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Kunio Murasugi
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**Abstract:** For a signed graph $G$, we define an invariant, called the index, $indG$ and prove several relationships between $ind\, G$ and other known invariants. Graphs with $ind\, G = 0$ or 1 are characterized. If $G$ is the Seifert graph of a diagram of a knot $K$, then $ind\, G$ is closely related to the braid index of $K$. We show that if $K$ is an alternating link and $ind G = 0$ for the Seifert graph $G$ associated with some alternating diagram of $K$, then the braid index of $K$ is completely determined by its skein polynomial. Moreover, the braid index of certain types of alternating links including alternating pretzel links is determined.

**Key words and phrases.**

Signed graph, bipartite graph, index of a graph, cycle index, knot, link, knot diagram, alternating knot or link, Seifert circle, Alexander polynomial, skein polynomial, braid index of a link, pretzel link, algebraic (or arborescent) link.

## Introduction

Every oriented link $L$ in a 3-sphere $S^3$ is represented as a closed braid with a finite number of strings [A]. The braid index of $L$, denoted by $\mathbf{b}(L)$, is defined as the minimum number of strings needed for $L$ to be represented as a closed braid. The braid index is a link type invariant, but generally it is extremely difficult to determine the braid index of a link.

However, the recent development on the polynomial invariants of links [J,FY, LM, PT], especially the invariant called the *skein* polynomial in this paper, has revealed a deep connection between these polynomials and the braid index of a link. On the other hand, Yamada [Y] proves that the number of Seifert circles, denoted by $s(D)$, of a link diagram $D$ of an oriented link $L$ is at least equal to the braid index of $L$. This remarkable theorem (combined with other results) makes it possible for us to determine the braid index of many links. In fact, the first author of the present paper, successfully determines, for the first time, the braid index of a certain type of alternating links [Mu 4]. However, in order to determine the braid index of more general links, we must determine the *deficit* $s(D) - \mathbf{b}(L)$ of the diagram $D$. Our study of the deficit leads to a new concept called the *index* of a graph $G$, which produces a direct correlate of the deficit for many (and probably all) alternating links. This relationship is the basis of our extensive investigation of the index of graphs. Using this concept, we can characterize the alternating links for which the deficit of an alternating diagram is equal to 0. [ Cf. Theorem 9.5.] Therefore, the braid index of these links is completely determined by counting Seifert circles in the diagram. An alternating fibred link is a typical example of the links with this property and therefore our theorem recovers one of the main theorems in [Mu 4]. We have also almost complete characterization of alternating links for which the deficit of an alternating diagram is one or two. (See Theorems 10.9 and 10.13.) For may familiar alternating links, like 2-bridge links or pretzel links, the deficit of an alternating diagram will be evaluated precisely and

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the braid index is therefore, completely determined.

Now we will briefly explain the contents of the paper.

The paper is divided into three chapters. Chapter I deals with the index of a graph. Since we are mainly interested in applications of graph theory to link theory, we concentrate on bipartite (and planar) graphs. A graph $G$ is called *bipartite* if every cycle in $G$ has an even number of edges. One of the useful properties of a bipartite graph is that the index is additive with respect to the block sum. (This is not true for non-bipartite graphs). In fact, we will prove in §2 the following theorem

**Theorem 1** (Cf. Theorem 2.4) *If $G_i$ is a bipartite graph, $i = 1, 2, \cdots, k$, then*

$$ind(G_1 * G_2 * \cdots * G_n) = \sum_{i=1}^{k} ind(G_i),$$

*where ind $X$ denotes the index of $X$ and $X * Y$ denotes the block sum (i.e. the one-point union ) of $X$ and $Y$.*

The index of a graph $G$ is also related to other invariants of $G$. For instance, the number of growing rooted spanning trees [1] $\lambda(G)$, in $G$ is well studied in the literature. We will prove in §4 the following theorem.

**Theorem 2** (Cf. Theorem 4.3 and Corollary 4.11) *Let $G$ be a plane bipartite connected graph without isthmuses. Let $G^*$ be the dual graph of $G$ (with a natural direction [2]). Then*

$$ind\ G \leq \lambda(G^*) - 1.$$

*If, moreover, $G$ has no cut vertices, then*

$$|V(G)| \leq 2\lambda(G^*),$$

*where $|V(G)|$ denotes the number of vertices in $G$.*

In §5, Theorems 1 and 2 stated above will be used to characterize the planar bipartite graphs with $\lambda(G^*) \leq 3$. Since our graph $G$ is finite, the index of $G$ can be decided

[1] For the definition, see §4

[2] For the definition, see §4.

algorithmically. In §6, we consider a special type of graph, called *reducible*, and express precisely the index of $G$ in terms of other numerical invariants of $G$. (See Theorem 6.5.) The graphs considered there correspond to a special type of algebraic links (in the sense of Conway), and their braid indices will be completely determined in §12.

In Chapter II, we will present a general strategy for determining the braid index of a link. The main purpose of Chapter II is to improve an inequality proven in [FW, Mo 1] and to find a sufficient condition for the equality.

The theorems proven in §8 and §9 are of fundamental importance in this paper. They not only determine the braid index of many links, but they also have many applications.

Chapter III will be devoted to the determination of the braid index of many links, i.e. algebraic links (in the sense of Conway) and pretzel links. In particular, the braid index of an alternating pretzel link is completely determined in §13. In §14 we will show that the braid index of an alternating link $L$ is determined by its skein polynomial if the leading coefficient $c_0$ of the Alexander polynomial is small, i.e. $|c_0| \leq 3$. If $c_0 = \pm 1$, $L$ is fibred and the braid index is already determined in [Mu 4]. (The same result also follows from Theorem 9.5.) Therefore, we only consider the links with $c_0 = \pm 2$ or $\pm 3$. Since links with this property are characterized by their Seifert graphs, the proof is not complicated. The original proof, however, has been simplified considerably by using the main result in [Mu 3]. In the last section, §15, we propose a few conjectures on the braid index.

There is one appendix in which we prove two technical lemmas needed in Chapter II.

Finally, we would like to express our deepest appreciation to J. Hoste who computed for us a part of the skein polynomials of two alternating links. Using his result, we were able to determine the braid index of these links which eventually disproved one of our original conjectures on the braid index. (See §15.) (A year after we submitted the paper, W. Menasco and M.B. Thistlethwaite announced a proof of the Tait flyping conjecture. (See [MT].) As a result, in this revised version of our paper, we have omitted some of the material relevant to this conjecture which was contained in the original version.)

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# Chapter I. Index of a graph.

## §1 Preliminaries and notations

Let $G$ be a graph. Let $V(G)$ and $E(G)$ be the sets of vertices and edges, respectively.

We restrict ourselves to finite graphs, that is, graphs for which $V(G)$ and $E(G)$ are both finite. In this paper, however, slightly more general graphs shall be considered.

A graph $G$ is said to be *signed* if either $+1$ or $-1$, called a *sign*, is assigned to each edge. More precisely, $G$ (or $(G, f_G)$ ) is a *signed graph* if $G$ is a graph equipped with a sign function $f_G : E(G) \to \{1, -1\}$. For convenience, we call an edge $e$ *positive* if $f_G(e) = +1$ and *negative* otherwise. Since a positive graph may be considered as an unsigned (or an ordinary undirected) graph, our results can be applied to ordinary undirected graphs.

A subgraph $H$ of $G$ has induced sign function $f_H = f_G|E(H)$. A subgraph $H$ is a *spanning* subgraph if $V(H) = V(G)$.

Throughout this paper, what is meant by a *graph* is frequently the geometric realization of a graph as a finite 1-dimensional CW-complex in $\mathbb{R}^3$. We are free to use many terminologies from algebraic topology.

For a set $X, |X|$ denotes the cardinality of $X$. $\beta_i(G)$ denotes the $i^{th}$ Betti number of a graph $G$ as a 1-complex.

In graph theory, $p_0(G)$ and $p_1(G)$ have been used instead of $\beta_0(G)$ and $\beta_1(G)$. $p_0(G)$ denotes the number of connected components of $G$, and $p_1(G)$ is called the cyclomatic number of $G$.

Let $H$ and $K$ be two graphs, both of which have at least one edge. Then the *one-point union* of $H$ and $K$ will be denoted by $H * K$.

We also refer to $[Be]$ for many standard terminologies in graph theory.

For $V \subset V(G)$ and $E \subset E(G)$, $G - (V, E)$ denotes the maximal subgraph of $G$ which does not contain vertices in $V$ and edges in $E$. In particular, $G - e$ is the subgraph of $G$ consisting of all vertices of $G$ and all edges but $e$. Therefore $G - e$ is the subgraph obtained from $G$ by deleting $e$. For a vertex $v$, $G - v$ is the subgraph consisting of all vertices but $v$ and edges of $G$ except those which are incident to $v$.

A graph $G$ is said to be *separable* if there are two subgraphs $H$ and $K$ such that $G = H \cup K$ and $H \cap K = \{v_0\}$, where $H$ and $K$ both have at least one edge and $v_0$ is a vertex. Otherwise, $G$ is non-separable. The vertex $v_0$ is called a *cut vertex*. If $G$ has no loops, then $G$ is separable when $\beta_0(G) < \beta_0(G - v)$ for some vertex $v$.

A *block* is a maximal non-separable connected subgraph of $G$. A connected graph is decomposed into finitely many blocks. Therefore, if $G_1, G_2, \dots, G_k$ are blocks of $G$, we can write $G = G_1 * G_2 * \cdots * G_k$ and $G$ is called the *block sum* of $G_1, G_2, \dots, G_k$.

$G$ is called *reduced* if $G$ has neither loops nor isthmuses. An *isthmus* is an edge $e$ such that $\beta_0(G) < \beta_0(G - e)$.

If two or more edges have the same ends, these edges are called *multiple-edges*. On the other hand, if two distinct vertices are joined by exactly one edge $e$, then $e$ is called a *singular* edge of $G$. A loop is not a singular edge.

A two-vertex graph $G$ is called a *multiple-edge graph* (or a *single-edge graph*) if all edges have the common (distinct) ends and $|E(G)| \geq 2$ (or $|E(G)| = 1$).

Let $G$ be a graph and $v$ a vertex of $G$. *star* $v$ is the smallest subgraph containing $v$ and all edges of $G$ which are incident to $v$. If $X$ is a connected subset of $G$, then $G/X$ is defined as the subgraph obtained from $G$ by identifying all points in $X$ to one point.

For convenience, for subgraphs $H$ and $K$ of a graph $G$, we define $H/K$ as $H/H \cap K$. Therefore, if $H \cap K = \phi$, then $H/K$ is $H$ itself. For an edge $e$, $G/(e)$ constructed from $G - e$ by identifying the ends of $e$ is said to be obtained by *contracting* $e$. If $e$ is a loop, then $G - e = G/(e)$.

An alternate sequence of vertices $v_i$ and edges $e_i$: $v_0, e_1, v_1, \dots, u_{n-1}, e_n, v_n$ is called a *chain* (connecting $v_0$ and $v_n$) of $G$ if $v_i$ and $v_{i+1}$ are ends of the edge $e_{i+1}$, for $i =$

$0, 1, \ldots, n-1$. The *length* of the chain is $n$.

A chain $C$ is called a *cycle* if $v_n = v_0$. The length of $C$, denoted by $|C|$, is $n$. A chain or a cycle is called *simple* if $e_i \neq e_j$ and $v_i \neq v_j$ for any $i$ and $j$, $i \neq j$, except possibly $v_n = v_0$. For simplicity, a cycle of length $n$ will be called an $n$-cycle. A chain (or a cycle) in which all the edges are distinct is called a *trail* (or a *closed trail*).

A graph $G$ is said to be *bipartite* if any cycle of $G$ has an even length. Equivalently, $G$ is bipartite if $V(G)$ can be decomposed into two disjoint (non-empty) sets $V_1$ and $V_2$ in such a way that each edge of $G$ has two distinct ends one of which belongs to $V_1$ and another to $V_2$. A bipartite graph cannot have a loop.

The valency of a vertex $v$, $val(v)$, is the number of edges incident to $v$. If a loop is incident to $v$, it is counted twice. Therefore, if $m$ loops and $k$ non-loop edges are incident to $v$, then $val(v) = 2m + k$. A graph is called an *even* graph if every vertex has an even valency.

A vertex of valency 1 is called a *stump*. A *twig* is a vertex of valency 2.

A graph $G$ is called *planar* if $G$ can be embedded into a plane $\mathbb{R}^2$ as a 1-complex. $G$ is called a *plane* graph if $G$ is a graph embedded in $\mathbb{R}^2$.

If $G$ is a connected plane graph, we can define the dual graph $G^*$. (Strictly speaking, $G^*$ is not unique as a plane graph, but if $G$ is imbedded in $S^2$, then $G^*$ is unique. However, non-uniqueness of $G^*$ in $\mathbb{R}^2$ does not cause any trouble in this paper.) $V(G^*)$ and the set $F(G)$ of domains in $\mathbb{R}^2 - G$ are in one-to-one correspondence, and, $E(G^*)$ and $E(G)$ are in one-to-one correspondence in such a way that $e^* \in E(G^*)$ and its partner have exactly one point, not a vertex, in common. We define the sign of $e^*$ as the opposite of its partner. If $G$ is a plane disconnected graph, then $G^*$ is a disjoint union of graphs dual to connected components of $G$.

**Example 1.1**

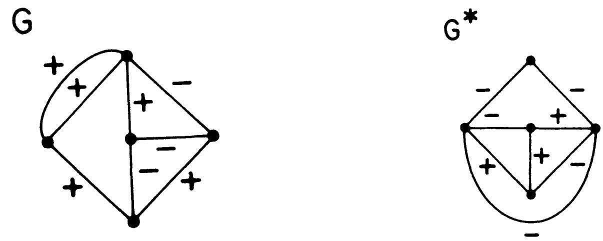



Fig. 1.1

## §2 Index of a graph

In this section, we introduce and analyze the concept of an index of a graph. The index will be further translated to an oriented link diagram and will provide an important tool to determine the braid index of a link.

**Definition 2.1** Let $G$ be a graph.

(1) A family $\mathcal{F} = \{e_1, \dots, e_k\}$ of edges of $G$ is said to be *independent* if (i) all $e_j$ ($j = 1, 2, \dots, k$) are singular and (ii) there is an edge $e_i$ in $\mathcal{F}$ and a vertex $v$, one of the ends of $e_i$, such that $\{\phi(e_1), \dots, \phi(e_{i-1}), \phi(e_{i+1}), \dots, \phi(e_k)\}$ is an independent set of $k-1$ edges in the graph $G/star\ v$, where $\phi : G \to G/star\ v$ is the collapsing map. (In the rest of the paper, we do not distinguish between $e_i$ and $\phi(e_i)$ unless confusion arises.) We define that the empty set of edges is independent.

(2) $ind\ (G)$ is defined to be the maximal number of independent edges in $G$.

(3) If $G$ is a signed graph, then $ind_+(G)$ (respectively $ind_-(G)$) is defined to be the maximal number of independent edges $\{e_1, \dots, e_k\}$ in $G$, where all $e_j (j = 1, \dots, k)$ are positive (respectively negative) and singular in $G$.

It is obvious that $ind\ (G) \leq\ ind_+(G) + ind_-(G)$.

**Example 2.2** For the graph $G$ depicted in Fig. 2.1, we see that $ind\ (G) = 1$, $ind_+(G) = 1$

and $ind_-(G) = 1$.

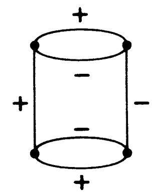



Fig. 2.1

From Definition 2.1, we have immediately the following

**Proposition 2.3** *If two graphs $G_1$ and $G_2$ are disjoint, then*

$$ind\,(G_1 \cup G_2) = ind\, G_1 + ind\, G_2,$$

$$ind_+(G_1 \cup G_2) = ind_+\,(G_1) + ind_+\,(G_2), \text{ and}$$

$$ind_-(G_1 \cup G_2) = ind_-\,(G_1) + ind_-\,(G_2).$$

One of the main theorems of this chapter is the following theorem.

**Theorem 2.4** *Let $G$ be a connected bipartite graph. If $G$ consists of blocks $G_1, G_2, \ldots, G_k$ then*

(1) $ind\, G = ind\,(G_1) + \cdots + ind\,(G_k)$.

*Furthermore, if $G$ is a signed graph, then*

(2) $ind_+(G) = \sum_{i=1}^{k} ind_+(G_i)$ *and* $ind_-(G) = \sum_{i=1}^{k} ind_-(G_i)$.

First we note that it suffices to show (1). Because if the graph $G'$ is obtained from $G$ by replacing all singular negative (or positive) edges by multiple-edges, then we see that $ind_+ G = ind\, G'$ (or $ind_- G = ind\, G'$) and apply (1) on $G'$.

Now Theorem 2.4 follows easily from the following

**Lemma 2.5** *Suppose that $G$ is the one-point union of two graphs $G_1$ and $G_2$ i.e. $G = G_1 * G_2$. If at least one of $G_1$ and $G_2$ is bipartite, then $ind\, G = ind\, G_1 + ind\, G_2$.*

Example 2.6 below shows that Lemma 2.5 (and hence Theorem 2.4) does not hold if $G_1$ and $G_2$ are non-bipartite.

**Example 2.6** Consider the graph $H$. (See Fig 2.2(a))

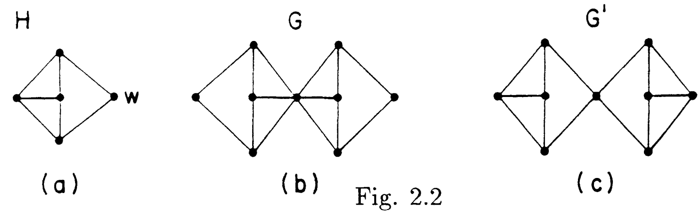



Fig. 2.2

Obviously $ind\ H = 2$ and we can easily check that $w$ is the only vertex of $H$ such that $ind(H/star\ w) = 1$. Now consider the graphs $G$ and $G'$ obtained as the one-point unions of two copies of $H$ along different vertices as depicted in Fig. 2.2(b) and (c). Then it is easy to see that $ind\ G = 4$, while $ind\ G' = 3$.

Now, Lemma 2.5 follows from Lemma 2.7 below

**Lemma 2.7** *Let $w$ be a vertex of a bipartite graph $G$ with $ind\ G = n \geq 1$. Then there exists a vertex $u$ that is an end of a singular edge of $G$ such that $u \neq w$ and $ind\ (G/star\ u) = n-1$.*

The lemma also does not hold for non-bipartite graphs. For example, for any vertex $u$ of $H$ (in Fig 2.2(a)) different from $w$, $ind\ (G/star\ v) = 0$

**Proof of Lemma 2.7 $\Longrightarrow$ Lemma 2.5.** Without loss of generality, we may assume that $G_1$ is bipartite. Then for any vertex $v$ of $G_1$, $G_1/star\ v$ is also bipartite. Observe that always $ind\ (G_1 * G_2) \leq ind\ G_1 + ind\ G_2$. Now we proceed by induction on $ind\ G_1$. If $ind\ G_1 = 0$, then $G_1$ has no singular edges and we can see easily that a family of edges in $G_2$ is independent iff it is independent in $G_1 * G_2$, and hence, $ind\ G = ind\ G_2$.

Suppose now that $ind\ G_1 = n \geq 1$ and that Lemma 2.5 holds for any graph $G'$ with $ind\ G' < ind\ G_1$. Let $v$ be the vertex of $G$, along which $G_1$ and $G_2$ are joined. By Lemma 2.7, we can choose a vertex $u$ of a singular edge of $G_1$ such that $u \neq v$ and $ind\ (G_1/star\ v) = n-1$. Then $(G_1 * G_2)/star\ u = (G_1/star\ u) * G_2$ and the induction hypothesis yields $ind\ (G_1/star\ u * G_2) = ind\ (G_1/star\ u) + ind\ G_2 = n - 1 + ind\ G_2$. Therefore, by the definition of the index, we have

$$ind\ (G_1 * G_2) \geq n + ind\ G_2 = ind\ G_1 + ind\ G_2.$$

Since the reverse inequality always holds, we have the equality. □

Now to prove Lemma 2.7 we need a few more definitions and a lemma.

**Definition 2.8** A sequence of vertices $w_1, w_2, \ldots, w_k$ of $G$ is called a *special sequence* if it satisfies the following conditions: For $i = 1, 2, \ldots, k$

(1) $w_i$ is an end of a singular edge in $G_{i-1}$ and $G_i = G_{i-1}/star\, w_i$ where, $G_0 = G$.

(2) $ind\ G_i = ind\ G - i$ for $i = 1, 2, \ldots, k$.

(3) For $i < k, dist_{G_{i-1}}(w_{i-1}, w_i) = 1$, where $dist_H(x, y)$ is the minimum of the length of all chains connecting $x$ and $y$ in $H$.

(4) $dist_{G_{k-1}}(w_{k-1}, w_k) \geq 2$ in $G_{k-1}$ or $w_{k-1}$ and $w_k$ are joined by a singular edge in $G_{k-1}$.

If $ind\ G = 1$, then an end of any singular edge of $G$ forms a special sequence.

**Lemma 2.9** *Let $w_1, \ldots, w_k$ be a special sequence in a graph $G$. Define a sequence of graphs $G'_1, \ldots, G'_k$ as follows: $G'_0 = G$, $G'_1 = G/star\, w_k$, and inductively, $G'_i = G'_{i-1}/star\, w_{i-1}$ for $i = 2, \ldots, k$. Then*

*(1) $w_i$ is an end of a singular edge in $G'_i$ for $i < k$ and $w_k$ is an end of a singular edge of $G'_0(= G)$.*

*(2) $ind\ G'_i = ind\ G - i$, for $i = 1, 2, \ldots, k$. In particular, $ind\ (G/star\, w_k) = ind\ G - 1$.*

**Proof** We only give a proof of Lemma 2.9 for $k = 2$, since the general case is completely analogous. First we show that $w_1$ is an end of a singular edge of $G'_1 = G/star\, w_2$.

Let $e$ be a singular edge of $G$ having $w_1$ as an end. Definition 2.8 (4) now ensures that $dist(w_1, w_2) \geq 2$ in $G$, and $w_1$ and $w_2$ cannot occur on a cycle in $G$ of length $\leq 4$. (Otherwise $w_1$ and $w_2$ would be joined by multiple edges in $G_1 = G/star\, w_1$. ) Therefore, $e$ is a singular edge in $G'_1(= G/star\, w_2)$ and $w_1$ is an end of $e$ in $G'_1$. This argument also shows that $w_2$ is an end of a singular edge in $G$, since $w_2$ is an end of a singular edge in $G_1$ by Definition 2.8 (1). On the other hand, since $dist_{G_1}(w_1, w_2) \geq 2$ it follows that

$$G'_2 = (G_0/star\, w_2)/star\, w_1 = (G/star\, w_1)/star\, w_2 = G_2$$

and hence, $ind\ G'_2 = ind\ G_2 = ind\ G - 2$. Finally, since $w_1$ is an end of a singular edge in $G'_1 = G/star\, w_2$, we see that $ind\ G'_1 = ind\ G - 1$. This proves Lemma 2.9. □

**Proof of Lemma 2.7** If $w$ is not an end of a singular edge of $G$ or $ind\,(G/star\, w) < n-1$, then Lemma 2.6 trivially holds by the definition of the index. Therefore, we assume that $w$ is an end of a singular edge $e$ of $G$ and $ind(G/star\, w) = n-1$. Now we proceed our proof by induction on $n$. For $n = 1$, the lemma holds by taking as $u$ the other end of $e$. Now suppose that $ind\ G = n \geq 2$ and Lemma 2.7 holds for any bipartite graph with a smaller index. Write $w_1 = w$. By induction hypothesis, there is a vertex $w_2$ in $G$ such that (1) $w_2$ is different from $w_1$ in $G_1 (= G/star\, w_1)$, (2) $w_2$ is an end of a singular edge of $G_1$ and (3) $ind\ (G_1/star\, w_2) = n-1$. If $w_1, w_2$ is a special sequence, then we have from Lemma 2.9 that $ind(G/star\, w_2) = n-1$ and hence, $w_2$ is what we want. If $w_1, w_2$ is not a special sequence, apply the induction hypothesis on $G_1 = G/star\, w_1$ to find the third vertex, say $w_3$, of $G$ such that $w_3$ is different from $w_2$ in $G_2 (= G_1/star\, w_2)$ and $ind\ G_3 (= ind\ (G_2/star\, w_3)) = ind\ G_2 - 1 = n-3$. Repeat the same argument as long as the sequence of vertices $w_1, w_2, \dots, w_m$ thus obtained is not a special sequence.

If for some $k \leq n$ the sequence $w_1, \dots, w_k$ is a special sequence, then $w_k$ is the vertex we sought. Suppose that $w_1, \dots, w_{n-1}$ is not a special sequence. This sequence satisfies the conditions (1) - (3) in Definition 2.8. In particular, from (3) we see that all $w_1, \dots, w_{n-1}$ collapse to the vertex $w_1$ in $G_{n-1}$. Let $e$ be a singular edge in $G_{n-1}$. Then choose $w_n$ as follows:

If $w_1$ is one of the ends of $e$, choose $w_n$ to be another end. ($w_n$ is also a vertex of $G$.)

If $w_1$ is not an end of $e$, then one of the ends, say $v_0$, of $e$ has $dist(w_1, v_0) \geq 2$, since $G_{n-1}$ is a bipartite graph and does not contain 3-cycles. Choose $v_0$ as $w_n$. Then $w_1, \dots, w_n$ is a special sequence in $G$, and $w_n$ is what we want. It now completes the proof of Lemma

2.7. □

## §3 Cycle index of a graph.

As the first approximation of *ind* $G$, in this section we define a *cycle index* of a graph $G$. Usually, the determination of the cycle index is much easier than that of the index and therefore, it provides a quite effective method to determine the index of a graph. In fact, the cycle index will be used to determine *ind* $G$ for a certain class of graphs. See Theorem 6.5.

**Definition 3.1** Let $\mathcal{S} = \{e_1, \ldots, e_n\}$ be a set of $n$ distinct edges in a graph $G$.

(1) $\mathcal{S}$ is said to be *cyclically independent* if no $k$ edges in $\mathcal{S}(1 \leq k \leq n)$ occur on a simple cycle of length at most $2k$. Otherwise $\mathcal{S}$ is called *cyclically dependent.*

(2) The cycle index of $G$, denoted by $\alpha(G)$, is defined as the maximal number of cyclically independent edges of $G$.

**Remark 3.2** In the Definition 3.1 (1), a simple cycle can be replaced by a closed trail.

**Example 3.3** For a graph $G$ depicted in Fig. 3.1, $\alpha(G) = 2$.

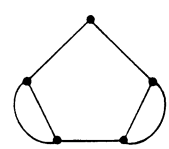


Fig. 3.1

**Theorem 3.4** *For a graph $G$, ind$(G) \leq \alpha(G)$.*

**Proof** We proceed by induction on $\alpha(G)$. If $\alpha(G) = 0$, then $G$ has no singular edges and hence, *ind* $G = 0$.

Let $\alpha(G) = n \geq 1$ and assume that Theorem 3.4 holds for a graph $H$ with $\alpha(H) < n$. Let $v(e)$ be an end of a singular edge $e$ in $G$. First we will show that $\alpha\big(G/star\ v(e)\big) \leq n-1$.

Take a family $\mathcal{S} = \{e_1, \ldots, e_n\}$ of $n$ distinct edges in $G/star\, v(e)$. $\mathcal{S}$ gives rise to a family $\mathcal{S}'$ of $n+1$ edges in $G$ by adding $e$ to $\mathcal{S}$. Since $\alpha(G) = n$, $\mathcal{S}'$ is cyclically dependent in $G$. Therefore, there are, say $k$, edges $e_{i_1}, \ldots, e_{i_k}$ in $\mathcal{S}'$ such that these edges occur on a simple cycle $C$ in $G$ of length at most $2k$. Let $U = \{e_{i_1}, \ldots, e_{i_k}\}$. We consider two cases.

**Case (1)** $e \notin U$. Then $U$ is also a family of $k$ edges, all of which occur on the closed trail $C/star\, v(e)$ in $G/star\, v(e)$, where $|C/star\, v(e)| \leq |C| \leq 2k$.

**Case (2)** $e \in U$. Then $U - \{e\}$ is a family of $k-1$ edges, all of which occur on the closed trail $C/star\, v(e)$ in $G/star\, v(e)$, where $|C/star\, v(e)| \leq |C| - 2 \leq 2k-2$.

