Functional Analysis and Evolution Equations

The Günter Lumer Volume

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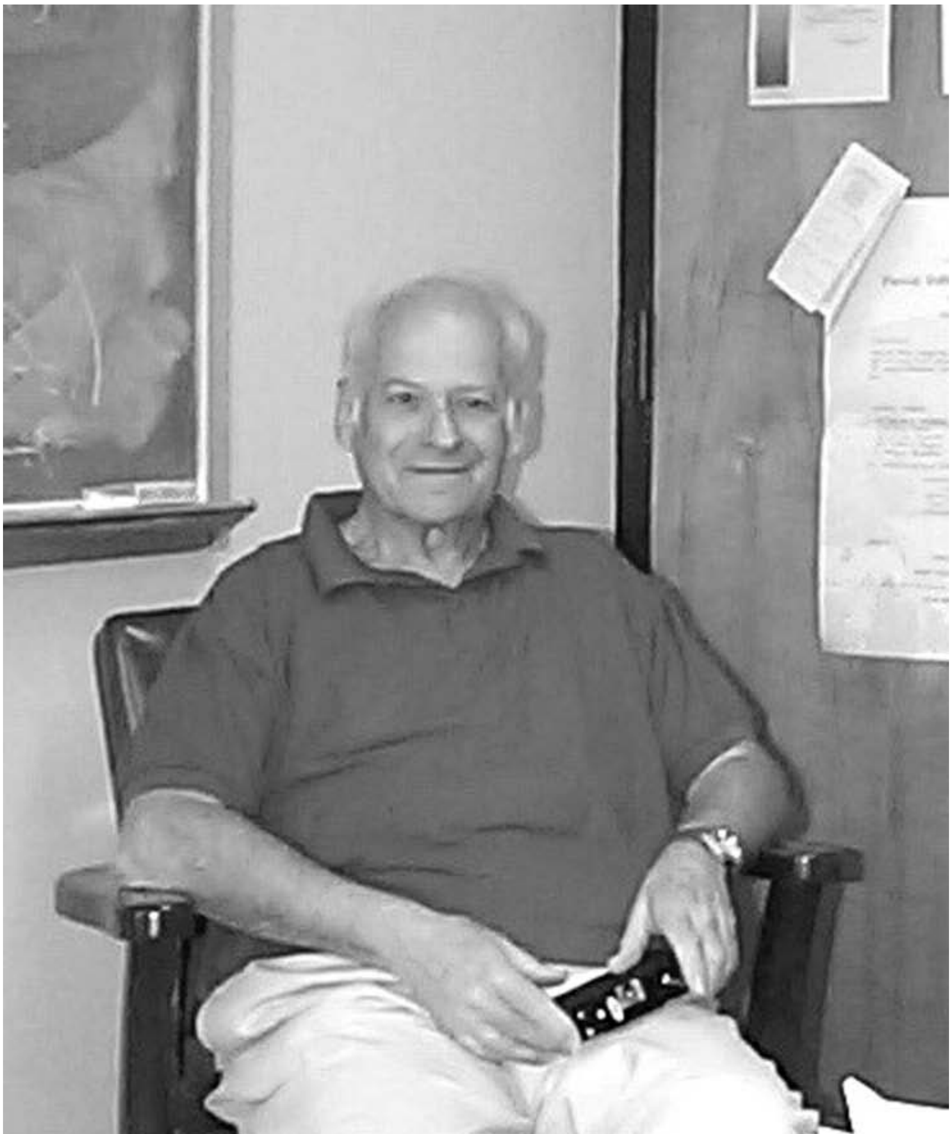

Günter Lumer (1929–2005)

Life and Work of Günter Lumer

Günter Lumer was born in Frankfurt, Germany in 1929. With Nazism on the rise, the Lumer family left Germany in 1933 and settled in France, where Günter received his early education. Then, in 1941, the Lumer family fled once again, this time to Uruguay, where Günter would become a citizen.

Possessing what would be a life-long passion for mathematics, Günter graduated in 1957 with a degree in electrical engineering from the University of Montevideo. In fact, while at Montevideo, he was in the research group of Paul Halmos, who would later dedicate a page to Günter in his book *I Want to be a Mathematician: an Automathography.* Günter's first paper "Square roots of operators," a joint work with P. Halmos and J.J. Schäffer, appeared in 1953 in the *Proceedings of the American Mathematical Society.*

In 1956, Günter received a Guggenheim fellowship to study at the University of Chicago. There he received his Ph.D. in Mathematics in 1959; his dissertation was entitled *Numerical Range and States* and was written under the supervision of Irving Kaplansky, thus earning himself a place among a long lineage of mathematicians connected to Kaplansky.

Following Chicago, Günter Lumer held positions at UCLA (1959–1960), Stanford University (1960–1961), University of Washington (1961–1974), University of Mons-Hainaut (1973–2005), and the International Solvay Institutes for Physics and Chemistry in Brussels (1999–2005).

Günter Lumer was a creative and prolific mathematician whose works have great influence on the research community in mathematical analysis and evolution equations. His scientific activities greatly contributed to the standing of the Belgian Universities in general and the University of Mons-Hainaut in particular. In 1976, supported by the Belgium National Science Foundation, Günter founded a contact group with the goal of organizing research and exchange meetings in the fields of Partial Differential Equations and Functional Analysis. From the 1990s on, building on the success of this group, Günter became a driving force and leading contributor to several large-scale projects sponsored by the European Community. The resulting conferences on *Evolution Equations* created a lasting network supporting international research collaboration. These activities, combined with Günter's relentless energy and love for mathematics, were at the origin of the breath-taking development of the field of evolution equations and the theory of operator semigroups after the pioneering book of Hille and Phillips from 1957.

In particular, between 1992 and 1997 he co-organized the *North West European Analysis Seminar* that was held in 1992 at Saint Amand les Eaux (France), in 1993 at Schloss Dagstuhl (Germany), in 1994 at Noordwijkerhout (The Netherlands), in 1995 at Lyon (France), in 1996 at Glasgow (United Kingdom) and in 1997 at Blaubeuren (Germany). Those seminars covered a broad range of topics in analysis and were a reflection of the true spirit of Günter Lumer, who always enjoyed bringing together and working with a wide range of mathematicians and scientists.

Although Günter Lumer's professional focus was on functional analysis, partial differential equations, and evolution equations, he nourished a broad interest for almost all areas of mathematics and for science in general. He published more than one hundred papers and edited many books. Probably his best known result is the celebrated Lumer-Phillips theorem, which gives necessary and sufficient conditions on an operator to generate a strongly continuous semigroup of contractions on a general Banach space. This result, published in the *Pacific Journal of Mathematics* in 1961, is a key contribution to the theory of operator semigroups.

Günter Lumer deeply loved mathematics. He considered his work as the most precious thing he could leave to future generations. He was an independent and original person, never influenced by fashion or convention. He used to say, "If a crowd of a thousand unanimously condemns someone, then he must be innocent. For it is unlikely for a thousand people to honestly agree on the same thing."

With Günter Lumer we miss an inspiring teacher, a mentor and friend of a generation of researchers, and a leader of our professional community. Günter Lumer: a mathematician to be honored.

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Charles Widger, *Multiplicative perturbations of generators of semigroups of operators*, U. Washington, 1970

David Neu, *Summability of the linear predictor*, U. Washington, 1972

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In Remembrance of Günter Lumer

Heinz König

Günter Lumer was a close friend of mine for several decades. We had the same age: our dates of birth were but 13 days apart. We met for the first time in the fall of 1962 at a functional analysis conference in Oberwolfach. The year before Günter had published two of his most important papers: the common paper with Ralph Phillips on dissipative operators and the paper on semi-inner products.

The subsequent years were the grand period in the development of the functional analytic theory of abstract analytic functions, known under the key words of uniform algebras and Hardy spaces. We were both deeply involved, with quite often different methods but close results. Günter obtained fundamental breakthroughs in two situations: The first time in Bulletin Amer. Math. Soc. 70(1964), where he was able to develop the abstract counterpart of the classical unit disk situation on an arbitrary uniform algebra and for an individual multiplicative linear functional, under the basic assumption that the functional in question has a unique representing measure. Before that one needed global assumptions on the algebra like to be Dirichlet or logmodular. After his work then 1965 Kenneth Hoffman-Hugo Rossi and myself independently obtained the final abstract version of the classical unit disk situation in terms of a fixed so-called Szegő measure for an individual multiplicative linear functional.

The second breakthrough was in his 1968 Lecture Notes, this time for an arbitrary multiplicative linear functional on any uniform algebra. Günter defined its universal Hardy class and was able to transfer the classical concepts and results to an amazing extent, in particular to establish an abstract conjugation operation via extension of the classical Kolmogorov estimations. He then left the field in the early seventies. I myself returned to it in a common frame with the extended concept of Daniell-Stone integration due to Michael Leinert 1982, which produced a definitive theory around 1990. But it is clear that to an essential extent the basic contributions are due to Günter Lumer in the sixties.

In all these years we had close contacts. During the academic year 1967/68 Günter stayed at Strasbourg University, thus close to my home University Saarbrücken. In the summer term 1967 he gave a series of lectures in Saarbrücken, and in the winter term 1967/68, which I spent at Caltech in Pasadena, a little bus supplied by our University brought my students to his lectures in Strasbourg every week. In the academic year 1969/70 Günter Lumer together with Irving Glicksberg organized a Research Seminar on function algebras at their home University, the University of Washington in Seattle. I had the good fortune to participate for three months on his invitation.

After his move to Belgium in 1973/74 Günter was a regular visitor to Saarbrücken, both private and for a further series of lectures and several colloquium talks. He wrote a comprehensive survey article on evolution equations for our Annales Universitatis Saraviensis and published several papers in the Archiv der Mathematik of which I had been the editor for abstract analysis. Our relations became even closer because of the sequence of the North-West European Analysis Seminars 1992–1997, of which Günter was the unique creator and driving force. We were common chairmen of the second seminar 1993 at Schloss Dagstuhl in the Saar State, which is the Informatics counterpart of the Oberwolfach Institute. Thus we two are in the tiny group of "outside" mathematicians who have ever been chairpersons of conferences at Schloss Dagstuhl. Unfortunately, in 1997 a serious hip joint operation forced Günter to discontinue the beautiful enterprise. There was no successor.

For me the first of the seminars 1992 in Saint-Amand-les-Eaux near Lille was a moving event: Near its end I fell into heart trouble, and my doctor said on the telephone that I should come to his hospital right away but must not drive a car. What then happened was that Günter asked Luc Paquet to place his own car next to his apartment in Brussels, and took the steering-wheel of my car (which was new at the time) to drive us for at least 400 kilometers to Saarbrücken. We arrived late at night, and my wife said later that I looked radiant with health but Günter grey with exhaustion. This was the deepest evidence of friendship which I ever experienced in my life.

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Expansions in Generalized Eigenfunctions of the Weighted Laplacian on Star-shaped Networks

Félix Ali Mehmeti, Robert Haller-Dintelmann and Virginie Régnier

In memory of Günter Lumer

Abstract. We are interested in evolution phenomena on star-shaped networks composed of n semi-infinite branches which are connected at their origins. Using spectral theory we construct the equivalent of the Fourier transform, which diagonalizes the weighted Laplacian on the n-star. It is designed for the construction of explicit solution formulas to various evolution equations such as the heat, wave or the Klein-Gordon equation with different leading coefficients on the branches.

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1. Introduction

We study the foundations for the understanding of evolution phenomena on star-shaped networks composed of n semi-infinite branches which are connected at their origins. To this end, we construct the equivalent of the Fourier transform which diagonalizes the weighted Laplacian on the n-star, using spectral theory. This allows us to formulate a functional calculus for the weighted Laplacian, designed to construct explicit solution formulas to various evolution equations such as the heat, wave or the Klein-Gordon equation with different leading coefficients on the branches. The model of the n-star should lead to a comprehension of the phenomena happening locally in time and space near the ramification nodes of

Parts of this work were done, while the second author visited the University of Valenciennes. He wishes to express his gratitude to F. Ali Mehmeti and the LAMAV for their hospitality.

more complicated networks. The investigation of evolution equations on networks starts with G. Lumer [17] and subsequent papers. See [1, 4, 9] and the references mentioned therein.

Let $N_1, \dots, N_n$ be n disjoint copies of $(0; +\infty)$ ($n \in \mathbb{N}$, $n \geq 2$) and $c_k > 0$, for $k \in \{1, \dots, n\}$. A vector $(u_1, \dots, u_n)$ of functions $u_k : \overline{N_k} \to \mathbb{C}$ is said to satisfy the transmission conditions

$$(T_0), \text{ if } u_i(0) = u_k(0) \text{ for all } (i,k) \in \{1, \dots, n\}^2,$$

$$(T_1), \text{ if } \sum_{k=1}^{n} c_k^2 \partial_x u_k(0^+) = 0.$$

A vector $(u_k)_{k=1,\dots,n}$ satisfying (T_0) can also be viewed as a function on $N := \bigcup_{k=1}^{n} \overline{N_k}$, where the n boundary points corresponding to $0 \in \overline{N_k}$ are identified. This domain is called a star-shaped network or n-star with the branches $N_1, \dots, N_n$.

In this paper, we study the weighted Laplacian submitted to (T_0) and (T_1):

$$\begin{cases} D(A) := \left\{ (u_k) \in \prod_{k=1}^{n} H^2(N_k) \mid (u_k) \text{ satisfies } (T_0) \text{ and } (T_1) \right\}, \\ A(u_k) := (-c_k^2 \cdot \partial_x^2 u_k)_{k=1,\dots,n}. \end{cases}$$

This operator can be inserted for example in the abstract wave equation

$$\begin{cases} \ddot{u}(t) + Au(t) = 0, \\ u(0) = u_0, \ \dot{u}(0) = v_0, \end{cases}$$

which means in concrete terms:

$$\begin{cases} [\partial_t^2 - c_k^2 \partial_x^2] u_k(t,x) = 0, & \forall\, k \in \{1, \dots, n\}, \\ u_i(t,0) = u_k(t,0), & \forall\, (i,k) \in \{1, \dots, n\}^2, \\ \sum_{k=1}^{n} c_k^2 \partial_x u_k(t, 0^+) = 0, & \\ u_k(0,x) = u_k^0(x), & \forall\, k \in \{1, \dots, n\}, \\ \partial_t u_k(0,x) = v_k^0(x), & \forall\, k \in \{1, \dots, n\} \end{cases}$$

for $x, t \geq 0$, where $u_0 = (u_k^0)_{k=1,\dots,n}$, $v_0 = (v_k^0)_{k=1,\dots,n}$ and $u(t) = (u_k(t, \cdot))_{k=1,\dots,n}$.

The operator A is self-adjoint, its spectrum is $[0; +\infty)$ and has multiplicity n (in the sense of ordered spectral representations, see Definition XII.3.15, p. 1216 of [14]). The analytical core of this paper is a representation of the kernel of the resolvent of A in terms of a special choice of a family of n generalized eigenfunctions parametrized by $\lambda \in [0; +\infty)$.

After having proved a limiting absorption principle for the resolvent, we insert A in Stone's formula to obtain a representation of the resolution of the identity of A in terms of the generalized eigenfunctions. This classical procedure (see for example [3]) should lead to an expansion formula for functions in $H = \prod_{k=1}^{n} L^2(N_k)$ in terms of the family of generalized eigenfunctions.

We observe that the transition from the formula for the resolution of the identity to an expansion formula involving a generalized Fourier transform, which diagonalizes A, is not straightforward in the case of the n-star. This comes from the fact that the resolvent kernel, which is defined on $N \times N$, changes its structure when crossing the n diagonals of $N_k \times N_k$, $k = 1, \ldots, n$. These diagonals cut $N \times N$ into n connected pieces in accordance with the structure of the resolvent. Our special choice of the generalized eigenfunctions allows us to recombine the inner integral of the formula for the resolution of the identity across the diagonals of $N_k \times N_k$ to an integral over all of N, furnishing the desired generalized Fourier transformation V as well as its left inverse Z. It is not obvious, whether this recombination is possible for all choices of generalized eigenfunctions, although theoretical results imply that an expansion in generalized eigenfunctions always exists [11, 19]. Now, V can be extended to an isometry on H, which diagonalizes A, and an explicit functional calculus for A can be given. We plan to give explicit expressions for the solutions of evolution equations like the weighted wave, heat and Klein-Gordon equations on the n-star and to derive results on their qualitative behaviour in a subsequent paper.

Such expressions can be obtained (at least formally) also from representations of the resolution of the identity which are not recombined to Fourier-type transformations. But these expressions would be sums of terms with very poor regularity although their sum, representing the solution, is regular (like a decomposition of a C^∞-function by multiplying it with characteristic functions on sub-domains). These artificial singularities are totally undesirable for any kind of investigations. They occur for example in [13], a pioneering paper of theoretical physics explaining the phenomenon of advanced transmission of dispersive wave packets crossing a potential barrier. The authors obtain a solution formula using Laplace transform in time, but which splits up into irregular terms. They do not attempt to prove that their formula represents a solution of the original problem, which should be possible only in some very weak sense. But this (artificial) lack of regularity permits only to study the advanced transmission phenomenon for gaussian wave packets using a highly special method.

In [7], the authors study the similar phenomenon of delayed reflection occurring at semi-infinite barriers. They construct an expansion in generalized eigenfunctions and thus avoid those artificial singularities. This expansion is used to define wave packets in frequency bands adapted to the transmission conditions. Thus it is possible to study the dependence of propagation patterns, in particular the delayed reflection, on the main frequency of the wave packets. In [8] it is pointed out using similar methods, that classical causality is valid for nonlinear dispersive waves hitting a semi-infinite barrier. In [6] a solution formula for the Klein-Gordon equation on the n-star but with one finite branch with an end with prescribed excitation is presented using Laplace transform in time. This result is not comparable with the present paper, because it does not concern an initial value problem.

There remains an unsatisfactory point in the present paper: our Fourier-type transformation V is not a spectral representation of A in the classical sense although it diagonalizes this operator: the natural norm on the range of V making

V an isometry, as in the theorem of Plancherel, is not just a weighted L^2-norm on some measure space. This is due to the fact that the back transformation Z has a different expression on each branch, and this is caused by the ramification of the domain.

It is not clear to us how one could find a family of generalized eigenfunctions leading to a spectral representation of A. The existing general literature on expansions in generalized eigenfunctions ([11, 19, 20] for example) does not seem to be helpful for this kind of problem: their constructions start from an abstractly given spectral representation. But in concrete cases you do not have an explicit formula for it at the beginning.

In [10] the relation of the eigenvalues of the Laplacian in a L^∞-setting on infinite, locally finite networks to the adjacency operator of the network is studied. The question of the completeness of the corresponding eigenfunctions, viewed as generalized eigenfunctions in an L^2-setting, could be asked. The n-star we consider is a particular case of the geometry studied by J. von Below and the completeness of the eigenfunctions is established in a way. In a recent paper ([15]), the authors consider general networks with semi-infinite ends. They give a construction to compute some generalized eigenfunctions from the coefficients of the transmission conditions (scattering matrix). The eigenvalues of the associated Laplacian are the poles of the scattering matrix and their asymptotic behaviour is studied. But no attempt is made to show the completeness of a given family of generalized eigenfunctions. Spectral theory for the Laplacian on finite networks has been studied since the 1980ies for example by J.P. Roth, J.v. Below, S. Nicaise, F. Ali Mehmeti (see [1]).

Natural perspectives for our expansion result are investigations on the qualitative behaviour of solutions of evolution equations on the n-star. For the weighted heat equation on the n-star, our expansion permits to prove Gaussian estimates (this feature shall be treated in a subsequent paper). For bounded networks and variable coefficients this has already been proved by D. Mugnolo ([18]) using different methods. In [16] the transport operator is considered on finite networks. The connection between the spectrum of the adjacency matrix of the network and the (discrete) spectrum of the transport operator is established. By adding semi-infinite branches to the finite network, continuous parts of the spectrum and generalized eigenfunctions might appear.

Many results have been obtained in spectral theory for elliptic operators on various types of unbounded domains in $\mathbb{R}^n$. Using the existing results on stratified bands [12] for example, one could reduce the spectral analysis of the Laplacian on networks of bands locally near the nodes to the case of the n-star. Time asymptotics for the associated evolution equations have also been studied extensively. For the Klein-Gordon equation on the n-star we conjecture that the maximum of the absolute value of the solutions decays as $t^{-1/2}$ when t tends to infinity as on the real line. For two branches with potential step this has been already proved using generalized eigenfunctions in [2]. An example for a three-dimensional coupled domain with singularities is treated in [5]. See also the other literature mentioned therein and in [3].

2. Data and functional analytic framework

Let us introduce some notation which will be used throughout the rest of the paper:

- **Domain and functions**: Let $N_1, \dots, N_n$ be n disjoint sets identified with $(0; +\infty)$ ($n \in \mathbb{N}$, $n \geq 2$) and put $N := \bigcup_{k=1}^{n} \overline{N_k}$. Furthermore, we write $[a,b]_{N_k}$ for the interval $[a,b]$ in the branch N_k. For the notation of functions two viewpoints are used:
  - functions f on the object N taking their values in $\mathbb{R}$ and f_k is then the restriction of f to N_k.
  - n-tuples of functions on the branches N_k; then sometimes we write $f = (f_1, \dots, f_n)$.
- **Transmission conditions**:

$$(T_0)\text{: } (u_k)_{k=1,\dots,n} \in \prod_{k=1}^{n} C^0(\overline{N_k}) \text{ satisfies } u_i(0) = u_k(0),\ \forall\ (i,k) \in \{1, \dots, n\}^2.$$

$$(T_1)\text{: } (u_k)_{k=1,\dots,n} \in \prod_{k=1}^{n} C^1(\overline{N_k}) \text{ satisfies } \sum_{k=1}^{n} c_k^2 \cdot \partial_x u_k(0^+) = 0.$$

- **Definition of the operator**: Define the real Hilbert space

$$H = \prod_{k=1}^{n} L^2(N_k) \text{ with scalar product } ((u_k),(v_k))_H = \sum_{k=1}^{n} (u_k, v_k)_{L^2(N_k)}$$

  and the operator $A : D(A) \longrightarrow H$ by

$$\begin{cases} D(A) = \left\{ (u_k) \in \displaystyle\prod_{k=1}^{n} H^2(N_k) \mid (u_k) \text{ satisfies } (T_0) \text{ and } (T_1) \right\}, \\ A(u_k) = (A_k u_k)_{k=1,\dots,n} = (-c_k^2 \cdot \partial_x^2 u_k)_{k=1,\dots,n}. \end{cases}$$

  Note that, if $c_k = 1$ for every $k \in \{1, \dots, n\}$, A is the Laplacian in the sense of the existing literature.
- **Notation for the resolvent**: The resolvent of an operator T is denoted by R, i.e., $R(z, T) = (zI - T)^{-1}$ for $z \in \rho(T)$.

Proposition 2.1 (spectrum of A). *The operator $A : D(A) \to H$ defined above is self-adjoint and satisfies $\sigma(A) = [0; +\infty)$.*

Proof. Simple adaptation of the proof of Lemma 1.1.5 in [3]. □

3. Expansion in generalized eigenfunctions

The aim of this section is to find an explicit expression for the kernel of the resolvent of the operator A on the star-shaped network defined in the previous section.

Definition 3.1 (generalized eigenfunction). Let $\lambda \in \mathbb{C}$ be fixed. An element $f \in \prod_{k=1}^{n} C^{\infty}(\overline{N_k})$ is called generalized eigenfunction of A if it satisfies (T_0), (T_1) and the formal differential expression $Af = \lambda f$.

Proposition 3.2 (an expression of the resolvent). *Let $\lambda \in \mathbb{C}$ be fixed. Let $\mathrm{Im}(\lambda) \neq 0$ and e_1^λ, e_2^λ be generalized eigenfunctions of A such that the Wronskian $w_{1,2}^\lambda(x)$ satisfies for every x in N*

$$w_{1,2}^\lambda(x) = \det W(e_1^\lambda(x), e_2^\lambda(x)) = e_1^\lambda(x) \cdot (e_2^\lambda)'(x) - (e_1^\lambda)'(x) \cdot e_2^\lambda(x) \neq 0.$$

If for some $k \in \{1, \ldots, n\}$ we have $e_1^\lambda|_{N_m} \in H^2(N_m)$ for all $m \neq k$ and $e_2^\lambda|_{N_k} \in H^2(N_k)$, then we have for any $f \in H$, $\lambda \in \rho(A)$ and $x \in N_k$

$$[R(\lambda, A)f](x) = \frac{1}{c_k^2 (w_{1,2}^\lambda)(x)} \cdot \left[\int_{[x;+\infty)_{N_k}} e_1^\lambda(x) e_2^\lambda(x') f(x')\, dx' + \int_{N \setminus [x;+\infty)_{N_k}} e_2^\lambda(x) e_1^\lambda(x') f(x')\, dx' \right]. \tag{1}$$

Note that by integral over N, we mean the sum of the integrals over N_k, $k = 1, \ldots, n$.

Proof. The arguments are the same as in the proof of Theorem 1.3.4 of [3] (see also [2]) and the calculations are analogous. The integration by parts is replaced here by the Green formula for the star-shaped network that is given in the next lemma. □

Lemma 3.3 (Green's formula on the star-shaped network with n semi-infinite branches). *Denote by $V_{a_1,\ldots,a_n}$ the subset of the network N defined by*

$$V_{a_1,\ldots,a_n} = \{x \in N \mid x \in [0; a_k), \text{ where } k \text{ is the index such that } x \in \overline{N_k}\}.$$

Then $u, v \in D(A)$ implies

$$\int_{V_{a_1,\ldots,a_n}} u''(x)v(x)\, dx = \int_{V_{a_1,\ldots,a_n}} u(x)v''(x)\, dx - \sum_{k=1}^{n} u(a_k)v'(a_k) + \sum_{k=1}^{n} u'(a_k)v(a_k).$$

Proof. Two successive integrations by parts are used and since both u and v belong to $D(A)$, they both satisfy the transmission conditions (T_0) and (T_1). So

$$\sum_{k=1}^{n} u_k(0)v_k'(0) = u_1(0) \sum_{k=1}^{n} v_k'(0) = 0.$$

Idem for $\sum_{k=1}^{n} u_k'(0)v_k(0)$. □

Definition 3.4 (generalized eigenfunctions of A). For $j \in \{1, \ldots, n\}$ let

$$s_j := -c_j^{-1} \cdot \sum_{l \neq j} c_l, \qquad d_{1,j} := (1 + s_j)/2 \qquad \text{and} \qquad d_{2,j} := (1 - s_j)/2.$$

The complex square root is chosen in such a way that $\sqrt{r \cdot e^{i\phi}} = \sqrt{r}e^{i\phi/2}$ with $r > 0$ and $\phi \in [-\pi;\pi)$. For $\lambda \in \mathbb{C}$ and $j, k \in \{1, \dots, n\}$, $F_\lambda^{\pm,j} : N \to \mathbb{C}$ is defined for $x \in \overline{N_k}$ by $F_\lambda^{\pm,j}(x) := F_{\lambda,k}^{\pm,j}(x)$ with

$$\begin{cases} F_{\lambda,j}^{\pm,j}(x) = d_{1,j} \cdot \exp(\pm i c_j^{-1}\sqrt{\lambda}x) + d_{2,j} \cdot \exp(\mp i c_j^{-1}\sqrt{\lambda}x), & \\ F_{\lambda,k}^{\pm,j}(x) = \exp(\pm i c_k^{-1}\sqrt{\lambda}x), & \text{for } k \neq j. \end{cases}$$

Remark 3.5.

- $F_\lambda^{\pm,j}$ satisfies the transmission conditions (T_0) and (T_1).
- Formally it holds $AF_\lambda^{\pm,j} = \lambda F_\lambda^{\pm,j}$.
- Clearly $F_\lambda^{\pm,j}$ does not belong to H, thus it is not a classical eigenfunction.
- For $\mathrm{Im}(\lambda) \neq 0$, the function $F_{\lambda,k}^{\pm,j}$, where the $+$-sign (respectively $-$-sign) is chosen if $\mathrm{Im}(\lambda) > 0$ (respectively $\mathrm{Im}(\lambda) < 0$), belongs to $H^2(N_k)$ for $k \neq j$. This feature is used in the formula for the resolvent of A.

Definition 3.6 (kernel of the resolvent). For any $\lambda \in \mathbb{C}$, $j \in \{1, \dots, n\}$ and $x \in \overline{N_j}$ we define

$$K(x, x', \lambda) = \begin{cases} \dfrac{1}{w(\lambda)} F_{\lambda,j}^{\pm,j}(x) F_{\lambda,j}^{\pm,j+1}(x'), \text{ for } x' \in \overline{N_j},\ x' > x, \\ \dfrac{1}{w(\lambda)} F_{\lambda,j}^{\pm,j+1}(x) F_\lambda^{\pm,j}(x'), \text{ for } x' \in \overline{N_k}, k \neq j \text{ or } x' \in \overline{N_j},\ x' < x, \end{cases}$$

where $w(\lambda) = \pm i\sqrt{\lambda} \cdot \sum_{j=1}^n c_j$. In the whole formula $+$ (respectively $-$) is chosen if $\mathrm{Im}(\lambda) > 0$ (respectively $\mathrm{Im}(\lambda) \leq 0$).

Here the index j is to be understood modulo n, that is to say, if $j = n$, then $j + 1 = 1$.

Note that in particular, if $c_j = c$ for all $j \in \{1, \dots, n\}$, then $w(\lambda) = \pm inc\sqrt{\lambda}$, for all $j \in \{1, \dots, n\}$.

Theorem 3.7 (expansion of the resolvent in the family $\{F_\lambda^{\pm,j}, j = 1, \dots, n\}$). *Let $f \in H$. Then, for $x \in N$ and $\lambda \in \rho(A)$*

$$[R(\lambda, A)f](x) = \int_N K(x, x', \lambda) f(x')\, dx'.$$

Proof. In (1), the generalized eigenfunction e_1^λ can be chosen to be $F_\lambda^{\pm,j}$. Then e_2^λ can be $F_\lambda^{\pm,l}$ with any $l \neq j$ so we have chosen $j+1$ to fix the formula. The choice has been done so that the integrands lie in $L^1(0, +\infty)$ (cf. the last item in Remark 3.5). □

4. Application of Stone's formula and limiting absorption principle

Let us first recall Stone's formula (see Theorem XII.2.11 in [14]).

Theorem 4.1 (Stone's formula). *Let E be the resolution of the identity of a linear unbounded self-adjoint operator $T : D(T) \to H$ in a Hilbert space H (i.e., $E(a,b) = \mathbf{1}_{(a,b)}(A)$ for $(a,b) \in \mathbb{R}^2$, $a < b$). Then, in the strong operator topology*

$$h(T)E(a,b) = \lim_{\delta\to 0^+} \lim_{\epsilon\to 0^+} \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} h(\lambda)[R(\lambda-\epsilon i, T) - R(\lambda+\epsilon i, T)]\, d\lambda$$

for all $(a,b) \in \mathbb{R}^2, a < b$ and for any continuous scalar function h defined on the real line.

To apply this formula we need to study the behaviour of the resolvent $R(\lambda, A)$ for λ approaching the spectrum of A.

Theorem 4.2 (limiting absorption principle for A). *For any $(x,x') \in N^2$ and $(\lambda,\epsilon) \in (\mathbb{R}^+)^2$, it holds with s_j, d_j as defined in Definition 3.4:*

1. $\lim_{\epsilon\to 0} K(x,x',\lambda - i\epsilon) = K(x,x',\lambda)$,
2. $|K(x,x',\lambda-i\epsilon)| \le M \cdot (\sqrt{\lambda})^{-1}$ *with*

$$M = \max_{j\in\{1,\dots,n\}} \left[\max(1; |d_{1,j}| + |d_{2,j}|) \cdot (\sum_{j=1}^n c_j)^{-1} \right].$$

Proof. 1. The complex square root is, by definition, continuous on $\{z \in \mathbb{C} \mid \mathrm{Im}(z) \le 0\}$ (cf. Definition 3.4), hence the continuity of $K(x,x',\lambda)$ at real positive numbers λ. (Note that x,x' are fixed parameters in this context.)

2. In concrete terms, the kernel is for $\mathrm{Im}(\mu) \le 0$ and $x \in \overline{N_j}$

$$K(x,x',\mu) = \frac{1}{w(\mu)} \begin{cases} e^{-i\sqrt{\mu}(c_j^{-1}x + c_k^{-1}x')}, & x' \in \overline{N_k},\ k \ne j, \\ d_{2,j}e^{-i\sqrt{\mu}c_j^{-1}(x-x')} + d_{1,j}e^{-i\sqrt{\mu}c_j^{-1}(x+x')}, & x' \in \overline{N_j},\ x' < x, \\ d_{2,j}e^{-i\sqrt{\mu}c_j^{-1}(x'-x)} + d_{1,j}e^{-i\sqrt{\mu}c_j^{-1}(x+x')}, & x' \in \overline{N_j},\ x' > x. \end{cases}$$

Now

$$\begin{aligned} \left|\frac{1}{w(\mu)}\right| &= \Big(\sum_{j=1}^n c_j \sqrt{|\lambda - i\epsilon|}\Big)^{-1} = \Big(\sum_{j=1}^n c_j\Big)^{-1} (\lambda^2+\epsilon^2)^{-1/4} \\ &\le \Big(\sum_{j=1}^n c_j\Big)^{-1} \lambda^{-1/2} \end{aligned}$$

for $\mu = \lambda - i\epsilon$, $\lambda > 0$, $\epsilon \ge 0$. Moreover, if $x' < x$,

$$\left| e^{-i(\sqrt{\lambda-i\epsilon})c_j^{-1}(x-x')} \right| = e^{\mathrm{Im}(\sqrt{\lambda-i\epsilon})c_j^{-1}(x-x')} \le 1,$$

since $\mathrm{sgn}(\mathrm{Im}(\sqrt{\lambda-i\epsilon})) = \mathrm{sgn}(\mathrm{Im}(\lambda-i\epsilon))$ (cf. Lemma 2.5.1 of [3], see also [2]). Idem for the other exponential terms. Hence the above estimate. $\square$

Remark 4.3. Note that, in particular, if $c_j = c$, $j = 1, \dots, n$, then $M = c(n-1)/n$.

Lemma 4.4. *For $(x, x') \in N^2$ and $\lambda \in \mathbb{C}$, it holds $K(x, x', \overline{\lambda}) = \overline{K(x, x', \lambda)}$.*

Proof. The choice of the branch cut of the complex square root has been made such that $\sqrt{\overline{\lambda}} = \overline{\sqrt{\lambda}}$ for all $\lambda \in \mathbb{C}$.

This implies $\overline{e^{i\sqrt{\lambda}x}} = e^{\overline{i\sqrt{\lambda}x}} = e^{-i\sqrt{\overline{\lambda}}x}$ for all $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$. Thus it holds

$$\overline{F_{\lambda}^{+,j}(x)} = F_{\overline{\lambda}}^{-,j}(x) \quad \text{and} \quad \overline{F_{\lambda}^{-,j}(x)} = F_{\overline{\lambda}}^{+,j}(x)$$

for all $\lambda \in \mathbb{C}$, $x \in N$ and $j \in \{1, \dots, n\}$. In the same way we have $\overline{w(\lambda)} = -w(\overline{\lambda})$. Observe, that switching from λ to $\overline{\lambda}$ the sign of the imaginary part is changing, so in the definition of $K(x, x', \lambda)$ we have to take the other sign whenever there is a $\pm$-sign in the formula. This gives the assertion. $\square$

Proposition 4.5 (rewriting of the resolution of the identity of A). *Take $f \in H = \prod_{j=1}^{n} L^2(N_j)$, vanishing almost everywhere outside a compact set $B \subset N$ and let $-\infty < a < b < +\infty$. Then, for $x \in N$*

$$(E(a,b)f)(x) = \mathrm{Re}\left\{ \frac{1}{\pi} \int_a^b \sum_{j=1}^{n} \sigma_j(\lambda, x) \cdot F_{\lambda}^{-,j+1}(x) \left(\int_N f(x') \cdot F_{\lambda}^{-,j}(x')\, dx' \right) d\lambda \right\},$$

where E is the resolution of the identity of A (cf. Theorem 4.1) and

$$\sigma_j(\lambda, x) := \frac{1}{\sqrt{\lambda}} \sigma_j(x), \text{ where } \sigma_j(x) := \mathbf{1}_{\overline{N_j}}(x) \cdot \frac{1}{C} \text{ for } j \in \{1, \dots, n\}.$$

Here $C = (\sum_k c_k)$ and the index j is to be understood modulo n, that is to say, if $j = n$, then $j + 1 = 1$.

Note that in particular if $c_j = c$ for all $j \in \{1, \dots, n\}$, then $C = nc$, for all $j \in \{1, \dots, n\}$.

Proof. The proof is analogous to that of Lemma 1.3.13 of [3] (see also [2]).

Let in addition $g \in H$ be vanishing outside B. Then

$$\begin{aligned}
&(E(a,b)f, g)_H \\
&= \left(\lim_{\delta \to 0^+} \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} [R(\lambda - \varepsilon i, A) - R(\lambda + \varepsilon i, A)]\, d\lambda\, f, g \right)_H && (2) \\
&= \lim_{\delta \to 0^+} \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \left(\int_{a+\delta}^{b-\delta} [R(\lambda - \varepsilon i, A) - R(\lambda + \varepsilon i, A)]\, d\lambda\, f, g \right)_H && (3) \\
&= \lim_{\delta \to 0^+} \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} ([R(\lambda - \varepsilon i, A) - R(\lambda + \varepsilon i, A)]f, g)_H\, d\lambda && (4)
\end{aligned}$$

$$= \lim_{\delta\to 0^+} \lim_{\varepsilon\to 0^+} \frac{1}{2\pi i} \int\limits_{a+\delta}^{b-\delta} \left(\int\limits_N f(x')[K(\cdot,x',\lambda-i\varepsilon) - K(\cdot,x',\lambda+i\varepsilon)]\, dx', g(\cdot) \right)_H d\lambda \tag{5}$$

$$= \lim_{\delta\to 0^+} \lim_{\varepsilon\to 0^+} \frac{1}{2\pi i} \int\limits_{a+\delta}^{b-\delta} \left(\int\limits_N f(x')[K(\cdot,x',\lambda-i\varepsilon) - \overline{K(\cdot,x',\lambda-i\varepsilon)}]\, dx', g(\cdot) \right)_H d\lambda \tag{6}$$

$$= \lim_{\delta\to 0^+} \lim_{\varepsilon\to 0^+} \frac{1}{2\pi i} \int\limits_{a+\delta}^{b-\delta} \left(\int\limits_N f(x')\, 2i\, \mathrm{Im}(K(\cdot,x',\lambda-i\varepsilon))\, dx', g(\cdot) \right)_H d\lambda \tag{7}$$

$$= \lim_{\delta\to 0^+} \frac{1}{\pi} \int\limits_{a+\delta}^{b-\delta} \left(\int\limits_N f(x')[\lim_{\varepsilon\to 0^+} \mathrm{Im}(K(\cdot,x',\lambda-i\varepsilon))]\, dx', g(\cdot) \right)_H d\lambda \tag{8}$$

$$= \left(\frac{1}{\pi} \int\limits_a^b \int\limits_N f(x')\mathrm{Im}(K(\cdot,x',\lambda-i0))\, dx'\, d\lambda, g(\cdot) \right)_H \tag{9}$$

$$\begin{aligned} &= \int\limits_N \frac{1}{\pi} \int\limits_a^b \Big[\int\limits_N f(x')\mathrm{Im}\Big[\frac{1}{\sqrt{\lambda}} \sum_{j=1}^n \frac{\mathbf{1}_{\overline{N_j}}(x)}{-iC} \Big(\mathbf{1}_{\{x'\in\overline{N_j}, x'>x\}}(x') F_\lambda^{-,j}(x) F_\lambda^{-,j+1}(x') \\ &\qquad + \mathbf{1}_{N\setminus\{x'\in\overline{N_j}, x'>x\}}(x') F_\lambda^{-,j+1}(x) F_\lambda^{-,j}(x') \Big) \Big] dx' \Big]\, d\lambda\, g(x)\, dx \end{aligned} \tag{10}$$

$$\begin{aligned} &= \int\limits_N \frac{1}{\pi} \int\limits_a^b \Big[\int\limits_N f(x') \frac{1}{\sqrt{\lambda}} \sum_{j=1}^n \mathbf{1}_{\overline{N_j}}(x) \mathrm{Re}\Big[\frac{1}{C} \big(\mathbf{1}_{\{x'\in\overline{N_j}, x'>x\}}(x') F_\lambda^{-,j}(x) F_\lambda^{-,j+1}(x') \\ &\qquad + \mathbf{1}_{N\setminus\{x'\in\overline{N_j}, x'>x\}}(x') F_\lambda^{-,j+1}(x) F_\lambda^{-,j}(x') \big) \Big] dx' \Big]\, d\lambda\, g(x)\, dx \end{aligned} \tag{11}$$

$$= \int\limits_N \frac{1}{\pi} \int\limits_a^b \Big[\int\limits_N f(x') \frac{1}{C\sqrt{\lambda}} \sum_{j=1}^n \mathbf{1}_{\overline{N_j}}(x) \mathrm{Re}\Big[F_\lambda^{-,j+1}(x) F_\lambda^{-,j}(x') \Big]\, dx' \Big]\, d\lambda\, g(x)\, dx \tag{12}$$

$$= \int\limits_N \mathrm{Re}\Big[\frac{1}{\pi} \int\limits_a^b \frac{1}{C\sqrt{\lambda}} \sum_{j=1}^n \mathbf{1}_{\overline{N_j}}(x) F_\lambda^{-,j+1}(x) \Big(\int\limits_N f(x') F_\lambda^{-,j}(x')\, dx' \Big) \Big]\, d\lambda\, g(x)\, dx.$$

Here, the justifications for the equalities are the following:

(2): Stone's formula (Theorem 4.1) applied with $h(\lambda) \equiv 1$.

(3): After applying the operator-valued integral to f, the two limits are in H. So they commute with the scalar product in H.

(4): $(\cdot f, g)_H$ is a continuous linear form on $\mathcal{L}(H)$, and can therefore be commuted with the vector-valued integration.

(5): Theorem 3.7.
(6): Lemma 4.4.
(7): $z - \overline{z} = 2i \cdot \operatorname{Im} z\ \forall z \in \mathbb{C}$.
(8): Dominated convergence. Note that $\operatorname{supp} f$, $\operatorname{supp} g$ and $[a, b]$ are compact and use the limiting absorption principle (Theorem 4.2).
(9): Fubini.
(10): Definition 3.6.
(11): $\operatorname{Im}(z) = \operatorname{Re}(z/i)$ for all $z \in \mathbb{C}$. Note that, if $\lambda \in \mathbb{R}^-$, then $\lambda \in \rho(A)$ and thus the integrand in Stone's formula is zero.
(12): Note that

$$\begin{cases} (F^{-,j}_{\lambda,j})(x)(F^{-,j+1}_{\lambda,j})(x') = d_{2,j} e^{-ic_j^{-1}\sqrt{\lambda}(x-x')} + d_{1,j} e^{-ic_j^{-1}\sqrt{\lambda}(x+x')}, \\ (F^{-,j+1}_{\lambda,j})(x)(F^{-,j}_{\lambda,j})(x') = d_{2,j} e^{-ic_j^{-1}\sqrt{\lambda}(x'-x)} + d_{1,j} e^{-ic_j^{-1}\sqrt{\lambda}(x+x')}. \end{cases}$$

Since $e^{-ic_j^{-1}\sqrt{\lambda}(x-x')}$ and $e^{-ic_j^{-1}\sqrt{\lambda}(x'-x)}$ are conjugated for real λ, both expressions have the same real part. Thus the integrals on $\{x' \in \overline{N_j}, x' > x\}$ and its complement $N \setminus \{x' \in \overline{N_j}, x' > x\}$ recombine to a single integral on N. The formula of the theorem follows.

The assertion follows, because g was arbitrary with compact support. □

5. A Plancherel-type formula and a functional calculus for the operator

Now we use the explicit formula for the resolution of the identity of the operator A obtained in Proposition 4.5 to prove a Plancherel-type formula. As in [3] (see also [2]), we define the Fourier-type transformation V associated with the system of generalized eigenfunctions $\{F^{-,j}_{\lambda} \mid \lambda \in [0; +\infty), j \in \{1, \dots, n\}\}$ on regular functions using Proposition 4.5.

The main difficulty here is that the coefficient $\sigma_j(x)$ appearing in Proposition 4.5 depends on $x \in N$: it is different on each branch of the star, unlike the situation in [2] and [3]. Thus $\sigma_j(x)$ does not commute with V and therefore the scalar product making the range of V a Hilbert space and V an isometry cannot be directly defined as in [2] and [3], but must be transferred from H via V. This introduces some additional technicalities. Apart from this we follow the lines of [2] and [3].

Definition 5.1.

1. For $f \in L^1(N)$ define $V_j f : [0; +\infty) \to \mathbb{R}$ by

$$V_j f(\lambda) = \int_N f(x) \cdot F^{-,j}_{\lambda}(x)\, dx,\ j = 1, \dots, n$$

and $Vf : [0; +\infty) \to \mathbb{C}^n$ by $Vf = (V_j f)_{j \in \{1,\dots,n\}}$.

2. Let σ be defined as in Proposition 4.5 and $\chi \in C^\infty(\mathbb{R})$ be such that $\chi \equiv 0$ on $(-\infty, 1)$ and $\chi \equiv 1$ on $(2, +\infty)$. For $K_j \in C^\infty((0, +\infty), \mathbb{C})$ such that $\chi K_j \in \mathcal{S}(\mathbb{R})$, for $j \in \{1, \ldots, n\}$ define $Z(K) : N \to \mathbb{R}$ by

$$Z(K_1, \ldots, K_n)(x) = \frac{1}{\pi}\mathrm{Re}\Big\{\int_0^{+\infty} \frac{1}{\sqrt{\lambda}} \sum_{j=1}^n \sigma_j(x) K_j(\lambda) F_\lambda^{-,j+1}(x)\, d\lambda\Big\},\ x \in N.$$

Note that the integral on the right-hand side is absolutely convergent because $\lambda \mapsto 1/\sqrt{\lambda}$ is L^1_{loc}, K_j is continuous and rapidly decreasing at $+\infty$ and $|F_\lambda^{-,k}(x)| \le Const$, for all $\lambda \in (0; +\infty)$, $x \in N$, $k \in \{1, \ldots, n\}$.

Remark 5.2. Unlike W in [3], Z is not injective: an easy computation shows that $Z(K) = 0$ is equivalent to

$$\mathrm{Re}\left[\mathcal{F}(K_j(\cdot^2) \cdot \mathbf{1}_{[0;+\infty)}(\cdot))(x)\right] = 0,\ \forall\, x \in N_j,\ \forall\, j \in \{1, \ldots, n\},$$

where $\mathcal{F}$ denotes the Fourier transform. And there exist non-vanishing functions K_j satisfying this equation.

Lemma 5.3 (asymptotic behaviour of $V_j f$). *Consider $f \in \prod_{k=1}^n \mathcal{D}(N_k)$. Then $V_j f \in C^0([0; +\infty)) \cap C^\infty((0; +\infty))$ and $\chi V_j f \in \mathcal{S}(\mathbb{R})$ for any $j \in \{1, \ldots, n\}$ with χ as in Definition* 5.1.

Proof. For $\lambda \in [0; +\infty)$ and $j \in \{1, \ldots, n\}$, it holds

$$V_j f(\lambda) = \int_N f(x) F_\lambda^{-,j}(x)\, dx = \sum_{k=1}^n \int_{N_k} f_k(x) F_{\lambda,k}^{-,j}(x)\, dx.$$

Due to the definition of $F_{\lambda,k}^{-,j}$ and due to the fact that f_k is a test function having its support in $(0; +\infty)$, each term of the right-hand side is the Fourier transform of a test function and thus C^∞ and rapidly decreasing in λ. □

Proposition 5.4 (left inverse of V). *For $f \in H$ with f vanishing almost everywhere outside a compact set $B \subset N$ and $-\infty < a < b < +\infty$, it holds*

1. $E(a, b)f = Z\mathbf{1}_{(a,b)} V f$, [1]
2. $f = \lim\limits_{\substack{a\to-\infty \\ b\to+\infty}} E(a, b)f = ZVf$, *if* $f \in \prod_{k=1}^n \mathcal{D}(N_k)$,
3. V *is injective on* $\prod_{k=1}^n \mathcal{D}(N_k)$.

Proof. 1. Follows directly from Proposition 4.5, using Definition 5.1.

2. Let us fix $f \in \prod_{k=1}^n \mathcal{D}(N_k)$. Lemma 5.3 implies that there exists $M_1(f) \ge 0$, such that

$$|V_j f(\lambda)| \le \frac{M_1(f)}{1 + \lambda^2},\ \forall\, \lambda > 0,\ j \in \{1, \ldots, n\}.$$

[1]This formula is well defined using the expression for Z as defined in 5.1 in spite of the discontinuities introduced by the characteristic function

Clearly there exists $M_2 \geq 0$, such that

$$\left| \frac{1}{\sqrt{\lambda}} \sum_{j=1}^{n} \sigma_j(x) F_\lambda^{-,j+1}(x) \right| \leq \frac{M_2}{\sqrt{\lambda}}, \ \forall\, \lambda > 0,\ x \in N.$$

Thus the theorem of Lebesgue implies that

$$\frac{1}{\pi}\mathrm{Re}\left\{ \int_0^{+\infty} \frac{1}{\sqrt{\lambda}} \mathbf{1}_{(a,b)}(\lambda) \sum_{j=1}^{n} \sigma_j(x) V_j f(\lambda) F_\lambda^{-,j+1}(x)\, d\lambda \right\}$$

converges for $a \longrightarrow -\infty$ and $b \longrightarrow +\infty$ and almost every $x \in N$ towards the same expression with $\mathbf{1}_{(a,b)}$ replaced by 1.

3. Direct consequence of 2. □

Now we shall introduce a structure on the range of V which shall be later on identified as a scalar product.

Theorem 5.5 (Plancherel-type formula). *Let σ be defined as in the end of Proposition* 4.5 *and χ as in Definition* 5.1. *Let $f \in \prod_{k=1}^{n} \mathcal{D}(N_k)$ and $G = (G_1, \ldots, G_n) \in (C^\infty(0; +\infty))^n \cap (C^0[0; +\infty))^n$ such that $\chi G_l \in \mathcal{S}(\mathbb{R})$ for $l \in \{1, \ldots, n\}$. Define*

$$\langle Vf, G \rangle_{\sigma,V} = \frac{1}{\pi}\mathrm{Re}\left\{ \sum_{j=1}^{n} \int_0^{+\infty} \frac{1}{\sqrt{\lambda}} V_{j+1}(\sigma_j(\cdot) f(\cdot))(\lambda) G_j(\lambda)\, d\lambda \right\}.$$

Then the integrals on the right-hand side are absolutely convergent and it holds $\langle Vf, G \rangle_{\sigma,V} = (f, Z(G))_H$.

Proof. For $\lambda \in (0; +\infty)$, it holds

$$\left| \frac{1}{\sqrt{\lambda}} V_{j+1}(\sigma_j(\cdot) f(\cdot))(\lambda) \right| = \left| \frac{1}{\sqrt{\lambda}} \int_N \sigma_j(x) f(x) F_\lambda^{-,j+1}(x)\, dx \right| \leq C \frac{1}{\sqrt{\lambda}} \int_N |f(x)|\, dx. \tag{13}$$

Together with the fact that G_j is rapidly decreasing and continuous for any $j \in \{1, \ldots, n\}$, the latter estimate ensures the absolute convergence of the integrals.

Estimate (13) also allows the application of the theorem of Fubini:

$$\begin{aligned}
\langle Vf, G \rangle_{\sigma,V} &= \frac{1}{\pi}\mathrm{Re}\left\{ \sum_{j=1}^{n} \int_0^{+\infty} \frac{1}{\sqrt{\lambda}} \left(\int_N \sigma_j(x) f(x) F_\lambda^{-,j+1}(x)\, dx \right) G_j(\lambda)\, d\lambda \right\} \\
&= \frac{1}{\pi}\mathrm{Re}\left\{ \sum_{j=1}^{n} \int_N \frac{1}{\sqrt{\lambda}} \left(\int_0^{+\infty} \sigma_j(x) F_\lambda^{-,j+1}(x) G_j(\lambda)\, d\lambda \right) f(x)\, dx \right\} \\
&= \int_N Z(G)(x) f(x)\, dx = (f, Z(G))_H.
\end{aligned}$$

□

This Plancherel formula can now be combined with the fact that Z is the left inverse of V to prove that $\langle \cdot, \cdot \rangle_{\sigma,V}$ is a scalar product and that V is an isometry.

Corollary 5.6.

1. *Let* $(F, G) \in (V(\prod_{k=1}^{n} \mathcal{D}(N_k)))^2$ *and* $(f, g) \in (\mathcal{D}(N_k))^2$*, such that* $F = Vf$ *and* $G = Vg$*. Then* $\langle F, G\rangle_{\sigma,V} = \langle Vf, Vg\rangle_{\sigma,V} = (f, g)_H$*.*
2. $\langle\cdot,\cdot\rangle_{\sigma,V}$ *is a scalar product on* $V(\prod_{k=1}^{n} \mathcal{D}(N_k))$*.*
3. *Let* $L^2_{\sigma,V}$ *be the completion of* $V(\prod_{k=1}^{n} \mathcal{D}(N_k))$ *with respect to* $\langle\cdot,\cdot\rangle_{\sigma,V}$*. We denote the extended scalar product by the latter bracket as well. Thus* $(L^2_{\sigma,V}, \langle\cdot,\cdot\rangle_{\sigma,V})$ *is a Hilbert space.*
4. $V : \prod_{k=1}^{n} \mathcal{D}(N_k) \longrightarrow V(\prod_{k=1}^{n} \mathcal{D}(N_k))$ *extends to a surjective isometry* $\tilde{V} : H \longrightarrow L^2_{\sigma,V}$*.*
5. $Z = V^{-1} : V(\prod_{k=1}^{n} \mathcal{D}(N_k)) \longrightarrow \prod_{k=1}^{n} \mathcal{D}(N_k)$ *extends to a surjective isometry* $\tilde{Z} : L^2_{\sigma,V} \longrightarrow H$*. Thus* $\tilde{Z} = \tilde{V}^{-1}$*.*

Proof. 1. Lemma 5.3 implies that Vg is rapidly decreasing and thus Theorem 5.5 is applicable:

$$\langle F, G\rangle_{\sigma,V} = \langle Vf, Vg\rangle_{\sigma,V} = (f, Z(Vg))_H = (f, g)_H.$$

The last equality comes from Proposition 5.4.

2. $V : \prod_{k=1}^{n} \mathcal{D}(N_k) \longrightarrow \mathrm{Ran}V$ is linear and bijective (for the injectivity see Part 3 of Proposition 5.4). Thus $\langle\cdot,\cdot\rangle_{\sigma,V}$ inherits the property of being a scalar product from $(\cdot,\cdot)_H$.

3. and 4. Clear by construction.

5. Theorem 5.5 implies $\langle Vf, G\rangle_{\sigma,V} = (f, Z(G))_H$ for all $f \in \prod_{k=1}^{n} \mathcal{D}(N_k)$ and $G \in V(\prod_{k=1}^{n} \mathcal{D}(N_k))$. Thus it follows from 1.

$$|(f, Z(G))_H| = |\langle Vf, G\rangle_{\sigma,V}| \le \|G\|_{\sigma,V}\|Vf\|_{\sigma,V} = \|G\|_{\sigma,V}\|f\|_H. \tag{14}$$

Due to the denseness of $\prod_{k=1}^{n} \mathcal{D}(N_k)$ in H, inequality (14) is valid for all $f \in H$. Thus

$$\|Z(G)\|_H \le \|G\|_{\sigma,V}.$$

Therefore Z extends by density-continuity to a continuous operator $\tilde{Z}$ on $L^2_{\sigma,V}$. □

Theorem 5.7. *Let* $h \in C(\mathbb{R})$ *and* $f \in H$*, such that* $\lambda \mapsto (h(\lambda)/\sqrt{\lambda})\tilde{V}f(\lambda)$ *is absolutely integrable on* $[0; +\infty)$*. Then we have for* $x \in N$

$$h(A)f(x) = \frac{1}{\pi}\mathrm{Re}\left\{\int_0^{+\infty} \frac{h(\lambda)}{\sqrt{\lambda}} \sum_{j=1}^{n} \sigma_j(x)V_j f(\lambda)F_\lambda^{-,j+1}(x)\, d\lambda\right\}. \tag{15}$$

Proof. The same proof as in Proposition 4.5, but this time using Stone's formula (Theorem 4.1) with arbitrary $h \in C(\mathbb{R})$, yields

$$h(A)E(a,b)f(x) = \frac{1}{\pi}\mathrm{Re}\left\{\int_0^{+\infty} \frac{h(\lambda)}{\sqrt{\lambda}} \mathbf{1}_{(a,b)}(\lambda) \sum_{j=1}^{n} \sigma_j(x)V_j f(\lambda)F_\lambda^{-,j+1}(x)\, d\lambda\right\}.$$

Now, the assertion follows from dominated convergence and the fact that $E(a,b)$ commutes with $h(A)$ and tends to the identity if $a \to -\infty$ and $b \to \infty$. □

Remark 5.8.

1. Formally (15) reads like
$$h(A)f = \tilde{Z} M_h \tilde{V} f, \tag{16}$$
where $(M_h K)(\lambda) := h(\lambda)K(\lambda)$. It should be investigated, if under the hypotheses of Theorem 5.7 we have $M_h \tilde{V} f \in L^2_{\sigma,V}$, and thus (16) is rigorously valid.
2. Using Theorem 5.7, we can represent solutions of evolution equations involving A (heat, wave, Klein-Gordon, ...) in view of obtaining qualitative information like decay properties in time on the n-star. It remains the open problem of describing the relation of the belonging of f to $D(A^s)$ and the decay of $\tilde{V} f$ at infinity. This is important, because for example $f \in D(A)$ ensures the twice differentiability of $u(t) = \cos(\sqrt{A}t)f$ and thus the validity of the abstract wave equation $\ddot{u}(t) + Au(t) = 0$.

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Diffusion Equations with Finite Speed of Propagation

Fuensanta Andreu, Vicent Caselles and José M. Mazón

Dedicated to the memory of Günter Lumer

Abstract. In this paper we summarize some of our recent results on diffusion equations with finite speed of propagation. These equations have been introduced to correct the infinite speed of propagation predicted by the classical linear diffusion theory.

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1. Introduction

The speed of light c is the highest admissible velocity for transport of radiation in transparent media, and, to ensure it, J.R. Wilson (in an unpublished work, see [27]) proposed to use a flux limiter. The flux limiter merely enforces the physical restriction that the flux cannot exceed energy density times the speed of light, that is, the flux cannot violate causality. The basic idea is to modify the diffusion-theory formula for the flux in a way that gives the standard result in the high opacity limit, while simulating free streaming (at light speed) in transparent regions. As an example, one of the expressions suggested for the flux of the energy density u is

$$F = -\nu u \frac{Du}{u + \nu c^{-1}|Du|} \tag{1.1}$$

(where ν is a constant representing a kinematic viscosity and c the speed of light) which yields in the limit $\nu \to \infty$ the flux $F = -cu\frac{Du}{|Du|}$. Observe also that when $c \to \infty$, the flux tends to $F = -\nu Du$, and the corresponding diffusion equation becomes the heat equation, which has an infinite speed of propagation.

The diffusion equation corresponding to (1.1) is

$$u_t = \nu \operatorname{div}\left(\frac{uDu}{u + \frac{\nu}{c}|Du|}\right) \tag{1.2}$$

and is one among the various *flux limited diffusion equations* used in the theory of radiation hydrodynamics [27].

The speed of sound is the highest admissible free velocity in a medium. This property is lost in the classical transport theory that predicts the nonphysical divergence of the flux with the gradient (as it happens with Fourier's law). To overcome this problem Ph. Rosenau ([32]) proposed to change the classical flux

$$\mathbf{q} = -D_0 u_x$$

associated with the Fokker-Plank equation

$$u_t = [D_0 u_x]_x, \tag{1.3}$$

by a flux that saturates as the gradient becomes unbounded. To do that, he associated u and the flux $\mathbf{q}$ through the velocity v defined by

$$\mathbf{q} = uv.$$

Then

$$v = -D_0 \frac{u_x}{u}. \tag{1.4}$$

According to (1.4), if $|\frac{u_x}{u}| \uparrow \infty$, so will do v. However, the inertia effects impose a macroscopic upper bound on the allowed free speed, namely, the acoustic speed C. With this aim, Rosenau modified (1.4) by taking

$$D_0 \frac{u_x}{u} = \frac{-v}{\sqrt{1 - \frac{v^2}{C^2}}}. \tag{1.5}$$

The postulate (1.5) forces v to stay in the subsonic regime. The sonic limit is approached only if $|\frac{u_x}{u}| \uparrow \infty$. Solving (1.5) for v, we obtain

$$\mathbf{q} = uv = \frac{-D_0 u_x}{\sqrt{1 + \left(\frac{D_0 u_x}{Cu}\right)^2}}. \tag{1.6}$$

Using this new flux (1.6) in the conservation energy equation,

$$u_t = \left[\frac{D_0 u u_x}{\sqrt{u^2 + \frac{D_o^2}{C^2} u_x^2}}\right]_x \tag{1.7}$$

is obtained. Equation (1.7) is the main result of [32].

Equation (1.7) was derived by Y. Brenier by means of Monge-Kantorovich's mass transport theory ([17]) and he named it as the *relativistic heat equation*. Many well-known equations for probability densities can be recovered in the formalism of gradient flows with respect to the optimal transport differential structure. This point of view was introduced by F. Otto in a series of pioneering papers [28, 29, 30] and there are at least two different approaches to make it rigorous. One of them is to decide that a gradient flow is an equation of the form

$$\frac{d\rho_t}{dt} \in \partial^- F(\rho_t),$$

where ∂^- stands for some appropriate notion of subdifferential. This approach was considered in [3]. Another strategy is to proceed with a time-discretization. This was the approach first used by Jordan, Kinderlehrer and Otto [26] (see also [18]) for the linear Fokker-Planck equation and it does not require any study of tangent spaces, subdifferentiability or related concepts. This subject has been considered in depth in Agueh's PhD thesis [1], where the following general equation

$$u_t + \operatorname{div}(uV_u) = 0 \tag{1.8}$$

is studied. Here

$$V_u := -\nabla k^*[\nabla(F'(u))]$$

denotes the vector field describing the average velocity of a fluid evolving with the continuity equation (1.8), and the unknown $u(t, x)$ is the mass density of the fluid at time t and position x. k^* denotes the Legendre-Fenchel transform of the cost function $k : \mathbb{R}^N \to [0, \infty)$, that is,

$$k^*(z) = \sup_{x\in\mathbb{R}^N} \{x \cdot z - k(x)\}.$$

The free energy associated with the fluid at time $t \in [0, \infty)$ is given by

$$E(u(t)) := \int_{\mathbb{R}^N} F(u(t, x))\, dx.$$

In [1] is assumed that the cost function $k : \mathbb{R}^N \to [0, \infty)$ is strictly convex, $0 = k(0) < k(z)$, for $z \neq 0$, k is coercive, and verifies

$$\beta|z|^q \leq k(z) \leq \alpha(|z|^q + 1), \quad \text{for } z \in \mathbb{R}^N, \quad \alpha, \beta > 0, \text{ and } q > 1.$$

Besides the most important cost functions, namely $k(z) = |z|$, which corresponds to the original Monge problem, and $k(z) = \frac{|z|^2}{2}$, which corresponds to the Monge-Ampère equations – and is related to PDEs as different as the Euler equations of incompressible flows [16] and the heat equation [26] – more general cost functions have been considered in the literature (see for instance [25], [1] or [33]). Surprisingly, an important cost function had not been considered, in spite of its obvious geometric and relativistic flavor, namely

$$k(z) := \begin{cases} c^2\left(1 - \sqrt{1 - \frac{|z|^2}{c^2}}\right) & \text{if } |z| \leq c \\ \\ +\infty & \text{if } |z| > c. \end{cases} \tag{1.9}$$

This cost function was considered by Y. Brenier in [17], where he derived a *relativistic heat equation* as a gradient flow of the Boltzmann entropy for the metric corresponding to the cost (1.9). More precisely, since

$$k^*(z) = c^2 \left(\sqrt{1 + \frac{|z|^2}{c^2}} - 1 \right),$$

if we take

$$F(x) = \nu(\log(x) - 1)x,$$

we have

$$V_u := -\nabla k^*[\nabla(F'(u))] = -\nabla k^*[\nabla(\nu \log(u))] = -\nu \left(\frac{Du}{\sqrt{u^2 + \frac{\nu^2}{c^2}|Du|^2}} \right),$$

and, consequently, for this cost function equation (1.8) becomes

$$u_t = \nu \operatorname{div} \left(\frac{uDu}{\sqrt{u^2 + \frac{\nu^2}{c^2}|Du|^2}} \right). \tag{1.10}$$

Observe that in the one-dimensional case equation (1.10) is similar to the equation (1.7) derived by P. Rosenau.

Recently, in [5, 6, 7, 8, 9], we have studied the Neumann and Cauchy problem for the quasi-linear parabolic equation

$$\frac{\partial u}{\partial t} = \operatorname{div} \mathbf{a}(u, Du), \tag{1.11}$$

where $\mathbf{a}(z,\xi) = \nabla_\xi f(z,\xi)$ and f being a function with linear growth as $\|\xi\| \to \infty$, satisfying other additional assumptions, which are satisfied, in particular, by the relativistic heat equation (1.10) and the flux limited diffusion equation (1.2). The aim of this paper is to summarize some of our recent results about this type of equations.

2. The Cauchy problem for a strongly degenerate quasi-linear equation

2.1. Introduction

Consider the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div} \mathbf{a}(u, Du) & \text{in} \quad Q_T = (0,T) \times \mathbb{R}^N \\ u(0,x) = u_0(x) & \text{in} \quad x \in \mathbb{R}^N, \end{cases} \tag{2.1}$$

where $0 \leq u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $\mathbf{a}(z,\xi) = \nabla_\xi f(z,\xi)$ and f is a function with linear growth as $\|\xi\| \to \infty$.

Particular instances of problem (2.1) have been studied in [14] and [21], when $N = 1$. Let us describe their results in some detail. In these papers the authors considered the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = (\varphi(u)\mathbf{b}(u_x))_x & \text{in} \quad (0,T) \times \mathbb{R} \\ u(0,x) = u_0(x) & \text{in} \quad x \in \mathbb{R} \end{cases} \tag{2.2}$$

corresponding to (2.1) when $N = 1$ and $\mathbf{a}(u, u_x) = \varphi(u)\mathbf{b}(u_x)$, where $\varphi : \mathbb{R} \to \mathbb{R}^+$ is smooth and strictly positive, and $\mathbf{b} : \mathbb{R} \to \mathbb{R}$ is a smooth odd function such that $\mathbf{b}' > 0$ and $\lim_{s\to\infty} \mathbf{b}(s) = \mathbf{b}_\infty$. Such models appear as models for heat and mass transfer in turbulent fluids [12], or in the theory of phase transitions where the corresponding free energy functional has a linear growth rate with respect to the gradient [31]. As the authors observed, in general, there are no classical solutions of (2.1), indeed, the combination of the dependence on u in $\varphi(u)$ and the constant behavior of $\mathbf{b}(u_x)$ as $u_x \to \infty$ can cause the formation of discontinuities in finite time (see [14], Theorem 2.3). As noticed in [14], the parabolicity of (2.2) is so weak when $u_x \to \infty$ that solutions become discontinuous and behave like solutions of the first-order equation $u_t = \mathbf{b}_\infty(\varphi(u))_x$ (which can be formally obtained differentiating the product in (2.2) and replacing $\mathbf{b}(u_x)$ by $\mathbf{b}_\infty$). For this reason, they defined the notion of entropy solution and proved existence ([14]) and uniqueness ([21]) of entropy solutions of (2.2). Existence was proved for bounded strictly increasing initial conditions $u_0 : \mathbb{R} \to \mathbb{R}$ such that $\mathbf{b}(u_0') \in C(\mathbb{R})$ (where $\mathbf{b}(u_0'(x_0)) = \mathbf{b}_\infty$ if u_0 is discontinuous at x_0), $\mathbf{b}(u_0'(x)) \to 0$ as $x \to \pm\infty$ [14]. The entropy condition was written in Oleinik's form and uniqueness was proved using suitable test functions constructed by regularizing the sign of the difference of two solutions. Moreover, the authors showed that there exist functions φ and initial conditions u_0 for which there exist solutions of (2.2) which do not satisfy the entropy condition ([14], Theorem 2.2). Thus, uniqueness cannot be guaranteed without an additional condition like the entropy condition.

In [15], the author considered the Neumann problem in an interval of $\mathbb{R}$

$$\begin{cases} \dfrac{\partial u}{\partial t} = (\mathbf{a}(u, u_x))_x & \text{in} \quad (0,T) \times (0,1) \\ u_x(t,0) = u_x(t,1) = 0 & \\ u(0,x) = u_0(x) & \text{in} \quad x \in (0,1) \end{cases} \tag{2.3}$$

for functions $\mathbf{a}(u,v)$ of class $C^{1,\alpha}([0,\infty) \times \mathbb{R})$ such that $\frac{\partial}{\partial v}\mathbf{a}(u,v) < 0$ for any $(u,v) \in [0,\infty) \times \mathbb{R}$, $\mathbf{a}(u,0) = 0$ (and some other additional assumptions). After observing that there are no, in general, classical solutions of (2.1), the author associated an m-accretive operator to $-(\mathbf{a}(u,u_x))_x$ with Neumann boundary conditions, and proved the existence and uniqueness of a semigroup solution of (2.3). However, the accretive operator generating the semigroup was not characterized in distributional terms. An example of the equations considered in [15] is the so-called *plasma equation* (see [24])

$$\frac{\partial u}{\partial t} = \Big(\frac{u^{5/2} u_x}{1 + u|u_x|} \Big)_x \quad \text{in} \quad (0,T) \times (0,1), \tag{2.4}$$

where the initial condition u_0 is assumed to be positive. In this case u represents the temperature of electrons and the form of the conductivity $\mathbf{a}(u, u_x) = \frac{u^{5/2} u_x}{1+u|u_x|}$ has the effect of limiting heat flux. Thus, existence and uniqueness results for higher-dimensional problems were not considered. This was the purpose of our papers [5] and [6] in which we studied the Neumann problem for Lagrangians f satisfying the coercivity and linear growth condition

$$C_0 \|\xi\| - D_0 \leq f(z,\xi) \leq M_0(1 + \|\xi\|) \tag{2.5}$$

for some positive constants C_0, D_0, M_0.

Now, there are some relevant cases like the *relativistic heat equation* (1.10) for which the Lagrangian $f(z,\xi) = \frac{c^2}{\nu}|z|\sqrt{z^2 + \frac{\nu^2}{c^2}|\xi|^2}$ does not satisfy (2.5) but verifies

$$C_0(z)\|\xi\| - D_0(z) \leq f(z,\xi) \leq M_0(z)(\|\xi\| + 1) \tag{2.6}$$

for any $(z,\xi) \in \mathbb{R} \times \mathbb{R}^N$, and some positive and continuous functions C_0, D_0, M_0, such that $C_0(z) > 0$ for any $z \neq 0$. The purpose of the papers [7] and [8] was to extend the results of [5] and [6] to the case of Lagrangians satisfying (2.6).

We proved in [6] existence and uniqueness results for the Cauchy problem (2.1). For that, we considered in [5] the elliptic problem

$$u - \operatorname{div} \mathbf{a}(u, Du) = v \quad \text{in} \quad \mathbb{R}^N, \tag{2.7}$$

we defined a notion of entropy solution for it, and we proved existence and uniqueness results when the right-hand side $0 \leq v \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. With this, we associated an accretive operator B in $L^1(\mathbb{R}^N)$ whose domain is contained in $\big(L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)\big)^+$ (which amounts to consider the right-hand side v of (2.7) in $\big(L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)\big)^+$) whose closure $\mathcal{B}$ is accretive in $L^1(\mathbb{R}^N)$ and generates a non-linear contraction semigroup $T(t)$ in $L^1(\mathbb{R}^N)$ ([13], [19]). However, $\mathcal{B}$ is not characterized in distributional terms. In spite of this, the knowledge of the operator B and the fact that, if u is the entropy solution of (2.7), we have $\|u\|_\infty \leq \|v\|_\infty$, allowed to use Crandall-Ligget's iteration scheme and define

$$u(t) := T(t)u_0 = \lim_{n\to\infty} \Big(I + \frac{t}{n} B \Big)^{-n} u_0, \qquad u_0 \in \big(L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)\big)^+.$$

Then we proved that $u(t)$ is an entropy solution of (2.1) (a notion that will be defined later), and that entropy solutions are unique. As a technical tool we used some lower semi-continuity results for energy functionals whose density is a function $g(x, u, Du)$ convex in Du with a linear growth rate as $|Du| \to \infty$, which were proved in [20] and [22].

2.2. Basic assumptions

Assume that the Lagrangian $f : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^+$ satisfies the following assumptions, which we shall refer collectively as (H):

(H_1) f is continuous on $\mathbb{R}\times\mathbb{R}^N$ and is a convex differentiable function of ξ such that $\nabla_\xi f(z,\xi) \in C(\mathbb{R}\times\mathbb{R}^N)$. Further we require f to satisfy the linear growth condition

$$C_0(z)\|\xi\| - D_0(z) \le f(z,\xi) \le M_0(z)(\|\xi\|+1) \tag{2.1}$$

for any $(z,\xi)\in\mathbb{R}\times\mathbb{R}^N$, and some positive and continuous functions C_0, D_0, M_0, such that $C_0(z) > 0$ for any $z \neq 0$. Let f^0 denote the recession function of f, defined by

$$f^0(z,\xi) = \lim_{t\to 0^+} tf\left(z,\frac{\xi}{t}\right), \tag{2.2}$$

We consider the function $\mathbf{a}(z,\xi) = \nabla_\xi f(z,\xi)$ associated to the Lagrangian f. By the convexity of f, we have

$$\mathbf{a}(z,\xi)\cdot(\eta-\xi) \le f(z,\eta) - f(z,\xi), \tag{2.3}$$

and the following monotonicity condition is satisfied

$$(\mathbf{a}(z,\eta) - \mathbf{a}(z,\xi))\cdot(\eta-\xi) \ge 0. \tag{2.4}$$

Moreover, it is easy to see that for each $R>0$, there is a constant $M = M(R) > 0$, such that

$$\|\mathbf{a}(z,\xi)\| \le M \qquad \forall\ (z,\xi)\in\mathbb{R}\times\mathbb{R}^N,\ |z|\le R. \tag{2.5}$$

We also assume that $\mathbf{a}(z,0) = 0$ for all $z\in\mathbb{R}$, and $\mathbf{a}(z,\xi) = z\mathbf{b}(z,\xi)$ with

$$\|\mathbf{b}(z,\xi)\| \le M_0 \qquad \forall\ (z,\xi)\in\mathbb{R}\times\mathbb{R}^N,\ |z|\le R. \tag{2.6}$$

We consider the function $h : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ defined by

$$h(z,\xi) := \mathbf{a}(z,\xi)\cdot\xi.$$

By (2.4), we have

$$h(z,\xi) \ge 0 \qquad \forall \xi\in\mathbb{R}^N,\ z\in\mathbb{R}. \tag{2.7}$$

Moreover we assume that

$$h(z,\xi) \le M(z)\|\xi\| \tag{2.8}$$

for some positive continuous function $M(z)$ and for any $(z,\xi)\in\mathbb{R}\times\mathbb{R}^N$. On the other hand, from (2.3) and (2.1), it follows that

$$C_0(z)\|\xi\| - D_1(z) \le h(z,\xi) \tag{2.9}$$

for any $(z,\xi)\in\mathbb{R}\times\mathbb{R}^N$ where $D_1(z) = D_0(z) + f(z,0)$. We assume that there exist constants $A, B > 0$ and $\alpha,\beta \ge 1$, such that

$$|D_1(z)| \le A|z|^\alpha + B|z|^\beta \quad \text{for any } z\in\mathbb{R}^N. \tag{2.10}$$

(H_2) We assume that $\frac{\partial \mathbf{a}}{\partial \xi_i}(z,\xi) \in C(\mathbb{R}\times\mathbb{R}^N)$ for any $i = 1,\ldots,N$.

(H_3) $h(z,\xi) = h(z,-\xi)$, for all $z\in\mathbb{R}$ and $\xi\in\mathbb{R}^N$ and h^0 exists.

Observe that we have

$$C_0(z)\|\xi\| \le h^0(z,\xi) \le M(z)\|\xi\| \quad \text{for any } (z,\xi) \in \mathbb{R}\times\mathbb{R}^N,\ |z| \le R.$$

(H_4) $f^0(z,\xi) = h^0(z,\xi)$, for all $\xi \in \mathbb{R}^N$ and all $z \in \mathbb{R}$.

(H_5) $\mathbf{a}(z,\xi)\cdot\eta \le h^0(z,\eta)$ for all $\xi,\eta \in \mathbb{R}^N$, and all $z \in \mathbb{R}$.

(H_6) We assume that $h^0(z,\xi)$ can be written in the form $h^0(z,\xi) = \varphi(z)\psi^0(\xi)$ with φ a Lipschitz continuous function such that $\varphi(z) > 0$ for any $z \ne 0$, and ψ^0 being a convex function homogeneous of degree 1.

(H_7) For any $R > 0$, there is a constant $C > 0$ such that

$$|(\mathbf{a}(z,\xi) - \mathbf{a}(\hat{z},\xi))\cdot(\xi - \hat{\xi})| \le C|z-\hat{z}|\,\|\xi - \hat{\xi}\| \tag{2.11}$$

for any $z,\hat{z} \in \mathbb{R}$, $\xi,\hat{\xi} \in \mathbb{R}^N$, with $|z|,|\hat{z}| \le R$.

Observe that, by the monotonicity condition (2.4) and using (2.11), it follows that

$$(\mathbf{a}(z,\xi) - \mathbf{a}(\hat{z},\hat{\xi}))\cdot(\xi - \hat{\xi}) \ge -C|z-\hat{z}|\,\|\xi - \hat{\xi}\| \tag{2.12}$$

for any $(z,\xi),(\hat{z},\hat{\xi}) \in \mathbb{R}\times\mathbb{R}^N$, $|z|,|\hat{z}| \le R$.

Remark 2.1. The function $f(z,\xi) = \frac{c^2}{\nu}|z|\sqrt{z^2 + \frac{\nu^2}{c^2}|\xi|^2}$ satisfies the assumptions (H_1)–(H_7), with $\mathbf{a}(z,\xi) = \nu\frac{|z|\xi}{\sqrt{z^2+\frac{\nu^2}{c^2}|\xi|^2}}$. This particular case corresponds to the *relativistic heat equation* (1.10). The Lagrangian

$$f(z,\xi) := cz\left(|\xi| - \frac{cz}{\nu}\log\left(1 + \frac{\nu}{cz}|\xi|\right)\right)$$

is associated with the flux limited diffusion equation (1.2) and satisfies also the assumptions (H_1)-(H_7).The Lagrangian $f(z,\xi) = |z|^{3/2}|\xi| - |z|^{1/2}\ln(1 + |z||\xi|)$, which corresponds *plasma equation* (2.4) satisfies also the assumptions (H_1)-(H_7).

The notion of solution of the Cauchy problem (2.1) is certainly complex due to assumption (2.1) and to give it we need some preliminaries.

2.3. Functions of bounded variations and some generalization

Due to the linear growth condition on the Lagrangian, the natural energy space to study the problems we are interested in is the space of functions of bounded variation. Recall that if Ω is an open subset of $\mathbb{R}^N$, a function $u \in L^1(\Omega)$ whose gradient Du in the sense of distributions is a vector-valued Radon measure with finite total variation in Ω is called a *function of bounded variation*. The class of such functions will be denoted by $BV(\Omega)$. For $u \in BV(\Omega)$, the vector measure Du decomposes into its absolutely continuous and singular parts $Du = D^a u + D^s u$. Then $D^a u = \nabla u\ \mathcal{L}^N$, where ∇u is the Radon-Nikodym derivative of the measure Du with respect to the Lebesgue measure $\mathcal{L}^N$. We also split $D^s u$ in two parts: the *jump* part $D^j u$ and the *Cantor* part $D^c u$. It is well known (see for instance [2]) that

$$D^j u = (u^+ - u^-)\nu_u \mathcal{H}^{N-1} \llcorner J_u,$$

where J_u denotes the set of approximate jump points of u, and $\nu_u(x) = \frac{Du}{|Du|}(x)$, $\frac{Du}{|Du|}$ being the Radon-Nikodym derivative of Du with respect to its total variation $|Du|$.

Due to the lack of coercivity, we need to consider the following truncature functions. For $a < b$, let $T_{a,b}(r) := \max(\min(b,r),a)$. As usual, we denote $T_k = T_{-k,k}$. We also consider truncature functions of the form $T^l_{a,b}(r) := T_{a,b}(r) - l$ ($l \in \mathbb{R}$). We denote

$$\mathcal{T}_r := \{T_{a,b} \ : \ 0 < a < b\},$$

$$\mathcal{T}^+ := \{T^l_{a,b} \ : \ 0 < a < b, \ l \in \mathbb{R}, \ \ T^l_{a,b} \geq 0\}$$

and

$$\mathcal{T}^- := \{T^l_{a,b} \ : \ 0 < a < b, \ l \in \mathbb{R}, \ \ T^l_{a,b} \leq 0\}.$$

We need to consider the function space

$$TBV^+(\mathbb{R}^N) := \left\{u \in L^1(\mathbb{R}^N)^+ \ : \ \ T(u) \in BV(\mathbb{R}^N), \ \ \forall\, T \in \mathcal{T}_r\right\},$$

and to give a sense to the Radon-Nikodym derivative (with respect to the Lebesgue measure) ∇u of Du for a function $u \in TBV^+(\mathbb{R}^N)$. Using chain's rule for BV-functions we obtain the following result.

Lemma 2.2. *For every $u \in TBV^+(\mathbb{R}^N)$ there exists a unique measurable function $v : \mathbb{R}^N \to \mathbb{R}^N$ such that*

$$\nabla T_{a,b}(u) = v\chi_{[a<u<b]} \qquad \mathcal{L}^N - \text{a.e.}, \ \ \forall\, T_{a,b} \in \mathcal{T}_r. \tag{2.13}$$

Thanks to this result we define ∇u for a function $u \in TBV^+(\mathbb{R}^N)$ as the unique function v which satisfies (2.13). This notation will be used throughout in the sequel.

To prove the existence of solution of the Cauchy problem (2.1) we use the nonlinear semigroup theory, so we need to study first the elliptic problem

$$u - \lambda \operatorname{div} \mathbf{a}(u, Du) = v \in (L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))^+. \tag{2.14}$$

To give a sense to the differential operator $\operatorname{div} \mathbf{a}(u, Du)$ we need several ingredients.

2.4. A generalized Green's formula

In order to give a meaning to integrals of bounded vector fields with divergence in L^1 integrated with respect to the gradient of a BV function, we need several results from Anzellotti [11] (see also [4]). Following Anzellotti, we denote

$$X_1(\mathbb{R}^N) = \left\{\mathbf{z} \in L^\infty(\mathbb{R}^N, \mathbb{R}^N) \ : \ \operatorname{div}(\mathbf{z}) \in L^1(\mathbb{R}^N)\right\}. \tag{2.15}$$

If $\mathbf{z} \in X_1(\mathbb{R}^N)$ and $w \in BV(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, we define the functional $(\mathbf{z}, Dw) : \mathcal{D}(\mathbb{R}^N) \to \mathbb{R}$ by the formula

$$\langle (\mathbf{z}, Dw), \varphi \rangle := - \int_{\mathbb{R}^N} w \, \varphi \, \mathrm{div}(\mathbf{z}) \, dx - \int_{\mathbb{R}^N} w \, \mathbf{z} \cdot \nabla \varphi \, dx. \tag{2.16}$$

Then $(\mathbf{z}, Dw)$ is a Radon measure in $\mathbb{R}^N$, and

$$\int_{\mathbb{R}^N} (\mathbf{z}, Dw) = \int_{\mathbb{R}^N} \mathbf{z} \cdot \nabla w \, dx, \qquad \forall \ w \in W^{1,1}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \tag{2.17}$$

Moreover, $(\mathbf{z}, Dw)$ is absolutely continuous with respect to $|Dw|$. We have the following *Green's formula* for $\mathbf{z} \in X_1(\mathbb{R}^N)$ and $w \in BV(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} w \ \mathrm{div}(\mathbf{z}) \ dx + \int_{\mathbb{R}^N} (\mathbf{z}, Dw) = 0. \tag{2.18}$$

To prove uniqueness of solutions of the elliptic problem (2.14), we use Kruzkov's technique of doubling variables. For that we need some entropy inequalities, and to derive these entropy inequalities we multiply (2.14) by $T(u)S(u)\phi$ and integrate by parts. So, we need to give sense to expressions of the form

$$\mathbf{a}(u, Du) \cdot D(S(u)T(u)) = S(u)\mathbf{a}(u, Du) \cdot DT(u) + T(u)\mathbf{a}(u, Du) \cdot DS(u).$$

This is possible if we observe that if

$$J_q(r) = \int_0^r q(s) ds,$$

then

$$S(u)\mathbf{a}(u, Du) \cdot DT(u) = \mathbf{a}(u, Du) \cdot DJ_{T'S}(u)$$

$$T(u)\mathbf{a}(u, Du) \cdot DS(u) = \mathbf{a}(u, Du) \cdot DJ_{S'T}(u)$$

and we use Anzellotti's results to give sense to pairings between gradients of BV functions and bounded measurable vector fields with divergence in $L^1(\mathbb{R}^N)$. Now, to do all this rigorously we need to introduce the following functional calculus.

2.5. A functional calculus

Inspired in the relaxed energy functionals introduced by Dal Maso ([20]) for functions with linear growth, we define the following Radon measures.

Assume that $g : \mathbb{R} \times \mathbb{R}^N \to [0, \infty[$ is a Borel function such that

$$C\|\xi\| - D \le g(z, \xi) \le M(1 + \|\xi\|) \qquad \forall (z, \xi) \in \mathbb{R}^N, \ |z| \le R, \tag{2.19}$$

for some constants $C, D, M \geq 0$ which may depend on R. Given a function $u \in BV(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, we define the Radon measure $g(u, Du)$ in $\mathbb{R}^N$ by

$$\langle g(u, Du), \phi \rangle := \int_\Omega g(u(x), \nabla u(x))\phi(x)\, dx + \int_\Omega g^0 \left(\tilde{u}(x), \frac{Du}{|Du|}(x) \right) \phi(x)\, |D^c u|$$

$$+ \int_{J_u} \left(\int_{u_-(x)}^{u_+(x)} g^0 s, \nu_u(x))\, ds \right) \phi(x)\, d\mathcal{H}^{N-1}(x), \qquad \forall\, \phi \in C_c(\mathbb{R}^N), \tag{2.20}$$

where g^0 is the recession function of g, and $\tilde{u}$ is the approximated limit of u.

Let us observe that if $g^0(z, \xi) = \varphi(z)\psi^0(\xi)$, where φ is Lipschitz continuous and ψ^0 is an homogeneous function of degree 1, by applying the chain rule for BV-functions, we have

$$\langle g(u, Du), \phi \rangle = \int_{\mathbb{R}^N} \phi(x) g(u, \nabla u) dx + \int_{\mathbb{R}^N} \phi(x)\psi^0 \left(\frac{Du}{|Du|} \right) |D^s J_\varphi(u)|. \tag{2.21}$$

In this case we have

$$g(u, Du)^s = \psi^0 \left(\frac{Du}{|Du|} \right) |D^s J_\varphi(u)|. \tag{2.22}$$

Let $T \in \mathcal{T}^+ \cup \mathcal{T}^-$. Then there is some $T_{a,b} \in \mathcal{T}_r$ and a constant $c \in \mathbb{R}$ such that $T = T_{a,b} - c$. For $u \in TBV^+(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $T = T_{a,b} - c$, we define the Radon measure $g(u, DT(u))$ in $\mathbb{R}^N$ by

$$\langle g(u, DT(u)), \phi \rangle := \langle g(u, DT_{a,b}(u)), \phi \rangle \; + \int_{[u \leq a]} \phi(x) \left(g(u(x), 0) - g(a, 0) \right) \, dx$$

$$+ \int_{[u \geq b]} \phi(x) \left(g(u(x), 0) - g(b, 0) \right) \, dx. \qquad \forall \phi \in C_c(\mathbb{R}^N). \tag{2.23}$$

We denote by $\mathcal{P}$ the set of Lipschitz continuous functions $p : [0, +\infty[\to \mathbb{R}$ satisfying $p'(s) = 0$ for s large enough. We write $\mathcal{P}^+ := \{ p \in \mathcal{P} \;:\; p \geq 0 \}$.

Let $u \in TBV^+(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $S \in \mathcal{P}^+$ and $T \in \mathcal{T}^+ \cup \mathcal{T}^-$. We denote by $h_S(u, DT(u))$, the Radon measure defined by (2.23) with $g(z, \xi) := S(z)h(z, \xi)$, being $h(z, \xi) := \mathbf{a}(z, \xi) \cdot \xi$. If $-S \in \mathcal{P}^+$ and $T \in \mathcal{T}^+ \cup \mathcal{T}^-$, by definition we set $h_S(u, DT(u)) := -h_{(-S)}(u, DT(u))$.

We observe that formally we have $\mathbf{a}(u, Du) \cdot DJ_{T'S}(u) = h_S(u, DT(u))$. When it comes to a rigorous proof, we have been able to prove only that $\mathbf{a}(u, Du) \cdot DJ_{T'S}(u) \geq h_S(u, DT(u))$, but this is sufficient to derive Kruzkov's inequalities and prove uniqueness of entropy solutions with Kruzkov's technique.

2.6. An existence and uniqueness result for the elliptic problem

We give the following concept of solution for the elliptic problem

$$v = -\mathrm{div}\; \mathbf{a}(u, Du) \quad \text{ in } \quad \mathbb{R}^N. \tag{2.24}$$

Definition 2.3. Given $v \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, $v \geq 0$, we say that $u \geq 0$ is an *entropy solution* of (2.24) if $u \in TBV^+(\mathbb{R}^N)$, and $\mathbf{a}(u, \nabla u) \in X_1(\mathbb{R}^N)$ satisfies

$$v = -\text{div } \mathbf{a}(u, \nabla u)) \qquad \text{in } \mathcal{D}'(\mathbb{R}^N), \tag{2.25}$$

$$h_S(u, DT(u)) \leq (\mathbf{a}(u, \nabla u), DJ_{T'S}(u)) \quad \text{as measures } \ \forall\, S \in \mathcal{P}^+,\, T \in \mathcal{T}^+, \tag{2.26}$$

$$h(u, DT(u)) \leq (\mathbf{a}(u, \nabla u), DT(u)) \quad \text{as measures } \ \forall\, T \in \mathcal{T}^+. \tag{2.27}$$

In [7] we obtain the following result.

Theorem 2.4 **([7]).** *Assume that assumptions* (H) *hold. Then, for any* $0 \leq v \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ *there exists a unique entropy solution* $u \in TBV^+(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *of the problem*

$$u - \text{div } \mathbf{a}(u, Du) = v \qquad in \qquad \mathbb{R}^N. \tag{2.28}$$

Moreover, given $v, \overline{v} \in (L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))^+$, *if* $u, \overline{u}$ *are bounded entropy solutions of the problems*

$$u - \text{div } \mathbf{a}(u, Du) = v \qquad in \qquad \mathbb{R}^N$$

and

$$\overline{u} - \text{div } \mathbf{a}(\overline{u}, D\overline{u}) = \overline{v} \qquad in \qquad \mathbb{R}^N,$$

respectively, then

$$\int_{\mathbb{R}^N} (u - \overline{u})^+ \leq \int_{\mathbb{R}^N} (v - \overline{v})^+.$$

From the above result, using Crandall-Liggett's Theorem we obtain that problem (2.1) has a unique mild-solution. The main difficulty is to characterize these mild-solutions in more classical terms.

2.7. The notion of entropy solution. Existence and uniqueness

To explain the notion of entropy solution, let us collect several observations:

a) We know that there are solutions which are discontinuous on a front which moves at the speed of light. In that case u_t is not a function and the best regularity we can expect is that u_t is a Radon measure(see [10]).

b) Admitting that we were able to prove that u_t is a Radon measure, we would obtain that $\text{div}\, \mathbf{a}(u, Du)$ is a Radon measure. In the formal computations we required the use of test functions of the form $T(u)$ for some Lipschitz function T. Observe that $T(u)$ is at most in $BV(\mathbb{R}^N)$, hence we need that the Radon measure $\text{div}\, \mathbf{a}(u, Du)$ can be integrated against BV functions. To be able to circumvent this difficulty we observe that, being the divergence of a bounded measurable vector field, the expression $\text{div}\, \mathbf{a}(u, Du)$ defines an element of $BV(\mathbb{R}^N)^*$, i.e., the dual of $BV(\mathbb{R}^N)$, and we can use test functions in $BV(\mathbb{R}^N)$. To be more precise, the time dependence has to be included and we have that $\text{div}\, \mathbf{a}(u, Du) \in \big(L^1(0, T; BV(\mathbb{R}^N))\big)^*$ and we can use test functions in $L^1(0, T; BV(\mathbb{R}^N))$. To integrate by parts we have to extend Anzellotti's integration by parts formula to the time-dependent case.

The above remarks explain the requirements in the definition of entropy solution.

To make precise our notion of solution we need to recall several definitions. We define the space

$$Z(\mathbb{R}^N) := \left\{(\mathbf{z},\xi) \in L^\infty(\mathbb{R}^N,\mathbb{R}^N) \times BV(\mathbb{R}^N)^* \ : \ \operatorname{div}(\mathbf{z}) = \xi \ \text{ in } \ \mathcal{D}'(\mathbb{R}^N)\right\}.$$

We need to consider the space $BV(\mathbb{R}^N)_2$, defined as $BV(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ endowed with the norm

$$\|w\|_{BV(\mathbb{R}^N)_2} := \|w\|_{L^2(\mathbb{R}^N)} + |Dw|(\mathbb{R}^N).$$

Definition 2.5. Let $\xi \in \left(L^1(0,T;BV(\mathbb{R}^N)_2)\right)^*$. We say that ξ is *the time derivative* in the space $\left(L^1(0,T;BV(\mathbb{R}^N)_2)\right)^*$ of a function $u \in L^1((0,T)\times\mathbb{R}^N)$ if

$$\int_0^T \langle \xi(t), \Psi(t)\rangle dt = -\int_0^T \int_{\mathbb{R}^N} u(t,x)\Theta(t,x)dxdt$$

for all test functions $\Psi \in L^1(0,T;BV(\mathbb{R}^N))$ with compact support in time, which admit a weak derivative $\Theta \in L^1_w(0,T;BV(\mathbb{R}^N)) \cap L^\infty(Q_T)$, that is, $\Psi(t) = \int_0^t \Theta(s)ds$, the integral being taken as a Pettis integral [23].

Note that if $w \in L^1(0,T;BV(\mathbb{R}^N)) \cap L^\infty(Q_T)$ and $\mathbf{z} \in L^\infty(Q_T,\mathbb{R}^N)$ is such that there exists $\xi \in \left(L^1(0,T;BV(\mathbb{R}^N))\right)^*$ with $\operatorname{div}(\mathbf{z}) = \xi$ in $\mathcal{D}'(Q_T)$, we can define, associated to the pair $(\mathbf{z},\xi)$, the distribution $(\mathbf{z},Dw)$ in Q_T by

$$\langle(\mathbf{z},Dw),\phi\rangle := -\int_0^T \langle\xi(t), w(t)\phi(t)\rangle\, dt - \int_0^T \int_{\mathbb{R}^N} \mathbf{z}(t,x)w(t,x)\nabla_x\phi(t,x)\, dxdt. \tag{2.29}$$

for all $\phi \in \mathcal{D}(Q_T)$.

Definition 2.6. Let $\xi \in \left(L^1(0,T;BV(\mathbb{R}^N)_2)\right)^*$ and $\mathbf{z} \in L^\infty(Q_T,\mathbb{R}^N)$. We say that $\xi = \operatorname{div}(\mathbf{z})$ in $\left(L^1(0,T;BV(\mathbb{R}^N)_2)\right)^*$ if $(\mathbf{z},Dw)$ is a Radon measure in Q_T such that

$$\int_{Q_T} (\mathbf{z},Dw) + \int_0^T \langle\xi(t), w(t)\rangle dt = 0,$$

for all $w \in L^1(0,T;BV(\mathbb{R}^N)) \cap L^\infty(Q_T)$.

Our concept of solution for problem (2.1) is the following one.

Definition 2.7. A measurable function $u : (0,T)\times\mathbb{R}^N \to \mathbb{R}$ is an *entropy solution* of (2.1) in $Q_T = (0,T)\times\mathbb{R}^N$ if $u \in C([0,T];L^1(\mathbb{R}^N))$, $T_{a,b}(u(\cdot)) \in L^1_{loc,w}(0,T;BV(\mathbb{R}^N))$ for all $0 < a < b$, and there exists $\xi \in \left(L^1(0,T;BV(\mathbb{R}^N)_2)\right)^*$ such that

(i) $(\mathbf{a}(u(t),\nabla u(t)),\xi(t)) \in Z(\mathbb{R}^N)$ a.e. in $t \in [0,T]$,

(ii) ξ is the time derivative of u in $\left(L^1(0,T;BV(\mathbb{R}^N)_2)\right)^*$ in the sense of Definition 2.5,

(iii) $\xi = \operatorname{div}\mathbf{a}(u(t),\nabla u(t))$ in the sense of Definition 2.6, and

(iv) the following inequality is satisfied

$$\int_0^T \int_{\mathbb{R}^N} \phi h_S(u, DT(u))\, dt + \int_0^T \int_{\mathbb{R}^N} \phi h_T(u, DS(u))\, dt$$

$$\leq \int_0^T \int_{\mathbb{R}^N} J_{TS}(u(t))\phi_t(t)\, dxdt - \int_0^T \int_{\mathbb{R}^N} \mathbf{a}(u(t), \nabla u(t)) \cdot \nabla\phi\; T(u(t))S(u(t))\, dxdt$$

for truncatures $S, T \in \mathcal{T}^+$ and any nonnegative smooth function ϕ of compact support, in particular of the form $\phi(t,x) = \phi_1(t)\rho(x)$, $\phi_1 \in \mathcal{D}((0,T))$, $\rho \in \mathcal{D}(\mathbb{R}^N)$.

In [8] we have obtained the following existence and uniqueness result.

Theorem 2.8 ([8])**.** *Assume we are under assumptions* (H)*. Then, for any initial datum $0 \leq u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ there exists a unique entropy solution u of* (2.1) *in $Q_T = (0,T) \times \mathbb{R}^N$ for every $T > 0$ such that $u(0) = u_0$. Moreover, if $u(t)$, $\overline{u}(t)$ are the entropy solutions corresponding to initial data u_0, $\overline{u}_0 \in \left(L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)\right)^+$, respectively, then*

$$\|(u(t) - \overline{u}(t))^+\|_1 \leq \|(u_0 - \overline{u}_0)^+\|_1 \qquad \textit{for all} \;\; t \geq 0. \tag{2.30}$$

3. The evolution of the support of the solutions of the relativistic heat equation

In [9] we have proved that the support of solutions of the relativistic heat equation evolves at constant speed, identified as light's speed c. For that we constructed entropy sub- and super-solutions which are fronts evolving at speed c and proved the corresponding comparison principle between entropy solutions and sub- and super-solutions, respectively. This enables us to prove the existence of discontinuity fronts moving at light's speed.

3.1. Sub and super-solutions. Comparison principles

Definition 3.1. Given $0 \leq u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, we say that a measurable function $u : (0,T)\times\mathbb{R}^N \to \mathbb{R}$ is an *entropy super-solution* (respectively, *entropy sub-solution*) of the Cauchy problem (2.1) in $Q_T = (0,T)\times\mathbb{R}^N$ if $u \in C([0,T]; L^1(\mathbb{R}^N))$, $u(0) \geq u_0$ (resp. $u(0) \leq u_0$), $T_{a,b}(u(\cdot)) \in L^1_{loc,w}(0,T,BV(\mathbb{R}^N))$ for all $0 < a < b$, $\mathbf{a}(u(\cdot), \nabla u(\cdot)) \in L^\infty(Q_T)$, and the following inequality is satisfied:

$$\int_{Q_T} h_S(u, DT(u))\phi + \int_{Q_T} h_T(u, DS(u))\phi \tag{3.1}$$

$$\leq \quad \int_{Q_T} J_{TS}(u)\phi_t - \int_0^T \int_{\mathbb{R}^N} \mathbf{a}(u(t), \nabla u(t)) \cdot \nabla\phi T(u(t))S(u(t))dxdt,$$

(resp. with $\geq$ sign instead of $\leq$) for any $\phi \in \mathcal{D}((0,T) \times \mathbb{R}^N)$, $\phi \geq 0$, and any $T \in \mathcal{T}^+$, $S \in \mathcal{T}^-$.

Note that taking $T(r) = 1$ and $S(r) = -1$, for all $r \in \mathbb{R}$, from (3.1), we get

$$\frac{\partial u}{\partial t} \geq \operatorname{div} \mathbf{a}(u(\cdot), \nabla u(\cdot)) \quad \text{in} \quad \mathcal{D}'(Q_T). \tag{3.2}$$

We cannot take these truncation functions directly, instead we can use $T = T_{\frac{1}{n},\frac{2}{n}} + 1$ and $S = T_{\frac{1}{n},\frac{2}{n}} - 1$, and then obtain (3.2) by a limit process.

We have the following comparison principle between entropy super-solutions and entropy solutions.

Theorem 3.2 ([9]). *Assume that there is some constant $C > 0$ such that the function $M(z)$ in* (2.8) *satisfies $M(z) \leq Cz$ for $z \geq 0$ small enough.*

Assume that u is an entropy solution of (2.1) *corresponding to initial datum $u_0 \in \left(L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)\right)^+$, and $\overline{u}$ is an entropy super-solution* (*or an entropy sub-solution*) *of* (2.1) *corresponding to initial datum $\overline{u}_0 \in \left(L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)\right)^+$ such that $\overline{u}(t) \in BV(\mathbb{R}^N)$ for almost all $0 < t < T$. Then*

$$\|(u(t) - \overline{u}(t))^+\|_1 \leq \|(u_0 - \overline{u}_0)^+\|_1 \qquad \textit{for all} \quad t \geq 0. \tag{3.3}$$

3.2. Some entropy super-solutions and sub-solutions of the relativistic heat equation

To study the evolution of the support of entropy solutions of the relativistic heat equation, we need to compute some explicit entropy super and sub-solutions.

Proposition 3.3 ([9]). *Let $C \subset \mathbb{R}^N$ a compact set, $0 < \alpha \leq \beta$. For $s > 0$, let $C(s) := \{x \in \mathbb{R}^N : d(x, C) \leq s\}$. Then $u(t,x) := \beta \chi_{C(ct)}(x)$ is an entropy super-solution of the Cauchy problem for the relativistic heat equation* (1.10) *with $u_0 = \alpha \chi_C$ as initial datum.*

Proposition 3.4 ([9]). *Given $R_0, \alpha_0 > 0$ and $\gamma_0 \geq 0$, there are values $\beta_1, \beta_2 > 0$ large enough such that*

$$u(t,x) = \begin{cases} e^{-\beta_1 t - \beta_2 t^2} \left(\alpha_0 \frac{c}{\nu} \sqrt{R(t)^2 - |x|^2} + \gamma_0 \right) & \text{if} \quad |x| < R(t) \\ 0 & \text{if} \quad |x| \geq R(t), \end{cases}$$

where $R(t) = R_0 + ct$, is an entropy sub-solution of (1.10).

3.3. The evolution of the support

Theorem 3.5 ([9]). *Let C be an open bounded set in $\mathbb{R}^N$. Let $u_0 \in (L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))^+$ with support equals to $\overline{C}$. Let $u(t)$ be the entropy solution of the Cauchy problem for the relativistic heat equation* (1.10) *with u_0 as initial datum. Then*

$$\operatorname{supp}(u(t)) \subset C(ct) \qquad \text{for all } t \geq 0. \tag{3.4}$$

Moreover, if we assume that for any closed set $F \subseteq C$, there is a constant $\alpha_F > 0$ such that $u_0 \geq \alpha_F$ in F, then

$$\operatorname{supp}(u(t)) = C(ct) \qquad \text{for all} \quad t \geq 0.$$

Finally, the following result can be derived from Proposition 3.4 and the comparison principle with sub-solutions.

Proposition 3.6 **([9]).** *Let* $u_0 \in (L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))^+$ *and let* u *be the entropy solution of the Cauchy problem for the equation* (1.10) *with* u_0 *as initial datum. Assume that* $u_0(y) \geq \alpha > 0$ *for any* $y \in B_R(x)$*,* $R > 0$*. Then* $u(t, y) \geq \alpha(t)$ *for any* $y \in B_{R+ct}(x)$ *and any* $t > 0$*, for some function* $\alpha(t) > 0$*. In particular, if* u_0 *is continuous at* $x \in \mathbb{R}^N$ *and* $u_0(x) > 0$*, then* $u(t, x) > 0$ *for any* $t > 0$*.*

This result implies the propagation of discontinuity fronts for any $t > 0$.

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Subordinated Multiparameter Groups of Linear Operators: Properties via the Transference Principle

Boris Baeumer, Mihály Kovács and Mark M. Meerschaert

Dedicated to Günter Lumer;
You were quite an inspiration!

Abstract. In this article we explore properties of subordinated d-parameter groups. We show that they are semi-groups, inheriting the properties of the subordinator via a transference principle. Applications range from infinitely divisible processes on a torus to the definition of inhomogeneous d-dimensional fractional derivative operators.

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1. Introduction

Let S be a strongly continuous uniformly bounded semigroup on $L_1(\mathbb{R}^d)$ that commutes with the translation operator $(T_a f)(x) = f(x+a)$ for all $a \in \mathbb{R}^d$ and let $\mathcal{B}(X)$ denote the algebra of bounded linear operators on a general Banach space X. Then, by [11, Theorem 1.4], S is given by

$$[S(t)f](x) = \int_{\mathbb{R}^d} f(x-s)\,\mu_S^t(ds),\ \ f \in L_1(\mathbb{R}^d) \tag{1.1}$$

where $\{\mu_S^t\}_{t\geq 0}$ is a family of bounded complex regular measures on $\mathbb{R}^d$ with

$$||S(t)||_{\mathcal{B}(L_1(\mathbb{R}^d))} = |\mu_S^t|(\mathbb{R}^d) \leq M_S \tag{1.2}$$

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for all $t \geq 0$. The Fourier transform of $S(t)f$ is given by

$$\widehat{S(t)f}(k) = \int_{\mathbb{R}^d} e^{-i\langle k,s\rangle} S(t)f(s)\, ds = e^{t\psi(k)} \hat{f}(k) \tag{1.3}$$

If, in addition, S is also positive; i.e., $S(t)f \geq 0$ for all $f, t \geq 0$, then S has a Lévy-Khintchine representation (see, for example, [13]), namely ψ is given via[1]

$$\psi(k) = -c^2 - i\langle k, a\rangle - \frac{1}{2}\langle k, Qk\rangle + \int_{x\neq 0} \Big(e^{-i\langle k,x\rangle} - 1 + \frac{i\langle k, x\rangle}{1 + |x|_2^2} \Big)\, \phi(dx) \tag{1.4}$$

where $a \in \mathbb{R}$, $Q = \{q_{ij}\}_{i,j=1}^d$ is a symmetric non-negative definite $d \times d$ matrix with real entries, and the *Lévy measure* ϕ is a σ-finite Borel measure on $\mathbb{R}^d \setminus \{0\}$ such that

$$\int_{x\neq 0} \min\{1, |x|_2^2\}\, \phi(dx) < \infty. \tag{1.5}$$

Let G be a d-parameter C_0-group of operators on a Banach space X generated by $\{(A_i, \mathcal{D}(A_i)) : i = 1 \dots d\}$. In this article we investigate the properties of semigroups obtained by subordinating G by S; i.e., we investigate

$$G_S(t)x = \int_{\mathbb{R}^d} G(s)x\, \mu_S^t(ds),\ \ x \in X, t \geq 0. \tag{1.6}$$

We develop a general theory of these subordinated d-parameter groups, including a powerful transference principle Theorem 2.8 that can be used to show how the subordinated group inherits many useful properties of the subordinator S in $L_1(\mathbb{R}^d)$.

In particular, we first show that G_S is indeed a bounded semigroup and give a core for its infinitesimal generator A_S. Let $(\mathcal{M}_\psi, \mathcal{D}(\mathcal{M}_\psi))$ denote the generator of S on $L_1(\mathbb{R}^d)$ with Fourier transform (1.3). We then prove a transference principle; i.e., we establish that

$$\|g(A_S)\|_{\mathcal{B}(X)} \leq C\|g(\mathcal{M}_\psi)\|_{\mathcal{B}(L_1(\mathbb{R}^d))}$$

for all allowable functions g in the Hille-Phillips functional calculus. The constant C does *not* depend on g. We thereby show, for example, how regularity of S translates into regularity of G_S.

Of special interest to applications is the case where S is positive and $\|S\| = 1$. Note that if $\psi(0) = 0$, then, using (1.1) and (1.2), it is easy to see that the two properties are equivalent. Then subordination has a stochastic interpretation as randomising time (velocity) in each component against an infinitely divisible distribution and ψ is given by the Lévy-Khintchine formula above with $c = 0$. If $\{(A_i, \mathcal{D}(A_i)),\ i = 1, 2, \dots, d\}$ denotes the set of generators of the multi-parameter

[1] Throughout the paper for $x \in \mathbb{R}^d$ we denote $|x|_2 := \left(\sum_{i=1}^d |x_i|^2\right)^{1/2}$ in order to distinguish it from the norm in an arbitrary Banach space.

group G, we give in Theorem 2.12 the proof of an explicit generator formula for all $x \in \bigcap_{i,j=1}^{d} \mathcal{D}(A_iA_j)$; i.e.,

$$A_S x = \sum_{i=1}^{d} a_i A_i x + \frac{1}{2} \sum_{i,j=1}^{d} q_{ij} A_i A_j x + \int_{s \neq 0} \Big(G(s)x - x - \sum_{i=1}^{d} \frac{s_i A_i x}{1 + |s|_2^2} \Big) \, \phi(ds),$$

extending the result of Phillips [20]. In case that S is unilateral; i.e., for all $t \geq 0$, $\mu_S^t(\Omega \cap \mathbb{R}_i^{d-}) = 0$ for all measurable $\Omega \subset \mathbb{R}^d$ and all $\mathbb{R}_i^{d-} := \{s \in \mathbb{R}^d : s_i \leq 0\}$, our theory readily applies to subordinating semi-groups and we generalise $d = 1$ results by Phillips [20], Carasso and Kato [6], Berg, Boyadzhiev and DeLaubenfels [4] and Baeumer and Kovács [2].

2. The transference principle and generator formulas

Since the treatment of multi-parameter semigroups and groups are not standard we first summarise some of their basic properties in the following proposition (see, for example, [5, Propositions 1.1.8 and 1.1.9]).

Proposition 2.1. *If T is a d-parameter C_0-semigroup on X, then T is the product of d one-parameter C_0-semigroups T_i with generators $(A_i, \mathcal{D}(A_i))$; i.e., for $t = (t_1, \ldots, t_d)$ we have $T(t) = \prod_{i=1}^{d} T_i(t_i)$ and the operators $T_i(t_i)$ commute with each other, $0 \leq t_i < \infty$ and $i = 1, \ldots, d$. Moreover,*

(i) *If $x \in \mathcal{D}(A_i)$ then $T(t)x \in D(A_i)$ and $A_i T(t)x = T(t)A_i x$, $t \in \mathbb{R}_+^d$.*
(ii) *The set $\bigcap_{i=1}^{d} \mathcal{D}(A_i)$ is a dense subspace of X and furthermore a Banach space with norm $||x||_{\bigcap_{i=1}^{d} \mathcal{D}(A_i)} := ||x|| + \sum_{i=1}^{d} ||A_i x||$.*
(iii) *If $x \in \mathcal{D}(A_i)$ and $x \in \mathcal{D}(A_iA_j)$, then $x \in \mathcal{D}(A_jA_i)$ and $A_jA_i x = A_iA_j x$.*

For a d-parameter C_0-semigroup T on X the set $\{(A_i, \mathcal{D}(A_i)),\ i = 1, 2, \ldots, d\}$ is called the *set of generators*.

Recall that X is a Banach space and G a bounded d-parameter C_0-group on X with set of generators $\{(A_i, \mathcal{D}(A_i)),\ i = 1, \ldots, d\}$. We define the *subordinated group G_S* via equation

$$G_S(t)x = \int_{\mathbb{R}^d} G(s)x \, \mu_S^t(ds), \;\; x \in X, t \geq 0.$$

where $\mu_S^t(ds)$ is given by (1.1). We will show first that G_S is a bounded C_0-semigroup. The proof (and basically all of the proofs for the theorems that follow) is based on the following commonly used construction.

Definition 2.2. Let G be a bounded d-parameter C_0 -group on X, $f \in L_1(\mathbb{R}^d)$, and $x \in X$. We say that an element $x_f \in X$ is *G-mollified* or short *mollified* if it is obtained by mollifying $\mathbb{R}^d \ni t \mapsto G(t)x$ with f; i.e., if

$$x_f = \int_{\mathbb{R}^d} f(r)G(r)x \, dr. \tag{2.1}$$

The following lemma shows that the set of mollified elements is sufficiently rich for our purposes.

Lemma 2.3. *Consider the set of mollified elements* $\mathcal{B} := \{x_f, x \in Y, f \in \mathcal{C} \subset L_1(\mathbb{R}^d)\}$*. If* $\mathcal{C}$ *is dense in* $L_1(\mathbb{R}^d)$ *and* Y *is dense in* X*, then* $\mathcal{B}$ *is dense in* X*.*

Proof. Assume that $\mathcal{B}$ is not dense in X. Then there is $0 \neq x^* \in X^*$ such that $\langle x_f, x^* \rangle = 0$ for all $x_f \in \mathcal{B}$; i.e.,

$$\left\langle \int_{\mathbb{R}^d} G(s)xf(s)\,ds, x^* \right\rangle = \int_{\mathbb{R}^d} \langle G(s)x, x^* \rangle f(s)\,ds = 0,\ \forall f \in \mathcal{C},\ x \in Y.$$

Now $\mathbb{R}^d \ni s \mapsto \langle G(s)x, x^* \rangle$ is continuous and bounded and hence belongs to $(L_1(\mathbb{R}^d))^* = L_\infty(\mathbb{R}^d)$. This implies that if $\mathcal{C}$ is dense in $L_1(\mathbb{R}^d)$, $\langle G(s)x, x^* \rangle \equiv 0$ for all $x \in Y$ and $s \in \mathbb{R}^d$. But $G(0) = I$ and therefore $\langle x, x^* \rangle = 0$ for all $x \in Y$ which implies that Y cannot be dense in X as $x^* \neq 0$. Hence $\mathcal{C}$ and Y being dense implies that $\mathcal{B}$ has to be dense. □

Proposition 2.4. G_S *given by* (1.6) *is a bounded* C_0*-semigroup on* X*.*

Proof. The operator family G_S is well defined since $\mathbb{R}^d \ni s \mapsto G(s)x$ is continuous and μ_S^t is a bounded measure. If G is bounded by M_G, then G_S is bounded by $M_S M_G$, where M_S is the constant from (1.2). Let

$$\mathcal{A} := \{x_f,\ x \in X, f \in L_1(\mathbb{R}^d)\} \tag{2.2}$$

be the set of mollified elements. By Fubini's theorem, for $x_f \in \mathcal{A}$,

$$\begin{aligned} G_S(t)x_f &= \int_{\mathbb{R}^d} G(s) \int_{\mathbb{R}^d} f(r)G(r)x\,dr\,\mu_S^t(ds) \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(r)G(s+r)x\,dr\,\mu_S^t(ds) \\ &= \int_{\mathbb{R}^d} G(v)x \int_{\mathbb{R}^d} f(v-s)\,\mu_S^t(ds)\,dv \\ &= \int_{\mathbb{R}^d} [S(t)f](v)G(v)x\,dv = x_{S(t)f}. \end{aligned} \tag{2.3}$$

This shows that for $x_f \in \mathcal{A}$,

$$G_S(0)x_f = x_{S(0)f} = x_f,$$

and

$$G_S(t+s)x_f = x_{S(t+s)f} = x_{S(t)S(s)f} = G_S(t)x_{S(s)f} = G_S(t)G_S(s)x_f.$$

The map $L_1(\mathbb{R}^d) \ni f \mapsto x_f \in X$ is clearly linear, and since

$$\|x_f\| \le M_G \|f\|_{L_1(\mathbb{R}^d)} \|x\|, \tag{2.4}$$

it is also continuous. Therefore,

$$\begin{aligned} \|G_S(t)x_f - x_f\| =& \|x_{S(t)f} - x_f\| = \|x_{S(t)f-f}\| \\ \le& M_G \|S(t)f - f\|_{L_1(\mathbb{R}^d)} \|x\|, \end{aligned} \tag{2.5}$$

and since S is a C_0-semigroup on $L_1(\mathbb{R}^d)$, we obtain that $t \to G_S(t)x_f$ is continuous at $t = 0+$. Finally, by Lemma 2.3, $\mathcal{A}$ is dense in X and since $\|G_S(t)\| \leq M_S M_G$, $t \geq 0$, the above holds for all $x \in X$ and the proof is complete. □

Next we identify a reasonably large subset, i.e., a core of the domain of the generator A_S of G_S.

