Core Books in Advanced Mathematics

Methods of Trigonometry

J E Hebborn C Plumpton

Core Books in Advanced Mathematics

Methods of Trigonometry

Core Books in Advanced Mathematics

General Editor: C. PLUMPTON, Moderator in Mathematics,
University of London School Examinations Department;
formerly Reader in Engineering Mathematics,
Queen Mary College, University of London.

Titles available

Methods of Trigonometry
Coordinate Geometry and Complex Numbers
Newton's Laws and Particle Motion
Mechanics of Groups of Particles
Proof
Differentiation
Integration
Vectors
Curve Sketching

Core Books in Advanced Mathematics

Methods of Trigonometry

J. E. Hebborn
Chief Examiner in Advanced Level Mathematics, University of London School Examinations Department; Senior Lecturer in Mathematics, Westfield College, University of London.

C. Plumpton
Moderator in Mathematics, University of London School Examinations Department; formerly Reader in Engineering Mathematics, Queen Mary College, University of London.

Macmillan Education
London and Basingstoke

ISBN 978-0-333-36537-3 ISBN 978-1-349-07109-8 (eBook)
DOI 10.1007/978-1-349-07109-8

First published 1984

Published by
Macmillan Education Limited
Houndmills Basingstoke Hampshire RG21 2XS
and London
Associated companies throughout the world

Typeset in Hong Kong by Asco Trade Typesetting Ltd.

Contents

Preface

Advanced level mathematics syllabuses are once again undergoing changes in content and approach following the revolution in the early 1960s which led to the unfortunate dichotomy between 'modern' and 'traditional' mathematics. The current trend in syllabuses for Advanced level mathematics now being developed and published by many GCE Boards is towards an integrated approach, taking the best of the topics and approaches of modern and traditional mathematics, in an attempt to create a realistic examination target through syllabuses which are maximal for examining and minimal for teaching. In addition, resulting from a number of initiatives, core syllabuses are being developed for Advanced level mathematics consisting of techniques of pure mathematics as taught in schools and colleges at this level.

The concept of a core can be used in several ways, one of which is mentioned above, namely the idea of a core syllabus to which options such as theoretical mechanics, further pure mathematics and statistics can be added. The books in this series are core books involving a different use of the core idea. They are books on a range of topics, each of which is central to the study of Advanced level mathematics, which together cover the main areas of any single-subject mathematics syllabus at Advanced level.

Particularly at times when economic conditions make the problems of acquiring comprehensive textbooks giving complete syllabus coverage acute, schools and colleges and individual students can collect as many of the core books as they need to supplement books they already have, so that the most recent syllabuses of, for example, the London, Cambridge, AEB and JMB GCE Boards can be covered at minimum expense. Alternatively, of course, the whole set of core books gives complete syllabus coverage of single-subject Advanced level mathematics syllabuses.

The aim of each book is to develop a major topic of the single-subject syllabuses giving essential book work, worked examples and numerous exercises arising from the authors' vast experience of examining at this level. Thus, as well as using the core books in either of the above ways, they are ideal for supplementing comprehensive textbooks by providing more examples and exercises, so necessary for the preparation and revision for examinations.

In this particular book we cover the requirements of non-specialist mathematicians in basic trigonometry in accordance with the core syllabus of pure mathematics now being included by GCE Examining Boards. It also meets the requirements of the polytechnics and universities for entrants to degree courses in mathematics-related subjects. No previous knowledge of trigonometry is assumed; that is, the basic definitions are stated and the standard results are derived, some by means of worked examples. While inevitably lacking experience, the student should try to acquire and appreciate good technique, so that more difficult problems may be tackled with confidence. Only the most elementary knowledge of coordinates and the theorem of Pythagoras have been assumed.

Confidence in using calculators which contain the trigonometric functions and their inverses is essential both at 'A' level and beyond. No reference is made to trigonometric tables in this book. The calculator is here to stay and we must use it accurately where appropriate. Plenty of GCE-type examples are provided throughout the book, both as exercises and as part of the text.

J. E. Hebborn
C. Plumpton

1 Trigonometric functions

1.1 Some definitions

There are two distinct ways of measuring angles. Since the days of the Babylonians, angles have been measured by using a subdivision of the circle into 360 parts or *degrees*. Another unit, the *radian*, is often used with advantage: this is the angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle (see Fig. 1.1(a)).

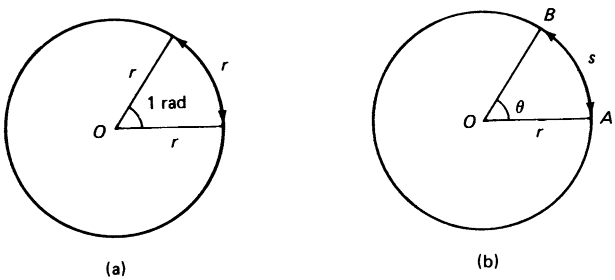

Fig. 1.1

Since the circumference of the circle is of length $2\pi r$, the total angle at the centre O is 2π radians. Therefore 2π radians $= 360°$ giving

$$1 \text{ rad} = \frac{360°}{2\pi} \approx 57{\cdot}3° \; (57° \; 18'),$$

$$\text{or} \quad 1° = \frac{\pi}{180} \approx 0{\cdot}0175 \text{ rad}.$$

Further $180° = \pi$ rad, $90° = \pi/2$ rad, $60° = \pi/3$ rad, $45° = \pi/4$ rad and $30° = \pi/6$ rad. Sometimes the index ^{c} is used to denote radians but we usually just write $180° = \pi$, and so on, rads being understood.

It follows from the definition of radian measure that the length s of the arc subtending an angle θ (radians) at the centre O of a circle of radius r is given by (see Fig. 1.1(b))

$$s = r\theta. \tag{1.1}$$

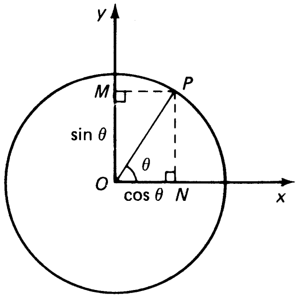

Fig. 1.2

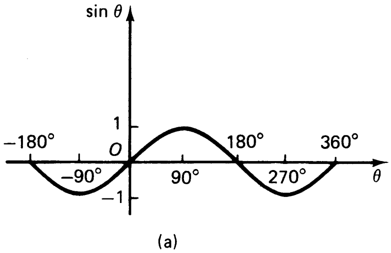

Fig. 1.3(a)

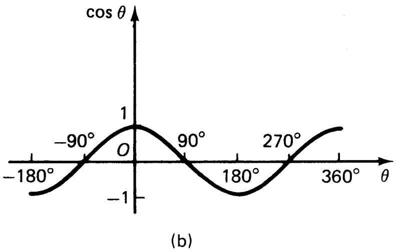

Fig. 1.3(b)

Also the area of the (minor) sector AOB is

$$\frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2} r^2 \theta. \tag{1.2}$$

In Fig. 1.2 the point P is describing a circle of radius 1 and centre O in the x, y plane, its position being specified by the angle $xOP = \theta$ measured anti-clockwise. The projection (ON) of OP onto the x-axis is *defined* to be $\cos\theta$ and its projection (OM) onto the y-axis is *defined* to be $\sin\theta$ for all values of θ. It follows immediately that

$$\cos\theta = \cos(-\theta), \quad \sin\theta = -\sin(-\theta), \tag{1.3}$$

so that $\cos\theta$ is an *even* function and $\sin\theta$ is an *odd* function and further

$$\cos(\theta + 2n\pi) = \cos\theta, \quad \sin(\theta + 2n\pi) = \sin\theta, \quad n \in \mathbb{Z}, \tag{1.4}$$

where $\mathbb{Z}$ is the set of integers $\{0, \pm 1, \pm 2, \pm 3, \ldots\}$.
Note that $\cos\theta$ and $\sin\theta$ are periodic functions of period 2π or 360°. The latter is illustrated by sketching the graphs of $\sin\theta$ and $\cos\theta$ (see Figs. 1.3).

The other frequently used trigonometric function is $\tan\theta$ which, with reference to Fig. 1.2, is defined by

$$\tan\theta = \frac{PN}{ON} = \frac{\sin\theta}{\cos\theta}. \tag{1.5}$$

The graph of $\tan\theta$ is shown in Fig. 1.4.

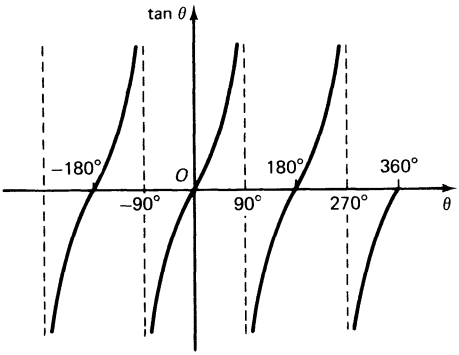

Fig. 1.4

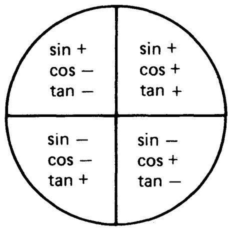

Fig. 1.5

Clearly $\tan\theta = -\tan(-\theta)$ and is periodic, but the period is now π or 180°, that is

$$\tan(\theta + n\pi) = \tan\theta, \quad n \in \mathbb{Z}. \tag{1.6}$$

From the definitions of the trigonometric functions and their graphs it is seen that

$$\begin{aligned} \sin(\pi/2 - \theta) &= \cos\theta, & \sin(\pi - \theta) &= \sin\theta, \\ \cos(\pi/2 - \theta) &= \sin\theta, & \cos(\pi - \theta) &= -\cos\theta, \\ & & \tan(\pi - \theta) &= -\tan\theta. \end{aligned} \tag{1.7}$$

These are very useful.

A useful aid for remembering the signs of the trigonometric functions for angles in the four quadrants is shown in Fig. 1.5. Three other trigonometric functions are *defined* as the reciprocals of those above, namely

$$\cot\theta = \frac{1}{\tan\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \operatorname{cosec}\theta = \frac{1}{\sin\theta}. \tag{1.8}$$

1.2 Set-square angles

The angles 30°, 45° and 60° occur frequently and you should be familiar with their sines, cosines and tangents which are summarised in the Table and Fig. 1.6.

Degrees	0°	30°	45°	60°	90°	180°	270°	360°
Radians	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$	2π
sine	0	$\frac{1}{2}$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	0	-1	0
cosine	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$\frac{1}{2}$	0	-1	0	1
tangent	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞	0	∞	0

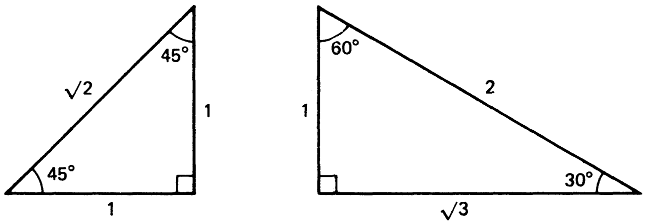

Fig. 1.6

1.3 Consequences of Pythagoras' theorem

It is clear by applying Pythagoras' theorem to Fig. 1.2 that

$$\cos^2\theta + \sin^2\theta \equiv 1. \tag{1.9}$$

Dividing (1.9) in turn by $\cos^2\theta$ and $\sin^2\theta$ we obtain two further identities:

$$1 + \tan^2\theta \equiv \sec^2\theta, \quad \cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta. \tag{1.10}$$

Example 1 Using the fact $\sin 30° = \frac{1}{2}$, find $\sin(-30°)$, $\sin 150°$, $\sin 210°$ and $\sin 330°$.

We draw the unit circle centre O and compare the projections of the different radii onto Oy (see Fig. 1.7).

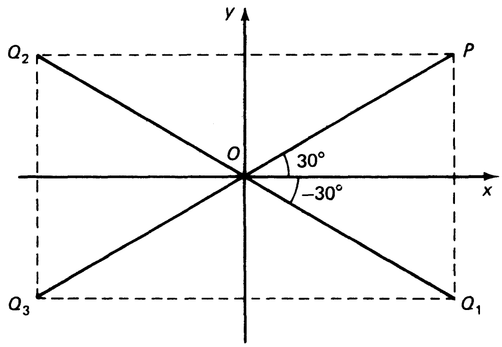

Fig. 1.7

The projection of OQ_1 onto Oy has the same length but the opposite sense to that of OP. Hence $\sin(-30°) = -\frac{1}{2}$. (This also follows from (1.3).)