In either case $\mathcal{S}$ is cyclically dependent in $G/star\, v(e)$, and therefore, $\alpha\big(G/star\, v(e)\big) \leq n-1$. But the inductive assumption yields $ind\ \big(G/star\, v(e)\big) \leq \alpha\big(G/star\, v(e)\big) \leq n-1$ and hence, $ind\ (G) \leq n$. □

**Corollary 3.5**

(1) *If $ind\ (G) \leq 1$, then $ind\ (G) = \alpha(G)$. In particular, $ind\ G = 1$ iff $G$ has singular edges and each pair of singular edges in $G$ occurs on a simple 3 - or 4 - cycle in $G$.*

(2) *Suppose that there are no simple 3-cycles in $G$. Then $ind\ G = 2$ iff $\alpha(G) = 2$.*

**Remark 3.6** Corollary 3.5(2) is false if $G$ has a 3-cycle. The graph $G$ in Fig. 3.2

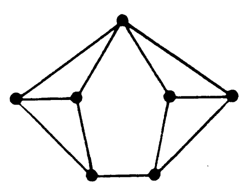


Fig. 3.2

has $ind\ G = 2$, but $\alpha(G) = 3$. However, for a bipartite graph $G$, we conjecture that $ind\ G = \alpha(G)$.

The proof of Corollary 3.5 is elementary but tedious and hence, the details will be omitted.

Finally from the definition of $\alpha(G)$, the following proposition is immediate. (Cf. Theorem 2.3.)

**Proposition 3.7**

*(1) If $G$ consists of $k$ blocks $G_1, \ldots, G_k$, then* $\alpha(G) = \alpha(G_1) + \cdots + \alpha(G_k)$.

*(2) If $G_1$ and $G_2$ are 2-isomorphic, then* $\alpha(G_1) = \alpha(G_2)$.

For the definition of 2-isomorphism of graphs, see [Wh].

## §4 Index and other invariants

Bipartite graphs play a very important role in link theory. Fortunately, they have many useful properties, as was shown in the previous sections. In this section, we will give the second approximation of the index for a planar bipartite graph using the familiar invariants in graph theory. The results in this section will be used in Chapter II.

Let $G$ be a connected plane bipartite graph and $G^*$ the dual of $G$. Since $G$ is bipartite, $G^*$ is an even graph. Therefore, we can define a direction (or put an arrow) to each edge of $G^*$ in such a way that the boundary of each domain in $\mathbb{R}^2 - G^*$ is an oriented cycle. Such a direction is called a *natural direction* of $G^*$. There are exactly two and only two natural directions. One is the complete reverse of the other. If an edge $e$ is directed from an end $v_1$ to another end $v_2$, then $v_1$ is called the *initial* end and $v_2$ the *terminal* end of $e$.

Now let $K$ be a connected plane even graph with a natural direction. Take a vertex $v_0$ from $K$ and fix it. $v_0$ will be called a *root* of $K$.

**Definition 4.1** A (directed) spanning tree $T$ in $K$ is called a *growing* spanning tree rooted at $v_0$ (or a growing rooted spanning tree) if every vertex except $v_0$ is the terminal end of exactly one edge of $T$ and if $v_0$ is never the terminal end of any edge in $T$.

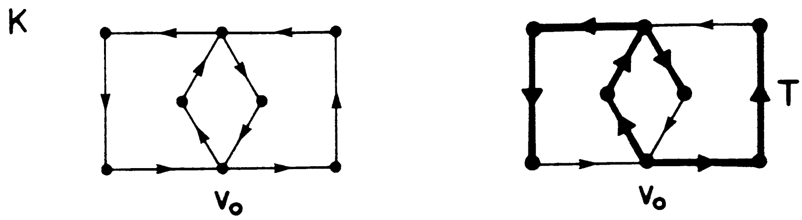



Fig. 4.1

Now it is well known that the following proposition is true for directed even (not necessarily plane) graphs. For a proof, see [Be]

**Proposition 4.2** *Let $K$ be a directed even connected graph such that each vertex has the same number of inputs and outputs. Let $\lambda(K, v_0)$ denote the number of growing spanning trees in $K$ rooted at $v_0$. Then*

*(1)* $\lambda(K, v_0) > 0$.

*(2)* $\lambda(K, v_0)$ *does not depend on the root $v_0$. Therefore, we write $\lambda(K)$ without reference to $v_0$.*

*(3) If $K_1$ and $K_2$ are connected even graphs, then*

$$\lambda(K_1 * K_2) = \lambda(K_1)\lambda(K_2).$$

*(4) Let $\overline{K}$ be the directed graph obtained from $K$ by reversing the orientation of each edge. Then*

$$\lambda(K, v_0) \ = \ \lambda(\overline{K}, v_0)\ .$$

If $K$ has $n$ connected components, $K_1, K_2, \ldots, K_n$, then we define $\lambda(K) = \lambda(K_1)\,\lambda(K_2)\ldots\lambda(K_n)$. The purpose of this section is to prove the following theorem.

**Theorem 4.3** *Let $G$ be a plane bipartite connected graph and $G^*$ its dual graph with a natural direction. Suppose that $G$ consists of $n$ blocks, $G_1, \ldots, G_n$. Then*

$$|V(G)| - 1 \leq n\{2\prod_{i=1}^{n}\lambda(G_i^*) - 1\}, \tag{4.4}$$

*where $G_i^*$ denotes the dual of $G_i$.*

We should note that $\lambda(G^*)$ does not depend on choice of natural directions of $G^*$. Cf. Proposition 4.2 (4).

To prove Theorem 4.3, we need the following lemma.

**Lemma 4.5** *Let $C$ be the boundary cycle of a domain in $\mathbb{R}^2 - G^*$. Let $n$ be the number of those vertices on $C$ whose valencies $\geq 4$. Then*

$$\lambda(G^*) \geq \lambda(G^* - C) + n - 1 \ . \tag{4.6}$$

**Proof** If $n = 0$, then, since $G^*$ is connected, $G^*$ is $C$ itself, and hence $\lambda(G^*) = 1$, while $\lambda(G^* - C) + n - 1 = -1$. If $n = 1$, then we see that $G^* = (G^* - C) * C$ and hence $\lambda(G^*) = \lambda(G^* - C)\lambda(C)$. Choose as a root the vertex common to $G^* - C$ and $C$. Then, since $\lambda(C) = 1$, (4.6) follows.

Now assume $n \geq 2$. Let $v_1, \ldots, v_n$ be non-twigs on $C$. $C$ is decomposed into $n$ chains $P_1, P_2, \ldots, P_n$ by $v_1, v_2, \ldots, v_n$ where $P_i (1 \leq i \leq n-1)$ connects $v_i$ and $v_{i+1}$. Choose $v_1$ as a root of $G^*$. Let $G_0^* = G^* - \dot{C}$. We claim then

$$\lambda(G^*) \geq \lambda(G_0^*) + \lambda(G^*/P_1) + \lambda(G^*/P_1 \cup P_2) + \cdots + \lambda(G^*/P_1 \cup \cdots \cup P_{n-1}). \tag{4.7}$$

Since $\lambda(G^*/P_1 \cup \cdots \cup P_i) > 0$, (4.7) yields (4.6) immediately.

Now to prove (4.7) it is enough to show that (1) to each growing rooted spanning tree $T$ in $G_0^*$ or $G_i^* (= G^*/P_1 \cup \cdots \cup P_i)$, there is a growing rooted spanning tree $T^*$ in $G^*$, called an associate of $T$, and (2) two trees associated with two distinct trees $T$ and $T'$ are distinct.

Now $T^*$ will be defined as follows. Suppose that $T$ is a tree in $G_i^*$. Let $\phi_i : G^* \longrightarrow G_i^* (= G^*/P_1 \cup \cdots \cup P_i)$ be the collapsing map. Since all $v_1, \cdots, v_{i+1}$ are identified to $v_1$ in $G_i^*$, two vertices $v_\ell$ and $v_m$, $1 \leq \ell,\ m \leq i+1$ are not connected in $\phi_i^{-1}(T)$. Define $T^* = \phi^{-1}(T) \cup P_1 \cup \cdots \cup P_i$. Then $T^*$ is a growing rooted spanning tree in $G^*$. If $T$ is a tree in $G_0^*$, then $T^*$ is the tree obtained from $T$ by adding appropriate simple chains in $P_1$, $P_2$,...,$P_n$ to $T$. From this construction, a proof of the other property (2) is obvious. It proves (4.7) and hence (4.6). □

Now we return to the proof of Theorem 4.3.

Suppose first that (4.4) holds for each block $G_i$. Since $|V(G)|-1 = \sum_{i=1}^{n}(|V(G_i)|-1)$, it follows from our assumption that

$$|V(G)|-1 \leq \sum_{i=1}^{n}\{2\lambda(G_i^*)-1\} = \sum_{i=1}^{n} 2\lambda(G_i^*) - n \leq 2n\lambda(G_1^*)\dots\lambda(G_n^*) - n.$$

The last inequality is implied from the following simple calculation:

$$\begin{aligned}2n\lambda(G_1^*)\dots\lambda(G_n^*) - \sum_{i=1}^{n} 2\lambda(G_i^*) &= 2\sum_{i=1}^{n}\{\lambda(G_1^*)\dots\widehat{\lambda(G_i^*)}\dots\lambda(G_n^*)-1\}\lambda(G_i^*)\\ &\geq 0, \quad \text{since} \quad \lambda(G_i^*)\geq 1.\end{aligned}$$

Therefore, it suffices to prove (4.4) for a non-separable graph $G$.

Consider $\mathbb{R}^2 - G^*$. Domains in $\mathbb{R}^2 - G^*$ are classified by black and white in such a way that no two domains of the same colour have edges in common. Let $\alpha$ and $\beta$, respectively, be the number of white and black domains in $\mathbb{R}^2 - G^*$. Since $G^*$ is connected, white domains $\{W_i\}$ can be numbers in such a way that for $i = 1,2,\dots,\alpha-1$, $(\partial W_1 \cup \dots \cup \partial W_i) \cap \partial W_{i+1}$ has at least one vertex. Let $q_{i+1}$ be the number of vertices that occur on $(\partial W_1 \cup \dots \cup \partial W_i) \cap \partial W_{i+1}$. Let $\Gamma_i = \partial W_1 \cup \dots \cup \partial W_i$. Then for $i = 1,2,\cdots,\alpha-1$, (4.6) implies that

$$\lambda(\Gamma_{i+1}) \geq \lambda(\Gamma_i) + q_{i+1} - 1.$$

Since $\lambda(\Gamma_1) = 1$ and $\lambda(\Gamma_\alpha) = \lambda(G^*)$, it follows that

$$\lambda(G^*) \geq 1 + (q_2-1) + \dots + (q_\alpha - 1) = \sum_{i=2}^{\alpha} q_i - (\alpha-2). \tag{4.8}$$

Now $|V(G)|$ is exactly the number of domains in $\mathbb{R}^2 - G^*$, and hence $|V(G)| = \alpha+\beta$. We may assume without loss of generality that $\alpha \leq \beta$. (Otherwise, exchange the colours of the domains.) Therefore $\alpha \leq \frac{|V(G)|}{2}$. Let $d_i$, $i = 1,2,\cdots,\alpha$, be the number of domains in which $\mathbb{R}^2 - \Gamma_i$ is divided. Then $d_1 = 2$ and an easy induction proves that for $i \geq 2$, $d_i \leq 2 + q_2 + \cdots + q_i$. Therefore, we have

$$|V(G)| = d_\alpha \leq 2 + \sum_{i=2}^{\alpha} q_i. \tag{4.9}$$

Combining (4.8) and (4.9), we see

$$|V(G)| \leq \lambda(G^*) + \alpha. \tag{4.10}$$

Since $\alpha \leq \frac{|V(G)|}{2}$, (4.10) yields $|V(G)| \leq \lambda(G^*) + \frac{|V(G)|}{2}$ and hence $|V(G)| \leq 2\lambda(G^*)$.

A proof of Theorem 4.3 is now complete. □

As an easy consequence of Theorem 4.3, we obtain the following corollary.

**Corollary 4.11** *Let $G$ be a plane bipartite connected graph without isthmuses and $G^*$ the dual of $G$ with a natural direction. Then*

$$ind\ G \leq \lambda(G^*) - 1. \tag{4.12}$$

**Proof** Since $G$ has no isthmuses, the dual $G^*$ has no loops.

Suppose that $G$ has $k$ blocks $H_1, \ldots, H_k$. Then the dual $G^*$ also has $k$ blocks, $H_1^*, \ldots, H_k^*$ and $H_i^*$ is the dual of $H_i$.

Assume first that the corollary holds for each block $H_i$. Using the fact that $\lambda(H_i^*) \geq 1$ for any $i$ and $\lambda(G^*) = \prod_{i=1}^{k} \lambda(H_i^*)$, an easy induction on $k$ proves that $\lambda(G^*) - 1 \geq \sum_{i=1}^{k} \{\lambda(H_i^*) - 1\} \geq \sum_{i=1}^{k} ind\ (H_i) = ind\ G$.

Now we may assume that $G$ has not cut vertices. It suffices to show that if $|V(G)| \geq 2$, then

$$ind\ G \leq \frac{|V(G)|}{2} - 1\ . \tag{4.13}$$

Note that if $|V(G)| = 1$, (4.13) is immediate. Now since $G$ is bipartite, the largest simple cycle in $G$ has length at most $|V(G)|$ and hence the cycle index $\alpha(G)$ of $G$ is at most $\frac{|V(G)|}{2} - 1$. Since $\alpha(G) \geq ind\ G$ by Theorem 3.4, (4.13) follows □

**Proposition 4.14** *Let $K$ be an even non-separable connected plane graph with a natural direction. Then*

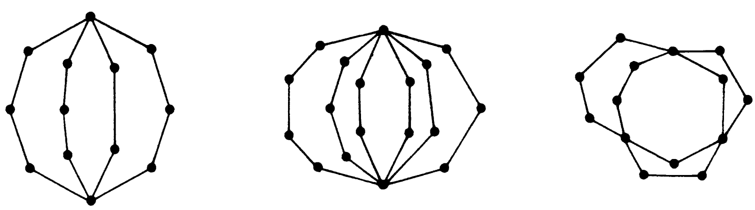
*(1)* $\lambda(K) = 1$ *iff $K$ is a cycle.*

*(2)* $\lambda(K) = 2$ *iff all but two vertices of $K$ are twigs and each of non-twig has valency 4. (See Fig.* a. 4.2(a))

(3) $\lambda(K) = 3$ *iff*

*(i) all but two vertices of $K$ are twigs and each of non-twigs has valency 6, (see Fig. 4.2 (b)) or*

*(ii) all but three vertices of $K$ are twigs and each of non-twigs has valency 4. (See Fig. 4.2. (c))*



(a) (b) (c)

Fig. 4.2

Proposition 4.14 follows easily from Lemma 4.5 and hence a proof is omitted. □

The index of the dual of the graph considered in Proposition 4.14 is at most two.

## §5 Graphs with small indices

By definition, a graph $G$ has index 0 iff $G$ has no singular edges. The main purpose of this section is to characterize the graphs with index 1 or 2. Since we are interested in the application to link theory, we are mainly concerned with plane bipartite graphs.

Now we begin with a definition.

**Definition 5.1** A subgraph $H$ of a graph $G$ is said to be *locally maximal* if

(1) $H$ contains all singular edges of $G$,

(2) $H$ has no multiple edges, i.e. all edges of $H$ are singular,

(3) $H$ has no isolated vertices,

(4) $ind\ H = ind\ G$ and

(5) for any edge $e \in G - H$ which is singular in $H \cup \{e\}$, $ind\ (H \cup e) > ind\ H$.

There is no guarantee that $G$ has locally maximal subgraphs. In fact, some (plane bipartite) graph does not have a locally maximal subgraph. See Example 5.2 below.

**Example 5.2** Consider two graphs $G_1$ and $G_2$ depicted in Fig. 5.1.

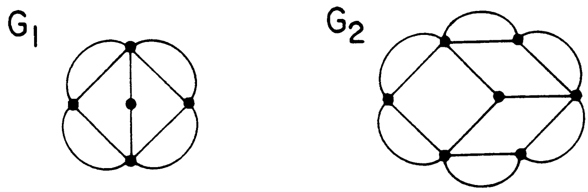



Fig. 5.1

$G_1$ has three locally maximal subgraphs $H_{1,1}H_{1,2}H_{1,3}$, but $G_2$ has no locally maximal subgraphs.

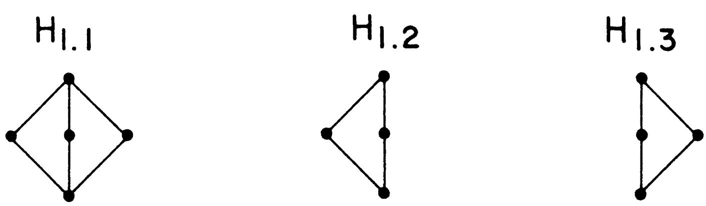



Fig. 5.2

The only locally maximal subgraph of a graph of index 0 is an empty graph.

If $G$ is a plane bipartite graph of index 1, then a locally maximal subgraph of $G$ (if it exists) is either the single-edge graph or a graph $H_1$ depicted in Fig. 5.3. More precisely, $H_1$ has $k+2$ vertices and $2k$ edges for some $k$, and all but two vertices are twigs and each of non-twigs has valency $k$. Such a graph $H_1$ will be denoted by $H_1^k$

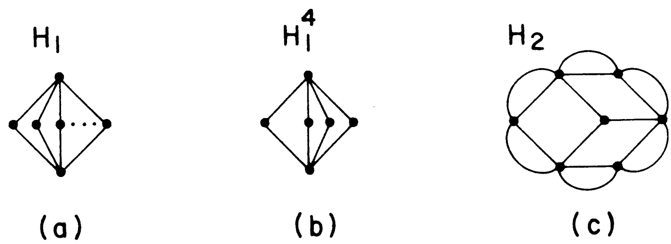



Fig. 5.3

A proof of the above statement is included in the following more general theorem.

**Theorem 5.3** *Let $G$ be a plane bipartite graph. Then ind $G = 1$ iff*

*(1) $G$ has a singular edge, and*

*(2) $G$ has a subgraph $H$ such that*

*(i) $H$ contains all singular edges of $G$, and*

*(ii) $H$ is one of the following graphs,*

*(a) a single edge graph,*

*(b) a graph of type $H_1$ (Fig. 5.3 (a)),*

*(c) a graph of type $H_2$ (Fig. 5.3 (c)).*

**Remark 5.4** If $G$ has a subgraph $H$ of type $H_2$, but not a subgraph of type $H_1$, and if $H$ contains all singular edges of $G$, then $G$ has no locally maximal subgraphs and conversely.

**Proof of Theorem 5.3**

Since "if part" is obvious, we only prove "only if part". Suppose that *ind* $G = 1$ and $G$ has at least two singular edges, say $e_1$ and $e_2$. We consider the following two cases:

**Case 1** $e_1$ and $e_2$ have no ends in common.

**Case 2** $e_1$ and $e_2$ have one end in common.

As a main tool of our proofs, we use Corollary 3.5 which shows that each pair of singular edges occurs on a simple 4-cycle in $G$. (Note that $G$ is bipartite.)

Consider Case 1. We claim that $G$ has a subgraph $H$ of type $H_1$. If $G$ has only two singular edges, then we are done, since $H = H_1^2$.

Let $e_3$ be another singular edges of $G$. Then $e_3$ cannot be disjoint from $e_1$ and $e_2$. Suppose the contrary. Since each pair $\{e_1, e_3\}$ and $\{e_2, e_3\}$ occurs on a simple 4-cycle in $G$, it follows that $G$ has a subgraph depicted in Fig. 5.4.

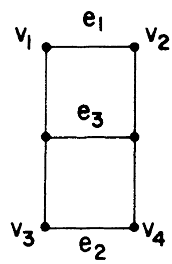



Fig. 5.4

Furthermore, $e_1$ and $e_2$ must occur on some simple 4-cycle. If $v_1$ has been connected to $v_3$ by an edge, then we would have a 3-cycle in $G$. This is impossible, because $G$ is bipartite. If there are two edges in $G$, one of which connects $v_1$ and $v_4$ and another connects $v_2$ and $v_3$, then $G$ could not be planar, a contradiction. Therefore, $e_1$ or $e_2$, say $e_2$, and $e_3$ have a common end $w$. If $e_1$ and $e_3$ have also a common end, then $e_1, e_2$ and $e_3$ occur on a 4-cycle (Fig. 5.5 (a))

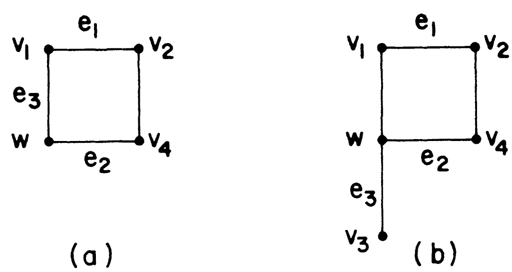



Fig. 5.5

On the other hand, if $e_3$ is disjoint from $e_1$, then we have the subgraph depicted in Fig 5.5 (b). Since $e_1$ and $e_3$ occur on a 4-cycle in $G$, $v_3$ and $v_2$ are connected by an edge. In either case, $e_1$, $e_2$ and $e_3$ occur in a subgraph $H$ of type $H_1$.

Next, consider another singular edge $e_4$ in $G$. $e_4$ must have a common end with either

$e_1$ or $e_2$, as was proved before. If the common end is $w$ or $v_2$, then it is easy to see that another end of $e_4$ must be joined to $v_2$ or $w$ by an edge, and hence all four singular edges $e_1, e_2, e_3$ and $e_4$ occur on a subgraph of type $H_1$. Suppose that the common end of $e_4$ and $e_1$ or $e_2$ is either $v_1$ or $v_4$. If $e_1, e_2$ and $e_3$ occur on a 4-cycle (Fig. 5.5 (a)), then four edges, $e_1, e_2, e_3$ and $e_4$ must occur on a subgraph of type $H_1$. On the other hand, suppose that $e_1, e_2$ and $e_3$ occur on a subgraph in Fig. 5.5 (b). Then $v_4$ cannot be a common end of $e_4$ and $e_2$. Otherwise $e_1, e_3$ and $e_4$ would be pairwise disjoint. Furthermore, $v_1$ cannot be a common end of $e_4$ and $e_1$. Otherwise, the other end of $e_4$ should be connected to $v_3$ and $v_4$ by edges. It is impossible, however, since $v_2$ and $v_3$ have been connected by an edge and $G$ is planar. The same argument eventually proves that all singular edges of $G$ occur on the subgraph of type $H_1$.

Now consider Case 2, where no two singular edges are disjoint. If $G$ has only two singular edges, then we are done. Suppose that $G$ has at least three singular edges. They have a common end $w$, since $G$ has no 3-cycles. See Fig. 5.6 (a)

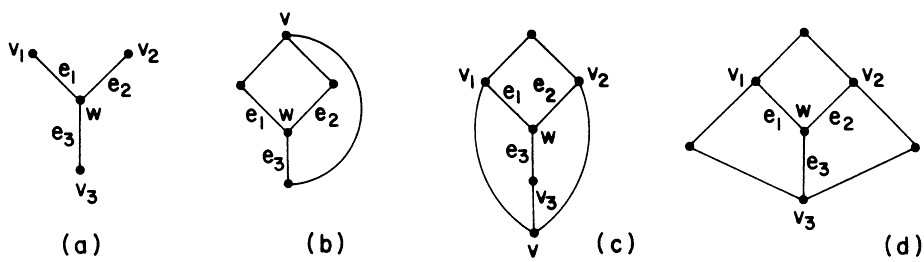



Fig. 5.6

Since each pair of edges occurs on a 4-cycle, there are two possibilities.

(i) $G$ has a subgraph depicted in Fig. 5.6 (b) or (c). Then any other singular edge $e$ of $G$ has also $w$ as one end and the second end of $e$ is connected to $v$ by an edge. Otherwise $G$ could not be planar. See Fig. 5.7 (c) and (d). The same argument eventually proves that all singular edges of $G$ occur on the subgraph of type $H_1$. See Fig. 5.7

(a) and (b).

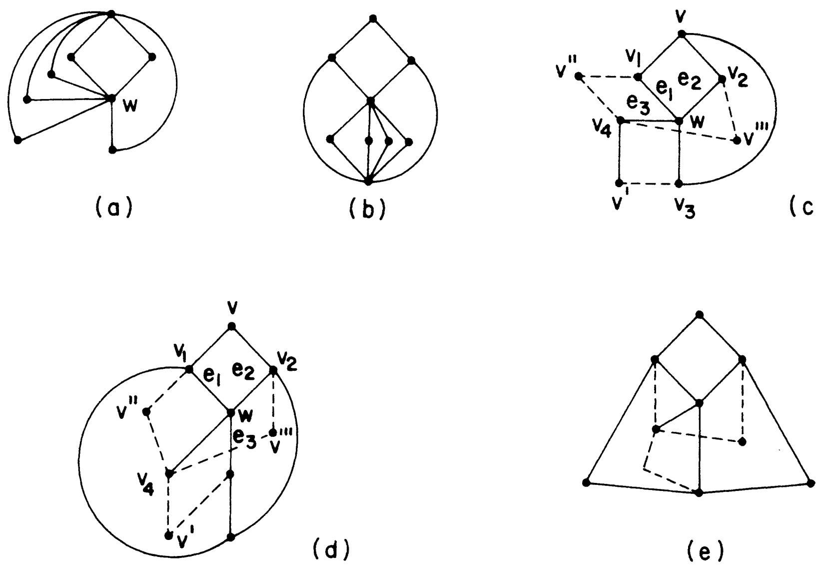



Fig. 5.7

(ii) If (i) did not occur, then there is a subgraph $H$ in $G$ depicted in Fig. 5.6 (d). Then $G$ cannot have any other singular edges, as is seen in Fig. 5.7 (e).

This completes the proof of Theorem 5.3. □

The following theorem characterizes a certain class of graphs with index 2.

**Theorem 5.5** *Let $G$ be a connected plane bipartite graph. Suppose that $G$ has no multiple edges and no isolated vertices. Furthermore, assume that $G$ is non-separable. Then, if $G$ has index 2, $G$ is one of the following graphs: $H_3, H_4$ or $H_5$ (depicted in Fig. 5.8) or a* *properly chosen subgraph of* $H_4$ *or* $H_5$.

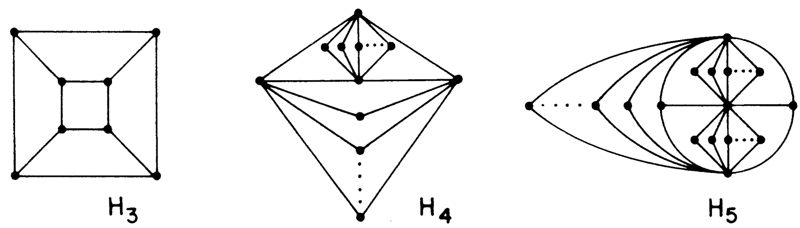



Fig. 5.8

Since the full proof of Theorem 5.5 is easy but tedious, we omit the details.

## §6 Index of a reducible graph

In the final section of Chapter I, we will determine the index of a particular type of graphs, called reducible. This is one of a few classes of graphs for which their indices are described in a precise formula.

**Definition 6.1** A connected plane graph $G$ is called *reducible* if $G$ has the following property. Let $\{D_0, D_1, \ldots, D_n\}$ be the set of domains in which $\mathbb{R}^2$ is divided by $G$, where $D_0$ is the unbounded domain. Then $D_1, \ldots, D_n$ can be renumbered, if necessary , in such a way that for $i = 1, 2, \ldots, n-1$, $\partial D_1 \cup \cdots \cup \partial D_i$ and $\partial D_{i+1}$ have *at most* one edge in common. [3]

**Example 6.2**

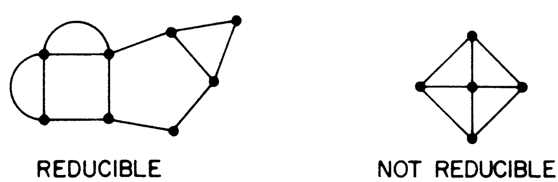



Fig. 6.1

Using the dual graph we can easily state the reducibility of $G$ as follows.

[3] In [Mu 2] a reducible graph is called a *collapsible graph.*

**Proposition 6.3** *A connected plane graph $G$ is reducible iff there is a vertex $v_*$ in the dual $G^*$ of $G$ such that $G^* - v_*$ is a tree. (In fact, $v_*$ is the vertex corresponding to $D_0$.)*

Now, let $G$ be a reducible plane graph. In the rest of this section, we assume that $G$ has no isthmuses.

**Definition 6.4** An edge in $\partial D_0$ is called *free*, where $D_0$ is the unbounded domain.

$G$ is called strongly *excessive* if for any bounded domain $D_i$, $i = 1, 2, \ldots, n$, the number of free edges on $\partial D_i$ is at least $\frac{1}{2}|\partial D_i| - 1$, where $|\partial D_i|$ denotes the length of a simple cycle $\partial D_i$.

The index of a reducible plane bipartite graph that is also strongly excessive is completely determined by the following theorem.