Theorem 2.5. *The set*

$$\mathcal{C} := \{x_f,\ x \in X,\ f \in \mathcal{D}(\mathcal{M}_\psi)\} \subset \mathcal{D}(A_S)$$

is a core for A_S *and*

$$A_S x_f = x_{\mathcal{M}_\psi f},\ x_f \in \mathcal{C},\ \textit{with } \|A_S x_f\| \leq M_G \|\mathcal{M}_\psi f\|_{L_1(\mathbb{R}^d)} \|x\|.$$

Proof. Let $x_f \in \mathcal{C}$. Then $f \in \mathcal{D}(\mathcal{M}_\psi)$ and

$$\begin{aligned}\left\| \frac{G_S(h)x_f - x_f}{h} - x_{\mathcal{M}_\psi f} \right\| &= \left\| x_{\left(\frac{S(h)f-f}{h} - \mathcal{M}_\psi f\right)} \right\| \\ &\leq M_G \left\| \frac{S(h)f - f}{h} - \mathcal{M}_\psi f \right\|_{L_1(\mathbb{R}^d)} \|x\| \to 0\end{aligned}$$

as $h \searrow 0$. This proves that $x_f \in \mathcal{D}(A_S)$ and $A_S x_f = x_{\mathcal{M}_\psi f}$. By (2.5) we see that $\|A_S x_f\| \leq M_G \|\mathcal{M}_\psi f\|_{L_1(\mathbb{R}^d)} \|x\|$. If $f \in \mathcal{D}(\mathcal{M}_\psi)$, then $S(t)f \in \mathcal{D}(\mathcal{M}_\psi)$. Therefore, in view of (2.3), G_S leaves $\mathcal{C}$ invariant. Also, $\mathcal{D}(\mathcal{M}_\psi)$ is dense in $L_1(\mathbb{R}^d)$ and thus $\mathcal{C}$ is dense in X by Lemma 2.3. Therefore, $\mathcal{C}$ is a core for A_S by [8, Chapter II, Proposition 1.7]. □

As a consequence we can transfer the action of A_i on mollified elements of X to actions on their mollifiers.

Corollary 2.6. *Let* $f \in W^{2,1}(\mathbb{R}^d)$, $x \in X$. *Then for all* $i = 1 \ldots d$, $x_f \in \mathcal{D}(A_i)$ *and*

$$A_i x_f = -\int_{\mathbb{R}^d} \frac{\partial f(s)}{\partial s_i} G(s)x\, ds = x_{-\frac{\partial f}{\partial s_i}}.$$

Furthermore, for all $i, j = 1, \ldots, d$, $x_f \in \mathcal{D}(A_i A_j) \cap \mathcal{D}(A_j A_i)$ *and*

$$A_i A_j x_f = A_j A_i x_f = \int_{\mathbb{R}^d} \frac{\partial^2 f(s)}{\partial s_i \partial s_j} G(s)x\, ds = x_{\frac{\partial^2 f}{\partial s_i \partial s_j}}.$$

Proof. It is well known that for the ith-coordinate wise right-translation semigroup

$$[T_{r,i}(t)f](s_1, \ldots, s_i, \ldots, s_n) = f(s_1, \ldots, s_i - t, \ldots, s_n)$$

acting on $L_1(\mathbb{R}^d)$ the generator is given by $\mathcal{M}_\psi f = -\frac{\partial f}{\partial x_i}$ with domain $\mathcal{D}(\mathcal{M}_\psi) = \{f \in L_1(\mathbb{R}^d) :\ x_i \mapsto f(x_1, \ldots, x_i, \ldots, x_n) \text{ abs. cont. with } -\frac{\partial f}{\partial x_i} \in L_1(\mathbb{R}^d)\}$. On the other hand, $T_{r,i}$ can be represented by equation (1.1), where $\mu_S^t = \delta_0 \times \cdots \times \delta_t \times \ldots \delta_0$ where δ_t is the Dirac measure on $\mathbb{R}$ concentrated at t. Hence $S = T_{r,i}$, $G_S = G$, $A_S = A_i$ and the first statement follows directly from Theorem 2.5. The proof is easily completed by repeating the above argument. □

Now we are in a position to prove our main theorem, the transference of the Hille-Phillips functional calculus, which has remarkable consequences. The transference principle is a powerful tool which has been used in many fields of analysis. For a general reference, see [7]. We recall a form of the Hille-Phillips functional calculus briefly. Let $(C, \mathcal{D}(C))$ generate a C_0-semigroup T_C bounded by $M \geq 1$. For a function $g : \mathbb{C} \hookrightarrow \mathbb{C}$ with representation

$$g(z) := \int_0^\infty e^{zt}\, d\alpha(t) \ (\operatorname{Re} z \leq 0) \tag{2.6}$$

where $\alpha : [0, \infty) \to \mathbb{C}$ is a normalised[2] function of bounded total variation, define

$$g(C)x := \int_0^\infty T_C(t)x\, d\alpha(t), \ \ x \in X$$

where the integral can be understood either in the Riemann-Stieltjes or in the Lebesgue-Stieltjes (Bochner) sense. In the latter case α is then replaced by the complex Borel measure induced by α. It turns out that functions with representation (2.6) form an algebra and the map $\Psi : g \mapsto g(C) \in \mathcal{B}(X)$ is an algebra homomorphism (see, for example, [10, Chapter XV] and [15]), where $\mathcal{B}(X)$ is the algebra of bounded linear operators on X. In particular, for $g_\lambda(z) = 1/(\lambda - z)$ and each λ with $\operatorname{Re}\lambda > 0$, the resolvent is an element of the algebra; i.e., for $x \in X$ and $\operatorname{Re}\lambda > 0$,

$$g_\lambda(C)x = R(\lambda, C)x = \int_0^\infty T_C(t)xe^{-\lambda t}\, dt.$$

Definition 2.7. A collection of functions $\{g_n\}_{n\in\mathbb{N}} \subset L_1(\mathbb{R}^d)$ is called an *approximate identity* if $g_n \geq 0$, $\|g_n\|_{L_1(\mathbb{R}^d)} = 1$ and $\lim_{n\to\infty} \int_{|s|_2 \geq \delta} g_n(s)\, ds = 0$ for any fixed $\delta > 0$.

It is straightforward to show that if $x \in X$ and $\{f_n\}_{n\in\mathbb{N}}$ is an approximate identity, then $x_{f_n} \to x$ as $n \to \infty$.

Theorem 2.8 (Transference Principle). *Let A_S be the generator of the semigroup G_S and $\mathcal{M}_\psi$ be the generator of the semigroup S. If g has representation* (2.6), *then*

$$\|g(A_S)\|_{\mathcal{B}(X)} \leq M_G \|g(\mathcal{M}_\psi)\|_{\mathcal{B}(L_1(\mathbb{R}^d))} \tag{2.7}$$

where M_G is independent of g.

[2] One may, for example, normalise α by setting $\alpha(0) = 0$ and $\alpha(u) = \frac{\alpha(u+)+\alpha(u-)}{2}$ for all $u > 0$, see, for example [15],

Proof. Let $x_f \in \mathcal{A}$ where $\mathcal{A}$ is defined in (2.2) and g given by (2.6). Then, since both G_S and S are bounded C_0-semigroups,

$$\begin{aligned} g(A_S)x_f &= \int_0^\infty G_S(t)x_f \, d\alpha(t) = \int_0^\infty x_{S(t)f} \, d\alpha(t) \\ &= \int_0^\infty \int_{\mathbb{R}^d} [S(t)f](r)G(r)x \, dr \, d\alpha(t) \qquad (2.8) \\ &= \int_{\mathbb{R}^d} \int_0^\infty [S(t)f](r) \, d\alpha(t)G(r)x \, dr = x_{g(\mathcal{M}_\psi)f}. \end{aligned}$$

where the interchange of the integrals is justified by Fubini's theorem. Therefore, by (2.4),

$$\begin{aligned} \|g(A_S)x_f\| &\le M_G \|g(\mathcal{M}_\psi)f\|_{L_1(\mathbb{R}^d)} \|x\| \\ &\le M_G \|g(\mathcal{M}_\psi)\|_{\mathcal{B}(L_1(\mathbb{R}^d))} \|f\|_{L_1(\mathbb{R}^d)} \|x\|. \end{aligned}$$

Finally, take $\{f_n\}_{n\in\mathbb{N}}$ to be an approximate identity. Then $x_{f_n} \to x$ and since $g(A_S) \in \mathcal{B}(X)$ we also have that $g(A_S)x_{f_n} \to g(A_S)x$ for all $x \in X$. Thus,

$$\|g(A_S)x\| \le M_G \|g(\mathcal{M}_\psi)\|_{\mathcal{B}(L_1(\mathbb{R}^d))} \|x\|, \; x \in X. \qquad \square$$

We immediately obtain an important corollary which shows an example of transference of regularity under subordination in the group case. Recall that a C_0-semigroup T_C generated by $(C, \mathcal{D}(C))$ is called a *bounded analytic semigroup of angle* $\theta \in (0, \frac{\pi}{2}]$ if T_C has a bounded analytic extension to a sectorial region $\{z \in \mathbb{C} : |\arg z| < \theta'\}$ for all $\theta' \in (0, \theta)$. This is equivalent to $(C, \mathcal{D}(C))$ being *a sectorial operator of angle* θ; that is, the resolvent set of C, $\rho(C)$, contains the sectorial region

$$\Sigma_\theta := \left\{\lambda \in \mathbb{C} : |\arg \lambda| < \frac{\pi}{2} + \theta\right\} \setminus \{0\} \subset \rho(C)$$

and $\|R(\lambda, C)\|_{\mathcal{B}(X)} \le \frac{M_\varepsilon}{|\lambda|}$ for all $\lambda \in \Sigma_{\theta-\varepsilon}$ and $\varepsilon \in (0, \delta)$ for some $M_\varepsilon \ge 1$ (see, for example [1, Theorem 3.7.11]).

Corollary 2.9. *If S is a bounded analytic semigroup of angle θ on $L_1(\mathbb{R}^d)$, then G_S is a bounded analytic semigroup of angle θ on X.*

Proof. Since both G_S and S are bounded semigroups it follows that $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$ is contained in both, the resolvent set $\rho(A_S)$ and $\rho(\mathcal{M}_\psi)$. If $g(z) := (\lambda - z)^{-1}$ ($\operatorname{Re}\lambda > 0$), then for the resolvent operators of A_S and $\mathcal{M}_\psi$ we obtain, by Theorem 2.8,

$$\|R(\lambda, A_S)\|_{\mathcal{B}(X)} \le M_G \|R(\lambda, \mathcal{M}_\psi)\|_{\mathcal{B}(L_1(\mathbb{R}^d))}, \; \operatorname{Re}\lambda > 0. \qquad (2.9)$$

Therefore,

$$\sup_{\operatorname{Re}\lambda>0} \|\lambda R(\lambda, A_S)\|_{\mathcal{B}(X)} \le M_G \sup_{\operatorname{Re}\lambda>0} \|\lambda R(\lambda, \mathcal{M}_\psi)\|_{\mathcal{B}(L_1(\mathbb{R}^d))} < \infty$$

where the finite supremum on the right-hand side follows from the assumption that S is an analytic semigroup on $L_1(\mathbb{R}^d)$ (see, for example [1, Corollary 3.7.12]). Even more, the finiteness of above supremum is a necessary and sufficient condition for the analyticity of a semigroup [1, Corollary 3.7.12]. Thus G_S is indeed a bounded analytic semigroup . Assume that S is a bounded analytic semigroup of angle θ, or, equivalently, that $\mathcal{M}_\psi$ is a sectorial operator of angle θ. Let $\lambda \in \Sigma_{\theta-\varepsilon}$. Then there is $\lambda_0 \in i\mathbb{R}$ and $r > 0$ such that $\lambda \in D(\lambda_0, r) \subset \rho(\mathcal{M}_\psi)$, where $D(\lambda_0, r)$ is the open disc with centre λ_0 and radius r. It follows from (2.9) that $\lambda_0 \in \rho(A_S)$. Therefore, by Theorem 2.8, the algebra homomorphism property of the Hille-Phillips functional calculus and the continuity of the resolvent,

$$\left\|R(\lambda_0, A_S)^{k+1}\right\|_{\mathcal{B}(X)} \leq M_G \left\|R(\lambda_0, \mathcal{M}_\psi)^{k+1}\right\|_{\mathcal{B}(L_1(\mathbb{R}^d))}.$$

Hence absolute convergence of $\sum(\lambda - \lambda_0)^k R(\lambda_0, \mathcal{M}_\psi)^{k+1}$ implies absolute convergence of $\sum(\lambda-\lambda_0)^k R(\lambda_0, A_S)^{k+1}$. Therefore, using the Taylor series representation of the resolvent, $\lambda \in \rho(A_S)$ and $\|R(\lambda, A_S)\|_{\mathcal{B}(X)} \leq \frac{M_\varepsilon M_G}{|\lambda|}$. □

2.1. Positive subordinators

Of particular interest is the case where the subordinator S is positive as it applies to stochastic models and the subordinator takes on the purpose of randomising the time variable (or velocity) in each component. In this case we prove an explicit generator formula using rather elementary tools. We need two preparatory results.

Proposition 2.10. *Let T be a d-parameter C_0-semigroup on a Banach space X with set of generators $\{(A_i, \mathcal{D}(A_i))\ i = 1, \dots, d\}$. A subspace of $\bigcap_{i=1}^d \mathcal{D}(A_i)$ which is $\|\cdot\|$-dense in X and invariant under T is $\|\cdot\|_{\bigcap_{i=1}^d \mathcal{D}(A_i)}$-dense in $\bigcap_{i=1}^d \mathcal{D}(A_i)$.*

Proof. The proof is a straightforward generalisation of the proof of the well-known one-parameter result (see, for example, [8, Chapter II, Proposition 1.7]) in view of Proposition 2.1. □

Consider $\bigcap_{i,j=1}^d \mathcal{D}(A_iA_j)$ with norm

$$||x||_{\bigcap_{i,j=1}^d \mathcal{D}(A_iA_j)} := ||x|| + \sum_{i=1}^d ||A_ix|| + \sum_{i,j=1}^d ||A_iA_jx||.$$

Note that, by Proposition 2.1 (iii), similarly to Sobolev spaces, one could use an equivalent norm by summing up the last term for $1 \leq j \leq i \leq d$, only.

Corollary 2.11. *Let T be a d-parameter C_0-semigroup on a Banach space X with set of generators $\{(A_i, \mathcal{D}(A_i))\ i = 1, \dots, d\}$. A subspace $\mathcal{D}$ of $X_{\bigcap_{i,j=1}^d \mathcal{D}(A_iA_j)}$ which is $\|\cdot\|$-dense in X and invariant under T is $\|\cdot\|_{\bigcap_{i,j=1}^d \mathcal{D}(A_iA_j)}$-dense in $X_{\bigcap_{i,j=1}^d \mathcal{D}(A_iA_j)}$.*

Proof. Clearly, T is a d-parameter C_0-semigroup on the Banach space

$$\left(\bigcap_{i=1}^d \mathcal{D}(A_i), \|\cdot\|_{\bigcap_{i=1}^d \mathcal{D}(A_i)}\right)$$

in view of Proposition 2.1, with set of generators $(A_i, \bigcap_{j=1}^d \mathcal{D}(A_iA_j))$. Therefore, by Proposition 2.10, $\mathcal{D}$ is dense in $\bigcap_{i,j=1}^d \mathcal{D}(A_iA_j)$ with respect to the norm

$$|||x||| = \|x\|_{\bigcap_{i=1}^d \mathcal{D}(A_i)} + \sum_{j=1}^d \|A_jx\|_{\bigcap_{i=1}^d \mathcal{D}(A_i)} \geq ||x||_{\bigcap_{i,j=1}^d \mathcal{D}(A_iA_j)},$$

which finishes the proof. $\square$

Next we prove the generator formula.

Theorem 2.12. *Let S be positive; i.e., the log-characteristic function is given by the Lévy-Khintchine formula* (1.4). *Then $\bigcap_{i,j=1}^d \mathcal{D}(A_iA_j) \subset \mathcal{D}(A_S)$ and*

$$A_Sx = -c^2x+\sum_{i=1}^d a_iA_ix+\frac{1}{2}\sum_{i,j=1}^d q_{ij}A_iA_jx+\int_{s\neq 0}\Big(G(s)x-x-\sum_{i=1}^d \frac{s_iA_ix}{1+|s|_2^2}\Big)\,\phi(ds), \tag{2.10}$$

for all $x \in \bigcap_{i,j=1}^d \mathcal{D}(A_iA_j)$.

Proof. In [3, Theorem 2.2] it is shown that $W^{2,1}(\mathbb{R}^d) \subset \mathcal{D}(\mathcal{M}_\psi)$ and

$$\begin{aligned}(\mathcal{M}_\psi f)\,(s) = &-c^2 f(s) - a\cdot\nabla f(s) + \tfrac{1}{2}\nabla\cdot Q\nabla f(s)\\ &+\int_{y\neq 0}\Big(f(s-y)-f(s)+\frac{\nabla f(s)\cdot y}{1+|y|_2^2}\Big)\,\phi(dy),\ \ f\in W^{2,1}(\mathbb{R}^d).\end{aligned}$$

Let $\mathcal{D} := \{x_f : x \in X,\ f \in W^{2,1}(\mathbb{R}^d) \cap C^2(\mathbb{R}^d)\}$. By Theorem 2.5, $\mathcal{D} \subset \mathcal{D}(A_S)$. For $x_f \in \mathcal{D}$ we have, using Corollary 2.6,

$$\begin{aligned} A_Sx_f &= x_{\mathcal{M}_\psi f}\\ &= \int_{\mathbb{R}^d}\big[-c^2f(s)-a\cdot\nabla f(s)+\tfrac{1}{2}\nabla\cdot Q\nabla f(s)\\ &+\int_{y\neq 0}\Big(f(s-y)-f(s)+\frac{\nabla f(s)\cdot y}{1+|y|_2^2}\Big)\ \phi(dy)\big]\,G(s)x\,ds\\ &= -c^2x_f+\sum_{i=1}^d a_iA_ix_f+\frac{1}{2}\sum_{i,j=1}^d q_{ij}A_iA_jx_f\\ &+\int_{\mathbb{R}^d}\Big(\int_{v\neq 0}\Big(f(s-v)-f(s)+\frac{\nabla f(s)\cdot v}{1+|v|_2^2}\Big)\,\phi(dv)\Big)\,G(s)x\,ds\\ &= -c^2x_f+\sum_{i=1}^d a_iA_ix_f+\frac{1}{2}\sum_{i,j=1}^d q_{ij}A_iA_jx_f\\ &+\int_{v\neq 0}\Big(G(v)x_f-x_f-\sum_{i=1}^d\frac{v_iA_ix_f}{1+|v|_2^2}\Big)\ \phi(dv), \end{aligned} \tag{2.11}$$

where the interchange of the integrals is justified because $\|G(s)x_f\| \le M_G\|x_f\|$ and because $f \in W^{2,1}(\mathbb{R}^d) \cap C^2(\mathbb{R}^d) \subset \mathcal{D}(\mathcal{M}_\psi)$ and thus

$$s \mapsto \int_{v\neq 0} \left(f(s-v) - f(s) + \frac{\nabla f(s)\cdot v}{1+|v|_2^2} \right) \phi(dv) \in L_1(\mathbb{R}^d).$$

Taylor's formula for $f \in W^{2,1}(\mathbb{R}^d) \cap C^2(\mathbb{R}^d)$ yields

$$f(s-v) = f(s) - v\cdot\nabla f(s) + \int_0^1 (1-t)\langle v, M_{s-tv}v\rangle\, dt$$

where M_r is the Hessian matrix of f at $r \in \mathbb{R}^d$. Thus for $x_f \in \mathcal{D}$, by Corollary 2.6 and Fubini's theorem,

$$\begin{aligned} G(v)x_f &= x_{f(\cdot - v)} = x_{f - v\cdot\nabla f + \int_0^1 (1-t)\langle v, M_{\cdot - tv}v\rangle\, dt} \\ &= x_f + \sum_{i=1}^d v_i A_i x_f + \int_0^1 (1-t) G(tv) \sum_{i,j=1}^d v_i v_j A_i A_j x_f\, dt. \end{aligned}$$

Hence, for $|v|_2 \le 1$,

$$\begin{aligned} &\left\| G(v)x_f - x_f - \sum_{i=1}^d \frac{v_i A_i x_f}{1+|v|_2^2} \right\| \\ &\le \|G(v)x_f - x_f - \sum_{i=1}^d v_i A_i x_f\| + \left\| \sum_{i=1}^d v_i A_i x_f - \sum_{i=1}^d \frac{v_i A_i x_f}{1+|v|_2^2} \right\| \\ &\le \frac{1}{2} M_G \sum_{i,j=1}^d |v_i v_j|\, \|A_i A_j x_f\| + \frac{|v|_2^2}{1+|v|_2^2} \sum_{i=1}^d |v_i|\, \|A_i x_f\| \\ &\le C_d^2 \frac{|v|_2^2}{2} M_G \sum_{i,j=1}^d \|A_i A_j x_f\| + C_d \frac{|v|_2^3}{1+|v|_2^2} \sum_{i=1}^d \|A_i x_f\| \\ &\le C_d^2 \frac{|v|_2^2}{1+|v|_2^2} M_G \sum_{i,j=1}^d \|A_i A_j x_f\| + C_d \frac{|v|_2^2}{1+|v|_2^2} \sum_{i=1}^d \|A_i x_f\| \\ &\le K_d M_G ||x_f||_{\bigcap_{i,j=1}^d \mathcal{D}(A_i A_j)} \frac{|v|_2^2}{1+|v|_2^2}. \end{aligned}$$

If $|v|_2 \ge 1$, then

$$\begin{aligned} &\left\| G(v)x_f - x_f - \sum_{i=1}^d \frac{v_i A_i x_f}{1+|v|_2^2} \right\| \le (M_G+1)\|x_f\| + C_d \frac{|v|_2}{1+|v|_2^2} \sum_{i=1}^d \|A_i x_f\| \\ &\le (M_G+1)\|x_f\| + C_d \frac{|v|_2^2}{1+|v|_2^2} \sum_{i=1}^d \|A_i x_f\| \end{aligned}$$

$$\le (M_G+1)\frac{2|v|_2^2}{1+|v|_2^2}\|x\| + C_d\frac{|v|_2^2}{1+|v|_2^2}\sum_{i=1}^d \|A_i x_f\|$$
$$\le L_d(M_G+1)||x_f||_{\bigcap_{i,j=1}^d \mathcal{D}(A_iA_j)}\frac{|v|_2^2}{1+|v|_2^2}.$$

Therefore, for all $v \in \mathbb{R}^d$ and $x_f \in \mathcal{D}$,

$$\left\| G(v)x_f - x_f - \sum_{i=1}^d \frac{v_i A_i x_f}{1+|v|_2^2}\right\| \le (C_d+K_d)(M_G+1)||x_f||_{\bigcap_{i,j=1}^d \mathcal{D}(A_iA_j)}\frac{|v|_2^2}{1+|v|_2^2}, \tag{2.12}$$

which implies that

$$\begin{aligned}\int_{v\neq 0}&\left\| G(v)x_f - x_f - \sum_{i=1}^d \frac{v_i A_i x_f}{1+|v|_2^2}\right\| \phi(dv)\\ &\le (C_d+K_d)(M_G+1)\int_{v\neq 0}\frac{|v|_2^2}{1+|v|_2^2}\phi(dv)||x_f||_{\bigcap_{i,j=1}^d \mathcal{D}(A_iA_j)}.\end{aligned} \tag{2.13}$$

The set $\mathcal{C}$ is dense in $L_1(\mathbb{R})$ and therefore by Lemma 2.3, $\mathcal{D}$ is dense in X. It is easy to see that G leaves $\mathcal{D}$ invariant and by Corollary 2.6, $\mathcal{D} \subset \bigcap_{i,j=1}^d \mathcal{D}(A_iA_j)$. Thus by Corollary 2.11, $\mathcal{D}$ is $\|\cdot\|_{\bigcap_{i,j=1}^d \mathcal{D}(A_iA_j)}$-dense in $\bigcap_{i,j=1}^d \mathcal{D}(A_iA_j)$. Therefore, if $x \in \bigcap_{i,j=1}^d \mathcal{D}(A_iA_j)$, there is a sequence $\{(x_n)_{f_n}\} \subset \mathcal{D}$ such that $(x_n)_{f_n} \to x$, $A_i(x_n)_{f_n} \to A_i x$, $i = 1,\dots,d$, and $A_iA_j(x_n)_{f_n} \to A_iA_jx$, $i,j = 1,\dots,d$, as $n \to \infty$. In particular, $\sup_n \|(x_n)_{f_n}\|_{\bigcap_{i,j=1}^d \mathcal{D}(A_iA_j)} < \infty$. Therefore, using (2.13) and (2.12), the right-hand side of (2.11) converges to

$$-c^2x + \sum_{i=1}^d v_iA_ix + \frac{1}{2}\sum_{i,j=1}^d q_{ij}A_iA_jx + \int_{s\neq 0}\Big(G(v)x - x - \sum_{i=1}^d \frac{v_iA_ix}{1+|v|_2^2}\Big)\,\phi(dv).$$

Since $x_{f_n} \to x$ and A_S is closed, $x \in \mathcal{D}(A_S)$ and

$$\begin{aligned}A_Sx = &-c^2x + \sum_{i=1}^d v_iA_ix + \frac{1}{2}\sum_{i,j=1}^d q_{ij}A_iA_jx\\ &+ \int_{s\neq 0}\Big(G(v)x - x - \sum_{i=1}^d \frac{v_iA_ix}{1+|v|_2^2}\Big)\,\phi(dv),\ x \in \bigcap_{i,j=1}^d \mathcal{D}(A_iA_j). \qquad \square\end{aligned}$$

2.2. The semigroup case

In case that S is unilateral; i.e., for all $t \geq 0$, $\mu_S^t(\Omega \cap \mathbb{R}_i^{d-}) = 0$ for all measurable $\Omega \subset \mathbb{R}^d$ and all $\mathbb{R}_i^{d-} := \{s \in \mathbb{R}^d : s_i \leq 0\}$, the above theory readily extends to the point where we can allow G to be a bounded d-parameter semigroup T; i.e., we define

$$T_S(t)x = \int_{\mathbb{R}_+^d} T(s)x\,\mu_S^t(ds),\ x \in X, \tag{2.14}$$

with generator $(A_S, \mathcal{D}(A_S))$. We denote, as above, the set of generators of T by $\{(A_i, \mathcal{D}(A_i)),\ i = 1, 2, \ldots, d\}$ and the generator of S, which is now considered as a C_0-semigroup on $L_1(\mathbb{R}^d_+)$, by $(\mathcal{M}_\psi, \mathcal{D}(\mathcal{M}_\psi))$. With the obvious modifications all of the above theorems hold, in particular, we would like to highlight the Transference Principle for semigroups.

Theorem 2.13 (Transference Principle for semi-groups). *If g has the form* (2.6), *then*

$$\|g(A_S)\|_{\mathcal{B}(X)} \leq M_T \|g(\mathcal{M}_\psi)\|_{\mathcal{B}(L_1(\mathbb{R}^d_+))} \tag{2.15}$$

where M_T is independent of g. □

The following corollary is more general then the one-dimensional result in [6], as it also shows the transference of the angle of analyticity and is more general then the corresponding one-dimensional statement in [4] as there is no restriction on the measures. For the one-dimensional case, see also [2].

Corollary 2.14. *If S is a bounded analytic semigroup on $L_1(\mathbb{R}^d)$ of angle θ on $L_1(\mathbb{R}^d_+)$, then T_S is a bounded analytic semigroup of angle θ on X.*

The other special case we would like to highlight, is the case when S is positive and unilateral. Then the Lévy-Khintchine representation simplifies to

$$\psi(k) = -c^2 - i\langle k, a\rangle + \int_{x\in\mathbb{R}^d_+\setminus\{0\}} \left(e^{-i\langle k,x\rangle} - 1\right) \phi(dx) \tag{2.16}$$

with $a \in \mathbb{R}^d_+$ and ϕ is a σ-finite Borel measure on $\mathbb{R}^d_+ \setminus \{0\}$ such that

$$\int \min\{1, |x|_2\}\phi\{dx\} < \infty.$$

This result is originally due to Paul Lévy. The proof for $d = 1$ is outlined in Feller [9, XVII.4(c) p. 571] but the proof extends immediately to the multivariate case (see also [20]). The proof of the generator formula below is analogous to the proof of Theorem 2.12 and therefore we only give an outline.

Theorem 2.15. *Let S be positive and unilateral with log characteristic function given by the Lévy-Khintchine formula* (2.16). *Then $\bigcap_{i=1}^d \mathcal{D}(A_i) \subset \mathcal{D}(A_S)$ and*

$$A_S x = -c^2 x + \sum_{i=1}^d a_i A_i x + \int_{\mathbb{R}^d_+} \left(T(s)x - x\right) \phi(ds),\ x \in \bigcap_{i=1}^d \mathcal{D}(A_i). \tag{2.17}$$

Proof. The proof, the same way as in the group case, uses the result on $L_1(\mathbb{R}^d_+)$, namely, $W^{1,1}_0(\mathbb{R}^d_+) \subset \mathcal{D}(\mathcal{M}_\psi)$ and

$$\mathcal{M}_\psi f(s) = -c^2 f(s) - a \cdot \nabla f(s) + \int_{\mathbb{R}^d_+} \left(f(s-y) - f(s)\right) \phi(dy),\ f \in W^{1,1}_0(\mathbb{R}^d_+).$$

This can be shown exactly the same fashion as [3, Theorem 2.2] using the simplified form of the Lévy-Kintchine formula (2.16). Then it is easily verified that $\mathcal{D} := \{x_f :$

$x \in X,\ f \in \mathcal{C}\} \subset \mathcal{D}(A_S)$, where $\mathcal{C} := W_0^{1,1}(\mathbb{R}_+^d) \cap C^1(\mathbb{R}_+^d)$, is $||\cdot||_{\bigcap_{i=1}^d \mathcal{D}(A_i)}$-dense in $\bigcap_{i=1}^d \mathcal{D}(A_i)$ and that (2.17) holds for $x_f \in \mathcal{D}$. Finally, Taylor's formula

$$T(v)x_f = x_f + \sum_{i=1}^{d} \int_0^1 T(tv)v_i A_i x_f \, dt,\ x_f \in \mathcal{D},\ v \in \mathbb{R}_+^d,$$

yields the estimate

$$\|(T(v)x_f - x_f)\| \le K_d(M_T+1)\frac{|v|_2}{1+|v|_2}\, ||x_f||_{\bigcap_{i=1}^d \mathcal{D}(A_i)},$$

and the proof can be completed the same way as the proof of Theorem 2.12. □

3. Examples

3.1. Subordinating the d-parameter translation semigroup

Let X be any of the spaces $C_0(\mathbb{R}^d)$, $\mathrm{UCB}(\mathbb{R}^d)$ or $L_p(\mathbb{R}^d)$ $(1 \le p < \infty)$.[3] Then with μ^t from (1.1) the semigroup

$$[S(t)f](x) = \int_{\mathbb{R}^d} f(x-y)\, \mu^t(dy),\ f \in X, \tag{3.1}$$

is strongly continuous on X and its generator is given by

$$\begin{aligned}(\mathcal{M}_\psi f)(s) = &- c^2 f(s) - a\cdot\nabla f(s) + \tfrac{1}{2}\nabla\cdot Q\nabla f(s)\\ &+ \int_{y\neq 0}\left(f(s-y) - f(s) + \frac{\nabla f(s)\cdot y}{1+|y|_2^2}\right)\phi(dy),\ f \in D,\end{aligned}$$

where $\mathcal{D} = \{f \in X :\ D^\alpha f \in X$ with multi-index $|\alpha| \le 2\}$ for $X = C_0(\mathbb{R}^d)$ or $X = \mathrm{UCB}(\mathbb{R}^d)$ and $\mathcal{D} = W^{2,p}(\mathbb{R}^d)$ for $X = L_p(\mathbb{R}^d)$. The statement for $X = C_0(\mathbb{R}^d)$ can be found in [22, Theorem 31.5] (for the variable coefficient version, see, for example, [21]) and now it is a corollary of Theorem 2.12 which is based on the $L_1(\mathbb{R}^d)$-result [3, Theorem 2.2]. Thus, the semigroup S on the function space X inherits several useful properties of the corresponding semigroup on the space $L_1(R^d)$ as long as the d-parameter translation is strongly continuous on X.

3.2. Infinitely divisible processes on a d-dimensional torus

Let X be the space of continuous functions on a d-dimensional torus, i.e.,

$$X = C_\pi([0,2\pi]^d)$$

with $f \in C_\pi([0,2\pi]^d)$ if and only if f is continuous and periodic in each dimension (i.e., in each component the value at 2π has to agree with the value at zero). Take $A_i = -\frac{\partial}{\partial x_i}$ with

$$\mathcal{D}(A_i) = \left\{ f \in C_\pi([0,2\pi]^d) : \frac{\partial}{\partial x_i} f \in C_\pi([0,2\pi]^d)\right\}.$$

[3] Here, $\mathrm{UCB}(\mathbb{R}^d)$ denotes the uniformly bounded and uniformly continuous functions on $\mathbb{R}^d$ and $C_0(\mathbb{R}^d)$ the continuous functions on $\mathbb{R}^d$ with zero limit as $|x|_2 \to \infty$.

The d-parameter group is then given by

$$G(t_1,\dots,t_d)f(x_1,\dots,x_d) = f(x_1-t_1,\dots,x_d-t_d) \tag{3.2}$$

using the periodic extension of f. If S is positive with $\|S\|=1$, i.e., S corresponds to an infinitely divisible process in $\mathbb{R}^d$, then G_S is the corresponding infinitely divisible process on the d-dimensional torus and Theorem 2.12 yields the generator formula for this semigroup. This provides a simple proof of the formula in [12] in this special case. Infinitely divisible processes on cylinders are defined analogously.

3.3. Inhomogeneous fractional derivatives

Consider $X = L_1(\mathbb{R}^2)$ and

$$A_x f := -\frac{\partial}{\partial x}\left(v_1(x)f(x,y)\right); A_y f := -\frac{\partial}{\partial y}\left(v_2(y)f(x,y)\right)$$

for continuously differentiable v_i with $v_i(x)>0$ for all x and $\int_{-\infty}^0 1/v_i(x)\,dx = \int_0^\infty 1/v_i(x)\,dx = \infty$. It is easy to check that the two-parameter flow group is then given by

$$T(t_1,t_2)f(x,y) = f(h_1(x,t_1),h_2(y,t_2))v_1(h_1(x,t_1))v_2(h_2(y,t_2))/(v_1(x)v_2(y)) \tag{3.3}$$

where $h_i(x,t)$ are implicitly defined via

$$\int_{h_i(x,t)}^x \frac{1}{v_i(s)}\,ds = t.$$

Suppose that S is defined by (1.1) where μ_S^t is a strictly operator stable probability measure on $\mathbb{R}^d$. Operator stable laws are infinitely divisible laws with nice scaling properties, see [14, 17]. For a certain range of the scaling parameters, it is shown in [18] that

$$\psi(k) = -\mu\cdot ik + \int_{\|\theta\|=1}\int_0^\infty \left(e^{-ik\cdot r^H\theta} - 1 + ik\cdot r^H\theta\right)\frac{dr}{r^2}\lambda(d\theta)$$

for some scaling matrix H and spectral measure λ. Operator stable semigroups are important in applications to physics [16], hydrology [23], and finance [18], partly because their generators involve multidimensional analogues of fractional derivatives. Subordinating the flow group leads to the generator formula, using substitution in (2.10),

$$A_S f = \mu_1 A_x f + \mu_2 A_y f + \int_{\|\theta\|=1}\int_0^\infty \left(G(r^H\theta)f - f - \langle r^H\theta,(A_x f, A_y f)\rangle\right)\frac{dr}{r^2}\lambda(d\theta)$$

for all $f\in\mathcal{D}(A_x^2)\cap\mathcal{D}(A_y^2)\cap\mathcal{D}(A_xA_y)$, generalising the fractional derivative operator defined in [19] for $v_1=v_2=1$.

In particular, if $\mu=0$ and $H=\begin{pmatrix}1/\alpha & 0\\ 0 & 1/\alpha\end{pmatrix}$ for some $1<\alpha<2$, using the same argument as in Lemma 7.3.8 in [17], the log-characteristic function reduces to

$$\psi(k) = -\Gamma(1-\alpha) \int_{\|\theta\|=1} \langle ik, \theta \rangle^\alpha \, \lambda(d\theta).$$

Hence we obtain that in this case, using the functional calculus for group generators developed in [2],

$$A_S f = -\Gamma(1-\alpha) \int_{\|\theta\|=1} (-\theta_1 A_x - \theta_2 A_y)^\alpha \, f \, \lambda(d\theta), \; f \in \mathcal{D}(A_x^2) \cap \mathcal{D}(A_y^2) \cap \mathcal{D}(A_x A_y).$$

Similarly, if λ is concentrated on the axes and $H = \begin{pmatrix} 1/\alpha_1 & 0 \\ 0 & 1/\alpha_2 \end{pmatrix}$ we obtain in the simplest case

$$A_S f = -|A_x|^{\alpha_1} f - |A_y|^{\alpha_2} f.$$

For example, if $v_1 = v_2 \equiv 1$, then the generator A_S is a mixture of one variable fractional derivatives.

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An Integral Equation in AeroElasticity

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Abstract. The integral equation that plays a key role in AeroElasticity is known as the Possio Integral Equation, named after its discoverer. From its inception in 1938, this equation was formulated in the Fourier Transform domain using divergent integrals, until 2002 when a more precise formulation valid in a right half-plane was given. In this paper we express it in the time-domain, which requires the language of Functional Analysis, L_p spaces, $1 < p < 2$, and Semigroup Theory. A key role is played by the Finite Hilbert Transform and the Tricomi-Sohngen airfoil equation, which may actually be considered a special case.

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1. Introduction

In this paper we treat the Possio Integral Equation purely from the mathematical point of view, eschewing any discussion of its origin or application in AeroElasticity except to refer the interested reader to [1, 2, 3; cf. 4, 5]. We begin in Section 2 with the formulation in the Laplace Domain (right half-plane), a little more than replacing $i\omega$ by λ! We show how the integral kernel can be simplified by taking the $(L_p - L_q)$ Fourier Transform in the spatial domain [4], from which it follows in particular that the integral operator is a Mikhlin multiplier. An explicit representation of the multiplier is developed that allows us, in Section 3, to obtain the time-domain representations of the integral equation in terms of the shift-semigroup on $L_p(R^1)$ and the Hilbert Transform. By using the Tricomi operator we convert this equation to a Volterra-type equation, a type not covered by the literature extant [6].

Hence we have to develop techniques special to our problem. Actually we have to take Laplace Transforms, the familiar tool for solving Volterra equations.

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We illustrate this by considering a special case, $M = 0$, in Section 4. A key result is due to Sears [7] on inverse Laplace Transforms.

In Section 5 we consider the general case, again taking Laplace Transforms, imitating the technique used for $M = 0$ as far as we can. Although we are unable to attain closure, we do wind up with a general formulation and obtain two useful approximations on the way.

In Section 6 we indicate two generalizations of the problem, still largely unsolved.

2. The Possio Equation

The original version [1] in Italian was published in 1938.We begin with the version, circa 1954, quoted in the classic text on AeroElasticity [2], extending the Fourier Transform to the Laplace Transform,

$$\hat{w}(\lambda\cdot, x) = \int_{-b}^{b} \hat{P}(\lambda\cdot, x-\xi)\hat{A}(\lambda, \xi)\, d\xi, \quad |x| < b < \infty \tag{2.1}$$

where $\text{Re}\lambda > 0$ (more generally, in a half-plane). The hat is there to indicate that it is the Laplace Transform of a function of time t, $0 < t < \infty$. We have an integral equation for $\hat{A}(\lambda, \cdot)$ for each λ in the half-plane, for given

$$\hat{w}(\lambda, \cdot) \in C_1[-b, b] \text{ for each } \lambda,$$

and the kernel

$$\begin{aligned}
\hat{P}(\lambda, x) = \frac{k}{\sqrt{1-M^2}} \Bigg[& \frac{M|x|}{x} K_1\Big(\frac{Mk}{1-M^2}|x|\Big) \exp\Big(\frac{M^2 k}{1-M^2}x\Big) \\
& + K_0\Big(\frac{Mk}{1-M^2}|x|\Big) \exp\Big(\frac{M^2 k}{1-M^2}x\Big) \\
& - k\int_0^\infty K_0\Big(\frac{k|s|M}{1-M^2}\exp\Big(-kx + \frac{ks}{1-M^2}\Big)ds \\
& - \sqrt{1-M^2}\log\Big(\frac{1+\sqrt{1-M^2}}{M}\Big)\exp(kx)\Bigg], \\
& -\infty < x < \infty,\ k = \lambda/U,\ U > 0 \text{ given },\ 0 \le M < 1
\end{aligned} \tag{2.2}$$

The integrals in the kernel are convergent and the kernel may be defined on the imaginary axis by taking limits,to obtain the original frequency version in [1, 2]. The function space for the solution $\hat{A}(\lambda, \cdot)$ was not specified, but it is required that

$$\hat{A}(\lambda, x) \to 0 \text{ as } x \to b.$$

From the presence of the Bessel $K_1(\cdot)$ function we can see that the kernel is singular at $x = 0$,so that we have a 'singular convolution' integral equation.

This was the state of affairs until 2002 when it was shown [4] that by going back to where the Possio Integral came from viz. the Neumann type boundary-value problem for a linear partial differential equation, and showing that $\hat{P}(\lambda, \cdot)$ is in $L_p(R^1),\ 1 < p < 2$, one could consider the $L_p - L_q$ Spatial Fourier Transform of $\hat{P}(\lambda, \cdot)$. This yields

$$\begin{aligned}\tilde{P}(\lambda, i\omega) &= \int_{-\infty}^{\infty} \hat{P}(\lambda, x) e^{-i\omega x} dx, \quad -\infty < \omega < \infty \\ &= \frac{1}{2}\frac{D(k, i\omega)}{k + i\omega}, \quad k = \lambda/U \end{aligned} \tag{2.3}$$

where

$$D(k, i\omega) = \sqrt{k^2M^2 + 2kM^2 i\omega + (1 - M^2)\omega^2} \tag{2.4}$$

where the root in the right half-plane is to be taken.

Not only did this provide an enormous simplification over the original representation but also made apparent the non-analyticity in M at 0 or 1. More importantly, one could readily calculate that this is a Mikhlin multiplier $L_p - L_q$. Moreover, setting $M = 0$ in 2.3, we see that it reduces to

$$\tilde{P}(\lambda, i\omega) = \frac{1}{2}\frac{|\omega|}{k + i\omega}. \tag{2.5}$$

In particular for $k = 0$ the equation is no more than the finite Hilbert Transform or the 'air-foil', equation, whose solution was established by Tricomi [5], showing in particular the need for the limit condition specified at $x = b-$, for uniqueness. We shall return to this in the next section.

To obtain the equation in (2.1) in multiplier form, let $\Phi(M, \lambda)$ denote the operator corresponding to the multiplier

$$\frac{1}{2}\frac{\sqrt{D(k, i\omega)}}{k + i\omega}.$$

Then requiring that $\hat{A}(\lambda, \cdot)$ be in $L_p(-b, b),\ 1 < p < 2$ (which we shall require from now on), we can consider

$$\Phi(M, \lambda)\hat{A}(\lambda, \cdot).$$

Since

$$L_p(-b, b) \subset L_p(R^1)$$

and denoting by $\mathcal{P}$ the projection on $L_p(R^1)$ into $L_p(-b, b)$, we may express (2.1) in the 'multiplier' form as

$$\hat{w}(\lambda, \cdot) = \mathcal{P}\Phi(M, \lambda)\hat{A}(\cdot, \lambda). \tag{2.6}$$

We note that

$$\tilde{P}(\lambda, 0) = \frac{1}{2}M = \underset{\mathrm{Re}\lambda \to \infty}{\text{limit}} \tilde{P}(\lambda, i\omega) \tag{2.7}$$

3. Possio Integral: Time domain version

We note that showing existence of solution to (2.6) for each λ, Re $\lambda > 0$, is not enough. We need to show that $\hat{A}(\lambda, \cdot)$ is the Laplace Transform of a time-domain function. This raises the question, can we express (2.1) or (2.4) as an equation in the time-domain? This is done here for the first time, apparently! For this purpose we need the representation theory in [4]. It is shown there that we can express

$$\tilde{P}(\lambda, i\omega) = \frac{1}{2}\frac{\sqrt{D(k, i\omega)}}{k + i\omega}$$

equivalently as

$$\frac{1}{2}\frac{|\omega|}{i\omega}\sqrt{1-M^2}\big(1 + k\tilde{q}(k, i\omega)\big), \quad -\infty < \omega < \infty,\ \omega \neq 0 \tag{3.1}$$

where

$$\tilde{q}(k, i\omega) = \frac{-1}{\sqrt{1-M^2}}\frac{1}{k+i\omega} - \int_{-\alpha_2}^{\alpha_1} \frac{1}{ks + i\omega} a(s)ds \tag{3.2}$$

where

$$\begin{aligned} a(s) &= \frac{1}{\pi}\frac{\sqrt{(\alpha_1 - s)(\alpha_2 + s)}}{1-s} \\ \alpha_1 &= \frac{M}{1+M} \qquad \alpha_2 = \frac{M}{1-M}. \end{aligned} \tag{3.3}$$

Note that $\alpha_2 \geq \alpha_1$. Moreover it is shown in [4] that $\tilde{q}(k, i\omega)$ is the $L_p - L_q$ Fourier Transform of a function

$$\hat{q}(k, x), \quad -\infty < x < \infty$$

$$\hat{q}(k, \cdot) \in L_p(R^1), \quad p > 1$$

$$\left.\begin{aligned} \hat{q}(k, x) &= -\frac{e^{-kx}}{\sqrt{1-M^2}} - \int_0^{\alpha_1} e^{-ksx} a(s)ds, \quad x > 0 \\ &= \int_0^{\alpha_2} e^{-ks|x|} a(-s)ds, \quad x < 0 \end{aligned}\right\}. \tag{3.4}$$

Note that for $M = 0$, there is considerable simplification in that the integral terms vanish.

Now the convolution, for f in $L_p(-b, b)$:

$$\hat{g}(k, x) = \int_{-b}^{b} \hat{q}(k, x - \xi) f(\xi) d\xi, \quad -\infty < x < \infty$$

can be expressed

$$\hat{g}(k, \cdot) = \int_{-b}^{b} S(\xi)\hat{q}(k, \cdot) f(\xi) d\xi$$

where $S(\cdot)$ is the shift group on $L_p(R^1)$ given by

$$S(t)q = q(\cdot - t)$$

and since

$$L_p(-b, b) \subset L_1(-b, b), \quad b < \infty$$

we have the familiar result (see[8]) ($||\cdot||_p$ denoting the p-norm, as usual)

$$\begin{aligned} ||\hat{g}(k,\cdot)||_p &\leq ||\hat{q}(k,\cdot)||_p \int_{-b}^{b} |f(\xi)|d\xi \\ &\leq (2b)^{1/q}||\hat{q}(k,\cdot)||_p||f||_p. \end{aligned}$$

Hence

$$\hat{g}(k,\cdot) = \hat{q}(k,\cdot)f \tag{3.5}$$

defines $\hat{q}(k,\cdot)$ as a linear bounded operator on $L_p(-b,b)$ into $L_p(R^1),\ 1<p<2$.

Moreover, we can write

$$\hat{g}(k,\cdot) = \frac{R(k)f}{\sqrt{1-M^2}} - \int_{-\alpha_2}^{\alpha_1} R(ks)fa(s)ds \tag{3.6}$$

where $R(\lambda)$ is the resolvent of $S(\cdot)$:

$$\begin{aligned} R(\lambda)f &= \int_0^\infty e^{-\lambda t}S(t)fdt \\ R(-\lambda)f &= -\int_0^\infty e^{-\lambda t}S(-t)fdt. \end{aligned}$$

Hence

$$\hat{g}(\lambda,\cdot) = \int_0^\infty e^{-\lambda t}L(t)fdt$$

where

$$L(t)f = -\frac{US(Ut)f}{\sqrt{1-M^2}} - \int_{-\alpha_2}^{\alpha_1} US(Ut/s)f\frac{a(s)ds}{s}. \tag{3.7}$$

This is formal in that the question of whether the integrals in (3.6), (3.7) are well defined, needs to be resolved, because of the singularity at $s=0$. To remove the singularity, we may proceed as follows:

$$\int_{-\alpha_2}^{\alpha_1} R(\lambda s)f\,a(s)ds = \int_{-\alpha_2}^{\alpha_1} R(\lambda s)f\,a(0)ds + \int_{-\alpha_2}^{\alpha_1} sR(\lambda s)f\big(a(s)-a(0)\big)/s\;ds$$

where in the second term we have only to note that

$$||R(\lambda s)s|| \leq \frac{1}{\mathrm{Re}\cdot\lambda}$$

while the first term

$$\begin{aligned} &= a(0)\int_0^{\alpha_1}\int_0^\infty e^{-\lambda st}S(t)fdtds - a(0)\int_0^{\alpha_2}\int_0^\infty e^{-\lambda st}S(-t)fdtds \\ &= a(0)\int_0^\infty (1-e^{-\lambda\alpha_1 t})\left(\frac{S(t)f-S(-t)f}{\lambda t}\right)dt - a(0)\int_{\alpha_1}^{\alpha_2}\int_0^\infty e^{-\lambda st}S(-t)fdtds \end{aligned}$$

and

$$\frac{1}{\pi}\int_0^\infty \frac{S(t)f-S(-t)f}{t}dt = Hf$$

by the Hille-Phillips operational calculus [9]. In other words, we need to interpret the integral in (3.6) as a Cauchy Integral.

The integral in the second term in (3.7) may be treated in a similar way by expressing it as:

$$\int_0^{\alpha_1} \frac{\big(\mathcal{S}(Ut/s)fa(s) - \mathcal{S}(-Ut/s)fa(-s)\big)ds}{s} - \int_{\alpha_1}^{\alpha_2} \frac{\mathcal{S}(-Ut/s)fa(-s)ds}{s}$$

where the first term can be expressed

$$\begin{aligned} a(0)\int_0^{\alpha_1} &\frac{\big(\mathcal{S}(Ut/s)f - \mathcal{S}(-Ut/s)f\big)ds}{s} \\ &+ \int_0^{\alpha_1} \mathcal{S}(Ut/s)f\left[\frac{a(s)-a(0)}{s}\right]ds \\ &- \int_0^{\alpha_1} \mathcal{S}(-Ut/s)f\left(\frac{a(-s)-a(0)}{s}\right)ds. \end{aligned}$$

We only need to examine the first term which can be expressed by a change of variable:

$$\begin{aligned} a(0)\int_{t/\alpha_1}^{\infty} &\frac{\big(\mathcal{S}(U\sigma)f - \mathcal{S}(-U\sigma)f\big)}{\sigma}d\sigma \\ &= \pi a(0)\left(Hf - \int_0^{t/\alpha_1} \frac{\big(\mathcal{S}(U\sigma)f - \mathcal{S}(-U\sigma)f\big)}{\sigma}d\sigma\right) \end{aligned}$$

and the integral in the second term is taken in the Cauchy sense at zero, a_0 as in the definition of H.

Hence finally we obtain the time-domain representation for the Possio Equation as a Volterra-type equation in $L_p(-b,b)$:

$$\mathcal{P}HA(t) - \frac{d}{dt}\int_0^t \mathcal{P}HL(t-\sigma)A(\sigma)d\sigma = \frac{2w(t)}{\sqrt{1-M^2}}, \quad 0 < t \tag{3.8}$$

where

$$w(\cdot) \in \Big(L(0,T),\ L_p(-b,b)\Big), \quad 0 < T < \infty$$

and

$$\begin{gathered} L(t)f = \frac{S(Ut)f}{\sqrt{1-M^2}} + \int_{-\alpha_2}^{\alpha_1} S(Ut/s)f\frac{a(s)}{s}ds, \quad t > 0 \\ f \in L_p(-b,b);\ \ L(t)f \in L_p(R^1). \end{gathered} \tag{3.9}$$

We can convert (3.8) into a more familiar Volterra-type equation by invoking a key result due to Tricomi-Sohngen (see [5]).

Theorem 3.1. *Let $w(\cdot) \in C_1(-b,b)$. Then the integral equation*

$$w(\cdot) = \mathcal{P}HA$$

where

$$A \in L_p(-b,b) \subset L_p(R^1), \quad 1 < p < 2$$

has a unique solution such that $A(x) \to 0$ as $x \to b-$. This solution is given by $A = \mathcal{T}w$ where $\mathcal{T}$ is the Tricomi operator, a linear bounded operator on $L_p(-b, b)$, $p = 4/3+$, into $L_p(-b, b)$, $1 < p < 4/3$:

$$\mathcal{T}f = g$$

$$g(x) = \frac{1}{\pi}\sqrt{\frac{b-x}{b+x}} \int_{-b}^{b} \sqrt{\frac{b-x}{b+x}} \frac{f(\xi)}{\xi - x} ds, \quad |x| < b. \tag{3.10}$$

Proof. See [4].

From now on we assume that

$$w(t) \in c_1(-b, b)$$

$$\int_0^\infty ||w(t)|| e^{-\sigma t} dt < \infty, \quad \sigma > 0$$

where the norm is the L_p norm, $1 < p < 2$, to be consistent. Then 'operating' on both sides of (3.8) with $\mathcal{T}$, noting that $A(\cdot) \in L_p(-b, b)$, $1 < p < 2$, we obtain

$$2\frac{\mathcal{T}w(t)}{\sqrt{1-M^2}} = A(t) - \mathcal{T}\frac{d}{dt}\int_0^t \mathcal{P}HL(t-s)A(s)ds, \quad t > 0. \tag{3.11}$$

We note in particular that the solution, if any, is in the range of $\mathcal{T}$:

$$A(t, x) = 1/\pi\sqrt{\frac{b-x}{b+x}} \int_{-b}^{b} \sqrt{\frac{b+s}{b-s}} \frac{A_1(t, s)}{s - x} ds, \quad |x| < b \tag{3.12}$$

where

$$A_1(t, \cdot) \in L_p(-b, b), \quad 1 < p < 2.$$

4. Special case $M = 0$

In this section we shall specialize $\Phi(M, \lambda)$ to the case $M = 0$, where the solution has been known since the 1940's (see [1]). Here we follow a different technique of solution based on Laplace Transformation. We shall extend this technique as far as we can go, to the more general case $0 < M < 1$, which is still not far enough for solution.

With $M = 0$, the time-domain Possio Equation simplifies to

$$2\mathcal{T}w(t) = A(t) - \mathcal{T}\frac{d}{dt}\int_0^t \mathcal{P}HS\big(U(t-\sigma)\big)A(\sigma)d\sigma, \quad 0 < t. \tag{4.1}$$

The first step here is clear: we decompose the integral:

$$\mathcal{P}HS\big(U(t-\sigma)\big)A(\sigma) = \mathcal{P}H\mathcal{P}S\big(U(t-\sigma)\big)A(\sigma) + \mathcal{P}H(I-\mathcal{P})S\big(U(t-\sigma)\big)A(\sigma)$$

so that

$$\mathcal{T}\mathcal{P}HS\big(U(t-\sigma)\big)A(\sigma) = S\big(U(t-\sigma)\big)A(\sigma) + \mathcal{T}\mathcal{P}H(I-\mathcal{P})S\big(U(t-\sigma)\big)A(\sigma).$$

Now for any f in $L_p(-b, b)$, we note that denoting

$$g = S(Ut)f$$

we have

$$\begin{aligned} g(x) &= f(x - Ut) \\ &= 0 \qquad x < -b \ \text{ for all } t \geq 0. \end{aligned}$$

And for $f(\cdot)$ in $L_p(R^1)$ such that $f(x) = 0,\ x < -b$, we have

$$g = \mathcal{TPH}(I - \mathcal{P})f$$

$$g(x) = \frac{1}{\pi}\sqrt{\frac{b-x}{b+x}} \int_{-b}^{b} \sqrt{\frac{b+\xi}{b-\xi}} \frac{d\xi}{\xi - x} \int_{b}^{\infty} \frac{f(\sigma)}{\xi - \sigma} d\sigma$$

and as in [10]

$$\int_{-b}^{b} \sqrt{\frac{b+\xi}{b-\xi}} \frac{d\xi}{(\xi - x)(\xi - \sigma)} = \sqrt{\frac{\sigma + b}{\sigma - b}} \cdot \frac{1}{x - \sigma}, \quad \text{for } \sigma > b.$$

Hence we have

$$g(x) = \frac{1}{\pi}\sqrt{\frac{b-x}{b+x}} \int_{-b}^{\infty} \sqrt{\frac{\sigma+b}{\sigma-b}} \cdot \frac{1}{x-\sigma} f(\sigma) d\sigma, \quad |x| < b.$$

Denoting

$$Gf = \mathcal{TPH}(I - \mathcal{P})f, \quad f \text{ in } L_p(R^1), \tag{4.2}$$

G linear bounded on $L_p(R^1)$ into $L_p(-b, b)$, we have

$$\mathcal{TPH}(I - \mathcal{P})S\big(U(t-\sigma)\big)A(\sigma) = GS\big(U(t-\sigma)\big)A(\sigma).$$

Hence we can express (4.1) as

$$\begin{aligned} 2\mathcal{T}w(t) = A(t) &- \frac{d}{dt}\int_0^t \mathcal{P}S\big(U(t-\sigma)\big)A(\sigma)d\sigma \\ &- \frac{d}{dt}\int_0^t GS\big(U(t-\sigma)\big)A(\sigma)d\sigma, \quad t > 0. \end{aligned} \tag{4.3}$$

We stop here because it is as far as we can go in the time domain.

To go further, we need to switch to the Laplace Domain, which every treatise on Volterra Equations [6] utilizes.

We may take Laplace Transforms in (4.3) formally, or require further that

$$\int_0^\infty \big(||w(t)||_p + ||A(t)||_p\big)e^{-\sigma t} dt \ < \infty, \quad \sigma > 0$$

so that, defining

$$\begin{aligned} \hat{w}(\lambda) &= \int_0^\infty e^{-\lambda t} w(t) dt, \quad \operatorname{Re}\lambda > 0 \\ \hat{A}(\lambda) &= \int_0^\infty e^{-\lambda t} A(t) dt, \quad \operatorname{Re}\lambda > 0, \end{aligned}$$

we have

$$2\mathcal{T}\hat{w}(\lambda) = \hat{A}(\lambda) - k\mathcal{P}R(k)\hat{A}(\lambda) - kGR(k)\hat{A}(\lambda), \quad \operatorname{Re}\lambda > 0. \tag{4.4}$$

Let

$$\mathcal{L}(k) = \mathcal{P}R(k)\mathcal{P}.$$

Then $\mathcal{L}(k)$ is Volterra on $L_p(-b, b)$ into itself and in fact,

$$\mathcal{L}(k)f = g; \quad g(x) = \int_{-b}^{x} e^{-k(x-\sigma)} f(\sigma)d\sigma, \quad |x| < b$$

and it is quickly verified that

$$\big(I - k\mathcal{L}(k)\big)^{-1} = I + k\mathcal{L}(0). \tag{4.5}$$

Further we see a remarkable simplification in that we can calculate:

$$GR(k)\hat{A}(\lambda) = -g(k) \int_{-b}^{b} e^{-k(b-\xi)} \hat{A}(\lambda, \xi)d\xi$$

where $g(k)$ is the function in $L_p(-b, b)$:

$$g(k, x) = \frac{1}{\pi}\sqrt{\frac{b-x}{b+x}} \int_0^\infty \sqrt{\frac{2b+\sigma}{\sigma}} \frac{1}{b-x+\sigma} e^{-k\sigma} d\sigma, \quad |x| < b. \tag{4.6}$$

We have a fixed element of $L_p(-b, b)$, multiplied by a continuous Linear Functional on $\hat{A}(\lambda, \cdot)$, which is a key simplification of the problem. Thus (4.4) becomes:

$$2\mathcal{T}\hat{w}(\lambda) = \hat{A}(\lambda) - k\mathcal{L}(k)\hat{A}(\lambda) + kg(k)L\big(k, \hat{A}(\lambda)\big) \tag{4.7}$$

where $L(k)$ is a linear functional on $L_p(-b, b)$ defined by

$$L(k, f) = \int_{-b}^{b} e^{-k(b-\xi)} f(\xi)d\xi.$$

Using (4.5), we can express (4.7) equivalently as:

$$\hat{\nu}(\lambda) = \hat{A}(\lambda) - kh(k)L\big(k, \hat{A}(\lambda)\big) \tag{4.8}$$

where

$$\hat{\nu}(\lambda) = (I + k\mathcal{L}(0))2\mathcal{T}\hat{w}(\lambda)$$
$$h(k) = (I + k\mathcal{L}(0))2\mathcal{T}g(k).$$

We can solve for $\hat{A}(\lambda)$ in (4.8) by noting that

$$L\big(k, \hat{\nu}(\lambda)\big) = L\big(k, \hat{A}(\lambda)\big)\big(1 + kL(k, h(k)\big).$$

and assuming

$$1 + kL\big(k, h(k)\big) \neq 0, \quad \mathrm{Re}\,\mathrm{k} > 0. \tag{4.9}$$

We see that the unique solution to (4.8) is given by:

$$\hat{A}(\lambda) = \hat{\nu}(\lambda) - \frac{1}{1 + kL\big(k, (h)k\big)} kh(k). \tag{4.10}$$

To prove (4.9), we calculate first that

$$1 + kL\big(k, (h)k\big) = k \int_0^\infty e^{-kt} \sqrt{\frac{2b+t}{t}} dt$$

and prove the assertion (see [6], p. 151). Next, to prove that

$$\hat{A}(\lambda, x) \to 0 \text{ as } x \to b$$

we have only to note that

$$\hat{\nu}(\lambda, x) \text{ and } h(k, x) \to 0 \text{ as } x \to b.$$

Thus (4.10) provides an explicit solution of (4.4)

Finally we need to show that the inverse Laplace Transform of $\hat{A}(\lambda)$ provides the required solution for the time-domain equation we started with. Here all we need is to invoke the key result due to Sears [7], who shows that

$$\frac{1}{1 + kL\big(k, h(k)\big)} = \int_0^\infty e^{-kt} c_1(t)dt, \quad c_1(t) \geq 0.$$

We refer to [7] for the proof.

5. The general case $M \neq 0,\ 0 < M < 1$

In this section we treat the general case, $M \neq 0,\ M < 1$ (cf. (3.10)):

$$\begin{aligned} 2\frac{\mathcal{T}w(t)}{\sqrt{1-M^2}} &= A(t) - \frac{d}{dt}\mathcal{T}\mathcal{P}H\int_0^t S(U(t-\sigma))A(\sigma)d\sigma \qquad (5.1)\\ &\quad - \frac{d}{dt}\mathcal{T}\mathcal{P}H\int_0^t\int_{-\alpha_2}^{\alpha_1} a(s)S(U(t-\sigma)/s)dsA(\sigma)d\sigma, \quad t > 0. \end{aligned}$$

It should be noted that (5.1) is not a generalization in the spirit of pure mathematics. Indeed, without the physical problem, it would have been impossible to conceive what a meaningful generalization would be.

Lacking any general method to be found in the vast relevant literature to tackle this 'Volterra' equation, we seek to generalize the technique we employed for the case $M = 0$. We shall go as far as we can go, but not, unfortunately, far enough for a solution to the problem.

Thus we begin by taking the Laplace Transformation and obtain:

$$\begin{aligned} \frac{2\mathcal{T}\hat{w}(\lambda)}{\sqrt{1-M^2}} &= \hat{A}(\lambda) - \mathcal{T}\mathcal{P}H\hat{L}(\lambda)\hat{A}(\lambda)\\ &= \hat{A}(\lambda) - \mathcal{T}\mathcal{P}H\left(\frac{kR(k)\hat{A}(\lambda)}{\sqrt{1-M^2}} + k\int_{-\alpha_2}^{\alpha_1} a(s)R(ks)\hat{A}(\lambda)ds\right)\\ &= \hat{A}(\lambda) - \mathcal{P}\left(\frac{kR(k)}{\sqrt{1-M^2}} + k\int_{-\alpha_2}^{\alpha_1} a(s)R(ks)\hat{A}(\lambda)ds\right) \qquad (5.2)\\ &\quad - \mathcal{T}\mathcal{P}H(I-\mathcal{P})\left(\frac{kR(k)\hat{A}(\lambda)}{\sqrt{1-M^2}} + k\int_{-\alpha_2}^{\alpha_1} a(s)R(ks)\hat{A}(\lambda)ds\right). \end{aligned}$$

Let as before

$$\mathcal{L}(k) = \mathcal{P}R(k)\mathcal{P}.$$

Then, modelling on (4.4), we can express (5.2) as

$$\frac{2\mathcal{T}\hat{w}(\lambda)}{\sqrt{1-M^2}} = A - kV(k)A + kW(k)A \tag{5.3}$$

where

$$V(k)A = \frac{1}{1-M^2}\mathcal{L}(k)A + \int_{-\alpha_2}^{\alpha_1} a(s)\mathcal{L}(ks)Ads \tag{5.4}$$

$$\begin{aligned} W(k)A = \frac{1}{1-M^2}g_-(k,\cdot)L_-(k,A) &+ \int_0^{\alpha_1} g_-(ks,\cdot)L_-(ks,A)a(s)ds \\ &+ \int_0^{\alpha_2} \big(g_+(ks,\cdot)j(ks,\cdot)\big)a(-s)L_+(ks,A)ds \end{aligned}$$

where the functionals

$$L_+(k,A) = \int_{-b}^{b} e^{-k(b+\xi)}A(\xi)d\xi$$

$$L_-(k,A) = \int_{-b}^{b} e^{-k(b-\xi)}A(\xi)d\xi$$

and finally the functions

$$g_+(k,x) = \frac{1}{\pi}\sqrt{\frac{b-x}{b+x}}\int_0^\infty \sqrt{\frac{\sigma}{2b+\sigma}}\frac{1}{b+\sigma+x}e^{-k\sigma}d\sigma, \quad |x|<b$$

$$g_-(k,x) = \frac{1}{\pi}\sqrt{\frac{b-x}{b+x}}\int_0^\infty \sqrt{\frac{2b+\sigma}{\sigma}}\frac{1}{b+\sigma-x}e^{-k\sigma}d\sigma, \quad |x|<b$$

$$j(k,x) = e^{k(b+x)}, \quad |x|<b.$$

Here the main thing to note is that $V(k)$ is Volterra on $L_p(-b,b)$ into itself but $W(k)$ is quite a bit more complicated than the last term in (4.7).

Approximation for small k

One result that is immediate is that with

$$\hat{\nu}(\lambda) = (I - kV(k))^{-1}\frac{2\mathcal{T}\hat{w}(\lambda)}{\sqrt{1-M^2}}$$

we have

$$\hat{\nu}(\lambda) = A - k(I-kV(k))^{-1}W(k)A$$

and for small enough k we can obtain a solution in terms of a Neumann series:

$$\begin{aligned} A &= \big(I - k(I-kV(k))^{-1}W(k)\big)^{-1}\hat{\nu}(\lambda) \\ &= \sum_0^\infty k^n\big(I-kV(k))^{-1}W(k)\big)^n\hat{\nu}(\lambda). \end{aligned}$$

This can be useful because in the applications in [11] we are primarily interested in small k.

Approximation for large k

We can also obtain a useful approximation for large k, $U \to 0$ for fixed λ for $M \neq 0$. For this purpose, let

$$Q(k) = \frac{-kR(k)}{\sqrt{1-M^2}} - \int_{-\alpha_2}^{\alpha_1} kR(ks)a(s)ds, \quad \operatorname{Re} k > 0$$

on $L_p(R^1)$ into $L_p(R^1)$. We note that as Re $k \to \infty$, $Q(k)$ converges strongly (strongly only!) to

$$Q(\infty) = -I - \frac{M}{\sqrt{1-M^2}}H.$$

This can be seen readily from the corresponding multiplier (defined in (3.2))

$$\hat{\nu}(k, iw) = \frac{-k}{\sqrt{1-M^2}}\frac{1}{k+iw} - \int_{-\alpha_2}^{\alpha_1} \frac{k}{ks+iw}a(s)ds$$

converges to

$$-1 - \frac{|w|}{iw}\frac{M}{\sqrt{1-M^2}}.$$

Hence for $M \neq 0$, we can rewrite (2.6) where we started as:

$$\frac{2}{M}\hat{w}(\lambda) = \hat{A} + \frac{\sqrt{1-M^2}}{M}\mathcal{P}H\big(Q(k) - Q(\infty)\big)\hat{A}.$$

Let us denote

$$K(k) = \frac{\sqrt{1-M^2}}{M}\mathcal{P}H\big(Q(k) - Q(\infty)\big).$$

Then

$$K(k)^n\hat{w}(\lambda) \to 0 \text{ as } \operatorname{Re} k \to \infty, \text{ in any subsector.}$$

Setting

$$\hat{A}_n = \Big(\sum_0^{n-1}(-1)^m K(k)^m\Big)\frac{2\hat{w}(\lambda)}{M}$$

we have that

$$\Big(I + K(k)\Big)\hat{A}_n = (I + (-1)^n K(k)^n)\frac{2}{M}\hat{w}(\lambda)$$

where

$$||K(k)^n \cdot \frac{2}{M}\hat{w}(\lambda)|| \to 0 \text{ as } \operatorname{Re} k \to \infty, \text{ in any subsector.}$$

Hence $\hat{A}_n$ yields the nth-order approximation to the solution for Re k large enough.

This result is useful also for the case $\hat{w}(\lambda) =$ nonzero constant.

General solution. Getting back now to (5.3) written as

$$\frac{2\mathcal{T}\hat{w}(\lambda)}{\sqrt{1-M^2}} = A - k\big(V(k) - W(k)\big)A$$

we note that

$$k\big(V(k) - W(k)\big)$$

is compact on $L_p(-b,b)$ into $L_p(-b,b)$ and hence for existence and uniqueness of solution we only need to prove that 1 is *not* in the Point Spectrum of $k\big(V(k) - W(k)\big)$ or equivalently of $(I - kV(k))^{-1}W(k)$. Then we need to prove that $(I - kV(k) - W(k))^{-1}$ is the Laplace Transform of a time domain function. This is an open problem.

We can go a bit further if we wish to imitate the successful procedure in Section 4. To take advantage of the special nature of the operator $W(k)$, let $\wedge$ denote the set

$$\wedge = \{(-\alpha_2, \alpha_1)U(1)\}$$

and redefine the generalized function:

$$a(s) = \frac{1}{\sqrt{1-M^2}}\delta(s-1) + \frac{1}{\pi}\frac{\sqrt{(\alpha_1 - s)(\alpha_2 + s)}}{1-s}$$

and

$$\begin{aligned} g(ks) &= g_-(ks, \cdot), \quad 0 < s < \alpha_1 \\ g(ks) &= g_+(-ks, \cdot) + j(-ks, \cdot), \quad -\alpha_2 < s < 0 \\ L(ks, A) &= L_-(ks, A), \quad 0 < s < \alpha_1 \\ &= L_+(-ks, A), \quad -\alpha_2 < s < 0 \end{aligned}$$

so that we can express

$$W(k)A = \int_\wedge g(ks)L(ks, A)a(s)ds,$$

obtaining

$$\frac{2\mathcal{T}\hat{w}(\lambda)}{\sqrt{1-M^2}} = A - k\big(V(k)A\big) + \int_\wedge g(ks)L(ks, A)a(s)ds.$$

Using the notations we have

$$\begin{aligned} h(ks) &= (I - kV(k))^{-1}g(ks) \\ \hat{\nu}(\lambda) &= \frac{2}{\sqrt{1-M^2}}(I - kV(k))^{-1}\mathcal{T}\hat{w}(\lambda) \\ \hat{\nu}(\lambda) &= A + k\int_\wedge h(ks)L(ks, A)a(s)ds. \end{aligned}$$

Then, imitating the procedure in Section 4, we can take the linear functional on both sides,

$$\int_\wedge L(ks, \hat{\nu}(\lambda))ds = F(ks) + k\int_\wedge L(ks, h(\sigma))F(k\sigma)a(\sigma)d\sigma, \tag{5.5}$$

yielding an integral equation for the function

$$F(ks), \quad s \in \wedge.$$

We need to prove existence and uniqueness of solution and then follow with a generalization of the Sears result in Section 4 for $M = 0$.

6. Generalization

In this section we indicate two generalizations of the Possio Equation that arise in AeroElasticity. These are not just mathematical generalizations!

6.1. Generalization to nonZero angle of attack

Given angle $\alpha,\ 0 < \alpha < \pi/4$

$$\hat{w}(\lambda, x) = \int_{-b}^{b} \hat{P}(\lambda, x-\xi)\hat{A}(\lambda, \xi)d\xi, \quad \operatorname{Re}\lambda > 0$$

as before, but now

$$\begin{aligned}&\int_{-\infty}^{\infty} e^{-i\omega x}\hat{P}(\lambda, x)dx \\ &\quad = \frac{1}{2}\frac{1}{(k+iw\cos\alpha)}\frac{k^2M^2 + 2M^2ki\omega\cos\alpha + \omega^2(1-M^2\cos^2\alpha)}{\sqrt{k^2M^2 + 2M^2ki\omega\cos\alpha + \omega^2(1-M^2)}}.\end{aligned} \tag{6.1}$$

Remark. For $k = 0$, (6.1) becomes

$$\frac{1}{2}\frac{|\omega|}{i\omega}\frac{(1-M^2\cos^2\alpha)}{\sqrt{1-M^2}}\sec\alpha \tag{6.2}$$

leading to the dichotomy ('Transonic Dip'; see [12]):

$$\begin{aligned}\text{as } M \to 1\ (6.2) &\to\ \ 0 \text{ for } \alpha = 0 \\ (6.2) &\to \infty \text{ for } \alpha \neq 0.\end{aligned}$$

6.2. Generalization to 2 dimensions

Generalization in a different direction is the extension to 2 dimensions. Let

$$\Omega = [\xi, \eta\ (-b < \xi < b;\ 0 < \eta < \ell)]. \tag{6.3}$$

The (3D Possio) equation is:

$$\hat{w}(\lambda, x, y) = \int_{\Omega} \hat{P}(\lambda, x-\xi, y-\eta)A(\lambda, s, \eta)d\xi d\eta, \quad x, y \in \Omega \tag{6.4}$$

where the $L_p - L_q$ transform

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\omega_1 x - i\omega_2 y}\hat{P}(\lambda, x, y)dxdy, \quad -\infty < \omega \\ &\quad = \frac{\sqrt{M^2k^2 + 2M^2ki\omega_1 + (1-M^2)\omega_1^2 + \omega_2^2}}{(k+i\omega_1)} \\ &A(\lambda, x, y) \to 0 \,\text{as}\, x \to b- .\end{aligned} \tag{6.5}$$

We note that for $k = 0$, (6.5) becomes

$$\frac{\sqrt{\omega_1^2(1-M^2) + \omega_2^2}}{i\omega_1}, \quad -\infty < \omega_1, \omega_2 < \infty.$$

Unfortunately nothing is known about this equation (for details, see [13]).

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Eigenvalue Asymptotics Under a Non-dissipative Eigenvalue Dependent Boundary Condition for Second-order Elliptic Operators

Joachim von Below and Gilles François

Abstract. The asymptotic behavior of the eigenvalue sequence of the eigenvalue problem

$$-\Delta\varphi + q(x)\varphi = \lambda\varphi$$

in a bounded Lipschitz domain $D \subset \mathbb{R}^N$ under the eigenvalue dependent boundary condition

$$\varphi_n = \sigma\lambda\varphi$$

with a continuous function σ is investigated in the case $\sigma^- \not\equiv 0$, the dissipative one $\sigma \geq 0$ having been settled in [6]. For $N = 1$ the eigenvalues grow like k^2 with leading asymptotic coefficient equal to the Weyl constant. For $N \geq 2$ the positive eigenvalues grow like $k^{2/N}$, while the negative eigenvalues grow in absolute value like $|k|^{1/(N-1)}$. Moreover, asymptotic bounds in dependence on the dynamical coefficient function σ are derived, firstly in the constant case, secondly for σ of constant sign, and finally for a function σ changing sign.

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1. Introduction

Let $D \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary ∂D, and let n denote its outer normal vector field. Let $q(x)$ be a nonnegative bounded function and σ be an arbitrary real number or a continuous function defined on ∂D. In this paper we investigate the asymptotic behavior of the eigenvalues of the eigenvalue problem

$$\begin{cases} -\Delta\varphi + q(x)\varphi = \lambda\varphi & \text{in } D, \\ \varphi_n = \sigma\lambda\varphi & \text{on } \partial D. \end{cases} \tag{1.1}$$

The *dissipative* case $\sigma \geq 0$ has to be distinguished from the non-dissipative one $\sigma^- \not\equiv 0$. The λ-dependent boundary condition stems, e.g., from parabolic problems under *dynamical* time lateral boundary conditions $\sigma\partial_t v + v_n = 0$. The consideration of these boundary conditions, including a coefficient σ changing sign, is highly motivated by various applications in control theory, conductor physics, chemical kinetics a.m.o. They lead to greater flexibility in problems involving mixed boundary conditions with energy decay or gain on the boundary, especially in ramified structures, see, e.g., [4], [7], [14] and [16] and the references therein.

By a compact resolvent argument and extremal variational principles, it has been shown in [1], [2], [5], [6] and [12] that the eigenvalues of (1.1) are real and form an increasing sequence $\Lambda = (\lambda_k)_{k\in I}$ with $\mathbb{N}^* \subset I \subset \mathbb{Z}$ and

$$\lim_{k\to+\infty} \lambda_k = +\infty.$$

In the dissipative case, $\Lambda \geq 0$, while for $\sigma^- \not\equiv 0$, negative eigenvalues exist, finitely many for $N = 1$, and with an index set satisfying $I \supset -\mathbb{N}^*$ for $N \geq 2$. Moreover, in the latter case

$$\lim_{k\to-\infty} \lambda_k = -\infty.$$

Throughout we adopt here the index convention that a negative (resp. positive) index will stand for a negative (resp. positive) eigenvalue. If zero is an eigenvalue then it will be denoted by λ_0. Moreover, a Hilbert basis of eigenfunctions in $H^1(D)$ exists except in the *resonance case* $|D| + \int_{\partial D} \sigma\, ds = 0$ where it has to be supplemented by an additional element, see [2] for more details. The Rayleigh quotient corresponding to Problem (1.1) reads

$$\mathcal{R}(u;\sigma) = \frac{\displaystyle\int_D |\nabla u|^2 dx + \int_D q|u|^2\, dx}{\displaystyle\int_D |u|^2\, dx + \int_{\partial D} \sigma|u|^2\, ds},$$

which is not positive definite for negative σ. Since $q \in L^\infty(D)$, the asymptotic eigenvalue behavior is the same as in the case $q \equiv 0$ by extremal variational principles. Thus, for the asymptotic analysis it suffices to consider the case of a vanishing potential term. By the same argument it follows that the asymptotic results presented here remain valid for any $q \in L^\infty(D)$ taking also negative values. To overcome the difficulty of the indefinite denominator, we treat the asymptotic behavior of the positive and the negative eigenvalues separately and decompose $\mathcal{R}(u;\sigma)^{-1}$ in the latter case, up to some technical refinements, into the inverse Rayleigh quotients stemming from the Steklov problem and from Problem (1.1) under the Neumann boundary condition, i.e., $\sigma \equiv 0$:

$$\mathcal{R}(u;\sigma)^{-1} = \sigma\frac{\displaystyle\int_{\partial D} |u|^2\, ds}{\displaystyle\int_D |\nabla u|^2\, dx} + \frac{\displaystyle\int_D |u|^2\, dx}{\displaystyle\int_D |\nabla u|^2\, dx}.$$

For $N = 1$, the eigenvalues grow like k^2 in all the cases with leading asymptotic coefficient equal to the Weyl constant, see [6] and Sections 2 and 4 below. For $N \geq 2$, the eigenvalues for a positive continuous function σ grow also like $k^{1/(N-1)}$. This is part of the following result in the dissipative case using the Weyl and the Steklov constants of the domain.

Theorem 1. ([6]) *Let σ be a positive continuous function. Then the eigenvalue sequence $(\lambda_k)_{k\in I}$ of Problem* (1.1) *satisfies*

$$\frac{1}{2^{1/(N-1)}} \frac{C_{\mathrm{Stek}}(D)}{\max\limits_{\partial D} \sigma} \leq \liminf_{k\to\infty} \frac{\lambda_k}{k^{1/(N-1)}} \leq \limsup_{k\to\infty} \frac{\lambda_k}{k^{1/(N-1)}} \leq \frac{C_{\mathrm{Stek}}(D)}{\min\limits_{\partial D} \sigma}$$

for $N \geq 3$ and

$$\begin{aligned} \frac{1}{2} \frac{C_{\mathrm{Weyl}}(D)\, C_{\mathrm{Stek}}(D)}{C_{\mathrm{Weyl}}(D) \max\limits_{\partial\Omega} \sigma + C_{\mathrm{Stek}}(D)} &\leq \liminf_{k\to\infty} \frac{\lambda_k}{k} \leq \limsup_{k\to\infty} \frac{\lambda_k}{k} \\ &\leq \min\left\{ C_{\mathrm{Weyl}}(D), \frac{C_{\mathrm{Stek}}(D)}{\min\limits_{\partial D} \sigma} \right\} \end{aligned}$$

for $N = 2$.

Note that in [6], Theorem 1 was shown for $q \equiv 0$, but by the aforementioned zero potential reduction it holds also in the present case. For further references and related topics we refer to [1]–[6] and [13].

The present contribution deals with the asymptotic behavior of the eigenvalues in the non-dissipative case and is organized as follows. In Section 2 it is shown that for constant negative coefficient σ, the positive eigenvalues of (1.1) grow like $k^{2/N}$, which is in contrast to the dissipative case $\sigma > 0$ for $N \geq 2$. Moreover, it is shown that the negative eigenvalues grow in absolute value like $|k|^{1/(N-1)}$ for $N \geq 2$ with asymptotic bounds similar to the ones (Theorem 1) of the (positive) eigenvalues in the dissipative case. In Section 3 the results are readily generalized to the case of negative continuous σ. The final Section 4 is devoted to dynamical coefficients changing sign. It turns out that mutatis mutandis the asymptotic behavior splits into different growth orders as above, but with asymptotic bounds involving constants depending on the domain and σ as the ones of the Steklov problem with a coefficient changing sign that has been investigated by Sandgren [15].

Remark 1. *For the sake of simplicity we treat only differential operators of the form $\varphi \mapsto -\Delta\varphi + q(x)\varphi$ in this contribution. The results presented here for negative σ can be generalized to symmetric uniformly elliptic principal parts of the form $-\sum_{i,j} \partial_i(a^{ij}(x)\partial_j\varphi)$. In the dissipative case this has been done in* [5] *and* [13].

2. Eigenvalue asymptotics for constant dynamical coefficient

The aim of this section is to determine the growth order of the eigenvalue sequence of Problem (1.1) if σ is constant. Since the case $\sigma \geq 0$ is well known [6], we assume here that

$$\sigma = \text{ const. } < 0.$$

With the reduction $q \equiv 0$ we can confine ourselves to the problem

$$-\Delta\varphi = \lambda\varphi \text{ in } D, \quad \varphi_n = \sigma\lambda\varphi \text{ on } \partial D. \tag{2.1}$$

whose corresponding Rayleigh quotient reads

$$\mathcal{R}(u;\sigma) = \frac{\displaystyle\int_D |\nabla u|^2\,dx}{\displaystyle\int_D |u|^2\,dx + \sigma\int_{\partial D} |u|^2\,ds}.$$

The positive eigenvalues of (2.1) and of (1.1) grow like $k^{2/N}$ and behave like those under Dirichlet or Neumann boundary conditions. This is part of the following

Theorem 2. *Suppose $\sigma < 0$. Then the sequence $(\lambda_k)_{k\in\mathbb{N}}$ of nonnegative eigenvalues of Problem* (2.1) *satisfies*

$$\alpha_k \leq \lambda_k \leq \omega_k \qquad \text{for all } k \in \mathbb{N},$$

where $(\alpha_k)_{k\in\mathbb{N}}$ denotes the eigenvalue sequence of the Laplacian in $H^1(D)$ under the Neumann boundary condition and $(\omega_k)_{k\in\mathbb{N}}$ the eigenvalue sequence of the Laplacian in $H_0^1(D)$. Thus, the positive part of the eigenvalue sequence of Problem (1.1) *satisfies*

$$\lim_{k\to+\infty} \frac{\lambda_k}{k^{2/N}} = C_{\text{Weyl}}(D),$$

where $C_{\text{Weyl}}(D) = \frac{4\pi^2}{\left(v_N|D|\right)^{2/N}}$ denotes the Weyl constant of the domain D and v_N the volume of the Euclidean unit ball in $\mathbb{R}^N$.

Proof. If $\mathcal{R}(u;\sigma) \geq 0$ then $\mathcal{R}(u;\sigma) \geq \mathcal{R}(u;0)$, which leads to $\alpha_k \leq \lambda_k$. For the other inequality recall that by the Courant–Fischer max-min-principle

$$\lambda_k = \min_{E\in\mathcal{H}_{k+1}} \max_{u\in E\setminus\{0\}} \frac{\displaystyle\int_D |\nabla u|^2\,dx}{\displaystyle\int_D |u|^2\,dx + \int_{\partial D} \sigma|u|^2\,ds},$$

where $\mathcal{H}_k$ denotes the class of the k-dimensional subspaces of $H^1(D)$. By restriction to those subspaces of $H_0^1(D)$ we deduce $\lambda_k \leq \omega_k$. Since $\lim_{k\to+\infty} \frac{\alpha_k}{k^{2/N}} = \lim_{k\to+\infty} \frac{\omega_k}{k^{2/N}} = C_{\text{Weyl}}(D)$ the assertion is shown. □

Thus, in contrast to the case $\sigma \geq 0$, for $\sigma < 0$ the leading asymptotic coefficient does not depend on σ and amounts to the classical Weyl constant. For higher dimensions the sequence of positive eigenvalues is thinned out when compared to the density in the dissipative case while negative eigenvalues occur of the latter density as will be shown next.

For $N = 1$ there are at most finitely many negative eigenvalues, see [2]. Therefore, the asymptotic behavior of the sequence $(\lambda_k)_{k\in-\mathbb{N}^*}$ of negative eigenvalues of Problem (2.1) is only of interest in the case $N \geq 2$. If $\mathcal{R}(u;\sigma) < 0$ then

$$\mathcal{R}(u;\sigma) \leq -\frac{1}{|\sigma|}\frac{\displaystyle\int_D |\nabla u|^2\,dx}{\displaystyle\int_{\partial D} |u|^2\,ds}.$$

Thus we conclude that

$$\lambda_k \leq -\frac{1}{|\sigma|}\mu_{|k|} \qquad \text{for all } k \in -\mathbb{N}, \tag{2.2}$$

where $(\mu_i)_{i\in\mathbb{N}}$ denotes the eigenvalue sequence of the Steklov problem

$$\begin{cases} \Delta u = 0 & \text{in } D,\\ u_n = \mu u & \text{on } \partial D. \end{cases} \tag{2.3}$$

It is well known that

$$\lim_{k\to+\infty}\frac{\mu_k}{k^{1/(N-1)}} = C_{\mathrm{Stek}}(D) > 0$$

with the Steklov constant $C_{\mathrm{Stek}}(D)$ depending only on N and ∂D, see [15]. Thus we are led to the following

Theorem 3. *Suppose $\sigma < 0$. Then the sequence $(\lambda_k)_{k\in-\mathbb{N}^*}$ of negative eigenvalues of Problem* (2.1) *satisfies*

$$\limsup_{k\to-\infty}\frac{\lambda_k}{|k|^{1/(N-1)}} \leq \frac{C_{\mathrm{Stek}}(D)}{\sigma} < 0.$$

In fact, $(|\lambda_k|)_{k\in-\mathbb{N}^*}$ grows like $|k|^{1/(N-1)}$ for $N \geq 2$. This is the contents of the following theorems.

Theorem 4. *Suppose $\sigma < 0$ and $N \geq 3$. Then the sequence $(\lambda_k)_{k\in-\mathbb{N}^*}$ of negative eigenvalues of Problem* (2.1) *satisfies*

$$\liminf_{k\to-\infty}\frac{\lambda_k}{|k|^{1/(N-1)}} \geq 2^{1/(N-1)}\frac{C_{\mathrm{Stek}}(D)}{\sigma}.$$

Proof. Consider $\mathcal{Q}(u;\sigma) = -\mathcal{R}(u;\sigma)^{-1}$ in the orthogonal space of the constant functions $H^1(D)\ominus 1\mathbb{R}$. Then

$$|\sigma|\frac{\displaystyle\int_{\partial D}|u|^2\,ds}{\displaystyle\int_D |\nabla u|^2\,dx} = \mathcal{Q}(u;\sigma) + \frac{\displaystyle\int_D |u|^2\,dx}{\displaystyle\int_D |\nabla u|^2\,dx}. \tag{2.4}$$

As the variation of the three quadratic forms involved in (2.4) takes place in the same space $H^1(D) \ominus 1\mathbb{R}$, the positive eigenvalues of the corresponding compact selfadjoint operators satisfy

$$\forall k, m \in \mathbb{N}\backslash\{0\}: \qquad \frac{|\sigma|}{\mu_{k+m-1}} \leq \frac{1}{-\lambda_{-k}} + \frac{1}{\alpha_m}$$

by a well-known spectral estimate for sums of compact hermitian operators, see, e.g., [9], p. 925. For $N \geq 3$ and for $k > 0$ sufficiently large, this yields

$$\lambda_{-k} \geq \frac{\mu_{2k-1}}{\sigma} \frac{1}{\frac{\mu_{2k-1}}{\sigma\alpha_k} + 1}.$$

Moreover, in connection with

$$\lim_{k\to\infty} \mu_{2k-1}\alpha_k^{-1} = 0, \tag{2.5}$$

it follows that

$$\liminf_{k\to\infty} \frac{\lambda_{-k}}{k^{1/(N-1)}} \geq 2^{1/(N-1)} \frac{C_{\mathrm{Stek}}(D)}{\sigma}. \qquad \square$$

For $N = 2$ the growth order amounts to $|k|$, but the proof is more complicated and requires the following simple preparation.

Lemma 5. *Suppose that $u \in H^1(D)$, $\sigma \leq \tau < 0$, $\mathcal{R}(u;\sigma) < 0$ and $\mathcal{R}(u;\tau) < 0$. Then $\mathcal{R}(u;\tau) \leq \mathcal{R}(u;\sigma)$.*

This applies especially to $\sigma \leq -\left\lceil \frac{1}{|\sigma|} \right\rceil^{-1}$, where $\lceil r \rceil$ denotes the smallest natural number bigger or equal to $r \in [0,\infty)$. Now we can state the

Theorem 6. *Suppose $\sigma < 0$ and $N = 2$. Then the sequence $(\lambda_k)_{k\in-\mathbb{N}^*}$ of negative eigenvalues of Problem (2.1) satisfies*

$$\liminf_{k\to-\infty} \frac{\lambda_k}{|k|} \geq C(\sigma, D) > -\infty$$

with a negative constant

$$C(\sigma, D) = \begin{cases} \frac{2C_{\mathrm{Weyl}}(D)\, C_{\mathrm{Stek}}(D)}{\sigma C_{\mathrm{Weyl}}(D) + 2C_{\mathrm{Stek}}(D)} & \text{for } \sigma < \sigma_1, \\ \\ C_{\mathrm{Stek}}(D)\sigma_2 \left\lceil \frac{\sigma_2}{\sigma} \right\rceil & \text{for } \sigma_1 \leq \sigma < 0, \end{cases} \tag{2.6}$$

where

$$\sigma_1 := -\frac{2C_{\mathrm{Stek}}(D)}{C_{\mathrm{Weyl}}(D)} \quad \text{and} \quad \sigma_2 := \min\left\{-2, -\frac{C_{\mathrm{Stek}}(D)}{C_{\mathrm{Weyl}}(D)} - \sqrt{\frac{C_{\mathrm{Stek}}^2(D)}{C_{\mathrm{Weyl}}^2(D)} + 2}\right\}.$$

Proof. For $N = 2$, Formula (2.5) reads

$$\lim_{k\to\infty} \mu_{2k-1}\alpha_k^{-1} = 2\frac{C_{\mathrm{Stek}}(D)}{C_{\mathrm{Weyl}}(D)}.$$

Thus, for $\sigma < \sigma_1$ we can follow the proof of Theorem 4 and can choose

$$C(\sigma, D) = 2\,\frac{C_{\mathrm{Weyl}}(D)\,C_{\mathrm{Stek}}(D)}{\sigma C_{\mathrm{Weyl}}(D) + 2C_{\mathrm{Stek}}(D)}.$$

For the remaining case $\sigma_1 \le \sigma < 0$, by Lemma 5 it suffices to show the assertion for all coefficients of the form

$$\sigma = \frac{1}{M}(\sigma_1 - \varepsilon), \quad M \in \mathbb{N}^*$$

with some fixed

$$\varepsilon \ge \max\left\{2 + \sigma_1, \frac{\sigma_1}{2} + \sqrt{\frac{\sigma_1^2}{4} + 2}\right\}. \tag{2.7}$$

For $M = 1$ the case shown applies to $\sigma_1 - \varepsilon$ and yields

$$\liminf_{k\to\infty} \frac{\lambda_{-k}[\sigma_1 - \varepsilon]}{k} \ge -2\frac{C_{\mathrm{Stek}}(D)}{\varepsilon} \ge C_{\mathrm{Stek}}(D)(\sigma_1 - \varepsilon),$$

where the brackets indicate the dependence on the dynamical coefficient. For $M \ge 2$, suppose for an induction argument that

$$\liminf_{k\to\infty} \frac{\lambda_{-k}[\frac{\sigma_1-\varepsilon}{M}]}{k} \ge M\,C_{\mathrm{Stek}}(D)(\sigma_1 - \varepsilon). \tag{2.8}$$

With the notation $\mathcal{Q}(u;\sigma) = -\mathcal{R}(u;\sigma)^{-1}$ we get

$$\frac{\varepsilon + |\sigma_1|}{M(M+1)}\,\frac{\displaystyle\int_{\partial D} |u|^2\,ds}{\displaystyle\int_D |\nabla u|^2\,dx} + \mathcal{Q}\left(u; \frac{\sigma_1 - \varepsilon}{M+1}\right) = \mathcal{Q}\left(u; \frac{\sigma_1 - \varepsilon}{M}\right).$$

Using a similar argument for sums of compact operators associated to the involved quadratic forms, it follows as above that for k sufficiently large

$$\frac{1}{-\lambda_{-2k+1}[\frac{\sigma_1-\varepsilon}{M}]} \le \frac{1}{-\lambda_{-k}[\frac{\sigma_1-\varepsilon}{M+1}]} + \frac{\varepsilon + |\sigma_1|}{M(M+1)\,\mu_k},$$

and

$$\frac{\lambda_{-k}[\frac{\sigma_1-\varepsilon}{M+1}]}{k} \ge \frac{1}{\frac{k(\varepsilon+|\sigma_1|)}{M(M+1)\,\mu_k} + \frac{k}{\lambda_{-2k+1}[\frac{\sigma_1-\varepsilon}{M}]}}.$$

Thus, by (2.8),

$$\liminf_{k\to\infty} \frac{\lambda_{-k}[\frac{\sigma_1-\varepsilon}{M+1}]}{k} \ge \frac{1}{\frac{\varepsilon+|\sigma_1|}{M(M+1)\,C_{\mathrm{Stek}}(D)} - \frac{\varepsilon+|\sigma_1|}{2M\,C_{\mathrm{Stek}}(D)}} = \frac{2M(M+1)\,C_{\mathrm{Stek}}(D)}{(\sigma_1 - \varepsilon)(M-1)}.$$

But $\frac{2M}{(\sigma_1-\varepsilon)(M-1)} \ge (\sigma_1 - \varepsilon)$ due to (2.7) which permits to conclude

$$\liminf_{k\to\infty} \frac{\lambda_{-k}[\frac{\sigma_1-\varepsilon}{M+1}]}{k} \ge (M+1)\,C_{\mathrm{Stek}}(D)(\sigma_1 - \varepsilon).$$

Thus, (2.8) holds for all $M \in \mathbb{N}^*$ and shows the assertion by Lemma 5 with the minimal choice for ε and $M = \lceil \frac{\sigma_2}{\sigma} \rceil$ bearing in mind that $\sigma_1 - \varepsilon = \sigma_2$ in that case. □

Corollary 7. *Suppose $\sigma < 0$. Then the sequence $(\lambda_k)_{k \in I}$ of eigenvalues of Problem* (1.1) *satisfies*

(i) $\lim\limits_{k \to +\infty} \dfrac{\lambda_k}{k^{2/N}} = C_{\text{Weyl}}(D)$,

(ii) $\limsup\limits_{k \to -\infty} \dfrac{\lambda_k}{|k|^{1/(N-1)}} \le \dfrac{C_{\text{Stek}}(D)}{\sigma} < 0$,

(iii) $\liminf\limits_{k \to -\infty} \dfrac{\lambda_k}{|k|^{1/(N-1)}} \ge 2^{1/(N-1)} \dfrac{C_{\text{Stek}}(D)}{\sigma}$, *for $N \ge 3$,*

(iv) $\liminf\limits_{k \to -\infty} \dfrac{\lambda_k}{|k|} \ge C(\sigma, D)$ *for $N = 2$ with the negative constant defined in* (2.6).

It should be noted that the aforementioned asymptotic bounds for negative σ seem not to be optimal yet.

Example 1. *For the unit disk D in $\mathbb{R}^2$ it is well known that*

$$\lim_{k \to \infty} \frac{\alpha_k}{k} = C_{\text{Weyl}}(D) = 4,$$

see, e.g., [8], *and that $C_{\text{Stek}}(D) = \frac{1}{2}$, see* [10]. *If $\sigma > 0$, then the eigenvalue sequence $(\lambda_k)_{k \in I}$ of Problem* (1.1) *satisfies*

$$\frac{2}{8\sigma + 1} \le \liminf_{k \to \infty} \frac{\lambda_k}{k} \le \limsup_{k \to \infty} \frac{\lambda_k}{k} \le \min \left\{ 4, \frac{1}{2\sigma} \right\}.$$

If $\sigma < 0$, then the eigenvalue sequence $(\lambda_k)_{k \in I}$ of Problem (1.1) *satisfies*

$$\lim_{k \to \infty} \frac{\lambda_k}{k} = 4 \quad \textit{and} \quad \frac{4}{4\sigma + 1} \le \liminf_{k \to \infty} \frac{\lambda_k}{k} \le \limsup_{k \to \infty} \frac{\lambda_k}{k} \le \frac{1}{2\sigma}$$

for $\sigma < -\frac{1}{4}$ and

$$-\left\lceil \frac{2}{|\sigma|} \right\rceil \le \liminf_{k \to \infty} \frac{\lambda_k}{k} \le \limsup_{k \to \infty} \frac{\lambda_k}{k} \le \frac{1}{2\sigma}$$

for $-\frac{1}{4} \le \sigma < 0$.

3. Dynamical coefficient of constant sign

The results of Section 2 can easily be generalized to the case of a negative continuous function σ by using Lemma 5. Omitting the details we are led to the

Theorem 8. *Let $N \ge 2$ and σ be a continuous negative function. Then the sequence $(\lambda_k)_{k \in I}$ of eigenvalues of Problem* (1.1) *satisfies*

(i) $\lim\limits_{k \to +\infty} \dfrac{\lambda_k}{k^{2/N}} = C_{\text{Weyl}}(D)$,

(ii) $\limsup_{k\to-\infty} \dfrac{\lambda_k}{|k|^{1/(N-1)}} \leq \dfrac{C_{\mathrm{Stek}}(D)}{\min\limits_{\partial\Omega} \sigma} < 0,$

(iii) $\liminf_{k\to-\infty} \dfrac{\lambda_k}{|k|^{1/(N-1)}} \geq 2^{1/(N-1)} \dfrac{C_{\mathrm{Stek}}(D)}{\max\limits_{\partial\Omega} \sigma}$, *for* $N \geq 3$,

(iv) $\liminf_{k\to-\infty} \dfrac{\lambda_k}{|k|} \geq C(\max\limits_{\partial\Omega} \sigma, D)$ *for* $N = 2$ *with the negative constant* $C(\max_{\partial\Omega} \sigma, D)$ *as defined in* (2.6).

4. Dynamical coefficient changing sign

In this section, we suppose that σ is a continuous function on ∂D with a sign change. To be more specific, introduce the decomposition

$$\partial D = \delta^+ \uplus \delta_0 \uplus \delta^-$$

such that $\sigma > 0$ on δ^+, $\sigma = 0$ on δ_0 and $\sigma < 0$ on δ^-. In view of Section 3 and the results of [6], we can suppose that the positive part σ^+ and the negative part σ^- of σ have each a positive maximum, writing $\sigma = \sigma^+ - \sigma^-$.
In dimension 1, Problem (2.1) in the interval $J = (0, 1)$ reads

$$\begin{cases} -\varphi'' = \lambda\varphi & \text{in } J, \\ \varphi'(0) = -\sigma_0\lambda\varphi(0) \\ \varphi'(1) = \sigma_1\lambda\varphi(1) \end{cases} \tag{4.1}$$

where $\sigma_0\sigma_1 < 0$. Problem (4.1) has a finite number of negative eigenvalues, see [2]. The characteristic equation for the positive eigenvalues is given by

$$\tan\sqrt{\lambda} = \frac{(\sigma_0 + \sigma_1)\sqrt{\lambda}}{\sigma_0\sigma_1\lambda - 1}$$

from which we deduce

$$\lim_{k\to+\infty} \frac{\lambda_k}{k^2} = \pi^2 = C_{\mathrm{Weyl}}(J).$$

In the particular case $\sigma_0 + \sigma_1 = 0$, the sequence $(\lambda_k)_{k\in I}$ coincides with the eigenvalue sequence of the Neumann problem.

In higher dimension the situation is more complicated. The eigenvalues of Problem (2.1) will be related to the ones of following modified Steklov problem. Let $(\mu_i)_{i\in I}$ denote the eigenvalue sequence of the problem

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u_n = \sigma\mu u & \text{on } \partial D. \end{cases} \tag{4.2}$$

It is well known [15] that

$$\lim_{i\to+\infty} \frac{\mu_i}{i^{1/(N-1)}} = C^+_{\mathrm{Sand}}(D, \sigma) > 0 \quad\text{and}\quad \lim_{i\to-\infty} \frac{\mu_i}{|i|^{1/(N-1)}} = C^-_{\mathrm{Sand}}(D, \sigma) < 0$$

with the Sandgren constants $C^+_{\text{Sand}}(D,\sigma)$ and $C^-_{\text{Sand}}(D,\sigma)$ depending on N and ∂D, but on σ too. We note in passing that a comparison of the corresponding Rayleigh quotient yields

$$C^+_{\text{Sand}}(D,\sigma) \geq \frac{C_{\text{Stek}}(D)}{\max\limits_{\partial D} \sigma} \quad \text{and} \quad C^-_{\text{Sand}}(D,\sigma) \leq \frac{C_{\text{Stek}}(D)}{\min\limits_{\partial D} \sigma} = -\frac{C_{\text{Stek}}(D)}{\max\limits_{\partial D} \sigma^-}. \tag{4.3}$$

For Problem (2.1), the Rayleigh quotient in question reads here

$$\mathcal{R}(u;\sigma) = \frac{\displaystyle\int_D |\nabla u|^2\,dx}{\displaystyle\int_D |u|^2\,dx + \int_{\partial D} \sigma |u|^2\,ds}.$$

An upper bound for the positive eigenvalues of Problem (1.1) can be obtained by using the Dirichlet eigenvalues $(\omega_i)_{i\in\mathbb{N}^*}$ in $H^1_0(D)$.

Theorem 9. *Under the aforementioned conditions on σ, the sequence $(\lambda_k)_{k\in\mathbb{N}^*}$ of positive eigenvalues of Problem (2.1) satisfies*

$$\lambda_k \leq \omega_k \quad \textit{for all } k \in \mathbb{N}^*.$$

Thus, the positive part of the eigenvalue sequence of Problem (1.1) satisfies

$$\limsup_{k\to+\infty} \frac{\lambda_k}{k^{2/N}} \leq C_{\text{Weyl}}(D).$$

Proof. As in [6], the min-max-principle applies

$$\lambda_k = \min_{E\in\mathcal{H}_{k+1}} \max_{u\in E\setminus\{0\}} \frac{\displaystyle\int_D |\nabla u|^2\,dx}{\displaystyle\int_D |u|^2\,dx + \int_{\partial D} \sigma|u|^2\,ds},$$

and a restriction to the k-dimensional subspaces of $H^1_0(D)$ accomplishes the proof. □

Before proving a result about a lower bound for the positive eigenvalue sequence $(\lambda_k)_{k\in I}$, we need the following result that generalizes the lower bound obtained in the dissipative case in [6].

Theorem 10. *Under the aforementioned conditions on σ, the sequence $(\lambda_k[\sigma^+])_{k\in\mathbb{N}^*}$ satisfies*

$$\liminf_{k\to+\infty} \frac{\lambda_k[\sigma^+]}{k^{1/(N-1)}} \geq C_{WS}(D) \tag{4.4}$$

with a positive constant

$$C_{WS}(D) = \begin{cases} \frac{C^+_{\text{Sand}}(D,\sigma)}{2^{1/(N-1)}} & \textit{for } N \geq 3, \\ \frac{1}{2}\frac{C_{\text{Weyl}}(D)\,C^+_{\text{Sand}}(D,\sigma)}{C_{\text{Weyl}}(D)+C^+_{\text{Sand}}(D,\sigma)} & \textit{for } N = 2. \end{cases}$$

Proof. The spectral estimate for sums of compact operators already used yields

$$\forall k, m \in \mathbb{N}\backslash\{0\}: \quad \frac{1}{\lambda_{k+m}} \leq \frac{1}{\alpha_k} + \frac{1}{\mu_m[\sigma^+]}.$$

Thus, for all $k > 0$

$$\frac{\lambda_{2k}[\sigma^+]}{(2k)^{1/(N-1)}} \geq \frac{1}{2^{1/(N-1)}} \frac{\mu_k[\sigma^+]}{k^{1/(N-1)}} \frac{1}{1 + \mu_k[\sigma^+]\alpha_k^{-1}}$$

and

$$\frac{\lambda_{2k+1}[\sigma^+]}{(2k+1)^{1/(N-1)}} \geq \frac{1}{\left(2 + \frac{1}{k}\right)^{1/(N-1)}} \frac{\mu_k[\sigma^+]}{k^{1/(N-1)}} \frac{1}{1 + \mu_k[\sigma^+]\alpha_{k+1}^{-1}}.$$

For $N \geq 3$, both *r.h.s.* have the same limit $\frac{1}{2^{1/(N-1)}} C^+_{\mathrm{Sand}}(D, \sigma)$, since

$$\lim_{k \to +\infty} \mu_k[\sigma^+]\alpha_k^{-1} = \lim_{k \to +\infty} \mu_k[\sigma^+]\alpha_{k+1}^{-1} = 0. \tag{4.5}$$

As

$$\liminf_{k \to +\infty} \frac{\lambda_{2k}[\sigma^+]}{(2k)^{1/(N-1)}}, \liminf_{k \to +\infty} \frac{\lambda_{2k+1}[\sigma^+]}{(2k+1)^{1/(N-1)}} \geq \frac{1}{2^{1/(N-1)}} C^+_{\mathrm{Sand}}(D, \sigma),$$

and both sequences cover the eigenvalue sequence $(\lambda_k)_{k \in I}$, the first part of the theorem is shown.
For $N = 2$, Formula (4.5) reads

$$\lim_{k \to +\infty} \mu_k[\sigma^+]\alpha_k^{-1} = \lim_{k \to +\infty} \mu_k[\sigma^+]\alpha_{k+1}^{-1} = \frac{C^+_{\mathrm{Sand}}(D, \sigma)}{C_{\mathrm{Weyl}}(D)}$$

and it suffices to follow the proof of the first part by modifying the cofactor in the above formulae correspondingly. □

The following theorem concerns a lower bound for the positive eigenvalues $(\lambda_k)_{k \in \mathbb{N}^*}$. Since the function σ has a positive part, the result cannot be as sharp as Theorem 2.

Theorem 11. *Under the aforementioned condition on σ, the sequence $(\lambda_k)_{k \in \mathbb{N}^*}$ of positive eigenvalues of Problem* (1.1) *satisfies Formula* (4.4).

Proof. If $\mathcal{R}(u; \sigma) > 0$ then

$$\mathcal{R}(u; \sigma) \geq \frac{\displaystyle\int_D |\nabla u|^2 \, dx}{\displaystyle\int_D |u|^2 \, dx + \int_{\partial D} \sigma^+ |u|^2 \, ds}$$

which yields

$$\lambda_k \geq \lambda_k[\sigma^+] \quad \text{for all } k \in I.$$

Formula (4.4) being true for the eigenvalue sequence $(\lambda_k[\sigma^+])_{k \in \mathbb{N}^*}$, by Theorem 10, the assertion is proved. □

For the negative eigenvalues of Problem (1.1), we observe that the inequality $\mathcal{R}(u;\sigma) < 0$ implies $\int_{\partial D} \sigma |u|^2\, ds < 0$ and

$$\mathcal{R}(u;\sigma) \leq \frac{\displaystyle\int_D |\nabla u|^2\, dx}{\displaystyle\int_{\partial D} \sigma |u|^2\, ds}.$$

This yields immediately the

Theorem 12. *Under the aforementioned condition on σ and for $N \geq 2$, the sequence $(\lambda_k)_{k\in -\mathbb{N}^*}$ of negative eigenvalues of Problem (2.1) satisfies*

$$\lambda_k \leq \mu_k \quad \textit{for all } k \in -\mathbb{N}^*.$$

Thus, the sequence $(\lambda_k)_{k\in -\mathbb{N}^*}$ of negative eigenvalues of Problem (1.1) satisfies

$$\limsup_{k\to -\infty} \frac{\lambda_k}{|k|^{1/(N-1)}} \leq C^-_{\mathrm{Sand}}(D,\sigma) < 0.$$

The following theorem concerns an asymptotic lower bound for the negative eigenvalues.

Theorem 13. *Under the aforementioned condition on σ and for $N \geq 3$, the sequence $(\lambda_k)_{k\in -\mathbb{N}^*}$ of negative eigenvalues of Problem (1.1) satisfies*

$$\liminf_{k\to -\infty} \frac{\lambda_k}{|k|^{1/(N-1)}} \geq -2^{1/(N-1)} C^+_{\mathrm{Sand}}(D,\sigma^-).$$

Proof. By using the decomposition

$$\frac{\displaystyle\int_{\partial D} \sigma^- |u|^2\, ds}{\displaystyle\int_D |\nabla u|^2\, dx} = \mathcal{Q}(u;\sigma) + \frac{\displaystyle\int_D |u|^2\, dx + \int_{\partial D} \sigma^+ |u|^2\, ds}{\displaystyle\int_D |\nabla u|^2\, dx}$$

in the orthogonal space of the constant functions, we are led to the spectral inequality

$$\forall k, m \in \mathbb{N}\backslash\{0\}: \quad \frac{1}{\mu_{k+m}[\sigma^-]} \leq \frac{1}{-\lambda_{-k}} + \frac{1}{\lambda_m[\sigma^+]}.$$

Therefore, we obtain for $k > 0$ sufficiently large

$$\lambda_{-k} \geq \mu_{2k}[\sigma^-] \frac{1}{\frac{\mu_{2k}[\sigma^-]}{\lambda_k[\sigma^+]} - 1}. \tag{4.6}$$

Since

$$\lim_{k\to\infty} \frac{\mu_{2k}[\sigma^-]}{\lambda_k[\sigma^+]} = 0, \tag{4.7}$$

Inequality (4.6) implies

$$\liminf_{k\to -\infty} \frac{\lambda_k}{|k|^{1/(N-1)}} \geq -2^{1/(N-1)} C^+_{\mathrm{Sand}}(D,\sigma^-). \qquad \square$$

For $N = 2$ the limit in (4.7) does not vanish, since

$$
\begin{aligned}
0 < \frac{2C^+_{\mathrm{Sand}}(D,\sigma^-)}{C_{WS}(D)} &\le \liminf_{k\to\infty} \frac{\mu_{2k}[\sigma^-]}{\lambda_k[\sigma^+]} \\
&\le \limsup_{k\to\infty} \frac{\mu_{2k}[\sigma^-]}{\lambda_k[\sigma^+]} \le \frac{2C^+_{\mathrm{Sand}}(D,\sigma^-)}{C_{\mathrm{Weyl}}(D)}.
\end{aligned} \tag{4.8}
$$

Thus, an argument using Inequality (4.6) must take into account the value of the superior limit in (4.8), which is done in the next result in the form of Condition (4.9).

Theorem 14. *Suppose $N = 2$ and that*

$$C^+_{\mathrm{Sand}}(D,\sigma^-) < \frac{C_{\mathrm{Weyl}}(D)}{2}. \tag{4.9}$$

Then, under the aforementioned conditions on σ, the sequence $(\lambda_k)_{k\in-\mathbb{N}^*}$ of negative eigenvalues of Problem (1.1) *satisfies*

$$\liminf_{k\to-\infty} \frac{\lambda_k}{|k|} \ge C_2(\sigma^-, D)$$

with a negative constant

$$C_2(\sigma^-, D) = 2\,\frac{C^+_{\mathrm{Sand}}(D,\sigma^-)\,C_{\mathrm{Weyl}}(D)}{2C^+_{\mathrm{Sand}}(D,\sigma^-) - C_{\mathrm{Weyl}}(D)}\,.$$

Proof. Inequalities (4.8) and (4.9) imply that

$$\limsup_{k\to\infty} \frac{\mu_{2k}[\sigma^-]}{\lambda_k[\sigma^+]} < 1.$$

Then we can follow the proof of Theorem 12 and conclude with (4.6) that

$$\liminf_{k\to-\infty} \frac{\lambda_k}{|k|} \ge 2C^+_{\mathrm{Sand}}(D,\sigma^-) \frac{1}{\frac{2C^+_{\mathrm{Sand}}(D,\sigma^-)}{C_{\mathrm{Weyl}}(D)} - 1}\,. \qquad \square$$

Note that by (4.3), Condition (4.9) corresponds to the constant case condition $\sigma < \sigma_1$. The case $C^+_{\mathrm{Sand}}(D,\sigma^-) \ge \frac{C_{\mathrm{Weyl}}(D)}{2}$ corresponds to small values σ^- and is much more complicated. It turns out that the geometry of the domain $D \subset \mathbb{R}^2$ can play a crucial role in that case. For instance, for simply connected domains, a technique analogous to the induction argument in the proof of Theorem 6 for small negative constant dynamical coefficients can never work for a continuous function σ under the hypotheses of this section. But for doubly connected domains D with a boundary decomposition $\partial D = \delta^+ \uplus \delta^-$ into two connected parts δ^+ and δ^-, Theorem 6 holds mutatis mutandis by comparison of the Rayleigh quotients, since the induction argument can be applied owing to Inequality (4.3). The same holds in n-fold connected domains, if σ has constant sign in each boundary component. But for general domains in $\mathbb{R}^2$, this technique does not seem to apply for small σ^-. We omit the details here and refer to a work in progress about this parameter gap for the dynamical coefficient function in two dimensions and discuss here only the case of an annulus.

Example 2. *Let* $A = \{x \in \mathbb{R}^2 \,|\, r^2 < x_1^2 + x_2^2 < R^2\}$ *denote the annulus in* $\mathbb{R}^2$ *with radii* $0 < r < R$*. Then*

$$C_{\mathrm{Weyl}}(A) = \frac{4}{R^2 - r^2}, \quad C_{\mathrm{Stek}}(A) = \frac{1}{2R + r}.$$

If σ *is a positive function, then the eigenvalue sequence* $(\lambda_k)_{k\in I}$ *of Problem* (1.1) *satisfies*

$$\frac{2}{R^2 - r^2 + (8R + 4r)\max\limits_{\partial A}\sigma} \le \liminf_{k\to\infty}\frac{\lambda_k}{k} \le \limsup_{k\to\infty}\frac{\lambda_k}{k}$$
$$\le \min\left\{\frac{4}{R^2 - r^2}, \frac{1}{(2R + r)\min\limits_{\partial A}\sigma}\right\}.$$

If

$$\delta^- = \{x \in \mathbb{R}^2 \,|\, x_1^2 + x_2^2 = r^2\} \quad \text{and} \quad \delta^+ = \{x \in \mathbb{R}^2 \,|\, x_1^2 + x_2^2 = R^2\},$$

then a comparison argument for the corresponding Rayleigh quotients using (4.3) *shows that the eigenvalue sequence* $(\lambda_k)_{k\in I}$ *of Problem* (1.1) *satisfies*

$$\frac{2}{R^2 - r^2 + (8R + 4r)\max\limits_{\partial A}\sigma} \le \liminf_{k\to+\infty}\frac{\lambda_k}{k} \le \limsup_{k\to+\infty}\frac{\lambda_k}{k} \le \frac{4}{R^2 - r^2}$$

and

$$\frac{4}{R^2 - r^2 - (4R + 2r)\max\limits_{\partial A}\sigma^-} \le \liminf_{k\to-\infty}\frac{\lambda_k}{|k|} \le \limsup_{k\to-\infty}\frac{\lambda_k}{|k|} \le \frac{1}{(2R + r)\min\limits_{\partial A}\sigma}$$

for $C^+_{\mathrm{Sand}}(A, \sigma^-) < \frac{2}{R^2 - r^2}$ *and*

$$-\frac{\sigma_2}{2R + r}\left\lceil\frac{\sigma_2}{\min\limits_{\partial A}\sigma^-}\right\rceil \le \liminf_{k\to-\infty}\frac{\lambda_k}{|k|} \le \limsup_{k\to-\infty}\frac{\lambda_k}{|k|} \le \frac{1}{(2R + r)\min\limits_{\partial A}\sigma}$$

for $C^+_{\mathrm{Sand}}(A, \sigma^-) \ge \frac{2}{R^2 - r^2}$ *, where*

$$\sigma_2 = \min\left\{-2, -\frac{R^2 - r^2}{8R + 4r} - \sqrt{\frac{(R^2 - r^2)^2}{16(2R + r)^2} + 2}\right\}.$$

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Feynman-Kac Formulas, Backward Stochastic Differential Equations and Markov Processes

Jan A. Van Casteren

This article is written in honor of G. Lumer
whom I consider as my semi-group teacher

Abstract. In this paper we explain the notion of stochastic backward differential equations and its relationship with classical (backward) parabolic differential equations of second order. The paper contains a mixture of stochastic processes like Markov processes and martingale theory and semi-linear partial differential equations of parabolic type. Some emphasis is put on the fact that the whole theory generalizes Feynman-Kac formulas. A new method of proof of the existence of solutions is given. All the existence arguments are based on rather precise quantitative estimates.