To find $\sin 150°$ we consider OQ_2. The projection of OQ_2 is the same as that of $OP \Rightarrow \sin 150° = \frac{1}{2}$.

Consideration of the point Q_3 enables us to see that

$$\sin 210° = \sin(-30°) = -\tfrac{1}{2}.$$

Finally it is clear that $\sin 330°$ is the same as

$$\sin(-30°) \quad \Rightarrow \quad \sin 330° = -\tfrac{1}{2}.$$

This result also follows from (1.4) since

$$\sin(-30°) = \sin(-30° + 360°).$$

These results can also be obtained from the graph of $\sin\theta$ (see Fig. 1.3(a)) by considering the intersection of the lines $y = \frac{1}{2}$ and $y = -\frac{1}{2}$ with the curve.

Example 2 Sketch graphs of (i) $2\sin\theta°$, (ii) $\cos 3\theta°$, (iii) $\sin(\theta + \alpha)°$, where $\alpha > 0$.

(i) This graph is easily obtained from $y = \sin\theta$ by multiplying the corresponding y value by 2 (see Fig. 1.8).

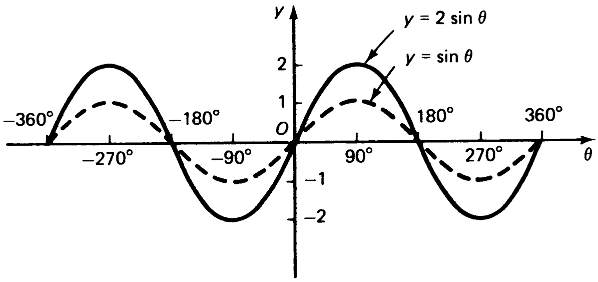

Fig. 1.8

The curve still crosses the axis at the same values of θ and the maxima and minima are achieved at the same values of θ as before. Notice that the maximum value of $2\sin\theta$ is 2 and the minimum value is -2.

(ii) In this case it is the scale along the θ-axis which is changed. Note that when $\theta = 30°$, $3\theta = 90°$, and since $\cos 90° = 0$, the graph crosses the θ-axis when $\theta = 30°$. Similarly, the graph crosses the θ-axis at $\theta = 90°$. We see therefore that the whole curve is compressed and appears as in Fig. 1.9. The maximum value is still 1 and the minimum value is still -1, but they now occur at different values of θ, namely the original ones *divided* by 3.

(iii) It is clear that when $\theta = -\alpha$, $\sin(\theta + \alpha)° = 0$ and when $\theta = (180° - \alpha)$, $\sin(\theta + \alpha)° = 0$. The effect of α here is to move the graph, without change of shape, to the *left*. Thus we have the situation shown in Fig. 1.10. (If α had been negative, the graph would have been moved to the right.)

If we take (i), (ii) and (iii) together, we see that it is possible, without much difficulty, to sketch any expression of the form $A \sin(k\theta + \alpha)$ or $A \cos(k\theta + \alpha)$, where A, k and α are given constants. In the early stages at least, it is useful to break the problem down into the three stages typified by (i), (ii) and (iii).

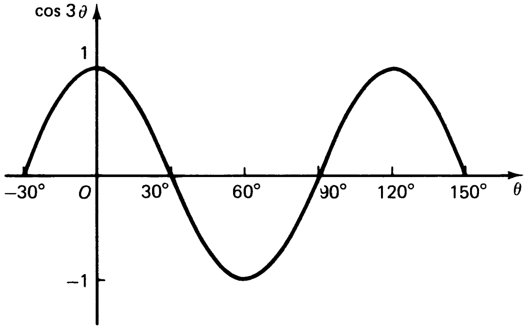

Fig. 1.9

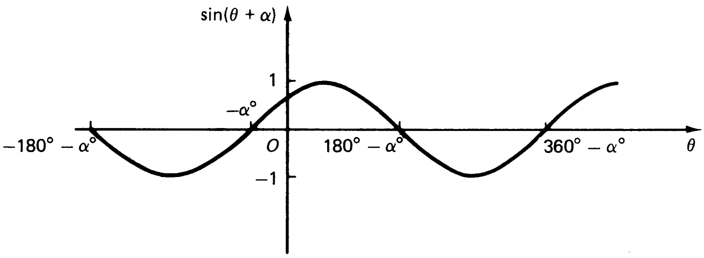

Fig. 1.10

Example 3 Find the values of θ between 0 and 360° for which (i) $\sin\theta° = 0{\cdot}37$, (ii) $\cos\theta° = -0{\cdot}61$, (iii) $\tan\theta° = 2{\cdot}72$. In each case find also, to two decimal places, the solution in radians.

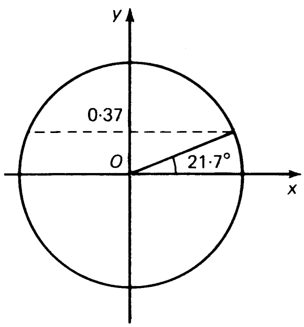

Fig. 1.11

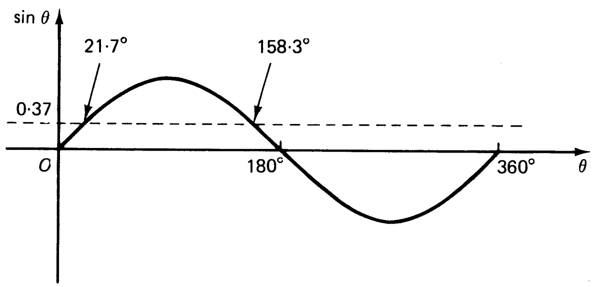

Fig. 1.12

(i) Using a calculator or tables, we find that $\sin\theta° = 0{\cdot}37 \Rightarrow \theta = 21{\cdot}7°$ (or 21° 43′) is a solution. If we draw Fig. 1.11 we see that another possible solution is 158·3° (158° 17′). These are the only two solutions in the given range. (Other solutions outside this range may be obtained by adding or subtracting integral multiples of 360°.) The solutions may also be estimated from the graph of $\sin\theta$ (see Fig. 1.12).

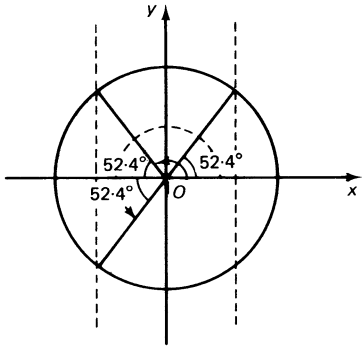

Fig. 1.13

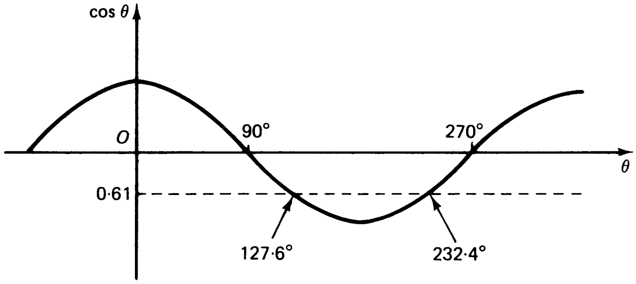

Fig. 1.14

We may obtain the solution in radians by recalling that $1^\circ \approx 0{\cdot}0175^c$. The solutions are then $0{\cdot}38^c$ and $2{\cdot}77^c$.

(ii) Using a calculator or tables, we find that $\cos\theta^\circ = 0{\cdot}61$ has a solution $\theta^\circ = 52{\cdot}4^\circ$ (52° 24′). Hence the required values are obtained from a modified Fig. 1.2 (see Fig. 1.13), namely $180^\circ - 52{\cdot}4^\circ$ and $180^\circ + 52{\cdot}4^\circ$, that is 127·6° (127° 36′) and 232·4° (232° 24′).

Again it is useful to see how these solutions could have been estimated from a graph (see Fig. 1.14). Using the result quoted earlier, it is easy to see that the solutions in radians are $2{\cdot}23^c$ and $4{\cdot}07^c$.

(iii) One value of θ satisfying $\tan\theta^\circ = 2{\cdot}72$ is found, using a calculator or tables, to be 69·8° (69° 48′). In this case the other values are easily obtained using the fact that the tangent is periodic with period 180°. Hence by adding or subtracting integral multiples of 180° all other solutions may be obtained. The only other one in the given range is 249·8° (249° 48′), as may also be seen from the graph of $\tan\theta$ (see Fig. 1.15). The solutions in radians are $1{\cdot}22^c$ and $4{\cdot}36^c$.

Example 4 Sketch the graph of cosec θ.

We recall the definition cosec $\theta = 1/\sin\theta$. Since $\sin\theta$ lies between -1 and $+1$, that is $|\sin\theta| \leqslant 1$, clearly cosec θ lies outside this region, that is $|\text{cosec}\,\theta| \geqslant 1$.

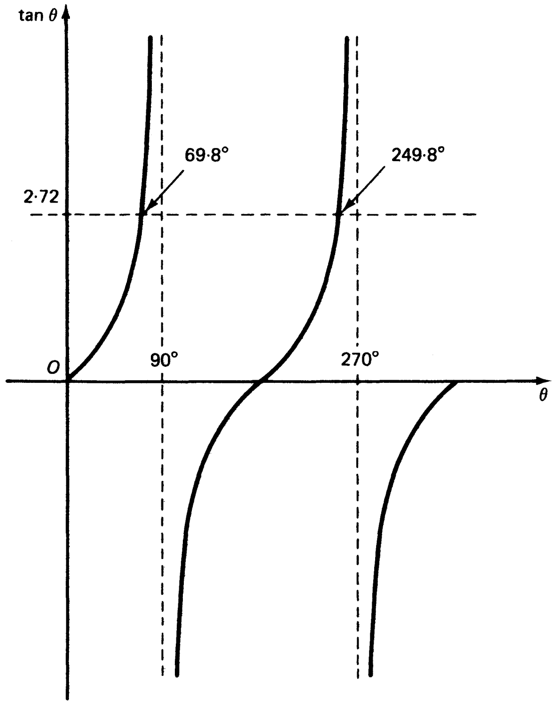

Fig. 1.15

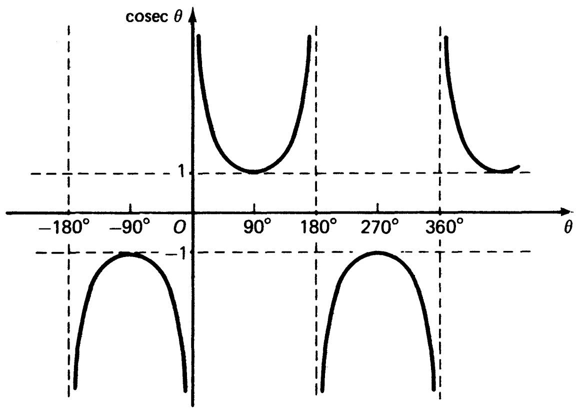

Fig. 1.16

By taking a series of values for θ, for example those in the table on p. 4, it is straightforward to obtain the graph shown in Fig. 1.16. Note the asymptotes at $\theta = 0°, \pm 180°, \ldots$.

Example 5 Given that $\sec\theta + \tan\theta = \frac{1}{2}$, find the values of $\sec\theta$ and $\tan\theta$.

To solve this problem we need to obtain an equation in either $\sec\theta$ or $\tan\theta$ only. We may do this by using the identity relating $\sec\theta$ and $\tan\theta$, namely the first identity in (1.10), rewritten in the form

$$\sec^2\theta - \tan^2\theta \equiv 1.$$

We factorise the left-hand side to give

$$(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) \equiv 1.$$

Using the given condition $\sec\theta + \tan\theta = \frac{1}{2}$, we have

$$\sec\theta - \tan\theta = 2.$$

Solving these simultaneous equations we find

$$\sec\theta = 5/4, \quad \tan\theta = -3/4.$$

An alternative method of solution is as follows. The given equation can be written as

$$2\tan\theta - 1 = -2\sec\theta.$$

Squaring

$$\Rightarrow \quad (2\tan\theta - 1)^2 = 4\sec^2\theta = 4(1 + \tan^2\theta)$$

using the first identity of (1.10),

$$\Rightarrow \quad -4\tan\theta = 3 \quad \Rightarrow \quad \tan\theta = -\tfrac{3}{4}.$$

Note that squaring $2\tan\theta - 1 = -2\sec\theta$ could introduce the roots of

$$2\tan\theta - 1 = 2\sec\theta.$$

You should always check your solutions if you square an equation. The next example illustrates this.