**Theorem 6.5** *Let $G$ be reducible strongly excessive plane bipartite graph. Then*

$$ind\ G = \sum_{i=1}^{n} \{\frac{1}{2}|\partial D_i| - 1\} \tag{6.6}$$

**Proof** By assumption, each cycle $\partial D_i$ has at least $\frac{1}{2}|\partial D_i| - 1$ free edges. Choose arbitrarily $\frac{1}{2}|\partial D_i| - 1(= \lambda_i)$ free edges $e_{i,1}, \ldots, e_{i,\lambda_i}$ from $\partial D_i$, $i = 1, 2, \ldots, n$. We claim that the collection of these edges $\mathcal{S} = \{e_{1,1} \ldots, e_{1,\lambda_1}, \ldots, e_{n,1} \ldots, e_{n,\lambda_n}\}$ is a maximal set of independent singular edges in $G$.

However, first we claim that $\mathcal{S}$ is cyclically independent.

**Lemma 6.7** $\alpha(G) = \sum_{i=1}^{n}\{\frac{1}{2}|\partial D_i| - 1\}$.

**Proof** Denote $\rho = \sum_{i=1}^{n}\{\frac{1}{2}|\partial D_i| - 1\}$. First we show that $\alpha(G) \leq \rho$. Let $\mathcal{S}_0$ be a maximal set of cyclically independent singular edges in $G$. Then $|\mathcal{S}_0| = \alpha(G)$. $\mathcal{S}_0$ cannot contain more than $\frac{1}{2}|\partial D_i| - 1$ singular edges on $\partial D_i$ for each $i = 1, 2, \ldots, n$, and hence, a possible maximal number of singular edge in $\mathcal{S}_0$ is $\rho$. Therefore $\alpha(G) \leq \rho$.

Now to prove the reverse inequality, take, say $k$, edges $e_1, \ldots, e_k$ from $\mathcal{S}$. We need the following easy lemma.

**Lemma 6.8** *Let $D_{j_1}, \ldots, D_{j_\ell}$ be the bounded domains such that $\bigcup_{m=1}^{\ell} D_{j_m}$ is connected and has no cut-vertices. Then*

$$|\partial(\bigcup_{m=1}^{\ell} D_{j_m})| = \sum_{m=1}^{\ell} |\partial D_{j_m}| - 2(\ell - 1). \tag{6.9}$$

**Proof** Since $D_{j_p}$ and $D_{j_q}$ ($j_p \neq j_q$) have at most one edge in common, and since $\bigcup_{m=1}^{k} D_{j_m}$ is connected, an easy induction on $\ell$ proves (6.9). Details will be omitted. □

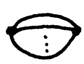
Now let $C$ be a simple cycle of $G$ of the smallest length on which all edges $e_1, \ldots, e_k$ occur. Suppose that the interior of $C$ consists of bounded domains, say $D_{\mu_1}, \ldots, D_{\mu_\ell}$, where $\mu_i \neq 0$ for $i = 1, 2, \ldots, \ell$. Then from Lemma 6.8, we see that

$$|C| = |\partial(\bigcup_{m=1}^{\ell} D_{\mu_m})| = \sum_{m=1}^{\ell} |\partial D_{\mu_m}| - 2(\ell - 1).$$

Since $e_j$, $j = 1, 2, \ldots, k$, is a free edge on some $D_{\mu_m}$ and the number of free edges of $\partial D_{\mu_m}$ is at least $\frac{1}{2}|\partial D_{\mu_m}| - 1$, it follows that

$$2k \leq \sum_{m=1}^{\ell} 2\{\frac{|\partial D_{\mu_m}|}{2} - 1\} = \sum_{m=1}^{\ell} \{|\partial D_{\mu_m}| - 2\} < |C|.$$

This proves that $e_1, \ldots, e_k$ are cyclically independent, and hence $\alpha(G) = \rho$. This proves Lemma 6.7. □

The final step of the proof of Theorem 6.5 is to show that the set of edges in $\mathcal{S}$ is independent in $G$. Now the smallest strongly excessive plane bipartite reducible graph is the multiple-edge graph  , for which Theorem 6.5 trivially holds. Furthermore, if $G$ is a $2k$-cycle, $k \geq 2$, then $\mathcal{S}$ is obviously independent in $G$. Therefore, we will prove the independence of $\mathcal{S}$ in $G$ by induction on $|E(G)|$. Suppose that $|E(G)| \geq 3$. We note that it suffices to show that there is an edge $e'$ in $\mathcal{S}$ such that at least one end of $e'$, say $v'$, is not an end of any (free) edge in $\mathcal{S}$. In fact, then, $G' = G/star\, v'$ is a strongly excessive plane bipartite reducible graph with $|E(G')| < |E(G)|$. Therefore, by induction on $|E(G)|$, $\mathcal{S}' = \mathcal{S} - \{e'\}$ is independent in $G'$ and hence $\mathcal{S}$ is independent in $G$.

Now to show the existence of such a free edge $e'$ in $\mathcal{S}$, we consider the subgraph $G_0$ of $G$ obtained from $G$ by removing all but one edge connecting two vertices. $G_0$ is a spanning subgraph of $G$, and furthermore, $G_0$ is strongly excessive. Consider the dual graph $G_0^*$ of $G_0$. Since $G$ is reducible, there is a vertex $\hat{v}$ such that $G^* - \hat{v}$ is a tree. But from our construction of $G_0$, we see easily that $G_0^* - \hat{v}$ is also a tree, say $T_0$.

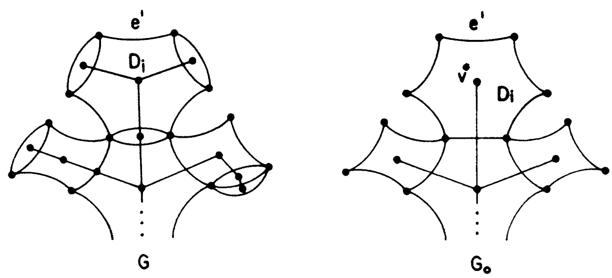



Fig. 6.2

Take a stump $v^*$ in $T_0$. Let $D_i$ be the domain corresponding to $v^*$. Then all but one edge of $\partial D_i$ are free in $G_0$ and hence $\frac{|\partial D_i|}{2} - 1$ free edges of $\partial D_i$ belong to $\mathcal{S}$. Note that any domain in $\mathbb{R}^2 - G_0$ has more than 2 sides, and hence $\frac{|\partial D_i|}{2} - 1 \neq 0$. Therefore there is a free edge $e'$ on $\partial D_i$ which satisfies our requirements.

This proves Theorem 6.5. □

# Chapter II. Link Theory

## §7 Preliminaries and the index of a link

This chapter will be devoted to the application of results obtained in Chapter I to knots or links in $S^3$. In particular, we will utilize these results to determine the braid index of many links.

We consider only oriented links in this chapter, and hence, a diagram $D$ of a link $L$ has always orientation which is induced from that of $L$. We assign $+1$ or $-1$ called the sign $w(c)$, to each crossing $c$ as is depicted in Fig. 7.1. A crossing $c$ is said to be *positive* (or *negative* ) if $w(c) > 0$ (or $w(c) < 0$ ).


Fig. 7.1

$n_+(D)$ and $n_-(D)$ will denote, respectively, the number of positive crossings and negative crossings in $D$. Therefore $n(D) = n_+(D) + n_-(D)$ is the total number of crossings in $D$. On the other hand, the sum of all signs on $D$, $\tilde{n}(D) = \sum_{c \in D} w(c)$, is called the *Tait number* of $D$. Note that $\tilde{n}(D) = n_+(D) - n_-(D)$.

Now if we split $D$ at each crossing of $D$ according to the orientation of $D$, $D$ is decomposed into finitely many simple closed curves, called *Seifert circles* of $D$. Using Seifert circles of $D$, we can associate with $D$ a signed graph $\Gamma(D)$ as follows. Each vertex of $\Gamma(D)$ corresponds to each Seifert circle of $D$, and each edge of $\Gamma(D)$ corresponds to each crossing of $D$. The ends of an edge $e$ of $\Gamma(D)$ correspond to Seifert circles connected by the crossing corresponding to $e$. See Fig. 7.2.

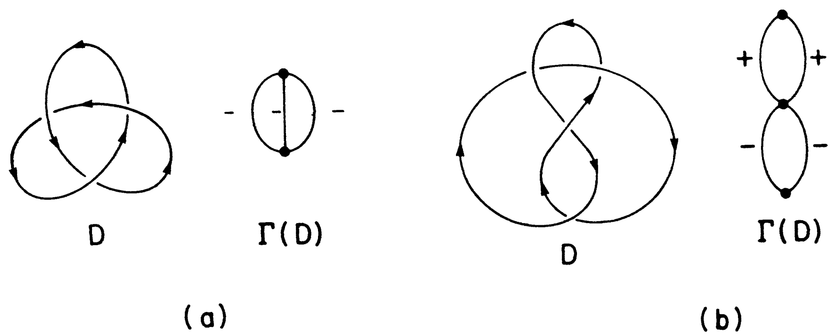



Fig. 7.2

The sign of edge of $\Gamma(D)$ is the same as that of the corresponding crossing of $D$. A signed graph $\Gamma(D)$ is called a *Seifert graph* of $D$. It is easy to see that $\Gamma(D)$ is a planar bipartite graph.

We should note that two non-equivalent links can have link diagrams that associate with the same Seifert graph.

A link diagram $D$ of a link $L$ is called a *positive* (or *negative*) diagram if each crossing of $D$ is positive (or negative). Therefore $D$ is positive iff the signed graph $\Gamma(D)$ is a positive (or negative) graph.

A link $L$ is called a positive (or negative) link if $L$ admits a positive (or negative) diagram. A crossing $c$ in $D$ is said to be *nugatory* or *removable* if a smoothing at $c$ makes $D$ split, or equivalently, if the edge in $\Gamma(D)$ corresponding to $c$ is an isthmus. A link diagram $D$ is called *reduced* if $D$ has no nugatory crossings.

Let $D$ be a link diagram of $L$. Let $\mathcal{S} = \{S_1, \dots, S_m\}$ be the set of all Seifert circles of $D$. If for each $i = 1, 2, \dots, m$, at least one of the connected components of $\mathbb{R}^2 - S_i$ does not have Seifert circles, then $D$ is called a *special* diagram. Any link has at least one special diagram [BZ, Mu 1]. If $D$ is a special alternating diagram, then the associated Seifert graph $\Gamma(D)$ is a plane and bipartite positive (or negative) graph, and conversely, and furthermore $\Gamma(D)$ determines $D$.

**Definition 7.1** For an oriented diagram $D$ of an (oriented) link $L$, we define

$$ind\ D = ind\ \Gamma(D).$$

Similarly, we define $ind_+D = ind_+\Gamma(D)$ and $ind_-D = ind_-\Gamma(D)$.

Suppose that $\Gamma(D) = \Gamma_1 * \Gamma_2$. Then $\Gamma_i(i = 1,2)$ uniquely determines a link diagram, denoted by $D_i$, and we say that $D$ is a planar *star* (or *Murasugi) product* of $D_1$. and $D_2$, denoted by $D = D_1 * D_2$. In particular, $D_1$ and $D_2$ have one Seifert circle in common.

A link diagram $D$ is written (not necessarily uniquely) as a $*$-product of finitely many special link diagrams $D_1, \ldots, D_m$ [Mu 1]. Therefore, the Seifert graph $\Gamma(D)$ is written as $\Gamma(D) = \Gamma_1 * \cdots * \Gamma_m$, where $\Gamma_i$ is the Seifert graph of $D_i$. Since $D_i$ is a link diagram, $\Gamma_i$ is bipartite. If each $D_i$ is either a positive or negative diagram, then $D$ is called a homogeneous diagram [C]. If a link admits a homogeneous diagram, it is called a *homogeneous link.* An alternating diagram is homogeneous, but not conversely.

Now suppose $D = D_1 * D_2$. Since the Seifert graph $\Gamma_i$ of $D_i$ is bipartite, Theorem 2.4 implies the following proposition.

**Proposition 7.2** *Let $D$ be a link diagram and $D = D_1 * D_2$. Then $ind\,(D_1 * D_2) = ind\,D_1 + ind\,D_2$. If $D$ is a homogeneous diagram, then $ind\,D = ind_+D + ind_-D$.*

**Proof** If $D$ is a homogeneous diagram, then $D$ is written as $D_1 \cdots * D_p * D'_1 * \cdots * D'_n$, where $D_i(i = 1,2,\ldots,p)$ is positive and $D'_j$ $(j = 1, 2, ..., n)$ is negative. Since $\sum_{i=1}^{p} ind\ D_i = ind_+D$ and $\sum_{j=1}^{n} ind\,D'_j = ind_-D$, the second equality follows. □

Now to each oriented link $L$ the integer polynomial $P_L(v,z)$, called the *skein* (named also Homfly, Flypmoth, generalized Jones, 2-variable Jones, Jones-Conway, twisted Alexander, oriented) polynomial is defined recursively as follows [FY, LM, PT].

Let $D_+, D_-, D_0$ be the diagrams of links which are identical except in the neighborhood of a crossing, where they look like

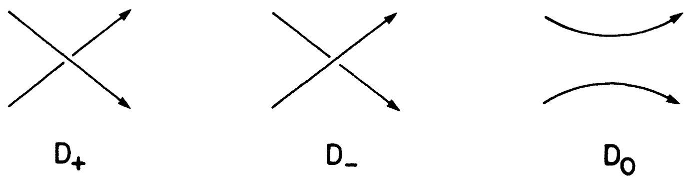



Fig. 7.3

then $P_L(v,z)$ satisfies the following formula

(7.3) (1) $\frac{1}{v}P_{D_+}(v,z) - vP_{D_-}(v,z) = zP_{D_0}(v,z)$

(2) If $L$ is a trivial knot then $P_L(v,z) = 1$.

We denote $P_D(v,z)$ for $P_L(v,z)$, if necessary, to emphasize that a diagram $D$ has been used to evaluate $P_L(v,z)$. We will call the equation (7.3)(1) the *skein relation* in this paper.

$P_L(v,z)$ is an integer Laurent polynomial on two variables $v$ and $z$. Generally, let $f(v,z)$ be an integer Laurent polynomial on $v, z$, i.e. $f(v,z) \in \mathbb{Z}[v, v^{-1}, z, z^{-1}]$. We write $f(v,z) = \sum_{i=a}^{b} \phi_i(z)v^i$, where $\phi_a(z) \neq 0 \neq \phi_b(z)$, $a \leq b$ and $\phi_i(z) \in \mathbb{Z}[z, z^{-1}]$. We denote

(7.4)
$$b = max\, deg_v\ f(v,z)$$
$$a = min\, deg_v\ f(v,z)$$
$$b - a = v - span\ f(v,z)$$

Similarly, we can define $max\, deg_z\ f(v,z)$, $min\, deg_z f(v,z)$ and $z - span\ f(v,z)$. Furthermore, let $A_\alpha z^\alpha$ and $B_\beta z^\beta$ be, respectively, the highest terms in $\phi_a(z)$ and $\phi_b(z)$. Then, we define

(7.5) $B_\beta v^b z^\beta = max - max\ f(v,z)$ and $A_\alpha v^a z^\alpha = min - max\ f(v,z)$.

These terms are called the *extremal terms* of $f(v,z)$.

**Example 7.6** Let $f(v,z) = (z^3 - 2z^5 + 3z^7)v - z^{-1}v^3 + (z + 2z^3)v^5$. Then $max - max\ f(v,z) = 2z^3v^5$, while $min - max\ f(v,z) = 3z^7v$.

## §8 Improvement of Morton-Frank-Williams inequalities

Let $D$ be an oriented link diagram of $L$. Let $s(D)$ be the number of Seifert circles in $D$. We begin with the following well-known theorem.

**Theorem 8.1** [FW, Mo 2] *For any link diagram $D$ of a link $L$,*

$$(8.2)\quad \tilde{n}(D) - s(D) + 1 \leq \ min\,deg_v\ P_L(v,z) \leq \ max\,deg_v\ P_L(v,z) \leq \tilde{n}(D) + s(D) - 1.$$

Equalities in either side hold for some links, but for many links, inequalities are sharp.

In this section we will prove a considerable improvement of these inequalities which, combined with Yamada's Theorem [Y], enables us to determine the braid index of many links. In fact, we prove

**Theorem 8.3** *For any link diagram $D$ and the associated Seifert graph $\Gamma(D)$, we have*

$$(8.4)\qquad max\,deg_v\ P_L(v,z) \leq \tilde{n}(D) + s(D) - 1 - 2\ ind_+\Gamma(D), \qquad and$$

$$min\,deg_v\ P_L(v,z) \geq \tilde{n}(D) - s(D) + 1 + 2\ ind_-\Gamma(D)$$

*and hence*

$$(8.5)\qquad v - span\ P_L(v,z) \leq 2\{s(D) - 1 - ind_+\Gamma(D) - ind_-\Gamma(D)\}.$$

Now, to prove Theorem 8.3, the following lemma is crucial.

**Lemma 8.6** *Given an oriented link diagram $D$ of a link $L$, there are new link diagrams $D', D''$ and $D'''$ of $L$ such that*

$$(8.7)\qquad \begin{array}{lllll} (1) & \tilde{n}(D') = \tilde{n}(D) - ind_+(D) & and & s(D') = s(D) - ind_+(D), \\ (2) & \tilde{n}(D'') = \tilde{n}(D) + ind_-(D) & and & s(D'') = s(D) - ind_-(D) \\ (3) & s(D''') = s(D) - ind(D) & & \end{array}$$

*and hence*

$$\mathbf{b}(L) \le s(D) - ind(D) \tag{8.8}$$

**Remark 8.9** (1) It may not exist a diagram $D'$ such that $s(D') = s(D) - (ind_+(D) + ind_-(D))$. (2) If $D$ is an alternating diagram, then $ind\ D = ind_+ D + ind_- D$ and we can choose $D'''$ so that $\tilde{n}(D''') = \tilde{n}(D) - ind_+\Gamma(D) + ind_-\Gamma(D)$.

**Proof of Lemma 8.6** First we note that it suffices to prove (8.7) (1).

Write $D = D_1 * D_2 * \cdots * D_m$ as a $*$-product of $D_i$. Then $\Gamma(D) = \Gamma(D_1) * \cdots * \Gamma(D_m)$. Since $\Gamma(D_i)$ is a plane bipartite graph, we have $ind_+(D) = \sum_{i=1}^{m} ind_+(D_i)$.

Now consider $\Gamma(D_1)$. If $ind_+(D_1) = 0$, we have nothing to do on $D_1$. Suppose $ind_+ D_1 = k > 0$. Then there exists a singular positive edge $e$ and a vertex $v$, one of two ends of $e$, such that $ind_+\big(\Gamma(D_1)/star\ v\big) = k - 1$. $e$ corresponds to a crossing $c$ of $D_1$.

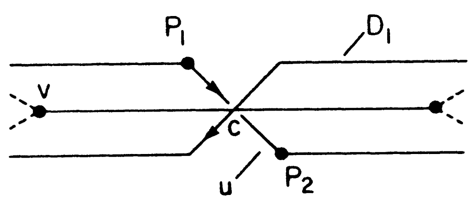



Fig. 8.1

Let $u$ be a small part of $D_1$ that crosses under the other part of $D_1$ at $c$. Let $P_1$ and $P_2$ be the end points of $u$. See Fig. 8.1. We will deform isotopically the short path $u$ to a long under-crossing path $\ell$.

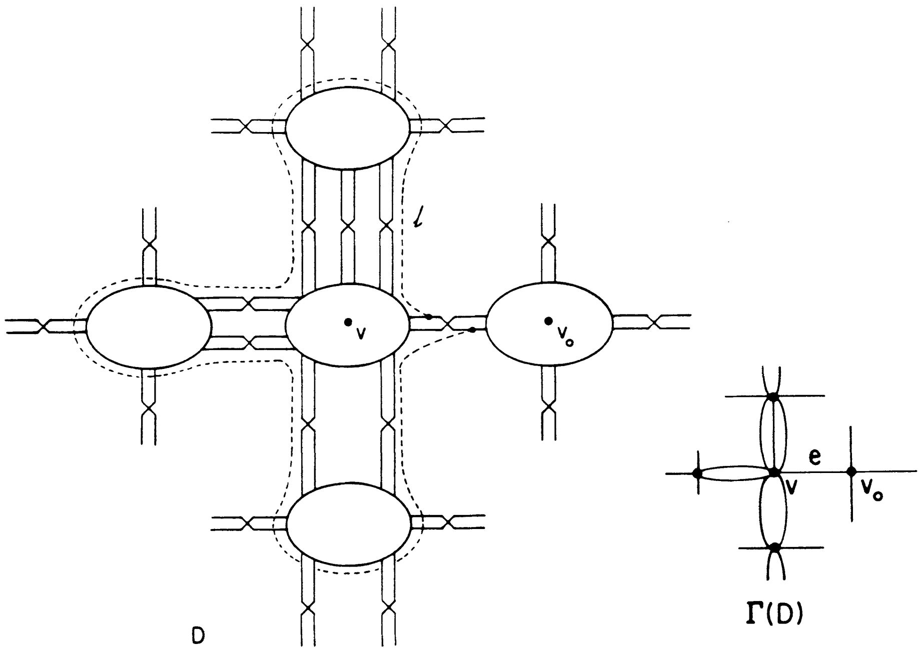



Fig. 8.2

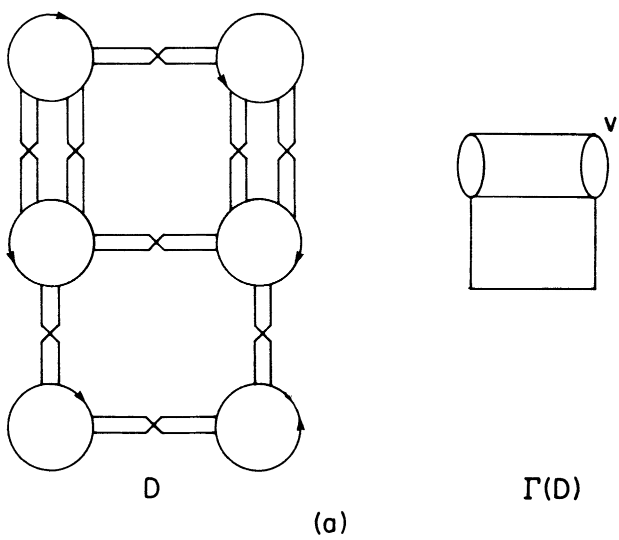
$\ell$ is depicted by a dotted line in Fig. 8.2. $\ell$ crosses under those "bands" which are *not* connected to $v$. To be more precise, let $v_0, v_1, \ldots, v_r$ be vertices in $\Gamma(D)$, each of which is connected to $v$, where $v_0(\neq v)$ is another end of $e$. Then $\ell$ is a path which does not intersect any "bands" in $D$ corresponding to edges connecting $v$ and $v_j$, $j = 0, 1, \ldots, m$, except $P_1$ and $P_2$, but $\ell$ crosses under all "bands" that connect $v_j (j \neq 0)$ and other vertices $(\neq v)$ at the place close to the Seifert circle represented by $v_j$. Since $D_1$ is bipartite, $v_i$ and $v_j$ $(0 \neq i \neq j \neq 0)$ are not connected by any "bands".

In this new diagram $D_1'$, two Seifert circles represented by $v$ and $v_0$ are amalgamated to one circle and hence $s(D_1') = s(D_1) - 1$.

Now we see that $\Gamma(D_1')$ is the one-point union of $\Gamma(D_1)/star\ v$ and some multiple-edge graph $K$, where $K$ contains *star* $v - e$ as a subgraph and $ind_+\Gamma(D_1') = k - 1$. We can repeat the same argument $k$ times so that finally $\Gamma(D_1)$ is reduced to the block sum of $\Gamma(D_1^{(k)})$ and $k$ multiple-edge graphs $K, K', \ldots, K^{(k-1)}$, where $ind_+\Gamma(D_1^{(k)}) = 0$. Apply the same argument on each $\Gamma(D_i)$ and eventually $\Gamma(D)$ is reduced to the block sum of $\Gamma(D_i^{(k_i)})$, $i = 1, \ldots, m$, where $ind\ \Gamma(D_i^{(k_i)}) = 0$ and multiple-edge graphs $K_i, K_i', \ldots, K_i^{(k_i-1)}$, $i = 1, 2, \ldots, m$.

The final link diagram $\hat{D}$ corresponding to this graph has $s(\hat{D}) = s(D) - \sum_{i=1}^{m} ind_+(D_i)$ and $\tilde{n}(\hat{D}) = \tilde{n}(D) - \sum_{i=1}^{m} ind_+(D_i)$. Since $\sum_{i=1}^{n} ind_+(D_i) = ind_+(D)$, $\hat{D}$ is what we sought. Since $\mathbf{b}(L) \leq s(D''')$, (8.8) follows from (8.7) (3). It completes a proof of Lemma 8.6.

**Example 8.10** The series of diagrams $(a)-(d)$ in Fig. 8.3 illustrates our proof of Lemma 8.6 for some link. A diagram $D$ has index 2 and $s(D) = 6$, but $s(\hat{D}) = s(D'') = 4$. Note that $D'$ and $\tilde{D}'$ are the same diagram.




(a)

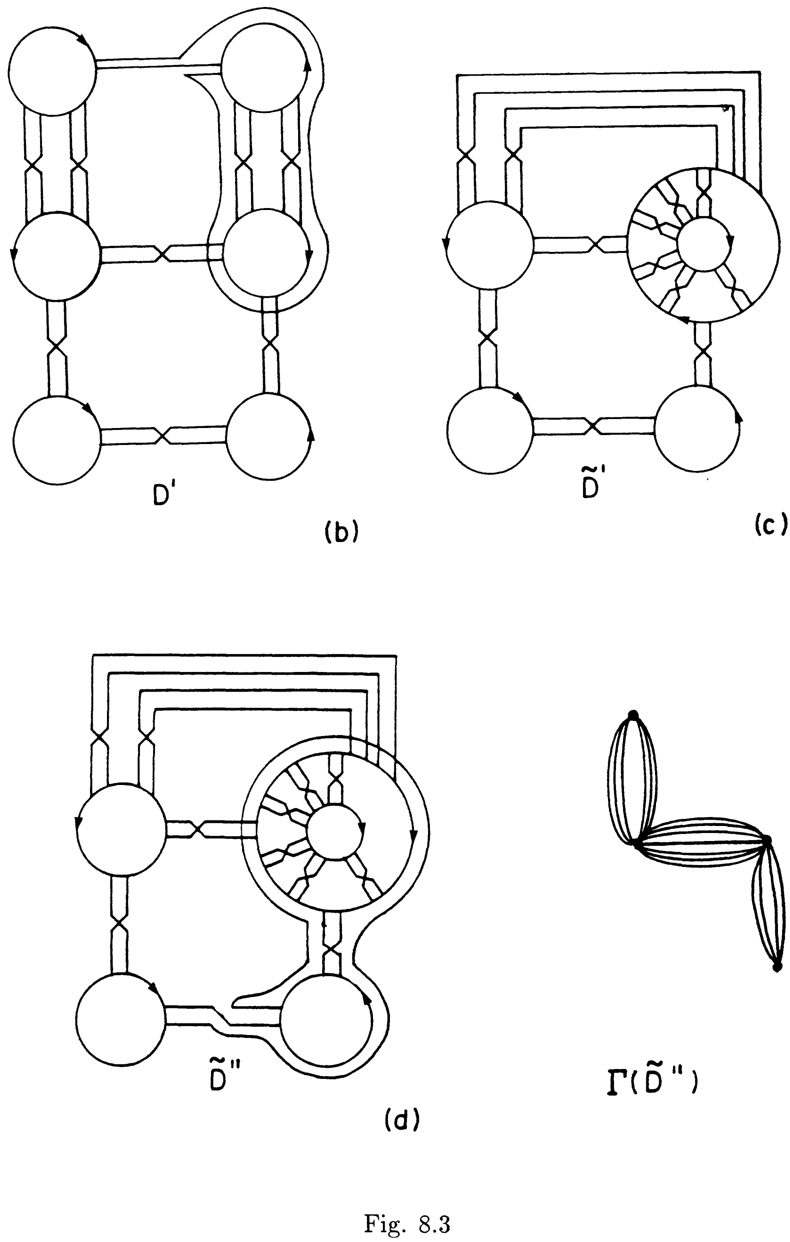



Fig. 8.3

We are now in position to prove Theorem 8.3. Using the diagrams $D'$ and $D''$ in Lemma 8.6, we have from Theorem 8.1

$$max\, deg_v\ P_L(v,z) \leq \tilde{n}(D') + s(D') - 1 = \tilde{n}(D) + s(D) - 1 - 2\ ind_+\Gamma(D), \quad \text{and}$$

$$min\, deg_v\ P_L(v,z) \geq \tilde{n}(D'') - s(D'') + 1 = \tilde{n}(D) - s(D) + 1 + 2\ ind_-\Gamma(D)\ .$$

It proves Theorem 8.3. □

As a consequence of Theorem 8.3, we have

**Corollary 8.11** *Suppose that the equalities hold in (8.4). Then if* $ind\ D = ind_+D + ind_-D$ *, we have*

$$\mathbf{b}(L) = s(D) - ind\ D.$$

**Proof** It follows from Theorem 8.1 and the theorem in [Y]

$$v - span\ P_L(v,z) \leq 2\{\mathbf{b}(L) - 1\},$$

and hence $s(D) - 1 - ind\ D \leq \mathbf{b}(L) - 1$, i.e. $s(D) - ind\ D \leq \mathbf{b}(L)$. However, Lemma 8.6 shows that there is a diagram $D'''$ of $L$ such that $s(D''') = s(D) - ind\ D$, and hence, $s(D''') \leq \mathbf{b}(L)$. Since $\mathbf{b}(L) \leq s(\tilde{D})$ for any diagram $\tilde{D}$ of $L$, it follows that $\mathbf{b}(L) = s(D''') = s(D) - ind\ D$. □

Proposition 7.2 now implies the following theorem.

