)^2 - 4a^2\mathcal{M}r\right]}{r^2(r^3 + a^2 r + 2\mathcal{M}a^2)}, \\
\theta(n)^{\hat{r}\hat{\phi}} &= \frac{\mathcal{M}a(3r^2 + a^2)}{r^2(r^3 + a^2 r + 2\mathcal{M}a^2)}, \\
k_{(\mathrm{lie})}(n)^{\hat{r}} &= -\frac{(r^3 - a^2\mathcal{M})\sqrt{\Delta}}{r^2(r^3 + a^2 r + 2a^2\mathcal{M})},
\end{aligned} \tag{8.157}$$

by

$$a(n)^{\hat{r}} = k_{(\mathrm{lie})}(n)^{\hat{r}}\,\nu_+\nu_-, \qquad \theta(n)^{\hat{\phi}\hat{r}} = -k_{(\mathrm{lie})}(n)^{\hat{r}}\,\nu_{(\mathrm{gmp})}, \tag{8.158}$$

where $k_{(\mathrm{lie})}(n)^{\hat{r}} = -\partial_{\hat{r}} \ln\sqrt{g_{\phi\phi}}$, as follows from the definition (3.169).

[3] To simplify notation in this section we use $U_\pm$ to denote the 4-velocity of corotating and counter-rotating geodesics as well as $\nu_\pm$ and $\zeta_\pm$ for their relative and angular velocities, dropping the symbol K.

On the equatorial plane, the expressions of the gravitational force given in (8.129) and (8.130) on general spatially circular orbits reduce to

$$F^{(G)}_{(\mathrm{fw},U,m)} = -\gamma(U,m)[a(m)^{\hat r} + \omega(m)^{\hat\theta}\nu(U,m)^{\hat\phi}]e(m)_{\hat r} \tag{8.159}$$

as measured by the static observer m, and

$$F^{(G)}_{(\mathrm{fw},U,n)} = -\gamma(U,n)[a(n)^{\hat r} + \theta(n)^{\hat r\hat\phi}\nu(U,n)^{\hat\phi}]e(n)_{\hat r} \tag{8.160}$$

as measured by a ZAMO observer n.

We now have expressions for all the geometrical and kinematical quantities which are needed to specify the Frenet-Serret, Fermi-Walker, and parallel propagated frames.

The Frenet-Serret frame is given by (8.131) with $\chi = 0$, that is

$$\begin{aligned} E_0 &= \cosh\alpha\, n + \sinh\alpha\, e_{\hat\phi}, \qquad & E_1 &= e_{\hat r},\\ E_2 &= \sinh\alpha\, n + \cosh\alpha\, e_{\hat\phi}, \qquad & E_3 &= -e_{\hat\theta}, \end{aligned} \tag{8.161}$$

recalling that $\nu = \tanh\alpha, \gamma = \cosh\alpha$. The second torsion τ_2 vanishes, while the geodesic curvature κ and the first torsion τ_1 are given by

$$\begin{aligned} \kappa &= \frac{\sqrt{\Delta}\mathcal{M}}{r^3}\Gamma^2\zeta^2\left[\left(a-\frac{1}{\zeta}\right)^2 - \frac{r^3}{\mathcal{M}}\right],\\ \tau_1 &= \frac{\Gamma^2}{r}\left\{\frac{\mathcal{M}a}{r^2} - \zeta\left[\frac{\mathcal{M}(r^2+2a^2)}{r^2} - r\left(1-\frac{2\mathcal{M}}{r}\right)\right]\right.\\ &\quad\left. +\,\zeta^2\frac{\mathcal{M}a(3r^2+a^2)}{r^2}\right\}. \end{aligned} \tag{8.162}$$

Equivalently, we have

$$\begin{aligned} \kappa &= k_{(\mathrm{lie})}(n)_{\hat r}\gamma^2(\nu-\nu_+)(\nu-\nu_-),\\ \tau_1 &= k_{(\mathrm{lie})}(n)_{\hat r}\,\nu_{(\mathrm{gmp})}\gamma^2(\nu-\nu_{(\mathrm{crit})+})(\nu-\nu_{(\mathrm{crit})-}), \end{aligned} \tag{8.163}$$

where

$$\begin{aligned} \nu_{(\mathrm{crit})\pm} &= \frac{\gamma_-\nu_- \mp \gamma_+\nu_+}{\gamma_- \mp \gamma_+}\\ &= -\frac{1}{2\mathcal{M}a(3r^2+a^2)\sqrt{\Delta}}\Big[-2a^2\mathcal{M}(a^2-3\mathcal{M}r)\\ &\quad + r^2(r^2+a^2)(r-3\mathcal{M})\\ &\quad \pm(r^3+a^2r+2a^2\mathcal{M})\sqrt{r}\sqrt{r(r-3\mathcal{M})^2-4a^2\mathcal{M}}\Big], \end{aligned} \tag{8.164}$$

or equivalently

$$\zeta_{(\mathrm{crit})\pm} = \frac{\Gamma_-\zeta_- \mp \Gamma_+\zeta_+}{\Gamma_- \mp \Gamma_+}, \tag{8.165}$$

identify those orbits on which the first torsion vanishes. The notation "crit" for the observers moving along equatorial circular orbits comes from the relation

$$\tau_1 = -\frac{1}{2\gamma^2}\frac{d\kappa}{d\nu}, \tag{8.166}$$

which can be easily verified; critical observers have vanishing first torsion and hence extreme acceleration (in magnitude) and are termed *extremely accelerated observers.*

The time-like condition $|\nu_{(\text{crit})+}| < 1$ is satisfied when $r_+ < r < r^{(\text{null})}_{K+}$ while $|\nu_{(\text{crit})-}| < 1$ when $r > r^{(\text{null})}_{K-}$.

We can now specify the geometrical properties of extremely accelerated observers. Consider for example the $U_{(\text{crit})-}$ orbits (analogous considerations hold for the $U_{(\text{crit})+}$ orbits). From Eq. (8.164) it follows that

$$\begin{aligned}\nu_+ - \nu_{(\text{crit})-} &= \frac{\gamma_-}{\gamma_- - \gamma_+}(\nu_+ - \nu_-),\\ \nu_- - \nu_{(\text{crit})-} &= \frac{\gamma_+}{\gamma_- - \gamma_+}(\nu_+ - \nu_-),\end{aligned} \tag{8.167}$$

and therefore

$$\frac{\nu_+ - \nu_{(\text{crit})-}}{\nu_- - \nu_{(\text{crit})-}} = \frac{\gamma_-}{\gamma_+}. \tag{8.168}$$

Similarly we find

$$\begin{aligned}1 - \nu_+\nu_{(\text{crit})-} &= \frac{\gamma_+\gamma_-(1-\nu_+\nu_-) - 1}{\gamma_+(\gamma_- - \gamma_+)},\\ 1 - \nu_-\nu_{(\text{crit})-} &= -\frac{\gamma_+\gamma_-(1-\nu_+\nu_-) - 1}{\gamma_-(\gamma_- - \gamma_+)},\end{aligned} \tag{8.169}$$

so that

$$\frac{1 - \nu_+\nu_{(\text{crit})-}}{1 - \nu_-\nu_{(\text{crit})-}} = -\frac{\gamma_-}{\gamma_+}. \tag{8.170}$$

Combining (8.168) and (8.170) one has

$$\frac{\nu_+ - \nu_{(\text{crit})-}}{1 - \nu_+\nu_{(\text{crit})-}} = -\frac{\nu_- - \nu_{(\text{crit})-}}{1 - \nu_-\nu_{(\text{crit})-}}. \tag{8.171}$$

Recalling the relativistic formula for the addition of velocities, the above relation implies

$$\nu(U_+, U_{(\text{crit})-}) = -\nu(U_-, U_{(\text{crit})-}), \tag{8.172}$$

that is, the $U_{(\mathrm{crit})-}$ observers are so special that with respect to them the spatial geodesic velocities differ only by a sign. Therefore, splitting of co- and counter-rotating geodesics by $U_{(\mathrm{crit})-}$ observers gives

$$\begin{aligned} U_+ &= \gamma(U_+, U_{(\mathrm{crit})-})[U_{(\mathrm{crit})-} + \nu(U_+, U_{(\mathrm{crit})-})\bar{U}_{(\mathrm{crit})-}], \\ U_- &= \gamma(U_+, U_{(\mathrm{crit})-})[U_{(\mathrm{crit})-} - \nu(U_+, U_{(\mathrm{crit})-})\bar{U}_{(\mathrm{crit})-}], \end{aligned} \tag{8.173}$$

where $\bar{U}_{(\mathrm{crit})-}$ is the unit spatial vector orthogonal to $U_{(\mathrm{crit})-}$ in the Killing 2-plane $(\partial_t, \partial_\phi)$. By adding these equations one gets

$$U_+ + U_- = 2\gamma(U_+, U_{(\mathrm{crit})-})U_{(\mathrm{crit})-} \tag{8.174}$$

or, equivalently,

$$U_{(\mathrm{crit})-} = \frac{U_+ + U_-}{||U_+ + U_-||}, \tag{8.175}$$

which is the most significant characterization of these observers. Since a similar result is obtained by the observers $U_{(\mathrm{crit})+}$ when measuring the spatial velocities of the circular geodesics U_+ and U_-, we deduce that, as a sort of compensation, the gravitational drag does not affect the measurements themselves. Repeating the above derivation for the $U_{(\mathrm{crit})+}$ observers, we can write the more general relation

$$U_{(\mathrm{crit})\pm} = \frac{U_+ \mp U_-}{||U_+ \mp U_-||}. \tag{8.176}$$

Extremely accelerated observers on the equatorial plane are also special because the vanishing of the first torsion of their orbits together with the identical vanishing of the second torsion makes the whole Frenet-Serret angular velocity vanish. In other words the natural Frenet-Serret frame adapted to $U_{(\mathrm{crit})\pm}$ observers is also a Fermi-Walker frame, i.e. with spatial axes aligned with the axes of a gyroscope. In particular the unit vector of the ϕ direction is such that

$$\frac{D_{(\mathrm{fw}, U_{(\mathrm{crit})\pm})}}{d\tau_{U_{(\mathrm{crit})\pm}}} \bar{U}_{(\mathrm{crit})\pm} = 0, \tag{8.177}$$

or, equivalently,

$$\nabla_{U_{(\mathrm{crit})\pm}} \bar{U}_{(\mathrm{crit})\pm} = 0, \tag{8.178}$$

because $\bar{U}_{(\mathrm{crit})\pm} \cdot U_{(\mathrm{crit})\pm} = 0$ and $a(U_{(\mathrm{crit})\pm}) \cdot \bar{U}_{(\mathrm{crit})\pm} = 0$, since the acceleration has only a radial component.

Another interesting geometrical property of the extremely accelerated observers – a consequence of the above result – concerns their relationship with any other equatorial circular orbit u. One can prove the following statement: "As seen by extremely accelerated observers, any equatorial circular orbit u is relatively straight," that is, its relative Fermi-Walker curvature is zero. In fact, we have $\hat{\mathcal{V}}(u, U_{(\text{crit})\pm}) = -\hat{\nu}(u, U_{(\text{crit})\pm}) = \bar{U}_{(\text{crit})\pm}$. Therefore, from (4.46) for the comoving relative Frenet-Serret frames and taking into account (8.177), we have

$$\begin{aligned}\frac{D_{(\text{fw},U_{(\text{crit})\pm})}}{d\ell_{U_{(\text{crit})\pm}}}\bar{U}_{(\text{crit})\pm} &= \mathcal{K}_{(\text{fw},u,U_{(\text{crit})\pm})}\mathcal{N}_{(\text{fw},u,U_{(\text{crit})\pm})}\\ &= 0,\end{aligned} \tag{8.179}$$

where the factor $\gamma(u, U_{(\text{crit})\pm})\nu(u, U_{(\text{crit})\pm})$ has been re-absorbed in the parameter $\ell_{U_{(\text{crit})\pm}}$; hence, $\mathcal{K}_{(\text{fw},u,U_{(\text{crit})\pm})} = 0$.

Observer-dependent embedding diagrams in Kerr space-time

Two-dimensional embedding diagrams have proven to be very valuable in visualizing certain aspects of space-time geometry in stationary axially symmetric space-times. However, an embedding diagram, as well as the associated space-time image, is observer-dependent. In what follows we shall consider families of observers in motion along spatially circular orbits on the equatorial plane of Kerr space-time and study how they would *see* the black hole space-time geometry. Some of these, including static, ZAMO, geodesic, Carter, geodesic meeting point, Lie/Fermi-Walker relatively and comoving relatively straight, and extremely accelerated, have already been extensively studied. In the analysis we are going to perform, certain "new" special families of observers arise.

Let us consider spatially circular orbits in the equatorial plane of Kerr space-time, with tangent vector fields written with respect to ZAMO observers as

$$U = \gamma(U, n)[n + ||\nu(U, n)||\hat{\nu}(U, n)], \tag{8.180}$$

with

$$\begin{aligned}\hat{\nu}(U, n) &= \frac{1}{\sqrt{g_{\phi\phi}}}\partial_\phi, \qquad \hat{\nu}(U, n)^\flat = \sqrt{g_{\phi\phi}}(d\phi + N^\phi dt),\\ ||\nu(U, n)|| &= \sqrt{-g^{tt}g_{\phi\phi}}\left[\zeta + \frac{g_{t\phi}}{g_{\phi\phi}}\right].\end{aligned} \tag{8.181}$$

Let $\bar{U}$ be (minus) the unit vector of the relative velocity of n with respect to U,

$$\bar{U}^\flat = -\hat{\nu}(n, U)^\flat = \gamma(U, n)[||\nu(U, n)||n^\flat + \hat{\nu}(U, n)^\flat], \tag{8.182}$$

and $\omega^{\hat{r}} = \sqrt{g_{rr}}dr$ a unit 1-form in the radial direction. Then the set $\{U^\flat, \omega^{\hat{r}}, \bar{U}^\flat\}$ forms a U-adapted tetrad in the equatorial plane. With respect to the above frame the metric element can be written as

$$ds^2 = -(U^\flat)^2 + (\bar{U}^\flat)^2 + (\omega^{\hat{r}})^2. \tag{8.183}$$

Since $n^\flat = -Ndt$, on a t = constant and θ = constant slice of LRS_U, $(\bar{U}^\flat)^2$ reduces to

$$(\bar{U}^\flat)^2|_{t,\theta=\text{const.}} = [\gamma_{(U,n)}\sqrt{g_{\phi\phi}}d\phi]^2, \tag{8.184}$$

so that the induced metric becomes

$$
\begin{aligned}
{}^{(2)}ds^2_{t,\theta=\text{const.}} &= (\bar{U}^\flat)^2 + (\omega^{\hat{r}})^2|_{t,\theta=\text{const.}} \\
&= \gamma(U,n)^2 g_{\phi\phi} d\phi^2 + g_{rr} dr^2 .
\end{aligned} \tag{8.185}
$$

This in turn can be interpreted as the metric of a 2-surface corresponding to the "image" of the space-time the observers U will construct once the embedding of this surface in a flat Euclidean or Minkowskian three-dimensional space is considered.

Notice that the effect of the relative velocity on the spatial geometry is to increase the circumferences of circles by a gamma factor while keeping radial distances fixed.

To embed the 2-metric (8.185) in a flat 3-metric, let us consider the flat spatial line element written in polar-like coordinates as

$$
{}^{(3)}ds^2 = \pm dZ^2 + dR^2 + R^2 d\phi^2 , \tag{8.186}
$$

where the plus sign refers to the Euclidean case and the minus sign to the Minkowskian one. Let the embedding 2-surface be of the form $Z = Z(R)$ so that the corresponding induced metric becomes

$$
{}^{(2)}ds^2 = h_{RR} dR^2 + R^2 d\phi^2 , \tag{8.187}
$$

where

$$
h_{RR} = \pm \left(\frac{dZ}{dR} \right)^2 + 1 , \tag{8.188}
$$

where the $+$ sign refers to the Euclidean embedding and the $-$ sign to the Minkowskian one. Comparing (8.187) with (8.185), we deduce that

$$
\begin{aligned}
R(r) &= \gamma(U,n)\sqrt{g_{\phi\phi}} , \\
h_{RR} &= g_{rr} \left(\frac{dR}{dr} \right)^{-2} .
\end{aligned} \tag{8.189}
$$

Integrating over the embedding surface $Z = Z(R)$ and using as integration variable the radial Boyer-Lindquist coordinate r, requiring $R = R(r)$, we have

$$
Z_\pm(r) \equiv \int_{r_{(\text{ss})}}^{r} \sqrt{\pm(h_{RR} - 1)} \frac{dR}{dr} dr , \tag{8.190}
$$

where $r_{(\text{ss})}$ is a solution of the equation $h_{RR} = 1$ and therefore marks the signature-switch point. Clearly $Z_\pm(r_{(\text{ss})}) = 0$. A general solution of (8.190) is

$$
Z(r) = H(r - r_{(\text{ss})}) Z_+(r) + H(r_{(\text{ss})} - r) Z_-(r) , \tag{8.191}
$$

where H is the Heaviside step function. Numerical integration easily gives the form of the embedding diagram. This is shown in Fig. 8.3 for ZAMOs, static, and Carter observers.

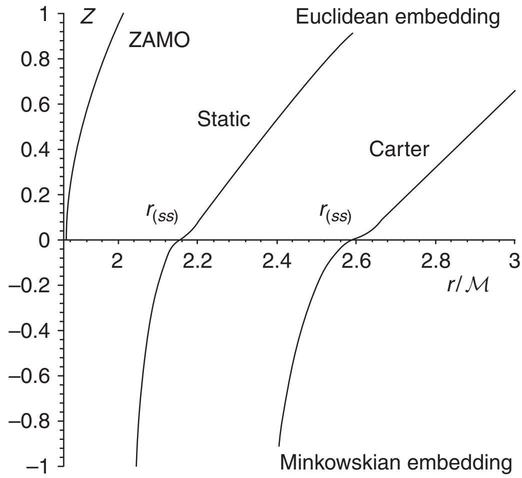

Fig. 8.3. Relative embedding for selected families of observers on the equatorial plane of Kerr space-time.

In the case of Kerr, the (ss) condition $h_{RR} = 1$, namely

$$g_{rr}\left(\frac{dR}{dr}\right)^{-2} = 1, \tag{8.192}$$

can be analytically integrated to get the following expression for R:

$$R_{(\mathrm{ss})}(r) = F(r; \mathcal{M}, a, b) \equiv \sqrt{\Delta} + \mathcal{M}\log\left[\frac{r - \mathcal{M} + \sqrt{\Delta}}{r_+ - \mathcal{M}}\right] + b, \tag{8.193}$$

where r_+ is the outer horizon and b is an arbitrary integration constant. From $(8.189)_1$ and (8.193) we can define a new class of observers $U_{(\mathrm{ss})}$ characterized by

$$\nu(U_{(\mathrm{ss})}, n) = \pm\sqrt{1 - \frac{g_{\phi\phi}}{F(r; \mathcal{M}, a, b)^2}}. \tag{8.194}$$

Assuming we have a maximally extended family of $U_{(\mathrm{ss})}$ observers (from the horizon to spatial infinity), the integration constant b is fixed at $b = 2\mathcal{M}$ since $\nu_{(U_{(\mathrm{ss})}, n)}|_{r=r_+} = 0$. For a non-maximally extended family (i.e. from some observer horizon radius $r_* > r_+$ to infinity) b would depend trivially on r_*.

8.5 Gravitational plane-wave space-time

The metric of a plane monocromatic gravitational wave, elliptically polarized and propagating along a direction which we fix as the x coordinate direction, can be written in *transverse-traceless* (TT) gauge as

$$ds^2 = -dt^2 + dx^2 + (1 - h_+)dy^2 + (1 + h_+)dz^2 - 2h_\times dy dz\,, \tag{8.195}$$

where $h_{+/\times}$ are functions only of $(t-x)$. A physically reasonable observer who could make a measurement is a geodesic one. The time-like geodesics of this metric have been deduced in de Felice (1979); their 4-velocity has the form

$$\begin{aligned} U_{(\mathrm{g})} = & \frac{1}{2E}[(1+f+E^2)\partial_t + (1+f-E^2)\partial_x] \\ & + \frac{1}{1-h_+^2-h_\times^2}\{[\alpha(1+h_+)+\beta h_\times]\partial_y \\ & + [\beta(1-h_+)+\alpha h_\times]\partial_z\}, \end{aligned} \tag{8.196}$$

where α, β, and E are conserved Killing quantities and $f = g_{AB}U^A_{(\mathrm{g})}U^B_{(\mathrm{g})}$ with $A, B = (2, 3)$, is equal to

$$\begin{aligned} f &= (1-h_+)\left(U^y_{(\mathrm{g})}\right)^2 + (1+h_+)\left(U^z_{(\mathrm{g})}\right)^2 - 2h_\times U^y_{(\mathrm{g})}U^z_{(\mathrm{g})} \\ &\simeq \alpha^2(1+h_+) + \beta^2(1-h_+) + 2\alpha\beta h_\times\,, \end{aligned} \tag{8.197}$$

where $\simeq$ denotes the corresponding weak-field limit, i.e. up to first order in h_+ and $h_\times$.

9

Measurements in physically relevant space-times

The aim of modern astronomy is to uncover the properties of cosmic sources by measuring their key parameters and deducing their dynamics. Black holes are targets of particular interest for the role they have in understanding the cosmic puzzles and probing the correctness of current theories. Black holes can be considered simply as deep gravitational potential wells; therefore their existence can only be inferred by observing the behavior of the surrounding medium. The latter can be made of gas, dust, star fields, and obviously light, but all suffer tidal strains and deformations which herald, out of the observer's perspective, the black hole's existence and type. Essential tools for the acquisition of this knowledge are the equations of relative acceleration which stand as basic seeds for any physical measurement. We shall revisit them for specific applications, but will always neglect electric charge in our discussion.

9.1 Measurements in Schwarzschild space-time

Consider a collection of particles undergoing tidal deformations; we shall deduce how these would be measured by any particle of the collection, taken as a fiducial observer. Let us assume that the test particles of the collection move in spatially circular orbits in Schwarzschild space-time whose metric is given by (8.1). Indeed, the physical measurements which can be made in the rest frame of the fiducial observer in the collection are the most natural to be performed in satellite experiments.

Strain-induced rigidity

If U is a unitary time-like vector field whose integral curves form a congruence $\mathcal{C}_U$ parameterized by the proper time τ_U, then a connecting vector field Y over $\mathcal{C}_U$ satisfies the condition

$$\pounds_U Y = 0, \tag{9.1}$$

and is a solution of Eq. (7.64), which we recall here:

$$\frac{D^2 Y^\alpha}{d\tau_U^2} = -R^\alpha{}_{\beta\gamma\delta} U^\beta Y^\gamma U^\delta + Y^\sigma \nabla_\sigma a(U)^\alpha. \tag{9.2}$$

Given a field of orthonormal frames $\{E^{\hat\alpha}\}$, (9.2) can be written in tetrad components as (see (7.78))

$$\ddot{Y}^{\hat a} + \mathcal{K}_{(U,\,E)}{}^{\hat a}{}_{\hat b} Y^{\hat b} = 0, \tag{9.3}$$

where the deviation matrix $\mathcal{K}_{(U,\,E)}{}^{\hat a}{}_{\hat b}$ is given by

$$\mathcal{K}_{(U,\,E)}{}^{\hat a}{}_{\hat b} = [T_{(\mathrm{fw},\,U,\,E)} - S(U) + \mathcal{E}(U)]^{\hat a}{}_{\hat b}. \tag{9.4}$$

All quantities in (9.4) bear a physical meaning as we have already discussed in Chapter 7: $T_{(\mathrm{fw},\,U,\,E)}$ is the twist tensor (defined in (7.77)), $S(U)$ is the Fermi-Walker strain tensor (defined in (7.76)), and $\mathcal{E}(U)$ is the electric part of the Riemann tensor.

Here we specify the vector field U as being tangent to a family of equatorial spatially circular orbits as in (8.110), that is

$$U = \Gamma(\partial_t + \zeta\partial_\phi), \tag{9.5}$$

with Γ given by (8.27), with $\theta = \pi/2$ and $\zeta =$ constant over $\mathcal{C}_U$.

Let us choose as a tetrad field adapted to U the following (see (8.53)):

$$\begin{aligned} E_{\hat t} &= U = \Gamma(\partial_t + \zeta\partial_\phi),\\ E_{\hat r} &= \left(1 - \frac{2\mathcal{M}}{r}\right)^{1/2} \partial_r,\\ E_{\hat\theta} &= \frac{1}{r}\partial_\theta,\\ E_{\hat\phi} &= \bar\Gamma(\partial_t + \bar\zeta\partial_\phi), \end{aligned} \tag{9.6}$$

where

$$\bar\Gamma = \frac{\Gamma\zeta r}{[1 - (2\mathcal{M}/r)]^{1/2}}, \qquad \bar\zeta = \frac{1}{\zeta r^2}\left(1 - \frac{2\mathcal{M}}{r}\right). \tag{9.7}$$

The various quantities involved are listed below.

(i) The physical components of the acceleration:

$$a(U)_{\hat a} = \Gamma^2 \left(1 - \frac{2\mathcal{M}}{r}\right)^{1/2} \left(\frac{\mathcal{M}}{r^2} - \zeta^2 r\right) \delta_{\hat a \hat r}\,. \tag{9.8}$$

(ii) The expansion tensor:

$$\theta(U)_{\hat a\hat b} \equiv 0, \tag{9.9}$$

showing that the above congruence is Born-rigid.

(iii) The vorticity vector, whose only non-vanishing component is

$$\omega(U)^{\hat\theta} = -\Gamma^2\zeta\left(1-\frac{3\mathcal{M}}{r}\right), \tag{9.10}$$

and is coincident with the Fermi rotation vector $\zeta_{(\mathrm{fw})\hat a}$.

(iv) The Fermi-Walker strain tensor $S(U)$, which, from (7.76), is given by

$$\begin{aligned} S(U)_{\hat a\hat b} &= \nabla(U)_{\hat b}a(U)_{\hat a} + a(U)_{\hat a}a(U)_{\hat b} \\ &= \mathrm{diag}[S(U)_{\hat r\hat r}, S(U)_{\hat\theta\hat\theta}, S(U)_{\hat\phi\hat\phi}], \end{aligned} \tag{9.11}$$

where

$$\begin{aligned} S(U)^{\hat r\hat r} &= \Gamma^4\left\{(\zeta_K^2-\zeta^2)^2\mathcal{M}r + (\zeta_K^2-\zeta^2)\left[\left(1-\frac{3\mathcal{M}}{r}\right)\left(1-\frac{\mathcal{M}}{r}\right)\right.\right.\\ &\quad \left.\left. -3\zeta_K^2 r^2\left(1-\frac{2\mathcal{M}}{r}\right)\right] - 3\zeta_K^2\left(1-\frac{2\mathcal{M}}{r}\right)\left(1-\frac{3\mathcal{M}}{r}\right)\right\},\\ S(U)^{\hat\theta\hat\theta} &= \Gamma^2\left[\zeta_K^2 - \frac{2\mathcal{M}}{r}(\zeta_K^2-\zeta^2)\right],\\ S(U)^{\hat\phi\hat\phi} &= -\mathcal{M}r\Gamma^4(\zeta^2-\tilde\zeta^2)(\zeta_K^2-\zeta^2), \end{aligned} \tag{9.12}$$

with

$$\zeta_K^2 = \frac{\mathcal{M}}{r^3}, \qquad \tilde\zeta^2 = \frac{1}{\mathcal{M}r}\left(1-\frac{2\mathcal{M}}{r}\right)^2. \tag{9.13}$$

(v) The twist tensor $T_{(\mathrm{fw},\,U,\,E)}$ which in our case, from (7.77), is given by

$$\begin{aligned} T_{(\mathrm{fw},\,U,\,E)}{}^{\hat a}{}_{\hat b} &= \delta^{\hat a}_{\hat b}\zeta^2_{(\mathrm{fw})} - \zeta^{\hat a}_{(\mathrm{fw})}\zeta_{(\mathrm{fw})\hat b} - 2\epsilon^{\hat a}{}_{\hat f\hat c}\zeta^{\hat f}_{(\mathrm{fw})}\omega(U)^{\hat c}{}_{\hat b} \\ &= \mathrm{diag}[C, 0, C], \end{aligned} \tag{9.14}$$

where

$$C = \zeta^2\Gamma^4\left(1-\frac{3\mathcal{M}}{r}\right)^2 = ||\zeta_{(\mathrm{fw})}||^2. \tag{9.15}$$

(vi) The electric part of the Riemann tensor $\mathcal{E}(U)^{\hat a}{}_{\hat b}$ (restricted to the equatorial plane) which has only the following non-vanishing components:

$$\begin{aligned} \mathcal{E}(U)_{\hat r\hat r} &= -\frac{\mathcal{M}}{r^3}\Gamma^2\left[2\left(1-\frac{2\mathcal{M}}{r}\right)+\zeta^2r^2\right],\\ \mathcal{E}(U)_{\hat\theta\hat\theta} &= \frac{\mathcal{M}}{r^3}\Gamma^2\left[\left(1-\frac{2\mathcal{M}}{r}\right)+2\zeta^2r^2\right],\\ \mathcal{E}(U)_{\hat\phi\hat\phi} &= \frac{\mathcal{M}}{r^3}. \end{aligned} \tag{9.16}$$

After some algebra, we obtain

$$
\begin{aligned}
\frac{d^2Y^{\hat r}}{d\tau_U^2} &= \left\{\frac{\mathcal{M}}{r^3}\Gamma^2\left[2\left(1-\frac{2\mathcal{M}}{r}\right)+\zeta^2 r^2\right]+S(U)^{\hat r\hat r}+C\right\}Y_{\hat r}\equiv 0,\\
\frac{d^2Y^{\hat \theta}}{d\tau_U^2} &= \left\{-\frac{\mathcal{M}}{r^3}\Gamma^2\left[2\zeta^2 r^2+\left(1-\frac{2\mathcal{M}}{r}\right)\right]+S(U)^{\hat\theta\hat\theta}\right\}Y_{\hat\theta}\equiv 0,\\
\frac{d^2Y^{\hat \phi}}{d\tau_U^2} &= \left[-\frac{\mathcal{M}}{r^3}+S(U)^{\hat\phi\hat\phi}+C\right]Y_{\hat\phi}\equiv 0,
\end{aligned}
\tag{9.17}
$$

consistent with the rigidity condition (9.9). The terms which ensure this rigidity are the components of the Fermi-Walker strain tensor $S(U)^{\hat i\hat i}(\hat i = r,\,\theta,\,\phi)$ which balance the effects of both the curvature and the centrifugal effects generated by the Fermi rotation of the tetrad (term C in (9.15) and (9.17)). Crucial to this compensation is the requirement that the physical frame carried by the observer (the fiducial particle of the system) is exactly the frame (9.6) all along the orbit. The operational fulfillment of this requirement can only be achieved if the observer is able to identify in his rest frame and without ambiguity the radial, azimuthal, and latitudinal directions. A tetrad whose spatial axes remain parallel to the above directions is termed *phase-locked*, as stated in (8.53) (de Felice, 1991; de Felice and Usseglio-Tomasset, 1991). The relative strains $S(U)^{\hat i\hat i}$ can be balanced by springs, for example, connecting discrete particles, or by the internal structure of a configuration like a star in the case of a fluid. Let us now see what information arises from a measurement of strains.

The radial direction is identified by the direction of the thrust when the orbit is not a geodesic. In the case of a geodesic motion the thrust is zero; hence the identification of the frame (9.6) would require an approach different from what we shall pursue here. If the observer moves along a general spatially circular orbit, the thrust is constant and, as stated, fixes locally the radial direction with respect to the center of the gravitational potential. Along this direction, the relative strain $S(U)^{\hat r\hat r}$ in (9.12) is always negative, meaning that the spring – or whatever other mechanism one considers to ensure rigidity – exerts a compression (see Fig. 9.1).

This behavior is expected since in our case both the curvature and the Fermi rotation cause a stretch at all values of r. However, the measurement of the radial strain alone does not allow the observer to decide whether he is actually moving in a circular orbit or is at rest with respect to the coordinate grid. This uncertainty can be overcome with other types of measurements.

The azimuthal strain $S(U)^{\hat\phi\hat\phi}$ can be written from (9.12) as

$$
S(U)^{\hat\phi\hat\phi} = -\zeta_K^2\frac{(\zeta^2-\tilde\zeta^2)(\zeta_K^2-\zeta^2)}{[(1-3\mathcal{M}/r)/r^2+(\zeta_K^2-\zeta^2)]^2},
\tag{9.18}
$$

and its plot is shown in Fig. 9.1. It vanishes when $\zeta^2=\zeta_K^2$ and $\zeta^2=\tilde\zeta^2$, this being the manifestation of an exact balancing of the curvature (tidal) compression with the centrifugal stretch induced by the Fermi rotation. It is negative (meaning a

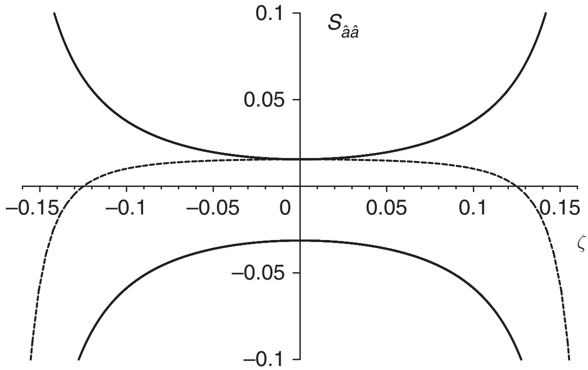

Fig. 9.1. Behavior of the relative strains $S(U)^{\hat r\hat r}$ (lower curve), $S(U)^{\hat\theta\hat\theta}$ (upper curve), and $S(U)^{\hat\phi\hat\phi}$ (middle curve) as functions of ζ for fixed values of $r = 4\mathcal{M}$. The physical region is $\zeta^2 < \zeta_c^2$.

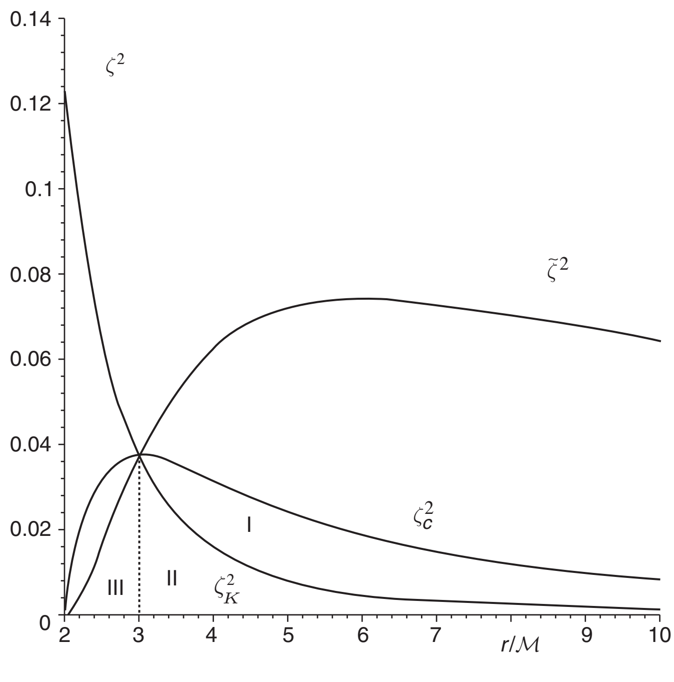

Fig. 9.2. The points with $\zeta_c^2 > \zeta^2 \geq 0$ are physically allowed equatorial circular orbits in Schwarzschild space-time. The relative strains have different signs in the areas delimited by the curves ζ_c^2, ζ_K^2, $\tilde\zeta^2$, and $r = 3\mathcal{M}$.

compression) when the centrifugal stretch overcomes the tidal compression, and this occurs when $\tilde\zeta^2 > \zeta^2 > \zeta_K^2$ and $\tilde\zeta^2 < \zeta^2 < \zeta_K^2$; it is positive (meaning a stretch) when the tidal compression overcomes the centrifugal stretch, and this occurs when $\zeta^2 < \zeta_K^2 < \tilde\zeta^2$ for $r > 3\mathcal{M}$ and $\zeta^2 < \tilde\zeta^2 < \zeta_K^2$ for $r < 3\mathcal{M}$ (Fig. 9.2).

At $r = 3\mathcal{M}$ we have $\zeta_K = \tilde{\zeta}$; hence $S(U)^{\hat{\phi}\hat{\phi}}|_{3\mathcal{M}} = \zeta_K^2|_{3\mathcal{M}}$, independent of ζ. Since the Fermi rotation of the frame is zero at $r = 3\mathcal{M}$, the azimuthal strain only needs to balance a tidal compression; hence the $\hat{\phi}$-component of the tidal curvature tensor is itself independent of ζ. The above result is consistent with the independence of the thrust from ζ, as first noticed by Abramowicz and Lasota (1974). If the observer is at rest with respect to infinity, i.e. if $\zeta = 0$, then $S(U)^{\hat{\phi}\hat{\phi}} = \zeta_K^2 > 0$, meaning a stretch, so the measurement of this strain would not help us understand whether the observer is at rest or not.

The relative strain in the latitudinal direction is, from (9.12), given by

$$S(U)^{\hat{\theta}\hat{\theta}} = \frac{\zeta_K^2 - \frac{2\mathcal{M}}{r}(\zeta_K^2 - \zeta^2)}{r^2(\zeta_c^2 - \zeta^2)}. \tag{9.19}$$

As can be seen from Fig. 9.1, $S(U)^{\hat{\theta}\hat{\theta}}$ is always positive, meaning that in the θ direction a stretch is needed to ensure rigidity. If the observer is at rest, then from (9.12) we would have $S(U)^{\hat{\theta}\hat{\theta}} = \zeta_K^2 = S(U)^{\hat{\phi}\hat{\phi}} > 0$. In this case, if one knows the local radial direction, the measurement of the strains in any two directions orthogonal to each other and to the radial one would unambiguously indicate that the observer is at rest, if these two strains are positive and equal to each other, independent of rotation in the plane orthogonal to the radial direction.

From the above analysis it follows that a direct measurement of the radial $S(U)^{\hat{r}\hat{r}}$, latitudinal $S(U)^{\hat{\theta}\hat{\theta}}$, and azimuthal $S(U)^{\hat{\phi}\hat{\phi}}$ strains allows one to deduce in general that the observer is orbiting around a gravitational source (a black hole, say), but they are not sufficient to let the observer recognize where in the (ζ^2, r)-plane of Fig. 9.2 he is actually orbiting. In fact, a measurement of the azimuthal strain $S(U)^{\hat{\phi}\hat{\phi}}$ would not allow the observer to distinguish between orbiting with ζ in the range $\zeta_c^2 > \zeta^2 > \zeta_K^2$ and in the range $\zeta_c^2 > \zeta^2 > \tilde{\zeta}^2$. In both cases, in fact, $S(U)^{\hat{\phi}\hat{\phi}}$ is negative (meaning a compression). Clearly, if he can vary ζ and r then he would recognize that he was crossing the line $\zeta^2 = \tilde{\zeta}^2$ in the (ζ^2, r)-plane if $S(U)^{\hat{\phi}\hat{\phi}}$ vanishes but *not* the thrust, implying that the corresponding circular orbit is not a geodesic.

Partially constrained circular motion

Let us now relax the rigidity condition imposed on the collection of particles and allow free motion in the $\hat{\theta}$ direction only. This condition is ensured by requiring that $a(U)^{\hat{\theta}} = 0$ and $\partial_{\hat{\theta}} a(U)^{\hat{\theta}} = 0$ for any θ. This particular state of motion is operationally set up by forcing the monitored particle to move inside a frictionless narrow pipe fixed in the θ direction. Calculating the above constraints from (8.28) and imposing the resulting condition in $(9.17)_2$, the relative acceleration in the $\hat{\theta}$ direction becomes (de Felice and Usseglio-Tomasset, 1992)

$$\frac{d^2Y^{\hat{\theta}}}{d\tau_U^2} = \left\{-\frac{\mathcal{M}}{r^3}\Gamma^2\left[2\zeta^2r^2 + \left(1-\frac{2\mathcal{M}}{r}\right)\right]\right.$$
$$\left. +\Gamma^2\left(1-\frac{2\mathcal{M}}{r}\right)(\zeta_K^2-\zeta^2)\right\}Y^{\hat{\theta}} = -\zeta^2\Gamma^2Y^{\hat{\theta}}. \tag{9.20}$$

The particle appears to be acted on by an elastic type of force which pulls it towards the equatorial plane if it was initially out of it. Then, a particle constrained to move inside a pipe without friction, aligned in an $r =$ constant line and perpendicular to the plane $\theta = \pi/2$, will undergo harmonic oscillations with a frequency equal to the proper frequency of the orbital revolution,

$$|\overset{p}{\zeta}| \equiv \left|\frac{d\phi}{d\tau_U}\right| = |\zeta|\Gamma. \tag{9.21}$$

Let us see what type of trajectory is described by the particle constrained in the pipe.

In Schwarzschild space-time a geodesic, not confined to the equatorial plane, will lie on a plane which has some inclination with respect to the equatorial one. The orbital plane is fixed by the initial conditions. These can be expressed in terms of:

(a) the conserved Killing quantities, namely $\tilde{\gamma}$, the total energy in units of μ_0c^2 (μ_0 being the particle mass and c the speed of light in vacuum), and λ, the azimuthal angular momentum in units of μ_0c;

(b) the separation constant of the Hamilton-Jacobi equation Λ, which is the total angular momentum in units of μ_0c.