1. Introduction

Backward stochastic differential equations, in short BSDEs, have been well studied during the last ten years or so. They were introduced by Pardoux and Peng [20], who proved existence and uniqueness of adapted solutions, under suitable square-integrability assumptions on the coefficients and on the terminal condition. They provide probabilistic formulas for solution of systems of semi-linear partial differential equations, both of parabolic and elliptic type. The interest for this kind of stochastic equations has increased steadily; this is due to the strong connections of these equations with mathematical finance and the fact that they provide a generalization of the well-known Feynman-Kac formula to semi-linear partial differential equations. In the present paper we will concentrate on the relationship between time-dependent strong Markov processes and abstract backward stochastic differential equations. The equations are phrased in terms of a martingale type problem, rather than a strong stochastic differential equation. They could be called weak backward stochastic differential equations. Emphasis is put on existence and uniqueness of solutions. The paper in [27] deals with the same subject, but it concentrates on comparison theorems and viscosity solutions.

The notion of squared gradient operator is implicitly used by Bally at al in [4]. The latter paper was one of the motivations to write the present paper with an emphasis on the squared gradient operator. In addition, our results are presented in such a way that the state space of the underlying Markov process, which in most of the other papers on BSDEs is supposed to be $\mathbb{R}^n$, can be any diffusion with an abstract state space, which throughout our text is denoted by E. In fact in the existing literature the underlying Markov process is a (strong) solution of a (forward) stochastic differential equation: see, e.g., [4], [8] and [7] and [19]. For more on this see Remark 2.9 below. In particular our results are applicable in case the Markov process under consideration is Brownian motion on a Riemannian manifold. Our condition on the generator (or coefficient) of the BSDE f in terms of the squared gradient is very natural. In the Lipschitz context it is more or less optimal. Moreover, our proof of existence is not based on standard regularization methods by using convolution products with smooth functions, but on a homotopy argument due to Crouzeix [11], which seems more direct than the classical approach. We also obtain rather precise quantitative estimates. Only very rudimentary sketches of proofs are given; details will appear elsewhere.

For examples of strong solutions which are driven by Brownian motion the reader is referred to, e.g., Section 2 in Pardoux [19]. If the coefficients $x \mapsto b(s, x)$ and $x \mapsto \sigma(s, x)$ of the underlying (forward) stochastic differential equation are linear in x, then the corresponding forward-backward stochastic differential equation is related to option pricing in financial mathematics. A BSDE may serve as a model for a hedging strategy. For more details on this interpretation see, e.g., El Karoui and Quenez [16], pp. 198–199. Pardoux and Zhang [21] use BSDEs to give a probabilistic formula for the solution of a system of Parabolic or elliptic semi-linear partial differential equation with Neumann boundary condition. The first author who discussed BSDEs was probably Bismut: see [6].

In Section 2 we introduce the relevant notions about time-dependent Markov processes and generators of diffusions including the abstract notion of squared gradient operator and (some) of its properties. It also contains the necessary terminology and results on martingales and their L^p-properties which includes the inequality of Burkholder-Davis-Gundy. In Section 3 we formulate one of the main results of the paper: see Remarks 2.3 and 2.9 below as well. It includes a discussion of the result by Crouzeix and the way it is applied here: see [11] and [10]. Section 4 contains the precise conditions under which we have existence and uniqueness of solutions. It also contains a mathematical description of the stochastic phase space in which solutions to our BSDE exist. In Section 5 a single probability space is replaced by a Markov family of probability spaces.

2. Preliminary results and auxiliary notation

In this paper we want to consider the situation where the family of operators $L(s)$, $0 \leq s \leq T$, generates a time-inhomogeneous Markov family of probability spaces or a Markov process

$$\left\{\left(\Omega, \mathcal{F}_T^{\tau}, \mathbb{P}_{\tau,x}\right)_{(\tau,x)\in[0,T]\times E}, \left(X(t) : T \geq t \geq 0\right), (E, \mathcal{E})\right\} \tag{2.1}$$

in the sense of Definition 2.4. The Markov property and a Markov process are defined in Definition 2.3. We consider the operators $L(s)$ as operators on (a subspace of) the space of bounded continuous functions on E, i.e., on $C_b(E)$ equipped with the supremum norm: $\|f\|_\infty = \sup_{x\in E} |f(x)|$, $f \in C_b(E)$.

2.1. Definition. With the Markov process (2.1) the squared gradient operator Γ_1 defined by

$$\Gamma_1(f,g)(\tau,x) = \lim_{s\downarrow\tau} \frac{\mathbb{E}_{\tau,x}\left[\left(f(s,X(s)) - f(\tau,X(\tau))\right)\left(g(s,X(s)) - g(\tau,X(\tau))\right)\right]}{s-\tau}, \tag{2.2}$$

for f, $g \in D(\Gamma_1)$, is associated. A function $f : [0,T]\times E \to \mathbb{R}$ is said to belong to $D(\Gamma_1) \subset C_b([0,T]\times E)$, if $\Gamma_1(f,f)(s,x)$ exists for all pairs $(s,x) \in [0,T]\times E$, and if the resulting function is a member of $C_b([0,T]\times E)$.

It is assumed that $D(\Gamma_1)$ is dense in $C_b([0,T]\times E)$ for the topology of uniform convergence on compact subsets. These squared gradient operators are also called energy operators: see, e.g., Barlow, Bass and Kumagai [5]. In the sequel it is assumed that the family of operators $\{L(s) : 0 \leq s \leq T\}$ possesses the property that the space of functions $u : [0,T]\times E \to \mathbb{R}$ for which the function $(s,x) \mapsto \dfrac{\partial u}{\partial s}(s,x) + L(s)u(s,\cdot)(x)$ belongs to $C_0([0,T]\times E) := C_0([0,T]\times E;\mathbb{R})$ is dense in the space $C_0([0,T]\times E)$. This subspace of functions is denoted by $D(L)$, and the operator L is defined by $Lu(s,x) = L(s)u(s,\cdot)(x)$, $u \in D(L)$. We assume that the operator L, or that the family of operators $\{L(s) : 0 \leq s \leq T\}$, generates a diffusion in the sense of the following definition.

2.2. Definition. A family of operators $\{L(s) : 0 \leq s \leq T\}$ is said to generate a *diffusion* if for every C^∞-function $\Phi : \mathbb{R}^n \to \mathbb{R}$, with $\Phi(0,\ldots,0) = 0$, and every pair $(s,x) \in [0,T]\times E$ the following identity is valid

$$\begin{aligned} &L(s)\left(\Phi(f_1,\ldots,f_n)\right)(s,x) \\ &= \sum_{j=1}^n \frac{\partial\Phi}{\partial x_j}(f_1,\ldots,f_n)\, L(s)f_j(s,x) \\ &\quad + \frac{1}{2}\sum_{j,k=1}^n \frac{\partial^2\Phi}{\partial x_j\partial x_k}(f_1,\ldots,f_n)(s,x)\Gamma_1(f_j,f_k)(s,x) \end{aligned} \tag{2.3}$$

for all functions $f_1,\ldots,f_n$ in an algebra of functions $\mathcal{A}$, contained in the domain of the operator L, which forms a core for L.

Generators of diffusions for single operators are described in Bakry's lecture notes [1]. For more information on the squared gradient operator see, e.g., [3] and [2] as well. Put $\Phi(f,g) = fg$. Then (2.3) implies

$$L(s)\,(fg)\,(s,\cdot)(x) = L(s)f(s,\cdot)(x)g(s,x) + f(s,x)L(s)g(s,\cdot)(x) + \Gamma_1\,(f,g)\,(s,x),$$

provided that the three functions f, g and fg belong to $\mathcal{A}$. Instead of using the full strength of (2.3), i.e., with a general function Φ, we just need it for the product $(f,g) \mapsto fg$: see Proposition 2.12.

By definition the gradient of a function $u \in D\,(\Gamma_1)$ in the direction of $v \in D\,(\Gamma_1)$ is the function $(\tau,x) \mapsto \Gamma_1\,(u,v)\,(\tau,x)$. For given $(\tau,x) \in [0,T] \times E$ the functional $v \mapsto \Gamma_1\,(u,v)\,(\tau,x)$ is linear: its action is denoted by $\nabla_u^L\,(\tau,x)$. Hence, for $(\tau,x) \in [0,T] \times E$ fixed, we can consider $\nabla_u^L\,(\tau,x)$ as an element in the dual of $D\,(\Gamma_1)$. The pair $(\tau,x) \mapsto \left(u\,(\tau,x)\,,\nabla_u^L\,(\tau,x)\right)$ may be called an element in the phase space of the family $L(s)$, $0 \le s \le T$, (see Prüss [22]), and the process $s \mapsto \left(u\,(s,X(s))\,,\nabla_u^L\,(s,X(s))\right)$ will be called an element of the stochastic phase space.

2.3. Definition. The family of probability spaces and state variables

$$\left\{(\Omega,\mathcal{F}_T^\tau,\mathbb{P}_{\tau,x})_{(\tau,x)\in[0,T]\times E}\,,(X(t): T \ge t \ge 0)\,,(E,\mathcal{E})\right\} \tag{2.4}$$

is called a time-inhomogeneous Markov family or Markov process if

$$\mathbb{E}_{\tau,x}\left[f(X(t)) \;\middle|\; \mathcal{F}_s^\tau\right] = \mathbb{E}_{s,X(s)}\left[f(X(t))\right], \quad \mathbb{P}_{\tau,x}\text{-almost surely.} \tag{2.5}$$

Here f is a bounded Borel measurable function defined on the state space E and $\tau \le s \le t \le T$.

Suppose that the process $X(t)$ in (2.4) has paths which are right-continuous and have left limits in E. Then it can be shown that the Markov property for fixed times carries over to stopping times in the sense that (2.5) may be replaced with

$$\mathbb{E}_{\tau,x}\left[Y \;\middle|\; \mathcal{F}_S^\tau\right] = \mathbb{E}_{S,X(S)}\left[Y\right], \quad \mathbb{P}_{\tau,x}\text{-almost surely.} \tag{2.6}$$

Here $S : E \to [\tau,T]$ is an $\mathcal{F}_t^\tau$-adapted stopping time and Y is a bounded stochastic variable which is measurable with respect to the future (or terminal) σ-field after S, i.e., the one generated by $\{X\,(t \vee S) : \tau \le t \le T\}$. For this type of result the reader is referred to Chapter 2 in Gulisashvili et al. [12]. Markov processes for which (2.6) holds are called strong Markov processes.

2.4. Definition. The family of operators $L(s)$, $0 \le s \le T$, is said to generate a time-inhomogeneous Markov process, as described in (2.4) in Definition 2.3, if for all functions $u \in D(L)$, for all $x \in E$, and for all pairs (τ,s) with $0 \le \tau \le s \le T$ the following equality holds:

$$\frac{d}{ds}\mathbb{E}_{\tau,x}\left[u\,(s,X(s))\right] = \mathbb{E}_{\tau,x}\left[\frac{\partial u}{\partial s}\,(s,X(s)) + L(s)u\,(s,\cdot)\,(X(s))\right]. \tag{2.7}$$

Let the process $X(t)$, $t \in [0,T]$, in (2.4) be a Markov process in the sense of Definition 2.3, and put

$$\begin{aligned}&M_u(s) - M_u(\tau)\\&= u(s, X(s)) - u(\tau, X(\tau)) - \int_\tau^s \left(L(\rho)u(\rho, X(\rho)) + \frac{\partial u}{\partial \rho}(\rho, X(\rho)) \right) d\rho.\end{aligned} \tag{2.8}$$

In the following proposition we write $\mathcal{F}_s^\tau$, $s \in [\tau, T]$, for the σ-field generated by $X(\rho)$, $\rho \in [\tau, s]$. It shows that under rather general conditions the process $s \mapsto M_u(s) - M_u(\tau)$, $\tau \le s \le T$, as defined in (2.8) is a $\mathbb{P}_{\tau,x}$-martingale. The proof is left to the reader.

2.5. Proposition. Fix $t \in [\tau, T)$. Let the function $u : [t,T] \times E \to \mathbb{R}$ be such that

$$(s,x) \mapsto \frac{\partial u}{\partial s}(s,x) + L(s)u(s,\cdot)(x)$$

belongs to $C_0([t,T] \times E)$ Then the process $s \mapsto M_u(s) - M_u(\tau)$ is adapted to the filtration of σ-fields $(\mathcal{F}_s^\tau)_{s \in [\tau,T]}$. Moreover, it is a $\mathbb{P}_{\tau,x}$-martingale if and only if (2.7) is satisfied.

As explained in Definition 2.2 it is assumed that the subspace $D(L)$ contains an algebra $\mathcal{A}$ of functions which forms a core for the operator L.

2.6. Proposition. Let the family of operators $L(s)$, $0 \le s \le T$, generate a time-inhomogeneous Markov process in the sense of Definition 2.4: see equality (2.7). Then the process $X(t)$ has a modification which is right-continuous and has left limits.

In view of Proposition 2.6 we will assume that our Markov process has left limits and is continuous from the right.

Proof. Let the function $u : [0,T] \times E \to \mathbb{R}$ belong to the space $D(L)$. Then the process $s \mapsto M_u(s) - M_u(t)$, $t \le s \le T$, as defined in (2.8), is a $\mathbb{P}_{t,x}$-martingale. The proof of Proposition 2.6 is based on the fact that the subspace $D(L)$ is dense in $C_0([0,T] \times E)$, and that martingales have left and right limits, as explained in, e.g., Chapter II in Revuz and Yor [23]. Details are left to the reader. □

The hypotheses in Proposition 2.7 below are the same as in Proposition 2.6. Its proof is skipped.

2.7. Proposition. Let the continuous function $u : [0,T] \times E \to \mathbb{R}$ be such that for every $s \in [t,T]$ the function $x \mapsto u(s,x)$ belongs to $D(L(s))$ and suppose that the function $(s,x) \mapsto [L(s)u(s,\cdot)](x)$ is bounded and continuous. In addition suppose that the function $s \mapsto u(s,x)$ is continuously differentiable for all $x \in E$. Then the process $s \mapsto M_u(s) - M_u(t)$ is a $\mathcal{F}_s^t$-martingale with respect to the probability $\mathbb{P}_{t,x}$. If v is another such function, then the (right) derivative of the quadratic co-variation of the martingales M_u and M_v is given by:

$$\frac{d}{dt} \langle M_u, M_v \rangle (t) = \Gamma_1(u,v)(t, X(t)).$$

In fact the following identity holds as well:

$$\begin{aligned} & M_u(t)M_v(t) - M_u(0)M_v(0) \\ & = \int_0^t M_u(s)dM_v(s) + \int_0^t M_v(s)dM_u(s) + \int_0^t \Gamma_1(u,v)(s,X(s))\,ds. \end{aligned} \tag{2.9}$$

Here $\mathcal{F}_s^t$, $s \in [t,T]$, is the σ-field generated by the state variables $X(\rho)$, $t \le \rho \le s$. Instead of $\mathcal{F}_s^0$ we usually write $\mathcal{F}_s$, $s \in [0,T]$. The formula in (2.9) is known as the integration by parts formula for stochastic integrals.

2.1. Remark. The quadratic variation process of the (local) martingale $s \mapsto M_u(s)$ is given by the process $s \mapsto \Gamma_1(u(s,\cdot),u(s,\cdot))(X(s))$, and therefore

$$\mathbb{E}_{s_1,x}\left[\left|\int_{s_1}^{s_2} dM_u(s)\right|^2\right] = \mathbb{E}_{s_1,x}\left[\int_{s_1}^{s_2} \Gamma_1(u(s,\cdot),u(s,\cdot))(X(s))\,ds\right] < \infty \tag{2.10}$$

under appropriate conditions on the function u. The formula in (2.10) is closely related to a formula which occurs in Malliavin calculus: see Nualart [17] and [18].

2.2. Remark. It is worthwhile to observe that for Brownian motion $(W(s),\mathbb{P}_x)$ the martingale difference $M_u(s_2) - M_u(s_1)$, $s_1 \le s_2 \le T$, is given by a stochastic integral $M_u(s_2) - M_u(s_1) = \int_{s_1}^{s_2} \nabla u(\tau,W(\tau))\,dW(\tau)$. Its increment of the quadratic variation process is given by

$$\langle M_u, M_u\rangle(s_2) - \langle M_u, M_u\rangle(s_1) = \int_{s_1}^{s_2} |\nabla u(\tau,W(\tau))|^2\,d\tau.$$

Next suppose that the function u solves the equation:

$$f\left(s,x,u(s,x),\nabla_u^L(s,x)\right) + L(s)u(s,x) + \frac{\partial}{\partial s}u(s,x) = 0. \tag{2.11}$$

If moreover, $u(T,x) = \varphi(T,x)$, $x \in E$, is given, then we have

$$\begin{aligned} & u(t,X(t)) \\ & = \varphi(T,X(T)) + \int_t^T f\left(s,X(s),u(s,X(s)),\nabla_u^L(s,X(s))\right)ds - \int_t^T dM_u(s), \end{aligned} \tag{2.12}$$

with $M_u(s)$ as in (2.8). From (2.12) we get

$$\begin{aligned} u(t,x) & = \mathbb{E}_{t,x}[u(t,X(t))] \\ & = \mathbb{E}_{t,x}[\varphi(T,X(T))] + \int_t^T \mathbb{E}_{t,x}\left[f\left(s,X(s),u(s,X(s)),\nabla_u^L(s,X(s))\right)\right]ds. \end{aligned} \tag{2.13}$$

2.8. Theorem. Let $u : [0,T] \times E \to \mathbb{R}$ be a continuous function with the property that for every $(t,x) \in [0,T] \times E$ the function $s \mapsto \mathbb{E}_{t,x}[u(s,X(s))]$ is differentiable and that

$$\frac{d}{ds}\mathbb{E}_{t,x}[u(s,X(s))] = \mathbb{E}_{t,x}\left[L(s)u(s,X(s)) + \frac{\partial}{\partial s}u(s,X(s))\right], \quad t < s < T.$$

Then the following assertions are equivalent:

(a) The function u satisfies the following differential equation:

$$L(t)u(t,x) + \frac{\partial}{\partial t}u(t,x) + f\left(t,x,u(t,x),\nabla_u^L(t,x)\right) = 0.$$

(b) The function u satisfies the following type of Feynman-Kac integral equation:

$$u(t,x) = \mathbb{E}_{t,x}\left[u(T,X(T)) + \int_t^T f\left(\tau, X(\tau), u(\tau, X(\tau)), \nabla_u^L(\tau, X(\tau))\right) d\tau\right].$$

(c) For every $t \in [0,T]$ the process

$$s \mapsto u(s,X(s)) - u(t,X(t)) + \int_t^s f\left(\tau, X(\tau), u(\tau, X(\tau)), \nabla_u^L(\tau, X(\tau))\right) d\tau$$

is an $\mathcal{F}_s^t$-martingale with respect to $\mathbb{P}_{t,x}$ on the interval $[t,T]$.

(d) For every $s \in [0,T]$ the process

$$t \mapsto u(T,X(T)) - u(t,X(t)) + \int_t^T f\left(\tau, X(\tau), u(\tau, X(\tau)), \nabla_u^L(\tau, X(\tau))\right) d\tau$$

is an $\mathcal{F}_T^t$-backward martingale with respect to $\mathbb{P}_{s,x}$ on the interval $[s,T]$.

2.3. Remark. Suppose that the function u is a solution to the following terminal value problem:

$$\begin{cases} L(s)u(s,\cdot)(x) + \dfrac{\partial}{\partial s}u(s,x) + f\left(s,x,u(s,x),\nabla_u^L(s,x)\right) = 0; \\ u(T,x) = \varphi(T,x). \end{cases} \tag{2.14}$$

Then the pair $\left(u(s,X(s)), \nabla_u^L(s,X(s))\right)$ can be considered as a weak solution to a backward stochastic differential equation. More precisely, for every $s \in [0,T]$ the process

$$t \mapsto u(T,X(T)) - u(t,X(t)) + \int_t^T f\left(\tau, X(\tau), u(\tau, X(\tau)), \nabla_u^L(\tau, X(\tau))\right) d\tau$$

is an $\mathcal{F}_T^t$-backward $\mathbb{P}_{s,x}$-martingale on the interval $[s,T]$. The symbol $\nabla_u^L v(s,x)$ stands for the functional $v \mapsto \nabla_u^L v(s,x) = \Gamma_1(u,v)(s,x)$, where Γ_1 is the squared gradient operator, which is defined in Definition 2.1. Possible choices for the function f are for example

$$f\left(s,x,y,\nabla_u^L\right) = -V(s,x)y \quad \text{and} \tag{2.15}$$

$$f\left(s,x,y,\nabla_u^L\right) = \frac{1}{2}\left|\nabla_u^L(s,x)\right|^2 - V(s,x) = \frac{1}{2}\Gamma_1(u,u)(s,x) - V(s,x). \tag{2.16}$$

2.4. Example. The choice in (2.15) turns equation (2.14) into the following heat equation:

$$\begin{cases} \dfrac{\partial}{\partial s}u(s,x) + L(s)u(s,\cdot)(x) - V(s,x)u(s,x) = 0; \\ u(T,x) = \varphi(T,x). \end{cases} \tag{2.17}$$

The function $v(s,x)$ defined by the Feynman-Kac formula

$$v(s,x) = \mathbb{E}_{s,x}\left[e^{-\int_s^T V(\rho,X(\rho))d\rho}\varphi\left(T,X(T)\right)\right] \tag{2.18}$$

is a solution candidate to equation (2.17).

In the next example we see how the classical Feynman-Kac formula is related to backward stochastic differential equations.

2.5. Example. This example is copied from Remark 2.5 in Pardoux [19]. An important example is one in which the function f is linear: $f(t,x,r,z) = c(t,x)r + h(t,x)$ and $X(s) = X^{t,x}(s)$ is a solution to a stochastic differential of the form below:

$$\begin{aligned} X^{t,x}(s) - X^{t,x}(t) &= \int_t^s b\left(\tau, X^{t,x}(\tau)\right) d\tau + \int_t^s \sigma\left(\tau, X^{t,x}(\tau)\right) dW(\tau), \quad t \le s \le T; \\ X^{t,x}(s) &= x, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le s \le t. \end{aligned}$$

In this case the linear BSDE

$$\begin{aligned} &Y^{t,x}(s) \\ &= g(X^{t,x}(T)) + \int_s^T \left[c(r,X^{t,x}(r))Y^{t,x}(s) + h(r,X^{t,x}(r))\right] dr - \int_s^T Z^{t,x}(r)\, dW(r), \end{aligned}$$

has an explicit solution. From an extension of the classical "variation of constants formula" (see the argument in the proof of the comparison Theorem 1.6 in Pardoux [19]) or by direct verification we get:

$$\begin{aligned} Y^{t,x}(s) = g\left(X^{t,x}(T)\right) e^{\int_s^T c\left(r,X^{t,x}(r)\right) dr} + \int_s^T h\left(r, X^{t,x}(r)\right) e^{\int_s^r c(\alpha,X^{t,x}(\alpha))\, d\alpha}\, dr \\ - \int_s^T e^{\int_s^r c\left(\alpha,X^{t,x}(\alpha)\right)d\alpha} Z^{t,x}(r)\, dW(r). \end{aligned}$$

Hence $Y^{t,x}(t) = \mathbb{E}\left[Y^{t,x}(t)\right]$, so that

$$Y^{t,x}(t) = \mathbb{E}\left[g(X^{t,x}(T))e^{\int_t^T c(s,X^{t,x}(s))\, ds} + \int_t^T h\left(s, X^{t,x}(s)\right) e^{\int_t^s c\left(r,X^{t,x}(r)\right)dr} ds\right],$$

which is the well-known Feynman-Kac formula. For more details and explicit formulas see Remark 2.5 in Pardoux [19].

2.6. Example. The choice in (2.16) turns equation (2.14) into the following Hamilton-Jacobi-Bellmann equation of Riccati type:

$$\begin{cases} \dfrac{\partial}{\partial s}u\left(s,x\right) + L(s)u\left(s,X(s)\right) - \dfrac{1}{2}\Gamma_1\left(u,u\right)\left(s,x\right) + V(s,x) = 0; \\ u\left(T,x\right) = -\log\varphi(T,x), \end{cases} \tag{2.19}$$

where $-\log\varphi(T,x)$ replaces $\varphi(T,x)$. The function S_L defined by the genuine non-linear Feynman-Kac formula

$$S_L(s,x) = -\log \mathbb{E}_{s,x}\left[e^{-\int_s^T V(\rho,X(\rho))d\rho}\varphi\left(T,X(T)\right)\right] \tag{2.20}$$

is a solution candidate to (2.19). Often these "solution candidates" are viscosity solutions. However, this will be the main topic of [27]. For more details on the equation in (2.19) in case of a diffusion on a manifold see Theorem 2.4 in Zambrini [28]. The result in [28] is put in the (present) framework of diffusions with $L(s) = L$ in [25].

2.7. Remark. Suppose that the function $u(t,x)$ satisfies one of the equivalent conditions in Theorem 2.8. Put $Y(\tau) = u(\tau, X(\tau))$, and let $M(s)$ be the martingale determined by $M(0) = Y(0) = u(0, X(0))$ and by

$$M(s) - M(t) = Y(s) + \int_t^s f\left(\tau, X(\tau), Y(\tau), \nabla_u^L(\tau, X(\tau))\right) d\tau.$$

Then the expression $\nabla_u^L(\tau, X(\tau))$ only depends on the martingale part M of the process $s \mapsto Y(s)$. This entitles us to write $Z_M(\tau)$ instead of $\nabla_u^L(\tau, X(\tau))$. The mapping $Z_M(\tau) : \mathcal{M}^2(\Omega, \mathcal{F}_T^\tau, \mathbb{P}_{\tau,x}) \to \mathbb{R}$ is then to be interpreted as the linear functional $N \mapsto \frac{d}{d\tau}\langle M, N\rangle(\tau)$, where the process $t \mapsto N(t) - N(\tau)$, $t \in [\tau, T]$, is a $\mathbb{P}_{\tau,x}$-martingale in $\mathcal{M}^2(\Omega, \mathcal{F}_T^\tau, \mathbb{P}_{\tau,x})$. Here a process $N(\cdot) - N(\tau)$ belongs to $\mathcal{M}^2(\Omega, \mathcal{F}_T^\tau, \mathbb{P}_{\tau,x})$ whenever it is martingale in $L^2(\Omega, \mathcal{F}_T^\tau, \mathbb{P}_{\tau,x})$. Notice that the functional $Z_M(\tau)$ is known as soon as the martingale $M(\cdot) - M(\tau) \in \mathcal{M}^2(\Omega, \mathcal{F}_T^\tau, \mathbb{P}_{\tau,x})$ is known. From our definitions it also follows that

$$M(T) = Y(T) + \int_0^T f(\tau, X(\tau), Y(\tau), Z_M(\tau))\, d\tau,$$

where we used the fact that $Y(0) = M(0)$. For closely related notions see equalities (5.4) and (5.5) in Section 5.

2.8. Remark. Let the notation be as in Remark 2.7. Then the variables $Y(t)$ and $Z_M(t)$ only depend on the space variable $X(t)$, and as a consequence the martingale increments $M(t_2) - M(t_1)$, $0 \le t_1 < t_2 \le T$, only depend on $\mathcal{F}_{t_2}^{t_1} = \sigma(X(s) : t_1 \le s \le t_2)$. In Section 3 we give Lipschitz type conditions on the function f in order that the BSDE

$$Y(t) = Y(T) + \int_t^T f(s, X(s), Y(s), Z_M(s))\, ds + M(t) - M(T), \quad \tau \le t \le T, \tag{2.21}$$

possesses a unique pair of solutions

$$(Y, M(\cdot) - M(\tau)) \in L^2(\Omega, \mathcal{F}_T^\tau, \mathbb{P}_{\tau,x}) \times \mathcal{M}^2(\Omega, \mathcal{F}_T^\tau, \mathbb{P}_{\tau,x}).$$

For an explanation of the functional $Z_M(s)$ see Remark 2.7. Here $\mathcal{M}^2(\Omega, \mathcal{F}_T^t, \mathbb{P}_{t,x})$ stands for the space of all $(\mathcal{F}_s^t)_{s\in[t,T]}$-martingales in the space $L^2(\Omega, \mathcal{F}_T^t, \mathbb{P}_{t,x})$. Suppose that the $\sigma(X(T))$-measurable variable $Y(T) \in L^2(\Omega, \mathcal{F}_T^\tau, \mathbb{P}_{\tau,x})$ is given. In fact we will prove that the solution (Y, M) of the equation in (2.21) belongs to the space $\mathcal{S}^2 \times \mathcal{M}^2 := \mathcal{S}^2\left(\Omega, \mathcal{F}_T^\tau, \mathbb{P}_{\tau,x}; \mathbb{R}^k\right) \times \mathcal{M}^2\left(\Omega, \mathcal{F}_T^t, \mathbb{P}_{\tau,x}; \mathbb{R}^k\right)$. For more details see Definitions 2.9 and 4.1, and Theorem 5.1. We will also consider the bilinear mapping $Z(s)$ which associates with a pair of local semi-martingales $(Y_1, Y_2) \in$

$\mathcal{S}^2 \times \mathcal{S}^2$ a process which is to be considered as the right derivative of the covariation process: $\langle Y_1, Y_2 \rangle (s)$. We write

$$Z_{Y_1}(s)(Y_2) = Z(s)(Y_1, Y_2) = \frac{d}{ds} \langle Y_1, Y_2 \rangle (s). \tag{2.22}$$

Since this covariation process only depends on the martingale parts M_1 and M_2 of the processes Y_1 and Y_2 we also use the notation Z_{M_1} instead of Z_{Y_1}. The function f (i.e., the generator of the backward differential equation) will then be of the form: $f(s, X(s), Y(s), Z_Y(s)) = f(s, X(s), Y(s), Z_M(s))$; the deterministic phase $\left(u(s,x), \nabla^L u(s,x)\right)$ is replaced with the stochastic phase $(Y(s), Z_Y(s))$. We should find an appropriate stochastic phase $s \mapsto (Y(s), Z_Y(s)) = (Y(s), Z_M(s))$, which we identify with the process $s \mapsto (Y(s), M(s))$ in the stochastic phase space $\mathcal{S}^2 \times \mathcal{M}^2$, such that (2.21) is satisfied. The stochastic phase space $\mathcal{S}^2 \times \mathcal{M}^2$ plays a role in stochastic analysis very similar to the role played by the first Sobolev space $H^{1,2}$ in the theory of deterministic partial differential equations.

2.9. Remark. In case we deal with strong solutions driven by standard Brownian motion the martingale difference $M_Y(s_2) - M_Y(s_1)$ can be written in the form (martingale representation theorem) $\int_{s_1}^{s_2} Z_Y(s) dW(s)$, provided that the martingale $M_Y(s)$ belongs to $\mathcal{M}^2\left(\Omega, \mathcal{G}_T^0, \mathbb{P}\right)$. Here $\mathcal{G}_T^0$ is the σ-field generated by $W(s)$, $0 \le s \le T$. If $Y(s) = u(s, X(s))$, then this stochastic integral satisfies:

$$\begin{aligned}&\int_{s_1}^{s_2} Z_Y(s) dW(s)\\ &= u(s_2, X(s_2)) - u(s_1, X(s_1)) - \int_{s_1}^{s_2} \left(L(s) + \frac{\partial}{\partial s}\right) u(s, X(s))\, ds.\end{aligned} \tag{2.23}$$

Such stochastic integrals are for example defined if the process $X(t)$ is a solution to a stochastic differential equation (in Itô sense):

$$X(s) = X(t) + \int_t^s b(\tau, X(\tau))\, d\tau + \int_t^s \sigma(\tau, X(\tau))\, dW(\tau), \quad t \le s \le T. \tag{2.24}$$

Here the matrix $\left(\sigma_{j,\ell}(\tau, x)\right)_{j,\ell=1}^d$ is chosen in such a way that

$$a_{j,k}(\tau, x) = \sum_{\ell=1}^d \sigma_{j\ell}(\tau, x)\, \sigma_{k,\ell}(\tau, x) = \left(\sigma(\tau, x)\sigma^*(\tau, x)\right)_{j,k}. \tag{2.25}$$

The process $W(\tau)$ is Brownian motion or Wiener process. It is assumed that operator $L(\tau)$ has the form

$$L(\tau)u(x) = b(\tau, x) \cdot \nabla u(x) + \frac{1}{2} \sum_{j,k=1}^d a_{j,k}(\tau, x) \frac{\partial^2}{\partial x_j x_k} u(x). \tag{2.26}$$

Then from Itô's formula together with (2.23), (2.24) and (2.26) it follows that the process $Z_Y(s)$ has to be identified with $\sigma(s, X(s))^* \nabla u(s, \cdot)(X(s))$. For more details see, e.g., Pardoux and Peng [20], Pardoux [19], and Bally et al. [4]. In this case the squared gradient operator is given by

$$\Gamma_1 (u, v) (s, x) = \sum_{j,k=1}^{d} a_{j,k}(s, x) \frac{\partial u}{\partial x_j}(s, x) \frac{\partial v}{\partial x_k}(s, x).$$

Since in our setup the squared gradient operator only depends on the coefficients $a_{j,k}(s, x)$, $1 \leq j,\ k \leq d$, and not on a matrix $\sigma(s, x)$ satisfying (2.25) we see that the solutions of the BSDE in (2.21) should be considered as weak solutions. In the literature on this subject strong solutions depend on the matrix $\sigma(s, x)$. In the present paper that need not be the case.

2.10. Remark. Backward doubly stochastic differential equations (BDSDEs) could have been included in the present paper: see Boufoussi, Mrhardy and Van Casteren [8]. In our notation a BDSDE may be written in the form:

$$\begin{aligned} Y(t) - Y(T) = &\int_t^T f\left(s, X(s), Y(s), Z_M(s)\right) ds \\ &+ \int_t^T g\left(s, X(s), Y(s), Z_M(s)\right) d\overleftarrow{B}(s) + M(t) - M(T). \end{aligned} \tag{2.27}$$

For an explanation of the functional $Z_M(s)$ see Remark 2.7. Here the expression

$$\int_t^T g\left(s, X(s), Y(s), Z_M(s)\right) d\overleftarrow{B}(s)$$

represents a backward Itô integral. The symbol $\langle M, N\rangle$ stands for the covariation process of the (local) martingales M and N; it is assumed that this process is absolutely continuous with respect to Lebesgue measure. Moreover,

$$\left\{\left(\Omega, \mathcal{F}_T^\tau, \mathbb{P}_{\tau,x}\right), \left(X(t) : T \geq t \geq 0\right), \left(E, \mathcal{E}\right)\right\}$$

is a Markov process generated by a family of operators $L(s)$, $0 \leq s \leq T$, and

$$\mathcal{F}_t^\tau = \sigma\left\{X(s) : \tau \leq s \leq t\right\}.$$

The process $X(t)$ could be the (unique) weak or strong solution to a (forward) stochastic differential equation (SDE):

$$X(t) = x + \int_\tau^t b\left(s, X(s)\right) ds + \int_\tau^t \sigma\left(s, X(s)\right) dW(s). \tag{2.28}$$

Here the coefficients b and σ have certain continuity or measurability properties, and $\mathbb{P}_{\tau,x}$ is the distribution of the process $X(t)$ defined as being the unique weak solution to the equation in (2.28). The authors want to find a pair

$$(Y, M) \in \mathcal{S}^2\left(\Omega, \mathcal{F}_t^\tau, \mathbb{P}_{\tau,x}\right) \times \mathcal{M}^2\left(\Omega, \mathcal{F}_t^\tau, \mathbb{P}_{\tau,x}\right)$$

which satisfies (2.27).

We first give some definitions. Fix $(\tau, x) \in [0,T] \times E$. In Definitions 2.9 and 2.10 the probability measure $\mathbb{P}_{\tau,x}$ is defined on the σ-field $\mathcal{F}_T^\tau$. In Definition 4.1 we return to these notions. The following definition and implicit results described therein shows that, under certain conditions, by enlarging the sample space a family of processes may be reduced to just one process without losing the $\mathcal{S}^2$-property.

2.9. Definition. Fix $(\tau, x) \in [0,T] \times E$. An $\mathbb{R}^k$-valued process Y is said to belong to the space $\mathcal{S}^2\left(\Omega, \mathcal{F}_T^\tau, \mathbb{P}_{\tau,x}; \mathbb{R}^k\right)$ if $Y(t)$ is $\mathcal{F}_t^\tau$-measurable $(\tau \leq t \leq T)$ and if $\mathbb{E}_{\tau,x}\left[\sup_{\tau \leq t \leq T} |Y(t)|^2\right] < \infty$. It is assumed that $Y(s) = Y(\tau)$, $\mathbb{P}_{\tau,x}$-almost surely, for $s \in [0,\tau]$. The process $Y(s)$, $s \in [0,T]$, is said to belong to the space $\mathcal{S}^2_{\text{unif}}\left(\Omega, \mathcal{F}_T^\tau, \mathbb{P}_{\tau,x}; \mathbb{R}^k\right)$ if

$$\sup_{(\tau,x) \in [0,T] \times E} \mathbb{E}_{\tau,x}\left[\sup_{\tau \leq t \leq T} |Y(t)|^2\right] < \infty,$$

and it belongs to $\mathcal{S}^2_{\text{loc,unif}}\left(\Omega, \mathcal{F}_T^\tau, \mathbb{P}_{\tau,x}; \mathbb{R}^k\right)$ provided that

$$\sup_{(\tau,x) \in [0,T] \times K} \mathbb{E}_{\tau,x}\left[\sup_{\tau \leq t \leq T} |Y(t)|^2\right] < \infty$$

for all compact subsets K of E.

If the σ-field $\mathcal{F}_t^\tau$ and $\mathbb{P}_{\tau,x}$ are clear from the context we write $\mathcal{S}^2\left([0,T], \mathbb{R}^k\right)$ or sometimes just $\mathcal{S}^2$. A similar convention is used for the space $\mathcal{M}^2$.

2.10. Definition. Let the process M be such that the process $t \mapsto M(t) - M(\tau)$, $t \in [\tau, T]$, is a $\mathbb{P}_{\tau,x}$-martingale with the property that the stochastic variable $M(T) - M(\tau)$ belongs to $L^2\left(\Omega, \mathcal{F}_T^\tau, \mathbb{P}_{\tau,x}\right)$. Then M is said to belong to the space $\mathcal{M}^2\left(\Omega, \mathcal{F}_T^\tau, \mathbb{P}_{\tau,x}; \mathbb{R}^k\right)$. By the Burkholder-Davis-Gundy inequality (see inequality (4.2) below) it follows that $\mathbb{E}_{\tau,x}\left[\sup_{\tau \leq t \leq T} |M(t) - M(\tau)|^2\right]$ is finite if and only if $M(T) - M(\tau)$ belongs to the space $L^2\left(\Omega, \mathcal{F}_T^\tau, \mathbb{P}_{\tau,x}\right)$. Here an $\mathcal{F}_t^\tau$-adapted process $M(\cdot) - M(\tau)$ is called a $\mathbb{P}_{\tau,x}$-martingale provided that $\mathbb{E}_{\tau,x}\left[|M(t) - M(\tau)|\right] < \infty$ and

$$\mathbb{E}_{\tau,x}\left[M(t) - M(\tau) \mid \mathcal{F}_s^\tau\right] = M(s) - M(\tau), \quad \mathbb{P}_{\tau,x}\text{-almost surely, for } T \geq t \geq s \geq \tau.$$

The martingale difference $s \mapsto M(s) - M(0)$, $s \in [0,T]$, is said to belong to the space $\mathcal{M}^2_{\text{unif}}\left(\Omega, \mathcal{F}_T^\tau, \mathbb{P}_{\tau,x}; \mathbb{R}^k\right)$ if

$$\sup_{(\tau,x) \in [0,T] \times E} \mathbb{E}_{\tau,x}\left[\sup_{\tau \leq t \leq T} |M(t) - M(\tau)|^2\right] < \infty,$$

and it belongs to $\mathcal{M}^2_{\text{loc,unif}}\left(\Omega, \mathcal{F}_T^\tau, \mathbb{P}_{\tau,x}; \mathbb{R}^k\right)$ provided that

$$\sup_{(\tau,x) \in [0,T] \times K} \mathbb{E}_{\tau,x}\left[\sup_{\tau \leq t \leq T} |M(t) - M(\tau)|^2\right] < \infty$$

for all compact subsets K of E. From the Burkholder-Davis-Gundy inequality (see inequality (4.2) below) it follows that the process $M(s) - M(\tau)$ belongs to the space $\mathcal{M}^2_{\text{unif}}\left(\Omega, \mathcal{F}^\tau_T, \mathbb{P}_{\tau,x}; \mathbb{R}^k\right)$ if and only if

$$\begin{aligned}&\sup_{(\tau,x)\in[0,T]\times E} \mathbb{E}_{\tau,x}\left[\left|M(T) - M(\tau)\right|^2\right] \\ &= \sup_{(\tau,x)\in[0,T]\times E} \mathbb{E}_{\tau,x}\left[\langle M, M\rangle (T) - \langle M, M\rangle (\tau)\right] < \infty.\end{aligned}$$

Here $\langle M, M\rangle$ stands for the quadratic variation process of the process $t \mapsto M(t)$.

The notions in Definitions 2.9 and 2.10 will exclusively be used in case the family of measures $\{\mathbb{P}_{\tau,x} : (\tau, x) \in [0, T] \times E\}$ constitutes the family of distributions of a Markov process which was defined in Definition 2.3.

2.11. Remark. Again let the Markov process, with right-continuous sample paths and with left limits, be generated by the family of operators $\{L(s) : 0 \le s \le t\}$: see Definitions 2.3 equality (2.5), and 2.4 equality (2.7). We define the family of operators $\{Q\,(t_1, t_2) : 0 \le t_1 \le t_2 \le T\}$ by

$$Q\,(t_1, t_2)\, f(x) = \mathbb{E}_{t_1,x}\left[f\,(X\,(t_2))\right], \quad f \in C_0\,(E)\,,\ 0 \le t_1 \le t_2 \le T.$$

Fix $\varphi \in D(L)$. Then this family satisfies the Chapman-Kolmogorov identity:

$$Q\,(s, t')\, \varphi\,(t', \cdot)\,(x) = Q\,(s, t)\, Q\,(t, t')\, \varphi\,(t', \cdot)\,(x), \quad 0 \le s \le t \le t' \le T,\ x \in E.$$

If $\varphi \in D(L)$ is such that $L(\rho)\varphi\,(\rho, \cdot)\,(y) = -\dfrac{\partial\varphi}{\partial\rho}\,(\rho, y)$, then it can be proved that

$$\varphi\,(s, x) = Q\,(s, t)\, \varphi\,(t, \cdot)\,(x) = \mathbb{E}_{s,x}\left[\varphi\,(t, X(t))\right]. \tag{2.29}$$

For more details on propagators or evolution families see [12].

As a corollary to Theorems 2.8 and 4.6 we have the following result.

2.11. Corollary. Suppose that the function u solves the following

$$\begin{cases} \dfrac{\partial u}{\partial s}(s, y) + L(s)u(s, \cdot)\,(y) + f\left(s, y, u(s, y), \nabla^L_u(s, y)\right) = 0; \\ u\,(T, X(T)) = \xi \in L^2\left(\Omega, \mathcal{F}^\tau_T, \mathbb{P}_{\tau,x}\right). \end{cases} \tag{2.30}$$

Let the pair (Y, M) be a solution to

$$Y(t) = \xi + \int_t^T f\,(s, X(s), Y(s), Z_M(s))\, ds + M(t) - M(T), \tag{2.31}$$

with $M(\tau) = 0$. Then $(Y(t), M(t)) = (u\,(t, X(t))\,, M_u(t))$, where

$$M_u(t) = u\,(t, X(t)) - u\,(\tau, X(\tau)) - \int_\tau^t L(s)u\,(s, \cdot)\,(X(s))\, ds - \int_\tau^t \frac{\partial u}{\partial s}\,(s, X(s))\, ds.$$

Notice that the processes $s \mapsto \nabla^L_u\,(s, X(s))$ and $s \mapsto Z_{M_u}(s)$ may be identified and that $Z_{M_u}(s)$ only depends on $(s, X(s))$.

The decomposition

$$u(t, X(t)) - u(\tau, X(\tau))$$
$$= \int_\tau^t \left(\frac{\partial u}{\partial s}(s, X(s)) + L(s)u(s,\cdot)(X(s))\right) ds + M_u(t) - M_u(\tau) \tag{2.32}$$

splits the process $t \mapsto u(t, X(t)) - u(\tau, X(\tau))$ into a part which is of bounded variation (i.e., the part which is absolutely continuous with respect to the Lebesgue measure on $[\tau, T]$) and a $\mathbb{P}_{\tau,x}$-martingale part $M_u(t) - M_u(\tau)$ (which in fact is a martingale difference part). The following proposition connects the family of generators of a Markov process and the family of squared gradient operators. Due to lack of space its proof is omitted.

2.12. Proposition. Let the functions f, $g \in D(L)$ be such that their product fg also belongs to $D(L)$. Then $\Gamma_1(f, g)$ is well defined and for $(s, x) \in [0, T] \times E$ the following equality holds:

$$L(s)(fg)(s,\cdot)(x) - f(s,x)L(s)g(s,\cdot)(x) - L(s)f(s,\cdot)(x)g(s,x) = \Gamma_1(f,g)(s,x).$$

2.12. Remark. Suggestions for further research:

(a) Find "explicit solutions" to BSDEs with a linear drift part. This should be a type of Cameron-Martin formula or Girsanov transformation.
(b) Treat weak (and strong) solutions BDSDEs in a manner similar to what is presented here for BSDEs.
(c) Treat weak (strong) solutions to BSDEs generated by a function f which is not necessarily of linear growth but for example of quadratic growth in one or both of its entries $Y(t)$ and $Z_M(t)$.
(d) Can anything be done if f depends not only on s, x, $u(s, x)$, $\nabla_u(s, x)$, but on $L(s)u(s,\cdot)(x)$ as well?

3. A probabilistic approach: weak solutions

In Section 5 we want treat equation (3.1). This means that we want to find a function $u(t, x)$ which satisfies the following partial differential equation:

$$\begin{cases} \dfrac{\partial u}{\partial s}(s,x) + L(s)u(s,x) + f\left(s, x, u(s,x), \nabla_u^L(s,x)\right) = 0; \\ u(T,x) = \varphi(T,x), \quad x \in E. \end{cases} \tag{3.1}$$

Here $\nabla_{f_2}^L(s, x)$ is the linear functional $f_1 \mapsto \Gamma_1(f_1, f_2)(s, x)$ for smooth enough functions f_1 and f_2. For $s \in [0, T]$ fixed the symbol $\nabla_{f_2}^L$ stands for the linear mapping $f_1 \mapsto \Gamma_1(f_1, f_2)(s,\cdot)$. The equation in (3.1) can be phrased in a semi-linear stochastic setting as follows. Find a pair of adapted processes $(Y, M) \in \mathcal{S}^2 \times \mathcal{M}^2 := \mathcal{S}^2\left(\Omega, \mathcal{F}_T^\tau, \mathbb{P}_{\tau,x}; \mathbb{R}^k\right) \times \mathcal{M}^2\left(\Omega, \mathcal{F}_T^t, \mathbb{P}_{\tau,x}; \mathbb{R}^k\right)$ with $Y(0) = M(0)$ and $Y(T) = \varphi(T, X(T))$ and which satisfies

$$Y(t) = Y(T) + \int_t^T f(s, X(s), Y(s), Z_M(s))\, ds + M(t) - M(T), \quad t \in [\tau, T]. \tag{3.2}$$

For a preliminary discussion on this topic see Theorem 2.8. Under certain Lipschitz type hypotheses on the function f we will give existence and uniqueness results. In this section and also in Section 4 we will study BSDEs on a single probability space. In Section 5 we will consider Markov families of probability spaces. In the present section we write $\mathbb{P}$ instead of $\mathbb{P}_{0,x}$, and similarly for the expectations $\mathbb{E}$ and $\mathbb{E}_{0,x}$. We are discussing the martingale problem, which basically means that only the distributions of the process $t \mapsto X(t)$, $t \in [0,T]$, are involved. Consequently, the solutions we obtain are of weak type. In most of the existing literature (see, e.g., [4], [21], [19]) strong solutions driven by Brownian motion are considered and a martingale representation theorem (in terms of Brownian motion) is employed: see, e.g., [24] Theorem 5.4.2. In Section 5 we will use the results of this section for probability measures of the form $\mathbb{P}_{\tau,x}$; the interval $[0,T]$ is then replaced with $[\tau,T]$. We consider a pair of $(\mathcal{F}_t)_{t\in[0,T]} = \left(\mathcal{F}_t^0\right)_{t\in[0,T]}$-adapted processes $(Y,M) \in L^2\left(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R}^k\right) \times L^2\left(\Omega,\mathcal{F}_T,\mathbb{P}:\mathbb{R}^k\right)$ such that $Y(0) = M(0)$ and such that (3.2) is satisfied. In [27] and in Section 5 we employ the results of the present section with $\mathbb{P} = \mathbb{P}_{\tau,x}$, where $(\tau,x) \in [0,T] \times E$. In Section 5 we will see that, in case we are dealing with a Markov family of probability spaces, or what amounts to the same, a Markov process, the variable $Y(s)$ is measurable with respect to $X(s)$. Hence it follows that in the Markov case, the process $Y(s)$ is of the form $Y(s) = u\left(s,X(s)\right)$, where in many interesting cases the function $u : [0,T]\times E \to \mathbb{R}$ is continuous. From Theorem 2.8 it follows that the process $s \mapsto Y(s) = u\left(s,X(s)\right)$ with $Y(T) = u\left(T,X(T)\right)$ satisfies (3.2) provided the function u satisfies the equation in (3.1). Conversely, if the process $s \mapsto Y(s)$ satisfies (3.2), then $u(s,x) = \mathbb{P}_{s,x}\left[Y(s)\right]$ satisfies (3.1). Since we almost exclusively deal with the Markov property of the process $X(t)$ solutions to (3.2) are weak solutions.

3.1. Proposition. Let the pair (Y,M) be as in (3.2), and suppose that $Y(0) = M(0)$. Then

$$Y(t) = M(t) - \int_0^t f\left(s,X(s),Y(s),Z_M(s)\right) ds, \tag{3.3}$$

$$Y(t) = \mathbb{E}\left[Y(T) + \int_t^T f\left(s,X(s),Y(s),Z_M(s)\right) ds \;\middle|\; \mathcal{F}_t\right], \quad \text{and} \tag{3.4}$$

$$M(t) = \mathbb{E}\left[Y(T) + \int_0^T f\left(s,X(s),Y(s),Z_M(s)\right) ds \;\middle|\; \mathcal{F}_t\right]. \tag{3.5}$$

The equality in (3.3) shows that the process M is the martingale part of the semi-martingale Y.

Proof. The equality in (3.4) follows from (3.2) and from the fact that M is a martingale. A calculation, in which (3.2) is used, then implies (3.5). Since

$$M(T) = Y(T) + \int_0^T f\left(s,X(s),Y(s),Z_M(s)\right) ds$$

the equality in (3.3) follows. □

In the following theorem z denotes $Z_M(s)$, (s,x) belongs to $[0,T]\times E$, and y to $\mathbb{R}^k$.

3.2. Theorem. Suppose that there exist finite constants C_1 and C_2 such that

$$\langle y_2 - y_1, f(s,x,y_2,z) - f(s,x,y_1,z)\rangle \le C_1 \left|y_2 - y_1\right|^2; \tag{3.6}$$

$$\left|f(s,x,y,Z_{M_2}(s)) - f(s,x,y,Z_{M_1}(s))\right|^2 \le C_2^2 \frac{d}{ds}\langle M_2 - M_1, M_2 - M_1\rangle (s). \tag{3.7}$$

Then there exists a unique pair of adapted processes (Y,M) such that $Y(0) = M(0)$ and such that the process M is the martingale part of the semi-martingale Y:

$$\begin{aligned} Y(t) &= M(t) - M(T) + Y(T) + \int_t^T f(s, X(s), Y(s), Z_M(s))\, ds \\ &= M(t) - \int_0^t f(s, X(s), Y(s), Z_M(s))\, ds. \end{aligned} \tag{3.8}$$

Outline of a proof of Theorem 3.2. The uniqueness follows from Corollary 4.4 of Theorem 4.3 below. In the existence part of the proof of Theorem 3.2 the function f is approximated by Lipschitz continuous functions f_δ, $0 < \delta < (2C_1)^{-1}$, where each function f_δ has Lipschitz constant δ^{-1}, but at the same time the martingale M is fixed. So the inequality (3.7) remains valid for fixed second variable with $C_2 = 0$. It follows that for the functions f_δ (3.7) remains valid and that (3.6) is replaced with

$$\left|f_\delta(s,x,y_2,z) - f_\delta(s,x,y_1,z)\right| \le \frac{1}{\delta}\left|y_2 - y_1\right|. \tag{3.9}$$

In the uniqueness part of the proof it suffices to assume that (3.6) holds. In Theorem 4.6 we will see that the monotonicity condition (3.6) also suffices to prove the existence. For details the reader is referred to the propositions 4.7 and 4.8, Corollary 4.9, and to Proposition 4.10. In fact for $M \in \mathcal{M}^2$ fixed, and the function $y \mapsto f(s,x,y,Z_M(s))$ satisfying (3.6) the function $y \mapsto y - \delta f(s,x,y,Z_M(s))$ is surjective as a mapping from $\mathbb{R}^k$ to $\mathbb{R}^k$ and its inverse exists and is Lipschitz continuous with constant $\dfrac{1}{1-\delta C_1}$. The Lipschitz continuity is part of Proposition 4.8. The surjectivity of this mapping is a consequence of Theorem 1 in [10]. As pointed out by Crouzeix et al. the result follows from a non-trivial homotopy argument. A relatively elementary proof of Theorem 1 in [10] can be found for a continuously differentiable function in Hairer and Wanner [13]: see Theorem 14.2 in Chapter IV. For a few more details see Remark 4.2. Let $f_{s,M}$ be the mapping $y \mapsto f(s,y,Z_M(s))$, and put

$$f_\delta(s,x,y,Z_M(s)) = f\left(s,x,(I - \delta f_{s,x,M})^{-1}, Z_M(s)\right). \tag{3.10}$$

Then the functions f_δ, $0 < \delta < (2C_1)^{-1}$, are Lipschitz continuous with constant δ^{-1}. Proposition 4.10 treats the transition from solutions of BSDE's with generator or coefficient f_δ with fixed martingale $M \in \mathcal{M}^2$ to solutions of BSDE's driven by f with the same fixed martingale M. Proposition 4.7 contains the passage from solutions $(Y, N) \in \mathcal{S}^2 \times \mathcal{M}^2$ to BBSDE's with generators of the form $(s, y) \mapsto f(s, y, Z_M(s))$ for any fixed martingale $M \in \mathcal{M}^2$ to solutions for BSDE's of the form (3.8) where the pair (Y, M) belongs to $\mathcal{S}^2 \times \mathcal{M}^2$. By hypothesis the process $s \mapsto f(s, x, Y(s), Z_M(s))$ satisfies (3.6) and (3.7). Essentially speaking a combination of these observations show the result in Theorem 3.2. □

3.1. Remark. In the literature functions with the monotonicity property are also called one-sided Lipschitz functions. In fact Theorem 3.2, with $f(t, x, \cdot, \cdot)$ Lipschitz continuous in both variables, will be superseded by Theorem 4.5 in the Lipschitz case and by Theorem 4.6 in case of monotonicity in the second variable and Lipschitz continuity in the third variable. The proof of Theorem 3.2 is part of the results in Section 4. Theorem 5.1 contains a corresponding result for a Markov family of probability measures. Its proof is omitted, it follows the same lines as the proof of Theorem 4.6.

4. Existence and uniqueness of solutions to BSDEs

As explained in the beginning of Section 3, the equation in (3.1) can be phrased in a semi-linear stochastic setting as follows. Find a pair of adapted processes $(Y, M) \in \mathcal{S}^2\left([0, T], \mathbb{R}^k\right) \times \mathcal{M}^2\left([0, T], \mathbb{R}^k\right)$ satisfying (3.2), i.e.,

$$Y(t) = Y(T) + \int_t^T f\left(s, X(s), Y(s), Z_M(s)\right) ds + M(t) - M(T).$$

In fact in the present section we will also suppress the dependence of the generator f as a function of the Markov process X. Instead of a family of measure spaces $(\Omega, \mathcal{F}_T^\tau, \mathbb{P}_{\tau,x})$, like in Section 5, we will consider a single measure space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{0 \le t \le T} = \left(\mathcal{F}_t^0\right)_{0 \le t \le T}$. In Section 5 we will employ the results of this Section 4 to obtain results for Markov processes. As a consequence we write $f(s, Y(s), Z_M(s))$ instead of $f(s, X(s), Y(s), Z_M(s))$. Next we define the spaces $\mathcal{S}^2\left([0, T], \mathbb{R}^k\right)$ and $\mathcal{M}^2\left([0, T], \mathbb{R}^k\right)$: compare with Definitions 2.9 and 2.10.

4.1. Definition. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\mathcal{F}_t$, $t \in [0, T]$, be a filtration on $\mathcal{F}$. Let $t \mapsto Y(t)$ be a stochastic process with values in $\mathbb{R}^k$ which is adapted to the filtration $\mathcal{F}_t$ and which is $\mathbb{P}$-almost surely continuous. Then Y is said to belong to $\mathcal{S}^2\left([0, T], \mathbb{R}^k\right)$ provided that $\mathbb{E}\left[\sup_{t \in [0,T]} |Y(t)|^2\right] < \infty$.

4.2. Definition. The space of $\mathbb{R}^k$-valued martingales in $L^2\left(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^k\right)$ is denoted by $\mathcal{M}^2\left([0, T], \mathbb{R}^k\right)$. So that a continuous martingale $t \mapsto M(t) - M(0)$ belongs to $\mathcal{M}^2\left([0, T], \mathbb{R}^k\right)$ if $\mathbb{E}\left[|M(T) - M(0)|^2\right] < \infty$. Since the process

$$t \mapsto |M(t)|^2 - |M(0)|^2 - \langle M, M \rangle (t) + \langle M, M \rangle (0)$$

is a martingale difference we see that

$$\mathbb{E}\left[|M(T)-M(0)|^2\right]=\mathbb{E}\left[\langle M,M\rangle(T)-\langle M,M\rangle(0)\right], \tag{4.1}$$

and hence a martingale difference $t\mapsto M(t)-M(0)$ in $L^2\left(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^k\right)$ belongs to $\mathcal{M}^2\left([0,T],\mathbb{R}^k\right)$ if and only if $\mathbb{E}\left[\langle M,M\rangle(T)-\langle M,M\rangle(0)\right]$ is finite. By the Burkholder-Davis-Gundy inequality this is the case if and only if

$$\mathbb{E}\left[\sup_{0<t<T}|M(t)-M(0)|^2\right]<\infty.$$

To be precise, let $M(s)$, $t\le s\le T$, be a continuous local L^2-martingale taking values in $\mathbb{R}^k$. Put $M^*(s)=\sup_{t\le\tau\le s}|M(\tau)|$. Fix $0<p<\infty$. The Burkholder-Davis-Gundy inequality says that there exist universal finite and strictly positive constants c_p and C_p such that

$$c_p\mathbb{E}\left[(M^*(s))^{2p}\right]\le\mathbb{E}\left[\langle M(\cdot),M(\cdot)\rangle^p(s)\right]\le C_p\mathbb{E}\left[(M^*(s))^{2p}\right],\quad t\le s\le T. \tag{4.2}$$

If $p=1$, then $c_p=\frac14$, and if $p=\frac12$, then $c_p=\frac18\sqrt2$. For more details and a proof see, e.g., Ikeda and Watanabe [14].

The following theorem will be employed to prove continuity of solutions to BSDEs. It also implies that BSDEs as considered by us possess at most unique solutions. The variables (Y,M) and (Y',M') attain their values in $\mathbb{R}^k$ endowed with its Euclidean inner-product $\langle y',y\rangle=\sum_{j=1}^k y'_jy_j$, y', $y\in\mathbb{R}^k$. Processes of the form $s\mapsto f\left(s,Y(s),Z_M(s)\right)$ are progressively measurable processes whenever the pair (Y,M) belongs to the space mentioned in (4.3) of the next theorem.

4.3. Theorem. Let the pairs (Y,M) and (Y',M'), which belong to the space

$$L^2\left([0,T]\times\Omega,\mathcal{F}_T^0,dt\times\mathbb{P}\right)\times\mathcal{M}^2\left(\Omega,\mathcal{F}_T^0,\mathbb{P}\right), \tag{4.3}$$

be solutions to the following BSDEs:

$$Y(t)=Y(T)+\int_t^T f\left(s,Y(s),Z_M(s)\right)ds+M(t)-M(T),\quad\text{and} \tag{4.4}$$

$$Y'(t)=Y'(T)+\int_t^T f'\left(s,Y'(s),Z_{M'}(s)\right)ds+M'(t)-M'(T) \tag{4.5}$$

for $0\le t\le T$. In particular this means that the processes (Y,M) and (Y',M') are progressively measurable and are square integrable. Suppose that the coefficient f' satisfies the following monotonicity and Lipschitz condition. There exist some positive and finite constants C_1' and C_2' such that the following inequalities hold for all $0\le t\le T$:

$$\langle Y'(t)-Y(t),f'\left(t,Y'(t),Z_{M'}(t)\right)-f'\left(t,Y(t),Z_{M'}(t)\right)\rangle\le\left(C_1'\right)^2|Y'(t)-Y(t)|^2, \tag{4.6}$$

and

$$\left|f'\left(t,Y(t),Z_{M'}(t)\right)-f'\left(t,Y(t),Z_M(t)\right)\right|^2 \leq \left(C_2'\right)^2 \frac{d}{dt}\left\langle M'-M,M'-M\right\rangle(t). \tag{4.7}$$

Then the pair $(Y'-Y,M'-M)$ belongs to $\mathcal{S}^2\left(\Omega,\mathcal{F}_T^0,\mathbb{P};\mathbb{R}^k\right)\times\mathcal{M}^2\left(\Omega,\mathcal{F}_T^0,\mathbb{P};\mathbb{R}^k\right)$, and there exists a constant C' which depends on C_1', C_2' and T such that

$$\begin{aligned}&\mathbb{E}\left[\sup_{0<t<T}\left|Y'(t)-Y(t)\right|^2+\left\langle M'-M,M'-M\right\rangle(T)\right]\\&\leq C'\mathbb{E}\left[\left|Y'(T)-Y(T)\right|^2+\int_0^T\left|f'\left(s,Y(s),Z_M(s)\right)-f\left(s,Y(s),Z_M(s)\right)\right|^2 ds\right].\end{aligned}$$

4.1. Remark. From the proof it follows that for C' we may choose $C'=260e^{\gamma T}$, where $\gamma=1+2\left(C_1'\right)^2+2\left(C_2'\right)^2$.

By taking $Y(T)=Y'(T)$ and $f\left(s,Y(s),Z_M(s)\right)=f'\left(s,Y(s),Z_M(s)\right)$ it also implies that BSDEs as considered by us possess at most unique solutions. A precise formulation reads as follows.

4.4. Corollary. Suppose that the coefficient f satisfies the monotonicity condition (4.6) and the Lipschitz condition (4.7). Then there exists at most one pair $(Y,M)\in L^2\left([0,T]\times\Omega,\mathcal{F}_T^0,dt\times\mathbb{P}\right)\times\mathcal{M}^2\left(\Omega,\mathcal{F}_T^0,\mathbb{P}\right)$ which satisfies the backward stochastic differential equation (4.4).

Proof of Theorem 4.3. Put $\overline{Y}=Y'-Y$ and $\overline{M}=M'-M$. The proof follows from Itô's formula applied to the process $\left|\overline{Y}(t)\right|^2-\left\langle\overline{M},\overline{M}\right\rangle(t)$ in conjunction with the Burkholder-Davis-Gundy inequality (4.2) for $p=\frac{1}{2}$. For more details on the Burkholder-Davis-Gundy inequality, see, e.g., Ikeda and Watanabe [14]. □

In Definitions 4.1 and 4.2 the spaces $\mathcal{S}^2\left([0,T],\mathbb{R}^k\right)$ and $\mathcal{M}^2\left([0,T],\mathbb{R}^k\right)$ are defined.

In Theorem 4.6 we will replace the Lipschitz condition (4.8) in Theorem 4.5 of the function $Y(s)\mapsto f\left(s,Y(s),Z_M(s)\right)$ with the (weaker) monotonicity condition (4.14). Here we write y for the variable $Y(s)$ and z for $Z_M(s)$. It is noticed that we consider a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with a filtration $\left(\mathcal{F}_t\right)_{t\in[0,T]}=\left(\mathcal{F}_t^0\right)_{t\in[0,T]}$ where $\mathcal{F}_T=\mathcal{F}$.

4.5. Theorem. Let $f:[0,T]\times\mathbb{R}^k\times\left(\mathcal{M}^2\right)^*\to\mathbb{R}^k$ be a Lipschitz continuous in the sense that there exists finite constants C_1 and C_2 such that for any two pairs of processes (Y,M) and $(U,N)\in\mathcal{S}^2\left([0,T],\mathbb{R}^k\right)\times\mathcal{M}^2\left([0,T],\mathbb{R}^k\right)$ the following inequalities hold for all $0\leq s\leq T$:

$$\left|f\left(s,Y(s),Z_M(s)\right)-f\left(s,U(s),Z_M(s)\right)\right|\leq C_1\left|Y(s)-U(s)\right|, \text{ and} \tag{4.8}$$

$$\left|f\left(s,Y(s),Z_M(s)\right)-f\left(s,Y(s),Z_N(s)\right)\right|\leq C_2\left(\frac{d}{ds}\left\langle M-N,M-N\right\rangle(s)\right)^{1/2}. \tag{4.9}$$

Suppose that $\mathbb{E}\left[\int_0^T |f(s,0,0)|^2\,ds\right] < \infty$. Then there exists a unique pair $(Y, M) \in \mathcal{S}^2\left([0,T],\mathbb{R}^k\right) \times \mathcal{M}^2\left([0,T],\mathbb{R}^k\right)$ such that

$$Y(t) = \xi + \int_t^T f\left(s, Y(s), Z_M(s)\right) ds + M(t) - M(T), \tag{4.10}$$

where $Y(T) = \xi \in L^2\left(\Omega, \mathcal{F}_T, \mathbb{R}^k\right)$ is given and where $Y(0) = M(0)$.

For brevity we write

$$\mathcal{S}^2\times\mathcal{M}^2 = \mathcal{S}^2\left([0,T],\mathbb{R}^k\right)\times\mathcal{M}^2\left([0,T],\mathbb{R}^k\right) = \mathcal{S}^2\left(\Omega,\mathcal{F}_T^0,\mathbb{P};\mathbb{R}^k\right)\times\mathcal{M}^2\left(\Omega,\mathcal{F}_T^0,\mathbb{P};\mathbb{R}^k\right).$$

In fact we employ this theorem with the function f replaced with f_δ defined by

$$f_\delta\left(s, y, Z_M(s)\right) = f\left(s, \left(I - \delta f_{s,M}\right)^{-1}, Z_M(s)\right), \quad 0 < \delta < \frac{1}{2C_1}. \tag{4.11}$$

Here $f_{s,M}(y) = f\left(s, y, Z_M(s)\right)$. If the function f is monotone (or one-sided Lipschitz) in the second variable with constant C_1, and Lipschitz in the second variable with constant C_2, then the function f_δ is Lipschitz in y with constant δ^{-1}.

Proof. The proof of the uniqueness part follows from Corollary 4.4.

In order to prove existence we proceed as follows. By induction we define a sequence (Y_n, M_n) in the space $\mathcal{S}^2 \times \mathcal{M}^2$ as follows.

$$Y_{n+1}(t) = \mathbb{E}\left[\xi + \int_t^T f\left(s, Y_n(s), Z_{M_n}(s)\right) ds \;\middle|\; \mathcal{F}_t\right], \text{ and} \tag{4.12}$$

$$M_{n+1}(t) = \mathbb{E}\left[\xi + \int_0^T f\left(s, Y_n(s), Z_{M_n}(s)\right) ds \;\middle|\; \mathcal{F}_t\right], \tag{4.13}$$

Then, since the process $s \mapsto f\left(s, Y_n(s), M_n(s)\right)$ is adapted and belongs to $\mathcal{S}^2\times\mathcal{M}^2$, it is possible to prove, using for example Itô calculus and Burkholder-Davis-Gundy inequality with $p = \frac{1}{2}$ (see (4.2)), that the sequence (Y_n, M_n) converges in the space $\mathcal{S}^2 \times \mathcal{M}^2$. □

In the following theorem we replace the Lipschitz condition (4.8) in Theorem 4.5 for the function $Y(s) \mapsto f\left(s, Y(s), Z_M(s)\right)$ with the (weaker) monotonicity condition (4.14). Here we write y for the variable $Y(s)$ and z for $Z_M(s)$.

4.6. Theorem. Let $f : [0,T] \times \mathbb{R}^k \times \left(\mathcal{M}^2\right)^* \to \mathbb{R}^k$ be monotone in the variable y and Lipschitz in z. More precisely, suppose that there exist finite constants C_1 and C_2 such that for any two pairs of processes (Y, M) and (U, N) in the space $\mathcal{S}^2\left([0,T],\mathbb{R}^k\right) \times \mathcal{M}^2\left([0,T],\mathbb{R}^k\right)$ the following inequalities hold for all $0 \le s \le T$:

$$\left\langle Y(s) - U(s), f\left(s, Y(s), Z_M(s)\right) - f\left(s, U(s), Z_M(s)\right)\right\rangle \le C_1 \left|Y(s) - U(s)\right|^2, \tag{4.14}$$

$$\left|f\left(s, Y(s), Z_M(s)\right) - f\left(s, Y(s), Z_N(s)\right)\right| \le C_2 \left(\frac{d}{ds}\left\langle M - N, M - N\right\rangle(s)\right)^{1/2},$$

and

$$|f(s, Y(s), 0)| \leq \overline{f}(s) + K |Y(s)|. \tag{4.15}$$

If $\mathbb{E}\left[\int_0^T \left|\overline{f}(s)\right|^2 ds\right] < \infty$, then there exists a unique pair $(Y, M) \in \mathcal{S}^2\left([0,T], \mathbb{R}^k\right) \times \mathcal{M}^2\left([0,T], \mathbb{R}^k\right)$ such that

$$Y(t) = \xi + \int_t^T f(s, Y(s), Z_M(s))\, ds + M(t) - M(T), \tag{4.16}$$

where $Y(T) = \xi \in L^2\left(\Omega, \mathcal{F}_T, \mathbb{R}^k\right)$ is given and where $Y(0) = M(0)$.

The proof of Theorem 4.6 may be based on the next proposition. Its proof uses the monotonicity condition (4.14) in an explicit manner.

4.7. Proposition. Suppose that for every $\xi \in L^2\left(\Omega, \mathcal{F}_T^0, \mathbb{P}\right)$ and $M \in \mathcal{M}^2$ there exists a pair $(Y, N) \in \mathcal{S}^2 \times \mathcal{M}^2$ such that

$$Y(t) = \xi + \int_t^T f(s, Y(s), Z_M(s))\, ds + N(t) - N(T). \tag{4.17}$$

Then for every $\xi \in L^2\left(\Omega, \mathcal{F}_T^0, \mathbb{P}\right)$ there exists a unique pair $(Y, M) \in \mathcal{S}^2 \times \mathcal{M}^2$ which satisfies (4.16).

The following proposition can be viewed as a consequence of Theorem 12.4 in [13]. The result is due to Burrage and Butcher [9] and Crouzeix [11]. The obtained constants are somewhat different from ours. If $C_1 = 0$, then they agree. The proof is omitted. The surjectivity of the mapping $y \mapsto y - \delta f(t, y, Z_M(t))$ is a consequence of Theorem 1 in Croezeix et al. [10].

4.8. Proposition. Fix a martingale $M \in \mathcal{M}^2$, and choose $\delta > 0$ in such a way that $\delta C_1 < 1$. Here C_1 is the constant which occurs in inequality (4.14). Choose, for given $y \in \mathbb{R}^k$, the stochastic variable $\widetilde{Y}(t) \in \mathbb{R}^k$ in such a way that $y = \widetilde{Y}(t) - \delta f\left(t, \widetilde{Y}(t), Z_M(t)\right)$. Then the mapping $y \mapsto f\left(t, \widetilde{Y}(t), Z_M(t)\right)$ is Lipschitz continuous with a Lipschitz constant equal to $\dfrac{1}{\delta} \max\left(1, \dfrac{\delta C_1}{1 - \delta C_1}\right)$. Moreover, the mapping $y \mapsto y - \delta f(t, y, Z_M(t))$ is surjective and has a Lipschitz continuous inverse with Lipschitz constant $\dfrac{1}{1 - \delta C_1}$.

4.9. Corollary. For $\delta > 0$ such that $\delta C_1 < 1$ there exist processes Y_δ and $\widetilde{Y}_\delta \in \mathcal{S}^2$ and a martingale $M_\delta \in \mathcal{M}^2$ such that the following equalities are satisfied:

$$\begin{aligned} Y_\delta(t) &= \widetilde{Y}_\delta(t) - \delta f\left(t, \widetilde{Y}_\delta(t), Z_M(t)\right) \\ &= Y_\delta(T) + \int_t^T f\left(s, \widetilde{Y}_\delta(s), Z_M(s)\right) ds + M_\delta(t) - M_\delta(T). \end{aligned} \tag{4.18}$$

Proof. Suppose $0 < \delta C_1 < 1$. From Theorem 1 (page 87) in Crouzeix et al. [10] it follows that the mapping $y \mapsto y - \delta f\left(t, y, Z_M(t)\right)$ is a surjective map from $\mathbb{R}^k$ onto itself. If y_2 and y_1 in $\mathbb{R}^k$ are such that $y_2 - \delta f\left(t, y_2, Z_M(t)\right) = y_1 - \delta f\left(t, y_1, Z_M(t)\right)$. Then

$$\left|y_2 - y_1\right|^2 = \left\langle y_2 - y_1, \delta f\left(t, y_2, Z_M(t)\right) - \delta f\left(t, y_1, Z_M(t)\right)\right\rangle \leq \delta C_1 \left|y_2 - y_1\right|^2,$$

and hence $y_2 = y_1$. It follows that the continuous mapping $y \mapsto y - \delta f\left(t, y, Z_M(t)\right)$ has a continuous inverse. Denote this inverse by $\left(I - \delta f_{t,M}\right)^{-1}$. Moreover, from Proposition 4.8 it follows that the mapping $y \mapsto f\left(t, \left(I - \delta f_{t,M}\right)^{-1} y, Z_M(t)\right)$ is Lipschitz continuous with Lipschitz constant δ^{-1} where $\delta > 0$ is such that $0 < 2\delta C_1 < 1$. The remaining assertions in Corollary 4.9 are consequences of Theorem 4.5 where the Lipschitz condition in (4.8) was used with δ^{-1} instead of C_1. This establishes the proof of Corollary 4.9. □

4.2. Remark. The surjectivity property of the mapping $y \mapsto y - \delta f\left(s, y, Z_M(s)\right)$ follows from Theorem 1 in [10]. The authors use a homotopy argument to prove this theorem for $C_1 = 0$. Upon replacing $f\left(t, y, Z_M(t)\right)$ with

$$f_{C_1}\left(t, y, Z_M(t)\right) = e^{tC_1} f\left(t, e^{-tC_1} y, e^{-tC_1} Z_M(t)\right) - C_1 y$$

the result follows in our version: see Theorem 4.11. An elementary proof of Theorem 1 in [10] can be found for a continuously differentiable function in Hairer and Wanner [13]: see Theorem 14.2 in Chapter IV. The author is grateful to Karel in't Hout (University of Antwerp) for pointing out Runge-Kutta type results and these references.

Proof of Proposition 4.7. The proof of the uniqueness part follows from Corollary 4.4.

Fix $\xi \in L^2\left(\Omega, \mathcal{F}_T^0, \mathbb{P}\right)$, and let the martingale $M_{n-1} \in \mathcal{M}^2$ be given. Then by hypothesis there exists a pair $\left(Y_n, M_n\right) \in \mathcal{S}^2 \times \mathcal{M}^2$ which satisfies:

$$Y_n(t) = \xi + \int_t^T f\left(s, Y_n(s), Z_{M_{n-1}}(s)\right) ds + M_n(t) - M_n(T). \tag{4.19}$$

Another use of this hypothesis yields the existence of a pair $\left(Y_{n+1}, M_{n+1}\right) \in \mathcal{S}^2 \times \mathcal{M}^2$ which again satisfies (4.19) with $n+1$ instead of n. Then it can be proved that the sequence $\left(Y_n, M_n\right)$ is a Cauchy sequence in the space $\mathcal{S}^2 \times \mathcal{M}^2$. Again the Burkholder-Davis-Gundy inequality with $p = \frac{1}{2}$ (see (4.2)) is required. □

4.10. Proposition. Let the notation and hypotheses be as in Theorem 4.6. Let for $\delta > 0$ with $2\delta C_1 < 1$ the processes Y_δ, $\widetilde{Y}_\delta \in \mathcal{S}^2$ and the martingale $M_\delta \in \mathcal{M}^2$ be such that the equalities of (4.18) in Corollary 4.9 are satisfied. Then the family $\left\{\left(Y_\delta, M_\delta\right) : 0 < \delta < \dfrac{1}{2C_1}\right\}$ converges in the space $\mathcal{S}^2 \times \mathcal{M}^2$ if δ decreases to 0, provided that the terminal value $\xi = Y_\delta(T)$ is given.

Let (Y, M) be the limit in the space $\mathcal{S}^2 \times \mathcal{M}^2$. In fact from the proof of Proposition 4.10 it may be deduced that the speed of convergence is $\mathcal{O}(\delta)$ provided that $\|Y_{\delta_2}(T) - Y_{\delta_1}(T)\|_{L^2\left(\Omega, \mathcal{F}_T^0, \mathbb{P}\right)} = \mathcal{O}\left(|\delta_2 - \delta_1|\right)$.

Outline of a proof of Proposition 4.10. Let C_1 be the constant which occurs in inequality (4.14) in Theorem 4.6, and fix $0 < \delta_2 < \delta_1 < (2C_1)^{-1}$. Our estimates give quantitative bounds in case we restrict the parameters δ, δ_1 and δ_2 to the interval $\left(0, (4C_1+4)^{-1}\right)$. From the equalities in (4.18) we infer

$$Y_\delta(t) = \widetilde{Y}_\delta(t) - \delta \widetilde{f}_\delta(t) = Y_\delta(T) + \int_t^T \widetilde{f}_\delta(s) ds + M_\delta(t) - M_\delta(T). \tag{4.20}$$

First we prove that the family $\left\{(Y_\delta, M_\delta) : 0 < \delta < (4C_1+4)^{-1}\right\}$ is bounded in the space $\mathcal{S}^2 \times \mathcal{M}^2$. Then it can also be shown that this family converges in the space $\mathcal{S}^2 \times \mathcal{M}^2$ as $\delta \downarrow 0$. The continuity of the functions $y \mapsto f(s, y, Z_M(s))$, $y \in \mathbb{R}^k$. The fact that the convergence of the family (Y_δ, M_δ), $0 < \delta \le (4C_1+4)^{-1}$ is of order δ, as $\delta \downarrow 0$, can be shown as well. □

Proof of Theorem 4.6. The proof of the uniqueness part follows from Corollary 4.4. The existence is a consequence of Theorem 4.5, Proposition 4.10 and Corollary 4.9. □

The following result shows that in the monotonicity condition we may always assume that the constant C_1 can be chosen as we like provided in Theorem 4.11 we replace the equation in (1) by the one in (2) and adapt its solution.

4.11. Theorem. Let the pair (Y, M) belong to $\mathcal{S}^2\left([0,T], \mathbb{R}^k\right) \times \mathcal{M}^2\left([0,T], \mathbb{R}^k\right)$. Fix $\lambda \in \mathbb{R}$ and put $(Y_\lambda(t), M_\lambda(t)) = \left(e^{\lambda t} Y(t), Y(0) + \int_0^t e^{\lambda s} dM(s)\right)$. Then the pair (Y_λ, M_λ) belongs to $\mathcal{S}^2 \times \mathcal{M}^2$. Moreover, the following assertions are equivalent:

1. The pair $(Y, M) \in \mathcal{S}^2 \times \mathcal{M}^2$ satisfies $Y(0) = M(0)$ and
$$Y(t) = Y(T) + \int_t^T f\left(s, Y(s), Z_M(s)\right) ds + M(t) - M(T).$$
2. The pair (Y_λ, M_λ) satisfies $Y_\lambda(0) = M_\lambda(0)$ and
$$Y_\lambda(t) = Y_\lambda(T) + \int_t^T e^{\lambda s} f_\lambda\left(s, e^{-\lambda s} Y_\lambda(s), e^{-\lambda s} Z_{M_\lambda}(s)\right) ds \tag{4.21}$$
where $f_\lambda(s, y, z) = e^{\lambda s} f\left(s, e^{-\lambda s} y, e^{-\lambda s} z\right) - \lambda y$.

4.3. Remark. If the function $y \mapsto f(s,y,z)$ has monotonicity constant C_1, then the function $y \mapsto f_\lambda(s,y,z)$ has monotonicity constant $C_1 - \lambda$. It follows that by reformulating the problem one always may assume that the monotonicity constant is 0.

Proof of Theorem 4.11. First notice the equality $e^{-\lambda s} Z_{M_\lambda}(s) = Z_M(s)$: see Remark 2.7. The equivalence of (1) and (2) follows by considering the equalities in (1) and (2) in differential form. □

5. Backward stochastic differential equations and Markov processes

In this section the coefficient f of our BSDE is a mapping from $[0,T]\times E\times\mathbb{R}^k\times\left(\mathcal{M}^2\right)^*$ to $\mathbb{R}^k$. Theorem 5.1 below is the analogue of Theorem 4.6 with a Markov family of measure spaces $\{\mathbb{P}_{\tau,x}:(\tau,x)\in[0,T]\times E\}$ instead of a single measure space. Put

$$f_n(s)=f\left(s,X(s),Y_n(s),Z_{M_n}(s)\right),$$

and suppose that the processes $Y_n(s)$ and $Z_{M_n}(s)$ only depend of the state-time variable $(s,X(s))$. Suppose that for every $f\in C_0(E)$ the function $(\tau,x,t)\mapsto\mathbb{E}_{\tau,x}\left[f\left(X(t)\right)\right]$ is continuous on the set $\{(\tau,x,t)\in[0,T]\times E\times[0,T]:\tau\le t\le T\}$. Then it can be proved that the Markov process

$$\left\{\left(\Omega,\mathcal{F}_T^\tau,\mathbb{P}_{\tau,x}\right),\left(X(t):T\ge t\ge 0\right),(E,\mathcal{E})\right\} \tag{5.1}$$

has left limits and is right-continuous: see, e.g., Theorem 2.22 in [12], and Proposition 2.6 in Section 1. Suppose that the $\mathbb{P}_{\tau,x}$-martingale $t\mapsto N(t)-N(\tau)$, $t\in[\tau,T]$, belongs to the space $\mathcal{M}^2\left([\tau,T],\mathbb{P}_{\tau,x},\mathbb{R}^k\right)$ (see Definition 2.10). It follows that the quantity $Z_M(s)(N)$ is measurable with respect to $\sigma\left(\mathcal{F}_{s+}^s,N(s+)\right)$: see equalities the (5.4), (5.5) and (5.6) below. The following iteration formulas play an important role:

$$\begin{aligned}Y_{n+1}(t)&=\mathbb{E}_{t,X(t)}\left[\xi\right]+\int_t^T\mathbb{E}_{t,X(t)}\left[f_n(s)\right]ds,\quad\text{and}\\ M_{n+1}(t)&=\mathbb{E}_{t,X(t)}\left[\xi\right]+\int_0^t f_n(s)ds+\int_t^T\mathbb{E}_{t,X(t)}\left[f_n(s)\right]ds.\end{aligned}$$

Then the processes Y_{n+1} and M_{n+1} are related as follows:

$$Y_{n+1}(T)+\int_t^T f_n(s)ds+M_{n+1}(t)-M_{n+1}(T)=Y_{n+1}(t).$$

Moreover, by the Markov property, the process

$$\begin{aligned}&t\mapsto M_{n+1}(t)-M_{n+1}(\tau)\\ &=\mathbb{E}_{\tau,X(\tau)}\left[\xi+\int_\tau^T f_n(s)ds\;\Big|\;\mathcal{F}_t^\tau\right]-\mathbb{E}_{\tau,X(\tau)}\left[\xi+\int_\tau^T f_n(s)ds\right]\end{aligned}$$

is a $\mathbb{P}_{\tau,x}$-martingale on the interval $[\tau,T]$ for every $(\tau,x)\in[0,T]\times E$.

In Theorem 5.1 below we replace the Lipschitz condition (4.8) in Theorem 4.5 for the function $Y(s)\mapsto f\left(s,Y(s),Z_M(s)\right)$ with the (weaker) monotonicity condition (5.7) for the function $Y(s)\mapsto f\left(s,X(s),Y(s),Z_M(s)\right)$. Sometimes we write y for the variable $Y(s)$ and z for $Z_M(s)$. Notice that the functional $Z_{M_n}(t)$ only depends on $\mathcal{F}_{t+}^t:=\bigcap_{h:T\ge t+h>t}\sigma\left(X(t+h)\right)$ and that this σ-field belongs the $\mathbb{P}_{t,x}$-completion of $\sigma\left(X(t)\right)$ for every $x\in E$. This is the case, because by assumption the process $s\mapsto X(s)$ is right-continuous at $s=t$: see Proposition 2.7. In order to show this we have to prove equalities of the following type:

$$\mathbb{E}_{s,x}\left[Y\;\big|\;\mathcal{F}_{t+}^s\right]=\mathbb{E}_{t,X(t)}\left[Y\right],\quad\mathbb{P}_{s,x}\text{-almost surely,} \tag{5.2}$$

for all bounded stochastic variables which are $\mathcal{F}_T^t$-measurable. By the monotone class theorem and density arguments the proof of (5.2) reduces to showing these equalities for $Y = \prod_{j=1}^n f_j\left(t_j, X\left(t_j\right)\right)$, where $t = t_1 < t_2 < \cdots < t_n \leq T$, and the functions $x \mapsto f_j\left(t_j, x\right)$, $1 \leq j \leq n$, belong to the space $C_0(E)$. Next suppose that the bounded stochastic variable Y is measurable with respect to $\mathcal{F}_{t+}^t$. From (5.2) with $s = t$ it follows that $Y = \mathbb{E}_{t,X(t)}\left[Y\right]$, $\mathbb{P}_{t,x}$-almost surely. Hence, essentially speaking, such a variable Y only depends on the space-time variable $(t, X(t))$. Since $X(t) = x$ $\mathbb{P}_{t,x}$-almost surely it follows that the variable $\mathbb{E}_{t,x}\left[Y \,\middle|\, \mathcal{F}_{t+}^t\right]$ is $\mathbb{P}_{t,x}$-almost equal to the deterministic constant $\mathbb{E}_{t,x}\left[Y\right]$. A similar argument shows the following result. Let $0 \leq s < t \leq T$, and let Y be a bounded $\mathcal{F}_T^s$-measurable stochastic variable. Then the following equality holds $\mathbb{P}_{s,x}$-almost surely:

$$\mathbb{E}_{s,x}\left[Y \,\middle|\, \mathcal{F}_{t+}^s\right] = \mathbb{E}_{s,x}\left[Y \,\middle|\, \mathcal{F}_t^s\right] \tag{5.3}$$

In particular it follows that an $\mathcal{F}_{t+}^s$-measurable bounded stochastic variable coincides with the $\mathcal{F}_t^s$-measurable variable $\mathbb{E}_{s,x}\left[Y \,\middle|\, \mathcal{F}_t^s\right]$ $\mathbb{P}_{s,x}$-almost surely for all $x \in E$. Hence (5.3) implies that the σ-field $\mathcal{F}_{t+}^s$ is contained in the $\mathbb{P}_{s,x}$-completion of the σ-field $\mathcal{F}_t^s$.

In addition, notice that the functional $Z_M(s)$ is defined by

$$Z_M(s)(N) = \lim_{t \downarrow s} \frac{\langle M, N\rangle(t) - \langle M, N\rangle(s)}{t - s} \tag{5.4}$$

where

$$\langle M, N\rangle(t) - \langle M, N\rangle(s) = \lim_{n \to \infty} \sum_{j=0}^{2^n - 1} \left(M\left(t_{j+1,n}\right) - M\left(t_{j,n}\right)\right)\left(N\left(t_{j+1,n}\right) - N\left(t_{j,n}\right)\right). \tag{5.5}$$

For this the reader is referred to the remarks 2.7, 2.8, and to formula (2.22). The symbol $t_{j,n}$ represents the real number $t_{j,n} = s + j2^{-n}(t - s)$. The limit in (5.5) exists $\mathbb{P}_{\tau,x}$-almost surely for all $\tau \in [0, s]$. As a consequence the process $Z_M(s)$ is $\mathcal{F}_{s+}^\tau$-measurable for all $\tau \in [0, s]$. It follows that the process $N \mapsto Z_M(s)(N)$ is $\mathbb{P}_{\tau,x}$-almost surely equal to the functional $N \mapsto \mathbb{E}_{\tau,x}\left[Z_M(s)(N) \,\middle|\, \sigma\left(\mathcal{F}_s^\tau, N(s)\right)\right]$ provided that $Z_M(s)(N)$ is $\sigma\left(\mathcal{F}_{s+}^\tau, N(s+)\right)$-measurable. If the martingale M is of the form $M(s) = u\left(s, X(s)\right) + \int_0^s f(\rho)d\rho$, then the functional $Z_M(s)(N)$ is automatically $\sigma\left(\mathcal{F}_{s+}^s, N(s+)\right)$-measurable. It follows that, for every $\tau \in [0, s]$, the following equality holds $\mathbb{P}_{\tau,x}$-almost surely:

$$\mathbb{E}_{\tau,x}\left[Z_M(s)(N) \,\middle|\, \sigma\left(\mathcal{F}_{s+}^\tau, N(s+)\right)\right] = \mathbb{E}_{\tau,x}\left[Z_M(s)(N) \,\middle|\, \sigma\left(\mathcal{F}_s^\tau, N(s+)\right)\right]. \tag{5.6}$$

In the next Theorem 5.1 the filtered probability space $\left(\Omega, \mathcal{F}, \left(\mathcal{F}_t^0\right)_{t \in [0,T]}, \mathbb{P}\right)$ of Section 4 is replaced with a Markov family of measure spaces

$$\left(\Omega, \mathcal{F}_T^\tau, \left(\mathcal{F}_t^\tau\right)_{\tau \leq t \leq T}, \mathbb{P}_{\tau,x}\right), \quad (\tau, x) \in [0, T] \times E.$$

Its proof follows the lines of the proof of Theorem 4.6: it will not be given here. At the relevant places the measure $\mathbb{P}_{\tau,x}$ replaces $\mathbb{P}$ and $f(s, Y(s), Z_M(s))$ is replaced with the coefficient $f(s, X(s), Y(s), Z_M(s))$.

5.1. Theorem. Let $f : [0,T] \times E \times \mathbb{R}^k \times \left(\mathcal{M}^2\right)^* \to \mathbb{R}^k$ be monotone in the variable y and Lipschitz in z. More precisely, suppose that there exist finite constants C_1 and C_2 such that for any two pairs of processes (Y, M) and (U, N) in the space $\mathcal{S}^2\left([0,T], \mathbb{R}^k\right) \times \mathcal{M}^2\left([0,T], \mathbb{R}^k\right)$ the following inequalities hold for all $0 \le s \le T$:

$$\begin{aligned}
&\langle Y(s) - U(s), f(s, X(s), Y(s), Z_M(s)) - f(s, X(s), U(s), Z_M(s))\rangle \\
&\quad \le C_1 \left|Y(s) - U(s)\right|^2, \qquad (5.7) \\
&\left|f(s, X(s), Y(s), Z_M(s)) - f(s, X(s), Y(s), Z_N(s))\right| \\
&\quad \le C_2 \left(\frac{d}{ds}\langle M - N, M - N\rangle (s)\right)^{1/2}, \qquad (5.8)
\end{aligned}$$

and

$$\left|f(s, X(s), Y(s), 0)\right| \le \overline{f}(s, X(s)) + K\left|Y(s)\right|. \tag{5.9}$$

Fix $(\tau, x) \in [0,T] \times E$ and let $Y(T) = \xi \in L^2\left(\Omega, \mathcal{F}_T^\tau, \mathbb{P}_{\tau,x}; \mathbb{R}^k\right)$ be given. In addition, suppose $\mathbb{E}_{\tau,x}\left[\int_\tau^T \left|\overline{f}(s, X(s))\right|^2 ds\right] < \infty$. Then there exists a unique pair

$$(Y, M) \in \mathcal{S}^2\left([\tau,T], \mathbb{P}_{\tau,x}, \mathbb{R}^k\right) \times \mathcal{M}^2\left([\tau,T], \mathbb{P}_{\tau,x}, \mathbb{R}^k\right)$$

with $Y(\tau) = M(\tau)$ such that

$$Y(t) = \xi + \int_t^T f(s, X(s), Y(s), Z_M(s))\, ds + M(t) - M(T). \tag{5.10}$$

Next let $\xi = \mathbb{E}_{T,X(T)}[\xi] \in \bigcap_{(\tau,x)\in[0,T]\times E} L^2\left(\Omega, \mathcal{F}_T^\tau, \mathbb{P}_{\tau,x}\right)$ be given. Suppose that the functions $(\tau, x) \mapsto \mathbb{E}_{\tau,x}\left[|\xi|^2\right]$ and $(\tau, x) \mapsto \mathbb{E}_{\tau,x}\left[\int_\tau^T \left|\overline{f}(s, X(s))\right|^2 ds\right]$ are locally bounded. Then there exists a unique pair

$$(Y, M) \in \mathcal{S}^2_{\mathrm{loc,unif}}\left([\tau,T], \mathbb{R}^k\right) \times \mathcal{M}^2_{\mathrm{loc,unif}}\left([\tau,T], \mathbb{R}^k\right)$$

with $Y(0) = M(0)$ such that equation (5.10) is satisfied.

Again let $\xi = \mathbb{E}_{T,X(T)}[\xi] \in \bigcap_{(\tau,x)\in[0,T]\times E} L^2\left(\Omega, \mathcal{F}_T^\tau, \mathbb{P}_{\tau,x}\right)$ be given. Suppose that the functions $(\tau, x) \mapsto \mathbb{E}_{\tau,x}\left[|\xi|^2\right]$ and $(\tau, x) \mapsto \mathbb{E}_{\tau,x}\left[\int_\tau^T \left|\overline{f}(s, X(s))\right|^2 ds\right]$ are uniformly bounded. Then there exists a unique pair

$$(Y, M) \in \mathcal{S}^2_{\mathrm{unif}}\left([\tau,T], \mathbb{R}^k\right) \times \mathcal{M}^2_{\mathrm{unif}}\left([\tau,T], \mathbb{R}^k\right)$$

with $Y(0) = M(0)$ such that equation (5.10) is satisfied.

The notations

$$\begin{aligned}
\mathcal{S}^2\left([\tau,T], \mathbb{P}_{\tau,x}, \mathbb{R}^k\right) &= \mathcal{S}^2\left(\Omega, \mathcal{F}_T^\tau, \mathbb{P}_{\tau,x}; \mathbb{R}^k\right) \quad \text{and} \\
\mathcal{M}^2\left([\tau,T], \mathbb{P}_{\tau,x}, \mathbb{R}^k\right) &= \mathcal{M}^2\left(\Omega, \mathcal{F}_T^\tau, \mathbb{P}_{\tau,x}; \mathbb{R}^k\right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{S}^2_{\text{loc,unif}}\left([0,T],\mathbb{R}^k\right) &= \mathcal{S}^2_{\text{loc,unif}}\left(\Omega,\mathcal{F}^\tau_T,\mathbb{P}_{\tau,x};\mathbb{R}^k\right), \quad \text{and}\\
\mathcal{M}^2_{\text{loc,unif}}\left([0,T],\mathbb{R}^k\right) &= \mathcal{M}^2_{\text{loc,unif}}\left(\Omega,\mathcal{F}^\tau_T,\mathbb{P}_{\tau,x};\mathbb{R}^k\right),\\
\mathcal{S}^2_{\text{unif}}\left([0,T],\mathbb{R}^k\right) &= \mathcal{S}^2_{\text{unif}}\left(\Omega,\mathcal{F}^\tau_T,\mathbb{P}_{\tau,x};\mathbb{R}^k\right), \quad \text{and}\\
\mathcal{M}^2_{\text{unif}}\left([0,T],\mathbb{R}^k\right) &= \mathcal{M}^2_{\text{unif}}\left(\Omega,\mathcal{F}^\tau_T,\mathbb{P}_{\tau,x};\mathbb{R}^k\right)
\end{aligned}$$

are explained in Definitions 2.9 and 2.10 respectively. The probability measure $\mathbb{P}_{\tau,x}$ is defined on the σ-field $\mathcal{F}^\tau_T$. Since the existence properties of the solutions to backward stochastic equations are based on explicit inequalities, the proofs carry over to Markov families of measures. Ultimately these inequalities imply that boundedness and continuity properties of the function $(\tau,x)\mapsto \mathbb{E}_{\tau,x}\left[Y(t)\right]$, $0\le\tau\le t\le T$, depend the continuity of the function $x\mapsto \mathbb{E}_{T,x}\left[\xi\right]$, where ξ is a terminal value function which is supposed to be $\sigma\left(X(T)\right)$-measurable. In addition, in order to be sure that the function $(\tau,x)\mapsto \mathbb{E}_{\tau,x}\left[Y(t)\right]$ is continuous, functions of the form $(\tau,x)\mapsto \mathbb{E}_{\tau,x}\left[f\left(t,u\left(t,X(t)\right),Z_M(t)\right)\right]$ have to be continuous, whenever the following mappings

$$(\tau,x)\mapsto \mathbb{E}_{\tau,x}\left[\int_\tau^T \left|u(s,X(s))\right|^2 ds\right] \quad \text{and} \quad (\tau,x)\mapsto \mathbb{E}_{\tau,x}\left[\langle M,M\rangle(T)-\langle M,M\rangle\right]$$

represent finite and continuous functions.

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Generation of Cosine Families on $L^p(0,1)$ by Elliptic Operators with Robin Boundary Conditions

Ralph Chill, Valentin Keyantuo and Mahamadi Warma

We dedicate this work to Günter Lumer in admiration

Abstract. Let $a \in W^{1,\infty}(0,1)$, $a(x) \geq \alpha > 0$, $b, c \in L^\infty(0,1)$ and consider the differential operator A given by $Au = au'' + bu' + cu$. Let $\alpha_j,\ \beta_j$ $(j = 0,1)$ be complex numbers satisfying $(\alpha_j, \beta_j) \neq (0,0)$ for $j = 0,1$. We prove that a realization of A with the boundary conditions

$$\alpha_j u'(j) + \beta_j u(j) = 0, \quad j = 0,1,$$

generates a cosine family on $L^p(0,1)$ for every $p \in [1,\infty)$. This result is obtained by an explicit calculation, using simply d'Alembert's formula, of the solutions in the case of the Laplace operator.

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Keywords. Wave equation, cosine function, second-order elliptic operators, Robin boundary conditions.

1. Introduction

We study well-posedness of the linear, one-dimensional wave equation with initial and boundary conditions given by

$$\begin{cases} u_{tt} = a(x)u_{xx} + b(x)u_x + c(x)u =: Au, & x \in (0,1), \quad t \geq 0, \\ u(0,x) = u_0(x), & x \in (0,1), \\ u_t(0,x) = u_1(x), & x \in (0,1), \\ \alpha_0 u_x(t,0) + \beta_0 u(t,0) = 0, & t \geq 0, \\ \alpha_1 u_x(t,1) + \beta_1 u(t,1) = 0, & t \geq 0. \end{cases} \tag{1.1}$$

We suppose that the coefficients b and c belong to $L^\infty(0,1)$ and that $a \in W^{1,\infty}(0,1)$ satisfies $a(x) \geq \alpha > 0$ for some constant α. The initial values u_0 and u_1 are given in some function spaces to be made precise, and the constants α_j and β_j are complex numbers satisfying $(\alpha_j, \beta_j) \neq (0,0)$ for $j = 0, 1$.

The case $\alpha_j \neq 0$ corresponds to Neumann or Robin boundary conditions. Dirichlet boundary conditions are represented by the choice $\alpha_j = 0$ (while then $\beta_j \neq 0$). The two boundary conditions at the end points of the interval $(0,1)$ may be of different type, that is, we allow also mixed boundary conditions.

The aim of this paper is to prove well-posedness of the problem (1.1) in $L^p(0,1)$ for every $p \in [1,\infty)$ and for every type of proposed boundary conditions. The well-posedness on a space of continuous functions will be also investigated. Recall that the problem (1.1) is well posed in some Banach function space if for every choice of initial values u_0, u_1 in that Banach space there exists a unique mild solution; see [1] or Section 2 below for precise definitions and equivalent formulations of well-posedness.

It is well known that the wave equation with $Au = u_{xx}$ and with Dirichlet boundary conditions is well posed on $C_0((0,1))$; see [2], [6], [14], [16]. For more general second-order differential operators with Dirichlet boundary conditions on $C_0((0,1))$ we refer to [2] or [16]. Generation results for second-order differential operators on $W^{1,p}(0,1)$ resp. $L^p(0,1)$ $(1 \leq p < \infty)$ with Dirichlet or on $W^{1,p}(0,1)$ resp. $L^p(0,1)\times\mathbb{C}^2$ with Wentzell type boundary conditions are contained in [3] resp. [9], [13] and [17]. Note that in the case of the Laplace operator with Dirichlet or Neumann boundary conditions, it is a classical fact that solutions can be computed explicitly by considering first odd (resp. even) and then 2-periodic extensions of the initial values and by using d'Alembert's formula. This fact was mainly used in the above mentioned articles. In [8], solutions in $L^p(0,1)$ were obtained in terms of Fourier series. Their summability in L^p was derived from Gaussian estimates with the advantage that the technique generalizes to the higher-dimensional case.

As in the literature cited above, our approach to proving well-posedness is based on two ideas: first, in the case of the Laplace operator, we will obtain an explicit local solution of (1.1) by simply using d'Alembert's formula and by solving suitable ordinary differential equations; that is, we show that d'Alembert's formula is also applicable when considering Robin boundary conditions. Second, we show that the general case follows from the case of the Laplace operator by a similarity argument and by perturbation. Such similarity and perturbation arguments were used in [2] for Dirichlet boundary conditions, but see also [4, Proof of Theorem 4.3, p. 387] (for the similarity argument) and [3], [9].

Although we are able to treat all types of proposed boundary conditions, we concentrate on the case of Robin (and mixed) boundary conditions.

The paper is organized as follows. In Section 2, we recall some well-known results on cosine families and we state our main results. In Section 3, we investigate explicitly the Laplace operator. We show that a realization of the Laplace operator with Robin boundary conditions generates a cosine function in $L^p(0,1)$ and in

$C([0, 1])$. We also obtain an explicit representation of the associated cosine function for small times. Section 4 contains the proofs of the main results. Finally, in Section 5, we show that the results obtained on the interval $(0, 1)$ can be extended to unbounded intervals.

2. Preliminaries and results

If X is a Banach space, then we denote by $\mathcal{L}(X)$ the space of all bounded linear operators on X.

In a Banach space X, a *cosine family* is a strongly continuous family $(C(t))_{t\in\mathbb{R}} \subset \mathcal{L}(X)$ which satisfies $C(0) = I$ and the abstract functional equation known as *d'Alembert's equation*:

$$C(t + s) + C(t - s) = 2C(t)C(s), \; t, s \in \mathbb{R}.$$

The infinitesimal generator A of a cosine family $(C(t))_{t\in\mathbb{R}}$ is defined by:

$$D(A) := \left\{ x \in X, \lim_{t\to 0} \frac{C(t)x - x}{t^2} \text{ exists} \right\},$$
$$Ax := 2 \lim_{t\to 0} \frac{C(t)x - x}{t^2} \quad \text{for} \quad x \in D(A).$$

It is known that A is closed and densely defined. The operator family $(S(t))_{t\in\mathbb{R}}$ given by

$$S(t) := \int_0^t C(s)ds, \quad t \in \mathbb{R},$$

is called the associated *sine family*.

The interest in cosine and sine families stems from the study of the abstract second-order Cauchy problem

$$\begin{cases} \ddot{u}(t) = Au(t), & t \in \mathbb{R}, \\ u(0) = u_0, \quad \dot{u}(0) = u_1. \end{cases} \tag{2.1}$$

In fact, a closed, densely defined, linear operator A on a Banach space X is the generator of a cosine family if and only if for every u_0, $u_1 \in X$ the second-order problem (2.1) admits a unique *mild solution* $u \in C(\mathbb{R}; X)$ which by definition is a function satisfying $\int_0^t (t - s)u(s)\, ds \in D(A)$ and

$$u(t) = u_0 + tu_1 + A \int_0^t (t - s)u(s)\, ds \text{ for every } t \in \mathbb{R}.$$

The unique mild solution is given by $u(t) = C(t)u_0 + S(t)u_1$.

The following fundamental characterization of the generation of a cosine family was proved by Kisyński (see [1, Theorem 3.14.11], [5, Theorem 1.3, Chapter III] or [11]).

Theorem 2.1. *Let X be a Banach space, and let A be a closed linear operator in X. Then the following assertions are equivalent:*

(i) *The operator A is the generator of a cosine family $(C(t))_{t\in\mathbb{R}}$.*

(ii) *There exists a Banach space V such that $D(A) \subset V \subset X$ with continuous embeddings (where $D(A)$ is equipped with the graph norm) and such that the operator $\mathcal{A} := \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$ with domain $D(A) \times V$ generates a strongly continuous group in $V \times X$.*

The Banach space V is uniquely determined by (ii). *The space $V \times X$ is called the* phase space *associated with $(C(t))_{t\in\mathbb{R}}$.*

We note that if A generates a cosine family on X with phase space $V \times X$, then for every $u_0 \in D(A)$ and $u_1 \in V$ the second-order problem (2.1) admits a unique classical solution u, that is, the mild solution is twice continuously differentiable (see, e.g., [1, Chapter 3]). For more details on cosine functions and the second-order Cauchy problem, we refer to [1], [4], [5], [7], [10], [11] and [15].

Our main results of this paper show that the wave equation (1.1) is well posed in the scale of the $L^p(0,1)$ spaces ($1 \le p < \infty$) and in a space of continuous functions, no matter the boundary conditions. In the formulation of our results, however, we exclude the case of pure Dirichlet boundary conditions which has already been studied in the literature. One will see from our proofs that the case of Dirichlet boundary conditions is even easier to study.

In our first main result we state well-posedness in $L^p(0,1)$ for every $p \in [1,\infty)$.

Theorem 2.2. *Let α_j, β_j ($j = 0, 1$) be complex numbers satisfying $(\alpha_j, \beta_j) \neq (0,0)$. Let $a \in W^{1,\infty}(0,1)$ be such that $a(x) \ge \alpha > 0$ for every $x \in [0,1]$, and let b, $c \in L^\infty(0,1)$. Let $1 \le p < \infty$.*

(a) (*Robin and Neumann boundary conditions*) *Assume that α_0, $\alpha_1 \neq 0$. Then the operator $A_{R,p}$ defined by*

$$\begin{aligned} D(A_{R,p}) &:= \{u \in W^{2,p}(0,1) :\ \alpha_j u'(j) + \beta_j u(j) = 0,\ j = 0,1\}, \\ A_{R,p} u &:= a u'' + b u' + c u, \end{aligned}$$

generates a cosine family on $L^p(0,1)$ with phase space $W^{1,p}(0,1) \times L^p(0,1)$.

(b) (*Mixed boundary conditions*) *Assume that $\alpha_0 = 0$ and $\alpha_1 \neq 0$. Then the operator $A_{M,p}$ defined by*

$$\begin{aligned} D(A_{M,p}) &:= \{u \in W^{2,p}(0,1) :\ u(0) = 0 \ \text{ and } \ \alpha_1 u'(1) + \beta_1 u(1) = 0\}, \\ A_{M,p} u &:= a u'' + b u' + c u, \end{aligned}$$

generates a cosine function on $L^p(0,1)$ with phase space $W_0 \times L^p(0,1)$, where $W_0 := \{u \in W^{1,p}(0,1) :\ u(0) = 0\}$.

The second main result concerns generation of cosine functions on the space of continuous functions. Compared to Theorem 2.2, we only require more regularity of the coefficients a, b and c.

Theorem 2.3. *Let α_j, β_j $(j = 0, 1)$ be complex numbers satisfying $(\alpha_j, \beta_j) \neq (0, 0)$. Let $a \in C^1([0,1])$ be such that $a(x) \geq \alpha > 0$ for every $x \in [0,1]$ and let b, $c \in C([0,1])$.*

(a) (*Robin and Neumann boundary conditions*) *Assume that α_0, $\alpha_1 \neq 0$. Then the operator $A_{R,\infty}$ defined by*

$$\begin{aligned} D(A_{R,\infty}) &:= \{u \in C^2([0,1]) : \ \alpha_j u'(j) + \beta_j u(j) = 0, \ j = 0, 1\}, \\ A_{R,\infty} u &:= a u'' + b u' + c u, \end{aligned}$$

generates a cosine function on $C([0,1])$ with phase space $C^1([0,1]) \times C([0,1])$.

(b) (*Mixed boundary conditions*) *Assume that $\alpha_0 = 0$ and $\alpha_1 \neq 0$. Then the operator $A_{M,\infty}$ defined by*

$$\begin{aligned} D(A_{M,\infty}) &:= \{u \in C^2([0,1]) : \ u(0) = a(0)u''(0) + b(0)u'(0) = 0 \\ &\qquad\qquad \text{and } \ \alpha_1 u'(1) + \beta_1 u(1) = 0\}, \\ A_{M,\infty} u &:= a u'' + b u' + c u, \end{aligned}$$

generates a cosine function on $X_0 := \{u \in C([0,1]) : \ u(0) = 0\}$ with phase space $(C^1[0,1] \cap X_0) \times X_0$.

3. The case of the Laplacian

In this section, we investigate explicitly the Laplace operator on the interval, that is, we will prove our main theorems in the case $a = 1$ and b, $c = 0$. In this particular situation, we write $\Delta_{R,p}$ instead of $A_{R,p}$, so that $D(\Delta_{R,p}) = D(A_{R,p})$ and $\Delta_{R,p} u = u''$. Similarly, we write $\Delta_{M,p}$ instead of $A_{M,p}$.

We first concentrate on Robin or Neumann boundary conditions.

Lemma 3.1 (Robin and Neumann boundary conditions). *Assume α_0, $\alpha_1 \neq 0$, and let $1 \leq p \leq \infty$. Let $u_0 \in D(\Delta_{R,p})$ and $u_1 \in W^{1,p}(0,1)$ ($u_1 \in C^1([0,1])$ in case $p = \infty$) be given, and define the extensions*

$$w_0(s) := \begin{cases} u_0(-s) + 2\frac{\beta_0}{\alpha_0} e^{-\frac{\beta_0}{\alpha_0} s} \int_0^{-s} e^{-\frac{\beta_0}{\alpha_0}\tau} u_0(\tau)\, d\tau, & s \in [-1, 0), \\ u_0(s), & s \in [0,1], \\ u_0(2-s) - 2\frac{\beta_1}{\alpha_1} e^{-\frac{\beta_1}{\alpha_1}(s-1)} \int_0^{s-1} e^{\frac{\beta_1}{\alpha_1}\tau} u_0(1-\tau)\, d\tau, & s \in (1, 2], \end{cases} \tag{3.1}$$

and

$$w_1(s) := \begin{cases} u_1(-s) + 2\frac{\beta_0}{\alpha_0} e^{-\frac{\beta_0}{\alpha_0} s} \int_0^{-s} e^{-\frac{\beta_0}{\alpha_0}\tau} u_1(\tau)\, d\tau, & s \in [-1, 0), \\ u_1(s), & s \in [0,1], \\ u_1(2-s) - 2\frac{\beta_1}{\alpha_1} e^{-\frac{\beta_1}{\alpha_1}(s-1)} \int_0^{s-1} e^{\frac{\beta_1}{\alpha_1}\tau} u_1(1-\tau)\, d\tau, & s \in (1, 2]. \end{cases} \tag{3.2}$$

For every $t \in [0,1]$ *and* $x \in [0,1]$ *we set*

$$u(t)(x) := \frac{1}{2}(w_0(x+t) + w_0(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} w_1(\tau)\, d\tau. \tag{3.3}$$

Then

$$u \in C([0,1]; D(\Delta_{R,p})) \cap C^1([0,1]; W^{1,p}(0,1)) \cap C^2([0,1]; L^p(0,1)) \text{ if } 1 \le p < \infty,$$
$$u \in C([0,1]; D(\Delta_{R,p})) \cap C^1([0,1]; C^1([0,1])) \cap C^2([0,1]; C([0,1])) \text{ if } p = \infty,$$

and u *is a classical solution of the second-order problem*

$$\begin{cases} \ddot{u}(t) = \Delta_{R,p} u(t), & t \in [0,1], \\ u(0) = u_0, \ \dot{u}(0) = u_1. \end{cases} \tag{3.4}$$

Proof. In this proof, we will sometimes identify u with a function depending on the two variables t and x. There will be no danger of confusion. We fix $p \in [1, \infty)$. The proof in the case $p = \infty$ is very similar and requires only an appropriate change of the spaces.

First, we assume that $u_1 = 0$, so that $w_1 = 0$.

By assumption, $u_0 \in W^{2,p}(0,1)$. This implies that w_0 is of class $W^{2,p}$ on each of the intervals $(-1,0)$, $(0,1)$ and $(1,2)$. Since w_0 is in addition continuous, we find $w_0 \in W^{1,p}(-1,2)$.

By assumption, $u_0 \in D(\Delta_{R,p})$, which implies $\alpha_0 u_0'(0) + \beta_0 u_0(0) = 0$ and $\alpha_1 u_0'(1) + \beta_1 u_0(1) = 0$. Using these two equalities, it is then easy to check that w_0' is continuous and therefore $w_0 \in W^{2,p}(-1,2)$. As a consequence, by the strong continuity of the left and right shift semigroups in L^p, $W^{1,p}$, and $W^{2,p}$, $u \in C([0,1]; W^{2,p}(0,1)) \cap C^1([0,1]; W^{1,p}(0,1)) \cap C^2([0,1]; L^p(0,1))$.

It is also easy to check that on the interval $[-1,0]$ the function w_0 is the unique solution of the linear first-order initial value problem

$$\begin{aligned} &\alpha_0(u_0'(t) + w_0'(-t)) + \beta_0\,(u_0(t) + w_0(-t)) = 0, \quad t \in [0,1], \\ &w_0(0) = u_0(0). \end{aligned}$$

Since $w_0 = u_0$ on $[0,1]$, this implies

$$\alpha_0 u_x(t,0) + \beta_0 u(t,0) = 0 \text{ for every } t \in [0,1].$$

Similarly, on the interval $[1,2]$ the function w_0 is the unique solution of

$$\begin{aligned} &\alpha_1(u_0'(1-t) + w_0'(1+t)) + \beta_1(u_0(1-t) + w_0(1+t)) = 0, \quad t \in [0,1], \\ &w_0(1) = u_0(1). \end{aligned}$$

This implies

$$\alpha_1 u_x(t,1) + \beta_1 u(t,1) = 0 \text{ for every } t \in [0,1].$$

Hence, for every $t \in [0,1]$ we have $u(t) \in D(\Delta_{R,p})$ and thus $u \in C([0,1]; D(\Delta_{R,p}))$.

Next, we assume that $u_0 = 0$, so that $w_0 = 0$.

By assumption, $u_1 \in W^{1,p}(0,1)$. This implies that w_1 is of class $W^{1,p}$ on each of the intervals $(-1,0)$, $(0,1)$ and $(1,2)$. Since w_1 is in addition continuous, we find $w_1 \in W^{1,p}(-1,2)$. As a consequence, $u \in C([0,1]; W^{2,p}(0,1)) \cap C^1([0,1]; W^{1,p}(0,1)) \cap C^2([0,1]; L^p(0,1))$.

It is straightforward to check that on the interval $[-1,0]$ the function w_1 is the unique solution of the linear first-order initial value problem

$$\alpha_0(u_1(t) - w_1(-t)) + \beta_0 \left(\int_0^t u_1(\tau)\, d\tau + \int_{-t}^0 w_1(\tau)\, d\tau \right) = 0, \quad t \in [0,1],$$
$$w_1(0) = u_1(0).$$

Since $w_1 = u_1$ on $[0,1]$, this implies

$$\alpha_0 u_x(t,0) + \beta_0 u(t,0) = 0 \text{ for every } t \in [0,1].$$

Similarly, on the interval $[1,2]$ the function w_1 is the unique solution of

$$\alpha_1(u_1(1-t) - w_1(1+t)) + \beta_1 \left(\int_{1-t}^1 u_1(\tau)\, d\tau + \int_1^{1+t} w_1(\tau)\, d\tau \right) = 0, \quad t \in [0,1],$$
$$w_1(1) = u_1(1).$$

This implies

$$\alpha_1 u_x(t,1) + \beta_1 u(t,1) = 0 \text{ for every } t \in [0,1].$$

Hence, for every $t \in [0,1]$ we have $u(t) \in D(\Delta_{R,p})$ and thus $u \in C([0,1]; D(\Delta_{R,p}))$.

Taking the two cases $u_0 \in D(\Delta_{R,p})$ and $u_1 \in W^{1,p}(0,1)$ together, by linearity, we have proved the required regularity for the function u.

Formula (3.3) is nothing else than d'Alembert's formula. It is straightforward to see that u is a classical solution of (3.4). □

An analogous existence lemma is true for mixed Dirichlet/Robin boundary conditions, that is, for example, when $\alpha_0 = 0$ and $\alpha_1 \neq 0$. The case $\alpha_0 \neq 0$ and $\alpha_1 = 0$ is very similar.