**Theorem 8.12** *Let* $L$ *be a homogeneous (or, in particular, an alternating ) link. If the equalities hold in (8.4), then we have*

$$\mathbf{b}(L) = s(D) - ind\ D.$$

**Remark 8.13** The converse of Theorem 8.12 need not be true even for alternating links. See §15 for an example.

## §9 Extremal terms of $P_L(v,z)$

The second, but important step to determine the braid index is careful evaluations of $z$-degrees of some terms in $P_L(v,z)$. It is already known [LM, Mo 1] that $mindeg_z P_L(v,z) = -(\mu - 1)$ for any $\mu$ component link $L$ and $max\,deg_z\ P_L(v,z) \leq n(D) - s(D) + 1$ for any diagram $D$ of $L$. However, we need the maximal $z$-degree among the terms in $P_L(v,z)$ with the maximal $v$-degree or the minimal $v$-degree. After we have determined the $z$-degrees of these terms, we are able to prove that the equality holds in (8.4) for an alternating link with $ind\ D = 0$, and hence, the braid index has been completely determined for these links. See Theorem 9.5.

We now begin with a few definitions.

Let $D$ be an oriented link diagram of $L$.

**Definition 9.1** $J(D)$ denotes the number of pairs of Seifert circles of $D$ which are connected by a crossing. In other words, $J(D)$ is the number of those pairs of vertices in the Seifert graph $\Gamma(D)$ which are connected by at least one edge. Similarly $J_+(D)$ (or $J_-(D)$ ) denote the number of those pairs of Seifert circles of $D$ which are connected by at least one positive (or negative) crossings.

We use the following notation in the rest of the paper.

(9.2) For a link diagram $D$,

$$\begin{cases} \phi_+(D) &= \tilde{n}(D) + s(D) - 1, \text{ and} \\ \phi_-(D) &= \tilde{n}(D) - s(D) + 1, \end{cases}$$
$$\begin{cases} \psi_+(D) &= n(D) - s(D) + 1 - 2J_+(D) \text{ and} \\ \psi_-(D) &= n(D) - s(D) + 1 - 2J_-(D). \end{cases}$$

Now we write

$$P_L(v,z) = \sum_{i,j} c_{ij} v^i z^j = \sum_i a_i(z) v^i$$

where $a_i(z)$ is a Laurent polynomial in $z$ and $c_{ij}$ is an integer. Sometimes we write $c_{ij}(D)$ (or $c_{ij}(L)$) for $c_{ij}$ to emphasize the diagram $D$ (or a link $L$).

The main purpose of this section is to prove the following theorems.

**Theorem 9.3** *For any oriented link diagram $D$ of a link $L$,*

$$\text{(9.4)} \qquad \begin{aligned} &(i) \quad max\,deg_z\ a_{\phi_+(D)}(z) \leq \psi_+(D) \\ &(ii) \quad max\,deg_z\ a_{\phi_-(D)}(z) \leq \psi_-(D). \end{aligned}$$

Note that $max\,deg_v\ P_L(v,z) \leq \phi_+(D)$ and $mindeg_v\ P_L(v,z) \geq \phi_-(D)$.

The next theorem shows that the equalities hold in (9.4) for alternating links with index 0.

**Theorem 9.5** *Suppose that $D$ is an alternating link diagram. Then*

$$\text{(9.6)} \qquad \begin{aligned} &(i) \quad c_{\phi_+(D),\psi_+(D)} = \begin{cases} (-1)^{n_-(D)+s(D)-1} & \text{if } ind_+(D) = 0 \\ 0, & \text{otherwise} \end{cases} \\ &(ii) \quad c_{\phi_-(D),\psi_-(D)} = \begin{cases} (-1)^{n_-(D)} & \text{if } ind_-(D) = 0 \\ 0, & \text{otherwise} \end{cases} \\ &(iii) \quad \mathbf{b}(L) = s(D) \quad iff \quad ind\ D = 0. \end{aligned}$$

**Remark 9.7** (1) If $ind_+D \neq 0$, then it follows from Theorem 8.3 that $a_{\phi_+(D)}(z) = 0$ and hence (9.4) (i) holds trivially. Similarly, if $ind_-D \neq 0$, then (ii) holds trivially, since $a_{\phi_-(D)}(z) = 0$. Therefore, we may assume henceforth that $ind_+D = 0$ in a proof of (i). This assumption makes our proof considerably simpler. (2) If $D$ is an alternating diagram, then whenever two Seifert circles are connected by crossings, these crossings are either all positive or all negative. (3) Theorem 9.5 will be extended to some non-alternating links. See Theorem 10.8.

Now, to prove Theorems 9.3 and 9.5, we need the following crucial lemma (which is a special case of Theorem 9.3).

**Lemma 9.8** *Let $D$ be a positive diagram of an oriented link. Then*

$$max\,deg_z\ a_{\phi_+(D)}(z) \leq \psi_+(D).$$

Before we proceed to prove Lemma 9.8, we give a few remarks. Firstly, (9.6) (iii) follows from (9.6) (i) and (ii) by Corollary 8.11. Secondly, it suffices to prove (9.4) (i) and

(9.6) (i). In fact, (9.4) (ii) follows from (9.4) (i) by considering the mirror image $\overline{D}$ of $D$ and by applying (9.4) (i) on $\overline{D}$. Similarly, (9.6)(ii) follows from (9.6) (i). Therefore we only need to prove (9.4) (i) and (9.6) (i), which, in fact, follows from Lemma 9.8.

**Proof. Lemma 9.8 $\Longrightarrow$ (9.4) (i)** We proceed by induction on $n_-(D)$. If $n_-(D) = 0$, then (9.4) (i) is Lemma 9.8 itself. Now let $c$ be a negative crossing in $D$. Denote by $D_+^c$ the diagrams obtained from $D$ by changing the crossing at $c$, and $D_0^c$ the diagram obtained from $D$ by smoothing the crossing $c$. Then the skein equation (7.3) (1) yields

$$a_{\phi_+(D_+^c)}(z) - a_{\phi_+(D)}(z) = z a_{\phi_+(D_0^c)}(z). \tag{9.9}$$

On the other hand, since $n(D_+^c) = n(D) = n(D_0^c) + 1$ and $s(D_+^c) = s(D) = s(D_0^c)$ and $J_+(D_+^c) \geq J_+(D) = J_+(D_0^c)$, we see that $\psi_+(D_+^c) \leq \psi_+(D)$ and $\psi_+(D_0^c) + 1 = \psi_+(D)$. Therefore (9.9) and the induction hypothesis for $D_+^c$ and $D_0^c$ yield $max\, deg_z\, a_{\phi_+(D)}(z) \leq \psi_+(D)$. $\square$

**Proof. Lemma 9.8 $\Longrightarrow$ (9.6)(i)** If $n_-(D) = 0$, then $D$ is a collection of disjoint special alternating diagrams. Then (9.6) (i) follows from Theorem 8.1 in [Mu 4]. (Another proof can be found in Theorem 10.8.) We now proceed by induction on $n_-(D)$. Suppose $n' = n_-(D) \neq 0$. Let $c_1, c_2, \ldots, c_{n'}$ be all negative crossings of $D$. Consider the binary resolving tree of $D$ using $c_1, \ldots, c_{n'}$. See Fig. 9.1

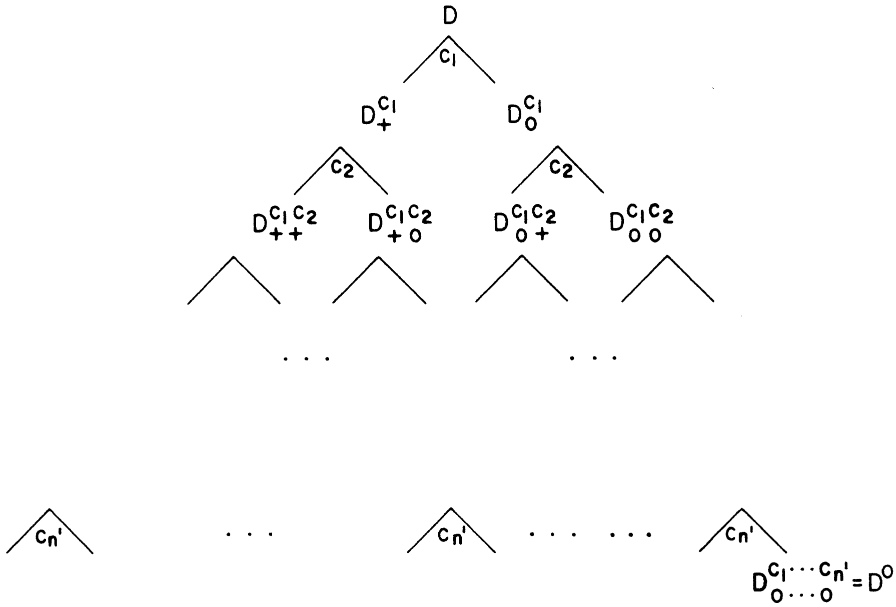



Fig. 9.1

where, for example, $D^{c_1c_2c_3}_{0++}$ denotes the diagram obtained from $D$ by smoothing $c_1$ and changing crossings at $c_2$, and $c_3$. Thus $D^0 = D^{c_1\dots c_{n'}}_{0\dots 0}$ is the leaf (i.e. diagram) obtained from $D$ by smoothing only. Let $D^\ell$ denote any leaf different from $D^0$. Note that $D^0$ and $D^\ell$ are positive. Now since $D^0$ is positive and alternating, the induction hypothesis yields

$$max\,deg_z\ a_{\phi_+(D^0)}(z) = \psi_+(D^0).$$

Since $J_+(D^0) = J_+(D)$ and $ind_+(D^0) = ind_+(D) = 0$ by Remark 9.7 (1), we see that

$$\psi_+(D^0) + n_-(D) = \psi_+(D^0) + n(D) - n_+(D) = \psi_+(D).$$

Since $\phi_+(D^0) - n_-(D) = \phi_+(D)$, $D^0$ contributes a maximal term $z^{\psi_+(D)}$ in $a_{\phi_+(D)}(z)$. On the other hand, the maximal term in $a_{\phi_+(D)}(z)$ $D^\ell$ can contribute is

$z^{\psi_+(D^\ell)+n(D)-n(D^\ell)}$. However, since $J_+(D^\ell) > J_+(D)$, we see that $\psi_+(D^\ell)+n(D)-n(D^\ell) < \psi_+(D)$. Therefore, only $D^0$ can contribute the maximal term $z^{\psi_+(D)}$ in $a_{\phi_+(D)}(z)$. Finally, using the induction hypothesis, we can see that

$$c_{\phi_+(D),\psi_+(D)}(D) = (-1)^{n_-(D)} c_{\phi_+(D),\psi_+(D)}(D^0) = (-1)^{n_-(D)+s(D)-1}.$$

It proves (9.6) (i) □

Now to prove Lemma 9.8 by induction, we need a slightly more general formulation of the lemma.

A (connected) arc $\gamma$ (with a base point if $\gamma$ is closed) in an oriented link diagram $D$ is called a *descending part* of $D$ if $\gamma$ satisfies the following property: If one travels along $\gamma$ (according to the orientation of $D$) starting from the beginning of $\gamma$ (or the base point), then each crossing which is met for the first time is crossed by an overcrossing. An oriented link diagram $D$ is called *quasi-positive* if there is a descending part of $D$ on which all negative crossings occur. A positive link diagram is quasi-positive.

**Example 9.10** A knot diagram $K_1$ and a 2-component link diagram $K_1 \cup K_2$ are quasi-positive, but a 3-component link diagram $K_1 \cup K_2 \cup K_3$ is not quasi-positive.

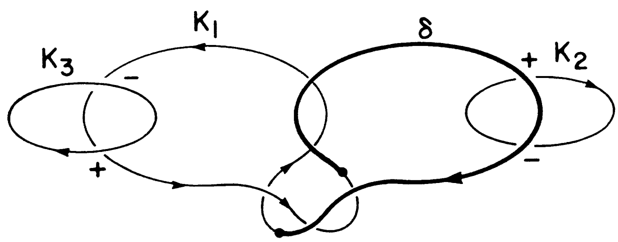



Fig. 9.2

Now Lemma 9.8 is replaced by the lemma below

**Lemma 9.11** *Let $D$ be a quasi-positive diagram of an oriented link. Then*

$$max\, deg_z\, a_{\phi_+(D)}(z) \leq \psi_+(D).$$

To prove Lemma 9.11, we need two technical lemmas whose proofs will be postponed to the appendix.

**Lemma 9.12** *Let $D$ be an oriented link diagram, $\hat{D}$ a simple closed curve that is a part of $D$, and $E = D - \hat{D}$. Then*

*(1)* $s(D) \geq s(E) + 1$ *(Cf. [Mo 1]).*

*(2)* *If* $s(D) = s(E) + 1$, *then* $J_+(D) \leq J_+(E) + \frac{1}{2}cr(\hat{D}, E)$,

*where $cr(\hat{D}, E)$ is the number of crossings between $\hat{D}$ and $E$.*

*(3)* *If $\hat{D}$ cuts each Seifert circle in $E$ and $s(D) = s(E)+1$, then the reduced Seifert graph of $D, \hat{\Gamma}(D)$, is a tree. Here $\hat{\Gamma}(D)$ is the graph obtained from the Seifert graph $\Gamma(D)$ by removing all but one edge connecting two vertices. In particular, if traveling along $D$ one leaves a Seifert circle $S$ for $S'$ then one goes again to $S$ through $S'$.*

A simple arc $\gamma$ in $D$ is called a *bridge* of $D$ if $\gamma$ never crosses under other parts of $D$.

**Lemma 9.13** *Let $D$ be an oriented link diagram. Suppose that there are three Seifert circles $S_0, S_1$ and $S_2$, and a (oriented) bridge $\gamma$ in $D$ such that*

*(1) there are crossings $p_1$ between $S_0$ and $S_1$ and $p_2$ between $S_0$ and $S_2$,*

*(2) $\gamma$ connects two points $q_1$ (close to $p_1$) and $q_2$ (close to $p_2$), but $\gamma$ never crosses $p_1$ and $p_2$, and*

*(3) $\gamma$ is disjoint from $S_0$. (See Fig. 9.3) Then*

$$max\, deg_v\ P_D(v, z) < \phi_+(D).$$

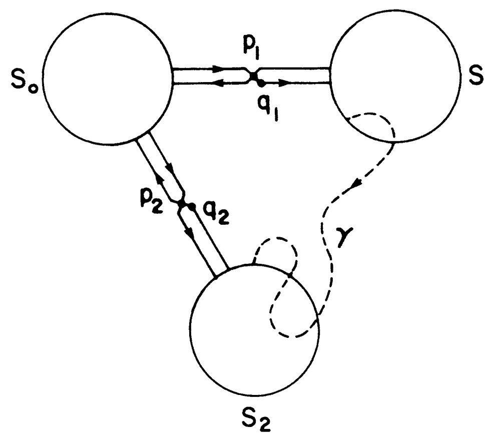



Fig. 9.3

Now we will prove Lemma 9.11 using Lemmas 9.12 and 9.13.

Since $D$ is quasi-positive, $D$ has at least one descending part on which all negative crossing occur. Let $b_\gamma(D)$ denote the number of crossings of $D$ which are not on a descending part $\gamma$. Define $b(D) = min\ b_\gamma(D)$, where the minimum is taken over all descending parts $\gamma$ of $D$ which contain all negative crossings. Now the proof of Lemma 9.11 will be given by induction on a pair $(n(D), b(D))$, ordered lexicographically.

If $n(D) = 0$, then $P_D(v,z) = \left(\frac{v^{-1}-v}{z}\right)^{s-1}$ and hence $a_{\phi_+(D)}(z) = (-z)^{-s+1}$. Therefore Lemma 9.11 holds for this case.

Suppose inductively that Lemma 9.11 holds for any link diagram $D'$ with

$$(n(D'),\ b(D')) < (n(D),\ b(D)).$$

Let $\gamma$ be a descending part of $D$ on which all negative crossings occur and $\gamma$ is maximal in the sense that the number of crossings which do not occur on $\gamma$ is minimal (i.e. equal to $b(D)$).

There are three cases to be considered.

**Case 1** $\gamma$ contains self-crossings.

Let $p$ be the *first* self-crossing of $\gamma$.

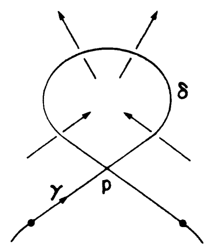



Fig. 9.4

Let $\delta$ be a simple closed curve containing $p$. Obviously, $\delta$ is eliminated by an isotopy to obtain a new quasi-positive diagram $E$. Now (an obvious modification of) Lemma 9.12 (1) yields $s(E) \leq s(D) - 1$.

We now need to consider three cases:

(a) $p$ is positive. Then $\tilde{n}(E) = \tilde{n}(D) - 1$ and hence $\phi_+(E) \leq \phi_+(D) - 2$. Therefore $a_{\phi_+(D)}(z) = 0$.

(b) $p$ is negative, and $s(E) < s(D) - 1$. Then $\tilde{n}(E) = \tilde{n}(D) + 1$ and hence $\phi_+(E) < \phi_+(D)$. Therefore again $a_{\phi_+(D)}(z) = 0$.

(c) $p$ is negative and $s(E) = s(D) - 1$. Then $\phi_+(E) = \phi_+(D)$. Since $E$ is quasi-positive, it follows from induction hypothesis that $max\, deg_z\ a_{\phi_+(E)}(z) \leq n(E) - s(E) + 1 - 2J_+(E)$. On the other hand, by Lemma 9.12 (2), we have

$$\begin{aligned} J_+(D_0^p) &\leq J_+(E) + \frac{1}{2}cr(D_0^p - E, E) \\ &= J_+(E) + \frac{1}{2}(n(D) - n(E) - 1). \end{aligned}$$

However, since $J_+(D) = J_+(D_0^p)$, $p$ being a negative crossing, it follows that

$$\begin{aligned} max\, deg_z a_{\phi_+(E)}(z) &\leq n(E) - s(E) + 1 - 2J_+(D) + n(D) - n(E) - 1 \\ &= n(D) - s(D) + 1 - 2J_+(D) = \psi_+(D). \end{aligned}$$

This proves Lemma 9.11 for Case 1.

**Case 2** $\gamma$ is a simple closed curve.

We can use a similar argument employed in the first case. Since $\tilde{n}(D-\gamma)=\tilde{n}(D)$, a direct application of Lemma 9.12 proves the lemma. The details will be omitted.

**Case 3** $\gamma$ is a bridge on $D$.

Let $q$ be the last crossing over which $\gamma$ cannot be extended. (See Fig. 9.5)

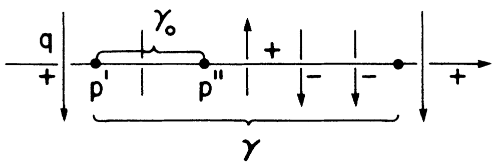



Fig. 9.5

Then $q$ is a positive crossing. Therefore we have an equation

$$v^{-1}P_D(v,z)-vP_{D^q_-}(v,z)=zP_{D^q_0}(v,z).$$

Since $q$ becomes a negative crossing in $D^q_-$ we can extend $\gamma$ a bit to $\gamma'$ contains the newly created negative crossing. Then, $b(D^q_-)<b(D)$ and hence $(n(D^q_-),b(D^q_-))<(n(D),b(D))$. The induction hypothesis now yields

$$\begin{aligned} max\, deg_z a_{\phi_+(D^q_-)}(z) &\le \psi(D^q_-) \\ &= n(D^q_-)-s(D^q_-)+1-2J_+(D^q_-) \\ &= n(D)-s(D)+1-2J_+(D^q_-). \end{aligned}$$

Therefore, if $J_+(D)=J_+(D^q_-)$, then

(9.14) $$max\, deg_z\ a_{\phi_+(D^q_-)}(z)\le\psi_+(D).$$

On the other hand, since $n(D^q_0)<n(D)$, it follows from the induction assumption that

$$\begin{aligned} max\, deg_z\ a_{\phi_+(D^q_0)}(z) &\le \psi_+(D^q_0) \\ &= n(D^q_0)-s(D^q_0)+1-2J_+(D^q_0) \\ &= n(D)-1-s(D)+1-2J_+(D^q_0). \end{aligned}$$

Therefore, if $J_+(D^q_0)=J_+(D)$, then we see that

(9.15) $$max\, deg_z\ a_{\phi_+(D^q_0)}\le\psi_+(D)-1.$$

Suppose now that $J_+(D^q_-) = J_+(D^q_0) = J_+(D)$ . Then $\psi_+(D) = \psi_+(D^q_-) = \psi_+(D^q_0) + 1$ , but the skein relation yields

$$a_{\phi_+(D)}(z) - a_{\phi_+(D^q_-)}(z) = za_{\phi_+(D^q_0)}(z)$$

and hence we have from (9.14) and (9.15)

$$max\, deg_z\ a_{\phi_+(D)}(z) \leq \psi_+(D).$$

If $J_+(D^q_-) = J_+(D^q_0) < J_+(D)$ , then either $q$ is the only crossing between $S_1$ and $S_2$ , and then by Theorem 8.3, $a_{\phi_+(D)}(z) = 0$ , or there are other negative crossings between $S_1$ and $S_2$ (and no positive crossings). Since $\gamma$ must cross other negative crossing between $S_1$ and $S_2$ , $\gamma$ eventually returns to $S_1$ . Consider the largest part of $\gamma$ , say $\gamma_0$ which starts at $p_1$ but does not cross any crossing connected to $S_1$ . Let $q'$ be the first crossing that prevents the extension of $\gamma_0$ any further. Then $q'$ must be a crossing connecting $S_1$ and another Seifert circle, say $S_0$ . Then $S_0 \neq S_2$ , otherwise $\gamma$ would cross under a negative crossing, since $S_1$ and $S_2$ are connected by negative crossing except $q$ . Let $p''$ be the terminal point of $\gamma_0$ . See Fig. 9.6.

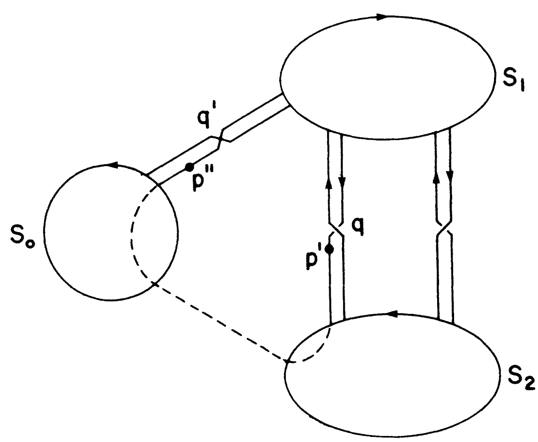



Fig. 9.6

Now $S_0, S_1, S_2$ and $\gamma_0$ satisfy all conditions in Lemma 9.13, putting $p' = q_1$, $p'' =$

$q_2$, $q = p_1$ and $q' = p_2$. Therefore, $a_{\phi_{+(D)}}(z) = 0$. A proof of Lemma 9.11 is now complete. □

## §10 Braid index of special alternating links

Since an alternating link is a $*$-product of special alternating links, it is natural to expect that the braid index of an alternating link is completely determined by those of its $*$-components. (See Conjecture 15.4.) Although this is not proved yet, the determination of the braid index of a special alternating link will be the first step toward the complete determination of the braid index of an alternating link. However, it is by no means easy to determine the braid index of a special alternating link.

In this section, we will determine the braid index of most of special alternating links whose alternating diagrams have index at most 2. In the case that diagrams have index 1, the only links that are left undecided are those whose diagrams have Seifert graphs without locally maximal subgraphs. In fact we prove the following theorem.

**Theorem 10.1** *Let $D$ be a special alternating diagram of a (special alternating) oriented link $L$. Let $\Gamma(D)$ be the Seifert graph associated with $D$. Then $\mathbf{b}(L) = s(D) - ind\ D$ if*

*(1) $ind\ D \leq 1$ and $\Gamma(D)$ has locally maximal subgraphs, or*

*(2) $ind\ D = 2$ and $\Gamma(D)$ has local maximal subgraphs, all of which have the same number of isthmuses (mod 2).*

Unfortunately, Theorem 10.1 cannot be extended immediately to an alternating link $L$ with $ind\ D \leq 2$. There are several difficulties which we must overcome in order to prove Theorem 10.1 for an alternating link. However, we have shown in Theorem 9.5. that if $ind\ D = 0$ then Theorem 10.1 holds for any alternating link.

Now we need a few preparations before we begin to prove Theorem 10.1.

First we note that if $D$ is a special diagram, then its Seifert graph $\Gamma(D)$ is a plane graph determined from $D$. In fact $\Gamma(D)$ coincides with the classical graph of a link [Ba, Mu 1].

Now a (local) deformation of a link diagram $D$ as shown in Fig. 10.1 (i.e. 180° rotation keeping end points $a, a', b, b'$ fixed) will be called the *Tait flype*.

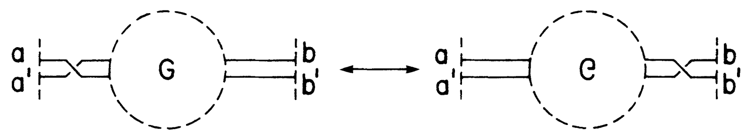



Fig. 10.1

The Tait flype preserves the isotopy class of a link, the property "being alternating", the Tait number $w(D)$ of $D$ and the number of Seifert circles of $D$.

By applying Tait flypes if necessary, we can transform a special link diagram $D$ into a *nice* special link diagram $D'$. A special diagram is said to be *nice* if a disk in $\mathbb{R}^2$ bounded by a 2-cycle $c = \{v_0, e_1, v_1, e_2, v_0\}$ in $\Gamma(D)$ has only edges (of the same sign) connecting two vertices $v_0$ and $v_1$.

**Lemma 10.2** *Any special diagram can be transformed into a nice special diagram by Tait flypes and obvious isotopy.*

**Proof** A proof is seen from Fig. 10.2 below. □

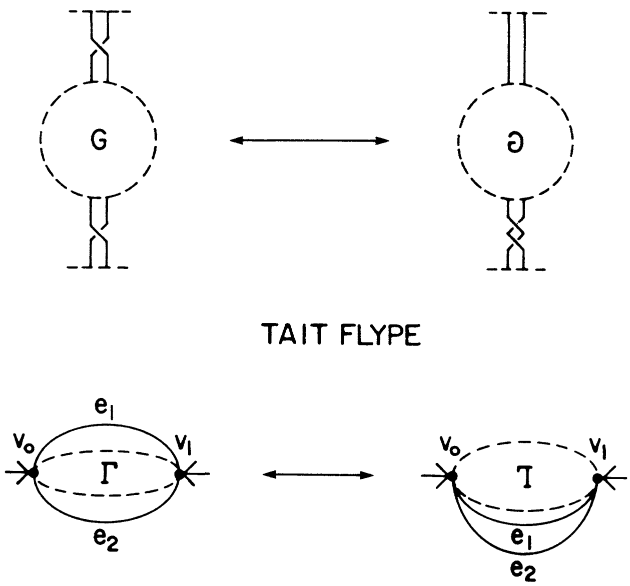



Fig. 10.2.

We may assume therefore that any special diagram is always nice.

Let $x$ and $y$ be two vertices of a signed graph $G$. Denote by $n_+(x,y)$ and $n_-(x,y)$, respectively, the number of positive and negative edges connecting $x$ and $y$. Let $n(x,y) = n_+(x,y) + n_-(x,y)$ and $\tilde{n}(x,y) = n_+(x,y) - n_-(x,y)$. Sometimes we write $\tilde{n}_G(x,y)$ for $\tilde{n}(x,y)$ to emphasize the graph $G$.

Now we define, for an integer $k$, a polynomial $w^{(k)}(z)$ as follows:

$$w^{(-1)}(z) = 1 \qquad \text{and} \qquad w^{(0)}(z) = 0. \tag{10.3}$$

For an integer $n \geq 1$, we define inductively

$$w^{(n)}(z) = zw^{(n-1)}(z) + w^{(n-2)}(z), \quad \text{and}$$

finally, we set, for $n \geq 0$,

$$w^{(-n)}(z) = w^{(n)}(-z).$$

If $n \neq 0$, then $w^{(n)}(z)$ is a polynomial of degree $|n|-1$. In particular, it is shown [P] that

(10.4) if $z = \sqrt{-1}(r + r^{-1})$, then $w^{(n)}(z) = (\sqrt{-1})^{n-1}(r^n - r^{-n})/(r - r^{-1})$.