These enter the geodesic equations, giving (de Felice and Clarke, 1990)

$$\begin{aligned}
\dot{t} &= \tilde{\gamma}\left(1-\frac{2\mathcal{M}}{r}\right)^{-1},\\
\dot{r} &= \pm\left[\tilde{\gamma}^2 - 1 + \frac{2\mathcal{M}}{r} - \frac{\Lambda^2}{r^2}\left(1-\frac{2\mathcal{M}}{r}\right)\right]^{1/2},\\
\dot{\theta} &= \pm\frac{1}{r^2}\left[\Lambda^2 - \frac{\lambda^2}{\sin^2\theta}\right]^{1/2},\\
\dot{\phi} &= \frac{\lambda}{r^2\sin^2\theta},
\end{aligned} \tag{9.22}$$

dot meaning derivative with respect to the proper time on the geodesics. The requirement that the observer (the fiducial particle of the collection) moves on a spatially circular geodesic confined to the equatorial plane and with radius r_0, say, is summarized by the following conditions:

$$r = r_0, \quad \dot{r} = 0, \quad \theta = \frac{\pi}{2}, \quad \dot{\theta} = 0,$$
$$\tilde{\gamma}_0^2 = \frac{\left(1-\frac{2\mathcal{M}}{r_0}\right)^2}{\left(1-\frac{3\mathcal{M}}{r_0}\right)}, \quad \Lambda_0^2 = \frac{\mathcal{M}r_0}{\left(1-\frac{3\mathcal{M}}{r_0}\right)}, \quad \lambda_0 = \Lambda_0. \tag{9.23}$$

Let the monitored particle of the collection move on a spatially circular geodesic with $r' = r_0$, $\tilde{\gamma}' = \tilde{\gamma}_0$, $\Lambda' = \Lambda_0$ but on a plane inclined with respect to the equatorial one by an angle $\theta' = \sin^{-1}(\lambda'/\Lambda_0)$, for any chosen $\lambda' < \Lambda_0$. In this case we would have, from (9.22) and recalling that $\zeta = d\phi/dt = \dot{\phi}/\dot{t}$,

$$\begin{aligned}\zeta_0 &= \left(1 - \frac{2\mathcal{M}}{r_0}\right) \frac{\lambda_0}{r_0^2 \tilde{\gamma}_0}, \\ \zeta' &= \left(1 - \frac{2\mathcal{M}}{r_0}\right) \frac{\Lambda_0 \sin\theta'}{r_0^2 \tilde{\gamma}_0 \sin^2\theta} = \zeta_0 \frac{\sin\theta'}{\sin^2\theta}\,;\end{aligned} \tag{9.24}$$

hence ζ' varies with θ as the particle moves in its orbit. We require, however, that the particle move rigidly with the observer in the r and ϕ directions, that is, $r = r_0$, $\dot{r} = 0$, and $\zeta' \equiv \zeta'_{(\mathrm{rig})} = \zeta_0$. Hereafter in this section $\zeta_0 \equiv \zeta_\phi$ unless otherwise specified. It follows from $\zeta' = \zeta_0$ that $\lambda' = \Lambda_0 \sin^2\theta \equiv \lambda'_{(\mathrm{rig})}$, which breaks the geodesic character of the particle's orbit. The physical properties of the constrained particle are then given by

$$\begin{aligned}{\gamma'_{(\mathrm{rig})}}^2 &= \tilde{\gamma}_0^2 = \frac{\left(1 - \frac{2\mathcal{M}}{r_0}\right)^2}{\left(1 - \frac{3\mathcal{M}}{r_0}\right)}, \\ \Lambda'^2_{(\mathrm{rig})} &= \Lambda_0^2 = \frac{\mathcal{M} r_0}{\left(1 - \frac{3\mathcal{M}}{r_0}\right)}, \\ \lambda'^2_{(\mathrm{rig})} &= \Lambda_0^2 \sin^4\theta = \frac{\mathcal{M} r_0 \sin^4\theta}{\left(1 - \frac{3\mathcal{M}}{r_0}\right)}.\end{aligned} \tag{9.25}$$

Using these quantities in (9.22) yields the non-geodesic trajectory which is followed by the particle inside the pipe:

$$\begin{aligned}\dot{t}' &= \left(1 - \frac{3\mathcal{M}}{r_0}\right)^{-1/2}, \\ \dot{\theta}' &= \pm\left(\frac{\mathcal{M}}{r_0^3}\right)^{1/2} \left(1 - \frac{3\mathcal{M}}{r_0}\right)^{-1/2} \cos\theta, \\ \dot{\phi}' &= \left(\frac{\mathcal{M}}{r_0^3}\right)^{1/2} \left(1 - \frac{3\mathcal{M}}{r_0}\right)^{-1/2} = \text{constant}, \\ \dot{r}' &= 0.\end{aligned} \tag{9.26}$$

These are the components of a unitary 4-velocity

$$\begin{aligned}u^\alpha &= \Gamma' \left(\delta^\alpha_t + \zeta'_\theta\, \delta^\alpha_\theta + \zeta_\phi\, \delta^\alpha_\phi\right) \\ &= \left(\sqrt{1 - \frac{3\mathcal{M}}{r_0}}\right)^{-1} \left[\delta^\alpha_t \pm \sqrt{\frac{\mathcal{M}}{r_0^3}} \cos\theta\, \delta^\alpha_\theta + \sqrt{\frac{\mathcal{M}}{r_0^3}}\, \delta^\alpha_\phi\right].\end{aligned} \tag{9.27}$$

Here we recognize the latitudinal frequency $\zeta'_\theta = \zeta_\phi \cos\theta$, which justifies the harmonic oscillations seen inside the pipe.

9.2 The problem of space navigation

The oscillations of the particle in the pipe allow us to know the magnitude of the proper angular velocity of the orbital revolution, $|\overset{p}{\zeta}| = \Gamma|\zeta|$. We shall now solve the problem of determining the sign of $\overset{p}{\zeta}$. The first obvious consideration is that a measurement of $|\overset{p}{\zeta}|$ will directly tell whether the observer is moving or at rest with respect to infinity, but a direct measurement of $|\overset{p}{\zeta}|$ is not sufficient to determine ζ, unless one knows the factor Γ. Suppose that a photon strikes the orbiting frame along the radial direction with a measurable frequency shift $1+z$; then by definition we have

$$\Gamma = \frac{1}{1+z} \tag{9.28}$$

and so, from (9.21),

$$|\zeta| = |\overset{p}{\zeta}|(1+z). \tag{9.29}$$

Expression (9.29) provides a relation between a quantity which can only be measured at infinity, namely ζ, and quantities which can be measured *in the vicinity* of a black hole, namely $\overset{p}{\zeta}$ and z; the first of these can be measured by reading the clock of the orbiting observer, and the second by means of a spectrograph. Indeed, relations of this type are of crucial importance in astrophysics.

Relation (9.29) provides only the magnitude of ζ, but it would be more useful to know the angular velocity of revolution *with its sign* relative to a local clockwise direction, for example. The operational acquisition of this information will now be discussed.

Let us assume that the collection of particles considered in the previous section represents a space-ship and that the fiducial particle which moves along a circular orbit on the plane $\theta = \pi/2$ is the ship commander, our observer. When this observer is unable to interact with anything outside the space-ship, then his orientation in the space-time relies entirely on the measurements which can be performed within the space-ship itself.

We then pose the question: what is the minimum amount of a-priori information about the global space-time structure which is necessary to avoid getting lost? In de Felice (1991) the problem of recognizing from within the space-ship the direction in which the black hole lies was solved by making the concept of *inward* operationally well defined. The generalization to an arbitrary stationary space-time is conceptually very similar to the present discussion.

As a starting point one realizes that at least two pieces of a-priori information are needed, namely the background metric and the type of orbits in which the observer is moving; the former will be the Schwarzschild metric and the orbits will be the spatially circular ones. It is possible to perform a set of measurements which enable the observer to deduce the above information; we shall not analyze them here but will assume for simplicity that they are given a priori. In what follows we shall describe the measurements which can be performed within an orbiting space-ship on the basis of the above-mentioned information. The first step is to fix a frame adapted to the given observer.

Setting the frame

A suitable frame of reference which can be adapted to an observer described by the 4-velocity (9.5) and moving around a Schwarzschild black hole, for example, is a phase-locked one given by (9.6). We shall now show how the observer can fix this frame at any point of his orbit in the plane $\theta = \pi/2$.

As stated, the observer can directly measure the thrust needed to remain in a circular orbit; hence the direction of the thrust identifies the local radial direction, namely that of the $E_{\hat{r}}$-leg of the frame. In order to fix the $E_{\hat{\theta}}$-leg one needs to monitor the behavior of a particle free to move in a pipe with negligible internal friction and perpendicular to the local radial direction, being otherwise constrained to move rigidly with the rest of the ship. The $E_{\hat{\theta}}$-leg of the frame is identified as the direction of the pipe, on the equatorial plane, when the particle inside is seen to perform harmonic oscillations. In this case one also identifies the plane of the orbit as the plane which is perpendicular to the local θ direction and contains the radial direction. The direction of the remaining $E_{\hat{\phi}}$-leg of the frame is unambiguously fixed orthogonally to the others. Of course the complete setting of the frame requires that one fix a positive sense on the axes, for example on each of the spatial legs.

Determining where the black hole is

Let the space-ship be a box with an engine applied to each of the sides perpendicular to the orbital plane. Since the orbit is not in general a geodesic, one of the engines on a side perpendicular to the radial direction will be on. If the space-time is given by the Schwarzschild solution exterior to a black hole, then the circular orbit will be in one of the following regions of the permitted (ζ^2, r)-plane (area below the curve ζ_c^2 in Fig. 9.2):

- Region I: $\zeta_K^2 < \zeta^2 < \zeta_c^2$;
- Region II: $\zeta^2 < \zeta_K^2 < \zeta_c^2$;
- Region III: $\zeta^2 < \zeta_c^2 < \zeta_K^2$.

Let us now remember that in a Schwarzschild background the frequency of the harmonic oscillations inside the pipe is equal to the proper frequency of revolution $|\overset{p}{\zeta}|$, as shown in (9.20). The a-priori knowledge of being in Schwarzschild space-time allows one to interpret the following behavior of the thrust: if the radius of the orbit is kept constant, an *increase* in the magnitude of the proper angular velocity of revolution $|\overset{p}{\zeta}|$ – equivalently an increase in $|\zeta|$ – will cause

- an increase in the thrust if the observer is in region I; then the thrust points to the center of symmetry;
- a decrease in the thrust if the observer is in region II; then the thrust points away from the center of symmetry;
- an increase in the thrust if the observer is in region III; then the thrust points away from the center of symmetry.

We see clearly that monitoring the thrust alone would not permit the observer to distinguish between regions I and III, although he could recognize at once that he was in region II and in that case deduce the local inward direction. One must have an independent way to recognize which of the above regions the space-ship occupies. As implied by the previous discussion, the observer must be able to change the angular velocity of revolution without changing the radius of the orbit, at least not in an appreciable way. To this end we shall describe the following experimental device. The main thrust fixes the radial direction from within the space-ship while the direction of the harmonic oscillations inside the pipe fixes the plane of the orbit as the one orthogonal to it. Then let a set of rails go across the space-ship orthogonal to the radial direction and parallel to the orbital plane. The rails will be approximately parallel to the orbit, at least within the space-ship. Let us have a small box, of negligible mass, which slides freely on these rails. Inside the box we have a test mass μ_0 linked by a spring to one side of the box which is perpendicular to the direction of the main thrust. The spring is rigidly fastened to that side of the box, so it can stretch or compress but not bend transversely. The acceleration produced by the main thrust is transferred to the mass μ_0 by the spring, which is then stretched or compressed by a given amount and is regulated so that the test mass remains at the center of the box. We assume that this position of the test mass corresponds to the average radius of the orbit, although its actual coordinate value is not known. The whole set orbits rigidly with proper angular velocity $|\overset{p}{\zeta}|$. Let us now give an impulse to the small box parallel to the rails; the small box will then slide on the rails and consequently the mass μ_0 will acquire a new angular velocity ζ with respect to infinity or, equivalently, will be acted upon by a Coriolis force with respect to the orbiting frame. In this case the initial acceleration exerted by the spring on the mass will no longer be sufficient to keep it in equilibrium at the center of the box. The spring, in fact, will either stretch or compress with respect to its initial state. If it is stretched, then it means that in order to remain at the center of the box the particle needs a bigger acceleration if it was initially stretched or a smaller one if it was initially compressed. If on the contrary the spring is compressed, then it means that the mass needs either a bigger or a smaller acceleration according to whether it was initially compressed or stretched, respectively.

In all cases, by readjusting the elastic properties of the spring, one can measure the extra acceleration needed to keep the mass μ_0 at the center of the box. This test case can be exploited to vary the angular velocity of revolution of the entire space-ship, keeping the radius of the orbit unchanged. To do this, let us give the space-ship an impulse transverse to the main thrust and in the same sense of the one given to the box, but with a magnitude corrected by the ratio between the mass of the space-ship and that of the test mass, to ensure the same acceleration. Then, simultaneously vary the main thrust in the same sense along which we accelerated the spring to keep the mass μ_0 at rest at the center of the box, but again corrected by the same mass ratio. With this procedure one can modify the angular velocity of revolution, keeping the radius of the orbit unchanged. Once stabilized in the new regime which follows the application of the transverse impulse and the correction of the main thrust, one can measure the new angular velocity of revolution, monitoring the harmonic oscillation of the particle in the pipe, and determine whether $|\overset{p}{\zeta}|$ has increased or decreased. The knowledge of the thrust as function of $|\overset{p}{\zeta}|$ at a fixed orbital radius allows the observer to decide, with no ambiguity, in which of the regions I, II, or III the orbit is. In fact, the orbit will be in

- Region I, if a decrease in $|\overset{p}{\zeta}|$ corresponds a decrease in the thrust, which can be reduced to zero (the orbit reduces to a geodesic) before $|\overset{p}{\zeta}|$ vanishes.
- Region II, if a decrease in $|\overset{p}{\zeta}|$ corresponds an increase in the thrust.
- Region III, if a decrease in $|\overset{p}{\zeta}|$ corresponds a decrease in the main thrust, which nonetheless remains non-zero when $|\overset{p}{\zeta}| \to 0$.

From the above experimental results, we can give the following definition: the *inward* direction to the center of symmetry is

(i) concordant with the sense of the thrust if the observer is in region I;
(ii) opposite to the sense of the thrust if the observer is in regions II or III.

This local definition of inwards to the center of symmetry is consistent with the global definition of inwards as it would be decided upon by an observer at infinity.

The unambiguous identification of the inward direction allows the observer to fix the sense of the orbital revolution with respect to a local clockwise direction. The sign of $\overset{p}{\zeta}$ is uniquely defined by the simultaneous knowledge of the direction along which $|\overset{p}{\zeta}|$ increases and the direction to the center of symmetry. One can then set up a dial parallel to the orbital plane and comoving with the space-ship; moreover a pointer can be set to rotate on the dial with the angular velocity $|\overset{p}{\zeta}|$ in the sense opposite to that of the space-ship as experimentally established. In the rest frame of the observer a 2π turn of the pointer on the dial will correspond to a complete orbital revolution of the space-ship. Indeed, this apparatus permits the local determination of the gyroscopic precession.

Probing the strength of the gravitational field

The thrust needed to keep the orbit circular is the result of a lack of balance between a gravitational attraction and a centrifugal repulsion with respect to the center of symmetry. The observer, however, cannot distinguish between these two components; hence he would find it difficult to decide how *strong* the gravitational field is at the given orbital radius as compared to the centrifugal one. However, a signature of the gravitational strength is provided by the behavior of the axis of a gyroscope (de Felice, 1991).

Postponing to Chapter 10 a detailed analysis of gyroscopic precession, we consider here the implications of knowing it a priori. From (10.66) we can write the angular velocity of gyroscopic precession as (Rindler and Perlick, 1990)

$$\overset{g}{\zeta} = \overset{p}{\zeta}\Gamma\left(\frac{3\mathcal{M}}{r} - 1\right). \tag{9.30}$$

We see that, while in a flat space-time the special relativistic Thomas precession of a gyroscope moving in a spatially circular orbit is backward with respect to the angular velocity of revolution (Eq. (9.30) with $\mathcal{M} = 0$), the contribution to the precession from a gravitational source of mass $\mathcal{M}$ is such as to make it increase from backward to forward.

From (9.30) we see that in regions I and II where $r > 3\mathcal{M}$ the ratio $\overset{g}{\zeta}/\overset{p}{\zeta}$ is negative, so the axis of the gyroscope rotates in a sense opposite to that of the orbital revolution, and the precession is said to be *backward*. In region III where $r < 3\mathcal{M}$ the ratio $\overset{g}{\zeta}/\overset{p}{\zeta}$ is positive, so the axis of the gyroscope rotates in the same sense as $\overset{p}{\zeta}$, and the precession is said to be *forward*. Moreover, since the gyroscopic precession increases from backward to forward as we approach the source (r decreases), while it does the opposite when we move away from it, we can establish a correspondence between the precession of a gyroscope and the strength of the gravitational field, namely: *an increase in the gyroscopic precession from backward to forward with respect to the proper frequency of revolution implies an increase in the gravitational strength.*

Since (time-like) circular geodesics are allowed only at $r > 3\mathcal{M}$ but such that $\zeta_K^2 \to \zeta_c^2$ as $r \to 3\mathcal{M}$, we can then establish a correspondence between this and the gyroscopic precession becoming less and less backward ($\overset{g}{\zeta} \to 0$) as $r \to 3\mathcal{M}$. As a consequence, no centrifugal compensation to the gravitational component of the thrust, to allow for geodesics, is possible with any $\overset{p}{\zeta}$. Hence the behavior of the gyroscope seems to suggest that when its precession becomes forward, i.e. in the region III with $r < 3\mathcal{M}$, the angular velocity of revolution $\overset{p}{\zeta}$ contributes to the gravitational component of the thrust more than it does to the centrifugal one. The opposite holds elsewhere (see de Felice, 1991; de Felice and Usseglio-Tomasset, 1992; Semerák, 1996). The above considerations show how important it is to measure the gyroscopic precession and decide whether it is backward or forward with respect to the sense of the orbital revolution. Since the latter can be determined, then the following experimental device permits a direct measurement of $\overset{g}{\zeta}$.

Consider again the pointer rotating with frequency $-\overset{p}{\zeta}$. If there was no gravitational source ($\mathcal{M} = 0$) then the axis of the gyroscope, which was taken to coincide with the pointer at some instant, would be seen to rotate with respect to the dial with the same angular frequency $-\overset{p}{\zeta}$ as the pointer; hence it would be at rest with respect to the pointer itself. This is the case of maximum backward precession. In the presence of gravity ($\mathcal{M} \neq 0$) the axis of the gyroscope will be seen to *delay* with respect to the pointer; in particular,

(a) the value of $|\overset{g}{\zeta}|$ can be read directly, comparing the rate of rotation of the axis of the gyroscope with respect to the dial;
(b) the sign of $\overset{g}{\zeta}$, i.e. the type of precession, will be: *backwards* with respect to the orbital sense of revolution if the axis of the gyroscope rotates with respect to the dial in the same sense as that of the pointer (remember that the latter is kept rotating with frequency $-\overset{p}{\zeta}$); *forward* if the axis of the gyroscope is seen to rotate in the opposite sense to that of the pointer on the dial.

Equation (9.30) establishes a relation among the proper angular velocity of revolution, the normalization factor of the observer 4-velocity, and the angular velocity of the gyroscopic precession. This implies that the simultaneous knowledge of $\overset{g}{\zeta}$, $\overset{p}{\zeta}$, and Γ allows an observer orbiting a Schwarzschild black hole to deduce the ratio

$$\frac{2\mathcal{M}}{r} = \frac{2}{3}\left[1 + \frac{\overset{g}{\zeta}(1+z)}{\overset{p}{\zeta}}\right]. \tag{9.31}$$

A measurement of this ratio, from the point of view of an observer at infinity, is very important in astrophysics.

9.3 Measurements in Kerr space-time

Let the background metric be given by the Kerr solution. In Boyer-Lindquist coordinates it is given by

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2\mathcal{M}r}{\Sigma}\right) dt^2 - \frac{4a\mathcal{M}r\sin^2\theta}{\Sigma} dt d\phi + \frac{A}{\Sigma}\sin^2\theta d\phi^2 \\ &+ \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2, \end{aligned} \tag{9.32}$$

where $\mathcal{M}$ and a are the total mass and specific angular momentum of the metric source, respectively, and Σ, Δ, and A have been introduced in (8.74) and (8.75), i.e.

$$
\begin{aligned}
\Sigma &= r^2 + a^2\cos^2\theta, \\
A &= (r^2+a^2)^2 - a^2\Delta\sin^2\theta, \\
\Delta &= r^2 + a^2 - 2\mathcal{M}r.
\end{aligned}
\tag{9.33}
$$

A black hole solution is characterized by $\mathcal{M} > a$.

We consider the same situation as described in the previous section for Schwarzschild space-time, namely a set of particles orbiting in spatially circular trajectories, all moving with the same angular velocity ζ. The associated 4-velocity field is given by

$$
U = \Gamma(\partial_t + \zeta\partial_\phi)\,, \tag{9.34}
$$

where the normalization factor is now given by

$$
\Gamma = \left[1 - \frac{2\mathcal{M}r}{\Sigma}(1 - a\zeta\sin^2\theta)^2 - (r^2+a^2)\zeta^2\sin^2\theta\right]^{-1/2}. \tag{9.35}
$$

The integral curves of U form a congruence $\mathcal{C}_U$, with ζ assumed to be constant over $\mathcal{C}_U$. Let us consider the tetrad frame (8.117), i.e.

$$
\begin{aligned}
E_{\hat t} &= U = \Gamma(\partial_t + \zeta\partial_\phi), \\
E_{\hat r} &= (\Delta/\Sigma)^{1/2}\partial_r, \\
E_{\hat\theta} &= \Sigma^{-1/2}\partial_\theta, \\
E_{\hat\phi} &= \bar\Gamma(\partial_t + \bar\zeta\partial_\phi),
\end{aligned}
\tag{9.36}
$$

with $\bar\Gamma$ and $\bar\zeta$ defined by (8.118). This is phase-locked to the coordinate directions and reduces to (9.6) when $a = 0$. As in the Schwarzschild case, the assumption $\zeta =$ constant over $\mathcal{C}_U$ guarantees that $\ddot{Y}^{\hat a} \equiv 0$ identically for all ζ and all indices $\hat a$; $Y^{\hat a}$ are the tetrad components of the connecting vectors over the given congruence. Moreover, consistently with the above assumptions, we deduce from (9.36) that $\theta(U)_{\hat a\hat b} \equiv 0$, i.e. the Born rigidity condition holds and $\zeta_{(\mathrm{fw})} = \omega(U)$. After some algebra, Eqs. (9.3) in the equatorial plane ($\theta = \pi/2$) become

$$
\begin{aligned}
\frac{d^2Y^{\hat r}}{d\tau^2} &= \left\{-\mathcal{E}(U)_{\hat r\hat r} + \partial_{\hat r}a(U)_{\hat r} + (a(U)_{\hat r})^2 + (\zeta_{(\mathrm{fw})\,\hat\theta})^2\right\}Y^{\hat r}, \\
\frac{d^2Y^{\hat\theta}}{d\tau^2} &= \left\{-\mathcal{E}(U)_{\hat\theta\hat\theta} + \partial_{\hat\theta}a(U)_{\hat\theta} + \frac{1}{2}(E_{\hat\theta}^{\theta})^2(\partial_{\hat r}g_{\theta\theta})a(U)_{\hat r}\right\}Y^{\hat\theta}, \\
\frac{d^2Y^{\hat\phi}}{d\tau^2} &= \left\{-\mathcal{E}(U)_{\hat\phi\hat\phi} - \Gamma_{\hat r\hat\phi\hat\phi}a(U)_{\hat r} + (\zeta_{(\mathrm{fw})\,\hat\theta})^2\right\}Y^{\hat\phi},
\end{aligned}
\tag{9.37}
$$

where the components of the spatial acceleration are given by (8.123); we recall them here for convenience:

$$a(U)^{\hat r} = \frac{\Gamma^2\sqrt{\Delta}}{\sqrt{\Sigma}}\left[\frac{\mathcal{M}(r^2-a^2\cos^2\theta)}{\Sigma^2}(1-a\zeta\sin^2\theta)^2 - r\zeta^2\sin^2\theta\right],$$
$$a(U)^{\hat\theta} = -\frac{\Gamma^2\sin\theta\cos\theta}{\sqrt{\Sigma}}\left[\frac{2\mathcal{M}r}{\Sigma^2}[(r^2+a^2)\zeta-a]^2+\Delta\zeta^2\right]. \tag{9.38}$$

The components of the Riemann tensor and those of the Fermi rotation coefficients which enter Eqs. (9.37) can be evaluated; see Exercise 41.

Response of the internal structure

If we require that the collection of particles moves rigidly then the tidal strains induced by the curvature ($\mathcal{E}(U)$-terms) and the Fermi rotation of the tetrad ($\zeta_{(\mathrm{fw})}$-terms) are to be balanced by the internal structure of the orbiting system. The latter responds by *generating* the Fermi-Walker strain tensor $S(U)$ defined in (7.76). Collecting all terms different than the $\mathcal{E}(U)$-terms and the $\zeta_{(\mathrm{fw})}$-terms in (9.37) and introducing the reduced frequency

$$y = \frac{\zeta}{1-a\zeta}, \tag{9.39}$$

we obtain, from (9.37) and (9.11),

$$\begin{aligned} S(U)^{\hat r\hat r} &= \Gamma^4(1+ay)^{-4}\Big\{(y_K^2-y^2)^2(\mathcal{M}r-a^2) \\ &\quad + (y_K^2-y^2)\left[\left(1-\frac{\mathcal{M}}{r}\right)\left(1-\frac{3\mathcal{M}}{r}+2ay\right)-\frac{3\mathcal{M}\Delta}{r^3}\right] \\ &\quad - \frac{3\mathcal{M}\Delta}{r^5}\left(1-\frac{3\mathcal{M}}{r}+2ay\right)\Big\}, \\ S(U)^{\hat\theta\hat\theta} &= \frac{\Gamma^2\mathcal{M}}{r^5}(1+ay)^{-2}[r^2+3a^2-2r^4(y_K^2-y^2)-4ar^2y], \\ S(U)^{\hat\phi\hat\phi} &= \Gamma^4(1+ay)^{-4}(y_K^2-y^2)(a^2-\mathcal{M}r)(y-\bar y_+)(y-\bar y_-), \end{aligned} \tag{9.40}$$

where

$$y_{K\pm} \equiv \pm\sqrt{\frac{\mathcal{M}}{r^3}}, \qquad \bar y_\pm = \frac{-a(1-\mathcal{M}/r)\pm\left(\mathcal{M}/r^3\right)^{1/2}\Delta}{a^2-\mathcal{M}r}. \tag{9.41}$$

Their plots are shown in Fig. 9.3. Knowledge of the internal strains in a collection of spatially circular non-geodesic orbits relative to an equatorial one is useful. This orbit, in fact, approximates a general equatorial geodesic at its turning points. At these points ($r = r_{(\mathrm{tp})}$, $\dot r = 0$) the azimuthal angular momentum λ (in units

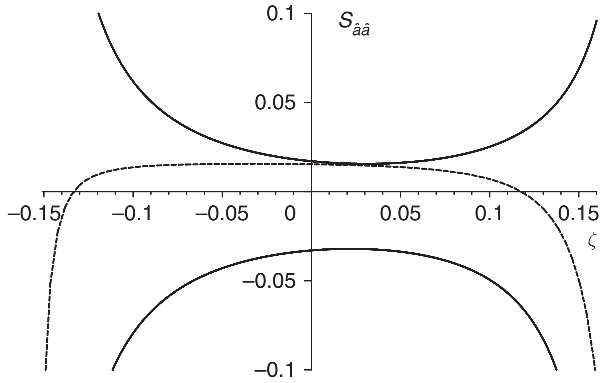

Fig. 9.3. Behavior of the relative strains $S(U)^{\hat{r}\hat{r}}$ (lower curve), $S(U)^{\hat{\theta}\hat{\theta}}$ (upper curve), and $S(U)^{\hat{\phi}\hat{\phi}}$ (middle curve) as functions of ζ for fixed values of $r = 4\mathcal{M}$, with $a = 0.5\mathcal{M}$ and $\mathcal{M} = 1$.

of $\mu_0 c$, μ_0 being the mass of the particle), written as a function of the particle's total energy $\tilde{\gamma}$ (in units of $\mu_0 c^2$), is equal to

$$\lambda_\pm = \frac{-2\mathcal{M}a\tilde{\gamma} \mp \Delta^{1/2} r(\tilde{\gamma}^2 - 1 + 2\mathcal{M}/r)^{1/2}}{r - 2\mathcal{M}}. \tag{9.42}$$

Then, at $r = r_{(\mathrm{tp})}$, a general geodesic approximates a non-geodesic spatially circular orbit (the local osculating circle) having orbital frequency of revolution

$$\zeta'_\pm = \frac{r(\tilde{\gamma}^2 - 1 + 2\mathcal{M}/r)^{1/2}\left[2\mathcal{M}a(\tilde{\gamma}^2 - 1 + 2\mathcal{M}/r)^{1/2} \mp \tilde{\gamma} r \Delta^{1/2}\right]}{\tilde{\gamma}^2 A + 4\mathcal{M}^2 a^2} \tag{9.43}$$

and normalization factor

$$\Gamma' = \left(1 - \frac{2\mathcal{M}}{r}\right)^{-1}\left[\tilde{\gamma} \mp \frac{2\mathcal{M}a}{r\Delta^{1/2}}\left(\tilde{\gamma}^2 - 1 + \frac{2\mathcal{M}}{r}\right)^{1/2}\right]. \tag{9.44}$$

This analysis has a clear astrophysical relevance. A star approaching a rotating black hole will suffer the maximum tidal perturbation at the point of maximum approach to the black hole. If we want the star to survive after a close encounter with a black hole it must balance the tidal strains, relying entirely on its internal structure. The induced strains can be given by (9.40) if we approximate the orbit by its osculating circle at the periastron.

Measurements and ambiguities

Let us consider the collection of particles as being a space-ship. This has to support the strains (9.40) in order to support its structure. The question is whether the measurements which were possible in the case of a Schwarzschild background could also be made in a Kerr background and still help the observer (the ship commander) to orient himself in the vicinity of a rotating black hole. We assume, as is known a priori, that the background geometry is that of the Kerr solution and that the orbit of the fiducial particle is equatorial and spatially circular and the frame of reference is the one described in (9.36).

As in the Schwarzschild case, consider a particle which can only move in the $\hat\theta$ direction in a frictionless pipe, perpendicular to the orbital plane. In order to know what kind of motion a test particle undergoes in that direction, being otherwise constrained to rotate rigidly with the space-ship, we impose in the second equation of (9.37) the conditions $a(U)_{\hat\theta} = 0$ and $\partial_{\hat\theta} a(U)_{\hat\theta} = 0$ for all values of θ. As a consequence, recalling that the observer moves in the equatorial plane, we obtain the equation of motion

$$\frac{d^2 Y^{\hat\theta}}{d\tau^2} = -\mathcal{G}^2(r;\zeta,a,\mathcal{M})Y^{\hat\theta}, \tag{9.45}$$

where

$$\mathcal{G}^2 = \frac{2\mathcal{M}[\zeta r^2 - a(1-a\zeta)]^2 + \zeta^2 r^3\Delta}{r^5[1-\frac{2\mathcal{M}}{r}(1-a\zeta)^2 - \zeta^2(r^2+a^2)]}, \tag{9.46}$$

from (9.37).

Function $\mathcal{G}^2$ is positive in the region of physical interest, namely for $\zeta \in (\zeta_{c-},\zeta_{c+})$, where $\zeta_{c,\pm}$ are the zeroes of the denominator of (9.46). This justifies the choice of a square, as $\mathcal{G}^2$, in (9.46); see Fig. 9.4. Clearly equation (9.45) describes a harmonic motion in the $\hat\theta$ direction with frequency $|\mathcal{G}|$, and when that happens the direction of the oscillations fixes the $\hat\theta$ direction of the frame and, at the same time, the equatorial plane.

In the Schwarzschild case the frequency of the harmonic oscillations was just the proper angular velocity of the orbital revolution, a value which $|\mathcal{G}|$ reduces to when $a = 0$. Contrary to that case, however, the quantity $|\mathcal{G}|$ does not give direct information about the angular velocity of the orbital revolution. Following de Felice and Usseglio-Tomasset (1996) we write (9.46) in terms of the reduced frequency y and obtain

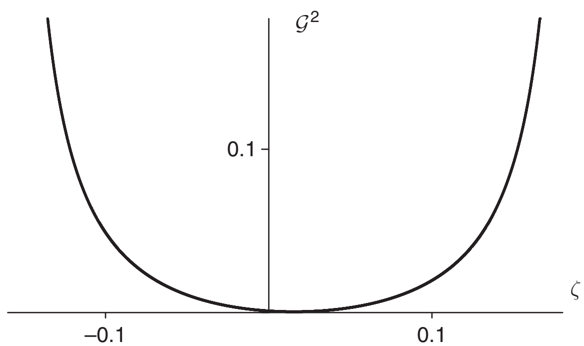

Fig. 9.4. Behavior of $\mathcal{G}^2$ as a function of ζ for fixed values of $r = 4\mathcal{M}$, with $a = 0.5\mathcal{M}$ and $\mathcal{M} = 1$. Note that when $\zeta = 0$, with the choice of parameters used, one has $\mathcal{G}^2 = 2^{-10} \neq 0$, invisible on this scale.

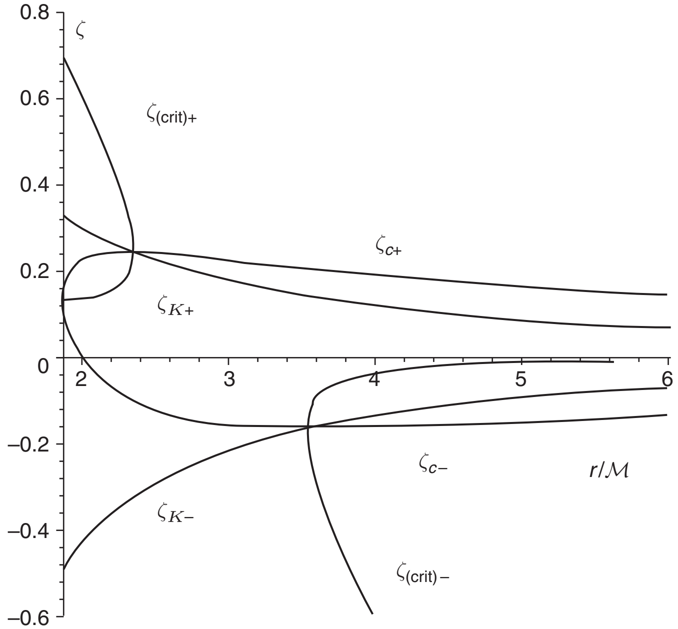

Fig. 9.5. The map of the permitted equatorial circular orbits in the Kerr metric. ζ_{c+} and ζ_{c-} are the limits of permitted angular velocities. ζ_{K+} and ζ_{K-} are the corotating and counter-rotating circular geodesics. $\zeta_{(\mathrm{crit})-}$ and $\zeta_{(\mathrm{crit})+}$ are the counter-rotating and corotating extremely accelerated circular orbits.

$$\mathcal{G}^2 = \frac{2\mathcal{M}(yr^2 - a)^2 + y^2 r^3 \Delta}{r^7 (y_{c+} - y)(y - y_{c-})}. \tag{9.47}$$

Here $y_{c\pm}$ are solutions of the equation $\Gamma^{-1} = 0$ with $\theta = \pi/2$; from (9.35) these are

$$y_{c\pm} = (a \pm \sqrt{\Delta})/r^2. \tag{9.48}$$

Spatially circular equatorial motion is then allowed in the range $y_{c-} < y < y_{c+}$ (de Felice, 1994). An intriguing implication of (9.47) is that an observer at rest with respect to infinity ($\zeta = 0 = y$) would still see a harmonic oscillation in the $\hat{\theta}$ direction, with a proper frequency square

$$\mathcal{G}^2\Big|_{y=0} = \frac{2\mathcal{M}a^2}{r^4(r - 2\mathcal{M})}. \tag{9.49}$$

The value (9.49), however, is not the minimum which can be attained by $|\mathcal{G}|$. Differentiating (9.47) with respect to y, we obtain

$$\frac{\partial \mathcal{G}^2}{\partial y} = \frac{2\Delta}{r^9} \frac{ay^2 r^3 + r^3 y - 2\mathcal{M}a}{(y_{c+} - y)^2 (y - y_{c-})^2}. \tag{9.50}$$

This vanishes at

$$y_{\pm} = \frac{1}{2a}\left(-1 \pm \sqrt{1 + \frac{8\mathcal{M}a^2}{r^3}}\right) \equiv \overset{*}{y}_{\pm}. \tag{9.51}$$

It is easily proven (de Felice and Usseglio-Tomasset, 1996; see Exercise 43) that

(i) $\overset{*}{y}_{-} < y_{c-}$; this is disregarded, being outside the range of the permitted angular velocities for circular motion;
(ii) $y_{c-} \leq \overset{*}{y}_{+} \leq y_{c+}$, the equality sign holding only on the event horizons (i.e. at $r = r_{\pm}$ solutions of $\Delta = 0$) and for $r \to \infty$.

From (ii) one is sure that the orbiting observer can always measure the minimum frequency of the harmonic oscillation by varying y in the permitted range of its values. Because $\overset{*}{y}_{+} > 0$, the minimum of $\mathcal{G}^2$ only occurs at a positive value of ζ; hence the observer who brings the angular frequency of the harmonic oscillations to a minimum would deduce that its trajectory is corotating with the black hole. However the property of $\overset{*}{y}_{+}$ of being always positive has an intriguing consequence. In the range $0 < y < \overset{*}{y}_{+}$, an increase of y causes a decrease in $|\mathcal{G}|$; hence the orbiting observer who sees $|\mathcal{G}|$ decrease as a consequence of a variation of y from within his space-ship is unable to decide whether he was decelerating the space-ship – a decrease of $|y|$ – in which case the angular velocity of revolution would have been in the ranges $y_{c-} < y < 0$ or $\overset{*}{y}_{+} < y < y_{c+}$, or was accelerating it – an increase of $|y|$ – meaning that y was in the range $0 < y < \overset{*}{y}_{+}$. Even assuming a priori a certain amount of information, e.g. being in the Kerr metric in a spatially equatorial circular orbit and with a phase-locked frame, a measurement of $|\mathcal{G}|$ leads to ambiguous information. This ambiguity could probably be eliminated with the help of a larger set of measurements, but this will not be considered here.

Finally, let us now compare $\overset{*}{y}_{+}$ with $y_{(\text{ZAMO})}$, the reduced angular velocity corresponding to $\zeta_{(\text{ZAMO})} = 2\mathcal{M}ra/A$ (de Felice, 1994):

$$y_{(\text{ZAMO})} = \frac{2\mathcal{M}a}{r(r^2 + a^2)}.$$

Simple algebra shows that $\overset{*}{y}_{+} \geq y_{(\text{ZAMO})}$ for $\Delta \geq 0$, the equality sign holding also at the limit $r \to \infty$. In particular we deduce that as $r \to \infty$, $\overset{*}{y}_{+} \approx 2\mathcal{M}a/r^3 - 4\mathcal{M}^2a^3/r^6 + \cdots$ while $y_{(\text{ZAMO})} \approx 2\mathcal{M}a/r^3 - 2\mathcal{M}a^3/r^5 + \cdots$, showing that to the lowest order in a the two functions asymptotically coincide. In slow rotation, then, an orbiting observer would detect the minimum frequency of the harmonic oscillation about the orbital plane when the angular velocity of the orbital revolution was equal to the gravitational drag ($\zeta_{(\text{ZAMO})}$).

A behavior which appears to be induced by gravitational drag without being completely justified by it is encountered in the effect we discuss next.

9.4 Relativistic thrust anomaly

In Schwarzschild space-time, a collection of particles moving in spatially circular orbits have an acceleration relative to the phase-locked frame (9.6) given by

$$a(U)_{\hat{a}} = \delta_{\hat{a}\hat{r}} \frac{1}{r} \left(1 - \frac{2\mathcal{M}}{r}\right)^{1/2} \frac{(\zeta_K^2 - \zeta^2)}{(\zeta_c^2 - \zeta^2 \sin^2\theta)}. \tag{9.52}$$

This quantity is the specific thrust which acts on each particle to keep its orbit spatially circular. It is directly measurable and, by convention, acts outwardly if it is positive and inwardly if it is negative. In the plane $\theta = \pi/2$ the thrust becomes independent of ζ at $r = 3\mathcal{M}$, where $\zeta_K^2 = \zeta_c^2$. The behavior of the thrust as a function of ζ is shown in Fig. 9.6; clearly at any radius $r \neq 3\mathcal{M}$ the thrust takes its extreme value at $\zeta = 0$, as is easily inferred from

$$\frac{\partial}{\partial\zeta} a(U)_{\hat{a}} = \frac{2}{r} \delta_{\hat{a}\hat{r}} \left(1 - \frac{2\mathcal{M}}{r}\right)^{1/2} \zeta \frac{\zeta_K^2 - \zeta_c^2}{(\zeta_c^2 - \zeta^2)^2}, \tag{9.53}$$

while at $r = 3\mathcal{M}$ it vanishes identically for any ζ. Moreover, at that radius all derivatives of the thrust with respect to ζ vanish; hence the thrust is constant and equal to $1/(3\sqrt{3}\mathcal{M})$. At $r < 3\mathcal{M}$ the thrust is always positive, meaning that it acts outwardly for all ζ and, most peculiarly, increases with $|\zeta|$ in the outward direction (Abramowicz and Lasota, 1974). This behavior suggests, as already pointed out, that an increase in the angular velocity contributes to the

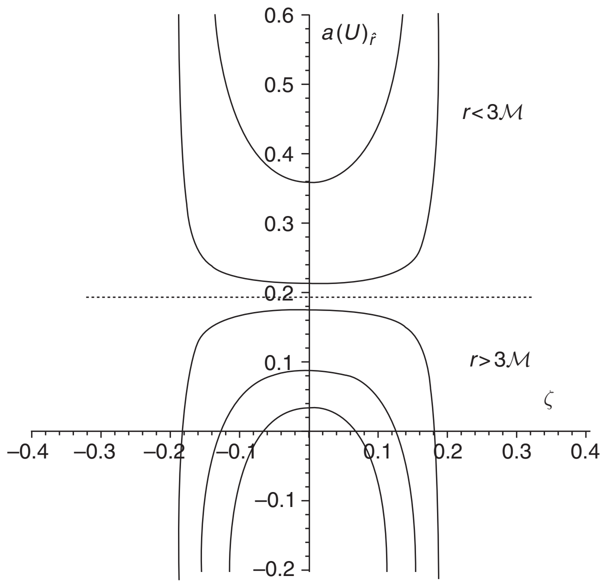

Fig. 9.6. The behavior of the thrust as a function of the angular velocity of revolution ζ at a fixed radius r in Schwarzschild space-time.

gravitational component of the thrust more than it does to the centrifugal one (de Felice, 1991; Semerák, 1994; 1995).

As already established, there is a surprising analogy between the behavior of the gradient of the thrust with respect to ζ and that of a gyroscopic precession. Recalling its expression in the Schwarzschild metric (9.30), we find that the gyroscopic precession is forward at $r < 3\mathcal{M}$, while it is backward at $r > 3\mathcal{M}$, being zero at $r = 3\mathcal{M}$. This behavior is consistent with an increasing gravitational strength as one approaches the horizon at $r = 2\mathcal{M}$; however, a more careful analysis shows that the phenomenon is related to a form of gravitational drag which is not induced by the rotation of the metric source. We shall refer to it as a gravitational *grip*. Clearly in Schwarzschild space-time we have only gravitational grip, while both types of effect exist in Kerr space-time. Since the axis of a gyroscope is Fermi-Walker transported along its own trajectory (the spatially circular orbits in our case), the absence of precession at $r = 3\mathcal{M}$, that is $\overset{g}{\zeta} = 0$, implies that the gravitational grip forces the phase-locked frame to become a Fermi frame at that radius.

Let us now see how the situation changes if we consider the Kerr background. If the metric source is a rotating black hole then the above effect manifests itself not only at small coordinate distances from the outer event-horizon on corotating equatorial circular orbits, but also arbitrarily far away from the source on counter-rotating circular orbits with a finite range of angular velocities which vanish at infinity.

This behavior has no Newtonian analog and its occurrence at asymptotic distances from a rotating source is a combination of grip and drag, giving rise to a measurable new test of general relativity.