Example 6 Solve the equation

$$\sec\theta = 3\tan\theta - 1, \quad \text{for } 0 \leqslant \theta \leqslant 2\pi.$$

Squaring $\Rightarrow$ $\sec^2\theta = 9\tan^2\theta - 6\tan\theta + 1$,

and by (1.10) $\Rightarrow$ $1 + \tan^2\theta = 9\tan^2\theta - 6\tan\theta + 1$,

$\Rightarrow$ $8\tan^2\theta - 6\tan\theta = 0$

$$\Rightarrow \quad \tan\theta(4\tan\theta - 3) = 0$$

$$\Rightarrow \quad \tan\theta = 0 \text{ or } \tan\theta = \tfrac{3}{4}.$$

$$\tan\theta = 0 \quad \Rightarrow \quad \theta = 0 \text{ or } \pi,$$

$$\tan\theta = \tfrac{3}{4} \quad \Rightarrow \quad \theta \approx 0{\cdot}644 \text{ or } 3{\cdot}785.$$

But $\theta = 0$ does *not* satisfy the given equation. Similarly the solution $\theta \approx 3{\cdot}785$ does not satisfy the equation. In fact these are solutions of

$$-\sec\theta = 3\tan\theta - 1.$$

The only solutions in the given range are $0{\cdot}644$ and π.

Example 7 Eliminate θ from the equations

$$x = 4 + 2\cos\theta, \quad y = 1 + 3\sin\theta.$$

To eliminate θ from these equations, which are the parametric equations of a curve, we again use one of the identities. The obvious identity to use in this case is (1.9) since it involves $\sin\theta$ and $\cos\theta$. From the given equations

$$\frac{(x-4)}{2} = \cos\theta, \quad \frac{(y-1)}{3} = \sin\theta.$$

Substituting in (1.9) we obtain

$$\frac{(x-4)^2}{4} + \frac{(y-1)^2}{9} = 1,$$

which is the equation of an ellipse.

Example 8 Simplify $\dfrac{\sec^2\theta + 2\tan\theta}{(\cos\theta + \sin\theta)^2} \quad (1 + \sin 2\theta \neq 0)$

We begin by expanding the denominator

$$(\cos\theta + \sin\theta)^2 \equiv \cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta \equiv 1 + 2\sin\theta\cos\theta,$$

using the identity (1.9).

To proceed further we write the numerator in terms of sine and cosine only, that is,

$$\sec^2\theta + 2\tan\theta \equiv \frac{1}{\cos^2\theta} + \frac{2\sin\theta}{\cos\theta} \equiv \frac{1}{\cos^2\theta}(1 + 2\sin\theta\cos\theta).$$

The given expression is now

$$\frac{(1 + 2\sin\theta\cos\theta)}{(1 + 2\sin\theta\cos\theta)}\frac{1}{\cos^2\theta} \equiv \frac{1}{\cos^2\theta} = \sec^2\theta,$$

provided that $1 + 2\sin\theta\cos\theta \neq 0$, that is, $1 + \sin 2\theta \neq 0$. [See equation (2.7).]

Example 9 Find all solutions of $\sin\theta = \sin\alpha$.

The solutions of this equation are obtained by reading off the points of intersection of the graphs of $y = \sin\theta$ and $y = \sin\alpha$. The graph of $y = \sin\theta$ is given in Fig. 1.3(a). The graph of $y = \sin\alpha$ is just a line drawn parallel to the θ-axis making an intercept $\sin\alpha$ on the y-axis.

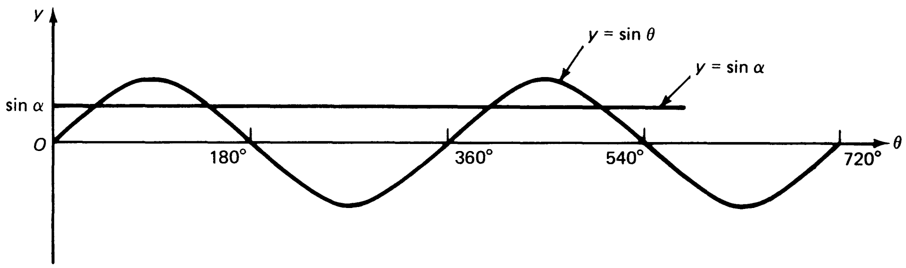

Fig. 1.17

From Fig. 1.17 we see that there are two sets of values given by $\alpha + 2m\pi$ and $(\pi - \alpha) + 2m\pi$ where $m \in \mathbb{Z}$. We may combine these in the single expression

$$\theta = n\pi + (-1)^n\alpha, \quad n \in \mathbb{Z}.$$

Example 10 Find all the values of x in the range $0 \leqslant x \leqslant 360$ for which

$$\cos(x^\circ + 15^\circ) = \sin(2x^\circ - 15^\circ).$$

We first write this in the form

$$\cos(x^\circ + 15^\circ) = \sin(90^\circ - x^\circ - 15^\circ) = \sin(2x^\circ - 15^\circ).$$

Using the result of Example 9 we have

$$90 - x - 15 = 180n + (-1)^n(2x - 15) \quad \text{for all } n \in \mathbb{Z}.$$

$$n \text{ odd} \Rightarrow x = 180n - 60,$$

$$n \text{ even} \Rightarrow 3x = 90 - 180n \Rightarrow x = 30 - 60n.$$

When n is odd, the only solution in the range occurs when $n = 1 \Rightarrow x = 120^\circ$.
When n is even, the solutions in the range occur when

$$n = 0, -2, -4 \Rightarrow x = 30^\circ, 150^\circ \text{ and } 270^\circ.$$

Exercise 1

1 Given that $\cos 30^\circ = \sqrt{3}/2$, find $\cos(-30^\circ)$, $\cos 150^\circ$, $\cos 210^\circ$ and $\cos 330^\circ$, leaving surds (roots) in your answer.

2 Sketch the graphs of (a) $3\cos\theta$, (b) $\cos 2\theta$, (c) $\cos(\theta + \pi/4)$, (d) $3\cos(2\theta + \pi/4)$.

3 Sketch on the same axes the graphs of $y = \sin 2\theta$ and $y = \cos\theta$ in the interval $-\pi \leqslant \theta \leqslant \pi$ and deduce the set of values of θ for which $\sin 2\theta \geqslant \cos\theta$ in this interval.

4 Find, to the nearest tenth of a degree, the values of θ between 0° and 360° for which (a) $\sin\theta = -0{\cdot}74$, (b) $\cos\theta = 0{\cdot}45$, (c) $\tan\theta = -2{\cdot}72$, (d) $\cos\frac{1}{2}\theta = -0{\cdot}61$.

5 Sketch the graphs of (a) $\sec\theta$, (b) $\cot\theta$.

6 Solve for values of θ between 0 and 360 (a) $\operatorname{cosec}\theta° = 2$, (b) $\cot\theta° = -1$.

7 Solve, for values of θ between 0 and 360, the equation

$$3\cos^2\theta° - \cos\theta° + 1 = 3\sin^2\theta°.$$

8 Simplify (a) $\dfrac{\cos^2\theta}{1+\sin\theta} + \dfrac{\cos^2\theta}{1-\sin\theta}$, (b) $\dfrac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta - \sin\theta}$.

9 Use the fact that $\sin\theta = \cos(\frac{1}{2}\pi - \theta)$ for any angle θ to solve, for values of x lying between 0 and 2π, the equation

$$\sin x = \cos 4x.$$

10 Find, in radians, the general solution of the equation $\cos\theta = \sin 2\theta$.

11 Find the values of θ between 0 and 2π for which $\sin 3\theta = \sin(\pi/6)$.

12 Show that the general solution of the equation $\cos x = \cos\alpha$ is $x = 2n\pi \pm \alpha$, where $n \in \mathbb{Z}$.

13 Show that the general solution of the equation $\tan x = \tan\beta$ is $x = n\pi + \beta$, where $n \in \mathbb{Z}$.

14 Find the general solution, in radians, of the equation $\sin 2x = \cos 2\beta$.

15 Find, in radians, the value of x between 0 and π which satisfies the equation

$$\sin(x + \pi/3) = \cos(x - \pi/3).$$

16 Solve the equation

$$\cos\tfrac{3}{4}x = \tan 163°,$$

giving all solutions between 0° and 360°.

17 Find, in radians, the general solution of

$$2\operatorname{cosec} x - 8\sin x = 0.$$

18 Find, in radians, the general solution of

$$2\cot x = \sqrt{3}\operatorname{cosec} x.$$

19 By drawing suitable graphs, show that the equation $\cos\theta = \theta$ has one positive root and that this root is approximately $0{\cdot}74$.

2 Trigonometric identities

2.1 Compound-angle identities

It is often necessary to express the trigonometric ratios of angles such as $\theta + \phi$ and $\theta - \phi$ in terms of the trigonometric ratios of θ and ϕ.

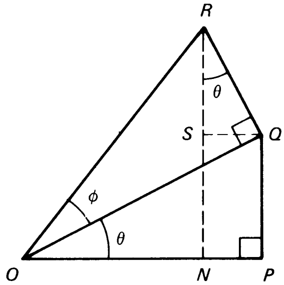

Fig. 2.1

Consider the case where the angles θ and ϕ are both acute and $(\theta + \phi)$ is also acute. From the definition of sine given in Chapter 1,

$$\sin(\theta + \phi) \equiv \frac{NR}{OR} = \frac{NS + SR}{OR} = \frac{PQ + SR}{OR} = \frac{PQ}{OQ} \times \frac{OQ}{OR} + \frac{SR}{QR} \times \frac{QR}{OR}$$

$$\Rightarrow \quad \sin(\theta + \phi) \equiv \sin\theta\cos\phi + \cos\theta\sin\phi. \qquad (2.1)$$

If we replace ϕ by $-\phi$ in (2.1), we obtain

$$\sin(\theta - \phi) \equiv \sin\theta\cos\phi - \cos\theta\sin\phi. \qquad (2.2)$$

Replacing θ by $(\pi/2 - \theta)$ in (2.2) leads to

$$\cos(\theta + \phi) \equiv \cos\theta\cos\phi - \sin\theta\sin\phi. \qquad (2.3)$$

Finally, if we replace ϕ by $-\phi$ in (2.3), we have

$$\cos(\theta - \phi) \equiv \cos\theta\cos\phi + \sin\theta\sin\phi. \qquad (2.4)$$

It can be shown that the identities (2.1)–(2.4) hold for all values of θ and ϕ and not just for acute angles. (These general results are easily proved using matrix methods.)

If we divide (2.1) by (2.3) and use the definition of $\tan\theta$, we obtain

$$\tan(\theta + \phi) \equiv \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}. \quad (2.5)$$

Similarly, dividing (2.2) by (2.4) we obtain

$$\tan(\theta - \phi) \equiv \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}. \quad (2.6)$$

2.2 The double-angle identities

Useful formulae (identities) are obtained from (2.1), (2.3) and (2.5), respectively, when $\theta = \phi$.

$$\sin 2\theta \equiv 2\sin\theta\cos\theta, \quad (2.7)$$

$$\cos 2\theta \equiv \cos^2\theta - \sin^2\theta, \quad (2.8)$$

$$\tan 2\theta \equiv \frac{2\tan\theta}{1 - \tan^2\theta}. \quad (2.9)$$

If we use the relation (1.9), $\cos^2\theta + \sin^2\theta \equiv 1$, we can obtain two alternative forms of (2.8). Since $\sin^2\theta \equiv 1 - \cos^2\theta$, we have

$$\cos 2\theta \equiv 2\cos^2\theta - 1. \quad (2.10)$$

Similarly, since $\cos^2\theta \equiv 1 - \sin^2\theta$, we have

$$\cos 2\theta \equiv 1 - 2\sin^2\theta. \quad (2.11)$$

These alternative expressions for $\cos 2\theta$ can themselves be rearranged to give

$$2\sin^2\theta \equiv 1 - \cos 2\theta, \quad (2.12)$$

$$2\cos^2\theta \equiv 1 + \cos 2\theta. \quad (2.13)$$

Familiarity with these double-angle formulae is essential.