In fact, in the analogous formulation, it suffices to assume in addition that the initial values satisfy $u_0(0) = 0$ and $u_1(0) = 0$, and to replace the extensions w_0 and w_1 by the following extensions

$$w_0(s) := \begin{cases} -u_0(-s), & s \in [-1,0), \\ u_0(s), & s \in [0,1], \\ u_0(2-s) - 2\frac{\beta_1}{\alpha_1} e^{-\frac{\beta_1}{\alpha_1}(s-1)} \int_0^{s-1} e^{\frac{\beta_1}{\alpha_1}\tau} u_0(1-\tau)\, d\tau, & s \in (1,2], \end{cases}$$

and

$$w_1(s) := \begin{cases} -u_1(-s), & s \in [-1,0), \\ u_1(s), & s \in [0,1], \\ u_1(2-s) - 2\frac{\beta_1}{\alpha_1} e^{-\frac{\beta_1}{\alpha_1}(s-1)} \int_0^{s-1} e^{\frac{\beta_1}{\alpha_1}\tau} u_1(1-\tau)\, d\tau, & s \in (1,2]. \end{cases}$$

Remark 3.2. Lemma 3.1 includes the case of Neumann boundary conditions. It suffices to take $\beta_0 = \beta_1 = 0$. In this case, the extensions w_0 resp. w_1 are even extensions of u_0 resp. u_1. Clearly, also the case of mixed Robin/Neumann boundary conditions is included ($\beta_0 = 0$ and $\beta_1 \neq 0$, for example).

It is known that the wave equation on the interval $(0, 1)$ with Dirichlet boundary conditions can be locally solved by considering odd extensions w_0 and w_1 of the initial values u_0 and u_1, respectively (compare the extensions w_0 and w_1 defined above). The odd extensions to the interval $(-1, 0)$ appear as a limiting case in Lemma 3.1 if one assumes $\beta_0 > 0$ and if one lets $\alpha_0 \to 0$ while keeping α_0 negative, and similarly for the interval $(1, 2)$.

Global solutions in the case of Dirichlet or Neumann boundary conditions are obtained by considering 2-periodic extensions of w_0 and w_1, and by using d'Alembert's formula. We are not trying to give explicit *global* solutions in the case of Robin boundary conditions. Obviously, it does not suffice to consider 2-periodic extensions of w_0 and w_1.

From the proof of Lemma 3.1, it is not difficult to see that classical solutions of the second-order problem (3.4) *which are given by d'Alembert's formula* (3.3) (for some functions w_0 and w_1) are necessarily unique.

The following lemma implies uniqueness of classical solutions of (3.4) without any further condition. Indeed, uniqueness is shown in a slightly different and more general context. Note that classical solutions of (3.4), when considered as functions of two variables, are weak C^1 solutions of the wave equation as considered in the following lemma.

Lemma 3.3. *Assume $(\alpha_j, \beta_j) \neq (0, 0)$ $(j = 0,\ 1)$. For every $u_0 \in C^1([0, 1])$ and every $u_1 \in C([0, 1])$, there exists at most one function*

$$u \in C^1([0, 1] \times [0, 1])$$

which is a weak solution of the wave equation $u_{tt} = u_{xx}$ on $(0, 1) \times (0, 1)$, that is,

$$\int_0^1 \int_0^1 u\,(\varphi_{tt} - \varphi_{xx})\,dx\,dt = 0 \text{ for every } \varphi \in \mathcal{D}((0, 1) \times (0, 1)),$$

and which satisfies the initial and boundary conditions

$$\begin{cases} u(0, x) = u_0(x), & x \in [0, 1], \\ u_t(0, x) = u_1(x), & x \in [0, 1], \\ \alpha_0 u_x(t, 0) + \beta_0 u(t, 0) = 0, & t \in [0, 1], \\ \alpha_1 u_x(t, 1) + \beta_1 u(t, 1) = 0, & t \in [0, 1]. \end{cases} \tag{3.5}$$

Proof. We assume that $\alpha_j \neq 0$ $(j = 0,\ 1)$, the case $\alpha_j = 0$ (and $\beta_j \neq 0$) being very similar.

By linearity, it suffices to prove that the only solution for the initial values $u_0 = u_1 = 0$ is the trivial solution $u = 0$. So we assume that $u_0 = u_1 = 0$.

Put $v := u_t + u_x$. Then $v \in C([0,1] \times [0,1])$, and for every $\varphi \in \mathcal{D}((0,1) \times (0,1))$ one has

$$\int_0^1 \int_0^1 v(\varphi_t - \varphi_x) = \int_0^1 \int_0^1 (u_t + u_x)(\varphi_t - \varphi_x)$$
$$= -\int_0^1 \int_0^1 u(\varphi_{tt} - \varphi_{tx} + \varphi_{xt} - \varphi_{xx}) = -\int_0^1 \int_0^1 u\,(\varphi_{tt} - \varphi_{xx}) = 0,$$

i.e., v is a continuous weak solution of the linear transport equation

$$\begin{cases} v_t - v_x = 0, & (t,x) \in [0,1] \times [0,1], \\ v(0,x) = 0, & x \in [0,1]. \end{cases} \tag{3.6}$$

The initial value of v follows from the definition of v and the assumption that $u_0 = u_1 = 0$. Unique solvability of the linear transport equation implies that $v(t,x) = 0$ for every $(t,x) \in [0,1] \times [0,1]$ satisfying $t \leq 1 - x$. In particular, $v(t,0) = 0$ for every $t \in [0,1]$. By the definition of v, this means that $u_t(t,0) + u_x(t,0) = 0$ for every $t \in [0,1]$. Using the Robin boundary condition satisfied by u, we obtain

$$u_t(t,0) = \frac{\beta_0}{\alpha_0} u(t,0) \text{ for every } t \in [0,1] \text{ and } u(0,0) = 0.$$

Hence, $u(t,0) = 0$ for every $t \in [0,1]$.

Next we put $w := u_t - u_x$. Similarly as above, one shows that $w \in C([0,1] \times [0,1])$ is a weak solution of the transport equation $w_t + w_x = 0$, $w(0,x) = 0$, so that $w(t,x) = 0$ for every $(t,x) \in [0,1] \times [0,1]$ satisfying $t \leq x$. In particular, $w(t,1) = 0$ for every $t \in [0,1]$. By the definition of w, this means that $u_t(t,1) - u_x(t,1) = 0$, and when we use again the Robin boundary condition satisfied by u, then we obtain

$$u_t(t,1) = -\frac{\beta_1}{\alpha_1} u(t,1) \text{ for every } t \in [0,1] \text{ and } u(0,1) = 0.$$

Hence, $u(t,1) = 0$ for every $t \in [0,1]$.

By the definition of v, this implies

$$v(t,1) = u_t(t,1) + u_x(t,1) = u_t(t,1) - \frac{\beta_1}{\alpha_1} u(t,1) = 0 \text{ for every } t \in [0,1].$$

This boundary condition and the transport equation (3.6) imply that $v = 0$ on $[0,1] \times [0,1]$. By the definition of v, the function u thus solves

$$u_t + u_x = 0 \text{ on } [0,1] \times [0,1],$$

with boundary conditions $u(0,x) = 0$ and $u(t,0) = 0$. Hence, $u = 0$. □

Lemma 3.4. *Let α_j, β_j $(j = 0,\ 1)$ be complex numbers such that $(\alpha_j, \beta_j) \neq (0,0)$. Then there exists $\omega > 0$ such that for every $f \in L^p(0,1)$ $(1 \leq p \leq \infty)$ the problem*

$$\omega^2 u + u'' = f \text{ on } (0,1), \quad \alpha_j u'(j) + \beta_j u(j) = 0 \text{ for } j = 0,\ 1, \tag{3.7}$$

admits a unique solution $u \in W^{2,p}(0,1)$.

Proof. Fix $1 \le p \le \infty$, and let $f \in L^p(0,1)$ be given. Every solution of the ordinary differential equation $\omega^2 u + u'' = f$ is of the form

$$u(x) = \cos(\omega x)u(0) + \frac{1}{\omega}\sin(\omega x)u'(0) + \frac{1}{\omega}\int_0^x \sin(\omega(x-s))f(s)\,ds, \quad x \in [0,1].$$

The free variables $u(0)$ and $u'(0)$ are to be determined in a unique way such that the boundary conditions in (3.7) are satisfied.

Assume first $\alpha_j \neq 0$ and set $\lambda_j := \frac{\beta_j}{\alpha_j}$. From the preceding representation of the solution (and thus also its derivative) and from the required Robin boundary conditions we obtain the following system to be solved:

$$\begin{aligned}\left(-\cos\omega + \frac{\lambda_0}{\omega}\sin\omega\right)u(0) + u(1) &= \frac{1}{\omega}\int_0^1 \sin(\omega(1-s))f(s)\,ds,\\ (\omega\sin\omega + \lambda_0\cos\omega)u(0) - \lambda_1 u(1) &= \int_0^1 \cos(\omega(1-s))f(s)\,ds.\end{aligned}$$

This system in the unknowns $u(0)$ and $u(1)$ is uniquely solvable if and only if

$$\lambda_1 \neq \frac{\omega\sin\omega + \lambda_0\cos\omega}{\cos\omega - \frac{\lambda_0}{\omega}\sin\omega}.$$

From this relation for λ_0 and λ_1 it follows easily that there exists $\omega > 0$ with the required property.

The case $\alpha_0 = 0$ or $\alpha_1 = 0$ is solved similarly. □

Theorem 3.5. *Let α_j, β_j, $j = 0, 1$ be complex numbers verifying $(\alpha_j, \beta_j) \neq (0,0)$.*

(a) (*Robin and Neumann boundary conditions*) *Assume that α_0, $\alpha_1 \neq 0$. For every $1 \le p < \infty$, the operator $\Delta_{R,p}$ generates a cosine function on $L^p(0,1)$ with phase space $W^{1,p}(0,1) \times L^p(0,1)$.*

(b) (*Robin and Neumann boundary conditions*) *Assume that α_0, $\alpha_1 \neq 0$. The operator $\Delta_{R,\infty}$ generates a cosine function on $C([0,1])$ with phase space $C^1([0,1]) \times C([0,1])$.*

(c) (*Mixed boundary conditions*) *Assume that $\alpha_0 = 0$ and $\alpha_1 \neq 0$. For every $1 \le p < \infty$, the operator $\Delta_{M,p}$ generates a cosine function on $L^p(0,1)$ with phase space $W_0 \times L^p(0,1)$, where $W_0 := \{W^{1,p}(0,1) : \ u(0) = 0\}$.*

(d) (*Mixed boundary conditions*) *Assume that $\alpha_0 = 0$ and $\alpha_1 \neq 0$. The operator $\Delta_{M,\infty}$ generates a cosine function on $C_0((0,1])$ with phase space $C^1([0,1]) \cap C_0((0,1]) \times C_0((0,1])$.*

Proof. (a) Fix $1 \le p < \infty$ and consider the operator $A = \begin{pmatrix} 0 & I \\ \Delta_{R,p} & 0 \end{pmatrix}$ on $W^{1,p}(0,1) \times L^p(0,1)$ with natural domain $D(A) := D(\Delta_{R,p}) \times W^{1,p}(0,1)$. The operator A is clearly closed and densely defined. Moreover, since $\Delta_{R,p}$ has non-empty resolvent set by Lemma 3.4, the operator A has non-empty resolvent set.

By Lemma 3.1 and 3.3, for every $U_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in D(\Delta_{R,p}) \times W^{1,p}(0,1) = D(A)$ there exists a unique classical solution

$$u \in C([0,1]; D(\Delta_{R,p})) \cap C^1([0,1]; W^{1,p}(0,1)) \cap C^2([0,1]; L^p(0,1))$$

of the second-order Cauchy problem (3.4). As a consequence, the function

$$U := \begin{pmatrix} u \\ \dot{u} \end{pmatrix} \in C([0,1]; D(A)) \cap C^1([0,1]; W^{1,p}(0,1) \times L^p(0,1))$$

is a classical solution of the first-order Cauchy problem

$$\begin{cases} \dot{U} = AU, & t \in [0,1], \\ U(0) = U_0. \end{cases} \tag{3.8}$$

Conversely, if $U = \begin{pmatrix} u \\ v \end{pmatrix}$ is a classical solution of this latter Cauchy problem, then $\dot{u} = v$ and u is a classical solution of the second-order problem (3.4). Hence, the solution of (3.8) is unique.

It is a standard argument which implies that the solution of (3.8) can be extended to a unique classical solution defined on $\mathbb{R}_+$. Hence, by [1, Theorem 3.1.12], the operator A is the generator of a C_0-semigroup. However, the operator A is the generator of a C_0-semigroup if and only if $-A$ is the generator of a C_0-semigroup. In fact, $\begin{pmatrix} u \\ v \end{pmatrix}$ is a solution of $\dot{U} = AU$ if and only if $\begin{pmatrix} u \\ -v \end{pmatrix}$ is a solution of $\dot{U} = -AU$. Therefore, the operator A generates a C_0-group.

By Kisynski's theorem, the operator $\Delta_{R,p}$ is thus the generator of a cosine function, and the associated phase space is $W^{1,p}(0,1) \times L^p(0,1)$.

The statements (b), (c) and (d) are proved similarly. □

4. Proof of the main results

In this section we prove the main results stated in Section 2. One key is the following lemma (compare with Step (ii) in the proof of [2, Lemma 4.2] or with the proof of Theorem 4.3 in [4, p. 387]).

Lemma 4.1. *Let α_j, β_j $(j = 0, 1)$ be complex numbers such that $(\alpha_j, \beta_j) \neq (0,0)$. Let $a \in W^{1,\infty}(0,1)$ satisfy $a(x) \geq \alpha > 0$ for every $x \in [0,1]$, and assume in addition that $\int_0^1 \frac{1}{a} = 1$. Let $1 \leq p < \infty$ and consider the operators $A_{a,p}$ and $\Delta_{R,p}$ in $L^p(0,1)$ defined by*

$$\begin{aligned} D(A_{a,p}) &:= \{u \in W^{2,p}(0,1) : \alpha_j u'(j) + \beta_j u(j) = 0, \ j = 0,1\}, \\ A_{a,p} u &:= a\,(au')', \end{aligned}$$

and

$$\begin{aligned} D(\Delta_{R,p}) &:= \{u \in W^{2,p}(0,1) : \frac{\alpha_j}{a(j)} u'(j) + \beta_j u(j) = 0,\ j = 0,1\}, \\ \Delta_{R,p} u &:= u''. \end{aligned}$$

Then $A_{a,p}$ *and* $\Delta_{R,p}$ *are similar.*

Proof. Define

$$\varphi(s) := \int_0^s \frac{1}{a(r)}\, dr, \quad s \in [0,1].$$

By the assumptions on a, the function φ is strictly increasing and $\varphi(1) = 1$. Hence, φ is invertible from $[0,1]$ onto $[0,1]$. Since $a \in W^{1,\infty}(0,1)$, and since a is strictly positive, we have φ, $\varphi^{-1} \in W^{2,\infty}(0,1)$.

As a consequence, for every $1 \le p < \infty$, the linear operator $Q_\varphi \in \mathcal{L}(L^p(0,1))$ given by

$$Q_\varphi u := u \circ \varphi$$

is invertible with inverse $Q_\varphi^{-1} = Q_{\varphi^{-1}}$.

Fix $1 \le p < \infty$. We first show that

$$D(Q_\varphi \Delta_{R,p} Q_\varphi^{-1}) = D(A_{a,p}). \tag{4.1}$$

Let $u \in D(Q_\varphi \Delta_{R,p} Q_\varphi^{-1}) = Q_\varphi D(\Delta_{R,p})$. Then $u = v \circ \varphi$ for some $v \in D(\Delta_{R,p})$. Since $v \in W^{2,p}(0,1)$ and $\varphi \in W^{2,\infty}(0,1)$, we have $u \in W^{2,p}(0,1)$. Moreover, since

$$u'(s) = v'(\varphi(s)) \cdot \varphi'(s) = v'(\varphi(s)) \cdot \frac{1}{a(s)}$$

and since $\varphi(j) = j$ for $j = 0,1$, it follows that

$$\alpha_j u'(j) + \beta_j u(j) = \frac{\alpha_j}{a(j)} v'(\varphi(j)) + \beta_j v(\varphi(j)) = \frac{\alpha_j}{a(j)} v'(j) + \beta_j v(j) = 0.$$

Hence, $u \in D(A_{a,p})$.

Conversely, let $u \in D(A_{a,p})$ and set $v := u \circ \varphi^{-1}$. Then $v \in W^{2,p}(0,1)$. Since $(\varphi^{-1})'(x) = \frac{1}{\varphi'(\varphi^{-1}(x))}$ and $\varphi^{-1}(j) = j$ for $j = 0,1$, we have that, $(\varphi^{-1})'(j) = \frac{1}{\varphi'(j)} = a(j)$. Since

$$v'(x) = u'(\varphi^{-1}(x)) \cdot (\varphi^{-1})'(x),$$

it follows that for $j = 0,1$,

$$\frac{\alpha_j}{a(j)} v'(j) + \beta_j v(j) = \alpha_j u'(\varphi^{-1}(j)) + \beta_j u(\varphi^{-1}(j)) = \alpha_j u'(j) + \beta_j u(j) = 0.$$

Hence, $u \in D(Q_\varphi \Delta_{R,p} Q_\varphi^{-1}) = Q_\varphi D(\Delta_{R,p})$. We have shown (4.1).

Next, we show that for $u \in D(A_{a,p})$, we have

$$Q_\varphi \Delta_{R,p} Q_\varphi^{-1} u = A_{a,p} u. \tag{4.2}$$

Let $u \in D(A_{a,p})$. Since $(\varphi^{-1})'(s) = a \circ \varphi^{-1}(s)$ and $u \circ \varphi^{-1} \in C^1([0,1])$, we have that for all $s \in [0,1]$,

$$\begin{aligned}(u \circ \varphi^{-1})'(s) &= u'(\varphi^{-1}(s)) \cdot (\varphi^{-1})'(s) = u'(\varphi^{-1}(s)) \cdot (a \circ \varphi^{-1})(s) \\ &= (a \cdot u') \circ \varphi^{-1}(s).\end{aligned}$$

Since $(u \circ \varphi^{-1})' \in W^{1,p}(0,1)$, it follows that $(u \circ \varphi^{-1})'$ is absolutely continuous on $(0,1)$ and the classical derivative of $(u \circ \varphi^{-1})'$ agrees almost everywhere with the weak derivative of $(u \circ \varphi^{-1})'$ (see [12, Theorem 1.41]). Hence, for almost all $s \in (0,1)$,

$$(u \circ \varphi^{-1})''(s) = \left[(a \cdot u') \circ \varphi^{-1}\right]' = \left(a^2 \cdot u'' + aa'u'\right) \circ \varphi^{-1}.$$

This implies that for every $u \in D(A_{a,p})$, we have

$$Q_\varphi \Delta_{R,p} Q_\varphi^{-1} u = \varphi \circ (u \circ \varphi^{-1})'' = a^2 u'' + aa' \cdot u' = a(au')' = A_{a,p}u. \tag{4.3}$$

It follows from (4.1) and (4.2) that the operators $A_{a,p}$ and $\Delta_{R,p}$ are similar. □

Now we are ready to give a proof our main results; for the perturbation argument used below, see also the proof of [2, Lemma 4.2].

Proof of Theorem 2.2.

(a) Fix $1 \le p < \infty$, assume α_0, $\alpha_1 \ne 0$, and let the operator $A_{R,p}$ on $L^p(0,1)$ be defined as in the statement.

(i) Set $\tilde{a} := \sqrt{a}$. By the assumptions on a, $\tilde{a} \in W^{1,\infty}(0,1)$. Let the operator $A_{\tilde{a},p}$ in $L^p(0,1)$ be defined by

$$D(A_{\tilde{a},p}) := D(A_{R,p}) \text{ and } A_{\tilde{a},p}u := \tilde{a}(\tilde{a}u')'.$$

Let $c = \int_0^1 \frac{1}{\tilde{a}} > 0$. By Lemma 4.1, the operator $c^2 A_{\tilde{a},p}$ is similar to the operator $\Delta_{R,p}$ defined in Lemma 4.1. By Theorem 3.5 (a), the operator $\Delta_{R,p}$ generates a cosine function on $L^p(0,1)$ with phase space $W^{1,p}(0,1) \times L^p(0,1)$. Hence, the operator $A_{\tilde{a},p}$ also generates a cosine function on $L^p(0,1)$ with phase space $Q_\varphi(W^{1,p}(0,1)) \times L^p(0,1) = W^{1,p}(0,1) \times L^p(0,1)$ (the constant c^2 does not change the generation property).

(ii) By the definition of $A_{R,p}$ and $A_{\tilde{a},p}$, for every $u \in D(A_{R,p})$

$$A_{R,p}u = au'' + bu' + cu = A_{\tilde{a},p}u - (\tilde{a}\tilde{a}' - b)u' + cu.$$

Define $B_{R,p} : W^{1,p}(0,1) \to L^p(0,1)$ by

$$B_{R,p}u := -(\tilde{a}\tilde{a}' - b)u' + cu, \quad u \in W^{1,p}(0,1).$$

The operator $B_{R,p}$ is clearly bounded and $A_{R,p}u = A_{\tilde{a},p}u + B_{R,p}u$. Since $A_{\tilde{a},p}$ generates a cosine function on $L^p(0,1)$ with phase space $W^{1,p}(0,1) \times L^p(0,1)$, it follows from [1, Corollary 3.14.13] that $A_{R,p}$ also generates a cosine function on $L^p(0,1)$ with phase space $W^{1,p}(0,1) \times L^p(0,1)$.

(b) The proof of this part is similar to the proof of the first part by using Lemma 4.1 and Theorem 3.5 (c). □

Proof of Theorem 2.3. The proof of Theorem 2.3 is similar to the proof of Theorem 2.2. One proves, for example, that the operators $A_{a,\infty}$ and $\Delta_{R,\infty}$ on $C([0,1])$ given by

$$\begin{aligned} D(A_{a,\infty}) &:= \{u \in C^2[0,1] : \alpha_j u'(j) + \beta_j u(j) = 0,\ j = 0,1\}, \\ A_{a,\infty} u &:= a\,(au')', \end{aligned}$$

and

$$\begin{aligned} D(\Delta_{R,\infty}) &:= \{u \in C^2[0,1] : \frac{\alpha_j}{a(j)} u'(j) + \beta_j u(j) = 0,\ j = 0,1\}, \\ \Delta_{R,\infty} u &:= u'', \end{aligned}$$

are similar if $a \in C^1([0,1])$ and if $\int_0^1 \frac{1}{a} = 1$. Then one proceeds as in the proof of Theorem 2.2, using now Theorem 3.5 (b) or (d). □

5. Second-order elliptic operators on unbounded intervals

We remark that our generation results remain true if the interval $(0,1)$ is replaced by the unbounded interval $(0,\infty)$, that is, if one considers the following wave equation:

$$\begin{cases} u_{tt} = a(x)u_{xx} + b(x)u_x + c(x)u, & x \in (0,\infty),\ t \geq 0 \\ u(0,x) = u_0(x), & x \in (0,\infty), \\ u_t(0,x) = u_1(x), & x \in (0,\infty), \\ \alpha_0 u_x(t,0) + \beta_0 u(t,0) = 0, & t \geq 0. \end{cases} \tag{5.1}$$

Here α_0 and β_0 are complex numbers. We will concentrate on Robin boundary conditions and will therefore assume $\alpha_0 \neq 0$, knowing that this case also includes the case of Neumann boundary conditions.

By following our approach in the previous sections, that is, by solving explicitly the above wave equation in the case of the Laplace operator, and by a similarity and perturbation argument, we obtain the following result which, in the case of the Laplace operator, contains also global norm estimates on the corresponding cosine family.

Theorem 5.1. *Let α_0, β_0 be complex numbers such that $\alpha_0 \neq 0$. Let $a \in W^{1,\infty}(0,\infty)$ be such that $a(x) \geq \alpha > 0$ for every $x \geq 0$ and some constant α, and let b, $c \in L^\infty(0,\infty)$. Let $1 \leq p < \infty$, and consider the operator $A_{R,p}$ in $L^p(0,\infty)$ defined by*

$$\begin{aligned} D(A_{R,p}) &:= \{u \in W^{2,p}(0,\infty) : \alpha_0 u'(0) + \beta_0 u(0) = 0\}, \\ A_{R,p} u &:= au'' + bu' + cu. \end{aligned}$$

Then the following are true:

(a) *The operator $A_{R,p}$ generates a cosine family $(C(t))_{t\in\mathbb{R}}$ on $L^p(0,\infty)$ with phase space $W^{1,p}(0,\infty) \times L^p(0,\infty)$.*

(b) *Assume $a = 1$ and $b,\ c = 0$. If* $\operatorname{Re}\frac{\beta_0}{\alpha_0} < 0$, *then the corresponding cosine family $(C(t))_{t\in\mathbb{R}}$ is bounded, i.e., there exists $M \geq 0$ such that $\|C(t)\| \leq M$ for every $t \in \mathbb{R}$.*

(c) *Assume $a = 1$ and $b,\ c = 0$. If* $\operatorname{Re}\frac{\beta_0}{\alpha_0} > 0$, *then the corresponding cosine family is (exponentially) unbounded.*

Proof. (a) Let $\tilde{a} := \sqrt{a}$. By considering the bijection $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$\varphi(s) := \int_0^s \frac{1}{\tilde{a}(\tau)}\, d\tau, \quad s \geq 0,$$

and by considering the isomorphism $Q_\varphi \in \mathcal{L}(L^p(0,\infty))$ given by $Q_\varphi u := u \circ \varphi$, one shows as in Lemma 4.1 that the operators $A_{\tilde{a},p}$ and $\Delta_{R,p}$ in $L^p(0,\infty)$ given by

$$\begin{aligned} D(A_{\tilde{a},p}) &:= \{u \in W^{2,p}(0,\infty) :\ \alpha_0 u'(0) + \beta_0 u(0) = 0\}, \\ A_{\tilde{a},p} u &:= \tilde{a}\,(\tilde{a} u')', \\ \text{and} \qquad D(\Delta_{R,p}) &:= \{u \in W^{2,p}(0,\infty) :\ \frac{\alpha_0}{\tilde{a}(0)} u'(0) + \beta_0 u(0) = 0\}, \\ \Delta_{R,p} u &:= u'' \end{aligned}$$

are similar.

We show that $\Delta_{R,p}$ is the generator of a cosine family by giving an explicit formula for the cosine family. Assume for simplicity that $a(0) = 1$. Let $u_0 \in D(\Delta_{R,p})$ and $u_1 \in W^{1,p}(0,\infty)$ be given, and define the extensions w_0 and w_1 by

$$w_0(s) := \begin{cases} u_0(-s) + 2\frac{\beta_0}{\alpha_0} e^{-\frac{\beta_0}{\alpha_0}s} \int_0^{-s} e^{-\frac{\beta_0}{\alpha_0}\tau} u_0(\tau)\, d\tau, & s < 0, \\ u_0(s), & s \geq 0, \end{cases} \tag{5.2}$$

and

$$w_1(s) := \begin{cases} u_1(-s) + 2\frac{\beta_0}{\alpha_0} e^{-\frac{\beta_0}{\alpha_0}s} \int_0^{-s} e^{-\frac{\beta_0}{\alpha_0}\tau} u_1(\tau)\, d\tau, & s < 0, \\ u_1(s), & s \geq 0, \end{cases}$$

For every $t \geq 0$ and every $x \in (0,\infty)$ we set

$$u(t)(x) := \frac{1}{2}(w_0(x+t) + w_0(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} w_1(s)\, ds.$$

Similarly as in the proof of Lemma 3.1 one shows that

$$u \in C(\mathbb{R}_+; D(\Delta_{R,p})) \cap C^1(\mathbb{R}_+; W^{1,p}(0,\infty)) \cap C^2(\mathbb{R}_+; L^p(0,\infty)),$$

and that u is a classical solution of the abstract second-order problem

$$\begin{cases} \ddot{u} = \Delta_{R,p} u, & t \geq 0, \\ u(0) = u_0,\ \dot{u}(0) = 0. \end{cases}$$

Uniqueness of classical solutions is shown by showing uniqueness of weak C^1 solutions of the wave equation $u_{tt} = u_{xx}$ on $(0,\infty) \times (0,\infty)$ with Robin boundary conditions, as in Lemma 3.3.

By passing to a first-order problem on the product space $W^{1,p}(0,\infty) \times L^p(0,\infty)$ and by using Kisynski's theorem (Theorem 2.1), one deduces that $\Delta_{R,p}$ generates a cosine family, and the associated phase space is $W^{1,p}(0,\infty)\times L^p(0,\infty)$.

Hence, by similarity, the operator $A_{\tilde{a},p}$ generates a cosine family, and one proves that the associated phase space is $W^{1,p}(0,\infty) \times L^p(0,\infty)$.

The rest of the proof follows exactly as in the proof of Theorem 2.2.

(b) If $\operatorname{Re}\frac{\beta_0}{\alpha_0} < 0$, then the function $s \mapsto e^{\frac{\beta_0}{\alpha_0}s}$ is integrable on $\mathbb{R}_+$. This property, formula (5.2), and Young's inequality for convolutions imply that for every $u_0 \in D(\Delta_{R,p})$ the extension w_0 belongs to $L^p(\mathbb{R})$ and

$$\|w_0\|_{L^p(\mathbb{R})} \le 2\|u_0\|_{L^p(0,\infty)} + |2\frac{\beta_0}{\alpha_0}\operatorname{Re}\frac{\alpha_0}{\beta_0}|\,\|u_0\|_{L^p(0,\infty)}.$$

Hence,

$$\|C(t)u_0\|_{L^p(0,\infty)} = \|\frac{1}{2}(w_0(\cdot+t)+w_0(\cdot-t))\|_{L^p(0,\infty)} \le \|w_0\|_{L^p(\mathbb{R})} \le C\,\|u_0\|_{L^p(0,\infty)},$$

where $C = 2 + |2\frac{\beta_0}{\alpha_0}\operatorname{Re}\frac{\alpha_0}{\beta_0}|$. Since $D(\Delta_{R,p})$ is dense in $L^p(0,\infty)$, this proves that $(C(t))_{t\in\mathbb{R}}$ is bounded.

(c) Let $u_0(x) := e^{-\frac{\beta_0}{\alpha_0}x}$ for $x \in (0,\infty)$. Since $\operatorname{Re}\frac{\beta_0}{\alpha_0} > 0$, it is clear that $u_0 \in D(\Delta_{R,p})$ for every $1 \le p < \infty$. Moreover, the extension w_0 defined in (5.2) is given by

$$w_0(s) = e^{-\frac{\beta_0}{\alpha_0}s}, \quad s \in \mathbb{R}.$$

In particular,

$$\begin{aligned}
\|C(t)u_0\|^p_{L^p} &= \frac{1}{2}\int_0^\infty |e^{-\frac{\beta_0}{\alpha_0}(x+t)} + e^{-\frac{\beta_0}{\alpha_0}(x-t)}|^p\,dx \\
&= \frac{1}{2}\,\|u_0\|^p_{L^p}\left|e^{-\frac{\beta_0}{\alpha_0}t} + e^{\frac{\beta_0}{\alpha_0}t}\right|^p \\
&\to \infty \quad \text{as } t \to \infty.
\end{aligned}$$

This shows that $(C(t))_{t\in\mathbb{R}}$ is (exponentially) unbounded. □

We conclude our article by formulating the theorem of generation of cosine families in $C_0([0,\infty))$, the space of continuous functions on $[0,\infty)$ vanishing at infinity.

Theorem 5.2. *Let α_0, β_0 be complex numbers such that $\alpha_0 \neq 0$. Let $a \in C_b^1([0,\infty))$ be such that $a(x) \ge \alpha > 0$ for every $x \ge 0$, and let b, $c \in C_b([0,\infty))$. Consider the operator $A_{R,\infty}$ defined on $C_0([0,\infty))$ by*

$$\begin{aligned}
D(A_{R,\infty}) &:= \{u \in C^2([0,\infty)) :\ u, u', u'' \in C_0([0,\infty)),\ \alpha_0 u'(0) + \beta_0 u(0) = 0\}, \\
A_{R,\infty}u &:= au'' + bu' + cu.
\end{aligned}$$

Then $A_{R,\infty}$ generates a cosine family on $C_0([0,\infty))$.

A similar generation result holds in $C_{ub}([0,\infty))$, the space of bounded and uniformly continuous functions, if the coefficients a, a', b and c are not only bounded but also uniformly continuous.

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Global Smooth Solutions to a Fourth-order Quasilinear Fractional Evolution Equation

Philippe Clément and Rico Zacher

Dedicated to the memory of Günter Lumer

Abstract. We study a quasilinear fractional evolution equation, which is of order four in space and $1 + \beta$ in time, where $\beta \in (0, 1)$. Under the restriction $\beta < 3/5$ we are able to prove existence and uniqueness of global smooth solutions. This result can be seen as the analogue of a result obtained by Engler for a related problem of second order.

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1. Introduction

In this paper we investigate the existence and uniqueness of global smooth solutions to the problem

$$\left\{ \begin{array}{ll} \partial_t^\beta (u_t - u_1) + \sigma(u_{xx})_{xx} = f(t,x), & t \in (0,T],\ x \in [0,L] \\ u(t,0) = u(t,L) = 0, & t \in [0,T] \\ u_{xx}(t,0) = u_{xx}(t,L) = 0, & t \in [0,T] \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in [0,L]. \end{array} \right. \tag{1.1}$$

Here ∂_t^β denotes the Riemann-Liouville fractional derivation operator of order $\beta \in (0, 1)$ defined by

$$\partial_t^\beta u(t) = \partial_t \int_0^t g_{1-\beta}(t-\tau) u(\tau)\, d\tau, \tag{1.2}$$

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where

$$g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad t > 0, \quad \alpha > 0.$$

The nonlinearity σ is a smooth real-valued function satisfying the condition

$$0 < \kappa_1 \le \sigma'(s) \le \kappa_2, \quad s \in \mathbb{R}. \tag{1.3}$$

The functions f, u_0, and u_1 are given data.

The corresponding second-order problem, that is,

$$\left\{ \begin{array}{ll} \partial_t^\beta (u_t - u_1) - \sigma(u_x)_x = f(t,x), & t \in (0,T],\ x \in [0,L] \\ u(t,0) = u(t,L) = 0, & t \in [0,T] \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in [0,L], \end{array} \right. \tag{1.4}$$

as well as variants of it have been studied by many authors. Existence of global weak solutions but not uniqueness has been obtained in [10] for all $\beta \in (0,1)$. Existence of global strong solutions for all $\beta \in (0,1)$ has been established in [8] and [3] by means of a perturbation argument requiring the smallness of the number

$$\frac{\kappa_2 - \kappa_1}{\kappa_1}. \tag{1.5}$$

We further mention Gripenberg's result [9], where global weak solutions u with u_{xx} square integrable but no uniqueness were obtained under the condition $\beta \le 1/2$. Restricting further β to be less than $1/3$, Engler [7] was able to show existence and uniqueness of global smooth solutions for a variant of (1.4) without smallness condition on the number in (1.5).

In this paper we prove an analogue of Engler's result [7] in the 'fourth-order case', see Theorem 4.2. We also need to impose a restriction on β, which is $\beta < 3/5$. Assuming this condition together with (1.3) and suitable smoothness and compatibility conditions on the data and the nonlinearity (see (H1)–(H4) below), we establish global existence and uniqueness of smooth solutions of (1.1) (with $u_1 = 0$, see below).

Our proof consists of two parts. In the first step we obtain the local well-posedness of (1.1) for all $\beta \in (0,1)$ in the framework of continuous interpolation spaces, see Theorem 3.2. Here we make use of a recent result on abstract quasilinear fractional evolution equations, [4, Theorem 13]. This result requires $u_1 = 0$, which will be assumed throughout this paper. We remark that by using the results in [11], it is also possible to treat the case $u_1 \ne 0$. We recall that the method of continuous interpolation spaces has been introduced by Da Prato and Grisvard in [5] and extended by Angenent [2], Lunardi [12], and Simonett [13].

In the second part of our proof we derive a priori estimates which imply the global well-posedness of (1.1). A crucial step here is to obtain an a priori bound for u_{xx} in a Hölder space, which is achieved by using Engler's method, see [7]. In order to justify the corresponding computations, we are forced to work with higher regularity. So the parameters in our setting are chosen in such a way that a solution on a time-interval $[0,T]$, say, necessarily belongs to the space $C^1([0,T]; C^4([0,L]))$.

We remark that short-time existence and uniqueness of smooth solutions can be shown under weaker assumptions on the function σ and the data.

Taking the a priori Hölder estimate for u_{xx} as a starting point, we then carry out a bootstrap argument, which eventually yields the global well-posedness of (1.1). Note that in contrast to the second-order case, problem (1.4), here one is confronted with an extra nonlinear term, which is of third order, as can be seen by writing the first equation in (1.1) as

$$\partial_t^\beta (u_t - u_1) + \sigma'(u_{xx})u_{xxxx} = -\sigma''(u_{xx})u_{xxx}^2 + f(t,x), \;\; t > 0, \; x \in [0, L].$$

The paper is organized as follows. In Section 2 we fix some notation. Section 3 is devoted to the local well-posedness, while Section 4 is on a priori estimates and global existence. Finally, in Section 5 we prove an auxiliary result, which is needed in Section 3.

2. Preliminaries

By $f * g$ we mean the convolution defined by $(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$, $t \geq 0$, of two functions f, g supported on the positive half-line.

Let X be a Banach space and $T > 0$. We say that a function $u \in L_1((0,T);X)$ has a fractional derivative of order $\beta \in (0,1)$ if $u = g_\beta * f$ for some $f \in L_1((0,T);X)$. In this case we write $\partial_t^\beta u = f$.

We next consider functions defined on $J_0 := (0,T]$ which have (at most) a singularity of prescribed order at $t = 0$. Letting $J = [0,T]$ and $\mu \in (0,1)$ we define the space

$$\begin{aligned} BUC_{1-\mu}(J;X) = \{u \in & C(J_0;X) : \; t^{1-\mu}u(t) \in BUC(J_0;X) \\ & \text{and } \lim_{t\to 0+} t^{1-\mu}|u(t)|_X = 0\}, \end{aligned}$$

which becomes a Banach space when endowed with the norm

$$|u|_{BUC_{1-\mu}(J;X)} = \sup_{t\in J_0} t^{1-\mu}|u(t)|_X.$$

We further introduce the following subspace of $BUC_{1-\mu}(J;X)$. For $\beta \in (0,1)$ we set (cf. [4, p. 423])

$$\begin{aligned} BUC_{1-\mu}^{1+\beta}(J;X) = \{u \in BUC_{1-\mu}(J;X) : \; & u = x + g_{1+\beta} * f, \\ & \text{for some } x \in X \text{ and } f \in BUC_{1-\mu}(J;X)\}. \end{aligned}$$

3. Local well-posedness

In order to prove existence and uniqueness of local (in time) smooth solutions of (1.1) with $u_1 = 0$, we will apply [4, Theorem 13]. In what follows we explain the underlying setting and verify the assumptions needed in this result.

Let $L > 0$ and set $I = [0, L]$. Let

$$F_0 = \{v \in C(I) :\ v(0) = v(L) = 0\}$$

and

$$F_1 = \{v \in C^4(I) :\ v^{(i)}(0) = v^{(i)}(L) = 0,\ i = 0, 2, 4\},$$

endowed with the canonical norms. For $s \in (0, 8)$ with $s \notin \mathbb{N}$ we define

$$h^s_{bc}(I) = \{v \in h^s(I) :\ v^{(i)}(0) = v^{(i)}(L) = 0 \text{ for all } i \in \{0, 2, 4, 6\} \text{ with } i < s\},$$

where $h^s(I)$ stands for the little Hölder space with exponent s. It is well known, see [12], that the continuous interpolation space

$$F_\theta := (F_0, F_1)^0_{\theta,\infty} = h^{4\theta}_{bc}(I), \quad \theta \in (0, 1),\ 4\theta \notin \mathbb{N}.$$

Putting

$$E_0 = h^{4\theta}_{bc}(I), \quad E_1 = h^{4+4\theta}_{bc}(I), \quad \theta \in (0, 1),\ 4\theta \notin \mathbb{N},$$

the following embeddings hold true:

$$E_1 \hookrightarrow F_1 \hookrightarrow E_0 \hookrightarrow F_0.$$

We further set

$$E_\eta = (E_0, E_1)^0_{\eta,\infty}, \quad \eta \in (0, 1).$$

Then

$$E_\eta = h^{4\eta+4\theta}_{bc}(I), \quad \theta \in (0, 1),\ 4\theta \notin \mathbb{N},\ \eta \in (0, 1),\ 4(\eta + \theta) \notin \mathbb{N}. \tag{3.1}$$

We next put

$$\hat{\mu} = \frac{\mu + \beta}{1 + \beta},$$

and assuming that $E_{\hat{\mu}} \hookrightarrow C^3(I)$ (cp. (3.8) below) we may define

$$\mathcal{A}(v)w = \sigma'(v_{xx})w_{xxxx}, \quad v \in E_{\hat{\mu}},\ w \in E_1,$$

and

$$\mathcal{F}(v) = -\sigma''(v_{xx})v^2_{xxx}, \quad v \in E_{\hat{\mu}}.$$

Then (1.1) with $u_1 = 0$ can be written as an abstract quasilinear problem of the form

$$\left\{ \begin{array}{c} \partial^\beta_t u_t + \mathcal{A}(u)u = \mathcal{F}(u) + f(t), \quad t > 0 \\ u(0) = u_0, \quad u_t(0) = 0. \end{array} \right. \tag{3.2}$$

Letting μ, $\beta \in (0, 1)$ and $J = [0, T]$, we choose

$$\tilde{E}_0(J) := BUC_{1-\mu}(J; E_0)$$

as the base space for the fractional differential equation in (3.2) and seek solutions in the corresponding maximal regularity class

$$\tilde{E}_1(J) := BUC^{1+\beta}_{1-\mu}(J; E_0) \cap BUC_{1-\mu}(J; E_1).$$

It has been shown in [4, Theorem 10], cf. also [11], that

$$\tilde{E}_1(J) \hookrightarrow BUC^{(1+\beta)(1-\eta)-(1-\mu)}(J; E_\eta), \quad 0 \le \eta \le \hat{\mu}. \tag{3.3}$$

In particular, we have

$$\tilde{E}_1(J) \hookrightarrow BUC(J; E_{\hat{\mu}}). \tag{3.4}$$

Note that if $\mu+\beta>1$, then the Hölder exponent in (3.3) exceeds 1, provided $\eta>0$ is sufficiently small.

We will next fix the parameters μ, $\theta\in(0,1)$ appropriately, ensuring among other things that $E_{\hat\mu}\hookrightarrow C^3(I)$ and $\tilde E_1(J)\hookrightarrow C^1(J;C^4(I))$.

Let $\varepsilon\in(0,\frac{\beta}{4(1+\beta)})$ and set

$$\mu=1-\varepsilon(1+\beta),\quad \theta=\frac{1}{1+\beta}+3\varepsilon,\quad \eta=\frac{\beta}{1+\beta}-2\varepsilon. \tag{3.5}$$

Here we exclude those values of ε, for which the condition

$$4\theta\notin\mathbb{N},\quad 4(\eta+\theta)\notin\mathbb{N},\quad \text{and}\quad 4(\hat\mu+\theta)\notin\mathbb{N}$$

is violated. Then

$$\hat\mu=\frac{\mu+\beta}{1+\beta}=1-\varepsilon,$$

and it is readily checked that $\eta\in(0,\hat\mu)$. We further have $\theta+\eta=1+\varepsilon$, and

$$\begin{aligned}(1+\beta)(1-\eta)-(1-\mu)&=(1+\beta)(\frac{1}{1+\beta}+2\varepsilon)-\varepsilon(1+\beta)\\&=1+\varepsilon(1+\beta).\end{aligned}$$

Using (3.1) and (3.3), we thus see that

$$\begin{aligned}\tilde E_1(J)&\hookrightarrow BUC^{(1+\beta)(1-\eta)-(1-\mu)}(J;h_{bc}^{4\eta+4\theta}(I))\\&= BUC^{1+\varepsilon(1+\beta)}(J;h_{bc}^{4(1+\varepsilon)}(I))\hookrightarrow C^1(J;C^4(I)).\end{aligned}\tag{3.6}$$

Notice as well that

$$\hat\mu+\theta=\frac{2+\beta}{1+\beta}+3\varepsilon-\varepsilon\in\Big(\frac{3}{2}+2\varepsilon,2-\varepsilon\Big).\tag{3.7}$$

In particular, we have

$$E_{\hat\mu}=h_{bc}^{4\hat\mu+4\theta}(I)\hookrightarrow C^{6+8\varepsilon}(I).\tag{3.8}$$

We will assume that the data and the nonlinearity in (1.1) are subject to the following conditions:

- **(H1)** $\sigma\in C^7(\mathbb{R})$, $\sigma^{(k)}(0)=0$, $k=0,2,4$;
- **(H2)** $0<\kappa_1\le\sigma'(s)\le\kappa_2,\quad s\in\mathbb{R}$;
- **(H3)** $f\in C^1(\mathbb{R}_+;C(I))\cap C(\mathbb{R}_+;C^4(I))$,
  $f(t,0)=f(t,L)=f_{xx}(t,0)=f_{xx}(t,L)=0$, $t\ge0$;
- **(H4)** $u_0\in C^8(I)$, $u_0^{(k)}(0)=u_0^{(k)}(L)=0$, $k=0,2,4,6$; $u_1=0$.

Observe that (H3) implies that $f\in BUC_{1-\mu}([0,T];E_0)$, for any $T>0$, while (H4) and (3.7) ensure that $u_0\in E_{\hat\mu}=h_{bc}^{4\hat\mu+4\theta}(I)$. Therefore condition (47) in [4] is satisfied.

We remark that for the Theorems 3.2 and 4.2 below we do not need the full regularity of u_0 required in (H4). In fact, $u_0\in h_{bc}^{4(\hat\mu+\theta+\varepsilon)}(I)$ would be sufficient.

For Banach spaces X, Y, and a mapping $\mathcal{G}$ of X into Y, we write $\mathcal{G} \in C^{1-}_{\rm loc}(X;Y)$, if every point $x \in X$ has a neighbourhood U such that $\mathcal{G}$ restricted to U is globally Lipschitz continuous. By $\mathcal{B}(X,Y)$ we mean the space of bounded linear operators from X into Y. We write $\mathcal{B}(X) = \mathcal{B}(X,X)$ for short.

In order to be able to apply [4, Theorem 13], it remains to verify that (cf. [4, condition (46)])

$$(\mathcal{A}, \mathcal{F}) \in C^{1-}_{\rm loc}(E_{\hat{\mu}}; \mathcal{M}_{\beta,\,\mu}(E_1, E_0) \times E_0). \tag{3.9}$$

Here $\mathcal{M}_{\beta,\,\mu}(E_1,E_0)$ denotes the space of all operators $A \in \mathcal{B}(E_1,E_0)$ satisfying the following two conditions: (i) $\exists \omega \geq 0$ such that $A_\omega := A + \omega I$ is a nonnegative closed operator in E_0 with spectral angle $\varphi_{A_\omega} < \frac{\pi}{2}(1-\beta)$; (ii) $\partial_t^\beta u_t + Au = h(t)$, $u(0) = 0$, $u_t(0) = 0$, has maximal regularity in $\tilde{E}_0(J)$, i.e., there exists $C > 0$ such that for any $h \in \tilde{E}_0(J)$,

$$|u|_{\tilde{E}_1(J)} \leq C|h|_{\tilde{E}_0(J)},$$

where u solves $\partial_t^\beta u_t + Au = h(t)$, $u(0) = 0$, $u_t(0) = 0$. $\mathcal{M}_{\beta,\,\mu}(E_1,E_0)$ is equipped with the topology of $\mathcal{B}(E_1,E_0)$.

Let $v \in E_{\hat{\mu}}$ and $w \in E_1$. Then, obviously, $w_{xxxx} \in E_0 = h^{4\theta}_{bc}(I)$ and $v_{xx} \in h^{4\hat{\mu}+4\theta-2}_{bc}(I)$. Note that $2 < 4\theta < 4 < 4\hat{\mu} + 4\theta - 2$, due to (3.5) and (3.7). Since

$$\begin{aligned}\Big(\sigma'(v_{xx})w_{xxxx}\Big)_{xx} &= \sigma'(v_{xx})_{xx}w_{xxxx} + 2\sigma''(v_{xx})v_{xxx}w_{xxxxx} \\ &\quad + \sigma'(v_{xx})w_{xxxxxx},\end{aligned}$$

and $\sigma''(0) = 0$, by (H1), we see that $(\sigma'(v_{xx})w_{xxxx})_{xx}$ vanishes at $x = 0$ and $x = L$. In view of (H1) ($\sigma \in C^6$ is enough) it is then clear that $\mathcal{A} \in C^{1-}_{\rm loc}(E_{\hat{\mu}}; \mathcal{B}(E_1,E_0))$. Similarly, one checks that $\mathcal{F} \in C^{1-}_{\rm loc}(E_{\hat{\mu}}, E_0)$. Note that here one needs one derivative more for σ; $\sigma \in C^6$ does not suffice. Notice also that

$$\begin{aligned}\Big(\sigma''(v_{xx})v_{xxx}^2\Big)_{xx} &= \sigma''''(v_{xx})v_{xxx}^4 + 5\sigma'''(v_{xx})v_{xxx}^2 v_{xxxx} \\ &\quad + 2\sigma''(v_{xx})[v_{xxxx}^2 + v_{xxx}v_{xxxxx}],\end{aligned}$$

which shows that $(\sigma''(v_{xx})v_{xxx}^2)_{xx}$ vanishes at $x = 0$ and $x = L$, by (H1).

Finally, let $v \in E_{\hat{\mu}}$ be fixed and define the operators

$$\tilde{A}w = \sigma'(v_{xx})w_{xxxx}, \quad w \in F_1,$$

and

$$Aw = \mathcal{A}(v)w = \sigma'(v_{xx})w_{xxxx}, \quad w \in E_1.$$

Then it follows from (H2) and the preceding considerations that $\tilde{A}$ and A are isomorphisms mapping F_1 into F_0 and E_1 into E_0, respectively. Note that $Av = \tilde{A}v$ for all $v \in E_1$. Furthermore, $\tilde{A}$ as an operator in F_0 is nonnegative with spectral angle $\phi_{\tilde{A}} = 0$. The latter property is a consequence of the following result.

Lemma 3.1. *Let $L > 0$ and $F_0 = \{v \in C([0,L];\mathbb{C}) : \ v(0) = v(L) = 0\}$ equipped with the supremum norm. Suppose further that $m \in C([0,L])$ is strictly positive. Then the operator $\tilde{A} : D(\tilde{A}) \subset F_0 \to F_0$ defined by*

$$D(\tilde{A}) = F_1 = \{v \in C^4([0,L];\mathbb{C}) : \ v^{(i)}(0) = v^{(i)}(L) = 0, \ i = 0, 2, 4\},$$

and

$$\tilde{A}w = mw'''', \quad w \in D(\tilde{A}),$$

is invertible and sectorial with spectral angle $\phi_{\tilde{A}} = 0$. We have $\mathbb{C} \setminus (0,\infty) \subset \rho(\tilde{A})$ and for any $\vartheta \in [0,\pi)$ there exists $M_1(\vartheta) > 0$ such that

$$|(\lambda + \tilde{A})^{-1}|_{\mathcal{B}(F_0)} \leq \frac{M_1(\vartheta)}{1 + |\lambda|}, \quad \lambda \in \mathbb{C} \setminus \{0\}, \ |\arg \lambda| \leq \vartheta. \tag{3.10}$$

A proof of Lemma 3.1 is given in Section 5.

It follows now from [4, Theorem 11] applied to the operators $\tilde{A}$ and A, that $A \in \mathcal{M}_{\beta,\mu}(E_1, E_0)$. This shows that $\mathcal{A}(v) \in \mathcal{M}_{\beta,\mu}(E_1, E_0)$ for all $v \in E_{\hat{\mu}}$. Hence condition (3.9) is satisfied.

We are now in a position to apply [4, Theorem 13]. This establishes the local well-posedness of (1.1) in the described setting.

Theorem 3.2. *Let the assumptions* (H1)–(H4) *be satisfied. Let $\beta \in (0,1)$ and assume that the parameters μ, $\theta \in (0,1)$ are chosen as in* (3.5)*. Then there exists a unique maximal solution u defined on the maximal interval of existence $[0, T_0)$, where $T_0 \in (0,\infty]$, and such that for any $T \in (0, T_0)$ one has*

$$u \in Z^T := BUC^{1+\beta}_{1-\mu}([0,T]; h^{4\theta}_{bc}([0,L])) \cap BUC_{1-\mu}([0,T]; h^{4+4\theta}_{bc}([0,L]))$$

and u solves (1.1) *on $[0,T]$. Further, for any $T \in (0,T_0)$, $u \in C^1([0,T]; C^4([0,L]))$. If $T_0 < \infty$ then*

$$\limsup_{t \uparrow T_0} |u(t)|_{h^{4\delta+4\theta}_{bc}([0,L])} = \infty, \quad \text{for any } \delta \in (\hat{\mu}, 1), \text{ where } \hat{\mu} = \frac{\mu + \beta}{1 + \beta}.$$

4. A priori estimates and global well-posedness

In this section we will prove that the solution u of (1.1) constructed in Theorem 3.2 exists globally, i.e., $T_0 = \infty$. We will make use of the following simple lemma.

Lemma 4.1. *Let $T > 0$, $\beta \in (0,1)$, and $v \in \mathrm{Lip}(-\infty, T]$ with $v(t) = v(0)$, $t < 0$. Then*

$$\int_{-\infty}^{t} g_\beta(t-s)[v(s) - v(t)]_s \, ds = \int_{-\infty}^{t} \dot{g}_\beta(t-s)[v(s) - v(t)] \, ds, \quad 0 < t \leq T. \tag{4.1}$$

Proof. We split the integral on the right-hand side of (4.1) and integrate by parts. This gives for $t \in (0,T]$,

$$\int_{-\infty}^{t} \dot{g}_\beta(t-s)[v(s)-v(t)]\,ds = \int_{-\infty}^{0} \dots\, ds \quad + \int_{0}^{t} \dots\, ds$$
$$= -\,[v(0)-v(t)]\,g_\beta(t) + \Big[(v(t)-v(s))\,g_\beta(t-s)\Big]_{s=0}^{s=t}$$
$$+ \int_0^t g_\beta(t-s)[v(s)-v(t)]_s\,ds$$
$$= \int_0^t g_\beta(t-s)[v(s)-v(t)]_s\,ds = \int_{-\infty}^{t} g_\beta(t-s)[v(s)-v(t)]_s\,ds.$$

Note that the first line shows that the integral on the right-hand side of (4.1) is well defined. In the step before last we used the Lipschitz continuity of v to conclude that $\lim_{s\uparrow t}(v(t)-v(s))\,g_\beta(t-s)=0$. □

The main result of the present paper is now the following.

Theorem 4.2. *Let the assumptions* (H1)–(H4) *be satisfied. Assume that*

$$0 < \beta < \frac{3}{5},$$

and suppose that the parameters μ, $\theta \in (0,1)$ *are chosen as in* (3.5). *Then the unique maximal solution* u *of* (1.1) *constructed in Theorem* 3.2 *exists globally, that is,* $T_0=\infty$*: For any* $T>0$ *one has*

$$u \in Z^T = BUC^{1+\beta}_{1-\mu}([0,T]; h^{4\theta}_{bc}([0,L])) \cap BUC_{1-\mu}([0,T]; h^{4+4\theta}_{bc}([0,L]))$$

and u *solves* (1.1) *on* $[0,T]$.

Proof. Suppose that $T_0<\infty$ and let $T\in[T_0/2,T_0)$. By means of a series of estimates for u on $[0,T]\times[0,L]$ (uniform with respect to T), we will show that

$$\limsup_{t\uparrow T_0} |u(t)|_{h^{4(\bar{\mu}+\theta+\varepsilon)}_{bc}([0,L])} < \infty, \tag{4.2}$$

where ε is the positive number that was used in the definition of θ and μ in (3.5). By the blow up criterion given in Theorem 3.2, (4.2) leads to a contradiction, which will imply that $T_0=\infty$.

The proof of (4.2) proceeds in four steps. In the first step we will obtain the basic a priori bound for u_{xx} in a space of Hölder continuous functions. In the Steps 2–4 we will carry out a bootstrap argument which eventually yields (4.2).

Step 1: An estimate for u_{xx} in $C^\delta([0,T];C^\delta(I))$ with some $\delta>0$. Since $u\in C^1([0,T];C^4([0,L]))$, by Theorem 3.2, and $f\in C^1([0,T];C([0,L]))$, due to (H3), we may convolve the first equation in (1.1) with g_β and differentiate with respect to time, thereby obtaining

$$u_{tt} + g_\beta * [\sigma(u_{xx})_{xxt}] = g_\beta * (f_t) + g_\beta(t)\varphi(x), \tag{4.3}$$

where $\varphi(x) = f(0,x) - \sigma(u_0''(x))_{xx}$. Setting $u(t,x) = u_0(x)$ for $t < 0$, (4.3) can be written as

$$\begin{aligned} u_{tt}(t,x) + \int_{-\infty}^{t} g_\beta(t-s)[\sigma(u_{xx}(s,x)) - \sigma(u_{xx}(t,x))]_{xxs}\, ds \\ = (g_\beta * f_t)(t,x) + g_\beta(t)\varphi(x), \end{aligned}$$

which after an integration by parts, cf. Lemma 4.1, appears in the form

$$\begin{aligned} u_{tt}(t,x) + \int_{-\infty}^{t} \dot{g}_\beta(t-s)[\sigma(u_{xx}(s,x)) - \sigma(u_{xx}(t,x))]_{xx}\, ds \\ = (g_\beta * f_t)(t,x) + g_\beta(t)\varphi(x). \end{aligned} \tag{4.4}$$

We multiply (4.4) by u_t, integrate over $[0,L]$, and integrate by parts. This gives ($\sigma(0) = 0$, by (H1))

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^L u_t(t,x)^2\, dx - \int_{-\infty}^{t} \dot{g}_\beta(t-s)\, \frac{\partial}{\partial t} H(t,s)\, ds \\ = \int_0^L [(g_\beta * f_t)(t,x) + g_\beta(t)\varphi(x)]u_t(t,x)\, dx, \quad t \in (0,T] \end{aligned} \tag{4.5}$$

where

$$H(t,s) = \int_0^L \int_{u_{xx}(s,x)}^{u_{xx}(t,x)} [\sigma(y) - \sigma(u_{xx}(s,x))]\, dy\, dx.$$

Since σ is strictly increasing, we see that $H(t,s) \geq 0$ for all $0 \leq s,\, t \leq T$. Also, by continuity of u_{xxt} on $[0,T] \times [0,L]$, there exists a constant $M > 0$ such that

$$H(t,s) \leq M|t-s|^2, \qquad |\frac{\partial}{\partial t} H(t,s)| \leq M|t-s|, \quad 0 \leq s,\, t \leq T.$$

Therefore, setting

$$W(t) = \frac{1}{2}\int_0^L u_t(t,x)^2\, dx - \int_{-\infty}^{t} \dot{g}_\beta(t-s)H(t,s)\, ds, \quad t \in [0,T],$$

we may rewrite (4.5) as

$$\begin{aligned} \dot{W}(t) = \int_0^L [(g_\beta * f_t)(t,x) + g_\beta(t)\varphi(x)]u_t(t,x)\, dx \\ - \int_{-\infty}^{t} \ddot{g}_\beta(t-s)\, H(t,s)\, ds, \quad t \in (0,T]. \end{aligned}$$

Since $\ddot{g}_\beta$ and H are nonnegative and by using Young's inequality, we then obtain

$$\dot{W}(t) \leq W(t) + \frac{1}{2}\int_0^L [(g_\beta * f_t)(t,x) + g_\beta(t)\varphi(x)]^2\, dx, \quad t \in (0,T],$$

which yields the estimate

$$W(t)=\frac{1}{2}\int_0^L u_t(t,x)^2\,dx-\int_{-\infty}^t \dot g_\beta(t-s)H(t,s)\,ds\le C,\quad t\in[T_0/2,T],\tag{4.6}$$

where the constant C depends only on $W(T_0/2)$ and the data.

It is not difficult to see (cf. [7, Lemma 3.3]) that

$$\frac{1}{2\kappa_2}\,|\sigma(u_{xx}(t,x))-\sigma(u_{xx}(s,x))|^2\le\int_{u_{xx}(s,x)}^{u_{xx}(t,x)}[\sigma(y)-\sigma(u_{xx}(s,x))]\,dy.\tag{4.7}$$

Proceeding as in Engler [7] (cf. Lemma 3.4), it follows from (4.6) and (4.7) that

$$|\sigma(u_{xx})(t,\cdot)-\sigma(u_{xx})(s,\cdot)|_{L_2(I)}\le C_1|t-s|^{(1-\beta)/2},\quad t,\,s\in[T_0/2,T],\tag{4.8}$$

where the constant C_1 depends only on $W(T_0/2)$ and the data. In fact, writing $\xi(t)=\sigma(u_{xx})(t,\cdot)\in L_2(I)$ for $t\in[T_0/2,T]$, (4.6) and (4.7) imply that

$$\int_{t-h}^t (t-\tau)^{\beta-2}|\xi(t)-\xi(\tau)|^2_{L_2(I)}\,d\tau\le\tilde C,\quad t\in[T_0/2,T],\,h\in(0,T_0/2],\tag{4.9}$$

where the constant $\tilde C$ depends only on $W(T_0/2)$ and the data. From (4.9) and Hölder's inequality we then obtain

$$\int_{t-h}^t|\xi(t)-\xi(\tau)|_{L_2(I)}\,d\tau\le\sqrt{\tilde C}\,\Big(\int_{t-h}^t(t-\tau)^{2-\beta}\,d\tau\Big)^{\frac12}\le\tilde C_1h^{\frac{3-\beta}{2}}$$

for all $t\in[T_0/2,T]$ and $h\in(0,T_0/2]$, where $\tilde C_1=\tilde C_1(\tilde C,\beta)$. Setting

$$\xi_h(t)=\frac{2}{h^2}\int_{t-h}^t(\tau-t+h)\xi(\tau)\,d\tau,\quad t\in[T_0/2,T],\,h\in(0,T_0/2],$$

this yields

$$\begin{aligned}|\xi(t)-\xi_h(t)|_{L_2(I)}&=\frac{2}{h^2}\,|\int_{t-h}^t(\tau-t+h)(\xi(t)-\xi(\tau))\,d\tau|_{L_2(I)}\\&\le\frac{2}{h}\int_{t-h}^t|\xi(t)-\xi(\tau)|_{L_2(I)}\,d\tau\ \le 2\tilde C_1h^{\frac{1-\beta}{2}}\end{aligned}$$

as well as

$$|\dot\xi_h(t)|_{L_2(I)}\le\frac{2}{h^2}\int_{t-h}^t|\xi(t)-\xi(\tau)|_{L_2(I)}\,d\tau\le 2\tilde C_1h^{-\frac{1+\beta}{2}}.$$

Hence, for $T_0/2\le s<t\le T$ and $h:=t-s\in(0,T_0/2)$ we have

$$\begin{aligned}|\xi(t)-\xi(s)|_{L_2(I)}&\le|\xi(t)-\xi_h(t)|_{L_2(I)}+|\xi_h(t)-\xi_h(s)|_{L_2(I)}+|\xi_h(s)-\xi(s)|_{L_2(I)}\\&\le 4\tilde C_1h^{\frac{1-\beta}{2}}+2\tilde C_1(t-s)h^{-\frac{1+\beta}{2}}\le 6\tilde C_1(t-s)^{\frac{1-\beta}{2}},\end{aligned}$$

which proves (4.8).

Thanks to (4.6), (4.8), and the smoothness of σ we obtain bounds for

$$u_t\in L_\infty([0,T];L_2(I))\quad\text{and}\quad\sigma(u_{xx})\in C^{\frac{1-\beta}{2}}([0,T];L_2(I)),\tag{4.10}$$

which depend only on the data, $W(T_0/2)$, and the corresponding bounds on the interval $[0, T_0/2]$.

We now put $v = 1 * \sigma(u_{xx})$. Then we have a bound for v in the space $C^{1+(1-\beta)/2}([0,T]; L_2(I))$ in terms of the bound for $\sigma(u_{xx}) \in C^{\frac{1-\beta}{2}}([0,T]; L_2(I))$. On the other hand, we may integrate the first equation in (1.1) with respect to time to the result

$$v_{xx} = -g_{1-\beta} * u_t + 1 * f,$$

which yields a bound for v in the space $C^{1-\beta}([0,T]; H_2^2(I))$ in terms of the bound for $u_t \in L_\infty([0,T]; L_2(I))$ and the data. By means of interpolation (cp. [7, pp. 283–284]) and Sobolev embedding, we have the embeddings

$$\begin{aligned} C^{\frac{3-\beta}{2}}([0,T]; L_2(I)) \cap C^{1-\beta}([0,T]; H_2^2(I)) &\hookrightarrow C^{(1-\tau)\frac{3-\beta}{2}+\tau(1-\beta)}([0,T]; H_2^{2\tau}(I)) \\ &\hookrightarrow C^{1+\delta}([0,T]; C^\delta(I)) \end{aligned}$$

for some $\delta > 0$ and some $\tau \in (\frac{1}{4}, \frac{1-\beta}{1+\beta})$, the latter being possible, since $\beta < 3/5$. Hence we obtain an a priori estimate for $\sigma(u_{xx})$ in $C^\delta([0,T]; C^\delta(I))$. Since σ is strictly increasing and smooth, we get also an a priori bound for u_{xx} itself in the space $C^\delta([0,T]; C^\delta(I))$. Note that this bound is uniform with respect to $T \in [T_0/2, T_0)$.

Step 2: An estimate for u in $BUC([0,T]; C^{4+\delta_1}(I))$ with $\delta_1 \in (0,\delta)$. We write the first equation in (1.1) as

$$\partial_t^\beta u_t + \sigma'(u_{xx})u_{xxxx} = f - \sigma''(u_{xx})u_{xxx}^2, \quad t \in (0,T],\ x \in [0,L], \tag{4.11}$$

and view it as a linear equation for u of the form

$$\partial_t^\beta u_t + m(t,x)u_{xxxx} = \tilde{f}, \tag{4.12}$$

where $m = \sigma'(u_{xx})$ and $\tilde{f} = f - \sigma''(u_{xx})u_{xxx}^2$. Note that $\tilde{f}(t,0) = \tilde{f}(t,L) = 0$, $t \in [0,T]$, since, by assumptions, f enjoys the same property and $\sigma''(0) = 0$. We use then maximal regularity of (4.12) together with the boundary and initial conditions as in (1.1), in the space $\tilde{E}_0([0,T]) = BUC_{1-\mu}([0,T]; h_{bc}^{4\theta}(I))$ with $\mu \in (0,1)$ and $\theta \in (0, \delta/4)$. Letting

$$\tilde{E}_1([0,T]) = BUC_{1-\mu}^{1+\beta}([0,T]; h_{bc}^{4\theta}(I)) \cap BUC_{1-\mu}([0,T]; h_{bc}^{4+4\theta}(I)),$$

this gives the estimate

$$|u|_{\tilde{E}_1([0,T])} \le C\Big(|\tilde{f}|_{\tilde{E}_0([0,T])} + |u_0|_{h_{bc}^{4\theta+4\hat{\mu}}(I)}\Big), \tag{4.13}$$

where C is a positive constant which depends only on the parameters and the bound for u_{xx} in $C^\delta([0,T]; C^\delta(I))$. The space $C^{4\theta}(I)$ forms an algebra with respect to pointwise multiplication, and we have

$$|\sigma''(u_{xx}(t,\cdot))u_{xxx}(t,\cdot)^2|_{C^{4\theta}(I)} \le |\sigma''(u_{xx}(t,\cdot))|_{C^{4\theta}(I)}|u_{xxx}(t,\cdot)|^2_{C^{4\theta}(I)}, \quad t \in [0,T].$$

Since $4\theta < \delta$, there exists $\eta \in (0, 1/2)$ such that

$$|u_{xxx}(t,\cdot)|_{C^{4\theta}(I)} \le C_1 |u_{xx}(t,\cdot)|^\eta_{C^{2+4\theta}(I)} |u_{xx}(t,\cdot)|^{1-\eta}_{C^\delta(I)}, \quad t \in [0,T],$$

where $C_1 > 0$ is a positive constant. Using these inequalities we may estimate

$$\begin{aligned}
|\tilde{f}|_{\tilde{E}_0([0,T])} &\leq |\sigma''(u_{xx})u_{xxx}^2|_{\tilde{E}_0([0,T])} + |f|_{\tilde{E}_0([0,T])} \\
&\leq C_2 \sup_{t\in(0,T]} t^{1-\mu}|u_{xx}(t,\cdot)|^{2\eta}_{C^{2+4\theta}(I)}|u_{xx}(t,\cdot)|^{2(1-\eta)}_{C^{\delta}(I)} + |f|_{\tilde{E}_0([0,T])} \\
&\leq C_3|u|^{2\eta}_{BUC_{1-\mu}([0,T];h^{4+4\theta}_{bc}(I))} + |f|_{\tilde{E}_0([0,T])}, \qquad (4.14)
\end{aligned}$$

where the constants C_2, C_3 depend on T_0 and the bound for u_{xx} in

$$C^{\delta}([0,T];C^{\delta}(I)).$$

It follows then from (4.13) and (4.14) that

$$|u|_{\tilde{E}_1([0,T])} \leq C_2 C|u|^{2\eta}_{\tilde{E}_1([0,T])} + C\Big(|f|_{\tilde{E}_0([0,T])} + |u_0|_{h^{4\theta+4\hat{\mu}}_{bc}(I)}\Big).$$

Thanks to $2\eta < 1$ and by Young's inequality, this yields a bound for u in $\tilde{E}_1([0,T])$ in terms of T_0, the parameters, the data, and $|u|_{C^{\delta}([0,T];C^{\delta}(I))}$. In view of (3.4), (3.8), and $\theta < \delta/4$ the space Z^T embeds into $BUC([0,T];C^{4+4\theta}(I))$. We thus obtain a bound for u in the latter space in terms of the preceding set of quantities and the corresponding bound on the interval $[0,T_0/2]$. Putting $\delta_1 = 4\theta$, this is the desired bound of Step 2.

Step 3: An estimate for u in $BUC([0,T];C^{6+\delta_2}(I))$ with some $\delta_2 \in (0,\delta_1)$. We differentiate the first equation in (1.1) twice with respect to x, which is possible since f and $\sigma(u_{xx})_{xx}$ belong to the space $C([0,T];C^2(I))$ (by (3.4), (3.8), (H1), (H3)) and thus $\partial_t^{\beta}u_t$ does so, by (1.1). Letting $w = u_{xx}$ we obtain

$$\begin{aligned}
\partial_t^{\beta}w_t + \sigma'(u_{xx})w_{xxxx} =& f_{xx} - 4\sigma''(u_{xx})u_{xxx}w_{xxx} - 3\sigma''(u_{xx})u_{xxxx}^2 \\
&- 6\sigma'''(u_{xx})u_{xxx}^2u_{xxxx} - \sigma''''(u_{xx})u_{xxx}^4. \qquad (4.15)
\end{aligned}$$

Furthermore

$$w(t,0) = w(t,L) = w_{xx}(t,0) = w_{xx}(t,L) = 0, \;\; t \in (0,T],$$

and

$$w(0,x) = u_0''(x), \quad w_t(0,x) = 0, \;\; x \in [0,L].$$

Denoting the right-hand side of (4.15) by $\tilde{f}$, we have $\tilde{f}(t,0) = \tilde{f}(t,L) = 0$, $t \in (0,T]$, as f_{xx}, u_{xx}, and u_{xxxx} enjoy this property, and $\sigma''(0) = \sigma''''(0) = 0$, by assumption. By means of maximal regularity in the space $\tilde{E}_0([0,T]) = BUC_{1-\mu}([0,T];h^{\delta_2}_{bc}(I))$ with $\mu \in (0,1)$ as in (3.5) and $\delta_2 \in (0,\min\{\delta_1, 8\varepsilon\})$, there is a positive constant C depending only on the parameters and the a priori bound for u_{xx} in $C^{\delta}([0,T];C^{\delta}(I))$ such that with

$$\tilde{E}_1([0,T]) = BUC^{1+\beta}_{1-\mu}([0,T];h^{\delta_2}_{bc}(I)) \cap BUC_{1-\mu}([0,T];h^{4+\delta_2}_{bc}(I))$$

there holds the estimate

$$|w|_{\tilde{E}_1([0,T])} \leq C\Big(|\tilde{f}|_{\tilde{E}_0([0,T])} + |u_0''|_{h^{\delta_2+4\hat{\mu}}_{bc}(I)}\Big). \qquad (4.16)$$

Since

$$|w_{xxx}(t,\cdot)|_{C^{\delta_2}(I)} \le C_1 |w_{xx}(t,\cdot)|^{1/2}_{C^{2+\delta_2}(I)} |u_{xxxx}(t,\cdot)|^{1/2}_{C^{\delta_2}(I)},$$

we obtain, similarly to the argument in Step 2,

$$|\tilde{f}|_{\tilde{E}_0([0,T])} \le C_2 \Big(|w|^{1/2}_{\tilde{E}_1([0,T])} + |f_{xx}|_{\tilde{E}_0([0,T])} + 1 \Big), \tag{4.17}$$

where the constant C_2 depends on T_0, the parameters, and the a priori bounds for u from Step 1 and Step 2. Combining (4.16) and (4.17), and employing Young's inequality, we find a bound for u_{xx} in $\tilde{E}_1([0,T])$ in terms of T_0, the parameters, the data, and the a priori estimates for u from Step 1 and Step 2. In view of (3.4), (3.8), and $\delta_2 < 8\varepsilon$, the space Z^T embeds into $BUC([0,T]; C^{6+\delta_2}(I))$. Therefore we obtain a bound for u in the latter space in terms of the preceding set of quantities and the corresponding bound on the interval $[0, T_0/2]$. This completes Step 3.

Step 4: An estimate for u in $BUC([0,T]; C^{4(\theta+\hat{\mu}+\varepsilon)}(I))$ with θ, μ as in (3.5). Letting again $w = u_{xx}$, it follows from (4.15) and Step 3 that

$$\partial_t^{\beta} w_t + \sigma'(u_{xx}) w_{xxxx} + 4\sigma''(u_{xx}) u_{xxx} w_{xxx} = \tilde{g}, \tag{4.18}$$

where $\tilde{g}$ is a function which is a priori bounded in $C([0,T]; C^2(I))$, uniform with respect to $T \in [T_0/2, T_0)$, and satisfies $\tilde{g}(t,0) = \tilde{g}(t,L) = 0$, $t \in (0,T]$.

Let θ and μ as in (3.5) and define

$$\gamma = 4\theta + 4\varepsilon - 2. \tag{4.19}$$

Note that $\gamma \in (0,2)$, because $\beta \in (0, \frac{3}{5})$ and $\varepsilon \in (0, \frac{\beta}{4(1+\beta)})$.

We then consider (4.18) as an equation for w in the space $\tilde{E}_0([0,T]) = BUC_{1-\mu}([0,T]; h^{\gamma}_{bc}(I))$. The coefficients are a priori bounded in $C([0,T]; C^2(I))$, uniform with respect to $T \in [T_0/2, T_0)$. The linear term $4\sigma''(u_{xx}) u_{xxx} w_{xxx}$ is of lower order, and hence by a perturbation argument, one has maximal regularity in $\tilde{E}_0([0,T])$, that is, with

$$\tilde{E}_1([0,T]) = BUC^{1+\beta}_{1-\mu}([0,T]; h^{\gamma}_{bc}(I)) \cap BUC_{1-\mu}([0,T]; h^{4+\gamma}_{bc}(I))$$

we get the estimate

$$|w|_{\tilde{E}_1([0,T])} \le C \Big(|\tilde{g}|_{\tilde{E}_0([0,T])} + |u_0''|_{h^{\gamma+4\hat{\mu}}_{bc}(I)} \Big),$$

where C depends only on T_0, the parameters, the data, and the bound for u from Step 3. By the embedding

$$\tilde{E}_1([0,T]) \hookrightarrow BUC([0,T]; h^{\gamma+4\hat{\mu}}_{bc}(I)),$$

we thus obtain a uniform bound for u in $BUC([0,T]; h^{2+\gamma+4\hat{\mu}}_{bc}(I))$. In view of (4.19) this means we have a bound for u in $BUC([0,T]; h^{4(\theta+\hat{\mu}+\varepsilon)}_{bc}(I))$, uniform with respect to $T \in [T_0/2, T_0)$. This contradicts the hypothesis that $T_0 < \infty$. Hence we have global existence. $\square$

5. Proof of Lemma 3.1

Proof. Define the operator B by means of

$$D(B) = \{v \in C^2([0,L];\mathbb{C}) : \ v^{(i)}(0) = v^{(i)}(L) = 0, \ i = 0,2\},$$
$$(Bu)(x) = -u''(x), \quad u \in D(B).$$

Then $B : D(B) \subset F_0 \to F_0$ is invertible, i.e., $0 \in \rho(B)$, and it is sectorial with spectral angle $\phi_B = 0$. The same then holds for $B^2 : D(B^2) \subset F_0 \to F_0$. We have $\mathbb{C} \setminus (0,\infty) \subset \rho(B^2)$ and for any $\vartheta \in [0,\pi)$ there exists $C_0(\vartheta) > 0$ such that

$$|(\lambda + B^2)^{-1}|_{\mathcal{B}(F_0)} \le \frac{C_0(\vartheta)}{1+|\lambda|}, \quad \lambda \in \mathbb{C}\setminus\{0\}, \ |\arg\lambda| \le \vartheta.$$

Moreover, $D(B^2) = F_1 = \{v \in C^4([0,L];\mathbb{C}) : \ v^{(i)}(0) = v^{(i)}(L) = 0, \ i = 0,2,4\}$, and by using standard interpolation inequalities we see that the graph norm of B^2 on $D(B^2)$ is equivalent to the usual norm of $C^4([0,L])$. Again from standard interpolation inequalities and from the identity

$$B^2(\lambda + B^2)^{-1} = I - \lambda(\lambda + B^2)^{-1}$$

we obtain for any $\vartheta \in [0,\pi)$ and $0 \le k \le 4$,

$$|D^k(\lambda + B^2)^{-1}|_{\mathcal{B}(F_0)} \le C_k(\vartheta)\Big(\frac{1}{1+|\lambda|}\Big)^{1-\frac{k}{4}}, \ \lambda \in \mathbb{C}\setminus\{0\}, \ |\arg\lambda| \le \vartheta, \tag{5.1}$$

where $D = \frac{d}{dx}$, and $C_k(\vartheta) > 0$ are constants depending only on k and ϑ.

Let now $m \in C([0,L])$ be strictly positive and set

$$m_1 := \min_{x\in[0,L]} m(x), \qquad m_2 := \max_{x\in[0,L]} m(x).$$

Define the operator $M : F_0 \to F_0$ by means of

$$(Mu)(x) = m(x)u(x), \quad x \in [0,L], \ u \in F_0.$$

Then $M \in \mathrm{Isom}(F_0)$, $|M|_{\mathcal{B}(F_0)} \le m_2$, and $|M^{-1}|_{\mathcal{B}(F_0)} \le 1/m_1$. Furthermore we have $\tilde{A} = MB^2$ with $D(\tilde{A}) = D(B^2)$, and so clearly $0 \in \rho(\tilde{A})$ and $\tilde{A}^{-1} = (B^2)^{-1}M^{-1}$.

In order to show that $\mathbb{C}\setminus(0,\infty) \subset \rho(\tilde{A})$, it is sufficient to prove that $\lambda + \tilde{A} : D(\tilde{A}) \to F_0$ is bijective for any $\lambda \in \mathbb{C}\setminus(-\infty,0)$. To this end, for such a λ, we define the operator $K : D(\tilde{A}) \to F_0$ ($D(\tilde{A})$ equipped with the graph norm) as follows:

$$(Ku)(x) = \lambda u(x), \quad u \in D(\tilde{A}).$$

Then K is compact, and since $\tilde{A} \in \mathrm{Isom}(D(\tilde{A}), F_0)$, it follows then from the stability of the index under compact perturbations that the index of $K + \tilde{A}$ is zero.

The null space of $K+\tilde{A}$ is $\{0\}$. Indeed, let $u \in D(\tilde{A})$ be such that $Ku+\tilde{A}u = 0$. We divide this equation by m, multiply by $\bar{u}$, and integrate over $[0,L]$; this yields

$$\lambda\int_0^L \frac{1}{m(x)}\,|u(x)|^2\,dx + \int_0^L |u''(x)|^2\,dx = 0.$$

In view of the positivity of m and due to $\lambda \in \mathbb{C} \setminus (-\infty, 0)$, it follows that $u = 0$. We conclude that $K + \tilde{A} \in \mathrm{Isom}(D(\tilde{A}), F_0)$. Hence $\lambda + \tilde{A} \in \mathrm{Isom}(D(\tilde{A}), F_0)$ for all $\lambda \in \mathbb{C} \setminus (-\infty, 0)$.

As to (3.10), observe that by continuity of the resolvent and as a consequence of what we have just proved, (3.10) holds provided $|\lambda| \leq \rho$ for $\rho > 0$, with $M_1(\vartheta)$ replaced with some $M_1(\vartheta, \rho)$. Therefore it remains to show the following:

$$\left\{ \begin{array}{l} \quad \forall \vartheta \in [0, \pi) \ \exists \rho > 0 \ \exists M(\vartheta, \rho) > 0 \text{ such that} \\ |(\lambda + \tilde{A})^{-1}|_{\mathcal{B}(F_0)} \leq \frac{M(\vartheta,\rho)}{1+|\lambda|}, \ |\lambda| \geq \rho, \ |\arg \lambda| \leq \vartheta. \end{array} \right. \tag{5.2}$$

Employing the resolvent estimates (5.1) for the operator B^2 and using the continuity of m, (5.2) can be proved by the method of localization and perturbation arguments, see, e.g., [1, pp. 479–480] or [6]. Due to limitations of space, we do not carry out the details. $\square$

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Positivity Property of Solutions of Some Quasilinear Elliptic Inequalities

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Abstract. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function.We prove that under some additional assumptions on f and $A : \mathbb{R} \to \mathbb{R}_+$, weak $\mathscr{C}^1$ solutions of the differential inequality $-\mathrm{div}(A(|\nabla u|)\nabla u) \geq f(u)$ on $\mathbb{R}^N$ are nonnegative. Some extensions of the result in the framework of subelliptic operators on Carnot Groups are considered.

1. Introduction

In this paper we shall study the following problem.

Let L be a second-order differential operator and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Find additional assumptions on (L, f) that imply the positivity of the possible solutions of the differential inequality

$$L(u) \geq f(u) \quad \text{on} \quad \mathbb{R}^N. \tag{1.1}$$

Some partial answers to this problem have been obtained in [4, 5]. In those papers, the authors deal with elliptic inequalities of the form (1.1) in the case when L is the Laplacian operator or the polyharmonic operator $(-\Delta)^k$ in the Euclidean setting or, more generally L is a sub elliptic Laplacian on a Carnot group and f is non negative. The main strategy used in [4, 5] for proving positivity results was via integral representation formulae. One essential difficulty using this approach is that no assumptions on the behavior of the solutions at infinity are known. A typical example in this direction is given by,

$$-\Delta u \geq |u|^q \quad \text{on} \quad \mathbb{R}^N, \tag{1.2}$$

where $N \geq 3$ and $q > 1$.

The following result holds (see [4]).

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Theorem 1.1. *Let $N \geq 3$ and $q > 1$. Let $u \in L^q_{\mathrm{loc}}(\mathbb{R}^N)$ be a distributional solution of* (1.2) *and let $Leb(u)$ be the set of its Lebesgue points. If $x \in Leb(u)$, then*

$$u(x) \geq C_N \int_{\mathbb{R}^N} \frac{|u(y)|^q}{|x-y|^{N-2}}\, dy,$$

where C_N is an explicit positive constant.

From this result it follows that, if u is a solution of (1.2) then, either $u(x) = 0$ a.e. on $\mathbb{R}^N$, or $u(x) > 0$ a.e. on $\mathbb{R}^N$.

Obviously, the approach via representation formulae cannot be applied to quasilinear problems. In this paper we shall consider a class of quasilinear model problems for which the positivity property mentioned at the beginning of this introduction holds.

More precisely, we shall deal with the case when L is the p-Laplacian operator, namely $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, or the p sub-Laplacian operator on the Heisenberg group, $\Delta_{H,p} u = \mathrm{div}_H(|\nabla_H u|^{p-2}\nabla_H u)$ where div_H and ∇_H are, respectively, the horizontal divergence and horizontal gradient on Heisenberg group and f is a continuous function. In this cases some results on positivity of solutions of (1.1) are proved by using a suitable comparison Lemma (see Lemma 2.5 below). This paper is organized as follows. In Section 2 we state and prove our main result and present some special cases, while in Section 3 we briefly indicate some possible generalizations.

2. Main result

In this section we shall study the main problem in two cases. Namely in the Euclidean and in the Heisenberg group setting. In what follows, ∇_L stands, either for the usual gradient on $\mathbb{R}^N$ or, for the horizontal vector field

$$\nabla_H = (X_1, \ldots, X_n, Y_1 \ldots Y_n)$$

in $\mathbb{H}^n \equiv \mathbb{R}^{2n+1} \equiv \mathbb{R}^N$. That is, for $x = (\xi_1, \ldots, \xi_n, \eta_1 \ldots \eta_n, \tau) \in \mathbb{R}^{2n+1}$,

$$X_1, \ldots, X_n, Y_1 \ldots Y_n$$

are defined as follows,

$$X_i := \frac{\partial}{\partial \xi_1} + 2\eta_1 \frac{\partial}{\partial \tau}, \quad Y_i := \frac{\partial}{\partial \eta_1} - 2\xi_1 \frac{\partial}{\partial \tau}, \qquad i = 1, \ldots, n.$$

We shall denote by div_L the formal adjoint of ∇_L, that is the usual divergence operator in the Euclidean setting or, in the Heisenberg case, $\mathrm{div}_L(h) = \sum_{i=1}^n X_i h_i + Y_i h_{n+i}$ for any smooth vector field $h = (h_1, \ldots, h_{2n}) : \Omega \subset \mathbb{R}^N \to \mathbb{R}^{2n}$.

Definition 2.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set, let $A : \mathbb{R} \to \mathbb{R}$ and $h : \Omega \to \mathbb{R}$ be a continuous functions. We say that* u is a solution *of*

$$\mathrm{div}_L(A(|\nabla_L u|)\nabla_L u) \geq h \quad \text{on} \quad \Omega,$$

if $u \in \mathscr{C}^1(\Omega)$ *and for any nonnegative* $\phi \in \mathscr{C}^1_0(\Omega)$*, we have*

$$-\int_\Omega A(|\nabla_L u|)\nabla_L u \cdot \nabla_L \phi \geq \int_\Omega h\phi.$$

In a similar manner we can define solutions of the inequalities

$$-\mathrm{div}_L(A(|\nabla_L u|)\nabla u) \geq h \quad \text{and} \quad \mathrm{div}_L(A(|\nabla_L u|)\nabla u) \leq h.$$

Our main result is the following.

Theorem 2.2. *Let* $p > 1$*. Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be a continuous function such that*

$$f(t) > 0 \quad \text{if} \quad t < 0, \quad f \quad \text{is non increasing on} \quad]-\infty, 0[\tag{2.3}$$

and

$$\int_{-\infty}^{-1} \left(\int_t^{-1} f(s)\, ds \right)^{-\frac{1}{p}} dt < +\infty. \tag{2.4}$$

If u *is a solution of*

$$-\mathrm{div}_L(|\nabla_L u|^{p-2}\, \nabla_L u) \geq f(u) \quad \text{on} \quad \mathbb{R}^N, \tag{2.5}$$

then $u \geq 0$*. Moreover if* $f(t) \geq 0$ *for* $t \geq 0$ *then, either* $u \equiv 0$ *or* $u > 0$*.*

Corollary 2.3. *Let* $p > 1$ *and* $q > 1$*. Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be a continuous function such that* $f(t) \geq C\,|t|^q$ *for* $t < 0$*. Let* u *be a solution of*

$$-\mathrm{div}_L(|\nabla_L u|^{p-2}\, \nabla_L u) \geq f(u) \quad \text{on} \quad \mathbb{R}^N. \tag{2.6}$$

If $q > p-1$ *then* $u \geq 0$*. Moreover if* $f(t) \geq 0$ *for* $t \geq 0$ *then, either* $u \equiv 0$ *or* $u > 0$*.*

Remark 2.4. The above assumptions on f are sharp in the following sense. If $p = 2$ and $q = 1 = p - 1$ the result is false. Indeed the equation

$$-\Delta u = |u| \quad \text{on} \quad \mathbb{R}^N,$$

admits the explicit negative solution

$$u(x) := -Exp(x_1), \quad x \in \mathbb{R}^N,$$

or solutions that changes sign (see [4]).

In the general case $q = p - 1$ the equation

$$\Delta_p u = u^{p-1} \quad \text{on} \quad \mathbb{R}^N$$

admits a positive solution (see for instance [6]). Therefore the equation

$$-\Delta_p u = |u|^{p-1} \quad \text{on} \quad \mathbb{R}^N,$$

has a negative solution.

Let us briefly describe the idea of the proof of our main result. Let u be a solution of (2.5). Without loss of generality we will show that $u(0) \geq 0$. The function $U := -u$ satisfies the inequality

$$\mathrm{div}_L(|\nabla_L U|^{p-2}\, \nabla_L U) \geq f(-U) \quad \text{on} \quad \mathbb{R}^N.$$

Let v be a positive solution of

$$\mathrm{div}_L(|\nabla_L v|^{p-2}\nabla_L v) = f(-v) \quad \text{on} \quad B_R,$$

such that $v(0) = a > 0$ and $v(x) \to +\infty$ as $|x| \to R$. The assumptions on f imply the existence of v. Since $U(x) \leq v(x)$ for $|x|$ close to R, by a comparison Lemma (see Lemma below) it follows that $U(x) \leq v(x)$ for any $|x| < R$. In particular $U(0) \leq v(0) = a$. Letting $a \to 0$ we have $U(0) \leq 0$. Hence $u(0) \geq 0$. Finally, if $f(t) \geq 0$ for $t \geq 0$, by the weak Harnack inequality we get that, either $u \equiv 0$ or $u > 0$.

2.1. A comparison Lemma

In this section, we shall prove a comparison Lemma that it is useful when considering solutions of inequalities of the form,

$$\mathrm{div}_L(A_1(|\nabla_L u|)\nabla u) \geq g_1(x, u) \quad \text{on} \quad \Omega, \tag{2.7}$$

and

$$\mathrm{div}_L(A_2(|\nabla_L v|)\nabla v) \leq g_2(x, v) \quad \text{on} \quad \Omega. \tag{2.8}$$

Here, for $i = 1, 2$, A_i is a continuous function such that $A_i(t) > 0$ for $t > 0$ and $g_i : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous.

Lemma 2.5. *Let Ω be a bounded open set and let u and v be solutions of* (2.7) *and* (2.8) *respectively. Assume that*

1. (a) *for any $x \in \Omega$, $t \geq s \geq 0$ there holds $g_1(x, t) \geq g_2(x, s)$, $g_1(x, \cdot)$ is not decreasing on $]0, +\infty[$ and $v \geq 0$;*
   or
   (b) *for any $x \in \Omega$, $t \geq s$ there holds $g_1(x, t) \geq g_2(x, s)$ and $g_1(x, \cdot)$ is not decreasing;*
2. (a) *$A_1(t) \geq A_2(t)$ for $t > 0$ and the function $tA_2(t)$ is increasing for $t > 0$;*
   or
   (b) *$A_2(t) \geq A_1(t)$ for $t > 0$ and the function $tA_1(t)$ is increasing for $t > 0$;*
3. *$u \leq v$ on $\partial\Omega$.*

Then $u \leq v$ on Ω.

Proof. Let u and v be solutions of (2.7) and (2.8) respectively. Let $\epsilon > 0$ be fixed and set $v_\epsilon := v + \epsilon$. It is a simple matter to check that the function v_ϵ satisfies the inequality

$$\mathrm{div}_L(A_2(|\nabla_L w|)\nabla v) \leq g_1(x, w) \quad \text{on} \quad \Omega.$$

Therefore, for any nonnegative $\phi \in \mathscr{C}_0^1(\Omega)$ we have

$$-\int_\Omega (A_1(|\nabla_L u|)\nabla u - A_2(|\nabla_L v_\epsilon|)\nabla v_\epsilon) \cdot \nabla_L \phi \geq \int_\Omega (g_1(x, u) - g_1(x, v_\epsilon))\phi. \tag{2.9}$$

Next we choose ϕ as follows: $\phi := ((u - v_\epsilon)_+)^2$. It is clear that ϕ is nonnegative and $\phi \in \mathscr{C}^1(\Omega)$. Moreover, since $v_\epsilon - u \geq \epsilon > 0$ on $\partial\Omega$, it follows that ϕ has compact support. Substituting ϕ in (2.9), we obtain

$$-\int_\Omega (A_1(|\nabla_L u|)\nabla u - A_2(|\nabla_L v|)\nabla v)\cdot(\nabla_L u - \nabla_L v)2(u-v_\epsilon)_+ \geq \int_\Omega (g_1(x,u)-g_1(x,v_\epsilon))\phi.$$

We claim that

$$(A_1(|\nabla_L u|)\nabla u - A_2(|\nabla_L v|)\nabla v)\cdot(\nabla_L u - \nabla_L v) \geq 0. \tag{2.10}$$

Indeed,

$$\begin{aligned}
&(A_1(|\nabla_L u|)\nabla u - A_2(|\nabla_L v|)\nabla v)\cdot(\nabla_L u - \nabla_L v)\\
&\quad = A_1(|\nabla_L u|)\,|\nabla u|^2 + A_2(|\nabla_L v|)\,|\nabla v|^2 - (A_1(|\nabla_L u|) + A_2(|\nabla_L v|))(\nabla_L u\cdot\nabla_L v)\\
&\quad = \Big(A_1(|\nabla_L u|)\,|\nabla u| - A_2(|\nabla_L v|)\,|\nabla v|\Big)\Big(\,|\nabla_L u| - |\nabla_L v|\,\Big) +\\
&\qquad +\Big(A_1(|\nabla_L u|) + A_2(|\nabla_L v|)\Big)\Big(\,|\nabla_L u|\,|\nabla_L v| - \nabla_L u\cdot\nabla_L v\Big) =: I_1 + I_2. \quad (2.11)
\end{aligned}$$

Since $A_i \geq 0$, we have $I_2 \geq 0$. Assume first that the case 2.(a) holds and $g_1(x,.)$ is strictly increasing. The other cases are similar. From the inequality $A_1 \geq A_2$ and from the monotonicity of $tA_2(t)$, it follows that

$$\begin{aligned}
I_1 &= \Big(A_1(|\nabla_L u|)\,|\nabla u| - A_2(|\nabla_L v|)\,|\nabla v|\Big)\Big(\,|\nabla_L u| - |\nabla_L v|\,\Big)\\
&\geq \Big(A_2(|\nabla_L u|)\,|\nabla u| - A_2(|\nabla_L v|)\,|\nabla v|\Big)\Big(\,|\nabla_L u| - |\nabla_L v|\,\Big) \geq 0.
\end{aligned}$$

Therefore, since $g_1(x,\cdot)$ is increasing, the inequality $(g_1(x,u) - g_1(x,v_\epsilon)((u-v_\epsilon)_+)^2 \geq 0$ holds for every $x \in \Omega$. As a consequence, $(g_1(x,u) - g_1(x,v_\epsilon)((u-v_\epsilon)_+)^2 = 0$ on Ω.

This completes the proof in case (a) holds and $g_1(x,.)$ is strictly increasing. For the general case we need an extra argument. Indeed, from (2.10) and (2.11) we have that $\int_\Omega (I_1+I_2)(u-v_\epsilon)_+ = 0$. Let $x \in \Omega$ be such that $u(x) \geq v_\epsilon(x)$. Since $I_1 \geq 0$ and $I_2 \geq 0$ we have $I_1(x) = 0 = I_2(x)$.

We claim that $\nabla_L u(x) = \nabla_L v(x)$. Indeed, if $\nabla_L u(x) \neq \nabla_L v(x)$, from $I_2(x) = 0$, we deduce that $|\nabla_L u(x)| \neq |\nabla_L v(x)|$[1]. Thus from $I_1(x) = 0$, the injectivity of $tA_2(t)$, it follows that

$$0 = A_1(|\nabla_L u|)\,|\nabla u| - A_2(|\nabla_L v|)\,|\nabla v| \geq A_2(|\nabla_L u|)\,|\nabla u| - A_2(|\nabla_L v|)\,|\nabla v| \neq 0.$$

This implies $\nabla_L((u-v_\epsilon)_+)^2 = 0$ on Ω, that is $((u-v_\epsilon)_+)^2 = \phi = 0$. Therefore, letting $\epsilon \to 0$ in $u \leq v + \epsilon$ the claim follows. □

Remark 2.6. The above lemma enables to compare solutions of differential inequalities involving different operators. As an example, consider the following situation.

[1]if t, s are two different vectors in a Hilbert space such that $(s\cdot t) = |t|\,|s|$, then $0 < |t-s| = |t|^2 + |s|^2 - 2(s\cdot t) = |t|^2 + |s|^2 - 2\,|s|\,|t| = (|s|-|t|)^2$.

Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous and non decreasing function on $]0, +\infty[$. Let $\gamma : \Omega \to \mathbb{R}$ be a bounded continuous function such that $\|\gamma\|_\infty \leq 1$. Let u be a solution of

$$\operatorname{div}_L\left(\frac{\nabla_L u}{\sqrt{1 + |\nabla_L u|^2}}\right) \geq g(u) \quad \text{on} \quad \Omega,$$

and let v be a positive solution of

$$\operatorname{div}_L(\nabla_L v) \leq \gamma g(v) \quad \text{on} \quad \Omega$$

with $u \leq v$ on $\partial\Omega$. Then $u \leq v$ on Ω.

The proof of Theorem 2.2 relies on the following.

Theorem 2.7. *Let $p > 1$ and let $g\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that, $g(t) > 0$ if $t > 0$, g is non decreasing on $]0, +\infty[$, and*

$$\int_1^{+\infty} \left(\int_1^t g(s)\; ds\right)^{-\frac{1}{p}} dt < +\infty. \tag{2.12}$$

For any $a > 0$, $D > 1$, there exists a function φ and $R > 0$ such that, φ is a solution of

$$\left(r^{D-1} |\varphi'(r)|^{p-2} \varphi'(r)\right)' = r^{D-1} g(\varphi(r)), \quad \varphi(0) = a, \quad \varphi'(0) = 0, \tag{2.13}$$

ϕ is increasing on $]0, R[$ and $\varphi(r) \to +\infty$ as $r \to R$.

See [7] for a proof in the case $p = 2$ and [6] for the quasilinear case $p \neq 2$.

Proof of Theorem 2.2. Let u be a solution of (2.5). Since the inequality is invariant under translations, it is sufficient to prove that $u(0) \geq 0$.

Let $g(t) := f(-t)$. The function g satisfies the assumptions of Theorem 2.7. Let $D = Q > 1$ be the homogeneous dimension. We know that $D = Q = N$ if we are dealing with the Euclidean case and $D = Q = 2n + 2$ in the Heisenberg setting. Let $a > 0$ and let φ be a solution of (2.13) such that $\varphi(r) \to +\infty$ as $r \to R$. We set $v(x) := \varphi(N_2(x))$, where $N_2(x) := |x|$ in the Euclidean case and $N_2(x) := |x|_H := ((\sum_{i=1}^n \xi_i^2 + \eta_i^2)^2 + \tau^2)^{1/4}$ in the Heisenberg setting. By computation we have,

$$\begin{aligned}\operatorname{div}_L(|\nabla_L v|^{p-2} \nabla_L v) &= (p-1)\psi^p |v'|^{p-2} \left(v''(r) + \frac{Q-1}{r} v'(r)\right)_{r=N_2} \\ &= \psi^p N_2^{1-Q} \left(r^{Q-1} |v'(r)|^{p-2} v'(r)\right)'_{r=N_2}\end{aligned}$$

where $\psi = 1$ in the Euclidean case or $\psi = \left|\nabla_L |x|_H\right| \leq 1$. Therefore, the function v satisfies the differential equation

$$\operatorname{div}_L(|\nabla_L v|^{p-2} \nabla_L v) = g_2(x, v) := \psi^p g(v) \quad \text{on} \quad \Omega_R,$$

where $\Omega_R := \{x \mid N_2(x) < R\}$. On the other hand the function $U := -u$ satisfies the inequality

$$\mathrm{div}_L(|\nabla_L U|^{p-2}\,\nabla_L U) \geq g_1(U) := g(U) \quad \text{on} \quad \mathbb{R}^N.$$

Since $g_1 \geq g_2$ and $U(x) \leq v(x)$ for $|x|$ close to R we are in the position to apply the comparison Lemma 2.5. As a consequence, $U(x) \leq v(x)$ for any $x \in \Omega_R$. In particular $U(0) \leq v(0) = a$. Letting $a \to 0$ it follows that $U(0) \leq 0$. Hence $u(0) \geq 0$.

Next, if $f \geq 0$, then u is a non negative super solution of the equation, $-\mathrm{div}_L(|\nabla_L u|^{p-2}\,\nabla_L u) = 0$ on $\mathbb{R}^N$. Hence, by the weak Harnack inequality (see [1]) it follows that, either $u \equiv 0$ or $u > 0$. □

3. Some extensions of the main result

3.1. Carnot Groups

Let $\mathbb{R}^N \equiv \mathbb{G}$ be a Carnot group and let ∇_L be the horizontal gradient on G. Let Γ_p be the fundamental solution of the quasilinear operator

$$-\Delta_{L,p} u = -\mathrm{div}_L(|\nabla_L u|^{p-2}\,\nabla_L u)$$

at the origin. Set

$$N_p := \begin{cases} \Gamma_p^{\frac{p-1}{p-Q}} & p > 1, p \neq Q \\ \exp(-\Gamma_p) & p = Q \end{cases}$$

It is known that N_p is a homogeneous norm on $\mathbb{G}$. To the authors knowledge, the best result on the regularity of N_p is that it is Hölder continuous, see [1, 2]. In what follows we shall assume that N_p is smooth. This assumption is satisfied for example for "Heisenberg type" groups. See for instance [2].

With the above notation, we have that if $\zeta\colon \mathbb{R} \to \mathbb{R}$ is a smooth function, then the *radial* function $v := \zeta \circ N_p\colon \mathbb{G} \to \mathbb{R}$ satisfies

$$\Delta_{L,p} v = \mathrm{div}_L(|\nabla_L v|^{p-2}\,\nabla_L v) = (p-1)\psi^p\,|\zeta'|^{p-2}\left(\zeta''(r) + \frac{Q-1}{r}\zeta'(r)\right)_{r=N_P},$$

where $\psi := |\nabla_L N_p|$ (see [3]). Hence we can apply the same arguments used in the preceding section obtaining an analog of Theorem 2.2 in this more general setting.

Theorem 3.1. *Let $p > 1$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying* (2.3) *and* (2.4)*. Let u be a solution of*

$$-\mathrm{div}_L(|\nabla_L u|^{p-2}\,\nabla_L u) \geq f(u) \quad \text{on} \quad \mathbb{R}^N, \tag{3.14}$$

then $u \geq 0$. Moreover if $f(t) \geq 0$ for $t \geq 0$ then, either $u \equiv 0$ or $u > 0$.

Other extensions in this setting are also possible for quasilinear inequalities of the type

$$-\mathrm{div}_L(A(|\nabla_L u|)\nabla_L u) \geq f(u). \tag{3.15}$$

Theorem 3.2. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying* (2.3) *and*

$$\int_{-\infty}^{-1} \left(\int_{t}^{-1} f(s)\ ds \right)^{-\frac{1}{2}} dt < +\infty. \tag{3.16}$$

Let A be a positive continuous function such that, either

1. *$A(t) \geq c$ for a positive constant c*
   or
2. *$A(t) \leq c$ for a positive constant c and $tA(t)$ is increasing for $t > 0$.*

Let u be a solution of (3.15), *then $u \geq 0$.*

We leave the proof of the above result to the interested reader. Let us just observe that the proof is based on the possibility to compare the solutions of

$$\mathrm{div}_L(A(|\nabla_L u|)\nabla_L u) \geq g(u),$$

and

$$\Delta_{2.L} u = \mathrm{div}_L(\nabla_L u) \geq g(u),$$

and applying Lemma 2.5.

Corollary 3.3. *Let u be a solution of*

$$-\mathrm{div}_L\Big(\frac{\nabla_L u}{\sqrt{1+|\nabla_L u|^2}}\Big) \geq f(u) \quad \text{on} \quad \mathbb{R}^N,$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying (2.3) *and* (3.16). *Then $u \geq 0$.*

3.2. A differential inequality related to the mean curvature operator

In the Euclidean case, $\nabla_L = \nabla$, the above corollary can be improved. Indeed, the claim follows without the assumption (3.16) on f.

Theorem 3.4. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying* (2.3). *Let u be a solution of*

$$-\mathrm{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) \geq f(u) \quad \text{on} \quad \mathbb{R}^N.$$

Then $u \geq 0$.

The argument for proving the above result is exactly the same as in the proof of Theorem 2.2, so we shall be brief.

Proof. Let $g(t) := f(-t)$. Under the assumptions of Theorem 3.4, there exists a radial solution v of $\mathrm{div}(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}) \geq g(v)$ such that $v(0) = a > 0$ and $v(r) \to +\infty$ as $r \to R$, see [6]. By the comparison Lemma 2.5 we get $u(0) \geq -a$. Letting $a \to 0$, the claim follows. $\square$

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On a Stochastic Parabolic Integral Equation

Wolfgang Desch and Stig-Olof Londen

To the memory of Günter Lumer

Abstract. In this article we analyze the stochastic parabolic integral equation

$$u(t,x,\omega) = c_\alpha t^{-1+\alpha} * \Delta u + \sum_{k=1}^{\infty} \int_0^t g^k(s,x,\omega)\, dw_s^k,$$

where $t \geq 0$, $x \in \mathbb{R}^d$, $\alpha \in (\frac{1}{2}, 1)$ and $\omega \in \Omega$. We take $\{w_t^k \mid k = 1, 2, \dots\}$ to be a family of independent $\mathcal{F}_t$-adapted Wiener processes defined on a probability space $(\Omega, \mathcal{F}, P)$. Here $\mathcal{F}_t \subset \mathcal{F}$ and $\mathcal{F}_t$ is an increasing filtration.

By applying and modifying the method of Krylov we obtain existence and regularity results in L_p-spaces, $p \geq 2$.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space, with $\{\mathcal{F}_t\}_{t\geq 0}$ an increasing filtration of σ-algebras satisfying $\mathcal{F}_t \subset \mathcal{F}$. Let $\mathcal{P}$ denote the predictable σ-algebra on $\mathbb{R}_+ \times \Omega$ generated by $\{\mathcal{F}_t\}_{t\geq 0}$, and assume $\{w_t^k \mid k = 1, 2, \dots\}$ is a family of independent one-dimensional $\mathcal{F}_t$-adapted Wiener processes defined on $(\Omega, \mathcal{F}, P)$.

In this setting, we consider the stochastic parabolic integral equation

$$u(t,x,\omega) = \int_0^t k(t-s)\Delta u(s,x,\omega)\, ds + \sum_{k=1}^{\infty} \int_0^t g^k(s,x,\omega)\, dw_s^k, \tag{1}$$

where the variables satisfy $t \geq 0$, $x \in \mathbb{R}^d$, $\omega \in \Omega$, and $k(t) = c_\alpha t^{-1+\alpha}$, with c_α, α given constants; $\alpha \in (\frac{1}{2}, 1)$; and g^k given functions. The infinite series of stochastic integrals on the right side of (1) converges in a weak sense made precise below. By modifying the analytic approach of Krylov [11], developed for stochastic parabolic partial differential equations, we obtain an existence and uniqueness result on (1). As in [11], the setting is L_p, with $p \geq 2$, thus a Hilbert space framework is not needed.

While in [11], starting from a linear heat equation with a stochastic nonhomogeneous term, finally very general problems with space-dependent coefficients and nonlinear (multiplicative) stochastic perturbations are treated, this paper is considered to be just a first step in this direction, solving the linear equation with Laplacian.

Before outlining the paper, we make some brief comments on the range of α-values.

With $\alpha = 1$, the equation (1) is a (much studied) parabolic stochastic partial differential equation. See, e.g., [11], for further references. Our proofs require $k \in L_2(0,1)$, thus $\alpha > \frac{1}{2}$. For small α one may however formally argue as follows.

The equation (1) can be inverted to give

$$D_t^\alpha u = \Delta u + F, \tag{2}$$

where $D_t^\alpha u \overset{\text{def}}{=} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt}(t^{-\alpha} * u)$, $t > 0$, is the fractional time derivative of order α of u (with $u(0) = 0$), and where $F = \frac{d}{dt}(t^{-\alpha} * G)$, with $G = \sum_k \int_0^t g^k \, dw_s^k$. Suppose that, in some sense, $G \in C^\delta$; then $F \in C^{\delta-\alpha}$. Assume that $\delta - \alpha > 0$. Equations of this type have been treated in Bessel potential spaces in [15], [16], and in Hölder spaces in [3] and [4].

The case $\alpha \in (1,2)$ will be included in future work.

Equations of type (1) have been considered in Hilbert spaces in [1] and [2] by applying methods of [5]. In particular, certain regularity results on the stochastic convolution associated with (1) were obtained in [1].

Stochastic integral equations of type (1) or (2) occur in models of anomalous diffusion.

In Section 2, we introduce the necessary machinery and show how the stochastic Banach spaces developed in [11] can be modified in order to apply to the equations we consider.

In Section 3 we state and prove an existence result on (1). The fact that $\alpha < 1$ allows us to obtain additional time-regularity on the solution as compared to the case $\alpha = 1$. This we do in Section 4.

We will develop the present approach further in forthcoming work.

2. The stochastic machinery

Below, everywhere, $p \geq 2$.

Let $n \in \mathbb{R}$, and let $H_p^n(\mathbb{R}^d)$ be the Bessel potential space of distributions u such that $(1-\Delta)^{\frac{n}{2}} u \in L_p(\mathbb{R}^d)$, with norm

$$\|u\|_{n,p} \overset{\text{def}}{=} \|(1-\Delta)^{\frac{n}{2}} u\|_p.$$

Denote by l_2 the set of real-valued sequences $g = \{g^k \mid k = 1, 2, \dots\}$ with norm $|g|_{l_2}^2 = \sum_k |g^k|^2$, and, for a function $g : \mathbb{R}^d \to l_2$, $\|g\|_p \overset{\text{def}}{=} \||g|_{l_2}\|_p$; $\|g\|_{n,p} \overset{\text{def}}{=} \||(1-\Delta)^{\frac{n}{2}} g|_{l_2}\|_p$.

For τ a bounded stopping time, write

$$(0,\tau]] \stackrel{\text{def}}{=} \{(\omega,t) \mid 0 < t \le \tau(\omega)\},$$
$$\mathcal{H}_p^n(\tau) \stackrel{\text{def}}{=} L_p((0,\tau]],\mathcal{P},H_p^n),$$
$$\mathcal{H}_p^n(\tau,l_2) \stackrel{\text{def}}{=} L_p((0,\tau]],\mathcal{P},H_p^n(\mathbb{R}^d;l_2)).$$

The stochastic solution spaces $\hat{\mathcal{H}}_p^n(\tau)$ of (1) are then defined as follows.

Definition 1. *Let $u \in \cap_{T>0}\mathcal{H}_p^n(\tau\wedge T)$. Then $u \in \hat{\mathcal{H}}_p^n(\tau)$ if $u_{xx} \in \mathcal{H}_p^{n-2}(\tau)$, and there exist $f \in \mathcal{H}_p^{n-2}(\tau)$, $g \in \mathcal{H}_p^{n-1}(\tau,l_2)$ such that for any $\phi \in C_0^\infty(\mathbb{R}^d)$, the equality*

$$(u(t,\cdot),\phi(\cdot)) = \int_0^t k(t-s)(f(s,\cdot),\phi(\cdot))\,ds + \sum_{k=1}^\infty \int_0^t (g^k(s,\cdot),\phi(\cdot))\,dw_s^k, \tag{3}$$

holds for all $t \le \tau$, a.s. The norm in the solution space is

$$\|u\|_{\hat{\mathcal{H}}_p^n(\tau)} \stackrel{def}{=} \|u_{xx}\|_{\mathcal{H}_p^{n-2}(\tau)} + \|f\|_{\mathcal{H}_p^{n-2}(\tau)} + \|g\|_{\mathcal{H}_p^{n-1}(\tau,l_2)}.$$

In (3), for $v \in H_p^n$, $\phi \in C_0^\infty$,

$$(v,\phi) \stackrel{\text{def}}{=} \big((1-\Delta)^{\frac{n}{2}}v,(1-\Delta)^{-\frac{n}{2}}\phi\big) = \int_{\mathbb{R}^d} \big((1-\Delta)^{\frac{n}{2}}v(x)\big)\big((1-\Delta)^{-\frac{n}{2}}\phi(x)\big)\,dx.$$

By the assumption on g, the series of stochastic integrals in (3) does converge (uniformly in t) in probability on $[0,\tau\wedge T]$, $T<\infty$.

Thus, if $u \in \hat{\mathcal{H}}_p^n(\tau)$, then u can be represented as the sum (in the weak sense (3)), of a Lebesgue convolution integral and a series of stochastic integrals. (For simplicity, we take $u(t=0)=0$.)

An obvious question is whether this representation is unique. For $\alpha=1$ the well-known answer is yes. Below, in Lemma 2, we show that uniqueness holds also for $\alpha \in (\frac{1}{2},1)$.

Lemma 2. *Take $T>0$, $\alpha \in (\frac{1}{2},1)$. Let f, $\{g^k\}$ satisfy*

$$f \in L_2((0,T)\times\Omega), \quad \{g^k\} \in L_2((0,T)\times\Omega,l_2),$$

and let both be adapted to $\{\mathcal{F}_t\}_{t\ge 0}$. Suppose that for $t \in [0,T]$,

$$\int_0^t (t-s)^{\alpha-1} f(s,\omega)\,ds = \sum_k \int_0^t g^k(s,\omega)\,dw_s^k,$$

a.s. Then $f = g^k = 0$ a.s.

Proof of Lemma 2. Both $\|f(t,\cdot)\|^2_{L_2(\Omega)}$ and $\|g(t,\cdot)\|^2_{L_2(\Omega;l_2)}$ are integrable over $(0,T)$. Let t_0 be a Lebesgue point of both functions. Consider the orthogonal projection P in $L_2(\Omega)$:

$$Pu = u - E(u \mid \mathcal{F}_{t_0}).$$

If $f_1(s,\cdot) \stackrel{\text{def}}{=} Pf(s,\cdot)$, then

$$P\Big(\int_0^t (t-s)^{\alpha-1} f(s)\,ds\Big) = \int_0^t (t-s)^{\alpha-1} f_1(s)\,ds = \int_{t_0}^t (t-s)^{\alpha-1} f_1(s)\,ds,$$

where we used the fact that since f is adapted to $\mathcal{F}_t$,

$$f(t) = E\big(f(t) \mid \mathcal{F}_{t_0}\big), \quad t \le t_0.$$

The series $\sum_k \int_0^t g^k(s)\,dw_s^k$ has the martingale property:

$$E\Big(\sum_k \int_0^t g(s)\,dw_s^k \mid \mathcal{F}_{t_0}\Big) = \sum_k \int_0^{t_0} g^k(s)\,dw_s^k, \quad t \ge t_0.$$

We conclude that

$$P\Big(\sum_k \int_0^t g^k(s)\,dw_s^k\Big) = \sum_k \int_{t_0}^t g^k(s)\,dw_s^k, \quad t \ge t_0,$$

and therefore, a.s.,

$$\int_{t_0}^t (t-s)^{\alpha-1} f_1(s)\,ds = \sum_k \int_{t_0}^t g^k(s)\,dw_s^k, \quad t \in [t_0, T]. \tag{4}$$

Use Hölder and the fact that P is an orthogonal projection in $L_2(\Omega)$, to estimate the L_2-norms:

$$\begin{aligned} &\| \int_{t_0}^t (t-s)^{\alpha-1} f_1(s)\,ds\|^2_{L_2(\Omega)} \le \Big(\int_{t_0}^t (t-s)^{2\alpha-2}\,ds\Big)\Big(\int_{t_0}^t \|f_1(s)\|^2_{L_2(\Omega)}\,ds\Big) \\ &\le \frac{1}{2\alpha-1}(t-t_0)^{2\alpha-1}\int_{t_0}^t \|f(s)\|^2_{L_2(\Omega)}\,ds \le M(t-t_0)^{2\alpha}, \end{aligned} \tag{5}$$

with $M = \frac{1}{2\alpha-1}\sup_{t>t_0}\frac{1}{t-t_0}\int_{t_0}^t \|f(s)\|^2_{L_2(\Omega)}\,ds$, which is finite since t_0 is a Lebesgue point of $\|f\|^2_{L_2(\Omega)}$. By Itô's identity,

$$\|\sum_k \int_{t_0}^t g^k(s)\,dw_s^k\|_{L_2(\Omega)} = \int_{t_0}^t \sum_k \|g^k(s)\|^2_{L_2(\Omega)}\,ds. \tag{6}$$

Combine (4), (5) and (6), and use the fact that t_0 is a Lebesgue point of $\sum_k \|g^k(s)\|^2_{L_2(\Omega)}$, to get

$$\begin{aligned} &\|g(t_0)\|^2_{L_2(\Omega;l_2)} = \lim_{t\to t_0}(t-t_0)^{-1}\int_{t_0}^t \|g(s)\|^2_{L_2(\Omega;l_2)}\,ds \\ &\le \lim_{t\to t_0}(t-t_0)^{-1}M(t-t_0)^{2\alpha} = 0, \end{aligned}$$

where $2\alpha > 1$ was used. Lemma 2 follows.

To show that $\hat{\mathcal{H}}_p^n(\tau)$ is a Banach space, proceed as in [11], Theorem 3.7, and use $k \in L_2(0,1)$. We also recall the density result proved in [11], Theorem

3.10: If $g \in \mathcal{H}_p^n(\tau, l_2)$, then there exist $g_j \in \mathcal{H}_p^n(\tau, l_2)$; $j = 1, 2, \dots$, such that $\|g - g_j\|_{\mathcal{H}_p^n(\tau, l_2)} \to 0$, as $j \to \infty$, and such that

$$g_j^k(t, x) = \sum_{i=1}^{j} I_{(\tau_{i-1}^j, \tau_i^j]}(t) g_j^{ik}(x), \quad k \leq j, \tag{7}$$

and $g_j^k = 0$, for $k > j$. Here $g_j^{ik} \in C_0^\infty(\mathbb{R}^d)$ and $\tau_0^j \leq \tau_1^j \leq \dots \leq \tau_j^j$ are bounded stopping times.