The specific thrust associated with a general non-geodesic equatorial spatially circular orbit in the Kerr metric is given, from de Felice and Usseglio-Tomasset (1991) and de Felice (1994), by

$$a(U) = \frac{\Delta^{1/2}}{r^2}\frac{(y - y_{K+})(y - y_{K-})}{(y - y_{c+})(y - y_{c-})}, \qquad \theta = \pi/2, \tag{9.54}$$

where y is the reduced frequency (9.39), while $y_{K\pm}$ and $y_{c\pm}$ are given by Eqs. (9.41) and (9.48) respectively. The response of the thrust to a change in the reduced angular velocity y at a fixed r (and a) is illustrated by the function

$$\left.\frac{\partial a(U)}{\partial y}\right|_r = -\frac{2a\Delta^{1/2}}{r^4}\frac{(y - y_{(\mathrm{crit})+})(y - y_{(\mathrm{crit})-})}{(y - y_{c+})^2(y - y_{c-})^2}, \tag{9.55}$$

where

$$y_{(\mathrm{crit})\pm} = -\frac{1}{2a}\left[1 - \frac{3\mathcal{M}}{r} \mp \sqrt{\left(1 - \frac{3\mathcal{M}}{r}\right)^2 - \frac{4\mathcal{M}a^2}{r^3}}\right] \tag{9.56}$$

identify the orbits with extreme acceleration. Clearly $y_{(\mathrm{crit})-}$ is always negative and vanishes at infinity as $\sim r^{-3}$, remaining larger than y_{K-}, which vanishes

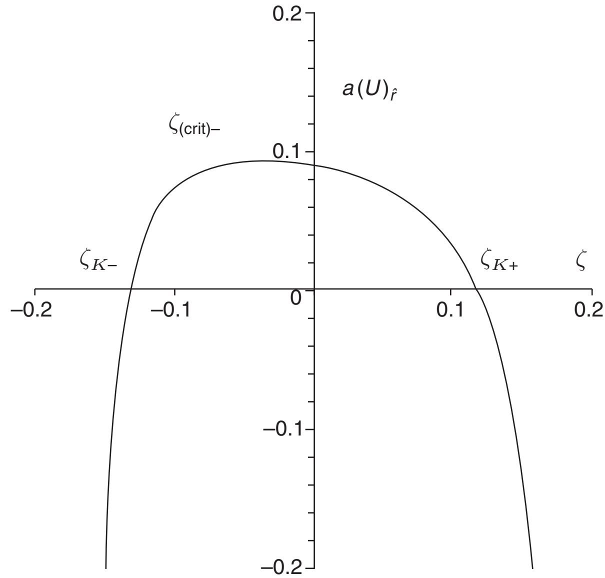

Fig. 9.7. The behavior of the thrust as a function of the angular velocity of revolution ζ in equatorial spatially circular orbits at $r = 4\mathcal{M}$ in the Kerr metric with $a = 0.5\mathcal{M}$.

asymptotically as $\sim r^{-3/2}$. In Fig. 9.5 we show the behavior of $\zeta_{c\pm}, \zeta_{K\pm}$, and $\zeta_{\rm (crit)\pm}$ (equivalently, of $y_{c\pm}$, $y_{K\pm}$, $y_{\rm (crit)\pm}$) as functions of r. From (9.54), (9.55), and (9.56) we deduce how the specific thrust $a(U)$ behaves with y at a fixed r. This is shown in Fig. 9.7. Setting $y = y_{\rm (crit)-}$, the maximum of $a(U)$ takes the value

$$a(U_{\rm (crit)-}) = \frac{\Delta^{1/2}}{r^2} \frac{\sqrt{1 - \frac{4\mathcal{M}a^2}{r^3}\left(1 - \frac{3\mathcal{M}}{r}\right)^{-2}} - 1}{\sqrt{1 - \frac{4\mathcal{M}a^2}{r^3}\left(1 - \frac{3\mathcal{M}}{r}\right)^{-2} - \frac{2a^2}{r^2}\left(1 - \frac{3\mathcal{M}}{r}\right)^{-1}} - 1}. \qquad (9.57)$$

As we see, when the metric source is rotating, the maximum of the thrust at asymptotic distances is displaced to negative values of y; hence its *anomalous* behavior occurs in the small interval $(y_{\rm (crit)-}, 0)$. In fact, an increase of $|y|$ from 0 to $|y_{\rm (crit)-}|$ implies, contrary to intuition, an increase of the thrust outwardly (being $a(U) > 0$). It appears that an increase of $|y|$ in the above range causes a loss of energy, which would let the orbit plunge into the source unless a larger radial thrust were applied outwardly. The behavior of the thrust appears to be a response to both gravitational drag and grip, since the same behavior is met in corotating circular orbits in the Kerr metric (de Felice, 1994) and in the Schwarzschild metric, where no drag exists at all.

In both the rotating and the non-rotating cases the condition of extreme acceleration corresponds to the vanishing of the precession of a gyroscope. From de Felice (1994) we deduce the angular velocity of precession of a gyroscopic when the latter moves in equatorial spatially circular orbits. In the Kerr metric this is given by

$$\overset{g}{\zeta} = \frac{a(y - y_{(\mathrm{crit})+})(y - y_{(\mathrm{crit})-})}{r^2(y - y_{c+})(y - y_{c-})}. \tag{9.58}$$

At each value of the radius r, then, a Fermi frame counter-rotating with the metric source and with angular velocity of revolution equal to $|y_{(\mathrm{crit})-}|$ is made to coincide to a phase-locked frame. This effect can in principle be tested, as we shall see.

Let us suppose that the rotating source is the Earth. Neglecting deviations from sphericity, its space-time is described by the Lense-Thirring metric (Lense and Thirring, 1918; de Felice, 1968). The latter coincides with the Kerr metric in the weak field limit $\mathcal{M}/r \ll 1$ and $a/r \ll 1$. At the Earth's surface we have

$$\frac{\mathcal{M}_\oplus}{R_\oplus} \approx 6.9 \times 10^{-10}, \qquad \frac{a_\oplus}{R_\oplus} \approx 5.4 \times 10^{-7}, \tag{9.59}$$

where $R_\oplus \approx 6.37103 \times 10^8$cm, $\mathcal{M}_\oplus \equiv GM_\oplus/c^2 = 0.443\,\mathrm{cm}$, $M_\oplus \approx (5.977 \pm 0.004) \times 10^{27}$g, $a_\oplus \equiv \mathcal{J}_\oplus/(c\mathcal{M}_\oplus) \approx 3.4 \times 10^2$ cm, taking $\mathcal{J}_\oplus \approx 5.9 \times 10^{40}\mathrm{g\,cm^2\,s^{-1}}$.

In conventional units ($\mathrm{cm\,s^{-2}}$), the specific thrust is given by $\tilde{f} \equiv c^2 a(U)$; its extreme values are, from (9.57),

$$\tilde{f}_{(\mathrm{crit})-} \approx \frac{GM_\oplus}{r_\oplus^2}\left(1 + \frac{\mathcal{M}_\oplus}{r_\oplus} + \frac{1}{2}\frac{a_\oplus^2}{r_\oplus^2} + \cdots\right), \tag{9.60}$$

where $r_\oplus = \alpha R_\oplus$, with $\alpha > 1$, is the radius of the orbit of the space-ship. Let this orbit be a counter-rotating equatorial spatially circular geodesic with radius $r_\oplus$. We then consider the device described in Section 8.2, namely a rigid box free to slide with negligible friction on rails which run across the ship tangentially to the orbit. We require that the center of the box follows the trajectory of the baricenter of the ship, moving therefore on a geodesic. A test point-like mass is held at the center of the box by sensors which trigger small thrusters. As stated, the test mass moves initially on a geodesic; therefore the sensors exert no net acceleration on the mass. Now let the box be set in motion within the space-ship, along the rails in the direction opposite to the ship's motion. The latter being counter-rotating, the test mass will acquire with respect to infinity an angular velocity $\check{\zeta}$ in magnitude smaller than that allowed for a geodesic. As a consequence, the test mass will be acted upon by a thrust which keeps it to the center of the box on the same orbital radius. If we decrease the value of $|\check{\zeta}|$ of the test mass by increasing the velocity of the box within the space-ship, the thrust will ultimately reach a maximum value at $\check{\zeta} = \zeta_{(\mathrm{crit})-}$. In the non-rotating case $(a = 0)$ the maximum occurs when $\check{\zeta} = 0$, i.e. when the velocity of the test mass, as seen from infinity, is equal in magnitude but opposite to that of the ship. If rotational effects are taken into account $(a \neq 0)$, then the maximum occurs when the moving test mass has an angular velocity with respect to infinity equal to

$$c\zeta_{(\text{crit})-} \approx -c\frac{a_\oplus \mathcal{M}_\oplus}{r_\oplus^3} \approx 1.75 \times 10^{-14}\alpha^{-3}\,\text{s}^{-1}, \tag{9.61}$$

from (9.56) and to the lowest order in the relativistic corrections.

While the observer comoving with the ship is able to recognize when the thrust acting on the test mass reaches a maximum, he cannot distinguish whether that occurs when $\check{\zeta} = 0$ or when $\check{\zeta} = \zeta_{(\text{crit})-}$. The latter case would signal a new relativistic effect. To detect the anomalous behavior of the thrust with $|\zeta|$, one has to measure, at the maximum of the thrust, the (linear) velocity of the test mass relative to the space-ship.

Consider an observer who is comoving with the space-ship. If the thrust acting on the test mass in the box is zero, then he knows that the angular velocity of the ship with respect to infinity is, from (9.54) and (9.39),

$$\zeta_{K_-} \approx -\left(\frac{\mathcal{M}_\oplus}{r_\oplus^3}\right)^{\frac{1}{2}}\left[1 + \frac{a_\oplus}{r_\oplus}\left(\frac{\mathcal{M}_\oplus}{r_\oplus}\right)^{\frac{1}{2}}\right]. \tag{9.62}$$

If the test mass moves within the ship as indicated above, then it acquires an angular velocity $\check{\zeta}$ with respect to infinity, which is smaller in magnitude than the ship's ζ_{K_-}. Therefore it will have, relative to the latter, a velocity ν and a Lorentz factor

$$\gamma \equiv \gamma(U_-, u) = (1 - \nu(U_-, u)^2)^{-1/2} = -g_{\alpha\beta}u^\alpha U_-^\beta, \tag{9.63}$$

where

$$U_-^\alpha = \Gamma(\zeta_{K-})(\delta_0^\alpha + \zeta_{K-}\delta_\phi^\alpha), \qquad u^\beta = \Gamma(\check{\zeta})(\delta_0^\beta + \check{\zeta}\delta_\phi^\beta) \tag{9.64}$$

are the 4-velocities of the ship and test mass, respectively, with their normalization factors given in general by

$$\Gamma(\zeta) = \left[1 - \frac{2\mathcal{M}}{r}(1 - a\zeta)^2 - \zeta^2(r^2 + a^2)\right]^{-1/2}. \tag{9.65}$$

To lowest order in the parameters a and $\mathcal{M}$ and in the equatorial plane, the weak-field limit of the Kerr metric is given by

$$\begin{aligned} ds^2 \approx &-\left(1 - \frac{2\mathcal{M}}{r}\right)dt^2 - \frac{4\mathcal{M}a}{r}d\phi dt + r^2\left(1 + \frac{a^2}{r^2} + \frac{2\mathcal{M}a^2}{r^3}\right)d\phi^2 \\ &+\left(1 + \frac{2\mathcal{M}}{r} - \frac{a^2}{r^2}\right)dr^2. \end{aligned} \tag{9.66}$$

From (9.62), (9.64), and (9.66), the Lorentz factor is, to the lowest order in a/r,

$$\begin{aligned} \gamma \approx \Gamma(\zeta_{K-})\Gamma(\check{\zeta})\Bigg[1 &- \frac{2\mathcal{M}}{r} - r^2\left(1 + \frac{a^2}{r^2} + \frac{2\mathcal{M}a^2}{r^3}\right)\zeta_{K-}\check{\zeta} \\ &+ \frac{2\mathcal{M}a}{r}(\zeta_{K-} + \check{\zeta})\Bigg]. \end{aligned} \tag{9.67}$$

On the Earth we have $2(\mathcal{M}/r)(a^2/r^2) \sim 10^{-23}$ and therefore we shall neglect that term. From (9.62) and (9.65) we have

$$\Gamma(\zeta_{K-}) \approx 1 + \frac{3\mathcal{M}}{2r} + \frac{3\mathcal{M}^2}{2r^2} + 3\frac{a}{r}\left(\frac{\mathcal{M}}{r}\right)^{3/2}. \tag{9.68}$$

If the test mass were at rest with respect to infinity then $\check{\zeta} = 0$ and therefore

$$\Gamma(0) \approx 1 + \frac{\mathcal{M}}{r} + \frac{3\mathcal{M}^2}{2r^2}; \tag{9.69}$$

hence, from (9.67) and to the lowest order in a/r,

$$\gamma|_{\check{\zeta}=0} \approx 1 + \frac{\mathcal{M}}{2r} - \frac{\mathcal{M}^2}{2r^2} + \frac{a}{r}\left(\frac{\mathcal{M}}{r}\right)^{3/2} - \frac{8a}{r}\left(\frac{\mathcal{M}}{r}\right)^{5/2}. \tag{9.70}$$

From the latter, and recalling the definition of the Lorentz factor, we deduce the velocity of the test mass relative to the space-ship when it is at rest with respect to infinity:

$$\nu_0 \approx c\sqrt{\frac{\mathcal{M}}{r}}\left(1 - \frac{\mathcal{M}}{2r} + \frac{a}{r}\sqrt{\frac{\mathcal{M}}{r}}\right). \tag{9.71}$$

This differs from the Keplerian velocity by a fractional change

$$\frac{1}{2}\frac{\mathcal{M}_\oplus}{r_\oplus} \approx \alpha^{-1}\, 3.45 \times 10^{-9}, \tag{9.72}$$

which is independent of rotation, and a rotationally induced correction given by

$$\frac{a_\oplus}{r_\oplus}\left(\frac{\mathcal{M}_\oplus}{r_\oplus}\right)^{1/2} \approx \frac{1}{\alpha^{3/2}}\, 1.41 \times 10^{-11}. \tag{9.73}$$

The maximum thrust, however, occurs when the test mass has angular velocity (9.61). In this case it would be

$$\Gamma(\zeta_{(\mathrm{crit})-}) \approx 1 + \frac{\mathcal{M}}{r} + \frac{3\mathcal{M}^2}{2r^2} + \frac{5\mathcal{M}^2 a^2}{2r^4}. \tag{9.74}$$

Hence we neglect the contribution from rotation and set $\Gamma(\zeta_{(\mathrm{crit})-}) \approx \Gamma(0)$. From Eqs. (9.61), (9.62), and (9.67) we deduce

$$\gamma|_{\zeta_{(\mathrm{crit})-}} \approx 1 + \frac{\mathcal{M}}{2r} - \frac{\mathcal{M}^2}{2r^2} - \frac{21a}{2r}\left(\frac{\mathcal{M}}{r}\right)^{5/2}. \tag{9.75}$$

As expected, the relative linear velocity of the test mass within the ship is now lower and is given by

$$\nu_{(\mathrm{crit})-} \approx c\sqrt{\frac{\mathcal{M}}{r}}\left(1 - \frac{\mathcal{M}}{2r} - \frac{21}{2}\frac{a}{r}\left(\frac{\mathcal{M}}{r}\right)^{3/2}\right). \tag{9.76}$$

In this case the correction due to the Earth's rotation is of the order of 1.02×10^{-19}, so it can be neglected. In order to detect the general relativistic anomaly in the behavior of the thrust, the observer in the ship needs to identify the maximum thrust acting on the test mass when the velocity of the latter is $\nu_{(\text{crit})-}$ (i.e. when $\check{\zeta} = \zeta_{(\text{crit})-}$) and not ν_0 (i.e. when $\check{\zeta} = 0$) as would be natural in the field of a non-rotating source. This requires measuring linear velocities within the ship to better than $\sim 10^{-11}$. As we can see, in fact, the fractional change of the velocity $\nu_{(\text{crit})-}$ relative to ν_0 is given by the factor $(a/r)\sqrt{\mathcal{M}/r}$. Clearly, at this level of precision the deviation from spherical symmetry of the Earth's potential must be known with an accuracy better than $\sim 10^{-11}$.

The obvious conclusion is that the proposed experiment is hardly feasible at present. Summarizing, one would have to measure the velocity of the test mass relative to the space-ship to 1 part in 10^{11}, and measure the radial force on the test mass with an accuracy of 1 part in 10^{13}, in the short interval of time that the mass could be kept in motion within the ship. Moreover, one has to be sure that the mass moves within the ship at a constant radius relative to the Earth and control with high precision the gravity gradients along the path of the test mass resulting from the mass distribution of the space-ship itself.

9.5 Measurements of black-hole parameters

An important correspondence can be established between measurements performed within a space-ship, as discussed above, and data which can be gathered with observations "at infinity" (de Felice and Usseglio-Tomasset, 1996; Semerák and de Felice, 1997).

There is considerable observational evidence that most active galactic nuclei (AGN) and X-ray binaries host black holes (Rees, 1988; 1998). Essential to a relativistic modeling of these sources is knowledge of the properties of black holes and of the matter distribution around them. An estimate of their mass has mainly been based on the energy emission and on the time-scale of their variability (Begelman, Blandford, and Rees, 1984). In Semerák, Karas, and de Felice (1999) a method was presented for determining, from a set of observations, the parameters of a system made of a rotating black hole and an accretion disk interacting in such a way as to power sources with a periodic modulation of variability. In the case of AGNs or stellar-size X-ray sources, a star orbiting freely around such a system on a spherical and almost equatorial circular geodesic will play the role of the test particle which was free to move in a frictionless pipe oriented in the $\hat{\theta}$ direction but constrained to rotate rigidly around the compact source. Such a particle was found to perform in the pipe harmonic oscillations about the equatorial plane with "proper" frequency $|\mathcal{G}|$. In the astrophysical situation, the star in a spherical orbit will cross the disk with a frequency which is just twice $|\mathcal{G}|$. Moreover, from photometric observations one can deduce the frequency of the orbital revolution; what is observed at infinity, however, would not be the proper frequencies $|\mathcal{G}|$ and $|\overset{p}{\zeta}|$, but rather their values measured *at infinity* given by the general relations $|\mathcal{G}_\infty| = \Gamma^{-1}|\mathcal{G}|$ and $|\zeta| = |\Gamma^{-1}\overset{p}{\zeta}|$.

For a spherical geodesic in Kerr space-time, the frequencies of the azimuthal and latitudinal motion are given by rather complicated formulas involving elliptic integrals (Wilkins, 1972; Karas and Vokrouhlický, 1994). However, those formulas simplify considerably in the case of a nearly equatorial geodesic. The azimuthal angular velocity $\zeta = d\phi/dt$ with respect to an observer at rest at infinity can be approximated by that of an equatorial circular geodesic (Bardeen, Press, and Teukolsky, 1972):

$$\zeta_{K\pm} = (a + 1/y_{K\pm})^{-1}, \tag{9.77}$$

where $y_{K\pm} = \pm\sqrt{\mathcal{M}/r^3}$ are the corresponding values of the reduced frequency, and the upper/lower sign corresponds to a corotating/counter-rotating orbit.[1] The "proper" angular frequency $|\mathcal{G}|$ of (small) harmonic latitudinal oscillations about the equatorial plane of a spherical orbit with general and steady radial component of the acceleration is given by[2]

$$\mathcal{G}^2 = \left(\frac{\Gamma}{r}\right)^2 \left\{\Delta\zeta^2 + 2y_{K\pm}^2\left[a - (r^2 + a^2)\,\zeta\right]^2\right\}, \tag{9.78}$$

where ζ is a constant and Γ is the normalization factor (9.35) which, written in terms of the reduced frequency y and setting $\theta = \pi/2$, is given by

$$\frac{1}{\Gamma^2} = \frac{1}{(1+ay)^2}\left(1 - \frac{2\mathcal{M}}{r} + 2ay - y^2r^2\right). \tag{9.79}$$

As stated, at infinity we observe $|\mathcal{G}_\infty|$ which, in the case of geodesic motion, satisfies the equation

$$\mathcal{G}_\infty^2 = \zeta_{K\pm}^2(1 - 4ay_{K\pm} + 3a^2/r^2). \tag{9.80}$$

We have then identified two "observables," $|\zeta_{K\pm}|$ and $|\mathcal{G}_\infty|$, which are both expressed in terms of the black-hole mass $\mathcal{M}$, its specific angular momentum a, and the radius r of the most active part of the disk, through Eqs. (9.77) and (9.78). They provide two equations, namely (9.77) and (9.80); hence they are not sufficient to determine the above three parameters. In order to obtain them in an unambiguous way we need at least one more observable. This comes from the analysis made by Fanton *et al.* (1997) relating those parameters to quantities which are observable in the integrated spectrum of an accretion disk. A stationary disk produces a double-horn line profile which is presumably modulated by

[1] We allow for both ($\pm$) cases for completeness, but only the corotating trajectory can be considered; the accretion disk, in fact, is more likely to be corotating with the central black hole, and it has been shown that the interaction with the disk also makes the star corotate eventually.

[2] A simpler derivation, based on the perturbation of an equatorial circular orbit, can be found in Semerák and de Felice (1997).

the star–disk interaction at a radius r (see Semerák, Karas, and de Felice 1999; see Appendix of astro-ph/9802025 for details). Treating the star as if it were a point-like emitting source moving on a circular equatorial geodesic, the observed frequency shift $h = (1 + z)_0$ of each emitted photon can be written in terms of its direction cosine $e_{\hat{\phi}}$, the azimuthal component of the unit vector along the direction of emission of a given photon, measured in the emitter's local rest frame:

$$h = \frac{\left[1 - 2\mathcal{M}/r + y_{K\pm}\left(a + \sqrt{\Delta}\, e_{\hat{\phi}}\right)\right]}{(1 - 3\mathcal{M}/r + 2ay_{K\pm})^{1/2}}. \tag{9.81}$$

One of the most important attributes of a spectral line is its width; this arises from the different frequency shifts h carried by the photons which reach the observer at infinity. Since h depends on the direction cosine $e_{\hat{\phi}}$ at emission, the spectral line would appear broadened, the maximum extent of which, as a result of integration over one entire orbit, is a measure of the variation δh corresponding to the largest possible variation of $e_{\hat{\phi}}$ compatible with detection at infinity. If we call the latter $\delta e_{\hat{\phi}}$, we have, from (9.81),

$$(\delta h) = (\delta e_{\hat{\phi}})\frac{\sqrt{\Delta} y_{K\pm}}{(1 - 3\mathcal{M}/r + 2ay_{K\pm})^{1/2}}, \tag{9.82}$$

and thus

$$\delta^2 \equiv \frac{(\delta h)^2}{(\delta e_{\hat{\phi}})^2} = y_{K\pm}^2 r^2\, \frac{1 - 2\mathcal{M}/r + a^2/r^2}{1 - 3\mathcal{M}/r + 2ay_{K\pm}}\,. \tag{9.83}$$

It is clear that $\delta e_{\hat{\phi}}$ can be at most 2 but, in realistic situations, it varies significantly with the inclination angle θ_o of the black-hole–disk system with respect to the line of sight. It also depends, though only weakly, on the rotational parameter a and the radius of emission r. From a numerical ray-tracing analysis it is found that $\delta e_{\hat{\phi}}$ ranges from $(\delta e_{\hat{\phi}})_{\min} \simeq 0.4$ to $(\delta e_{\hat{\phi}})_{\max} \simeq 2$ as θ_o goes from $\simeq 0°$ to $\simeq 90°$; hence one can fix as observables the extreme values of δ, $\delta_{\max} = 2.5\,\delta h$ and $\delta_{\min} = 0.5\,\delta h$, corresponding to a line of sight nearly polar in the former case and nearly equatorial in the latter one. If the line width δh is measured, then formulas (9.77), (9.78), and (9.83) provide a closed system of ordinary equations which yield the parameters $\mathcal{M}$, a, and r in terms of the observable quantities $\zeta_{K\pm}$, $|\mathcal{G}_\infty|$, and δ.

These equations can be solved for a, r^2, and $y_{K\pm}$:

$$a = \zeta_{K\pm}^{-1} - y_{K\pm}^{-1}, \tag{9.84}$$

$$r^2 = \frac{3\,(1 - \zeta_{K\pm}/y_{K\pm})^2}{4\zeta_{K\pm}y_{K\pm} + \mathcal{G}_\infty^2 - 5\zeta_{K\pm}^2}, \tag{9.85}$$

where $y_{K\pm}$ is determined by the quartic equation from (9.83)–(9.85) and assuming geodesic motion, i.e.

$$34y_{K\pm}^4 - By_{K\pm}^3 + Cy_{K\pm}^2 - Dy_{K\pm} + E = 0, \tag{9.86}$$

with

$$\begin{aligned}
B &= \frac{4}{\zeta_{K\pm}}\,(\delta^2\zeta_{K\pm}^2 + 23\zeta_{K\pm}^2 - 2\mathcal{G}_\infty^2),\\
C &= 21\delta^2\zeta_{K\pm}^2 + 76\zeta_{K\pm}^2 + 7\delta^2\mathcal{G}_\infty^2 + 3\mathcal{G}_\infty^2 + \mathcal{G}_\infty^4/\zeta_{K\pm}^2,\\
D &= \frac{2}{\zeta_{K\pm}}\,\left[18\delta^2\zeta_{K\pm}^4 + 12\zeta_{K\pm}^4 + 5\delta^2\zeta_{K\pm}^2\mathcal{G}_\infty^2 + 11\zeta_{K\pm}^2\mathcal{G}_\infty^2 - (\delta^2+1)\,\mathcal{G}_\infty^4\right],\\
E &= 5\zeta_{K\pm}^4(4\delta^2+3) + (\zeta_{K\pm}^2 - \mathcal{G}_\infty^2)(\delta^2\mathcal{G}_\infty^2 - 7\zeta_{K\pm}^2) - \mathcal{G}_\infty^4.
\end{aligned} \tag{9.87}$$

Clearly a numerical solution of Eq. (9.86) always exists for each given set of data $\zeta_{K\pm}$, $|\mathcal{G}_\infty|$, and $|\delta|$.

Let us note that in the Schwarzschild case, $a = 0$, Eq. (9.77) reduces to

$$\zeta_{K\pm} = y_{K\pm} = \pm|\mathcal{G}_\infty| = \pm\sqrt{\mathcal{M}/r^3}, \tag{9.88}$$

and (9.83) becomes

$$\delta^2 = r^2\zeta_{K\pm}^2\,\frac{1-2r^2\zeta_{K\pm}^2}{1-3r^2\zeta_{K\pm}^2}\,. \tag{9.89}$$

The physical solution of this equation is

$$r^2 = (4\zeta_{K\pm}^2)^{-1}\left[3\delta^2 + 1 - \sqrt{(3\delta^2+1)^2 - 8\delta^2}\right]; \tag{9.90}$$

hence $\mathcal{M}$ follows from (9.88) as $\mathcal{M} = r^3\zeta_{K\pm}^2$.

The analysis in this section is a clear example of how one can link measurements which could have been made in the vicinity of a black hole, such as $\tilde{\zeta}$ and $\overset{p}{\zeta}$, to observations made at *infinity*.

9.6 Gravitationally induced time delay

Measurements of time made by two different observers depend on their relative motion but also on their geometrical environment. We shall see in a simple situation how this time rate difference arises.

The Sagnac effect and its time-like analog measure the difference between the revolution time of a pair of particles orbiting in the opposite sense in time-like or null spatially circular orbits, as seen by an observer orbiting on a similar type of trajectory. If (ζ_1, ζ_2) are the coordinate angular velocities of such a pair (either (ζ_-, ζ_+) for photons or (ζ_{K-}, ζ_{K+}) for massive particles), and ζ is the angular velocity of the given observer with 4-velocity $U = \Gamma(\partial_t + \zeta\partial_\phi)$ distinct from the pair, one finds that the difference in the coordinate orbital times after one complete revolution with respect to the observer is

$$\begin{aligned}
\Delta t = t_2 - t_1 &= 2\pi\,[1/(\zeta_2 - \zeta) - 1/(\zeta - \zeta_1)]\\
&= -4\pi[\zeta - (\zeta_1+\zeta_2)/2]/[(\zeta-\zeta_1)(\zeta-\zeta_2)].
\end{aligned} \tag{9.91}$$

For the pair of oppositely rotating time-like geodesics one has

$$\Delta t_K(U) = -4\pi[\zeta - \zeta_{(\mathrm{gmp})}]/[(\zeta-\zeta_{K-})(\zeta-\zeta_{K+})], \tag{9.92}$$

while for the pair of oppositely rotating null orbits one has

$$\Delta t_{(\mathrm{null})}(U) = -4\pi[\zeta - \zeta_{(\mathrm{nmp})}]/[(\zeta - \zeta_-)(\zeta - \zeta_+)], \tag{9.93}$$

where

$$\zeta_{(\mathrm{gmp})} = \frac{\zeta_{K-} + \zeta_{K+}}{2} \tag{9.94}$$

is the angular velocity associated with the geodesic meeting point (gmp) orbits, defined in (8.155) as the orbits which contain the meeting points of co- and counter-rotating circular geodesics. Analogously,

$$\zeta_{(\mathrm{nmp})} = \frac{\zeta_- + \zeta_+}{2} \tag{9.95}$$

is the angular velocity associated with the *null meeting point* (nmp) orbits, which contain the meeting points of co- and counter-rotating photons.

When the observer is static, i.e. has 4-velocity $m = (-g_{tt})^{-1/2}\partial_t$ and vanishing angular velocity $\zeta = 0$, then the Sagnac time difference and its time-like geodesic analog are given by

$$\begin{aligned} \Delta t_{(\mathrm{null})}(m) &= 4\pi({\zeta_-}^{-1} + {\zeta_+}^{-1})/2, \\ \Delta t_K(m) &= 4\pi({\zeta_{K-}}^{-1} + {\zeta_{K+}}^{-1})/2. \end{aligned} \tag{9.96}$$

Let us now specify the above general relations to the case of spatially circular orbits in Kerr space-time.

Consider three families of observers in Kerr space-time, the first made up of static observers and the others made up of those moving on equatorial spatially circular geodesics corotating and counter-rotating with the metric source. The latters are described by the 4-velocities

$$U_\pm^\alpha = \Gamma_\pm(\delta_t^\alpha + \zeta_{K\pm}\delta_\phi^\alpha), \tag{9.97}$$

where

$$\begin{aligned} \zeta_{K\pm} &= \pm\sqrt{\mathcal{M}/r^3}\left(1 \pm a\sqrt{\mathcal{M}/r^3}\right)^{-1}, \\ \Gamma_\pm &= \left[1 - \zeta_{K\pm}^2(r^2 + a^2) - \frac{2\mathcal{M}}{r}(1 - a\zeta_{K\pm})^2\right]^{-1/2}. \end{aligned} \tag{9.98}$$

From the above relations the Lorentz factor is given by

$$\gamma_\pm = -U_\pm^\alpha m_\alpha = \Gamma_\pm\left(1 - \frac{2\mathcal{M}}{r}\right)^{1/2}\left[1 + \frac{2\mathcal{M}a}{r}\left(1 - \frac{2\mathcal{M}}{r}\right)^{-1}\zeta_{K\pm}\right]. \tag{9.99}$$

The static observers are spatially fixed at each point of space-time; hence the orbiting particles $U_\pm$ meet one static observer at each point of their orbit. Then, at each point, the local m-observer will judge a small interval of the proper time of $U_\pm$ as corresponding to an interval of his own proper time equal to

$$d\tau_{m\pm} = \gamma_\pm d\tau_{U_\pm} = (U_\pm^\phi)^{-1}\gamma_\pm d\phi. \tag{9.100}$$

Each m-observer will make the same measurement between any pair of events along the trajectories of $U_\pm$. Therefore, the radius r of the orbit being constant, we can evaluate the proper time elapsed on the clock of the static observer at some initial event after $U_\pm$ has made one revolution around the metric source until they cross the same initial m-observer. As stated, the particle U_+ makes a round trip along a corotating geodesic and the other one, U_-, makes a round trip along a counter-rotating geodesic. Let us then recall that

$$\frac{1}{\zeta_{K\pm}} = a \pm \sqrt{r^3/\mathcal{M}}. \tag{9.101}$$

Hence, from (9.100) and after a $\pm 2\pi$ turn of the azimuthal angle (+ for corotating, – for counter-rotating), we have the following relations (Cohen and Mashhoon, 1993; Lichtenegger, Gronwald, and Mashhoon, 2000):

$$\begin{aligned}
\Delta\tau_{m_+} &= +2\pi \left(1 - \frac{2\mathcal{M}}{r}\right)^{1/2} \left[\frac{2\mathcal{M}a}{r}\left(1 - \frac{2\mathcal{M}}{r}\right)^{-1} + a + \sqrt{\frac{r^3}{\mathcal{M}}}\right], \\
\Delta\tau_{m_-} &= -2\pi \left(1 - \frac{2\mathcal{M}}{r}\right)^{1/2} \left[\frac{2\mathcal{M}a}{r}\left(1 - \frac{2\mathcal{M}}{r}\right)^{-1} + a - \sqrt{\frac{r^3}{\mathcal{M}}}\right],
\end{aligned}$$

so that

$$\delta\tau_m \equiv \Delta\tau_{m_+} - \Delta\tau_{m_-} = 4\pi a \left(1 - \frac{2\mathcal{M}}{r}\right)^{-1/2}. \tag{9.102}$$

The above relation implies

$$\Delta\tau_{m_+} > \Delta\tau_{m_-}. \tag{9.103}$$

Like a viscous fluid, the gravitational drag *helps* the corotating particles to go faster than the counter-rotating ones; hence the former suffer a larger time dilation than the latter, justifying (9.103).

In the gravitational field of the Earth we have

$$\frac{2\mathcal{M}_\oplus}{r_\oplus} \approx 10^{-9}, \qquad a_\oplus \approx 3.4 \times 10^2\,\mathrm{cm}; \tag{9.104}$$

hence

$$\delta\tau_m \approx \frac{4\pi}{c} a_\oplus \approx 4.2 \times 10^{-8}\,\mathrm{s}, \tag{9.105}$$

a value easy to measure with modern technology.

Clearly a direct measurement of this time delay would unambiguously show that the metric source is rotating with a well-defined value of the rotational parameter $a = c\delta\tau_m/(4\pi)$. This is entirely due to gravitational drag; in fact it vanishes when $a = 0$. In this case there would be no difference between the revolution time of orbits covered in the opposite sense, although the gravitational grip still acts as a result of the dependence of the revolution time on the source mass $\mathcal{M}$, as expected.

9.7 Ray-tracing in Kerr space-time

Most of our knowledge of the universe comes from *what we see*; hence deducing the optical appearance of cosmic sources is of paramount importance. What we see, however, depends on how a light ray reaches us after being emitted by the source; therefore it may happen that the real universe hides itself behind a curtain of illusions. To be free of uncertainties and ambiguities one needs to recognize the actual light trajectory, taking into account the geometrical environment it propagates through. This type of analysis is known as *ray-tracing*. Among the extensive literature on this topic it is worth mentioning the earliest works by Polnarev (1972), Cunningham and Bardeen (1973), and de Felice, Nobili, and Calvani (1974), in which the Kerr metric was assumed as the background geometry.

Here we shall deduce the shape of a luminous ring surrounding a rotating black hole as it would appear to a distant observer (Li *et al.*, 2005).

Null geodesics

Let us briefly recall the equations for null geodesics in Kerr space-time; the coordinate components of the tangent vector are

$$
\begin{aligned}
\frac{dt}{d\lambda} &= \frac{1}{\Sigma}\left[-a(aE\sin^2\theta - L) + \frac{(r^2+a^2)}{\Delta}P\right],\\
\frac{dr}{d\lambda} &= \epsilon_r \frac{1}{\Sigma}\sqrt{R},\\
\frac{d\theta}{d\lambda} &= \epsilon_\theta \frac{1}{\Sigma}\sqrt{\Theta},\\
\frac{d\phi}{d\lambda} &= -\frac{1}{\Sigma}\left[\left(aE - \frac{L}{\sin^2\theta}\right) + \frac{a}{\Delta}P\right],
\end{aligned}
\tag{9.106}
$$

where ϵ_r and ϵ_θ are sign indicators, and

$$
\begin{aligned}
P &= E(r^2+a^2) - aL,\\
R &= P^2 - \Delta[\mathcal{K} + (L-aE)^2],\\
\Theta &= \mathcal{K} - \cos^2\theta\left[-a^2E^2 + \frac{L^2}{\sin^2\theta}\right].
\end{aligned}
\tag{9.107}
$$

Here the quantities E, L, and $\mathcal{K}$ are constants of the motion representing respectively the total energy, the azimuthal angular momentum, and the separation constant of the Hamilton-Jacobi equation. It is convenient to introduce the notation

$$
b = \frac{L}{E}, \qquad q^2 = \frac{\mathcal{K}}{E^2},
\tag{9.108}
$$

so that

$$
\begin{aligned}
R &= (r^2+a^2-ab)^2 - \Delta[q^2 + (b-a)^2],\\
\Theta &= q^2 - \cos^2\theta\left(-a^2 + \frac{b^2}{\sin^2\theta}\right),
\end{aligned}
\tag{9.109}
$$

and

$$
\begin{aligned}
\frac{dt}{d\lambda} &= \frac{E}{\Sigma}\left[a(b - a\sin^2\theta) + \frac{(r^2+a^2)}{\Delta}(r^2+a^2-ab)\right], \\
\frac{dr}{d\lambda} &= \epsilon_r \frac{E}{\Sigma}\sqrt{R}, \\
\frac{d\theta}{d\lambda} &= \epsilon_\theta \frac{E}{\Sigma}\sqrt{\Theta}, \\
\frac{d\phi}{d\lambda} &= -\frac{E}{\Sigma}\left[\left(a - \frac{b}{\sin^2\theta}\right) + \frac{a}{\Delta}(r^2+a^2-ab)\right].
\end{aligned}
\tag{9.110}
$$

Note that q^2 can be positive, negative, or eventually zero.

The geodesic equations can be formally integrated by eliminating the affine parameter as follows:

$$
\begin{aligned}
&\epsilon_r \int^r \frac{dr}{\sqrt{R(r)}} = \epsilon_\theta \int^\theta \frac{d\theta}{\sqrt{\Theta(\theta)}}, \\
&t = \epsilon_r \int^r \frac{r^2(r^2+a^2) + 2a\mathcal{M}r(a-b)}{\Delta\sqrt{R(r)}}\,dr + \epsilon_\theta \int^\theta \frac{a^2\cos^2\theta}{\sqrt{\Theta(\theta)}}\,d\theta, \\
&\phi = \epsilon_r \int^r \frac{r^2 b + 2\mathcal{M}r(a-b)}{\Delta\sqrt{R(r)}}\,dr + \epsilon_\theta \int^\theta \frac{b\cot^2\theta}{\sqrt{\Theta(\theta)}}\,d\theta.
\end{aligned}
\tag{9.111}
$$

The integrals are along the photon path.

Consider now a typical ray-tracing problem, i.e. a photon emitted at the point $r_{\rm em}$, $\theta_{\rm em}$, and $\phi_{\rm em}$ at the coordinate time $t_{\rm em}$, which reaches an observer located at $r_{\rm obs}$, $\theta_{\rm obs}$, and $\phi_{\rm obs}$ at the coordinate time $t_{\rm obs}$. From $(9.111)_1$ we see that the photon trajectories, originating at the emitter, must satisfy the integral equation

$$
\epsilon_r \int_{r_{\rm em}}^r \frac{dr}{\sqrt{R(r)}} = \epsilon_\theta \int_{\theta_{\rm em}}^\theta \frac{d\theta}{\sqrt{\Theta(\theta)}}. \tag{9.112}
$$

The signs ϵ_r and ϵ_θ change when a turning point is reached. Turning points in r and θ are solutions of the equations $R = 0$ and $\Theta = 0$ respectively. To find out which photons actually reach the observer one should find all pairs (b, q^2) satisfying (9.112). We shall consider the case of an emitting source moving along spatially circular orbits confined to the equatorial plane (i.e. $\theta_{\rm em} = \pi/2$) and a distant observer located far away from the black hole (i.e. $r_{\rm obs} \to \infty$). Since in this case the system is stationary and axisymmetric, only motions in the r and θ directions are required in the calculation of the radiation spectrum from the emitting source.

Both integrals (9.112) can be expressed in terms of standard elliptic functions of the first kind, and classified in terms of different values of the parameters b and q^2 corresponding to different kinds of orbits. This was first done by Rauch and Blandford (1994), who presented tables of reductions of these integrals by using the new variables $u = 1/r$ and $\mu = \cos\theta$.

We will proceed by retaining instead the variable r, so that (9.112) becomes

$$\epsilon_r \int_{r_{\rm em}}^{r} \frac{dr}{\sqrt{R(r)}} = \epsilon_\mu \int_{\mu_{\rm em}}^{\mu} \frac{d\mu}{\sqrt{\Theta_\mu(\mu)}}, \tag{9.113}$$

where

$$\Theta_\mu = q^2 + (a^2 - q^2 - b^2)\mu^2 - a^2\mu^4 \equiv a^2(\mu_-^2 + \mu^2)(\mu_+^2 - \mu^2), \tag{9.114}$$

with

$$\mu_\pm^2 = \frac{1}{2a^2}\left\{\left[(b^2+q^2-a^2)^2 + 4a^2q^2\right]^{1/2} \mp (b^2+q^2-a^2)\right\}. \tag{9.115}$$

In the case of a photon crossing the equatorial plane, we have $q^2 > 0$; hence both μ_+^2 and μ_-^2 are non-negative. Note that $\mu_+^2\mu_-^2 = q^2/a^2$.

For a photon emitted by the orbiting source, we have $\mu_{\rm em} = 0$, so μ^2 can never exceed μ_+^2. The integral over μ can thus be worked out with the inverse Jacobian elliptic integral

$$\int_{\mu}^{\mu_+} \frac{d\mu}{\sqrt{\Theta_\mu}} = \frac{1}{\sqrt{a^2(\mu_+^2 + \mu_-^2)}}\,{\rm cn}^{-1}\left(\frac{\mu}{\mu_+}\middle| m_\mu\right), \tag{9.116}$$

where $0 \le \mu < \mu_+$ and

$$m_\mu = \frac{\mu_+^2}{\mu_+^2 + \mu_-^2}. \tag{9.117}$$

The integral over r can also be solved in terms of inverse Jacobian elliptic integrals. Let us denote the four roots of $R(r) = 0$ by r_1, r_2, r_3, and r_4. There are two relevant cases to be considered.

Case A: $R(r) = 0$ has four real roots.

Let the roots be ordered so that $r_1 \ge r_2 \ge r_3 \ge r_4$, with $r_4 \le 0$. Physically allowed regions for photons are given by $R \ge 0$, i.e. $r \ge r_1$ (region I) and $r_3 \ge r \ge r_2$ (region II). In region I the integral over r has the solution

$$\int_{r_1}^{r} \frac{dr}{\sqrt{R(r)}} = \frac{2}{\sqrt{(r_1-r_3)(r_2-r_4)}}\,{\rm sn}^{-1}\left[\sqrt{\frac{(r_2-r_4)(r-r_1)}{(r_1-r_4)(r-r_2)}}\middle| m_4\right], \tag{9.118}$$

where

$$m_4 = \frac{(r_1-r_4)(r_2-r_3)}{(r_1-r_3)(r_2-r_4)}, \qquad 0 \le m_4 \le 1, \tag{9.119}$$

when $r_1 \ne r_2$. The case of two equal roots $r_1 = r_2$ should be treated separately, but it is of no practical interest here, since it corresponds to unstable circular orbits.

In region II the integral over r has the solution

$$\int_r^{r_2} \frac{dr}{\sqrt{R(r)}} = \frac{2}{\sqrt{(r_1 - r_3)(r_2 - r_4)}} \, \mathrm{sn}^{-1}\left[\sqrt{\frac{(r_1 - r_3)(r_2 - r)}{(r_2 - r_3)(r_1 - r)}}\middle| m_4\right], \tag{9.120}$$

when $r_1 \neq r_2$.