Next, we need the following technical lemma.

**Lemma 10.5** *Let $D$ be a special diagram of a link $L$ and $\Gamma$ be a Seifert graph of $D$. Let $x$ and $y$ be two vertices of $\Gamma$ such that $\tilde{n}(x,y) \neq 1$. Define $\Gamma'$ as the graph obtained from $\Gamma$ by replacing all edges connecting $x$ and $y$ by a single positive edge. Let $\Gamma''$ be the graph obtained from $\Gamma$ by removing all edges connecting $x$ and $y$. Let $D'$ and $D''$ be the link diagrams associated with $\Gamma'$ and $\Gamma''$, respectively. For a diagram $D$, let $E(D) = \phi_+(D) - 2\, ind_+ D$. Suppose that $ind_+\Gamma' > ind_+\Gamma$. Then*

$$a_{E(D)}(z) = \begin{cases} w^{(\tilde{n}(x,y)-1)}(z) a_{E(D'')}(z) & \text{if } ind_+ D = ind_+ D'' \\ 0 & \text{if } ind_+ D < ind_+ D'' \end{cases} \tag{10.6}$$

Note that $ind_+ D \leq ind_+ D''$.

**Proof** By Lemma 10.2, we may assume that $n(x,y) = |\tilde{n}(x,y)|$. Applications of skein relations on the crossings corresponding to the edges connecting $x$ and $y$ yield the following formula. (Use an induction on $\tilde{n}(x,y)$.)

$$P_D(v,z) = v^{\tilde{n}(x,y)-1} w^{(\tilde{n}(x,y))}(z) P_{D'}(v,z) + v^{\tilde{n}(x,y)} w^{(\tilde{n}(x,y)-1)}(z) P_{D''}(v,z). \tag{10.7}$$

(Cf. [P, Theorem 1.1].)

Suppose that $ind_+ D < ind_+ D''$. Then

$$\begin{aligned} \phi_+(D) - 2\, ind_+ D &= \phi_+(D'') + \tilde{n}(x,y) - 2\, ind_+ D \\ &> \phi_+(D'') - 2\, ind_+ D'' + \tilde{n}(x,y). \end{aligned}$$

Since $ind_+\Gamma < ind_+\Gamma'$ by assumption, it follows that

$$\begin{aligned} \phi_+(D) - 2\, ind_+ D &= \phi_+(D') + \tilde{n}(x,y) - 1 - 2\, ind_+ D \\ &> \phi_+(D') - 2\, ind_+\Gamma' + \tilde{n}(x,y) - 1. \end{aligned}$$

Therefore, we have from (10.7)

$$max\,deg_v\ P_D(v,z) < \phi_+(D) - 2\ ind_+(D),$$

and hence $a_{E(D)}(z) = 0$. On the other hand, if $ind_+ D = ind_+ D''$, then

$$\phi_+(D'') - 2\ ind_+ D'' + \tilde{n}(x,y) > \phi_+(D') - 2\ ind_+ D' + \tilde{n}(x,y) - 1$$

and hence (10.7) yields

$$a_{E(D)}(z) = w^{(\tilde{n}(x,y)-1)}(z) a_{E(D'')}(z).$$

This proves Lemma 10.5. □

The following theorem, a generalization of Theorem 9.3 to a special (not necessarily alternating) link, is an easy consequence of Lemma 10.5. Therefore, a proof will be omitted.

**Theorem 10.8** *Let $D$ be a special diagram of an oriented link $L$. Then*

(1)
$$a_{\phi_+(D)}(z) = (-z)^{-s(D)+1} \prod_{(x,y)} w^{(\tilde{n}(x,y)-1)}(z),$$

*where the product is taken over all pairs of vertices in $\Gamma(D)$. In particular, $max\,deg_v\ P_D(v,z) = \phi_+(D)$ if and only if $\tilde{n}(x,y) \neq 1$ for every pair of vertices $x$ and $y$ in $\Gamma(D)$. Furthermore, if $max\ deg_v\ P_D(v,z) = \phi_+(D)$, then*

(2)
$$max\,deg_z\ a_{\phi_+(D)}(z) = \psi_+(D)\ \textit{and}$$

$$c_{\phi_+(D),\psi_+(D)} = (-1)^{n_-(D)-s(D)+1}$$

(3)
$$a_{\phi_-(D)}(z) = z^{-s(D)+1} \prod_{(x,y)} w^{(-\tilde{n}(x,y)-1)}(z)$$

*where $(x,y)$ runs over all pairs of vertices in $\Gamma(D)$. In particular $min\,deg_v\ P_D(v,z) = \phi_-(D)$ iff $\tilde{n}(x,y) \neq -1$ for every pair of vertices $x$ and $y$ in $\Gamma(D)$. Furthermore, if $max\,deg_v\ P_D(v,z) = \phi_-(D)$, then*

*(4)* $\max deg_z\, a_{\phi_-(D)}(z) = \psi_-(D)$ *and* $c_{\phi_-(D),\psi_-(D)} = (-1)^{n_-(D)}$

*(5)* $s(D) = \frac{1}{2}v - spanP_D(v,z) + 1$ *iff* $|\tilde{n}(x,y)| \neq 1$ *for every pair of vertices any* $x$ *and* $y$ *in* $\Gamma(D)$.

*(6)* $\mathbf{b}(L) = s(D)$ *iff* $|\tilde{n}(x,y)| \neq 1$ *for every pair of vertices* $x$ *and* $y$ *in* $\Gamma(D)$.

Now the rest of this section will be devoted to prove Theorem 10.1 for the case where $ind\ D = 1$ or 2.

If the Seifert graph of a special alternating diagram $D$ has a locally maximal subgraph, we will obtain a lot of information about its skein polynomial. In fact we can prove the following

**Theorem 10.9** *Let* $D$ *be a special alternating (positive) diagram of an oriented link. Then*

*(1)* $\max deg_v\, P_D(v,z) = \phi_+(D) - 2$ *iff* $ind\ D = 1$ *and* $\Gamma(D)$ *has locally maximal subgraphs. Therefore, if* $ind\ D = 1$ *and* $\Gamma(D)$ *has locally maximal subgraphs, then* $\mathbf{b}(L) = s(D) - 1$.

*(2) Suppose that a locally maximal subgraph of* $\Gamma(D)$ *is a single-edge graph. Then*

*(i)* $\max deg_z a_{\phi_+(D)-2}(z) = \psi_+(D) + 2$

*(ii)* $c_{\phi_+(D)-2,\psi_+(D)+2} = (-1)^{s(D)}$

*(iii) All the roots of* $a_{\phi_+(D)-2}\left(\sqrt{-1}\,(r + r^{-1})\right)$ *are roots of unity.*

*(3) If a locally maximal subgraph of* $\Gamma(D)$ *is of type* $H_1$, *then*

*(i)* $\max deg_z\, a_{\phi_+(D)-2}(z) = \psi_+(D) + E_{max}(D) + 2$, *where* $E_{max}(D)$ *is the maximal number of edges any locally maximal subgraph (or* $\Gamma(D)$*) of type* $H_1$ *can have.*

*(ii) Let* $\alpha_+(D) = \max deg_z a_{\phi_+(D)-2}(z)$. *Then*

$$c_{\phi_+(D)-2,\alpha_+(D)} = (-1)^{s(D)-1}\beta_{max}(D),$$

*where* $\beta_{max}(D)$ *is the number of locally maximal subgraphs (of* $\Gamma(D)$*) of type* $H_1$ *with* $E_{max}(D)$ *edges.*

**Proof** First we prove (2). Without loss of generality, we may assume that $D$ is a nice diagram. $\Gamma(D)$ has only one singular edge say $e_0$. If another edge $e$ and $e_0$ occur on a simple 4-cycle, then $\Gamma(D)$ would have a locally maximal subgraph of type $H_1$. Therefore, $e_0$ never lie on a simple 4-cycle. Then it follows from Lemma 10.5 that for adjacent vertices $x$ and $y$ (not connected by $e_0$) of $\Gamma(D)$ the following formula holds

$$a_{\phi_+(D)-2}(z) = w^{(k-1)}(z)a_{\phi_+(D'')-2}(z), \tag{10.10}$$

where $k = \tilde{n}(x,y)$. Using induction on $J_+(D)$, we have

$$\begin{aligned} max\ deg_z\ a_{\phi_+(D)-2}(z) &= k - 2 + max\ deg_z\ a_{\phi_+(D'')-2}(z) \\ &= k - 2 + \psi_+(D'') + 2 \\ &= k + n(D'') - s(D'') + 1 - 2J_+(D'') \\ &= n(D) - s(D) + 1 - 2(J_+(D) - 1) \\ &= \psi_+(D) + 2. \end{aligned}$$

Since the other parts of (2) also follow immediately from (10.10), we omit details.

**Proof of (3)** First suppose that $\Gamma(D)$ is $H_1^k,\ k \geq 1$. Then the skein relation gives the following formula:

$$P_D(v,z) = v^2 P_{D''}(v,z) + vzP_{D'}(v,z), \tag{10.11}$$

where $\Gamma(D')$ and $\Gamma(D'')$ are depicted in Fig. 10.3.

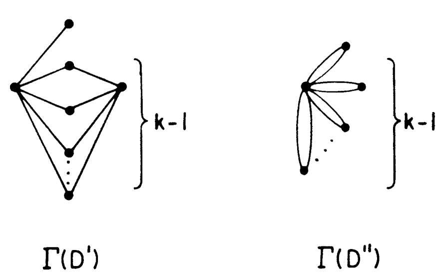



Fig. 10.3

Since $ind\,\Gamma(D') = 2$ and $ind\,\Gamma(D'') = 0$, we see from (10.11) that

$$max\, deg_v P_{D'}(v,z) \leq n(D') + s(D') - 1 - 2\, ind\, D' = 2(k-1) + 1 + k + 2 - 1 - 4$$
$$= 3k - 4,$$

while

$$max\, deg_v P_{D''}(v,z) = n(D'') + s(D'') - 1$$
$$= 2(k-1) + k - 1 = 3k - 3$$

and hence

$$max\, deg_v\ P_D(v,z) = 3k - 1 = \phi_+(D) - 2.$$

Furthermore, a simple computation shows that

$$a_{\phi_+(D)-2}(z) = a_{\phi_+(D'')}(z) = (-1)^{k-1} z^{-(k-1)}. \tag{10.12}$$

Now, since $E_{max}(D) = 2k$, it follows that

$$\alpha_+(D) = \psi_+(D) + E_{max}(D) + 2$$
$$= 2k - (k+2) + 1 - 2J_+(D) + 2k + 2$$
$$= k - 1 - 2\cdot 2k + 2k + 2 = -k + 1.$$

Also, since $\beta_{max}(D) = 1$ and $s(D) = k+2$, we see that $c_{\phi_+(D)-2,\alpha_+} = (-1)^{k+1}$. This proves (3) for the special case where $\Gamma(D) = H_1^k$.

Now consider the general case. We may assume that $D$ is a nice special alternating diagram.

First we build a (partial) resolving binary tree for $D$ (to evaluate $P_D(v,z)$) in such a way that only crossing changes and smoothings are applied at crossings on multiple edges. After we change a crossing, we eliminate simultaneously both a new negative crossing and one positive crossing by an obvious isotopy so that we keep the diagram nice and alternating. Now the resulting leaves of the tree have only singular crossings. Since we are interested in $a_{\phi_+(D)-2}(z)$, we ignor all leaves with index $\geq 2$ and we are left with diagrams $D_1^\ell, D_2^\ell, \ldots, D_t^\ell$, whose graphs $\Gamma(D_j^\ell)$ are of type $H_1$ or (possibly) single-edge graphs with isolated vertices. From the previous computation, we know that each of these graphs contributes some term to $a_{\phi_+(D)-2}(z)$. In fact, a single-edge graph contributes $(-1)^{s(D)}z^{\psi_+(D)+2-2m}$ and graph $H_1^k$ does $(-1)^{s(D)-1}z^{\psi_+(D)+2k+2-2m}$, where $m$ is the number of crossing changes performed on the way from $D$ to $D_j^\ell$. Observe that there are no cancellations between two different terms contributed by $H_1^k$. The highest exponent of $z$ is achieved if a leaf is of type $H_1^k$ for maximal $k$ and only smoothing were performed on the way from $D$ to $H_1^k$, i.e. $m = 0$. This completes the proof of (3).

**Proof of (1)** If $ind\ D = 1$, it follows from Theorem 8.3 that $max\,deg_v\ P_D(v,z) \leq \phi_+(D) - 2$. Furthermore, if $\Gamma(D)$ has locally maximal subgraphs, then we see from Theorem 5.3 that a locally maximal subgraph of $\Gamma(D)$ is either a single-edge graph or it is of type $H_1$. Therefore, Theorem 10.9 (2) and (3), just proven above, show that $max\,deg_v\ P_D(v,z) = \phi_+(D) - 2$. Conversely, suppose that $max\,deg_v\ P_D(v,z) = \phi_+(D) - 2$. If $ind_+D (= ind\ D) = 0$, then it follows from Theorem 9.5 that $max\,deg_v\ P_L(v,z) = \phi_+(D)$. If $ind_+D \geq 2$, then Theorem 8.3 implies that $max\,deg_v\ P_D(v,z) \leq \phi_+(D) - 4$. Therefore, we have $ind\ D = 1$. Suppose now that $\Gamma(D)$ has no locally maximal subgraphs. Since $ind\ D = 1$, it follows from Theorem 5.3 that $\Gamma(D)$ has a subgraph of type $H_2$. Then the argument used in the proof of (3) shows that none of the leaves in the resolving binary tree contributes the term $a_{\phi_+(D)-2}(z)$, since the graphs associated with these leaves are neither of type $H_1$ nor single-edge graphs. Therefore $max\,deg_v\ P_L(v,z) < \phi_+(D) - 2$.

□

**Theorem 10.13** *Let $D$ be a special alternating (positive) diagram of an oriented link $L$. Suppose $ind\,\Gamma(D) = 2$.*

*(1) If $max\,deg_v\ P_D(v,z) = \phi_+(D) - 4$, then $\Gamma(D)$ has a locally maximal subgraph.*

*(2) If $\Gamma(D)$ has a local maximal subgraph and all locally maximal subgraphs have the same number of isthmuses, $(mod\ 2)$, then*

$$max\,deg_v\ P_D(v,z) = \phi_+(D) - 4.$$

*Therefore, under the assumption,* $\mathbf{b}(L) = s(D) - 2$.

*(3) Under the assumptions of (2), the sign of the coefficient of the highest term in* $z$ *in* $a_{\phi_+(D)-4}(z)$ *is* $(-1)^{s(D)-1+ist(D)}$ *where* $ist(D) = 0$ *or 1 according as a local maximal subgraph of* $\Gamma(D)$ *has an even or odd number of isthmuses.*

Since a proof of Theorem 10.13 is elementary but tedious, we will omit the details.

**Proof of Theorem 10.1** Since $L$ is a special alternating (positive) link, we have $ind\_D = 0$ and hence $min\ deg_v\ P_L(v,z) = \phi_-(D)$ by (9.6) (ii). Theorem 10.1 now follows from Theorems 10.9 (1), 10.13 (2) and Theorem 8.12. □

**Example 10.14** Consider the following special alternating positive diagram $D$ of an oriented link $L$. See Fig. 10.4. Note that $ind\ D = 2$.

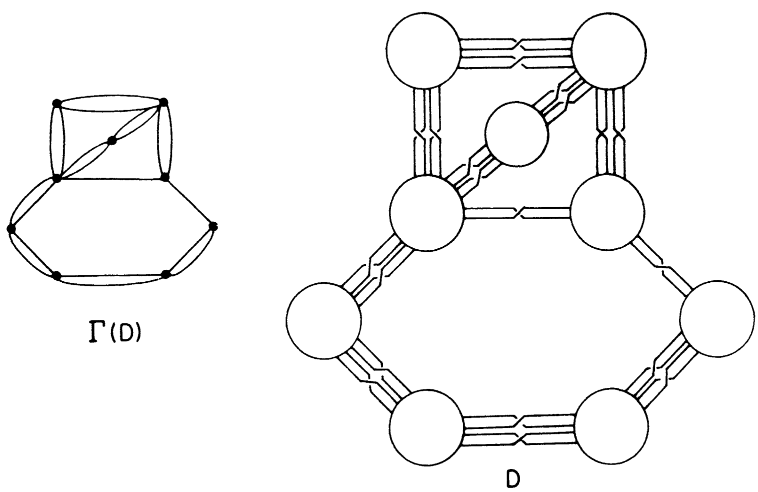



Fig. 10.4

We can prove that $max\ deg_v\ P_D(v,z) = \phi_+(D) - 4 = 24$. In fact, $L$ consists of 7 trivial knots and since the total linking number $Lk(L) = 9$, $P_D(v,z) = z^{-6}(v^{-1} - v)^6 v^{18} + z^{-4}(\ldots)\ldots$ [LM]. The example is interesting, because $D$ does not satisfy the assumptions in Theorem 10.13. There are 6 locally maximal subgraphs of $\Gamma(D)$ each of which has index 2 and has only singular edges. See Fig. 10.5.

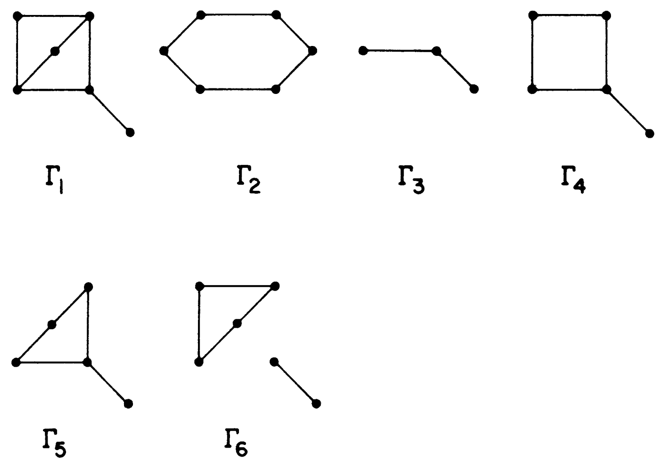



Fig. 10.5.

Each subgraph contributes some term to $a_{\phi_+(D)-4}(z)$. Now a simple computation shows that to $a_{\phi_+(D)-4}(z)$, $\Gamma_1$ contributes $-z^0$, $\Gamma_2$ does $z^0$, $\Gamma_3$ does $z^{-6}$ and each of $\Gamma_4, \Gamma_5$ and $\Gamma_6$ contributes $-z^{-2}$. Therefore the *potentially* highest terms in $z$ contributed by $\Gamma_1$ and $\Gamma_2$ have been cancelled, and finally we have $a_{\phi_+(D)-4}(z) = z^{-6} - 3z^{-2}$.

## §11 Braid index and other invariants

In §4 we proved a relationship between the index of a plane bipartite graph $G$ and the number $\lambda(G^*)$ of the directed growing spanning tree in the dual $G^*$ of $G$. For an alternating link $L$, the absolute value of the leading coefficient of the (reduced) Alexander polynomial of $L$ is exactly $\lambda(G^*)$. Therefore, many theorems proved in §4 can be restated in term of these link invariants. In particular, Theorem 4.3 will give an upper bound of the braid index of an alternating link. In fact, we have

**Theorem 11.1** *Let $c_0(L)$ be the leading coefficient of the (reduced) Alexander polynomial* *of a non-split link* $L$ *. If* $L$ *is a special alternating link, then*

$$\mathbf{b}(L) \leq 2|c_0(L)|.$$

*In general, if an alternating link* $L$ *is a* $*$ *-product of special alternating links* $L_1, \dots, L_m$ *, then*

$$\mathbf{b}(L) - 1 \leq m(2|c_0(L)| - 1). \tag{11.2}$$

**Proof** Let $D$ and $D_i$ be alternating diagrams of $L$ and $L_i$, respectively. Then $\mathbf{b}(L) \leq s(D)$. Let $G$ and $G_i$ denote the graphs associated with $D$ and $D_i$, $i = 1, 2, \dots, m$. Since $|V(G)| = s(D)$ and $\prod_{i=1}^{m} \lambda(G_i^*) = \prod_{i=1}^{m} |c_0(L_i)| = |c_0(L)|$ [Mu 1], (4.4) yields (11.2) □

The following theorem is an immediate consequence of Corollary 4.11.

**Theorem 11.3** *Let L be an alternating link. Let* $D$ *be a reduced alternating diagram of* $L$ *. Then*

$$\mathit{ind}\ D \leq |c_0(L)| - 1. \tag{11.4}$$

**Remark 11.5** (11.4) is the best possible.

These theorems can be used to determine the braid index of an alternating link $L$ for which $|c_0(L)|$ is small. In fact, in §14, we will determine the braid index of an alternating link $L$ with $|c_0(L)| \leq 3$.

# Chapter III. Braid index of alternating links

## §12 Algebraic links

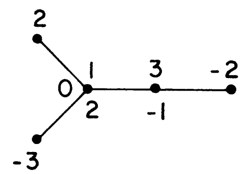
Three sections of this chapter will be devoted to the determination of the braid index of some familiar links, including some alternating algebraic (or arborescent) links and most of the pretzel links.

First, as a straightforward generalization of 2-bridge links, we will consider alternating algebraic links and determine the braid index for a certain type of alternating algebraic links. The family of these links include 2-bridge links and alternating Montesinos links.

Now, as is well known, an algebraic link $L$ is associated with a weighted tree $T$. A tree is called a *weighted* tree if for each pair of adjacent edges $e$ and $e'$ emanating from a vertex $v$, there is an integer $w(e,e';v)$, called a *weight.* If $v$ adjoins $k$ edges $e_1, e_2, \ldots, e_k$, in counter-clockwise, there are $k$ weights $w(e_1,e_2;v)$, $w(e_2,e_3;v)$, $\ldots$, $w(e_k,e_1;v)$ assigned to $v$. The sum of these weights $\sum_{i=1}^{k} w(e_i,e_{i+1};v)$, $e_{k+1} = e_1$, will be called the *weight* of $v$, denoted by $w(v)$. A vertex $v$ of $T$ is called positive (or negative) if $w(v) > 0$ ( or $w(v) < 0$ ). $T$ is called *positive* (or *negative*) if all the vertices are positive (or negative). $T$ is called *even* if $w(v)$ is even for all vertices. Now the algebrac link $L$ associated with a weighted tree $T$ is the boundary of a (not necessarily orientable) surface constructed by plumbing as specified by $T$. $L$ will be denoted by $L(T)$.

**Example 12.1** $T$ is a weighted tree and $L(T)$ is the link associated with $T$.

T



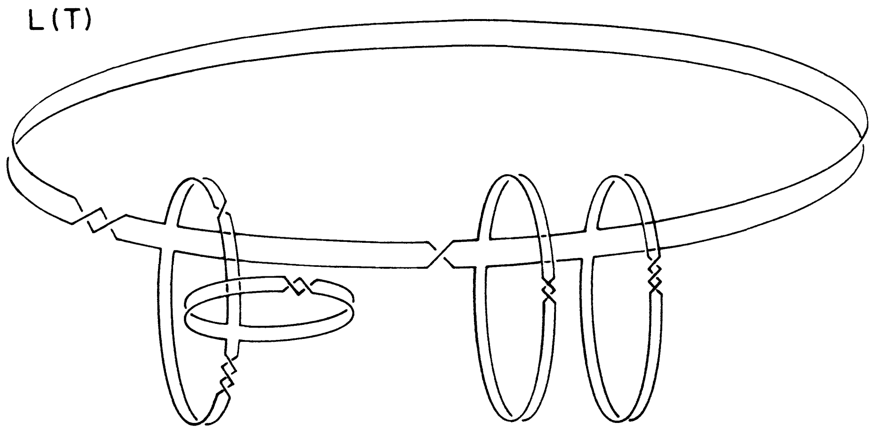



Fig. 12.1

Now, $L$ is not oriented and there is no natural way to define an orientation to $L$. Since the braid index of $L$ depends on the orientation, we must find a natural way to define an orientation of the algebraic link. This problem can be avoided if the surface $F$ constructed from $T$ is orientable. In fact, $F$ is orientable iff $w(v_i)$ is even for each vertex of $T$, i.e., $T$ is an even tree. Therefore, we assume hereafter that $T$ is an even tree. Furthermore, since our links are alternating, we may need another restriction on $T$. A sufficient (but not necessary) condition for an algebraic link to be alternating is that $T$ is positive (or negative) and excessive, i.e. $|w(v_i)| \geq val(v_i)$ for all vertices $v_i$.

In fact, it is easy to find alternating diagrams of this type of algebraic links. Given a positive excessive even tree $T$, there is another positive excessive even tree $T'$ such that

(12.2) (1) $T$ is isomorphic to $T'$ (as weighted graphs, but not necessarily weighted plane graphs),

(2) every weight $w(e, e'; v')$ is positive for $v' \in V(T')$,

(3) $L(T')$ is ambient isotopic to $L(T)$.

Therefore we may assume without loss of generality that every weight of a positive excessive tree is positive.

**Example 12.3** $T$ and $T'$ in Fig. 12.2(a) are isomorphic.

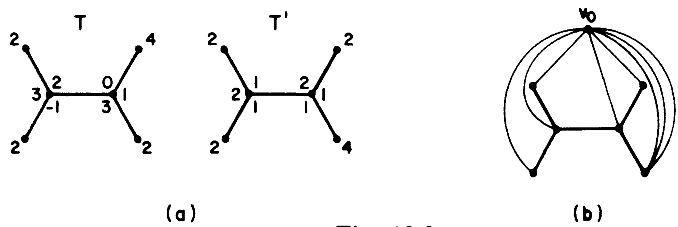



Fig. 12.2

Now we construct a plane graph $K$ from $T$ by adding a finite number of edges to $T$ as follows.

Take a point $v_0$, not on $T$, from the plane. If $w(e, e'; v_i) = k > 1$, then join $v_0$ and $v_i$ by $k-1$ disjoint simple arcs in such a way that these arcs divide the angle, where the weight is assigned, into $k$ parts. (See Fig. 12.2 (b).) Thus we obtain a plane graph $K$, called a *completion* of $T$. By assigning a negative sign to every edge, $K$ becomes a negative plane graph. Since $T$ is excessive and even, every vertex of $K$ has even valencies, and, moreover, $K$ is reduced, i.e. $K$ has no loops and no isthmuses.

Next, consider the signed dual graph $G$ of $K$. Proposition 6.3 then implies that $G$ is reducible. $G$ is a positive bipartite graph.

From this plane graph $G$, we can easily construct a link diagram $D$ whose Seifert graph $\Gamma(D)$ is $G$ (as a plane graph). Since $G$ is bipartite, $D$ is a special alternating diagram and the orientation of $D$ is induced from that of the Seifert surface. $G$ is called a graph associated with $T$, and is denoted by $G(T)$.

It is evident that $D$ represents a link that is ambient isotopic to the algebraic link associated with $T$.

Analogously, we can construct a negative (bipartite) plane graph $G$ associated with a negative excessive even tree, and a special (negative) alternating diagram $D$.

These constructions can be extended to a slightly more general weighted tree. Consider an arbitrary weighted tree $T$. Let $A(T)$ be the set of those edges in $T$ which joint positive vertices and negative vertices. If all the edges in $A(T)$ are removed from $T$, $T$ will split into finitely many subtrees $T_1, T_2, \dots T_k$, each of which is either strictly positive or negative. A collection $\{T_1, T_2, \dots, T_k\}$ will be called a *uniform decomposition* of $T$.

**Proposition 12.4** *Let $\{T_1, T_2, \dots T_k\}$, be the uniform decomposition of a tree $T$. If each of $T_1, \dots, T_k$ is excessive, then the link $L(T)$ is alternating.*

(For a proof, see [Mu 2, Proposition 4.2].)

It is easily seen that the link $L(T)$ is a $*$-product of $k$ algebraic links $L(T_1), \dots, L(T_k)$.

Since $T_j$ is a positive (or negative) excessive even tree, then $L(T_j)$ is a special alternating link (and is the boundary of the Seifert surface $F$ constructed before).

Unfortunately, we are unable to determine the braid index for all alternating algebraic link, but we can determine the braid index of those links, each of whose $*$-components is so-called strongly excessive.

**Definition 12.5** Let $T$ be a positive (or negative) tree. $T$ is called *strongly excessive* if $|w(v_i)| \neq 0$ and $|w(v_i)| \geq 2[val(v_i) - 1]$ for any vertex $v_i$. If $\{T_1, \dots, T_k\}$ is a uniform decomposition of $T$, then $T$ is called *strongly excessive* if each tree $T_j$ is strongly excessive.

We note that if a weighted positive (or negative) tree is strongly excessive, then the plane graph $G(T)$ is strongly excessive. (See Definition 6.4.)