3. Existence of solutions

Our goal is now to prove the existence result Theorem 4, formulated at the end of this Section. In this proof, c will always denote a generic positive constant which may vary from line to line.

Take $n = 1$ in the definition of $\hat{\mathcal{H}}_p^n(\tau)$. Thus $g \in L_p = \mathcal{H}_p^0(\tau, l_2)$. Consider (1) with finitely many stochastic terms, each g^k being of the simple structure (7):

$$u(t, x, \omega) = \int_0^t k(t - s) \Delta u(s, x, \omega)\, ds + \sum_{k=1}^{m} \int_0^t g^k(s, x, \omega)\, dw_s^k. \tag{8}$$

Define

$$u(t, x, \omega) \stackrel{\text{def}}{=} \sum_{k=1}^{m} \int_0^t S(t - s) g^k(s, x, \omega)\, dw_s^k. \tag{9}$$

The resolvent $S(t) \subset B(X)$ (take, e.g., $X = L_p(\mathbb{R}^d)$) satisfies

$$S(t)y = y + \int_0^t k(t - s) \Delta S(s) y\, ds, \quad y \in D(\Delta), \quad t \geq 0. \tag{10}$$

In fact, see [13], one has a kernel representation for S, such that $S(t-s)g^k(x)$ is bounded in $x \in \mathbb{R}^d$, $t \in [0, T]$. Hence u is well defined. By the stochastic Fubini theorem, see, e.g., p. 159 of [12], and by (10), it follows that u as defined in (9) satisfies (8) a.s., $t \geq 0$.

Our next purpose is to obtain a priori bounds on u. In the case $\alpha = 1$, these are implied by the key result of [10]. This result is not immediately applicable in the case $\alpha < 1$, and so, to prove the needed estimates, we proceed differently.

Lemma 3. *Let $\alpha \in (\frac{1}{2}, 1)$, $g \in L_p([0, T] \times \mathbb{R}^d; l_2)$. Then*

$$\int_{\mathbb{R}^d} \int_0^T \Big(\int_0^t |\nabla S(t - s) g(s, x)|_{l_2}^2\, ds \Big)^{\frac{p}{2}} dt\, dx \leq c \int_{\mathbb{R}^d} \int_0^T |g(t, x)|_{l_2}^p\, dt\, dx, \tag{11}$$

where $c = c(d, p, \alpha, T)$.

Proof of Lemma 3. Take the subadditive map

$$g \mapsto \Big(\int_0^t |\nabla S(t - s) g(s, x)|_{l_2}^2\, ds \Big)^{\frac{1}{2}}.$$

If this is shown to map

$$L_\infty((0,T)\times\mathbb{R}^d;l_2) \quad \to \quad L_\infty((0,T)\times\mathbb{R}^d;\mathbb{R}), \tag{12}$$

and

$$L_2((0,T)\times\mathbb{R}^d;l_2) \quad \to \quad L_2((0,T)\times\mathbb{R}^d;\mathbb{R}); \tag{13}$$

then, by the Marcinkiewicz interpolation theorem, (11) follows.

To prove (12), one argues as follows.

Suppose we can show that for any $h^k \in L^\infty(\mathbb{R}^d;l_2)$, and for $i=1,\dots,d$;

$$\sup_{x\in\mathbb{R}^d}\Big|\frac{\partial}{\partial x_i}S(t)h^k(x)\Big|_{l_2}^2 \le ct^{-\alpha}\sup_{x\in\mathbb{R}^d}|h^k(x)|_{l_2}^2, \tag{14}$$

with $c=c(\alpha,d)$. Replace t by $t-s$ in (14), and integrate in s over $[0,t]$. This gives

$$\sup_{x\in\mathbb{R}^d,0\le t\le T}\int_0^t|\nabla S(t-s)g(s,x)|_{l_2}^2\,ds \le c_1 \sup_{x\in\mathbb{R}^d,0\le t\le T}|g(t,x)|_{l_2}^2, \tag{15}$$

with $c_1 = cdT^{1-\alpha}/(1-\alpha)$, which is (12).

To prove (14), take Laplace transforms in t in the resolvent equation (10), solve for the transform of $S(t)h^k(x)$, and invert. This results in

$$S(t)h^k(x) = (2\pi i)^{-1}\int_{\Gamma_{1,\psi}} e^{\lambda t}[I-\lambda^{-\alpha}\Delta]^{-1}\lambda^{-1}h^k(x)\,d\lambda, \tag{16}$$

where

$$\Gamma_{1,\psi} = \{e^{it}\mid |t|\le\psi\}\cup\{\rho e^{i\psi}\mid 1<\rho<\infty\}\cup\{\rho e^{-i\psi}\mid 1<\rho<\infty\},$$

and $\psi\in(\frac{\pi}{2},\pi)$. In (16), use analyticity, change variables and apply $\frac{\partial}{\partial x_i}$. This gives

$$\frac{\partial}{\partial x_i}S(t)h^k(x) = (2\pi i)^{-1}t^{-\alpha}\int_{\Gamma_{1,\psi}} e^s s^{\alpha-1}\frac{\partial}{\partial x_i}(\mu-\Delta)^{-1}h^k(x)\,ds, \tag{17}$$

where $\mu=(\frac{s}{t})^\alpha$ is complex-valued. Consequently, $\frac{\partial}{\partial x_i}(\mu-\Delta)^{-1}h^k(x)$ needs to be evaluated. One obtains, after some calculations,

$$(\mu-\Delta)^{-1}h^k(x) = \phi_\mu * h^k \text{ with } \phi_\mu(x) = c(d)\frac{\mu^{\frac{\nu}{2}}}{r^\nu}K_\nu(\mu^{\frac12}r), \tag{18}$$

where $\nu=\frac{d}{2}-1$, $r^2=\sum_{i=1}^d x_i^2$, and where $K_\nu(z)$ is the modified Bessel function of second kind of order ν.

For infinite rays Γ_τ originating at the origin one has

$$|\tau|^\nu K_\nu(\tau)\in L_1(\Gamma_\tau); \quad |\tau|^{\nu+1}K_\nu'(\tau)\in L_1(\Gamma_\tau), \tag{19}$$

uniformly in $|\arg\Gamma_\tau|\le\theta<\frac{\pi}{2}$.

Now use (18) and (19) in (17), recall Hölder's inequality, estimate, and sum in k. The relation (14) follows – hence also (12).

To obtain (13) one argues in much the same way. Lemma 3 is proved.

To proceed, fix some $t > 0$ and observe that

$$\sum_{k=1}^{m} \int_0^r \nabla S(t-s) g^k(s,x,\omega)\, dw_s^k,$$

considered as a stochastic process with respect to the time variable $r \in [0,t]$, is a martingale. Burkholder-Davis-Gundy's inequality yields then

$$\begin{aligned} E \int_{\mathbb{R}^d} \Big|\sum_k \int_0^r \nabla S(t-s) g^k(s,x,\omega)\, dw_s^k\Big|^p\, dx & \\ \leq cE \int_{\mathbb{R}^d} \left(\int_0^r \sum_k |\nabla S(t-s) g^k(s,x,\omega)|^2\, ds \right)^{\frac{p}{2}} dx & \\ = cE \int_{\mathbb{R}^d} \left(\int_0^r |\nabla S(t-s) g(s,x,\omega)|_{l_2}^2\, ds \right)^{\frac{p}{2}} dx. & \end{aligned}$$

Now let $r = t$ and use (11):

$$\begin{aligned} E \int_{\mathbb{R}^d} \int_0^T |\nabla u(t,x)|^p\, dt\, dx &= E \int_{\mathbb{R}^d} \int_0^T \Big|\sum_k \int_0^t \nabla S(t-s) g^k(s,x,\omega)\, dw_s^k\Big|^p\, dt\, dx \\ &\leq cE \int_0^T \int_{\mathbb{R}^d} |g(s,x,\omega)|_{l_2}^p\, dx\, ds. \end{aligned}$$

The solution u can be estimated in an analogous fashion, using modified Bessel functions, to obtain

$$E \int_0^T \int_{\mathbb{R}^d} |u(t,x)|^p\, dx\, dt \leq c(p,\alpha,d,T) E \int_{\mathbb{R}^d} \int_0^T |g(s,x,\omega)|_{l_2}^p\, ds\, dx. \tag{20}$$

In addition, observe that $\|u_{xx}\|_{\mathcal{H}_p^{-1}}^p \leq c\|u_x\|_{L_p}^p$, and so the right side of (20) dominates $\|u_{xx}\|_{\mathcal{H}_p^{-1}}^p$.

Finally take an arbitrary $g \in \mathcal{H}_p^0(l_2)$, and approximate this g in the manner above by simpler functions g_j. Each g_j gives a solution u_j, and by the convergence of $\{g_j\}$ in $L_p(\Omega \times (0,T) \times \mathbb{R}^d, l_2)$, one has that $\{u_j\}$ is a Cauchy-sequence in $\hat{\mathcal{H}}_p^1$. By completeness, there exists some u to which $\{u_j\}$ converges. Some additional analysis yields that u solves (1) in the sense of (3). One has proved the existence part of the following Theorem 4.

In the proof of uniqueness, however, the stochastic forcing is cancelled immediately, and one is left with the well-known uniqueness of weak solutions for the deterministic integral equation

$$u(t,x) = c_\alpha \int_0^t (t-s)^{-1+\alpha} \Delta u(s,x)\, ds.$$

Theorem 4. *Let $\alpha \in (\frac{1}{2}, 1)$; $p \geq 2$. Assume that*

$$g \in L_p\big((0,T) \times \Omega, \mathcal{P}, L_p(\mathbb{R}^d; l_2)\big).$$

Then there exists a unique $u \in \hat{\mathcal{H}}_p^1$ *such that*

$$E\int_0^T \int_{\mathbb{R}^d} |u|^p \, dx \, dt + E\int_0^T \int_{\mathbb{R}^d} |\nabla u|^p \, dx \, dt + \|u_{xx}\|^p_{\mathcal{H}_p^{-1}} \\ \leq cE\int_0^T \int_{\mathbb{R}^d} |g(t,x,\omega)|^p_{l_2} \, dx \, dt,$$

and such that, for $\phi \in C_0^\infty(\mathbb{R}^d)$,

$$\big(u(t,\cdot),\phi(\cdot)\big) = \int_0^t k(t-s)\big(\Delta u(s,\cdot),\phi(\cdot)\big)\,ds + \sum_{k=1}^\infty \int_0^t \big(g^k(s,\cdot),\phi(\cdot)\big)\,dw_s^k,$$

a.s. for all $t \in [0,T]$.

Remark 5. *Since* $(1-\Delta)^{\frac{\beta}{2}}$ *is an isomorphism from* $H_p^n(\mathbb{R}^d)$ *to* $H_p^{n-\beta}(\mathbb{R}^d)$, *which commutes with all operators in this setting, all regularity results in the theorem above may be shifted. Thus, if* $g \in L_p((0,T), H_p^\beta(\mathbb{R}^d, l_2))$, *then* $u \in \hat{\mathcal{H}}_p^{1+\beta}$.

4. Additional time-regularity

It is not difficult to observe that some time-regularity is lacking in Theorem 4 above. To see this, argue heuristically as follows. In (1), a time-derivative of order α corresponds to a second-order derivative in space. The stochastic series in (1) is, roughly, $C^{\frac{1}{2}}((0,T), L_p(\Omega\times\mathbb{R}^d))$. But, by Theorem 4, $\Delta u \in \mathcal{H}_p^{-1}$ and the smoothing out (in time) by the kernel $t^{-1+\alpha}$ is not enough to give the deterministic integral the same degree of smoothness as the stochastic series. One therefore conjectures that Δu has some additional time-regularity. This is, in fact, the case:

Theorem 6. *Let* p, α, g *be as in the assumptions of Theorem* 4. *Let* u *be the solution given by Theorem* 4. *Take* $\epsilon > 0$ *arbitrary, but such that* $\frac{1}{2}-\epsilon \neq \frac{1}{p}$, $\frac{1}{2}-\frac{\alpha}{2}-\epsilon \neq \frac{1}{p}$. *Then*

$$\text{(i)} \quad u \in L_p\Big(\Omega; H_p^{\frac{1}{2}-\epsilon}\big([0,T]; L_p(\mathbb{R}^d)\big)\Big),$$

$$\text{(ii)} \quad u \in L_p\Big(\Omega; H_p^{\frac{1}{2}-\frac{\alpha}{2}-\epsilon}\big([0,T]; H_p^1(\mathbb{R}^d)\big)\Big),$$

$$\text{(iii)} \quad u \in L_p\Big(\Omega; H_p^{\frac{1}{2}-\alpha-\epsilon}\big([0,T]; H_p^2(\mathbb{R}^d)\big)\Big).$$

The norm of u *in the respective space is bounded by* (*a constant times*) *the norm of* g *in* $L_p\big((0,T)\times\Omega\times\mathbb{R}^d; l_2\big)$.

An interpolation between (ii) and (iii) in Theorem 6 yields

$$\text{(iv)} \quad u \in L_p\big(\Omega; L_p([0,T]; H_p^{\frac{1}{\alpha}-\epsilon}(\mathbb{R}^d))\big).$$

This is an immediate consequence of the following interpolation result whose proof we defer to the end of the paper:

Lemma 7. *For* $\alpha \in (\frac{1}{2}, 1)$ *and sufficiently small* $0 < \epsilon < \delta$*, the following inclusion holds* (*as a continuous embedding*)

$$L_p\Big(\Omega; H_p^{\frac{1}{2}-\frac{\alpha}{2}-\epsilon}\big([0,T]; H_p^1(\mathbb{R}^d)\big)\Big) \cap L_p\Big(\Omega; H_p^{\frac{1}{2}-\alpha-\epsilon}\big([0,T]; H_p^2(\mathbb{R}^d)\big)\Big)$$
$$\subset L_p\Big(\Omega; L_p\big([0,T]; H_p^{\frac{1-2\delta}{\alpha}}(\mathbb{R}^d)\big)\Big)$$

For $\alpha = 1$ (the stochastic heat equation) the result obtained in [11] is $u \in L_p\big(\Omega; L_p([0,T]; H_p^1)\big)$. In forthcoming work we will analyze the apparent loss of regularity when moving from the stochastic heat equation to the stochastic integral equation.

Outline of proof of Theorem 6 (ii). Again, throughout this proof, c will denote a generic constant, which depends only on $d, \alpha, p, T, \epsilon$ and may vary from line to line. Let $\epsilon > 0$ be such that $\frac{1}{2} - \frac{\alpha}{2} - \epsilon > 0$. We claim that, for fixed ω, $u_x \in H_p^{\frac{1}{2}-\frac{\alpha}{2}-\epsilon}\big([0,T]; L_p(\mathbb{R}^d)\big)$. By [15], p. 29, this amounts to showing that

$$v \overset{\text{def}}{=} \big(\frac{d}{dt}\big)^{\frac{1}{2}-\frac{\alpha}{2}-\epsilon} u_x = \frac{d}{dt}\sum_k \big(t^{-\frac{1}{2}+\frac{\alpha}{2}+\epsilon} * S * g_x^k\big) \in L_p\big((0,T)\times\mathbb{R}^d\big)$$

with $\|v\|_{L_p((0,T)\times\mathbb{R}^d)}$ being equivalent to $\|u_x\|_{H_p^{\frac{1}{2}-\frac{\alpha}{2}-\epsilon}([0,T];L_p(\mathbb{R}^d))}$.

Let $F(t) \overset{\text{def}}{=} \frac{d}{dt}\big(t^{-\frac{1}{2}+\frac{\alpha}{2}+\epsilon} * S\big)$. The convolution $F * g_x$ is well defined as an Itô integral, since $E\big\{\int_0^t |F(t-s)g_x(s)|_{l_2}^2\, ds\big\} < \infty$. Computing the Laplace transform of $F(t)$ gives

$$\tilde{F}(\lambda) = c\lambda^{-\frac{1}{2}-\frac{\alpha}{2}-\epsilon}\big(I - \lambda^{-\alpha}\Delta\big)^{-1},$$

with $c = \Gamma(\frac{1}{2}+\frac{\alpha}{2}-\epsilon)$. The complex inversion formula implies for any $h \in L_p(\mathbb{R}^d, l_2)$

$$(F(t)h)(x) = c(2\pi i)^{-1}\int_{\Gamma_{1,\psi}} e^s\big(st^{-1}\big)^{-\frac{1}{2}+\frac{\alpha}{2}-\epsilon}[(\mu-\Delta)^{-1}h](x)t^{-1}\, ds,$$

where, as in the proof of Theorem 4, $\mu = \big(st^{-1}\big)^{\alpha}$.

We use again the representation (18) for $(\mu - \Delta)^{-1}$ and observe that ϕ_μ satisfies the following estimates:

$$\int_{\mathbb{R}^d} |\phi_\mu(x)|\, dx \le c|\mu|^{-1} \quad \text{and} \quad \int_{\mathbb{R}^d} |\frac{\partial}{\partial x_i}\phi_\mu(x)|\, dx \le c|\mu|^{-\frac{1}{2}}. \tag{21}$$

Therefore

$$\|\frac{\partial}{\partial x_i}(\mu-\Delta)^{-1}h\|_{L_p(\mathbb{R}^d,l_2)} \le \|\frac{\partial}{\partial x_i}\phi_\mu\|_{L_1(\mathbb{R}^d)}\|h\|_{L_p(\mathbb{R}^d,l_2)} \le c|\mu|^{-\frac{1}{2}}\|h\|_{L_p(\mathbb{R}^d,l_2)}.$$

We infer that

$$\|\frac{\partial}{\partial x_i}F(t)h\|_{L_p(\mathbb{R}^d,l_2)} \le ct^{-1}\int_{\Gamma_{1,\psi}} e^{\Re(s)}\big(\frac{|s|}{t}\big)^{-\frac{1}{2}+\frac{\alpha}{2}-\epsilon}\big(\frac{|s|}{t}\big)^{-\frac{\alpha}{2}}\|h\|_{L_p(\mathbb{R}^d,l_2)}\, ds$$
$$\le ct^{-\frac{1}{2}+\epsilon}\|h\|_{L_p(\mathbb{R}^d,l_2)}.$$

The case $p = \infty$ yields

$$\sup_{x\in\mathbb{R}^d} \Big(\int_0^t |\frac{\partial}{\partial x_i} F(t-s)g(s,x)|_{l_2}^2\, ds\Big)^{\frac{1}{2}} \le ct^{\epsilon}\|g\|_{L_\infty((0,t)\times\mathbb{R}^d;l_2)},$$

while the case $p = 2$ yields

$$\left[\int_0^T \int_{\mathbb{R}^d}\int_0^t |\frac{\partial}{\partial x_i} F(t-s)g(s,x)|_{l_2}^2\, ds\, dx\, dt\right]^{\frac{1}{2}} \le cT^{\epsilon}\|g\|_{L_2((0,T)\times\mathbb{R}^d;l_2)}.$$

The Marcinkiewicz interpolation theorem yields

$$\left[\int_0^T \int_{\mathbb{R}^d}\Big(\int_0^t |\frac{\partial}{\partial x_i} F(t-s)g(s,x)|_{l_2}^2\, ds\Big)^{\frac{p}{2}}\, dx\, dt\right]^{\frac{1}{p}} \le cT^{\epsilon}\|g\|_{L_p((0,T)\times\mathbb{R}^d;l_2)}.$$

Hence, by the Burkholder-Davis-Gundy inequality,

$$\begin{aligned} E\,\|u_x\|^p_{H_p^{\frac{1}{2}-\frac{\alpha}{2}-\epsilon}([0,T];L_p(\mathbb{R}^d))} &\le cE\int_{\mathbb{R}^d}\int_0^T |v|^p\, dt\, dx \\ &= c\int_{\mathbb{R}^d}\int_0^T E\Big(\sum_k \int_0^t \nabla F(t-s)g^k(s,x,\omega)\, dw_s^k\Big)^p\, dt\, dx \\ &\le c\int_{\mathbb{R}^d}\int_0^T E\Big(\int_0^t |\nabla F(t-s)g(s,x,\omega)|_{l_2}^2\, ds\Big)^{\frac{p}{2}}\, dt\, dx \\ &= c\,E\int_{\mathbb{R}^d}\int_0^T \Big(\int_0^t |\nabla F(t-s)g(s,x,\omega)|_{l_2}^2\, ds\Big)^{\frac{p}{2}}\, dt\, dx \\ &\le c\,E\int_{\mathbb{R}^d}\int_0^T |g(t,x,\omega)|_{l_2}^p\, dt\, dx, \end{aligned}$$

which is (ii).

The relations (i), (iii) are proved in much the same fashion. Notice that in the proof of (iii) we make use of the following fact which is not hard to prove using the Marcinkiewicz multiplier theorem: Let $\eta \in (0,1)$, E any UMD-space, and $p \in (1,\infty)$. Then, provided $t^{-1+\eta} * u \in L_p((0,T);E)$, one has $u \in H_p^{-\eta}((0,T);E)$.

We finally remark that the statements (i)–(iii) of Theorem 5 can be slightly strengthened as follows. Take, e.g., (i), which states that

$$D_t^{\frac{1}{2}-\epsilon} S * g \in L_p\big((0,T)\times\Omega\times\mathbb{R}^d\big).$$

An examination of the proof reveals that one in fact has somewhat more:

Remark 8. *With the assumptions of Theorem* 4, *let* $M(t) > 0$ *be such that*

$$\int_0^T (tM^2(t))^{-1}\, dt < \infty.$$

Then

$$\Big[\big(M(t)\big)^{-1} D_t^{\frac{1}{2}} S\Big] * g \in L_p((0,T)\times\Omega\times\mathbb{R}^d, l_2).$$

Outline of the proof of Remark 8*:* Let $F_1 = \left(\frac{d}{dt}\right)^{\frac{1}{2}} S$. The Laplace transform is

$$\tilde{F}_1(\lambda) = c\lambda^{-\frac{1}{2}}(1-\lambda^{-\alpha}\Delta)^{-1}$$

and the complex inversion formula yields for $h \in L_p(\mathbb{R}^d, l_2)$

$$F_1(t)h = c\int_{\Gamma_{1,\psi}} e^s \left(\frac{s}{t}\right)^{-\frac{1}{2}+\alpha} (\mu-\Delta)^{-1} h t^{-1}\, ds,$$

with $\mu = \left(\frac{s}{t}\right)^{\alpha}$. By (21) we obtain

$$\|F_1(t)h\|_{L_p(\mathbb{R}^d,l_2)} \leq ct^{-1}\int_{\Gamma_{1,\psi}} e^{\Re s}\left(\frac{|s|}{t}\right)^{-\frac{1}{2}} \|h\|_{L_p(\mathbb{R}^d,l_2)} \leq\ ct^{-\frac{1}{2}}\|h\|_{L_p(\mathbb{R}^d,l_2)}.$$

The case $p=\infty$ yields

$$\begin{aligned}&\|\int_0^t M^{-1}(t-s)F_1(t-s)g(s)\,ds\|_{L_\infty(\mathbb{R}^d,l_2)}\\ &\qquad \leq c\int_0^t M^{-1}(s)s^{-\frac{1}{2}}\,ds\ \|g\|_{L_\infty((0,T)\times\mathbb{R}^d,l_2)},\end{aligned}$$

while the case $p=2$ yields by Hölder's inequality

$$\begin{aligned}&\int_0^T \|\int_0^t M^{-1}(t-s)F_1(t-s)g(s)\,ds\|^2_{L_2(\mathbb{R}^d,l_2)}\,dt\\ &\qquad \leq c\int_0^T \left(\int_0^t M(s)^{-2}s^{-1}\,ds\right)\ \|g\|^2_{L_2((0,t)\times\mathbb{R}^d,l_2)}\,dt.\end{aligned}$$

The Marcinkiewicz interpolation theorem concludes the proof of the remark.

Proof of Lemma 7. Let A be an UMD-space. For $s\in\mathbb{R}$, $\epsilon>0$, and $p\in(1,\infty)$, $q\in[1,\infty)$,

$$H_p^{s+\epsilon}(\mathbb{R};A)\subset B^s_{p,q}(\mathbb{R};A). \tag{22}$$

([7], see also [8].) Define

$$\begin{aligned} H_p^{s+\epsilon}([0,T];A) &\stackrel{\text{def}}{=} \{g\mid_{[0,T]}\ \mid\ g\in H_p^{s+\epsilon}(\mathbb{R};A)\}\\ \text{with}\quad \|f\|_{H_p^{s+\epsilon}([0,T];A)} &\stackrel{\text{def}}{=} \inf_{g\in S_{0,f}} \|g\|_{H_p^{s+\epsilon}(\mathbb{R};A)},\\ \text{where}\quad S_{0,f} &\stackrel{\text{def}}{=} \{g\in H_p^{s+\epsilon}(\mathbb{R},A)\ \big|\ g\mid_{[0,T]}=f\}.\end{aligned}$$

Similarly, we define

$$\begin{aligned} B^s_{p,q}([0,T];A) &\stackrel{\text{def}}{=} \{g\mid_{[0,T]}\ \mid\ g\in B^s_{p,q}(\mathbb{R};A)\}\\ \text{with}\quad \|f\|_{B^s_{p,q}([0,T];A)} &\stackrel{\text{def}}{=} \inf_{g\in S_{1,f}} \|g\|_{B^s_{p,q}(\mathbb{R};A)},\\ \text{where}\quad S_{1,f} &\stackrel{\text{def}}{=} \{g\in B^s_{p,q}(\mathbb{R},A)\ \big|\ g\mid_{[0,T]}=f\}.\end{aligned}$$

Given $f \in H_p^{s+\epsilon}([0,T];A)$ and $g \in S_{0,f}$, we infer by (22) with some constant c that $g \in B_{p,q}^s(\mathbb{R};A)$ and $\|g\|_{B_{p,q}^s(\mathbb{R};A)} \le c\|g\|_{H_p^{s+\epsilon}(\mathbb{R};A)}$, thus $g \in S_{1,f}$, hence $f = g\big|_{[0,T]} \in B_{p,q}^s([0,T];A)$, and

$$\|f\|_{B_{p,q}^s([0,T];A)} = \inf_{g\in S_{1,f}} \|g\|_{B_{p,q}^s([0,T];A)} \le c \inf_{g\in S_{0,f}} \|g\|_{H_p^{s+\epsilon}(\mathbb{R};A)} = c\,\|f\|_{H_p^{s+\epsilon}([0,T];A)}.$$

Thus, if $0 < \epsilon < \delta$,

$$\begin{aligned} &L_p\Big(\Omega; H_p^{\frac{1}{2}-\frac{\alpha}{2}-\epsilon}\big([0,T]; H_p^1(\mathbb{R}^d)\big)\Big) \cap L_p\Big(\Omega; H_p^{\frac{1}{2}-\alpha-\epsilon}\big([0,T]; H_p^2(\mathbb{R}^d)\big)\Big) \\ &\subset L_p\Big(\Omega; B_{p,q}^{\frac{1}{2}-\frac{\alpha}{2}-\delta}\big([0,T]; H_p^1(\mathbb{R}^d)\big)\Big) \cap L_p\Big(\Omega; B_{p,q}^{\frac{1}{2}-\alpha-\delta}\big([0,T]; H_p^2(\mathbb{R}^d)\big)\Big). \end{aligned}$$

Let δ be sufficiently small so that

$$\theta \overset{\text{def}}{=} \frac{1}{\alpha} - 1 - \frac{2\delta}{\alpha} \in (0,1).$$

Use complex interpolation. From [14, p. 185,(11)] we have

$$\Big[H_p^1(\mathbb{R}^d), H_p^2(\mathbb{R}^d)\Big]_\theta = H_p^{(1-\theta)+2\theta}(\mathbb{R}^d) = H_p^{\frac{1-2\delta}{\alpha}}(\mathbb{R}^d).$$

Now use [14, p. 128,(4)] and subsequently [6, p. 179, (6.8)] to show that

$$\begin{aligned} &L_p\Big(\Omega; B_{p,q}^{\frac{1}{2}-\frac{\alpha}{2}-\delta}\big([0,T]; H_p^1(\mathbb{R}^d)\big)\Big) \cap L_p\Big(\Omega; B_{p,q}^{\frac{1}{2}-\alpha-\delta}\big([0,T]; H_p^2(\mathbb{R}^d)\big)\Big) \\ &\subset L_p\Big(\Omega; \Big[B_{p,q}^{\frac{1}{2}-\frac{\alpha}{2}-\delta}\big([0,T]; H_p^1(\mathbb{R}^d)\big)\,,\, B_{p,q}^{\frac{1}{2}-\alpha-\delta}\big([0,T]; H_p^2(\mathbb{R}^d)\big)\Big]_\theta\Big) \\ &\subset L_p\Big(\Omega; B_{p,q}^0\big([0,T]; H_p^{\frac{1-2\delta}{\alpha}}(\mathbb{R}^d)\big)\Big). \end{aligned}$$

Choose $q = 1$ and note that $B_{p,1}^0(\mathbb{R}, A) \subset L_p(\mathbb{R}, A)$, [9]. Now argue as above to obtain that $B_{p,1}^0([0,T], A) \subset L_p([0,T], A)$. This proves Lemma 7.

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Resolvent Estimates for a Perturbed Oseen Problem

Paul Deuring

Abstract. We consider a resolvent equation arising from a stability problem for exterior Navier-Stokes flows with nonzero velocity at infinity.

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Keywords. Stability, Oseen system, resolvent estimate.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be an exterior domain in $\mathbb{R}^3$. Consider the Navier-Stokes system

$$\partial_t u - \Delta u + \tau \cdot (u \cdot \nabla)u + \nabla p = h, \ \ \operatorname{div} u = 0 \quad \text{in } \ \Omega \times (0, \infty), \qquad (1.1)$$

with the boundary conditions

$$u \,|\, \partial\Omega \times (0, \infty) = 0, \quad u(x, t) \to e_1 \ \ (|x| \to \infty) \ \text{ for } \ t \in (0, \infty), \qquad (1.2)$$

where $e_1 := (1, 0, 0)$, and where the data h do not depend on the time variable t. Let (u, p) be a solution to problem (1.1), (1.2), and let (U, P) be a solution of the corresponding stationary boundary value problem

$$-\Delta U + \tau \cdot (U \cdot \nabla)U + \nabla P = h, \ \ \operatorname{div} U = 0 \quad \text{in } \ \Omega, \qquad (1.3)$$

$$U \,|\, \partial\Omega = 0, \quad U(x) \to e_1 \ \ (|x| \to \infty). \qquad (1.4)$$

In this situation, the question arises as to whether $u(t) - U$ tends to zero in some sense for t tending to infinity, provided $u(0) - U$ is small in a suitable way. This “stability problem” attracted much attention for some time now; see [2], [8], [9], [11], [12], [17], for example. Most of the results in these references are based on smallness assumptions on U. However, as explained in [13], [14], one would also like to find a criterion related to the spectrum of a suitable linear operator, similar to the situation with ODE. Recently Neustupa [15] came rather close to such a criterion. His result may be stated as follows:

Write P_2 for the usual Helmholtz operator on $L^2(\Omega)^3$. Define the operator L by

$$L(v) := P_2\big(\,\Delta v - \tau \cdot (U \cdot \nabla)v - \tau \cdot (v \cdot \nabla)U\,\big),$$

with v from a suitable function space. Let $\mathfrak{B}_{\mathrm{sym}}$ denote the symmetric part of an operator $\mathfrak{B}$ given by $\mathfrak{B}(v) := -(U \cdot \nabla)v - (v \cdot \nabla)U$, and let H_0' be the finite-dimensional subspace of $L^2(\Omega)^3$ consisting of the eigenfunctions associated to the positive eigenvalues of the operator $P_2\big(\,\Delta + \widetilde{a} \cdot \tau \cdot \mathfrak{B}_{\mathrm{sym}}\,\big)$, where $\widetilde{a}$ is some fixed real number. (For rigorous definitions see Section 2.) Suppose there is some $R > 0$ and some non-increasing, integrable and square-integrable function $\varphi : [0, \infty) \mapsto [0, \infty)$ such that

$$\|\nabla e^{Lt}(f)\,|\,B_R\|_2 \leq \varphi(t) \cdot \|f\|_2 \quad \text{for } \ t \in (0, \infty),\ f \in H_0'. \tag{1.5}$$

Then Neustupa [15] could show that for a strong solution (u, p) of (1.1), (1.3), the relation $\|\nabla(u(t) - U)\|_2 \to 0$ holds for $t \to \infty$ if $\|u(0) - U\|_{1,2}$ is small. Neustupa considers (1.5) as a substitute of the assumption that all eigenvalues of L have negative real part.

In the work at hand, we show that this point of view is justified at least in the case $\Omega = \mathbb{R}^3$ (the case of the whole space). It turned out that for such Ω, inequality (1.5) is valid provided all the eigenvalues of L have negative real part and the point 0 is almost in the resolvent of L, in the same sense as the point 0 is almost in the resolvent of respectively the Stokes and the Oseen operator. A precise statement of these conditions may be found in assertion (C1) and (C2) in Section 2; our results are stated in Theorem 2.3. These results are by no means obvious since the spectrum of L touches the imaginary axis from the left, independently of the concrete form of the function U.

In this article, we are only able to indicate the way we proceed and elaborate some selected points. More detailed proofs will be given in [5].

2. Notations, definitions and main result

For $\epsilon > 0$, we put $B_\epsilon(x) := \big\{\, y \in \mathbb{R}^3 \;:\; |y - x| < \epsilon \,\big\}$. Set $B_\epsilon := B_\epsilon(0)$. For $A \subset \mathbb{R}^3$, we abbreviate $A^c := \mathbb{R}^3 \backslash A$. The length $\alpha_1 + \alpha_2 + \alpha_3$ of a multi-index $\alpha \in \mathbb{N}_0^3$ is denoted by $|\alpha|_1$.

If $\sigma \in \mathbb{N}$, and if $f : \mathbb{R}^3 \mapsto \mathbb{R},\ g : \mathbb{R}^3 \mapsto \mathbb{R}^\sigma$ are measurable functions, with

$$\int_{\mathbb{R}^3} |f(x - y)| \cdot |g(y)|\, dy < \infty \quad \text{for a.e. } \ x \in \mathbb{R}^3,$$

then we set

$$(f * g)(x) := \Big(\int_{\mathbb{R}^3} f(x - y) \cdot g_j(y)\, dy\Big)_{1 \leq j \leq 3} \quad \text{for a.e. } \ x \in \mathbb{R}^3.$$

We define $\mathfrak{D}_0^{1,2}(\mathbb{R}^3)$ as the space of all functions $v \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)$ such that $\nabla v \in L^2(\mathbb{R}^3)^3$; see [6, Remark II.5.2, Theorem II.5.1, II.6.1]. This space is equipped with the gradient norm. Furthermore, put $\mathfrak{D} := [\mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3]'$. We define the norm $\| \,\|_{-1,2}$ on $\mathfrak{D}$ by setting

$$\|F\|_{-1,2} := \sup\big\{ \, |F(v)|/\|\nabla v\|_2 \; : \; v \in \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3, \; \nabla v \neq 0 \, \big\}. \tag{2.1}$$

We note that in (2.1), it is sufficient to take the sup with respect to all functions $v \in C_0^\infty(\mathbb{R}^3)^3$ with $\nabla v \neq 0$. Any function $f \in L^1_{\text{loc}}(\mathbb{R}^3)^3$ with

$$\gamma_f := \sup\big\{ \, \Big| \int_{\mathbb{R}^3} f \cdot v \, dx \Big| / \|\nabla v\|_2 \; : \; v \in C_0^\infty(\mathbb{R}^3)^3, \; \nabla v \neq 0 \, \big\} < \infty$$

defines an element of $\mathfrak{D}$, which we also denote by f, and which verifies the relation $\|f\|_{-1,2} = \gamma_f$. This is true in particular for $f \in L^{6/5}(\mathbb{R}^3)^3$, due to the standard Sobolev estimate $\|v\|_6 \leq C \cdot \|\nabla v\|_2$ for $v \in C_0^\infty(\mathbb{R}^3)^3$.

For $p \in (1, \infty)$, let $H_p(\mathbb{R}^3)$ denote the closure of the set $\big\{ \varphi \in C_0^\infty(\mathbb{R}^3)^3 \; : \; \operatorname{div} \varphi = 0 \big\}$ with respect to the norm $\| \,\|_p$. Then, for any element $f \in L^p(\mathbb{R}^3)^3$, there is a unique function $P_p f \in H_p(\mathbb{R}^3)$ and some $g \in W^{1,p}_{\text{loc}}(\mathbb{R}^3)$ with $P_p f + \nabla g = f$. The mapping $P_p : H_p(\mathbb{R}^3) \mapsto L^p(\mathbb{R}^3)^3$ is linear and bounded. We refer to [6, Section III.1] for these results. Since for any $p, q \in (1, \infty)$ and any $f \in L^p(\mathbb{R}^3)^3 \cap L^q(\mathbb{R}^3)^3$, we have $P_p f = P_q f$, we will only write P instead of P_q in the following.

We fix $\tau \in (0, \infty)$. Put $s(x) := \tau \cdot (|x| - x_1)$ for $x \in \mathbb{R}^3$, and define

$$E^{(0)}(z) := (4 \cdot \pi)^{-1} \cdot |z|^{-1} \cdot e^{-s(z)/2},$$
$$E^{(\lambda)}(z) := (4 \cdot \pi)^{-1} \cdot |z|^{-1} \cdot e^{-\sqrt{\lambda + (\tau/2)^2} \cdot |z| + \tau \cdot z_1/2}$$

for $z \in \mathbb{R}^3 \backslash \{0\}$, $\lambda \in \mathbb{C} \backslash \{0\}$. Then $E^{(\varrho)}$, for $\varrho \in \mathbb{C}$ with $\Re \varrho \geq 0$, is a fundamental solution of the equation $-\Delta v + \tau \cdot \partial_1 v + \varrho \cdot v = g$.

Concerning the function h in (1.1) and (1.3), we suppose that $h \in L^s(\mathbb{R}^3)^3$ for $s \in (1, 3 + \epsilon]$, with some $\epsilon > 0$. Then there is a pair $(U, P) \in H^2_{\text{loc}}(\mathbb{R}^3)^3 \times H^1_{\text{loc}}(\mathbb{R}^3)$ which solves (1.3) with $\Omega = \mathbb{R}^3$, and which verifies the relations

$$\begin{aligned} &U - e_1 \in L^s(\mathbb{R}^3)^3 \;\text{ for }\; s \in (2, 3 + \epsilon], \\ &\nabla U \in L^s(\mathbb{R}^3)^9 \;\text{ for }\; s \in [4/3, \, 3 + \epsilon], \quad \text{with some } \epsilon > 0. \end{aligned} \tag{2.2}$$

For this result, see [7, Section IX.7], [4, Theorem 4.9]. For the rest of this article, we fix such a solution (U, P).

For $v \in W^{1,1}_{\text{loc}}(\mathbb{R}^3)^3$, we put

$$\mathfrak{B}(v) := \Big(- \sum_{k=1}^3 \big(\, \partial_k U_j \cdot v_k + (U - e_1)_k \cdot \partial_k v_j \, \big) \Big)_{1 \leq j \leq 3},$$

$$\mathfrak{B}_{\text{sym}}(v) := \Big((-1/2) \cdot \sum_{k=1}^3 v_k \cdot (\partial_k U_j + \partial_j U_k) \Big)_{1 \leq j \leq 3}.$$

Further put $\mathfrak{D}(L) := H_2(\mathbb{R}^3) \cap H^2(\mathbb{R}^3)^3$, and define an operator $L : \mathfrak{D}(L) \mapsto H_2(\mathbb{R}^3)$ by setting

$$Lv := P\big(\, \Delta v - \tau \cdot \partial_1 v + \tau \cdot \mathfrak{B}(v) \, \big) \quad \text{for } \; v \in \mathfrak{D}(L),$$

where we used implicitly that $\mathfrak{B}(v) \in L^p(\mathbb{R}^3)^3$ for some $p \in (1, \infty)$. In fact, the relation $\mathfrak{B}(v) \in L^2(\mathbb{R}^3)^3$ holds for $v \in \mathfrak{D}$. It should further be noted that

$$P\big(\,\Delta v - \tau \cdot \partial_1 v\,\big) = \Delta P v - \tau \cdot \partial_1 P v = \Delta v - \tau \cdot \partial_1 v$$

for $v \in \mathfrak{D}(L)$, a relation which is not valid for functions in a corresponding space on Ω with $\Omega \neq \mathbb{R}^3$. The ensuing theorem holds according to [1], [13], [14].

Theorem 2.1. *The set $\mathfrak{D}(L)$ is dense in $H_2(\mathbb{R}^3)$. The operator L is closed. Let $\varrho(L)$ denote the resolvent set of L, and $\sigma(L)$ the spectrum of L. Then there is a countable set $\mathfrak{K}$ of isolated eigenvalues of L such that*

$$\sigma(L)\backslash\mathfrak{K} \subset \big\{\,\lambda \in \mathbb{C} \;:\; \Re\lambda \leq -(\Im\lambda)^2/\tau^2\,\big\}. \tag{2.3}$$

Moreover, there are $a \in (0, \infty)$, $\vartheta \in (\pi/2,\, \pi)$ such that

$$S_{\vartheta,a} := \big\{\,\lambda \in \mathbb{C}\backslash\{a\} \;:\; |\arg(\lambda - a)| \leq \vartheta\,\big\} \subset \varrho(L),$$

and there is $C_1 > 0$ with

$$\|(\lambda \cdot I - L)^{-1}(\Phi)\|_2 \leq C_1 \cdot |\lambda - a|^{-1} \cdot \|\Phi\|_2 \tag{2.4}$$

for $\Phi \in H_2(\mathbb{R}^3),\ \lambda \in S_{\vartheta,a}$.

We require that the spectrum of L satisfies the following two conditions:

(C1) $\Re\lambda < 0$ for $\lambda \in \mathfrak{K}$.

(C2) For any $G \in \mathfrak{D}$, there is one and only one function $u \in \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3$ with $\operatorname{div} u = 0$ and

$$\int_{\mathbb{R}^3} \big(\,\nabla u \cdot \nabla v + \tau \cdot \partial_1 u \cdot v - \tau \cdot P\mathfrak{B}(u) \cdot v\,\big)\, dx = G(v)$$

for $v \in C_0^\infty(\mathbb{R}^3)^3$ with $\operatorname{div} v = 0$.

Note that an existence and uniqueness result as in (C2) is valid for the Oseen system ([7, Theorem IX.4.1]). Thus condition (C2) may be interpreted in the sense that the term $\tau \cdot P\mathfrak{B}(u)$ should not destroy this existence and uniqueness result for the Oseen system.

We fix some $\widetilde{a} \in \mathbb{R}$. For the ensuing theorem, we refer to [15].

Theorem 2.2. *The set of all numbers $\lambda \in (0, \infty)$ with $\Delta f + \widetilde{a} \cdot \tau \cdot P\mathfrak{B}_{\mathrm{sym}}(f) = \lambda \cdot f$ for some $f \in \mathfrak{D}(L)$ with $f \neq 0$ is finite.*

Let H_0' be the set consisting of these functions f and of the zero function. Then H_0' is a vector space of finite dimension.

We remark that according to Lemma 5.1, the term $P\mathfrak{B}_{\mathrm{sym}}(f)$ is well defined for functions f as in Theorem 2.2. By Theorem 2.1, the operator $-L$ is sectorial ([10, Definition 1.3.1]), and thus generates an analytic semigroup $(e^{Lt})_{t\geq 0}$ of linear operators from $H_2(\mathbb{R}^3)$ into $\mathfrak{D}(L)$; see [10, Theorem 1.3.4]. Our aim is to show the following

Theorem 2.3. *Let $R \in (0, \infty)$. Then there is some $C > 0$ depending on τ, U, a, ϑ, C_1, $\widetilde{a}$ and R such that*

$$\|\nabla e^{Lt}(f) \,|\, B_R\|_2 \leq C \cdot (1+t)^{-9/8} \cdot \|f\|_2 \quad \textit{for } f \in H_0', \;\; t \in (0, \infty).$$

Theorem 2.3 will be proved via resolvent estimates related to the operator L. These estimates are stated in Theorem 5.3, 5.4 and Lemma 5.5 below. Since L may be considered as a perturbed Oseen operator, we thus establish resolvent estimates for a perturbed Oseen system.

Concerning the constants appearing in the following, the symbol $\mathfrak{C}$ will denote constants only depending on τ, U, the parameters a, ϑ and C_1 from Theorem 2.1, and on the constant $\widetilde{a}$ appearing in Theorem 2.2. We will write $\mathfrak{C}(\gamma_1, \ldots, \gamma_n)$ for constants depending on the preceding quantities as well as on the parameters $\gamma_1, \ldots, \gamma_n$. A constant which only depends on $\gamma_1, \ldots, \gamma_n$ and on no other quantity will be denoted by $C(\gamma_1, \ldots, \gamma_n)$.

3. Convolutions of $E^{(\varrho)}$; estimates of $P\mathfrak{B}(E^{(\varrho)} * \Phi)$

We begin by stating some estimates of the fundamental solution $E^{(\varrho)}$ of the equation $-\Delta v + \tau \cdot \partial_1 v + \varrho \cdot v = g$. We remark that in this section and in the following, we will use the letter ϱ to denote complex numbers including 0, whereas the variable λ stands for non-vanishing complex numbers.

Theorem 3.1. *Let $\kappa, \gamma \in [0, \infty)$. Then*

$$\begin{aligned}|\partial_z^\alpha E^{(\lambda)}(z)| \leq C(\tau, \kappa, \gamma) \cdot |\lambda|^{-2\cdot\gamma} \cdot \big(|z|^{-\gamma-1-|\alpha|_1/2} + |z|^{-\gamma-1-|\alpha|_1}\big) \\ \cdot \big(1 + s(z)\big)^{-\kappa} \cdot e^{-\mu \cdot |\lambda|^2 \cdot |z|}\end{aligned} \tag{3.1}$$

for $z \in \mathbb{R}^3 \backslash \{0\}$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha|_1 \leq 1$, $\lambda \in \mathbb{C} \backslash \{0\}$ with $\Re\lambda \geq 0$ and $|\lambda| \leq (\tau/2)^2$, where μ is a constant only depending on τ. Moreover,

$$|\partial_z^\alpha E^{(\varrho)}(z)| \leq C(\tau) \cdot \big(|z|^{-1-|\alpha|_1/2} + |z|^{-1-|\alpha|_1}\big) \cdot \big(1 + s(z)\big)^{-1-|\alpha|_1/2} \tag{3.2}$$

for z, α as in (3.1)*, and for $\varrho \in \mathbb{C}$ with $\Re\varrho \geq 0$, $|\varrho| \leq (\tau/2)^2$.*

Theorem 3.1, Young's and Minkowski's inequality and the Hardy-Littlewood-Sobolev inequality yield

Theorem 3.2. *Let $p \in (1, 2]$, $q \in [1, p]$ with $p < 2$ or $q > 1$. Then*

$$\|\, |E^{(\lambda)}| * |f| \,\|_p \leq C(\tau, p, q) \cdot |\lambda|^{2-4\cdot(1-1/q+1/p)} \cdot \|f\|_q \tag{3.3}$$

for $f \in L^q(\mathbb{R}^3)$, $\lambda \in \mathbb{C} \backslash \{0\}$ with $\Re\lambda \geq 0$, $|\lambda| \leq (\tau/2)^2$.

Let $q \in [1, 2)$ and

$$p \in \big((1/q - 1/2)^{-1}, \infty\big] \;\; \textit{if } q \geq 3/2,$$
$$p \in \big((1/q - 1/2)^{-1}, (1/q - 2/3)^{-1}\big) \;\; \textit{if } q < 3/2.$$

Then, for $f \in L^q(\mathbb{R}^3)$, $\varrho \in \mathbb{C}$ with $\Re\varrho \geq 0$ and $|\varrho| \leq (\tau/2)^2$,

$$\|\, |E^{(\varrho)}| * |f| \,\|_p \leq C(\tau, p, q) \cdot \|f\|_q. \tag{3.4}$$

Let $p, q \in [1, \infty]$ *with* $1/q - 1/3 < 1/p < 1/q - 1/4$. *Then*

$$\| \, |\partial_l E^{(\varrho)}| * |f| \, \|_p \le C(\tau, p, q) \cdot \|f\|_q \tag{3.5}$$

for $1 \le l \le 3$ *and for* f *and* ϱ *as in* (3.4). *Finally*

$$\| \, |E^{(\varrho)}| * |f| \, \|_6 + \| \, |\partial_l E^{(\varrho)}| * |f| \, \|_2 \le C(\tau) \cdot \|f\|_{6/5}$$

for $1 \le l \le 3,\ f \in L^{6/5}(\mathbb{R}^3),\ \varrho \in \mathbb{C}$ *with* $\Re\varrho \ge 0,\ |\varrho| \le (\tau/2)^2$.

Among other results, the next theorem gives a precise form of the assertion that $E^{(\varrho)}$ is a fundamental solution of the equation $-\Delta v + \tau \cdot \partial_1 v + \varrho \cdot v = g$.

Theorem 3.3. *Let* $q \in (1, 2),\ f \in L^q(\mathbb{R}^3),\ \varrho \in \mathbb{C}$ *with* $\Re\varrho \ge 0,\ |\varrho| \le (\tau/2)^2$. *Then*

$$\begin{aligned}
&E^{(\varrho)} * f \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^3), \quad \partial_l(E^{(\varrho)} * f) = (\partial_l E^{(\varrho)}) * f \ \ (1 \le l \le 3),\\
&-\Delta(E^{(\varrho)} * f) + \tau \cdot \partial_1(E^{(\varrho)} * f) + \varrho \cdot (E^{(\varrho)} * f) = f,\\
&\|\partial_l \partial_m (E^{(\varrho)} * f) \,|\, B_R\|_q \le C(\tau, q, R) \cdot \|f\|_q \quad \big(1 \le l, m \le 3,\ R > 0\big).
\end{aligned} \tag{3.6}$$

We note a consequence of Theorem 3.2 and a remark in [6, p. 391/392]:

Lemma 3.4. $\|\nabla E^{(\varrho)} * w\|_2 \le C(\tau) \cdot \|w\|_{-1,2}$ *for* $\varrho \in \mathbb{C}$ *with* $\Re\varrho \ge 0,\ |\varrho| \le (\tau/2)^2$, $w \in C_0^\infty(\mathbb{R}^3)^3$.

Due to this lemma, we may define convolutions of $E^{(\varrho)}$ with elements of $\mathfrak{D}$:

Corollary 3.5. *Let* $\varrho \in \mathbb{C}$ *with* $\Re\varrho \ge 0,\ |\varrho| \le (\tau/2)^2$. *Then there is a linear mapping* $\Gamma := \Gamma_\varrho : \mathfrak{D} \mapsto L^6(\mathbb{R}^3)^3$ *with*

$$\begin{aligned}
&\Gamma(\Phi) \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^3)^3, \quad \nabla\Gamma(\Phi) \in L^2(\mathbb{R}^3)^9, \quad \partial_1\Gamma(\Phi) \in \mathfrak{D},\\
&\|\nabla\Gamma(\Phi)\|_2 \le C(\tau) \cdot \|\Phi\|_{-1,2} \quad \textit{for } \Phi \in \mathfrak{D},\\
&\Gamma(\Phi) = E^{(\varrho)} * \Phi \quad \textit{for } \Phi \in C_0^\infty(\mathbb{R}^3)^3.
\end{aligned}$$

Moreover, $\Gamma(w) = E^{(\varrho)} * w$ *if* $w \in \mathfrak{D} \cap L^q(\mathbb{R}^3)^3$ *for some* $q \in (1, 2)$, *or if* $\varrho \ne 0$ *and* $w \in \mathfrak{D} \cap L^2(\mathbb{R}^3)^3$.

If $w \in \mathfrak{D} \cap L^2(\mathbb{R}^3)^3$, *then*

$$\begin{aligned}
&\partial_l\Gamma(w) = (\partial_l E^{(\varrho)}) * w \ \ (1 \le l \le 3), \quad \Gamma(w) \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}^3)^3,\\
&\partial_l\partial_m\Gamma(w) \in L^2(\mathbb{R}^3)^3 \ \ (1 \le l, m \le 3), \quad -\Delta\Gamma(w) + \tau \cdot \partial_1\Gamma(w) + \varrho \cdot \Gamma(w) = w.
\end{aligned}$$

Finally, if $w \in \mathfrak{D} \cap H_2(\mathbb{R}^3)$, *the equation* $\operatorname{div}\Gamma(w) = 0$ *holds.*

The last statement of Corollary 3.5 means that for $w \in \mathfrak{D} \cap H_2(\mathbb{R}^3)$, the pair (v, π) with $v = \Gamma(w),\ \pi = 0$ is a solution in $\mathbb{R}^3$ of the resolvent problem

$$-\Delta v + \tau \cdot \partial_1 v + \varrho \cdot v + \nabla\pi = w, \quad \operatorname{div} v = 0$$

associated to the Oseen operator. This observation explains why convolutions of $E^{(\varrho)}$ are studied here.

In the ensuing theorem, which is a consequence of (2.2), we evaluate the operator $P\mathfrak{B}$ applied to convolutions of $E^{(\varrho)}$.

Theorem 3.6. *Let $q \in (1,2)$, $p \in (1,2]$. Then $\mathfrak{B}(E^{(\varrho)} * \Phi) \in L^q(\mathbb{R}^3)^3$ for $\Phi \in L^q(\mathbb{R}^3)^3$, and $\mathfrak{B}\big(\Gamma_\varrho(w)\big) \in L^p(\mathbb{R}^3)^3$ for $w \in \mathfrak{D}$, where $\varrho \in \mathbb{C}$ with $\Re\varrho \geq 0$ and $|\varrho| \leq (\tau/2)^2$. Moreover,*

$$\|P\mathfrak{B}(E^{(\varrho)} * \Phi)\|_q \leq \mathfrak{C}(q) \cdot \|\Phi\|_q, \quad \|P\big(\Gamma_\varrho(w)\big)\|_p \leq \mathfrak{C}(p) \cdot \|w\|_{-1,2} \tag{3.7}$$

for Φ, w, ϱ as above. In addition, there are non-increasing functions $\mathfrak{D}_1^{(q)}, \mathfrak{D}_2^{(p)} : [0,\infty) \mapsto (0,\infty)$ such that $\mathfrak{D}_1^{(q)}(R) \to 0$, $\mathfrak{D}_2^{(p)}(R) \to 0$ for $R \to \infty$, and

$$\begin{aligned} \|P\big(\chi_{B_R^c} \cdot \mathfrak{B}(E^{(\varrho)} * \Phi)\big)\|_q &\leq \mathfrak{D}_1^{(q)}(R) \cdot \|\Phi\|_q, \\ \|P\big(\chi_{B_R^c} \cdot \mathfrak{B}(\Gamma_\varrho(w))\big)\|_p &\leq \mathfrak{D}_2^{(p)}(R) \cdot \|w\|_{-1,2} \end{aligned} \tag{3.8}$$

for Φ, w, ϱ as above, and for $R \in [R_0, \infty)$.

The following theorem, although technical, is a crucial part of our theory. Its significance will become apparent in the next section.

Theorem 3.7. *Let $q \in (1,2)$, $p \in (1,2]$. Then there are constants $\delta_1^{(q)} = \delta_1(\tau, U, q)$, $\delta_2^{(p)} = \delta_2(\tau, U, p) \in (0,1)$ and non-decreasing functions $\gamma_1^{(q)}, \gamma_2^{(p)} : (0,\infty) \mapsto (0,\infty)$ such that the following holds:*

Let $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda \geq 0$ and $|\lambda| \leq (\tau/2)^2$. Let $R \in [R_0, \infty)$, $\widetilde{R} \in [2 \cdot R + 1, \infty)$. Then

$$\begin{aligned} &\|P\mathfrak{B}(E^{(\lambda)} * w) - P\mathfrak{B}(E^{(0)} * w)\|_q \\ &\leq \Big(2 \cdot \mathfrak{D}_1^{(q)}(R) + \mathfrak{C}(q) \cdot \gamma_1^{(q)}(R) \cdot \widetilde{R}^{-\delta_1^{(q)}} + \gamma_1^{(q)}(\widetilde{R}) \cdot |\lambda|^{1/3}\Big) \cdot \|w\|_q \\ &\text{for } \ w \in L^q(\mathbb{R}^3)^3, \end{aligned}$$

$$\begin{aligned} &\|P\mathfrak{B}(E^{(\lambda)} * w) - P\mathfrak{B}\big(\Gamma_0(w)\big)\|_p \\ &\leq \Big(2 \cdot \mathfrak{D}_2^{(p)}(R) + \mathfrak{C}(p) \cdot \gamma_2^{(p)}(R) \cdot \Big(\widetilde{R}^{-\delta_2^{(p)}} + \big(\ln(\widetilde{R}/(\widetilde{R}-1))\big)^{1/2}\Big) \\ &\qquad + \gamma_2^{(p)}(\widetilde{R}) \cdot |\lambda|^{1/3}\Big) \cdot (\|w\|_2 + \|w\|_{1,2}) \quad \text{for } \ w \in L^2(\mathbb{R}^3)^3 \cap \mathfrak{D}. \end{aligned}$$

4. Solving a perturbed Oseen system

We are now looking for solutions in $H_2(\mathbb{R}^3) \cap H^2(\mathbb{R}^3)^3$ of the perturbed Oseen system

$$-\Delta v + \tau \cdot \partial_1 v + \lambda v - \tau \cdot P\mathfrak{B}(v) = f, \quad \operatorname{div} v = 0. \tag{4.1}$$

To this end, we intend to use the formula

$$\begin{aligned} &(-\Delta + \tau \cdot \partial_1 + \lambda \cdot I - \tau \cdot P\mathfrak{B})^{-1} \\ =\ &(-\Delta + \tau \cdot \partial_1 + \lambda \cdot I)^{-1} \circ \big(I \ - \ \tau \cdot P\mathfrak{B} \circ (-\Delta + \tau \cdot \partial_1 + \lambda \cdot I)^{-1}\big)^{-1}. \end{aligned} \tag{4.2}$$

Of course, this equation is only formal, and the problem consists in giving it a sense. Our idea is to start with $\lambda = 0$. In that case, equation (4.2) may be transformed into something rigorous due to our assumption (C2) in Section 2. Then we use a perturbation argument and Theorem 3.7 in order to deal with the case $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda \geq 0$ and $|\lambda|$ small. The results for this case will be used in Section 5 in order to derive estimates of solutions of (4.1) (that is, resolvent estimates of the operator L), under the assumptions that $|\lambda|$ small, $\Re\lambda \geq 0$, and the right-hand side of f in (4.1) belongs to the space H_0' introduced in Theorem 2.2.

We begin by looking for a rigorous form of (4.2) in the case $\lambda = 0$. In a first step, we deduce from Corollary 3.5 and a standard uniqueness result (see [6, p. 397 and p. 391]):

Theorem 4.1. *Define $D(A)$ as the space of all functions $v \in L^6(\mathbb{R}^3)^3 \cap W^{2,1}_{\mathrm{loc}}(\mathbb{R}^3)^3$ such that $\partial_l v$, $\partial_m\partial_l v \in L^2(\mathbb{R}^3)^3$ for $1 \leq l, m \leq 3$, $\partial_1 v \in \mathfrak{D}$ and* $\operatorname{div} v = 0$.

Put $Av := -\Delta v + \tau \cdot \partial_1 v$ for $v \in D(A)$. Then the operator $A : D(A) \mapsto \mathfrak{D} \cap H_2(\mathbb{R}^3)$ is linear and bijective, with $A^{-1} = \Gamma_0$, where Γ_0 was introduced in Corollary 3.5.

Theorem 4.1 and assumption (C2) imply

Corollary 4.2. *The operator $\widetilde{A} : D(A) \mapsto \mathfrak{D} \cap H_2(\mathbb{R}^3)$, with $\widetilde{A}v := Av - \tau \cdot P\mathfrak{B}(v)$ for $v \in D(A)$, is linear and bijective.*

In view of Corollary 4.2, we may use the simple operator calculus indicated by (4.2). It follows with Corollary 3.5 and Theorem 3.6:

Corollary 4.3. *The mapping*

$$\widetilde{Z}_0 : \mathfrak{D} \cap H_2(\mathbb{R}^3) \ni w \mapsto w - \tau \cdot P\mathfrak{B}\big(\Gamma_0(w)\big) \in \mathfrak{D} \cap H_2(\mathbb{R}^3)$$

is linear, bounded, and bijective, and

$$\|w\|_{-1,2} + \|w\|_2 \leq \mathfrak{C} \cdot \big(\|\widetilde{Z}_0(w)\|_{-1,2} + \|\widetilde{Z}_0(w)\|_2\big) \quad \textit{for } w \in \mathfrak{D} \cap H_2(\mathbb{R}^3).$$

But the invertibility $\widetilde{Z}_0$ implies that the operator $Z_0^{(q)}$ defined below and acting on L^q-spaces is also invertible:

Theorem 4.4. *Let $q \in (1, 2)$. Then the operator*

$$Z_0^{(q)} : L^q(\mathbb{R}^3)^3 \ni w \mapsto w - \tau \cdot P\mathfrak{B}(E^{(0)} * w) \in L^q(\mathbb{R}^3)^3$$

is linear, bounded and bijective, with $\|w\|_q \leq \mathfrak{C}(q) \cdot \|Z_0^{(q)}(w)\|_q$ for $w \in L^q(\mathbb{R}^3)^3$.

The idea of the proof of Theorem 4.4 consists in showing that Z_q is Fredholm with index zero. This may be done by a compactness argument involving the spaces $W^{2,q}(B_R)$ and $L^q(B_R)$ for $R \in (0, \infty)$ as well as inequality (3.6), and by referring to a contraction principle and inequality (3.8) with large R. On the other hand, it may be shown that $Z_0^{(q)}$ is one-to-one because $\widetilde{Z}_0$ has the same property. Theorem 4.4 then follows.

Now we use a perturbation argument together with Theorem 3.7 in order to show that corresponding operators $\widetilde{Z}_\lambda$ and $Z_\lambda^{(q)}$, with $\lambda \neq 0$ but $|\lambda|$ small, are bijective, too. We get

Theorem 4.5. *There is* $\epsilon_1 \in (0, (\tau/2)^2]$ *depending on* τ *and* U *such that for* $\lambda \in \mathbb{C}\backslash\{0\}$ *with* $\Re\lambda \geq 0$ *and* $|\lambda| \leq \epsilon_1$*, the operator*

$$\widetilde{Z}_\lambda : \mathfrak{D} \cap H_2(\mathbb{R}^3) \ni w \mapsto w - \tau \cdot P\mathfrak{B}(E^{(\lambda)} * w) \in \mathfrak{D} \cap H_2(\mathbb{R}^3)$$

is linear, bounded, and bijective, with

$$\|w\|_{-1,2} + \|w\|_2 \leq \mathfrak{C} \cdot \big(\|\widetilde{Z}_\lambda(w)\|_{-1,2} + \|\widetilde{Z}_\lambda(w)\|_2 \big)$$

for $w \in \mathfrak{D} \cap H_2(\mathbb{R}^3)$ *and for* λ *as before.*

Let $q \in (1, 2)$*. Then there is* $\epsilon_2 = \epsilon_2(q) = \epsilon_2(\tau, U, q) \in (0, \epsilon_1]$ *such that for* $\lambda \in \mathbb{C}\backslash\{0\}$ *with* $\Re\lambda \geq 0$ *and* $|\lambda| \leq \epsilon_2$*, the operator*

$$Z^{(q)}_\lambda : L^q(\mathbb{R}^3)^3 \ni w \mapsto w - \tau \cdot P\mathfrak{B}(E^{(\lambda)} * w) \in L^q(\mathbb{R}^3)^3$$

is linear, bounded and bijective, with $\|w\|_q \leq \mathfrak{C}(q) \cdot \|Z^{(q)}_\lambda(w)\|_q$ *for* $w \in L^q(\mathbb{R}^3)^3$ *and for* $\lambda \in \mathbb{C}\backslash\{0\}$ *with* $\Re\lambda \geq 0,\ |\lambda| \leq \epsilon_2$.

Now, for the second time, we apply the straightforward operator calculus implicit in (4.2). This argument combined with Theorem 4.5, Corollary 3.5 and Theorem 3.2 (which yields that $E^{(\lambda)} * w \in L^2(\mathbb{R}^3)^3$ for $w \in L^2(\mathbb{R}^3)^3,\ \lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda \geq 0$ and $|\lambda| < (\tau/2)^2$) implies the ensuing corollary.

Corollary 4.6. *Let* $g \in \mathfrak{D} \cap H_2(\mathbb{R}^3),\ \lambda \in \mathbb{C}\backslash\{0\}$ *with* $\Re\lambda \geq 0,\ |\lambda| \leq \epsilon_1$*. Put* $\psi := \widetilde{Z}^{-1}_\lambda(g),\ u := E^{(\lambda)} * \psi$*. Then* $u \in H^2(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$,

$$-\Delta u + \tau \cdot \partial_1 u + \lambda \cdot u - \tau \cdot P\mathfrak{B}(u) = g, \quad \operatorname{div} u = 0.$$

If in addition $q \in (1, 2),\ |\lambda| \leq \epsilon_2(q)$ *and* $g \in L^q(\mathbb{R}^3)^3$*, we have* $\psi \in L^q(\mathbb{R}^3)^3$ *and* $Z^{(q)}_\lambda(\psi) = g$.

In view of (C1) and Theorem 2.1, the preceding corollary implies

Corollary 4.7. *Let* $\lambda \in \mathbb{C}\backslash\{0\}$ *with* $\Re\lambda \geq 0$ *and* $|\lambda| \leq \epsilon_1$*, with* ϵ_1 *from Theorem* 4.5*. Then* $\lambda \in \varrho(L)$*, the operator* $\widetilde{Z}_\lambda$ *(Theorem* 4.5*) is bijective, and* $(\lambda \cdot I - L)^{-1}(g) = E^{(\lambda)} * \widetilde{Z}^{-1}_\lambda(g)$.

5. Some resolvent estimates for a perturbed Oseen system

The aim of this section is to present some estimates of solutions of equation (4.1). We begin by an observation with respect to the operator $\mathfrak{B}_{\mathrm{sym}}$. Hölder's inequality, the Sobolev imbedding $\|w\|_\infty \leq C \cdot \|w\|_{2,2}$ for $w \in H^2(\mathbb{R}^3)$, and the assumptions in (2.2), imply

Lemma 5.1. *Let* $q \in [1, 6/5]$*. Then* $\|\mathfrak{B}_{\mathrm{sym}}(w)\|_q \leq \mathfrak{C}(q) \cdot \|w\|_2$ *for* $w \in L^2(\mathbb{R}^3)^3$.

Let $q \in (6/5, 2]$*. Then* $\|\mathfrak{B}_{\mathrm{sym}}(w)\|_q \leq \mathfrak{C}(q) \cdot \|w\|_{2,2}$ *for* $w \in H^2(\mathbb{R}^3)^3$.

The special role of the exponent 6/5 in Lemma 5.1 is due to the inequality $(1 - q/2)^{-1} \cdot q \leq 3$, which is valid for $q \in [1, 6/5]$. This inequality and (2.2) yield $\|\nabla U\|_{(1-q/2)^{-1} \cdot q} < \infty$.

Next we observe that for $f \in \mathfrak{D}(L)$, $\sigma \in (0,\infty)$ with $\Delta f + \widetilde{a} \cdot \tau \cdot P\mathfrak{B}_{\mathrm{sym}}(f) = \sigma \cdot f$, this function f verifies the Stokes resolvent system $-\Delta f + \sigma \cdot f = g$, $\operatorname{div} f = 0$ with a right-hand side g given by $g = \widetilde{a} \cdot \tau \cdot P\mathfrak{B}_{\mathrm{sym}}(f)$. (Note that $f \in \mathfrak{D}(L) \subset H_2(\mathbb{R}^3)$, so that it is in fact the Stokes resolvent system which appears here.) Thus we may combine Lemma 5.1 and the regularity theory for the Stokes resolvent problem. The latter theory yields the inequality $\|\nabla f\|_2 \leq C \cdot \|\widetilde{a} \cdot \tau \cdot P\mathfrak{B}(f)\|_{6/5}$, and much more deep-lying $\mathrm{W}^{2,q}$-estimates; see [3], for example. As a consequence, we get

Theorem 5.2. *Let $f \in \mathfrak{D}(L)$, and suppose that $\Delta f + \widetilde{a} \cdot \tau \cdot P\mathfrak{B}_{\mathrm{sym}}(f) = \sigma \cdot f$ for some $\sigma \in (0,\infty)$. (These assumptions are verified by the functions f from the space H_0' introduced in Theorem* 2.2.) *Let $s \in (1, 2]$.*

Then $f \in W^{2,s}(\mathbb{R}^3)^3$ and $\|f\|_{2,s} \leq \mathfrak{C}(s,\sigma) \cdot \|f\|_{2,2}$. In the case $s \leq 6/5$, the estimate $\|f\|_{2,s} \leq \mathfrak{C}(s,\sigma) \cdot \|f\|_2$ holds. Moreover, $\|\nabla f\|_2 \leq \mathfrak{C} \cdot \|f\|_2$.

The next theorem, which exploits Theorem 4.5, Corollary 4.6 and 4.7, as well as Theorem 5.2, is the principal tool in the proof of Theorem 2.3. It yields resolvent estimates for the operator L under the assumptions that the resolvent parameter λ is small and the right-hand side in the resolvent equation (4.1) belongs to the space H_0' from Theorem 2.2.

Theorem 5.3. *Let f and σ be given as in Theorem* 5.2. *Let $R \in (0,\infty)$, $\delta \in (0,1)$.*

Then there is $\epsilon_3 = \epsilon_3(\delta) = \epsilon_3(\tau, U, \sigma, \widetilde{a}, \delta) \in (0, \epsilon_1]$ (the constant ϵ_1 was introduced in Theorem 4.5) *such that for $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda \geq 0$, $|\lambda| \leq \epsilon_3$, the ensuing inequalities hold:*

$$\|\nabla(\lambda \cdot I - L)^{-1}(f)\|_2 \leq \mathfrak{C}(\sigma) \cdot \|f\|_2, \tag{5.1}$$

$$\|\nabla(\lambda \cdot I - L)^{-2}(f) \,|\, B_R\|_2 \leq \mathfrak{C}(\sigma,\delta,R) \cdot |\lambda|^{-\delta} \cdot \|f\|_2, \tag{5.2}$$

$$\begin{aligned}&\|\nabla(\overline{\lambda} \cdot I - L)^{-1} \circ (\lambda \cdot I - L)^{-1}(f) \,|\, B_R\|_2 \\ &\quad \leq \ \mathfrak{C}(\sigma,\delta,R) \cdot |\lambda|^{-\delta} \cdot \|f\|_2,\end{aligned} \tag{5.3}$$

$$\|\nabla(\lambda \cdot I - L)^{-3}(f) \,|\, B_R\|_2 \leq \mathfrak{C}(\sigma,\delta,R) \cdot |\lambda|^{-2-\delta} \cdot \|f\|_2. \tag{5.4}$$

Next we state a resolvent estimate valid for large values of $|\lambda|$.

Theorem 5.4. *There is $\widetilde{C} = \widetilde{C}(\tau, U, a, \vartheta, C_1) > 0$ such that for $\lambda \in S_{\vartheta,a}$ with $|\lambda| \geq \widetilde{C}$, and for $g \in H_2(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)^3$, the inequality*

$$|\lambda| \cdot \|\nabla(\lambda \cdot I - L)^{-1}(g)\|_2 \leq \mathfrak{C} \cdot \|\nabla g\|_2$$

is valid.

This theorem may be established by multiplying equation (4.1) by $-\Delta\overline{v}$, with v an abbreviation for $(\lambda \cdot I - L)^{-1}(g)$, and then integrating by parts.

For values of $|\lambda|$ which may be considered as neither large nor small, we exploit the continuity of the resolvent, to obtain

Lemma 5.5. *Let $\kappa_1, \kappa_2 \in (0, \infty)$ with $\kappa_1 < \kappa_2$. Put*

$$\mathfrak{M} := \{\, \lambda \in \mathbb{C} \,:\, \Re\lambda \geq 0,\ \kappa_1 \leq |\lambda| \leq \kappa_2 \,\}.$$

Then $\mathfrak{M} \subset \varrho(L)$ and $\|(\lambda \cdot I - L)^{-1}(w)\|_{1,2} \leq \mathfrak{C}(\kappa_1, \kappa_2) \cdot \|w\|_2$ for $w \in H_2(\mathbb{R}^3)$.

Note that the relation $\mathfrak{M} \subset \varrho(L)$ holds by (2.3) and assumption (C1).

6. Estimate of the semigroup e^{Lt}

Put $\overline{C} := \max\{\, \widetilde{C},\, \epsilon_3(1/16),\, 2^{-1/2},\, a,\, 2 \cdot a \cdot \tan(\pi - \vartheta) \,\}$, where $\widetilde{C}$ was introduced in Theorem 5.4, $\epsilon_3(1/16)$ in Theorem 5.3 (with $\delta = 1/16$), and a and ϑ in Theorem 2.1.

Since $\overline{C} \geq 2 \cdot a \cdot \tan(\pi - \vartheta)$ and $\overline{C} \geq a$, we may choose $\vartheta_0 \in (\pi/2, \vartheta)$ so close to $\pi/2$ that for any $s \in [\overline{C}, \infty)$, the relation

$$\{\, s \cdot e^{i \cdot \varphi} \,:\, \varphi \in [-\vartheta_0, \vartheta_0] \,\} \cup \{\, r \cdot e^{i \cdot \vartheta_0} \,:\, r \in [s, \infty) \,\} \subset S_{\vartheta, a} \tag{6.1}$$

holds. Let $\alpha, \beta \in (0, \infty)$ with $\alpha < \beta,\ \beta \geq \overline{C}$. Then we define the curves $\Gamma_1^{(\alpha,\beta)}, \dots, \Gamma_5^{(\alpha,\beta)} \subset \mathbb{C}$ by

$$\begin{aligned}
&\Gamma_1^{(\alpha,\beta)} := \{\, \alpha \cdot e^{i \cdot \varphi} \,:\, \varphi \in [-\pi/2, \pi/2] \,\},\ \ \Gamma_2^{(\alpha,\beta)} := \{\, i \cdot r \,:\, r \in [\alpha, \beta] \,\},\\
&\Gamma_3^{(\alpha,\beta)} := \{\, i \cdot \beta + r \cdot e^{i \cdot \vartheta} \,:\, r \in [0, \infty) \,\},\\
&\Gamma_i^{(\alpha,\beta)} := \{\, \overline{y} \,:\, y \in \Gamma_{i-2}^{(\alpha,\beta)} \,\} \ \text{ for } \ i \in \{4, 5\}.
\end{aligned}$$

Let $s \in [\overline{C}, \infty)$ and define

$$\begin{aligned}
&\Lambda_1^{(s)} := \{\, s \cdot e^{i \cdot \varphi} \,:\, \varphi \in [-\vartheta_0, \vartheta_0] \,\}, \quad \Lambda_2^{(s)} := \{\, r \cdot e^{i \cdot \vartheta_0} \,:\, r \in [s, \infty) \,\},\\
&\Lambda_3^{(s)} := \{\, \overline{y} \,:\, y \in \Lambda_2^{(s)} \,\}.
\end{aligned}$$

Then, in view of (6.1), (2.3), (C1) and the choice of $\overline{C}$, the curves $\Gamma_\nu^{(\alpha,\beta)}$ and $\Lambda_\mu^{(s)}$ are contained in $\varrho(L)$ $(1 \leq \nu \leq 5,\ 1 \leq \mu \leq 3)$, and we have by [10, Theorem 1.3.4], for $t \in (0, \infty),\ w \in H_2(\mathbb{R}^3)$:

$$\begin{aligned}
e^{Lt}(w) &= (2 \cdot \pi \cdot i)^{-1} \cdot \sum_{\nu=1}^{5} \int_{\Gamma_\nu^{(\alpha,\beta)}} e^{\lambda \cdot t} \cdot (\lambda \cdot I - L)^{-1}(w)\, d\lambda \qquad (6.2)\\
&= (2 \cdot \pi \cdot i)^{-1} \cdot \sum_{\mu=1}^{3} \int_{\Lambda_\mu^{(s)}} e^{\lambda \cdot t} \cdot (\lambda \cdot I - L)^{-1}(w)\, d\lambda.
\end{aligned}$$

A remark is perhaps in order with respect to the difficulties we have to face in this section. In Theorem 2.3, it is claimed that for large t, the term $\|\nabla e^{Lt}(f) \,|\, B_R\|_2$ is bounded by $t^{-1-\epsilon} \cdot \|f\|_2$, for some $\epsilon > 0$, times a factor independent of t and f. (Incidentally we chose $\epsilon = 1/8$, but this is only for definiteness.) We will obtain such an estimate by considering the first sum on the right-hand side of (6.2). This means in particular that we have to show that

$$\Big\| \int_{\Gamma_1^{(\alpha,\beta)}} e^{\lambda\cdot t} \cdot \nabla(\lambda\cdot I - L)^{-1}(f)\,|\,B_R\,d\lambda \Big\|_2 \le \mathfrak{C}(R,\vartheta,\sigma)\cdot \|f\|_2 \cdot t^{-1-\epsilon}.$$

In view of (5.1), this should require $\alpha \le t^{-1-\epsilon}$. On other hand, in order to produce a factor $t^{-\mu}$ for some $\mu > 0$ in the estimate of $\int_{\Gamma_\nu^{(\alpha,\beta)}} e^{\lambda\cdot t}\cdot\nabla(\lambda\cdot I - L)^{-1}(f)\,|\,B_R\,d\lambda$ for $\nu = 2$ and $\nu = 4$, we integrate by parts after introducing the local parameter $\varphi(r) := i\cdot r$ ($r \in [\alpha,\beta]$), so that the factor $e^{i\cdot r\cdot t}$ is transformed into $e^{i\cdot r\cdot t}\cdot(i\cdot t)^{-1}$. But this means that a single partial integration does not suffice to generate a factor $t^{-1-\epsilon}$. On the other hand, after two such integrations, we obtain a term $\nabla(i\cdot r\cdot I - L)^{-3}(f)\,|\,B_R$, which gives rise to a factor $r^{-2-\delta}$ for some $\delta > 0$ (see (5.4)). Integrating this term on the interval $[\alpha,\beta]$ leads to a factor $\alpha^{-1-\delta} = t^{(1+\epsilon)(1+\delta)}$ which cancels the effect of the second partial integration. Therefore, in view of the fact that the term $\nabla(i\cdot r\cdot I - L)^{-2}(f)\,|\,B_R$ only produces a factor $r^{-\delta}$ (see (5.2)), we perform some kind of interpolation between one and two partial integrations. To this end, we use fractional derivatives, as introduced in the next lemma.

Lemma 6.1. *Let $\kappa, b \in \mathbb{R}$ with $\kappa < b$, $\mu \in (0,1)$, $h \in C^1([\kappa,b])$ with $h(b) = 0$. Define $\overline{h} : [\kappa,b] \mapsto \mathbb{C}$ by*

$$\overline{h}(r) := \Gamma(1-\mu)^{-1}\cdot \int_r^b (s-r)^{-1+\mu}\cdot h(s)\,ds \quad \textit{for } r \in [\kappa,b].$$

Then $\overline{h} \in C^1([\kappa,b])$ with

$$\overline{h}'(r) = \Gamma(1-\mu)^{-1}\cdot \int_r^b (\alpha - r)^{-1+\mu}\cdot h'(\alpha)\,d\alpha \quad \textit{for } r \in [\kappa,b]. \tag{6.3}$$

Define $\gamma : [\kappa,b] \ni r \mapsto \Gamma(\mu)^{-1}\cdot\int_r^b (s-r)^{-\mu}\cdot \overline{h}'(s)\,ds \in \mathbb{C}$. Then $h = -\gamma$.

Now we prove an inequality which will be the key element in the estimate of the integrals over $\Gamma_2^{(\alpha,\beta)}$ and $\Gamma_4^{(\alpha,\beta)}$ in (6.2).

Lemma 6.2. *Let f and σ be given as in Theorem 5.2. Let $R \in (0,\infty)$, $\delta \in (0, 1/4)$. Abbreviate $b := \min\{\, 2^{-1/2},\, \epsilon_3(\delta)\,\}$, with $\epsilon_3(\delta)$ from Theorem 5.3. Let $\kappa \in (0,b)$, $t \in (0,\infty)$. Then*

$$\Big\| \int_\kappa^b e^{i\cdot r\cdot t}\cdot \nabla(i\cdot r\cdot I - L)^{-2}(f)\,|\,B_R\,dr \Big\|_2 \le \mathfrak{C}(\sigma,\delta,R)\cdot t^{-1/4}\cdot\kappa^{-\delta}\cdot\|f\|_2.$$

To give some indications on the proof of this lemma, we first observe that by (2.3) and (C1), we have $\{\, i\cdot r \,:\, r \in [\kappa,b]\,\} \subset \varrho(L)$. Therefore the mapping

$$g : [\kappa,b] \ni r \mapsto \nabla(i\cdot r\cdot I - L)^{-1}(f)\,|\,B_R \in L^2(B_R)^9$$

is in particular twice continuously differentiable, with

$$g^{(\nu)}(r) = (-i)^\nu\cdot\nu\cdot\nabla(i\cdot r\cdot I - L)^{-(\nu+1)}(f)\,|\,B_R \quad \text{for } \nu \in \{1,\,2\},\ r \in [\kappa,b].$$

Thus, due to the assumption $b \le \epsilon_3(\delta)$, inequalities (5.2) and (5.4) yield

$$\|g'(r)\|_2 \le \mathfrak{C}(\sigma,\delta,R)\cdot\|f\|_2\cdot r^{-\delta}, \quad \|g''(r)\|_2 \le \mathfrak{C}(\sigma,\delta,R)\cdot\|f\|_2\cdot r^{-2-\delta} \tag{6.4}$$

for $r \in [\kappa, b]$. Put $h(r) := (i\cdot t)^{-1}\cdot(e^{i\cdot r\cdot t} - e^{i\cdot b\cdot t})$ for $r \in [\kappa, b]$. Define $\overline{h}$ and γ as in Lemma 6.1, with $\mu = 1/4$. Then we get by partial integration

$$\begin{aligned}\int_\kappa^b e^{i\cdot r\cdot t}\cdot\nabla(i\cdot r\cdot I - L)^{-2}(f)\,|\,B_R\,dr &= -i\cdot\int_\kappa^b \gamma'(r)\cdot g'(r)\,dr \\ &= \ i\cdot\Gamma(1/4)^{-1}\cdot\int_\kappa^b \overline{h}'(s)\cdot\Big(\int_\kappa^s (s-r)^{-1/4}\cdot g''(r)\,dr\Big)\,ds \\ &\quad +\ i\cdot\Gamma(1/4)^{-1}\cdot\int_\kappa^b (s-\kappa)^{-1/4}\cdot\overline{h}'(s)\,ds\cdot g'(\kappa).\end{aligned} \tag{6.5}$$

Take $s \in (\kappa, b)$. If $s - s^3 > \kappa$, we have

$$\begin{aligned}&\int_\kappa^s (s-r)^{-1/4}\cdot g''(r)\,dr \\ &\quad = \int_{s-s^3}^s (s-r)^{-1/4}\cdot g''(r)\,dr - (1/4)\cdot\int_\kappa^{s-s^3} (s-r)^{-5/4}\cdot g'(r)\,dr \\ &\qquad + s^{-3/4}\cdot g'(s-s^3) - (s-\kappa)^{-1/4}\cdot g'(\kappa).\end{aligned}$$

Further observe that $s \le b \le 2^{-1/2}$, hence $s^3 \le s/2$. Now we find with (6.4), in the case $s - s^3 > \kappa$,

$$\begin{aligned}&\Big\|\int_\kappa^s (s-r)^{-1/4}\cdot g''(r)\,dr\Big\|_2 \\ &\le\ \mathfrak{C}(\sigma,\delta,R)\cdot\|f\|_2\cdot\Big(\int_{s-s^3}^s (s-r)^{-1/4}\cdot r^{-2-\delta}\,dr \\ &\quad +\ \int_\kappa^{s-s^3} (s-r)^{-5/4}\cdot r^{-\delta}\,dr\ +\ s^{-3/4}\cdot(s-s^3)^{-\delta} + (s-\kappa)^{-1/4}\cdot\kappa^{-\delta}\Big) \\ &\le \mathfrak{C}(\sigma,\delta,R)\cdot\|f\|_2\cdot\Big(s^{-2-\delta}\cdot\int_{s-s^3}^s (s-r)^{-1/4}\,dr\ + \kappa^{-\delta}\cdot\int_\kappa^{s-s^3} (s-r)^{-5/4}\,dr \\ &\quad + s^{-3/4}\cdot\kappa^{-\delta} + (s-\kappa)^{-1/4}\cdot\kappa^{-\delta}\Big) \\ &\le \mathfrak{C}(\sigma,\delta,R)\cdot\|f\|_2\cdot(s^{1/4-\delta} + \kappa^{-\delta}\cdot s^{-3/4} + (s-\kappa)^{-1/4}\cdot\kappa^{-\delta}),\end{aligned} \tag{6.6}$$

where we used the inequality $s^3 \le s/2$ and the assumption $s - s^3 > \kappa$ in the last but one inequality. If $s - s^3 \le \kappa$, we argue as follows, again using (6.4),

$$\begin{aligned}&\Big\|\int_\kappa^s (s-r)^{-1/4}\cdot g''(r)\,dr\Big\|_2\ \le\ \mathfrak{C}(\sigma,\delta,R)\cdot\|f\|_2\cdot\int_\kappa^s (s-r)^{-1/4}\cdot r^{-2-\delta}\,dr \\ &\le\ \mathfrak{C}(\sigma,\delta,R)\cdot\|f\|_2\cdot\kappa^{-2-\delta}\cdot\int_{s-s^3}^s (s-r)^{-1/4}\,ds \\ &\le\ \mathfrak{C}(\sigma,\delta,R)\cdot\|f\|_2\cdot\kappa^{-2-\delta}\cdot s^{9/4} \\ &\le\ \mathfrak{C}(\sigma,\delta,R)\cdot\|f\|_2\cdot\kappa^{1/4-\delta}\ \le\ \mathfrak{C}(\sigma,\delta,R)\cdot\|f\|_2\cdot s^{1/4-\delta},\end{aligned}$$

where the last but one inequality holds because $s^3 \leq s/2$ (see above), so that $s \leq s^3 + \kappa \leq s/2 + \kappa$, hence $s \leq 2 \cdot \kappa$. Therefore we see that inequality (6.6) holds in any case. Starting from (6.5), and applying (6.6), we now find

$$\begin{aligned}
&\Big\| \int_\kappa^b e^{i \cdot r \cdot t} \cdot \nabla(i \cdot r \cdot I - L)^{-2}(f) \,|\, B_R \, dr \Big\|_2 \qquad (6.7)\\
&\leq \mathfrak{C}(\sigma, \delta, R) \cdot \max\{|\overline{h}'(s)| \,:\, s \in [\kappa, b]\}\\
&\quad \cdot \Big(\int_\kappa^b \Big\| \int_\kappa^s (s-r)^{-1/4} \cdot g''(r)\, dr \Big\|_2 \, ds \,+\, \|f\|_2 \cdot \kappa^{-\delta} \cdot \int_\kappa^b (s-\kappa)^{-1/4}\, ds \Big)\\
&\leq \mathfrak{C}(\sigma, \delta, R) \cdot \max\{|\overline{h}'(s)| \,:\, s \in [\kappa, b]\} \cdot \|f\|_2\\
&\quad \cdot \Big(\int_\kappa^b \big(s^{1/4-\delta} + \kappa^{-\delta} \cdot s^{-3/4} + \kappa^{-\delta} \cdot (s-\kappa)^{-1/4} \big)\, ds + \kappa^{-\delta} \Big)\\
&\leq \mathfrak{C}(\sigma, \delta, R) \cdot \max\{|\overline{h}'(s)| \,:\, s \in [\kappa, b]\} \cdot \|f\|_2 \cdot \kappa^{-\delta}.
\end{aligned}$$

This leaves us to consider $|\overline{h}'(s)|$, for $s \in [\kappa, b]$. In this respect, we observe that $|h(s)| \leq 2 \cdot t^{-1}$, $|h'(s)| \leq 2$ for $s \in [\kappa, b]$. Thus, in the case $s + 1/t < b$, by (6.3) and a partial integration,

$$\begin{aligned}
|\overline{h}'(s)| \;&=\; \Gamma(3/4)^{-1} \cdot \Big| \int_s^{s+1/t} (\alpha - s)^{-3/4} \cdot h'(\alpha)\, d\alpha\\
&\quad + (3/4) \cdot \int_{s+1/t}^b (\alpha - s)^{-7/4} \cdot h(\alpha)\, d\alpha \;-\; t^{3/4} \cdot h(s + 1/t) \Big|\\
&\leq \mathfrak{C} \cdot \Big(\int_s^{s+1/t} (\alpha - s)^{-3/4}\, d\alpha + \int_{s+1/t}^b (\alpha - s)^{-7/4}\, d\alpha \cdot t^{-1} + t^{-1/4} \Big)\\
&\leq \mathfrak{C} \cdot t^{-1/4}.
\end{aligned}$$

If $s \in [\kappa, b]$ with $s + 1/t \geq b$, we deduce from (6.3) and the inequality $|h'(s)| \leq 2$ for $s \in [\kappa, b]$ that $|\overline{h}'(s)| \leq \mathfrak{C} \cdot (b-s)^{1/4} \leq \mathfrak{C} \cdot t^{-1/4}$. The estimate $|\overline{h}'(s)| \leq \mathfrak{C} \cdot t^{-1/4}$ thus holds in any case. When we insert this estimate into (6.7), we obtain the inequality stated in the lemma. ◇

Lemma 6.2 enters into the proof of

Theorem 6.3. *Let* $R \in (0, \infty)$, $t \in [\max\{\epsilon_3(1/16)^{-1}, 2^{1/2}\}, \infty)$, *with* $\epsilon_3(1/16)$ *from Theorem* 5.3 *with* $\delta = 1/16$. *Let* f, σ *be given as in Theorem* 5.2. *Then*

$$\|\nabla e^{Lt}(f) \,|\, B_R\|_2 \leq \mathfrak{C}(\sigma, R, \vartheta) \cdot \|f\|_2 \cdot t^{-9/8}.$$

Let us give some indications on the proof of this theorem. We consider the first sum in (6.2), with $\alpha = t^{-2}$ (hence $\alpha = t^{-2} \leq t^{-1} \leq \min\{2^{-1/2}, \epsilon_3(1/16)\}$) and $\beta \in [\overline{C}, \infty)$, where $\overline{C}$ was introduced at the beginning of this section. Then the gradient of the integral over $\Gamma_1^{(\alpha,\beta)}$ in (6.2) may be estimated in the L^2-norm on B_R by a constant times $t^{-2} \cdot \|f\|_2$, as follows from (5.1).

The same norm of the gradient of the integrals over $\Gamma_3^{(\alpha,\beta)}$ and $\Gamma_5^{(\alpha,\beta)}$ is evaluated by referring to Theorem 5.4 and to the last inequality in Theorem 5.2. (Recall that $\beta \geq \overline{C} \geq \widetilde{C}$.) We obtain the upper bound $\mathfrak{C}(\sigma) \cdot (\beta \cdot t)^{-1} \cdot \|f\|_2$. This leaves us to consider the integrals over $\Gamma_2^{(\alpha,\beta)}$ and $\Gamma_4^{(\alpha,\beta)}$. In this respect, we observe that after a partial integration,

$$\int_{\Gamma_2^{(\alpha,\beta)}} e^{\lambda \cdot t} \cdot \nabla(\lambda \cdot I - L)^{-1}(f) \,|\, B_R \, d\lambda = \sum_{j=1}^{5} N_j,$$

where

$$\begin{aligned}
N_1 &:= t^{-1} \cdot e^{i \cdot t \cdot \beta} \cdot \nabla(i \cdot \beta \cdot I - L)^{-1}(f) \,|\, B_R, \\
N_2 &:= -t^{-1} \cdot e^{i \cdot t \cdot \alpha} \cdot \nabla(i \cdot \alpha \cdot I - L)^{-1}(f) \,|\, B_R, \\
N_3 &:= (i/t) \cdot \int_\alpha^b e^{i \cdot t \cdot r} \cdot \nabla(i \cdot r \cdot I - L)^{-2}(f) \,|\, B_R \, dr, \\
N_4 &:= (i/t) \cdot \int_b^\beta e^{i \cdot t \cdot r} \cdot \nabla(i \cdot r \cdot I - L)^{-2}(f) \,|\, B_R \, dr,
\end{aligned}$$

with $b := \min\{\epsilon_3(1/16),\, 2^{-1/2}\}$. Note that $b \leq \overline{C} \leq \beta$. The integral over $\Gamma_4^{(\alpha,\beta)}$ is split into a sum $\sum_{j=1}^{5} \overline{N}_j$, where $\overline{N}_j$ is defined in an analogous way as N_j ($1 \leq j \leq 5$). Now the terms $\|N_1\|_2$ and $\|\overline{N}_1\|_2$ are estimated by Theorem 5.4 and 5.2; we obtain the upper bound $\mathfrak{C}(\sigma) \cdot \beta^{-1} \cdot \|f\|_2$. Moreover, the resolvent formula and (5.3) yield

$$\|N_2 + \overline{N}_2\|_2 \leq \mathfrak{C} \cdot (\alpha + \alpha^{15/16} \cdot t^{-1}) \cdot \|f\|_2.$$

Concerning N_3 and $\overline{N}_3$, we get by Lemma 6.2:

$$\|N_3\|_2 + \|\overline{N}_3\|_2 \leq \mathfrak{C} \cdot t^{-5/4} \cdot \alpha^{-1/16} \cdot \|f\|_2.$$

Finally, in the integrals defining N_4 and $\overline{N_4}$, we perform another partial integration in order to generate an additional term t^{-1}. The term $\|\nabla(i \cdot r \cdot I - L)^{-3}(f) \,|\, B_R\|_2$ arising in this way is evaluated for $r \in [b, \overline{C}]$ by referring to Lemma 5.5, and for $r \in [\overline{C}, \beta]$ by applying Theorem 5.4 and 5.2. Combining all these estimates and letting β tend to infinity, we arrive at Theorem 6.3

Theorem 6.3 dealt with the case of large t. This leaves us to consider small and intermediate values of t. To this end, we use the representation of $e^{Lt}(\Phi)$ by the second sum in (6.2), with

$$s = \overline{C} \ \text{ if } \ t \in [\overline{C}^{-1},\, \max\{\epsilon_3(1/16)^{-1},\, 2^{1/2}\}\,], \quad s = 1/t \ \text{ if } \ t \in (0, \overline{C}^{-1}].$$

By referring to Theorem 5.4 and 5.2, we then obtain

Theorem 6.4. *Let* $t \in (0, \max\{\epsilon_3(1/16)^{-1},\, 2^{1/2}\}\,]$*, and let* f, σ *be given as in Theorem* 5.2*. Then* $\|\nabla e^{Lt}(f)\|_2 \leq \mathfrak{C}(\sigma) \cdot \|f\|_2$.

Combining Theorem 6.3 and 6.4 yields Theorem 2.3.

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Abstract Delay Equations Inspired by Population Dynamics

Odo Diekmann and Mats Gyllenberg

To the memory of Günter Lumer, a source of inspiration to both of us.

Abstract. In this short note we show that delay equations can be reformulated as abstract weak*-integral equations (AIE) involving dual semigroups, even in the case of infinite delay and/or when the solution takes values in a non-reflexive Banach space. The advantage is that for such (AIE) the standard local stability and bifurcation results are already available, see [8]. Our motivation derives from models of physiologically structured populations, as explained in more detail in [12].

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1. Introduction

The perturbation theory for dual semigroups, as developed in the series [3, 4, 5, 6, 7] of papers, turned out to be a very powerful tool in the local stability and bifurcation theory of delay differential equations (DDE) [8]. The key step is the reformulation of the initial value problem for the DDE as an abstract integral equation

$$u(t) = T_0(t)\varphi + j^{-1}\left(\int_0^t T_0^{\odot *}(t-s)G(u(s))ds\right). \tag{AIE}$$

Here T_0 is a strongly continuous semigroup of bounded linear operators on a Banach space X with sun-dual space $X^\odot$ (the subspace of the dual space X^* on

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which the adjoint (or dual) semigroup T_0^* is strongly continuous), $T_0^{\odot *}$ is the adjoint semigroup of $T_0^{\odot} := (T_0^*)_{|X^{\odot}}$, G is a nonlinear mapping from X into $X^{\odot *}$ and j is the natural injection of X into $X^{\odot *}$ defined by

$$\langle \varphi^{\odot}, j\varphi \rangle = \langle \varphi, \varphi^{\odot} \rangle, \quad \varphi \in X,\ \varphi^{\odot} \in X^{\odot}. \tag{1.1}$$

We refer to [2, 3, 8, 18] for more background information about dual semigroups.

Recently, it has been shown [12] that the sun-star-calculus based on (AIE) is equally efficient for treating delay equations (DE) which are functional equations of Volterra type prescribing the value of the function itself in the right end point, rather than the value of its derivative. The only real difference between the treatment of (DDE) and of (DE) is the choice of the underlying function space.

In order for (AIE) to make sense, the convolution integral (which by definition is a weak*-Riemann integral on $X^{\odot *}$) should take values in $j(X)$. It is known [3] that it takes values in $X^{\odot\odot}$. So whenever X is sun-reflexive, that is, whenever $j(X) = X^{\odot\odot}$, this is automatically guaranteed.

The theory developed in [3, 4, 5, 8] concentrates on the sun-reflexive case. As a consequence, the application to delay equations requires a finite delay and that the functions take values in a reflexive space. The aim of the present note is to show that delay equations with infinite delay and involving functions that take values in arbitrary Banach spaces can still be written in the form of an abstract integral equation of the form (AIE). Because in the non-sun-reflexive case the convolution integral in (AIE) need not belong to $j(X)$, we have instead to impose a range condition that for functions f taking values in an appropriate subspace of $X^{\odot *}$, which contains the range of the function G, it is true that

$$\int_0^t T_0^{\odot *}(t-s) f(s) ds \in j(X). \tag{1.2}$$

It turns out that for delay equations it is easy to verify by direct computation that (1.2) holds. Once (AIE) is justified, the methods and results of [8, 12] become available and one obtains the principle of linearized stability, the centre manifold theorem and the Hopf bifurcation theorem essentially for free ('essentially', because the spectral analysis of $T(t)$ is a bit more complicated in the case of infinite delay).

It was already noted in [3, 5, 8] that (AIE) also covers age-dependent population models. More recently, Hans Metz and the present authors found a way to formulate population models that incorporate more general physiological structure (e.g., size structure) as abstract integral equations of type (AIE). This formulation employs delay equations (DE) which do not involve any derivative. In a recent joint work with Philipp Getto [12] we elaborated the details of the reformulation as an (AIE) and its analysis in an L^1-setting, assuming sun-reflexivity. In the present paper we consider the same setting, but we neither impose an upper bound on the delay (i.e., on the maximal attainable age), nor assume that the number of possible states-at-birth is finite. Our results also allow the so-called *interaction variables* [9, 10, 11] to take values in an infinite-dimensional space.

As a general reference concerning (DDE) with infinite delay we mention [16], while for (DDE) in infinite-dimensional spaces we refer to [1], [14, Ch. VI.6] and [21].

2. The abstract setting

Let Y be a Banach space and let $\varrho \geq 0$. As the state space we choose the space $X = L^1(\mathbf{R}_-; Y)$ of all measurable functions $\varphi : \mathbf{R}_- = (-\infty, 0] \to Y$ such that the weighted Bochner integral

$$\|\varphi\|_1 = \int_{\mathbf{R}_-} e^{\varrho\theta} \|\varphi(\theta)\| \, d\theta \tag{2.1}$$

is finite. On X we consider the strongly continuous semigroup T_0 defined by translation and extension by zero:

$$(T_0(t)\varphi)(\theta) = \begin{cases} \varphi(t+\theta), & -\infty < \theta \leq -t, \\ 0, & -t < \theta \leq 0, \end{cases} \qquad \varphi \in X, \quad t \geq 0. \tag{2.2}$$

The reason that we chose $X = L^1(\mathbf{R}_-; Y)$ and not a space of continuous functions as state space is that in applications to delay equations the semigroup T_0 occurs and it does not leave the continuous functions invariant. It is also the right choice for our biological applications, which is not the case of L^p, $1 < p < \infty$, which from a purely mathematical point of view could have been used.

It does not seem possible to give the dual space X^* a representation in terms of familiar functions or measures unless Y^* has the Radon-Nikodym property, in which case X^* is isometrically isomorphic to $L^\infty(\mathbf{R}_+; Y^*)$ [13, Theorem 1, p. 98]. And for the function spaces Y that most frequently occur in our applications, viz. C and L^1, the dual space Y^* does *not* possess the Radon-Nikodym property. However, this is no problem because Greiner and van Neerven [15] (see also [18, Theorem 7.3.11, p. 135]) have characterized the sun-dual $X^\odot = L^1(\mathbf{R}_-; Y)^\odot$ with respect to the translation semigroup (2.2).

Proposition 2.1. *Let Y be a Banach space and let the semigroup T_0 be defined on $X = L^1(\mathbf{R}_-; Y)$ by (2.2). Then $X^\odot$ is isometrically isomorphic to the space of all functions $\varphi^\odot : \mathbf{R}_+ \to Y^*$ such that $\theta \mapsto e^{\rho\theta}\varphi^\odot(\theta)$ is bounded and uniformly continuous with the norm*

$$\left\|\varphi^\odot\right\|_\infty = \sup_{\theta\in\mathbf{R}_+} e^{\varrho\theta} \left\|\varphi^\odot(\theta)\right\| < \infty \tag{2.3}$$

and the pairing

$$\left\langle \varphi, \varphi^\odot \right\rangle = \int_{\mathbf{R}_+} \left\langle \varphi(-\theta), \varphi^\odot(\theta) \right\rangle d\theta. \tag{2.4}$$

The sun-dual semigroup $T^\odot$ is given by

$$\left(T_0^\odot(t)\varphi^\odot\right)(\theta) = \varphi^\odot(t+\theta), \quad 0 \leq \theta < \infty, \quad \varphi^\odot \in X^\odot, \quad t \geq 0. \tag{2.5}$$

Note that on the right-hand side of (2.4) we have the duality pairing between the spaces Y and Y^*.

Proof. In [15] and [18] it was proven (without weights, $\varrho = 0$) that $L^1(\mathbf{R}; Y)^\odot = BUC(\mathbf{R}; Y^*)$ with respect to the translation semigroup on the whole real line. The proof for the half-line case is identical. Because we work on weighted spaces, the exponential weight enters in the characterization of $X^\odot$. □

Because by Proposition 2.1 the elements of $X^\odot$ are represented by continuous functions, we can unambiguously talk about the value $\varphi^\odot(\theta)$, $\theta \in \mathbf{R}_+$, of any element $\varphi^\odot \in X^\odot$. In particular, the *evaluation-in-zero map* $\delta : X^\odot \to Y^*$ is well defined through

$$\delta\varphi^\odot = \varphi^\odot(0), \quad \varphi^\odot \in X^\odot. \tag{2.6}$$

The adjoint δ^* of δ maps Y^{**} into $X^{\odot *}$. By restricting δ^* to Y (using the canonical embedding of a Banach space into its second dual) we obtain a linear mapping $\ell : Y \to X^{\odot *}$. Explicitly, it is defined via

$$\left\langle \varphi^\odot, \ell y \right\rangle = \left\langle y, \varphi^\odot(0) \right\rangle, \quad y \in Y, \quad \varphi^\odot \in X^\odot. \tag{2.7}$$

Obviously, ℓ is an isometric isomorphism of Y onto a closed subspace of $X^{\odot *}$.

Lemma 2.2. *For every $y \in Y$ and $\varphi^\odot \in X^\odot$ one has*

$$\left\langle T_0^{\odot *}(t)\ell y, \varphi^\odot \right\rangle = \left\langle y, \varphi^\odot(t) \right\rangle, \quad t \geq 0.$$

Proof. $\left\langle T_0^{\odot *}(t)\ell y, \varphi^\odot \right\rangle = \left\langle \ell y, T_0^\odot(t)\varphi^\odot \right\rangle = \left\langle y, \left(T_0^\odot(t)\varphi^\odot\right)(0) \right\rangle = \langle y, \varphi^\odot(t) \rangle .$ □

Lemma 2.3. *Let $h : \mathbf{R}_+ \to Y$ be a continuous function. Then, for every $\varphi^\odot \in X^\odot$ one has*

$$\left\langle \int_0^t T_0^{\odot *}(t-\tau)\ell h(\tau)\, d\tau, \varphi^\odot \right\rangle = \int_0^t \left\langle h(t-\tau), \varphi^\odot(\tau) \right\rangle d\tau, \quad t \geq 0.$$

Proof. Using Lemma 2.2 one gets

$$\left\langle \int_0^t T_0^{\odot *}(t-\tau)\ell h(\tau)\, d\tau, \varphi^\odot \right\rangle = \int_0^t \left\langle T_0^{\odot *}(t-\tau)\ell h(\tau), \varphi^\odot \right\rangle d\tau =$$
$$\int_0^t \left\langle \ell h(\tau), T_0^\odot(t-\tau)\varphi^\odot \right\rangle d\tau = \int_0^t \left\langle h(\tau), \varphi^\odot(t-\tau) \right\rangle d\tau$$
$$\int_0^t \left\langle h(t-\tau), \varphi^\odot(\tau) \right\rangle d\tau.$$ □

As a corollary, we get the result alluded to in the introduction: the convolution integral $\int_0^t T_0^{\odot *}(t-\tau) f(\tau)\, d\tau$ belongs to $j(X)$, whenever $f : \mathbf{R}_+ \to X^{\odot *}$ is continuous with values in $\ell(Y)$.

Corollary 2.4. *Let $h : \mathbf{R}_+ \to Y$ be a continuous function and define $\varphi \in X = L^1(\mathbf{R}_-; Y)$ by*

$$\varphi(\theta) = \begin{cases} h(t+\theta) & -t \leq \theta \leq 0, \\ 0, & -\infty < \theta < -t. \end{cases} \tag{2.8}$$

Then

$$\int_0^t T_0^{\odot *}(t-\tau)\ell h(\tau)\, d\tau = j\varphi. \tag{2.9}$$

In particular, $\int_0^t T_0^{\odot *}(t-\tau)\ell h(\tau)\, d\tau \in j(X)$ *and*

$$\left\| j^{-1}\left(\int_0^t T_0^{\odot *}(t-\tau)\ell h(\tau)\, d\tau\right)\right\|_1 \le \frac{1}{\varrho}\left(1-e^{-\varrho t}\right) \sup_{0\le\tau\le t} \|h(\tau)\|, \quad t\ge 0. \tag{2.10}$$

(If $\varrho = 0$*, the factor* $\left(1-e^{-\varrho t}\right)/\varrho$ *has to be interpreted as the limiting value* t*.)*

Proof. For each $\varphi^{\odot} \in X^{\odot}$ we have by the definition of φ and Lemma 2.3:

$$\left\langle \varphi, \varphi^{\odot}\right\rangle = \int_{-\infty}^0 \left\langle \varphi(\theta), \varphi^{\odot}(-\theta)\right\rangle\, d\theta = \int_{-t}^0 \left\langle h(t+\theta), \varphi^{\odot}(-\theta)\right\rangle\, d\theta$$
$$\int_0^t \left\langle h(t-\theta), \varphi^{\odot}(\theta)\right\rangle\, d\theta = \left\langle \int_0^t T_0^{\odot *}(t-\tau)\ell h(\tau)\, d\tau, \varphi^{\odot}\right\rangle.$$

The definition (1.1) of the embedding $j : X \to X^{\odot *}$ now yields (2.9). The estimate (2.10) follows readily:

$$\left\| j^{-1}\left(\int_0^t T_0^{\odot *}(t-\tau)\ell h(\tau)\, d\tau\right)\right\|_1 = \|\varphi\|_1 = \int_{-\infty}^0 e^{\rho\theta}\, \|\varphi(\theta)\|\, d\theta =$$
$$\int_{-t}^0 e^{\varrho\theta}\, \|h(t+\theta)\|\, d\theta = e^{-\varrho t}\int_0^t e^{\varrho\tau}\, \|h(\tau)\|\, d\tau \le \frac{1}{\varrho}\left(1-e^{-\varrho t}\right) \sup_{0\le\tau\le t} \|h(\tau)\|. \quad \square$$

Theorem 2.5. *Let* $F : X \to Y$ *be Lipschitz continuous. Then the abstract integral equation*

$$u(t) = T_0(t)\varphi + j^{-1}\left(\int_0^t T_0^{\odot *}(t-s)G(u(s))ds\right), \tag{AIE}$$

with T_0 *defined by* (2.2) *and* $G = \ell \circ F$*, has a unique solution on* $[0,\infty)$.

Proof. With Corollary 2.4 at hand, the proof is identical to the proof of the corresponding result in the sun-reflexive case [5, 8]. □

Next we consider steady states of the dynamical system $\Sigma(t)$ induced by (AIE) by declaring $\Sigma(t)\varphi$ to be the solution $u(t)$ of (AIE). We now realize why we have to use *weighted* L^1-spaces: Without a weight, nonzero constant functions on an infinite interval do not belong to L^1. Linearization around a steady state works exactly as in the sun-reflexive case [5]:

Theorem 2.6. *Let* $\Sigma(t)\overline{\varphi} = \overline{\varphi}$ *and assume that the nonlinear operator* $F : X \to Y$ *is continuously Fréchet differentiable. Then for every* $t > 0$ *the nonlinear operator* $\Sigma(t)$ *is Fréchet differentiable at* $\overline{\varphi}$*. Its Fréchet derivative*

$$T(t) = (D\Sigma(t))(\overline{\varphi}) \tag{2.11}$$

defines a strongly continuous semigroup of bounded linear operators with generator A given by

$$\begin{aligned}\mathcal{D}(A) &= \{\varphi \in X : j\varphi \in \mathcal{D}(A_0^{\odot *}), A_0^{\odot *} j\varphi + \ell F'(\overline{\varphi})\varphi \in j(X)\},\\ A\varphi &= j^{-1}(A_0^{\odot *} j\varphi + \ell F'(\overline{\varphi})\varphi).\end{aligned}$$

Moreover, for every $\varphi \in X$, $T(t)\varphi$ is the unique solution of the linear abstract integral equation

$$T(t)\varphi = T_0(t)\varphi + j^{-1}\left(\int_0^t T_0^{\odot *}(t-s)\ell F'(\overline{\varphi})T(s)\varphi ds\right). \tag{LAIE}$$

The proofs of the principle of linearized stability, the centre manifold theorem and the Hopf bifurcation theorem depend essentially on the linearization described in Theorem 2.6.

3. Delay equations as abstract integral equations

We consider the initial value problem

$$x(t) = F(x_t), \quad t > 0 \tag{DE}$$

$$x_0(\theta) = \varphi(\theta), \quad \theta \in (-\infty, 0], \tag{IC}$$

consisting of a delay equation (DE) specifying the rule for extending the unknown function x from the history given by (IC). Here the unknown function x takes values in a Banach space Y and x_t denotes for each $t \geq 0$ the translated function defined by

$$x_t(\theta) := x(t+\theta), \quad -\infty < t \leq 0. \tag{3.1}$$

As state space (history space) we choose the space $X = L^1(\mathbf{R}_-; Y)$ of Bochner integrable (with respect to the weight function $\theta \mapsto e^{\rho\theta}$) functions on $\mathbf{R}_-$, see Section 2. We therefore assume that F maps X into Y and that the initial value φ belongs to X. In this section we show that the problem (DE) & (IC) is equivalent to (AIE) with $G = \ell \circ F$ and T_0 defined by (2.2). We shall always assume that T_0 and G are chosen in this way.

An application of Corollary 2.4 to the function $h = F \circ u$ for a continuous function $u : \mathbf{R}_+ \to X$ immediately gives the following result:

Lemma 3.1.

$$\left(j^{-1}\int_0^t T_0^{\odot *}(t-s)\ell F(u(s))ds\right)(\theta) = \begin{cases} F(u(t+\theta)), & -t \leq \theta \leq 0,\\ 0, & -\infty < \theta < -t.\end{cases}$$

We are now ready to state and prove the equivalence of (DE) & (IC) and (AIE).

Theorem 3.2. *Let $\varphi \in X = L^1(\mathbf{R}_-; Y)$ be given.*

(a) *Suppose that $x \in L^1_{\text{loc}}((-\infty,\infty); Y)$ satisfies* (DE) *&* (IC). *Then the function $u : [0,\infty) \to X$ defined by $u(t) := x_t$ is continuous and satisfies* (AIE).

(b) *If $u : [0,\infty) \to X$ is continuous and satisfies* (AIE), *then the function x defined by*

$$x(t) := \begin{cases} \varphi(t) & \text{for } -\infty < t < 0, \\ u(t)(0) & \text{for } t \geq 0 \end{cases} \tag{3.2}$$

is an element of $L^1_{\text{loc}}((-\infty,\infty); Y)$ and satisfies (DE) *&* (IC).

Proof. (a) The continuity of $u(t) = x_t$ follows from the continuity of translation in L^1. Fix $t \geq 0$. By the definition of T_0 one has for $-t \leq \theta \leq 0$

$$u(t)(\theta) - (T_0(t)\varphi)(\theta) = x(t+\theta) - 0 = F(x_{t+\theta}) = F(u(t+\theta))$$

and for $-\infty < \theta < -t$

$$u(t)(\theta) - (T_0(t)\varphi)(\theta) = x(t+\theta) - \varphi(t+\theta) = \varphi(t+\theta) - \varphi(t+\theta) = 0$$

Lemma 3.1 shows that in both cases $u(t)(\theta) - (T_0(t)\varphi)(\theta)$ equals $\left(j^{-1}\int_0^t T_0^{\odot *}(t-s)\ell F(u(s))ds\right)(\theta)$ and thus u satisfies (AIE).

(b) Lemma 3.1 shows that for $t > 0$,

$$\begin{aligned} x(t) &= u(t)(0) = (T_0(t)\varphi)(0) + \left(j^{-1}\int_0^t T_0^{\odot *}(t-s)\ell F(u(s))ds\right)(0) \\ &= F(u(t)). \end{aligned} \tag{3.3}$$

It thus remains to be shown that $u(t) = x_t$. For $-t < \theta \leq 0$, (3.3) gives

$$x_t(\theta) = x(t+\theta) = u(t+\theta)(0) = F(u(t+\theta)) = u(t)(\theta)$$

and for $-\infty < \theta < -t$, Lemma 3.1 gives

$$x_t(\theta) = x(t+\theta) = \varphi(t+\theta) = (T_0(t)\varphi)(\theta) = u(t)(\theta)$$

so indeed $u(t) = x_t$. □

4. A model involving cannibalistic behaviour

Consider a population structured by the size of individuals. We assume that individuals eat their conspecifics and that this cannibalistic behaviour is modelled through the *attack rate* $\alpha(\xi,\eta)$, which is the rate at which individuals of size η kill and eat individuals of size ξ. Usually the victim of cannibalism is smaller than the attacker, so $a(\xi,\eta)$ should be zero for $\xi > \eta$, but we will make no explicit use of this assumption in what follows. We assume that all individuals are born with the same size ξ_b.

Cannibalism leads to an extra mortality in the population. If $n(t,\cdot)$ denotes the density of the size-distribution of the population at time t, then the extra size-specific mortality rate due to cannibalism at time t is

$$M(t,\xi) = \int_{\xi_b}^{\infty} \alpha(\xi,\eta) n(t,\eta) d\eta. \tag{4.1}$$

Let $c(\eta)$ be the energetic value of an individual of size η. Then the extra energy intake due to cannibalism per unit of time of an individual of size ξ is

$$E(t,\xi) = \int_{\xi_b}^{\infty} c(\eta)\alpha(\eta,\xi) n(t,\eta) d\eta. \tag{4.2}$$

We assume that E is channelled into growth and affects "ordinary" mortality, that is, mortality not due to cannibalism but due, e.g., to starvation. The traditional PDE formulation then takes the form of the boundary value problem

$$\begin{aligned} &\frac{\partial}{\partial t} n(t,\xi) + \frac{\partial}{\partial \xi}\left(g(\xi, E(t,\xi)) n(t,\xi)\right) = \\ &-\left(\mu(\xi, E(t,\xi)) + M(t,\xi)\right) n(t,\xi), \quad \xi > \xi_b \\ &g(\xi_b, E(t,\xi_b)) n(t,\xi_b) = \int_{\xi_b}^{\infty} \beta(\xi) n(t,\xi) d\xi, \end{aligned} \tag{4.3}$$

where $\beta(\xi)$ is the size-specific fecundity. If some of the extra energy intake is also channelled into reproduction, then β depends also on $E(t,\xi)$. Nothing essential would change in the sequel, only the notation would be more cumbersome.

Next we want to write the model as a delay equation (DE) for the unknown

$$x(t) = \begin{pmatrix} b(t) \\ I(t) \end{pmatrix},$$

where $b(t)$ is the population birth rate and $I(t)$ is some conveniently chosen interaction variable. To this end, let $I^1(t,a)$ be the total per capita death rate and let $I^2(t,a)$ be the individual growth rate of an individual of age a at time t:

$$I^1(t,a) = \mu(\xi, E(t,\xi)) + M(t,\xi), \tag{4.4}$$

$$I^2(t,a) = g(\xi, E(t,\xi)). \tag{4.5}$$

Note that we use superscripts as indices because subscripts are reserved for translation, cf. (3.1).

We emphasize that *age* does not occur in the original model formulation and that Eqs. (4.4) and (4.5) are meaningless as they stand. So for the time being we *assume* that $I^1(t,a)$ and $I^2(t,a)$ are given. More precisely, we consider the mappings $t \mapsto I^1(t,\cdot)$ and $t \mapsto I^2(t,\cdot)$ as mappings from $\mathbf{R}_-$ to $C(\mathbf{R}_+)$, the Banach space of bounded continuous scalar-valued functions on $\mathbf{R}_+$. Later, when we close the feedback loop, we shall see how the original ingredients, given in terms of size, transform into quantities defined in terms of age.