Case B: $R(r) = 0$ has two complex roots and two real ones.
Let us assume that r_1 and r_2 are complex, r_3 and r_4 are real, and $r_3 > r_4$. Then, we must have $r_1 = \bar{r}_2$, whereas $r_3 \geq 0$ and $r_4 \leq 0$. The physically allowed region for photons is given by $r \geq r_3$. The integral over r has the solution

$$\int_{r_3}^{r} \frac{dr}{\sqrt{R(r)}} = \frac{1}{\sqrt{AB}} \, \mathrm{cn}^{-1}\left[\sqrt{\frac{(A-B)r + r_3 B - r_4 A}{(A+B)r - r_3 B - r_4 A}}\middle| m_2\right], \tag{9.121}$$

where

$$A^2 = (r_3 - u)^2 + v^2, \qquad B^2 = (r_4 - u)^2 + v^2, \tag{9.122}$$

with $u = \mathrm{Re}(r_1)$ and $v = \mathrm{Im}(r_1)$, and

$$m_2 = \frac{(A+B)^2 - (r_3 - r_4)^2}{4AB}, \qquad 0 \leq m_2 \leq 1. \tag{9.123}$$

Images

The apparent position of the image of the emitting source on the celestial sphere is represented by two impact parameters, α and β, measured on a plane centered about the observer location and perpendicular to the direction $\theta_{\rm obs}$. The impact parameter α is the apparent displacement of the image perpendicular to the projected axis of symmetry of the black hole, while β is the apparent displacement of the image parallel to the axis of symmetry in the sense of the angular momentum of the black hole. They are defined by

$$\begin{aligned}
\alpha &= \lim_{r_{\rm obs}\to\infty} -r_{\rm obs}\frac{k^{\hat{\phi}}}{k^{\hat{t}}} = -\frac{b}{\sin\theta_{\rm obs}} = -\frac{b}{\sqrt{1-\mu_{\rm obs}^2}}, \\
\beta &= \lim_{r_{\rm obs}\to\infty} r_{\rm obs}\frac{k^{\hat{\theta}}}{k^{\hat{t}}} = \epsilon_{\theta_{\rm obs}}\sqrt{q^2 + a^2\cos^2\theta_{\rm obs} - b^2\cot^2\theta_{\rm obs}} \\
&= -\epsilon_{\mu_{\rm obs}}\sqrt{q^2 - \mu_{\rm obs}^2[b^2/(1-\mu_{\rm obs}^2) - a^2]},
\end{aligned} \tag{9.124}$$

where the $k^{\hat{\alpha}}$ are the frame components of k with respect to ZAMOs. The line of sight to the black hole's center marks the origin of the coordinates, where $\alpha = 0 = \beta$. Now imagine a source of illumination behind the black hole whose angular size is large compared with the angular size of the black hole. As seen by the distant observer the black hole will appear as a black region in the middle of the larger bright source. No photons with impact parameters in a certain range about $\alpha = 0 = \beta$ reach the observer. The rim of the black hole corresponds to photon trajectories which are marginally trapped by the black hole; they spiral around many times before they reach the observer. The calculation of the precise apparent shape of the black–hole has been done by Cunningham and Bardeen (1973) and by Chandrasekhar (1983).

The shape of the image is thus obtained by determining all pairs (b, q^2) satisfying (9.112) (or equivalently (9.113)), then substituting back into (9.124) to get the corresponding coordinates on the observer's photographic plate. Alternatively, one can solve (9.124) for b and q^2, i.e.

$$b = -\alpha \sin\theta_{\rm obs}, \qquad q^2 = \beta^2 + (\alpha^2 - a^2)\cos^2\theta_{\rm obs}, \tag{9.125}$$

then substitute back into (9.113) and solve for all allowed pairs of impact parameters (α, β).

The images of the source so obtained can be classified according to the number of times the photon trajectory crosses the equatorial plane between the emitting source and the observer. The trajectory of the "direct" image does not cross the equatorial plane; that of a "first-order" image crosses once; and so on. The shapes of direct and first-order images are shown in Fig. 9.8 for $r_{\rm em} = 10M$ and $\theta_{\rm obs} = 85°$ as an example.

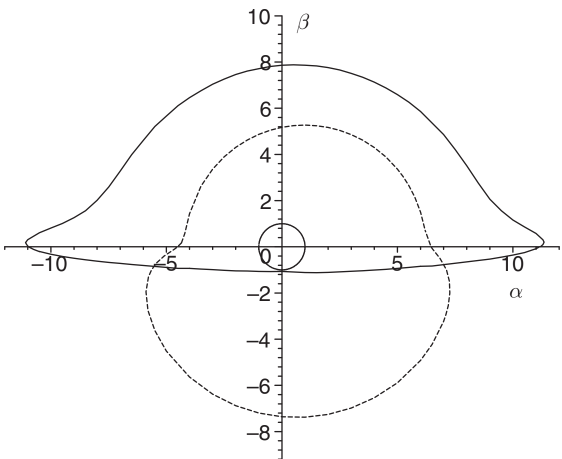

Fig. 9.8. Apparent positions of direct (solid line) and first-order (dashed line) images are shown for the emitting orbital radius $r_{\rm em} = 10M$ and an observer at the polar angle $\theta_{\rm obs} = 85°$. The small circle is the locus $\alpha^2 + \beta^2 = 1$ and gives the scale of the plot. The cross at the origin marks the position of the black hole, whose spin parameter has been chosen to be $a/M = 0.5$.

Direct image

As the photon reaches the observer, on the photon orbit we have $d\theta/dr > 0$ (i.e. $d\mu/dr < 0$) if $\beta > 0$, and $d\theta/dr < 0$ (i.e. $d\mu/dr > 0$) if $\beta < 0$. Therefore, when $\beta > 0$ the photon must encounter a turning point at $\mu = \mu_+$: μ starts from 0, goes up to μ_+, then goes down to $\mu_{\rm obs}$ (which is $\leq \mu_+$). When $\beta < 0$, the photon must not encounter a turning point at $\mu = \mu_+$: μ starts from 0 and monotonically increases to $\mu_{\rm obs}$.

The total integration over μ along the path of the photon from the emitting source to the observer is thus given by

$$I_\mu = \begin{cases} \left[\int_0^{\mu_+} + \int_{\mu_{\rm obs}}^{\mu_+}\right] \frac{d\mu}{\sqrt{\Theta_\mu(\mu)}} & (\beta > 0), \\ \\ \int_0^{\mu_{\rm obs}} \frac{d\mu}{\sqrt{\Theta_\mu(\mu)}} = \left[\int_0^{\mu_+} + \int_{\mu_{\rm obs}}^{\mu_+}\right] \frac{d\mu}{\sqrt{\Theta_\mu(\mu)}} & (\beta < 0). \end{cases} \tag{9.126}$$

By using (9.116) we get

$$I_\mu = \frac{1}{\sqrt{a^2(\mu_+^2 + \mu_-^2)}} \left[K(m_\mu) + \epsilon_\beta \mathrm{cn}^{-1}\left(\frac{\mu_{\rm obs}}{\mu_+}\middle| m_\mu\right)\right], \tag{9.127}$$

where $\epsilon_\beta = 1$ if $\beta > 0$, 0 if $\beta = 0$, and -1 if $\beta < 0$, and $K(m)$ is the complete elliptic integral of the first kind.

Now let us consider the integration over r. Since the observer is at infinity, the photon reaching him must have been moving in the allowed region defined by $r \geq r_1$ when $R(r) = 0$ has four real roots (case A), or the allowed region defined by $r \geq r_3$ when $R(r) = 0$ has two complex roots and two real roots (case B). There are then two possibilities for the photon during its trip: it has encountered a turning point at $r = r_t$ ($r_t = r_1$ in case A, $r_t = r_3$ in case B), or it has not encountered any turning point in r. Define

$$I_\infty \equiv \int_{r_t}^{\infty} \frac{dr}{\sqrt{R(r)}}, \qquad I_{r_{\rm em}} \equiv \int_{r_t}^{r_{\rm em}} \frac{dr}{\sqrt{R(r)}}. \tag{9.128}$$

Obviously, according to (9.113), a necessary and sufficient condition for the occurrence of a turning point in r on the path of the photon is that $I_\infty < I_\mu$. Therefore, the total integration over r along the path of the photon from the emitting source to the observer is

$$I_r = \begin{cases} I_\infty + I_{r_{\rm em}} & (I_\infty < I_\mu), \\ \\ \int_{r_{\rm em}}^{\infty} \frac{dr}{\sqrt{R(r)}} & (I_\infty \geq I_\mu). \end{cases} \tag{9.129}$$

By definition, I_∞, $I_{r_{\rm em}}$, and I_r are all positive. According to (9.113), we must have $I_r = I_\mu$ for the orbit of a photon. The relevant cases to be considered are the following.

Case A: $R(r) = 0$ has four real roots.
When $r_1 \neq r_2$, by using Eq. (9.118) to evaluate integrals in (9.128) with $r_t = r_1$, we get

$$
\begin{aligned}
I_\infty &= \frac{2}{\sqrt{(r_1 - r_3)(r_2 - r_4)}} \operatorname{sn}^{-1}\left[\sqrt{\frac{r_2 - r_4}{r_1 - r_4}}\middle| m_4\right], \\
I_{r_{\rm em}} &= \frac{2}{\sqrt{(r_1 - r_3)(r_2 - r_4)}} \\
&\quad \times \operatorname{sn}^{-1}\left[\sqrt{\frac{(r_2 - r_4)(r_{\rm em} - r_1)}{(r_1 - r_4)(r_{\rm em} - r_2)}}\middle| m_4\right].
\end{aligned}
\tag{9.130}
$$

Substitute these expressions into (9.129), then let $I_r = I_\mu$, and finally solve for $r_{\rm em}$:

$$
r_{\rm em} = \frac{r_1(r_2 - r_4) - r_2(r_1 - r_4)\operatorname{sn}^2(\xi_4|m_4)}{(r_2 - r_4) - (r_1 - r_4)\operatorname{sn}^2(\xi_4|m_4)}, \tag{9.131}
$$

where

$$
\xi_4 = \frac{1}{2}(I_\mu - I_\infty)\sqrt{(r_1 - r_3)(r_2 - r_4)}. \tag{9.132}
$$

Since $\operatorname{sn}^2(-\xi_4|m_4) = \operatorname{sn}^2(\xi_4|m_4)$, the solution given by (9.131) applies whether $I_\mu - I_\infty$ is positive or negative, i.e. whether or not there is a turning point in r along the path of the photon.

Case B: $R(r) = 0$ has two complex roots and two real roots.
By using (9.121) to evaluate integrals in (9.128) with $r_t = r_3$, we get

$$
\begin{aligned}
I_{r_{\rm em}} &= \frac{1}{\sqrt{AB}} \operatorname{cn}^{-1}\left[\sqrt{\frac{(A - B)r_{\rm em} + r_3 B - r_4 A}{(A + B)r_{\rm em} - r_3 B - r_4 A}}\middle| m_2\right], \\
I_\infty &= \frac{1}{\sqrt{AB}} \operatorname{cn}^{-1}\left[\sqrt{\frac{A - B}{A + B}}\middle| m_2\right],
\end{aligned}
\tag{9.133}
$$

where A and B are given by (9.122). Substitute these expressions into (9.129), then let $I_r = I_\mu$, and finally solve for $r_{\rm em}$:

$$
r_{\rm em} = \frac{r_4 A - r_3 B - (r_4 A + r_3 B)\operatorname{cn}(\xi_2|m_2)}{(A - B) - (A + B)\operatorname{cn}(\xi_2|m_2)}, \tag{9.134}
$$

where

$$
\xi_2 = (I_\mu - I_\infty)\sqrt{AB}. \tag{9.135}
$$

Since $\operatorname{cn}(-\xi_2|m_2) = \operatorname{cn}(\xi_2|m_2)$, the solution given by Eq. (9.134) applies whether $I_\mu - I_\infty$ is positive or negative, i.e. whether or not there is a turning point in r along the path of the photon.

First-order image

In this case the photon trajectory crosses the equatorial plane once.

The integration over μ along the path of the photon from the emitting source to the observer is given by

$$I_\mu = \begin{cases} \left[-\int_0^{-\mu_+} + \int_{-\mu_+}^{\mu_+} - \int_{\mu_+}^{\mu_{\rm obs}}\right] \frac{d\mu}{\sqrt{\Theta_\mu(\mu)}} \\ = \left[-\int_0^{\mu_+} + 2\int_{-\mu_+}^{\mu_+} + \int_{\mu_{\rm obs}}^{\mu_+}\right] \frac{d\mu}{\sqrt{\Theta_\mu(\mu)}} \quad (\beta > 0), \\ \left[-\int_0^{-\mu_+} + \int_{-\mu_+}^{\mu_{\rm obs}}\right] \frac{d\mu}{\sqrt{\Theta_\mu(\mu)}} \\ = \left[-\int_0^{\mu_+} + 2\int_{-\mu_+}^{\mu_+} - \int_{\mu_{\rm obs}}^{\mu_+}\right] \frac{d\mu}{\sqrt{\Theta_\mu(\mu)}} \quad (\beta < 0). \end{cases} \tag{9.136}$$

By using (9.116) we get

$$I_\mu = \frac{1}{\sqrt{a^2(\mu_+^2 + \mu_-^2)}} \left[3K(m_\mu) + \epsilon_\beta {\rm cn}^{-1}\left(\frac{\mu_{\rm obs}}{\mu_+}\middle| m_\mu\right)\right]. \tag{9.137}$$

Now let us consider the integration over r. The discussion holding for the case of the direct image applies also in this case. The solution is thus given by (9.131) and (9.134), with I_μ given by (9.137).

9.8 High-precision astrometry

Modern space technology allows one to measure with high accuracy the general relativistic corrections to light trajectories due to the background curvature. Astrometric satellites like GAIA and SIM, for example, are expected to provide measurements of the position and motion of a star in our galaxy with an accuracy of the order of $O(3)$, recalling that we set $O(n) \equiv O(1/c^n)$. With such accuracy, we must model and interpret the satellite observations in a general relativistic context. In particular the measurement conditions as well as the determination of the satellite rest frame must be modeled with equal accuracy. This frame consists of a clock and a triad of orthonormal axes which is adapted to the satellite attitude. We shall give here two examples of frames which best describe the satellite's measuring conditions. First we construct a Fermi frame which can be operationally fixed by a set of three mutually orthogonal gyroscopes; then we find a frame which mathematically models a given satellite attitude.

Since we have in mind applications to satellite missions, we fix the background geometry as that of the Solar System, assuming that it is the only source of gravity. Moreover we assume that it generates a weak gravitational field, so we shall retain only terms of first order in the gravitational constant G and consider these terms only up to order $O(3)$.

The rest frame of a satellite consists of a *clock* which measures the satellite proper time and a triad of orthonormal axes. The latter are described by 4-vectors whose components are referred to a coordinate system which in general is not connected to the satellite itself. Moreover, they are defined up to spatial rotations; there are infinitely many possible orientations of the spatial triad which can be fixed to a satellite, so our task is to identify those which correspond to actual attitudes.

The space-time metric is given by

$$ds^2 \equiv g_{\alpha\beta}dx^\alpha dx^\beta = [\eta_{\alpha\beta} + h_{\alpha\beta} + O(h^2)]dx^\alpha dx^\beta, \tag{9.138}$$

where $O(h^2)$ denotes non-linear terms in h, the coordinates are $x^0 = t, x^1 = x$, $x^2 = y, x^3 = z$, the origin being fixed at the center of mass of the Solar System, and $\eta_{\alpha\beta}$ is the Minkowski metric, so that the metric components are

$$g_{00} = -1 + \underset{2}{h}_{00} + O(4), \quad g_{0a} = \underset{3}{h}_{0a} + O(5), \quad g_{ab} = 1 + \underset{2}{h}_{00}\delta_{ab} + O(4). \tag{9.139}$$

Here $\underset{2}{h}_{00} = 2U$, where U is the gravitational potential generated by the bodies of the Solar System, and the subscripts indicate the order of $1/c$ (e.g. $\underset{3}{h}_{0a} \sim O(3)$). Following Bini and de Felice (2003) and Bini, Crosta, and de Felice (2003), we shall not specify the metric coefficients, in order to ensure generality.

Let us fix the satellite's trajectory by the time-like, unitary 4-vector u' ($u'^\alpha u'_\alpha = -1$), given by

$$u' = T_s(\partial_t + \beta_1\partial_x + \beta_2\partial_y + \beta_3\partial_z), \tag{9.140}$$

where the ∂_α are the coordinate basis vectors relative to the coordinate system, and β_i are the *coordinate* components of the satellite 3-velocity with respect to a local static observer, recalling that we use subscripts here to refer to contravariant components, so as not to confuse them with power indices. Finally we define $T_s = 1 + (U + \frac{1}{2}\beta^2)$ and $\beta^2 = \beta_1^2 + \beta_2^2 + \beta_3^2$.

Fermi frame

A Fermi frame adapted to a given observer can be obtained from *any* frame adapted to that observer, provided we know its Fermi coefficients. The latter tell how much the spatial axes of the given triad must rotate in order to be reduced to a Fermi frame.

We shall apply this procedure to a tetrad adapted to u'. Although we use the simplest possible frame, we find that an algebraic solution for a Fermi frame is possible only if we confine ourselves to terms of the order $O(2)$ and set $\beta_3 = 0$. To this order a tetrad solution is given by Bini and de Felice (2003) as

$$
\begin{aligned}
\lambda_{\hat{0}} &= \left[1 + U + \frac{1}{2}\beta^2\right]\partial_t + \beta[\cos\zeta(t)\partial_x + \sin\zeta(t)\partial_y],\\
\lambda_{\hat{1}} &= (1-U)[\sin\zeta(t)\partial_x - \cos\zeta(t)\partial_y],\\
\lambda_{\hat{2}} &= \beta\partial_t + \left[\frac{1}{2}\beta^2 - U + 1\right](\cos\zeta(t)\partial_x + \sin\zeta(t)\partial_y),\\
\lambda_{\hat{3}} &= (1-U)\partial_z,
\end{aligned}
\tag{9.141}
$$

where we have set $\beta_1 = \beta\cos\zeta(t)$, $\beta_2 = \beta\sin\zeta(t)$, and $\beta_3 = 0$, ζ being the angular velocity of orbital revolution of the given observer. This tetrad is not a Fermi tetrad because one Fermi coefficient is different from zero, namely

$$
C_{(\mathrm{fw})\hat{2}\hat{1}} = -\dot{\zeta},
\tag{9.142}
$$

a dot meaning derivative with respect to the coordinate time. Subtracting the Fermi rotation at each time, the triad $\{\lambda_{\hat{a}}\}$ reduces to a Fermi triad:

$$
\begin{aligned}
R_{\hat{1}} &= -\beta\sin\zeta(t)\partial_t - \frac{1}{4}\beta^2\sin 2\zeta(t)\partial_x\\
&\quad - \left[\frac{\beta^2}{2}\sin^2\zeta(t) - U + 1\right]\partial_y,\\
R_{\hat{2}} &= \beta\cos\zeta(t)\partial_t + \left[1 - U + \frac{\beta^2}{2}\cos^2\zeta(t)\right]\partial_x\\
&\quad + \frac{1}{4}\sin 2\zeta(t)\beta^2\partial_y,\\
R_{\hat{3}} &= (1-U)\partial_z.
\end{aligned}
\tag{9.143}
$$

It is easy to verify that all the Fermi coefficients of the triad (9.143) vanish identically. The operational setting of a Fermi triad by means of three mutually orthogonal gyroscopes has a degree of arbitrariness which arises from the freedom one has in fixing a spatial triad; a Fermi triad, in fact, is defined up to a constant rotation. In most cases it is more convenient to define a frame which is fixed to the satellite and is constrained according to criteria of best efficiency for the mission goal.

Attitude frame

Let us find now a frame adapted to the satellite's attitude, keeping the approximation to the order $O(3)$. We first fix a coordinate system centered at the baricenter of the Solar System, with the spatial axes pointing to distant sources; the latter identify a *global* Cartesian-like spatial coordinate representation $(x,\, y,\, z)$ with respect to which the space-time metric takes the form (9.138). The world line of an observer at rest with respect to the chosen coordinate grid is given by the unit 4-vector

$$
u = (g_{tt})^{-1/2}\partial_t \approx (1+U)\partial_t,
\tag{9.144}
$$

where t is a coordinate time. The observer u, together with the spatial axes as specified, is a static observer, termed *baricentric*, and the parameter on its world line is the *baricentric proper time*. Obviously, at each point in space-time there exists a static observer u who carries a triad of spatial and mutually orthogonal unitary vectors which point to the same distant sources as for the baricentric frame.

As shown in Bini, Crosta, and de Felice (2003), the spatial triad of a static observer at each space-time point is given to $O(3)$ by the following vectors:

$$\begin{aligned}\lambda_{\hat{1}} &= h_{01}\partial_t + (1-U)\partial_x,\\ \lambda_{\hat{2}} &= h_{02}\partial_t + (1-U)\partial_y,\\ \lambda_{\hat{3}} &= h_{03}\partial_t + (1-U)\partial_z.\end{aligned} \tag{9.145}$$

We need to identify the spatial direction to the geometrical center of the Sun as seen from within the satellite. To this end we first identify this direction with respect to the local static observer which is defined at each point of the satellite's trajectory, then we boost the corresponding triad to adapt it to the motion of the satellite.

Let $x_0(t)$, $y_0(t)$, $z_0(t)$ be the coordinates of the satellite's center of mass with respect to the baricenter of the Solar System, and $x_\odot(t)$, $y_\odot(t)$, $z_\odot(t)$ those of the Sun at the same coordinate time t. Here the time dependence is assumed to be known. The relative spatial position of the Sun with respect to the satellite at the time t is then

$$\begin{aligned}x'_\odot &= x_\odot - x_0,\\ y'_\odot &= y_\odot - y_0,\\ z'_\odot &= z_\odot - z_0.\end{aligned} \tag{9.146}$$

With respect to a local static observer, the Sun direction is fixed, rotating the triad (9.145) by an angle ϕ_s around $\lambda_{\hat{3}}$ and then by an angle θ_s around the vector image of $\lambda_{\hat{2}}$ under the above ϕ_s-rotation, where

$$\phi_s = \tan^{-1}\frac{y'_\odot}{x'_\odot}\quad,\quad \theta_s = \tan^{-1}\frac{z'_\odot}{\sqrt{x'^2_\odot + y'^2_\odot}}. \tag{9.147}$$

Thus we have the new triad adapted to the observer u,

$$\underset{s}{\lambda}_{\hat{a}} = \mathcal{R}_2(\theta_s)\mathcal{R}_3(\phi_s)\lambda_{\hat{a}}, \tag{9.148}$$

where

$$\mathcal{R}_2(\theta_s) = \begin{pmatrix}\cos\theta_s & 0 & \sin\theta_s\\ 0 & 1 & 0\\ -\sin\theta_s & 0 & \cos\theta_s\end{pmatrix} \tag{9.149}$$

and

$$\mathcal{R}_3(\phi_s) = \begin{pmatrix} \cos\phi_s & \sin\phi_s & 0 \\ -\sin\phi_s & \cos\phi_s & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{9.150}$$

It should be noted here that, since the Sun is an extended body, its geometrical center may be difficult to determine with great precision. The uncertainty in this measurement may affect the precision in fixing the angles ϕ_s and θ_s from on board the satellite, contributing to the determination of the error box.

From Eqs. (9.148)–(9.150), the explicit expressions for the coordinate components of the vectors of the new triad are

$$\begin{aligned} \underset{s}{\lambda}_{\hat{1}} = & [\cos\theta_s(\cos\phi_s h_{01} + \sin\phi_s h_{02}) + \sin\theta_s h_{03}]\partial_t \\ & + \cos\phi_s \cos\theta_s(1-U)\partial_x \\ & + \sin\phi_s \cos\theta_s(1-U)\partial_y \\ & + \sin\theta_s(1-U)\partial_z, \end{aligned} \tag{9.151}$$

$$\begin{aligned} \underset{s}{\lambda}_{\hat{2}} = & -(\sin\phi_s h_{01} + \cos\phi_s h_{02})\partial_t \\ & - \sin\phi_s(1-U)\partial_x \\ & - \cos\phi_s(1-U)\partial_y, \end{aligned} \tag{9.152}$$

$$\begin{aligned} \underset{s}{\lambda}_{\hat{3}} = & -[\sin\theta_s(\cos\phi_s h_{01} + \sin\phi_s h_{02}) - \cos\theta_s h_{03}]\partial_t \\ & - \cos\theta_s \sin\theta_s(1-U)\partial_x \\ & - \sin\phi_s \sin\theta_s(1-U)\partial_y \\ & + \cos\theta_s(1-U)\partial_z. \end{aligned} \tag{9.153}$$

It is easy to verify that the set $\{u, \underset{s}{\lambda}_{\hat{a}}\}$ forms an orthonormal tetrad; moreover, it is equally straightforward to see that

$$\cos\theta_s \, \underset{s}{\lambda}_{\hat{2}} = \frac{d}{d\phi_s} \underset{s}{\lambda}_{\hat{1}}, \quad \underset{s}{\lambda}_{\hat{3}} = \frac{d}{d\theta_s} \underset{s}{\lambda}_{\hat{1}}. \tag{9.154}$$

All quantities in equations (9.151)–(9.153) are defined at the position $(x_0, \, y_0, \, z_0)$ of the satellite at time t.

Let us remember that our aim here is to identify a tetrad frame which is adapted to the satellite and whose spatial triad mirrors its attitude. Recalling that the satellite 4-velocity is given by u' as in (9.140), we boost the vectors of the triad $\{\underset{s}{\lambda}_{\hat{a}}\}$ along the satellite's relative motion to obtain the following boosted triad (see (3.143)):

$$\underset{bs}{\lambda}{}^{\alpha}_{\hat{a}} = P(u')^{\alpha}{}_{\sigma} \left[\underset{s}{\lambda}{}^{\sigma}_{\hat{a}} - \frac{\gamma}{\gamma+1} \nu^{\sigma} \left(\nu^{\rho} \underset{s}{\lambda}_{\rho\hat{a}} \right) \right]_{\hat{a}=1,2,3} , \tag{9.155}$$

where

$$P(u')^{\alpha}{}_{\sigma} = \delta^{\alpha}_{\sigma} + u'^{\alpha} u'_{\sigma} \tag{9.156}$$

is the operator which projects onto the rest-space of u', $\nu^\alpha \equiv \nu(u',u)^\alpha$ is the relative spatial velocity of u' with respect to the local static observer u, defined as

$$\nu^\alpha = \frac{1}{\gamma}(u'^\alpha - \gamma u^\alpha), \tag{9.157}$$

and $\gamma \equiv \gamma(u',u) = -u'^\alpha u_\alpha$ is the relative Lorentz factor. The vector $\underset{bs}{\lambda}_{\hat{1}}$ identifies the direction to the Sun as seen from within the satellite. The other vectors of the boosted triad are related to $\underset{bs}{\lambda}_{\hat{1}}$ by the simple relations

$$\underset{bs}{\lambda}_{\hat{2}} = \frac{d}{d\phi_s}\underset{bs}{\lambda}_{\hat{1}}, \quad \underset{bs}{\lambda}_{\hat{3}} = \frac{d}{d\theta_s}\underset{bs}{\lambda}_{\hat{1}}. \tag{9.158}$$

The tetrad $\{\underset{bs}{\lambda}_{\hat{0}} \equiv u', \underset{bs}{\lambda}_{\hat{a}}\}$ will be referred to as the *Sun-locked frame*. The relation between the components $\nu(u',u)^\alpha$ of the spatial velocity $\nu(u',u)$ and the components β_i appearing in (9.140) is easily established from (9.140) itself and (9.157), and is given by

$$\nu(u',u)^\alpha = \frac{1}{\gamma}\left[T_s\left(\beta_i\delta^{i\alpha} + \delta^{0\alpha}\right) - u^\alpha\gamma\right]. \tag{9.159}$$

The explicit expressions for the components of the vectors $\underset{bs}{\lambda}_{\hat{a}}$ can be found in the cited literature (Bini, Crosta, and de Felice, 2003; de Felice and Preti, 2008).

10
Measurements of spinning bodies

A *test gyroscope* is a point-like massive particle having an additional "structure" termed *spin*. A spinning body is not, strictly speaking, point-like because its average size cannot be less than the ratio between its spin and its mass, in geometrized units. This guarantees that no point of the spinning body moves with respect to any observer at a velocity larger than c.

Rotation is a common feature in the universe so knowing the dynamics of rotating bodies is almost essential in modern physics. Although in most cases the assumption of no rotation is necessary to make the equations tractable, the existence of a strictly non-rotating system should be considered a rare event, and a measurement which revealed one would be of great interest. This is the case for the massive black holes which appear to exist in the nuclei of most galaxies. Detailed measurements are aimed at detecting their intrinsic angular momentum from the behavior of the surrounding medium. A black hole, however, could also be *seen* directly if we were able to detect the gravitational radiation it would emit after being perturbed by an external field. A direct measurement of gravitational waves is still out of reach for earthbound detectors; nevertheless they are still extensively searched for since they are the ultimate resource to investigate the nature of space-time. Because of the vast literature available on this topic, in what follows we shall limit ourselves to the interaction of gravitational waves with a spinning body, with the aim of searching for new ways to detect them.

10.1 Behavior of spin in general space-times

As already stated, we shall distinguish between a point-like gyroscope and an extended spinning body. In the first case, let us consider the world line γ of an observer U who carries the gyroscope. We denote the spin of the gyroscope by a vector S defined along γ, everywhere orthogonal to its tangent vector U and undergoing Fermi-Walker transport:

$$\frac{D_{(\mathrm{fw},U)}S}{d\tau_U} = \nabla_U S - (a(U)\cdot S)U = 0, \qquad S\cdot U = 0. \tag{10.1}$$

In the second case, the behavior of an extended spinning body is described by the Mathisson-Papapetrou equations (Mathisson, 1937; Papapetrou, 1951), later generalized by Dixon (1970a; 1970b; 1979):

$$\frac{D}{d\tau_U}P^\mu = -\frac{1}{2}R^\mu{}_{\alpha\beta\gamma}U^\alpha S^{\beta\gamma} \equiv F^{(\mathrm{spin})\,\mu}\,, \tag{10.2}$$

$$\frac{D}{d\tau_U}S^{\mu\nu} = 2P^{[\mu}U^{\nu]}, \tag{10.3}$$

where U is the unit time-like vector tangent to the line representative of the body (hereafter, the *center of mass line*), and P_μ and $S^{\mu\nu}$ are its momentum 4-vector and antisymmetric spin tensor, satisfying the additional conditions

$$S^{\mu\nu}P_\nu = 0. \tag{10.4}$$

Because of the spin–curvature coupling appearing in (10.2), U is in general accelerated, with $a(U) = f(U)$; therefore, following the notation of Section 6.7 (see (6.75)), we set

$$P(u,U)f(U) = \gamma F(U,u). \tag{10.5}$$

In the limit of small spin relative to the length scale of the background curvature, Eqs. (10.2)–(10.4) imply (10.1), so the two descriptions agree.

In what follows we shall consider first the motion of a test gyroscope.

Test gyroscope: the projected spin vector

Let u be a vector field tangent to a congruence $\mathcal{C}_u$ of curves which cross the world line of U. With respect to $\mathcal{C}_u$ the spin vector S admits the representation

$$S = \Sigma(U,u) + [\nu(U,u)\cdot\Sigma(U,u)]u, \qquad \Sigma(U,u) = P(u,U)S. \tag{10.6}$$

Since S is Fermi-Walker transported along U, its magnitude is constant, whereas the projected vector

$$\Sigma(U,u) = ||\Sigma(U,u)||\hat\Sigma(U,u) \tag{10.7}$$

varies. The evolution of $\Sigma(U,u)$ along U, as measured by the observer u, can be obtained as follows. Starting from (10.1) and using (10.6), we find

$$\begin{aligned}\frac{DS}{d\tau_U} &= \frac{D}{d\tau_U}\Sigma(U,u) + (\nu(U,u)\cdot\Sigma(U,u))\frac{Du}{d\tau_U}\\ &\quad + u\left[\frac{D}{d\tau_U}\nu(U,u)\cdot\Sigma(U,u) + \nu(U,u)\cdot\frac{D}{d\tau_U}\Sigma(U,u)\right],\end{aligned} \tag{10.8}$$

so that, acting on both sides with $P(u)$, we find, from (10.1),

$$(a(U) \cdot S)\gamma\nu = \frac{D_{(\mathrm{fw},U,u)}}{d\tau_U}\Sigma(U,u) - (\nu(U,u) \cdot \Sigma(U,u))F^{(G)}_{(\mathrm{fw},U,u)}, \tag{10.9}$$

recalling the definition (3.156) of the Fermi-Walker gravitational force

$$\frac{Du}{d\tau_U} = -F^{(G)}_{(\mathrm{fw},U,u)}. \tag{10.10}$$

Let us evaluate the term $(a(U) \cdot S)$. We have

$$\begin{aligned} a(U) \cdot S &= a(U) \cdot \Sigma(U,u) + (\nu(U,u) \cdot \Sigma(U,u))u \cdot a(U) \\ &= \gamma F(U,u) \cdot \Sigma(U,u) \\ &\quad + (\nu(U,u) \cdot \Sigma(U,u))u \cdot \nabla_U U. \end{aligned} \tag{10.11}$$

But

$$u \cdot \nabla_U U = -\frac{d\gamma}{d\tau_U} + \gamma\nu \cdot F^{(G)}_{(\mathrm{fw},U,u)} = -\gamma F(U,u) \cdot \nu, \tag{10.12}$$

where we have used Eq. (6.85), that is,

$$\frac{d\gamma}{d\tau_U} = \gamma[F(U,u) + F^{(G)}_{(\mathrm{fw},U,u)}] \cdot \nu(U,u). \tag{10.13}$$

Hence

$$a(U) \cdot S = \gamma\left[F(U,u) \cdot \Sigma(U,u) - (\nu(U,u) \cdot \Sigma(U,u))F(U,u) \cdot \nu\right]. \tag{10.14}$$

Using this in (10.9) and setting $\nu(U,u) = \nu$ and $F(U,u) = F$ to simplify notation, we obtain

$$\begin{aligned} \frac{D_{(\mathrm{fw},U,u)}}{d\tau_U}\Sigma(U,u) &= (\nu \cdot \Sigma(U,u))F^{(G)}_{(\mathrm{fw},U,u)} + (a(U) \cdot S)\gamma\nu \\ &= (\nu \cdot \Sigma(U,u))F^{(G)}_{(\mathrm{fw},U,u)} \\ &\quad + (F \cdot \Sigma(U,u) - (\nu \cdot \Sigma(U,u))F \cdot \nu)\gamma^2\nu. \end{aligned} \tag{10.15}$$

Let us now consider the evolution of the magnitude of $\Sigma(U,u)$, that is $||\Sigma(U,u)||$, and of its unit direction $\hat{\Sigma}(U,u) \equiv \hat{\Sigma}$ separately. Using (10.7) we find

$$\frac{d}{d\tau_U}\ln||\Sigma(U,u)|| = (\nu \cdot \hat{\Sigma})\left[\gamma^2\chi + F^{(G)}_{(\mathrm{fw},U,u)} \cdot \hat{\Sigma}\right], \tag{10.16}$$

where

$$\chi = F \cdot \hat{\Sigma} - (\nu \cdot \hat{\Sigma})(F \cdot \nu). \tag{10.17}$$

Using the relative standard time parameterization, we deduce

$$\begin{aligned} \frac{D_{(\mathrm{fw},U,u)}}{d\tau_{(U,u)}}\hat{\Sigma} &= \gamma\chi\left[\hat{\Sigma} \times_u (\nu \times_u \hat{\Sigma})\right] \\ &\quad + \frac{1}{\gamma}(\hat{\Sigma} \cdot \nu)\left[\hat{\Sigma} \times_u (F^{(G)}_{(\mathrm{fw},U,u)} \times_u \hat{\Sigma})\right]. \end{aligned} \tag{10.18}$$

We can also cast (10.18) in the form

$$\frac{D_{(\mathrm{fw},U,u)}}{d\tau_{(U,u)}}\hat{\Sigma} = \zeta_{(\mathrm{fw})}^{(\mathrm{project})} \times_u \hat{\Sigma}, \tag{10.19}$$

where

$$\begin{aligned}\zeta_{(\mathrm{fw})}^{(\mathrm{project})} &= \gamma\chi(\hat{\Sigma} \times_u \nu) + \frac{1}{\gamma}(\nu \cdot \hat{\Sigma})\left(\hat{\Sigma} \times_u F_{(\mathrm{fw},U,u)}^{(G)}\right)\\ &= \hat{\Sigma} \times_u \left[\gamma\chi\nu + \frac{1}{\gamma}(\nu \cdot \hat{\Sigma})F_{(\mathrm{fw},U,u)}^{(G)}\right]\end{aligned} \tag{10.20}$$

is the precession rate of the gyroscope as it would be locally measured by u.

Boosted spin vector

Let us examine now the complementary problem of describing the precession of a gyroscope as it would be measured by the observer U who carries it, relative to a frame, say $\{e(u)_a\}$, adapted to the observers of the congruence $\mathcal{C}_u$. According to U, the axes $e(u)_a$ are moving axes; hence their orientation in LRS_U is defined by boosting them onto LRS_U itself, that is

$$E(U)_a = B_{(\mathrm{lrs})}(U,u)e(u)_a. \tag{10.21}$$

Our aim here is to study the precession rate of the spin vector S with respect to the axes $E(U)_a$. However, since the boost is an isometry, the precession of S with respect to the axes $E(U)_a$ is the same as the precession of the boosted vector

$$\mathcal{S}(U,u) = B_{(\mathrm{lrs})}(u,U)S, \tag{10.22}$$

momentarily at rest with respect to u, as measured with respect to the axes $e(u)_a$. Using the representation (3.143) of the boost, we have

$$\mathcal{S}(U,u) = \Sigma(U,u) - \frac{\gamma}{1+\gamma}(\nu(U,u)\cdot\Sigma(U,u))\nu(U,u). \tag{10.23}$$

The evolution of $\mathcal{S}(U,u)$ along U is a consequence of the similar result for $\Sigma(U,u)$ and can be cast in the form

$$\frac{D_{(\mathrm{fw},U,u)}}{d\tau_{(U,u)}}\mathcal{S}(U,u) = \zeta_{(\mathrm{fw})}^{(\mathrm{boost})} \times_u \mathcal{S}(U,u), \tag{10.24}$$

where $\zeta_{(\mathrm{fw})}^{(\mathrm{boost})}$ is made up of two terms:

$$\zeta_{(\mathrm{fw})}^{(\mathrm{boost})} = \zeta_{(\mathrm{Thomas})} + \zeta_{(\mathrm{geo})}, \tag{10.25}$$

where

$$\zeta_{(\mathrm{Thomas})} = -\frac{\gamma}{1+\gamma}\nu(U,u) \times_u F(U,u), \tag{10.26}$$

and

$$\zeta_{(\mathrm{geo})} = \frac{1}{1+\gamma}\nu(U,u) \times_u F^{(G)}_{(\mathrm{fw},U,u)}. \qquad (10.27)$$

The Thomas precession arises from the acceleration of the world line of the observer who carries the gyroscope and is due to the non-gravitational force $F(U,u)$ defined in (6.75). The geodesic precession is due to the gravitational force $F^{(G)}_{(\mathrm{fw},U,u)}$ only.

Let us derive (10.24). Start from (10.23) and replace $\Sigma(U,u)$ using (10.6) to obtain

$$\begin{aligned}\mathcal{S}(U,u) &= S - \frac{[\nu(U,u)\cdot\Sigma(U,u)]}{1+\gamma}(U+u) \\ &= S - \frac{\gamma[\nu(U,u)\cdot\mathcal{S}(U,u)]}{1+\gamma}(U+u),\end{aligned} \qquad (10.28)$$

because

$$\nu(U,u)\cdot\Sigma(U,u) = \gamma[\nu(U,u)\cdot\mathcal{S}(U,u)]. \qquad (10.29)$$

Let f denote the quantity

$$f = \frac{\gamma[\nu(U,u)\cdot\mathcal{S}(U,u)]}{1+\gamma}. \qquad (10.30)$$

Differentiate both sides of (10.28) along U:

$$\begin{aligned}\frac{D}{d\tau_U}\mathcal{S}(U,u) &= \frac{DS}{d\tau_U} - (u+U)\frac{df}{d\tau_U} - \left(a(U) + \frac{D}{d\tau_U}u\right) f \\ &= (a(U)\cdot S)U - (u+U)\frac{df}{d\tau_U} \\ &\quad - \left(a(U) - F^{(G)}_{(\mathrm{fw},U,u)}\right) f,\end{aligned} \qquad (10.31)$$

where we have used the properties of S of being orthogonal to U and of being Fermi-Walker transported along it, that is

$$\frac{D_{(\mathrm{fw},U)}}{d\tau_U}S = \frac{D}{d\tau_U}S - (a(U)\cdot S)U = 0. \qquad (10.32)$$

From (10.28) we have

$$a(U)\cdot S = a(U)\cdot\mathcal{S}(U,u) + (u\cdot a(U))f. \qquad (10.33)$$

But

$$u\cdot a(U) = u\cdot\nabla_U U = -\frac{d\gamma}{d\tau_U} + \gamma\nu(U,u)\cdot F^{(G)}_{(\mathrm{fw},U,u)}, \qquad (10.34)$$

so that Eq. (10.13) gives

$$u\cdot a(U) = -\gamma\nu(U,u)\cdot F(U,u). \qquad (10.35)$$

Therefore

$$\begin{aligned}\frac{D}{d\tau_U}\mathcal{S}(U,u) &= [a(U)\cdot\mathcal{S}(U,u) - f\gamma\nu(U,u)\cdot F(U,u)]\,U\\ &\quad -\frac{df}{d\tau_U}(u+U) - \left(a(U) - F^{(G)}_{(\mathrm{fw},U,u)}\right)f\\ &= \gamma\,[F(U,u)\cdot\mathcal{S}(U,u) - f\nu(U,u)\cdot F(U,u)]\,U\\ &\quad -(u+U)\frac{df}{d\tau_U} - \left(a(U) - F^{(G)}_{(\mathrm{fw},U,u)}\right)f. \end{aligned}\tag{10.36}$$

First project both sides of (10.36) orthogonally to u, obtaining

$$\begin{aligned}\frac{D_{(\mathrm{fw},U,u)}}{d\tau_U}\mathcal{S}(U,u) = \gamma^2\,&[F(U,u)\cdot\mathcal{S}(U,u)\\ &- f\nu(U,u)\cdot F(U,u)]\,\nu(U,u)\\ &-\gamma\nu\frac{df}{d\tau_U} + \left(-\gamma F + F^{(G)}_{(\mathrm{fw},U,u)}\right)f. \end{aligned}\tag{10.37}$$

Then take the scalar product of both sides of (10.36) with u, to give

$$\begin{aligned}u\cdot\frac{D}{d\tau_U}\mathcal{S}(U,u) &= -\gamma^2(F(U,u)\cdot\mathcal{S}(U,u) - f\nu(U,u)\cdot F(U,u))\\ &\quad +\frac{df}{d\tau_U}(\gamma+1) - (u\cdot a(U))\,f, \end{aligned}\tag{10.38}$$

that is, since $u\cdot\mathcal{S}(U,u)=0$,

$$\begin{aligned}F^{(G)}_{(\mathrm{fw},U,u)}\cdot\mathcal{S}(U,u) &= -\gamma^2(F(U,u)\cdot\mathcal{S}(U,u)\\ &\quad - f\nu(U,u)\cdot F(U,u))\\ &\quad +\frac{df}{d\tau_U}(\gamma+1)\\ &\quad + f\gamma\nu(U,u)\cdot F(U,u)\\ &= -\gamma^2 F(U,u)\cdot\mathcal{S}(U,u)\\ &\quad +\gamma(f\nu(U,u)\cdot F(U,u))(1+\gamma)\\ &\quad +\frac{df}{d\tau_U}(\gamma+1). \end{aligned}\tag{10.39}$$

From this it follows that

$$\begin{aligned}\frac{df}{d\tau_U} &= (1+\gamma)^{-1}\left(F^{(G)}_{(\mathrm{fw},U,u)} + \gamma^2 F(U,u)\right)\cdot\mathcal{S}(U,u)\\ &\quad -\gamma f\nu(U,u)\cdot F(U,u). \end{aligned}\tag{10.40}$$

Substituting this expression into (10.37), we obtain the final result,

$$\begin{aligned}\frac{D_{(\mathrm{fw},U,u)}}{d\tau_U}\mathcal{S}(U,u) &= \frac{\gamma}{1+\gamma}\,[\nu(U,u)\otimes\mathcal{S}(U,u) - \nu(U,u)\cdot\mathcal{S}(U,u)P(u)]\\ &\quad \llcorner\left(\gamma F(U,u) - F^{(G)}_{(\mathrm{fw},U,u)}\right), \end{aligned}\tag{10.41}$$

which is equivalent to (10.24).