2.3 The half-angle identities

We already know from (2.9) that $\tan 2\theta \equiv 2\tan\theta/(1 - \tan^2\theta)$, but we can also write $\sin 2\theta$ and $\cos 2\theta$ in terms of $\tan\theta$. Using (1.9), identities (2.7) and (2.8), respectively, may be written as

$$\sin 2\theta \equiv \frac{2\sin\theta\cos\theta}{\cos^2\theta + \sin^2\theta} \equiv \frac{2\tan\theta}{1 + \tan^2\theta},$$

$$\cos 2\theta \equiv \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin^2\theta} \equiv \frac{1 - \tan^2\theta}{1 + \tan^2\theta}.$$

Writing $\phi = 2\theta$ we obtain what are often called the half-angle identities

$$\tan\phi \equiv \frac{2t}{1 - t^2}, \quad (2.14)$$

$$\sin\phi \equiv \frac{2t}{1 + t^2}, \quad (2.15)$$

$$\cos\phi \equiv \frac{1-t^2}{1+t^2}, \tag{2.16}$$

where $t = \tan(\phi/2)$. These three results enable us to write *all* the trigonometric ratios of any one angle in terms of a common variable t.

2.4 The factor formulae

Identities (2.1) and (2.2) are

$$\sin(\theta+\phi) \equiv \sin\theta\cos\phi + \cos\theta\sin\phi,$$

$$\sin(\theta-\phi) \equiv \sin\theta\cos\phi - \cos\theta\sin\phi.$$

Adding, we obtain

$$2\sin\theta\cos\phi \equiv \sin(\theta+\phi) + \sin(\theta-\phi),$$

and subtracting, we obtain

$$2\cos\theta\sin\phi \equiv \sin(\theta+\phi) - \sin(\theta-\phi).$$

Writing

$$\left.\begin{aligned}\theta+\phi&=\alpha\\ \theta-\phi&=\beta\end{aligned}\right\} \Rightarrow \quad \begin{aligned}\theta&=\tfrac{1}{2}(\alpha+\beta),\\ \phi&=\tfrac{1}{2}(\alpha-\beta),\end{aligned}$$

we find

$$\sin\alpha + \sin\beta \equiv 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right), \tag{2.17}$$

$$\sin\alpha - \sin\beta \equiv 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right). \tag{2.18}$$

Using (2.3) and (2.4), and proceeding in a similar way, we obtain

$$\cos\alpha + \cos\beta \equiv 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right), \tag{2.19}$$

$$\cos\alpha - \cos\beta \equiv -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right). \tag{2.20}$$

Example 1 Given that $\sin\alpha = 3/5$ and $\cos\beta = -4/5$, where $0 < \alpha < \pi/2$ and $0 < \beta < \pi$, without using a calculator or tables find the values of (i) $\sin(\alpha+\beta)$, (ii) $\cos(\beta-\alpha)$, (iii) $\tan(\alpha+\beta)$.

Since $\sin\alpha = 3/5$ and $0 < \alpha < \pi/2$,

$$\cos\alpha = +\sqrt{(1-\sin^2\alpha)} = +(1-9/25)^{1/2} = 4/5.$$

(The positive value for the square root is chosen since $0 < \alpha < \pi/2$.) Similarly, $\sin\beta = +\sqrt{(1-16/25)} = 3/5$ since $0 < \beta < \pi$.

(i) $$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\Rightarrow \quad \sin(\alpha+\beta) = (3/5)(-4/5) + (4/5)(3/5) = 0.$$

(ii) $$\cos(\beta-\alpha) = \cos\beta\cos\alpha + \sin\beta\sin\alpha$$

$$\Rightarrow \quad \cos(\beta-\alpha) = (-4/5)(4/5) + (3/5)(3/5) = (-7/25).$$

(iii) $$\tan(\alpha+\beta) \equiv \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \quad \text{and } \tan\alpha = \tfrac{3}{4}, \quad \tan\beta = -\tfrac{3}{4},$$

$$\Rightarrow \quad \tan(\alpha+\beta) = 0.$$

Example 2 Without using a calculator or tables, find, in surd form, the values of (i) $\sin 75°$, (ii) $\cos 75°$, (iii) $\tan 75°$.

(i) By (2.1), $$\sin 75° = \sin(45° + 30°)$$
$$= \sin 45°\cos 30° + \cos 45°\sin 30°$$
$$= \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}.$$

(ii) By (2.3), $$\cos 75° = \cos(45° + 30°)$$
$$= \cos 45°\cos 30° - \sin 45°\sin 30°$$
$$= \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}.$$

(iii) By (2.5), $$\tan 75° = \tan(45° + 30°)$$
$$= \frac{\tan 45° + \tan 30°}{1 - \tan 45°\tan 30°}$$
$$= \frac{1 + 1/\sqrt{3}}{1 - 1/\sqrt{3}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{(\sqrt{3}+1)^2}{(\sqrt{3}-1)(\sqrt{3}+1)}$$
$$= \frac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3}.$$

Example 3 Prove that $\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$.

$$\sin 3\theta \equiv \sin(2\theta + \theta)$$
$$\equiv \sin 2\theta\cos\theta + \cos 2\theta\sin\theta, \quad \text{by (2.1)},$$
$$\equiv (2\sin\theta\cos\theta)\cos\theta + (1 - 2\sin^2\theta)\sin\theta.$$

We have used (2.11) for $\cos 2\theta$ since the answer is required in terms of $\sin\theta$. It remains now to replace $\cos^2\theta$ by $1 - \sin^2\theta$ and we obtain

$$\sin 3\theta \equiv 2\sin\theta(1 - \sin^2\theta) + (1 - 2\sin^2\theta)\sin\theta$$

$$\Rightarrow \quad \sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta.$$

The corresponding result for $\cos 3\theta$ is set as question 5 in Exercise 2, p. 18. It should be known.

Example 4 Eliminate θ from the equations

$$x = \cos 2\theta, \quad y = \sec\theta.$$

Since $\sec\theta = 1/\cos\theta$ and $\cos 2\theta \equiv 2\cos^2\theta - 1$, then, in terms of $\cos\theta$,

$$x = 2\cos^2\theta - 1, \quad y = \frac{1}{\cos\theta}.$$

Eliminating $\cos\theta$

$$x = 2(1/y)^2 - 1,$$

$$\Rightarrow \quad (x+1)y^2 = 2.$$

Example 5 Show that

$$\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta} + \frac{\sin(\beta-\gamma)}{\cos\beta\cos\gamma} + \frac{\sin(\gamma-\alpha)}{\cos\gamma\cos\alpha} = 0. \ (\cos\alpha, \cos\beta, \cos\gamma \neq 0.)$$

Using the result (2.2) we have for the left-hand side

$$\frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta} + \frac{\sin\beta\cos\gamma - \cos\beta\sin\gamma}{\cos\beta\cos\gamma} + \frac{\sin\gamma\cos\alpha - \cos\gamma\sin\alpha}{\cos\gamma\cos\alpha}$$

$$\equiv \frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} - \frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta} + \frac{\sin\beta\cos\gamma}{\cos\beta\cos\gamma}$$

$$- \frac{\cos\beta\sin\gamma}{\cos\beta\cos\gamma} + \frac{\sin\gamma\cos\alpha}{\cos\gamma\cos\alpha} - \frac{\cos\gamma\sin\alpha}{\cos\gamma\cos\alpha}$$

$$\equiv \tan\alpha - \tan\beta + \tan\beta - \tan\gamma + \tan\gamma - \tan\alpha \equiv 0.$$

Example 6 Show that $\dfrac{\cos\theta + \sin\theta - 1}{\cos\theta - \sin\theta + 1} \equiv \tan\frac{1}{2}\theta$.

Using the half-angle identities (2.15) and (2.16), the left-hand side is

$$\frac{\dfrac{1-t^2}{1+t^2} + \dfrac{2t}{1+t^2} - 1}{\dfrac{1-t^2}{1+t^2} - \dfrac{2t}{1+t^2} + 1},$$

where $t = \tan\frac{1}{2}\theta$,

$$\Rightarrow \quad \frac{1 - t^2 + 2t - (1+t^2)}{1 - t^2 - 2t + (1+t^2)} \equiv \frac{2t - 2t^2}{2 - 2t} \equiv t, \quad \text{as required.}$$

Example 7 Show that

$$\frac{\sin\theta + 2\sin 3\theta + \sin 5\theta}{\sin 3\theta + 2\sin 5\theta + \sin 7\theta} \equiv \frac{\sin 3\theta}{\sin 5\theta}.$$

Using the factor formula we have

$$\sin\theta + \sin 5\theta \equiv 2\sin 3\theta\cos 2\theta,$$

$$\sin 3\theta + \sin 7\theta \equiv 2\sin 5\theta\cos 2\theta.$$

The left-hand side may therefore be written

$$\frac{2\sin 3\theta\cos 2\theta + 2\sin 3\theta}{2\sin 5\theta\cos 2\theta + 2\sin 5\theta} \equiv \frac{2\sin 3\theta(\cos 2\theta + 1)}{2\sin 5\theta(\cos 2\theta + 1)}$$

$$\equiv \frac{\sin 3\theta}{\sin 5\theta},$$

provided that $\cos 2\theta + 1 \neq 0$.

Exercise 2

1 Given that $\sin x = (8/17)$, $\cos x = (-15/17)$, $\sin y = (-24/25)$, and $\cos y = (7/25)$, find, without the use of a calculator or tables,
(a) $\sin(x+y)$, (b) $\cos(x+y)$, (c) $\tan(x+y)$, (d) $\sin(x-y)$,
(e) $\cos(x-y)$, (f) $\tan(x-y)$.

2 Show that $\sin(A+B)/\cos A\cos B \equiv \tan A + \tan B$.

3 Show that

$$(\sin A + \cos A)(\sin B + \cos B) \equiv \sin(A+B) + \cos(A-B).$$

4 Without using a calculator or tables, find, leaving surds in your answers,
(a) $\cos 15°$, (b) $\sin 165°$, (c) $\tan(-15°)$.

5 Show that $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$.

6 Show that $\tan 3\theta \equiv (3\tan\theta - \tan^3\theta)/(1 - 3\tan^2\theta)$.

7 Show that $\sin 2\theta/(1-\cos 2\theta) \equiv \cot\theta$, $(\theta \neq n\pi, n \in \mathbb{Z})$.

8 Show that $(1+\cos x)/(1-\cos x) \equiv \cot^2(\frac{1}{2}x)$, $(x \neq 2n\pi, n \in \mathbb{Z})$.

9 By eliminating θ from the following equations, find the corresponding cartesian equation (a) $x = \cos 2\theta, y = \cos\theta$; (b) $x = \tan 2\theta, y = \tan\theta$.

10 Show that $\dfrac{\sin\theta - \sin 2\theta + \sin 3\theta}{\cos\theta - \cos 2\theta + \cos 3\theta} \equiv \tan 2\theta$.

11 Given that $\sin\alpha = 0{\cdot}75$ and $0 < \alpha < \pi/2$, calculate, without using a calculator or tables, $\cos 2\alpha$ and $\cos 4\alpha$ and deduce that $45° < \alpha < 67\frac{1}{2}°$.

12 Given that α and β are acute angles and that $\tan\alpha = \frac{1}{3}$ and $\tan\beta = \frac{1}{2}$, prove, without the use of calculator or tables, that $\alpha + \beta = \pi/4$.

13 Given that $\sin x = 0{\cdot}6$ and that $\cos x = -0{\cdot}8$, evaluate, without using tables or a calculator, $\cos(x + 270°)$ and $\cos(x + 540°)$.

14 Given that X is an acute angle such that $\sin X = 4/5$ and Y is an obtuse angle such that $\sin Y = 12/13$, find, without using tables or a calculator, the exact value of $\tan(X+Y)$.