Consider an individual of age a at time t. It was born at time $t-a$. By definition, it has grown according to

$$\begin{aligned}\frac{d\xi}{d\tau} &= I^2(t-a+\tau,\tau), \quad 0<\tau\le a, &(4.6)\\ \xi(0) &= \xi_b. &(4.7)\end{aligned}$$

The solution evaluated at $\tau = a$ gives the size of the individual at time t:

$$\xi(a) = \xi_b + \int_0^a I^2(t-a+\tau,\tau)d\tau = \xi_b + \int_0^a I_t^2(\tau-a,\tau)d\tau =: X^2\left(I_t^2\right)(a). \quad (4.8)$$

Notice that the size of an individual of age a at time t is an *affine* (that is, constant plus linear) mapping X^2 of $L^1(\mathbf{R}_-;C(\mathbf{R}_+))$ into $C(\mathbf{R}_+)$.

The probability that an individual that was born at time $t-a$ survives to age a, given the history of I, is

$$e^{-\int_0^a I^1(t-a+\tau,\tau)d\tau} = e^{-X^1\left(I_t^1\right)(a)},$$

where, in analogy with the definition of X^2, we have defined the linear mapping $X^1 : L^1(\mathbf{R}_-;C(\mathbf{R}_+)) \to C(\mathbf{R}_+)$ by

$$X^1\left(I_t^1\right)(a) := \int_0^a I_t^1(\tau-a,\tau)d\tau.$$

Therefore the birth rate

$$b(t) := g(\xi_b, E(t,\xi_b))n(t,\xi_b)$$

satisfies the renewal equation

$$b(t) = \int_0^a \beta\left(X^2\left(I_t^2\right)(a)\right)e^{-X^1\left(I_t^1\right)(a)}b_t(-a)da. \quad (4.9)$$

Alternatively and equivalently, the renewal equation (4.9) could have been obtained from the boundary condition in (4.3) by the change $\xi = X^2\left(I_t^2\right)(a)$ of variables. Similarly, we get from (4.1) and (4.2), respectively:

$$M(t,\xi) = \int_0^\infty \alpha\left(\xi, X^2\left(I_t^2\right)(a)\right)e^{-X^1\left(I_t^1\right)(a)}b_t(-a)da \quad (4.10)$$

and

$$E(t,\xi) = \int_0^\infty c\left(X^2\left(I_t^2\right)(a)\right)\alpha\left(X^2\left(I_t^2\right)(a),\xi\right)e^{-X^1\left(I_t^1\right)(a)}b_t(-a)da. \quad (4.11)$$

We now substitute (4.8), (4.10) and (4.11) into (4.4) and (4.5) and obtain

$$I^1(t,a)=$$
$$\mu\Big(X^2\big(I_t^2\big)(a),\int_0^\infty c\big(X^2\big(I_t^2\big)(\tau)\big)\,\alpha\big(X^2\big(I_t^2\big)(\tau),X^2\big(I_t^2\big)(a)\big)\,e^{-X^1\left(I_t^1\right)(\tau)}b_t(-\tau)d\tau\Big)$$
$$+\int_0^\infty \alpha\big(X^2\big(I_t^2\big)(a),X^2\big(I_t^2\big)(\tau)\big)\,e^{-X^1\left(I_t^1\right)(\tau)}b_t(-\tau)d\tau \tag{4.12}$$
$$I^2(t,a)=$$
$$g\Big(X^2\big(I_t^2\big)(a),\int_0^\infty c\big(X^2\big(I_t^2\big)(\tau)\big)\,\alpha\big(X^2\big(I_t^2\big)(\tau),X^2\big(I_t^2\big)(a)\big)\,e^{-X^1\left(I_t^1\right)(\tau)}b_t(-\tau)d\tau\Big) \tag{4.13}$$

Equations (4.9), (4.12) and (4.13) form a delay equation (DE) for the unknown

$$x(t)=\begin{pmatrix} b(t)\\ I^1(t)\\ I^2(t)\end{pmatrix},$$

with $F : L^1(\mathbf{R}_-;Y) \to Y$ and $Y = \mathbf{R}\times C(\mathbf{R}_+)\times C(\mathbf{R}_+)$. The function F is of course defined by declaring

$$F\begin{pmatrix} b_t\\ I_t^1\\ I_t^2\end{pmatrix}(a)$$

to be the vector with the right-hand sides of (4.9), (4.12) and (4.13) as components.

The formulation of the principle of linearized stability, the centre manifold theorem and the Hopf bifurcation theorem involves the linearization described in Theorem 2.6 as well as the location of the spectrum of the generator of the linearized semigroup. Linearization is possible only if F is continuously Fréchet differentiable. It is a pleasant fact that F is indeed continuously differentiable under very natural conditions.

Theorem 4.1. *Let g, β, μ, α and c have continuous partial derivatives with respect to all variables. Then the mapping $F : L^1(\mathbf{R}_-;Y) \to Y$ is continuously Fréchet differentiable.*

Proof. F is linear in b_t and hence continuously differentiable in b_t. As noted above, X^1 and X^2 are affine mappings, and hence continuously differentiable with values in the continuous functions. $X^1(I_t^1)$ and $X^2(I_t^2)$ appear everywhere as arguments of continuously differentiable mappings. Because the Nemytskiĭ operator $N_g : f \mapsto g\circ f$ is continuously differentiable from C to C if g is continuously differentiable, the conclusion follows. $\square$

5. Conclusions

The reformulation of delay differential equations [8] and delay equations [12] as abstract integral equations has proven to be useful because standard results from the theory of ordinary differential equations such as linearized (in)stability and Hopf bifurcation can easily be extended to this class of problems using the so-called sun-star calculus of adjoint semigroups. In the references mentioned above, the analysis was restricted to the case of delay (differential) equations with finite delay and unknowns taking values in finite-dimensional spaces. The reason is that in this case the state space is sun-reflexive with respect to the unperturbed semigroup and standard results concerning adjoint semigroups show that the abstract integral equation makes sense and has a unique solution. In this paper we have shown that the assumption of sun-reflexivity can be relaxed. Indeed, we have shown that the abstract integral equation (AIE) is well posed if the nonlinear operator G is restricted to take on values in a certain subspace of $X^{\odot *}$. This is a very natural approach because when the delay is infinite, one cannot give an easy representation of $X^{\odot *}$, so one is anyhow forced to define the operator G as taking values in a subspace that can be given a representation.

The natural state space is the space $X = L^1(\mathbf{R}_-; Y)$ of suitably weighted Bochner integrable functions. One cannot work with continuous functions because they are not invariant under the unperturbed semigroup, which is translation and extension by zero. A weight is needed to have nonzero steady states in the state space. In applications to population problems, the components of the unknown are typically *rates*, which integrated over a finite time interval yield finite numbers. So L^1 (and not, e.g., L^p) is the right state space.

From certain points of view the space L^1 is not particularly nice. One complication is that the Nemytskiĭ (or substitution) operator $N_g : f \mapsto g \circ f$ is differentiable in L^1 if and only if g is affine, that is, a constant plus a linear map [17]. This appears to be a severe restriction, at least when the space Y is chosen in what at first thought seems the most natural way. For instance, in [12] the principle of linearized stability for the well-known Gurtin-MacCamy model

$$b(t) = \int_0^\infty \beta(a, N(t)) e^{-\int_0^a \mu(N(t-a+\tau,\tau)d\tau} b(t-a)da, \tag{5.1}$$

$$N(t) = \int_0^\infty e^{-\int_0^a \mu(N(t-a+\tau,\tau)d\tau} b(t-a)da \tag{5.2}$$

with one-dimensional interaction variable N, could be established only if the per capita death rate μ was affine: $\mu(a, N) = \mu_0(a) + \mu_1(N)$. This is somewhat unsatisfactory because the principle of linearized stability has been proven in much greater generality in [19, 20].

But in the present paper we allow for infinite-dimensional Y and hence the Gurtin-MacCamy model (5.1) & (5.2) can be rewritten using the infinite-dimensional interaction variable

$$I(t,a) = \mu(a, N(t))$$

as

$$b(t) = \int_0^\infty \beta \left(a, \int_0^\infty e^{-\int_0^\sigma I_t(\tau-\sigma,\tau)d\tau} b_t(-\sigma)d\sigma \right) e^{-\int_0^a I_t(\tau-a,\tau)d\tau} b_t(-a)da, \tag{5.3}$$

$$I(t,a) = \mu \left(\int_0^\infty e^{-\int_0^\sigma I_t(\tau-\sigma,\tau)d\tau} b_t(-\sigma)d\sigma, a \right). \tag{5.4}$$

If β and μ are continuously differentiable, then the right-hand sides of (5.3) and (5.4) are continuously differentiable in I_t as compositions of continuously differentiable mappings on C and affine mappings $L^1 \to C$. For the Gurtin-MacCamy model the shift from one-dimensional to infinite-dimensional interaction variable, just to make the abstract framework functioning, may seem artificial because the problem can be, and has been, solved by other means. But for the cannibalism model treated in Section 4, in which the more obvious candidates for interaction variables, viz. $M(t,x)$ and $E(t,x)$, already are infinite dimensional, the choice of $I^1(t,a)$ and $I^2(t,a)$ as interaction variables is very natural.

The setting of this paper with an infinite-dimensional Y also allows for infinitely many states at birth in contrast to the assumption of only finitely many states at birth made in [12].

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Weak Stability for Orbits of C_0-semigroups on Banach Spaces

Tanja Eisner and Bálint Farkas

Dedicated to the memory of Günter Lumer

Abstract. A result of Huang and van Neerven [12] establishes weak individual stability for orbits of C_0-semigroups under boundedness assumptions on the local resolvent of the generator. We present an elementary proof for this using only the inverse Fourier-transform representation of the orbits of the semigroup in terms of the local resolvent.

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1. Introduction

This paper is originally motivated by the structure theory of relatively weakly compact semigroups on Banach spaces as presented, for example, in Engel, Nagel [6, Ch. V]. Suppose that a C_0-semigroup $(T(t))_{t\geq 0}$, with generator $(A, D(A))$, is *relatively weakly compact*, that is each of the orbits $\{T(t)x : t \geq 0\}$ is a relatively weakly compact subset of the Banach space X. Then the Jacobs–Glicksberg–de Leeuw decomposition yields the existence of a projection $Q \in \mathcal{L}(X)$ commuting with the semigroup $(T(t))_{t\geq 0}$ such that

$$\ker Q = \big\{x \in X : 0 \in \overline{\{T(t)x : t \geq 0\}}^{\sigma}\big\},$$
$$\operatorname{rg} Q = \overline{\operatorname{lin}}\big\{x \in D(A) : \exists\, \alpha \in \mathbb{R} \text{ with} Ax = i\alpha x\big\}.$$

In particular, if $(T(t))_{t\geq 0}$ is a bounded semigroup on a reflexive Banach space X, then the semigroup is of course relatively weakly compact, and we always have the existence of such a projection. If now the generator does not have point spectrum

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on the imaginary axis, then we have $\ker Q = X$. So 0 belongs to the weak closure of *each* orbit. There are however examples showing that generally we can not expect weak stability, i.e., that all orbits converge to 0 in the weak topology (see [6, ExampleV.2.11 ii)]). In fact, the "no eigenvalues on the imaginary axis" assumption is roughly speaking equivalent to *almost weak stability* (i.e., convergence to zero along a large set of time values) but, in general, not to weak stability, see [7], [9], and also [8], [10].

Concerning stability questions for bounded semigroups the size of the spectrum on the imaginary line and the growth of the resolvent $R(\lambda, A)$ in a neighbourhood of it play an important role. The celebrated theorem of Arendt, Batty [1] and Lyubich, Vũ [15] gives a sufficient condition on the boundary spectrum for *strong stability.* They show that, in case of reflexive X, countable spectrum $\sigma(A)$ on the imaginary axis and no eigenvalues on $i\mathbb{R}$ imply strong stability of bounded C_0-semigroups. Later Batty [2] gave similar results for *weak individual stability* of the orbit $T(t)x_0$ under the above spectral assumptions and the boundedness of the orbit (see also Batty, Vũ[5]).

In connection with individual stability or growth of orbits, the boundedness of the *local resolvent* has gained wide recognition. In the sequel, we will say, with a slight abuse of terminology, that a bounded local resolvent $R(\lambda)x_0$ exists on $\mathbb{C}_+$ if the function $\rho(A) \ni \lambda \mapsto R(\lambda, A)x_0$ admits a bounded, holomorphic extension $R(\lambda)x_0$ to the whole right half-plane $\mathbb{C}_+ := \{\mu : \Re\mu > 0\}$.

Huang and van Neerven [12] proved that if the Banach space X has Fourier type $p \in (1, 2]$, then the existence of a bounded local resolvent $R(\lambda)x_0$ on $\mathbb{C}_+$ already implies the strong convergence $T(t)R(\mu, A)^\alpha x_0 \to 0$ as $t \to +\infty$, for all $\mu > \omega_0(A)$ and $\alpha > 1$ (see also [11]).

Interestingly enough, *weak convergence* of the orbit may be also concluded from the existence of bounded local resolvent. In [4] a functional calculus method was developed for investigating asymptotic behaviour of C_0-semigroups with bounded local resolvents. A corollary of this approach is an alternative proof of the next theorem (see[12] Theorem 0.3]).

Theorem (Huang, van Neerven [12]**, Theorem 0.3).** *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup with generator $(A, D(A))$, $x_0 \in X$, and suppose that the local resolvent $R(\lambda)x_0$ exists on the open right half-plane and that it is bounded, i.e., there exists some $M > 0$ such that*

$$\|R(\lambda)x_0\| \leq M \quad \textit{for all } \lambda \in \mathbb{C}_+.$$

Then it holds

$$T(t)R(\mu, A)^\alpha x_0 \to 0 \quad \textit{weakly as } t \to +\infty \textit{ forall } \alpha > 1 \textit{ and } \mu > \omega_0(A).$$

In [18] van Neerven obtains even the exponent$\alpha = 1$ under an additional positivity assumption.

Theorem (van Neerven [18]**).** *Suppose that X is an ordered Banach space with weakly closed normal cone C. If for some $x_0 \in X$*

i) $T(t)x_0 \in C$ *for all sufficiently large* t, *and*
ii) $R(\cdot, A)x_0$ *has a bounded holomorphic extension to* $\mathbb{C}_+$,

then for all $\mu \in \rho(A)$ *and* $y \in X'$

$$\langle T(t)R(\mu, A)x_0, y\rangle \to 0 \qquad \textit{as } t \to +\infty.$$

It is also known that the above *eventual positivity* assumption *cannot* be omitted (see Batty [3], van Neerven [18]). Reformulating van Neerven's assertion we can write

$$\langle T(t)x_0, y\rangle \to 0 \qquad \text{as } t \to +\infty \text{ for all } y \in D(A'). \tag{1}$$

This is an individual stability result for the orbit of x_0 under the semigroup. Our aim is to give an *elementary* proof of such convergence in the presence of bounded local resolvent without assumption on the Banach space, but only for $y \in D({A'}^2)$. This is the above mentioned result in the case $\alpha = 2$. That we assume $\alpha = 2$ instead of $\alpha > 1$ is only technical to keep the arguments the simplest possible.

At the end, we formulate the analogous individual stability result for bi-continuous semigroups (see [14] for general theory).

2. The result

Theorem 1. *Let* $(T(t))_{t\geq 0}$ *be a* C_0*-semigroup with generator* $(A, D(A))$, $x_0 \in X$, *and suppose that the local resolvent* $R(\lambda)x_0$ *exists on the open right half-plane and that it is bounded, i.e., there exists some* $M > 0$ *such that*

$$\|R(\lambda)x_0\| \leq M \quad \textit{for all } \lambda \in \mathbb{C}_+.$$

Then the convergence

$$\langle T(t)x_0, y\rangle \to 0 \quad \textit{as } t \to +\infty \textit{ for all } y \in D({A'}^2)$$

holds.

Remark. To make distinction between the resolvent operator and the local resolvent, for the latter we will use the notation $R(\mu)x_0$, while the use of the symbol $R(\lambda, A)$ tacitly assumes that λ belongs to the resolvent set $\rho(A)$, hence$(\lambda - A)^{-1}$ is a bounded linear operator.

To prove the theorem we need the following series of lemmas.

Lemma 1. *For all* $\lambda \in \rho(A)$ *and* $\mu \in \mathbb{C}_+$

$$R(\lambda, A)x_0 - R(\mu)x_0 = (\mu - \lambda)R(\lambda, A)R(\mu)x_0 \tag{2}$$

holds.

Proof. For a fixed $\lambda \in \rho(A)$ both functions on the two sides of (2) are analytic on $\mathbb{C}_+$. For large $\Re\mu$ the resolvent identity holds, so the assertion follows by uniqueness of analytic functions. □

Lemma 2. *For all $\lambda \in \rho(A)$ and $\mu \in \mathbb{C}_+$ we have*

$$\|R(\lambda, A)R(\mu)x_0\| \leq \frac{M + \|R(\lambda, A)x_0\|}{|\lambda - \mu|}.$$

Proof. Use Lemma 1. □

Lemma 3. *For $y \in D(A'^2)$ and $a > \omega_0(T)$ there exists a constant $c := c(y, a)$ such that*

$$\|R^2(a + is, A')y\| \leq \frac{c}{a^2 + s^2} \quad \textit{for all } s \in \mathbb{R}.$$

Proof. Let us write $\lambda = a + is$. Then we have

$$R(\lambda, A')y = \frac{1}{\lambda}\left(R(\lambda, A')A'y + y\right).$$

Thus

$$\begin{aligned} R(\lambda, A')^2 y &= \frac{1}{\lambda}\left(R(\lambda, A')R(\lambda, A')A'y + R(\lambda, A')y\right) \\ &= \frac{1}{\lambda^2}\left(R(\lambda, A')A'R(\lambda, A')A'y + R(\lambda, A')A'y + R(\lambda, A')A'y + y\right). \end{aligned}$$

The assertion follows by noticing that the terms in parenthesis are bounded. □

Lemma 4. *For $y \in D(A'^2)$, $x \in X$ and $a > \omega_0(T)$ we have*

$$\begin{aligned} \langle T(t)x, y\rangle &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{(a+is)t}\langle R(a + is, A)x, y\rangle \,\mathrm{d}s \\ &= \frac{1}{2\pi t}\int_{-\infty}^{+\infty} \mathrm{e}^{(a+is)t}\langle R^2(a + is, A)x, y\rangle \,\mathrm{d}s. \end{aligned} \tag{3}$$

Proof. The integral in (3) is just

$$\frac{1}{2\pi t}\int_{-\infty}^{+\infty} \mathrm{e}^{(a+is)t}\langle x, R^2(a + is, A')y\rangle \,\mathrm{d}s,$$

and it is absolutely convergent by Lemma 3. Integration by parts yields equality of the two integrals. In particular, since the first integral converges, we obtain immediately that it coincides with $\langle T(t)x, y\rangle$ as the inverse Laplace transform of the resolvent (see [13, Lemma 2.4]). □

Lemma 5. *For $y \in D(A'^2)$, $x \in X$, $a > \omega_0(T)$ and $0 < \delta < a$ we have*

$$\begin{aligned} \langle T(t)x_0, y\rangle &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{(a+is)t}\langle R(a + is, A)x_0, y\rangle \,\mathrm{d}s \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{(\delta+is)t}\langle R(\delta + is)x_0, y\rangle \,\mathrm{d}s. \end{aligned}$$

Proof. Let N be positive, then using the analyticity of $R(\lambda)x_0$ on $\mathbb{C}_+$ and Cauchy's theorem, we obtain for some $\mu \in \rho(A)$

$$\begin{aligned}
&\Big|\frac{1}{2\pi}\int_{-N}^{+N} \mathrm{e}^{(a+is)t}\langle R(a+is,A)x_0,y\rangle\,\mathrm{d}s - \frac{1}{2\pi}\int_{-N}^{+N} \mathrm{e}^{(\delta+is)t}\langle R(\delta+is)x_0,y\rangle\,\mathrm{d}s\Big| \\
&\le (a-\delta)\max_{b\in[\delta,a]}\big|\mathrm{e}^{(b+iN)}\langle R(b+iN)x_0,y\rangle\big| + (a-\delta)\max_{b\in[\delta,a]}\big|\mathrm{e}^{(b-iN)}\langle R(b-iN)x_0,y\rangle\big| \\
&= (a-\delta)\Big(\max_{b\in[\delta,a]}\big|\mathrm{e}^{b}\langle R(\mu,A)R(b+iN)x_0,(\mu-A')y\rangle\big| \\
&\qquad + \max_{b\in[\delta,a]}\big|\mathrm{e}^{b}\langle R(\mu,A)R(b-iN)x_0,(\mu-A')y\rangle\big|\Big),
\end{aligned}$$

but this converges to 0 by Lemma 2 as $N \to +\infty$. □

Proof of Theorem 1. According to (2) Lemma 1

$$\begin{aligned}
R(\delta+is)x_0 &= R(a+is,A)x_0 + (a-\delta)R(a+is,A)R(\delta+is)x_0 \\
&= R(a+is,A)x_0 + (a-\delta)R^2(a+is,A)x_0 \\
&\qquad\qquad + (a-\delta)^2R^2(a+is,A)R(\delta+is)x_0.
\end{aligned}$$

Using Lemma 5 we obtain for $y \in D({A'}^2)$

$$\begin{aligned}
2\pi\mathrm{e}^{-\delta t}\langle T(t)x_0,y\rangle &= \int_{-\infty}^{+\infty} \mathrm{e}^{ist}\langle R(\delta+is)x_0,y\rangle\,\mathrm{d}s \\
&= \int_{-\infty}^{+\infty} \mathrm{e}^{ist}\langle R(a+is,A)x_0,y\rangle\,\mathrm{d}s \\
&\quad + (a-\delta)\int_{-\infty}^{+\infty} \mathrm{e}^{ist}\langle R^2(a+is,A)x_0,y\rangle\,\mathrm{d}s \\
&\quad + (a-\delta)^2\int_{-\infty}^{+\infty} \mathrm{e}^{ist}\langle R^2(a+is,A)R(\delta+is)x_0,y\rangle\,\mathrm{d}s.
\end{aligned}$$

The functions $f_\delta(s) := \langle R^2(a+is,A)R(\delta+is)x_0,y\rangle$ form a relatively compact subset of $L^1(\mathbb{R})$. Indeed,we have

$$\begin{aligned}
|f_\delta(s)| &= |\langle R^2(a+is,A)R(\delta+is)x_0,y\rangle| = |\langle R(\delta+is)x_0,R^2(a+is,A')y\rangle| \\
&\le M\|R^2(a+is,A')y\|,
\end{aligned}$$

and the function on the right-hand side belongs to $L^1(\mathbb{R})$. This shows the family f_δ to be uniformly integrable (and bounded), thus relatively compact. So by compactness we find a sequence $\delta_n \to 0$ such that $f_{\delta_n} \to f$ in $L^1(\mathbb{R})(n \to \infty)$. Thus substituting δ_n in the above equality and letting $n \to \infty$ we obtain

$$\begin{aligned}2\pi\langle T(t)x_0, y\rangle = &\int_{-\infty}^{+\infty} \mathrm{e}^{ist}\langle R(a+is, A)x_0, y\rangle \,\mathrm{d}s\\ &+ a\int_{-\infty}^{+\infty} \mathrm{e}^{ist}\langle R^2(a+is, A)x_0, y\rangle \,\mathrm{d}s\\ &+ a^2\int_{-\infty}^{+\infty} \mathrm{e}^{ist} f(s)\,\mathrm{d}s =: I_1(t) + I_2(t) + I_3(t).\end{aligned}$$

It is easy to deal with the last term I_3. Since f belongs to $L^1(\mathbb{R})$ so by the Riemann–Lebesgue Lemma its Fourier transform vanishes at $+\infty$, i.e., $I_3(t) \to 0$ as $t \to +\infty$. Since $y \in D(A'^2)$, we can rewrite I_1 by Lemma 4 as

$$I_1(t) = \int_{-\infty}^{+\infty} \mathrm{e}^{ist}\langle x_0, R(a+is, A')y\rangle \,\mathrm{d}s = \frac{1}{t}\int_{-\infty}^{+\infty} \mathrm{e}^{ist}\langle x_0, R^2(a+is, A')y\rangle \,\mathrm{d}s.$$

The last integral is absolutely convergent by Lemma 3, hence

$$|I_1(t)| \leq \frac{1}{t}\int_{-\infty}^{+\infty} \|x_0\| \cdot \|R^2(a+is, A')y\| \,\mathrm{d}s \to 0 \qquad \text{as} t \to +\infty.$$

As for I_2 we first notice that $\langle x_0, R^2(a+i\cdot, A')y\rangle \in L^1(\mathbb{R})$, so by the Riemann–Lebesgue Lemma we have

$$I_2(t) = a\int_{-\infty}^{+\infty} \mathrm{e}^{ist}\langle x_0, R^2(a+is, A')y\rangle \,\mathrm{d}s \to 0 \qquad \text{as} t \to +\infty.$$

This concludes the proof. $\square$

Let us draw the following consequences of the above result.

Corollary 1. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup with generator $(A, D(A))$, and suppose that $\{T(t)x_0 : t \geq 0\}$ is bounded and that the local resolvent $R(\lambda)x_0$ exists and is bounded on $\mathbb{C}_+$. Then*

$$\langle T(t)x_0, y\rangle \to 0 \quad \text{as } t \to +\infty \text{ for all } y \in \overline{D(A')}.$$

Proof. Since $(A', D(A'))$ is a Hille–Yosida operator, its part A'_0 generates a C_0-semigroup on $\overline{D(A')}$. But $D({A'_0}^2) \subseteq D(A'^2) \subseteq D(A'_0)$, so $D(A'^2)$ is dense in $\overline{D(A')}$. Now let $\varepsilon > 0$. For $y \in \overline{D(A')}$ take $y' \in D(A'^2)$ with $\|y - y'\| \leq \varepsilon/2M$, where $\|T(t)x_0\| < M$, $t \geq 0$. For large t we have $|\langle T(t)x_0, y'\rangle| \leq \varepsilon/2$ by Theorem 1. So

$$|\langle T(t)x_0, y\rangle| \leq |\langle T(t)x_0, y'\rangle| + |\langle T(t)x_0, y - y'\rangle| \leq \varepsilon/2 + M \cdot \|y - y'\| \leq \varepsilon,$$

for large t. $\square$

Corollary 2. *Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup with generator $(A, D(A))$, and suppose $\sigma_p(A) \cap i\mathbb{R} = \emptyset$. If $(is - A)^{-1}x_0$ exists and is bounded in $s \in \mathbb{R}$ for some $x_0 \in X$, then*

$$\langle T(t)x_0, y\rangle \to 0 \quad \text{as } t \to +\infty \text{ for all } y \in \overline{D(A')}.$$

Proof. A version of the resolvent identity states that

$$(is - A)^{-1}x_0 - R(a + is, A)x_0 = (\lambda - is)R(a + is, A)(is - A)^{-1}x_0.$$

Here the right-hand side is bounded for $a > 0$ and $s \in \mathbb{R}$ by the Hille–Yosida theorem and by the assumption, so $R(a + is, A)x_0$ is bounded. The proof is concluded by applying Corollary 1. □

The above proof of Theorem 1 remains valid if the semigroup $(T(t))_{t \geq 0}$ is only strongly continuous for some coarser locally convex topology τ. More precisely, one has to assume that the semigroup is τ-*bi-continuous*, see [14] for the theory. Then the infinitesimal generator $(A, D(A))$ is a Hille–Yosida operator, but $D(A)$ is not necessarily dense with respect to the norm in X. It is dense however for the topology τ, so in the following the adjoint A' of A is understood with respect to τ. In addition, the resolvent identity, and replacing the vector-valued integrals by τ-strong integrals, all the above integral formulas remain valid, which were the essential ingredients of the proof. This proves the following.

Theorem 2. *For a bi-continuous semigroup $(T(t))_{t \geq 0}$ with generator $(A, D(A))$ and $x_0 \in X$ suppose that the local resolvent $R(\lambda)x_0$ exists on the open right half-plane and that it is bounded. Then for all $y \in D({A'}^2)$*

$$\langle T(t)x_0, y \rangle \to 0$$

holds for $t \to +\infty$.

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Contraction Semigroups on $L_\infty(\mathbf{R})$

Antonius F.M. ter Elst and Derek W. Robinson

Dedicated to the memory of Günter Lumer 1929–2005

Abstract. If X is a non-degenerate derivation on $\mathbf{R}$ and $H = -X^2$ we examine conditions for the closure of H to generate a weakly* continuous semigroup on L_∞ which extends to the L_p-spaces. We give an example which cannot be extended and an example which extends but for which the real part of the generator on L_2 is not lower semibounded.

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1. Introduction

The Lumer–Phillips theorem [LuP] is a cornerstone of the theory of one-parameter semigroups. The theorem characterizes the generator of a contraction semigroup with the aid of a dissipativity condition. The latter is an extension of the elementary properties of the operator $-d^2/dx^2$ of double differentiation acting on $C_0(\mathbf{R})$. In this note we analyze contraction semigroups S generated by squares $-X^2$ of derivations $X = a\,d/dx$ acting on $C_0(\mathbf{R})$, or $L_\infty(\mathbf{R})$. An integral part of the analysis consists of examining the one-parameter group T generated by X. Throughout we assume $a > 0$. If a is smooth this is the one-dimensional analogue of Hörmander's condition for vector fields [Hör] and the operator $-X^2$ is the simplest example of a Hörmander 'sum of squares of vector fields'.

First, we identify the kernel of S acting on $L_\infty(\mathbf{R})$. Secondly, T is a weakly* continuous group of contractions on L_∞ and we derive necessary and sufficient conditions for it to extend to a strongly continuous group on the $L_p(\mathbf{R}\,;\rho\,dx)$-spaces with $p \in [1,\infty\rangle$, where $\rho\colon \mathbf{R} \to \langle 0,\infty\rangle$ is a C^1-function. These conditions also ensure that S extends to a strongly continuous semigroup on $L_p(\mathbf{R}\,;\rho\,dx)$. Thirdly, we characterize those S, or T, which extend to a contraction semigroup, or group, on $L_p(\mathbf{R}\,;\rho\,dx)$ for some $p \in [1,\infty\rangle$. Fourthly, we give an example of a smooth derivation with a uniformly bounded coefficient for which neither T nor S can be extended to any of the L_p-spaces with $p < \infty$. Fifthly, we give an example of a smooth derivation with a uniformly bounded coefficient which is uniformly bounded away from zero for which T and S extend to all the L_p-spaces but the real part of the generator of S on $L_2(\mathbf{R}\,;\rho\,dx)$ is not lower semibounded. In particular the L_2-generator cannot satisfy a Gårding inequality. Since the Gårding inequality is the usual starting point for the analysis of elliptic divergence form operators on $L_2(\mathbf{R}\,;\rho\,dx)$, e.g., operators of the form X^*X, this example clearly demonstrates that the theory of 'non-divergent' form operators such as $-X^2$ on $L_\infty(\mathbf{R})$ is very different. Finally we discuss the volume doubling property for balls (intervals) whose radius (length) is measured by the distance associated with X.

2. Preliminaries

Let $a\colon \mathbf{R} \to \langle 0, \infty\rangle$ be a C^1-function. Further assume

$$\int_0^\infty dx\, a(x)^{-1} = \infty = \int_{-\infty}^0 dx\, a(x)^{-1} \quad . \tag{1}$$

Then define operators X_∞ and X_1 with action $a\,d/dx$ and with domains $D(X_\infty) = C_c^\infty(\mathbf{R})$ and $D(X_1) = C_c^1(\mathbf{R})$, respectively.

The ordinary differential equation $\dot{x} = a(x)$ has a unique maximal solution, for all initial data $x(0) = x_0 \in \mathbf{R}$ (see, for example, [Pla] Theorem 7.1.8). Let $t \mapsto e^{tX}x_0$ denote this solution. Since a satisfies (1) this maximal solution is defined for all $t \in \mathbf{R}$. Moreover, $e^{sX}e^{tX}x_0 = e^{(s+t)X}x_0$ and

$$\int_{x_0}^{e^{tX}x_0} dx\, a(x)^{-1} = t \tag{2}$$

for all $s, t \in \mathbf{R}$ and $x_0 \in \mathbf{R}$. In addition the map $(t, x) \mapsto e^{tX}x$ is continuous from $\mathbf{R}^2$ into $\mathbf{R}$. If $t \in \mathbf{R}$ then the map $y \mapsto e^{-tX}y$ is a C^1-diffeomorphism from $\mathbf{R}$ onto $\mathbf{R}$. Therefore one can define the map $T_t\colon L_\infty \to L_\infty$ by $(T_t\varphi)(y) = \varphi(e^{-tX}y)$. Then T_t is an isometry and T is a weakly* continuous group on L_∞. Clearly the generator of T is an extension of X_1.

Proposition 2.1.

I. *The operators X_∞ and X_1 are weakly* closable, the weak* closures $\overline{X_1}$ and $\overline{X_\infty}$ are equal and generate the weakly* continuous positive group T, i.e., $T_t = e^{-t\overline{X_1}}$.*

II. *The operator $H_1 = -X_1^2$ is weakly* closable and its weak* closure $\overline{H_1}$ generates a semigroup S, which is weakly* continuous, positive, contractive and holomorphic in the open right half-plane.*

III. *$\overline{H_1} = -\overline{X_1}^2$ and in particular $\overline{X_1}^2$ is weakly* closed.*

IV. *If $a \in C^\infty(\mathbf{R})$ then $\overline{H_\infty} = \overline{H_1}$ where $H_\infty = -X_\infty^2$.*

Proof. If $a = \mathbb{1}$ then T is the group of left translations and the proof of the proposition is straightforward. We next argue that one can reduce the general case to the case $a = \mathbb{1}$ using a coordinate transformation which goes back at least to Feller, [Fel] Section 7.

Define $\gamma\colon \mathbf{R} \to \mathbf{R}$ by $\gamma(t) = \int_0^t a^{-1}$. Then it follows from (1) that γ is bijective and since $a \in C^1(\mathbf{R})$ the map γ is a C^2-diffeomorphism. Moreover, $(\gamma^{-1})'(t) = a(\gamma^{-1}(t))$ for all $t \in \mathbf{R}$ and $\gamma^{-1}(0) = 0$. So $\gamma^{-1}(t) = e^{tX}0$ for all $t \in \mathbf{R}$.

Next define $U\colon L_\infty(\mathbf{R}) \to L_\infty(\mathbf{R})$ by $U\varphi = \varphi \circ \gamma$. Then U is an isometric isomorphism. If $\varphi \in L_\infty(\mathbf{R})$ then $\varphi \in C_c^1(\mathbf{R})$ if and only if $U\varphi \in C_c^1(\mathbf{R})$, and then $X_1 U\varphi = U d_1 \varphi$, where d_1 is the differential operator d/dx on L_∞ with domain $D(d_1) = C_c^1(\mathbf{R})$. Similarly, if $a \in C^\infty(\mathbf{R})$ then $X_\infty U = U d_\infty$, where $d_\infty = d_1|_{C_c^\infty}$. Since γ is a C^2-diffeomorphism one similarly obtains that $UD(d_1^2) = UC_c^2(\mathbf{R}) = C_c^2(\mathbf{R}) = D(X_1^2)$ and $X_1^2 U\varphi = U d_1^2 \varphi$ for all $\varphi \in D(d_1^2)$. If $\varphi \in L_\infty$ and $\psi \in L_1(\mathbf{R})$ then

$$\int \psi \, U\varphi = \int \big((a\,\psi) \circ \gamma^{-1}\big) \cdot \varphi$$

and $(a\,\psi) \circ \gamma^{-1} \in L_1(\mathbf{R})$. It is then easy to deduce that U and U^{-1} are weakly* continuous. Now the proposition follows easily. $\square$

Throughout this paper we will use the notation $e^{tX}x_0$, $\gamma(t)$ and U introduced above. Moreover, T and S denote the group and semigroup generated by $\overline{X_1}$ and $\overline{H_1}$, respectively.

Remark 2.2. The group T satisfies the property $T_t C_0(\mathbf{R}) = C_0(\mathbf{R})$ for all $t \in \mathbf{R}$, by definition. Moreover, the semigroup S is related to the group T through the integral algorithm

$$S_t = (4\pi t)^{-1/2} \int_{-\infty}^{\infty} ds\, e^{-s^2(4t)^{-1}} T_s \tag{3}$$

which is elementary if $a = \mathbb{1}$ and follows by use of the map U if $a \neq \mathbb{1}$. Therefore $S_t C_0(\mathbf{R}) \subseteq C_0(\mathbf{R})$ for all $t > 0$, i.e., S is a Feller semigroup.

One can associate a distance [JeS] with the derivation X by the definition

$$d(x\,;y) = \sup\{|\psi(x) - \psi(y)| \;:\; \psi \in C_c^\infty(\mathbf{R})\,,\; \|X\psi\|_\infty \le 1\,\} \quad .$$

Clearly one has

$$|\psi(x) - \psi(y)| = \Big|\int_x^y dz\, \psi'(z)\Big| \le \Big|\int_x^y dz\, a(z)^{-1}\Big|$$

for all $\psi \in C_c^\infty(\mathbf{R})$ with $\|X_\infty \psi\|_\infty \le 1$. So

$$d(x\,;y) \le \Big|\int_x^y dz\, a(z)^{-1}\Big| \quad .$$

But by regularizing a^{-1} on a compact interval one deduces that the inequality is in fact an equality, i.e.,

$$d(x\,;y) = \Big|\int_x^y a^{-1}\Big| = |\gamma(y) - \gamma(x)| \tag{4}$$

for all $x, y \in \mathbf{R}$. Note that by setting $y = e^{sX}x$ and using (2) one finds

$$d(x\,;e^{sX}x) = \Big| \int_x^{e^{sX}x} dz\, a(z)^{-1} \Big| = |s| \quad . \tag{5}$$

Moreover,

$$d(e^{tX}x\,;e^{tX}y) = d(e^{tX}x\,;e^{sX}e^{tX}x) = |s| = d(x\,;e^{sX}x) = d(x\,;y)$$

for all $t \in \mathbf{R}$, where we have used (5). Therefore the distance is invariant under the flow in the sense that

$$d(e^{tX}x\,;e^{tX}y) = d(x\,;y)$$

for all $x, y \in \mathbf{R}$ and $t \in \mathbf{R}$.

Equip $\mathbf{R}$ with the measure $\rho\,dx$ where $\rho\colon \mathbf{R} \to \langle 0, \infty\rangle$ is a C^1-function. Then one can calculate the kernel of the semigroup S.

Proposition 2.3. *The kernel K of the semigroup S on $L_\infty(\mathbf{R})$ is given by*

$$K_t(x\,;y) = (4\pi t)^{-1/2} \big(a(y)\rho(y)\big)^{-1} e^{-d(x;y)^2(4t)^{-1}} \tag{6}$$

for all $x, y \in \mathbf{R}$ and $t > 0$. Moreover, K_t is continuous and $\int dy\,\rho(y)\,K_t(x\,;y) = 1$ for all $x \in \mathbf{R}$.

Proof. First, if $a = \mathbb{1}$ then the proposition is well known.

In general, if $\varphi \in L_\infty(\mathbf{R})$ and $t > 0$ then

$$(S_t\varphi)(x) = (US_t^{(1)}\,U^{-1}\varphi)(x) = (4\pi t)^{-1/2} \int dy\, e^{-|\gamma(x)-y|^2(4t)^{-1}}\,(U^{-1}\varphi)(y)$$

$$= (4\pi t)^{-1/2} \int dy\,(a(y))^{-1}\, e^{-|\gamma(x)-\gamma(y)|^2(4t)^{-1}}\,\varphi(y)$$

for almost every $x \in \mathbf{R}$, where $S_t^{(1)} = e^{-t\overline{d_1}^2}$ is the semigroup corresponding to $a = \mathbb{1}$. The representation (6) then follows immediately from (4).

Clearly K_t is continuous and $H_1\mathbb{1} = 0$. So $S_t\mathbb{1} = \mathbb{1}$ as elements of L_∞. Therefore $\int dy\,\rho(y)\,K_t(x\,;y) = 1$ for all $t > 0$ and almost every $x \in \mathbf{R}$. Moreover, the map $x \mapsto \int dy\,\rho(y)\,K_t(x\,;y)$ is continuous. Hence $\int dy\,\rho(y)\,K_t(x\,;y) = 1$ for all $t > 0$ and $x \in \mathbf{R}$. $\square$

3. Extension properties

The group of isometries T and the contraction semigroup S are defined on L_∞ but they do not automatically extend to the L_p-spaces with $p \in [1, \infty\rangle$. This requires extra boundedness conditions on the coefficient function a and the density function ρ. The following proposition gives necessary and sufficient conditions for T to extend to a strongly continuous group and sufficient conditions for S to extend to a strongly continuous semigroup.

Proposition 3.1. *The following conditions are equivalent for all $C \geq 1$ and $\omega \geq 0$.*

I. *There exists a $p \in [1, \infty\rangle$ such that T extends to a strongly continuous group on $L_p(\mathbf{R}\,; \rho\, dx)$ satisfying the bounds $\|T_t\|_{p\to p} \leq C^{1/p}\, e^{\omega|t|/p}$ for all $t \in \mathbf{R}$.*

II. *For all $p \in [1, \infty\rangle$ the group T extends to a strongly continuous group on $L_p(\mathbf{R}\,; \rho\, dx)$ satisfying the bounds $\|T_t\|_{p\to p} \leq C^{1/p}\, e^{\omega|t|/p}$ for all $t \in \mathbf{R}$.*

III. $a(y)\rho(y) \leq C\, e^{\omega d(x;y)}\, a(x)\rho(x)$ *for all $x, y \in \mathbf{R}$.*

Moreover, if these conditions are satisfied then the semigroup S extends to a strongly continuous semigroup on all the L_p-spaces, $p \in [1, \infty\rangle$, satisfying the bounds

$$\|S_t\|_{p\to p} \leq \big((2\,C)^{1/p}\, e^{\omega^2 t/p}\big) \wedge \big(2\,C^{1/p}\, e^{\omega^2 t/p^2}\big)$$

if $\omega > 0$ and $\|S_t\|_{p\to p} \leq C^{1/p}$ if $\omega = 0$, for all $t > 0$.

Proof. First assume $a = \mathbb{1}$. If Condition I is satisfied then for all $\varphi \in L_p$ one has

$$\|T_t\varphi\|_p^p = \int_{\mathbf{R}} dy\, \rho(y)\, |\varphi(y - t)|^p = \int_{\mathbf{R}} dx\, \rho(x)\, \frac{\rho(x+t)}{\rho(x)}\, |\varphi(x)|^p \quad .$$

Therefore

$$\sup_{x\in\mathbf{R}} \Big(\frac{\rho(x+t)}{\rho(x)}\Big)^{1/p} = \|T_t\|_{p\to p} \leq C^{1/p} e^{\omega|t|/p}$$

for all $t \in \mathbf{R}$ and $x \in \mathbf{R}$. Hence $\rho(x + t) \leq C e^{\omega|t|}\rho(x)$ for all $t \in \mathbf{R}$ and $x \in \mathbf{R}$. Setting $y = x + t$ and noting that $d(x\,; y) = |t|$ one deduces that Condition III is satisfied. Conversely, the same calculation shows that if Condition III is satisfied then

$$\|T_t\varphi\|_p \leq C^{1/p} e^{\omega|t|/p} \|\varphi\|_p \tag{7}$$

for all $p \in [1, \infty\rangle$, $\varphi \in L_p \cap L_\infty$ and $t \in \mathbf{R}$. In addition if $\varphi \in C_c^\infty$ then obviously $\lim_{t\downarrow 0} \|(I - T_t)\varphi\|_p = 0$, so by the density of C_c^∞ in L_p one concludes that T extends to a strongly continuous group on L_p satisfying the bounds (7), i.e., Condition II is valid. The implication II⇒I is trivial. This proves the equivalence if $a = \mathbb{1}$.

In general, if $\varphi \in C_c(\mathbf{R})$ and $p \in [1, \infty\rangle$ then

$$\begin{aligned} \|U\varphi\|_{L_p(\rho)}^p &= \int dx\, \rho(x)\, |\varphi(\gamma(x))|^p \\ &= \int dx\, (a\rho)(\gamma^{-1}(x))\, |\varphi(x)|^p = \|\varphi\|^p_{L_p((a\rho)\circ\gamma^{-1})} \quad . \end{aligned} \tag{8}$$

Moreover, it follows from (4) that $((a\rho) \circ \gamma^{-1})(x) \leq C\, e^{\omega|x-y|}((a\rho) \circ \gamma^{-1})(y)$ for all $x, y \in \mathbf{R}$ if and only if Condition III is valid. This proves the general case.

If the conditions are satisfied then S extends to the L_p-spaces by (3). The estimates on the norms of S_t are established in two steps. First, if $\omega > 0$ then it follows from (3) and the estimates on $\|T_s\|_{1\to 1}$ that

$$\|S_t\|_{1\to 1} \leq 2\, C\, e^{\omega^2 t}$$

for all $t > 0$. Since S is contractive on L_∞ one deduces from interpolation that

$$\|S_t\|_{p\to p} \leq (2\, C)^{1/p}\, e^{\omega^2 t/p}$$

for all $p \in \langle 1, \infty \rangle$ and $t > 0$. Alternatively, one can reverse the reasoning and use the interpolated bounds $\|T_s\|_{p\to p} \le C^{1/p}\, e^{\omega|s|/p}$ together with (3) to calculate that

$$\|S_t\|_{p\to p} \le 2\, C^{1/p}\, e^{\omega^2 t/p^2}$$

for all $p \in [1, \infty]$ and $t > 0$.

If $\omega = 0$ similar arguments apply and both lead to the bounds $\|S_t\|_{p\to p} \le C^{1/p}$. □

Remark 3.2. It follows from (8) that U transforms the system with derivation $a\, d/dx$ and density function ρ, where a satisfies (1), into a new system with $a = \mathbb{1}$ and a different density function ρ_a. Moreover, if $\rho = \mathbb{1}$ then ρ_a satisfies

$$\int_0^\infty \rho_a = \infty = \int_{-\infty}^0 \rho_a \tag{9}$$

Conversely, every system with $a = \mathbb{1}$ and a density function ρ_a satisfying (9) is the image under U of a system with $\rho = \mathbb{1}$.

The situation described by the proposition simplifies if $C = 1$. Then Condition III together with (5) implies that

$$\begin{aligned} \pm(a\rho)'(y)\, a(y) &= \lim_{t\downarrow 0} t^{-1}\Big((a\rho)(e^{\pm tX}y) - (a\rho)(y)\Big) \\ &\le \limsup_{t\downarrow 0} t^{-1}(e^{\omega t} - 1)(a\rho)(y) = \omega\,(a\rho)(y) \end{aligned}$$

for all $y \in \mathbf{R}$. Thus $\|\rho^{-1}(a\rho)'\|_\infty \le \omega$. Conversely, if $\|\rho^{-1}(a\rho)'\|_\infty \le \omega$ then

$$\rho(e^{tX}y)^{-1}\frac{d}{dt}\Big(e^{-\omega t}\,(a\rho)(e^{\pm tX}y)\Big) \le 0$$

for all $t \ge 0$. Hence Condition III is satisfied with $C = 1$. But the condition $\|\rho^{-1}(a\rho)'\|_\infty \le \omega$ can be expressed in terms of the derivation. Therefore one has the following corollary.

Corollary 3.3. *The following conditions are equivalent for all* $\omega \ge 0$.

I. *There exists a* $p \in [1, \infty\rangle$ *such that* T *extends to a strongly continuous group on* $L_p(\mathbf{R}\,; \rho\, dx)$ *satisfying the bounds* $\|T_t\|_{p\to p} \le e^{\omega|t|/p}$ *for all* $t \in \mathbf{R}$.

II. *For all* $p \in [1, \infty\rangle$ *the group* T *extends to a strongly continuous group on* $L_p(\mathbf{R}\,; \rho\, dx)$ *satisfying the bounds* $\|T_t\|_{p\to p} \le e^{\omega|t|/p}$ *for all* $t \in \mathbf{R}$.

III. $\|\rho^{-1}(a\rho)'\|_\infty \le \omega$.

IV. $|(\psi, (X + X^*)\varphi)| \le \omega\, \|\psi\|_q\, \|\varphi\|_p$ *for all* $\varphi, \psi \in C_c^\infty(\mathbf{R})$ *and for one pair* (*for all pairs*) *of dual exponents* $p, q \in [1, \infty]$.

Moreover, if these conditions are satisfied then for all $p \in [1, \infty\rangle$ *the semigroup* S *extends to a strongly continuous semigroup on* L_p, *satisfying the bounds*

$$\|S_t\|_{p\to p} \le \big(2^{1/p} e^{\omega^2 t/p}\big) \wedge \big(2 e^{\omega^2 t/p^2}\big)$$

for all $t > 0$. In addition H_1 satisfies a Gårding inequality. More precisely,

$$\mathrm{Re}(\varphi, H_1\varphi) \geq (1-\varepsilon)\|X\varphi\|_2^2 - (4\varepsilon)^{-1}\|X + X^*\|_{2\to 2}^2\|\varphi\|_2^2$$

for all $\varphi \in C_c^\infty(\mathbf{R})$ and $\varepsilon > 0$.

Proof. The equivalence of the first three conditions and the existence of the extension of the semigroup S follow from Proposition 2.1 and the above discussion. Conditions III and IV are equivalent because

$$\begin{aligned}(\psi, X\varphi) + (X\psi, \varphi) &= \int_{\mathbf{R}} dx\, (a\rho)(x)\big(\psi(x)\,\varphi'(x) + \psi'(x)\,\varphi(x)\big)\\ &= -\int_{\mathbf{R}} dx\, \rho(x)\big(\rho(x)^{-1}(a\rho)'(x)\big)\psi(x)\,\varphi(x)\end{aligned}$$

for all $\varphi, \psi \in C_c^\infty(\mathbf{R})$. Finally if $\varphi \in C_c^\infty(\mathbf{R})$ then

$$\begin{aligned}\mathrm{Re}(\varphi, H_1\varphi) &= -\,\mathrm{Re}(X^*\varphi, X\varphi)\\ &= \|X\varphi\|_2^2 - \mathrm{Re}((X^* + X)\varphi, X\varphi)\\ &\geq \|X\varphi\|_2^2 - \|(X^*+X)\varphi\|_2\|X\varphi\|_2\\ &\geq (1-\varepsilon)\|X\varphi\|_2^2 - (4\varepsilon)^{-1}\|X + X^*\|_{2\to 2}^2\|\varphi\|_2^2\end{aligned}$$

for all $\varepsilon > 0$. □

The corollary, applied with $\omega = 0$, gives the following criteria for T or S to extend to a contraction group or semigroup on the L_p-spaces.

Proposition 3.4. *The following are equivalent.*

I. *There is a $p \in [1, \infty\rangle$ such that T extends to a strongly continuous contraction group on $L_p(\mathbf{R}\,; \rho\, dx)$.*

II. *For all $p \in [1, \infty\rangle$ the group T extends to a strongly continuous contraction group on $L_p(\mathbf{R}\,; \rho\, dx)$.*

III. *There is a $p \in [1, \infty\rangle$ such that S extends to a strongly continuous contraction semigroup on $L_p(\mathbf{R}\,; \rho\, dx)$.*

IV. *For all $p \in [1, \infty\rangle$ the semigroup S extends to a strongly continuous contraction semigroup on $L_p(\mathbf{R}\,; \rho\, dx)$.*

V. *The function $a\rho$ is constant.*

Proof. The implications V⇔I⇔II⇒IV follow from Corollary 3.3 and the implication IV⇒III is trivial.

If Condition III is valid for some $p \in [1, 2]$ then it follows by interpolation with the contraction semigroup on L_∞ that Condition III is valid for all $p > 2$. Hence it suffices to show that if $p \in \langle 2, \infty\rangle$ and S extends to a strongly continuous contraction semigroup on $L_p(\mathbf{R}\,; \rho\, dx)$ then Condition V is valid. Moreover, using the map U and (8), we may assume that $a = \mathbb{1}$, as in the proofs of Propositions 2.1 and 3.1.

Fix $p \in \langle 2,\infty\rangle$ and assume S extends to a strongly continuous contraction semigroup on $L_p(\mathbf{R}\,;\rho\,dx)$. Then it follows from the Lumer–Phillips theorem, [LuP] Theorem 3.1, that the generator H of the semigroup S on $L_p(\mathbf{R}\,;\rho\,dx)$ is accretive. Define the semi-inner product $[\,\cdot\,,\,\cdot\,]$ on $L_p(\mathbf{R}\,;\rho\,dx)$ by

$$[\psi,\varphi] = \int \rho\,\psi\,\|\varphi\|_p^{2-p}\,|\varphi|^{p-2}\,\overline{\varphi}$$

with obvious modifications if $\varphi = 0$. Then $\mathrm{Re}[H\varphi,\varphi] \geq 0$ for all $\varphi \in D(H)$. If $\varphi \in C_c^2(\mathbf{R})$ then $\varphi \in D(H_1)$ and $H_1\varphi \in L_p(\mathbf{R}\,;\rho\,dx)$. So $\varphi \in D(H)$ and $H_1\varphi = H\varphi$. Moreover, if in addition $\varphi \geq 0$, then

$$\int d(\rho\,\varphi^{p-1})\,(d\,\varphi) = \int \rho\,\varphi^{p-1}\,H_1\varphi = \int \rho\,\varphi^{p-1}\,H\varphi = \|\varphi\|_p^{p-2}[H\varphi,\varphi] \geq 0$$

where $d = d/dx$. Hence

$$\int d(\rho\,\varphi^{p-1})\,(d\,\varphi) \geq 0 \tag{10}$$

for all $\varphi \in C_c^1(\mathbf{R})$ with $\varphi \geq 0$, by approximation.

Next fix an even function $\tau \in C_c^\infty(\mathbf{R})$ such that $0 \leq \tau \leq 1$, $\tau(0) = \tau(1) = 1$ and τ is decreasing on $[0,\infty\rangle$. For all $n \in \mathbf{N}$ define $\varphi_n \in C_c^1(\mathbf{R})$ by

$$\varphi_n(x) = \rho(x)^{-p^{-1}}\,\tau(n^{-1}x) \quad .$$

Then

$$\varphi_n'(x) = -p^{-1}\rho(x)^{-1-p^{-1}}\,\rho'(x)\,\tau(n^{-1}x) + n^{-1}\rho(x)^{-p^{-1}}\,\tau'(n^{-1}x) \quad .$$

Similarly, $(\rho\,\varphi_n^{p-1})(x) = \rho(x)^{p^{-1}}\,\tau(n^{-1}x)^{p-1}$ and

$$\begin{aligned}(\rho\,\varphi_n^{p-1})'(x) &= p^{-1}\rho(x)^{-1+p^{-1}}\,\rho'(x)\,\tau(n^{-1}x)^{p-1}\\ &\qquad + n^{-1}(p-1)\rho(x)^{p^{-1}}\,\tau(n^{-1}x)^{p-2}\,\tau'(n^{-1}x) \quad .\end{aligned}$$

Then by (10) it follows that

$$\begin{aligned}0 \leq \int (\rho\,\varphi_n^{p-1})'\,\varphi_n' = \int dx\Big(&-p^{-2}\rho(x)^{-2}\,\big(\rho'(x)\big)^2\tau(n^{-1}x)^p\\ &- n^{-1}(1-2p^{-1})\,\rho(x)^{-1}\,\rho'(x)\,\tau(n^{-1}x)^{p-1}\,\tau'(n^{-1}x)\\ &+ n^{-2}(p-1)\,\tau(n^{-1}x)^{p-2}\,\big(\tau'(n^{-1}x)\big)^2\Big) \quad .\end{aligned}$$

Using the estimate $a\,b \leq \varepsilon a^2 + (4\varepsilon)^{-1}b^2$ for the second term, setting $\varepsilon = (2p(p-2))^{-1}$ and rearranging one finds

$$\begin{aligned}(2p^2)^{-1}\int dx\,\rho(x)^{-2}\,\big(\rho'(x)\big)^2\tau(n^{-1}x)^p &\leq n^{-2}\int dx\,p^2\tau(n^{-1}x)^{p-2}\,\big(\tau'(n^{-1}x)\big)^2\\ &= n^{-1}\int p^2\tau^{p-2}\,(\tau')^2\end{aligned}$$

for all $n \in \mathbf{N}$. Then the monotone convergence theorem establishes that

$$\int \rho^{-2}(\rho')^2 = \lim_{n\to\infty} \int dx\, \rho(x)^{-2} \left(\rho'(x)\right)^2 \tau(n^{-1}x)^p$$
$$\leq \lim_{n\to\infty} 2n^{-1}\, p^4 \int \tau^{p-2}\,(\tau')^2 = 0 \quad .$$

Therefore $\rho' = 0$ as required. □

In the unweighted case, i.e., $\rho = \mathbb{1}$, the proposition establishes that S extends to a contraction semigroups on one of the L_p-spaces with $p < \infty$ only in the case that X is proportional to d/dx.

4. Examples

Next we give two examples of rather unexpected properties although there is nothing inherently pathological about the weight ρ or the coefficient a. In fact in both examples $\rho = \mathbb{1}$ and the coefficient a of the derivation is strictly positive, smooth and uniformly bounded. The first example gives a weakly* continuous group T and semigroup S which do not extend from L_∞ to the other L_p spaces. The principal reason for this singular behaviour is the fact that $\inf a = 0$, i.e., there is a mild degeneracy at infinity.

Example 4.1. Let $\rho = 1$. For all $n \in \mathbf{N}_0$ define $h_n = n!^{-1}$. Define $y_n \in \mathbf{R}$ for all $n \in \mathbf{N}_0$ by $y_0 = 0$ and inductively

$$y_{n+1} = y_n + 4^{-1}(h_n + h_{n+1}) + 2^{-1}$$

for all $n \in \mathbf{N}$. Define $\tilde{a}\colon \mathbf{R} \to \langle 0, \infty\rangle$ by

$$\tilde{a}(x) = \begin{cases} h_n & \text{if } x \in [y_n - 4^{-1}h_n, y_n + 4^{-1}h_n\rangle \quad (n \in \mathbf{N}_0) \quad , \\ 1 & \text{if } x \in [y_n + 4^{-1}h_n, y_n + 4^{-1}h_n + 2^{-1}\rangle \quad (n \in \mathbf{N}_0) \quad , \\ 1 & \text{if } x \in \langle -\infty, 0] \quad . \end{cases}$$

Then $\tilde{a}(y_n) = h_n$ and $\int_{y_n}^{y_{n+1}} dx\, \tilde{a}(x)^{-1} = 1$ for all $n \in \mathbf{N}$. Next we regularize $\tilde{a}^{-1}$. For all $n \in \mathbf{N}_0$ let $\chi_n \in C_c^\infty(\mathbf{R})$ be such that $\chi_n \geq 0$, $\int \chi_n = 1$, $\operatorname{supp} \chi_n \subseteq [-8^{-1}h_n, 8^{-1}h_n]$ and $\chi_n(-x) = \chi_n(x)$ for all $x \in \mathbf{R}$. Define $a \in C^\infty(\mathbf{R})$ by

$$a(x)^{-1} = \begin{cases} (\chi_0 * \tilde{a}^{-1})(x) & \text{if } x \leq 0 \quad , \\ (\chi_n * \tilde{a}^{-1})(x) & \text{if } n \in \mathbf{N}_0 \text{ and} \\ & \qquad x \in [y_n - 4^{-1}h_n - 4^{-1}, y_n + 4^{-1}h_n + 4^{-1}\rangle \quad . \end{cases}$$

Then $a(y) = h_n$ for all $y \in [y_n - 8^{-1}h_n, y_n + 8^{-1}h_n]$ and $\int_{y_n}^{y_{n+1}} dx\, a(x)^{-1} = 1$ for all $n \in \mathbf{N}$. Hence $d(y_n\,; y_{n+1}) = 1$ for all $n \in \mathbf{N}$. But $a(y_n) = (n+1)\, a(y_{n+1})$ for all $n \in \mathbf{N}$. Therefore Condition III of Proposition 3.1 is not valid. In particular the group T does not extend to any of the other L_p spaces. Next we show that the semigroup S also does not extend to another L_p space.

Let $p \in [1, \infty\rangle$, $t > 0$ and let q be the dual exponent of p. For all $n \in \mathbf{N}$ set $I_n = [y_n - 8^{-1}h_n, y_n + 8^{-1}h_n]$. Let $n \in \mathbf{N}$. Set $\varphi = \mathbb{1}_{I_{n+1}}$ and $\psi = \mathbb{1}_{I_n}$. Then

$\|\varphi\|_p = |I_{n+1}|^{1/p}$ and $\|\psi\|_q = |I_n|^{1/q}$. Moreover, $d(x\,;y) \leq d(y_{n-1}\,;y_{n+2}) = 3$ for all $x \in I_n$ and $y \in I_{n+1}$. Therefore

$$\begin{aligned}(\psi, S_t\varphi) &= (4\pi t)^{-1/2} \int_{I_n} dx \int_{I_{n+1}} dy\, a(y)^{-1}\, e^{-d(x;y)^2(4t)^{-1}} \\ &\geq (4\pi t)^{-1/2} \int_{I_n} dx \int_{I_{n+1}} dy\, a(y)^{-1}\, e^{-3t^{-1}} \\ &= (4\pi t)^{-1/2} |I_n|\, |I_{n+1}|\, h_{n+1}^{-1}\, e^{-3t^{-1}} \quad .\end{aligned}$$

So

$$\|S_t\|_{p\to p} \geq (4\pi t)^{-1/2} |I_n|^{1/p}\, |I_{n+1}|^{1/q}\, h_{n+1}^{-1}\, e^{-3t^{-1}} = (64\pi t)^{-1/2}(n+1)^{1/p}\, e^{-3t^{-1}} \quad .$$

Hence the operator S_t on L_∞ does not extend to a continuous operator on L_p for any $p \in [1, \infty\rangle$ or $t > 0$. □

In the next example the coefficient a of X is uniformly bounded above and below by a positive constant but $\sup a' = \infty$. The semigroup S extends to a strongly continuous semigroup on all the L_p-spaces but the real part of the generator of S on L_2 is not lower semibounded. This contrasts with the case of continuous self-adjoint semigroups where boundedness of the semigroup immediately implies lower semiboundedness of the generator.

Example 4.2. First, let $\rho = 1$ and let $\chi \in C_c^\infty(\mathbf{R})$ be such that $0 \leq \chi \leq 3$, $\chi' \geq 0$, $\chi(x) = 0$ if $x \leq 0$, $\chi(x) = 3$ if $x \geq 3$ and $\chi(x) = x$ if $1 \leq x \leq 2$. Define $a\colon \mathbf{R} \to [1, 4]$ by

$$a(x) = 1 + \sum_{n=1}^{\infty} \big(\chi(n(x - 16n)) - \chi(n(x - (16n + 8))\big) \quad .$$

Thus $a = 1$ on an infinite sequence of intervals of length almost equal to 8 spaced at distance 8 one from the other. On the intermediate intervals a increases smoothly to the value 4 and then decreases in a similar fashion to the value 1. The rate of increase and decrease, however, becomes larger with the distance of the interval from the origin. Nevertheless $a \in C^\infty(\mathbf{R})$ and the bounds of Proposition 3.1.III are valid with $C = 4$ and $\omega = 0$. In particular S_t extends to the L_p-spaces and $\|S_t\|_{p\to p} \leq 8^{1/p}$.

Secondly, let $n \in \mathbf{N}$ with $n \geq 4$. Let $\psi \in C^\infty(\mathbf{R})$ be such that $\psi(x) = 3$ for all $x \leq 16n + 8$, $0 \leq \psi' \leq n^{1/2}$, $\psi'(x) = 0$ for all $x \geq 16n + 8 + 4n^{-1}$ and $\psi'(x) = n^{1/2}$ for all $x \in [16n + 8 + n^{-1}, 16n + 8 + 2n^{-1}]$. Then $3 \leq \psi(16n + 8 + 4n^{-1}) \leq 5$. Now define $\varphi \in C_c^\infty(\mathbf{R})$ by

$$\varphi(x) = \begin{cases} \chi(x - (16n + 4)) & \text{if } x \leq 16n + 8 \\ \psi(x) & \text{if } x \in [16n + 8, 16n + 8 + 4n^{-1}] \\ 3^{-1}\psi(16n + 8 + 4n^{-1})\big(3 - \chi(x - (16n + 8 + 4n^{-1})\big) & \\ & \text{if } x \geq 16n + 8 + 4n^{-1} \end{cases}$$

Then $\|\varphi\|_2 \le 5\cdot(12)^{1/2} = (300)^{1/2}$ and

$$\|\varphi'\|_2 \le 2\|\chi'\|_\infty + n^{1/2}(4n^{-1})^{1/2} + 3^{-1}\psi(16n+8+4n^{-1})\|\chi'\|_\infty \le 2+4\|\chi'\|_\infty \quad .$$

But $a'\,a\,\varphi\,\varphi' \le 0$ and

$$-(a'\varphi, X\varphi) \ge \int_{16n+8+n^{-1}}^{16n+8+2n^{-1}} (-a'\,a\,\varphi\,\varphi') \ge \int_{16n+8+n^{-1}}^{16n+8+2n^{-1}} n\cdot 2\cdot 3\cdot n^{1/2} = 6n^{1/2}$$

by the previous estimates. Therefore

$$\begin{aligned}\mathrm{Re}(\varphi, H_\infty\varphi) &= \|X\varphi\|_2^2 + \mathrm{Re}(a'\varphi, X\varphi)\\ &\le \|a\|_\infty^2(2+4\|\chi'\|_\infty)^2 - 8n^{1/2}\\ &\le -300^{-1}\big(6n^{1/2} - 16(2+4\|\chi'\|_\infty)^2\big)\|\varphi\|_2^2 \quad .\end{aligned}$$

Consequently, $\mathrm{Re}\,H_\infty$ is not lower semibounded. This is despite the uniform boundedness of S on L_2.

Next, since S is uniformly bounded on each of the L_p-spaces, the spectrum $\sigma(H)$ of the generator H of the semigroup on L_p is contained in the right half-plane. But $a(x) \in [1,4]$ for all $x \in \mathbf{R}$. Therefore $4^{-1}|x-y| \le d(x\,;y) \le |x-y|$ and Proposition 2.3 implies that

$$K_t(x\,;y) \le (4\pi t)^{-1/2}\, e^{-|x-y|^2(64t)^{-1}}$$

for all $x, y \in \mathbf{R}$ and $t > 0$. Hence it follows from [Kun] or [LiV] that $\sigma(H)$ is independent of $p \in [1,\infty]$. On the other hand $\mathrm{Re}\,H_\infty$ is not lower semibounded on L_2 and the above estimates establish that $\langle -\infty, 0] \subset \Theta(H)$, the L_2-numerical range of H. Therefore $\Theta(H) \neq \sigma(H)$ on L_2.

In fact this example illustrates the extreme situation that the spectrum of H is contained in the right half-plane but the numerical range is the whole complex plane. This follows since one can establish that the numerical range $\Theta(H) = \mathbf{C}$ by a small modification of the foregoing estimates applied to the function $\tilde\varphi \in C_c^\infty(\mathbf{R})$ defined by

$$\tilde\varphi(x) = e^{i\lambda x}\,\tau(x) + \varphi(x) \quad ,$$

where $\lambda \in \mathbf{R}$ and $\tau \in C_c^\infty(\langle -1, 4\rangle)$ is fixed such that $0 \le \tau \le 1$ and $\tau|_{[0,3]} = 1$. One also uses the observation that the numerical range is convex.

Finally note that the semigroup S has a bounded holomorphic extension to the open right half-plane on each of the L_p-spaces, $p \in [1,\infty\rangle$. This follows from the explicit form of the kernel given in Propositions 2.3. Therefore the operator H is of type S_{0+}. Nevertheless, since $\Theta(H) = \mathbf{C}$ the operator H is not sectorial. $\Box$

Example 4.2 has $\rho = \mathbb{1}$ and $a \neq \mathbb{1}$ but using Remark 3.2 one can convert it into an example with $a = \mathbb{1}$ and $\rho \neq \mathbb{1}$. Then ρ is an oscillating function satisfying (9). Alternatively one can construct an example of the latter type directly by analogous arguments with ρ bounded above and below, but now ρ' rather than a' has to increase appropriately.

5. Volume doubling

Let $V(x\,;r)$ denote the measure of the ball of radius r centred at x, i.e., the set $\{y : d(x\,;y) < r\} = \langle e^{-rX}x, e^{rX}x\rangle$. Then V is defined, as usual, to have the volume doubling property if there is a $c > 0$ such that

$$V(x\,;2r) \le c\,V(x\,;r)$$

for all $r > 0$. This property can be immediately related to the conditions of Proposition 3.1 which are necessary and sufficient for the continuous extension of T to the L_p-spaces.

Proposition 5.1.

I. *If the equivalent conditions of Proposition* 3.1 *are satisfied then*

$$V(x\,;2r) \le 2\,C^2\,e^{3\omega}\,V(x\,;r) \tag{11}$$

for all $x \in \mathbf{R}$ and $r \in \langle 0,1]$ where C and ω are the parameters of Proposition 3.1*. Moreover if $\omega = 0$ then* (11) *is valid for all $x \in \mathbf{R}$ and $r > 0$.*

II. *If there exist $c > 0$ and a function $v\colon \langle 0,\infty\rangle \to \mathbf{R}$ such that*

$$c^{-1}\,v(r) \le V(x\,;r) \le c\,v(r)$$

for all $x \in \mathbf{R}$ and $r \in \langle 0,1]$ then Condition III *of Proposition* 3.1 *is satisfied with $\omega = 0$.*

Proof. It follows by definition that

$$V(x\,;r) = \int_{e^{-rX}x}^{e^{rX}x} dy\,\rho(y) \quad .$$

But

$$\frac{d}{dr}V(x\,;r) = (a\rho)(e^{rX}x) + (a\rho)(e^{-rX}x) \quad .$$

Hence

$$V(x\,;r) = \int_0^r ds\,\Big((a\rho)(e^{sX}x) + (a\rho)(e^{-sX}x)\Big) = \int_{-r}^r ds\,(a\rho)(e^{sX}x) \quad .$$

Therefore if Condition III of Proposition 3.1 is satisfied one estimates that

$$2\,C^{-1}r\,e^{-\omega r}(a\rho)(x) \le V(x\,;r) \le 2\,C\,r\,e^{\omega r}(a\rho)(x)$$

for all $x \in \mathbf{R}$ and $r > 0$. These bounds imply (11) for all $x \in \mathbf{R}$ and $r \in \langle 0,1]$ or, if $\omega = 0$, for all $r > 0$.

If, however, the assumptions of the second statement are valid then

$$c^{-1}\,v(r) \le V(x\,;r) = \int_0^r ds\,\Big((a\rho)(e^{sX}x) + (a\rho)(e^{-sX}x)\Big) \le 2\,r \max_{y\in[e^{-X}x,e^{X}x]}(a\rho)(y)$$

for all $x \in \mathbf{R}$ and $r \in \langle 0,1]$. Similarly

$$c\,v(r) \ge r \min_{y\in[e^{-X}x,e^{X}x]}(a\rho)(y) \quad .$$

Hence there exists a $c_1 > 0$ such that $c_1^{-1} r \leq v(r) \leq c_1 r$ for all $r \in \langle 0, 1]$. But then

$$\begin{aligned} 2(a\rho)(x) &= \lim_{r\downarrow 0} r^{-1} \int_0^r ds \Big((a\rho)(e^{sX}x) + (a\rho)(e^{-sX}x) \Big) \\ &= \lim_{r\downarrow 0} r^{-1} V(x\,; r) \leq \limsup_{r\downarrow 0} r^{-1} c\, v(r) \leq c\, c_1 \end{aligned}$$

for all $x \in \mathbf{R}$. Similarly $2(a\rho)(x) \geq (c\, c_1)^{-1}$. Hence $(2c\, c_1)^{-1} \leq a\rho \leq 2^{-1} c\, c_1$ and Condition III of Proposition 3.1 is satisfied with $\omega = 0$. □

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On the Curve Shortening Flow with Triple Junction

Joachim Escher and Zhaoyong Feng

In memoriam Günter Lumer

Abstract. In this paper we show that the curve shorting flow with contact angle and triple junction in a mirror symmetric configuration is locally well posed in suitable Hölder spaces.

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Keywords. Analytic semigroups, curve shortening flow, triple junction, contact angle, Young's law.

1. Introduction

We study a diffusion model of a ternary alloy in a non-equilibrium state. Moreover we consider a symmetric and plane configuration. To describe the system precisely, let $\Omega \subset \mathbb{R}^2$ be a bounded domain in which the alloy is located. It is further assumed that three phases are separated by three moving boundaries $\Gamma_i(t)$, $i = 1, 2, 3$, where $t \geq 0$ denotes the time variable. It is assumed that these interfaces meet in a triple junction $m(t) \in \Omega$ at one of their end points and that they perpendicularly hit the boundary $\partial\Omega$ of Ω at their other end points $b_i(t)$. The evolution of the interfaces is governed by the following system:

$$\begin{cases} \text{for } t > 0 & \begin{cases} \text{along } \Gamma_i(t): & V_i = \sigma_i\left(\kappa_i - \bar{\kappa}_i\right), \\ \text{at } b_i(t): & \Gamma_i(t) \perp \partial\Omega, \\ \text{at } m(t): & \angle(\Gamma_i(t), \Gamma_j(t)) = \theta_k, \text{ for } i, j, k \\ & \in \{1, 2, 3\} \text{ mutually different}, \\ & \sigma_1\kappa_1 + \sigma_2\kappa_2 + \sigma_3\kappa_3 = 0, \end{cases} \\ \text{at } t = 0: \quad \Gamma_i(0) = \Gamma_{i0}, & m(0) = m_0. \end{cases} \tag{1.1}$$

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Here

$$\bar{\kappa}_i := \frac{1}{|\Gamma_i|} \int_{\Gamma_i} \kappa_i ds, \tag{1.2}$$

with $|\Gamma_i|$ being the length of Γ_i. Throughout this paper we write s for the arc length parameter of $\Gamma_i(t)$ running from $m(t)$ to $b_i(t)$. We denote by V_i and κ_i the normal velocity and the curvature of $\Gamma_i(t)$, respectively. Let $T_i(t)$ denote the unit tangent vector of $\Gamma_i(t)$ with respect to s and choose a unit normal vector $N_i(t)$ such that $(T_i(t), N_i(t))$ is a positive oriented Frenet frame. With this convention we can clarify the orientation of V_i and κ_i: both quantities are always determined with respect to $N_i(t)$. Moreover, σ_i and θ_i are positive constants with the constraints $0 < \theta_i < \pi,\ \theta_1 + \theta_2 + \theta_3 = 2\pi$ and satisfying Young's law:

$$\frac{\sigma_1}{\sin\theta_1} = \frac{\sigma_2}{\sin\theta_2} = \frac{\sigma_3}{\sin\theta_3}.$$

For a particular geometric configuration, system (1.1) has recently been investigated in [2]. In this paper we study (1.1) for a configuration proposed by K. Ito and Y. Kohsaka in [5] for the surface diffusion flow, see also [3]. Finally we also refer to [7] and [1] where curve shortening flows with triple junction are considered.

Let us now parameterize the geometric set up we are interested in. We first fix Ω by putting

$$\Omega := [-a, 0] \times [-b, b],$$

where a and b are positive sufficiently large constants, see Figure 1.

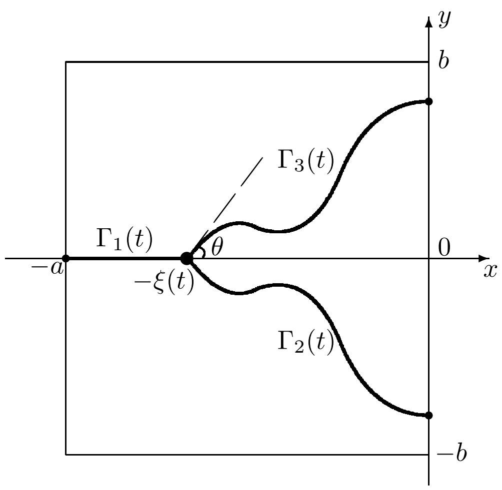

Figure 1

We consider an evolution such that $\Gamma_1(t)$ is always a segment on the x-axis, and $\Gamma_2(t)$ and $\Gamma_3(t)$ are symmetric with respect to the x-axis. For definiteness assume that $\Gamma_3(t)$ lies in $\{(x, y) \in \Omega;\ y \geq 0\}$. Let $\theta \in (0, \pi/2)$ be given and set $\theta_1 = 2\theta$ and $\theta_2 := \theta_3 := \pi - \theta$. For simplicity we put $\sigma_2 = \sigma_3 = 1$. Then Young's law implies that $\sigma_1 = 2\cos\theta$. Observe that the symmetry of the configuration also forces the relation $\kappa_2 = -\kappa_3$. Thus the continuity of the chemical potential $\sigma_1\kappa_1+\sigma_2\kappa_2+\sigma_3\kappa_3 = 0$ is automatically fulfilled. To have a neat representation, let us introduce the following notations: given $\xi \in (0,a)$ and $u : [-\xi, 0] \to [0,b)$, let

$$\begin{aligned}\Lambda_1[u,\xi] &:= \{(x,0); -a \le x \le -\xi\}\,,\\ \Lambda_2[u,\xi] &:= \{(x,-u(x)); -\xi \le x \le 0\}\,,\\ \Lambda_3[u,\xi] &:= \{(x,u(x)); -\xi \le x \le 0\}\,,\end{aligned}$$

and set $\Lambda[u,\xi] := \bigcup\limits_{i=1}^{3} \Lambda_i[u,\xi]$.

Definition 1.1.

(i) Let $\theta \in (0,\pi/2)$. We say that a union of three curves Γ belongs to $\mathcal{S}_\theta$, if there are $\xi \in (0,a)$ and non-negative function $u \in C^2[-\xi,0]$ with $u(-\xi) = 0$, $u_x(-\xi) = \tan\theta$, $u_x(0) = 0$, such that $\Gamma = \Lambda[u,\xi]$ and m is given by the point $(-\xi,0)$.

(ii) Let $A > 0$ be a given constant. We say that a union of three curves Γ belongs to $\mathcal{C}_A$ if there are $\xi \in (0,a)$ and non-negative function $u \in C^1[-\xi,0]$ with $u(-\xi) = 0$, $u_x(0) = 0$, and $\int_{-\xi}^{0} u(x)dx = A$ such that $\Gamma = \Lambda[u,\xi]$ and m is given by the point $(-\xi,0)$.

Note that any union of three curves in $\mathcal{S}_\theta$ or in $\mathcal{C}_A$ is symmetric with respect to the x-axis. In view of the structure of the evolution problem (1.1), it can be expected that if $\Gamma_0 = \bigcup_{i=1}^{3}\Gamma_{i0} \in \mathcal{S}_\theta$, then the solution $\Gamma(t)$ of (1.1) also belongs to $\mathcal{S}_\theta$ for all $t > 0$ as long as it exists. So we proceed with the evolution problem (1.1) on $\mathcal{S}_\theta$ and set $\Gamma(t) = \Lambda[u(t,\cdot),\xi(t)]$ with the moving triple point $m(t)$ given by $(-\xi(t),0)$ for $t > 0$. The initial conditions are denoted by $\Gamma_0 = \Lambda[u_0,\xi_0]$ with $m_0 = (-\xi_0, 0)$. Moreover, we also note that the spatial mean value of the curvature is given by

$$\bar{\kappa}_3(t) = -\frac{\theta}{L_3(t)}, \tag{1.3}$$

where $L_3(t)$ is the length of $\Lambda_3(t)$. Indeed, if we let $\omega(t,s)$ denote the angle between the unit normal of $\Lambda_3(t)$ and the x-axis, then $\kappa_3(t,s) = \partial\omega/\partial s(t,s)$, see [4]. Thus it follows from (1.2) that

$$\begin{aligned}\bar{\kappa}_3(t) &= \frac{1}{L_3(t)}\int_0^{L_3(t)} \frac{\partial\omega}{\partial s}ds\\ &= \frac{1}{L_3(t)}\left(\omega(t,L_3(t)) - \omega(t,0)\right).\end{aligned}$$

By the boundary conditions at $m(t)$ and $b_3(t)$, we have that $\omega(t,0) = \pi/2+\theta$ and $\omega(t,L_3(t)) = \pi/2$. This implies (1.3).

Let us now rewrite system (1.1) by using the new unknowns $(u(t,x),\xi(t))$ defined $t \ge 0$ and $-\xi(t) \le x \le 0$:

$$\begin{cases} u_t = \dfrac{u_{xx}}{1+u_x^2} + \dfrac{(1+u_x^2)^{1/2}\theta}{\displaystyle\int_{-\xi(t)}^0 (1+u_x^2)^{1/2}dx}, & -\xi(t) \le x \le 0, \\ u_x(t,-\xi(t)) = \tan\theta, \\ u_x(t,0) = 0, \\ u(t,-\xi(t)) = 0, \\ u(0,x) = u_0(x) & \text{for } -\xi_0 \le x \le 0, \\ \xi(0) = \xi_0. \end{cases} \tag{1.4}$$

2. Local existence

In this section, we study local existence of (1.4) for the initial data $\Gamma_0 \in C^{2+\alpha}$ with $0 < \alpha < 1$. We shall first derive the equation for ξ. To do so we assume that (u, ξ) is a solution of the problem

$$\begin{cases} u_t = \dfrac{u_{xx}}{1+u_x^2} + \dfrac{(1+u_x^2)^{1/2}\theta}{\displaystyle\int_{-\xi(t)}^0 (1+u_x^2)^{1/2}dx}, & -\xi(t) \le x \le 0, \\ u_x(t,-\xi(t)) = \tan\theta, \\ u_x(t,0) = 0, \\ u(0,x) = u_0(x) & \text{for } -\xi_0 \le x \le 0, \\ \xi(0) = \xi_0. \end{cases} \tag{2.1}$$

Observe that (2.1) and (1.4) coincide except for the condition $u(t,-\xi(t)) = 0$. For such a function the condition $u(t,-\xi(t)) = 0$ is equivalent to following equation

$$\dot{\xi}(t) = \frac{\cos^3\theta}{\sin\theta} u_{xx}(t,-\xi(t)) + \frac{\theta}{\sin\theta \displaystyle\int_{-\xi(t)}^0 (1+u_x^2(t,x))^{1/2}dx}, \tag{2.2}$$

where $\dot{\xi} := d\xi/dt$. Indeed, let us first assume that the equation $u(t,-\xi(t)) = 0$ is satisfied. By differentiation with respect to t, then we get

$$[u_t(t,x) - u_x(t,x)\dot{\xi}(t)]\Big|_{x=-\xi(t)} = 0.$$

Using the boundary condition $u_x(t,-\xi(t))) = \tan\theta$, it follows that

$$u_t(t,x)\Big|_{x=-\xi(t)} = \cos^2\theta u_{xx}(t,-\xi(t)) + \frac{\theta\sec\theta}{\displaystyle\int_{-\xi(t)}^0 (1+u_x^2(t,x))^{1/2}dx}.$$

Thus (2.2) holds true. Conversely, if (2.2) is satisfied, then we get

$$\frac{du(t,-\xi(t))}{dt} = 0.$$

By virtue of $\Gamma_0 \in \mathcal{S}_\theta$, we have $u(0,-\xi(0)) = u_0(-\xi_0) = 0$, so that $u(t,-\xi(t)) = 0$.

In order to normalize the x-coordinate, we introduce the following change of variables. Given $t \geq 0$, let:

$$\eta = 1 + \frac{x}{\xi(t)}, \qquad v(t,\eta) = u(t,(\eta-1)\xi(t)).$$

In these new coordinates $(t,\eta) \in (0,T] \times [0,1]$, system (2.1) and (2.2) is given by:

$$v_t = \frac{(\eta-1)\dot{\xi}v_\eta}{\xi} + \frac{v_{\eta\eta}}{\xi^2+v_\eta^2} + \frac{\theta(\xi^2+v_\eta^2)^{1/2}}{\displaystyle\int_0^1 (\xi^2+v_\eta^2)^{1/2}d\eta}, \tag{2.3}$$

$$v_\eta(t,0) = \xi(t)\tan\theta, \tag{2.4}$$

$$v_\eta(t,1) = 0, \tag{2.5}$$

$$v(0,\eta) = v_0(\eta) := u_0((\eta-1)\xi_0), \tag{2.6}$$

$$\dot{\xi}(t) = \frac{\cos^3\theta}{\xi^2\sin\theta}v_{\eta\eta}(t,0) + \frac{\xi\theta}{\sin\theta\displaystyle\int_0^1 (\xi^2(t)+v_\eta^2(t,\eta))^{1/2}d\eta}, \tag{2.7}$$

$$\xi(0) = \xi_0. \tag{2.8}$$

To solve this new system (2.3)–(2.8) locally in time, we need some preliminaries. We shall first linearize (2.3)–(2.8) about the initial data. For convenience, we introduce a parameter $\tau \geq 0$, which is regarded as initial time. Given a sufficiently regular pair of functions $(\bar{v},\bar{\xi})$, we define a linear differential operator $\mathcal{A}_\tau$, acting on a C^2-function $U : [0,1] \ni \eta \mapsto U(\eta) \in \mathbb{R}$, by

$$\begin{aligned}\mathcal{A}_\tau U := \frac{1}{g(\bar{v},\bar{\xi})(\tau,\eta)}\partial_\eta^2 U \quad &+ \quad \frac{\theta\,\bar{v}_\eta(\tau,\eta)}{L(\bar{v},\bar{\xi})(\tau)\,g^{1/2}(\bar{v},\bar{\xi})(\tau,\eta)}\,\partial_\eta U \\ &+ \quad \frac{(\eta-1)\cos^3\theta}{\sin\theta}\,\frac{\bar{v}_\eta(\tau,\eta)}{\bar{\xi}^3(\tau)}\partial_\eta^2 U(0),\end{aligned}$$

where

$$g(v,\xi)(t,\eta) = \xi^2(t) + v_\eta^2(t,\eta), \quad \text{and} \quad L(v,\xi)(t) = \int_0^1 g^{1/2}(v,\xi)(t,\eta)d\eta.$$

If no confusion seems likely, we abbreviate $g(v,\xi)(t,\eta)$ and $L(v,\xi)(t)$ by $g(t,\eta)$ and $L(t)$, respectively. In addition, we set

$$\begin{aligned}F_\tau(v,\xi)(t,\eta) := \quad & \left(\frac{1}{g(t,\eta)} - \frac{1}{g(\tau,\eta)}\right)v_{\eta\eta}(t,\eta) \\ + \quad & \left\{\frac{v_\eta(t,\eta)}{L(t)\,g^{1/2}(t,\eta)} - \frac{v_\eta(\tau,\eta)}{L(\tau)\,g^{1/2}(\tau,\eta)}\right\}\theta v_\eta(t,\eta) \\ + \quad & \frac{(\eta-1)\cos^3\theta}{\sin\theta}\left(\frac{v_\eta(t,\eta)}{\xi^3(t)} - \frac{v_\eta(\tau,\eta)}{\xi^3(\tau)}\right)v_{\eta\eta}(t,0) \\ + \quad & \frac{\theta}{L(t)}\left(\frac{(\eta-1)v_\eta(t,\eta)}{\sin\theta} + \frac{\xi^2(t)}{g^{1/2}(t,\eta)}\right).\end{aligned}$$

Again, for simplicity, we denote $F_\tau(v,\xi)(t,\eta)$ by $F_\tau(t,\eta)$ if no confusion is possible. Furthermore, we denote

$$\bar g(t,\eta) := g(\bar v,\bar\xi)(t,\eta),\quad \bar L(t) := L(\bar v,\bar\xi)(t)\quad \text{and}\quad \bar F_\tau(t,\eta) := F_\tau(\bar v,\bar\xi)(t,\eta).$$

Finally, we will introduce some function spaces we are concerned with. Let $I=[0,1]$. For $0\le t_0<t_1<\infty$ we set $R_{t_0,t_1}=(t_0,t_1]\times I$. Given $0<\gamma<1/2,\ 0<\alpha<1,\ 0<\mu<\min\{(1-\alpha)/2, 1/2-\gamma\}$ we define

$$\begin{aligned}\mathcal{Y}^{2+\alpha}_{\gamma,\mu}(\overline{R_{t_0,t_1}}) = \{&v\in C^{(1-\alpha)/2-\mu}([t_0,t_1],C^{1+\alpha}(I))\cap C^{1/2-\mu}([t_0,t_1],C^1(I))\\ &\cap C^{1,2+\alpha}(R_{t_0,t_1}); v(t_0)\in C^{2+\alpha}(I) \text{ and } \|v\|_{\mathcal{Y}^{2+\alpha}_{\gamma,\mu}(\overline{R_{t_0,t_1}})}<\infty\},\\ \mathcal{Z}^1_\gamma[t_0,t_1] = \{&\xi\in C[t_0,t_1]\cap C^1(t_0,t_1]; \xi(t)>0, \text{ for } t\in[t_0,t_1]\\ &\text{and } \|\xi\|_{\mathcal{Z}^1_\gamma[t_0,t_1]}<\infty\},\end{aligned}$$

where the norms of these spaces are defined by

$$\begin{aligned}\|v\|_{\mathcal{Y}^{2+\alpha}_{\gamma,\mu}(\overline{R_{t_0,t_1}})} &= \|v\|_{C^{(1-\alpha)/2-\mu}([t_0,t_1],C^{1+\alpha}(I))} + \|v\|_{C^{1/2-\mu}([t_0,t_1],C^1(I))}\\ &+ \sup_{0<\delta<t_1-t_0}\delta^\gamma\|v_{\eta\eta}\|_{C^{0,\alpha}(\overline{R_{t_0+\delta,t_1}})} + \sup_{0<\delta<t_1-t_0}\delta^\gamma\|v_t\|_{C^{0,\alpha}(\overline{R_{t_0+\delta,t_1}})}\\ \|\xi\|_{\mathcal{Z}^1_\gamma[t_0,t_1]} &= \|\xi\|_{C[t_0,t_1]} + \sup_{0<\delta<t_1-t_0}\delta^\gamma\|\dot\xi\|_{C[t_0+\delta,t_1]}.\end{aligned}$$

In the above definition the spaces such as $C^{0,\alpha}(\overline{R_{t_0+\delta,t_1}})$, $C^{1,2+\alpha}(R_{t_0,t_1})$ are the usual anisotropic parabolic Hölder spaces, see [6]. Let us study the nonlinear part $F_\tau(t,\eta)$.

Lemma 2.1. *If $0<T<\infty$ and $(v,\xi)\in\mathcal{Y}^{2+\alpha}_{\gamma,\mu}(\overline{R_{\tau,\tau+T}})\times\mathcal{Z}^1_\gamma[\tau,\tau+T]$, then $F_\tau(t,\eta)\in C(\overline{R_{\tau,\tau+T}})\cap C^{0,\alpha}(R_{\tau,\tau+T})$. Moreover, if $\|v\|_{\mathcal{Y}^{2+\alpha}_{\gamma,\mu}(\overline{R_{\tau,\tau+T}})}+\|\xi\|_{\mathcal{Z}^1_\gamma[\tau,\tau+T]}\le K$ and T satisfies the following estimates: $T\le1$, $KT^{1-\gamma}\le\xi(\tau)(1-\gamma)/2$ and $KT^{(1-\alpha)/2-\mu}\le1$, then we have*

$$\|F_\tau(t,\eta)\|_{C(\overline{R_{\tau,\tau+T}})}\le C_\Sigma(1+K^2T^{1/2-\mu-\gamma}),\tag{2.9}$$

$$\sup_{0<\delta<T}\delta^\gamma\|F_\tau(t,\eta)\|_{C^{0,\alpha}(\overline{R_{\tau+\delta,\tau+T}})}\le C_\Sigma(T^\gamma+K^2T^{(1-\alpha)/2-\mu}),\tag{2.10}$$

where C_Σ is dependent on $1/\xi(\tau)$, $\xi(\tau)$ and $\|v(\tau)\|_{C^{2+\alpha}(I)}$.

Proof. Clearly $F_\tau(t,\eta)\in C(R_{\tau,\tau+T})$. Since for $\tau\le s<t\le\tau+T$, we have that

$$\begin{aligned}|\xi(t)-\xi(s)| &= \left|\int_0^1\dot\xi\,(\beta t+(1-\beta)s)\ (t-s)d\beta\right|\\ &\le \int_0^1[\beta(t-s)]^{-\gamma}[\beta(t-s)]^\gamma\|\dot\xi\|_{C[\tau+\beta(t-s),\tau+T]}(t-s)d\beta\\ &\le \frac{(t-s)^{1-\gamma}}{1-\gamma}\|\xi\|_{\mathcal{Z}^1_\gamma}.\end{aligned}\tag{2.11}$$

The assumption of T implies that

$$\frac12\xi(\tau)\le\xi(t)\le\frac32\xi(\tau)\le C_{\tau,1},\tag{2.12}$$

where $C_{\tau,1}$ depends on $\xi(\tau)$. Since $v \in C^{1/2-\mu}([\tau, \tau+T], C^1(I))$, we know that

$$\|v(t)\|_{C^1(I)} \le \|v(\tau)\|_{C^1(I)} + KT^{1/2-\mu} \le \|v(\tau)\|_{C^1(I)} + 1 \le C_{\tau,2}, \tag{2.13}$$

where $C_{\tau,2}$ depends on $\|v(\tau)\|_{C^1(I)}$. Let $M = \xi(\tau)/2$, then

$$M^2 \le g(t,\eta) \le C_{\tau,3}^2 \text{ and } M \le L(t) \le C_{\tau,3}, \tag{2.14}$$

here, $C_{\tau,3}$ depends on $C_{\tau,1}$ and $C_{\tau,2}$. Similarly $v \in C^{(1-\alpha)/2-\mu}([\tau, \tau+T], C^{1+\alpha}(I))$ ensures that

$$\|v_\eta(t)\|_{C^\alpha(I)} \le \|v_\eta(\tau)\|_{C^\alpha(I)} + KT^{(1-\alpha)/2-\mu} \le \|v_\eta(\tau)\|_{C^\alpha(I)} + 1 \le C_{\tau,4} \tag{2.15}$$

where $C_{\tau,4}$ depends on $\|v(\tau)\|_{C^{1+\alpha}(I)}$. To simplify our notation, set

$$C_\tau := \max\{C_{\tau,1},\, C_{\tau,2},\, C_{\tau,3},\, C_{\tau,4}\}.$$

Recall that $v \in C^{(1-\alpha)/2-\mu}([\tau, \tau+T], C^{1+\alpha}(I)) \cap C^{1/2-\mu}([\tau, \tau+T], C^1(I))$. Thus, given $0 \le \eta_1 < \eta_2 \le 1$, we have

$$\begin{aligned}
&|v_\eta(t,\eta_1) - v_\eta(s,\eta_2)| \\
&\quad \le\ |v_\eta(t,\eta_1) - v_\eta(t,\eta_2)| + |v_\eta(t,\eta_2) - v_\eta(s,\eta_2)| \\
&\quad \le\ \|v_\eta(t)\|_{C^\alpha(I)}|\eta_1-\eta_2|^\alpha + \|v\|_{C^{1/2-\mu}([\tau,\,\tau+T],\,C^1(I))}|t-s|^{1/2-\mu} \\
&\quad \le\ C_\tau|\eta_1-\eta_2|^\alpha + K|t-s|^{1/2-\mu}.
\end{aligned} \tag{2.16}$$

Then, by (2.11)–(2.13), (2.16), and $\gamma < 1/2$, we get

$$\begin{aligned}
&|g(t,\eta_1) - g(s,\eta_2)| \\
&\quad \le\ \left|\xi^2(t) - \xi^2(s)\right| + \left|v_\eta^2(t,\eta_1) - v_\eta^2(s,\eta_2)\right| \\
&\quad \le\ 2C_\tau\left\{\frac{K|t-s|^{1-\gamma}}{1-\gamma} + C_\tau|\eta_1-\eta_2|^\alpha + K|t-s|^{1/2-\mu}\right\} \\
&\quad \le\ CC_\tau\left\{C_\tau|\eta_1-\eta_2|^\alpha + K|t-s|^{1/2-\mu}\right\}.
\end{aligned} \tag{2.17}$$

Moreover, by (2.14), we have

$$\begin{aligned}
|L(t) - L(s)| &\le \int_0^1 \left|\frac{g(t,\eta) - g(s,\eta)}{g^{1/2}(t,\eta) + g^{1/2}(s,\eta)}\right| d\eta \\
&\le \frac{CC_\tau}{M}K|t-s|^{1/2-\mu}.
\end{aligned} \tag{2.18}$$

Therefore, we obtain that

$$\left|\frac{1}{g(t,\eta_1)} - \frac{1}{g(s,\eta_2)}\right| \le \frac{CC_\tau}{M^4}\left\{C_\tau|\eta_1-\eta_2|^\alpha + K|t-s|^{1/2-\mu}\right\}, \tag{2.19}$$

$$\left|\frac{1}{g^{1/2}(t,\eta_1)} - \frac{1}{g^{1/2}(s,\eta_2)}\right| \le \frac{CC_\tau}{M^3}\left\{C_\tau|\eta_1-\eta_2|^\alpha + K|t-s|^{1/2-\mu}\right\}, \tag{2.20}$$

$$\left|\frac{1}{L(t)} - \frac{1}{L(s)}\right| \le \frac{CC_\tau}{M^3}K|t-s|^{1/2-\mu}. \tag{2.21}$$

Due to (2.11)–(2.21), we have

$$\begin{aligned}
|F_\tau(t,\eta)| \leq\ & \left|\frac{1}{g(t,\eta)}-\frac{1}{g(\tau,\eta)}\right||v_{\eta\eta}(t,\eta)|+|\theta v_\eta(t,\eta)|\left\{\left|\frac{v_\eta(t,\eta)-v_\eta(\tau,\eta)}{L(t)\,g^{\frac12}(t,\eta)}\right|\right.\\
&\left.+\left|\frac{v_\eta(\tau,\eta)}{g^{\frac12}(t,\eta)}\right|\left|\frac{1}{L(t)}-\frac{1}{L(\tau)}\right|+\left|\frac{v_\eta(\tau,\eta)}{L(\tau)}\right|\left|\frac{1}{g^{\frac12}(t,\eta)}-\frac{1}{g^{\frac12}(\tau,\eta)}\right|\right\}\\
&+\left|\frac{(\eta-1)\cos^3\theta}{\sin\theta}v_{\eta\eta}(t,0)\right|\left\{\left|\frac{v_\eta(t,\eta)-v_\eta(\tau,\eta)}{\xi^3(t)}\right|\right.\\
&\left.+\left|\frac{v_\eta(\tau,\eta)}{\xi^3(t)\,\xi^3(\tau)}\right||\xi^3(t)-\xi^3(\tau)|\right\}\\
&+\left|\frac{\theta}{L(t)}\right|\left(\left|\frac{(\eta-1)v_\eta(t,\eta)}{\sin\theta}\right|+\left|\frac{\xi^2(t)}{g^{1/2}(t,\eta)}\right|\right)\\
\leq\ & \frac{CC_\tau}{M^4}K(t-\tau)^{1/2-\mu}|v_{\eta\eta}(t,\eta)|+CC_\tau\left\{\frac{K(t-\tau)^{1/2-\mu}}{M^2}\right.\\
&\left.+\frac{CC_\tau^2K(t-\tau)^{1/2-\mu}}{M^4}+\frac{CC_\tau^2K(t-\tau)^{1/2-\mu}}{M^4}\right\}+C|v_{\eta\eta}(t,0)|\cdot\\
&\left\{\frac{K(t-\tau)^{1/2-\mu}}{M^3}+\frac{C_\tau^3K(t-\tau)^{1-\gamma}}{M^6}\right\}+\frac{C}{M}(C_\tau+\frac{C_\tau^2}{M})\\
\leq\ & C_\Sigma(1+KT^{1/2-\mu}+K^2T^{1/2-\mu-\gamma})\\
\leq\ & C_\Sigma(1+K^2T^{1/2-\mu-\gamma}),
\end{aligned}$$

where C_Σ increases with $1/M$ and C_τ. By definition,

$$F_\tau(\tau,\eta)=\frac{\theta}{L(\tau)}\left(\frac{(\eta-1)v_\eta(\tau,\eta)}{\sin\theta}+\frac{\xi^2(\tau)}{g^{1/2}(\tau,\eta)}\right).$$

Thus we get similarly that

$$\begin{aligned}
|F_\tau(t,\eta_1)-F_\tau(\tau,\eta_2)| \leq\ & C_\Sigma\Big\{K(t-\tau)^{1/2-\mu}(|v_{\eta\eta}(t,\eta_1)|+|v_{\eta\eta}(t,0)|+1)\\
&+|\eta_1-\eta_2|^\alpha\Big\}\\
\leq\ & C_\Sigma\left\{K^2(t-\tau)^{1/2-\mu-\gamma}+|\eta_1-\eta_2|^\alpha\right\}.
\end{aligned}$$

This means that $F_\tau(t,\eta)\in C(\overline{R_{\tau,\tau+T}})$ and

$$\|F_\tau(t,\eta)\|_{C(\overline{R_{\tau,\tau+T}})}\leq C_\Sigma(1+K^2T^{1/2-\mu-\gamma}). \tag{2.22}$$

This proves (2.9). Since $v\in C^{1,2+\alpha}(R_{\tau,\tau+T})$, it follows that $F_\tau\in C^{0,\alpha}(R_{\tau,\tau+T})$. Also, due to $v\in C^{(1-\alpha)/2-\mu}([\tau,\tau+T],C^{1+\alpha}(I))$, we have

$$[v_\eta(t,\cdot)-v_\eta(s,\cdot)]_\alpha\leq K|t-s|^{(1-\alpha)/2-\mu}, \tag{2.23}$$

where $[\cdot]_\alpha$ is the Hölder seminorm of the space $C^\alpha(I)$. Invoking (2.13), (2.15) and (2.16), we obtain

$$\begin{aligned}\left[v_\eta^2(t,\cdot)-v_\eta^2(s,\cdot)\right]_\alpha &\leq [v_\eta(t,\cdot)-v_\eta(s,\cdot)]_\alpha \|v_\eta(t,\cdot)+v_\eta(s,\cdot)\|_{C(I)} \\ &\quad +\|v_\eta(t,\cdot)-v_\eta(s,\cdot)\|_{C(I)}[v_\eta(t,\cdot)+v_\eta(s,\cdot)]_\alpha \\ &\leq 2C_\tau(K|t-s|^{(1-\alpha)/2-\mu}+K|t-s|^{1/2-\mu}) \\ &\leq C_\Sigma K|t-s|^{(1-\alpha)/2-\mu}.\end{aligned}$$

Observe that

$$\begin{gathered}[g(t,\cdot)]_\alpha=[v_\eta^2(t,\eta)]_\alpha\leq 2C_\tau^2\leq C_\Sigma, \\ [g(t,\cdot)-g(s,\cdot)]_\alpha\leq C_\Sigma K|t-s|^{(1-\alpha)/2-\mu}.\end{gathered} \tag{2.24}$$

Thus combining (2.14), (2.17), and (2.24), we find the estimates

$$\left[\frac{1}{g(t,\cdot)}-\frac{1}{g(s,\cdot)}\right]_\alpha\leq C_\Sigma K|t-s|^{\frac{1-\alpha}{2}-\mu}, \tag{2.25}$$

$$\left[\frac{1}{g^{1/2}(t,\cdot)}-\frac{1}{g^{1/2}(s,\cdot)}\right]_\alpha\leq C_\Sigma K|t-s|^{\frac{1-\alpha}{2}-\mu}, \tag{2.26}$$

where we used the following fact:

$$\left[\frac{1}{h}\right]_\alpha\leq\frac{[h]_\alpha}{Q^2},\ \text{provided the function } h \text{ satisfyies } |h|>Q>0.$$

Summarizing, (2.15), (2.19), (2.20), and (2.23)–(2.26) imply that

$$\begin{aligned}[F_\tau(t,,\cdot)]_\alpha &\leq C_\Sigma\Big\{K|t-\tau|^{(1-\alpha)/2-\mu}\|v_{\eta\eta}(t)\|_{C(I)} \\ &\quad +K|t-\tau|^{1/2-\mu}\|v_{\eta\eta}(t)\|_{C^\alpha(I)}+1+\big(K|t-\tau|^{(1-\alpha)/2-\mu} \\ &\quad +K|t-\tau|^{1/2-\mu}+K|t-\tau|^{1-\gamma}|\big)|v_{\eta\eta}(t,0)|\Big\} \\ &\leq C_\Sigma\Big(1+KT^{(1-\alpha)/2-\mu}\big(\|v_{\eta\eta}(t)\|_{C^\alpha(I)}+|v_{\eta\eta}(t,0)|\big)\Big).\end{aligned} \tag{2.27}$$

By means of (2.22) and (2.27), we conclude

$$\begin{aligned}\sup_{0<\delta<T}\delta^\gamma\|F_\tau\|_{C^{0,\alpha}(\overline{R_{\tau+\delta,\tau+T}})} &\leq C_\Sigma(T^\gamma+K^2T^{1/2-\mu}+K^2T^{(1-\alpha)/2-\mu}) \\ &\leq C_\Sigma(T^\gamma+K^2T^{(1-\alpha)/2-\mu}).\end{aligned}$$

This completes the proof. □

In the following, we will study the operator $\mathcal{A}_\tau$. We first decompose $\mathcal{A}_\tau$ into two operators $\mathcal{A}_\tau^{(1)}$ and $\mathcal{A}_\tau^{(2)}$, by setting

$$\begin{aligned}\mathcal{A}_\tau^{(1)}U &:= \frac{1}{\bar{g}(\tau,\eta)}\partial_\eta^2U+\frac{\theta\,\bar{v}_\eta(\tau,\eta)}{\bar{L}(\tau)\,\bar{g}^{1/2}(\tau,\eta)}\,\partial_\eta U, \\ \mathcal{A}_\tau^{(2)}U &:= \frac{(\eta-1)\cos^3\theta}{\sin\theta}\,\frac{\bar{v}_\eta(\tau,\eta)}{\bar{\xi}^3(\tau)}\partial_\eta^2U(0).\end{aligned}$$

Let $X:=C[0,1]$, and define

$$D(A_\tau^{(1)}):=\left\{U\in C^2[0,1];\partial_\eta U(0)=\partial_\eta U(1)=0\right\}.$$

Then $A_\tau^{(1)}\ :\ X \supset D(A_\tau^{(1)}) \ni U \mapsto \mathcal{A}_\tau^{(1)} U \in X$ is the realization of $\mathcal{A}_\tau^{(1)}$ in X. It is known that $A_\tau^{(1)}$ is sectorial in X, see Corollary 3.1.21 in [6]. Thus $A_\tau^{(1)}$ generates an analytic semigroup $e^{tA_\tau^{(1)}}$ in X for $t \geq 0$. The domain of definition $D(A_\tau^{(1)})$, endowed with the graph norm

$$\|v\|_{D(A_\tau^{(1)})} = \|v\| + \|A_\tau^{(1)} v\|,$$

is a Banach space. In addition, a family of intermediate spaces between $D(A_\tau^{(1)})$ and X can be defined by

$$D_{A_\tau^{(1)}}(\beta,\infty) = \left\{ \phi \in X; [\phi]_{D_{A_\tau^{(1)}}(\beta,\infty)} = \sup_{0<t\leq 1} \left\| t^{1-\beta} A_\tau^{(1)} e^{tA_\tau^{(1)}} \phi \right\| < \infty \right\}$$

for $0 < \beta < 1$. These spaces are Banach spaces with respect to the norm

$$\|\phi\|_{D_{A_\tau^{(1)}}(\beta,\infty)} = \|\phi\|_X + [\phi]_{D_{A_\tau^{(1)}}(\beta,\infty)}.$$

It is also known that

$$D_{A_\tau^{(1)}}(\beta,\infty) = \begin{cases} C^{2\beta}([0,1]), & if\ 0 < \beta < \dfrac{1}{2}, \\ C_B^{2\beta}([0,1]), & if\ \dfrac{1}{2} < \beta < 1, \end{cases} \tag{2.28}$$

with equivalence of the respective norms, see Theorem 3.1.30 in [6]. Here we used the notation

$$C_B^{2\beta}([0,1]) := \{U \in C^{2\beta}([0,1]); \partial_\eta U(0) = \partial_\eta U(1) = 0\}.$$

Now let $A_\tau^{(2)}\ :\ X \supset D(A_\tau^{(1)}) \ni U \mapsto \mathcal{A}_\tau^{(2)} U \in X$ be the realization of $\mathcal{A}_\tau^{(2)}$ in X. Then we have the following lemma.

Lemma 2.2. *The operator* $A_\tau := A_\tau^{(1)} + A_\tau^{(2)}$ *is sectorial in* X.

Proof. First we prove that $A_\tau^{(2)}$ is a bounded linear operator from $D(A_\tau^{(1)})$ into $C^\beta[0,1]$ for $0 < \beta < 1$. Indeed,

$$\begin{aligned} \|A_\tau^{(2)} U\|_{C^\beta[0,1]} &= \left\| \frac{(\eta-1)\cos^3\theta}{\sin\theta} \frac{\bar{v}_\eta(\tau,\eta)}{\bar{\xi}^3(\tau)} \partial_\eta^2 U(0) \right\|_{C^\beta[0,1]} \\ &\leq C_\theta \left\| \frac{\bar{v}_\eta(\tau,\eta)}{\bar{\xi}^3(\tau)} \right\|_{C^\beta[0,1]} \cdot |\partial_\eta^2 U(0)| \\ &\leq C\|U\|_{D(A_\tau^{(1)})}, \end{aligned}$$

where C increases with $1/\bar{\xi}(\tau)$ and $\|\bar{v}(\tau,\cdot)\|_{C^{2+\alpha}[0,1]}$. By (2.28), we know that $A_\tau^{(2)} \in \mathcal{L}((A_\tau^{(1)}), D_{A_\tau^{(1)}}(\beta/2,\infty))$. We already verified that $A_\tau^{(1)}$ is sectorial in X. Thanks to Proposition 2.4.1(ii) in [6], we conclude that A_τ is also a sectorial operator in X. $\square$

Remark 2.3. (i) The above Lemma implies that A_τ generates an analytic semigroup e^{tA_τ} for $t \geq 0$. Recall that A_τ only depends on $\bar{\xi}(\tau)$ and $\bar{v}(\tau, \cdot)$.

(ii) We have the following estimates: for $k \in \mathbb{N}$, $\beta_1, \beta_2 \in (0,1)$, there exists a constant $C = C(k, \beta_1, \beta_2, A_\tau)$ such that

$$\|t^{k-\beta_1+\beta_2} A_\tau^k e^{tA_\tau}\|_{\mathcal{L}(D_{A_\tau}(\beta_1,p), D_{A_\tau}(\beta_2,p))} \leq C \quad \text{for } 0 < t \leq 1,\ p \geq 1. \tag{2.29}$$

In particular,

$$\|t^{k-\beta_1+\beta_2} A_\tau^k e^{tA_\tau}\|_{\mathcal{L}(D_{A_\tau}(\beta_1,\infty), D_{A_\tau}(\beta_2,p))} \leq C \quad \text{for } 0 < t \leq 1,\ p \geq 1. \tag{2.30}$$

It can be shown that this constant C increases with $1/\bar{\xi}(\tau)$ and $\|\bar{v}(\tau,\cdot)\|_{C^2(I)}$.

Lemma 2.4. *Let A be a sectorial operator in the Banach space Y, and assume that $f \in C([\tau, \tau+T], Y)$. Let further*

$$\phi(t) := \int_\tau^t e^{(t-s)A} f(s)ds \quad \textit{for } \tau \leq t \leq \tau + T.$$

Then for every $\mu < \beta < 1$, we have that $\phi \in C^{\beta-\mu}([\tau, \tau+T], (Y, D(A))_{1-\beta,1})$, and there is a C independent of f such that

$$\|\phi\|_{C^{\beta-\mu}([\tau,\tau+T],\,(Y,D(A))_{1-\beta,1})} \leq C T^\mu \|f\|_{C([\tau,\,\tau+T],\,Y)}. \tag{2.31}$$

Proof. The proof of this lemma is similar to that of Proposition 4.2.1 in [6]. □

Next, we consider the following problem:

$$\begin{cases} v_t = A_\tau v + \bar{F}_\tau(t,\eta) \text{ in } R_{\tau,\tau+T}, \\ v_\eta(t,0) = \bar{\xi}(t)\tan\theta, \\ v_\eta(t,1) = 0, \\ v(\tau,\eta) = \bar{v}(\tau,\eta). \end{cases} \tag{2.32}$$

Lemma 2.5. *Let $0 < \gamma < 1/2$, $0 < \alpha < 1$, and $0 < \mu < \min\{1/2 - \gamma, (1-\alpha)/2\}$. Assume that $v_{0,\tau} \in C^{2+\alpha}(I)$ and $\xi_{0,\tau} > 0$ are given and satisfy*

$$v_{0,\tau}(0) = 0, \quad v'_{0,\tau}(0) = \xi_{0,\tau}\tan\theta, \quad v'_{0,\tau}(1) = 0.$$

Then, given $(\bar{v}, \bar{\xi}) \in \mathcal{Y}^{2+\alpha}_{\gamma,\mu}(\overline{R_{\tau,\,\tau+T}}) \times \mathcal{Z}^1_\gamma[\tau, \tau+T]$ with $(\bar{v}(\tau,\cdot), \bar{\xi}(\tau)) = (v_{0,\tau}(\cdot), \xi_{0,\tau})$, there exists a unique solution $v \in \mathcal{Y}^{2+\alpha}_{\gamma,\mu}(\overline{R_{\tau,\,\tau+T}})$ of (2.32).

Proof. By Lemma 2.1 and Lemma 2.2, we can invoke Theorem 5.1.2 and Theorem 5.1.4 in [6] to get a unique solution $v \in \mathcal{Y}^{2+\alpha}_{\gamma,\mu}(\overline{R_{\tau,\,\tau+T}})$ of (2.32). □

Suppose now that $v \in \mathcal{Y}^{2+\alpha}_{\gamma,\mu}(\overline{R_{\tau,\,\tau+T}})$ is a solution of (2.32), and consider the following initial value problem:

$$\begin{cases} \dot{\xi}(t) = \dfrac{\cos^3\theta}{\sin\theta\,\bar{\xi}^2(t)} v_{\eta\eta}(t,0) + \dfrac{\theta\bar{\xi}(t)}{\sin\theta\, L(v,\bar{\xi})(t)}, \\ \xi(\tau) = \bar{\xi}(\tau). \end{cases} \tag{2.33}$$

Then we have

Lemma 2.6. *Given $v \in \mathcal{Y}^{2+\alpha}_{\gamma,\mu}(\overline{R_{\tau,\tau+T}})$, there exists a unique solution $\xi \in \mathcal{Z}^1_\gamma[\tau,\tau+T]$ of* (2.33).

Proof. Let us define

$$h(t) = \frac{\cos^3\theta}{\sin\theta\bar{\xi}^2(t)} v_{\eta\eta}(t,0) + \frac{\theta\bar{\xi}(t)}{\sin\theta\, L(v,\bar{\xi})(t)}, \quad t \in (\tau,\tau+T].$$

Then $h \in C((\tau,\tau+T])$ and $\sup\limits_{0<\delta<T} \delta^\gamma \|h\|_{C[\tau+\delta,\tau+T]}$ is bounded. Hence we conclude that $h \in L_1(\tau,\tau+T)$. Thus, setting

$$\xi(t) = \bar{\xi}(\tau) + \int_\tau^t h(s)ds \qquad \text{for } \tau \le t \le \tau+T,$$

we see that ξ belongs to $\mathcal{Z}^1_\gamma[\tau,\tau+T]$ and satisfies (2.33). □

Before proving local existence, we shall obtain suitable a priori estimates for v. In order to reduce the problem (2.32) to a problem with homogeneous boundary conditions, we introduce an auxiliary function ψ by

$$\psi(t,\eta) := \bar{\xi}(t)\tan\theta h(\eta),$$

where $h \in C^\infty(I)$ is a cut-off function with $h'(\eta) < 0$ for $\eta \in (1/4,3/4)$, $h(\eta) \equiv 1$ on $\eta \in [0,1/4]$, and $h(\eta) \equiv 0$ on $\eta \in [3/4,1]$. Then $v - \psi$ satisfies homogeneous boundary conditions. From this fact and Lemma 2.2, we can represent $v - \psi$ as the variation of constant formula using the analytic semigroup e^{tA_τ}. A simple computation shows that

$$v(t,\cdot) = v_1(t,\cdot) + v_2(t,\cdot) + v_3(t,\cdot),$$

where

$$\begin{aligned} v_1(t,\cdot) &= e^{(t-\tau)A_\tau}(\bar{v}(\tau,\cdot) - \psi(\tau,\cdot)) + \psi(\tau,\cdot), \\ v_2(t,\cdot) &= \int_\tau^t e^{(t-\sigma)A_\tau}\left[\bar{F}_\tau(\sigma,\cdot) + A_\tau\psi(\sigma,\cdot)\right] d\sigma, \\ v_3(t,\cdot) &= -A_\tau \int_\tau^t e^{(t-\sigma)A_\tau}\left[\psi(\sigma,\cdot) - \psi(\tau,\cdot)\right] d\sigma \end{aligned}$$

for $\tau \le t \le \tau+T$. Clearly, $\psi(t,\cdot) \in C([\tau,\tau+T], C^\infty(I))$ and $\psi_t(t,\cdot) = \bar{\xi}\tan\theta h(\cdot)$. Moreover, using (2.11), we get

$$|\psi(\sigma,\eta) - \psi(s,\eta)| = |(\bar{\xi}(\sigma) - \bar{\xi}(s)|\tan\theta h(\eta) \le C|\sigma - s|^{1-\gamma}\|\bar{\xi}\|_{\mathcal{Z}^1_\gamma[\tau,\tau+T]}$$

for $\sigma, s \in [\tau,\tau+T]$. Applying the main results of Chapter 5 in [6], using Lemma 2.4, (2.29), and the following facts

$$D_{A_\tau}(\beta,1) \subset C^{2\beta}(I) \text{ for } 0<\beta<1, \text{ and } \|v\|_{D(A_\tau)} \le C\|v\|_{C^2(I)},$$

we get

$$
\begin{aligned}
\|v_1\|_{\mathcal{Y}^{2+\alpha}_{\gamma,\,\mu}(\overline{R_{\tau,\,\tau+T}})} &\le C\Big(\|\bar v(\tau,\cdot)-\psi(\tau,\cdot)\|_{C^{2+\alpha}(I)}+\|\psi(\tau,\cdot)\|_{C^{2+\alpha}(I)}\Big),\\
\|v_2\|_{\mathcal{Y}^{2+\alpha}_{\gamma,\,\mu}(\overline{R_{\tau,\,\tau+T}})} &\le C\Big(T^{\mu}\|\bar F_\tau(t,\cdot)+A_\tau\psi(t,\cdot)\|_{C[\tau,\tau+T]}\\
&\qquad + \sup_{0<\delta<T}\delta^{\gamma}\|\bar F_\tau(t,\cdot)+A_\tau\psi(t,\cdot)\|_{C^{0,\,\alpha}(\overline{R_{\tau+\delta,\,\tau+T}})}\Big),\\
\|v_3\|_{\mathcal{Y}^{2+\alpha}_{\gamma,\,\mu}(\overline{R_{\tau,\,\tau+T}})} &\le C\Big(T^{\mu}\|A_\tau(\psi(t,\cdot)-\psi(\tau,\cdot))\|_{C[\tau,\tau+T]}\\
&\qquad + \sup_{0<\delta<T}\delta^{\gamma}\|A_\tau(\psi(t,\cdot)-\psi(\tau,\cdot))\|_{C^{0,\,\alpha}(\overline{R_{\tau+\delta,\,\tau+T}})}\Big).
\end{aligned}
$$

If $\|\bar v\|_{\mathcal{Y}^{2+\alpha}_{\gamma,\,\mu}(\overline{R_{\tau,\,\tau+T}})}+\|\bar\xi\|_{\mathcal{Z}^1_\gamma[\tau,\,\tau+T]}\le K$ and T satisfies

$$
T\le 1,\quad KT^{1-\gamma}\le \bar\xi(\tau)(1-\gamma)/2\quad\text{and}\quad KT^{(1-\alpha)/2-\mu}\le 1, \tag{2.34}
$$

we obtain the estimates

$$
\begin{aligned}
&\|\psi(\tau,\cdot)\|_{C^{2+\alpha}(I)}\le C\bar\xi(\tau)\|h\|_{C^{2+\alpha}(I)},\\
&\|A_\tau\psi(t,\cdot)\|_{C[\tau,\tau+T]}\le C\|\bar\xi(t)\|_{C[\tau,\,\tau+T]}\|h\|_{C^2(I)}\le C\|h\|_{C^2(I)},\\
&\|A_\tau\psi(t,\cdot)\|_{C^{0,\,\alpha}(\overline{R_{\tau,\,\tau+T}})}\le C\|\bar\xi(t)\|_{C[\tau,\,\tau+T]}\|h\|_{C^{2+\alpha}(I)}\le C\|h\|_{C^{2+\alpha}(I)},\\
&\|A_\tau(\psi(t,\cdot)-\psi(\tau,\cdot))\|_{C[\tau,\tau+T]}\le CKT^{1-\gamma}\|h\|_{C^2(I)},\\
&\|A_\tau(\psi(t,\cdot)-\psi(\tau,\cdot))\|_{C^{0,\,\alpha}(\overline{R_{\tau,\,\tau+T}})}\le CKT^{1-\gamma}\|h\|_{C^{2+\alpha}(I)}.
\end{aligned}
$$

By Lemma 2.1, we know that

$$
\|v\|_{\mathcal{Y}^{2+\alpha}_{\gamma,\,\mu}(\overline{R_{\tau,\,\tau+T}})}\le M_\tau+P_\tau\left(T^{\mu}+T^{\gamma}+K^2T^{1/2-\gamma}+K^2T^{(1-\alpha)/2-\mu}\right)
$$

where M_τ, P_τ are constants depending on $\bar\xi(\tau), \|\bar v(\tau)\|_{C^{2+\alpha}(I)}, 1/\bar\xi(\tau), \theta, \gamma, \alpha$ and μ. Letting $0<\nu<\min\{\gamma,\mu,1/2-\gamma,(1-\alpha)/2-\mu\}$, we conclude that

$$
\|v\|_{\mathcal{Y}^{2+\alpha}_{\gamma,\,\mu}(\overline{R_{\tau,\,\tau+T}})}\le M_\tau+N_\tau T^{\nu}, \tag{2.35}
$$

where $N_\tau = P_\tau(T^{\mu-\nu}+T^{\gamma-\nu}+K^2T^{1/2-\gamma-\nu}+K^2T^{(1-\alpha)/2-\mu-\nu})$. This means that N_τ depends on $\bar\xi(\tau), \|\bar v(\tau)\|_{C^{2+\alpha}(I)}, 1/\bar\xi(\tau), \theta, \gamma, \alpha, \mu, \nu$, and K.