If the frame $\{e(u)_a\}$ is orthonormal, then we can further manipulate (10.24). In this case we have

$$\frac{D_{(\mathrm{fw},U,u)}}{d\tau_{(U,u)}}\mathcal{S}(U,u)^{\hat{a}} = \frac{d\mathcal{S}(U,u)^{\hat{a}}}{d\tau_{(U,u)}} + \left(C_{(\mathrm{fw})}{}^{\hat{a}}{}_{\hat{c}} + \Gamma^{\hat{a}}{}_{\hat{c}\hat{b}}\,\nu(U,u)^{\hat{b}}\right)\mathcal{S}(U,u)^{\hat{c}}. \qquad (10.42)$$

Introducing the quantities

$$\zeta^{\hat{a}}_{(\mathrm{fw})} = -\frac{1}{2}\eta(u)^{\hat{a}\hat{b}\hat{c}}C_{(\mathrm{fw})\,\hat{b}\hat{c}} \qquad (10.43)$$

and the spatial curvature angular velocity

$$\zeta^{\hat{a}}_{(\mathrm{sc})} = -\frac{1}{2}\eta(u)^{\hat{a}\hat{b}\hat{c}}\Gamma_{[\hat{c}|\hat{d}|\hat{b}]}\,\nu(U,u)^{\hat{d}}, \qquad (10.44)$$

already defined in (4.18) and (4.20) respectively, we have

$$\frac{D_{(\mathrm{fw},U,u)}}{d\tau_{(U,u)}}\mathcal{S}(U,u)^{\hat{a}} = \frac{d\mathcal{S}(U,u)^{a}}{d\tau_{(U,u)}} + [(\zeta_{(\mathrm{fw})} + \zeta_{(\mathrm{sc})})\times_u \mathcal{S}(U,u)]^{\hat{a}}, \qquad (10.45)$$

and hence

$$\begin{aligned}\frac{d\mathcal{S}(U,u)^{\hat{a}}}{d\tau_{(U,u)}} &= \left[\left(\zeta^{(\mathrm{boost})}_{(\mathrm{fw})} - \zeta_{(\mathrm{fw})} - \zeta_{(\mathrm{sc})}\right)\times_u \mathcal{S}(U,u)\right]^{\hat{a}} \\ &\equiv \left[\zeta^{(\mathrm{tot})}\times_u \mathcal{S}(U,u)\right]^{\hat{a}}, \end{aligned} \qquad (10.46)$$

or equivalently

$$\frac{d\mathcal{S}(U,u)^{\hat{a}}}{d\tau_{(U,u)}} = \eta(u)^{\hat{a}}{}_{\hat{b}\hat{c}}\zeta^{(\mathrm{tot})\,\hat{b}}\mathcal{S}(U,u)^{\hat{c}}. \qquad (10.47)$$

Note that a change of parameter along U implies a rescaling of the precession angular velocity. For instance, the above formula can also be written as

$$\frac{d\mathcal{S}(U,u)^{\hat{a}}}{d\tau_U} = \eta(u)^{\hat{a}}{}_{\hat{b}\hat{c}}[\gamma\zeta^{(\mathrm{tot})}]^{\hat{b}}\mathcal{S}(U,u)^{\hat{c}}, \qquad (10.48)$$

when the world line U is parameterized by the proper time τ_U. In this case the effective total angular velocity is

$$\zeta^{(\mathrm{tot})}_U = \gamma\zeta^{(\mathrm{tot})}. \qquad (10.49)$$

10.2 Motion of a test gyroscope in a weak gravitational field

Let us consider the post-Newtonian (PN) treatment of a weak gravitational field within general relativity and evaluate the orders of magnitude of the quantities we shall meet in our treatment in terms of powers of $1/c$, c being the velocity of light in vacuum. In a PN coordinate system, say (t, x^a) $(a = 1, 2, 3)$, the space-time metric can be written as

$$ds^2 = -(1-2\Phi)dt^2 + 2\Phi_i dt dx^i + (1+2\Phi)\delta_{ij}dx^i dx^j + O(4), \tag{10.50}$$

where $\Phi = O(2)$ and $\Phi_i = O(3)$. We use the three-dimensional notation

$$\vec{\Phi} = \Phi_1 \partial_x + \Phi_2 \partial_y + \Phi_3 \partial_z \tag{10.51}$$

and the ordinary Euclidean space operations for grad, curl, div, and vector product, unless otherwise specified. Let u be the family of observers at rest with respect to the coordinate grid

$$u = \frac{1}{\sqrt{-g_{tt}}}\partial_t = (1+\Phi)\partial_t + O(4), \tag{10.52}$$

so that

$$u^\flat = (-1+\Phi)dt + \Phi_a dx^a + O(4). \tag{10.53}$$

Let us fix an orthonormal frame adapted to u as follows:

$$e_{\hat{a}} = \Phi_a \partial_t + (1-\Phi)\partial_a, \qquad \omega^{\hat{a}} = (1+\Phi)\delta^{\hat{a}}_b dx^b. \tag{10.54}$$

The observers u have acceleration

$$a(u) = -\mathrm{grad}_u\, \Phi + O(4)\,; \tag{10.55}$$

they also have vorticity

$$\omega(u) = \frac{1}{2}\mathrm{curl}_u\, \vec{\Phi} + O(4) \tag{10.56}$$

and expansion[1]

$$\theta(u)_{ab} = \partial_t \Phi\, \delta_{ab} + O(4). \tag{10.57}$$

A direct calculation shows that

$$\begin{aligned}\nabla_u e_{\hat{a}} &= -\partial_a \Phi\, u + \omega(u) \times_u e_{\hat{a}} + O(4) \\ &= a(u)_a\, u + \omega(u) \times_u e_{\hat{a}} + O(4),\end{aligned} \tag{10.58}$$

and hence

$$C_{(\mathrm{fw})}{}^{\hat{b}}{}_{\hat{a}} e_{\hat{b}} = \omega(u) \times_u e_{\hat{a}}, \tag{10.59}$$

which implies

$$\omega(u) = \zeta_{(\mathrm{fw})}. \tag{10.60}$$

Similarly, the non-vanishing spatial Ricci rotation coefficients

$$\omega^{\hat{c}}\left(\nabla_{e_{\hat{b}}} e_{\hat{a}}\right) = \Gamma^{\hat{c}}{}_{\hat{a}\hat{b}} \tag{10.61}$$

[1] We note that for these three fields, that is acceleration, vorticity, and expansion, the coordinate and frame components coincide.

are as follows:

$$\begin{aligned}
\Gamma^{\hat{x}}{}_{\hat{y}\hat{y}} &= \Gamma^{\hat{x}}{}_{\hat{z}\hat{z}} = -\Gamma^{\hat{y}}{}_{\hat{x}\hat{y}} = -\Gamma^{\hat{z}}{}_{\hat{x}\hat{z}} = -\partial_x \Phi, \\
\Gamma^{\hat{y}}{}_{\hat{x}\hat{x}} &= \Gamma^{\hat{y}}{}_{\hat{z}\hat{z}} = -\Gamma^{\hat{x}}{}_{\hat{y}\hat{x}} = -\Gamma^{\hat{z}}{}_{\hat{y}\hat{z}} = -\partial_y \Phi, \\
\Gamma^{\hat{z}}{}_{\hat{x}\hat{x}} &= \Gamma^{\hat{z}}{}_{\hat{y}\hat{y}} = -\Gamma^{\hat{x}}{}_{\hat{z}\hat{x}} = -\Gamma^{\hat{y}}{}_{\hat{z}\hat{y}} = -\partial_z \Phi.
\end{aligned} \tag{10.62}$$

If $U = \gamma[u + \nu(U,u)^{\hat{a}} e_{\hat{a}}]$ is the 4-velocity of the observer carrying the gyroscope and we assume that it satisfies the slow motion condition $||\nu(U,u)|| \ll 1$, then we can evaluate the spatial curvature angular velocity

$$\nu^{\hat{c}} \Gamma^{\hat{d}}{}_{\hat{a}\hat{c}} = -\epsilon^{\hat{d}}{}_{\hat{a}\hat{f}} \zeta^{\hat{f}}_{(\text{sc})}, \tag{10.63}$$

and find that

$$\zeta_{(\text{sc})} = -\nu \times_u \operatorname{grad}_u \Phi. \tag{10.64}$$

Finally, we have

$$\zeta^{(\text{boost})}_{(\text{fw})} = -\frac{1}{2}\nu \times_u F - \frac{1}{2}\nu \times_u a(u) = -\frac{1}{2}\nu \times_u F + \frac{1}{2}\nu \times_u \operatorname{grad}_u \Phi, \tag{10.65}$$

so that the precession angular velocity of (10.46) becomes

$$\begin{aligned}
\zeta^{(\text{tot})} &= -\frac{1}{2}\nu \times_u F + \frac{1}{2}\nu \times_u \operatorname{grad}_u \Phi \\
&\quad + \nu \times_u \operatorname{grad}_u \Phi - \frac{1}{2}\operatorname{curl}_u \vec{\Phi} \\
&= -\frac{1}{2}\nu \times_u F + \frac{3}{2}\nu \times_u \operatorname{grad}_u \Phi \\
&\quad - \frac{1}{2}\operatorname{curl}_u \vec{\Phi}.
\end{aligned} \tag{10.66}$$

When a gyroscope moves along a geodesic ($F = 0$), Eq. (10.66) is known as the *Schiff formula*; that is,

$$\zeta^{(\text{tot})} = \frac{3}{2}\nu \times_u \operatorname{grad}_u \Phi - \frac{1}{2}\operatorname{curl}_u \vec{\Phi}. \tag{10.67}$$

The part of $\zeta^{(\text{tot})}$ which is contributed by the intrinsic spin of the gravity source is responsible for a relativistic effect implied by the Lense-Thirring metric, and known as the Lense-Thirring effect.

10.3 Motion of a test gyroscope in Schwarzschild space-time

We shall now study the motion of a test gyroscope in Schwarzschild space-time. Our aim here is to deduce the precession of a gyroscope carried by an observer U, but with respect to a frame $e(m)_a$ given by (8.18), adapted to the static observers. In this way we can specialize the general formulas given above.

The evolution of the boosted spin vector $\mathcal{S}(U,m)$ along U can be cast in the form

$$\frac{D_{(\text{fw},U,m)}}{d\tau_{(U,m)}} \mathcal{S}(U,m) = \zeta^{(\text{boost})}_{(\text{fw})}(U,m) \times_m \mathcal{S}(U,m), \tag{10.68}$$

where

$$\begin{aligned}\zeta_{(\mathrm{fw})}^{(\mathrm{boost})} &= \zeta_{(\mathrm{Thomas})} + \zeta_{(\mathrm{geo})}\\ &= \frac{\gamma}{1+\gamma}\nu(U,m)\times_m\left[-F(U,m)+\gamma^{-1}F^{(\mathrm{G})}_{(\mathrm{fw},U,m)}\right]\\ &= \frac{\gamma}{1+\gamma}\nu(U,m)\times_m\left[-F(U,m)-a(m)\right].\end{aligned}\tag{10.69}$$

$\zeta_{(\mathrm{fw})}^{(\mathrm{boost})}$ has a component only along the latitudinal frame direction $\hat\theta$,

$$\zeta_{(\mathrm{fw})}^{(\mathrm{boost})\hat\theta} = \frac{\gamma}{1+\gamma}\nu(U,m)^{\hat\phi}\left[-F(U,m)^{\hat r}-a(m)^{\hat r}\right],\tag{10.70}$$

with

$$\begin{aligned}F(U,m)^{\hat r} &= \Gamma\left(\frac{\mathcal{M}}{r^2}-r\zeta^2\right)=\frac{\gamma}{N}\left(\frac{\mathcal{M}}{r^2}-r\zeta^2\right),\\ a(m)^{\hat r} &= \left(1-\frac{2\mathcal{M}}{r}\right)^{-1/2}\frac{\mathcal{M}}{r^2}=\frac{\mathcal{M}}{Nr^2},\end{aligned}\tag{10.71}$$

with

$$N=\left(1-\frac{2\mathcal{M}}{r}\right)^{1/2}=\frac{r}{\mathcal{R}}.\tag{10.72}$$

We then have

$$\begin{aligned}\zeta_{(\mathrm{fw})}^{(\mathrm{boost})\hat\theta} &= \frac{\gamma}{1+\gamma}\nu(U,m)^{\hat\phi}\left[-\frac{\gamma}{N}\left(\frac{\mathcal{M}}{r^2}-r\zeta^2\right)-\frac{\mathcal{M}}{Nr^2}\right]\\ &= -\gamma\frac{\nu(U,m)^{\hat\phi}}{N}\frac{\mathcal{M}}{r^2}+\frac{r\gamma^2\zeta^2}{1+\gamma}\frac{\nu(U,m)^{\hat\phi}}{N}.\end{aligned}\tag{10.73}$$

Recalling that

$$\nu(U,m)^{\hat\phi}=\frac{r\zeta}{N},\tag{10.74}$$

we have

$$\zeta_{(\mathrm{fw})}^{(\mathrm{boost})\hat\theta} = -\gamma\frac{\mathcal{M}\zeta}{rN^2}+\frac{r^2\gamma^2\zeta^2}{1+\gamma}\frac{\zeta}{N^2}.\tag{10.75}$$

Moreover,

$$\frac{r^2\gamma^2\zeta^2}{1+\gamma}\frac{1}{N^2}=\frac{\gamma^2||\nu(U,m)||^2}{1+\gamma}=\frac{\gamma^2-1}{1+\gamma}=\gamma-1.\tag{10.76}$$

Therefore

$$\begin{aligned}\zeta_{(\mathrm{fw})}^{(\mathrm{boost})\hat\theta} &= -\gamma\frac{\mathcal{M}\zeta}{rN^2}+\zeta(\gamma-1)=\zeta\gamma\left[-\frac{\mathcal{M}}{rN^2}+1\right]-\zeta\\ &= \frac{\zeta\gamma}{N^2}\left(1-\frac{3\mathcal{M}}{r}\right)-\zeta.\end{aligned}\tag{10.77}$$

Taking into account now that $\zeta_{(\mathrm{fw})} = 0$ and that, from (8.48) with $\theta = \pi/2$,

$$\zeta_{(\mathrm{sc})} = -\zeta e_{\hat\theta}, \tag{10.78}$$

we have the Shiff formula in Schwarzschild space-time,

$$\zeta^{(\mathrm{tot})\,\hat\theta} = \zeta_{(\mathrm{fw})}^{(\mathrm{boost})\,\hat\theta} - \zeta^{\hat\theta}_{(\mathrm{sc})} = \frac{\zeta\gamma}{N^2}\left(1 - \frac{3\mathcal{M}}{r}\right). \tag{10.79}$$

Finally, from (8.69), we see that

$$\zeta^{(\mathrm{tot})\,\hat\theta} = \frac{\tau_1(U)}{\gamma}. \tag{10.80}$$

As a convention the physical (orthonormal) component along $-\partial_\theta$, perpendicular to the equatorial plane, is taken to be along the positive z-axis, denoted by $\hat z$. Therefore,

$$\zeta^{(\mathrm{tot})\,\hat z} = -\frac{\tau_1(U)}{\gamma}. \tag{10.81}$$

This result agrees with the expression for $\overset{g}{\zeta}$ given in Section 9.2. In fact, assuming U is a geodesic parameterized by the proper time and taking into account the discussion at the end of Section 10.1 we have

$$\overset{g}{\zeta} \equiv \zeta_U^{(\mathrm{tot})\,\hat z} = -\tau_1(U) = \Gamma^2\zeta\left(\frac{3\mathcal{M}}{r} - 1\right), \tag{10.82}$$

where the expression (8.69) for $\tau_1(U)$ has been used. Finally, introducing the proper frequency $\overset{p}{\zeta} = \Gamma\zeta$ yields

$$\overset{g}{\zeta} = \Gamma\overset{p}{\zeta}\left(\frac{3\mathcal{M}}{r} - 1\right). \tag{10.83}$$

10.4 Motion of a spinning body in Schwarzschild space-time

The equations of motion are given by (10.2) and (10.3) coupled to the supplementary condition (10.4), where, as stated, P_μ is the total 4-momentum of the particle and $S^{\mu\nu}$ is its (antisymmetric) spin tensor. In those equations, U is the time-like unit tangent vector of the center-of-mass line used to make the multipole reduction. Equation (10.2) is also referred to as the Riemann force equation.

Let the Schwarzschild metric be written in spherical-type coordinates (8.1), and introduce an orthonormal frame adapted to the static observers:

$$\begin{aligned} e_{\hat t} &= (1 - 2\mathcal{M}/r)^{-1/2}\,\partial_t, & e_{\hat r} &= (1 - 2\mathcal{M}/r)^{1/2}\,\partial_r,\\ e_{\hat\theta} &= \frac{1}{r}\,\partial_\theta, & e_{\hat\phi} &= \frac{1}{r\sin\theta}\,\partial_\phi, \end{aligned} \tag{10.84}$$

with dual

$$\omega^{\hat{t}} = (1 - 2\mathcal{M}/r)^{1/2}\, dt, \qquad \omega^{\hat{r}} = (1 - 2\mathcal{M}/r)^{-1/2} dr,$$
$$\omega^{\hat{\theta}} = r\, d\theta, \qquad \omega^{\hat{\phi}} = r \sin\theta\, d\phi. \tag{10.85}$$

Let us assume that U is tangent to a (time-like) spatially circular orbit, which we here denote as a U-orbit, with

$$U = \Gamma[\partial_t + \zeta \partial_\phi] = \gamma[e_{\hat{t}} + \nu e_{\hat{\phi}}], \qquad \gamma = (1 - \nu^2)^{-1/2}. \tag{10.86}$$

ζ is the angular velocity of the orbital revolution as it would be measured at infinity, ν is the magnitude of the local proper linear velocity measured in the frame (10.84), and Γ is a normalization factor given by

$$\Gamma = \left(-g_{tt} - \zeta^2 g_{\phi\phi}\right)^{-1/2} \tag{10.87}$$

in order to ensure that $U \cdot U = -1$. The angular velocity ζ is related to ν by

$$\zeta = \sqrt{-\frac{g_{tt}}{g_{\phi\phi}}}\, \nu. \tag{10.88}$$

Here ζ and therefore also ν are assumed to be constant along the U-orbit. We limit our analysis to the equatorial plane ($\theta = \pi/2$) of the Schwarzschild solution; again, as a convention, the physical (orthonormal) component along $-\partial_\theta$, perpendicular to the equatorial plane, will be taken to be along the positive z-axis and will be indicated by $\hat{z}$, as necessary.

Among the circular orbits analyzed in detail in Section 8.2, particular attention is devoted to the time-like spatially circular geodesics which corotate (ζ_+) and counter-rotate (ζ_-) relative to a pre-assigned positive sense of variation of the azimuthal coordinate ϕ. They have respective angular velocities $\zeta_\pm \equiv \pm\zeta_K = \pm(\mathcal{M}/r^3)^{1/2}$, so that

$$U_\pm = \Gamma_\pm[\partial_t + \zeta_\pm \partial_\phi] = \gamma_K[e_{\hat{t}} \pm \nu_K e_{\hat{\phi}}], \tag{10.89}$$

where

$$\nu_K = \left[\frac{\mathcal{M}}{r - 2\mathcal{M}}\right]^{1/2}, \qquad \gamma_K = \left[\frac{r - 2\mathcal{M}}{r - 3\mathcal{M}}\right]^{1/2}, \tag{10.90}$$

and with the time-like condition $\nu_K < 1$ satisfied if $r > 3\mathcal{M}$. Here γ_K is the local (geodesic) Lorentz factor relative to the frame (10.84).

Let us now introduce the Lie relative curvature (8.66) of each U-orbit,

$$k_{(\text{lie})} \equiv k_{(\text{lie})\hat{r}} = -\partial_{\hat{r}} \ln \sqrt{g_{\phi\phi}} = -\frac{1}{r}\left(1 - \frac{2\mathcal{M}}{r}\right)^{1/2} = -\frac{\zeta_K}{\nu_K}, \tag{10.91}$$

as well as a Frenet-Serret intrinsic frame along U, defined by

$$E_{\hat{t}} = U, \qquad E_{\hat{r}} = e_{\hat{r}}, \qquad E_{\hat{z}} = e_{\hat{z}}, \qquad E_{\hat{\phi}} = \gamma[\nu e_{\hat{t}} + e_{\hat{\phi}}], \tag{10.92}$$

satisfying the following system of evolution equations:

$$\frac{DU}{d\tau_U} \equiv a(U) = \kappa E_{\hat{r}}, \qquad \frac{DE_{\hat{r}}}{d\tau_U} = \kappa U + \tau_1 E_{\hat{\phi}},$$
$$\frac{DE_{\hat{\phi}}}{d\tau_U} = -\tau_1 E_{\hat{r}}, \qquad \frac{DE_{\hat{z}}}{d\tau_U} = 0, \tag{10.93}$$

where

$$\kappa = k_{(\mathrm{lie})}\gamma^2[\nu^2 - \nu_K^2], \qquad \tau_1 = -\frac{1}{2\gamma^2}\frac{d\kappa}{d\nu} = -k_{(\mathrm{lie})}\frac{\gamma^2}{\gamma_K^2}\nu\,; \tag{10.94}$$

in this case the second torsion τ_2 is identically zero. The dual of (10.92) is given by

$$\Omega^{\hat{t}} = -U, \qquad \Omega^{\hat{r}} = \omega^{\hat{r}}, \qquad \Omega^{\hat{z}} = \omega^{\hat{z}}, \qquad \Omega^{\hat{\phi}} = \gamma[-\nu\omega^{\hat{t}} + \omega^{\hat{\phi}}]. \tag{10.95}$$

To study the motion of spinning test particles in circular orbits let us consider first the evolution equation for the spin tensor (10.3). For simplicity, we search for solutions of the Riemann force equation (10.2) describing frame components of the spin tensor which are constant along the orbit. By contracting both sides of (10.3) with U_ν, one obtains the following expression for the total 4-momentum:

$$P^\mu = -(U \cdot P)U^\mu - U_\nu \frac{DS^{\mu\nu}}{d\tau_U} \equiv mU^\mu + P_s^\mu, \tag{10.96}$$

where m is the particle's *bare* mass, i.e. the mass it would have were it not spinning, and $P_s = U \llcorner DS/d\tau_U$ is a 4-vector orthogonal to U. As a consequence of (10.96), Eq. (10.3) is equivalent to

$$P(U)^\mu_\alpha P(U)^\nu_\beta \frac{DS^{\alpha\beta}}{d\tau_U} = 0, \tag{10.97}$$

where $P(U)^\mu_\alpha = \delta^\mu_\alpha + U^\mu U_\alpha$ projects into the local rest space of U; this implies

$$S_{\hat{t}\hat{\phi}} = 0, \qquad S_{\hat{r}\hat{\theta}} = 0, \qquad S_{\hat{t}\hat{\theta}} + S_{\hat{\phi}\hat{\theta}}\frac{\nu}{\nu_K^2} = 0. \tag{10.98}$$

From (10.92)–(10.95) it follows that

$$\frac{DS}{d\tau_U} = m_s[\Omega^{\hat{\phi}} \wedge U]\,; \tag{10.99}$$

hence P_s can be written as

$$P_s = m_s \Omega^{\hat{\phi}}, \tag{10.100}$$

where

$$m_s \equiv ||P_s|| = \gamma\frac{\zeta_K}{\nu_K}\left[-\nu_K^2 S_{\hat{r}\hat{\phi}} + \nu S_{\hat{t}\hat{r}}\right]. \tag{10.101}$$

From (10.96) and (10.100), and provided $m + \nu m_s \neq 0$, the total 4-momentum P can be written in the form $P = \mu U_p$, with

$$U_p = \gamma_p\,[e_{\hat t} + \nu_p e_{\hat\phi}], \quad \nu_p = \frac{\nu + m_s/m}{1 + \nu m_s/m}, \quad \mu = \frac{\gamma}{\gamma_p}(m + \nu m_s), \tag{10.102}$$

where $\gamma_p = (1-\nu_p^2)^{-1/2}$. U_p is a time-like unit vector; hence μ has the property of a physical mass. The first part of (10.102) tells us that, as the particle's center of mass moves along the U-orbit, the momentum P is instantaneously (i.e. at each point of the U-orbit) parallel to a unit vector which is tangent to a spatially circular orbit, hereafter denoted as a U_p-orbit, which intersects the U-orbit at each of its points. Although U- and U_p-orbits have the same spatial projections into the U-quotient space, there exists in the space-time one U_p-orbit for each point of the U-orbit where the two intersect.

Let us now consider the equation of motion (10.2). The spin-force is equal to

$$F^{(\mathrm{spin})} = \gamma\,\zeta_K^2\left[2S_{\hat t\hat r} + \nu S_{\hat r\hat\phi}\right]e_{\hat r} - \gamma\frac{\nu}{r^2}S_{\hat\theta\hat\phi}e_{\hat\theta}, \tag{10.103}$$

while the term on the left-hand side of Eq. (10.2) can be written, from (10.96) and (10.100), as

$$\begin{aligned}\frac{DP}{d\tau_U} &= ma(U) + m_s\frac{DE_{\hat\phi}}{d\tau_U},\\ &= (m\kappa - m_s\tau_1)e_{\hat r},\end{aligned} \tag{10.104}$$

where κ and τ_1 are given in (10.94) and the quantities μ, m, and m_s are constant along the world line of U. The acceleration κ of the U-orbit vanishes if the latter is a geodesic ($\nu = \nu_K$); the term $-m_s\tau_1 e_{\hat r}$ instead is a spin–rotation coupling force that was predicted by Mashhoon (1988; 1995; 1999), a sort of centrifugal force which of course vanishes when the particle is at rest ($\nu = 0$), as can be seen in $(10.94)_2$.

Since $DP/d\tau_U$ is directed radially, as (10.104) shows, Eq. (10.2) requires that $S_{\hat\theta\hat\phi} = 0$ (and therefore also $S_{\hat t\hat\theta} = 0$ from (10.98)); hence (10.2) can be written as

$$m\kappa - m_s\tau_1 - F_{\hat r}^{(\mathrm{spin})} = 0, \tag{10.105}$$

or, more explicitly,

$$0 = m\gamma[\nu^2 - \nu_K^2] + m_s\frac{\gamma\nu}{\gamma_K^2} + \nu_K\zeta_K\left[2S_{\hat t\hat r} + \nu S_{\hat r\hat\phi}\right]. \tag{10.106}$$

Summarizing, from the equations of motions (10.2) and (10.3) and before imposing (10.4), the spin tensor turns out to be completely determined by only two components, namely $S_{\hat t\hat r}$ and $S_{\hat r\hat\phi}$, related by (10.106). The spin tensor then takes the form

$$S = \omega^{\hat r}\wedge[S_{\hat r\hat t}\omega^{\hat t} + S_{\hat r\hat\phi}\omega^{\hat\phi}]. \tag{10.107}$$

It is useful to introduce, together with the quadratic invariant

$$s^2 = \frac{1}{2} S_{\mu\nu} S^{\mu\nu} = -S^2_{\hat{r}\hat{t}} + S^2_{\hat{r}\hat{\phi}}, \tag{10.108}$$

a frame adapted to U_p, given by

$$E^p_0 = U_p, \quad E^p_1 = e_{\hat{r}}, \quad E^p_2 = \gamma_p(\nu_p e_{\hat{t}} + e_{\hat{\phi}}), \quad E^p_3 = e_{\hat{z}}, \tag{10.109}$$

whose dual frame is denoted by $\Omega^{p\hat{a}}$. Imposing (10.4), one now gets $S_{\hat{r}\hat{t}} + S_{\hat{r}\hat{\phi}}\nu_p = 0$, or

$$S = s\,\omega^{\hat{r}} \wedge \Omega^{p\hat{\phi}}, \qquad \Omega^{p\hat{\phi}} = \gamma_p[-\nu_p\,\omega^{\hat{t}} + \omega^{\hat{\phi}}], \tag{10.110}$$

so that $(S_{\hat{r}\hat{t}}, S_{\hat{r}\hat{\phi}}) = (-s\gamma_p\nu_p, s\gamma_p)$ and (10.106) reduces to

$$0 = \gamma(\nu^2 - \nu_K^2) + \gamma_p \frac{\zeta_K}{\nu_K} \mathcal{M}\hat{s} \left[\frac{\gamma^2}{\gamma_K^2} \nu(\nu\nu_p - \nu_K^2) + \nu_K^2(2\nu_p + \nu) \right]. \tag{10.111}$$

Solving with respect to $\hat{s}$, we obtain

$$\hat{s} = -\frac{\nu_K}{\mathcal{M}\gamma_p\gamma\zeta_K} \frac{\nu^2 - \nu_K^2}{[(1 - 3\nu_K^2)\nu^2 + 2\nu_K^2]\nu_p - \nu\nu_K^2(\nu^2 - \nu_K^2)}. \tag{10.112}$$

Recalling the definition (10.101), m_s becomes

$$\frac{m_s}{m} = \gamma\gamma_p \frac{\zeta_K}{\nu_K} \mathcal{M}\hat{s}[\nu\nu_p - \nu_K^2], \tag{10.113}$$

and using (10.102) for ν_p, we obtain

$$\hat{s} = -\frac{\nu_K}{\mathcal{M}\gamma_p\gamma\zeta_K} \frac{\nu - \nu_p}{(1 - \nu\nu_p)(\nu\nu_p - \nu_K^2)}\ ; \tag{10.114}$$

this condition must be considered together with (10.112). Relations (10.112) and (10.114) imply that the spinless case ($\hat{s} = 0$) is compatible only with $\nu = \nu_p = \pm\nu_K$. Eliminating $\hat{s}$ from Eqs. (10.114) and (10.112) and solving with respect to ν_p, we have

$$\nu_p^{(\pm)} = \frac{1}{2} \frac{\nu_K}{\nu^2 + 2\nu_K^2} \{3\nu\nu_K \pm [\nu^2(13\nu_K^2 + 4\nu^2) - 8\nu_K^4]^{1/2}\}. \tag{10.115}$$

Since the case $\hat{s} \ll 1$ is the only physically relevant one, of the two branches of the solution (10.115) we shall consider only the branch $\nu_p^{(+)}$ when $\nu > 0$, and $\nu_p^{(-)}$ when $\nu < 0$. By substituting $\nu_p = \nu_p^{(\pm)}$ into (10.112), for instance, we obtain a relation between ν and $\hat{s}$. The reality condition (10.115) requires that ν take values outside the interval $(\bar{\nu}_-, \bar{\nu}_+)$, with

$$\bar{\nu}_\pm = \pm\nu_K\sqrt{2}\sqrt{-13 + 3\sqrt{33}}/4 \simeq \pm 0.727\nu_K;$$

moreover, the time-like condition for $|\nu_p| < 1$ is satisfied for all values of ν outside the same interval. A linear relation between ν and $\hat{s}$ can be obtained in the limit of small $\hat{s}$:

$$\nu = \pm\nu_K - \frac{3}{2}\zeta_K \nu_K \mathcal{M}\hat{s} + O(\hat{s}^2). \tag{10.116}$$

From this approximate solution for ν we also have that

$$\nu_p^{(\pm)} = \pm\nu_K - \frac{3}{2}\zeta_K \nu_K \mathcal{M}\hat{s} + O(\hat{s}^2), \tag{10.117}$$

and the total 4-momentum P is given by (10.102), with

$$\nu_p = \nu + O(\hat{s}^2). \tag{10.118}$$

The reciprocals of the angular velocities ζ, ζ_p also coincide to first order in $\hat{s}$, and are then given by

$$\frac{1}{\zeta} \equiv \frac{1}{\zeta_{(\pm,\pm)}} = \pm\frac{1}{\zeta_K} \pm \frac{3}{2}\mathcal{M}|\hat{s}|\,, \tag{10.119}$$

where the signs in front of $1/\zeta_K$ correspond to co-/counter-rotating orbits, while the signs in front of $\hat{s}$ refer to a positive or negative spin direction along the z-axis; for instance, the quantity $\zeta_{(+,-)}$ denotes the angular velocity of U corresponding to a corotating orbit $(+)$ with spin-down $(-)$ alignment, etc.

Clock effect for spinning bodies

One can measure the difference in the arrival times, after one complete revolution, with respect to a static observer, of two oppositely rotating spinning test particles (to first order in the spin parameter $\hat{s}$) with either spin orientation. From (10.119) we have that the coordinate time difference is given by

$$\Delta t_{(+,+;-,+)} = 2\pi\left(\frac{1}{\zeta_{(+,+)}} + \frac{1}{\zeta_{(-,+)}}\right) = 6\pi M|\hat{s}|, \tag{10.120}$$

and analogously for $\Delta t_{(+,-;-,-)}$. A similar result can be obtained, referring to any observer moving in spatially circular orbits, with a slight modification of the discussion. This effect creates an interesting parallelism with the clock effect in Kerr space-time, as we shall discuss next. In the present case, in fact, it is the spin of the particle which creates a non-zero clock effect, while in the Kerr metric this effect is induced by the *rotation* of the space-time even to geodesic spinless test particles. The latter have angular velocities

$$\frac{1}{\zeta_{(\mathrm{Kerr})\,\pm}} = \pm\frac{1}{\zeta_K} + a, \tag{10.121}$$

where a is the angular momentum per unit mass of the Kerr black hole; hence Kerr's clock effect is given by

$$\Delta t_{(+,-)} = 2\pi \left(\frac{1}{\zeta_{(\mathrm{Kerr})+}} + \frac{1}{\zeta_{(\mathrm{Kerr})-}} \right) = 4\pi a. \tag{10.122}$$

This complementarity suggests a sort of equivalence principle: to first order in the spin, a static observer cannot decide whether he measures a time delay of spinning clocks in a non-rotating space-time or a time delay of non-spinning clocks moving on geodesics in a rotating space-time.

10.5 Motion of a test gyroscope in Kerr space-time

Consider the gyroscope moving along a spatially circular orbit U on the equatorial plane and a static observer m, at rest with respect to the spatial coordinates, with an adapted frame given by (8.18). We have

$$\begin{aligned} \zeta^{(\mathrm{boost})}_{(\mathrm{fw})} &= \zeta_{(\mathrm{Thomas})} + \zeta_{(\mathrm{geo})} \\ &= \frac{\gamma}{1+\gamma} \nu(U,m) \times_m \left[-F(U,m) + \gamma^{-1} F^{(G)}_{(\mathrm{fw},U,m)} \right]. \end{aligned} \tag{10.123}$$

Repeating all the calculations done for the Schwarzschild case in Section 10.3, one gets

$$\zeta^{(\mathrm{tot})\,\hat\theta} = \zeta^{(\mathrm{boost})\,\hat\theta}_{(\mathrm{fw})} - \zeta^{\hat\theta}_{(\mathrm{sc})} = \frac{\tau_1(U)}{\gamma}, \tag{10.124}$$

where $\tau_1(U)$ is now given by Eq. (8.163). It is quite natural on the equatorial plane to have a vector $e_{\hat z} = -e_{\hat\theta}$. This simply changes the sign of $\zeta^{(\mathrm{tot})\,\hat\theta}$,

$$\zeta^{(\mathrm{tot})\,\hat z} = -\frac{\tau_1(U)}{\gamma}. \tag{10.125}$$

This result agrees with the expression for $\overset{g}{\zeta}$ given in Section 9.4.

Assuming $U = U_{K\pm}$ to be a geodesic parameterized by the proper time, and taking into account the discussion at the end of Section 10.1, we have

$$\overset{g}{\zeta} \equiv \zeta_U^{(\mathrm{tot})\,\hat z} = -\tau_1(U_{K\pm}) = \mp\sqrt{\mathcal{M}/r^3}. \tag{10.126}$$

This result gives the precession angle $\Delta\phi$ after a full revolution as

$$\begin{aligned} \Delta\phi &= \mp 2\pi \left[\frac{\tau_1(U_{K\pm})}{\Gamma_{K\pm}\zeta_{K\pm}} - 1 \right] \\ &= \mp 2\pi \left[\left(1 - \frac{3\mathcal{M}}{r} \pm 2a\sqrt{\frac{\mathcal{M}}{r^3}} \right)^{1/2} - 1 \right], \end{aligned} \tag{10.127}$$

where $\tau_1(U_{K\pm})$ is the first torsion for geodesics,

$$\tau_1(U_{K\pm}) = \pm\sqrt{\frac{\mathcal{M}}{r^3}}, \qquad (10.128)$$

and $\Gamma_{K\pm}$ is the normalization factor (see (8.111) with $\theta = \pi/2$ and $\zeta = \zeta_{K\pm}$). Equation (10.127) in the linear approximation in a reduces to the well-known Schiff precession formula.

10.6 Motion of a spinning body in Kerr space-time

Consider the Kerr metric written in standard Boyer-Lindquist coordinates (8.73) and introduce the ZAMO family of fiducial observers, with 4-velocity

$$n = N^{-1}(\partial_t - N^\phi \partial_\phi), \qquad n^\flat = -N dt\,; \qquad (10.129)$$

here $N = (-g^{tt})^{-1/2}$ and $N^\phi = g_{t\phi}/g_{\phi\phi}$ are the lapse and shift functions, respectively, explicitly given in (8.100). A suitable orthonormal frame adapted to ZAMOs is given by

$$e_{\hat t} = n, \quad e_{\hat r} = \frac{1}{\sqrt{g_{rr}}}\partial_r, \quad e_{\hat \theta} = \frac{1}{\sqrt{g_{\theta\theta}}}\partial_\theta, \quad e_{\hat \phi} = \frac{1}{\sqrt{g_{\phi\phi}}}\partial_\phi, \qquad (10.130)$$

with dual

$$\begin{aligned} \omega^{\hat t} &= N dt, & \omega^{\hat r} &= \sqrt{g_{rr}}\,dr, \\ \omega^{\hat \theta} &= \sqrt{g_{\theta\theta}}\,d\theta, & \omega^{\hat \phi} &= \sqrt{g_{\phi\phi}}(d\phi + N^\phi dt). \end{aligned} \qquad (10.131)$$

As for Schwarzschild space-time, the physical (orthonormal) component along $-\partial_\theta$, perpendicular to the equatorial plane, will be taken to be along the positive z-axis and will be indicated by $\hat z$. The space-time trajectory described by (10.86) will be termed a U-orbit. In terms of (10.129) the line element can be expressed in the form

$$ds^2 = -N^2 dt^2 + g_{\phi\phi}(d\phi + N^\phi dt)^2 + g_{rr} dr^2 + g_{\theta\theta} d\theta^2. \qquad (10.132)$$

We require here that the center of mass of the spinning body moves on a world line whose spatial projection on the equatorial plane of the Kerr metric is geometrically circular; we shall term this world line the U-orbit. Therefore we recall the following:

(i) The 4-velocity U of uniformly rotating spatially circular orbits can be parameterized either by ζ, the (constant) angular velocity with respect to infinity, or equivalently by ν, the (constant) linear velocity with respect to ZAMOs:

$$U = \Gamma[\partial_t + \zeta\partial_\phi] = \gamma[e_{\hat t} + \nu e_{\hat \phi}], \qquad \gamma = (1-\nu^2)^{-1/2}. \qquad (10.133)$$

Here Γ is a normalization factor which ensures that $U_\alpha U^\alpha = -1$ and is given by

$$\Gamma = \left[N^2 - g_{\phi\phi}(\zeta + N^\phi)^2\right]^{-1/2} = \frac{\gamma}{N}, \tag{10.134}$$

and

$$\zeta = -N^\phi + \frac{N}{\sqrt{g_{\phi\phi}}}\nu. \tag{10.135}$$

(ii) On the equatorial plane of the Kerr solution there exists a large variety of special circular orbits, already examined in Chapter 8. Particular interest is devoted to the corotating (+) and counter-rotating (−) time-like spatially circular geodesics whose angular and linear velocities (with respect to ZAMOs) are respectively

$$\begin{aligned} \zeta_{K\pm} &\equiv \zeta_\pm = \left[a \pm (r^3/\mathcal{M})^{1/2}\right]^{-1}, \\ \nu_{K\pm} &\equiv \nu_\pm = \frac{a^2 \mp 2a\sqrt{\mathcal{M}r} + r^2}{\sqrt{\Delta}(a \pm r\sqrt{r/\mathcal{M}})}. \end{aligned} \tag{10.136}$$

Other special orbits include geodesic meeting point orbits, with

$$\nu_{(\mathrm{gmp})} = \frac{\nu_+ + \nu_-}{2} = -\frac{a\mathcal{M}(3r^2 + a^2)}{\sqrt{\Delta}(r^3 - a^2\mathcal{M})}, \tag{10.137}$$

as already stated in (8.155), and those characterized by

$$\nu_{(\mathrm{pt})} = \frac{2}{\nu_+^{-1} + \nu_-^{-1}} = \frac{(r^2 + a^2)^2 - 4a^2\mathcal{M}r}{a\sqrt{\Delta}(3r^2 + a^2)}, \tag{10.138}$$

both of these playing a role in the study of parallel transport of vectors along circular orbits.

(iii) It is convenient to introduce the Lie relative curvature of each orbit,

$$k_{(\mathrm{lie})} \equiv k_{(\mathrm{lie})\hat{r}} = -\partial_{\hat{r}} \ln \sqrt{g_{\phi\phi}} = -\frac{(r^3 - a^2\mathcal{M})\sqrt{\Delta}}{r^2(r^3 + a^2 r + 2a^2\mathcal{M})}, \tag{10.139}$$

as well as a Frenet-Serret intrinsic frame along U, both well established in the literature.

(iv) It is well known that any circular orbit on the equatorial plane of Kerr space-time has zero second torsion τ_2, while the geodesic curvature κ and the first torsion τ_1 are simply related by

$$\tau_1 = -\frac{1}{2\gamma^2}\frac{d\kappa}{d\nu}, \tag{10.140}$$

so that

$$\begin{aligned} \kappa &= k_{(\mathrm{lie})}\gamma^2(\nu - \nu_+)(\nu - \nu_-), \\ \tau_1 &= k_{(\mathrm{lie})}\nu_{(\mathrm{gmp})}\gamma^2(\nu - \nu_{(\mathrm{crit})+})(\nu - \nu_{(\mathrm{crit})-}), \end{aligned} \tag{10.141}$$

where $\nu_{(\mathrm{crit})\pm}$ are the spatial 3-velocities associated with extremely accelerated observers. The Frenet-Serret frame along U is then given by

$$E_0 \equiv U = \gamma[e_{\hat{t}} + \nu e_{\hat{\phi}}], \quad E_1 = e_{\hat{r}},$$
$$E_2 \equiv E_{\hat{\phi}} = \gamma[\nu n + e_{\hat{\phi}}], \quad E_3 = e_{\hat{z}}, \tag{10.142}$$

satisfying the following system of evolution equations:

$$\frac{DE_0}{d\tau_U} = \kappa E_1, \qquad \frac{DE_1}{d\tau_U} = \kappa E_0 + \tau_1 E_2,$$
$$\frac{DE_2}{d\tau_U} = -\tau_1 E_1, \qquad \frac{DE_3}{d\tau_U} = 0. \tag{10.143}$$

To study the behavior of spinning test particles in spatially circular orbits, let us consider first the evolution equation for the spin tensor (10.3), assuming that the frame components of the spin tensor are constant along the orbit. Following the analysis for Schwarzschild space-time, the total 4-momentum can be written as

$$P^\mu = -(U \cdot P)U^\mu - U_\nu \frac{DS^{\mu\nu}}{d\tau_U} \equiv mU^\mu + P_s^\mu, \tag{10.144}$$

where $P_s = U \llcorner DS/d\tau_U$ and m is the *bare mass* of the particle, i.e. the mass it would have in the rest space of U if it were not spinning. From (10.144), Eq. (10.3) implies

$$S_{\hat{t}\hat{\phi}} = 0, \quad S_{\hat{r}\hat{\theta}} = 0, \quad S_{\hat{t}\hat{\theta}} + S_{\hat{\phi}\hat{\theta}} \frac{\nu - \nu_{(\mathrm{gmp})}}{\nu_{(\mathrm{gmp})}(\nu - \nu_{(\mathrm{pt})})} = 0. \tag{10.145}$$

It is clear from (10.144) that P_s is orthogonal to U; moreover, it turns out to be aligned also with $E_{\hat{\phi}}$:

$$P_s = m_s E_{\hat{\phi}}, \tag{10.146}$$

where $m_s \equiv ||P_s||$ is given by

$$m_s = -\gamma k_{(\mathrm{lie})} \left[S_{\hat{t}\hat{r}}(\nu - \nu_{(\mathrm{gmp})}) + S_{\hat{r}\hat{\phi}} \nu_{(\mathrm{gmp})}(\nu - \nu_{(\mathrm{pt})}) \right]. \tag{10.147}$$

From (10.144) and (10.146) the total 4-momentum P can be written in the form $P = \mu U_p$, with

$$U_p = \gamma_p [e_{\hat{t}} + \nu_p e_{\hat{\phi}}], \quad \nu_p = \frac{\nu + m_s/m}{1 + \nu m_s/m}, \quad \mu = \frac{\gamma}{\gamma_p}(m + \nu m_s), \tag{10.148}$$

and $\gamma_p = (1 - \nu_p^2)^{-1/2}$.