15 Given that $|\theta| < \pi/2$, find the set of values of θ for which $1/(1+\cos 2\theta) < 1$.

16 By expressing the numerator as a difference of sines, show that as θ varies the greatest

value of $2\sin\theta\cos(\theta+\alpha)/\cos^2\alpha$, where $0<\alpha<\pi/2$, is $(1+\sin\alpha)^{-1}$. Also give, in terms of α, the smallest positive value of θ for which the expression has this greatest value.

17 Given that $x = 1 + \sin\theta$, $y = 2\cos 2\theta - 1$, prove that

$$4x^2 - 8x + y + 3 = 0.$$

18 Given that $\sin x = 3/5$, find the possible values of $\tan\frac{1}{2}x$.

19 Express $\cos 2\theta$ and $\sin 2\theta$ in terms of t, where $t = \tan\theta$. Hence solve the equation $5\cos 2\theta - 2\sin 2\theta = 0$, for $0° < \theta < 360°$.

20 Show that

$$\frac{\sin(3A+B)+\sin(A-B)}{\cos(3A+B)+\cos(A-B)} \equiv \tan 2A.$$

21 Prove the identity

$$\sin A - \cos 3A \equiv (\sin A + \cos A)(\sin 2A - \cos 2A).$$

22 If $A + B + C = 180°$, prove that

$$\sin A + \sin B + \sin C = 4\cos(\tfrac{1}{2}A)\cos(\tfrac{1}{2}B)\cos(\tfrac{1}{2}C).$$

23 Given that $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$, use the substitution $x = 2\cos\theta$ in the equation $x^3 - 3x - 1 = 0$ and show that it reduces to $\cos 3\theta = \frac{1}{2}$. Hence find the 3 roots of this equation to 2 decimal places.

3 Trigonometric equations and their solution

3.1 The expression $a\cos\theta + b\sin\theta$

It is often useful to express $a\cos\theta + b\sin\theta$ as a single term such as $r\cos(\theta - \gamma)$, where r is positive. This is possible if we can find r and γ such that

$$r\cos(\theta - \gamma) \equiv r(\cos\theta\cos\gamma + \sin\theta\sin\gamma)$$
$$\equiv a\cos\theta + b\sin\theta.$$

Comparing the coefficients of $\cos\theta$ and $\sin\theta$

$$\Rightarrow \quad r\cos\gamma = a, \quad r\sin\gamma = b.$$

(It can be shown that, if

$$a_1\cos\theta + b_1\sin\theta \equiv a_2\cos\theta + b_2\sin\theta,$$

where a_1, a_2, b_1, b_2 are constants, then $a_1 = a_2$, $b_1 = b_2$. This should be compared with equating coefficients in polynomials.) Dividing, we have $\tan\gamma = b/a$ and the situation shown in Fig. 3.1 for the case when $a > 0$, $b > 0$.

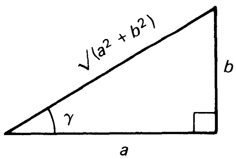

Fig. 3.1

In general, care is required in determining γ, and it should be noted that $\cos\gamma$ has the *same* sign as a and $\sin\gamma$ has the *same* sign as b. We have

$$\cos\gamma = \frac{a}{\sqrt{(a^2 + b^2)}}, \quad \sin\gamma = \frac{b}{\sqrt{(a^2 + b^2)}}, \quad r = (a^2 + b^2)^{1/2}. \qquad (3.1)$$

The first two of these equations define a unique γ for $0 \leqslant \gamma < 2\pi$. (It is sometimes more convenient to begin by comparing $a\cos\theta + b\sin\theta$ with $r\cos(\theta + \gamma)$, $r\sin(\theta + \gamma)$ or $r\sin(\theta - \gamma)$. There are no new principles involved and one may proceed as before using the appropriate formula.)

The graph of $a\cos\theta + b\sin\theta$ is obtained as follows. In Chapter 1 we saw that the graph of $r\cos\theta$ is simply related to the graph of $\cos\theta$, as shown in Fig. 3.2. Further, the graph of $\cos(\theta - \gamma)$ is obtained by translating the graph

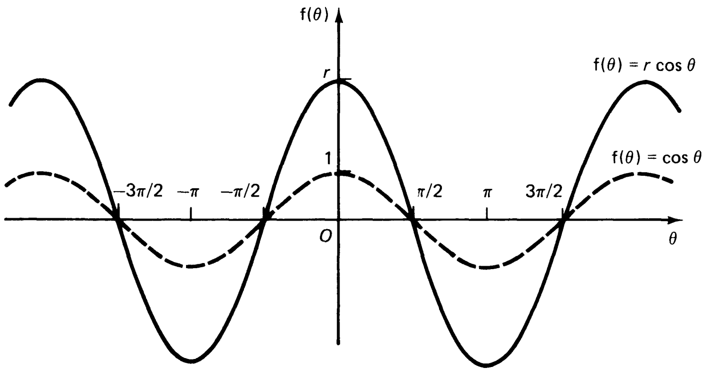

Fig. 3.2

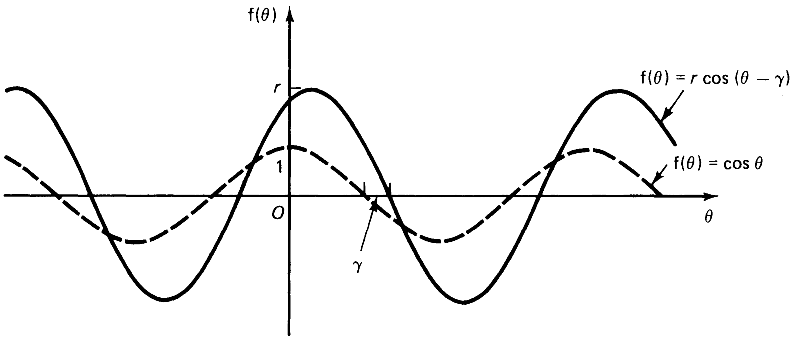

Fig. 3.3

of $\cos\theta$ a distance γ to the right. Combining these two ideas we obtain the graph of $r\cos(\theta - \gamma)$ as shown in Fig. 3.3.

In the expression $r\cos(\theta - \gamma)$, r is called the *amplitude*—it denotes the maximum value of the expression, the minimum being $-r$, and γ is the *phase shift* to the *right*.

3.2 The equation $a\cos\theta + b\sin\theta = c$

To solve this equation one of the following methods may be used.

(i) From the previous section, we may write

$$a\cos\theta + b\sin\theta = r\cos(\theta - \gamma) = c,$$

where r and γ are given by (3.1),

$$\Rightarrow \quad \cos(\theta - \gamma) = \frac{c}{r}.$$

The general solution of this equation can be found using a calculator or tables and hence the general value of θ satisfying the original equation is obtained (see Example 1, p. 22).

(ii) An alternative method is to use the half-angle formulae obtained in Chapter 2. By (2.15) and (2.16)

$$\begin{aligned} & a\cos\theta + b\sin\theta = c \\ \Rightarrow\quad & \frac{a(1-t^2)}{(1+t^2)} + \frac{2bt}{(1+t^2)} = c \\ \Rightarrow\quad & a(1-t^2) + 2bt = c(1+t^2) \\ \Rightarrow\quad & (a+c)t^2 - 2bt + (c-a) = 0. \end{aligned}$$

This is a quadratic equation in $t = \tan(\frac{1}{2}\theta)$ which may easily be solved and hence the required values obtained (see Example 2). A disadvantage of this method is that there is a loss of accuracy in some cases when calculation with four-figure tables results in only three significant figures being obtained in the answer for $\tan(\frac{1}{2}\theta)$. A further difficulty occurs when $\theta = \pi$ is one of the solutions—then $\tan(\frac{1}{2}\theta)$ is infinite and this root is apparent from the calculation only because of the absence of a term in t^2.

(iii) A graphical method of solution depends on the fact that $\cos^2\theta + \sin^2\theta = 1$ for all values of θ. Therefore $\cos\theta$ and $\sin\theta$ are the roots of the simultaneous equations $x^2 + y^2 = 1$ and $ax + by = c$. The points of intersection of the straight line $ax + by = c$ with the circle $x^2 + y^2 = 1$ give $\cos\theta$ and $\sin\theta$ and hence θ. An accurate drawing, as in the following Example 1, gives good approximations and the solution.

Example 1 Express $E(\theta) \equiv 3\cos\theta + 4\sin\theta$ in the form $r\cos(\theta - \gamma)$, where $r > 0$ and $0 \leqslant \gamma \leqslant \pi/2$. Hence (i) find the maximum and minimum values of $E(\theta)$, (ii) find the general solution of the equation $3\cos\theta + 4\sin\theta = 2$.

If

$$\begin{aligned} 3\cos\theta + 4\sin\theta &\equiv r\cos(\theta - \gamma) \\ &\equiv r(\cos\theta\cos\gamma + \sin\theta\sin\gamma), \end{aligned}$$

then $r\cos\gamma = 3$ and $r\sin\gamma = 4$. Thus $\tan\gamma \equiv 4/3$ with $\cos\gamma$ and $\sin\gamma$ both positive and $r = 5$,

$$\begin{aligned} \Rightarrow\quad & \gamma \approx 53{\cdot}1^\circ, \\ \Rightarrow\quad & E(\theta) = 5\cos(\theta - 53{\cdot}1^\circ). \end{aligned}$$

(i) The maximum value, 5, of $E(\theta)$ occurs at

$$\theta - 53{\cdot}1^\circ = 360n^\circ, \quad n \in \mathbb{Z}.$$

The minimum value, -5, of $E(\theta)$ occurs at

$$\theta - 53{\cdot}1^\circ = (2n+1)180^\circ, \quad n \in \mathbb{Z}.$$

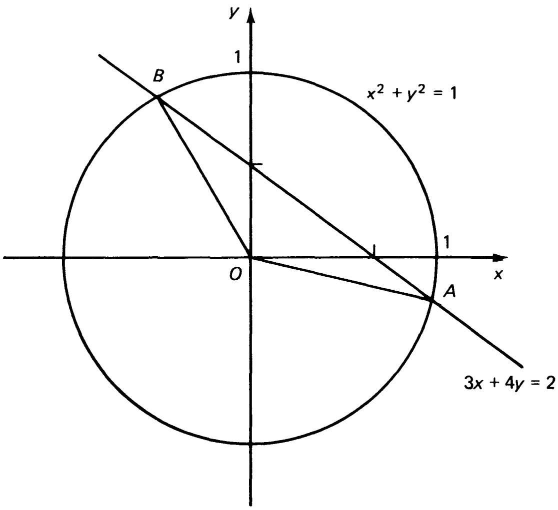

Fig. 3.4

(ii) $3\cos\theta + 4\sin\theta = 2$

$$\Rightarrow \quad 5\cos(\theta - 53{\cdot}1^\circ) = 2$$

$$\Rightarrow \quad \cos(\theta - 53{\cdot}1^\circ) = 2/5$$

$$\Rightarrow \quad \theta - 53{\cdot}1^\circ \approx 360n^\circ \pm 66{\cdot}4^\circ$$

$$\Rightarrow \quad \theta = 360n^\circ + 119{\cdot}5^\circ \text{ or } 360n^\circ - 13{\cdot}3^\circ.$$

This equation could be solved graphically by drawing the circle $x^2 + y^2 = 1$ and the straight line $3x + 4y = 2$ and measuring the angles $xOB \approx 119^\circ$ and $xOA \approx -13^\circ$ (see Fig. 3.4).

The general solutions are obtained by adding $360n^\circ$ to these values.

Example 2 Solve the equation $\sin x - 2\cos x = 1{\cdot}5$ using the half-angle formulae.

$$\sin x - 2\cos x = \frac{2t}{(1+t^2)} - \frac{2(1-t^2)}{(1+t^2)} = \frac{3}{2}$$

$$\Rightarrow \quad 2t - 2 + 2t^2 = (3/2)(1 + t^2)$$

$$\Rightarrow \quad t^2 + 4t - 7 = 0$$

$$\Rightarrow \quad t = \tan(\tfrac{1}{2}x) = -2 \pm \sqrt{11} \approx 1{\cdot}317 \text{ or } -5{\cdot}317$$

$$\Rightarrow \quad \tfrac{1}{2}x \approx 52{\cdot}8^\circ + 180n^\circ \text{ or } -79{\cdot}3^\circ + 180n^\circ, \quad n \in \mathbb{Z},$$

$$\Rightarrow \quad x \approx 360n^\circ + 105{\cdot}6^\circ \text{ or } 360n^\circ - 158{\cdot}6^\circ.$$

Example 3 Solve the equation $\cos\theta + \sqrt{2}\sin\theta = 1$ using the auxiliary equations $x^2 + y^2 = 1$ and $x + \sqrt{2}y = 1$.