Theorem 2.7 (Local well-posedness). *Let $0<\gamma<1/2$, $0<\alpha<1$, and $0<\mu<\min\{1/2-\gamma,(1-\alpha)/2\}$. Assume that $v_0\in C^{2+\alpha}(I)$, and $\xi_0>0$ are given and satisfy*

$$
v_0(0)=0,\qquad v_0'(0)=\xi_0\tan\theta,\qquad v_0'(1)=0.
$$

Then, there exists a $T_0=T_0(1/\xi_0,\|v_0\|_{C^{2+\alpha}(I)})>0$ such that (2.3)–(2.8) has a unique solution $(v,\xi)\in\mathcal{Y}^{2+\alpha}_{\gamma,\,\mu}(\overline{R_{0,\,T_0}})\times\mathcal{Z}^1_\gamma[0,T_0]$.

Proof. In order to obtain local existence result for the full problem (2.3)–(2.8), we shall use a fixed point argument. So we let

$$
\begin{aligned}
\mathcal{D}:=\Big\{&(v,\xi)\in\mathcal{Y}^{2+\alpha}_{\gamma,\,\mu}(\overline{R_{0,\,T}})\times\mathcal{Z}^1_\gamma[0,T];\\
&(v(0,\cdot),\xi(0))=(v_0(\cdot),\xi_0),\ \|v\|_{\mathcal{Y}^{2+\alpha}_{\gamma,\,\mu}(\overline{R_{0,\,T}})}+\|\xi\|_{\mathcal{Z}^1_\gamma[0,\,T]}\le K\Big\},
\end{aligned}
$$

for positive bounded parameters K, T satisfying (2.34). Moreover, we define a mapping Ψ by

$$\Psi:\mathcal{D}\ni(\bar{v},\bar{\xi})\mapsto(v,\xi)\in\mathcal{Y}^{2+\alpha}_{\gamma,\mu}(\overline{R_{0,T}})\times\mathcal{Z}^1_\gamma[0,T],$$

where (v,ξ) is the unique solution of (2.32) and (2.33), established in Lemma 2.5 and Lemma 2.6, respectively, for the case $\tau=0$.

We shall first prove that Ψ maps $\mathcal{D}$ into itself, provided we suitably choose K and T. By (2.35), we get

$$\|v\|_{\mathcal{Y}^{2+\alpha}_{\gamma,\mu}(\overline{R_{0,0+T}})}\le M_0+N_0T^\nu, \tag{2.36}$$

where M_0 is a constant depending on $\xi_0, \|v_0\|_{C^{2+\alpha}(I)}, 1/\xi_0, \theta, \gamma, \alpha, \mu$, and N_0 is a constant depending on $\xi_0, \|v_0\|_{C^{2+\alpha}(I)}, 1/\xi_0, \theta, \gamma, \alpha, \mu, \nu, K$ (for simplicity, throughout this section any constant depending on the preceding quantities will be denoted by N_0, whose value may be different on each occasion), and ν satisfies $0<\nu<\min\{\gamma,\mu,(1-\alpha)/2-\mu\}$.

According to (2.11), (2.12), (2.33) and (2.36), we know

$$\begin{aligned}\|\xi\|_{\mathcal{Z}^1_\gamma[0,T]} &\le \xi_0+\left(\frac{T^{1-\gamma}}{1-\gamma}+1\right)\left\{\frac{\cos^3\theta}{\sin\theta}(\frac{2}{\xi_0})^2(M_0+N_0T^\nu)+\frac{3\theta}{\sin\theta}\right\}\\ &\le \xi_0+(\frac{1}{1-\gamma}+1)\left\{\frac{\cos^3\theta}{\sin\theta}(\frac{2}{\xi_0})^2M_0+\frac{3\theta}{\sin\theta}\right\}+N_0T^\nu.\end{aligned}$$

Hence, letting

$$K=2\left\{\xi_0+(4(\frac{1}{1-\gamma}+1)\frac{\cos^3\theta}{\sin\theta}\xi_0^{-2}+1)M_0+(\frac{1}{1-\gamma}+1)\frac{3\theta}{\sin\theta}\right\}, \tag{2.37}$$

and choosing $\tilde{T}$ satisfying (2.34) and $4N_0\tilde{T}^\nu\le K$, we obtain that

$$\|v\|_{\mathcal{Y}^{2+\alpha}_{\gamma,\mu}(\overline{R_{0,T}})}+\|\xi\|_{\mathcal{Z}^1_\gamma[0,T]}\le K \qquad \text{for}\quad T\le\tilde{T}. \tag{2.38}$$

Thus, Ψ maps $\mathcal{D}$ into itself.

Next we prove that Ψ is a contraction on $\mathcal{D}$ for a suitable choice of T. Let $(\bar{v}_1,\bar{\xi}_1)$, $(\bar{v}_2,\bar{\xi}_2)\in\mathcal{D}$ with $T\le\tilde{T}$, and put $(v_1,\xi_1)=\Psi(\bar{v}_1,\bar{\xi}_1)$, $(v_2,\xi_2)=\Psi(\bar{v}_2,\bar{\xi}_2)$. Moreover let $V=v_1-v_2$, $\Xi=\xi_1-\xi_2$, $\bar{V}=\bar{v}_1-\bar{v}_2$, $\bar{\Xi}=\bar{\xi}_1-\bar{\xi}_2$. Then the function V satisfies

$$\begin{cases} V_t=A_0V+\tilde{F}(t,\eta) \text{ in } R_{0,T},\\ V_\eta(t,0)=\bar{\Xi}(t)\tan\theta,\\ V_\eta(t,1)=0,\\ V(0,\eta)=0,\end{cases} \tag{2.39}$$

and the function Ξ satisfies

$$\begin{cases} \dfrac{d}{dt}\Xi(t)=\dfrac{\cos^3\theta}{\sin\theta\bar{\xi}_1^2(t)}V_{\eta\eta}(t,0)+\int_0^1 b_1(t,\eta)V_\eta(t,\eta)d\eta\\ \qquad\qquad +[b_2(t)v_{2\eta\eta}(t,0)+b_3(t)]\bar{\Xi}(t),\\ \Xi(0)=0,\end{cases} \tag{2.40}$$

where

$$\begin{aligned}\tilde{F}(t,\eta) &= F_0(\bar{v}_1,\bar{\xi}_1)(t,\eta) - F_0(\bar{v}_2,\bar{\xi}_2)(t,\eta) \\ &= \left\{\frac{1}{\bar{\xi}_1^2(t)+\bar{v}_{1\eta}^2(t)} - \frac{1}{\xi_0^2+v_{0\eta}^2}\right\}\bar{V}_{\eta\eta} + f_1(\bar{\xi}_i,\bar{v}_{i\eta})\bar{v}_{2\eta\eta}\bar{V}_\eta \\ &\quad + f_2(\bar{\xi}_i,\bar{v}_{i\eta})\bar{v}_{2\eta\eta}\bar{\Xi} + \cdots,\end{aligned}$$

and $b_1(t,\eta)$, $b_2(t), b_3(t)$ are functions without significant singularities. Similar to the priori estimate of v, we get

$$\|V\|_{\mathcal{Y}^{2+\alpha}_{\gamma,\mu}(\overline{R_{0,T}})} \leq N_0 T^\nu (\|\bar{V}\|_{\mathcal{Y}^{2+\alpha}_{\gamma,\mu}(\overline{R_{0,T}})} + \|\bar{\Xi}\|_{\mathcal{Z}^1_\gamma[0,T]}), \tag{2.41}$$

and then, by means of (2.40) and (2.41),

$$\|\Xi\|_{\mathcal{Z}^1_\gamma[0,T]} \leq (\|\bar{V}\|_{\mathcal{Y}^{2+\alpha}_{\gamma,\mu}(\overline{R_{0,T}})} + \|\bar{\Xi}\|_{\mathcal{Z}^1_\gamma[0,T]}).$$

Thus we derive at

$$\|V\|_{\mathcal{Y}^{2+\alpha}_{\gamma,\mu}(\overline{R_{0,T}})} + \|\Xi\|_{\mathcal{Z}^1_\gamma[0,T]} \leq N_0 T^\nu (\|\bar{V}\|_{\mathcal{Y}^{2+\alpha}_{\gamma,\mu}(\overline{R_{0,T}})} + \|\bar{\Xi}\|_{\mathcal{Z}^1_\gamma[0,T]}). \tag{2.42}$$

Consequently, Ψ is a contraction on $\mathcal{D}$ for $T \leq T_0$, where

$$T_0 = \min\left\{\left(\frac{1}{2N_0}\right)^{1/\nu}, \tilde{T}\right\}. \tag{2.43}$$

This completes the proof. □

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The Dual Mixed Finite Element Method for the Heat Diffusion Equation in a Polygonal Domain, I

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In memory of Günter Lumer

Abstract. The aim of this paper is to prove *a priori* error estimates for the semi-discrete solution of the dual mixed method for the heat diffusion equation in a polygonal domain. Due to the geometric singularities of the domain, the solution is not regular in the context of classical Sobolev spaces. Instead, one must use weighted Sobolev spaces. In order to recapture the optimal order of convergence, the meshes are refined in an appropriate fashion near the reentrant corners of the domain.

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1. Introduction

The purpose of this paper is to establish *a priori* error estimates for the dual mixed method for the heat diffusion equation in a polygonal domain of $\mathbb{R}^2$. In the dual mixed approach, additionally to the classical unknown the temperature u, one considers as an additional unknown the heat flux $\overrightarrow{p}$. In many applications, the knowledge of the heat flux $\overrightarrow{p}$ is of particular importance. In such cases, the use of a dual mixed finite element method might be preferred as long as it provides a better accuracy for $\overrightarrow{p}$. On the other hand, by using the Raviart–Thomas vector field of degree 0 as approximation of $\overrightarrow{p}$, the heat-balance equation is exactly satisfied in the mean on each triangle. Let us point out that the difference with Vidar Thomée's work [6] is our *a priori* error estimates do not suppose H^2-regularity for $u_t\,(s)$ for almost every s in the interval $[0,t]$ and H^3-regularity for $u\,(t)$ as it is supposed in Theorem 17.2 p. 276 of [6], regularity properties which are not true in general for solutions u of the heat diffusion equation in polygonal domains. Note

also that the approximating spaces are not the same in [6] p. 268 as ours. The same remarks apply when comparing with Claes Johnson and Vidar Thomée's work [8]; in their *a priori* error estimates, Theorem 2.1 p. 54, they do not clearly consider the lowest-order case which is the most pertinent case in the presence of singularities. In our context of polygonal domains, due to the presence of the singularities of the solution, we have to work instead with weighted Sobolev spaces such as $H^{2,\alpha}(\Omega)$ (see Pierre Grisvard's book, Section 8.4 [2]). Also due to the singularities of the temperature u and the heat flux $\overrightarrow{p}$ of the heat diffusion equation, we will need to refine adequately our meshes near the reentrant corner of the polygonal domain Ω, using the ideas of Geneviève Raugel [9], in order to obtain optimal order of convergence in our *a priori* error estimates.
Let us close this introduction with a remark about the notations: when a norm appears in our text without any index, it is assumed that it is the L^2 norm.

2. Regularity of the solution of the heat diffusion equation

2.1. Regularity in time

Let Ω be a bounded open set of $\mathbb{R}^2$. For fixed $T > 0$, let us set $Q := \Omega\times]0,T[$ and let us denote by $\Sigma := \Gamma\times]0,T\,[$ the lateral boundary of the cylinder Q. Let us consider the Cauchy problem for the heat diffusion equation in Ω up to time T: given the right-hand side $f \in L^2\left(0,T;L^2(\Omega)\right)$ and the initial condition $g \in H_0^1(\Omega)$, find $u \in H^1\left(0,T;L^2(\Omega)\right) \cap L^2(0,T;H_0^1(\Omega))$ solution of:

$$\left\{\begin{array}{l} u_t(x,t) - \Delta u(x,t) = f(x,t),\ \forall\,(x,t) \in Q \\ u(x,t) = 0,\ \forall\,(x,t) \in \Sigma \\ u(x,0) = g(x),\ \forall x \in \Omega \end{array}\right. \tag{2.1}$$

where u_t means $\dfrac{\partial u}{\partial t}$.
Let us note that as $u \in H^1\left(0,T;L^2(\Omega)\right)$ and due to the embedding,

$$H^1\left(0,T;L^2(\Omega)\right) \hookrightarrow C([0,T];L^2(\Omega))$$

that the initial condition $u(.,0) = g$ has sense. Let us now recall the following result [1] about existence, uniqueness and regularity of the solution of the Cauchy problem:

Proposition 1. *Problem* (2.1) *admits one and only one solution*

$$u \in\ H^1\left(0,T;L^2(\Omega)\right) \cap L^2(0,T;H_0^1(\Omega)).$$

Moreover, there exist positive constants C_1 and C_2 such that

$$\|u_t(.)\|_{L^2(0,T;L^2(\Omega))} \leq C_1(\|g\|_{H_0^1(\Omega)} + \|f\|_{L^2(0,T;L^2(\Omega))}) \tag{2.2}$$

and

$$\sup_{0\leq t\leq T} \|u(t)\| \leq C_2(\|g\|_{H_0^1(\Omega)} + \|f\|_{L^2(0,T;L^2(\Omega))}). \tag{2.3}$$

Remark 2. *From inequalities* (2.2) *and* (2.3), *we have the following one:*

$$\|u\|_{H^1(0,T;L^2(\Omega))} \leq C(\|g\|_{H_0^1(\Omega)} + \|f\|_{L^2(0,T;L^2(\Omega))}) \tag{2.4}$$

where C is a positive constant.

2.2. Regularity in the spatial variables in a polygonal domain of $\mathbb{R}^2$

In the sequel Ω will denote a bounded polygonal domain of $\mathbb{R}^2$. In particular the boundary of $\Omega : \partial\Omega = \cup_{j=1}^N \overline{\Gamma}_j$ for some $N \in \mathbb{N}$, where Γ_j is an open segment of a straight line of the plane $\mathbb{R}^2$, $\forall\ j = 1, 2, \ldots, N$. As is well known the geometric singularities of the domain (the angles) induce in general singularities on the solution of the Cauchy problem for the heat diffusion equation (see for example the books of P. Grisvard [1], [2]). As shown in [2], [1], we may suppose without loss of generality that Ω has only one nonconvex angle, in other words one reentrant corner, and that its vertex is located at the origin. In the following, we denote the measure of that angle by ω.

We introduce the following weighted Sobolev space (see [2], Definition 8.4.1.1 and Lemma 8.4.1.2 p. 388):

$$H^{2,\alpha}(\Omega) = \{v \in H^1(\Omega);\, r^\alpha D^\beta \in L^2(\Omega),\ \forall|\beta| = 2\},$$

which is a Hilbert space for the norm

$$\|v\|_{2,\alpha,\Omega} = (\|v\|_{1,\Omega}^2 + |v|_{2,\alpha,\Omega}^2)^{1/2},$$

where the semi-norm $|.|_{2,\alpha,\Omega}$ is defined by

$$|v|_{2,\alpha,\Omega} = \left(\sum_{|\beta|=2} \left\|r^\alpha D^\beta v\right\|_{0,\Omega}^2\right)^{1/2}$$

r denoting the distance to the origin of $\mathbb{R}^2$.

We prove now a regularity result in the spatial variables for the solution of the Cauchy problem for the heat diffusion problem.
Before proceeding, let us mention that in the sequel we will use the notation $\forall' t \in]0, T[$ which means for almost every $t \in]0, T[$.

Proposition 3. *Let u be the solution of the Cauchy problem* (2.1). *Then for any $\alpha > 1 - \frac{\pi}{\omega}$,*

$$\|u\|_{L^2(0,T;H^{2,\alpha}(\Omega))} \leq C\left(\|f\|_{L^2(0,T;L^2(\Omega))} + \|u\|_{H^1(0,T;L^2(\Omega))}\right), \tag{2.5}$$

where C is a positive constant.

Proof. Let us introduce the closed operator A in the space $L^2(\Omega)$ which is the realization of the operator $-\Delta$ in the space $L^2(\Omega)$; more precisely:

$$D(A) := \{v \in H_0^1(\Omega); \Delta v \in L^2(\Omega)\} \text{ and } Av = -\Delta v, \forall v \in D(A).$$

We know from Section 8.4 of [2] that $D(A) \hookrightarrow H^{2,\alpha}(\Omega)$ for $\alpha > 1 - \frac{\pi}{\omega}$ and that

$$\|v\|_{H^{2,\alpha}(\Omega)} \leq C\,\|\Delta v\|\,. \tag{2.6}$$

Let u be the solution of the Cauchy problem for the heat diffusion problem (2.1). The heat-balance equation

$$\Delta u(t) = -f(t) + u_t(t)\ , \quad \forall' t \in [0, T]$$

and the regularity of u (see Proposition 1) implies that $\Delta u(t) \in L^2(\Omega)$, $\forall' t \in]0, T[$. Thus by (2.6), $\forall' t \in]0, T[\, : u(t) \in H^{2,\alpha}(\Omega)$ for any fixed $\alpha > 1 - \frac{\pi}{\omega}$ and

$$\|u(t)\|_{H^{2,\alpha}(\Omega)} \leq C\,\|\Delta u(t)\| \leq C\,(\|f(t)\| + \|u_t(t)\|)\,. \tag{2.7}$$

Taking the squares of both members of the above inequality, then integrating both sides from 0 to T, it follows that $u \in L^2(0, T; H^{2,\alpha}(\Omega))$ and that

$$\|u\|_{L^2(0,T;H^{2,\alpha}(\Omega))} \leq C(\|f\|_{L^2(0,T;L^2(\Omega))} + \|u_t\|_{L^2(0,T;L^2(\Omega))}). \tag{2.8}$$

From this last inequality, (2.5) follows immediately. The proof is complete. □

Let us now introduce the dual mixed formulation for the heat diffusion equation.

3. The dual mixed formulation for the heat diffusion equation

In the following $H(\mathrm{div}; \Omega)$ denotes the space

$$H(\mathrm{div}; \Omega) := \{\overrightarrow{q} \in L^2(\Omega)^2; \mathrm{div}\,\overrightarrow{q} \in L^2(\Omega)\}$$

endowed with its natural norm (see for example [7]). Introducing the new variable $\vec{p} = \vec{\nabla} u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2})^T$, the heat-balance equation may be rewritten in the form:

$$\mathrm{div}\ \vec{p}(x,t) = u_t(x,t) - f(x,t).$$

Now, as the solution of (2.1) belongs to the space $H^1\big(0,T;L^2(\Omega)\big) \cap L^2(0,T;H_0^1(\Omega))$ (cf. Proposition 1), we have $\vec{p} \in L^2(0,T;H(\mathrm{div};\Omega))$ and then

$$(\vec{p}, u) \in L^2(0,T;H(\mathrm{div};\Omega)) \times H^1\big(0,T;L^2(\Omega)\big).$$

In the following, we will also denote by $X := H(\mathrm{div}, \Omega)$, by $M := L^2(\Omega)$ and by I the interval of time $[0, T]$. In terms of $(\vec{p}, u)$, we may rewrite the Cauchy problem (2.1) for the heat diffusion equation as follows

$$\left\{\begin{array}{c} \vec{p}(x,t) = \overrightarrow{\nabla} u(x,t), \forall' x \in \Omega, \forall' t \in I, \\ u_t(x,t) - \mathrm{div}\,\vec{p}(x,t) = f(x,t), \forall'\,(x,t) \in Q, \\ u(x,t) = 0, \forall'\,(x,t) \in \Sigma, \\ u(x,0) = g(x), \forall' x \in \Omega. \end{array}\right. \tag{3.1}$$

Let us set $\vec{p}(t)(x) = \vec{p}(x,t)$ and $u(t)(x) = u(x,t)$. Taking $\vec{q} \in X$, multiplying both sides of $(3.1)_{(\mathrm{i})}$ by $\vec{q}$ and using Green's formula, we obtain:

$$\begin{aligned} \int_\Omega \vec{p}(t).\vec{q}\,dx + \int_\Omega u(t)\,\mathrm{div}\,\vec{q}\,dx &= \int_\Omega (\vec{\nabla} u(t).\vec{q} + u(t)\,\mathrm{div}\,\vec{q})dx \\ &= \int_{\partial\Omega} u(t)\ \vec{q}.\vec{n}\ ds, \qquad \forall' t \in I. \end{aligned}$$

As $u \in L^2(0,T;H_0^1(\Omega))$, $u(t)_{/\partial\Omega} = 0$ for almost every $t \in I$, so that:

$$\int_\Omega \vec{p}(t).\vec{q}\,dx + \int_\Omega u(t)\operatorname{div}\vec{q}\,dx = 0,\ \forall\vec{q} \in X,\ \forall' t \in I.$$

On the other hand multiplying both sides of the heat-balance equation $(3.1)_{(\mathrm{ii})}$ by $v \in M$ and integrating over Ω, it follows that

$$\int_\Omega \operatorname{div}\vec{p}(t)\ v\ dx = -\int_\Omega (f(t) - u_t(t))\,v\ dx,\ \forall v \in M,\ \forall' t \in I,$$

where $f(t)(x) := f(x,t)$.

Thus if $u \in H^1\big(0,T;L^2(\Omega)\big) \cap L^2(0,T;H_0^1(\Omega))$ denotes the solution of the Cauchy problem for the heat diffusion equation (2.1), then

$$(\vec{p} := \overrightarrow{\nabla} u, u) \in L^2(0,T;H(\operatorname{div},\Omega)) \times H^1(0,T;L^2(\Omega))$$

and is a solution of the following system of equations

$$\begin{cases} \int_\Omega \vec{p}(t).\vec{q}\,dx + \int_\Omega u(t)\ \operatorname{div}\vec{q}\,dx = 0,\ \ \forall\vec{q} \in X,\ \forall' t \in I, \\ \int_\Omega \operatorname{div}\vec{p}(t)\ v\ dx = -\int_\Omega (f(t) - u_t(t))\ v\ dx,\ \ \forall v \in M,\ \forall' t \in I, \\ u(0) = g \in H_0^1(\Omega). \end{cases} \tag{3.2}$$

The system of equations (3.2) is called the dual mixed formulation for the heat diffusion equation.

Theorem 4. *For every initial condition $g \in H_0^1(\Omega)$ and every right-hand side $f \in L^2(0,T;L^2(\Omega))$, the dual mixed formulation* (3.2) *admits a unique solution $(\vec{p},u) \in L^2(0,T;H(\operatorname{div};\Omega)) \times H^1(0,T;L^2(\Omega))$.*

Proof. We have already proved the existence of a solution. It remains to prove uniqueness. Let us consider $(\vec{p},u) \in L^2(0,T;H(\operatorname{div};\Omega)) \times H^1(0,T;L^2(\Omega))$ verifying $u(0) = 0$ and

$$\begin{cases} \int_\Omega \vec{p}(t).\vec{q}\,dx + \int_\Omega u(t)\ \operatorname{div}\vec{q}\,dx = 0,\ \ \forall\vec{q} \in X,\ \forall' t \in I, \\ \int_\Omega \operatorname{div}\vec{p}(t)\ v\ dx = \int_\Omega u_t(t)\ v\ dx,\ \ \forall v \in M,\ \forall' t \in I. \end{cases} \tag{3.3}$$

Let us observe that $u \in H^1(0,T;L^2(\Omega))$ implies $u_t(t) \in L^2(\Omega)\ \forall' t \in I$ and then $\int_\Omega u_t(t)\ v\ dx$ has sense $\forall v \in M, \forall' t \in I$.

Taking $\vec{q} = \vec{p}(t)$ in equation $(3.3)_{(\mathrm{i})}$ and $v = u(t)$ in equation $(3.3)_{(\mathrm{ii})}$, for a fixed $t \in I$ except a subset of measure zero, it follows that:

$$\begin{cases} \int_\Omega |\vec{p}(t)|^2\ dx + \int_\Omega u(t)\ \operatorname{div}\vec{p}(t)dx = 0 \\ \int_\Omega \operatorname{div}\vec{p}(t)\ u(t)\ dx = \int_\Omega u_t(t)\ u(t)\ dx. \end{cases}$$

Therefore

$$\int_\Omega |\vec{p}(t)|^2\ dx + \int_\Omega u_t(t)\ u(t)\ dx = 0 \tag{3.4}$$

which implies

$$\int_\Omega |\vec{p}(t)|^2\ dx + \frac{1}{2}\frac{d}{dt}\int_\Omega u(t)^2 dx = 0, \qquad \forall' t \in I.$$

Consequently

$$\frac{d}{dt}\int_\Omega u(t)^2 dx = -2\int_\Omega |\vec{p}(t)|^2 \; dx \leq 0 \;, \quad \forall' t \in I$$

which allows us to conclude that the function $t \mapsto \int_\Omega u(t)^2 dx$ is decreasing. From $\int_\Omega u(0)^2 dx = 0$, it now follows that

$$\int_\Omega u(t)^2 dx = 0$$

for $\forall t \in I$ as $u \in C([0,T]; L^2(\Omega))$. Thus $u = 0$.

From (3.4), we conclude that

$$\int_\Omega |\vec{p}(t)|^2 \; dx = 0, \quad \forall' t \in I$$

which implies $\vec{p}(t) = 0$, $\forall' t \in I$. Thus $\vec{p} = 0$ as an element of $L^2(0,T; H(\mathrm{div};\Omega))$. We have thus proved uniqueness. □

4. Semi-discrete solution of the dual mixed method for the heat diffusion equation in a polygonal domain of $\mathbb{R}^2$

Let us consider a family of triangulations $(\mathcal{T}_h)_{h>0}$ on $\overline{\Omega}$. For K a triangle belonging to the triangulation $\mathcal{T}_h$, let us denote by h_K the diameter of K and by ρ_K the interior diameter of K, i.e., the diameter of the biggest disc included in K. As in Theorem 8.4.1.6 p. 392 of [2], we suppose that the family of triangulations $(\mathcal{T}_h)_{h>0}$ has the property that $\max\limits_{K\in\mathcal{T}_h} \frac{h_K}{\rho_K}$ is bounded by a positive constant independent of the parameter h; in that case, one says usually that the family of triangulations is regular (see for example [3] (17.1) p. 131). In accordance with the tradition (see [3] Remark 17.1 p. 131) the same letter h may have also another significance, it may denote instead:

$$h =: \max_{K\in\mathcal{T}_h} h_K \;.$$

The true significance of h is always clear from the context.

Let us now define the semi-discretized problem. Firstly, let us define the following finite-dimensional vector subspaces X_h of X respectively M_h of M:

$$\begin{aligned} X_h &:= \left\{ \vec{q}_h \in H(\mathrm{div};\Omega); \forall K \in \mathcal{T}_h \;:\; \vec{q}_{h/K} \in RT_0(K) \right\} \\ M_h &:= \left\{ v_h \in L^2(\Omega);\; v_{h/K} \in P_0\,(K) \;,\; \forall K \in \mathcal{T}_h \right\} \end{aligned}$$

where $RT_0(K) := P_0(K)^2 \oplus P_0(K)\binom{x_1}{x_2}$ denotes the real vectorial space of dimension three of the so-called Raviart-Thomas vector fields of degree 0 on the triangle K ($RT_0(K)$ is denoted $D_1(K)$ in [4] p. 550) and $P_0\,(K)$ the real vectorial space of dimension one of the constant functions on the triangle K. We are now in a position to define the semi-discretized problem:

Find $(\vec{p}_h, u_h) \in L^2(0,T;X_h) \times H^1(0,T;M_h)$ such that

$$\left\{\begin{array}{l} \int_\Omega \vec{p}_h(t).\vec{q}_h\ dx + \int_\Omega u_h(t)\ \operatorname{div} \vec{q}_h\ dx = 0,\ \forall \vec{q}_h \in X_h,\ \forall' t \in I, \\ \int_\Omega v_h\ \operatorname{div} \vec{p}_h(t)\ dx = -\int_\Omega (f(t) - u_{h,t}(t)) v_h\ dx,\ \forall v_h \in M_h,\ \forall' t \in I, \\ u_h(0) = g_h \in M_h\ . \end{array}\right. \tag{4.1}$$

The initial condition g_h in M_h will be precised later. Let us first show that the above problem (4.1) possesses one and only one solution $(\vec{p}_h, u_h) \in L^2(0,T;X_h) \times H^1(0,T;M_h)$:

Proposition 5. *Problem* (4.1) *possesses one and only one solution*

$$(\vec{p}_h, u_h) \in L^2(0,T;X_h) \times H^1(0,T;M_h).$$

Moreover $\vec{p}_h \in H^1(0,T;X_h)$.

Proof. Let $\vec{q}_h^{(1)}, \dots, \vec{q}_h^{(J)}$ a basis of X_h and $v_h^{(1)}, \dots, v_h^{(K)}$ a basis of M_h. Expanding $\vec{p}_h(t)$, respectively $u_h(t)$ in these respective basis, we obtain

$$\vec{p}_h(t) = \sum_{j=1}^{J} \alpha_j(t) \vec{q}_h^{(j)},\ u_h(t) = \sum_{k=1}^{K} \beta_k(t) v_h^{(k)}$$

where $\alpha_j(t)$ $(j = 1, \dots, J)$ and $\beta_k(t)$ $(k = 1, \dots, K)$ denote some real coefficients. The semi-discrete mixed formulation (4.1) is equivalent to:

$$\left\{\begin{array}{l} \int_\Omega \left(\sum_{j=1}^{J} \alpha_j(t) \vec{q}_h^{(j)}\right).\vec{q}_h^{(j')}\ dx \\ + \int_\Omega \left(\sum_{k=1}^{K} \beta_k(t) v_h^{(k)}\right) \operatorname{div} \vec{q}_h^{(j')}\ dx = 0, \quad \forall j' = 1, 2, \dots, J \\ \int_\Omega v_h^{(k')} (\sum_{j=1}^{J} \alpha_j(t) \operatorname{div} \vec{q}_h^{(j)}) dx \\ = -\int_\Omega (f(t) - \sum_{k=1}^{K} \dot{\beta}_k(t) v_h^{(k)})\ v_h^{(k')}\ dx, \quad \forall k' = 1, 2, \dots, K \end{array}\right.$$

which can be rewritten in the form:

$$\left\{\begin{array}{l} \sum_{j=1}^{J} (\int_\Omega \vec{q}_h^{(j)}.\vec{q}_h^{(j')}\ dx)\ \alpha_j(t) \\ + \sum_{k=1}^{K} (\int_\Omega v_h^{(k)}\ \operatorname{div} \vec{q}_h^{(j')}\ dx)\ \beta_k(t) = 0, \quad \forall j' = 1, 2, \dots, J, \\ \sum_{j=1}^{J} \left(\int_\Omega v_h^{(k')} \operatorname{div} \vec{q}_h^{(j)} dx\right)\ \alpha_j(t) \\ = -\int_\Omega f(t) v_h^{(k')} dx + \sum_{k=1}^{K} (\int_\Omega v_h^{(k)} v_h^{(k')} dx) \dot{\beta}_k(t), \quad \forall k' = 1, 2, \dots, K. \end{array}\right.$$

Let us now set:

$$\left\{\begin{array}{c} a_{kk'} = \int_\Omega v_h^{(k)} v_h^{(k')} dx,\ b_{jj'} = \int_\Omega \vec{q}_h^{(j)} \cdot \vec{q}_h^{(j')}\ dx,\ c_{j'k'} = \int_\Omega (\operatorname{div} \vec{q}_h^{(j')}) v_h^{(k')} dx \\ \forall\ j, j' = 1, 2, \dots, J\ ; \forall\ k, k' = 1, 2, \dots, K. \end{array}\right.$$

With these notations, the above differential system may be rewritten:

$$\left\{\begin{array}{c} \sum_{j=1}^{J}\ b_{j'j} \alpha_j(t) + \sum_{k=1}^{K} c_{j'k} \beta_k(t) = 0,\ \forall j' = 1, 2, \dots, J \\ \sum_{j=1}^{J} c_{jk'} \alpha_j(t) = -\int_\Omega f(t) v_h^{(k')} dx + \sum_{k=1}^{K} a_{kk'} \dot{\beta}_k(t),\ \forall k' = 1, 2, \dots, K. \end{array}\right. \tag{4.2}$$

Let us introduce the following matrices:

$$A = (a_{kk'})_{1\le k,k'\le K} \in \mathbb{R}^{K\times K};$$
$$B = (b_{j'j})_{1\le j',j\le J} \in \mathbb{R}^{J\times J};$$
$$C = (c_{j'k})_{1\le j'\le J, 1\le k\le K} \in \mathbb{R}^{J\times K}.$$

It is immediate that the matrices A and B are symmetric and positive definite. Let us also introduce the vectors:

$$\beta(t) = \begin{pmatrix} \beta_1(t) \\ \cdot \\ \cdot \\ \cdot \\ \beta_K(t) \end{pmatrix} \in \mathbb{R}^K, \qquad \alpha(t) = \begin{pmatrix} \alpha_1(t) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \alpha_J(t) \end{pmatrix} \in \mathbb{R}^J,$$

$$F(t) = \begin{pmatrix} \int_\Omega f(t) v_h^{(1)} dx \\ \cdot \\ \cdot \\ \cdot \\ \int_\Omega f(t) v_h^{(K)} dx \end{pmatrix} \in \mathbb{R}^K.$$

These matrices and vectors allow us to rewrite our differential system (4.2) in the matrix form:

$$\begin{cases} B\,\alpha(t) + C\,\beta(t) = 0 \\ A\,\dot\beta(t) = C^\intercal \alpha(t) + F(t)\,. \end{cases}$$

From the first matrix equation, it follows immediately that

$$\alpha(t) = -B^{-1} C\,\beta(t).$$

Thus:

$$\begin{cases} \alpha(t) = -B^{-1}C\,\beta(t), \\ A\,\dot\beta(t) = -C^\intercal B^{-1} C\,\beta(t) + F(t)\,. \end{cases}$$

It suffices thus to solve the following Cauchy problem

$$\begin{cases} A\,\dot\beta(t) + C^\intercal B^{-1} C\,\beta(t) = F(t),\ F \in L^2(0,T;\mathbb{R}^K) \\ \beta(0) = \beta_0 \in \mathbb{R}^K \end{cases}$$

where the vector of initial conditions β_0 is the vector of the components of g_h in the basis

$$v_h^{(1)}, \ldots, v_h^{(K)}$$

of M_h, i.e.,

$$\sum_{k=1}^{K} (\beta_0)_k v_h^{(k)} = g_h \in M_h.$$

This differential system of K equations in the K unknowns $\beta_1, \ldots, \beta_K$ may be rewritten in the form

$$\dot{\beta}(t) = -A^{-1}C^{\intercal}B^{-1}C\beta(t) + A^{-1}F(t).$$

Using the contraction semigroup $\left(e^{-tA^{-1}C^{\intercal}B^{-1}C}\right)_{t\geq 0}$ generated by the symmetric negative operator $-A^{-1}C^{\intercal}B^{-1}C$ on the Euclidean space $\mathbb{R}^K$, the solution of the above inhomogeneous Cauchy problem may be written [5]:

$$\beta(t) = e^{-tA^{-1}C^{\intercal}B^{-1}C}\,\beta_0 + \int_0^t e^{-(t-\tau)A^{-1}C^{\intercal}B^{-1}C}\,A^{-1}F(\tau)\,d\tau.$$

It follows immediately from the equation

$$\dot{\beta}(t) = -A^{-1}C^{\intercal}B^{-1}C\beta(t) + A^{-1}F(t)$$

that $\dot{\beta} \in L^2(0, T; \mathbb{R}^K)$ and thus that $\beta \in C([0, T]; \mathbb{R}^K)$. From $\alpha(t) = -B^{-1}C\beta(t)$, follows also that $\alpha \in C([0, T]; \mathbb{R}^J)$ and that $\dot{\alpha} \in L^2(0, T; \mathbb{R}^J)$. Therefore

$$u_h \in H^1(0, T; M_h) \text{ and } \vec{p}_h \in H^1(0, T; X_h).$$

This completes the proof of the result. □

5. A priori error estimates for the semi-discrete solution of the dual mixed method for the heat diffusion equation

To prove error estimates on $\vec{p_h}$ and u_h, we need to introduce an intermediate problem, the so-called elliptic projection problem, with which we are going to compare firstly the exact solution $(\vec{p}\,(t)\,, u\,(t))$. The definition of the elliptic projection problem is similar to that one given by Vidar Thomée in his book ([6], (17.27) p. 276). Let $t \in I$ **fixed** such that $f\,(t) - u_t\,(t) \in L^2(\Omega)$; we know that it is true for almost every $t \in I$.

Definition 6. *We call elliptic projection of $(\vec{p}(t), u(t))$, the solution denoted $(\tilde{\vec{p}}_h(t), \tilde{u}_h(t))$ of the stationary discrete mixed formulation with right-hand side $\triangle u(t) = \operatorname{div}\vec{p}(t) = u_t\,(t) - f\,(t) \in L^2(\Omega)$.*

In other words $(\tilde{\vec{p}}_h(t), \tilde{u}_h(t))$ is the solution of the system of equations:

$$\begin{cases} \int_\Omega \tilde{\vec{p}}_h(t).\vec{q}_h\,dx + \int_\Omega \tilde{u}_h(t)\,\operatorname{div}\vec{q}_h\,dx = 0,\ \forall \vec{q}_h \in X_h, \\ \int_\Omega v_h\,\operatorname{div}\tilde{\vec{p}}_h(t)\,dx = \int_\Omega \Delta u(t)\,v_h\,dx,\ \forall v_h \in M_h. \end{cases} \tag{5.1}$$

Note that $\Delta u(t) = u_t(t) - f(t)$ for almost every t in I, $u_t(t) - f(t) \in L^2(\Omega)$. Thus for almost every t in I, we may state problem (5.1).

Proposition 7. *For almost every $t \in I$, problem* (5.1) *admits one and only solution $(\tilde{\vec{p}}_h(t), \tilde{u}_h(t)) \in X_h \times M_h$. Moreover $\tilde{\vec{p}}_h \in L^2(0, T; X_h)$ and $\tilde{u}_h \in L^2(0, T; M_h)$.*

Proof. We use the same notations as those we have used in the proof of the existence and uniqueness of the semi-discrete solution. Thus, let us write $\vec{\tilde{p}}_h(t)$, respectively $\tilde{u}_h(t)$, in the basis $\left(\vec{q}_h^{(j)}\right)_{j=1,\ldots,J}$ of X_h, respectively $\left(v_h^{(k)}\right)_{k=1,\ldots,K}$ of M_h:

$$\vec{\tilde{p}}_h(t) = \sum_{j=1}^{J} \tilde{\alpha}_j(t)\vec{q}_h^{(j)},\ \tilde{u}_h(t) = \sum_{k=1}^{K} \tilde{\beta}_k(t)v_h^{(k)}$$

where $\tilde{\alpha}_j(t)$ and $\tilde{\beta}_k(t)$ denote certain real coefficients. These coefficients must satisfy the following system of equations:

$$\begin{cases} B\,\tilde{\alpha}(t) + C\,\tilde{\beta}(t) = 0 \\ C^{\intercal}\,\tilde{\alpha}(t) + \tilde{F}(t) = 0, \end{cases} \tag{5.2}$$

where

$$\tilde{F}(t) = \begin{pmatrix} \int_\Omega (f(t) - u_t(t))\, v_h^{(1)}\ dx \\ \cdot \\ \cdot \\ \cdot \\ \int_\Omega (f(t) - u_t(t))\, v_h^{(K)}\ dx \end{pmatrix} \in \mathbb{R}^K.$$

This last expression of $\tilde{F}(t)$ shows that $\tilde{F}(\cdot) \in L^2(0,T;\mathbb{R}^K)$. The system (5.2) is equivalent to

$$\begin{cases} \tilde{\alpha}(t) = -B^{-1}C\,\tilde{\beta}(t),\ \forall' t \in I, \\ C^{\intercal}\tilde{\alpha}(t) + \tilde{F}(t) = 0,\ \forall' t \in I. \end{cases}$$

Thus

$$(C^{\intercal}B^{-1}C)\tilde{\beta}(t) = \tilde{F}(t)$$

where the matrix $C^{\intercal}B^{-1}C \in \mathbb{R}^{K\times K}$. This last matrix is symmetric. Let us prove that this matrix is also positive definite which will prove that it is invertible. Let $\xi \in \mathbb{R}^K \setminus \{0\}$

$$(\left(C^{\intercal}B^{-1}C\right)\xi, \xi) = (\left(B^{-1}C\right)\xi, C\xi) \geq \frac{1}{\max\ \sigma(B)}\,\|C\ \xi\|^2$$

where $\max\sigma(B)$ denotes the maximum of the eigenvalues of B. Let us note that $\left(C^{\intercal}B^{-1}C\right)\xi \in \mathbb{R}^K$ and that $\left(B^{-1}C\right)\xi \in \mathbb{R}^J$.

The preceding inequality implies that

$$(C^{\intercal}B^{-1}C\xi, \xi) \geq 0,\ \forall \xi \in \mathbb{R}^K \setminus \{0\}\,.$$

To prove that $C^{\intercal}B^{-1}C$ is positive definite, it suffices to verify that the vector $C\xi \in \mathbb{R}^J$ is nonzero, $\forall \xi \in \mathbb{R}^K \setminus \{0\}$. To prove this, let us suppose one moment that $C\xi = 0$. This means that

$$\int_\Omega \left(\operatorname{div} \vec{q}_h^{(j')}\right) \left(\sum_{k=1}^{K} v_h^{(k)} \xi_k\right)\ dx = 0,\ \forall j' = 1,\ldots,J.$$

But $\vec{q}_h^{(1)}, \vec{q}_h^{(2)}, \ldots, \vec{q}_h^{(J)}$ is a basis of X_h. Thus $\forall \vec{q}_h \in X_h$:

$$\int_\Omega (\operatorname{div} \vec{q}_h) \left(\sum_{k=1}^{K} v_h^{(k)} \xi_k \right) \, dx = 0 \, .$$

But $\sum_{k=1}^{K} \xi_k v_h^{(k)} \in M_h$ so that by the inf-sup inequality (see for example Lemma (1.2) p. 612 of [7]) it follows that $\sum_{k=1}^{K} \xi_k v_h^{(k)} = 0$ which implies $\xi_1 = \xi_2 = \cdots = \xi_K = 0$ as $v_h^{(1)}, \ldots, v_h^{(K)}$ is a basis of M_h and thus $\xi = 0$. This proves that the matrix $C^\intercal B^{-1} C$ is positive definite and thus invertible. Therefore

$$\tilde{\beta}(t) = (C^\intercal B^{-1} C)^{-1} \, \tilde{F}(t).$$

As $\tilde{F} \in L^2(0, T; \mathbb{R}^K)$, it follows from the preceding formula that $\tilde{\beta} \in L^2(0, T; \mathbb{R}^K)$ and consequently $\tilde{\alpha} \in L^2(0, T; \mathbb{R}^J)$ by the first equation of (5.2). Thus

$$\tilde{u}_h \in L^2(0, T; M_h) \ \text{ and } \ \ \tilde{\vec{p}}_h \in L^2(0, T; X_h).$$

The proof is complete. $\square$

In the continuation, to obtain the error estimates, we will need also regularity on u_t, i.e., on the time derivative of u. To obtain it, we will assume more regularity on the data.

Proposition 8. *Let us suppose that:*

$$f \in H^1(0, T; L^2(\Omega)) \text{ and that } \Delta g + f(0) \in H_0^1(\Omega),$$

where g which belongs to $H_0^1(\Omega)$ denotes the initial condition of the heat diffusion equation. Then

$$u_t \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)) \cap L^2(0, T; H^{2,\alpha}(\Omega)), \tag{5.3}$$

and

$$\tilde{u}_h \in H^1(0, T; M_h) \text{ and } \tilde{\vec{p}}_h \in H^1(0, T; X_h). \tag{5.4}$$

Proof. By hypothesis $\frac{df}{dt} \in L^2(0, T; L^2(\Omega))$. Let

$$v \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$$

be the solution of the following heat diffusion problem:

$$\begin{cases} \frac{dv}{dt} = \Delta v + \frac{df}{dt}, \text{ in } Q \\ v(0) = \Delta g + f(0), \text{ in } \Omega \, . \end{cases}$$

As $\triangle g + f(0) \in H_0^1(\Omega)$ and $\frac{df}{dt} \in L^2(0, T; L^2(\Omega))$, by Proposition 1 v exists and is unique and by the regularity result (2.5):

$$v \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)) \cap L^2(0, T; H^{2,\alpha}(\Omega)).$$

Let us set

$$u\,(t) = \int_0^t v\,(s) \ ds \ + \ g, \ \forall t \in I.$$

One immediately checks that u so defined is the solution of (2.1) . Moreover $\frac{du}{dt} = v$. By the above regularity properties of v follows immediately (5.3). On the other hand, we have seen in the proof of the existence and uniqueness of the elliptic projection that

$$\tilde{\beta}(t) = (C^{\intercal} B^{-1} C)^{-1}\ \tilde{F}(t)$$

from which follows

$$\begin{aligned} \frac{d\tilde{\beta}}{dt}(t) &= (C^{\intercal} B^{-1} C)^{-1}\ \frac{d\tilde{F}}{dt}(t) \\ &= (C^{\intercal} B^{-1} C)^{-1} \begin{pmatrix} \int_\Omega (\frac{df}{dt}(t) - \frac{du_t}{dt}(t))\, v_h^{(1)}\ dx \\ \cdot \\ \cdot \\ \cdot \\ \int_\Omega (\frac{df}{dt}(t) - \frac{du_t}{dt}(t))\, v_h^{(K)}\ dx \end{pmatrix}. \end{aligned}$$

Thus

$$\frac{d\tilde{\beta}}{dt} \in L^2(0,T;\mathbb{R}^K),$$

and from $\tilde{\alpha}(t) = -B^{-1}C\tilde{\beta}(t)$, it now follows $\frac{d\tilde{\alpha}}{dt} \in L^2(0,T;\mathbb{R}^J)$. Consequently

$$\tilde{u}_h \in H^1(0,T;M_h) \ \text{ and } \ \vec{\tilde{p}}_h \in H^1(0,T;X_h).$$

The proof is complete. □

Observing, that the solution of the elliptic projection problem $\left(\vec{\tilde{p}}_h(t), \tilde{u}_h(t)\right)$ is nothing else than the solution of the discretized mixed formulation for the Laplacian with right-hand side

$$-\Delta u(t) = f(t) - u_t(t) \in L^2(\Omega),$$

it follows from Theorem 1.13 p. 619 and Theorem 1.17 p. 623 of [7]:

Proposition 9. *Let $\{\mathcal{T}_h\}$ be a regular family of triangulations on $\overline{\Omega}$ satisfying for some fixed $\alpha \in \left]1 - \frac{\pi}{\omega}, 1\right[$ (let us recall that ω denotes the measure of the angle of the reentrant corner located at the origin), the following rules of refinement:*

(i) *$h_K \leq \sigma h^{\frac{1}{1-\alpha}}$ for every triangle $K \in \mathcal{T}_h \in \{\mathcal{T}_h\}$ having one vertex at the origin (the vertex of the reentrant corner),*

(ii) *$h_K \leq c\,(\inf_{x\in K} r^\alpha(x))\, h$ for every triangle $K \in \mathcal{T}_h \in \{\mathcal{T}_h\}$ having no vertex at the origin.*

Then, there exists a constant $C > 0$ independent of the parameter h such that

$$\left\| \vec{p}(t) - \vec{\tilde{p}}_h(t) \right\|_{L^2(\Omega)^2} \leq C\ h\ |u(t)|_{H^{2,\alpha}(\Omega)} \tag{5.5}$$

and

$$\|u\,(t) - \tilde{u}_h(t)\| \leq C\ h(\ |u(t)|_{H^1(\Omega)}\ + |u(t)|_{H^{2,\alpha}(\Omega)}). \tag{5.6}$$

Proposition 10. *Let $\{\mathcal{T}_h\}$ be a regular family of triangulations on $\overline{\Omega}$ satisfying the same refinement rules as stated in Proposition 9 for some α fixed in the interval $\left]1-\frac{\pi}{w},1\right[$. There exists a positive constant β^* independent of h such that $\forall' t \in I$:*

$$\|u_h(t) - P_h\,u(t)\| \leq \frac{1}{\beta^*}\,\|\vec{p}(t) - \vec{p}_h(t)\| \tag{5.7}$$

where P_h denotes the orthogonal projection operator from M onto M_h (i.e., the mean operator on each triangle of $\mathcal{T}_h$).

Proof. Taking $\vec{q} = \vec{q}_h$ in the first equation of the mixed formulation for the heat diffusion problem (3.2), we obtain

$$\int_\Omega \vec{p}(t).\vec{q}_h dx + \int_\Omega u(t)\operatorname{div}\vec{q}_h\ dx = 0. \tag{5.8}$$

As div $\vec{q}_h$ is constant on each $K \in \mathcal{T}_h$, we have for every $\vec{q}_h \in X_h$:

$$\begin{aligned}\int_\Omega u(t)\ \operatorname{div}\vec{q}_h\ dx &= \sum_{K\in\mathcal{T}_h}\int_K u(t)\operatorname{div}\vec{q}_h\ dx = \sum_{K\in\mathcal{T}_h}\operatorname{div}(\vec{q}_{h|K})\int_K P_h\left(u(t)\right)\ dx\\ &= \int_\Omega P_h\left(u(t)\right)\operatorname{div}\vec{q}_h\ dx\ .\end{aligned}$$

Using this equality, equation (5.8) becomes:

$$\int_\Omega \vec{p}(t).\vec{q}_h dx + \int_\Omega P_h\left(u(t)\right)\operatorname{div}\vec{q}_h\ dx = 0,\ \forall\vec{q}_h \in X_h, \forall' t \in I, \tag{5.9}$$

and taking its difference with the first equation of the semi-discretized mixed formulation for the heat diffusion problem (4.1), we obtain

$$\int_\Omega \left(\vec{p}(t) - \vec{p}_h(t)\right).\vec{q}_h dx + \int_\Omega \left(P_h(u(t)) - u_h(t)\right)\operatorname{div}\vec{q}_h\ dx = 0,\ \forall\vec{q}_h \in X_h, \forall' t \in I. \tag{5.10}$$

Now applying the uniform inf-sup inequality (see for example Corollary (1.15) of [7]) and using (5.10), we obtain:

$$\|u_h(t) - P_h\,u(t)\| \leq \frac{1}{\beta^*}\,\|\vec{p}(t) - \vec{p}_h(t)\|\ ,\ \forall' t \in I,$$

where β^* denotes the positive constant appearing in the inf-sup inequality. The proof is complete. □

The following result is a straightforward consequence of a well-known error interpolation inequality (see for example inequality (45) p. 624 in [7]):

Lemma 11. *Let $\{\mathcal{T}_h\}$ be a regular family of triangulations on $\overline{\Omega}$. There exists a positive constant C independent of h such that for almost every $t \in I$:*

$$\|u(t) - P_h\,u(t)\| \leq C\ h\ |u(t)|_{H^1(\Omega)}. \tag{5.11}$$

Proposition 12. *Let $\{\mathcal{T}_h\}$ be a regular family of triangulations on $\overline{\Omega}$ satisfying the same refinement rules as stated in Proposition 9 for some α fixed in the interval $\left]1-\frac{\pi}{w},1\right[$. Assuming that the data f and g satisfy the hypotheses of Proposition 8, there exists a constant $C>0$ independent of h such that for every $t\in I$:*

$$\|u(t)-u_h(t)\| \leq C\ h\ |u(t)|_{H^1(\Omega)} + \frac{1}{\beta^*}\left(C\ h|u(t)|_{H^{2,\alpha}(\Omega)} + \left\|\vec{\tilde{p}}_h(t)-\vec{p}_h(t)\right\|\right). \tag{5.12}$$

Proof. Applying the inequalities (5.11) and (5.7), we obtain for almost every $t\in I$:

$$\begin{aligned}\|u(t)-u_h(t)\| &\leq \|u(t)-P_hu(t)\| + \|P_hu(t)-u_h(t)\| \qquad \forall' t\in I,\\ &\leq C\ h|u(t)|_{H^1(\Omega)} + \frac{1}{\beta^*}\left(\|\vec{p}(t)-\vec{p}_h(t)\|\right)\\ &\leq C\ h|u(t)|_{H^1(\Omega)} + \frac{1}{\beta^*}\left(\left\|\vec{p}(t)-\vec{\tilde{p}}_h(t)\right\| + \left\|\vec{\tilde{p}}_h(t)-\vec{p}_h(t)\right\|\right)\ .\end{aligned}$$

Now applying inequality (5.5), we obtain for almost every $t\in I$:

$$\|u(t)-u_h(t)\| \leq C\ h|u(t)|_{H^1(\Omega)} + \frac{1}{\beta^*}\left(C\ h|u(t)|_{H^{2,\alpha}(\Omega)} + \left\|\vec{\tilde{p}}_h(t)-\vec{p}_h(t)\right\|\right).$$

But $u\in H^1\left(0,T;L^2(\Omega)\right)\hookrightarrow C([0,T];L^2(\Omega))$ and u_h is also a continuous function implying that $u-u_h$ is a continuous function. Thus, the preceding inequality is in fact true for all $t\in I$. □

We are now in a position to bound $\|\vec{p}(t)-\vec{p}_h(t)\|$ and $\|u(t)-u_h(t)\|$. We have the following a priori error estimates:

Theorem 13. *Firstly, we suppose that $f\in H^1\left(0,T;L^2(\Omega)\right)$, $g\in H_0^1(\Omega)$ and $\Delta g + f(0)\in H_0^1(\Omega)$. Now, let us choose $g_h=\tilde{u}_h(0)$ as initial condition for the semi-discrete mixed formulation and let $\{\mathcal{T}_h\}$ be a regular family of triangulations on $\overline{\Omega}$ satisfying the same refinement rules as stated in Proposition 9 for some α fixed in the interval $\left]1-\frac{\pi}{w},1\right[$. Then there exists a constant $C>0$ independent of h such that for every $t\in I$:*

$$\|\vec{p}(t)-\vec{p}_h(t)\| \leq C\ h\left(\ |u(t)|_{H^{2,\alpha}(\Omega)} + \left\|\frac{du}{dt}\right\|_{L^2(0,T;H^{2,\alpha}(\Omega))}\right) \tag{5.13}$$

and

$$\|u(t)-u_h(t)\| \leq C\ h\left(\ |u(t)|_{H^1(\Omega)} + |u(t)|_{H^{2,\alpha}(\Omega)} + \left\|\frac{du}{dt}\right\|_{L^2(0,T;H^{2,\alpha}(\Omega))}\right). \tag{5.14}$$

Proof. Subtracting the first equation of (5.1) from the first one of (4.1), we obtain:

$$\int_\Omega \left(\vec{p}_h(t)-\vec{\tilde{p}}_h(t)\right).\vec{q}_h\ dx + \int_\Omega (u_h(t)-\tilde{u}_h(t))\operatorname{div}\vec{q}_h\ dx = 0, \forall' t\in I\ ,\forall \vec{q}_h\in X_h.$$

Let us set: $\vec{\varepsilon}_h(t) = \vec{p}_h(t) - \vec{\tilde{p}}_h(t)$ and $\theta_h(t) = u_h(t) - \tilde{u}_h(t)$. With these new notations, the preceding equation may be rewritten:

$$\int_\Omega \vec{\varepsilon}_h(t).\vec{q}_h \; dx + \int_\Omega \theta_h(t) \; \operatorname{div} \vec{q}_h \; dx = 0 \;, \qquad \forall' t \in I \;, \forall \vec{q}_h \in X_h. \tag{5.15}$$

Due to our assumptions on the data, it follows from Proposition 5 and Proposition 8 that $\vec{\varepsilon}_h \in H^1(0,T;X_h)$ and $\theta_h \in H^1(0,T;M_h)$. We are thus allowed to derive the preceding equality with respect to the time variable and we obtain:

$$\int_\Omega \frac{d\vec{\varepsilon}_h}{dt}(t).\vec{q}_h \; dx + \int_\Omega \frac{d\theta_h}{dt}(t) \; \operatorname{div} \vec{q}_h \; dx = 0, \; \forall' t \in I, \; \forall \vec{q}_h \in X_h.$$

Choosing $\vec{q}_h = 2\,\vec{\varepsilon}_h(t)$, we obtain:

$$2\int_\Omega \frac{d}{dt}\vec{\varepsilon}_h(t).\vec{\varepsilon}_h(t) \; dx + \int_\Omega 2\frac{d\theta_h}{dt}(t) \; \operatorname{div} \vec{\varepsilon}_h(t) \; dx = 0, \; \forall' t \in I \;.$$

Thus

$$\frac{d}{dt}\int_\Omega |\vec{\varepsilon}_h(t)|^2 \, dx + \int_\Omega 2\frac{d\theta_h}{dt}(t) \; \operatorname{div} \vec{\varepsilon}_h(t) \; dx = 0, \; \forall' t \in I. \tag{5.16}$$

In a similar way, subtracting the second equation of (5.1) from the second one of (4.1), we obtain:

$$\int_\Omega v_h \; \operatorname{div} \vec{\varepsilon}_h(t) dx = \int_\Omega \frac{d(u_h - u)}{dt}(t) \; v_h \; dx, \; \forall v_h \in M_h, \forall' t \in I \;. \tag{5.17}$$

To obtain from the first term of (5.17) the second term of (5.16), we choose $v_h = 2\frac{d\theta_h(t)}{dt}$ in (5.17) :

$$\begin{aligned}
&\int_\Omega 2\frac{d\theta_h}{dt}(t) \; \operatorname{div}(\vec{\varepsilon}_h(t) dx \\
&\quad = 2\int_\Omega \frac{d}{dt}(u_h(t) - u(t)) \; \frac{d\theta_h}{dt}(t) \; dx, \; \forall' t \in I \\
&\quad = 2\int_\Omega \frac{d}{dt}(u_h(t) - \tilde{u}_h(t)) \; \frac{d\theta_h}{dt}(t) \; dx + 2\int_\Omega \frac{d}{dt}(\tilde{u}_h(t) - u(t)) \; \frac{d\theta_h}{dt}(t) \; dx \\
&\quad = 2\int_\Omega \left(\frac{d\theta_h}{dt}(t)\right)^2 \; dx + 2\int_\Omega \frac{d}{dt}(\tilde{u}_h(t) - u(t)) \; \frac{d\theta_h}{dt}(t) \; dx.
\end{aligned} \tag{5.18}$$

Equations (5.16), (5.18), and the Cauchy–Schwarz inequality implies

$$\frac{d}{dt}\int_\Omega |\vec{\varepsilon}_h(t)|^2 dx + 2\int_\Omega \left(\frac{d\theta_h}{dt}(t)\right)^2 dx = -2\int_\Omega \frac{d}{dt}(\tilde{u}_h(t) - u(t)) \, \frac{d\theta_h}{dt}(t) \, dx, \forall' t \in I$$

$$\leqslant 2\left[\int_\Omega\left(\frac{d}{dt}(\tilde{u}_h(t)-u(t))\right)^2\right]^{1/2}\left[\int_\Omega\left(\frac{d\theta_h}{dt}(t)\right)^2 dx\right]^{1/2}$$
$$\leq \int_\Omega\left(\frac{d}{dt}(\tilde{u}_h(t)-u(t))\right)^2 dx+\int_\Omega\left(\frac{d\theta_h}{dt}(t)\right)^2 dx.$$

After simplification, we obtain:

$$\frac{d}{dt}\int_\Omega |\vec{\varepsilon}_h(t)|^2\,dx \leq \int_\Omega\left(\frac{d}{dt}(\tilde{u}_h(t)-u(t))\right)^2 dx,\ \forall' t\in I.$$

Integrating the two sides of this last inequality from 0 to t, we obtain:

$$\int_\Omega |\vec{\varepsilon}_h(t)|^2\,dx \leq \int_\Omega |\vec{\varepsilon}_h(0)|^2\,dx + \int_0^t\int_\Omega\left(\frac{du}{dt}(t)-\frac{d\tilde{u}_h}{dt}(t)\right)^2 dx\ dt. \tag{5.19}$$

Since $u_h(0) = g_h = \tilde{u}_h(0)$, we have $\theta_h(0) = u_h(0) - \tilde{u}_h(0) = g_h - \tilde{u}_h(0) = 0$. Equation (5.15) with $t = 0$ gives us:

$$\int_\Omega \vec{\varepsilon}_h(0).\vec{q}_h\ dx = 0\ , \qquad \forall \vec{q}_h \in X_h.$$

Taking $\vec{q}_h = \vec{\varepsilon}_h(0)$ in this last equation, we obtain $\vec{\varepsilon}_h(0) = 0$. Inequality (5.19) becomes:

$$\int_\Omega |\vec{\varepsilon}_h(t)|^2\,dx \leq \int_0^t\left(\int_\Omega\left(\frac{du}{dt}(t)-\frac{d\tilde{u}_h}{dt}(t)\right)^2 dx\right)\ dt$$
$$\equiv \int_0^t\left(\int_\Omega\left[\frac{du}{dt}(t)-\left(\frac{du}{dt}(t)\right)_h^{\sim}\right]^2 dx\right)\ dt$$

as the operators $\frac{d}{dt}$ and the elliptic projection $(.)_h^{\sim}$ commute. Thus:

$$\int_\Omega |\vec{\varepsilon}_h(t)|^2\,dx \leq \int_0^t\left\|\frac{du}{dt}(t)-\left(\frac{du}{dt}(t)\right)_h^{\sim}\right\|^2 dt. \tag{5.20}$$

Therefore, it suffices to bound $\left\|\frac{du}{dt}(t)-\left(\frac{du}{dt}(t)\right)_h^{\sim}\right\|$. Due to our hypotheses on the data:

$$f\in H^1(0,T;L^2(\Omega)) \quad \text{and} \quad g\in H_0^1(\Omega),\ \Delta g + f(0)\in H_0^1(\Omega)$$

it follows from Proposition 8 that

$$u_t \in H^1(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))\cap L^2(0,T;H^{2,\alpha}(\Omega))\ .$$

As the elliptic projection of $\frac{du}{dt}(t)$ is nothing else than the solution of the stationary mixed discrete problem with datum $-\Delta\frac{du}{dt}(t)$, it follows from Proposition 9 that there exists a constant $C > 0$ independent of h such that

$$\left\| \frac{du}{dt}(t) - \left(\frac{du}{dt}(t)\right)_h^{\sim} \right\| \leq C\, h \left(\left| \frac{du(t)}{dt} \right|_{H^1(\Omega)} + \left| \frac{du(t)}{dt} \right|_{H^{2,\alpha}(\Omega)} \right).$$

From this last inequality and inequality (5.20), we get

$$\|\vec{\varepsilon}_h(t)\| \leq C\, h \left\| \frac{du}{dt} \right\|_{L^2(0,T;H^{2,\alpha}(\Omega))}, \ \forall' t \in [0,T].$$

Using the triangle inequality and Proposition 9, it follows that

$$\begin{aligned} \|\vec{p}(t) - \vec{p}_h(t)\| &\leq \left\| \vec{p}(t) - \tilde{\vec{p}}_h(t) \right\| + \left\| \tilde{\vec{p}}_h(t) - \vec{p}_h(t) \right\| \\ &\leq C\, h \left(|u(t)|_{H^{2,\alpha}(\Omega)} + \left\| \frac{du}{dt} \right\|_{L^2(0,T;H^{2,\alpha}(\Omega))} \right). \end{aligned}$$

Using inequality (5.12) and the above bound on $\|\vec{\varepsilon}_h(t)\|$, it follows that:

$$\|u(t) - u_h(t)\| \leq C\, h \left(|u(t)|_{H^1(\Omega)} + |u(t)|_{H^{2,\alpha}(\Omega)} + \left\| \frac{du}{dt} \right\|_{L^2(0,T;H^{2,\alpha}(\Omega))} \right).$$

The proof is complete. □

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Maximal Regularity of the Stokes Operator in General Unbounded Domains of $\mathbb{R}^n$

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Dedicated to our colleague Günter Lumer

Abstract. It is well known that the Helmholtz decomposition of L^q-spaces fails to exist for certain unbounded smooth domains unless $q = 2$. Hence also the Stokes operator and the Stokes semigroup are not well defined for these domains when $q \neq 2$. In this note, we generalize a new approach to the Stokes operator in general unbounded domains from the three-dimensional case, see [6], to the n-dimensional one, $n \geq 2$, by replacing the space $L^q, 1 < q < \infty$, by $\tilde{L}^q$ where $\tilde{L}^q = L^q \cap L^2$ for $q \geq 2$ and $\tilde{L}^q = L^q + L^2$ for $1 < q < 2$. As a main result we show that the nonstationary Stokes equation has maximal regularity in $L^s(0, T; \tilde{L}^q)$, $1 < s$, $q < \infty$, $T > 0$, for every unbounded domain of uniform $C^{1,1}$-type in $\mathbb{R}^n$.

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1. Introduction

Throughout this paper, $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, means a general unbounded domain with uniform $C^{1,1}$-boundary $\partial\Omega \neq \emptyset$, see Definition 1.1 below. As is well known, the standard approach to the stationary and nonstationary Stokes equations in L^q-spaces, $1 < q < \infty$, cannot be extended to general unbounded domains unless $q = 2$. One reason is the fact that the Helmholtz decomposition fails to exist for certain unbounded smooth domains in L^q, $q \neq 2$, see [4], [16]. On the other hand, in L^2 the Helmholtz projection and the Stokes operator are well defined for every domain, the Stokes operator is self-adjoint, generates a bounded analytic semigroup and has maximal regularity. This observation was used in [6] to consider in the three-dimensional case the Helmholtz decomposition in the space

$$\tilde{L}^q(\Omega) = \begin{cases} L^q(\Omega) \cap L^2(\Omega), & 2 \le q < \infty \\ L^q(\Omega) + L^2(\Omega), & 1 < q < 2 \end{cases},$$

and to define and to analyze the Stokes operator in the space

$$\tilde{L}^q_\sigma(\Omega) = \begin{cases} L^q_\sigma(\Omega) \cap L^2_\sigma(\Omega), & 2 \le q < \infty \\ L^q_\sigma(\Omega) + L^2_\sigma(\Omega), & 1 < q < 2 \end{cases}.$$

It was proved that for every unbounded domain $\Omega \subseteq \mathbb{R}^3$ of uniform C^2-type the Stokes operator in $\tilde{L}^q_\sigma$ satisfies the usual resolvent estimate, that it generates an analytic semigroup and has maximal regularity. Moreover, for every unbounded domain $\Omega \subseteq \mathbb{R}^n$, $n \ge 2$, of uniform $C^{1,1}$-type, the Helmholtz decomposition of $\tilde{L}^q(\Omega)$ exists, see [7], and the Stokes operator generates an analytic semigroup in $\tilde{L}^q_\sigma(\Omega)$, see [8].

To describe these results, we introduce the space of gradients,

$$\tilde{G}^q(\Omega) = \begin{cases} G^q(\Omega) \cap G^2(\Omega), & 2 \le q < \infty \\ G^q(\Omega) + G^2(\Omega), & 1 < q < 2 \end{cases},$$

where $G^q(\Omega) = \{\nabla p \in L^q(\Omega) : p \in L^q_{\mathrm{loc}}(\Omega)\}$ and recall the notion of domains of uniform C^k- and $C^{k,1}$-type.

Definition 1.1. A domain $\Omega \subseteq \mathbb{R}^n$, $n \ge 2$, is called a uniform C^k-domain of type (α, β, K), $k \in \mathbb{N}$, $\alpha > 0$, $\beta > 0$, $K > 0$, if for each $x_0 \in \partial\Omega$ we can choose a Cartesian coordinate system with origin at x_0 and coordinates $y = (y', y_n)$, $y' = (y_1, \dots, y_{n-1})$, and a C^k-function $h(y')$, $|y'| \le \alpha$, with C^k-norm $\|h\|_{C^k} \le K$ such that the neighborhood

$$U_{\alpha,\beta,h}(x_0) := \{y = (y', y_n) : \ |y_n - h(y')| < \beta, \ |y'| < \alpha\}$$

of x_0 implies $U_{\alpha,\beta,h}(x_0) \cap \partial\Omega = \{(y', h(y')) : \ |y'| < \alpha\}$ and

$$U^-_{\alpha,\beta,h}(x_0) := \{(y', y_n) : \ h(y') - \beta < y_n < h(y'), \ |y'| < \alpha\} = U_{\alpha,\beta,h}(x_0) \cap \Omega.$$

By analogy, a domain $\Omega \subseteq \mathbb{R}^n$, $n \ge 2$, is called a uniform $C^{k,1}$-domain of type (α, β, K), $k \in \mathbb{N} \cup \{0\}$, if the functions h mentioned above may be chosen in $C^{k,1}$ such that the $C^{k,1}$-norm satisfies $\|h\|_{C^{k,1}} \le K$.

Theorem 1.2. [7] *Let $\Omega \subseteq \mathbb{R}^n$, $n \ge 2$, be a uniform C^1-domain of type (α, β, K) and let $q \in (1, \infty)$. Then each $u \in \tilde{L}^q(\Omega)$ has a unique decomposition*

$$u = u_0 + \nabla p, \quad u_0 \in \tilde{L}^q_\sigma(\Omega), \ \nabla p \in \tilde{G}^q(\Omega),$$

satisfying the estimate

$$\|u_0\|_{\tilde{L}^q} + \|\nabla p\|_{\tilde{L}^q} \le c\|u\|_{\tilde{L}^q}, \tag{1.1}$$

where $c = c(\alpha, \beta, K, q) > 0$. In particular, the Helmholtz projection $\tilde{P}_q$ defined by $\tilde{P}_q u = u_0$ is a bounded linear projection on $\tilde{L}^q(\Omega)$ with range $\tilde{L}^q_\sigma(\Omega)$ and kernel

$\tilde{G}^q(\Omega)$. *Moreover,* $\tilde{L}^q_\sigma(\Omega)$ *is the closure in* $\tilde{L}^q(\Omega)$ *of the space* $C^\infty_{0,\sigma}(\Omega) = \{u \in C^\infty_0(\Omega)^n : \operatorname{div} u = 0\}$, $\big(\tilde{L}^q_\sigma(\Omega)\big)' = \tilde{L}^{q'}_\sigma(\Omega)$ *and* $\big(\tilde{P}_q\big)' = \tilde{P}_{q'}$, $q' = \frac{q}{q-1}$.

Using the Helmholtz projection $\tilde{P}_q$ we define the Stokes operator $\tilde{A}_q$ as an operator with domain

$$\mathcal{D}(\tilde{A}_q) = \begin{cases} D^q(\Omega) \cap D^2(\Omega), & 2 \le q < \infty \\ D^q(\Omega) + D^2(\Omega), & 1 < q < 2 \end{cases},$$

where $D^q(\Omega) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega)$, by setting

$$\tilde{A}_q u = -\tilde{P}_q \Delta u, \quad u \in \mathcal{D}(\tilde{A}_q).$$

Moreover, we define the space $\tilde{W}^{2,q} = \tilde{W}^{2,q}(\Omega)$ by $W^{2,q}(\Omega) \cap W^{2,2}(\Omega)$ when $2 \le q < \infty$ and $W^{2,q}(\Omega) + W^{2,2}(\Omega)$ when $1 < q < 2$. Let I be the identity and $\mathcal{S}_\varepsilon = \{0 \ne \lambda \in \mathbb{C};\ |\arg \lambda| < \frac{\pi}{2} + \varepsilon\}$, $0 < \varepsilon < \frac{\pi}{2}$, and let $\langle \cdot, \cdot \rangle$ be the usual L^q-$L^{q'}$-pairing. Then in [8] the authors of this paper proved the following theorem:

Theorem 1.3. *Let* $\Omega \subseteq \mathbb{R}^n$ *be a uniform* $C^{1,1}$*-domain of type* (α, β, K) *and let* $1 < q < \infty$.

(i) *The Stokes operator*

$$\tilde{A}_q = -\tilde{P}_q\, \Delta : \mathcal{D}(\tilde{A}_q) \subset \tilde{L}^q_\sigma(\Omega) \to \tilde{L}^q_\sigma(\Omega)$$

is a densely defined, closed operator and satisfies the duality relation

$$\langle \tilde{A}_q u, v \rangle = \langle u, \tilde{A}_{q'} v \rangle \quad \text{for all} \quad u \in \mathcal{D}(\tilde{A}_q),\ v \in \mathcal{D}(\tilde{A}_{q'}). \tag{1.2}$$

(ii) *For any* $0 < \varepsilon < \frac{\pi}{2}$ *and for all* $\lambda \in \mathcal{S}_\varepsilon$, *its resolvent* $(\lambda I + \tilde{A}_q)^{-1} : \tilde{L}^q_\sigma(\Omega) \to \tilde{L}^q_\sigma(\Omega)$ *is well defined and* $u = (\lambda I + \tilde{A}_q)^{-1} f$, $f \in \tilde{L}^q_\sigma(\Omega)$, *satisfies the resolvent estimate*

$$\|\lambda u\|_{\tilde{L}^q_\sigma} + \|\tilde{A}_q u\|_{\tilde{L}^q} \le C \|f\|_{\tilde{L}^q_\sigma}, \quad |\lambda| \ge \delta, \tag{1.3}$$

where $\delta > 0$ *and* $C = C(q, \varepsilon, \delta, \alpha, \beta, K) > 0$. *Hence* $-\tilde{A}_q$ *generates an analytic semigroup* $e^{-t\tilde{A}_q}$ *with bound*

$$\|e^{-t\tilde{A}_q} f\|_{\tilde{L}^q_\sigma} \le M e^{\delta t} \|f\|_{\tilde{L}^q_\sigma}, \quad f \in \tilde{L}^q_\sigma,\ t \ge 0, \tag{1.4}$$

where $M = M(q, \delta, \alpha, \beta, K) > 0$.

(iii) *Let* $f \in \tilde{L}^q(\Omega)$ *and* $\lambda \in \mathcal{S}_\varepsilon$. *Then the Stokes resolvent equation*

$$\lambda u - \Delta u + \nabla p = f,\ \operatorname{div} u = 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega,$$

has a unique solution $(u, \nabla p) \in \mathcal{D}(\tilde{A}_q) \times \tilde{G}^q(\Omega)$ *defined by* $u = (\lambda I + \tilde{A}_q)^{-1} \tilde{P}_q f$ *and* $\nabla p = (I - \tilde{P}_q)(f + \Delta u)$ *satisfying*

$$\|\lambda u\|_{\tilde{L}^q} + \|\nabla^2 u\|_{\tilde{L}^q} + \|\nabla p\|_{\tilde{L}^q} \le C \|f\|_{\tilde{L}^q}, \quad |\lambda| \ge \delta, \tag{1.5}$$

where $\delta > 0$ *and* $C = C(q, \varepsilon, \delta, \alpha, \beta, K) > 0$. *In particular, the norms* $\|u\|_{\tilde{W}^{2,q}}$ *and the graph norm* $\|u\|_{\mathcal{D}(\tilde{A}_q)} = \|u\|_{\tilde{L}^q} + \|\tilde{A}_q u\|_{\tilde{L}^q}$ *are equivalent with constants depending only on* q *and* (α, β, K).

Note that $\delta > 0$ in Theorem 1.3 may be chosen arbitrarily small, but that it is not clear whether $\delta = 0$ is allowed for a general unbounded domain and whether the semigroup $e^{-t\tilde{A}_q}$ is uniformly bounded in $\tilde{L}^q_\sigma$ for $0 \le t < \infty$.

Using the Stokes semigroup $\{e^{-t\tilde{A}_q};\ t \ge 0\}$ we solve the instationary Stokes system

$$\begin{array}{rclrcll} u_t - \Delta u + \nabla p &=& f, & \operatorname{div} u &=& 0 & \text{in} \quad \Omega \times (0,T) \\ u(0) &=& u_0, & u|_{\partial\Omega} &=& 0. & \end{array} \tag{1.6}$$

Now our main result reads as follows:

Theorem 1.4. *Let $\Omega \subseteq \mathbb{R}^n$ be a uniform $C^{1,1}$-domain of type (α, β, K), and let $0 < T < \infty$, $1 < q,\ s < \infty$.*

Then for each $f \in L^s(0,T; \tilde{L}^q_\sigma(\Omega))$ and each $u_0 \in \mathcal{D}(\tilde{A}_q)$ there exists a unique solution $u \in L^s\big(0,T; \mathcal{D}(\tilde{A}_q)\big)$, $u_t \in L^s(0,T; \tilde{L}^q_\sigma(\Omega))$, of the system (1.6)*, satisfying the estimates*

$$\begin{aligned} &\|u_t\|_{L^s(0,T;\tilde{L}^q_\sigma)} + \|u\|_{L^s(0,T;\tilde{L}^q_\sigma)} + \|\tilde{A}_q u\|_{L^s(0,T;\tilde{L}^q_\sigma)} \\ &\quad \le C\big(\|u_0\|_{D(\tilde{A}_q)} + \|f\|_{L^s(0,T;\tilde{L}^q_\sigma)}\big) \end{aligned} \tag{1.7}$$

and

$$\|u_t\|_{L^s(0,T;\tilde{L}^q_\sigma)} + \|u\|_{L^s(0,T;\tilde{W}^{2,q})} \le C\big(\|u_0\|_{D(\tilde{A}_q)} + \|f\|_{L^s(0,T;\tilde{L}^q_\sigma)}\big) \tag{1.8}$$

with $C = C(q, s, T, \alpha, \beta, K) > 0$.

Remark 1.5. (i) The assumption $u_0 \in \mathcal{D}(\tilde{A}_q)$ in this theorem is not optimal and may be replaced by the weaker properties $u_0 \in \tilde{L}^q_\sigma$ and $\int_0^T \|\tilde{A}_q e^{-t\tilde{A}_q} u_0\|^s_{\tilde{L}^q_\sigma}\, dt < \infty$. Then the term $\|u_0\|_{\mathcal{D}(\tilde{A}_q)}$ in (1.7), (1.8) may be substituted by the weaker norm

$$\Big(\int_0^T \|\tilde{A}_q e^{-t\tilde{A}_q} u_0\|^s_{\tilde{L}^q_\sigma}\, dt\Big)^{\frac{1}{s}},\ 1 < q < \infty. \tag{1.9}$$

(ii) Let $f \in L^s(0,T; \tilde{L}^q_\sigma)$ in Theorem 1.4 be replaced by $f \in L^s(0,T; \tilde{L}^q)$. Then $u \in L^s\big(0,T; \mathcal{D}(\tilde{A}_q)\big)$, defined by $u_t + \tilde{A}_q u = \tilde{P}_q f,\ u(0) = u_0$, and ∇p, defined by $\nabla p(t) = (I - \tilde{P}_q)\big(f + \Delta u\big)(t)$, is a unique solution pair of the system

$$u_t - \Delta u + \nabla p = f,\ u(0) = u_0,$$

satisfying

$$\begin{aligned} &\|u_t\|_{L^s(0,T;\tilde{L}^q_\sigma)} + \|u\|_{L^s(0,T;\tilde{W}^{2,q})} + \|\nabla p\|_{L^s(0,T;\tilde{L}^q)} \\ &\quad \le C\big(\|u_0\|_{D(\tilde{A}_q)} + \|f\|_{L^s(0,T;\tilde{L}^q)}\big) \end{aligned} \tag{1.10}$$

with $C = C(q, s, T, \alpha, \beta, K) > 0$.

Using (2.1) below we see that in the case $1 < q < 2$ the solution pair $u, \nabla p$ possesses a decomposition $u = u^{(1)} + u^{(2)}$, $\nabla p = \nabla p^{(1)} + \nabla p^{(2)}$ such that

$$\begin{aligned} &u^{(1)} \in L^s(0,T;W^{2,2}),\ u_t^{(1)} \in L^s(0,T;L^2_\sigma),\\ &u^{(2)} \in L^s(0,T;W^{2,q}),\ u_t^{(2)} \in L^s(0,T;L^q_\sigma),\\ &\nabla p^{(1)} \in L^s(0,T;L^2),\ \nabla p^{(2)} \in L^s(0,T;L^q), \end{aligned} \tag{1.11}$$

and

$$\begin{aligned} &\|u_t\|_{L^s(0,T;\tilde L^q_\sigma)} + \|u\|_{L^s(0,T;\tilde L^q_\sigma)} + \|\nabla^2 u\|_{L^s(0,T;\tilde L^q)} + \|\nabla p\|_{L^s(0,T;\tilde L^q)}\\ = \ &\|u_t^{(1)}\|_{L^{s,2}} + \|u^{(1)}\|_{L^{s,2}} + \|\nabla^2 u^{(1)}\|_{L^{s,2}} + \|\nabla p^{(1)}\|_{L^{s,2}} +\\ &\|u_t^{(2)}\|_{L^{s,q}} + \|u^{(2)}\|_{L^{s,q}} + \|\nabla^2 u^{(2)}\|_{L^{s,q}} + \|\nabla p^{(2)}\|_{L^{r,q}} \end{aligned}$$

where $L^{s,2} = L^s(0,T;L^2)$, $L^{s,q} = L^s(0,T;L^q)$.

(iii) Note that the constant C in (1.7), (1.8), (1.10) could depend on the given interval $(0,T]$. We do not know whether C can be chosen independently of T as in the usual L^q-theory in bounded and exterior domains as well as in aperture domains, see [14], [10].

Remark 1.6. The main application of these results concerns the instationary Navier-Stokes system in the three-dimensional case, see [6]. It is proved that under suitable assumptions on the external force f and the initial value u_0 the Navier-Stokes system has at least one (global) *suitable weak solution* u which satisfies even Leray's Structure Theorem. Using Theorems 1.2–1.4 this result, see [6], Theorem 2.7 and Remark 2.8, can be extended to unbounded domains with uniform $C^{1,1}$-boundary (instead of C^2) provided that $u_0 \in \mathcal{D}(\tilde A_2^{1/4})$; this last condition rather than $u_0 \in L^2_\sigma$ is needed to get [6], Remark 2.8.

2. Preliminaries

Let us recall some properties of sum and intersection spaces known from interpolation theory, cf. [3], [19].

Consider two (complex) Banach spaces X_1, X_2 with norms $\|\cdot\|_{X_1}$, $\|\cdot\|_{X_2}$, respectively, and assume that both X_1 and X_2 are subspaces of a topological vector space V with continuous embeddings. Further, we assume that $X_1 \cap X_2$ is a dense subspace of both X_1 and X_2. Then the sum space

$$X_1 + X_2 := \{u_1 + u_2;\ u_1 \in X_1,\ u_2 \in X_2\} \subseteq V$$

is a well-defined Banach space with the norm

$$\|u\|_{X_1+X_2} := \inf\{\|u_1\|_{X_1} + \|u_2\|_{X_2};\ u = u_1 + u_2,\ u_1 \in X_1,\ u_2 \in X_2\}.$$

The intersection space $X_1 \cap X_2$ is a Banach space with norm

$$\|u\|_{X_1 \cap X_2} = \max(\|u\|_{X_1}, \|u\|_{X_2}).$$

Suppose that X_1 and X_2 are reflexive Banach spaces. Then an argument using weakly convergent subsequences yields the following property: Given $u \in X_1 + X_2$

there exist $u_1 \in X_1$, $u_2 \in X_2$ with $u = u_1 + u_2$ such that

$$\|u\|_{X_1+X_2} = \|u_1\|_{X_1} + \|u_2\|_{X_2}. \tag{2.1}$$

The dual space $(X_1 + X_2)'$ of $X_1 + X_2$ is given by $X_1' \cap X_2'$, and we get

$$(X_1 + X_2)' = X_1' \cap X_2'$$

with the natural pairing $\langle u, f\rangle = \langle u_1, f\rangle + \langle u_2, f\rangle$ for all $u = u_1 + u_2 \in X_1 + X_2$, $f \in X_1' \cap X_2'$. Thus it holds

$$\|u\|_{X_1+X_2} = \sup\Big\{\frac{|\langle u_1, f\rangle + \langle u_2, f\rangle|}{\|f\|_{X_1'\cap X_2'}};\ 0 \neq f \in X_1' \cap X_2'\Big\}$$

and

$$\|f\|_{X_1'\cap X_2'} = \sup\Big\{\frac{|\langle u_1, f\rangle + \langle u_2, f\rangle|}{\|u\|_{X_1+X_2}};\ 0 \neq u = u_1 + u_2 \in X_1 + X_2\Big\};$$

see [3], [19]. By analogy,

$$(X_1 \cap X_2)' = X_1' + X_2'$$

with the natural pairing $\langle u, f_1 + f_2\rangle = \langle u, f_1\rangle + \langle u, f_2\rangle$ for $u \in X_1 \cap X_2$ and $f = f_1 + f_2 \in X_1' + X_2'$.

Consider closed subspaces $L_1 \subseteq X_1$, $L_2 \subseteq X$ with norms $\|\cdot\|_{L_1} = \|\cdot\|_{X_1}$, $\|\cdot\|_{L_2} = \|\cdot\|_{X_2}$ and assume that $L_1 \cap L_2$ is dense in both L_1 and L_2. Then $\|u\|_{L_1\cap L_2} = \|u\|_{X_1\cap X_2}$, $u \in L_1 \cap L_2$, and an elementary argument using the Hahn-Banach theorem shows that also

$$\|u\|_{L_1+L_2} = \|u\|_{X_1+X_2}, \quad u \in L_1 + L_2. \tag{2.2}$$

In particular, we need the following special case. Let $B_1 : \mathcal{D}(B_1) \to X_1$, $B_2 : \mathcal{D}(B_2) \to X_2$ be closed linear operators with dense domains $\mathcal{D}(B_1) \subseteq X_1$, $\mathcal{D}(B_2) \subseteq X_2$ equipped with graph norms

$$\|u\|_{\mathcal{D}(B_1)} = \|u\|_{X_1} + \|B_1 u\|_{X_1}, \quad \|u\|_{\mathcal{D}(B_2)} = \|u\|_{X_2} + \|B_2 u\|_{X_2}.$$

We assume that $\mathcal{D}(B_1) \cap \mathcal{D}(B_2)$ is dense in both $\mathcal{D}(B_1)$ and $\mathcal{D}(B_2)$ in the corresponding graph norms. Each functional $F \in \mathcal{D}(B_i)'$, $i = 1, 2$, is given by some pair $f, g \in X_i'$ in the form $\langle u, F\rangle = \langle u, f\rangle + \langle B_i u, g\rangle$. Using (2.2) with $L_i = \{(u, B_i u);\ u \in \mathcal{D}(B_i)\} \subseteq X_i \times X_i$, $i = 1, 2$, and, on $(X_1\times X_1)+(X_2\times X_2)$, the equality of norms $\|\cdot\|_{(X_1\times X_1)+(X_2\times X_2)}$ and $\|\cdot\|_{(X_1+X_2)\times(X_1+X_2)}$, we conclude that for each $u \in \mathcal{D}(B_1)+\mathcal{D}(B_2)$ with decomposition $u = u_1+u_2$, $u_1 \in \mathcal{D}(B_1)$, $u_2 \in \mathcal{D}(B_2)$,

$$\|u\|_{\mathcal{D}(B_1)+\mathcal{D}(B_2)} = \|u_1 + u_2\|_{X_1+X_2} + \|B_1 u_1 + B_2 u_2\|_{X_1+X_2}. \tag{2.3}$$

For instationary problems we need the usual Banach space $L^s(0, T; X)$, $0 < T \le \infty$, of measurable X-valued (classes of) functions u with norm

$$\|u\|_{L^s(0,T;X)} = \Big(\int_0^T \|u(t)\|_X^s\, dt\Big)^{\frac{1}{s}}, \quad 1 \le s < \infty,$$

where X is a Banach space. If X is reflexive and $1 < s < \infty$, then the dual space of $L^s(0, T; X)$ is given by $L^s(0, T; X)' = L^{s'}(0, T; X')$, $s' = \frac{s}{s-1}$, with the natural pairing $\langle u, f\rangle_T = \int_0^T \langle u(t), f(t)\rangle\, dt$, where $\langle\cdot,\cdot\rangle$ denotes the pairing between X and its dual X'.

Let $X = L^q(\Omega)$, $1 < q < \infty$. Then we use the notation

$$L^{s,q} := L^s(L^q(\Omega)) = L^s(0,T;L^q(\Omega)), \quad \|u\|_{L^{s,q}} = \left(\int_0^T \|u\|_q^s\, dt\right)^{1/s}.$$

The pairing of $L^s(0,T;L^q)$ with its dual $L^{s'}(0,T;L^{q'})$ is given by $\langle u, f\rangle_T = \langle u, f\rangle_{\Omega,T} = \int_0^T \big(\int_\Omega u\cdot f\, dx\big)dt$. Moreover, we see that

$$(L^{s,q} + L^{s,2})' = (L^{s,q})' \cap (L^{s,2})' = L^{s'}(0,T;L^{q'} \cap L^2) = L^s(0,T;L^q + L^2)',$$

where the pairing between $L^{s,q}+L^{s,2}$ and $(L^{s,q})'\cap(L^{s,2})'$ is given by $\langle u_1+u_2, f\rangle_T = \langle u_1, f\rangle_T + \langle u_2, f\rangle_T$ for $u_1 \in L^{s,q}$, $u_2 \in L^{s,2}$, $f \in (L^{s,q})' \cap (L^{s,2})'$. Furthermore, we can choose the decomposition $u = u_1 + u_2 \in L^s(0,T;L^q + L^2)$ in such a way that

$$\|u\|_{L^{s,q}+L^{s,2}} = \|u_1\|_{L^{s,q}} + \|u_2\|_{L^{s,2}}.$$

We conclude that

$$\|u_1 + u_2\|_{L^{s,q}+L^{s,2}} = \sup\Big\{\frac{|\langle u_1 + u_2, f\rangle_T|}{\|f\|_{(L^{s,q})'\cap(L^{s,2})'}};\ 0 \neq f \in L^{s'}(0,T;L^{q'} \cap L^2)\Big\}.$$

In view of the identities $L^{s,q} \cap L^{s,2} = L^s(0,T;L^q \cap L^2)$ and – see above – $L^{s,q} + L^{s,2} = L^s(0,T;L^q + L^2)$ we introduce the short notation

$$\tilde{L}^{s,q} = \begin{cases} L^{s,q} \cap L^{s,2}, & 2 \le q < \infty \\ L^{s,q} + L^{s,2}, & 1 < q < 2 \end{cases}.$$

Concerning Definition 1.1 for domains of uniform $C^{1,1}$-type we need further notations and discuss some properties. Obviously, the axes e_i, $i = 1,\dots,n$, of the new coordinate system (y', y_n) can be chosen in such a way that $e_1,\dots,e_{n-1}$ are tangential to $\partial\Omega$ at x_0. Hence at $y' = 0$ we have $h(y') = 0$ and $\nabla' h(y') = (\partial h/\partial y_1,\dots,\partial h/\partial y_{n-1})(y') = 0$. Since $h \in C^{1,1}$, for any given constant $M_0 > 0$, we may choose $\alpha > 0$ sufficiently small such that $\|h\|_{C^1} \le M_0$ is satisfied.

It is easily shown that there exists a covering of $\overline{\Omega}$ by open balls $B_j = B_r(x_j)$ of fixed radius $r > 0$ with centers $x_j \in \overline{\Omega}$, such that with suitable functions $h_j \in C^{1,1}$ of type (α, β, K)

$$\overline{B}_j \subset U_{\alpha,\beta,h_j}(x_j) \ \text{ if } x_j \in \partial\Omega, \quad \overline{B}_j \subset \Omega \ \text{ if } x_j \in \Omega. \tag{2.4}$$

Here j runs from 1 to a finite number $N = N(\Omega) \in \mathbb{N}$ if Ω is bounded, and $j \in \mathbb{N}$ if Ω is unbounded. Moreover, as an important consequence, the covering $\{B_j\}$ of Ω may be constructed in such a way that not more than a fixed number $N_0 = N_0(\alpha, \beta, K) \in \mathbb{N}$ of these balls have a nonempty intersection. Related to this covering, there exists a partition of unity $\{\varphi_j\}$, $\varphi_j \in C_0^\infty(\mathbb{R}^n)$, such that

$$0 \le \varphi_j \le 1, \ \ \operatorname{supp}\varphi_j \subset B_j, \quad \text{and} \quad \sum_{j=1}^N \varphi_j = 1 \ \text{ or } \ \sum_{j=1}^\infty \varphi_j = 1 \ \text{ on } \Omega. \tag{2.5}$$

The functions φ_j may be chosen so that $|\nabla\varphi_j(x)| + |\nabla^2\varphi_j(x)| \le C$ uniformly in j and $x \in \Omega$ with $C = C(\alpha, \beta, K)$.

If Ω is unbounded, then Ω can be represented as the union of an increasing sequence of bounded uniform $C^{1,1}$-domains $\Omega_k \subset \Omega$, $k \in \mathbb{N}$,

$$\Omega_1 \subset \cdots \subset \Omega_k \subset \Omega_{k+1} \subset \cdots, \quad \Omega = \bigcup_{k=1}^{\infty} \Omega_k, \tag{2.6}$$

where each Ω_k is of the same type (α', β', K'). Without loss of generality we assume that $\alpha = \alpha'$, $\beta = \beta'$, $K = K'$.

Using the partition of unity $\{\varphi_j\}$ we will perform the analysis of maximal regularity of the Stokes operator by starting from well-known results for certain bounded and unbounded domains. For this reason, consider a function $h \in C^{1,1}(\mathbb{R}^{n-1})$ satisfying $h(0) = 0$, $\nabla' h(0) = 0$ and with compact support $\operatorname{supp} h$ contained in the $(n-1)$-dimensional ball $B'_r(0)$ of radius $r = r(\alpha, \beta, K) \in (0, \alpha)$ and center 0. Then we introduce the bounded domain

$$H = H_{\alpha,\beta,h;r} = \{y = (y', y_n) \in \mathbb{R}^n : h(y') - \beta < y_n < h(y'),\ |y'| < \alpha\} \cap B_r(0)\,;$$

here we assume that $\overline{B_r(0)} \subset \{y \in \mathbb{R}^n : |y_n - h(y')| < \beta, |y'| < \alpha\}$.

On H we consider the classical Sobolev spaces $W^{k,q}(H)$ and $W_0^{k,q}(H)$, $k \in \mathbb{N}$, the dual space $W^{-1,q}(H) = \big(W_0^{1,q'}(H)\big)'$ and the space

$$L_0^q(H) = \Big\{u \in L^q(H) : \int_H u\,dx = 0\Big\}$$

of L^q-functions with vanishing mean on H.

Lemma 2.1. *Let $1 < q < \infty$ and $H = H_{\alpha,\beta,h;r}$.*

(i) *There exists a bounded linear operator*

$$R : L_0^q(H) \to W_0^{1,q}(H)$$

such that $\operatorname{div} \circ R = I$ *on* $L_0^q(H)$ *and a constant* $C = C(\alpha, \beta, K, q) > 0$ *such that*

$$\begin{array}{lcll} \|Rf\|_{W^{1,q}} & \le & C\|f\|_{L^q(H)} & \text{for all} \quad f \in L_0^q(H) \\ \|Rf\|_{W^{2,q}} & \le & C\|f\|_{W^{1,q}(H)} & \text{for all} \quad f \in L_0^q(H) \cap W_0^{1,q}(H) \end{array} \tag{2.7}$$

and $R\big(L_0^q(H) \cap W_0^{1,q}(H)\big) \subset W_0^{2,q}(H)$.

(ii) *There exists* $C = C(\alpha, \beta, K, q) > 0$ *such that for every* $p \in L_0^q(H)$

$$\|p\|_q \le C\|\nabla p\|_{W^{-1,q}} = C \sup\Big\{\frac{|\langle p, \operatorname{div} v\rangle|}{\|\nabla v\|_{q'}} : 0 \ne v \in W_0^{1,q'}(H)\Big\}. \tag{2.8}$$

Proof. (i) It is well known that there exists a bounded linear operator $R : L_0^q(H) \to W_0^{1,q}(H)$ such that $u = Rf$ solves the divergence problem $\operatorname{div} u = f$. Moreover, the estimate $(2.7)_1$ holds with $C = C(\alpha, \beta, K, q) > 0$, see [12], III, Theorem 3.1. The second part follows from [12], III, Theorem 3.2.

(ii) A duality argument and (i) yield (ii), see [7] and [17], II.2.1; we may set $p = R'(\nabla p)$ where $R' : W^{-1,q}(H) \to L^q_0(H)$ means the dual operator of $R : L^{q'}_0(H) \to W^{1,q'}_0(H)$. $\square$

The next lemma concerns the instationary Stokes systems

$$u_t - \Delta u + \nabla p = f, \quad u(0) = u_0 \quad \text{or} \quad -u_t - \Delta u + \nabla p = f, \quad u(T) = u_0, \tag{2.9}$$

in the domain H. To describe this crucial result we define the Stokes operator as usual by $A_q = -P_q\Delta$ with domain $\mathcal{D}(A_q) = L^q_\sigma(H) \cap W^{1,q}_0(H) \cap W^{2,q}(H)$.

Lemma 2.2. *Let $0 < T < \infty$, $u_0 \in \mathcal{D}(A_q)$ and $f \in L^q\big(0,T;L^q(H)\big)$ be given. Assume that $u \in L^q\big(0,T,\mathcal{D}(A_q)\big)$, $p \in L^q\big(0,T;W^{1,q}(H)\big)$ solve (2.9) and satisfy $\operatorname{supp} u_0 \cup \operatorname{supp} u(t) \cup \operatorname{supp} p(t) \subseteq B_r(0)$ for a.a. $t \in [0,T]$.*

Then there is a constant $C = C(q,\alpha,\beta,K,T) > 0$ such that

$$\begin{aligned} &\|u_t\|_{L^q(0,T;L^q(H))} + \|u\|_{L^q(0,T;W^{2,q}(H))} + \|\nabla p\|_{L^q(0,T;L^q(H))} \\ &\quad \le C\big(\|u_0\|_{W^{2,q}(H)} + \|f\|_{L^q(0,T;L^q(H))}\big). \end{aligned} \tag{2.10}$$

Proof. In the case $u(0) = u_0$ this estimate follows from [18], Theorem 4.1, (4.2) and (4.21′), see also [15]. A careful inspection of the proofs shows that the constant $C = C(\Omega)$ in (2.10) depends only on the type (α,β,K) and on q, T; actually, it suffices to assume the boundary regularity $C^{1,1}$ since only the boundedness of second order derivatives of functions locally describing the boundary is used.

The second case $-u_t - \Delta u + \nabla p = f$, $u(T) = u_0$, can be reduced to the first one by the transformation $\tilde u(t) = u(T-t)$, $\tilde f(t) = f(T-t)$, $\tilde p(t) = p(T-t)$. $\square$

We note that the relatively strong assumption $u_0 \in \mathcal{D}(A_q)$ is used for simplicity and can be weakened as in Remark 1.5, (i). Note that the conditions $u(0) = u_0$ or $u(T) = u_0$, resp., are well defined since $u_t \in L^q(0,T;L^q_\sigma)$.

Next we collect several results on Sobolev embeddding estimates for a *bounded* $C^{1,1}$-domain.

Lemma 2.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded $C^{1,1}$-domain.*

(i) *Let $1 < q < \infty$, $0 < \varepsilon \le 1$. Then there is a constant $C = C(q,\varepsilon,\alpha,\beta,K) > 0$ such that*
$$\|\nabla u\|_{L^q} \le \varepsilon\|\nabla^2 u\|_{L^q} + C\|u\|_{L^q} \tag{2.11}$$
for all $u \in W^{2,q}(\Omega)$.

(ii) *If $2 \le q < \infty$, $0 < \varepsilon \le 1$, then there is a constant $C = C(q,\varepsilon,\alpha,\beta,K) > 0$ such that*
$$\|u\|_{L^q} \le \varepsilon\|\nabla^2 u\|_{L^q} + C\big(\|\nabla^2 u\|_{L^2} + \|u\|_{L^2}\big) \tag{2.12}$$
for all $u \in W^{2,q}(\Omega)$.

(iii) *The Stokes operator $A_q = -P_q\Delta : \mathcal{D}(A_q) \to L^q_\sigma(\Omega)$ generates a bounded analytic semigroup e^{-tA_q}, $t \ge 0$, on $L^q_\sigma(\Omega)$. Moreover, $\langle A_q u, v\rangle = \langle u, A_{q'}v\rangle$ for all $u \in \mathcal{D}(A_q)$, $v \in \mathcal{D}(A_{q'})$ and $A'_q = A_{q'}$.*

(iv) *The graph norm of A_q is equivalent to the usual $W^{2,q}(\Omega)$-norm, i.e., there exists a constant $C = C(q, \alpha, \beta, K) > 0$ such that*

$$\frac{1}{C}\|u\|_{W^{2,q}} \leq \|A_q u\|_{L^q} \leq C\|u\|_{W^{2,q}}.$$

for all $u \in \mathcal{D}(A_q)$.

Proof. The proofs of (i), (ii) are easily reduced to the case $u \in W_0^{2,q}(\Omega')$, $\overline{\Omega} \subset \Omega'$, Ω' a bounded $C^{1,1}$-domain, using an extension operator on Sobolev spaces the norm of which is shown to depend only on q and (α, β, K). In (ii) we choose some $r \in [2, q)$ such that $\|u\|_{L^q} \leq \varepsilon\|\nabla^2 u\|_{L^r} + C_\varepsilon\|u\|_{L^r}$, $\varepsilon \in (0, 1)$ and use the interpolation inequality

$$\|v\|_{L^r} \leq \gamma\Big(\frac{1}{\varepsilon}\Big)^{1/\gamma}\|v\|_{L^2} + (1-\gamma)\varepsilon^{1/(1-\gamma)}\|v\|_{L^q}, \tag{2.13}$$

with $\gamma \in (0, 1)$, $\frac{1}{r} = \frac{\gamma}{2} + \frac{1-\gamma}{q}$, for $v = u$ and $v = \nabla^2 u$ for suitable $\varepsilon > 0$ to get (2.12). For basic details see [1], IV, Theorem 4.28, [11] and [17], II.1.3.

(iii) These assertions are well known, see, e.g., [9], [13], [18]. Part (iv) is proved in [8], Lemma 3.1. □

Lemma 2.4.

(i) *Let $1 < q$, $s < \infty$, $0 < T < \infty$ and let $\Omega \subseteq \mathbb{R}^n$ be a bounded $C^{1,1}$-domain. Define the operators $\mathcal{J}_{s,q}$, $\mathcal{J}'_{s,q}$ by*

$$(\mathcal{J}_{s,q}f)(t) = \int_0^t e^{-(t-\tau)A_q} f(\tau)\,d\tau, \quad (\mathcal{J}'_{s,q}f)(t) = \int_t^T e^{-(\tau-t)A_q} f(\tau)\,d\tau$$

for $f \in L^s(0, T; L^q_\sigma(\Omega))$ and $0 \leq t \leq T$. Then the nonstationary Stokes system

$$u_t + A_q u = f, \quad u(0) = u_0,$$

with initial value $u_0 \in \mathcal{D}(A_q)$ has a unique solution $u \in L^s\big(0, T; \mathcal{D}(A_q)\big)$ given by $u(t) = e^{-tA_q}u_0 + (\mathcal{J}_{s,q}f)(t)$ and satisfies the estimate

$$\|u_t\|_{L^{s,q}} + \|u\|_{L^{s,q}} + \|A_q u\|_{L^{s,q}} \leq C\big(\|u_0\|_{\mathcal{D}(A_q)} + \|f\|_{L^{s,q}}\big) \tag{2.14}$$

with a constant $C = C(q, s, T, \Omega)$. Analogously, the nonstationary Stokes system $-u_t + A_q u = f$, $u(T) = u_0$, has a unique solution $u \in L^s\big(0, T; \mathcal{D}(A_q)\big)$, namely, $u(t) = e^{-(T-t)A_q}u_0 + (\mathcal{J}'_{s,q}f)(t)$; this solution satisfies (2.14) with the same constant C. Moreover, there holds the duality relation

$$(\mathcal{J}_{s,q})' = \mathcal{J}'_{s',q'}. \tag{2.15}$$

(ii) *In the case $q = 2$ the constant $C = C(2, s, T, \Omega)$ in (2.14) does not depend on the domain Ω.*

Proof. For (i) [14], [18]. The assertions on $\mathcal{J}'_{s,q}$ follow from the transformation $\tilde{u}(t) = u(T-t)$, $\tilde{f}(t) = f(T-t)$ and by duality arguments. For (ii) – including even general unbounded domains – we refer to [17], IV.1.6. □

3. Proof of Theorem 1.4

Given $0 < T < \infty$, $1 < s, q < \infty$, and a bounded or unbounded domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, of $C^{1,1}$-type (α, β, K) we define the subspace $\tilde{L}^{s,q}_\sigma := L^s(0,T;\tilde{L}^q_\sigma(\Omega)$ of $\tilde{L}^{s,q} := L^s(0,T;\tilde{L}^q(\Omega))$ with norm $\|\cdot\|_{\tilde{L}^{s,q}_\sigma} = \|\cdot\|_{L^s(0,T;\tilde{L}^q(\Omega)_\sigma)}$. In addition to the operators $\mathcal{J}_{s,q}$, $\mathcal{J}'_{s,q}$ for bounded domains, see Lemma 2.4, we define $\tilde{\mathcal{J}}_{s,q}, \tilde{\mathcal{J}}'_{s,q}$ by

$$(\tilde{\mathcal{J}}_{s,q}f)(t) = \int_0^t e^{-(t-\tau)\tilde{A}_q} f(\tau)\, d\tau, \qquad (\tilde{\mathcal{J}}'_{s,q}f)(t) = \int_t^T e^{-(\tau-t)\tilde{A}_q} f(\tau)\, d\tau,$$

for $f \in \tilde{L}^{s,q}_\sigma$ and $0 \leq t \leq T$. Since $(\tilde{A}_q)' = \tilde{A}_{q'}$, we obtain for all $f \in \tilde{L}^{s,q}_\sigma$, $g \in \tilde{L}^{s',q'}_\sigma$ that

$$\langle \tilde{\mathcal{J}}_{s,q}f, g\rangle_T = \langle f, \tilde{\mathcal{J}}'_{s',q'}g\rangle_T.$$

3.1. Maximal regularity in a bounded domain Ω when $s = q \geq 2$

We consider the case $u_0 = 0$ and $s = q$. Then $u = \tilde{\mathcal{J}}_{q,q}f$ solves the equation $u_t + \tilde{A}_q u = f$, $u(0) = 0$, and $u = \tilde{\mathcal{J}}'_{q,q}f$ is the solution of the system $-u_t + \tilde{A}_q u = f$, $u(T) = 0$. Our aim is to prove in both cases the estimate (1.8) with a constant $C = C(T, q, \alpha, \beta, K) > 0$. Obviously it suffices to consider the case $u = \tilde{\mathcal{J}}_{q,q}f$ only since the other case follows using the transformation $\tilde{u}(t) = u(T-t)$, $\tilde{f}(t) = f(T-t)$. By Lemma 2.4 we know that $u = \tilde{\mathcal{J}}_{q,q}$ solves

$$u_t + \tilde{A}_q u = u_t - \Delta u + \nabla p = f \in L^q(0,T;\tilde{L}^q_\sigma), \quad u(0) = 0,$$

with $\nabla p = (I - \tilde{P}_q)\Delta u$, and that u satisfies (2.14) with a constant $C = C(\Omega, q) > 0$; note that the norms $\|u\|_{W^{2,q}}$ and $\|u\|_{\mathcal{D}(A_q)}$ are equivalent. Thus it remains to prove that C in (2.14) can be chosen depending only on T, q and (α, β, K).

For this reason, we use the system of functions $\{h_j\}$, $1 \leq j \leq N$, the covering of Ω by balls $\{B_j\}$, and the partition of unity $\{\varphi_j\}$ as described in Section 2. Let

$$U_j = U^-_{\alpha,\beta,h_j}(x_j) \cap B_j \;\text{ if }\; x_j \in \partial\Omega \;\text{ and } U_j = B_j \;\text{ if }\; x_j \in \Omega,\; 1 \leq j \leq N.$$

On U_j let $w = R\big((\nabla\varphi_j)\cdot u\big) \in L^q\big(0,T;W^{2,q}_0(U_j)\big)$, and let $M_j = M_j(p)$ be a constant depending on t defined by $p - M_j = R'(\nabla p) \in L^q\big(0,T;L^q_0(U_j)\big)$, see Lemma 2.3, and the proof of Lemma 2.1, (ii). Since $\operatorname{div} w = (\nabla\varphi_j)\cdot u$ and $\operatorname{div} w_t = (\nabla\varphi_j)\cdot u_t$ for a.a. $t \in (0,T)$, $(\varphi_j u - w)$ solves the local equation

$$\begin{aligned}&(\varphi_j u - w)_t - \Delta(\varphi_j u - w) + \nabla\big(\varphi_j(p - M_j)\big)\\ &\quad = \varphi_j f - w_t + \Delta w - 2\nabla\varphi_j\cdot\nabla u - (\Delta\varphi_j)u + (\nabla\varphi_j)(p - M_j)\end{aligned} \tag{3.1}$$

in U_j.

From (2.7), (2.8) using $w_t = R\big((\nabla\varphi_j)\cdot u_t\big)$ and $\nabla p = f - u_t + \Delta u$ we obtain for all $\varepsilon \in (0,1)$ the estimates

$$\begin{aligned}
\|w_t\|_{L^q(L^q(U_j))} &\le C\|u_t\|_{L^q(L^2(U_j))} + \varepsilon\|u_t\|_{L^q(L^q(U_j))},\\
\|\nabla^2 w\|_{L^q(L^q(U_j))} &\le C\big(\|u\|_{L^q(L^q(U_j))} + \|\nabla u\|_{L^q(L^q(U_j))}\big),\\
\|p - M_j\|_{L^q(L^q(U_j))} &\le C\big(\|f\|_{L^q(L^q(U_j))} + \|u_t\|_{L^q(L^2(U_j))} + \|\nabla u\|_{L^q(L^q(U_j))}\big)\\
&\quad +\varepsilon\|u_t\|_{L^q(L^q(U_j))}
\end{aligned} \tag{3.2}$$

with $C = C(q,T,\varepsilon,\alpha,\beta,K) > 0$. In fact, for the proof of $(3.2)_1$, choose $r \in [2,q)$ such that the embedding $W^{1,r}(U_j) \subset L^q(U_j)$ holds with an embedding constant $c = c(q,r,\alpha,\beta,K) > 0$ independent of j. Then

$$\|w_t\|_{L^q(U_j)} \le c\|w_t\|_{W^{1,r}(U_j)} \le c\|u_t\|_{L^r(U_j)}$$

for a.a. $t \in (0,t)$. Finally the interpolation inequality (2.13) proves $(3.2)_1$, and $(2.7)_2$ implies $(3.2)_2$. For the proof of $(3.2)_3$ we use (2.8), the embedding $W^{1,q'}(U_j) \subset L^{r'}(U_j)$ with an embedding constant $c = c(q,r,\alpha,\beta,K) > 0$ independent of j and apply the previous interpolation argument to u_t.

Applying the local estimate (2.10) to (3.1) and using (3.2) we get that

$$\begin{aligned}
&\|\varphi_j u_t\|^q_{L^q(L^q(U_j))} + \|\varphi_j u\|^q_{L^q(L^q(U_j))} + \|\varphi_j \nabla^2 u\|^q_{L^q(L^q(U_j))} + \|\varphi_j \nabla p\|^q_{L^q(L^q(U_j))}\\
&\quad \le C\big(\|f\|^q_{L^q(L^q(U_j))} + \|u\|^q_{L^q(W^{1,q}(U_j))} + \|u_t\|^q_{L^q(L^2(U_j))}\big) + \varepsilon\|u_t\|_{L^q(L^q(U_j))}
\end{aligned}$$

with $C = C(T,q,\varepsilon,\alpha,\beta,K) > 0$. Taking the sum over $j = 1,\dots,N$ and exploiting the crucial property of the number N_0 we are led to the estimate

$$\begin{aligned}
&\|u_t\|^q_{L^{q,q}} + \|u\|^q_{L^{q,q}} + \|\nabla^2 u\|^q_{L^{q,q}} + \|\nabla p\|^q_{L^{q,q}}\\
&= \int_0^T\int_\Omega \bigg(\Big|\sum_j \varphi_j u_t\Big|^q + \Big|\sum_j \varphi_j u\Big|^q + \Big|\sum_j \varphi_j \nabla^2 u\Big|^q\bigg)\,dx\,dt\\
&\quad + \int_0^T\int_\Omega \Big|\sum_j \varphi_j \nabla p\Big|^q\,dx\,dt\\
&\le \int_0^T\int_\Omega N_0^{\frac{q}{q'}}\Big(\sum_j |\varphi_j u_t|^q + \sum_j |\varphi_j u|^q + \sum_j |\varphi_j \nabla^2 u|^q + \sum_j |\varphi_j \nabla p|^q\Big)\,dx\,dt\\
&\le CN_0^{\frac{q}{q'}}\Big(\sum_j \|f\|^q_{L^q(0,T;L^q(U_j))} + \sum_j \|u\|^q_{L^q(0,T;W^{1,q}(U_j))} + \sum_j \|u_t\|^q_{L^q(0,T;L^2(U_j))}\Big)\\
&\quad + \varepsilon N_0^{\frac{q}{q'}} \sum_j \|u_t\|^q_{L^q(0,T;L^q(U_j))}\,.
\end{aligned} \tag{3.3}$$

Choosing $\varepsilon > 0$ sufficiently small in (2.11) and in (3.3), exploiting the absorption principle and again the property of the number N_0, (3.3) may be simplified to the estimate

$$\|u_t\|_{L^{q,q}}+\|u\|_{L^{q,q}}+\|\nabla^2 u\|_{L^{q,q}}+\|\nabla p\|_{L^{q,q}} \le C\big(\|f\|_{L^{q,q}}+\|u\|_{L^{q,q}}+\|u_t\|_{L^{q,2}}\big) \quad (3.4)$$

where $C = C(q,\alpha,\beta,K) > 0$; note that in order to deal with the sum of the terms $\|u_t\|_{L^q(0,T;L^2(U_j))}$ we also used the reverse Hölder inequality $\sum a_j^{q/2} \le \big(\sum a_j\big)^{q/2}$ for nonnegative real numbers a_j. Now, concerning the term $\|u\|^q_{L^{q,q}}$, we use (2.12) with $\varepsilon > 0$ sufficiently small and exploit the absorption principle. Finally we apply Lemma 2.4, i.e., we add the estimate (2.14) with $q = 2$ to (3.4), and use the equivalence of the norm $\|u\|_{W^{2,q}(\Omega)}$ to the graph norm $\|u\|_{\mathcal{D}(\mathcal{A}_q)}$, see Lemma 2.3 (iv). This argument proves estimate (1.8) for bounded domains when $s = q > 2$. Again using the equivalence of the norms $\|u\|_{W^{2,q}(\Omega)}$ and $\|u\|_{\mathcal{D}(\mathcal{A}_q)}$, we get (1.7) for $s = q$.