The same arguments apply here as in the Schwarzschild case. Since U_p is a unit vector, the quantity μ can be interpreted as the total mass of the particle in the rest frame of U_p. We see from (10.148) that the total 4-momentum P is parallel to the unit tangent of a spatially circular orbit, which we shall denote as a U_p-orbit. The latter intersects the U-orbit at only one point, where it makes sense to compare the vectors U and U_p and the physical quantities related to them. It is clear that there exists one U_p-orbit for each point of the U-orbit where the two intersect. Hence, along the U-orbit, we can only compare *at* the point of intersection the quantities defined in a frame adapted to U with those defined in a frame adapted to U_p.

Let us now consider the equation of motion (10.2). Direct calculation shows that the spin-force is equal to

$$F^{(\mathrm{spin})} = F^{(\mathrm{spin})\hat{r}} e_{\hat{r}} + F^{(\mathrm{spin})\hat{\theta}} e_{\hat{\theta}}, \tag{10.149}$$

with

$$\begin{aligned} F^{(\mathrm{spin})\hat{r}} &= \frac{\gamma}{r^4} \frac{\mathcal{M}}{r(r^2+a^2)+2a^2\mathcal{M}} \Big\{ [r^2(2r^2+5a^2) + a^2(3a^2-2\mathcal{M}r) \\ &\quad - 3a(r^2+a^2)\sqrt{\Delta}\nu] S_{\hat{t}\hat{r}} + \{[r^2(r^2+4a^2) + a^2(3a^2-4\mathcal{M}r)]\nu \\ &\quad - 3a(r^2+a^2)\sqrt{\Delta}\} S_{\hat{r}\hat{\phi}} \Big\}, \\ F^{(\mathrm{spin})\hat{\theta}} &= \frac{\gamma}{r^3} \frac{S_{\hat{\theta}\hat{\phi}}}{(r^2+a^2)^2 - 4a^2\mathcal{M}r - a(3r^2+a^2)\sqrt{\Delta}\nu} \Big\{ \\ &\quad - \nu[3a^2 r(a^2-2\mathcal{M}r) + r^3(r^2+4a^2) - 2\mathcal{M}a^4] \\ &\quad + a\sqrt{\Delta}\{-4\mathcal{M}a^2 + \nu^2[3r(r^2+a^2)+2a^2\mathcal{M}]\} \Big\}, \end{aligned} \tag{10.150}$$

while the term on the left-hand side of (10.2) can be written as

$$\frac{DP}{d\tau_U} = ma(U) + m_s \frac{DE_{\hat{\phi}}}{d\tau_U}, \tag{10.151}$$

where

$$a(U) = \kappa e_{\hat{r}}, \qquad \frac{DE_{\hat{\phi}}}{d\tau_U} = -\tau_1 e_{\hat{r}} = \frac{1}{2\gamma^2}\frac{d\kappa}{d\nu} e_{\hat{r}}, \tag{10.152}$$

μ, m, and m_s being constant along the U-orbit.

Since $DP/d\tau_U$ is directed radially, Eq. (10.149) requires that $S_{\hat{\theta}\hat{\phi}} = 0$ (and therefore, from (10.145), that $S_{\hat{t}\hat{\theta}} = 0$); Eqn. (10.2) then reduces to

$$m\kappa - m_s\tau_1 - F^{(\mathrm{spin})}_{\hat{r}} = 0. \tag{10.153}$$

Summarizing, from the equations of motion (10.2) and (10.3) we deduce that the spin tensor is completely determined by two components, namely $S_{\hat{t}\hat{r}}$ and $S_{\hat{t}\hat{\phi}}$, such that

$$S = \omega^{\hat{r}} \wedge [S_{\hat{r}\hat{t}}\omega^{\hat{t}} + S_{\hat{r}\hat{\phi}}\omega^{\hat{\phi}}]. \tag{10.154}$$

From the relations

$$\begin{aligned} \omega^{\hat{t}} &= \gamma[-U^\flat + \nu\Omega^{\hat{\phi}}], \\ \omega^{\hat{\phi}} &= \gamma[-\nu U^\flat + \Omega^{\hat{\phi}}], \qquad \Omega^{\hat{\phi}} = [E_{\hat{\phi}}]^\flat, \end{aligned} \tag{10.155}$$

one obtains the useful relation

$$S = \gamma \left[(S_{\hat{r}\hat{t}} + \nu S_{\hat{r}\hat{\phi}}) U^\flat \wedge \omega^{\hat{r}} + (\nu S_{\hat{r}\hat{t}} + S_{\hat{r}\hat{\phi}}) \omega^{\hat{r}} \wedge \Omega^{\hat{\phi}} \right]. \tag{10.156}$$

Since the components of S are constant along U, then from the Frenet-Serret formalism one finds

$$\frac{DS}{d\tau_U} = \gamma[(\tau_1 + \kappa\nu)S_{\hat{r}\hat{t}} + (\nu\tau_1 + \kappa)S_{\hat{r}\hat{\phi}}]U^\flat \wedge \Omega^{\hat{\phi}}, \tag{10.157}$$

or, from (10.3),

$$P_s = -\gamma[(\tau_1 + \kappa\nu)S_{\hat{r}\hat{t}} + (\nu\tau_1 + \kappa)S_{\hat{r}\hat{\phi}}]\Omega^{\hat{\phi}} \equiv m_s \Omega^{\hat{\phi}}. \tag{10.158}$$

Let us introduce the quadratic invariant

$$s^2 = \frac{1}{2} S_{\mu\nu} S^{\mu\nu} = -S_{\hat{r}\hat{t}}^2 + S_{\hat{r}\hat{\phi}}^2. \tag{10.159}$$

Equation (10.3) is identically satisfied if the only non-zero components of S are $S_{\hat{r}\hat{t}}$ and $S_{\hat{r}\hat{\phi}}$. To discuss the physical properties of the particle motion one needs to supplement (10.153) with the algebraic conditions $S^{\mu\nu}P_\nu = 0$, which are equivalent to

$$(S_{\hat{r}\hat{t}}, S_{\hat{r}\hat{\phi}}) = s(-\gamma_p \nu_p, \gamma_p). \tag{10.160}$$

Let us summarize the results. The quantity m_s in general is given by

$$m_s = -s\gamma_p\gamma[-\nu_p(\tau_1 + \kappa\nu) + (\nu\tau_1 + \kappa)], \tag{10.161}$$

and, once inserted in the equation of motion, it gives

$$m\kappa + s\gamma_p\gamma[-\nu_p(\tau_1 + \kappa\nu) + (\nu\tau_1 + \kappa)]\tau_1 - F_{\hat{r}}^{(\mathrm{spin})} = 0, \tag{10.162}$$

where

$$F_{\hat{r}}^{(\mathrm{spin})} = s\gamma\gamma_p[A\nu\nu_p + B\nu + C\nu_p + A], \tag{10.163}$$

with

$$\begin{aligned} A &= -\frac{3\mathcal{M}a(r^2 + a^2)\sqrt{\Delta}}{r^4(r^3 + a^2 r + 2a^2\mathcal{M})}, \\ B &= \frac{\mathcal{M}}{r^4} \frac{r^4 + 3a^4 + 4a^2 r(r - \mathcal{M})}{r^3 + a^2 r + 2a^2\mathcal{M}}, \\ C &= B + \frac{\mathcal{M}}{r^3}. \end{aligned} \tag{10.164}$$

Thus one can solve Eq. (10.162) for the quantity $\hat{s} = \pm|\hat{s}| = \pm|s|/(m\mathcal{M})$, which denotes the signed magnitude of the spin per unit (bare) mass m of the test particle and $\mathcal{M}$ of the black hole, obtaining

$$\hat{s} = -\frac{\kappa}{\mathcal{M}\gamma\gamma_p\mathcal{D}}, \tag{10.165}$$

Table 10.1. *The limiting values of ν are listed for particular values of the black-hole rotational parameter a and for a fixed radial distance $r/\mathcal{M} = 8$, with black-hole mass $\mathcal{M} = 1$.*

a	$\bar{\nu}_1$	$\bar{\nu}_2$
0	−0.2970	0.2970
0.2	−0.2955	0.2979
0.4	−0.2933	0.2984
0.6	−0.2902	0.2986
0.8	−0.2861	0.2984
1	−0.2808	0.2981

with

$$\mathcal{D} = \{[-\nu_p(\tau_1 + \kappa\nu) + (\nu\tau_1 + \kappa)]\tau_1 - (A\nu\nu_p + B\nu + C\nu_p + A)\}\,, \tag{10.166}$$

and with κ and τ_1 given by (10.141). Recalling its definition (10.161), m_s becomes

$$\frac{m_s}{m} = -\mathcal{M}\hat{s}\gamma\gamma_p[-\nu_p(\tau_1 + \kappa\nu) + (\nu\tau_1 + \kappa)]. \tag{10.167}$$

Using (10.148) for ν_p, we obtain

$$\hat{s} = \frac{1}{\mathcal{M}\gamma\gamma_p} \frac{\nu - \nu_p}{(1 - \nu\nu_p)[-\nu_p(\tau_1 + \kappa\nu) + (\nu\tau_1 + \kappa)]}, \tag{10.168}$$

which must be considered together with (10.165); of course, the case $\hat{s} = 0$ (absence of spin) is only compatible with geodesic motion: $\nu \equiv \nu_p = \nu_\pm$. By eliminating $\hat{s}$ from (10.165) and (10.168) and solving with respect to ν_p, we obtain

$$\begin{aligned}
\nu_p^{(\pm)} &= \frac{1}{2}\frac{(2\kappa\tau_1 + A)\nu^2 + [2(\kappa^2 + \tau_1^2) + \mathcal{M}/r^3]\nu + 2\kappa\tau_1 - A \pm \sqrt{\Psi}}{\kappa^2\nu^2 + (2\kappa\tau_1 + A)\nu + \tau_1^2 + C} \\
\Psi &= [A^2 + 4\kappa(\kappa B + \tau_1 A)]\nu^4 + [8A\kappa^2 + 2(B + C)(2\kappa\tau_1 + A)]\nu^3 \\
&\quad + [4\kappa^2\mathcal{M}/r^3 + (B + C)^2 + 2A^2]\nu^2 - [8A\kappa^2 \\
&\quad + 2(B + C)(2\kappa\tau_1 - A)]\nu - A(4\kappa\tau_1 - A) - 4C\kappa^2\,.
\end{aligned} \tag{10.169}$$

Now, by substituting $\nu_p = \nu_p^{(\pm)}$ for instance into (10.168), we obtain the main relation between $\hat{s}$ and ν.

The reality condition (10.169) requires that ν take values outside the interval $(\bar{\nu}_1, \bar{\nu}_2)$, with $\bar{\nu}_1$ and $\bar{\nu}_2$ listed in Table 10.1 for particular values of a ($r, \mathcal{M}$ fixed); the time-like condition for $|\nu_p| < 1$ is satisfied for all values of ν outside the same interval.

To first order in $\hat{s}$ we have

$$
\begin{aligned}
\nu &= \nu_{\pm} + \mathcal{N}\hat{s} + O(\hat{s}^2), \\
\mathcal{N} &= \frac{3}{2}\frac{\mathcal{M}(r^3 + a^2 r + 2a^2\mathcal{M})\{ar(r-5\mathcal{M}) \pm \sqrt{\mathcal{M}r}[2a^2 - r(r-3\mathcal{M})]\}^2}{r^2\sqrt{\Delta}\sqrt{\mathcal{M}r}(a \pm r\sqrt{r/\mathcal{M}})^2} \\
&\quad \times \{\sqrt{\mathcal{M}r}[4a^2(r-4\mathcal{M}) - r(r-3\mathcal{M})^2] \\
&\quad \pm a[4a^2\mathcal{M} + r(r-3\mathcal{M})(r-7\mathcal{M})]\}^{-1}. \qquad (10.170)
\end{aligned}
$$

Therefore, from the preceding approximate solution for ν we have that

$$
\nu_p^{(\pm)} = \nu + O(\hat{s}^2), \qquad (10.171)
$$

and the total 4-momentum P is given by (10.148) with $\nu_p = \nu_p^{(\pm)}$. The corresponding angular velocity ζ_p and its reciprocal are then given by (Abramowicz and Calvani, 1979)

$$
\begin{aligned}
\zeta_p &= \zeta + \frac{\mathcal{N} r\sqrt{\Delta}}{r^3 + a^2 r + 2a^2\mathcal{M}}\hat{s} + O(\hat{s}^2), \\
\frac{1}{\zeta_p} &= \frac{1}{\zeta} - \frac{\mathcal{N}}{\zeta_{\pm}^2}\frac{r\sqrt{\Delta}}{r^3 + a^2 r + 2a^2\mathcal{M}}\hat{s} + O(\hat{s}^2). \qquad (10.172)
\end{aligned}
$$

Clock effect for spinning bodies

Circularly rotating spinning bodies, to first order in the spin parameter $\hat{s}$ and the metric parameter a, and therefore neglecting terms containing $\hat{s}a$, have orbits close to being geodesics (as expected), with

$$
\frac{1}{\zeta_{(\pm,\pm)}} = \frac{1}{\zeta_{\pm}} \pm \mathcal{M}|\hat{s}|\mathcal{J}, \qquad (10.173)
$$

with $\mathcal{J} = 3/2$. Equation (10.173) defines such orbits with the various signs corresponding to co-/counter-rotating orbits with positive/negative spin direction along the z axis. For instance, $\zeta_{(+,-)}$ indicates the angular velocity of U corresponding to a corotating orbit $(+)$ with spin-down $(-)$ alignment. Therefore one can study the differences in the arrival times after one complete revolution with respect to a local static observer:

$$
\begin{aligned}
\Delta t_{(+,+;-,+)} &= 2\pi\left(\frac{1}{\zeta_{(+,+)}} + \frac{1}{\zeta_{(-,+)}}\right) = 4\pi\left(a + \mathcal{M}|\hat{s}|\mathcal{J}\right), \\
\Delta t_{(+,+;-,-)} &= \Delta t_{(+,-;-,+)} = 4\pi a, \\
\Delta t_{(+,-;-,-)} &= 4\pi\left(a - \mathcal{M}|\hat{s}|\mathcal{J}\right). \qquad (10.174)
\end{aligned}
$$

In the latter case, it is easy to see that if $a = \frac{3}{2}\mathcal{M}|\hat{s}|$ the clock effect can be made vanishing; this feature can be directly measured (Faruque, 2004). More interesting is the case of corotating spin-up against counter-rotating spin-down or, alternatively, corotating spin-down against counter-rotating spin-up. Now the compensating effect makes the spin contribution to the clock effect equal to zero; this case therefore appears indistinguishable from that of spinless particles.

10.7 Gravitational waves and the compass of inertia

The existence of gravitational waves as predicted by general relativity has been indirectly ascertained by observation of the binary pulsar PSR 1913+16 (Taylor and Weisberg, 1989). A large body of literature is now available on the properties of gravitational waves, and ways to detect them. Hence we shall confine our attention to a particular aspect of this problem which has been considered only recently, namely the dragging of inertial frames by a gravitational wave (Bini and de Felice, 2000; Sorge, Bini, and de Felice, 2001; Bičak, Katz, and Lynden-Bell, 2008). We then ask the question: what observable effects might be produced by a plane gravitational wave acting on a test gyroscope? It is well known that in the absence of significant coupling between the background curvature and the multipole moments of the energy-momentum tensor of an extended body, its spin vector is Fermi-Walker transported along its own trajectory (see de Felice and Clarke, 1990, and references therein); for measurable effects in a gravitational wave background, see also (Cerdonio, Prodi, and Vitale, 1988; Mashhoon, Paik, and Will, 1989; Krori, Chaudhury, and Mahanta, 1990; Fortini and Ortolan, 1992; Herrera, Paiva, and Santos, 2000).

The effects of a plane gravitational wave on a frame which is *not* Fermi-Walker transported are best appreciated by studying the precession of a gyroscope at rest in that frame (de Felice, 1991). The task, then, is to find a frame which is not Fermi-Walker transported *and* is also operationally well-defined so that, monitoring the precession of a gyroscope with respect to that frame, we can study the dragging induced on it by a plane gravitational wave. Our main purposes are:

(i) to establish the existence of a relativistic effect which is measurable, namely the gyroscopic precession induced by a plane gravitational wave;
(ii) to show that a frame can be selected with respect to which the precession is due to one polarization state only.
In the latter case we will have constructed a gravitational polarimeter.

The metric of a plane monochromatic gravitational wave, elliptically polarized and propagating along a direction which we fix as the x coordinate direction, can be written in transverse-traceless (TT) gauge, as in Chapter 8, Eq. (8.195). The time-like geodesics of this metric, deduced in de Felice (1979), are described by (8.196).

Test gyroscopes in motion along a geodesic

We consider a test gyroscope moving along a geodesic described by a tangent vector field $U = U_{(\mathrm{g})}$ given by (8.196). The spin vector $S(U)$ satisfies the equation

$$\frac{D_{(\mathrm{fw},U)}}{d\tau_U} S(U) = 0. \tag{10.175}$$

This property implies that $S(U)$ does not precess with respect to spatial axes which are Fermi-Walker transported along the world line of U.

If the observer U is comoving with the gyroscope and refers to a general orthonormal spatial frame $\{e(U)_{\hat a}\}|_{a=1,2,3}$ adapted to his world line, then Eq. (10.175) becomes

$$\left[\frac{dS(U)^{\hat a}}{d\tau_U} + \epsilon^{\hat a}{}_{\hat b\hat c}\zeta^{\hat b}_{(\mathrm{fw},\,U,\,e(U))}S(U)^{\hat c}\right] e(U)_{\hat a} = 0, \tag{10.176}$$

where $\zeta_{(\mathrm{fw},\,U,\,e(U))}$ is the *observed* rate of spin precession, defined by

$$\zeta^{\hat a}_{(\mathrm{fw},\,U,\,e(U))} = -\frac{1}{2}\epsilon^{\hat a\hat b\hat c} e(U)_{\hat b}\cdot \nabla(U)_{(\mathrm{fw})} e(U)_{\hat c} \equiv -{}^{*(U)}C^{\hat a}_{(\mathrm{fw},\,U,\,e(U))}. \tag{10.177}$$

Test gyroscopes at rest in the TT-grid of a gravitational wave

Let us now consider the case of gyroscopes carried by observers at rest in the TT-grid of a gravitational wave. These observers move along world lines which form a vorticity-free congruence of geodesics and whose tangent field is given by

$$u^\flat = -dt, \qquad u = \partial_t. \tag{10.178}$$

One can adapt to this field an infinite number of spatial frames by rotating any given one arbitrarily. For example, consider the following adapted frame $\{e_{\hat\alpha}\} = \{e_{\hat 0} = u, e_{\hat a} = e(u)_{\hat a}\}$ with its dual $\{\omega^{\hat\alpha}\} = \{\omega^{\hat 0} = -u^\flat, \omega^{\hat a} = \omega(u)^{\hat a}\}$:

$$\begin{aligned}
u &= \partial_t\,,\\
e(u)_{\hat 1} &= \partial_x\,,\\
e(u)_{\hat 2} &= (1-h_+)^{-1/2}\partial_y \simeq \left(1+\frac{1}{2}h_+\right)\partial_y\,,\\
e(u)_{\hat 3} &= (1-h_+^2-h_\times^2)^{-1/2}[(1-h_+)^{-1/2}h_\times\partial_y + (1-h_+)^{1/2}\partial_z]\\
&\simeq h_\times\partial_y + \left(1-\frac{1}{2}h_+\right)\partial_z\,,
\end{aligned} \tag{10.179}$$

$$\begin{aligned}
-u^\flat &= dt\,,\\
\omega(u)^{\hat 1} &= dx\,,\\
\omega(u)^{\hat 2} &= (1-h_+)^{1/2}\left[dy - \frac{h_\times}{1-h_+}dz\right] \simeq \left(1-\frac{1}{2}h_+\right)dy - h_\times dz\,,\\
\omega(u)^{\hat 3} &= \left(\frac{1-h_+^2-h_\times^2}{1-h_+}\right)^{1/2} dz \simeq \left(1+\frac{1}{2}h_+\right)dz.
\end{aligned} \tag{10.180}$$

This is not a Fermi-Walker frame; the Fermi rotation coefficients are given by

$$C_{(\mathrm{fw},\,u,\,e(u))\,\hat b\hat a} = e(u)_{\hat b}\cdot_u \nabla(u)_{(\mathrm{fw})} e(u)_{\hat a}, \tag{10.181}$$

and therefore, to first order in the metric perturbations h, one finds that the only independent non-zero component is[2]

$$-\zeta^{\hat{1}}_{(\mathrm{fw},\,u,\,e(u)_{\hat{a}})} = C_{(\mathrm{fw},\,u,\,e(u)_{\hat{a}})\hat{3}\hat{2}} = -\frac{[\dot{h}_\times(1-h_+) + \dot{h}_+ h_\times]}{2(1-h_+)\sqrt{1-h_+^2-h_\times^2}} \simeq -\frac{1}{2}\dot{h}_\times . \tag{10.182}$$

Clearly Eq. (10.182) shows that the Fermi nature of the frame only depends on the $h_\times$ component of the wave amplitude.

In fact, if we set $h_\times = 0$ and consider the weak-field limit, the spatial triad becomes

$$e(u)_{\hat{1}} = \partial_x, \quad e(u)_{\hat{2}} \simeq \left(1+\frac{1}{2}h_+\right)\partial_y, \quad e(u)_{\hat{3}} \simeq \left(1-\frac{1}{2}h_+\right)\partial_z\,, \tag{10.183}$$

showing that these axes do not rotate with respect to the coordinate directions. In this case ($h_\times = 0$) the axes $e(u)_{\hat{a}}$ are Fermi-Walker transported along u, as expected. In the complementary case ($h_+ = 0$) we have

$$e(u)_{\hat{1}} = \partial_x, \quad e(u)_{\hat{2}} \simeq \partial_y, \quad e(u)_{\hat{3}} \simeq h_\times \partial_y + \partial_z; \tag{10.184}$$

hence the wave rotates the $e(u)_{\hat{3}}$ axis and therefore a gyroscope must be seen to precess with respect to it. However, the frame $e(u)_{\hat{a}}$, although special in selecting only one state of polarization, cannot be operationally defined in a simple way.

Of course there exist infinitely many spatial frames $\{\tilde{e}(u)_{\hat{a}}\}$ which are adapted to the observers (10.178) and can be obtained from the one in (10.179) by a spatial rotation R:

$$\tilde{e}(u)_{\hat{a}} = e(u)_{\hat{b}} R^{\hat{b}}{}_{\hat{a}}\,. \tag{10.185}$$

Two of these are quite natural: a Fermi-Walker frame and a Frenet-Serret one. In the first case (Fermi-Walker) the spatial directions are easily fixed by three mutually orthogonal axes of comoving gyroscopes. Obviously this frame is not suitable for measuring the Fermi rotation itself, as already noted.

The second case (Frenet-Serret) does not correspond to a uniquely defined frame along u, since its world line is a geodesic. Properly speaking, in this degenerate situation (u geodesic and the 4-acceleration $||a(u)|| = \kappa = 0$) the first vector e_1 of a Frenet-Serret frame can be chosen orthogonally to u in an arbitrary way (i.e. according to ∞^2 different possibilities) and then the standard procedure for fixing a frame gives

[2] We use a dot notation for the derivative with respect to t.

$$\frac{De_1}{d\tau_u} = \tau_1 e_2,$$
$$\frac{De_2}{d\tau_u} = -\tau_1 e_1 + \tau_2 e_3,$$
$$\frac{De_3}{d\tau_u} = -\tau_2 e_2\,, \tag{10.186}$$

corresponding to a Frenet-Serret rotation angular velocity

$$\omega_{(\mathrm{FS})} = \tau_1 e_3 + \tau_2 e_1. \tag{10.187}$$

We recall here that $\omega_{(\mathrm{FS})}$ is the negative of the angular velocity of precession of a gyroscope. This arbitrariness can be eliminated by choosing e_1 to be aligned with the spin of a gyroscope. Then, τ_1 vanishes and the previous relations reduce to

$$\frac{De_1}{d\tau_u} = 0,$$
$$\frac{De_2}{d\tau_u} = \tau_2 e_3,$$
$$\frac{De_3}{d\tau_u} = -\tau_2 e_2\,, \tag{10.188}$$

where now

$$\omega_{(\mathrm{FS})} = \tau_2 e_1. \tag{10.189}$$

This corresponds to a second degenerate condition for the Frenet-Serret approach; in fact, as $\tau_1 = 0$, the Frenet-Serret procedure starts with e_2, which in turn can be any direction in the subspace orthogonal to u and e_1.

The form of the metric suggests the choice

$$e_1 = \partial_x\,, \tag{10.190}$$

which corresponds to a gyroscope pointing along the direction of propagation of the wave. Thus a parametric form for e_2 (and consequently for e_3) can be given in terms of the frame (10.179) by introducing an angle ϕ which is an arbitrary function along the world line

$$e_2 = \cos\phi\, e(u)_{\hat 2} + \sin\phi\, e(u)_{\hat 3},$$
$$e_3 = -\sin\phi\, e(u)_{\hat 2} + \cos\phi\, e(u)_{\hat 3}\,, \tag{10.191}$$

so that τ_2 becomes a function of ϕ. Obviously, particular choices for ϕ allow τ_2 (and $\omega_{(\mathrm{FS})}$) to depend only on h_+ or $h_\times$ or even to vanish ($\tau_2 = 0$), leading back to the Fermi-Walker case. In general, a straightforward calculation shows that, in the weak-field limit and for arbitrary ϕ,

$$\omega_{(\mathrm{FS})} = \left(\dot\phi - \frac{1}{2}\dot h_\times\right) e_1\,. \tag{10.192}$$

Assuming that ϕ depends only on h_+ and $h_\times$ along the world line, and expanding this for weak fields, one has

$$\phi = \phi_0 + \phi_+ h_+ + \phi_\times h_\times\,, \tag{10.193}$$

so

$$\omega_{(\mathrm{FS})} = \left[\phi_+ \dot h_+ + \left(\phi_\times - \frac{1}{2}\right)\dot h_\times\right] e_1. \tag{10.194}$$

Special cases are the following:

$$\begin{aligned}
\phi_+ &= 1, &\quad \phi_\times &= 1/2, &\quad \omega_{(\mathrm{FS})} &= \dot h_+\, e_1,\\
\phi_+ &= 0, &\quad \phi_\times &= 3/2, &\quad \omega_{(\mathrm{FS})} &= \dot h_\times\, e_1,\\
\phi_+ &= 0, &\quad \phi_\times &= 1/2, &\quad \omega_{(\mathrm{FS})} &= 0\,.
\end{aligned}$$

It is then clear that according to the way in which one selects the spatial axis e_2, the angular velocity of precession $\omega_{(\mathrm{FS})}$ (i.e. the gyro dragged along u) becomes a "filter" for the gravitational wave polarization.

Test gyroscopes in general geodesic motion

Let us now consider the case of a gyroscope in motion along a general geodesic with 4-velocity $U = U_{(\mathrm{g})}$, and assume that $u = \partial_t$ is a family of observers defined all along the world line of U. As discussed in detail in Chapter 3, when dealing with different families of observers (u and U in this case) the two local rest spaces LRS_u and LRS_U are naturally connected by two maps: the mixed projection map $P(U,u) = P(U)P(u) : LRS_u \to LRS_U$ and the local rest space boost map $B_{(\mathrm{lrs})}(U,u) = P(U)B(U,u)P(u) : LRS_u \to LRS_U$, where $B(U,u)$ is defined so that

$$U = \gamma[u + \nu\hat\nu_{(U,u)}] = B(U,u)u\,,$$

where $\nu \equiv ||\nu(U,u)|| = ||\nu(u,U)||$ is the magnitude of the relative spatial velocity, $\hat\nu(U,u)$ is the unit spatial direction of the velocity of u relative to U, $\gamma = \gamma(U,u) = \gamma(u,U)$ is the relative Lorentz factor, and the corresponding expression for $B_{(\mathrm{lrs})}(U,u)$ is given by (3.143). In this case we have, from (8.196) and (10.178),

$$\begin{aligned}
\gamma &= \frac{1+f+E^2}{2E},\\
\nu &= [(1+f+E^2)^2 - 4E^2]^{1/2}(1+f+E^2)^{-1},\\
\hat\nu &= [(1+f+E^2)^2 - 4E^2]^{-1/2}[(1+f-E^2)\, e(u)_{\hat 1}\\
&\quad + 2E\alpha(1-h_+)^{-1/2}\, e(u)_{\hat 2}\\
&\quad + \frac{2E[\beta(1-h_+) + \alpha h_\times]}{(1-h_+)^{1/2}(1-h_+^2-h_\times^2)^{1/2}}\, e(u)_{\hat 3}].
\end{aligned} \tag{10.195}$$

In order to study the spin precession as seen by an observer U comoving with the gyro, we must first choose axes with respect to which the precession will be measured. We then proceed to find a spatial frame which is suitable for actual measurements. It is clear that the observers with 4-velocity u can unambiguously determine in their rest frame a spatial direction given by that of the relative velocity ν of the gyroscope, $\hat{\nu}$. Let the direction of propagation of the gravitational wave be the x-axis with unit vector $e(u)_{\hat{1}}$. Then from these two directions, namely the (instantaneous) relative velocity of the gyro and the wave propagation direction, it is possible to construct the following spatial triad:

$$\begin{aligned}
\lambda(u)_1 &= e(u)_{\hat{1}} = \partial_x\,,\\
\lambda(u)_2 &= [(\hat{\nu}(U,u)^{\hat{2}})^2 + (\hat{\nu}(U,u)^{\hat{3}})^2]^{-1/2}[\hat{\nu}_{(U,u)} \times_u e(u)_{\hat{1}}]\,,\\
\lambda(u)_3 &= \lambda(u)_1 \times_u \lambda(u)_2.
\end{aligned} \tag{10.196}$$

This is a Frenet-Serret triad, which can be cast into the form (10.191) if

$$\tan\phi = -\frac{\hat{\nu}(U,u)^{\hat{2}}}{\hat{\nu}(U,u)^{\hat{3}}} = -\frac{\alpha\sqrt{1-h_+^2-h_\times^2}}{\beta(1-h_+)+\alpha h_\times}. \tag{10.197}$$

In the weak-field limit (see eq. (10.193)) we obtain

$$\phi_0 = -\arctan\frac{\alpha}{\beta},\quad \phi_+ = -\frac{\alpha\beta}{\alpha^2+\beta^2},\quad \phi_\times = \frac{\alpha^2}{\alpha^2+\beta^2}\,. \tag{10.198}$$

Since the family of observers u is defined along the world line of the observer U carrying the gyro, the latter can identify spatial directions in his own local rest space simply by boosting the directions of the Frenet-Serret generated frame $\{\lambda(u)_a\}$.

At each event along his world line, the observer U will see the axes $\lambda(u)_a$ defined by (10.196) to be in relative motion, whereas the boost of these axes,

$$\begin{aligned}
\lambda(U)_{\hat{a}} &= B_{(\mathrm{lrs})}(U,u)\lambda(u)_a\\
&= \lambda(u)_a + \frac{\gamma}{\gamma+1}[\nu_{(U,u)}\cdot\lambda(u)_a](u+U),
\end{aligned} \tag{10.199}$$

will be identified as the corresponding axes with the same orientation which are "momentarily at rest" with respect to U. The precession of the spin then corresponds to the spatial dual of the Fermi-Walker structure functions of $\lambda(U)_{\hat{a}}$, namely $C_{(\mathrm{fw},U,\lambda(U))\hat{b}\hat{a}}$, according to (10.177). To first order in h, let us consider the precession angular velocity of the triad $e(U)_{\hat{a}}$ which is the boosted Frenet-Serret e_a given by (10.191). We then have

$$
\begin{aligned}
-\zeta_{(\mathrm{fw},\,U,\,e(U)_{\hat a})}{}^{\hat 1} &\simeq -\frac{E}{2}\left[\dot h_\times\left(-1+2\phi_\times-\frac{2\alpha^2}{(E+1)^2+\alpha^2+\beta^2}\right)+2\phi_+\dot h_+\right]\,,\\
-\zeta_{(\mathrm{fw},\,U,\,e(U)_{\hat a})}{}^{\hat 2} &\simeq -\frac{E(E+1)}{(E+1)^2+\alpha^2+\beta^2}[\dot h_+(\beta\cos\phi_0+\alpha\sin\phi_0)\\
&\quad+\dot h_\times(\beta\sin\phi_0-\alpha\cos\phi_0)]\,,\\
-\zeta_{(\mathrm{fw},\,U,\,e(U)_{\hat a})}{}^{\hat 3} &\simeq \frac{E(E+1)}{(E+1)^2+\alpha^2+\beta^2}[\dot h_+(\beta\cos\phi_0+\alpha\sin\phi_0)\\
&\quad+\dot h_\times(\beta\sin\phi_0-\alpha\cos\phi_0)]\,.
\end{aligned}
\tag{10.200}
$$

Next we specialize these results to the operationally defined triad $\lambda(U)_a$, corresponding to the values of ϕ_0, ϕ_+, and $\phi_\times$ given in (10.198); in this special case, we finally have

$$
\begin{aligned}
-\zeta_{(\mathrm{fw},\,U,\,\lambda(U)_{\hat a})}{}^{\hat 1} &\simeq -\frac{E}{2}\left[\dot h_\times\left(-1+\frac{2\alpha^2(E+1)^2}{(\alpha^2+\beta^2)[(E+1)^2+\alpha^2+\beta^2]}\right)\right.\\
&\quad\left.+2\frac{\alpha\beta}{\alpha^2+\beta^2}\dot h_+\right]\,,\\
-\zeta_{(\mathrm{fw},\,U,\,\lambda(U)_{\hat a})}{}^{\hat 2} &\simeq -\frac{E(E+1)\sqrt{\alpha^2+\beta^2}}{(E+1)^2+\alpha^2+\beta^2}\left[\dot h_+\frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}+\dot h_\times\right]\,,\\
-\zeta_{(\mathrm{fw},\,U,\,\lambda(U)_{\hat a})}{}^{\hat 3} &\simeq \frac{E(E+1)\sqrt{\alpha^2+\beta^2}}{(E+1)^2+\alpha^2+\beta^2}\left[\dot h_+\frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}+\dot h_\times\right]\,.
\end{aligned}
\tag{10.201}
$$

Moreover, by rescaling the precession angular velocity $\zeta_{(\mathrm{fw},\,U,\,\lambda(U)_{\hat a})}$ by a γ-factor, i.e. $\gamma^{-1}\,\zeta_{(\mathrm{fw},\,U,\,\lambda(U)_{\hat a})}$, one refers the same angular velocity to the proper time of the observer u, obtaining a sort of reconstruction made by u of the precession seen by the observer U carrying the gyroscope. From these relations it is clear that the essential features we have found for a gyroscope at rest in the background of the wave are still valid when the gyro itself is moving.

Equations (10.200) represent the components of the precession angular velocity as functions of the parameters E, α, β, ϕ_0, ϕ_+, and $\phi_\times$. Special choices of these parameters can simplify the components or specialize them to particular physical conditions.

10.8 Motion of an extended body in a gravitational wave space-time

Consider an extended body as described by the Dixon model, Eqs. (10.2)–(10.4).

Introduce the unit tangent vector U_p aligned with the 4-momentum P, i.e. $P^\mu = mU_p^\mu$ (with $U_p\cdot U_p=-1$), as well as the spin vector

$$
S^\beta = \tfrac{1}{2}\eta_\alpha{}^{\beta\gamma\delta}U_p^\alpha S_{\gamma\delta}. \tag{10.202}
$$

Let the body move in the space-time (8.195) of a weak gravitational plane wave, propagating along the x direction, with metric functions given by

$$h_+ = A_+ \sin\omega(t-x), \qquad h_\times = A_\times \cos\omega(t-x). \tag{10.203}$$

The geodesics of this metric have already been presented in Eq. (8.196). For a weak gravitational wave they reduce to the following form:

$$\begin{aligned}
t(\lambda) &= E\lambda + t_0 + x(\lambda) - x_0,\\
x(\lambda) &= (\mu^2 + \alpha^2 + \beta^2 - E^2)\frac{\lambda}{2E}\\
&\quad - \frac{1}{2\omega E^2}[(\alpha^2-\beta^2)A_+ \cos\omega(E\lambda + t_0 - x_0)\\
&\quad - 2\alpha\beta A_\times \sin\omega(E\lambda + t_0 - x_0)] + x_0,\\
y(\lambda) &= \alpha\lambda + y_0\\
&\quad - \frac{1}{\omega E}[\alpha A_+ \cos\omega(E\lambda + t_0 - x_0) - \beta A_\times \sin\omega(E\lambda + t_0 - x_0)],\\
z(\lambda) &= \beta\lambda + z_0 + \frac{1}{\omega E}[\beta A_+ \cos\omega(E\lambda + t_0 - x_0)\\
&\quad + \alpha A_\times \sin\omega(E\lambda + t_0 - x_0)],
\end{aligned} \tag{10.204}$$

where λ is an affine parameter, x_0^α are integration constants, and α, β, and E are conserved Killing quantities; $\mu^2 = 1, 0, -1$ correspond to time-like, null, and space-like geodesics, respectively.

It is in general a hard task to solve the whole set of equations (10.2)–(10.4) in complete generality. However, looking for weak-field solutions, it is sufficient to search for changes in the mass parameter m of the body, its velocity and total 4-momentum, as well as the spin tensor, which are of the same order as the metric functions $h_{+,\times}$, i.e.

$$\begin{aligned}
m &= m_0 + \tilde{m}, \qquad U = U_0 + \tilde{U},\\
P &= P_0 + \tilde{P}, \quad S^{\mu\nu} = S_0^{\mu\nu} + \tilde{S}^{\mu\nu}.
\end{aligned} \tag{10.205}$$

The quantities with subscript "0" are the flat background ones, i.e. those corresponding to the initial values, before the passage of the wave. Let U and U_p be written in the form

$$\begin{aligned}
U &= \gamma(\partial_t + \tilde{\nu}^a \partial_a), \qquad \gamma = (1-\tilde{\nu}^2)^{-1/2},\\
U_p &= \gamma_p(\partial_t + \tilde{\nu}_p^a \partial_a), \quad \gamma_p = (1-\tilde{\nu}_p^2)^{-1/2}.
\end{aligned} \tag{10.206}$$

The zeroth-order set of equations turns out to be

$$\frac{DP_0^\mu}{d\tau_{U_0}} = 0, \qquad \frac{DS_0^{\mu\nu}}{d\tau_{U_0}} = 2P_0^{[\mu}U_0^{\nu]}. \tag{10.207}$$

Consider the body initially at rest at the origin of the coordinates, i.e.

$$t = \tau_{U_0}, \quad x(\tau_{U_0}) = 0, \quad y(\tau_{U_0}) = 0, \quad z(\tau_{U_0}) = 0, \tag{10.208}$$

where τ_{U_0} denotes the proper time parameter. In this case the unit tangent vector reduces to $U_0 = \partial_t$. The set of equations (10.207) is fulfilled, for instance, by taking $P_0 = m_0 U_0$, implying that the components of the background spin tensor $S_0^{\mu\nu}$ remain constant along the path. The conditions (10.4) simply give $S_0^{0a} = 0$. The remaining components are related to the background spin vector components by

$$S_0^{12} = S_0^3, \qquad S_0^{13} = -S_0^2, \qquad S_0^{23} = S_0^1. \tag{10.209}$$

The first-order system of equations can also be solved straightforwardly. Setting $\sigma^\alpha \equiv (S^\alpha/m_0)\omega$ and splitting $\sigma^\alpha = \sigma_0^\alpha + \tilde{\sigma}^\alpha$, according to (10.205) we have (Bini *et al.*, 2009)

$$\begin{aligned}
\tilde{m} &= 0, \\
\tilde{\nu}_p^1 &= -A_\times \sigma_0^2 \sigma_0^3, \\
\tilde{\nu}_p^2 &= -\frac{1}{2}\left[\sigma_0^2 A_\times \sin\omega\tau_U + \sigma_0^3 A_+ (\cos\omega\tau_U - 1)\right] + \frac{1}{2} A_\times \sigma_0^1 \sigma_0^3, \\
\tilde{\nu}_p^3 &= \frac{1}{2}\left[\sigma_0^3 A_\times \sin\omega\tau_U - \sigma_0^2 A_+ (\cos\omega\tau_U - 1)\right] + \frac{1}{2} A_\times \sigma_0^1 \sigma_0^2,
\end{aligned} \tag{10.210}$$

$$\begin{aligned}
\tilde{\nu}^1 &= \frac{1}{2}[(\sigma_0^2)^2 - (\sigma_0^3)^2] A_+ \sin\omega\tau_U + \sigma_0^2 \sigma_0^3 A_\times (\cos\omega\tau_U - 1), \\
\tilde{\nu}^2 &= -\frac{1}{2}\left[A_+ \sigma_0^1 \sigma_0^2 + A_\times \sigma_0^2\right] \sin\omega\tau_U \\
&\quad - \frac{1}{2}\left[A_\times \sigma_0^1 \sigma_0^3 + A_+ \sigma_0^3\right] (\cos\omega\tau_U - 1), \\
\tilde{\nu}^3 &= \frac{1}{2}\left[A_+ \sigma_0^1 \sigma_0^3 + A_\times \sigma_0^3\right] \sin\omega\tau_U \\
&\quad - \frac{1}{2}\left[A_\times \sigma_0^1 \sigma_0^2 + A_+ \sigma_0^2\right] (\cos\omega\tau_U - 1),
\end{aligned} \tag{10.211}$$

and

$$\begin{aligned}
\tilde{\sigma}^1 &= 0, \\
\tilde{\sigma}^2 &= \frac{1}{2} A_\times \sigma_0^3 (\cos\omega\tau_U - 1) - \frac{1}{2} A_+ \sigma_0^2 \sin\omega\tau_U, \\
\tilde{\sigma}^3 &= \frac{1}{2} A_\times \sigma_0^2 (\cos\omega\tau_U - 1) - \frac{1}{2} A_+ \sigma_0^3 \sin\omega\tau_U,
\end{aligned} \tag{10.212}$$

so that the spatial orbit is

$$
\begin{aligned}
\omega\, x &= -\frac{1}{2}[(\sigma_0^2)^2 - (\sigma_0^3)^2]A_+(\cos\omega\tau_U - 1) \\
&\quad + \sigma_0^2\sigma_0^3 A_\times(\sin\omega\tau_U - \omega\tau_U), \\
\omega\, y &= \frac{\sigma_0^2}{2}(A_+\sigma_0^1 + A_\times)(\cos\omega\tau_U - 1) \\
&\quad - \frac{\sigma_0^3}{2}(A_\times\sigma_0^1 + A_+)(\sin\omega\tau_U - \omega\tau_U), \\
\omega\, z &= -\frac{\sigma_0^3}{2}(A_+\sigma_0^1 + A_\times)(\cos\omega\tau_U - 1) \\
&\quad - \frac{\sigma_0^2}{2}(A_\times\sigma_0^1 + A_+)(\sin\omega\tau_U - \omega\tau_U),
\end{aligned}
\tag{10.213}
$$

in agreement with the results of Mohseni and Sepangi (2000).