Eliminating x between the given equations we obtain

$$y^2 + (1 - \sqrt{2}y)^2 = 1 \quad \Rightarrow \quad 3y^2 - 2\sqrt{2}y = 0$$
$$\Rightarrow \quad y = 0 \text{ or } y = (2\sqrt{2}/3).$$

The corresponding values of x are $x = 1$ and $-\frac{1}{3}$. The values of θ are obtained from $x = \cos\theta$, $y = \sin\theta$. Hence $\theta = 360n^\circ$ or $360n^\circ + 109{\cdot}5^\circ$, where $n \in \mathbb{Z}$.

Example 4 Find the angles between 0° and 360° which satisfy

$$\cos 2\theta = 7\cos\theta + 3.$$

If we write $\cos 2\theta = 2\cos^2\theta - 1$ (2.10), we obtain an equation containing $\cos\theta$ only.

$$\cos 2\theta = 2\cos^2\theta - 1 = 7\cos\theta + 3$$
$$\Rightarrow \quad 2\cos^2\theta - 7\cos\theta - 4 = 0$$
$$\Rightarrow \quad (2\cos\theta + 1)(\cos\theta - 4) = 0$$
$$\Rightarrow \quad \cos\theta = -\tfrac{1}{2} \text{ or } \cos\theta = 4 \quad \text{(which is impossible)}$$
$$\Rightarrow \quad \theta = 120^\circ \text{ or } 240^\circ.$$

Example 5 Solve the equation $\sin 5\theta + \sin 3\theta - \sin 4\theta = 0$.

The important point to note here is that by using the factor formulae we can write $\sin 5\theta + \sin 3\theta$ in a form which involves $\sin 4\theta$. Thus

$$\sin 5\theta + \sin 3\theta \equiv 2\sin 4\theta\cos\theta$$
$$\Rightarrow \quad 2\sin 4\theta\cos\theta = \sin 4\theta$$
$$\Rightarrow \quad \sin 4\theta(2\cos\theta - 1) = 0$$
$$\Rightarrow \quad \sin 4\theta = 0 \text{ or } 2\cos\theta = 1, \quad \text{i.e. } \cos\theta = \tfrac{1}{2},$$
$$\Rightarrow \quad 4\theta = 180n^\circ \text{ or } \theta = \pm 60^\circ + 360n^\circ$$
$$\Rightarrow \quad \theta = 45n^\circ \text{ or } \theta = \pm 60 + 360n^\circ, \quad n \in \mathbb{Z}.$$

These solutions may also be written

$$\theta = n\pi/4 \quad \text{or} \quad \theta = 2n\pi \pm \pi/3.$$

Example 6 Express $f(\theta) \equiv 5\sin\theta + 12\cos\theta$ in the form $r\sin(\theta + \gamma)$, giving the values of r and γ. Show that

$$5\sin\theta + 12\cos\theta + 5 \leqslant 18$$

and find the minimum value of $5\sin\theta + 12\cos\theta + 5$. Sketch the graph of the expression $1/\mathrm{f}(\theta)$.

Let

$$5\sin\theta + 12\cos\theta \equiv r\sin(\theta+\gamma)$$
$$\equiv r(\sin\theta\cos\gamma + \cos\theta\sin\gamma).$$

Then $r\cos\gamma = 5$, $r\sin\gamma = 12$ (see Fig. 3.5)

$$\Rightarrow \quad r = 13 \text{ and } \tan\gamma = (12/5) \quad \Rightarrow \quad \gamma \approx 67{\cdot}4°.$$

Hence

$$5\sin\theta + 12\cos\theta \equiv 13\sin(\theta + 67{\cdot}4°).$$

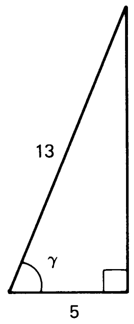

Fig. 3.5

But

$$-1 \leqslant \sin(\theta+\gamma) \leqslant 1$$
$$\Rightarrow \quad -13 \leqslant 13\sin(\theta+\gamma) \leqslant 13$$
$$\Rightarrow \quad -13 \leqslant 5\sin\theta + 12\cos\theta \leqslant 13.$$

Adding 5 throughout we have

$$-8 \leqslant 5\sin\theta + 12\cos\theta + 5 \leqslant 18.$$

This shows that $5\sin\theta + 12\cos\theta + 5 \leqslant 18$ and the minimum value of $5\sin\theta + 12\cos\theta + 5$ is -8.

Using the above

$$\frac{1}{\mathrm{f}(\theta)} = \frac{1}{5\sin\theta + 12\cos\theta} = \frac{1}{13\sin(\theta+\gamma)} = \frac{1}{13}\operatorname{cosec}(\theta+\gamma).$$

The general shape of the graph is a typical cosecant curve, as shown in Fig. 3.6, except that the excluded region in which $1/\mathrm{f}(\theta)$ cannot lie is $(-1/13, 1/13)$, and the curve is translated γ to the left.

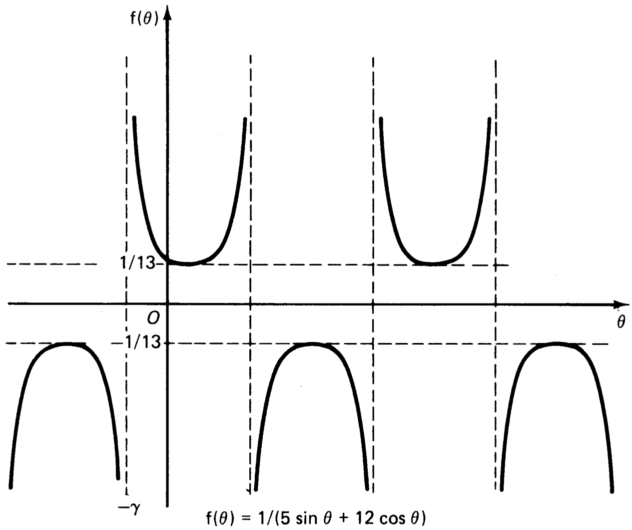

Fig. 3.6

Example 7 Solve, for $0 \leqslant \theta < 2\pi$, the equation $\cos\theta + \sin\theta = 1$, by using the identity $\cos^2\theta + \sin^2\theta = 1$.

We write the given equation in the form $\sin\theta = 1 - \cos\theta$. Squaring both sides of this equation we obtain

$$\sin^2\theta = (1 - \cos\theta)^2 = 1 - 2\cos\theta + \cos^2\theta.$$

But by the given identity $\sin^2\theta = 1 - \cos^2\theta$.
Hence

$$\begin{aligned} & 1 - \cos^2\theta = 1 - 2\cos\theta + \cos^2\theta \\ \Rightarrow\quad & 2\cos^2\theta - 2\cos\theta = 0 \\ \Rightarrow\quad & (\cos\theta - 1)\cos\theta = 0 \\ \Rightarrow\quad & \cos\theta = 1 \text{ or } 0 \\ \Rightarrow\quad & \theta = 0,\ \pi/2,\ 3\pi/2 \end{aligned}$$

in the given range. Clearly 0 and $\pi/2$ are solutions, but $\theta = 3\pi/2$ is a solution of

$$-\sin\theta = 1 - \cos\theta.$$

The only solutions are 0 and $\pi/2$.

Similar difficulties are encountered if one squares the original equation as it stands. We then obtain

$$\cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta = 1,$$

and using the given identity we have $\sin 2\theta = 0$

$$\Rightarrow\quad \theta = 0,\ \pi/2,\ \pi,\ 3\pi/2.$$

But the last two are roots of $-(\cos\theta + \sin\theta) = 1$.

Exercise 3

(All answers should be obtained to the nearest 1/10 of a degree.)

1 Transform the following into the form stated, giving the values of r and γ.
(a) $\cos\theta + \sin\theta$, $r\sin(\theta - \gamma)$;
(b) $\sqrt{3}\cos\theta - \sin\theta$, $r\cos(\theta + \gamma)$;
(c) $\cos 2\theta - \sin 2\theta$, $r\cos(2\theta + \gamma)$;
(d) $3\sin 3\theta - \cos 3\theta$, $r\sin(3\theta + \gamma)$.

2 Sketch the graphs of the expressions in Question 1.

3 Solve the following equations for values of θ between 0° and 360° by each of the methods described on pp. 21–2:
(a) $\sin\theta + \cos\theta = \sqrt{2}$; (b) $2\cos\theta - \sin\theta = 2$; (c) $7\sin\theta - 24\cos\theta = 10$.

4 Find the maximum and minimum values of the following expressions. Give the values of θ in the range 0–360° at which the maximum and minimum values occur.
(a) $3\cos\theta + 2\sin\theta$; (b) $5\sin\theta - 12\cos\theta + 5$;
(c) $1/(\cos 2\theta + \sin 2\theta)$; (d) $(3\sin\theta + 4\cos\theta)^2$.

5 Find the general solution of the following equations:
(a) $3\cos 2\theta - 13\cos\theta + 5 = 0$; (b) $\cos 2\theta + \sin\theta - 1 = 0$;
(c) $\sin 2\theta + \sin\theta = 0$; (d) $\tan\theta\tan 2\theta = 2$;
(e) $3\cos\theta\sin 2\theta + 4\sin^2\theta = 4$.

6 Solve the following equations for values of θ between 0° and 360°:
(a) $\sin 3\theta + \sin 2\theta = 0$; (b) $\cos\theta = \cos 2\theta + \cos 4\theta$;
(c) $\cos 5\theta - \cos\theta = \sin 3\theta$.

7 Show, with the help of a sketch, how the solutions of the simultaneous equations

$$ax + by = c, \quad x^2 + y^2 = 1,$$

are related to the solutions of the trigonometric equation $a\cos\theta + b\sin\theta = c$. Show that if there are two distinct angles between 0° and 360° which satisfy the equation $a\cos\theta + b\sin\theta = c$, then $c^2 < a^2 + b^2$. Show also that if $c^2 = \frac{1}{2}(a^2 + b^2)$, then these angles differ by 90°.

8 Find all the values of θ in the range $0 \leqslant \theta \leqslant 2\pi$ for which

$$\cos\theta + \cos 3\theta = \sin\theta + \sin 3\theta.$$

9 Given that $\cos\theta + \cos(\theta + \phi) + \cos(\theta + 2\phi) = 0$, whatever the value of θ, find the set of possible values of ϕ.

10 By expressing $\cot 2x$ and $\operatorname{cosec} 2x$ in terms of $\tan x$ find the values of $\tan x$ for which

$$5\operatorname{cosec} 2x = 3\cot 2x + 7\tan x.$$

11 Find the acute angle γ for which

$$4\cos\theta - 3\sin\theta \equiv 5\cos(\theta + \gamma).$$

Calculate the values of θ in the interval $-180° \leqslant \theta \leqslant 180°$ for which the function $f(\theta)$, where

$$f(\theta) = 4\cos\theta - 3\sin\theta - 4,$$

attains its least value, its greatest value and zero value.