An algebraic link $L$ is called *strongly excessive* if $L$ is associated with a strongly excessive weighted tree. For example, a 2-bridge link is strongly excessive. Now the purpose of this section is to prove the following

**Theorem 12.6** *Let $T$ be a strongly excessive even tree and $L$ the link associated with $T$. The orientation of $L$ is induced from the orientation of the (orientable) surface $F$ constructed from $T$. Then*

$$v - span \quad P_L(v,z) \;=\; 2\big(\mathbf{b}(L) - 1\big) . \tag{12.7}$$

Therefore, the braid index is completely determined by the skein polynomial.

First we prove (12.7) for special alternating links.

**Proposition 12.8** *Let $L$ be a strongly excessive positive algebraic link. Then (12.7) holds.*

**Proof** Let $G$ be the positive graph associated with $T$ and $D$ the special alternating diagram of $L$. When we need to emphasize the association of $G$ or $D$ with $T$, we will write $G(T)$ or $D(T)$.

Now, since $L$ is special (positive) alternating, we see that $min\, deg_v\, P_L(v,z) = \phi_-(D)$ and hence if suffices to prove

$$max\, deg_v \; P_L(v,z) \;=\; \phi_+(L) - 2\; ind\, G(T) \;=\; n(D) + s(D) - 1 - 2\; ind\, G(T) . \tag{12.9}$$

Since $n(D) = |E(G)|$ and $s(D) = |V(G)|$, (12.9) is equivalent to

$$max\, deg_v \; P_L(v,z) \;=\; |E(G)| + |V(G)| - 1 - 2\; ind\, G(T) . \tag{12.10}$$

To compute $|E(G)|$ and $|V(G)|$, consider the dual graph $G^*$ of $G$. Denote by $w(T)$ the total weight of $T$, i.e., $w(T) = \sum_{v_i \in V(T)} w(v_i)$. Then it follows from Theorem 6.5 that

$$2\; ind\, G \;=\; 2 \sum_{v_i \in V(T)} \left\{ \frac{w(v_i)}{2} - 1 \right\} \;=\; w(T) - 2|V(T)| .$$

Now it is easy to see that $|E(G)| = |E(G^*)| = \sum_{v_i \in V(T)} w(v_i) - |E(T)| = w(T) - \big(|V(T)| - 1\big)$ and $|V(G)| = |E(G)| - d(G) + 2$, where $d(G)$ is the number of connected components of $\mathbb{R}^2 - G$. Note that $d(G) = |V(G^*)| = |V(T)| + 1$, and hence $|V(G)| = |E(G)| - |V(T)| + 1$. Using these formulae, a simple computation shows that

$$\phi_+(D) - 2\, ind\, G(T) \;=\; |E(G)| + |V(G)| - 1 - 2\, ind\; G(T) \;=\; w(T) - |V(T)| + 2\,.$$

Therefore, to prove (12.10) it suffices to show that

$$\text{(12.11)} \qquad max\, deg_v\; P_L(v,z) \;=\; w(T) - |V(T)| + 2\,.$$

A similar computation will show that

$$\text{(12.12)} \qquad min\, deg_v\; P_L(v,z) \;=\; |V(T)|\,.$$

We prove (12.11) by induction on $\big(|V(T)|\,,\; w(T)\big)$ where the order is given lexicographically. We may assume without loss of generality that $T$ is connected.

Now consider the initial case, $\big(1, w(T)\big)$. Then $G$ is a polygon with $w(T)$ sides and hence $L$ is an (oriented) fibred torus link of type $\big(w(T), 2\big)$. It is known that $max\, deg_v\; P_L(v,z) \;=\; w(T) + 1$ and hence (12.11) hold trivially. Suppose (12.11) holds for any (strongly excessive even positive) tree $T'$ such that $\big(|V(T')|,\; w(T')\big) < \big(|V(T)|,\; w(T)\big)$.

**Lemma 12.13** *If $T$ has a stump $v_0$ with $w(v_0) > 2$. Then (12.11) holds by induction.*

**Proof** Since $w(v_0) > 2$, the boundary of the domain (in $\mathbb{R}^2 - G(T)$) corresponding to $v_0$ contains a free edge $e_0$. Apply the skein relation at the crossing corresponding to $e_0$, and we have

$$P_L(v,z) \;=\; v^2 P_{L_-}(v,z) + vzP_{L_0}(v,z).$$

For simplicity, we say that a crossing $c$ in a diagram $D$ *occurs* on an edge $e$ in the graph $\Gamma(D)$ if $e$ corresponds to $c$. Furthermore, we call the ordered pair $\big(|V(T)|,\; w(T)\big)$ the type of $T$. Let $T_-$ and $T_0$ denote the weighted trees associated with $L_-$ and $L_0$, respectively. For a subgraph $T'$ of a weighted tree $T$, the weight function is always defined by restriction. Both $L_-$ and $L_0$ are algebraic links. $T_-$ has the type $\big(|V(T)|,\ w(T)-2\big)$ while $T_0$ has the type $\big(|V(T)|-1,\ w(T)-w(v_0)\big)$. Note that $w(v_0) \geq 3$. Using the induction hypothesis, we can prove that

$$2 + max\, deg_v\ P_{L_-}(v,z) \ > \ 1 + max\, deg_v\ P_{L_0}(v,z)$$

and hence,

$$max\, deg_v\ P_L(v,z) \ = \ w(T) - |V(T)| + 2$$

which proves Lemma 12.13. □

Now we may assume henceforth that each stump of $T$ has weight 2. A chain $C$ : $v_0, e_1, v_1, \ldots, v_{m-1}, e_m, v_m$ is called *elementary* if it is simple and $val.(v_i) \leq 2$ for $i = 0, 1, \ldots, m-1$.

**Lemma 12.14** *If $T$ has an elementary chain $C$ (of length $\geq 2$ ) connecting a stump $v_0$ and another vertex, say $v_m$ such that some (intermediate) verteces $v_i$ $(0 < i < m)$ have weight $\geq 4$, then (12.11) holds by induction.*

**Proof** Let $v_i$ be the nearest vertex of $C$ to $v_0$ which has weight $\geq 4$. In other words, $w(v_j) = 2$ for $j = 0, 1, \ldots, i-1$ but $w(v_i) \geq 4$.

Let $T' = T - \bigcup_{j=0}^{i-1}$ *star* $v_j$ and $T^0 = T' -$ *star* $v_i$. Let $L'$ and $L^0$, respectively, denote the links associated to the weighted trees $T'$ and $T^0$. Then applications of the skein relation give us

$$P_L(v,z) \ = \ \alpha(v,z) P_{L'}(v,z) + \beta(v,z) P_{L^0}(v,z)\ , \tag{12.15}$$

where $\alpha(v,z)$ and $\beta(v,z)$ are polynomials such that $max\, deg_v\ \alpha(v,z) \ = \ i$ and $max\, deg_v\ \beta(v,z) = i+1$.

Now by applying the induction assumption on $L'$ and $L^0$, we see that

$$i + max\, deg_v\ P_{L'}(v,z) \ = \ w(T) - |V(T)| + 2\ ,$$

while

$$i + 1 + max\, deg_v\; P_{L^0}(v,z) \;=\; w(T) - |V(T)| - w(v_i) + 4\ .$$

Since $w(v_i) \geq 4$, Lemma 12.14 follows from (12.15). □

By Lemma 12.14, we may assume that any intermediate vertex and a stump on an elementary chain have always weight 2.

**Lemma 12.16** *Let $C_1, \ldots, C_k$ be $k$ simple chains each of which connects a stump and the common vertex $v_*$. Let $v_{i,\ell}$ be vertices which occur on $C_i$, $i = 1, \ldots, k$, $\ell = 0, 1, \ldots, \lambda_i + 1$, where $v_{i,\lambda_i+1} = v_*$. Let $T_* = T - \bigcup_{i=1}^{k} \bigcup_{\ell=0}^{\lambda_i}$ star $u_{i,\ell}$. Suppose that val $(v_*) = k+1$. Let $L_*$ be the alternating link associated with $T_*$. Then*

$$max\, deg_v\; P_L(v,z) \;=\; max\, deg_v\; P_{L_*}(v,z) + \sum_{i=1}^{k} \lambda_i\ .$$

We should note that $v_*$ is a stump in $T_*$.

**Proof** We prove the lemma by induction on $k$. If $k = 1$, Lemma 12.16 follows from Lemma 12.14. Now a repeated application of the skein relation gives

$$P_L(v,z) \;=\; \alpha(v,z)P_{L_{k-1}}(v,z) + \beta(v,z)P_{L^{(0)}}(v,z)\prod_{i=1}^{k} P_{L^{(i)}}(v,z)$$

where

(12.17) (1) $max\, deg_v\; \alpha(v,z) \;=\; \lambda_1$.

(2) $max\, deg_v\; \beta(v,z) \;=\; \lambda_1 + 1$.

(3) $L_{k-1}$ is the algebraic link associated with the weighted tree

$T_{k-1} = T - \bigcup_{j=1}^{\lambda_1}$ *star* $v_{1j}$

(4) $L^{(i)} (i = 2, \ldots, k)$ is the algebraic link associated with the

elementary chain $\hat{C}_i = C_i -$ *star* $v_*$ .

Note that $w(v_{i,j}) = 2$, for any $j = 1, \ldots, \lambda_i$. $L^{(0)}$ is the algebraic link associated with the (weighted) tree $T^{(0)} = T_* -$ *star* $v_*$ .

Now by the induction assumption on $L_{k-1}$, we have

$$\text{(12.18)} \qquad max\,deg_v\ \alpha(v,z)\ P_{L_{k-1}}(v,z) \ = \ \sum_{i=1}^{k} \lambda_i + max\,deg_v\ P_{L_*}(v,z)\ .$$

Furthermore, since $\big(|V(T^0)|,\ w(T^0)\big), \big(|V(T_*)|,\ w(T_*)\big) < \big(|V(T)|,\ w(T)\big)$, it follows from the induction assumption that

$$\begin{aligned} max\,deg_v\ P_{L^{(0)}}(v,z) \ &= \ w(T^{(0)}) - |V(T^{(0)})| + 2 \\ &= w(T_*) - w(v_*) - \big(|V(T_*)| - 1\big) + 2 \\ &= max\,deg_v\ P_{L_*}(v,z) - \big(w(v_*) - 1\big)\ . \end{aligned}$$

Since $T_*$ is strongly excessive, we see that $w(v_*) \geq 2[val\,(v_*) - 1]$ and hence

$$max\,deg_v\ P_{L^{(0)}}(v,z) \ \leq \ max\,deg_v P_{L_*} - (2k-1)\ .$$

Since $max\,deg_v\ P_{L^{(i)}}(v,z) = \lambda_i + 1$ and $k \geq 2$, we have finally

$$\begin{aligned} &max\,deg_v\ \Big[\beta(v,z) P_{L^{(0)}}(v,z) \prod_{i=2}^{k}\ P_{L^{(i)}}(v,z)\Big] \\ &= \sum_{i=1}^{k}\ \lambda_i + k + max\,deg_v\ P_{L^{(0)}}(v,z) \\ &< \sum_{i=1}^{k}\ \lambda_i + max\,deg_v\ P_{L_*}(v,z)\ . \end{aligned}$$ □

Now Lemmas 12.13-12.16 complete a proof of Proposition 12.8. □

We are now in a position to prove Theorem 12.6. Let $T$ be a strongly excessive even tree. Let $\{T_1, T_2, \ldots, T_m\}$ be the uniform decomposition of $T$. By the definition, $T_i (i = 1, \ldots, m)$ is either a positive or negative strongly excessive tree. For simplicity, we assume that $T_1, \ldots, T_p$ are positive and $T_{p+1}, \ldots, T_m$ are negative.

We have proved in Proposition 12.8 that if $T_i$ is positive, then

$$\text{(12.19)} \qquad \begin{aligned} max\,deg_v\ P_{L(T_i)}(v,z) \ &= \ w(T_i) - |V(T_i)| + 2 \\ min\,deg_v\ P_{L(T_i)}(v,z) \ &= \ |V(T_i)|, \end{aligned}$$

and if $T_j$ is negative, then

$$(12.20)\qquad \begin{aligned} max\, deg_v\ P_{L(T_j)}(v,z) &= -|V(T_j)| \\ min\, deg_v\ P_{L(T_j)}(v,z) &= -\Big\{|w(T_j)| - |V(T_j)| + 2\Big\} \end{aligned}$$

Now it suffices to prove

$$(12.21)\qquad \begin{aligned} &(1)\ \ max\, deg_v\ P_L(v,z) \\ &\quad = \sum_{i=1}^{p} \Big\{|w(T_i)| - |V(T_i)| + 2\Big\} - \sum_{j=p+1}^{m} |V(T_j)| \\ &(2)\ \ min\, deg_v\ P_L(v,z) \\ &\quad = -\sum_{j=p+1}^{m} \Big\{|w(T_j)| - |V(T_j)| + 2\Big\} + \sum_{i=1}^{p} |V(T_j)|\ . \end{aligned}$$

In fact, since $s(D) = \sum_{i=1}^{m} \Big\{|w(T_i)| - 2|V(T_i)|\Big\} + m + 1$ and $ind\ G = \sum_{i=1}^{m} \frac{|w(T_i)|}{2} - \sum_{i=1}^{m} |V(T_i)|$, (12.21) will yield

$$v - spanP_L(v,z) = 2\{s(D) - 1 - ind\ G\} = 2(\mathbf{b}(L) - 1)\ ,$$

which will prove Theorem 12.6.

Now it is enough to prove one of the formulas of (12.21), say (12.21) (1). Furthermore the induction argument on $|V(T)|$ easily shows that it only needs to prove (12.21) (1) for the case where one of the components, say $T_1$ consists of an isolated vertex $v_1$ and $w(v_1) = 2$. (See Fig. 12.3.)

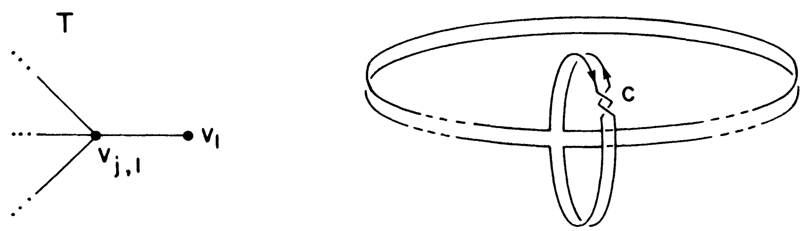



Fig. 12.3

Suppose that $v_1$ is connected to $v_{j,1}$ in $T_j$. Then $w(v_{j,1}) < 0$. Apply the skein relation at the crossing $c$ (Fig. 12.3) and we obtain

$$P_L(v,z) = v^2 P_{L_-}(v,z) + vzP_{L_0}(v,z)$$

where $L_-$ and $L_0$ are the links associated with $T_- = T - (star\ v_1) - (star\ v_{j,1})$ and $T_0 = T - (star\ v_1)$, respectively.

Note that $T_-$ is disconnected. Since $w(T_1) - |V(T_1)| + 2 = 3$, using the induction hypothesis, we have

$$\begin{aligned} max\, deg_v\{v^2\ P_{L(T_-)}(v,z)\} &= \sum_{i=2}^{p}[w(T_i) - |V(T_i)| + 2] - \Big\{\sum_{j=p+1}^{m} |V(T_j)| - 1\Big\} + 2\ , \\ &= \sum_{i=1}^{p}\Big[w(T_i) - |V(T_i)| + 2\Big] - \sum_{j=p+1}^{m} |V(T_j)|\ . \end{aligned}$$

On the other hand, we have by the induction hypothesis,

$$\begin{aligned} max\, deg_v\ \{vzP_{L(T_0)}(v,z)\} &= \sum_{i=2}^{p}[w(T_i) - |V(T_i)| + 2] - \sum_{j=p+1}^{m} |V(T_j)| + 1 \\ &= max\, deg_v\{v^2 P_{L_-}(v,z)\} - 2 \end{aligned}$$

and hence,

$$max\, deg_v\ P_L(v,z) = \sum_{i=1}^{p}\{w(T_i) - |V(T_i)| + 2\} - \sum_{j=p+1}^{m} |V(T_j)|\ .$$

Now the proof of Theorem 12.6 is complete. □

The following corollaries are easy consequences of Theorem 12.6 and (12.21).

**Corollary 12.22** *Let $L$ be a strongly excessive alternating algebraic link associated with a weighted tree $T$. Let $\{T_1, \ldots, T_m\}$ be the uniform decomposition of $T$. Then*

$$\mathbf{b}(L) = \sum_{i=1}^{m}\Big\{\frac{|w(T_i)|}{2} - |V(T_i)|\Big\} + m + 1\ .$$

**Corollary 12.23** *(Cf. [Mu 7, Theorem B].) Let $L$ be a 2-bridge link of type $(\alpha, \beta)$, where $0 < \beta < \alpha$ and $\beta$ is odd. Let*

$[2n_{11}, \ldots, 2n_{1,k_1}, -2n_{21}, \ldots, -2n_{2,k_2}, \ldots, (-1)^{t-1}2n_{t,1}, \ldots, (-1)^{t-1}2n_{t,k_t}]$, $n_{ij} > 0$, *be the continued faction form of $\frac{\alpha}{\alpha-\beta}$ if $\alpha$ is odd or $\frac{\alpha}{\beta}$ if $\alpha$ is even. Then*

$$\mathbf{b}(L) = \sum_{i=1}^{t}\sum_{j=1}^{k_i}(n_{ij} - 1) + t + 1 .$$

## §13 Pretzel links

In this section we will show that the braid index of alternating pretzel links is completely determined by their skein polynomials. Although the braid index of non-alternating pretzel links is not determined by their skein polynomials, it may be determined by evaluating the skein polynomials of appropriate cables of the links. However, we will not pursue these problems in this paper.

An (oriented) pretzel link with $k$ vertical strips is denoted by $L[c_1^{\varepsilon(1)}, \ldots, c_k^{\varepsilon(k)}]$, where $c_i$ denotes the number of half twists on the $i^{th}$ strip and $c_i$ is positive (or negative) if the twists are in a right-hand (or a left-hand) sense. The superscript $\varepsilon(i)$ is $+1$ (or $-1$ ) if all the crossings on the $i^{th}$ strip are positive (or negative). See the example below.

**Example 13.1** $L = L[4^{(-1)}, -4^{(-1)}, 4^{(-1)} - 3^{(+1)} - 3^{(+1)}]$ has a diagram shown in Fig. 13.1.

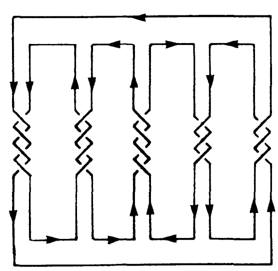
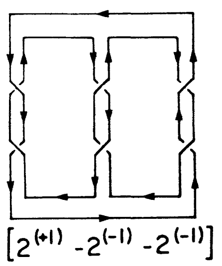


Fig. 13.1

Since $P_{L[c_{\sigma(1)}^{\varepsilon(\sigma(1))},\ldots,c_{\sigma(k)}^{\varepsilon(\sigma(k))}]}(v,z) = P_{L[c_1^{\varepsilon(1)},\ldots,c_k^{\varepsilon(k)}]}(v,z)$ for any permutation $\sigma$ on $\{1,2,\ldots,k\}$, we may change the order of $c_1,\ldots,c_n$ arbitrarily to evaluate the skein polynomial. Our theorem shows that the braid index of a pretzel link is independent of the order of $c_i$. However, we do not know whether or not the braid index of a link is invariant under mutation.

Now, if all $c_i$ are odd or all $c_i(\neq 0)$ are even, then we can change an orientation, if necessary, of some components of $L$ so that $\varepsilon(i) = sign\, c_i\left(= \frac{c_i}{|c_i|}\right)$.



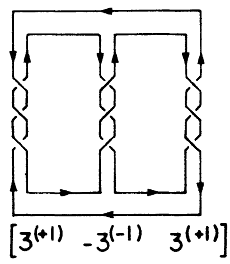




Fig. 13.2

And, as long as all $c_i$ are odd or all are even, we consider only these pretzel links. Therefore, we may drop the superscripts $\varepsilon(i)$ from the notation. Note that the diagram thus obtained is special and has $\sum_{i=1}^{k} |c_i| - k + 2$ Seifert circles.

There are a few more remarks. Since we are only interested in $P_L(v,z)$, we may assume without loss of generality that

$$(13.2) \qquad [c_1,\ldots,c_k] = [a_1,\ldots,a_p,-b_1,\ldots,-b_n]$$

where $0 < a_1 \leq a_2 \leq \cdots \leq a_p$, and $0 < b_1 \leq b_2 \leq \cdots \leq b_n$.

If all $c_i$ are odd, we may assume further that

$$(13.3) \qquad [c_1,\ldots,c_k] = [\overbrace{1,\ldots,1}^{r},a_1,\ldots,a_p,-b_1,\ldots,-b_n]$$

where $3 \leq a_1 \leq \ldots \leq a_p$ and $3 \leq b_1 \leq \ldots \leq b_n$.

The pretzel link of this form will be denoted by $L[r|a_1,\ldots,a_p;-b_1,\ldots,-b_n]$. Now to describe the maximal or minimal $v$-degree of $P_L(v,z)$, we recall the extremal terms of $P_L(v,z)$, i.e., $max - max\, P_L(v,z)$ and $min - max\, P_L(v,z)$. See (7.5).

We prove first the following theorem.

**Theorem 13.4** *Let* $L$ *be a pretzel link,* $L = L[a_1, a_2, \ldots, a_p, -b_1, \ldots, -b_n]$, *where all* $a_i$ *and* $b_j$ *are positive and even. Let* $D$ *be a special diagram of* $L$. *Denote* $a = \sum_{i=1}^{p} a_i$ *and* $b = \sum_{j=1}^{n} b_j$.

*(1)* *(i)(a)* $ind_+ D = \frac{a}{2} - p + 1$

*(b)* $ind_- D = \frac{b}{2} - n + 1$

*(c)* $ind\ D = \frac{1}{2}(a+b) - p - n + 1$.

*(ii)* $s(D) = a + b - p - n + 2$

*(2)* *If* $p$ *and* $n \geq 2$, *then*

*(i) (a)* $max - max\ P_L(v,z) = (-1)^{n+p+1}\ v^{a+p-n-1}\ z^{n-p+1}$

*(b)* $min - max\ P_L(v,z) = v^{-b+p-n+1}\ z^{p-n+1}$

*(ii)* $v$ *-span* $P_L(v,z) = a + b - 2$

*and hence* $\frac{1}{2}(a+b) \leq \mathbf{b}(L) \leq \frac{1}{2}(a+b) + 1$.

*(3)* *Suppose* $n = 0$ *and* $p \geq 1$, *(and hence* $L$ *is special alternating), then*

*(i) (a)* $max - max\ P_L(v,z) = (-1)^{p-1} v^{a+p-1}\ z^{-p-1}$

*(b)* $min - max\ P_L(v,z) = v^{p-1} z^{p-1}$

*(ii)* $\mathbf{b}(L) = s(D) - ind(D)$.

*(4)* *If* $p = 1$ *and* $n \geq 2$, *then*

*(i)* $min - max\ P_L(v,z) = v^{-b-n+2} z^{-n+2}$

*(ii) (A)* *If* $2 = b_1 = \cdots = b_m < b_{m+1}$, *then*

*(a)* $max - max\ P_L(v,z) = (-1)^n m v^{a-n} z^{n-2}$, *where* $a = a_1$, *and*

*(b)* $\mathbf{b}(L) \leq s(D) - ind\ D = \frac{1}{2}(a+b) + 1$.

*(B)* *Suppose* $4 \leq b_1 = \cdots = b_m < b_{m+1}$.

*(a) If* $a_1 - b_1 + 2 > 0$, *then*

$$max - max\ P_L(v,z) = (-1)^n m v^{a_1 - b_1 + 2 - n}\ \ z^{n-2}.$$

*(b) If* $a_1 - b_1 + 2 \leq 0$, *then*

$$max - max\, P_L(v,z) \;=\; (-1)^n v^{-n} z^n$$

(c) $\mathbf{b}(L) \leq s(D) - ind\ D = \frac{1}{2}(a+b)+1$.

**Remark 13.5.**

(1) For a pretzel link considered in Theorem 13.4, $max - max\, P_L(v,z)$ and $min - max\, P_L(v,z)$ are completely determined by $a_i$ and $b_j$.

(2) Theorem 13.4 (3) is not an immediate consequence of (2).

**Proof** Since the proof of Theorem 13.4 is a *model* of the proofs of other theorems discussed later in this section, we will give a detailed proof of this theorem. We only need to show (2)-(4).

We will use the idea employed in [LM 1] to evaluate the skein polynomial of a pretzel link. It should be noted, however, that there is a slight difference between our notation and their notation.

First we define, for any even integer $c$, a few particular polynomials

(13.6)
$$x_c^1 \;=\; v^c$$
$$x_c^0 \;=\; \mu^{-1}(1 - x_c^1), \qquad \text{where} \quad \mu = (v^{-1} - v)z^{-1}.$$

**Proposition 13.7** *Let* $L = L[a_1, \ldots, a_p, -b_1, \ldots, -b_n]$ *be a pretzel link, where* $a_i$ *and* $b_j$ *are even positive integers. Let* $\delta : \{1, 2, \ldots, p+n\} \to \{0,1\}$ *be* $2^{p+n}$ *functions. Then*

(13.8)
$$P_L(v,z) = \sum_{\delta} x_{a_1}^{\delta(1)} \ldots x_{a_p}^{\delta(p)} x_{-b_1}^{\delta(p+1)} \ldots x_{-b_n}^{\delta(p+n)} \mu^{\sum_1^{p+n} \delta(i) - 1},$$

*where the summation runs over the* $2^{n+p}$ *functions* $\delta$ *and define* $\mu^{-1} = \mu$.

Since a proof is easy, we omit it.

Now, to avoid the occurrence of $\mu^{-\lambda}$ in the summation in (13.8) we will evaluate $P_L(v,z)\mu^{p+n}$ instead of $P_L(v,z)$. Using a new notation $y_c^\delta = x_c^\delta \mu$, we have from (13.8)

$$\mu^{p+n} P_L(v,z) = \sum_{\Sigma\delta\neq 0} y_{a_1}^{\delta(1)} \cdots y_{a_p}^{\delta(p)} y_{-b_1}^{\delta(p+1)} y_{-b_n}^{\delta(p+n)} \mu^{\Sigma\delta(i)-1} + \left(y_{a_1}^0 \cdots y_{a_p}^0 y_{-b_1}^0 \cdots y_{-b_n}^0\right)\mu \,. \tag{13.9}$$

Denote

$$A = \sum_{\Sigma\delta\neq 0} y_{a_1}^{\delta(1)} \cdots y_{-b_n}^{\delta(p+n)} \mu^{\Sigma\delta(i)-1} \quad \text{and} \quad B = y_{a_1}^0 \cdots y_{-b_n}^0 \mu \,.$$

Note that $y_c^1 = x_c^1\mu = v^c(v^{-1}-v)z^{-1} = (v^{c-1}-v^{c+1})z^{-1}$ and $y_c^0 = x_c^0\mu = 1 - v^c$.

Now we will compute extremal terms of $A$ and $B$ separately, and then compare these terms. Since $a_i$ and $b_j \geq 2$, we see that $max - max\ A$ occurs in

$$\left(y_{a_1}^1 \cdots y_{a_p}^1 y_{-b_1}^0 \cdots y_{-b_n}^0\right)\mu^{p-1} = \prod_{i=1}^{p} \{(v^{a_i-1} - v^{a_i+1})z^{-1}\} \left[\prod_{j=1}^{n} (1 - v^{-b_j})\right] \cdot \left(\frac{v^{-1}-v}{z}\right)^{p-1}$$

and hence

$$\begin{aligned} max - max\ A &= (-1)^p v^{\sum_1^p (a_i+1)} z^{-p} (-1)^{p-1} v^{p-1} z^{-(p-1)} \\ &= (-1) v^{a+2p-1} z^{-2p+1} \,. \end{aligned}$$

On the other hand, since

$$B = \prod_{i=1}^{p} (1 - v^{a_i} \prod_{j=1}^{n} (1 - v^{-b_j}) \cdot (v^{-1} - v) z^{-1} \,,$$

we see that

$$max - max\ B = (-1)^p v^a (-1) v z^{-1} = (-1)^{p+1} v^{a+1} z^{-1} \,.$$

Therefore, if $p \geq 2$, then $max - max\ P_L(v,z)\mu^{n+p} = max - max\ A$ and hence

$$\begin{aligned} max - max\ P_{L(v,z)} &= (-1)^{1+n+p} v^{a+2p-1-n-p} z^{-2p+1+n+p} \\ &= (-1)^{n+p+1} v^{a+p-n-1} z^{n-p+1} \,. \end{aligned}$$

This proves (2) (i) (a).