Epilogue

The mathematical structure of general relativity makes its equations quite remote from a direct understanding of their content. Indeed, the combination of a covariant four-dimensional description of the physical laws and the need to cope with the relativity of the observations makes a physical measurement an elaborate procedure. The latter consists of a few basic steps:

(i) Identify the covariant equations which describe the phenomenon under investigation.
(ii) Identify the observer who makes the measurements.
(iii) Choose a frame adapted to that observer, allowing the space-time to be split into the observer's *space* and *time*.
(iv) Decide whether the intended measurement is local or non-local with respect to the background curvature.
(v) Identify the frame components of those quantities that are the observational targets.
(vi) Find a physical interpretation of the above components, following a suitable criterion such as a comparison with what is known from special relativity or from non-relativistic theories.
(vii) Verify the degree of residual ambiguity in the interpretation of the measurements and decide on a strategy to eliminate it.

Clearly, each step of the above procedure relies on the previous one, and the very first step provides the seed of a measurement despite the mathematical complexity.

- Fixing the observer is independent of the coordinate representation; we can have many observers with a given choice of the coordinate grid, but we can also deal with a given observer within many coordinate systems.
- Whatever the choice of the coordinate system, one may adapt to a given observer many *spatial* frames, each providing a different perspective.
- A measurement requires a mathematical modeling of the target, but also of the measuring conditions which account for the dynamical state of the observer and the level of accuracy of his measuring devices.
- The physical interpretation of a measurement requires some previous knowledge of the object of investigation, so that it can be identified as a *generalization* from

a restricted case as from special to general relativity, from empty to non-empty states, or from static to non-static cases, just to mention a few.

- Without a comparison cornerstone, the result of a measurement may reveal itself as a new general relativistic effect.
- General relativity stems from the Principle of Equivalence, which states a fundamental ambiguity, namely the impossibility of distinguishing true gravity from inertial accelerations. Like a kind of original sin, this ambiguity surfaces in many measuring operations, forcing the observer to envisage sets of independent measurements to cure this inadequacy and reach the required degree of certainty.

In this book we have illustrated all of the above steps, with the intention of highlighting the method rather than of considering all possible measurements, most of which are widely discussed in the literature. The notation used is meant to help the reader to recognize at each step who the observer is and what is being observed; clearly this distinction leads easily to confusion when more than two observers are involved.

Although we conclude here our bird's-eye view of the theory of measurements in general relativity, we hardly consider this subject closed. Indeed much remains to be done to understand how classical measurements in relativity theory would match with quantum measurements in the search for a unified theory.

Exercises

1. Consider the Euclidean 2-sphere V_2 with Riemannian metric

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2.$$

(a) Show that the orthonormal frame

$$e_{\hat{1}} = \partial_\theta, \qquad e_{\hat{2}} = \frac{1}{\sin\theta}\partial_\phi,$$

with dual

$$\omega^{\hat{1}} = d\theta, \qquad \omega^{\hat{2}} = \sin\theta d\phi,$$

admits a single non-vanishing structure function

$$C^{\hat{2}}{}_{\hat{1}\hat{2}} = -\cot\theta.$$

(b) Show that the only non-vanishing Christoffel symbols are

$$\Gamma^\phi{}_{\theta\phi} = \cot\theta, \qquad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta.$$

(c) Solve the Killing equation $\nabla_{(\beta}\xi_{\alpha)} = 0$ in the above metric and show that the general solution for the vector ξ is given by

$$\xi = c_1\xi_1 + c_2\xi_2 + c_3\xi_3,$$

where c_1, c_2, c_3 are constants and

$$\begin{aligned}
\xi_1 &= \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi,\\
\xi_2 &= \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi,\\
\xi_3 &= \partial_\phi,
\end{aligned}$$

are the well-known angular momentum operators or the generators of the rotation group.

(d) Show that the 4-acceleration of the curve with unit tangent vector $e_{\hat{\phi}} = 1/\sin\theta\partial_\phi$ is

$$a(e_{\hat{\phi}}) \equiv \nabla_{e_{\hat{\phi}}} e_{\hat{\phi}} = -\cot\theta\partial_\theta.$$

Is it possible to specify a priori the direction of $a(e_{\hat{\phi}})$?

(e) Evaluate the wedge product $a(e_{\hat{\phi}}) \wedge e_{\hat{\phi}}$.

(f) Discuss the parallel transport of a general vector

$$X = X^\theta \partial_\theta + X^\phi \partial_\phi$$

along a ϕ-loop, that is, the curve with parametric equations

$$\theta = \theta_0 = \text{constant}, \qquad \phi = \phi(\lambda),$$

i.e. with unit tangent vector $e_{\hat\phi} = (1/\sin\theta)\partial_\phi$.
Hint. Using ϕ (in place of λ) as a parameter along ℓ, the transport equations reduce to the system

$$\frac{dX^\theta}{d\phi} - \sin\theta\cos\theta X^\phi = 0, \qquad \frac{dX^\phi}{d\phi} + \frac{\cos\theta}{\sin^2\theta} X^\theta = 0.$$

It is also convenient to use frame components for X:

$$X^{\hat\theta} = X^\theta, \qquad X^{\hat\phi} = \sin\theta X^\phi.$$

In fact the above equations reduce to

$$\frac{dX^{\hat\theta}}{d\phi} - \cos\theta X^{\hat\phi} = 0, \qquad \frac{dX^{\hat\phi}}{d\phi} + \cos\theta X^{\hat\theta} = 0,$$

and can be easily solved:

$$\begin{pmatrix} X^{\hat\theta}(\phi) \\ X^{\hat\phi}(\phi) \end{pmatrix} = R(\phi\cos\theta) \begin{pmatrix} X^{\hat\theta}(0) \\ X^{\hat\phi}(0) \end{pmatrix},$$

where

$$R(\phi\cos\theta) = \begin{pmatrix} \cos(\phi\cos\theta) & \sin(\phi\cos\theta) \\ -\sin(\phi\cos\theta) & \cos(\phi\cos\theta) \end{pmatrix}.$$

(g) Study the conditions for *holonomy invariance* of the transported vector after one loop.
Hint. When $\phi = 2\pi$, for $\theta = \pi/2$ (equatorial orbit) as well as for $\theta = 0$ we have

$$R(2\pi\cos\theta) \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For a general value of θ the invariance is lost.

2. Repeat the above exercise for the 2-pseudosphere with metric

$$ds^2 = d\theta^2 + \sinh^2\theta d\phi^2.$$

3. Consider Schwarzschild space-time written in standard spherical-like coordinates. Discuss the parallel transport of a vector $X = X^\alpha\partial_\alpha$ along a circular orbit on the equatorial plane with unit tangent vector

$$U = \Gamma(\partial_t + \zeta\partial_\phi).$$

4. Considering the setting of the previous problem, discuss the parallel transport of the vector $X = X^\alpha \partial_\alpha$ along a ϕ-loop with parametric equations

$$t = t_0, \quad r = r_0, \quad \theta = \pi/2, \quad \phi = \phi(\lambda),$$

and unit tangent vector $e_{\hat\phi} = 1/r_0 \partial_\phi$.

(a) Discuss the conditions for *holonomy invariance* after a ϕ-loop.

(b) Compare with the limiting case $\zeta \to \infty$.

5. Consider the Schwarzschild metric.

(a) Show that the unit 4-velocity vector of an observer at rest with respect to the spatial coordinates is

$$m^\flat = -\sqrt{1 - \frac{2\mathcal{M}}{r}}\, dt.$$

(b) Show that

$$dm^\flat = m^\flat \wedge g(m), \qquad g(m)^\flat = \frac{\mathcal{M}}{r^2\left(1 - \frac{2\mathcal{M}}{r}\right)}\, dr.$$

(c) Show that $g(m)$ is a gradient, i.e. that $dg(m) = 0$, and find the potential.

(d) Show that $\pounds_m g(m)^\flat = 0$.

6. Discuss the geometric properties of the 2-metric

$$ds^2 = dv^2 - v^2 du^2.$$

7. Show that the Riemann tensor of a maximally symmetric space-time is given by

$$R^{\alpha\beta}{}_{\gamma\delta} = \frac{R}{12}\delta^{\alpha\beta}_{\gamma\delta},$$

where R is the curvature scalar.

8. Show that, for the unit 4-form η,

$$\eta^{\alpha\beta\gamma\delta}\eta_{\alpha\beta\gamma\delta} = -4!$$

9. Prove that $P(u)\eta = 0$, where $P(u)$ is the projection tensor onto LRS_u.
Hint. A tensor vanishing in a frame vanishes identically.

10. Prove Eq. (2.51).
Hint. First permute indices in Eq. (2.45) as follows:

$$\begin{aligned}
e_\gamma(g_{\delta\beta}) - 2\Gamma_{(\delta\beta)\gamma} &= 0,\\
e_\delta(g_{\beta\gamma}) - 2\Gamma_{(\beta\gamma)\delta} &= 0,\\
e_\beta(g_{\gamma\delta}) - 2\Gamma_{(\gamma\delta)\beta} &= 0,
\end{aligned}$$

then add and subtract these relations side by side so that

$$e_\gamma(g_{\delta\beta}) - e_\delta(g_{\beta\gamma}) + e_\beta(g_{\gamma\delta}) - 2\Gamma_{(\delta\beta)\gamma} + 2\Gamma_{(\beta\gamma)\delta} - 2\Gamma_{(\gamma\delta)\beta} = 0,$$

or, equivalently,

$$e_\gamma(g_{\delta\beta}) - e_\delta(g_{\beta\gamma}) + e_\beta(g_{\gamma\delta}) - 2\Gamma_{\delta(\beta\gamma)} + 2\Gamma_{\beta[\gamma\delta]} + 2\Gamma_{\gamma[\beta\delta]} = 0.$$

Noting that

$$C^\gamma{}_{\alpha\beta} = 2\Gamma^\gamma{}_{[\beta\alpha]},$$

one can cast the above relation in the form

$$e_\gamma(g_{\delta\beta}) - e_\delta(g_{\beta\gamma}) + e_\beta(g_{\gamma\delta}) - 2\Gamma_{\delta(\beta\gamma)} + C_{\beta\delta\gamma} + C_{\gamma\delta\beta} = 0,$$

which in turn gives

$$2\Gamma_{\delta(\beta\gamma)} = e_\gamma(g_{\delta\beta}) - e_\delta(g_{\beta\gamma}) + e_\beta(g_{\gamma\delta}) + C_{\beta\delta\gamma} + C_{\gamma\delta\beta}.$$

From this it follows that

$$\begin{aligned}\Gamma_{\delta\beta\gamma} &= \Gamma_{\delta(\beta\gamma)} + \Gamma_{\delta[\beta\gamma]} \\ &= \frac{1}{2}\left[e_\gamma(g_{\delta\beta}) - e_\delta(g_{\beta\gamma}) + e_\beta(g_{\gamma\delta})\right. \\ &\quad \left. + C_{\beta\delta\gamma} + C_{\gamma\delta\beta} + C_{\delta\gamma\beta}\right],\end{aligned}$$

as in (2.51).

11. Show that Fermi-Walker transport preserves orthogonality.

12. Let u be the time-like unit tangent vector to a world line γ parameterized by the proper time τ. Show that

$$\frac{D_{(\mathrm{fw},u)}}{d\tau} u^\alpha = 0.$$

13. Let u be the time-like unit tangent vector to a world line γ and let X be a spatial vector with respect to u ($X \cdot u = 0$) Fermi-Walker dragged along γ. Show that

$$\frac{D}{d\tau} X^\alpha = [a(u) \cdot X] u^\alpha.$$

14. Discuss the extension of Fermi-Walker transport to null curves. This result is discussed in Bini *et al.* (2006), generalizing previous results by Castagnino (1965).

15. Let u be the time-like unit vector tangent to a (time-like) world line γ parameterized by the proper time τ and the spatial vectors $e_{\hat a}$ ($a = 1, 2, 3$) mutually orthogonal and orthogonal to u, so that $e_{\hat\alpha} = \{e_{\hat 0} \equiv u, e_{\hat a}\}$ is an orthonormal tetrad. Show that

$$\frac{D_{(\mathrm{fw},u)}}{d\tau} e_{\hat{a}} = \nabla_u e_{\hat{a}} - a(u)_{\hat{a}} u \equiv C^{\hat{b}}_{(\mathrm{fw})\,\hat{a}}\, e_{\hat{b}},$$

where the coefficients $C^{\hat{b}}_{(\mathrm{fw})\,\hat{a}}$ are the Fermi rotation coefficients.

16. If T is a $\binom{0}{3}$-tensor, show that its fully symmetric and antisymmetric parts are given by

$$\begin{aligned} T_{(\mu\nu\lambda)} &= \frac{1}{3!}\left(T_{\mu\nu\lambda} + T_{\nu\lambda\mu} + T_{\lambda\mu\nu} + T_{\nu\mu\lambda} + T_{\mu\lambda\nu} + T_{\lambda\nu\mu}\right), \\ T_{[\mu\nu\lambda]} &= \frac{1}{3!}\left(T_{\mu\nu\lambda} + T_{\nu\lambda\mu} + T_{\lambda\mu\nu} - T_{\nu\mu\lambda} - T_{\mu\lambda\nu} - T_{\lambda\nu\mu}\right). \end{aligned}$$

17. If T is a $\binom{0}{4}$-tensor, evaluate its fully symmetric and antisymmetric parts.

18. Show that for a differential form S, the Lie derivative can be expressed in terms of the exterior derivative and the contraction operation:

$$\pounds_X S = X \rfloor dS + d(X \rfloor S).$$

19. Verify the following identity for a 1-form X:

$$\delta X\, \eta = {}^*\delta X = {}^{**}d^*X = -d^*X,$$

which can also be written in the form

$$d(X^\sharp \rfloor \eta) = [\operatorname{div} X]\, \eta.$$

20. Show that, in the Kerr metric, the Papapetrou field associated with the time-like Killing vector $\xi = \partial_t$, namely $F_{\alpha\beta} = \nabla_\alpha \xi_\beta$, has principal null directions which are aligned with those of the Weyl tensor.

21. Show that a constant curvature space-time is conformally flat, i.e. $C^{\alpha\beta}{}_{\gamma\delta} = 0$.

22. Show that

$$^*[u^\flat \wedge S] = {}^{*(u)}S,$$

and therefore that

$$^*[{}^{*(u)}S] = (-1)^p u \wedge S.$$

23. Consider the splitting of a 2-form F and that of its space-time dual ${}^*F$ with respect to an observer u. Show that

$$F = u \wedge E(u) + {}^{*(u)}B(u), \qquad {}^*F = u \wedge B(u) - {}^{*(u)}E(u),$$

where $E(u) = -u \rfloor F$, $B(u) = {}^{*(u)}[P(u)F]$.

24. Let X be non-spatial with respect to an observer u, so that

$$X = X^{||}\, u + X^\perp.$$

Show that its spatial curl as defined in Eq. (3.37) is given by

$$\operatorname{curl} X = 2X^{||}\omega(u)^\alpha + [\operatorname{curl}_u X^\perp]^\alpha.$$

25. Show that the symmetric spatial Riemann tensor $R_{(\mathrm{sym})}$ satisfies the algebraic conditions characterizing a Riemann tensor:

$$\begin{aligned} &R_{(\mathrm{sym})\,(ab)cd} = 0, \qquad R_{(\mathrm{sym})\,ab(cd)} = 0, \\ &R_{(\mathrm{sym})\,abcd} - R_{(\mathrm{sym})\,cdab} = 0, \\ &R_{(\mathrm{sym})\,a[bcd]} = 0. \end{aligned}$$

26. Show that the Lie derivative along u of the projection tensor $P(u)$ is given by

$$\begin{aligned} \pounds_u P(u)^\flat &= 2\theta(u), \\ \pounds_u P(u)^\sharp &= -2\theta(u)^\sharp, \\ \pounds_u P(u) &= u \otimes a(u)^\flat. \end{aligned}$$

27. Show that

$$\pounds_u \eta(u)_{\alpha\beta\gamma} = \Theta(u)\eta_{\alpha\beta\gamma}.$$

Hint. From the definition of the Lie derivative and that of the unit spatial volume 3-form $\eta(u)$, one gets

$$\pounds_u \eta(u)_{\beta\gamma\delta} = \eta(u)_{\mu\gamma\delta}\theta(u)^\mu{}_\beta - \eta(u)_{\mu\beta\delta}\theta(u)^\mu{}_\gamma + \eta(u)_{\mu\beta\gamma}\theta(u)^\mu{}_\delta.$$

For $\beta = 1$, $\gamma = 2$, and $\delta = 3$ (the only possible choice of indices apart from permutations), we have

$$\pounds_u \eta(u)_{123} = \Theta(u)\eta_{123}.$$

28. Consider an observer u with his adapted orthonormal spatial frame $e(u)_{\hat a}$, with $a = 1, 2, 3$. If another observer U is related to u by a boost in the generic direction $\hat\nu(U,u)$, i.e.

$$U = \gamma(U,u)[u + \nu(U,u)\hat\nu(U,u)] \equiv \gamma[u + \nu\hat\nu(U,u)],$$

show that the boost of the triad $e(u)_{\hat a}$ onto LRS_U gives the following orthonormal triad:

$$e(U)_{\hat a} = e(u)_{\hat a} + [\hat\nu(U,u)\cdot e(u)_{\hat a}]\,[\gamma\nu u + (\gamma - 1)\hat\nu(U,u)].$$

29. Show that if $X(U) \in LRS_U$ and $Y(u) \in LRS_u$ then

$$\begin{aligned} X(U) \times_U Y(u) = \gamma\,\{&[P(u,U)X(U) \times_u Y(u)] \\ &- (\nu \cdot P(u,U)X(U))(\nu \times_u Y(u)) \\ &- u[P(u,U)X(U) \cdot (\nu \times_u Y(u))]\}\,. \end{aligned}$$

Hint. From the definition of cross product in LRS_U we have

$$\begin{aligned}\eta(U)^{\alpha\beta\gamma}X(U)_\beta Y(u)_\gamma &= \eta^{\mu\alpha\beta\gamma}U_\mu X(U)_\beta Y(u)_\gamma\\ &= [-u^\mu\eta(u)^{\alpha\beta\gamma} + u^\alpha\eta(u)^{\mu\beta\gamma} - u^\beta\eta(u)^{\gamma\mu\alpha}\\ &\quad + u^\gamma\eta(u)^{\beta\mu\alpha}]U_\mu X(U)_\beta Y(u)_\gamma\\ &= \gamma[-u^\mu\eta(u)^{\alpha\beta\gamma} + u^\alpha\eta(u)^{\mu\beta\gamma} - u^\beta\eta(u)^{\gamma\mu\alpha}]\\ &\quad \times [u_\mu + \nu(U,u)_\mu]X(U)_\beta Y(u)_\gamma.\end{aligned}$$

30. Deduce the algebra of the mixed projection operators

$$P(u,U,u) \equiv P(u,U)P(U,u), \qquad P(u,U,u)^{-1}.$$

Show that these maps have the following expressions:

$$\begin{aligned}P(u,U,u) &= P(u) + \gamma^2\nu(U,u)\otimes\nu(U,u)^\flat,\\ P(u,U,u)^{-1} &= P(U,u)^{-1}P(u,U)^{-1}\\ &= P(u) - \nu(U,u)\otimes\nu(U,u)^\flat.\end{aligned}$$

31. Deduce the algebra of the mixed projection operators

$$P(u,U,u') \equiv P(u,U)P(U,u'), \qquad P(u,U,u')^{-1}.$$

Show that

$$\begin{aligned}P(u,U,u') &= P(u,u') + \gamma(U,u)\gamma(U,u')\nu(U,u)\otimes\nu(U,u')\\ P(u,U,u')^{-1} &= P(u',u) + \gamma(u,u')[(\nu(u,u') - \nu(U,u'))\otimes\nu(U,u)\\ &\quad + \nu(U,u')\otimes\nu(u',u)].\end{aligned}$$

Hint. For the various observers u, U, and u', we have

$$\begin{aligned}U &= \gamma(U,u)[u + \nu(U,u)],\\ U &= \gamma(U,u')[u' + \nu(U,u')],\\ u' &= \gamma(u',u)[u + \nu(u',u)],\\ u &= \gamma(u,u')[u' + \nu(u,u')].\end{aligned}$$

32. Prove the following relations:

$$P(U,u)^{-1}P(U,u') = P(u,u') + \gamma(u,u')\nu(U,u)\otimes\nu(u,u'),$$

$$P(u',u)P(U,u)^{-1}P(U,u') = P(u') + \delta(U,u,u')\nu(U,u')\otimes\nu(u,u'),$$

$$
\begin{aligned}
P(u', u)P(u', U, u)^{-1} &= P(u') \\
&\quad + \delta(U, u, u')\nu(U, u') \otimes [\nu(u, u') - \nu(U, u')],
\end{aligned}
$$

$$
\begin{aligned}
P(u', u)P(u, U, u)^{-1} &= P(u', u) - \frac{\delta(U, u, u')}{\gamma(u, u')}\nu(U, u') \otimes \nu(U, u) \\
&\quad + \gamma(u, u')\nu(u, u') \otimes \nu(U, u),
\end{aligned}
$$

$$
\begin{aligned}
P(u', u)P(u, U, u)^{-1}P(u, u') &= P(u') + \delta(U, u, u')[\nu(U, u') \otimes \nu(u, u') \\
&\quad + \nu(u, u') \otimes \nu(U, u')] \\
&\quad - \delta(U, u, u')^2\nu(U, u') \otimes \nu(U, u')/\gamma(u, u')^2,
\end{aligned}
$$

where

$$
\delta(U, u, u') = \frac{\gamma(U, u')\gamma(u', u)}{\gamma(U, u)}, \qquad \delta(U, u, u')^{-1} = \delta(u, U, u').
$$

33. Consider the Schwarzschild solution in standard coordinates (t, r, θ, ϕ), and an observer at rest at a fixed position on the equatorial plane.

Show that the Schwarzschild metric can be cast in the following form (see (5.41)):

$$
ds^2 = -(1 - 2\mathcal{A}X)dT^2 + dX^2 + dY^2 + dZ^2 + O(2),
$$

where (T, X, Y, Z) are Fermi coordinates associated with an observer at rest at $r = r_0$, $\theta = \theta_0 = \pi/2$, $\phi = \phi_0$ (Leaute and Linet, 1983; see also Bini, Geralico, and Jantzen, 2005 for the generalization to the case of a Kerr space-time and a uniformly rotating observer) and related to Schwarzschild coordinates (t, r, θ, ϕ) by the transformation

$$
\begin{aligned}
t &\simeq t_0 + \left(1 - \frac{2M}{r_0}\right)^{-1/2} T, \\
r &\simeq r_0 + \left(1 - \frac{2M}{r_0}\right)^{1/2} X + \frac{1}{2}\left[\frac{M}{r_0^2}X^2 + \frac{1}{r_0}\left(1 - \frac{2M}{r_0}\right)(Y^2 + Z^2)\right], \\
\theta &\simeq \frac{\pi}{2} + \frac{Y}{r_0} - \frac{1}{r_0^2}\left(1 - \frac{2M}{r_0}\right)^{1/2} XY, \\
\phi &\simeq \phi_0 + \frac{Z}{r_0} - \frac{1}{r_0^2}\left(1 - \frac{2M}{r_0}\right)^{1/2} XZ.
\end{aligned}
$$

All the above relations are valid up to $O(3)$ and with the uniform acceleration $\mathcal{A}$ in Eq. (5.41) given by

$$
\mathcal{A} = \frac{M}{r_0^2}\left(1 - \frac{2M}{r_0}\right)^{-1/2}.
$$

34. Find the transformation laws for the relative thermal stresses $(q(u))$ and the relative energy density $(\rho(u))$ passing to another observer U.

35. Deduce relations (7.40) and (7.41) and then combine them to obtain (7.46).

36. Deduce Eq. (7.50).

37. Show that by taking the covariant derivative of both sides of (7.59) one gets the relative acceleration equation.
Hint. The covariant derivative of both sides of the equation

$$\nabla_U Y = \nabla_Y U$$

gives

$$\begin{aligned}\nabla_{UU} Y &= \nabla_U \nabla_Y U = [\nabla_U, \nabla_Y]U + \nabla_Y \nabla_U U\\ &= [\nabla_U, \nabla_Y]U + \nabla_Y a(U)\\ &= R(U, Y)U + \nabla_Y a(U),\end{aligned}$$

since $[U, Y] = 0$.

38. Show that the coordinate transformation

$$r = \bar{r}\left(1 + \frac{\mathcal{M}}{2\bar{r}}\right)^2$$

leads to the isotropic form of the Schwarzschild metric,

$$\begin{aligned}ds^2 = &-\left(1 - \frac{\mathcal{M}}{2\bar{r}}\right)^2 \left(1 + \frac{\mathcal{M}}{2\bar{r}}\right)^{-2} dt^2\\ &+ \left(1 + \frac{\mathcal{M}}{2\bar{r}}\right)^4 (dx^2 + dy^2 + dz^2).\end{aligned}$$

39. In Schwarzschild space-time let U be the unit time-like vector tangent to a circular orbit, $U = \Gamma(\partial_t + \zeta\partial_\phi)$. Consider the vector fields $m = (1/\sqrt{-g_{tt}})\partial_t$ (congruence of static observers) and $e_{\hat{\phi}} = (1/\sqrt{g_{\phi\phi}})\partial_\phi$ (congruence of spatial ϕ-loops). Show that

$$\begin{aligned}\nabla_U m &= \frac{\Gamma\mathcal{M}}{r^2} e_{\hat{r}},\\ \nabla_U e_{\hat{\phi}} &= -\Gamma\zeta\left[\left(1 - \frac{2\mathcal{M}}{r}\right)^{1/2} \sin\theta\, e_{\hat{r}} + \cos\theta\, e_{\hat{\theta}}\right].\end{aligned}$$

40. With the Kerr metric written in Painlevé-Gullstrand coordinates X^α (see (8.79)), the locally non-rotating observers have 4-velocity $\mathcal{N}^\flat = -dT$.

(a) Show that they form a geodesic ($a(\mathcal{N}) = 0$) and vorticity-free ($\omega(\mathcal{N}) = 0$) congruence, with non-vanishing expansion tensor given by

$$\theta(\mathcal{N}) = \theta(\mathcal{N})^{\alpha\beta} \partial_{X^\alpha} \otimes \partial_{X^\beta},$$

with

$$\theta(\mathcal{N})^{RR} = \frac{r\Delta\sqrt{2\mathcal{M}r(r^2+a^2)}}{\Sigma^3} - \frac{(r^2+a^2)^2 - 4\mathcal{M}r^3}{\Sigma^2}\sqrt{\frac{\mathcal{M}}{2r(r^2+a^2)}},$$

$$\theta(\mathcal{N})^{\Theta\Theta} = -\frac{r\sqrt{2\mathcal{M}r(r^2+a^2)}}{\Sigma^3},$$

$$\theta(\mathcal{N})^{\Phi\Phi} = -\sqrt{\frac{2\mathcal{M}r}{r^2+a^2}}\frac{r}{(r^2+a^2)\Sigma\sin^2\theta},$$

$$\theta(\mathcal{N})^{R\Theta} = -\frac{a^2\sin\theta\cos\theta}{\Sigma^3}\sqrt{2\mathcal{M}r(r^2+a^2)},$$

$$\theta(\mathcal{N})^{R\Phi} = -\frac{2a\mathcal{M}r^2}{(r^2+a^2)\Sigma^2},$$

being the only non-zero components.

(b) Show that the trace is also non-zero, and given by

$$\Theta(\mathcal{N}) = -\frac{a^2+3r^2}{\Sigma}\sqrt{\frac{\mathcal{M}}{2r(r^2+a^2)}}.$$

41. Consider a set of particles orbiting on spatially circular trajectories in Kerr space-time, and let $U = \Gamma(\partial_t + \zeta\partial_\phi)$ be the unit time-like tangent vector.

(a) Show that

$$\Gamma = \left[1 - \frac{2\mathcal{M}r}{\Sigma}(1 - a\zeta\sin^2\theta)^2 - (r^2+a^2)\zeta^2\sin^2\theta\right]^{-1/2}.$$

Consider the following adapted frame:

$$\begin{aligned}
E_{\hat{t}} &= \Gamma(\partial_t + \zeta\partial_\phi),\\
E_{\hat{r}} &= (\Delta/\Sigma)^{1/2}\partial_r,\\
E_{\hat{\theta}} &= \Sigma^{-1/2}\partial_\theta,\\
E_{\hat{\phi}} &= \bar{\Gamma}(\partial_t + \bar{\zeta}\partial_\phi).
\end{aligned}$$

(b) Determine the expressions for $\bar{\Gamma}$ and $\bar{\zeta}$.

(c) Show that the frame components of the electric part of the Riemann tensor associated with U are given by

$$\begin{aligned}
\mathcal{E}(U)_{\hat{r}\hat{r}} &= \frac{\Gamma^2\mathcal{M}r}{\Sigma^4}(r^2 - 3a^2\cos^2\theta)[(3\Delta + 2\mathcal{M}r)J\\
&\quad + \Sigma(2\Delta\sin^2\theta\zeta^2 + 1)],\\
\mathcal{E}(U)_{\hat{\theta}\hat{\theta}} &= -\frac{\Gamma^2\mathcal{M}r}{\Sigma^4}(r^2 - 3a^2\cos^2\theta)[(3\Delta + 4\mathcal{M}r)J\\
&\quad + \Sigma(\Delta\sin^2\theta\zeta^2 + 2)],
\end{aligned}$$

$$\mathcal{E}(U)_{\hat{r}\hat{\theta}} = \frac{3a\sin\theta\cos\theta\mathcal{M}\Gamma^2\sqrt{\Delta}}{\Sigma^4}[a-\zeta(r^2+a^2)]$$
$$\times(1-a\sin^2\theta\zeta)(3r^2-a^2\cos^2\theta),$$
$$\mathcal{E}(U)_{\hat{\phi}\hat{\phi}} = \frac{\mathcal{M}r}{\Sigma^3}(r^2-3a^2\cos^2\theta),$$

where

$$J = -(1-a\zeta)^2\sin^2\theta - \zeta^2 r^2\sin^2\theta - \cos^2\theta.$$

(d) Show that the Fermi rotation of the above frame has components

$$\zeta_{(\mathrm{fw})\hat{r}} = -\frac{\Gamma^2\cos\theta}{\Sigma^2}\sqrt{\frac{\Delta}{\Sigma}}\,[2\mathcal{M}ra(1-a\zeta\sin^2\theta)^2-\zeta\Sigma^2],$$
$$\zeta_{(\mathrm{fw})\hat{\theta}} = -\frac{\Gamma^2\sin\theta}{a\Sigma^{5/2}}\left\{\mathcal{M}(r^2-a^2\cos^2\theta)[(r^2+a^2)\zeta-a]^2\right.$$
$$\left.+\zeta[\mathcal{M}(a^2-r^2)\zeta+\Sigma^2 a(r-\mathcal{M})]\right\}.$$

42. Deduce the 4-vector $U_{(\mathrm{g})}$ as in Eq. (8.196).
Hint. Use coordinates (u, v, y, z) and the Killing vectors.

43. Prove that (i) $\overset{*}{y}_- < y_{c-}$ and (ii) $y_{c-} \le \overset{*}{y}_+ < y_{c+}$, where $\overset{*}{y}$ is given by (9.51) and $y_{c\pm}$ are given by (9.48).
Hint. To prove point (i) let us first notice that

$$y_{c-} > -1/a, \qquad \lim_{r\to\infty} y_{c-} = 0_-,$$
$$y_{c-} \ge 0 \text{ for } r \le 2\mathcal{M},\ y_{c-} < 0 \text{ for } r > 2\mathcal{M}.$$

On the other hand, we have from (9.51) that $\overset{*}{y}_- < 0$ and $\partial\overset{*}{y}_-/\partial r > 0$ for $0 < r < \infty$, and

$$\lim_{r\to 0}\overset{*}{y}_- = -\infty, \qquad \lim_{r\to\infty}\overset{*}{y}_- = -\frac{1}{a}.$$

Then it is always the case that $\overset{*}{y}_- \le -1/a < y_{c-}$.

The function $\overset{*}{y}_+$ is always positive ($\overset{*}{y}_+ > 0$) and satisfies the condition $\overset{*}{y}_+ \le a/r^2$ for $\Delta \ge 0$, so, because $y_{c+} \ge a/r^2$ whenever $\Delta \ge 0$, we also have $\overset{*}{y}_+ \le y_{c+}$.

To prove point (ii) we need to show that $\overset{*}{y}_+ > y_{c-}$ in the range $r_+ < r \le 2\mathcal{M}$, where $y_{c-} > 0$. At $r = r_+$ (where $\Delta = 0$) we have, as stated, $y_{c-} = \overset{*}{y}_+$, but these two functions leave the point r_+ with different slopes; in fact $\partial y_{c-}/\partial r|_{r_+} = -\infty$, while

$$\left.\frac{\partial\overset{*}{y}_+}{\partial r}\right|_{r_+} = -\left(\frac{6\mathcal{M}a}{r^4}\right)\left(1+\frac{8\mathcal{M}a^2}{r_+^3}\right)^{-1/2}$$

is finite and negative. Hence for $r = r_+ + \epsilon$ ($\epsilon > 0$ and sufficiently small) we certainly have $\overset{*}{y}_+ > y_{c-}$. However, since y_{c-} goes to zero monotonically as $r \to 2\mathcal{M}$, the function $\overset{*}{y}_+$ could intersect y_{c-} (it should do it at least twice!) only if the plot of $\overset{*}{y}_+$ is convex somewhere in the range $r_+ < r < 2\mathcal{M}$. But this is never so because the second derivative of $\overset{*}{y}_+$ is always positive, being equal to

$$\frac{\partial^2 \overset{*}{y}_+}{\partial r^2} = \frac{24\mathcal{M}a}{r^5} \frac{1 + 5\mathcal{M}a^2/r^3}{(1 + 8\mathcal{M}a^2/r^3)^{3/2}}.$$

44. Show that the Riemann tensor $R^\alpha{}_{\beta\gamma\delta}$ has *in general* no specific symmetries with respect to the indices β and γ.

45. In Schwarzschild space-time, consider a particle moving in a circular orbit in the equatorial plane $\theta = \pi/2$ with 4-velocity

$$U^\alpha = \Gamma\left(\delta^\alpha_t + \zeta\delta^\alpha_\phi\right),$$

where Γ is the normalization factor given by

$$\Gamma = \left(1 - \frac{2M}{r} - \zeta^2 r^2\right)^{-1/2}.$$

If the magnitude of the 4-acceleration of the particle, namely $||a(U)|| = \sqrt{g_{\alpha\beta}a(U)^\alpha a(U)^\beta}$, where $a(U)^\alpha = U^\beta\nabla_\beta U^\alpha$, measures the effective specific weight of the particle with respect to a comoving observer, calculate the value of ζ at which the particle's weight is maximum.

46. Given the Friedmann-Robertson-Walker solution describing a spatially homogeneous and isotropic universe,

$$ds^2 = -dt^2 + R^2(t)\left[\frac{dr^2}{1 - \kappa r^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right],$$

where $R(t)$ is a differentiable function of time and $\kappa = \pm 1$ or 0 according to the type of spatial section, i.e. a pseudo-sphere ($\kappa = -1$), a flat space ($\kappa = 0$), or a sphere ($\kappa = 1$):

(a) Show that the cosmic observer with 4-velocity

$$u^\alpha = \delta^\alpha_t$$

is a geodesic for any κ.

(b) Given two cosmic observers very close to each other, their relative acceleration in the radial direction depends only on the component of the Riemann tensor $R^r{}_{trt}$. Calculate this component.

47. Given a null vector field with components ℓ^α, so that

$$\ell_\alpha \ell^\alpha = 0,$$

if it satisfies the condition

$$\nabla_{[\alpha}\ell_{\beta]} = 0,$$

show that it is tangent to a family of affine geodesics.

48. Given an electromagnetic field with energy-momentum tensor

$$T_{\alpha\beta} = \frac{1}{4\pi}\left[F_{\alpha\sigma}F_{\beta}{}^{\sigma} - \frac{1}{4}g_{\alpha\beta}F_{\rho\sigma}F^{\rho\sigma}\right],$$

show that the curvature it generates satisfies the condition

$$R = 0,$$

R being the curvature scalar.

49. In Schwarzschild space-time, consider two static observers with unitary 4-velocities

$$u_1 = \left(1 - \frac{2\mathcal{M}}{R_1}\right)^{-1/2} \partial_t$$

and

$$u_2 = \left(1 - \frac{2\mathcal{M}}{R_2}\right)^{-1/2} \partial_t,$$

with $R_2 > R_1$. Determine which of these observers ages faster.

50. Consider Kerr space-time written in standard Boyer-Lindquist coordinates. Discuss the parallel transport of a vector $X = X^\alpha \partial_\alpha$ along a circular orbit on the equatorial plane with unit tangent vector

$$U = \Gamma(\partial_t + \zeta\partial_\phi).$$

51. Considering the setting of the previous problem, discuss the parallel transport of the vector $X = X^\alpha\partial_\alpha$ along a ϕ-loop with parametric equations

$$t = t_0, \quad r = r_0, \quad \theta = \pi/2, \quad \phi = \phi(\lambda),$$

and unit tangent vector $e_{\hat{\phi}} = [r/(r^3 + a^2 r + 2\mathcal{M}a^2)]^{1/2}\partial_\phi$.

(a) Discuss the conditions for *holonomy invariance* after a ϕ-loop.

(b) Compare with the limiting case $\zeta \to \infty$.

52. Under what condition does the relation

$$(\pounds_X Y^\sharp)^\flat = (\pounds_X Y^\flat)$$

hold for any pair of vector fields X and Y?

53. Consider Minkowski space-time in cylindrical coordinates $\{t, r, \phi, z\}$:

$$ds^2 = -dt^2 + dr^2 + r^2 d\phi^2 + dz^2.$$

Let U be an observer in circular motion on the hyperplane $z = 0$, i.e. with 4-velocity

$$U = \gamma_\zeta[\partial_t + \zeta\partial_\phi], \qquad \gamma_\zeta = [1 - \zeta^2 r^2]^{-1/2}.$$

(a) Show that

$$a(U) = -\gamma_\zeta^2 \zeta^2 r \partial_r.$$

(b) Show that the triad

$$e_{\hat 1} = \partial_r, \quad e_{\hat 2} = \gamma_\zeta \left[\zeta r \partial_t + \frac{1}{r}\partial_\phi\right], \quad e_{\hat 3} = \partial_z,$$

with $e_{\hat 0} = U$, is identical to a Frenet-Serret frame along U.

(c) Show that the associated curvature and torsions are given by

$$\kappa = -\gamma_\zeta^2 \zeta^2 r, \quad \tau_1 = \gamma_\zeta^2 \zeta, \quad \tau_2 = 0.$$

54. Consider the setting of the above problem. Show that the triad

$$\begin{aligned} E_{\hat 1} &= \cos(\gamma_\zeta \zeta t) e_{\hat 1} - \sin(\gamma_\zeta \zeta t) e_{\hat 2}, \\ E_{\hat 2} &= \sin(\gamma_\zeta \zeta t) e_{\hat 1} + \cos(\gamma_\zeta \zeta t) e_{\hat 2}, \\ E_{\hat 3} &= \partial_z \end{aligned}$$

with $e_{\hat 0} = U$, is identical to a Fermi-Walker frame along U.

55. Consider Minkowski space-time in cylindrical coordinates $\{t, r, \phi, z\}$:

$$ds^2 = -dt^2 + dr^2 + r^2 d\phi^2 + dz^2.$$

Let $u = \partial_t$ be a static observer. Show that this observer is inertial, that is

$$a(u) = 0, \quad \omega(u) = 0, \quad \theta(u) = 0.$$

Consider then an observer in circular motion, i.e. with 4-velocity

$$U = \gamma_\zeta[\partial_t + \zeta \partial_\phi], \qquad \gamma_\zeta = [1 - \zeta^2 r^2]^{-1/2}.$$

Show that

$$F^{(G)}_{(\mathrm{fw},U,u)} = 0, \qquad \nabla_U e_{\hat\phi} = -\gamma_\zeta \zeta \partial_r.$$

Use this result to show that

$$a^{(C)}_{(\mathrm{fw},U,u)} = -[\,\mathrm{sign}\,\zeta]\, r\zeta^2 \partial_r.$$

56. Consider the setting of the above problem, but with two rotating observers:

$$\begin{aligned} U &= \gamma_\zeta[\partial_t + \zeta \partial_\phi], \qquad \gamma_\zeta = [1 - \zeta^2 r^2]^{-1/2}, \\ u &= \gamma_\omega[\partial_t + \omega \partial_\phi], \qquad \gamma_\omega = [1 - \zeta^2 r^2]^{-1/2}. \end{aligned}$$

Show that

$$F^{(G)}_{(\mathrm{fw},U,u)} = \gamma_\zeta \gamma_\omega \omega \zeta r \partial_r, \quad F^{(G)}_{(\mathrm{lie},U,u)} = \gamma_\zeta \gamma_\omega \omega r[(\zeta - \omega)\gamma_\omega^2 + \zeta]\partial_r.$$

Hint. For a detailed discussion, see Bini and Jantzen (2004).

57. Show that in a three-dimensional Riemannian manifold, conceived as the LRS of an observer u, one has the following identity for the Scurl operation:

$$\mathrm{Scurl}_u\, \nabla X = -X \times \mathrm{Ricci},$$

where X is a vector field.
Hint. Start from the Ricci identities,

$$2\nabla_{[b}\nabla_{c]}X^a = R^a{}_{dbc}X^d,$$

and take the symmetrized dual of each side:

$$\eta^{bc(a}\nabla_{[b}\nabla_{c]}X^{e)} = -X^d\eta^{(e}{}_{dg}R^{a)g}.$$

58. Consider the setting of the above problem. Show that if the vector field X is a gradient, $X_a = \nabla_a\Phi$, then the following identity holds:

$$\mathrm{Scurl}\,[X \otimes X] = -X \times \nabla X.$$

59. Consider Kerr space-time in standard Boyer-Lindquist coordinates, and let m denote the 4-velocity field of the static family of observers.

(a) Show that the projected metric onto LRS_m is given by

$$\gamma^\flat = \frac{\Sigma}{\Delta} dr \otimes dr + \Sigma d\theta \otimes d\theta - \frac{\Delta\Sigma\sin^2\theta}{\Sigma - 2\mathcal{M}r} d\phi \otimes d\phi.$$

(b) Evaluate the Ricci tensor R_{ab} associated with the above 3-metric.

(c) Show that the only non-vanishing components of the corresponding Cotton tensor are the following:

$$[y^{r\phi}, y^{r\theta}] = \frac{6a^2\mathcal{M}^2}{[\Sigma(\Sigma - 2\mathcal{M}r)]^{5/2}}[\Delta\cos\theta, -(r-\mathcal{M})\sin\theta],$$

where $y^{ab} = -[\mathrm{Scurl}_m\, R]^{ab}$.

(d) What conclusions can be drawn in the limiting case of Schwarzschild space-time?

60. Consider Kerr space-time in standard Boyer-Lindquist coordinates and let m denote the 4-velocity field of the static family of observers.

(a) Evaluate the acceleration vector $a(m)$ and the vorticity vector $\omega(m)$ associated with m.

(b) Evaluate the electric ($E(m)$) and magnetic ($H(m)$) parts of the Weyl tensor as measured by the m observers.

(c) Define the complex fields $Z(m) = E(m) - iH(m)$ and $z(m) = -a(m) - i\omega(m)$ and show that

$$S(m) = z(m) \times Z(m) = 0.$$

Note that $S(m)$ is proportional to the so-called Simon tensor (Simon, 1984), whose vanishing is a peculiar property of Kerr space-time.