4 Other topics

4.1 Approximations for small angles

Consider a small angle θ subtended by an arc AB at the centre O of a circle of radius r (see Fig. 4.1). Draw AC perpendicular to OA to cut OB at C. Then $AC = r\tan\theta$ and area $\triangle OAC = \frac{1}{2}r^2\tan\theta$. If we draw the chord AB then $\triangle OAB$ is an isosceles triangle of area $\frac{1}{2}r^2\sin\theta$. Now assuming that

$$\text{area } \triangle OAB < \text{area sector } OAB < \text{area } \triangle OAC$$

and also assuming the result (1.2) on p. 2, it follows that

$$\frac{1}{2}r^2\sin\theta < \frac{1}{2}r^2\theta < \frac{1}{2}r^2\tan\theta$$

$$\Rightarrow \quad \sin\theta < \theta < \tan\theta. \qquad (4.1)$$

Dividing by $\sin\theta$

$$\Rightarrow \quad 1 < \frac{\theta}{\sin\theta} < \sec\theta.$$

As $\theta \to 0$, $\sec\theta \to 1$ and hence

$$\frac{\theta}{\sin\theta} \to 1. \qquad (4.2)$$

Dividing (4.1) by $\tan\theta$

$$\Rightarrow \quad \cos\theta < \frac{\theta}{\tan\theta} < 1.$$

As $\theta \to 0$, $\cos\theta \to 1$ and hence

$$\frac{\theta}{\tan\theta} \to 1. \qquad (4.3)$$

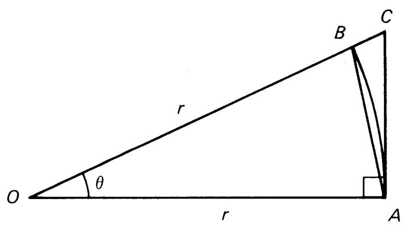

Fig. 4.1

Hence, for small positive values of θ,

$$\sin\theta \approx \theta, \quad \tan\theta \approx \theta. \tag{4.4}$$

In Chapter 2 we established the double-angle identity

$$\cos\theta \equiv 1 - 2\sin^2\tfrac{1}{2}\theta.$$

If θ is small, then $\sin\frac{1}{2}\theta \approx \frac{1}{2}\theta$,

$$\Rightarrow \quad \cos\theta \approx 1 - 2(\tfrac{1}{2}\theta)^2 \approx 1 - \tfrac{1}{2}\theta^2. \tag{4.5}$$

It is easily shown that (4.4) and (4.5) also apply to small negative values of θ. It is to be noted that in all these formulae θ is measured in *radians*.

It can be shown by use of the Maclaurin series that, when θ^4 and higher powers are negligible,

$$\cos\theta = 1 - \tfrac{1}{2}\theta^2, \quad \sin\theta = \theta - \theta^3/6.$$

Example 1 If θ is so small that θ^3 may be neglected, find approximations for the following:

(i) $\dfrac{3\theta}{\sin 4\theta}$, (ii) $\dfrac{\theta\tan 2\theta}{1-\cos\theta}$, (iii) $\sin\frac{1}{2}\theta\sec\theta$.

(i) If θ^3 may be neglected, $\sin 4\theta \approx 4\theta$

$$\Rightarrow \quad \frac{3\theta}{\sin 4\theta} \approx \frac{3\theta}{4\theta} = \frac{3}{4}.$$

(ii) To the given order of approximation

$$\tan 2\theta \approx 2\theta, \quad \cos\theta \approx 1 - \tfrac{1}{2}\theta^2$$

$$\Rightarrow \quad \frac{\theta\tan 2\theta}{1-\cos\theta} \approx \frac{\theta\,.\,2\theta}{1-(1-\frac{1}{2}\theta^2)} = \frac{2\theta^2}{\frac{1}{2}\theta^2} = 4.$$

(iii) Again neglecting θ^3,

$$\sin\tfrac{1}{2}\theta \approx \tfrac{1}{2}\theta, \quad \sec\theta = \frac{1}{\cos\theta} \approx \frac{1}{1-\frac{1}{2}\theta^2}$$

$$\Rightarrow \quad \sin\tfrac{1}{2}\theta\sec\theta \approx \frac{\frac{1}{2}\theta}{1-\frac{1}{2}\theta^2} = \tfrac{1}{2}\theta(1-\tfrac{1}{2}\theta^2)^{-1} \approx \tfrac{1}{2}\theta(1+\tfrac{1}{2}\theta^2+\ldots)$$

$$= \tfrac{1}{2}\theta \text{ (since } \tfrac{1}{4}\theta^3 \text{ is negligible).}$$

4.2 Inverse trigonometric functions

If $\sin\theta = x$, then θ is an angle with a sine value of x. We write $\theta = \text{Sin}^{-1}\,x$ or Arc $\sin x$. There are of course infinitely many values of θ corresponding to a given x in the range $[-1, 1]$ (see Fig. 4.2). If, however, we restrict θ to the range $[-\pi/2, \pi/2]$, there is a unique solution called the *principal value* denoted by

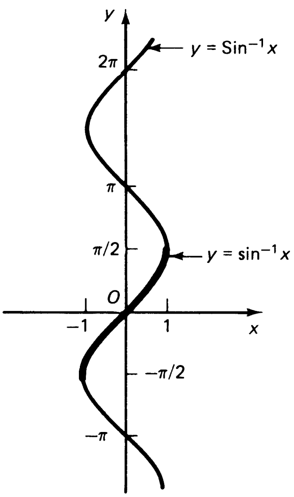

Fig. 4.2

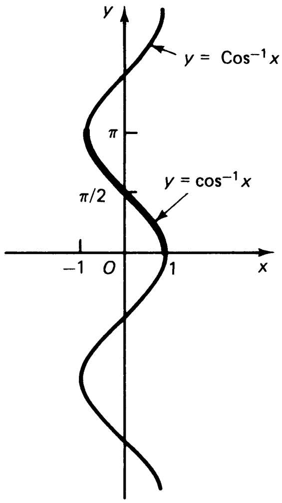

Fig. 4.3

$$\theta = \sin^{-1} x \qquad \text{(small s)}$$

$$\text{or} \quad \theta = \text{arc} \sin x \quad \text{(small a).}$$

This is indicated by the heavy line in Fig. 4.2. It follows that $\sin^{-1} x$ as defined in the range $[-\pi/2, \pi/2]$ is a function.

Similarly, if $\cos\theta = x$, we can obtain the general value $\theta = \text{Cos}^{-1} x$ (or Arc $\cos x$) and the principal value $\cos^{-1} x$ (or arc $\cos x$). The principal value is obtained when we restrict θ to the range $[0, \pi]$ (see Fig. 4.3). The principal value is again indicated by the heavy line.

The inverse tangent $\theta = \text{Tan}^{-1} x$ (or Arc $\tan x$) is obtained from $\tan\theta = x$ in general. If we restrict θ to $(-\pi/2, \pi/2)$, we obtain the principal value $\theta = \tan^{-1} x$ or arc $\tan x$ (see Fig. 4.4).

Example 2 Evaluate (i) $\cos^{-1}(\frac{1}{2})$, (ii) $\tan^{-1}(1/\sqrt{3})$, (iii) $\sin^{-1}(-1)$.

(i) We require the angle θ such that

$$\cos\theta = \tfrac{1}{2}(0 \leqslant \theta \leqslant \pi) \quad \Rightarrow \quad \theta = \pi/3.$$

(ii) The required angle ϕ satisfies $\tan\phi = 1/\sqrt{3}$ with $-\pi/2 \leqslant \phi \leqslant \pi/2$. It is therefore $\phi = \pi/4$.

(iii) If $\alpha = \sin^{-1}(-1)$, then $\sin\alpha = -1$. Since α is in the range $[-\pi/2, \pi/2]$, it follows that $\alpha = -\pi/2$.

Example 3 Prove that $\tan^{-1}(\frac{1}{3}) + 2\tan^{-1}(\frac{1}{4}) = \cot^{-1}(37/39)$.

Let

$$\tan^{-1}(\tfrac{1}{3}) = \theta \quad \Rightarrow \quad \tan\theta = \tfrac{1}{3},$$

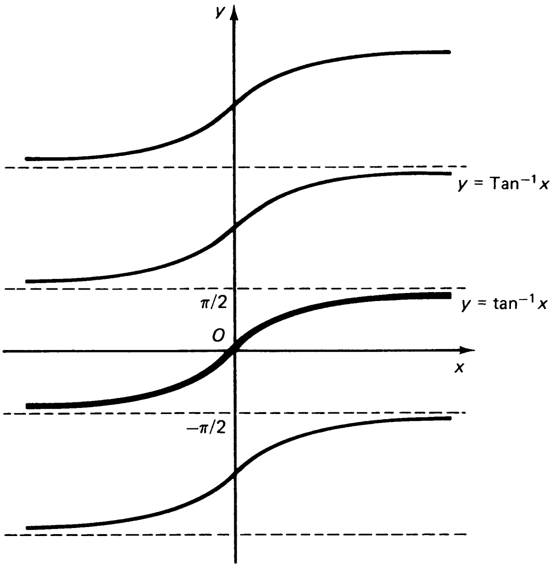

Fig. 4.4

$$\tan^{-1}(\tfrac{1}{4}) = \phi \quad \Rightarrow \quad \tan\phi = \tfrac{1}{4},$$

$$\Rightarrow \quad \tan^{-1}(\tfrac{1}{3}) + 2\tan^{-1}(\tfrac{1}{4}) = \theta + 2\phi.$$

By (2.5)

$$\tan(\theta + 2\phi) = \frac{\tan\theta + \tan 2\phi}{1 - \tan\theta\tan 2\phi}$$

and

$$\tan 2\phi = \frac{2\tan\phi}{1 - \tan^2\phi} = \frac{2 \times \frac{1}{4}}{1 - 1/16} = \frac{8}{15}$$

$$\Rightarrow \quad \tan(\theta + 2\phi) = \frac{\frac{1}{3} + 8/15}{1 - \frac{1}{3}(8/15)} = \frac{39}{37}$$

$$\Rightarrow \quad \theta + 2\phi = \tan^{-1}(39/37) = \cot^{-1}(37/39).$$

Note that here we have used the result $\tan^{-1} x = \cot^{-1}(1/x)$.

Example 4 Given that $0 \leqslant x \leqslant \pi/4$, solve the equation

$$\tan^{-1}(2x) - \tan^{-1}(x) = \tan^{-1}(\tfrac{1}{3}).$$

Let $\tan^{-1} 2x = \alpha$, $\tan^{-1} x = \beta$ and $\tan^{-1}\frac{1}{3} = \gamma$. Then the given equation is equivalent to

$$\alpha - \beta = \gamma,$$

where $\tan\alpha = 2x$, $\tan\beta = x$ and $\tan\gamma = \frac{1}{3}$.

Now

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \tan\gamma = \tfrac{1}{3}$$

$$\Rightarrow \quad \frac{2x - x}{1 + 2x^2} = \frac{x}{1 + 2x^2} = \frac{1}{3}$$

$$\Rightarrow \quad 2x^2 - 3x + 1 = 0 \quad \Rightarrow \quad (2x - 1)(x - 1) = 0$$

$$\Rightarrow \quad x = \tfrac{1}{2} \text{ or } x = 1.$$

The only solution in the given range is $x = \frac{1}{2}$.

4.3 The sine and cosine rules

We refer to Fig. 4.5 in which a, b and c are the lengths of the sides of a triangle in which A, B and C are the corresponding opposite angles. The *sine rule* is

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \tag{4.6}$$

This can be used to solve a triangle uniquely when two angles and a side are known. If two sides and a non-included angle are given, there may be a unique solution, or there may be two alternative solutions. The latter is known as the *ambiguous* case (see Example 5).

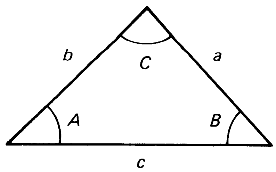

Fig. 4.5

If we are not given a side and the opposite angle, then we must use the appropriate form of the *cosine rule*:

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A, \\ b^2 &= c^2 + a^2 - 2ca\cos B, \\ c^2 &= a^2 + b^2 - 2ab\cos C. \end{aligned} \tag{4.7}$$

4.4 The area of a triangle

The area of the triangle in Fig. 4.5 can be calculated from

$$\triangle = \tfrac{1}{2}bc\sin A = \tfrac{1}{2}ac\sin B = \tfrac{1}{2}ab\sin C. \tag{4.8}$$

An alternative formula which must be used when all the sides are known is

$$\triangle = \sqrt{[s(s-a)(s-b)(s-c)]} \tag{4.9}$$

where $s = \frac{1}{2}(a + b + c)$.

Note. In all calculations for solving triangles, the use of a calculator is useful, but retention of more than four significant figures is not advised. The values of angles are usually required to the nearest tenth of a degree.

Example 5 Given that in the triangle ABC, $b = 3{\cdot}8$ cm, $A = 25°$ and $a = 1{\cdot}8$ cm, find C. Find also the area of the triangle.