Next, we will compute $min-max\, P_L(v,z)$. For convenience, we denote $y_{a_1}^{\delta(1)} \dots y_{a_p}^{\delta(p)}$ $y_{-b_1}^{\delta(p+1)}, \dots, y_{-b_n}^{\delta(p+n)}$ by $\big(\delta(1), \dots, \delta(p) \,\big|\, \delta(p+1), \dots, \delta(p+n)\big)$. Now $min-max\, A$ occurs in

$$\begin{aligned}
&(0, \dots, 0 \,\big|\, 1, \dots, 1)\mu^{n-1} \\
&= \prod_{i=1}^{p} (1 - v^{a_i}) \prod_{j=1}^{n} \cdot \Big[(v^{-b_j-1} - v^{-b_j+1})z^{-1}\Big] \cdot \Big(\frac{v^{-1} - v}{z}\Big)^{n-1}
\end{aligned}$$

and hence

$$\begin{aligned}
min - max\ A &= v^{-\sum_{1}^{n}(b_j+1)} z^{-n} \cdot v^{-(n-1)} z^{-(n-1)} \\
&= v^{-b-2n+1} z^{-2n+1} .
\end{aligned}$$

On the other hand, since $B = (0, \dots, 0 \,\big|\, 0, \dots, 0)\mu$, we see that

$$min - max\ B \ = \ (-1)^n v^{-b} v^{-1} z^{-1} \ = \ (-1)^n v^{-b-1} z^{-1} .$$

Therefore, if $n \geq 2$, then $min - max\ P(v,z)\mu^{n+p} \ = \ min - max\ A$ and hence

$$min - max\ P(v,z) = v^{-b-n+p+1} z^{-n+p+1} = v^{-b-2n+1+n+p} z^{-2n+1+n+p}$$

which proves (2)(i)(b). The other propositions in (2) are obvious.

**Proof of (3)** Suppose $n = 0$. Then, since $p \geq 2$, the previous argument shows easily (3)(i) (a) and (b). (ii) is immediate from (1).

**Proof of (4)** Suppose $p = 1$, i.e., $L = L[a, -b_1, \dots, -b_n]$. We assume $n \geq 2$. (If $n = 1$, then $L$ is an elementary torus link.)

Now, since $(0 \,\big|\, 0, \dots, 0)\mu = (- \,\big|\, 0, \dots, 0)\mu - (1 \,\big|\, 0, \dots, 0)$, where $(- \,\big|\, 0, \dots, 0) =$

$y^0_{-b_n} \cdots y^0_{-b_n}$, we can write

$$\begin{aligned} P(v,z)\mu^{n+1} &= \sum_{\Sigma\delta(i)\neq 0} \big(\delta(1) \,\big|\, \delta(2)\ldots\delta(n+1)\big)\mu^{\Sigma\delta(i)-1} + (0 \,|\, 0\ldots 0)\mu \\ &= \sum_{\Sigma\delta(i)\neq 0} \big(\delta(1)\big|\delta(2)\ldots\delta(n+1)\big)\mu^{\Sigma\delta(i)-1} + (-\big|0\ldots 0)\mu - (1\big|0\ldots 0) \\ &= \sum_{\Sigma\delta(i)\neq 0} \big(1\big|\delta(2)\ldots\delta(n+1)\big)\mu^{\Sigma\delta(i)} + \sum_{\Sigma\delta(i)\neq 0} \big(0\big|\delta(2)\ldots\delta(n+1)\big)\mu^{\Sigma\delta(i)-1} \\ &\qquad + (-\big|0,\ldots,0)\mu. \end{aligned}$$

For various $\delta(2),\ldots,\delta(n+1)$, we will evaluate

$$\begin{aligned} A' &= \big(1 \,\big|\, \delta(2)\ldots\delta(n+1)\big)\mu^{\Sigma\delta(i)}, \\ A'' &= \big(0 \,\big|\, \delta(2)\ldots\delta(n+1)\big)\mu^{\Sigma\delta(i)-1} \qquad \text{and} \\ B' &= (- \,|\, 0,\ldots,0)\mu\,. \end{aligned}$$

Since we may assume without loss of generality that $\delta(2) = \ldots = \delta(\lambda+1) = 1$, $\delta(\lambda+2) = \ldots = \delta(n+1) = 0$, where $\lambda \geq 1$, we will write

$$\begin{aligned} A'_\lambda &= \big(1 \,\big|\, \underbrace{1,\ldots,1}_{\lambda},0,\ldots,0\big)\mu^{\lambda} \qquad \text{and} \\ A''_\lambda &= \big(0 \,\big|\, \underbrace{1,\ldots,1}_{\lambda},0,\ldots,0\big)\mu^{\lambda-1}\,. \end{aligned}$$

Then we have

$$A'_\lambda = (v^{a-1} - v^{a+1})z^{-1} \prod_{j\geq\lambda+1} (1 - v^{-b_j}) \prod_{j=1}^{\lambda} \big[(v^{-b_j} - v^{-b_j+1})z^{-1}\big](v^{-1} - v)^{\lambda} z^{-\lambda}$$

and hence

$$\begin{aligned} max - max\ A'_\lambda &= (-1)v^{a+1}z^{-1}\cdot(-1)^{\lambda} v^{-\sum_1^{\lambda}(b_j-1)} z^{-\lambda}(-1)^{\lambda}v^{\lambda}z^{-\lambda} \\ &= (-1)v^{a-b+2\lambda+1}z^{-1-2\lambda}\,. \end{aligned}$$

Similarly, since

$$A'_\lambda = (1 - v^{a}) \prod_{j=1}^{\lambda} \big[(v^{-b_j-1} - v^{-b_j+1})z^{-1}\big] \prod_{j\geq\lambda+1} (1 - v^{-b_j})\cdot(v^{-1} - v)^{\lambda-1} z^{-(\lambda-1)}\,,$$

we have

$$\begin{aligned} max - max\ A'' &= (-1)v^a(-1)^\lambda v^{-\sum_1^\lambda (b_j-1)} z^{-\lambda}(-1)^\lambda v^{\lambda-1}z^{-\lambda+1} \\ &= (-1)v^{a-b+2\lambda-1}z^{-2\lambda+1} . \end{aligned}$$

Finally, since

$$B' = (- \mid 0, \ldots, 0)\mu = \prod_{i=1}^{p} (1-v^{-b_i})(v^{-1}-v)z^{-1} ,$$

we see that $max - max\ B' = -vz^{-1}$ .

Since $max\, deg_v\ A'_\lambda > max\, deg_v\ A''_\lambda$ , we only need to compare $max - max\, A'_\lambda$ and $max - max\, B'$ . And we conclude easily that if $a - \sum_1^\lambda b_j + 2\lambda > 0$ , then $max\, deg_v\ A'_\lambda > max\, deg_v\, B'$ , but if $a - \sum_1^\lambda b_j + 2\lambda \le 0$ , then $max - max\, P(v,z)\mu^{n+1} = max - max\, B'$ , because $\lambda \ge 1$ .

Now suppose that $a - \sum_{j=1}^{\lambda} b_j + 2\lambda > 0$ , i.e., $a - \sum_{j=1}^{\lambda}(b_j - 2) > 0$ . Since $b_j \ge 2$ for all $j \ge 1$ , it follows that $max\, deg_v A'_1 > max\, deg_v\ A'_\lambda$ for any $\lambda \ge 2$ . Therefore, if $2 = b_1, = \ldots = b_m < b_{m+1}$ , then

$$max - max\, P_L(v,z)\mu^{n+1} = max - max\, A'_1 = (-1)mv^{a+1}z^{-3} .$$

If $4 \le b_1 = \cdots = b_m < b_{m+1}$ , then $max - max\, P_L(v,z)\mu^{n+1} = max - max\, A'_1 = (-1)mv^{a-b_1+3}z^{-3}$ . Therefore, we have

$$max - max\, P_L(v,z) = (-1)^n mv^{a-n}z^{n-2}, \quad \text{if} \quad 2 = b_1 = \cdots = b_m < b_{m+1} ,$$

and

$$max - max\, P_L(v,z) = (-1)^n mv^{a-b_1-n+2}z^{n-2}$$

if, $4 \le b_1 = \cdots = b_m < a_{m+1}$ , and $a - b_1 + 2 > 0$ .

Suppose $a - \sum b_j + 2\lambda \le 0$ . Then

$$max - max\, P_L(v,z)\mu^{n+1} = max - max\, B' = -vz^{-1}$$

and hence $max - max\, P_L(v,z) = (-1)^n v^{-n} z^n$ . It completes the evaluation of $max - max\, P_L(v,z)$ .

$min - max\ P_L(v,z)$ follows from (2)(i), since $n \geq 2$ . (4)(ii)(A)(b) and (B)(c) follow from (8.8). □

Now we consider the case where all $c_i$ are odd.

**Theorem 13.10** *Let* $L = L[r \mid a_1, \ldots, a_p, -b_1, \ldots, -b_n]$ *be a pretzel link, where* $a_i$ *and* $b_j \geq 3$ . *Denote* $a = \sum_{i=1}^{p} a_i$ .

*(1) If* $r \geq 2$ *or if* $r = 1$ *and* $p \geq 1$ *, then*

$$max - max\, P_L(v,z)\mu^{n+1} = (-1)^{n+1} v^{a+r-n+1} z^{r+p+n-3} .$$

*(2) Suppose that* $r = 0$ *and* $p \geq 2$ *. Then*

$$max - max\, P_L(v,z) = (-1)^{n+1} v^{a-n+1} z^{p+n-3} .$$

*(3) Suppose that* $r = 0$ *and* $p = 1$ *. Suppose further that* $b_1 = \cdots = b_m < b_{m+1}$ .

*(i) If* $a + 2 > b_1$ *, then*

$$max - max\, P(v,z) = (-1)^{n+1} m v^{a-b_1+2-n} z^{n-2} .$$

*(ii) If* $a + 2 \leq b_1$ *, then*

$$max - max\, P(v,z) = (-1)^n v^{-n} z^n .$$

*(4) Suppose that* $r = 1$ *and* $p = 0$ *. Then*

$$max - max\, P_L(v,z) = (-1)^n v^{-n} z^n .$$

**Remark 13.11** If $n = 0$ , $L$ is special alternating and $\mathbf{b}(L)$ will be determined in Corollary 13.17. However, if $n > 0$ , then $min - max\, P_L(v,z)$ is generally quite complicated.

**Proof** Define the polynomials $x_c^\delta, y_c^\delta$ as follows

(13.12)
$$\begin{aligned} x_c^1 &= v^{c-1}, \\ x_c^0 &= \mu^{-1}(1-v^{c-1}) \\ y_c^1 &= \mu x_c^1 = (v^{c-2}-v^c)z^{-1} \\ y_c^0 &= \mu x_c^0 = 1-v^{c-1} \ , \end{aligned}$$

where $\mu = (v^{-1}-v)z^{-1}$.

Define $P(\lambda)$ as the skein polynomial of an elementary torus link of type $(\lambda, 2)$. To be more precise, $P(\lambda)$ is defined inductively as follows.

(13.13)
$$\begin{aligned} P(0) &= \mu \\ P(1) &= 1 \ , \\ \text{For} \quad \lambda \geq 1, & \qquad P(\lambda) = (1\ 0)\begin{pmatrix} vz & v^2 \\ 1 & 0 \end{pmatrix}^{\lambda-1}\begin{pmatrix} 1 \\ \mu \end{pmatrix} . \end{aligned}$$

For instance,

$$\begin{aligned} P(0) &= (v^{-1}-v)z^{-1} \ , \\ P(1) &= 1 \ , \\ P(\lambda) &= v^{\lambda-1}z^{\lambda-1} + \cdots - v^{\lambda+1}z^{\lambda-3}, \qquad \text{if} \quad \lambda \geq 2 \end{aligned}$$

and hence

$$max - max\ P(\lambda) \ = \ v^{\lambda+1}z^{\lambda-3} \ .$$

Now, it is proven in [LM] that

(13.14)
$$P_{L[r|a_1 \ldots a_p, -b_1 \ldots -b_n]} = \sum_{\delta(i)} x_1^{(1)} \ \ldots \ x_1^{\delta(r)} x_{a_1}^{\delta(r+1)} \ \ldots \ x_{-b_n}^{\delta(r+p+n)} P\Big(\sum_{i=1}^{r+p+n} \delta(i)\Big)$$

where the summation runs over the $2^{r+p+n}$ functions

$$\delta : \ \{1, 2, \ldots, r+p+n\} \to \{0, 1\} \ .$$

Since $x_1^0 = 0$, we may assume that $\delta(1) = \cdots = \delta(r) = 1$. Therefore (13.14) can be rewritten as

$$P_{L[r|a_1\ldots -b_n]} = \sum_{\delta(i)} x_{a_1}^{\delta(1)} \ldots x_{-b_n}^{\delta(p+n)} P\big(r + \Sigma\delta(i)\big) . \tag{13.15}$$

We consider $P_L(v,z)\mu^{p+n}$ instead of $P_L(v,z)$ as before. Then a simple computation shows that

$$P_L(v,z)\mu^{p+n} = \sum_{\delta} y_{a_1}^{\delta(1)} \ldots, y_{-b_n}^{\delta(p+n)} P\big(r + \Sigma\delta(i)\big) . \tag{13.16}$$

Now the rest of our proof is similar to that of Theorem 13.4, and hence, we will omit the details. □

**Corollary 13.17** *Let* $L = L[r \mid a_1, \ldots, a_p]$ *be a special alternating pretzel link, where* $r \geq 0$ *and all* $a_i$ *are positive odd integers and* $r + p \geq 2$. *Let* $D$ *be a special alternating diagram* $L$. *Denote* $a = \sum_{i=1}^{p} a_i$. *Then*

*(1)* $max - max\ P_L(v,z) = (-1)v^{a+r+1}z^{p+r-3}$

$min - max\ P_L(v,z) = \pm v^{r+p-1}z^{r+p-1}$

*(2)* $v - span\ P_L(v,z) = a - p + 2$

*(3)* $s(D) = a - p + 2$ *and* $ind\ D = \frac{1}{2}(a-p)$,

*(4)* $v - span P_L(v,z) = 2\{s(D) - 1 - ind D\}$, *and hence* $\mathbf{b}(L) = s(D) - ind D$.

**Proof** (1) follows from Theorem 13.10 (1)(2). (2) and (3) are easy. □

Now before we discuss the braid index of an arbitrary alternating pretzel link, we need to consider the other type of special alternating pretzel link.

Let $L(n_1, \ldots, n_{2k})$ be a pretzel link with even number of vertical strips, where $n_i$

are all positive.

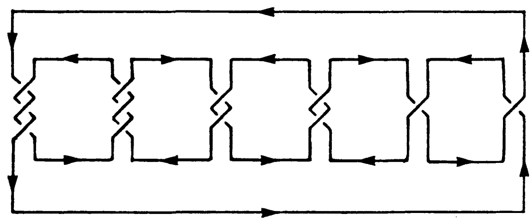


Fig. 13.3

Then it is possible to give orientation to each component in such a way that the resulting oriented link diagram is special and the boundary of each vertical strip belongs to distinct Seifert circles. For simplicity, such a special alternating pretzel link will be called a special alternating pretzel link of *even type.* (See Fig. 13.3).

**Theorem 13.18** *Let* $L = L(n_1, \dots, n_{2k})$ *be a special alternating pretzel link of even type, where* $0 < n_1 \le n_2 \le \cdots \le n_{2k}$ *. Denote* $N = \sum_{i=1}^{2k} n_i$ *.*

*Let* $D$ *be a special alternating diagram of* $L$ *. Then*

*(1)* $min - max\ P_L(v,z) = \pm v^{N-2k+1} z^{N-2k+1}$

*(2)* *(i) If* $k = 1$ *, then*

$$max - max\ P_L(v,z) = v^{n_1+n_2+1} z^{n_1+n_2-3}\ .$$

*(ii) Suppose that* $k \ge 2$ *. Assume furthermore that* $n_1 = n_2 = \cdots = n_m = 1$ *, but* $n_{m+1} \ge 2$ *.*

*(a) If* $k - 1 \le m$ *, then*

$$max - max\ P_L(v,z) = (-1)v^{N+1} z^{N-2k-1}\ .$$

*(b) If* $m \le k-2$ *, then*

$$max - max\ P_L(v,z) = (-1)^{m+1} v^{N+2k-2m-1} z^{N-6k+2m+1}\ .$$

*(3)* *(i)* $s(D) = 2k$

*(ii)* *If* $k-1 \leq m$ , *then* $ind_+(D) = k-1$ *and* $v-span\ P_L(v,z) = 2k$ , *and hence* $\mathbf{b}(L) = k+1$ .

*(iii)* *If* $m \leq k-2$ , *then* $ind_+(D) = m$ *and* $v-span\ P_L(v,z) = 2(2k-m-1)$ , *and hence* $\mathbf{b}(L) = 2k-m$

**Proof** Since $D$ is a special alternating positive diagram, it follows from (9.6) (ii) that $min-max\ P_L(v,z) = \pm v^{\phi_-(D)} z^{\psi_-(D)}$ . Since $J_-(D) = 0$ , we have $\phi_-(D) = \psi_-(D) = n(D) - s(D) + 1 = N - 2k + 1$ . It proves (1). (2) will be proven by induction on $m$ . If $m = 2k$ , then $L$ is a torus link of type $(2k, 2)$ for which Theorem 13.18 is already known.

For the general case, apply the skein relation at the crossings on the $(m+1)^{st}$ vertical strip. A careful comparison of various terms, using the induction hypothesis, will prove Theorem 13.18 (2). Since the argument is similar to that used in the proofs of the previous theorems, we will omit the details.

**Proof of (3)** Since the reduced Seifert graph of $D$ is a polygon with $2k$ sides, *ind* $D$ is easily evaluated from the definition. $\mathbf{b}(L)$ is determined by Theorem 8.12. □

Now, finally, we will consider an arbitrary alternating pretzel link.

**Proposition 13.19** *Let* $L$ *be an alternating pretzel link. Then* $L$ *has an alternating diagram* $D$ *such that*

*(1)* $D$ *has an even number, say* $p+1$ , *of major Seifert circles,* $S_0, S_1, \ldots, S_p$ .

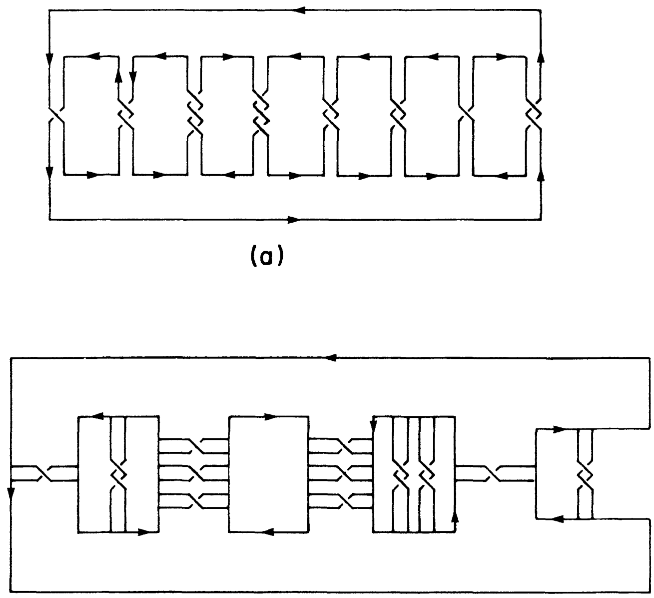
*(2)* $S_0$ *contains other* $S_i$ , $1 \leq i \leq p$.

*(3)* *each* $S_i$, $0 \leq i \leq p$ , *has finitely many vertical strips,* $B_{i,1}, \ldots, B_{i,k_i}$ *where* $B_{i,j}$ *has an even number, say* $n_{ij}$ *of positive half twists, and*

*(4)* $S_i$ *and* $S_{i+1}$, $0 \leq i \leq p$ $(S_{p+1} = S_0)$ , *are connected by finitely many, say* $r_{i+1}$ , *horizontal strips, each of which has only negative twist. For convenience,* $D$ *(or* $L$ *) will be called a link diagram (or a link) of type* $\{(n_{0,1}, \ldots, n_{0,k_0}), (n_{1,1}, \ldots, n_{1,k_1}), \ldots, (n_{p,1}, \ldots, n_{pk_p});\ (-r_1, \ldots, -r_{p+1})\}$ , *where* $n_{ij}$ *and* $r_i > 0$ . *See Fig. 13.4.*

A proof of Proposition 2.19 is easy and is omitted.

**Example 13.20** Let $L$ be a pretzel link $L$ shown in Fig. 13.4(a). Then $D$ has the type $\{(2),(2),(\phi),(2,2);(-1,-3,-3,-1)\}$. (Fig. 13.4(b).)



(a)

(b)

Fig. 13.4

**Theorem 13.21** *Let $L$ be an alternating pretzel link. Suppose that a link diagram $D$ of $L$ has the type*

$$\{(n_{0,1},\ldots,n_{0,k_0}),(n_{1,1},\ldots,n_{1,k_1}),\ldots,(n_{p,1},\ldots,n_{p,k_p});(-r_1,\ldots,-r_{p+1})\}$$

*Denote* $N=\sum_{ij} n_{ij}$, $K=\sum_{i=0}^{p} k_i$ *and* $R=\sum_{i=1}^{p+1} r_i$. *Then*

*(1)* $max-max\, P_L(v,z)=v^{N+K-R+p}\, z^{R-p-K}$

*(2)* $min-max\, P_L(v,z)$ *is given by the following formulae.*

*Let $m$ be the number of $r_i$ such that $r_i=1$.*

*(i) If* $p \leq 2m+1$ , *then*

$$min - max\ P_L(v,z) = v^{K-R-1} z^{K+R-2(p+1)-1}$$

*(ii) If* $p > 2m+1$ , *then*

$$min - max\ P_L(v,z) = v^{K-R-p+2m} z^{K+R-p+2m}$$

*(3)* $v - span\ P_L(v,z) = N + p + 1$ *if* $p \leq 2m+1$ , *and*

$v - span\ P_L(v,z) = N + 2p - 2m$ *if* $p > 2m+1$

*(4)* $s(D) = N - K + p + 1$ ,

$ind_+(D) = \frac{N}{2} - K$ , *and*

$ind_-(D) = \frac{p-1}{2}$ *if* $p \leq 2m+1$

$= m$ *if* $p > 2m+1$ .

*Therefore* $v - span\ P_L(v,z) = 2\{s(D) - 1 - ind\, D\}$ , *and hence we have*

$$\mathbf{b}(L) \ = \ s(D) - ind(D) \ .$$

A proof will be given by induction on $N$ . Since the argument is standard, we will omit the details.

**Example 13.22** Let $L$ be a pretzel link $L$ of type $\{(2),(2),(\phi),(2,2)\,;\,(-1,-3,-3,-1)\}$. Then we have

(13.23) (1) $max - max\ P_L(v,z) = v^7 z$

Since $m = 2$ and $p = 3$, we see that $p \leq 2m+1$ .

(2) $min - max\ P_L(v,z) = v^{-5} z^3$

(3) $v - span\ P_L(v,z) = 12$

(4) $s(D) = 8$, $ind_+(D) = 0$, $ind_- D = 1$ and $ind\, D = 1$ .

(5) $\mathbf{b}(L) = 7$

We note that $\Gamma(D)$ is a $*$-product of four subgraphs, three of which are positive and one is negative.

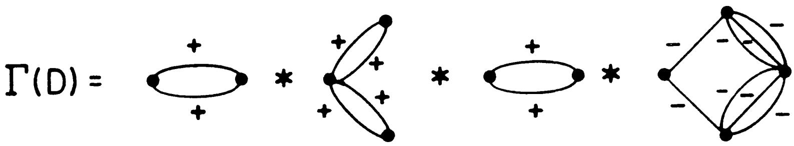


Fig. 13.5

## §14 Some other alternating links

In this section, we determine the braid index of an alternating link $L$ whose reduced Alexander polynomial $\Delta_L(t)$ has the small leading coefficient. The main theorem of this section is the following theorem

**Theorem 14.1** *Let $L$ be an alternating (non-split) link. Let $c_0(L)$ be the leading coefficient of the reduced Alexander polynomial $\Delta_L(t)$ of $L$. If $|c_0(L)| \leq 3$, then*

$$v - span\, P_L(v,z) \;=\; 2\big(\mathbf{b}(L) - 1\big)\,. \tag{14.2}$$

If $c_0(L) = \pm 1$, then $L$ is a fibred link for which (14.2) has already been proven [Mu 4]. Therefore we consider the case where $c_0(L) = \pm 2$ or $\pm 3$, but we will prove (14.2) for a slightly wider family of alternating links. (Cf. Theorem 14.4.)

Although our proof depends on the evaluation of $P_L(v,z)$, the method used here is completely different from the *standard* method employed in the previous sections.

First we introduce a new type of links.

A link depicted in Fig. 14.1 is called a *double pretzel link* of type $(a_1, \dots, a_k \mid b_1, \dots, b_k)$, and denoted by $L(a_1, \dots, a_k \mid b_1, \dots, b_k)$, where $a_i$ and $b_j$

are non-negative integers, and to each $i = 1, \ldots, k$, at least one of $a_i$ and $b_i$ is not zero.

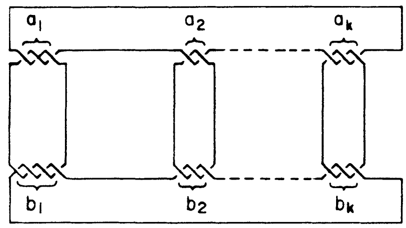



Fig. 14.1

If all $a_i = 1$ and all $b_j = 1$, or all $a_i = 0$ (or all $b_j = 0$), then it becomes an (ordinary) pretzel link.

Now if neither $a_i$ nor $b_j$ are zero, then we can give an orientation to each component of $L$, in such a way that the diagram of $L$ is positive and special alternating, and it has exactly $k+2$ Seifert circles. The special alternating (positive) link thus obtained will be called a special alternating *double pretzel link*. The Seifert graph $\Gamma$, then, is of the form depicted in Fig. 14.2.

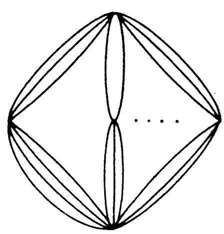


Fig. 14.2

**Proposition 14.3** *Let $L$ be a special alternating double pretzel link of type $(a_1, \ldots, a_k \mid b_1, \ldots, b_k)$, where $a_i$ and $b_j \geq 1$ for $i = 1, \ldots, k$. Let $D$ be a special alternating diagram of $L$.*

*(1) If all $a_i$ and $b_i \geq 2$, $i = 1, \ldots, k$, then $\operatorname{ind} D = 0$.*

*(2) If at least one of $a_i$ or $b_i$ is one, then $\operatorname{ind} D = 1$.*

A proof follows from the definition of the index.

Now the following theorem will prove (14.2) for $c_0(L) = \pm 2$.

**Theorem 14.4** *Let $L$ be an alternating link. Suppose that $L$ is a $*$-product of two alternating links $L_0$ and $L_1$, i.e. $L = L_0 * L_1$, where $L_0$ is an alternating fibred link and $L_1$, is either*

*(1) a special alternating (positive) pretzel link of even type $L_1 = L(n_1, \ldots, n_{2k})$, where*

*(i) $n_i \geq 2$ for all $i = 1, \ldots, 2k$, or*

*(ii) at least $k-1$ $n_i$'s are one, or*

*(2) a special alternating (positive) double pretzel link $L_1 = L(a_1, \ldots, a_k \mid b_1, \ldots, b_k)$, where all $a_i, b_i \geq 1$.*

*Let $D$ be the reduced alternating link diagram of $L$. Then*

$$(14.5) \qquad (i) \quad max\,deg_v\ P_L(v,z) = \phi_+(D) - 2\ ind\,D$$

$$(ii) \quad min\,deg_v\ P_L(v,z) = \phi_-(D)$$

*and hence*

$$v - span\ P_L(v,z) = 2\big(\mathbf{b}(L) - 1\big).$$

**Proof** Let $D_0$ and $D_1$ be the link diagrams of $L_0$ and $L_1$ respectively. Then $D = D_0 * D_1$ and $ind\,D = ind\,D_0 + ind\,D_1$ (by Proposition 7.2). First, since $ind_-D_0 = 0$, it follows that $ind_-D = 0$ and hence, $min\,deg_v\ P_L(v,z) = \phi_-(D)$ by Theorem 9.5. Therefore, it remains to prove (14.5) (i).

Now if $ind\,D_1 = 0$, then $ind\,D = 0$, and hence (14.5) (i) follows from Theorem 9.5. Therefore, we may assume henceforth that $ind\,D_1 \geq 1$, and hence $ind\,D \geq 1$.

Now we need a few technical lemmas due.