(d) Show that if in general $a \times A = 0$ (with a a spatial vector and A a spatial symmetric 2-tensor) then necessarily $A \propto [a \otimes a]^{(\mathrm{TF})}$.

61. Let u be a congruence of test observers in a general space-time. Show that the vorticity and expansion fields have the representations

$$\omega(u)^\flat = \frac{1}{2} d(u) u^\flat, \qquad \theta(u)^\flat = \frac{1}{2} \pounds(u)_u\, g^\flat.$$

62. Consider Minkowski space-time in standard Cartesian coordinates:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2.$$

Let

$$t = \frac{1}{\mathcal{A}} \sinh \mathcal{A}\tau, \quad x = 0, \quad y = 0, \quad z = -\frac{1}{\mathcal{A}} \cosh \mathcal{A}\tau,$$

the parametric equations of the world line u of an observer uniformly accelerated with acceleration $\mathcal{A}$ along the negative z-axis.

(a) Show that the frame

$$\begin{aligned} e_{\hat 0} &= \cosh \mathcal{A}\tau \partial_t - \sinh \mathcal{A}\tau \partial_z, \\ e_{\hat 1} &= \partial_x, \\ e_{\hat 2} &= \partial_y, \\ e_{\hat 3} &= -\sinh \mathcal{A}\tau \partial_t + \cosh \mathcal{A}\tau \partial_z \end{aligned}$$

is Fermi-Walker transported along u.

(b) Use this result to set up a system of Fermi coordinates $\{T, X, Y, Z\}$ on u.

Hint. The map between Cartesian and Fermi coordinates is given by

$$\begin{aligned} t &= \left(\frac{1}{\mathcal{A}} - Z\right) \sinh \mathcal{A}T, \\ x &= X, \\ y &= Y, \\ z &= -\left(\frac{1}{\mathcal{A}} - Z\right) \cosh \mathcal{A}T, \end{aligned}$$

and the Minkowski metric in coordinates $\{T, X, Y, Z\}$ is given by

$$ds^2 = -(1 - \mathcal{A}Z)^2 dT^2 + dX^2 + dY^2 + dZ^2.$$

63. Consider the Kasner vacuum metric (2.134),

$$ds^2 = -dt^2 + t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2,$$

with $p_1 + p_2 + p_3 = 1 = p_1^2 + p_2^2 + p_3^2$.

(a) Define the complex tensor $\tilde{C}_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} - i^*C_{\alpha\beta\gamma\delta}$ and form the two complex curvature invariants

$$I = \frac{1}{32}\tilde{C}_{\alpha\beta\gamma\delta}\tilde{C}^{\alpha\beta\gamma\delta}, \; J = \frac{1}{384}\tilde{C}_{\alpha\beta\gamma\delta}\tilde{C}^{\gamma\delta}{}_{\mu\nu}\tilde{C}_{\mu\nu\alpha\beta}.$$

Show that the ratio

$$\mathcal{S} = \frac{27J^2}{I^3} = -\frac{27}{4}p_1p_2p_3,$$

i.e. that it is a constant.
Hint. The quantity $\mathcal{S}$ is also known as the *spectral index* or *speciality index* of the space-time and plays a role in the Petrov classification of space-times as well as in perturbation theory.

(b) Consider the static family of observers $u = \partial_t$ with adapted orthonormal frame

$$e_{\hat{1}} = t^{-p_1}\partial_x, \quad e_{\hat{2}} = t^{-p_2}\partial_y, \quad e_{\hat{3}} = t^{-p_3}\partial_z.$$

Show that they see a purely electric Weyl tensor, i.e.

$$E(u) = \frac{p_1p_3}{t^2}e_{\hat{1}} \otimes e_{\hat{1}} + \frac{p_1p_2}{t^2}e_{\hat{2}} \otimes e_{\hat{2}} + \frac{p_2p_3}{t^2}e_{\hat{3}} \otimes e_{\hat{3}}, \quad H(u) = 0.$$

64. Consider Minkowski space-time with the metric written in cylindrical coordinates, i.e.

$$ds^2 = -dt^2 + dr^2 + r^2d\phi^2 + dz^2.$$

(a) Show that under the change of coordinates

$$t' = t, \quad r' = r, \quad \phi' = \phi - \Omega t, \quad z' = z$$

the metric becomes

$$ds^2 = -\gamma^{-2}dt'^2 + 2r'^2\Omega dt'd\phi' + r'^2d\phi'^2 + dr'^2 + dz'^2,$$

where $\gamma^{-2} = 1 - \Omega^2 r'^2$. Minkowski space-time endowed with the latter form of the metric will be referred to as *rotating Minkowski.* For convenience the space-time coordinates will be relabeled t, r, ϕ, z hereafter.

(b) In a rotating Minkowski space-time, consider the static family of observers, $m^\sharp = \gamma\partial_t$. Show that they are accelerated inward with acceleration

$$a(m) = -\gamma^2\Omega^2 r\partial_r\,;$$

that the vorticity vector is aligned with the z-axis,

$$\omega(m) = \gamma^2\Omega\partial_z\,;$$

and that the expansion vanishes identically, $\theta(m) = 0$.

(c) In a rotating Minkowski space-time, consider the ZAMO family of observers $n^\flat = -dt$ or $n^\sharp = \partial_t + \Omega\partial_\phi$. Show that they are geodesic ($a(n) = 0$), vorticity-free ($\omega(n) = 0$, by definition), and expansion-free ($\theta(n) = 0$).

(d) In a rotating Minkowski space-time, consider the family of circularly rotating orbits $U = \Gamma(\partial_t + \zeta\partial_\phi)$.

1. Show that $\Gamma^{-2} = 1 - r^2(\zeta - \Omega)^2$.
2. Show that for null circular orbits $\zeta = \zeta_\pm = -\Omega \pm 1/r$.

65. Consider a rotating Minkowski space-time,

$$ds^2 = -\gamma^{-2}dt^2 + 2r^2\Omega dt d\phi + r^2 d\phi^2 + dr^2 + dz^2,$$

where $\gamma^{-2} = 1 - \Omega^2 r^2$, and the family of static observers $m = \gamma\partial_t$.

(a) Show that the projected metric onto LRS_m is given by

$$P(m) = dr \otimes dr + \gamma^2 r^2 d\phi \otimes d\phi + dz \otimes dz.$$

(b) Show that the Ricci scalar associated with this 3-metric is given by

$$R = -6\Omega^2\gamma^4.$$

(c) Consider the $r - \phi$ part of this 3-metric,

$$P(m)|_{z=const.} = dr \otimes dr + \gamma^2 r^2 d\phi \otimes d\phi.$$

1. Show that the Ricci scalar associated with this 2-metric is again given by
$$R = -6\Omega^2\gamma^4.$$
2. Show that the embedding of this 2-metric is completely Minkowskian ($h_{RR} = \gamma^{-6} < 1$) and that the surface $Z = Z(R)$ can be explicitly obtained in terms of elliptic functions (Bini, Carini, and Jantzen, 1997a; 1997b).

66. Using the purely imaginary 4-form

$$E_{\alpha_1\ldots\alpha_4} = i\eta_{\alpha_1\ldots\alpha_4},$$
$$E^{\alpha_1\ldots\alpha_4} = -i\eta^{\alpha_1\ldots\alpha_4}$$

instead of η for the duality operation on 2-form index pairs defines the "hook" duality operation, with symbol $^{\smile}$. For example, for a 2-form F this leads to $^{\smile}F = i{}^*F$ and hence to $^{\smile\smile}S = S$. One can then find eigen-2-forms of the operation $^{\smile}$ with eigenvalues ± 1, called *self-dual* (+) and *anti-self-dual* (−), respectively.

(a) Show that the complex quantity

$$\mathcal{C}_+{}^{\alpha\beta}{}_{\gamma\delta} = C^{\alpha\beta}{}_{\gamma\delta} + i^*C^{\alpha\beta}{}_{\gamma\delta},$$

where $C = C_{\alpha\beta\gamma\delta}$ is the Weyl tensor, is anti-self-dual, i.e.

$${}^{\smile}\mathcal{C}_+{}^{\alpha\beta}{}_{\gamma\delta} = -\mathcal{C}_+{}^{\alpha\beta}{}_{\gamma\delta}.$$

(b) Show that the complex quantity

$$\mathcal{C}_-{}^{\alpha\beta}{}_{\gamma\delta} = C^{\alpha\beta}{}_{\gamma\delta} - i^*C^{\alpha\beta}{}_{\gamma\delta}$$

is self-dual, i.e.

$${}^{\smile}\mathcal{C}_-{}^{\alpha\beta}{}_{\gamma\delta} = \mathcal{C}_-{}^{\alpha\beta}{}_{\gamma\delta}.$$

Hint. Recall the property ${}^{**}C = -C$ of the Weyl tensor.

67. Suppose that S is a tensor-valued p-form, i.e. it has p antisymmetric indices (indices of a p-form) in addition to other tensorial indices:

$$S = S^{\alpha\dots}{}_{\beta\dots\alpha_1\dots\alpha_p} = S^{\alpha\dots}{}_{\beta\dots[\alpha_1\dots\alpha_p]}.$$

DS, the *covariant exterior derivative* of S, is defined so that it acts as the ordinary covariant derivative on the tensorial indices and as the exterior derivative on the p-form indices, that is

$$DS^{\alpha\dots}{}_{\beta\dots\alpha_1\dots\alpha_{p+1}} = (p+1)\nabla_{[\alpha_1}S^{\alpha}{}_{|\beta|\alpha_2\dots\alpha_{p+1}]}.$$

Show that the covariant exterior derivative of the curvature 2-form

$$\mathcal{R}^{\alpha}{}_{\beta} = \frac{1}{2}R^{\alpha}{}_{\beta\gamma\delta}\omega^{\gamma}\wedge\omega^{\delta}$$

vanishes identically, i.e. $D\mathcal{R} = 0$ or, in components,

$$[D\mathcal{R}]^{\alpha}{}_{\beta\gamma\delta\mu} = \nabla_{[\mu}R^{\alpha}{}_{|\beta|\gamma\delta]} = 0.$$

Hint. The right-hand side of the above equation is identically zero due to the Bianchi identities.

68. Show that Maxwell's equations,

$$\nabla_{\beta}{}^*F^{\alpha\beta} = 0, \qquad \nabla_{\beta}F^{\alpha\beta} = 4\pi J^{\alpha},$$

where $F_{\alpha\beta} = 2\nabla_{[\alpha}A_{\beta]}$ is the Faraday 2-form, A_{α} is the vector potential, and J^{α} is the 4-current vector, can be written in index-free form as follows:

$$dF^{\flat} = 0, \qquad \delta F^{\flat} = -4\pi J^{\flat},$$

with $\delta J^{\flat} = 0$.

69. Consider Kerr-Newman space-time in standard coordinates. The associated electromagnetic field can be written as

$$\begin{aligned} F^\flat &= \frac{Q}{\Sigma^2}(r^2 - a^2\cos^2\theta)\, dr \wedge [dt - a\sin^2\theta d\phi] \\ &+ \frac{2Q}{\Sigma^2} ar\sin\theta\cos\theta\, d\theta \wedge [(r^2+a^2)d\phi - adt]. \end{aligned}$$

Introduce the ZAMO family of observers, with 4-velocity

$$n = -Ndt, \qquad N = \sqrt{\frac{\Delta\Sigma}{A}},$$

and the following observer-adapted frame

$$\begin{aligned} e_{\hat r} &= \frac{1}{\sqrt{g_{rr}}}\partial_r = \sqrt{\frac{\Delta}{\Sigma}}\partial_r, \\ e_{\hat\theta} &= \frac{1}{\sqrt{g_{\theta\theta}}}\partial_\theta = \frac{1}{\sqrt{\Sigma}}\partial_\theta, \\ e_{\hat\phi} &= \frac{1}{\sqrt{g_{\phi\phi}}}\partial_\phi = \frac{1}{\sin\theta}\sqrt{\frac{\Sigma}{A}}\partial_\phi. \end{aligned}$$

(a) Show that the electric and magnetic fields as measured by ZAMOs are given by

$$\begin{aligned} E(n) &= \frac{Q}{\Sigma^2\sqrt{A}}[(r^2+a^2)(r^2 - a^2\cos^2\theta)e_{\hat r} \\ &\quad - 2a^2 r\sin\theta\cos\theta\sqrt{\Delta}e_{\hat\theta}], \\ B(n) &= -\frac{Q}{\Sigma^2\sqrt{A}}[2ar\cos\theta(r^2+a^2)e_{\hat r} \\ &\quad + a\sin\theta(r^2 - a^2\cos^2\theta)\sqrt{\Delta}e_{\hat\theta}]. \end{aligned}$$

(b) Show that electromagnetic energy density as measured by ZAMOs is given by

$$\mathcal{E}(n) = \frac{1}{8\pi}[E(n)^2 + B(n)^2] = \frac{1}{8\pi}\frac{Q^2}{\Sigma^2}\frac{(r^2+a^2)^2 + a^2\Delta\sin^2\theta}{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}.$$

Compute the limit of $\mathcal{E}(n)$ for $r \to r_+$.

70. In Kerr-Newman space-time, consider the Carter family of observers,

$$u^\flat_{(\mathrm{car})} = \sqrt{\frac{\Delta}{\Sigma}}(-dt + a\sin^2\theta d\phi).$$

Let

$$\bar u^\flat_{(\mathrm{car})} = \frac{\sin\theta}{\sqrt{\Sigma}}[-adt + (r^2+a^2)d\phi]$$

be the unit vector orthogonal to $u_{\rm car}$ in the $t-\phi$ plane. Show that the background electromagnetic field can be expressed as

$$F^\flat = \frac{Q}{\Sigma^{3/2}}\left[-\frac{1}{\sqrt{\Delta}}(r^2 - a^2\cos^2\theta)\, dr \wedge u^\flat_{(\rm car)} + 2ar\cos\theta\, d\theta \wedge \bar u^\flat_{(\rm car)}\right].$$

Using the splitting relation

$$F^\flat = u^\flat_{(\rm car)} \wedge E(u_{(\rm car)}) + {}^{*(u_{(\rm car)})}B(u_{(\rm car)}),$$

show that $E(u_{(\rm car)})$ and $B(u_{(\rm car)})$ are parallel.

71. Consider the setting of the above problem. Introduce the following orthonormal frame adapted to Carter's observers:

$$e_{\hat 0} = u_{(\rm car)}, \qquad e_{\hat 1} = e_{\hat r}, \qquad e_{\hat 2} = e_{\hat\theta}, \qquad e_{\hat 3} = \bar u_{(\rm car)}.$$

(a) Show that the energy momentum tensor of the field is given by

$$T = \mathcal{E}(u_{(\rm car)})[-e_{\hat 0}\otimes e_{\hat 0} - e_{\hat 1}\otimes e_{\hat 1} + e_{\hat 2}\otimes e_{\hat 2} + e_{\hat 3}\otimes e_{\hat 3}],$$

where $\mathcal{E}(u_{(\rm car)})$ is the energy density of the background electromagnetic field as measured by Carter observers.

(b) Show that

$$T^2 = \mathcal{E}(u_{(\rm car)})^2[-e_{\hat 0}\otimes e_{\hat 0} - e_{\hat 1}\otimes e_{\hat 1} + e_{\hat 2}\otimes e_{\hat 2} + e_{\hat 3}\otimes e_{\hat 3}],$$

so that the mixed form is such that $[T^2]^\mu{}_\nu = \mathcal{E}(u_{(\rm car)})^2\delta^\mu{}_\nu$.

72. In Kerr-Newman space-time, consider the two electromagnetic invariants

$$I_1 = \frac{1}{2}F_{\alpha\beta}F^{\alpha\beta}, \qquad I_2 = \frac{1}{2}{}^*F_{\alpha\beta}F^{\alpha\beta}.$$

(a) Show their expressions in terms of the electric and magnetic fields as measured by ZAMOs, $E(n)$ and $B(n)$.

(b) Evaluate them.
Hint. A straightforward calculation leads to

$$\begin{aligned} I_1 &= \frac{Q^2}{\Sigma^4}[4r^2a^2\cos^2\theta - (r^2 - a^2\cos^2\theta)^2],\\ I_2 &= \frac{Q^2}{\Sigma^4}[4ra\cos\theta(r^2 - a^2\cos^2\theta)]. \end{aligned}$$

73. Show that

$$A^\sharp = aB_0[\partial_t + \frac{1}{2a}\partial_\phi]$$

is a vector potential for an electromagnetic test field on Kerr background.

(a) Evaluate the electric and magnetic fields as measured by ZAMOs and make a plot of the lines of force of these fields.
Hint. A straightforward calculation shows that

$$\begin{aligned} E(n)_{\hat{r}} &= -a\sigma[(r^2+a^2)Y + r\Delta\Sigma\sin^2\theta], \\ E(n)_{\hat{\theta}} &= -a\sigma\sqrt{\Delta}[X - \Sigma(r^2+a^2)]\sin\theta\cos\theta, \\ B(n)_{\hat{r}} &= \sigma[(r^2+a^2)X - a^2\Delta\Sigma\sin^2\theta]\cos\theta, \\ B(n)_{\hat{\theta}} &= -\sigma\sqrt{\Delta}[a^2Y + r\Sigma(r^2+a^2)]\sin\theta, \end{aligned}$$

where

$$\begin{aligned} \sigma &= \frac{B_0}{\Sigma^2\sqrt{A}}, \\ Y &= 2r\cos^2\theta(r^2+a^2) - (r-\mathcal{M})(1+\cos^2\theta)(r^2-a^2\cos^2\theta), \\ X &= (r^2+a^2)(r^2-a^2\cos^2\theta) + 2a^2r(r-\mathcal{M})(1+\cos^2\theta). \end{aligned}$$

(b) Evaluate the invariants of this field.

Note that this electromagnetic field was found by Wald in 1974 (Wald, 1974).

74. Consider Kerr space-time. Show that the 2-form field

$$\begin{aligned} f = \frac{1}{2}f_{\mu\nu}dx^\mu \wedge dx^\nu &= a\,\cos\theta\,dr\wedge(dt - a\,\sin^2\theta\,d\phi) \\ &\quad + r\,\sin\theta\,d\theta\wedge[-a\,dt + (r^2+a^2)\,d\phi] \end{aligned}$$

satisfies the relations

$$\nabla_{(\gamma}f_{\beta)\alpha} = 0.$$

Note that the field f with this property is known as a *Killing-Yano tensor.*

75. Consider the metric (Euclidean Taub-NUT) with coordinates $\{\psi, r, \theta, \phi\}$:

$$\begin{aligned} ds^2 &= \left(1+\frac{2\ell}{r}\right)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \\ &\quad + \frac{4\ell^2}{1+2\ell/r}(d\psi + \cos\theta d\phi)^2, \end{aligned}$$

where ℓ is a parameter which can be either positive or negative.

(a) Solve the equations

$$\nabla_{(\gamma}f_{\beta)\alpha} = 0.$$

The tensor fields f are Killing-Yano tensors.

(b) Show that the above metric admits four Killing-Yano tensors.

Hint. The Killing-Yano tensors are

$$f_1 = X^\flat \wedge dr - \left(1 + \frac{2\ell}{r}\right) d\theta \wedge d\phi,$$

$$f_2 = X^\flat \wedge d\theta - \left(1 + \frac{2\ell}{r}\right) d\phi \wedge dr,$$

$$f_3 = X^\flat \wedge d\phi - \left(1 + \frac{2\ell}{r}\right) dr \wedge d\theta,$$

$$f_4 = X^\flat \wedge dr + 4r(r + \ell) \left(1 + \frac{r}{2\ell}\right) \sin\theta d\theta \wedge d\phi,$$

where $X^\flat = 4\ell(d\psi + \cos\theta d\varphi)$.

(c) Show that $\nabla_\alpha f_{i\beta} = 0$ when $i = 1, 2, 3$.

76. Consider the Reissner-Nordström space-time. Show that the equilibrium condition for a massive and charged particle to be at rest at a fixed point $r = b, \theta = 0$ on the z-axis is given by

$$m = qQ \frac{b\left(1 - \frac{2\mathcal{M}}{r} + \frac{Q^2}{r^2}\right)^{1/2}}{\mathcal{M}b - Q^2}.$$

What can one deduce from this relation? What happens when the black hole becomes extreme?

77. Consider the Friedmann-Robertson-Walker space-time, with metric

$$ds^2 = -dt^2 + R^2(t) \left[\frac{dr^2}{1 - \kappa r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right],$$

where $R(t)$ is a scale function and $\kappa = -1, 0, 1$ according to the type of spatial sections: a pseudo-sphere, a flat space, or a sphere, respectively.

(a) Compute the ratio between area and volume of the spatial sections in the three cases $\kappa = -1, 0, 1$.

(b) Discuss the geodesic deviation equation and evaluate the acceleration of the relative deviation vector in the three cases $\kappa = -1, 0, 1$.

78. Show that the two metrics

$$ds^2 = -dt^2 + e^{2H_0 t}(dx^2 + dy^2 + dz^2), \quad H_0^2 = \Lambda/3$$

and

$$ds^2 = \left[1 + \frac{H_0^2}{4}(x^2 + y^2 + z^2 - t^2)\right]^{-2} (-dt^2 + dx^2 + dy^2 + dz^2)$$

are equivalent. Note that both these metrics represent de Sitter space-time (see Section 5.4).

Hint. Introduce standard polar coordinates,

$$x = \rho \sin\theta \cos\phi, \qquad y = \rho \sin\theta \sin\phi, \qquad z = \rho \cos\theta.$$

The first metric thus takes the form

$$ds^2 = -dt^2 + e^{2H_0 t}[d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2)].$$

Applying the coordinate transformation

$$t = \frac{1}{H_0} \ln\left[e^{H_0\tau}\sqrt{1 - H_0^2 R^2}\right], \quad \rho = \frac{Re^{-H_0\tau}}{\sqrt{1 - H_0^2 R^2}}, \quad \theta = \theta, \quad \phi = \phi,$$

gives

$$ds^2 = -(1 - H_0^2 R^2)d\tau^2 + \frac{dR^2}{1 - H_0^2 R^2} + R^2(d\theta^2 + \sin^2\theta d\phi^2).$$

The further transformation

$$\tau = \frac{1}{2H_0} \ln\left[\frac{H_0^2\bar{\rho}^2 - (H_0\bar{t} - 2)^2}{H_0^2\bar{\rho}^2 - (H_0\bar{t} + 2)^2}\right], \quad R = \frac{\bar{\rho}}{1 + \frac{H_0^2}{4}(\bar{\rho}^2 - \bar{t}^2)},$$
$$\theta = \theta, \quad \phi = \phi,$$

finally gives

$$ds^2 = \left[1 + \frac{H_0^2}{4}(\bar{\rho}^2 - \bar{t}^2)\right]^{-2} [-d\bar{t}^2 + d\bar{\rho}^2 + \bar{\rho}^2(d\theta^2 + \sin^2\theta d\phi^2)],$$

which reduces to the second metric once the Cartesian coordinates are restored by using standard relations.

By combining the two transformations we get

$$t = \frac{1}{2H_0} \ln\chi,$$
$$\rho = \frac{\bar{\rho}}{1 + \frac{H_0^2}{4}(\bar{\rho}^2 - \bar{t}^2)}\chi^{-1/2},$$
$$\theta = \theta, \quad \phi = \phi,$$

where

$$\chi = \left[\frac{H_0^2\bar{\rho}^2 - (H_0\bar{t} - 2)^2}{H_0^2\bar{\rho}^2 - (H_0\bar{t} + 2)^2}\left(1 - \frac{H_0^2\bar{\rho}^2}{[1 + \frac{H_0^2}{4}(\bar{\rho}^2 - \bar{t}^2)]^2}\right)\right],$$

which allows us to pass directly from one metric to the other.

79. Show that the metric

$$ds^2 = -\left(1 - \frac{r^2}{\alpha^2}\right)dt^2 + \left(1 - \frac{r^2}{\alpha^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

represents de Sitter space-time. Consider spatially circular orbits $U = \Gamma(\partial_t + \zeta\partial_\phi)$ and show that the magnitude of the 4-acceleration is given by

$$||a(U)|| = \frac{r^2\left[\left(1-\frac{r^2}{\alpha^2}\right)\left(\frac{1}{\alpha^2}+\zeta^2\sin^2\theta\right)^2+\zeta^4\sin^2\theta\cos^2\theta\right]}{\left(1-\frac{r^2}{\alpha^2}-r^2\zeta^2\sin^2\theta\right)^2}.$$

80. Consider Gödel space-time, with the metric given by (2.133). Compute the vorticity of the congruence of time-like Killing vectors.

81. Consider the metric tensor of a general space-time. Show that the inverse metric is also a tensor.

82. Consider the Mathisson-Papapetrou equations for spinning test bodies (see Chapter 10, Eqs. 10.2, 10.3, 10.4),

$$\begin{aligned}\frac{DP^\mu}{d\tau} &= -\frac{1}{2}R^\mu{}_{\nu\alpha\beta}U^\nu S^{\alpha\beta}\,,\\ \frac{DS^{\mu\nu}}{d\tau} &= P^\mu U^\nu - P^\nu U^\mu\,,\\ S^{\mu\nu}P_\nu &= 0\,,\end{aligned}$$

where $D/d\tau = \nabla_U$ and with $P = mu$ (u time-like and unitary).

(a) Deduce the evolution equations for the spin vector $S^\alpha = \eta(u)^{\alpha\beta\gamma}S_{\beta\gamma}$.

(b) Show that the quadratic invariant

$$s^2 = \frac{1}{2}S_{\alpha\beta}S^{\alpha\beta}$$

is constant along U, that is

$$\nabla_U s = 0\,.$$

(c) Determine the evolution equation for the mass m along U.

(d) Show that

$$\nu(U,u)_\alpha = \frac{P(u)^\mu_\alpha[{}^*R^*]_{\mu\nu\rho\sigma}S^\nu S^\rho u^\sigma}{(1+[{}^*R^*]_{\mu\nu\rho\sigma}u^\mu S^\nu u^\rho S^\sigma)}\,.$$

Hint. See Tod, de Felice, and Calvani (1976) for details.

83. Consider the Friedmann-Robertson-Walker space-time with metric

$$ds^2 = -dt^2 + R^2(t)\left[\frac{dr^2}{1-\kappa r^2}+r^2(d\theta^2+\sin^2\theta d\phi^2)\right],$$

where $R(t)$ (which we leave as unspecified) is a scale function and $\kappa = -1, 0, 1$ is a sign indicator. Let U be the 4-velocity of a massive particle moving along a radial geodesic and let its velocity, relative to a static observer, be ν_0 at some initial time $t = t_0$.

(a) Show that $U = \cosh\alpha\partial_t + \sinh\alpha e_{\hat{r}}$, where $e_{\hat{r}} = (\sqrt{1-\kappa r^2}/R)\partial_r$ is the unit vector associated with the radial direction and $\sinh\alpha = \gamma_0\nu_0 R_0/R$, with $\gamma_0 = (1-\nu_0^2)^{-1/2}$.

(b) Show that the relative velocity as measured by another static observer at the generic value t of the coordinate time is

$$\nu \equiv \tanh\alpha = \frac{\nu_0\gamma_0 R_0}{\sqrt{R^2 + \nu_0^2\gamma_0^2 R_0^2}}.$$

(c) Discuss the limit $\nu_0 \to 0$.

84. Consider the case $\kappa = 0, R(t) = t^n$ of the Friedmann-Robertson-Walker metric of Exercise 83.

(a) Show that

$$R^\alpha{}_\beta = \frac{n}{t^2}\mathrm{diag}[3(n-1), 3n-1, 3n-1, 3n-1]\,.$$

(b) Discuss the two cases $n = 1$ and $n = 1/3$.

85. Consider the induced metric on the $t =$const hypersurfaces of a Friedmann-Robertson-Walker space-time which is the local rest space of the observers with 4-velocity $u = \partial_t$, namely

$$\gamma^\flat = \left[\frac{dr\otimes dr}{1-\kappa r^2} + r^2 d\theta\otimes d\theta + r^2\sin^2\theta d\phi\otimes d\phi\right],$$

conformally rescaled by the factor $R(t)$ (constant in this case).

(a) Show that the Ricci tensor associated with this metric in its mixed form is $R^a{}_b = -2\kappa\delta^a{}_b$.

(b) Show that the three-dimensional Cotton tensor $y^{ab} = -[\mathrm{Scurl}_u R]^{ab}$, where R is the Ricci tensor associated with the metric γ, for the three cases $\kappa = -1, 0, 1$ is always vanishing.

(c) Show that the geodesics of this metric when $\theta = \pi/2$ satisfy the following equations:

$$\frac{dr}{ds} = \pm\frac{1}{r}\sqrt{(r^2 - L^2)(1-\kappa r^2)}\,, \qquad \frac{d\phi}{ds} = \frac{L}{r^2},$$

where s is the curvilinear abscissa parameter and L is a Killing constant.

(d) Integrate the above equations.

86. Show that the equations of state $p = (1/3)\rho$ and $p = -\rho$, p and ρ being the isotropic pressure and the energy density of a perfect fluid, are Lorentz invariant.

87. Consider the 1-form

$$X^\flat = X_\alpha dx^\alpha \,, \qquad X_\alpha = X_\alpha(r,\theta)\,,$$

in the Kerr space-time with metric written in standard Boyer-Lindquist coordinates. Introduce the families of static observers (m, $m = M^{-1}\partial_t$, $M = \sqrt{-g_{tt}}$) and ZAMOs (n, $n^\flat = -N\, dt$, $N = 1/\sqrt{-g^{tt}}$).

(a) Evaluate $\mathrm{curl}_m X$ and $\mathrm{curl}_n X$.

(b) Compare these results in the special case $X^\flat = X_\phi(r,\theta)d\phi$.

88. Consider the vector X in a generic space-time. Let $u = e_0$ be a family of test observers with associated spatial frame e_a, so that

$$X = X^{||}\, u + X^\perp \,, \qquad X^\perp \cdot u = 0\,.$$

(a) Evaluate the components of $\nabla X = (\nabla X)_{\alpha\beta} \equiv (\nabla_\beta X_\alpha)$ in terms of the kinematical fields associated with u.

(b) Show that in the case $X^{||} = 0$, i.e. X spatial with respect to u, one has

$$\nabla_\alpha X^\alpha = [\nabla(u)_a + a(u)_a] X^a\,.$$

89. Repeat Exercise 88 in the case of a $\binom{1}{1}$-tensor $S \equiv S^\alpha{}_\beta$.

90. Consider a three-dimensional Riemannian space with metric $ds^2 = g_{ab}dx^a dx^b$ and a conformal transformation of the metric $\tilde g_{ab} = e^{2U} g_{ab}$.

(a) Show that the unit volume 3-form transforms as follows:

$$\tilde\eta_{abc} = e^{3U} \eta_{abc}\,, \qquad \tilde\eta^{abc} = e^{-3U} \eta^{abc}\,.$$

91. Vaidya's metric for a radiating spherical source is given by

$$ds^2 = -\left(1 - \frac{2M(u)}{r}\right) du^2 - 2du dr + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,,$$

where $M(u)$ is a non-increasing function of the retarded time $u = t - r$, often chosen in the form

$$M(u) = M_1 + \frac{M_2 - M_1}{2}(1 + \tanh \alpha u)\,, \quad M_1, M_2, \alpha = \text{constant.}$$

Consider the family of observers at rest with respect to the coordinates, i.e. with 4-velocity

$$\bar u = e_{\hat 0} = \frac{1}{\sqrt{-g_{uu}}}\partial_u$$

and associated spatial orthonormal triad

$$e_{\hat 1} = \sqrt{-g_{uu}}\left[\partial_r + \frac{1}{g_{uu}}\partial_u\right], \quad e_{\hat 2} = \frac{1}{\sqrt{g_{\theta\theta}}}\partial_\theta, \quad e_{\hat 3} = \frac{1}{\sqrt{g_{\phi\phi}}}\partial_\phi\,.$$

(a) Show that these observers are accelerated with acceleration

$$a(\bar{u}) = -\frac{M'r^2 - M(r-2M)}{[r(r-2M)]^{3/2}} e_{\hat{1}}, \qquad M' = \frac{dM}{du} .$$

(b) Show that the congruence $\mathcal{C}_{\bar{u}}$ of the integral curves of $\bar{u}$ has a shear given by

$$\theta(\bar{u}) = \frac{M'\sqrt{r}}{(r-2M)^{3/2}} e_{\hat{1}} \otimes e_{\hat{1}}.$$

(c) Show that the vorticity of $\mathcal{C}_{\bar{u}}$ is identically zero, $\omega(\bar{u}) = 0$.

(d) Show that

$$\pounds_{\bar{u}} e_{\hat{1}} = a(\bar{u})_{\hat{1}} \bar{u} - \theta(\bar{u})_{\hat{1}\hat{1}} e_{\hat{1}}, \qquad \pounds_{\bar{u}} e_{\hat{2}} = 0, \qquad \pounds_{\bar{u}} e_{\hat{3}} = 0.$$

Consider then the null vector $k = 1/\sqrt{2}\partial_r$ and show that Vaidya's metric is a solution of the Einstein equations with $T_{\mu\nu} = q k_\mu k_\nu$ where $q = \frac{1}{2\pi r^2} dM/du$.

92. The general relativistic motion of extended bodies with structure up to the quadrupole is described by Dixon's model (Dixon, 1964; 1970a; 1970b; 1974):

$$\begin{aligned}
\frac{DP^\mu}{d\tau_U} &= -\frac{1}{2} R^\mu{}_{\nu\alpha\beta} U^\nu S^{\alpha\beta} - \frac{1}{6} J^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}{}^{;\,\mu} \\
&\equiv F^{(\mathrm{spin})\,\mu} + F^{(\mathrm{quad})\,\mu} \,, \\
\frac{DS^{\mu\nu}}{d\tau_U} &= 2P^{[\mu}U^{\nu]} + \frac{4}{3} J^{\alpha\beta\gamma[\mu} R^{\nu]}{}_{\gamma\alpha\beta} \\
&\equiv 2P^{[\mu}U^{\nu]} + D^{(\mathrm{quad})\,\mu\nu} \,,
\end{aligned}$$

where P^α is the generalized linear momentum of the body, U^α is the unit tangent vector to the world line of the moment's reduction, $J^{\alpha\beta\gamma\delta}$ (a tensor with the same algebraic properties as the Riemann tensor, and hence with 20 independent components and support along U) is the space-time quadrupolar momentum tensor of the body, and $S^{\alpha\beta}$ (an antisymmetric tensor, also with support along U) is the proper angular momentum tensor of the body. The completeness of the model is assured if one requires the further conditions $S^{\alpha\beta}P_\beta = 0$ and specifies the type of body under consideration through the so-called constitutive equations.

(a) Show that the following two expressions for the torque $D^{(\mathrm{quad})\,\mu\nu}$, due to Dixon (1974) and Ehlers-Rudolph (1977), are equivalent:

$$D^{(\mathrm{quad})\,\mu\nu}_{\mathrm{Dixon}} = -\frac{4}{3} R^{[\mu}{}_{\alpha\beta\gamma} J^{\nu]\alpha\beta\gamma} \,, \quad D^{(\mathrm{quad})\,\mu\nu}_{\mathrm{E\_R}} = \frac{4}{3} J^{\alpha\beta\gamma[\mu} R^{\nu]}{}_{\gamma\alpha\beta} \,.$$

(b) Write down a 1+3 representation of the quadrupole momentum tensor with respect to an arbitrary observer family u, and identify the physical meaning of the various terms.

93. Starting from the definition of scalar expansion, $\Theta(u) = \nabla_\alpha u^\alpha$, show that the Raychaudhury equation,

$$\nabla_u \Theta(u) = \nabla_\alpha a(u)^\alpha - \mathrm{Tr}(k(u)^2) - R_{\alpha\beta} u^\alpha u^\beta \,,$$

holds identically. Consider then a full splitting of the terms $\nabla_\alpha a(u)^\alpha$ and show that it can be written as

$$\nabla_\alpha a(u)^\alpha = \nabla(u)_\alpha a(u)^\alpha + a(u)_\alpha a(u)^\alpha \,.$$

94. In the Schwarzschild space-time (with metric written in standard coordinates), consider the foliation

$$T = t - f(r)\,,$$

where $f(r)$ is an arbitrary function of the radial variable only.

(a) Show that the induced metric on each leaf is given by

$$^{(3)}ds^2 = \gamma_{rr} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \qquad \gamma_{rr} = \left(\frac{1}{N^2} - N^2 f'^2\right)\,,$$

where $N^2 = 1 - 2M/r$.

(b) Evaluate the three-dimensional Cotton tensor

$$y^{ab} = -[\mathrm{Scurl}_u R]^{ab}$$

of this 3-geometry, where R_{ab} is the Ricci tensor and f is arbitrary.

(c) Introduce the family $\mathcal{N}$ of observers whose world lines are orthogonal to the $T=$constant foliation and show that they are in radial motion with respect to the static observers n.
Hint. Put $\mathcal{N}^\flat = -LdT$ and determine L from the normalization condition $\mathcal{N}\cdot\mathcal{N} = -1$.

(d) Show that the relative velocity satisfies the condition $\nu(\mathcal{N}, n)^{\hat r} = N^2 f'$.

(e) Show that

$$\gamma_{rr} = \left(\frac{1}{N\gamma(\mathcal{N}, n)}\right)^2 .$$

(f) Under which conditions is the previous 3-metric is intrinsically flat? Is it possible to forecast such a condition without studying the induced Riemann tensor components?

(g) Prove that the family $\mathcal{N}$ of observers with associated intrinsically flat 3-metric are geodesic and coincide with the Painlevé-Gullstrand observers introduced in the text.

95. Consider Minkowski flat space-time in Cartesian coordinates.

(a) Write the 4-velocity of an observer U moving with constant speed ν along the positive direction of the x-axis until he reaches the distance ℓ (at the coordinate time $t = t_*$) from the origin (which he left at $t = 0$) and then suddenly reversing his direction to return to the origin with the same speed, during another interval t_* of coordinate time.

(b) Show that the associated 4-acceleration is

$$a(U) = -2\nu\gamma^2\delta(t - t_*)\partial_x \,,$$

where δ denotes the Dirac δ-function and γ is the Lorentz factor.

96. Consider Rindler's metric,

$$ds^2 = -(1 - gz)dt^2 + dx^2 + dy^2 + dz^2,$$

where g is a constant, and the family of observers at rest with respect to the coordinates, $U = (1 - gz)^{-1}\partial_t$.

(a) Show that $a(U) = -g/(1 - gz)\partial_z$.

(b) Write down the geodesic equations and find the corresponding solution for both the time-like and null cases when the motion is confined in the (t, z)-plane.

(c) Find a coordinate transformation mapping Rindler's metric in Minkowski space-time with Cartesian coordinates and show that Rindler's geodesics are mapped into straight lines, as expected.

97. In Kerr space-time in standard Boyer-Lindquist coordinates, consider the family of time-like "spherical orbits" parametrized by their coordinate angular velocities $\zeta = d\phi/dt$ and $\eta = d\theta/dt$,

$$U^\alpha = \Gamma[\delta^\alpha{}_t + \zeta\delta^\alpha{}_\phi + \eta\delta^\alpha{}_\theta] \,,$$

where $\Gamma = dt/d\tau_U > 0$ is defined by

$$\Gamma^{-2} = -[g_{tt} + 2\zeta g_{t\phi} + \zeta^2 g_{\phi\phi} + \eta^2 g_{\theta\theta}]$$

because of the condition $U_\alpha U^\alpha = -1$. Introduce also the family of static observers, $m = (-g_{tt})^{-1/2}\partial_t$.

(a) Evaluate the relative velocity $\nu(U, m)$.

(b) Evaluate the 4-acceleration $a(U)$ of these orbits.

(c) Evaluate the Fermi-Walker gravitational force

$$F^{(G)}_{(\mathrm{fw},U,m)} = -Dm/d\tau_U .$$

(d) Study the evolution of $\nu(U,m)$ along U and determine the relative acceleration $a_{(\mathrm{fw},U,m)}$.

(e) Determine the expressions for ζ, η, and Γ corresponding to the $U_{(\mathrm{geo})}$ geodesic and evaluate $F^{(G)}_{(\mathrm{fw},U_{(\mathrm{geo})},m)}$.

98. The general stationary cylindrically symmetric vacuum solution (see Iyer and Vishveshwara, 1993) can be written in the form

$$ds^2 = \lambda_{00}dt^2 + 2\lambda_{03}dtd\phi + \lambda_{33}d\phi^2 + e^{2\phi}(d\tau^2 + d\sigma^2),$$

where coordinates are such that $x^0 = t$, $x^1 = \tau$, $x^2 = \sigma$, $x^3 = \phi$, and

$$\begin{aligned} \lambda_\alpha &= A_\alpha \tau^{1+b} + B_\alpha \tau^{1-b}\,, \quad \alpha = 00, 03, 33,\\ e^{2\phi} &= c\tau^{(b^2-1)/2}\,, \quad \tau = \sqrt{2}\rho\,, \quad \sigma = \sqrt{2}z\,, \end{aligned}$$

b and c being constants. The coefficients A_α and B_α are related by the algebraic conditions

$$\begin{aligned} &A_{00}A_{33} - A_{03}^2 = 0\,,\\ &B_{00}B_{33} - B_{03}^2 = 0\,,\\ &A_{00}B_{33} + A_{33}B_{00} - 2A_{03}B_{03} = -\frac{1}{2}\,, \end{aligned}$$

and to the mass per unit length m and angular momentum per unit length j by

$$\begin{aligned} m &= \frac{1}{4} + \frac{1}{2}b(A_{33}B_{00} - A_{00}B_{33})\,,\\ j &= \frac{1}{2}b(A_{03}B_{33} - A_{33}B_{03})\,. \end{aligned}$$

Note that τ here is a radial coordinate while σ is the z coordinate (rescaled by a factor). Consider a time-like circular orbit with 4-velocity

$$U = \Gamma(\partial_t + \zeta\partial_\phi)\,,$$

with ζ constant along U, $\pounds_U\zeta = 0$.

(a) Show that the normalization factor is given by

$$\Gamma = [-(\lambda_{00} + 2\lambda_{03}\zeta + \lambda_{33}\zeta^2)]^{-1/2}\,.$$

(b) Show that these orbits are accelerated with purely radial acceleration

$$a(U) = -\frac{1}{2}e^{-2\phi}\Gamma^2(\dot\lambda_{00} + 2\dot\lambda_{03}\zeta + \dot\lambda_{33}\zeta^2)\partial_\tau,$$

where a dot denotes differentiation with respect to τ.

(c) Show that time-like circular geodesics correspond to

$$\zeta_{K\pm} = -\frac{\dot{\lambda}_{03}}{\dot{\lambda}_{33}} \pm \frac{\sqrt{\frac{1-b^2}{2}}}{\dot{\lambda}_{33}}\,.$$

(d) Show that geodesic meeting point observers have angular velocity

$$\zeta_{(\mathrm{gmp})} = -\frac{\dot{\lambda}_{03}}{\dot{\lambda}_{33}}\,.$$

99. In the context of Exercise 98, show that extremely accelerated observers $\zeta_{(\mathrm{crit})\pm}$ have angular velocities which are solutions of the following equation:

$$\lambda_{03}^2\zeta^2\frac{d}{d\tau}\left(\frac{\lambda_{33}}{\lambda_{03}}\right) + \lambda_{00}^2\zeta\frac{d}{d\tau}\left(\frac{\lambda_{33}}{\lambda_{00}}\right) + \lambda_{00}^2\frac{d}{d\tau}\left(\frac{\lambda_{33}}{\lambda_{00}}\right) = 0\,.$$

Show that extremely accelerated observers have constant angular velocities.

100. In the context of Exercise 98,

(a) study the kinematical properties of static observers, with

$$\zeta_{(\mathrm{stat})} = 0\,.$$

(d) study the kinematical properties of zero angular momentum observers, with

$$\zeta_{(\mathrm{ZAMO})} = -\frac{\lambda_{03}}{\lambda_{33}}\,.$$

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