By the sine rule,

$$\sin B = \frac{b \sin A}{a} = \frac{3{\cdot}8 \sin 25°}{1{\cdot}8} = 0{\cdot}8922.$$

If $\sin \theta° = 0{\cdot}8922$, there are two possible values of θ in the range $0° < \theta < 180°$; they are $116{\cdot}8°$ and $63{\cdot}2°$.

(i) If $B = 116{\cdot}8°$ and $A = 25°$, then $A + B = 141{\cdot}8°$, which is less than $180°$. This value of B is therefore possible, the remaining angle C being $38{\cdot}2°$. In this case the area of the triangle is most easily obtained from

$$\tfrac{1}{2}ab \sin C = \tfrac{1}{2}(1{\cdot}8)(3{\cdot}8)\sin 38{\cdot}2° = 2{\cdot}115 \text{ cm}^2.$$

(ii) If $B = 63{\cdot}2°$ and $A = 25°$, then $A + B = 88{\cdot}2°$, which is also less than $180°$. So $B = 63{\cdot}2°$ is also a solution corresponding to $C = 91{\cdot}8°$. The area in this case is

$$\tfrac{1}{2}(1{\cdot}8)(3{\cdot}8)\sin 91{\cdot}8° = 3{\cdot}418 \text{ cm}^2.$$

The two possible cases are shown in Fig. 4.6.

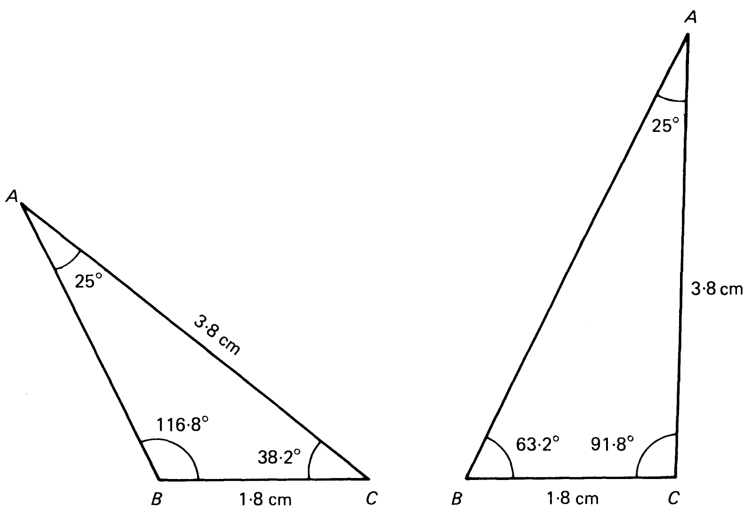

Fig. 4.6

Example 6 Given that in triangle ABC, $a = 4$, $b = 7$, $c = 5$, find the angles of the triangle and also the area of the triangle.

By the cosine rule,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{7^2 + 5^2 - 4^2}{2.7.5} = \frac{58}{70},$$

$$\Rightarrow \quad A \approx 34°.$$

Likewise

$$\cos B = \frac{c^2 + a^2 - b^2}{2ac} = \frac{25 + 16 - 49}{40} = -\frac{8}{40} = -\frac{1}{5},$$

$$\Rightarrow \quad B \approx 101{\cdot}5°.$$

Hence $C \approx 44{\cdot}5°$.

The area in this case is most easily obtained from (4.9)

$$\sqrt{[s(s-a)(s-b)(s-c)]} = \sqrt{(8 \times 4 \times 1 \times 3)}\ \text{cm}^2 = \sqrt{96}\ \text{cm}^2 = 9{\cdot}798\ \text{cm}^2.$$

Exercise 4

1 If θ is so small that θ^3 may be neglected find approximations for
(a) $\dfrac{3\sin(\frac{1}{2}\theta)}{\theta}$, (b) $\dfrac{\cos\theta - 1}{\sin(\frac{1}{3}\theta)}$, (c) $\dfrac{\theta\cot\theta}{1 + \cos\theta}$.

2 Find the limit, as $\theta \to 0$, of

$$\frac{\sin(x + \theta) - \sin x}{\sin 2\theta}.$$

3 Find the limit, as $\theta \to 0$, of

$$\frac{(\sin\theta + \cos\theta)^2 - 1}{\theta}.$$

4 Find the values of
(a) $\cos^{-1}(1/\sqrt{2})$, (b) $\tan^{-1}(\sqrt{3})$, (c) $\cos^{-1}(-1)$, (d) $\sin^{-1}(\frac{1}{3})$.

5 Show that $\tan^{-1}(\frac{1}{3}) + \tan^{-1}(\frac{1}{2}) = \pi/4$.

6 Solve the equation $\tan^{-1}(1 + x) + \tan^{-1}(1 - x) = \tan^{-1}(3)$.

7 In triangle ABC, $C = 52°$, $B = 49°$ and $c = 8{\cdot}62$ cm. Find b and also the area of the triangle.

8 Find the angles of the $\triangle ABC$ in which $a = 10$ cm, $b = 12$ cm and $c = 17$ cm. Find also the area of $\triangle ABC$.

9 In $\triangle ABC$, P, Q and R are the mid-points of BC, CA and AB respectively. Use the cosine rule to prove that

$$b^2 + c^2 = 2AP^2 + 2BP^2.$$

Write down two other similar results, and hence show that

$$AP^2 + BQ^2 + CR^2 = \tfrac{3}{4}(a^2 + b^2 + c^2).$$

10 In the convex quadrilateral $ABCD$, $AB = 5$ cm, $BC = 8$ cm, $CD = 3$ cm, $DA = 3$ cm and $BD = 7$ cm. Find the angles DAB and BCD and deduce that the quadrilateral is cyclic. Show that $AC = 39/7$ cm.

11 The sides BC, CA and AB of a triangle ABC are of length $(x + y)$ cm, x cm and $(x - y)$ cm respectively. Show that

$$\cos A = \frac{x - 4y}{2(x - y)}.$$

Show further that $\sin A - 2\sin B + \sin C = 0$.

12 Find

$$\lim_{x \to 0}\left(\frac{\cos x - \cos 2x}{\cos 2x - \cos 3x}\right).$$

13 If θ is so small that θ^3 may be neglected show that

$$2\cos(\pi/3 + \theta) = 1 - \theta\sqrt{3} - \theta^2/2.$$

14 Show that

$$\sin(\tan^{-1} x) = \frac{2x}{1 + x^2}, \qquad x > 0.$$

15 Solve for x the equation

$$\tan^{-1}\left(\frac{1 - x}{1 + x}\right) = \tfrac{1}{2}\tan^{-1} x.$$

16 Prove that

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{xy}{1 - xy}\right)$$

Use this result to show that

$$\tan^{-1} x + \tan^{-1}\left(\frac{1 - x}{1 + x}\right) = \pi/4.$$

17 The angle of elevation of the top of a vertical pole is 29° from a point A and 52° from another point B, 50 m nearer the base of the pole, which lies on the horizontal line AB produced. Calculate, to 2 decimal places, the height of the pole.

18 Using the sine rule show that for any plane triangle ABC

$$\tan\left(\frac{A}{2}\right)\tan\left(\frac{B - C}{2}\right) = \frac{b - c}{b + c}.$$

Answers

Exercise 1

1 $\sqrt{3}/2, -\sqrt{3}/2, -\sqrt{3}/2, \sqrt{3}/2$.
3 $(-180°, -90°)$; $(30°, 90°)$; $(150°, 180°)$; or $(-\pi, -\pi/2)$; $(\pi/6, \pi/2)$; $(5\pi/6, \pi)$.
4 (a) $227{\cdot}7°$ and $312{\cdot}3°$; (b) $63{\cdot}3°$ and $296{\cdot}7°$; (c) $110{\cdot}2°$ and $290{\cdot}2°$; (d) $255{\cdot}2°$.
6 (a) 30, 150; (b) 135, 315.
7 48·2, 311·8, 120, 240.
8 (a) $2(\sin^2\theta \neq 1)$; (b) $\sec^2\theta(\sin\theta \neq 0)$.
9 $\pi/10, \pi/2, 9\pi/10, 13\pi/10, 17\pi/10, 7\pi/6, 11\pi/6$.
10 $2n\pi/3 + \pi/6, (4n+1)\pi/2, (n \in \mathbb{Z})$.
11 $\pi/18, 5\pi/18, 13\pi/18, 17\pi/18, 25\pi/18, 29\pi/18$.
14 $n\pi + \pi/4 \pm \beta$.
15 $\pi/4$.
16 $336{\cdot}3°$; $143{\cdot}7°$.
17 $n\pi \pm \pi/6, n \in \mathbb{Z}$.
18 $2n\pi \pm \pi/6, n \in \mathbb{Z}$.

Exercise 2

1 (a) 416/425; (b) 87/425; (c) 416/87; (d) −304/425; (e) −297/425; (f) 304/297.
4 (a) $(1+\sqrt{3})/2\sqrt{2}$; (b) $(\sqrt{3}-1)/2\sqrt{2}$; (c) $(1-\sqrt{3})/(1+\sqrt{3})$.
9 (a) $x = 2y^2 - 1$; (b) $(1-y^2)x = 2y$.
11 $(-1/8), (-31/32)$.
13 0·6, 0·8.
14 −16/63.
15 $-\frac{1}{4}\pi < \theta < \frac{1}{4}\pi$.
16 $\frac{1}{4}\pi - \frac{1}{2}\alpha$.
18 3 or 1/3.
19 23·2°, 203·2°, 135°, 315°
23 −1·53, −0·35, 1·88

Exercise 3

1 (a) $\sqrt{2}, 7\pi/4$; (b) $2, \pi/6$; (c) $\sqrt{2}, \frac{1}{4}\pi$; (d) $\sqrt{10}$, $\tan\gamma = -1/3$ and $3\pi/2 \leqslant \gamma \leqslant 2\pi$.
3 (a) 45°; (b) 306·9°; (c) 97·3° and 230·2°.
4 (a) Max $\sqrt{13}$, 33·7°; Min $-\sqrt{13}$, 213·7°; (b) Max 18, 67·4°; Min 8, 337·4°, (c) Max $-1/\sqrt{2}$, 112·5°, 292·5°; Min $1/\sqrt{2}$, 22·5°, 202·5°; (d) Max 25, 36·9°, 216·9°; Min 0, 126·9°, 306·9°.
5 (a) $360\,n° \pm 80{\cdot}4°(n \in \mathbb{Z})$; (b) $180\,n°$ and $180\,n° + (-1)^n 30°(n \in \mathbb{Z})$; (c) $180\,n°$ and $360\,n° \pm 120°(n \in \mathbb{Z})$; (d) $180\,n° + 35{\cdot}3°$ and $180\,n° - 35{\cdot}3°(n \in \mathbb{Z})$; (e) $180\,n° + (-1)^n 90°$ and $180\,n° + (-1)^n 270°$ and $180\,n° + (-1)^n 41{\cdot}8°(n \in \mathbb{Z})$.
6 (a) 72°, 144°, 180°, 216°, 288°; (b) 20°, 90°, 100°, 140°, 220°, 260°, 270°, 340°; (c) 60°, 105°, 120°, 165°, 180°, 240°, 285°, 300°, 345°.
8 $\pi/2, 3\pi/2, \pi/8, 5\pi/8, 9\pi/8, 13\pi/8$.
9 $2\pi n \mp 2\pi/3, (n \in \mathbb{Z})$.
10 $\mp 1/\sqrt{3}$.
11 Least value at $\theta = 143{\cdot}1°$; greatest value at $\theta = -36{\cdot}9°$; zero at $\theta = 0°$.

Exercise 4

1 (a) $(3/2) - (\theta^2/16)$, (b) $-(3\theta/2)$, (c) $\frac{1}{2} - \theta^2/24$.
2 $\frac{1}{2}\cos x$.
3 2.
4 (a) 45°; (b) 60°; (c) 180°; (d) 19·5°.
6 $x = \pm\sqrt{(2/3)}$.
7 $b = 8{\cdot}26$ cm, area $= 34{\cdot}95$ cm^2.
8 $A = 35{\cdot}3°, B = 43{\cdot}9°, C = 100{\cdot}8°$, area $= 58{\cdot}9$ cm^2.
10 angle $DAB = 120°$, angle $BCD = 60°$.
12 $\frac{3}{5}$.
15 $\frac{1}{3}\sqrt{3}$.
17 48·89 m.

Index