**Lemma 14.6** *There exists a binary resolving tree for $D$ such that*

*(1) on the root-leaf path, no crossings are changed twice,*

*(2) in the diagram $D_0$, for any pair of Seifert circles connecting by crossing there is an unchanged connecting crossing in every leaf diagram, and*

*(3) at least $k$ crossings in $D_1$ are unchanged.*

**Proof** For a proof of (1), we refer to [C]. For (2), we refer to [Mu 7, Lemma 5.2].

Finally (3) follows from an easy geometric argument. We omit the detail. □

Let $f(x_1, \ldots, x_n)$ be a Laurent polynomial in $x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}$. Write

$$f(x_1, \ldots, x_n) = \sum_{-\infty < i_1, \ldots, i_n < \infty} a_{i_1 \ldots i_n} x_1^{i_1} \ldots x_n^{i_n} .$$

Define

$$max\, deg_{x_1,\ldots,x_n} f(x_1, \ldots, x_n) = max\{i_1 + \cdots + i_n \mid a_{i_1 \ldots i_n} \neq 0\} .$$

Now since $L_0$ is an alternating fibred link, $L_0$ is a $*$-product of, say $p$, positive elementary fibred torus links and, say $n$, negative elementary fibred torus links. We write

$$L_0 = L_{0,1} * \cdots * L_{0,p} * L'_{0,1} * \cdots * L'_{0,n}$$

**Lemma 14.7**

$$max\, deg_{v,z} P_L(v, z) \leq 2(n_+(D) - p - k) ,$$

*where $n_+(D)$ denotes the number of positive crossings in $D$.*

**Proof** Consider the resolving tree for $D$ found in Lemma 14.6. For each leaf $D^{(\ell)}$ of the tree, $max\, deg_{v,z} P_L(v, z) = 1$, since $P_{D^{(\ell)}}(v, z) = \left(\frac{v^{-1} - v}{z}\right)^{\lambda - 1}$, where $\lambda$ is the number of components of $D^{(\ell)}$. An application of a crossing change or smoothing at a positive crossing increases the degree by two, while the smoothing a negative crossing does not change the degree. On the other hand, a crossing change at a negative crossing decreases the degree by two. Therefore the maximal degree of the term associated to $D^{(\ell)}$ is $2(t_+ - k_-)$, where $t_+$ is the total number of crossing changes and smoothings at positive crossings and $k_-$ the number of crossing changes at negative crossings. Then, a possible maximal degree will be $2t_+ = 2(n_+(D) - p - k)$ (with $k_- = 0$). □

**Lemma 14.8** *Let $\sigma(L)$ denote the signature of $L$.*

*(1) If* $L_1 = P(n_1, \ldots, n_{2k})$, *then* $-\sigma(L) = \tilde{n}(D) - p + n - 2k + 1$

*(2) If* $L_1 = P(a_1, \ldots, a_k \mid b_1, \ldots, b_k)$, *then* $-\sigma(L) = \tilde{n}(D) - p + n - k - 1$.

**Proof** Let $m_i$ (or $m'_i$) be the number of crossings in a link diagram $D_{0i}$ (or $D'_{0i}$) of $L_{0i}$ (or $L'_{0i}$). Since $L$ is alternating, it follows from [Mu 2] that

$$\begin{aligned} \sigma(L) &= \sigma(L_0) + \sigma(L_1) \qquad \text{and} \\ \sigma(L_0) &= \sum_{i=1}^{p} \sigma(L_{0i}) + \sum_{j=1}^{n} \sigma(L'_{0j}) = -\sum_{i=1}^{p}(m_i - 1) + \sum_{j=1}^{n}(m'_j - 1) \\ &= -\tilde{n}(D_0) + p - n . \end{aligned}$$

Now since $L_1$ is a special alternating (positive) link, $-\sigma(L_0)$ is equal to the number of domains in $\mathbb{R}^2 - \Gamma(D_0)$ minus one. Therefore, if $L_1 = P(n_1, \ldots, n_{2k})$, then

$$\begin{aligned} -\sigma(L_1) &= \sum_{i=1}^{2k}(n_i - 1) + 1 = n_+(D_1) - 2k + 1 \qquad \text{and hence} \\ -\sigma(L) &= n_+(D_1) - 2k + 1 + \tilde{n}(D_0) - p + n = \tilde{n}(D) - p + n - 2k + 1 . \end{aligned}$$

If $L_1 = L(a_1, \ldots, a_k \mid b_1, \ldots, b_k)$, then

$$-\sigma(L_1) = \sum_{i=1}^{k}(a_i - 1) + \sum_{j=1}^{k}(b_j - 1) + k - 1 = n_+(D_1) - k - 1$$

and hence

$$-\sigma(L) = n_+(D_1) - k - 1 + \tilde{n}(D_0) - p + n = \tilde{n}(D) - p + n - k - 1 . \qquad \square$$

Now to prove (14.5), we need a few formulae involving the Jones polynomial. Let $V_L(t)$ be the Jones polynomial of a link $L$. It was proved in [Mu 6] that if $L$ is an alternating link and $D$ is an alternating diagram of $L$ then

$$\text{(14.9)} \qquad max\, deg\, V_L(t) = n_+(D) - \frac{1}{2}\sigma(L) .$$

Using (14.9) and Lemmas 14.7 and 14.8, we can prove (14.5)(i) as follows.

Let $L_1$ be a pretzel link of even type. Then we see from (14.9) that

$$\text{(14.10)} \qquad max\, deg V_L(t) = n_+(D) + \frac{1}{2}\big(\tilde{n}(D) - p + n - 2k + 1\big) \ .$$

Write

$$P_L(v,z) \ = \sum_{-\infty<i<\infty} a_i(z)v^i \ .$$

Since $P_L(t, \sqrt{t} - \frac{1}{\sqrt{t}}) \ = \ V_L(t)$ we have

$$\max_i \big\{\frac{1}{2} max\, deg_z\ a_i(z) + i\big\} \geq max\, deg\, V_L(t) \ .$$

Therefore, $P_L(v,z)$ contains some monomial $M(v,z)$ such that

$$\text{(14.11)} \qquad \frac{1}{2}\ max\, deg_z\ M(v,z) + max\, deg_v\ M(v,z) \geq maxdeg\, V_L(t)$$

and hence

$$max\, deg_z\ M(v,z) + 2\ max\, deg_v\ M(v,z) \geq 2\, maxdeg V_L(t) \ .$$

However, Lemma 14.7 yields

$$\text{(14.12)} \qquad max\, deg_z\ M(v,z) + max\, deg_v\ M(v,z) \leq 2\big(n_+(D) - p - k\big) \ .$$

Combining (14.10), (14.11) and (14.12), we can show that

$$\text{(14.13)} \qquad max\, deg_v\ M(v,z) \geq n_+(D) - n_-(D) + p + n + 1 = \tilde{n}(D) + p + n + 1 \ .$$

However, Theorem 8.3 shows that

$$max\, deg_v\ P_L(v,z) \leq \tilde{n}(D) + s(D) - 1 - 2\, ind_+ D \ .$$

Since $s(D) = p + n + 2k$ and $ind_+ D = ind\, D_1 = k - 1$ , it follows from (14.13) that

$$\begin{aligned} \tilde{n}(D) + p + n + 1 &\leq max\, deg_v\ M(v,z) \leq max\, deg_v\ P_L(v,z) \\ &\leq \tilde{n}(D) + s(D) - 1 - 2\ ind_+ D \ = \ \tilde{n}(D) + p + n + 1 \ , \end{aligned}$$

and hence

$$\begin{aligned} max\, deg_v\ M(v,z) &= max\, deg_v\ P_L(v,z) \\ &= \tilde{n}(D) + s(D) - 1 - 2\, ind_+ D \\ &= \phi_+(D) - 2\, ind_+ D \ . \end{aligned}$$

A similar argument works for $L_1 = L(a_1, \dots, a_k \mid b_1, \dots, b_k)$.

In fact, since

$$\begin{aligned} max\, deg\ V_L(t) &= n_+(D) - \frac{1}{2}\sigma(L) \\ &= n_+(D) + \frac{1}{2}\big(\tilde{n}(D) - p + n - k - 1\big), \end{aligned}$$

we have

$$\frac{1}{2} max\, deg_v\ M(v,z) + max\, deg_v M_L(v,z) \geq n_+(D) + \frac{1}{2}\big(\tilde{n}(D) - p + n - k - 1\big),$$

while

$$max\, deg_v M(v,z) + max\, deg_v M(v,z) \leq 2\big(n_+(D) - p - k\big) \ .$$

Therefore we see

$$max\, deg_v M(v,z) \geq \tilde{n}(D) + p + n + k - 1 \ .$$

Since $s(D) = p + n + k + 2$ and $ind_+ D (= ind\ D) = 1$, it follows that

$$\begin{aligned} \tilde{n}(D) + p + n + k - 1 &\leq max\, deg_v\ M(v,z) \\ &\leq \tilde{n}(D) + s(D) - 1 - 2\ ind_+ D \\ &= \tilde{n}(D) + p + n + k - 1 \\ &= \phi_+(D) - 2\ ind_+ D \ . \end{aligned}$$

This proves (14.5)(i). □

**Proposition 14.14** *Let $L$ be a special alternating link and $D$ a special diagram of $L$.*

*(1) If $c_0(L) = \pm 2$, then $L$ is an alternating pretzel link $P(n_1, n_2, n_3, n_4)$ of even type, where $n_i > 0$, $i = 1, 2, 3, 4$.*

*(2) If $c_0(L) = \pm 3$, then $L$ is either*

*(i) an alternating pretzel link* $P(n_1, n_2, n_3, n_6)$ *where* $n_i > 0$, $i = 1, 2, \ldots, 6$, *or*

*(ii) a special alternating double pretzel link* $L(a_1, a_2, a_3 \mid b_1, b_2, b_3)$, *where* $a_i$ *and* $b_i > 0$, $i = 1, 2, 3$.

**Proof** Since $D$ is special alternating, the Seifert graph $\Gamma$ of $D$ is a plane and bipartite graph. Let $\Gamma^*$ be the dual of $\Gamma$. Then $\Gamma^*$ is a plane even graph. Therefore, we can define $\lambda(\Gamma^*)$. (See Definition 4.1.) Since $D$ is a special alternating diagram, $\lambda(\Gamma^*)$ is equal to $|c_0(L)|$. (See §11.) Therefore the proposition is a consequence of Proposition 4.14. □

Since $|c_0(L_0 * L_1)| = |c_0(L_0)|\,|c_0(L_1)|$, it follows from Proposition 14.14 and Theorem 14.4 that (14.2) holds for an alternating link $L$ with $c_0(L) = \pm 2$ and for all alternating links with $c_0(L) = \pm 3$ except for those whose second $*$-component $L_1$ is an alternating pretzel link of even type $P(n_1, n_2, n_3, n_4, n_5, n_6)$ such that only one $n_i$ is 1. For this exceptional case, we cannot apply Theorem 14.4 (1) directly. However, we can use almost the same argument employed to prove (1). In fact, we can improve Lemma 14.6 ($k = 3$) in such a way that Lemma 14.6 (3) is replaced by a new statement: $(3)'$ at least $4 (= k + 1)$ crossings in $D_1$ are unchanged.

Then we can show that there is a monomial $M(v, z)$ in $P_L(v, z)$ such that

$$\begin{aligned} \tilde{n}(D) + p + n + 1 \;&\leq\; max\,deg_v\; M(v, z) \leq max\,deg_v\; P_L(v, z) \\ &\leq\; \tilde{n}(D) + s(D) - 1 - 2\; ind_+ D\,. \end{aligned}$$

Since $s(D) = p + n + 6$ and $ind_+ D = 1$, we see that

$$\tilde{n}(D) + p + n + 3 \;=\; \tilde{n}(D) + s(D) - 1 - 2 \;=\; \phi_+(D) - 2$$

and hence,

$$max\,deg_v\; P_L(v, z) \;=\; \phi_+(D) - 2\, ind_+ D\,.$$

Since $min\,deg_v\; P_L(v, z) = \phi_-(D)$, (14.2) follows. A proof of Theorem 14.1 is now complete. □

## §15 Concluding Remarks and Conjectures

In their paper [FW], Frank and Williams propose the following conjecture (that was disproved recently by Morton and Short [MS]):

**Conjecture 15.1** *[FW]. Let $\beta$ be a positive $n$-braid and $L$ the closure of $\beta$. Then*

$$v - span\ P_L(v, z) = 2(\mathbf{b}(L) - 1). \tag{15.2}$$

Our research has begun by trying to prove this conjecture for alternating links. We succeeded to prove (15.2) for many alternating links in Chapter III. Unfortunately, however, (15.2) does not hold, in general, for alternating links. The simplest counter-example we found is the 4-component link $L_1$ depicted in Fig. 15.1 (a). $L_1$ is the only link (up to mutation), for which (15.2) fails, among all links having special alternating positive diagrams with at most 15 crossings and index one. However, up to mutation, the simplest

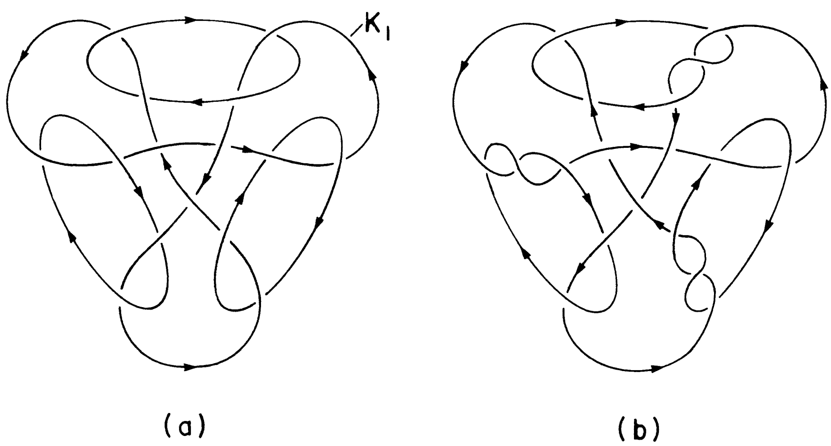



Fig. 15.1

special alternating positive *knot* (whose diagram has the index one), for which (15.2) fails, is the knot $L_2$ depicted in Fig. 15.1 (b). $L_2$ has 18 crossings. The Seifert graphs of (the diagrams of $D_1$ and $D_2$ ) of $L_1$ and $L_2$ are shown in Fig. 15.2.

(a) (b)

Fig. 15.2

(We should note that the Seifert graph of this link does not have locally maximal subgraphs.) Now a computation reveals that for $i = 1$ or $2$, $v - span\ P_{L_i}(v, z) = 8$, and hence $\frac{1}{2}\big(v - span\ P_{L_i}(v,z)\big) + 1 = 5$. On the other hand, since $ind_+ D = 1$, we see that $s(D_i) - ind\ D_i = 6$ and hence $5 \leq \mathbf{b}(L_i) \leq 6$. However, we can see that $\mathbf{b}(L_i) = 6$. To prove this, we compute the skein polynomial of the 2-cables of a link, as was seen in [MS]. First consider the knot $L_2$. The simplest 2-cable $L'$ of $L_2$ has 72 crossings. If $\mathbf{b}(L_2) = 5$, then $\mathbf{b}(L') = 10$ and hence $v - span\ P_{L'}(v,z) \leq 18$. Therefore, to prove $\mathbf{b}(L_2) = 6$, it suffices to show that $v - span\ P_{L'}(v,z) \geq 20$. However, to show this, it will not need to compute the whole polynomial $P_{L'}(v,z)$. In fact, write $P_{L'}(v,z) = \sum_{i=r}^{s} \lambda_i(v) z^i$, $r < s$. Then as is observed in [PP], the computation of the first few terms $\lambda_r(v), \ldots,$ is much faster than that of $P_{L'}(v,z)$, (approximately in time) $n(D)^{\log n(D)}$, where $D$ is the diagram of $L'$. See [PP]. J. Hoste has computed the first five (non-zero) terms $\lambda_{-1}(v)$, $\lambda_1(v), \ldots, \lambda_7(v)$ and found that $v - span\ P_{L'}(v,z) \geq 20$.

On the other hand, to show that $\mathbf{b}(L_1) = 6$, we consider the 5-component link $L''$ obtained from $L_1$ by taking the 2-cable of only one component $K_1$ (in Fig. 15.1(a)) and leaving the other component untouched. $L''$ has 36 crossings. The first five (non-zero) terms $\lambda_{-4}(v), \ldots, \lambda_4(v)$ of $P_{L''}(v,z)$ are enough to show that $v - span\ P_{L''}(v,z) \geq 14$, and hence $\mathbf{b}(L'') \geq 8$. Since $L_1$ has four components, each component of $L_1$ must be represented as a 1- or 2-braid in the (minimal) braid representation of $L_1$. Therefore,

$\mathbf{b}(L_1)$ cannot be equal to 5. (We are grateful to J. Hoste who wrote the computer program and carried out the computations of the major part of the proof.)

These examples, however, suggest the following

**Conjecture 15.3** *If $L$ is an alternating link and $D$ is an alternating diagram, then*

$$\mathbf{b}(L) = s(D) - ind\ D.$$

Finally many numerical link type invariants are additive with respect to the connected sum. According to [BM], $\mathbf{b}(L) - 1$ is additive with respect to the connected sum. If $\mathcal{F}$ is a family of links for which (15.2) holds, then the additivity of $\mathbf{b}(L) - 1$ is additive with respect to the connected sum of links in $\mathcal{F}$ follows from the fact that $v\text{-}spanP_L(v, z)$ is additive with respect to the connected sum [LM]. If $L$ is an alternating link, however, we would like to propose a much stronger conjecture

**Conjecture 15.4** *Let $L$ be an alternating link and let $L_1, L_2, \cdots, L_k$ be $*$-components of the alternating diagram of $L$. Then*

$$\mathbf{b}(L) - 1 = \sum_{i=1}^{k} \{\mathbf{b}(L_i) - 1\}.$$

*In other words, $\mathbf{b}(L) - 1$ is additive with respect to $*$-product for an alternating link.*

Conjecture 15.4 would follow from Conjecture 15.3, since $s(D) - 1 = \sum_{i=1}^{k}\{s(D_i) - 1\}$ and $ind\ D = \sum_{i=1}^{k} ind\ D_i$, where $D$ and $D_i$ are link diagrams of $L$ and $L_i$, $i = 1, 2, \cdots, k$.

**Remark 15.5** After distributing the preliminary version of our paper, P. Traczyk informed us that he proved our conjecture proposed in §3, i.e. for bipartite graphs, the index and the cycle index coincide. Furthermore, D. Welsh has proven that computing the cycle index of a graph is NP-hard. See [We].

# Appendix

## (I). Proof of Lemma 9.12

First we consider the case where all Seifert circles of $E$ are cut by $\hat{D}$.

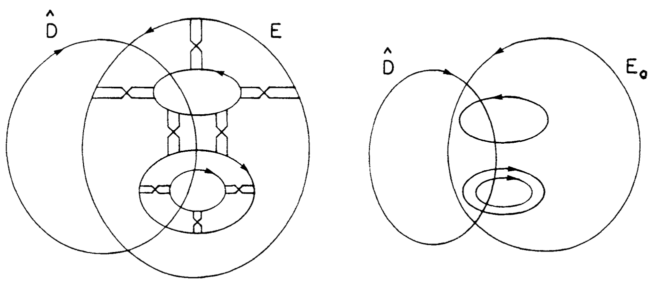



Fig. A.I.1

Let $E_0$ be the diagram consisting of $s(E)$ disjoint circles obtained from $E$ by smoothing all crossings in $E$. Let $D_0 = E_0 \cup \hat{D}$. Note that $s(E_0) = s(E)$.

Now $E_0$ divides $\mathbb{R}^2$ into $s(E)+1$ domains $V_1, V_2, \cdots, V_{m+1}$, where $m = s(E)$. By the assumption, $\hat{D}$ cuts each domain $V_i$. Furthermore, $\hat{D} \cap V_i$ and $\hat{D} \cap V_j (i \neq j)$ are parts of distinct Seifert circles in $D_0$. Therefore $s(D_0) \geq s(E_0) + 1$. If $k$ Seifert circles are not cut by $\hat{D}$, then these circles are not affected in the previous argument and hence we have an inequality $s(D) \geq s(E) + 1$. This proves (1).

To prove (3), we again assume that all Seifert circles of $E$ are cut by $\hat{D}$. We will use the same notation $E_0$ and $D_0$. Now we associate a graph $G$ with $E_0$ as follows. Each vertex $v_i$ of $G$ corresponds to each domain $V_i$, $i = 1, 2, \cdots, m+1$, and each edge $e_i$ of $G$ corresponds to each Seifert circle $S_i$ in $E$, and $e_i$ connects two vertices $v_j$ and $v_k$ iff $S_i$ is the common boundary of $V_j$ and $V_k$. (Fig. A. I. 2)

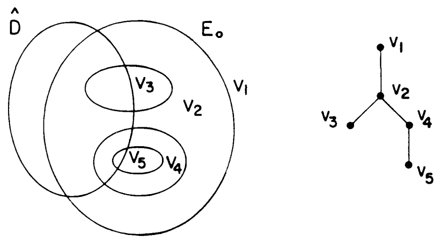



Fig. A. I. 2

Obviously, $G$ is a tree. Since $s(D_0) = s(E_0) + 1$, for any $i$ every arc in $\hat{D} \cap V_i$ must be a part of the same Seifert circle, say $S_i'$, in $D$. Then $S_i'$ and $S_j'$ are joined by a crossing in $D_0$ iff $V_i$ and $V_j$ have a common boundary. Therefore, $G$ is exactly the *reduced* Seifert graph $\hat{\Gamma}(D_0)$ of $D_0$.

Now to show that $\hat{\Gamma}(D_0) = \hat{\Gamma}(D)$, it suffices to prove that whenever $S_i'$ and $S_j'$ are joined in $D$, $V_i$ and $V_j$ have a common boundary. Suppose the contrary, i.e. $V_i$ and $V_j$ have no boundaries in common. Then there is a domain $V_k (k \neq i, j)$ such that $V_k$ has a common boundary to each $V_i$ and $V_j$. Therefore, Seifert circles, $S_i', S_j'$ and $S_k'$ in $D$ are connected with each other by crossings in $D$. This is impossible, since $\hat{\Gamma}(D)$ is bipartite. This proves the first part of (3). The second part of (3) follows immediately from the fact that $\hat{\Gamma}(D) = G$ is a tree.

To prove (2), first assume that every Seifert circle in $E$ is cut by $\hat{D}$. Then, we see that $J(D_0) \leq \frac{1}{2} cr(\hat{D}, E_0)$ and $J(D) = J(D_0)$ and hence

$$J(D) \leq \frac{1}{2} cr(\hat{D}, E_0).$$

Since $cr(\hat{D}, E_0) = cr(\hat{D}, E)$ and $J_+(D) \leq J(D)$, it follows that $J_+(D) \leq \frac{1}{2} cr(\hat{D}, E)$ and hence

$$J_+(D) \leq J_+(E) + \frac{1}{2} cr(\hat{D}, E).$$

Now suppose that there are Seifert circles in $E$ which are not cut by $\hat{D}$. To be more precise, let $S_1, S_2, \cdots, S_k$ be Seifert circles in $E$ which are cut by $\hat{D}$. Let $E'$ be the part of the diagram of $D$ which consists of $S_1, \cdots, S_k$ and crossings connecting these Seifert circles. Let $D' = E' \cup \hat{D}$. Then the previous argument shows that

$$J_+(D') \leq J_+(E') + \frac{1}{2}cr(\hat{D}, E').$$

Therefore, to prove (2), it suffices to show that

$$(A.1) \qquad J_+(D) - J_+(D') \leq J_+(E) - J_+(E').$$

In fact, since $J_+(D') \leq J_+(E') + \frac{1}{2}cr(\hat{D}, E)$, we will have

$$J_+(D) \leq J_+(E) + J_+(D') - J_+(E') \leq J_+(E) + \frac{1}{2}cr(\hat{D}, E) .$$

Now to prove (A.1) we must show that if two crossings of $E$ outside $E'$ join a pair of Seifert circles in $E$, then they join a pair of the Seifert circles in $D$.

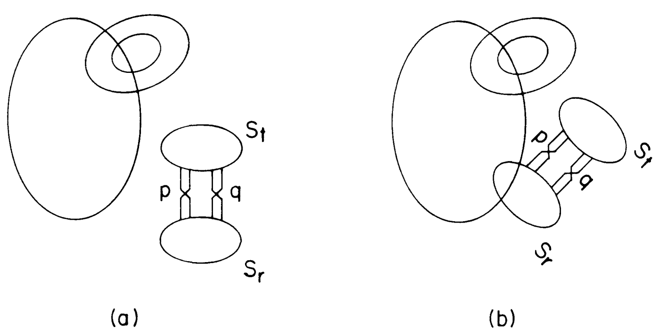



Fig. A.I.3

Assume that $p$ and $q$ are cossings of $E$ outside $E'$ joining Seifert circles $S_t$ and $S_r$ of $E$. Suppose that $S_t$ and $S_r$ are in $E - E'$. (Fig. A.I.3 (a)). Then obviously $p$ and $q$ connect between $S_t$ and $S_r$ in $D$. Suppose that $S_t$ occurs in $E - E'$ and $S_r$ in $E'$ (Fig. A.I.3 (b)). Then, since $S_t$ is disjoint from $\hat{D}$, $p$ and $q$ occur on the same side of $\hat{D}$. Let $S'_r$ be the (not necessarily connected) part of $S_r$ which is on the same side of $\hat{D}$ as $p$ and $q$. Since $s(D) = s(E) + 1$, $S'_r$ is a part of the unique Seifert circle in $D$

and this circle is connected to $S_t$ by $p$ and $q$. It proves (A.1). A proof of Lemma 9.12 is now complete. □

**(II) Proof of Lemma 9.13.**

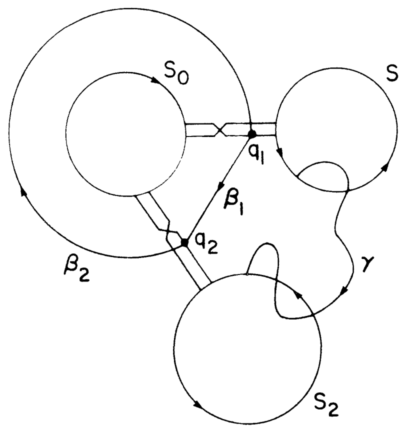



Fig. A.II.1

Let $U$ be a simple closed curve passing through two points $q_1$ and $q_2$ such that $U \cap \gamma = \{q_1, q_2\}$ and $U$ is close to $S_0$. (Fig. A. II.1). We also assume that $U$ lies above $D$. $U$ is decomposed into two simple arcs $\beta_1$ and $\beta_2$ by two points $q_1$ and $q_2$. (Fig. A.II.1). Then $s(D \cup U) = s(D) + 1$. If we smooth $D \cup U$ at $q_1$ and $q_2$, we have a new link diagram consisting of $D' = (D - \gamma) \cup \beta_1$ and $\hat{D} = \gamma \cup \beta_2$. (Fig. A.II.2).

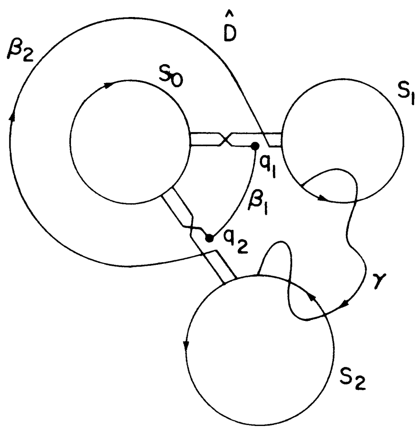



Fig. A.II.2

Since $D'$ is regularly isotopic to $D$, we see that $\tilde{n}(D) = \tilde{n}(D')$. Note that $\hat{D}$ is a simple closed curve lying above $D'$. It is easy to see that $D \cup U$ and $D' \cup \hat{D}$ have exactly the same set of Seifert circles and hence $s(D \cup U) = s(D' \cup \hat{D})$. Note that $U$ "forms" a new Seifert circle in $D \cup U$, denoted by $S_u$. Let $\overline{D}$ be the link diagram consisting of those Seifert circles in $D' \cup \hat{D}$ which are not disjoint from $\hat{D}$, and crossings of $D' \cup \hat{D}$ between these Seifert circles. Let $\overline{E} = \overline{D} - \hat{D}$. Then $\hat{D}$ cuts each Seifert circle in $\overline{E}$. However, if we travel along $\hat{D}$, we leave (before $q_1$) the Seifert circle $S_u$ for $S_1$ and go back to $S_u$ (after $q_2$) from $S_2$. Therefore, Lemma 9.12 (3) implies that $s(\overline{D}) > s(\overline{E}) + 1$ and hence $s(D') < s(D' \cup \hat{D}) - 1 = s(D \cup U) - 1 = s(D)$. Therefore, we have

$$\begin{aligned} max\, deg_v\ P_D(v,z) &= max\, deg_v\ P_{D'}(v,z) \leq \tilde{n}(D') + s(D') - 1 < \tilde{n}(D) + s(D) - 1 \\ &= \phi_+(D). \end{aligned}$$

This proves Lemma 9.13.

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