To prove (1.7) with $u_0 \in \mathcal{D}(\tilde{A}_q)$ we solve the system $\tilde{u}_t + \tilde{A}_q\tilde{u} = \tilde{f}$, $\tilde{u}(0) = 0$, with $\tilde{f} = f - \tilde{A}_q u_0$. Then $u(t) = \tilde{u}(t) + u_0$ yields the desired solution with $u_0 \in D(\tilde{A}_q)$. This proves Theorem 1.4 for bounded Ω and $s = q \ge 2$.

3.2. The case Ω bounded, $1 < s = q < 2$

In this case we consider for $f \in L^{q,q}_\sigma + L^{q,2}_\sigma = L^{q,q}_\sigma$ and the initial value $u_0 = 0$ the Stokes system $u_t + \tilde{A}_q u = f$, $u(0) = 0$. By Lemma 2.4 there exists a unique solution $u(t) = \mathcal{J}_{q,q} f(t) = \tilde{\mathcal{J}}_{q,q} f(t)$; here we used that $\tilde{P}_q = P_q$ and $\tilde{A}_q = A_q$. For the following duality argument we need that the space

$$C_0^\infty(C^\infty_{0,\sigma}) = \big\{v \in C_0^\infty(\Omega\times(0,T));\ \operatorname{div} v(x,t) = 0\ \ \forall t \in (0,T)\big\}$$

is dense in $L^{q',q'}_\sigma \cap L^{q',2}_\sigma = \big(L^{q,q}_\sigma + L^{q,2}_\sigma\big)'$. Then the identity

$$\langle u_t + \tilde{A}_q u, \tilde{A}_{q'} v\rangle = \langle u, (-\partial_t + \tilde{A}_{q'})\tilde{A}_{q'} v\rangle = \langle \tilde{A}_q u, (-\partial_t + \tilde{A}_{q'})v\rangle$$

holds for $u = \mathcal{J}_{q,q} f$ and every $v \in \tilde{A}^{-1}_{q'}\big(C_0^\infty(C^\infty_{0,\sigma})\big)$, since $(\tilde{\mathcal{J}}'_{q',q'})' = \tilde{\mathcal{J}}_{q,q}$. Let $g = -v_t + \tilde{A}_{q'} v$. Then we obtain by (1.7) with $s = q$ replaced by $s' = q' \ge 2$ and u replaced by v that

$$\begin{aligned}\|f\|_{L^{q,q}_\sigma + L^{q,2}_\sigma} &= \sup\left\{\frac{|\langle u_t + \tilde{A}_q u, \tilde{A}_{q'} v\rangle_T|}{\|\tilde{A}_{q'} v\|_{L^{q',q'}_\sigma \cap L^{q',2}_\sigma}};\ 0 \ne v \in \tilde{A}^{-1}_{q'} C_0^\infty(C^\infty_{0,\sigma})\right\}\\ &= \sup\left\{\frac{|\langle \tilde{A}_q u, g\rangle_T|}{\|\tilde{A}_{q'} v\|_{L^{q',q'}_\sigma \cap L^{q',2}_\sigma}};\ 0 \ne v \in \tilde{A}^{-1}_{q'} C_0^\infty(C^\infty_{0,\sigma})\right\};\\ &\ge \frac{1}{C}\|\tilde{A}_q u\|_{L^{q,q}_\sigma + L^{q,2}_\sigma},\end{aligned} \qquad (3.5)$$

where $C = C(T, q', \alpha, \beta, K) > 0$. Here we used that the estimate (1.7) also holds with u, u_0, f replaced by v, $v(T) = 0$, g due to the transformation in time in the proof of Lemma 2.4, (ii), and that due to Theorem 1.2 the norm $\|\cdot\|_{L^q_\sigma + L^2_\sigma}$ is equivalent to $\sup\big\{\frac{|\langle\cdot,h\rangle|}{\|h\|_{L^{q'}_\sigma \cap L^2_\sigma}};\ 0 \ne h \in L^{q'}_\sigma \cap L^2_\sigma\big\}$ with constants depending only on q and (α, β, K). Hence we obtain the estimate $\|\tilde{A}_q u\|_{L^{q,q}_\sigma + L^{q,2}_\sigma} \le C\|f\|_{L^{q,q}_\sigma + L^{q,2}_\sigma}$, and it follows

$$\|u_t\|_{L^{q,q}_\sigma + L^{q,2}_\sigma} + \|\tilde{A}_q u\|_{L^{q,q}_\sigma + L^{q,2}_\sigma} \le C\|f\|_{L^{q,q}_\sigma + L^{q,2}_\sigma}. \tag{3.6}$$

From the equivalence of norms $\|\cdot\|_{\mathcal{D}(A_q)}$ and $\|\cdot\|_{W^{2,q}}$ with constants depending only on q and (α, β, K), and from (2.3) with $B_1 = A_q, B_2 = A_2$, we conclude that also the norms $\|u\|_{W^{2,q}+W^{2,2}}$ and $\|u\|_{L^q_\sigma + L^2_\sigma} + \|A_q u\|_{L^q_\sigma + L^2_\sigma}$ are equivalent with constants depending only on q and (α, β, K). Then (3.6) and the identity $\nabla p = f - u_t + \Delta u$ lead to the estimate

$$\|u_t\|_{L^{q,q}_\sigma + L^{q,2}_\sigma} + \|u\|_{L^q(0,T;W^{2,q}+W^{2,2})} + \|\nabla p\|_{L^{q,q}+L^{q,2}} \le C\|f\|_{L^{q,q}_\sigma + L^{q,2}_\sigma} \tag{3.7}$$

with $C = C(q, \varepsilon, \alpha, \beta, K) > 0$.

Now the proof of Theorem 1.4 is complete for bounded domains in the case $s = q$.

3.3. The case Ω unbounded

Consider the sequence of bounded subdomains $\Omega_j \subseteq \Omega$, $j \in \mathbb{N}$, of uniform $C^{1,1}$-type as in (2.6), let $f \in \tilde{L}^{q,q}_\sigma$ and $f_j := \tilde{P}^{(j)}_q f_{|_{\Omega_j}}$ where $\tilde{P}^{(j)}_q$ denotes the Helmholtz projection in $\tilde{L}^q(\Omega_j)$. Then consider the solution $(u_j, \nabla p_j)$ of the instationary Stokes equation

$$\partial_t u_j - \tilde{P}_q \Delta u_j = \partial_t u_j - \Delta u_j + \nabla p_j = f_j, \quad \nabla p_j = (I - \tilde{P}_q)\Delta u_j \quad \text{in} \quad \Omega_j \times (0,T)$$

with initial condition $u_j(0) = 0$. From (1.7) with $s = q$ we obtain the estimate

$$\|\partial_t u_j\|_{\tilde{L}^{q,q}} + \|u_j\|_{L^q(0,T;\tilde{W}^{2,q}(\Omega_j))} + \|\nabla p_j\|_{\tilde{L}^{q,q}} \le C\|f\|_{\tilde{L}^{q,q}_\sigma} \tag{3.8}$$

on Ω_j with $C = C(T, q, \alpha, \beta, K) > 0$ independent of $j \in \mathbb{N}$. Extending u_j and ∇p_j for a.a. $t \in (0,T)$ from Ω_j by 0 to vector fields on Ω we find, suppressing subsequences, weak limits

$$u = w\text{–}\lim_{j\to\infty} u_j \quad \text{in } \tilde{L}^{q,q}_\sigma(\Omega), \quad \nabla p = w\text{–}\lim_{j\to\infty} \nabla p_j \quad \text{in } \tilde{L}^{q,q}(\Omega)$$

satisfying $u \in L^q(0,T; \tilde{L}^q_\sigma(\Omega))$, $\partial_t u - \Delta u + \nabla p = \partial_t u + \tilde{A}_q u = f$ in $\Omega \times (0,T)$ and the *a priori* estimate (1.7) with $u_0 = 0$; it follows (1.8) for this case. Note that each ∇p_j when extended by 0 need not be a gradient field in Ω; however, by de Rham's argument, the weak limit of the sequence $\{\nabla p_j\}$ is a gradient field in Ω. Hence we solved the instationary Stokes equation $\partial_t u + \tilde{A}_q u = \partial_t u - \Delta u + \nabla p = f$, $u(0) = u_0$, in $\Omega \times (0,T)$ and proved (1.7), (1.8).

Up to now we considered only the case when $s = q$. However, an abstract extrapolation argument shows that the validity of (1.7) with $s = q$ immediately extends to all $s \in (1, \infty)$, see [2], p. 191, and [5], (1.12), where A has to be replaced by $-\tilde{A}_q - \delta I$ with $\delta > 0$ as in (1.4). The case $u(0) = u_0 \neq 0$ can be reduced to the case $u_0 = 0$ in the same way as in Subsection 3.1.

Finally, to prove uniqueness let $v \in L^s(0,T;\tilde{W}^{2,q})$ satisfy $\partial_t v + \tilde{A}_q v = 0$ and $v(0) = 0$. Given $f' \in \tilde{L}^{s',q'}$ let $u \in L^{s'}(0,T;\tilde{W}^{2,q'})$ be a solution of $-u_t + \tilde{A}_{q'} u = \tilde{P}_{q'} f'$, $u(T) = 0$. Then

$$0 = \langle v_t + \tilde{A}_q v, u\rangle_T = \langle v, (-\partial_t + \tilde{A}_{q'})u\rangle_T = \langle v, \tilde{P}_{q'} f'\rangle_T = \langle v, f'\rangle_T$$

for all $f' \in \tilde{L}^{s',q'}$; hence, $v = 0$.

Now Theorem 1.4 is completely proved. □

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Linear Control Systems in Sequence Spaces

Hector O. Fattorini

Abstract. Pontryagin's maximum principle in its infinite-dimensional version provides (separate) necessary and sufficient conditions for both time and norm optimality for the system $y' = Ay + u$ (A an infinitesimal generator). The question whether targets in $D(A)$ guarantee a smooth costate has been open. We show the answer is "no" by means of a counterexample involving an analytic semigroup. Another analytic semigroup sheds some light on other subjects such as the existence of hypersingular time optimal controls (thus answering another open question) and the characterization of the reachable space and of singular functionals in its dual.

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1. Introduction

We consider the control system

$$y'(t) = Ay(t) + u(t)\,, \quad y(0) = \zeta \tag{1.1}$$

with controls $u(\cdot) \in L^\infty(0, T; E)$, where$A$ is the infinitesimal generator of a strongly continuous semigroup $S(t)$ in a Banach space E. In the *norm optimal* problem we drive the initial point ζ to a point target,

$$y(T) = \bar{y} \tag{1.2}$$

in a fixed time interval $0 \le t \le T$ minimizing $\|u(\cdot)\|_{L^\infty(0,T;E)}$, while in the *time optimal* problem we drive to the target with a bound on the norm of the control, say $\|u(\cdot)\|_{L^\infty(0,T;E)} \le 1$, in optimal time T. Solutions or *trajectories*

$$y(t) = S(t)\zeta + \int_0^t S(t-\sigma)u(\sigma)d\sigma \tag{1.3}$$

of the initial value problem (1.1) are denoted by $y(t) = y(t, \zeta, u)$. Controls in $L^\infty(0, T; E)$ with norm $\|u(\cdot)\|_{\in L^\infty(0,T;E)} \le 1$ are named *admissible*.

Separate necessary and sufficient conditions for norm or time optimality can be given in terms of the maximum principle, which requires as a preliminary the construction of a space of multipliers (final values of the costates). We summarize [6] or [10], Section 2.3 for the case (sufficient for this paper) where $S(t)$ is analytic and A has a bounded inverse. Let E^*_{-1} be the completion of E^* in the norm

$$\|y^*\|_{E^*_{-1}} = \|(A^{-1})^* y^*\|_{E^*} \,. \tag{1.4}$$

For each $t \geq 0$ $S(t)^*$ can be extended to an operator $S(t)^* : E^*_{-1} \to E^*$ (equally named) and the space $Z(T)$ consists of all $z \in E^*_{-1}$ such that

$$\|z\|_{Z(T)} = \int_0^T \|S(t)^* z\|_{E^*} dt < \infty \,. \tag{1.5}$$

Equipped with $\|\cdot\|_{Z(T)}$, $Z(T)$ is a Banach space. All spaces $Z(T)$ coincide and all norms $\|\cdot\|_{Z(T)}$ are equivalent for $T > 0$.

A control $\bar{u}(t)$ in the interval $0 \leq t \leq T$ satisfies the *strong maximum principle* (or simply the *maximum principle*) if there exists $z \in Z(T)$ such that $S(t)^* z$ is not identically zero in $0 < t \leq T$ and

$$\langle S(T-t)^* z, \bar{u}(t) \rangle = \max_{\|u\| \leq \rho} \langle S(T-t)^* z, u \rangle \quad \text{a. e. in } 0 \leq t \leq T \,, \tag{1.6}$$

where $\langle \cdot \,, \cdot \rangle$ is the duality of the space E and the dual E^*; we call z the *multiplier* and $z(t) = S(T-t)^* z$ the *costate.* It is known [6] Theorem 5.1, [10] Theorem 2.5.1 that if $\bar{u}(t)$ drives $\zeta \in E$ to $\bar{y} = y(T, \zeta, \bar{u})$ with[1]

$$\bar{y} - S(T)\zeta \in D(A) \tag{1.7}$$

then the strong maximum principle (1.6) is a necessary condition for norm optimality, with ρ the optimal norm. If $\rho = 1$, (1.6) is a necessary condition for time optimality. Conversely, the maximum principle is a sufficient condition for norm optimality and it is also sufficient for time optimality when $\rho = 1$ under suitable conditions on the initial condition ζ and the target $\bar{y}$.[2]

The following problem has remained open: *Does the maximum principle* (1.6) *under* (1.7) *imply that the multiplier* z *belongs to* E^* (rather than to the larger space $Z(T)$)? We show in this paper that the answer is negative. The counterexample (Theorem 3.6) involves a smooth (analytic) semigroup in the nonsmooth sequence space ℓ^1. We point out that the question has a trivial affirmative answer for special classes of semigroups; for instance, if $S(t)$ is a group or, more generally, if $S(T)E = E$ $(t > 0)$ then[3] $Z(T) = E^*$ [6], [10] Theorems 2.2.5 and 2.2.6.

A *multiplier space* $\mathfrak{Z} \supseteq E^*$ is a linear space $\mathfrak{Z}$ such that $S(t)^*$ can be extended to $\mathfrak{Z}$ for all $t \geq 0$ and $S(t)^* \mathfrak{Z} \subseteq E^*$ for $t > 0$. ($Z(T)$ is a particular multiplier space.) If $\bar{u}(t)$ satisfies (1.6) for some z in a multiplier space $\mathfrak{Z}$ such that $S(t)^* z$

[1] This condition can be relaxed to$\bar{y} - S(T)\zeta \in \overline{D(A)}$, where the bar indicates closure in $R^\infty(T)$. For the definition of this space see below.

[2] These conditions are $\bar{y} = y(T, \zeta, \bar{u}) \in D(A)$, $\|A\bar{y}\| < 1$ or $\zeta \in D(A)$, $\|A\zeta\| < 1$, $S(t)^* z \neq 0$ in $0 \leq t < T$. They are satisfied if either $\zeta = 0$ or $\bar{y} = 0$ [9], [10] Theorem 2.5.7.

[3] The condition $\bar{y} - S(T)\zeta \in D(A)$ is not needed in this case.

is not identically zero in $0 < t \le T$ we say that $\bar{u}(t)$ satisfies the *weak maximum principle* and that $\bar{u}(t)$ is *weakly regular;* if $z \in Z(T)$ the control is *regular,* and if $z \in E^*$ the control is *strongly regular.* It can be shown that, for a class of semigroups including analytic semigroups in Hilbert space every optimal control satisfies the weak maximum principle with no conditions on ζ, $\bar{y}$ [5], [10, Section 3.2], but the weak maximum principle is no longer a sufficient condition for optimality [7], [10] Section 3.5.

A time optimal control $\bar{u}(t)$ is called *singular* if it does not satisfy the maximum principle (1.6) for any z in any multiplier space $\mathfrak{Z}$, *hypersingular* if it does not satisfy the maximum principle in any interval $[t_0, t_1]$, $0 \le t_0 < t_1 \le T$; this means, there is no z in any multiplier space $\mathfrak{Z}$ such that $S(t)^* z$ is not identically zero in $t_0 \le t \le t_1$ and[4]

$$\langle S(t_1 - t)^* z, \bar{u}(t)\rangle = \max_{\|u\| \le \rho} \langle S(t_1 - t)^* z, u\rangle \quad \text{a. e. in } t_0 \le t \le t_1 \,. \tag{1.8}$$

Singular time optimal controls were constructed in [8], see also [10] Theorem 2.8.4, but existence of hypersingular time optimal controls was left open there. We construct in this paper a hypersingular control; the example provided by Corollary 6.3 involves a smooth (analytic) semigroup in the nonsmooth sequence space ℓ^0.

The examples in this paper throw also some light on the reachable space and on its dual. Given $t > 0$, the *reachable space* $R^\infty(t)$ (at time t) of the system (1.1) consists of all

$$y = y(t, 0, u) = \int_0^t S(t - \sigma) u(\sigma) d\sigma \ , \quad u(\cdot) \in L^\infty(0, t; E)\,, \tag{1.9}$$

and is equipped with the norm

$$\|y\|_{R^\infty(t)} = \inf \left\{ \|u\|_{L^\infty(0,t;E)}; \int_0^t S(t - \sigma) u(\sigma) d\sigma = y \right\} , \tag{1.10}$$

which makes $R^\infty(t)$ a Banach space. All spaces $R^\infty(t)$ coincide (with equivalent norms) for $t > 0$ [3], [10] Section 2.1. Characterization of the reachable space $R(T)$ and its dual $R^\infty(T)^\star$ lie at the heart of the theory of optimal problems for (1.1), in particular the time and norm optimal problems; in fact, the maximum principle (1.6) is obtained separating the target $\bar{y}$ from the ball $B^\infty_\rho(T)$ of radius ρ in $R^\infty(T)$ by means of functionals in $R^\infty(T)^\star$. A complete characterization of $R^\infty(T)$ and $R^\infty(T)^\star$ exists only when $S(t)E = E$ for $t > 0$, which is a necessary and sufficient condition for the equality $R^\infty(T) = E$ [6], [10], (2.2.2) and Theorem 2.2.3. Here, the norm of $R^\infty(T)$ and that of E are equivalent, which implies that $R^\infty(T)^\star = E^*$ with equivalent norms. In the general case, however there is no usable characterization of $R^\infty(T)$ and only some functionals in $R^\infty(T)^\star$ can be

[4]The notion of hypersingular control is only interesting in the time optimal problem. In fact, a control that is time optimal in $[0, T]$ is also time optimal in any $[t_0, t_1] \subseteq [0, T]$, while norm optimality is not inherited from $[0, T]$ by subintervals.

identified. These are the *regular* functionals ξ_z defined by

$$\left\langle\!\!\left\langle \xi_z \,, \int_0^T S(T-\sigma)u(\sigma)d\sigma \right\rangle\!\!\right\rangle = \int_0^T \langle S(T-\sigma)^* z, u(\sigma)\rangle d\sigma \tag{1.11}$$

where $z \in Z(T)$ and $\langle\!\langle \xi, y \rangle\!\rangle$ denotes the application of a functional $\xi \in R^\infty(T)^\star$ to an element $y \in R^\infty(T)$. The integral (1.11) exists and

$$\left| \int_0^T \langle S(T-\sigma)^* z, u(\sigma)\rangle d\sigma \right| \le \int_0^T \|S(T-\sigma)^* z\|_{E^*} d\sigma \cdot \|u(\cdot)\|_{L^\infty(0,T;E)}$$
$$= \int_0^T \|S(\sigma)^* z\|_{E^*} d\sigma \cdot \|u(\cdot)\|_{L^\infty(0,T;E)} = \|z\|_{Z(T)} \|u(\cdot)\|_{L^\infty(0,T;E)} \,,$$

which implies $\|\xi\|_{R^\infty(T)^\star} \le \|z\|_{Z(T)}$; it can be shown that $\|\xi\|_{R^\infty(T)^\star} = \|z\|_{Z(T)}$. We denote by $\mathfrak{R}(T) \subseteq R^\infty(T)^\star$ the space of all regular functionals.

We have $D(A) \subseteq R^\infty(T)$. Functionals $\xi_s \in R^\infty(T)^\star$ that vanish on $D(A)$ are called *singular* and $\mathfrak{S}(T) \subseteq R^\infty(T)$ denotes all of them. Of course, $\mathfrak{S}(T) \neq \{0\}$ if and only if $\overline{D(A)} \neq R^\infty(T)$, where the bar indicates closure in $R^\infty(T)$. There is a large class of infinitesimal generators that satisfy this condition [6], [10] Section 2.9 (among them, unbounded generators of analytic semigroups) but it seems to be an open problem whether there exists any unbounded infinitesimal generator with $R^\infty(T) \neq E$ such that $\overline{D(A)} = R^\infty(T)$. Regular and singular functionals totally describe the dual $R^\infty(T)^\star$; in fact, we have

$$R^\infty(T)^\star = \mathfrak{R}(T) \oplus \mathfrak{S}(T) \tag{1.12}$$

algebraic and metrically [10] Theorem 2.4.1; the direct sum is L^1 if $S(t)$ is an analytic semigroup [10] Theorem 2.4.3. The statement "totally describe" is of course, tempered by the fact that singular functionals are the spawn of the Hahn-Banach theorem in a nonseparable space, thus it is not surprising that all that can proved on singular functionals is their existence; actual examples of singular functionals are totally lacking. However, we examine an example in Section 7 where $R^\infty(T)$ can be easily described and where $R^\infty(T)^\star$ and the direct sum (1.12) can be "visualized" up to a point.

2. The first example

The space is ℓ^1, consisting of all sequences $y = \{y_1, y_2, \dots\} = \{y_n\}$ with

$$\|y\|_1 = \|\{y_n\}\|_1 = \sum_{n=1}^\infty |y_n| < \infty \,,$$

which is a Banach space equipped with $\|\cdot\|_1$. The (densely defined, unbounded) infinitesimal generator and the semigroup are

$$A\{y_n\} = \{-n y_n\} \,, \qquad S(t)\{y_n\} = \{e^{-nt} y_n\} \,, \tag{2.1}$$

$D(A)$ the set of all $\{y_n\} \in \ell^1$ with $\{ny_n\} \in \ell^1$. We have $(\ell^1)^* = \ell^\infty$, the space of all sequences $y^* = \{y_1^*, y_2^*, \dots\} = \{y_n^*\}$ with

$$\|y^*\|_\infty = \|\{y_n^*\}\|_\infty = \sup_{n \geq 1} |y_n^*| < \infty$$

equipped with $\|\cdot\|_\infty$, the duality of ℓ^1 and ℓ^∞ given by

$$\langle y^*, y \rangle = \langle \{y_n^*\}, \{y_n\} \rangle = \sum_{n=1}^{\infty} y_n^* y_n \, .$$

The adjoint semigroup $S(t)^*$is

$$S(t)^* \{y_n^*\} = \{e^{-nt} y_n^*\} \, , \tag{2.2}$$

analytic in $t > 0$ but not strongly continuous at $t = 0$; for instance, for $\{y_n^*\} = \{1, 1, \dots\}$ we have $\|S(t)^*\{y_n^*\} - \{y_n^*\}\|_\infty \to 1$ as $t \to 0$. The space $E_{-1}^* = (\ell^1)_{-1}^* = \ell_{-1}^\infty$ consists of all sequences $z = \{z_1, z_2, \dots\} = \{z_n\}$ with

$$\|z\|_{\infty,-1} = \|\{z_n\}\|_{\infty,-1} = \sup_{n \geq 1} \left| \frac{z_n}{n} \right| < \infty \, , \tag{2.3}$$

and $Z(T)$ is the subspace of ℓ_{-1}^∞ determined by the condition

$$\int_0^T \|\{z_n e^{-n\sigma}\}\|_\infty d\sigma < \infty \, . \tag{2.4}$$

The subspace $\ell^0 \subset \ell^\infty$ of all $x = \{x_n\}$ such that $\lim_{n\to\infty} x_n = 0$ is closed, hence a Banach space equipped with $\|\cdot\|_0 = \|\cdot\|_\infty$. We have $\ell^1 = (\ell^0)^*$ with duality

$$\langle y, x \rangle = \langle \{y_n\}, \{x_n\} \rangle = \sum_{n=1}^{\infty} y_n x_n \, .$$

We consider the control system (1.1)in ℓ^1 with A given by (2.1) and controls in $L^\infty(0, T; \ell^1)$. Since ℓ^1 is separable and the dual of the Banach space ℓ^0, the assumptions needed in existence theory [10], **3.1** apply and we have (a) if $\zeta \in \ell^1$ can be driven to a target $\bar{y} \in \ell^1$ in time T then there exists a control doing the same drive norm optimally in the same time, (b) if ζ can be driven to $\bar{y}$ in any time $t > 0$ by means of an admissible control, then there exists an admissible control doing the same drive in optimal time T.

3. The maximum principle and optimal controls

If $\bar{u}(t) = \{u_n(t)\}$ drives (norm, time) optimally $\zeta$to $\bar{y} \in D(A)$ then (1.7) holds and there exists $z = \{z_n\} \in Z(T)$ such that

$$\langle S(T-t)^* z, \bar{u}(t) \rangle = \max_{\|u\|_1 \leq \rho} \langle S(T-t)^* z, u \rangle \quad \text{a. e. in } 0 \leq t \leq T \tag{3.1}$$

where ρ is the optimal norm (for the time optimal problem $\rho = 1$). Conversely, (3.1) is a sufficient condition for optimality when $\zeta = 0$ or $\bar{y} = 0$. Let $\mathfrak{Z}^\infty$ be the set of all sequences $z = \{z_n\}$ such that $\{e^{-nt} z_n\} \in \ell^\infty$ for all $t > 0$. Obviously, we can extend $S(t)^*$ to $\mathfrak{Z}^\infty$ by $S^*(t)\{z_n\} = \{e^{-nt}z_n\} \in \ell^\infty$, thus $\mathfrak{Z}^\infty$ is a multiplier space[5] according to the definition in Section 1. The weak maximum principle (1.6) with $z \in \mathfrak{Z}^\infty$ is

$$\sum_{n=1}^{\infty} z_n e^{-n(T-t)} \bar{u}_n(t) = \max_{\|\{u_n\}\|_1 \le \rho} \sum_{n=1}^{\infty} z_n e^{-n(T-t)} u_n \,, \tag{3.2}$$

which characterizes uniquely the control $\{\bar{u}_n(t)\}$ for every t for which the sequence $\{z_n e^{-n(T-t)}\}$ has a unique maximum or, equivalently, for which the set

$$\begin{aligned} \mathfrak{M}(t,z) = \mathfrak{M}(t,\{z_n\}) = \big\{m; |z_m|e^{-m(T-t)} \\ = \|\{z_n e^{-n(T-t)}\}\|_\infty = \max_{n\ge 1} |z_n|e^{-n(T-t)}\big\} \end{aligned} \tag{3.3}$$

contains exactly one element $m = m(t,\{z_n\})$; in fact, for such a t we have

$$\{\bar{u}_n(t)\} = \operatorname{sign} z_n \rho\, \delta_{mn} \,, \tag{3.4}$$

δ_{mn} the Kronecker delta. Characterization of the optimal control $\{\bar{u}_n(t)\}$ for those values of t where $\mathfrak{M}(t,\{z_n\})$ contains more than one element is mostly a moot point in view of the next result, where we take $\rho = 1$.

Lemma 3.1. *Let $z \in \mathfrak{Z}^\infty$. Then $\mathfrak{M}(t,\{z_n\})$ consists of a single element $m(t,\{z_n\})$ except for a finite or infinite sequence $\{t_n\}$ accumulating, if infinite, at T.*

Proof. It is sufficient to show that every t for which $\mathfrak{M}(t,\{z_n\})$ has more than one element is isolated in the interval $0 \le t < T$. Let $t \ge 0$ be such that

$$\mathfrak{M}(t,\{z_n\}) = \{m_1,\, m_2, \dots, m_k\} \quad (m_1 < m_2 < \cdots < m_k,\ k > 1)\,,$$

which means

$$|z_{m_j}|e^{-m_j(T-t)} = \max_{n\ge 1} |z_n|e^{-n(T-t)} = M \quad (j = 1,\dots,k)\,.$$

We have $|z_n|e^{-n(T-t)} \to 0$, thus there exists $\epsilon > 0$ such that

$$|z_n|e^{-n(T-t)} \le M - \epsilon \quad (n \ne m_j,\ j = 1,\dots,k)\,.$$

Accordingly, if $t > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} |z_n|e^{-n(T-t-\tau)} &\le M - \frac{\epsilon}{3} \quad (|\tau| \le \delta,\ n \ne m_j,\ j = 1,\dots,k)\,, \\ |z_{m_j}|e^{-m_j(T-t-\tau)} &\ge M - \frac{\epsilon}{3} \quad (|\tau| \le \delta,\ j = 1,\dots,k)\,. \end{aligned}$$

This means

$$\mathfrak{M}(\tau,\{z_n\}) = \{m_1\} \quad (t \le \tau \le t+\delta)\,, \tag{3.5}$$

$$\mathfrak{M}(\tau,\{z_n\}) = \{m_k\} \quad (t-\delta \le \tau \le t)\,. \tag{3.6}$$

In case $t = 0$ we take $\tau \ge 0$ and obtain (3.5). □

[5]The space $\mathfrak{Z}^\infty$ is in fact the largest multiplier space.

We call $\{u_n(t)\}$ the control *associated with* the costate z. Lemma 3.1 says that $\{u_n(t)\}$ is uniquely defined except at the *switchings* or *switching points* t_n, $\{t_n\}$ the sequence in Lemma 3.1. In the intervals (t_n, t_{n+1}) between switchings we have $\mathfrak{M}(t, z) = \{m_n\}$ and equalities (3.5)–(3.6) imply that the sequence $\{m_n\}$ is increasing, thus

$$\mathfrak{M}(t, z) = \{m_n\} \quad (t_n < t < t_{n+1}) \quad m_1 < m_2 < \dots \tag{3.7}$$

It also follows from (3.5)–(3.6) that at the switching t_{n+1} we have

$$|z_{m_n}|e^{-m_n(T-t_{n+1})} = |z_{m_{n+1}}|e^{-m_{n+1}(T-t_{n+1})},$$

which can be written in the form

$$|z_{m_{n+1}}| = |z_{m_n}|e^{(m_{n+1}-m_n)(T-t_{n+1})} \tag{3.8}$$

and implies $|z_{m_{n+1}}| > |z_{m_n}|$. The control produced by (3.1) or, rather by its consequence (3.4) is

$$\bar{u}_{m_n}(t) = \begin{cases} \operatorname{sign} z_{m_n} & t \in [t_n, t_{n+1}] \\ 0 & t \notin [t_n, t_{n+1}] \end{cases} \qquad (n = 0, 1, \dots) \tag{3.9}$$

with $u_k(t) = 0$ for all k not included in (3.9), that is, for $k \neq m_n$ $(n = 1, 2, \dots)$. We call the m_n the *live coordinates* of the control and the $\{\bar{u}_{m_n}(t)\}$ the *live components.* Note that (3.7) implies that the index of the live coordinates increases with the index of the switchings; in other words *live coordinates come into play in ascending order as time increases.* Costates such that

$$\sup_{n\geq 1} |z_n| = |z_m| \quad \text{for some } m \geq 1 \tag{3.10}$$

produce controls $\{\bar{u}_n(t)\}$ with a finite number of switchings. In fact, if m is the smallest index such that (3.10) holds we have

$$|z_m|e^{-m(T-t)} > |z_n|e^{-n(T-t)} \quad (n \geq m\,, 0 \leq t < T)\,, \tag{3.11}$$

thus there are no live coordinates $> m$.

The following result goes in a direction opposite to that of Lemma 3.1: the costate is produced from the switchings.

Lemma 3.2. *Let* $\{t_n\}$ *be an arbitrary sequence with* $0 = t_1 < \cdots < t_n < \cdots < T$, *if infinite with* $t_n \to T$. *Then there exists* $\{z_n\} \in \mathfrak{Z}^\infty$, $z_n \geq 0$ *such that the control*

$$\bar{u}_n(t) = \begin{cases} 1 & t \in [t_n, t_{n+1}] \\ 0 & t \notin [t_n, t_{n+1}] \end{cases} \qquad (n = 0, 1, \dots) \tag{3.12}$$

produced by (3.1) $\Rightarrow$ (3.4) *has the* t_n *as switchings. If the number of switchings is infinite then* $z_n > 0$ *for all* n; *if the number of switchings is finite*$= N$ *then* $z_n > 0$ *for* $n = 1, 2, \dots, N$, $z_n = 0$ *for* $n > N$.

Proof. The sequence z_n is defined on the basis of (3.8) for the particular case $m_n = n$ and $z_n \geq 0$,

$$z_{n+1} = z_n e^{((n+1)-n)(T-t_{n+1})} = z_n e^{(T-t_{n+1})}\,.$$

We arbitrarily set $z_1 = 1$ and apply (3.8) inductively,

$$z_2 = z_1 e^{(T-t_2)} = e^{(T-t_2)}\,, \quad z_3 = z_2 e^{(T-t_3)} = e^{(T-t_3)} e^{(T-t_2)}\,, \dots$$

Putting all inductive steps together,

$$z_n = e^{(T-t_n)+(T-t_{n-1})+\cdots+(T-t_2)}\,. \tag{3.13}$$

We have

$$e^{-n(T-t)} z_n = e^{(T-t_n)+(T-t_{n-1})+\cdots+(T-t_2)-n(T-t)} = e^{\phi(n,t)}$$

with $\phi(n,t) = (T-t_2) + \cdots + (T-t_n) - n(T-t)$. Now, if $t_n < t < t_{n+1}$ we have

$$\phi(n,t) = \phi(n-1,t) + (T-t_n) - (T-t)\,,$$

and it follows that $\phi(n,t)$, as a function of n, is

$$\begin{aligned}&\text{increasing if } T - t_n > T - t \iff t_n < t\,,\\ &\text{decreasing if } T - t_n < T - t \iff t_n > t\,.\end{aligned}$$

Accordingly, $\phi(n,t)$ reaches its unique maximum in n at the largest n such that $t_n < t$, which means that $\{\bar{u}_n(t)\}$ satisfies (3.12). □

Corollary 3.3. *The multiplier* $\{z_n\} \in \ell^\infty$ *if and only if*

$$\sum_{n=1}^{\infty} (T - t_n) < \infty\,. \tag{3.14}$$

Proof. Follows from (3.13). □

Remark 3.4. Let $\{m_n\}$ be a sequence of integers such that

$$1 \le m_1 < m_2 < \cdots < m_n < \dots \tag{3.15}$$

The construction in Theorem 3.2 can be modified in such a way that the live coordinates of $\{z_n\}$ are the entire sequence $\{m_n\}$ in the case of infinite switchings or the finite sequence $\{m_1, m_2, \dots, m_N\}$ in the case of N switchings. The modifications are rather obvious. We set $= 0$ a priori all coordinates of z_n outside the sequence $\{m_n\}$. Then we use (3.8) for the inductive definition of the sequence, which leads to

$$z_n = e^{\phi(n,t)} \tag{3.16}$$

with

$$\begin{aligned}\phi(n,t) &= (m_2 - m_1)(T - t_2) + \cdots + (m_n - m_{n-1})(T - t_n) - m_n(T - t)\\ &= \phi(n-1,t) + (m_n - m_{n-1})(T - t_n) - (m_n - m_{n-1})(T - t)\end{aligned} \tag{3.17}$$

so that $\phi(n,t)$ reaches its unique maximum at the largest n such that $t_n < t$ and the construction of the control is

$$\bar{u}_{m_n}(t) = \begin{cases} 1 & t \in [t_n, t_{n+1}] \\ 0 & t \notin [t_n, t_{n+1}] \end{cases} \qquad (n = 0, 1, \dots)\,, \tag{3.18}$$

all other coordinates being zero. The companion of condition (3.14) for the costate to be in ℓ^∞ is

$$\sum_{n=1}^{\infty}(m_n - m_{n-1})(T - t_n) < \infty\,. \tag{3.19}$$

Remark 3.5. Obviously, condition (3.8) determines uniquely the live coordinates $z_{m(n)}$ of $\{z_n\}$. We have set the other coordinates $= 0$ for convenience's sake, but we can fix them in other ways as long as they don't interfere with the condition

$$\max z_n e^{-n(T-t)} = z_{m_n} e^{-m_n(T-t)} \quad (T - t_{n+1} \le t \le T - t_n)\,.$$

For instance, the two multipliers

$$\{z_n\} = \{1, 2, 0, 0, \dots\}\,, \quad \{\zeta_n\} = \{1, 2, 2, 2, \dots\}$$

produce via (3.1) $\Longrightarrow$ (3.4) the same control $\{u_n(t)\}$ in the interval $0 \le t \le 1$, where

$$u_n(t) = \left\{\begin{array}{ll} 1 & t \in [0, t_0) \\ 0 & t \notin [t_0, 1] \end{array}\right. , \quad u_2(t) = u_3(t) = \cdots = 0\,,$$

and $t_0 = 0.306853$ is the only solution of the equation

$$e^{-(1-t_0)} - 2e^{-2(1-t_0)} = 0\,.$$

In the case (covered by Lemma 3.2) where all coordinates of $\{\bar{u}_n(t)\}$ are live the multiplier $\{z_n\}$ is uniquely determined up to a constant (we can set z_1 arbitrary).

The counterexample announced in Section 1 is in Theorem 3.6. below. The sufficiency conditions there guarantee that this control will be time optimal.

Theorem 3.6. *There exists a control $\{\bar{u}_n(t)\}$ driving from the origin to a target $\{\bar{y}_n\} \in D(A)$ time optimally such that $\{\bar{u}_n(t)\}$ does satisfies the maximum principle* (1.6) *with $\{z_n\} \in Z(T)$ but not with $\{z_n\} \in \ell^\infty$.*

Proof. We construct the control from its switching points. Given an arbitrary sequence $\{t_n\}$, $0 < t_1 < t_2 < \cdots < T$, $t_n \to T$ we define

$$\bar{y}_n = \int_{t_n}^{t_{n+1}} e^{-n(T-\sigma)} d\sigma = \frac{e^{-n(T-t_{n+1})} - e^{-n(T-t_n)}}{n}\,. \tag{3.20}$$

We have

$$n\bar{y}_n = e^{-n(T-t_{n+1})} - e^{-n(T-t_n)} = ne^{-n\theta_n}(t_{n+1} - t_n) \tag{3.21}$$

with $T - t_{n+1} \le \theta_n \le T - t_n$. We take

$$t_n = T - \frac{\log(n-1)}{n-1} \quad (n \ge 4)\,. \tag{3.22}$$

Since $(\log x/x)' = -(\log x - 1)/x^2 < 0$ for $x > e$, $\{\log(n-1)/(n-1)\}$ is decreasing (so that $\{t_n\}$ is increasing) for $n \ge 4$ and $t_n \to T$. We pick t_1, t_2, t_3 arbitrary as long as $0 \le t_1 < t_2 < t_3 < t_4$. We have

$$T - t_{n+1} = \frac{\log n}{n} \le \theta_n \le \frac{\log(n-1)}{n-1} = T - t_n \quad (n \ge 4) \tag{3.23}$$

so that, using (3.21),

$$t_{n+1} - t_n = n\bar{y}_n \cdot \frac{e^{n\theta_n}}{n} \geq n\bar{y}_n \cdot \frac{e^{\log n}}{n} \cdot n\bar{y}_n = n\bar{y}_n \,,$$

and we obtain

$$\sum_{n=1}^{\infty} n\bar{y}_n \leq \sum_{n=1}^{\infty}(t_{n+1} - t_n) = \lim_{n\to\infty} t_n - t_1 = T - t_1 \,,$$

hence $\bar{y} = \{\bar{y}_n\} \in D(A)$. On the other hand, we have

$$\sum_{n=4}^{\infty}(T - t_n) \geq \sum_{n=4}^{\infty} \frac{\log(n-1)}{n-1} = \infty \,,$$

which, in view of Corollary 3.3 shows that if $\{z_n\}$ is the multiplier constructed from the switching points via Lemma 3.2, then $\{z_n\} \notin \ell^\infty$. We finally check that $\{z_n\} \in Z(T)$. The construction in Lemma 3.2 implies that

$$\|S(T-t)z\|_\infty = \|z_n e^{-n(T-t)}\|_\infty = z_n e^{-nt} \quad (t_n \leq t \leq t_{n+1}) \,,$$

thus

$$\begin{aligned}\int_0^T \|S(T-t)z\|_\infty dt &= z_n \int_{t_n}^{t_{n+1}} e^{-nt} dt \\ &= \sum_{n=1}^{\infty} z_n \frac{e^{-n(T-t_{n+1})} - e^{-n(T-t_n)}}{n} = \sum_{n=1}^{\infty} \frac{z_n}{n} \cdot n\bar{y}_n\end{aligned}$$

after (3.21), hence

$$\int_0^T \|S(T-t)z\|_\infty dt \leq \Big(\sup_{n\geq 1} \frac{z_n}{n}\Big) \sum_{n=1}^{\infty} n\bar{y}_n \leq \|z\|_{\infty,-1} \|\bar{y}\|_{D(A)} \,,$$

thus $\{z_n\} \in Z(T)$ as claimed. □

4. The second example

The (densely defined) infinitesimal generator and semigroup are

$$A\{y_n\} = \{-ny_n\} \,, \qquad S(t)\{y_n\} = \{e^{-nt}y_n\} \,, \tag{4.1}$$

in the space ℓ^0 (the semigroup in Sections 2 and 3 is actually the dual of this one). We consider the control system (1.1) in $E = \ell^0$ with the operator A in (4.1). The natural control space is $L^\infty(0,T;\ell^0)$, not a good space for existence of optimal controls. We use instead $L^\infty_w(0,T;\ell^\infty)$ as control space. Given a Banach space X, the space $L^\infty(0,T;X^*)$ consists of all X^*-valued functions $u(t)$ which are X-weakly measurable (this means $t \to \langle u(t), u\rangle$ is measurable for all $u \in X$) and such that

$$\langle u(t), u\rangle \leq C\|u\| \quad (u \in X) \,. \tag{4.2}$$

The norm of $u(\cdot)$ in $L^\infty_w(0,T;X^*)$ is the least C that does the job[6] in (4.2). This norm makes $L^\infty_w(0,T;X^*)$ a Banach space and we have $L^\infty_w(0,T;X^*) = L^1(0,T;X)^*$[11] Theorem 7, p. 94, which is the basis of existence theory for optimal controls for (1.1) [10] Theorem 4.8.1. Of course, we must show that the trajectories take values in ℓ^0. This follows from the result below, where it is shown that the reachable spaces corresponding to both choices of control space are the same.

Lemma 4.1. *$R^\infty(T) \subset \ell^0$ is the space ℓ^∞_1 of all sequences $\{y_n\}$ such that*

$$\|\{y_n\}\|_{\infty,1} = \sup_{n\geq 1} n|y_n| < \infty\,, \tag{4.3}$$

the norm (4.3) *equivalent to $\|\cdot\|_{R^\infty(T)}$. The reachable space is the same for controls in $L^\infty(0,T;\ell^0)$ or in $L^\infty(0,T;\ell^\infty)$.*

Proof. Let $\{u_n(\cdot)\} \in L^\infty_w(0,T;\ell^\infty) \supseteq L^\infty(0,T;\ell^0)$. The equation

$$\{y_n\} = \int_0^T S(T-\sigma)\{u_n(\sigma)\}d\sigma \tag{4.4}$$

splits into the separate (and independent) equations

$$\int_0^T e^{-n(T-\sigma)}u_n(\sigma)d\sigma = y_n \quad (n = 1,2,\dots)\,. \tag{4.5}$$

Accordingly, we have

$$\begin{aligned} |y_n| &\leq \int_0^T e^{-n(T-\sigma)}|u_n(\sigma)|d\sigma \\ &\leq \frac{1-e^{-nT}}{n}\|u(\cdot)\|_{L^\infty_w(0,T;\ell^\infty)} \leq \frac{1}{n}\|u(\cdot)\|_{L^\infty_w(0,T;\ell^\infty)}\,. \end{aligned} \tag{4.6}$$

Conversely, let $\{y_n\} \in \ell^\infty_1$. Set $u(t) = \{u_n(t)\}$ with

$$u_n(t) = \begin{cases} \dfrac{ny_n}{1-e^{-1}} & T - \dfrac{T}{n} \leq t < T \\ 0 & 0 < t < T - \dfrac{T}{n} \end{cases} \tag{4.7}$$

The control $\{u_n(t)\}$ is constant in the intervals $[T - T/n, T - T/(n+1))$ and $u_n(t) = 0$ for n large enough for each $t < T$, thus $\{u_n(\cdot)\} \in L^\infty(0,T;\ell^0)$ with norm

$$\|\{u_n(\cdot)\}\|_{L^\infty(0,T;\ell^0)} \leq \frac{1}{1-e^{-1}}\|\{y_n\}\|_{\infty,1}\,. \tag{4.8}$$

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[6]If X is separable the norm of $L^\infty_w(0,T;X^*)$ is the same as the ordinary L^∞ norm; we have $\|u(\cdot)\|_{L^\infty_w(0,T;X^*)} = \text{ess sup}\ \|u(t)\|_{X^*}$. This observation applies to the separable space $X = \ell^1$ and its dual $X^* = \ell^\infty$.

We have

$$\int_0^T S(T-\sigma)u(\sigma)d\sigma = \left\{\int_0^T e^{-n(T-\sigma)}u_n(\sigma)d\sigma\right\}$$
$$= \left\{\int_0^T e^{-n\sigma}u_n(T-\sigma)d\sigma\right\} = \left\{\frac{ny_n}{1-e^{-1}}\int_0^{1/n} e^{-n\sigma}d\sigma\right\} = \{y_n\}\,. \quad (4.9)$$

Inequality (4.6) implies the first of the estimates

$$\|\{y_n\}\|_{\infty,1} \le \|\{y_n\}\|_{R^\infty(T)}\,, \quad \|\{y_n\}\|_{R^\infty(T)} \le \frac{1}{1-e^{-1}}\|\{y_n\}\|_{\infty,1}\,, \qquad (4.10)$$

and (4.8) and (4.9) imply the second. □

5. The time optimal problem

To simplify, we limit ourselves to drives from $\zeta = 0$ to targets $\bar{y} = \{\bar{y}_n\} \in R^\infty(T)$. It follows from all equations (4.5) that if the admissible control $\{u_n(t)\}$ drives 0 to $\{\bar{y}_n\}$ in time T we have

$$|\bar{y}_n| \le \int_0^T e^{-n(T-\sigma)}|u_n(\sigma)|d\sigma$$
$$\le \int_0^T e^{-n(T-\sigma)}d\sigma = \frac{1-e^{-nT}}{n} \quad (n = 1, 2, \dots)\,, \qquad (5.1)$$

so that

$$\rho = \sup_{n\ge 1}\left\{\frac{n|\bar{y}_n|}{1-e^{-nT}}\right\} \le 1\,. \qquad (5.2)$$

On the other hand, if (5.2) holds, the admissible (constant) control

$$\{u_n(t)\} = \left\{\frac{n\bar{y}_n}{1-e^{-nT}}\right\} \qquad (5.3)$$

drives 0 to $\{\bar{y}_n\}$ in time T.

Lemma 5.1. *Assume that $\rho = 1$ and that*

$$\left\{\frac{m|\bar{y}_m|}{1-e^{-mT}}\right\} = 1 = \sup_{n\ge 1}\left\{\frac{n|\bar{y}_n|}{1-e^{-nT}}\right\} \qquad (5.4)$$

for some m. Then T is the optimal driving time from 0 to $\{\bar{y}_n\}$ and we have

$$\bar{u}_m(t) = \frac{m\bar{y}_m}{1-e^{-mT}} \qquad (5.5)$$

for any optimal control $\{u_m(t)\}$.

Proof. We can drive from 0 to $\{\bar{y}_n\}$ in time T with the constant admissible control (5.3), thus the optimal time is $\le T$. On the other hand, assuming that we can drive from 0 to $\{\bar{y}_n\}$ with an admissible control $\{u_n(t)\}$ in time $T' < T$ we have

$$\frac{1-e^{-mT}}{m} = |y_m| \le \frac{1-e^{-mT'}}{m}\,,$$

a contradiction, thus T is the optimal time as claimed. Obviously, (5.5) is the only solution of equation (4.5) for $n = m$ satisfying $|u_m(t)| \le 1$. $\square$

Lemma 5.1 produces extreme nonuniqueness examples of time optimal controls. For the target$\{y_n\} = \{\delta_{mn}(1 - e^{-m\pi})/m\}$ (m fixed) Lemma 5.1 says that any control $\{\bar{u}_n(t)\}$ with

$$\bar{u}_m(t) = 1\,, \quad \bar{u}_n(t) = \alpha_n \sin nt \quad (n \ne m) \tag{5.6}$$

(which is admissible if $|\alpha_n| \le 1$) drives 0 to $\{\bar{y}_n\}$ in optimal time π. The following is a criterion for optimality different from that in Lemma 5.1.

Theorem 5.2. *Let $\{T_n\}$ be a sequence*

$$0 = T_0 < T_1 < \cdots < T_n \cdots < T\,, \quad T_n \to T \tag{5.7}$$

and assume that

$$1 \ge \frac{n|\bar{y}_n|}{1 - e^{-nT}} > \rho_n = \frac{1 - e^{-nT_n}}{1 - e^{-nT}}\,. \tag{5.8}$$

Then the constant admissible control (5.3) *drives* 0 *to* $\{\bar{y}_n\}$ *time optimally.*

Proof. Assume we have an admissible control $\{u_n(t)\}$ driving 0 to $\{\bar{y}_n\}$in time $T' < T$. Then we can also drive from0 to$\{\bar{y}_n\}$ in time[7] $T_m > T'$. By virtue of (5.1) we have

$$\rho_n \frac{1 - e^{-nT}}{n} \le |\bar{y}_n| \le \frac{1 - e^{-nT_m}}{n}$$

or, using (5.8.)

$$1 - e^{-nT_n} < |\bar{y}_n| \le 1 - e^{-nT_m}$$

which is a contradiction for $n = m$. $\square$

6. Hypersingular controls

We have $E^* = (\ell^0)^* = \ell^1$. The space $E^*_{-1} = (\ell^0)^*_{-1}$ consists of all sequences $z = \{z_n\}$ such that

$$\|z\|_{1,-1} = \|\{z_n\}\|_{1,-1} = \sum_{n=1}^{\infty} \frac{|z_n|}{n} < \infty\,. \tag{6.1}$$

If $z = \{z_n\} \in (\ell^0)^*_{-1}$ then (1.5) is automatically satisfied: in fact,

$$\begin{aligned} \int_0^T \|S(T-\sigma)^* z\|_1 d\sigma &= \int_0^T \Big(\sum_{n=1}^{\infty} |z_n| e^{-n(T-\sigma)} \Big) d\sigma \\ &= \sum_{n=1}^{\infty} \frac{|z_n|}{n} (1 - e^{-nT}) \le \sum_{n=1}^{\infty} \frac{|z_n|}{n} = \|\{z_n\}\|_{1,-1}\,, \end{aligned} \tag{6.2}$$

[7] If $u(t)$ drives 0 to $\{y_n\}$ in time $T' < T_m$ the control $v(t) = 0$ $(0 \le t \le T_m - T')$, $v(t) = u(t - (T_m - T'))$ $(T_m - T' \le t \le T_m)$ drives 0 to $\{y_m\}$ in time T_m.

so that $Z(T) = (\ell^0)^*_{-1}$. Let $\mathfrak{Z}^1$ be the set of all sequences $z = \{z_n\}$ such that $\{e^{-nt}z_n\} \in \ell^1$ for all $t > 0$. Obviously, we can extend $S(t)^*$to $\mathfrak{Z}^1$ by $S^*(t)\{z_n\} = \{e^{-nt}z_n\} \in \ell^1$, thus $\mathfrak{Z}^1$ is a multiplier space[8] according to the definition in Section 1. The weak maximum principle (1.6) with $z \in \mathfrak{Z}^1$ and $\rho = 1$ is

$$\sum_{n=1}^{\infty} z_n e^{-n(T-t)} \bar{u}_n(t) = \max_{\|\{u_n\}\|_\infty \le 1} \sum_{n=1}^{\infty} z_n e^{-n(T-t)} u_n \,, \tag{6.3}$$

(the choice of $\{u_n\}$ in ℓ^∞ instead of in ℓ^0 is a consequence of the choice of $L^\infty(0, T; \ell^\infty)$ as control space). It follows from (6.3) that

$$\bar{u}_m(t) = \operatorname{sign} z_m \tag{6.4}$$

for all m for which $z_m \neq 0$. This equality has the following curious consequence:

Lemma 6.1. *An admissible control $\{\bar{u}_n(t)\}$ satisfies the weak maximum principle if and only if $\bar{u}_m(t) = 1$ $(0 \le t \le T)$ or $\bar{u}_m(t) = -1$ $(0 \le t \le T)$ for at least one $m \ge 1$.*

Proof. It suffices to set $\{z_n\} = \{\delta_{mn}\}$. □

It follows from Lemma 6.1 that the question formulated in Section 1 (which has a negative answer in general) can be conditionally answered "yes" (in a somewhat weird way) for the control system in the last two sections; the maximum principle (1.6) under (1.7) implies that (1.6) will be satisfied as well with a multiplier z having a single nonzero coordinate.

An admissible control $\{\bar{u}_n(t)\}$ that satisfies the maximum principle with a multiplier $z \in Z(T)$ drives 0 to $\{\bar{y}_n\} = y(t, 0, \{\bar{u}_n\})$ time optimally (Section 1) thus we have

Corollary 6.2. *Let $\{\bar{u}_n(t)\}$ be an admissible control. Assume that either $\bar{u}_m(t) = 1$ $(0 \le t \le T)$ or $\bar{u}_m(t) = -1$ $(0 \le t \le T)$ for at least one $m \ge 1$. Then $\{\bar{u}_n(t)\}$ drives 0 to$\{\bar{y}_n\} = y(T, 0, \{\bar{u}_n\})$ time optimally.*

Corollary 6.3. *Hypersingular time optimal controls exist.*

Proof. We use Theorem 5.2 for an arbitrary sequence $\{T_n\}$. All controls $\{\bar{u}_n(t)\}$ constructed there are time optimal. If the first inequality (5.8) is reinforced from $\le$ to $<$ then we have $|\bar{u}_n(t)| = |\bar{u}_n| < 1$ for all n, thus by Corollary 6.2 $\{\bar{u}_n(t)\}$ cannot satisfy (1.6) for any multiplier z. Clearly, the same argument shows that $\{\bar{u}_n(t)\}$ cannot satisfy the weak maximum principle (1.8) in any subinterval $t_0 \le t \le t_1$. □

Far more irregular hypersingular controls can be constructed using the following result, whose proof can be found in [12], Exercise 8, p. 88:

Lemma 6.4. *There exists disjoint measurable sets $d, e \subset [0, T]$, $d \cup e = [0, T]$ such that the intersection of any nonempty interval $(a, b) \subseteq [0, T]$ with both d and e has positive measure.*

[8]The space $\mathfrak{Z}^1$ is in fact the largest multiplier space.

This result granted, we "deform" one of the constant hypersingular controls constructed in Corollary 6.3. We take $\epsilon_n > 0$ and define

$$\begin{cases} \tilde{u}_n(t) = \bar{u}_n(t) + \chi_d(t)\epsilon_n \\ \tilde{u}_n(t) = \bar{u}_n(t) - \chi_e(t)\epsilon_n \end{cases} \tag{6.5}$$

where $\chi_d(t)$ (resp. $\chi_d(t)$) is the characteristic function of d (resp. of e) and $\delta_n > 0$ is determined from ϵ_n by the equation

$$\delta_n \int_e e^{-n(T-\sigma)} d\sigma = \epsilon_n \int_d e^{-n(T-\sigma)} d\sigma \,,$$

which implies

$$\int_d e^{-n(T-\sigma)} \tilde{u}_n(\sigma) d\sigma = \int_0^T e^{-n(T-\sigma)} \bar{u}_n(\sigma) d\sigma \quad (n = 1, 2. \dots) \,, \tag{6.6}$$

so that $\{\tilde{u}_n(t)\}$ and $\{\bar{u}_n(t)\}$ drive 0 to the same target in time T. For ϵ_n small enough, δ_n will be so small that $\{\tilde{u}_n(t)\}$ is admissible. Thus, $\{\tilde{u}_n(t)\}$ is time optimal, and no $\tilde{u}_n(t)$ is constant in subintervals, thus $\tilde{u}_n(t)$ is hypersingular as well. At this point, it is natural to ask *whether there exist targets* $\{\bar{y}_n\}$ *such that* 0 *can be driven to* $\{\bar{y}_n\}$ *time optimally only by a hypersingular control.* We don't know the answer to this question. If "hypersingular" is replaced by "singular", however, the answer is an easy "yes".

Corollary 6.5. *There exist targets* $\{\bar{y}_n\}$ *such that* 0 *can be driven time optimally to* $\{\bar{y}_n\}$ *only by a singular control.*

Proof. Just a rehash of Corollary 6.3. Let $\{\bar{y}_n\}$, $\{\bar{u}_n(t)\}$ be the target and the control in Corollary 6.3, with the first inequality (5.8) reinforced to $<$. Assume that 0 can be driven to$\{\bar{y}_n\}$ by another admissible control $\{\bar{v}(t)\}$ satisfying the maximum principle (1.6) with an arbitrary multiplier $z \in \mathfrak{Z}^1$. Then, by Lemma 6.1 there exists $m \geq 1$ such that $\bar{v}_m(t) = \pm 1$. Since both controls drive to the same target, all equations (4.5) have to be satisfied for both $\{\bar{u}_n(t)\}$ and $\{\bar{v}_n(t)\}$. In particular,

$$\int_0^T e^{-m(T-\sigma)} \bar{u}_m(\sigma) d\sigma = \int_0^T e^{-m(T-\sigma)} \bar{v}_m(\sigma) d\sigma$$

which is impossible since $\bar{u}_m(t)$ is a constant < 1. □

We check that "singular" cannot be upgraded to "hypersingular" in Corollary 6.5 making use of yet another deformation of the constant control $\{\bar{u}_n(t)\} = \{\bar{u}_n\}$ in Corollary 6.3. We select a sequence $\{\epsilon_n\}$, $\epsilon_n > 0$ from the equation

$$\begin{aligned} \frac{e^{-n\epsilon_n} - e^{-nT}}{n} &= \int_{\epsilon_n}^T e^{-n(T-\sigma)} d\sigma = \operatorname{sign} \bar{u}_n \int_0^T e^{-n(T-\sigma)} u_n d\sigma \\ &= |\bar{y}_n| = \frac{n|\bar{y}_n|}{1 - e^{-nT}} \cdot \frac{1 - e^{-nT}}{n} > \rho_n \frac{1 - e^{-nT}}{n} \,, \end{aligned} \tag{6.7}$$

which implies $e^{-n\epsilon_n} > \rho_n + (1-\rho_n)e^{-nT} > \rho_n$, thus $\epsilon_n \to 0$. It follows from (6.7) that the control $\{\bar{v}_n(t)\}$ defined by

$$\bar{v}_n(t) = \begin{cases} 0 & 0 \le t \le \epsilon_n \\ \operatorname{sign} z_n & \epsilon_n < t \le T \end{cases} \tag{6.8}$$

drives 0 to $\{\bar{y}_n\}$ in time T, accordingly it is time optimal. Lemma 6.1 says that $\{\bar{v}(t)\}$ cannot satisfy the weak maximum principle in $0 \le t \le T$, thus $\{\bar{v}_n(t)\}$ is singular. It is not hypersingular; in fact, it is "nearly regular" since, again by Lemma 6.1 it satisfies the maximum principle in each interval $\epsilon_n \le t \le T$.

7. Singular functionals

Since $\overline{D(A)} \ne R^\infty(T)$, singular functionals exist. We note as a curiosity that singular functionals can be "characterized" in this example although the characterization is far from constructive. We have $E^* = (\ell^0)^* = \ell^1$, $(\ell^1)^* = \ell^\infty$, and the construction requires the dual of ℓ^∞. Set $Z = \{1, 2, \dots\}$. A measure μ defined in all subsets of Z is *finitely additive* if $\mu(e_1 \cup e_2) = \mu(e_1) + \mu(e_2)$ for any two disjoint subsets of Z. The *absolute value* $|\mu|$ of μ is defined for an arbitrary $e \subseteq Z$ by $|\mu(e)| = \sup \sum_j |\mu(e_j)|$ the sup taken over all finite partitions $\{e_j\}$ of e; μ is a finitely additive measure as well. We say that μ is *bounded* if $|\mu|(Z) < \infty$. The integral $\int_Z y_n \mu(dn)$ can be defined using the standard theory in [2], Chapter III which is designed to accommodate finitely additive measures. It follows from the definition and that of the absolute value $|\mu|$ that

$$\left| \int_Z y_n \mu(dn) \right| \le \|\{y_n\}\|_\infty |\mu|(Z)\,.$$

We denote by $\Sigma(Z)$ the space of all finitely additive bounded measures in Z equipped with the norm $\|\mu\| = |\mu|(Z)$. The following result is a particular case of [2], Theorem 1, p. 258:

Theorem 7.1. *We have $(\ell^\infty)^* = \Sigma(Z)$ algebraically and metrically, the duality given by*

$$\langle \mu, \{y_n\} \rangle = \int_Z y_n \mu(dn)\,.$$

Corollary 7.2. *We have $R^\infty(T)^\star = \Sigma(Z)$, algebraically and metrically, the duality given by*

$$\langle \mu, \{y_n\} \rangle = \int_Z n y_n \mu(dn)\,.$$

Proof. Immediate consequence of Theorem 7.1 and of the identification of $R^\infty(T)$ in Lemma 4.1. $\square$

A finitely additive measure μ in Z is *locally null* if $\mu(\{n\}) = 0$ for $n = 1, 2, \ldots,$ and it is easily seen that $\mu \in \Sigma(Z)$ is locally null if and only if

$$\int_Z y_n \mu(dn) = 0 \quad (\{y_n\} \in \ell^0)\,.$$

The space of all locally null functionals is called $\Sigma_0(Z)$. An "example" of a locally null measure is

$$\mu(e) = \operatorname*{LIM}_{n\to\infty} \chi_e(n)$$

where $\operatorname*{LIM}_{n\to\infty}$ is a Banach limit [1], p. 34, [2], p. 73 and χ_e is the characteristic function of e. The space $\mathfrak{R}(T)$ of all regular functionals can be identified with ℓ^1 with duality $\langle\!\langle \xi_z, y \rangle\!\rangle = \sum_{n=1}^{\infty} n z_n y_n$ and the L^1 direct sum (1.12) has the following interpretation:

Lemma 7.3. *We have $R^\infty(T)^\star = \Sigma(Z) = \ell^1 \oplus \Sigma_0(Z)$ with L^1 direct sum.*

8. Conclusions and new questions

This paper answers two questions that were open by the time [10] was written. However, the solutions beg for new questions. It is clear that the example given in Theorem 3.6 on targets in $D(A)$ that require costates $S(T-t)^* z$ with $z \notin E^*$ depends on the extreme nonsmoothness of the space ℓ^1, which makes natural to ask: *is it possible to construct a similar example in a smooth space* (*for instance, in a Hilbert space*)?

In a similar vein, the example of time optimal hypersingular control provided in Corollary 6.3 can hardly be considered definitive. The main objection is that, due to the extreme nonuniquess of optimal controls (itself a consequence of the nonsmoothness of the space ℓ^0) all we can show is, among the controls that drive 0 to a given target $\{\bar{y}_n\}$ there are hypersingular controls. The strict convexity condition $\|x\| = \|y\| = 1\,,\ x \neq y \implies \|x + y\| < 2$ implies uniqueness of time optimal controls, thus we may ask: *are there examples of hypersingular time optimal controls in a strictly convex space* (*in particular in a Hilbert space*)? A "yes" answer to this question would imply that there exists a system (1.1) in a strictly convex space E and a target $\bar{y} \in E$ such that 0 can be driven to $\bar{y}$ *only* by a hypersingular time optimal control, a result stronger than Corollary 6.3, since the time optimal drive from 0 to $\{\bar{y}_n\}$ there can be performed by a hypersingular control but also by then early regular control (6.8).

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On the Motion of Several Rigid Bodies in a Viscous Multipolar Fluid

Eduard Feireisl and Šárka Nečasová

1. Introduction

The mathematical theory of viscous multipolar fluids, based on the general ideas of Green and Rivlin [8], was proposed by Nečas and Šilhavý [17] (see also Nečas et al. [15], [16] for relevant existence theory) in order to develop a general framework for studying viscous fluids and to present a suitable alternative to the boundary layer theory (see Bellout et al. [1]). The theory is compatible with the basic principles of thermodynamics as well as with the principle of material frame indifference. The present paper is concerned with the mathematical description of the motion of one or several rigid bodies immersed in a viscous multipolar fluid. The principal and very natural idea behind the analysis presented below is the fact that the dissipation of mechanical energy, being much stronger than for classical newtonian fluids, yields better estimates on the gradient of the velocity field, in particular, the streamlines are well defined, which seems crucial for this class of problems partially formulated in terms of the Lagrangian coordinate system.

1.1. Bodies and motions

From the mathematical viewpoint, a *rigid body* can be identified with a connected compact subset $\overline{S}$ of the Euclidean space R^3, the motion of which is represented as a mapping $\eta : (0,T) \times R^3 \to R^3$, where

$$\eta(t,\cdot) : R^3 \to R^3 \text{ is an isometry} \tag{1.1}$$

for any *time* $t \in (0,T)$. Throughout the whole text, we adopt the Eulerian (spatial) description of motion, where the coordinate system is attached to a fixed region of the physical space currently occupied by the fluid. The *place* $\mathbf{x}$ and the *time* $t \in (0,T)$ play the role of independent variables.

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As the mappings $\eta(t,\cdot)$ are isometries, we can write

$$\eta(t,\mathbf{x}) = \mathbf{X}_g(t) + \mathbb{O}(t)(\mathbf{x} - \mathbf{X}_g(0)),$$

where $\mathbf{X}_g$ stands for the position of the center of mass at a time t, and $\mathbb{O}(t)$ is a matrix satisfying $\mathbb{O}^T\mathbb{O} = \mathbb{I}$. Assuming the motion to be absolutely continuous with respect to time, we introduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}_g = \mathbf{U}_g \text{ – the translation velocity,} \tag{1.2}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{O}(t)\mathbb{O}^T(t) = \mathbb{Q}(t) \text{ – the angular velocity.} \tag{1.3}$$

Accordingly, the solid velocity in the Eulerian coordinate system can be written in the form

$$\mathbf{u}^S(t,\mathbf{x}) = \frac{\partial\eta}{\partial t}(t,\eta^{-1}(t,\mathbf{x})) = \mathbf{U}_g(t) + \mathbb{Q}(t)(\mathbf{x} - \mathbf{X}_g(t)),$$

where $\mathbf{X}_g$ is determined through (1.2).

The total force $\mathbf{F}^S$ acting on the body $\overline{S}$ can be written as a sum of the *body force* and the *contact force*, more specifically,

$$\mathbf{F}^S(t) = \int_{\overline{S}(t)} \varrho^S\mathbf{g}^S \ \mathrm{d}\mathbf{x} + \int_{\partial\overline{S}(t)} \mathbb{T}\mathbf{n} \ \mathrm{d}\sigma,$$

where $\mathbb{T}$ denotes the Cauchy stress, $\mathbf{g}^S$ is the specific body force, and

$$\overline{S}(t) = \eta(t,\overline{S}).$$

Thus Newton's second law gives rise to

$$m\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{U}_g(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\overline{S}(t)} \varrho^S\mathbf{u}^S \ \mathrm{d}\mathbf{x} = \int_{\overline{S}(t)} \varrho^S\mathbf{g}^S \ \mathrm{d}\mathbf{x} + \int_{\partial\overline{S}(t)} \mathbb{T}\mathbf{n} \ \mathrm{d}\sigma, \tag{1.4}$$

where m denotes the total mass of the body.

On the other hand, as the angular velocity $\mathbb{Q}$ is skew-symmetric, there exists a vector ω such that

$$\mathbb{Q}(t)(\mathbf{x} - \mathbf{X}_g) = \omega(t) \times (\mathbf{x} - \mathbf{X}_g).$$

The balance of moment of momentum reads

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{J}\omega) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\overline{S}(t)} \varrho^S(\mathbf{x} - \mathbf{X}_g) \times \mathbf{u}^S \ \mathrm{d}\mathbf{x} = \tag{1.5}$$

$$\int_{\partial\overline{S}(t)} (\mathbf{x} - \mathbf{X}_g) \times \mathbb{T}\mathbf{n} \ \mathrm{d}\sigma + \int_{\overline{S}(t)} \varrho^S(\mathbf{x} - \mathbf{X}_g) \times \mathbf{g}^S \ \mathrm{d}\mathbf{x},$$

where $\mathbb{J}$ is the *inertial tensor* that can be identified through formula

$$\mathbb{J}\mathbf{a}\cdot\mathbf{b} = \int_{\overline{S}(t)} \varrho^S(\mathbf{a} \times (\mathbf{x} - \mathbf{X}_g)) \cdot (\mathbf{b} \times (\mathbf{x} - \mathbf{X}_g)) \ \mathrm{d}\mathbf{x}.$$

Equations (1.4), (1.5) determine completely the motion of the rigid body initially occupying the spatial domain $\overline{S}$.

1.2. The fluid motion

In what follows, we shall assume that the *state* of the fluid is completely determined by its density ϱ^f and the velocity $\mathbf{u}^f$ satisfying the standard mass and momentum balance equations:

$$\partial \varrho^f + \mathrm{div}_x(\varrho^f \mathbf{u}^f) = 0, \tag{1.6}$$

$$\partial_t(\varrho^f \mathbf{u}^f) + \mathrm{div}_x(\varrho^f \mathbf{u}^f \otimes \mathbf{u}^f) + \nabla_x p = \mathrm{div}_x \mathbb{S} + \varrho^f \mathbf{g}^f, \tag{1.7}$$

where the symbol p denotes the pressure, $\mathbf{g}^f$ is the specific body force, and $\mathbb{S}$ denotes the viscous stress tensor related to the total stress through Stokes' law:

$$\mathbb{T} = \mathbb{S} - p\mathbb{I}. \tag{1.8}$$

In the present paper, the effect of the temperature on the motion will be ignored. On the other hand, we consider a general *compressible* fluid so that the state equation relates the pressure to the fluid density through an empirical formula

$$p = p(\varrho^f). \tag{1.9}$$

1.3. Viscosity

The heart of the theory of multipolar fluids lies in a particular choice of constitutive equations relating the fluid stress expressed through $\mathbb{S}$ to the symmetric component of the velocity gradient. Very roughly indeed, one can say that, in contrast to the classical theory of newtonian fluids, the stress tensor depends on higher-order gradients of the velocity field. This piece of information is sufficient in order to obtain *a priori* estimates yielding, in particular, strong compactness of the density ϱ.

Following the seminal paper by Nečas and Šilhavý [17] we assume the viscous stress tensor $\mathbb{S}$ to be given as

$$\mathbb{S}[\mathbf{u}] = \sum_{n=0}^{k-1}(-1)^n \Delta^n \Big[\mu_n\Big(\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^t - \frac{2}{3}\mathrm{div}_x \mathbf{u}\ \mathbb{I}\Big) + \zeta_n \mathrm{div}_x \mathbf{u}\mathbb{I}\Big],\ \mathbf{u} = \mathbf{u}^f, \tag{1.10}$$

where μ_n, ζ_n are (constant) viscosity coefficients, and the symbol Δ stands for the standard Laplace operator. Accordingly, one can speak about a k-polar fluid, the classical newtonian fluids being identified as monopolar with $k = 1$.

As expected, the presence of higher-order viscosities provides very strong *a priori* estimates on the velocity gradient, in particular, the streamlines are well defined allowing for the Lagrangian description of motion. Note that the theory of multipolar fluids requires additional "higher-order" stresses to be introduced in the energy equation in order to comply with the second law of thermodynamics (see Nečas and Šilhavý [17]).

1.4. Boundary conditions

A proper choice of the boundary conditions represents one of the most delicate issues of the present theory. We adopt the hypothesis of complete adherence of the fluid to the boundaries of rigid objects yielding the *full-stick* boundary conditions

$$D_x^j \mathbf{u}^f = D_x^j \mathbf{u}^S,\ j = 0, \dots, k-1,\ \text{on } \partial\overline{S}(t), \tag{1.11}$$

and

$$D_x^j \mathbf{u}^f = 0, \; j = 0, \ldots, k-1, \text{ on } \partial\Omega \tag{1.12}$$

provided the flow is confined to a fixed spatial domain $\Omega \subset R^3$.

Here, the symbol D_x^j denotes the vector of all spatial derivatives of order j, and conditions (1.11), (1.12) are *different* from those considered in [17], the latter being of "Neumann-type". Clearly, the boundary conditions depend on the physical properties of a given fluid and as such must be determined by experiments. In the present setting, the flowing rigid objects are supposed to be "sticky", in particular, they may be thought of as integral parts of the surrounding fluid of extremely high viscosity.

1.5. Global-in-time solutions and collisions of rigid objects

The motion of one or several rigid bodies in a viscous fluid has been a topic of numerous theoretical studies. Desjardins and Esteban [3] establish the existence of local-in-time solutions for incompressible newtonian fluids, where "local" is to be understood "up to the first collision of two rigid objects" if $\Omega \subset R^2$, or "up to the blow-up of the velocity gradient in a certain Sobolev norm" in the case $\Omega \subset R^3$. Similar methods are proposed in [4] in order to study both compressible and incompressible case. Similar existence results "up to the first collision" were obtained by Conca et al. [2], Gunzburger et al. [9], Hoffmann and Starovoitov [12], among others.

As we have seen, the problem of existence or rather non-existence of collisions is important not only because of its practical implications, but also from the purely theoretical point of view. To the best of our knowledge, this issue remains largely open even for a two-dimensional physical domain Ω. In the 2-D case, however, there is a remarkable result by San Martin et al. [18] stating, in particular, that possible collisions, if any, must be "smooth", that means, with zero relative velocities. Another strong evidence of absence of collisions in the 2-D geometry is provided independently by Hesla [10] and Hillairet [11]. These authors show, roughly speaking, that newtonian viscosity is strong enough to prevent collisions provided the rigid objects are discs. As already pointed out, the question is completely open for a linearly viscous fluid in the realistic situation $\Omega \subset R^3$ (for partial results see Starovoitov [20]).

The main objective of the present paper is to establish the existence of global-in-time solutions for problem (1.1–1.12) provided $k \geq 3$. In particular, we show that collisions cannot occur in a finite time unless they were already included in the initial data. The paper is organized as follows. In Section 2, we introduce a variational (weak) formulation of the problem in the spirit of Galdi [7], Hoffmann and Starovoitov [12], Serre [19]. The main results concerning global-in-time solutions are stated in Section 3. Suitable approximate solutions are constructed in Section 4 by means of a scheme similar to that used in [6]. In particular, the method of construction is based on the idea of San Martin et al. [18], where the rigid objects are approximated by a fluid of large viscosity. Such an approach is of course intimately related to our choice of the boundary conditions specified through (1.11), (1.12).

In Section 5, we derive suitable uniform estimates on the sequence of approximate solutions based on strong dissipation of the kinetic energy for multipolar fluids. In particular, the velocity field is uniformly Lipschitz continuous which facilitates the subsequent analysis considerably. Using the uniform energy estimates, we pass to the limit in the sequence of approximate solutions in order to obtain a suitable variational solution of the original problem (see Section 6).

2. Variational formulation

Similarly to a major part of the reference material mentioned above, our approach is based on a suitable variational formulation of the problem. In what follows, we shall assume that $\Omega \subset R^3$ is a bounded domain with smooth (C^∞) boundary.

2.1. Kinematics of the rigid bodies

The reference position S^i, $i = 1, \ldots, m$, of the ith rigid body is a bounded domain in R^3 with smooth boundary. The motions are described through affine isometries $\eta^i(t, \cdot) : R^3 \to R^3$, $i = 1, \ldots, m$, that are absolutely continuous as functions of the time $t \in [0, T]$. Furthermore, we set

$$S^i(t) = \eta^i(t, S^i),\ \ i = 1, \ldots, m,$$

and

$$Q^S = \Big\{(t, \mathbf{x}) \ \Big|\ t \in (0, T),\ \mathbf{x} \in \cup_{i=1}^m S^i(t)\Big\}.$$

2.2. Conservation of mass

The solid densities ϱ^{S^i} as well as the fluid density ϱ^f can be extended to be zero outside $S^i(t)$, and the fluid region $Q^f = ((0, T) \times \Omega) \setminus \overline{Q}^S$, respectively. In a similar way, we introduce a "global" velocity field $\mathbf{u}$,

$$\mathbf{u}(t, \mathbf{x}) = \begin{cases} \mathbf{u}^{S^i} \text{ for } t \in (0, T),\ \mathbf{x} \in \overline{S}^i(t), \\ \mathbf{u}^f \text{ for } t \in (0, T),\ \mathbf{x} \in \Omega \setminus \cup_{i=1}^m \overline{S}^i(t), \\ 0 \text{ for } t \in (0, T),\ \mathbf{x} \in R^3 \setminus \Omega. \end{cases}$$

The physical principle of mass conservation can be expressed through continuity equation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \text{ in } \mathcal{D}'((0, T) \times R^3). \tag{2.1}$$

2.3. Momentum equation

For a given family of motions η^i, $i = 1, \ldots, m$, we introduce the set of admissible velocity fields

$$V_{\mathrm{adm}}(t) = \{\mathbf{w} \in C^k(R^3; R^3) \mid \mathbf{w} \equiv 0 \text{ on } R^3 \setminus \Omega,\ \nabla_x \mathbf{w} + \nabla_x \mathbf{w}^t \equiv 0 \text{ on } \cup_{i=1}^m \eta^i(t, S^i)\}. \tag{2.2}$$

Following Nečas [14], we introduce a bilinear form $((\cdot,\cdot))$ associated to the stress tensor $\mathbb{S}$ given by (1.10), specifically,

$$((\mathbf{v},\mathbf{w})) = \int_\Omega \mathbb{S}[\mathbf{v}] : \nabla_x \mathbf{w} \ \mathrm{d}x \text{ for any } \mathbf{v},\ \mathbf{w} \in \mathcal{D}(\Omega; R^3). \tag{2.3}$$

Under the natural hypothesis

$$\mu_n,\ \eta_n \geq 0,\ \mu_{k-1} > 0,\ \eta_{k-1} > 0, \tag{2.4}$$

it is straightforward to see that $((\cdot,\cdot))$ can be extended to a scalar product on the Sobolev space $W_0^{k,2}(\Omega; R^3)$ defined as a completion of the set of compactly supported smooth functions with respect to the norm

$$\|\mathbf{v}\|^2_{W_0^{k,2}(\Omega;R^3)} = \sum_{n=0}^{k} \int_\Omega |D^n_x \mathbf{v}|^2 \ \mathrm{d}x$$

(see Nečas [14]).

Assuming continuity of the stresses on the boundaries of rigid objets, we can reformulate equations (1.4), (1.5), (1.7) in terms of ϱ and $\mathbf{u}$ as integral identity

$$\int_0^T \int_{R^3} \Big(\varrho \mathbf{u} \cdot \partial_t \mathbf{w} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \mathbf{w} + p \ \mathrm{div}_x \mathbf{w} \Big) \mathrm{d}\mathbf{x} \ \mathrm{d}t = \tag{2.5}$$

$$\int_0^T ((\mathbf{u},\mathbf{w})) \ \mathrm{d}t - \int_0^T \int_{R^3} \varrho \mathbf{g} \cdot \mathbf{w} \ \mathrm{d}\mathbf{x} \ \mathrm{d}t - \int_{R^3} \varrho_0 \mathbf{u}_0 \cdot \mathbf{w}(0) \ \mathrm{d}\mathbf{x}$$

to be satisfied for any test function

$$\mathbf{w} \in C^1([0,T]; C^k(R^3)),\ \mathbf{w}(T) = 0,\ \mathbf{w}(t) \in V_{\mathrm{adm}}(t) \text{ for any } t \in [0,T]. \tag{2.6}$$

Note that (2.5) includes the initial conditions

$$(\varrho \mathbf{u})(0,\cdot) = \varrho_0 \mathbf{u}_0. \tag{2.7}$$

2.4. Compatibility of the "global" velocity with rigid motions

We shall say that a velocity field $\mathbf{u}$ is compatible with the family of rigid motions η^i, $i = 1,\dots,m$ provided

$$\mathbf{u}(t,\mathbf{x}) = \mathbf{u}^{S^i}(t,\mathbf{x}) = \frac{\partial \eta^i}{\partial t}(t, (\eta^i)^{-1}(t,\mathbf{x})) \text{ for all } \mathbf{x} \in \eta^i(t,S^i),\ i = 1,\dots,m. \tag{2.8}$$

Relation (2.8) is to be satisfied for any $t \in [0,T]$.

2.5. Energy inequality

The velocity field associated to a multipolar fluid is expected regular because of the strong kinetic energy dissipation in the high velocity gradient regime. Indeed taking (formally) a test function $\mathbf{w} = -\psi(t)\mathbf{u}$, in (2.5) we obtain energy inequality

$$E(t_2) + \int_{t_1}^{t_2} ((\mathbf{u},\mathbf{u})) \ \mathrm{d}t \leq E(t_1) + \int_{t_1}^{t_2} \int_\Omega \varrho \mathbf{g} \cdot \mathbf{u} \ \mathrm{d}x \tag{2.9}$$

for any $0 \leq t_1 \leq t_2 \leq T$, with

$$E(t) = \int_\Omega \Big(\frac{1}{2}\varrho |\mathbf{u}|^2 + \varrho P(\varrho) \Big) \,\mathrm{d}x, \tag{2.10}$$

where P is related to the pressure p through

$$P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \,\mathrm{d}z. \tag{2.11}$$

2.6. Weak (variational) solutions

Having collected all the preliminary material, we are in a position to introduce the concept of weak solution to our problem referred to hereafter as **problem (P)**.

Definition 2.1. *Let the initial distribution of the density and the velocity field be determined through given functions ϱ_0 and $\mathbf{u}_0$, respectively; the initial position of the rigid bodies being $S^i \subset \Omega$, $i = 1, \dots, m$. We say that a family ϱ, $\mathbf{u}$, η^i, $i = 1, \dots, m$, represent a variational solution of* **problem (P)** *on a time interval $(0,T)$ provided the following conditions are satisfied:*

- *The density ϱ is a non-negative bounded function, the velocity field $\mathbf{u}$ belongs to the space $L^\infty(0,T; L^2(\Omega; R^3)) \cap L^2(0,T; W_0^{k,2}(\Omega; R^3))$, and they satisfy energy inequality* (2.9) *for $t_1 = 0$ and a.a. $t_2 \in (0,T)$, with*
$$E(0) = E_0 = \int_\Omega \Big(\frac{1}{2}\varrho_0 |\mathbf{u}_0|^2 + \varrho_0 P(\varrho_0) \Big) \,\mathrm{d}x.$$
- *We have $\varrho \in C([0,T]; L^1(\Omega))$, $\varrho(0) = \varrho_0$, and continuity equation* (2.1) *holds on $(0,T) \times R^3$ provided ϱ and $\mathbf{u}$ were extended to be zero outside Ω.*
- *Momentum equation* (*the integral identity*) (2.5) *holds for any admissible test function $\mathbf{w}$ satisfying* (2.6).
- *The mappings η^i, $i = 1, \dots, m$ are affine isometries of R^3 compatible with the velocity field $\mathbf{u}$ in the sense of* (2.8).

3. Global existence – main results

Our main goal is to prove the following existence result.

Theorem 3.1. *Let the viscous stress tensor $\mathbb{S}$ be given by* (1.10), *with $k \geq 3$. Let $\Omega \subset R^3$, $S^i \subset R^3$, $i = 1, \dots, m$ be a family of bounded domains with boundaries of class C^∞ such that*

$$\overline{S}^i \cap \overline{S}^j = \emptyset \text{ for } i \neq j,\ \overline{S}^i \subset \Omega \text{ for } i = 1, \dots, m. \tag{3.1}$$

Furthermore, assume that ϱ_0 is a measurable function such that

$$0 < \underline{\varrho} \leq \varrho_0(x) \leq \overline{\varrho} \text{ for a.a. } x \in \Omega, \tag{3.2}$$

and that

$$\mathbf{u}_0 \in W_0^{k,2}(\Omega; R^3) \cap V_{\mathrm{adm}}(0). \tag{3.3}$$

Finally, let $\mathbf{g} \in L^\infty(\Omega; R^3)$, and $p \in C[0,\infty)$ – a non-decreasing function be given.

Then **problem (P)** *admits a variational solution* ϱ, $\mathbf{u}$, η^i, $i = 1, \dots, m$, *in the sense of Definition 2.1 on an arbitrary time interval* $(0, T)$. *Moreover, we have*

$$\overline{S}^i(t) \cap \overline{S}^j(t) = \emptyset \text{ for } i \neq j, \ \overline{S}^i(t) \subset \Omega \text{ for } i = 1, \dots, m \tag{3.4}$$

for any $t \in [0, T]$, *that means, the motion is smooth without any collision of two or several rigid objects in a finite time.*

As already pointed out several times, the main ingredient of the proof are strong dissipation estimates resulting from energy inequality (2.9). Since $k \geq 3$, the velocity field $\mathbf{u}$ is *a priori* bounded in the space $L^2(0, T; C_0^1(\overline{\Omega}))$, in particular, the streamlines (characteristics) can be identified with the unique solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}(t, \mathbf{x}) = \mathbf{u}(t, \mathbf{X}(t, \mathbf{x})), \ t > 0, \ \mathbf{X}(0, \mathbf{x}) = \mathbf{x}. \tag{3.5}$$

Accordingly, the (unique) weak solution ϱ of (2.1) satisfying $\varrho(0) = \varrho_0$ is given by formula

$$\varrho(t, \mathbf{X}(t, \mathbf{x})) = \varrho_0(x) \exp\Big(- \int_0^t \mathrm{div}_x \mathbf{u}(s, \mathbf{X}(s, \mathbf{x})) \ \mathrm{d}s\Big), \ \mathbf{x} \in R^3. \tag{3.6}$$

Note that, in the absence of collisions, one can deduce from (2.5) that

$$\partial_t(\varrho\mathbf{u}) \in L^2(0, T; W^{-k,2}(\Omega; R^3)), \tag{3.7}$$

which, combined with (2.9), gives rise to

$$\mathbf{u} \in C([0, T]; L^2(\Omega; R^3)). \tag{3.8}$$

In particular, formula (3.6) makes sense for any weak solution in the sense of Definition 2.1.

The rest of the paper is devoted to the proof of Theorem 3.1.

4. Approximate problems

Let $\{\mathbf{v}_n\}_{n=1}^{\infty} \subset \mathcal{D}(\Omega; R^3)$ be a basis of the Hilbert space $W_0^{k,2}(\Omega; R^3)$. Following San Martin et al. [18] we introduce the approximate **problem (P)**$_{n,\varepsilon}$:

$$\int_0^T \int_{R^3} \Big(\varrho_{n,\varepsilon}\mathbf{u}_{n,\varepsilon} \cdot \partial_t \mathbf{w} + \varrho_{n,\varepsilon}\mathbf{u}_{n,\varepsilon} \otimes \mathbf{u}_{n,\varepsilon} : \nabla_x \mathbf{w} + p(\varrho_{n,\varepsilon}) \ \mathrm{div}_x \mathbf{w}\Big) \mathrm{d}\mathbf{x} \ \mathrm{d}t = \tag{4.1}$$

$$\int_0^T \int_\Omega M_\varepsilon(\chi_{n,\varepsilon})[\nabla_x \mathbf{u}_{n,\varepsilon} + \nabla_x^t \mathbf{u}_{n,\varepsilon}] : [\nabla_x \mathbf{w} + \nabla_x^t \mathbf{w}] \ \mathrm{d}x \ \mathrm{d}t +$$

$$\int_0^T ((\mathbf{u}_{n,\varepsilon}, \mathbf{w})) \ \mathrm{d}t - \int_0^T \int_{R^3} \varrho_{n,\varepsilon}\mathbf{g} \cdot \mathbf{w} \ \mathrm{d}\mathbf{x} \ \mathrm{d}t - \int_{R^3} \varrho_0 \mathbf{u}_0 \cdot \mathbf{w}(0) \ \mathrm{d}\mathbf{x}$$

to be satisfied for any test function

$$\mathbf{w} \in C^1([0, T]; X_n), \ \mathbf{w}(T) = 0, X_n = \mathrm{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}. \tag{4.2}$$

Here, $\varrho_{n,\varepsilon}$ is determined through formula

$$\varrho_{n,\varepsilon}(t, \mathbf{X}_{n,\varepsilon}(t,\mathbf{x})) = \varrho_0(x) \exp\Big(- \int_0^t \operatorname{div}_x \mathbf{u}_{n,\varepsilon}(s, \mathbf{X}_{n,\varepsilon}(s,\mathbf{x}))\ \mathrm{d}s\Big),\ \mathbf{x} \in R^3 \tag{4.3}$$

while

$$\chi_{n,\varepsilon}(t, \mathbf{X}_{n,\varepsilon}(t,\mathbf{x})) = \chi_{0,\varepsilon}(\mathbf{x}) \geq 0, \tag{4.4}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}_{n,\varepsilon}(t,\mathbf{x}) = \mathbf{u}_{n,\varepsilon}(t, \mathbf{X}_{n,\varepsilon}(t,\mathbf{x})),\ t > 0,\ \mathbf{X}_{n,\varepsilon}(0,\mathbf{x}) = \mathbf{x}. \tag{4.5}$$

Note that $\chi_{n,\varepsilon}$ is the unique distributional solution of the equation

$$\partial_t \chi_{n,\varepsilon} + \mathbf{u}_{n,\varepsilon} \cdot \nabla_x \chi_{n,\varepsilon} = 0,\ \chi_{n,\varepsilon}(0) = \chi_{0,\varepsilon}$$

(cf. DiPerna and Lions [5]).

The functions M_ε belong to the class $C^1[0,\infty) \cap BC[0,\infty)$ for any fixed $\varepsilon > 0$.

Problem (P)$_{n,\varepsilon}$ can be solved by means of the standard fixed-point argument used in [13, Section 2], the presence of the additional viscosity coefficient M_ε requiring only minor modifications.

5. Uniform estimates

Similarly to Section 2.5, one can show that the approximate solutions $\varrho_{n,\varepsilon}$, $\mathbf{u}_{n,\varepsilon}$ satisfy energy equality

$$E_{n,\varepsilon}(t_2) + \int_{t_1}^{t_2} \int_\Omega M_\varepsilon(\chi_{n,\varepsilon}) |\nabla_x \mathbf{u}_{n,\varepsilon} + \nabla_x^t \mathbf{u}_{n,\varepsilon}|^2 \ \mathrm{d}x\ \mathrm{d}t + \int_{t_1}^{t_2} ((\mathbf{u}_{n,\varepsilon}, \mathbf{u}_{n,\varepsilon}))\ \mathrm{d}t = \tag{5.1}$$

$$E_{n,\varepsilon}(t_1) + \int_{t_1}^{t_2} \int_\Omega \varrho_{n,\varepsilon} \mathbf{g} \cdot \mathbf{u}_{n,\varepsilon}\ \mathrm{d}x$$

for any $0 \leq t_1 \leq t_2 \leq T$, with

$$E_{n,\varepsilon}(t) = \int_\Omega \Big(\frac{1}{2} \varrho_{n,\varepsilon} |\mathbf{u}_{n,\varepsilon}|^2 + \varrho_{n,\varepsilon} P(\varrho_{n,\varepsilon}) \Big)\ \mathrm{d}x, \tag{5.2}$$

$$E_{n,\varepsilon}(0) = E_0 = \int_\Omega \Big(\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + \varrho_0 P(\varrho_0) \Big)\ \mathrm{d}x. \tag{5.3}$$

Under the hypotheses of Theorem 3.1, it is a routine matter to check that (5.1–5.3) give rise to uniform estimates:

$$\{\mathbf{u}_{n,\varepsilon}\}_{n,\varepsilon} \text{ bounded in } L^2(0,T; W_0^{k,2}(\Omega; R^3)), \tag{5.4}$$

$$\begin{aligned} \underline{\varrho} \exp\Big(- T \|\operatorname{div}_x \mathbf{u}_{n,\varepsilon}\|_{L^1(0,T;L^\infty(\Omega))}\Big) &\leq \varrho_{n,\varepsilon}(t,\mathbf{x}) \\ &\leq \overline{\varrho} \exp\Big(T \|\operatorname{div}_x \mathbf{u}_{n,\varepsilon}\|_{L^1(0,T;L^\infty(\Omega))}\Big) \end{aligned} \tag{5.5}$$

for a.a. $t \in (0,T)$, $\mathbf{x} \in \Omega$, and

$$\{\mathbf{u}_{n,\varepsilon}\}_{n,\varepsilon} \text{ bounded in } L^\infty(0,T; L^2(\Omega; R^3)). \tag{5.6}$$

In addition,

$$\int_0^T \int_\Omega M_\varepsilon(\chi_{n,\varepsilon}) |\nabla_x \mathbf{u}_{n,\varepsilon} + \nabla_x^t \mathbf{u}_{n,\varepsilon}|^2 \, \mathrm{d}x \, \mathrm{d}t \leq \text{const}, \tag{5.7}$$

where the bound is uniform with respect to n, ε.

6. Convergence

6.1. The limit $n \to \infty$

With $\varepsilon > 0$ fixed, our aim is to let $n \to \infty$ in the family of approximate solutions $\varrho_{n,\varepsilon}$, $\mathbf{u}_{n,\varepsilon}$ constructed in Section 4. To begin with, estimates (5.4), (5.6) yield

$$\mathbf{u}_{n,\varepsilon} \to \mathbf{u}_\varepsilon \text{ weakly in } L^2(0,T;W_0^{k,2}(\Omega;R^3)) \text{ and weakly-(*) in } L^\infty(0,T;L^2(\Omega;R^3)) \tag{6.1}$$

for $n \to \infty$, at least for a suitable subsequence.

Similarly, by virtue of (4.3), (5.5), we can assume that

$$\varrho_{n,\varepsilon} \to \varrho_\varepsilon \text{ in } C_{\text{weak}}([0,T];L^1(\Omega)) \text{ and weakly-(*) in } L^\infty(0,T;L^\infty(\Omega)). \tag{6.2}$$

Consequently, combining (6.1), (6.2), we conclude that

$$\varrho_{n,\varepsilon}\mathbf{u}_{n,\varepsilon} \to \varrho_\varepsilon \mathbf{u}_\varepsilon \text{ weakly-(*) in } L^\infty(0,T;L^2(\Omega;R^3)). \tag{6.3}$$

Moreover, it follows from (4.1) that

$$\{t \mapsto \int_\Omega (\varrho_{n,\varepsilon}\mathbf{u}_{n,\varepsilon})(t) \cdot \mathbf{v}_n \, \mathrm{d}x\} \to \{t \mapsto \int_\Omega (\varrho_\varepsilon \mathbf{u}_\varepsilon)(t) \cdot \mathbf{v}_n \, \mathrm{d}x\} \text{ in } C[0,T] \text{ for } n = 1,2,\ldots;$$

whence

$$\varrho_{n,\varepsilon}\mathbf{u}_{n,\varepsilon} \to \varrho_\varepsilon \mathbf{u}_\varepsilon \text{ in } C_{\text{weak}}([0,T];L^2(\Omega;R^3)). \tag{6.4}$$

Relations (6.1), (6.4) imply

$$\varrho_{n,\varepsilon}\mathbf{u}_{n,\varepsilon} \cdot \mathbf{u}_{n,\varepsilon} \to \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathbf{u}_\varepsilon \text{ weakly in } L^2((0,T) \times \Omega); \tag{6.5}$$

therefore, in view of (4.3),

$$\mathbf{u}_{n,\varepsilon} \to \mathbf{u}_\varepsilon \text{ in } L^2(0,T;L^2(\Omega;R^3)). \tag{6.6}$$

Consequently, (6.1), (6.6) and a simple interpolation argument yield

$$\mathbf{u}_{n,\varepsilon} \to \mathbf{u}_\varepsilon \text{ in } L^2(0,T;W_0^{k-1,2}(\Omega;R^3)), \tag{6.7}$$

in particular,

$$\mathrm{div}_x \mathbf{u}_{n,\varepsilon} \to \mathrm{div}_x \mathbf{u}_\varepsilon \text{ in } L^2(0,T;W_0^{k-2,2}(\Omega;R^3)). \tag{6.8}$$

Since $\varrho_{n,\varepsilon}$ satisfy continuity equation (2.1), one can deduce from (6.2), (6.8) that

$$\varrho_{n,\varepsilon} \to \varrho_\varepsilon \text{ in } L^1((0,T) \times \Omega). \tag{6.9}$$

In a similar way, one can show

$$\chi_{n,\varepsilon} \to \chi_\varepsilon \text{ in } L^1((0,T) \times \Omega). \tag{6.10}$$

Thus we have shown there are functions ϱ_ε, $\mathbf{u}_\varepsilon$ such that

$$\int_0^T \int_{R^3} \Big(\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \mathbf{w} + \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \mathbf{w} + p(\varrho_\varepsilon)\ \mathrm{div}_x \mathbf{w}\Big) \mathrm{d}\mathbf{x}\ \mathrm{d}t \tag{6.11}$$
$$= \int_0^T \int_\Omega M_\varepsilon(\chi_\varepsilon)[\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon] : [\nabla_x \mathbf{w} + \nabla_x^t \mathbf{w}]\ \mathrm{d}x\ \mathrm{d}t$$
$$+ \int_0^T ((\mathbf{u}_\varepsilon, \mathbf{w}))\ \mathrm{d}t - \int_0^T \int_{R^3} \varrho_\varepsilon \mathbf{g} \cdot \mathbf{w}\ \mathrm{d}\mathbf{x}\ \mathrm{d}t - \int_{R^3} \varrho_0 \mathbf{u}_0 \cdot \mathbf{w}(0)\ \mathrm{d}\mathbf{x}$$

to be satisfied for any test function

$$\mathbf{w} \in C^1([0,T]; W_0^{k,2}(\Omega, R^3)),\ \ \mathbf{w}(T) = 0. \tag{6.12}$$

Furthermore, we have

$$\varrho_\varepsilon(t, \mathbf{X}_\varepsilon(t,\mathbf{x})) = \varrho_0(x) \exp\Big(-\int_0^t \mathrm{div}_x \mathbf{u}_\varepsilon(s, \mathbf{X}_\varepsilon(s,\mathbf{x}))\ \mathrm{d}s\Big),\ \ \mathbf{x} \in R^3 \tag{6.13}$$

and

$$\chi_\varepsilon(t, \mathbf{X}_\varepsilon(t,\mathbf{x})) = \chi_{0,\varepsilon}(\mathbf{x}) \geq 0, \tag{6.14}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}_\varepsilon(t,\mathbf{x}) = \mathbf{u}_\varepsilon(t, \mathbf{X}_\varepsilon(t,\mathbf{x})),\ \ t > 0,\ \ \mathbf{X}_\varepsilon(0) = \mathbf{x}. \tag{6.15}$$

In addition, the energy inequality

$$E_\varepsilon(t_2) + \int_{t_1}^{t_2} \int_\Omega M_\varepsilon(\chi_\varepsilon)|\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon|^2\ \mathrm{d}x\ \mathrm{d}t + \int_{t_1}^{t_2} ((\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon))\ \mathrm{d}t \tag{6.16}$$
$$\leq E_\varepsilon(t_1) + \int_{t_1}^{t_2} \int_\Omega \varrho_\varepsilon \mathbf{g} \cdot \mathbf{u}_\varepsilon\ \mathrm{d}x$$

holds for any $0 \leq t_1 \leq T$ and a.a. $t_2 \in (t_1, T)$, where

$$E_\varepsilon(t) = \int_\Omega \Big(\frac{1}{2}\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \varrho_\varepsilon P(\varrho_\varepsilon)\Big)\ \mathrm{d}x,\ \ E_\varepsilon(0) = E_0 = \int_\Omega \Big(\frac{1}{2}\varrho_0 |\mathbf{u}_0|^2 + \varrho_0 P(\varrho_0)\Big)\ \mathrm{d}x.$$

6.2. The limit for $\varepsilon \to 0$

Adopting the idea of San Martin et al. [18] we take

$$M_\varepsilon(z) = \frac{1}{\varepsilon}\max\{z, 0\}, \tag{6.17}$$

$$\chi_{\varepsilon,0} = \chi_0 \in C^1(\overline{\Omega}),\ \ \chi_0(\mathbf{x}) = \begin{cases} 0 & \text{for } \mathbf{x} \in \overline{\Omega} \setminus \cup_{i=1}^m \overline{S}^i, \\ > 0 & \text{for } \mathbf{x} \in \cup_{i=1}^m S^i. \end{cases} \tag{6.18}$$

Our ultimate goal is to let $\varepsilon \to 0$ in (6.11–6.16) in order to recover the global-in-time solution of **problem (P)**, the existence of which is claimed in Theorem 3.1.

To begin with, we can assume, by virtue of (6.16), that

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } L^2(0,T; W_0^{k,2}(\Omega; R^3)) \tag{6.19}$$

passing to a suitable subsequence as the case may be. Moreover, as $\{\varrho_\varepsilon\}_{\varepsilon>0}$ is bounded below away from zero in view of (6.13), we have

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly-(*) in } L^\infty(0,T;L^2(\Omega;R^3)). \tag{6.20}$$

As stated in (6.13), $\{\varrho_\varepsilon\}_{\varepsilon>0}$ solve (in the sense of distributions) continuity equation (2.1) supplemented with the initial datum ϱ_0; whence we have

$$\varrho_\varepsilon \to \varrho \text{ in, say, } C_{\mathrm{weak}}([0,T];L^2(\Omega)), \tag{6.21}$$

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \varrho\mathbf{u} \text{ weakly-(*) in } L^\infty(0,T;L^2(\Omega;R^3)). \tag{6.22}$$

Thus we are allowed to conclude that ϱ, $\mathbf{u}$, extended to be zero outside Ω, solve equation (2.1). In addition, ϱ is uniquely determined by ϱ_0 and the velocity field $\mathbf{u}$, and $\varrho \in C([0,T];L^1(\Omega))$ (cf. DiPerna and Lions [5]).

In order to identify the family of isometries η^i, $i = 1,\dots,m$, we need the following auxiliary result proved in [6, Proposition 5.1].

Lemma 6.1. *Let* $\mathbf{u}_\varepsilon = \mathbf{u}_\varepsilon(t,\mathbf{x})$ *be a family of Carathéodory functions such that*

$$\int_0^T \|\mathbf{u}_\varepsilon(t)\|^2_{W^{1,\infty}(R^3;R^3)}\,\mathrm{d}t \le \text{const}$$

uniformly with respect to $\varepsilon \to 0$. *Let* $S_\varepsilon \subset R^3$ *be a family of open sets such that*

$$\mathbf{db}[S_\varepsilon] \to \mathbf{db}[S] \text{ in } C_{\mathrm{loc}}(R^3),$$

where $S \subset R^3$ *is an open set and the symbol* $\mathbf{db}$ *denotes the signed distance from the boundary:*

$$\mathbf{db}[S](\mathbf{x}) = \mathrm{dist}[\mathbf{x}, \overline{R^3 \setminus S}] - \mathrm{dist}[\mathbf{x}, \overline{S}],\ \mathrm{dist}[\mathbf{x},K] \equiv \min_{\mathbf{y}\in K}|\mathbf{x}-\mathbf{y}|.$$

Denote by $\mathbf{X}_\varepsilon$ *the unique solution of the problem*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}_\varepsilon(t,\mathbf{x}) = \mathbf{u}_\varepsilon(t,\mathbf{X}_\varepsilon(t,\mathbf{x})),\ 0<t<T,\ \mathbf{X}_\varepsilon(0,\mathbf{x}) = \mathbf{x}. \tag{6.23}$$

Then, extracting a suitable subsequence if necessary, we have

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly-(*) in } L^2(0,T;W^{1,\infty}(R^3;R^3)),$$

and

$$\mathbf{X}_\varepsilon(t,\cdot) \to \mathbf{X}(t,\cdot) \text{ in } C_{\mathrm{loc}}(R^3) \text{ uniformly in } t \in [0,T],$$

where $\mathbf{X}$ *solves*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}(t,\mathbf{x}) = \mathbf{u}(t,\mathbf{X}(t,\mathbf{x})),\ 0<t<T,\ \mathbf{X}(0,\mathbf{x}) = \mathbf{x}. \tag{6.24}$$

Moreover,

$$\mathbf{db}[S_\varepsilon(t)] \to \mathbf{db}[S(t)] \text{ in } C_{\mathrm{loc}}(R^3) \text{ uniformly in } t \in [0,T],$$

where we have set

$$S_\varepsilon(t) = \mathbf{X}_\varepsilon(t,S_\varepsilon),\ S(t) = \mathbf{X}(t,S).$$

Consider the domains $S^i(t)$, $i = 1, \ldots, m$ occupied by the images of the rigid bodies S^i, $i = 1, \ldots, m$ under the flow induced by (6.24):

$$S^i(t) = \mathbf{X}(t, S^i), \ i = 1, \ldots, m.$$

Since the velocity field $\mathbf{u}$ belongs to the class $L^2(0, T; C^1(R^3; R^3)$, we have, in accordance with hypothesis (3.1),

$$\overline{S}^i(t) \cap \overline{S}^j(t) = \emptyset \text{ for } i \neq j, \ \overline{S}^i \subset \Omega \text{ for } i = 1, \ldots, m \tag{6.25}$$

for all $t \in [0, T]$, that is to say, there is no collision of two or more "rigid" objects.

Let

$$(t, \mathbf{x}) \in \cup_{t \in (0,T)} S^i(t).$$

In accordance with Lemma 6.1, there is a small open neighbourhood V of $(t, \mathbf{x})$ such that

$$V \subset \overline{V} \subset \cup_{t \in (0,T)} \mathbf{X}_\varepsilon(t, S^i) \text{ for all } \varepsilon > 0 \text{ small enough,}$$

where $\mathbf{X}_\varepsilon$ are determined through (6.15).

Consequently, combining energy inequality (6.16) with (6.14), (6.17), (6.18) we conclude that

$$\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} = 0 \text{ a.a. on } V.$$

As the point $(t, \mathbf{x})$ was arbitrary, we have

$$\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} = 0 \text{ a.a. on } \cup_{i=1}^m \cup_{t \in (0,T)} S^i(t). \tag{6.26}$$

It is a routine matter to deduce from (6.24), (6.26) that

$$\mathbf{X}(t, \cdot) : S^i \to R^3 \text{ is an isometry for any } i = 1, \ldots, m \text{ and any fixed } t \in [0, T];$$

whence we can set

$$\eta^i(t, \cdot) : R^3 \to R^3, \ \eta^i(t, \mathbf{x}) = \mathbf{X}(t, \mathbf{x}) \text{ for all } \mathbf{x} \in S^i, \ t \in [0, T], \ i = 1, \ldots, m \tag{6.27}$$

Clearly, the family η^i, $i = 1, \ldots, m$ is compatible with the vector field $\mathbf{u}$ in the sense of Definition 2.1.

In order to complete the proof of Theorem 3.1, we have to pass to the limit for $\varepsilon \to 0$ in (6.11) to recover momentum equation (2.5). To this end, first observe that

$$\{\varrho_\varepsilon \mathbf{u}_\varepsilon\}_{\varepsilon > 0} \text{ is precompact in } C_{\text{weak}}([t_1, t_2], L^2(B; R^3)) \tag{6.28}$$
$$\text{whenever } ([t_1, t_2] \times \overline{B}) \cap \cup_{i=1}^m \cup_{t \in [0,T]} \overline{S}^i(t) = \emptyset.$$

In particular, in accordance with (6.20),

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \to \mathbb{Q} \text{ weakly in } L^2(0, T; L^2(\Omega; R^{3 \times 3})), \tag{6.29}$$

where

$$\mathbb{Q} = \varrho \mathbf{u} \otimes \mathbf{u} \text{ on } \Omega \setminus \cup_{i=1}^m \cup_{t \in (0,T)} S^i(t). \tag{6.30}$$

As a byproduct of (6.30) we get

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ in } L^2\Big(\Omega \setminus \cup_{i=1}^m \cup_{t \in (0,T)} S^i(t); R^3\Big); \tag{6.31}$$

whence, having used the estimates on $\mathrm{div}_x \mathbf{u}_\varepsilon$ resulting from (6.16), we conclude that

$$\mathrm{div}_x \mathbf{u}_\varepsilon \to \mathrm{div}_x \mathbf{u} \text{ in } L^1((0,T) \times \Omega). \tag{6.32}$$

Since ϱ_ε solve the continuity equation, relation (6.32) implies

$$\varrho_\varepsilon \to \varrho \text{ in } L^1((0,T) \times \Omega). \tag{6.33}$$

Using (6.29), (6.33) one can let $\varepsilon \to 0$ in (6.11) in order to recover (2.5) at least for any test function $\mathbf{w}$ such that

$$\mathbf{w} \in C^1([0,T]; C^k(R^3)),\ \mathbf{w}(T) = 0,\ \mathbf{w}(t) \in V^\delta_{\mathrm{adm}}(t) \text{ for any } t \in [0,T],$$

where

$$V^\delta_{\mathrm{adm}}(t) = \{\mathbf{w} \in C^k(R^3; R^3) \mid \mathbf{w} \equiv 0 \text{ on a } \delta\text{-neighbourhood of } \overline{R^3 \setminus \Omega},$$

$$\nabla_x \mathbf{w} + \nabla_x \mathbf{w}^t \equiv 0 \text{ on a } \delta\text{-neighbourhood of} \cup_{i=1}^m \overline{\eta^i(t, S^i)}\},\ \delta > 0.$$

Note that

$$\int_0^T \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \mathbf{w}\ \mathrm{d}x\ \mathrm{d}t = \frac{1}{2} \int_0^T \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : (\nabla_x \mathbf{w} + \nabla_x^t \mathbf{w})\ \mathrm{d}x\ \mathrm{d}t.$$

Since there is no contact of rigid objets (see (6.25)), it is easy to extend validity of (2.5) to all test functions in the class (2.6) via density argument.

Finally, by virtue of (6.19), (6.33), energy inequality (2.9) holds for $t_1 = 0$ and a.a. $t_2 \in (0,T)$ as required in Definition 2.1. Theorem 3.1 has been proved.

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On the Stokes Resolvent Equations in Locally Uniform L^p Spaces in Exterior Domains

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Abstract. The Stokes resolvent equations are studied in locally uniform L^p spaces where the domain is an exterior of a bounded domain. The unique existence of a solution of the Stokes resolvent equations is proved with a resolvent estimate. In particular, the analyticity of the Stokes semigroup is established. An interesting aspect of locally uniform L^p spaces is that these spaces contain non-decaying functions.

1. Introduction

In this note we consider the Stokes resolvent equations in locally uniform L^p spaces in an exterior domain, which is a complement of the closure of a bounded open set. We shall prove the analyticity of the Stokes semigroup in these spaces. Note that these spaces contain non-decaying functions. Although there is a huge literature for the analyticity of the Stokes semigroup, results are only known for spaces which exclude non-decaying functions if the domain is an exterior domain.

Throughout this note let $p \in (1,\infty)$ and $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an exterior domain with $C^{2+\mu}$-boundary for some $\mu \in (0,1)$ and let $G = \Omega$ or $G = \mathbb{R}^n$. We consider the Stokes equations

$$\begin{aligned} \lambda u - \Delta u + \nabla \pi = f, &\quad \text{in } G \\ \operatorname{div} u = 0, &\quad \text{in } G \\ u = 0, &\quad \text{on } \partial G \end{aligned} \tag{1}$$

in locally uniform spaces, i.e.,

$$L^p_{\mathrm{uloc}}(G) = \{u \in L^p_{\mathrm{loc}}(G) : \|u\|_{L^p_{\mathrm{uloc}}(G)} < \infty\},$$

where

$$\|u\|_{L^p_{\mathrm{uloc}}(G)} = \sup_{x_0 \in \mathbb{Z}^n} \|u\|_{L^p(B(x_0,2)\cap G)}.$$

Note that the choice of radius 2 for the balls is not important. Indeed, any radius r such that $\Omega \subset \bigcup_{i\in\mathbb{N}} B(x_i, r)$ leads to the same spaces $L^p_{\mathrm{uloc}}(G)$. There are even more possibilities to define locally uniform spaces, see [2] and [7].

Our aim is to show that (1) has a unique solution for solenoidal f in locally uniform L^p spaces in exterior domains and establish a resolvent estimate for large λ which yields analyticity of the Stokes semigroup (Theorem 3.1 and Theorem 3.4).

The advantage of locally uniform spaces is that $L^p_{\mathrm{uloc}}(\Omega)$ inherit many properties of the usual $L^p(\Omega)$ spaces but it contains non-decaying functions. In particular, $L^\infty(\Omega) \subset L^p_{\mathrm{uloc}}(\Omega)$.

Since locally uniform spaces coincide with the usual L^p-spaces if the domain is bounded, unbounded domains are of interest only. Unfortunately, we cannot expect the Helmholtz-projection to be bounded since it is unbounded in locally uniform spaces in $\mathbb{R}^n$. Up to now, [7] is the only work that deals with the Navier-Stokes equations in locally uniform spaces. The authors of [7] prove existence and uniqueness of a mild solution to the Navier-Stokes equations in $\mathbb{R}^n$ by using a variant of the Fujita-Kato iteration. In order to do so, they use kernel estimates for the heat-semigroup to show $L^p - L^q$ smoothing estimates. For further development see [8].

In contrast to the case $\mathbb{R}^n$ there are no kernel estimates for exterior domains available. However, we can construct a solution of (1) using the resolvent of the Laplacian in $\mathbb{R}^n$ in locally uniform spaces, see [2], and the solution of the generalized Stokes resolvent problem in $L^p(\Omega)$, see [4]. This is possible since the boundary of Ω is compact and thus $L^p(\partial\Omega) = L^p_{\mathrm{uloc}}(\partial\Omega)$, see the proof of Theorem 3.1 below.

The Stokes resolvent problem has not yet been studied much in a space which contains non-decaying functions if G is a domain with non-empty boundary. A few exception is a result by Desch, Hieber and Prüss [3] which established the boundedness and the analyticity of the Stokes semigroup in L^∞ space if the domain is a half-space by using an explicit representation of a solution. To show existence and uniqueness of a solution of the Navier-Stokes equations the analyticity of the semigroup is usually not enough so we do not touch this problem in this note.

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2. Preliminaries

Analogous to the homogeneous Sobolev space $\hat{W}^{1,p}(G)$ we define

$$\hat{W}^{1,p}_{\mathrm{uloc}}(G) = \{u \in L^p_{\mathrm{loc}}(\overline{G}) : \nabla u \in L^p_{\mathrm{uloc}}(G)\}.$$

Next, we define the space of solenoidal vector fields.

$$L^p_{\mathrm{ul}\sigma}(G) = \{u \in L^p_{\mathrm{uloc}}(G) : \mathrm{div}\ u = 0,\ u \cdot \nu = 0 \text{ on } \partial G\}.$$

Here, ν denotes the outer normal and the boundary condition $u \cdot \nu = 0$ on ∂G is understood in the sense of the trace theorem based on Gauss' divergence theorem similar as in the L^p-setting. For the convenience of the reader we discuss the differences to the proof for the L^p-setting given in [5, Chapter III.2]. A major difference to the usual L^p-setting is that $C_c^\infty(\overline{\Omega})$ is not dense in

$$H_p(\Omega) := \left\{u \in L^1_{\mathrm{loc}}(\Omega) : \|u\|_{H_p} < \infty\right\},$$

where $\|u\|_{H_p} = \|u\|_{L^p(\Omega)} + \|\mathrm{div}\ u\|_{L^p(\Omega)}$. But it is not difficult to show that $BC^\infty(\overline{\Omega}) = \{u \in C^\infty(\overline{\Omega}) : \partial^\alpha u \text{ is bounded for all } \alpha \in \mathbb{N}^n\}$ is dense in

$$H_{p,\mathrm{uloc}}(\Omega) := \left\{u \in L^1_{\mathrm{loc}}(\Omega) : \|u\|_{H_{p,\mathrm{uloc}}} < \infty\right\},$$

where $\|u\|_{H_{p,\mathrm{uloc}}} = \|u\|_{L^p_{\mathrm{uloc}}(\Omega)} + \|\mathrm{div}\ u\|_{L^p_{\mathrm{uloc}}(\Omega)}$. For $u \in BC^\infty(\overline{\Omega})$ we obtain

$$\int\limits_{\partial\Omega} u\nu\Psi \mathrm{d}x = \int\limits_{\Omega} u\nabla\Psi \mathrm{d}x + \int\limits_{\Omega} \Psi \mathrm{div}\ u \mathrm{d}x, \quad \Psi \in C_c^\infty(\mathbb{R}^n). \tag{2}$$

Obviously, the right-hand side does not make sense for all $\Psi \in W^{1,p'}(\Omega)$, where $1/p + 1/p' = 1$. Hence, we have to impose stronger decay properties on Ψ for $|x| \to \infty$ in order to makes sense out of (2). More precisely, let us define

$$L^p_{\mathrm{sum}}(G) = \{u \in L^p_{\mathrm{loc}}(G) : \|u\|_{L^p_{\mathrm{sum}}(G)} < \infty\},$$

where

$$\|u\|_{L^p_{\mathrm{sum}}(G)} = \sum_{x_0 \in \mathbb{Z}^n} \|u\|_{L^p(B(x_0,2)\cap G)}.$$

In contrast to the situation for locally uniform spaces, $C_c^\infty(G)$ is dense in $L^p_{\mathrm{sum}}(G)$. Furthermore, we have $L^p_{\mathrm{sum}}(G) \subsetneq L^p(G) \subsetneq L^p_{\mathrm{uloc}}(G)$.

Since $C_c^\infty(\overline{\Omega})$ is dense in $W^{1,p'}_{\mathrm{sum}}(\Omega)$, by Hölder's inequality, (2) is valid for $\varphi \in W^{1,p'}_{\mathrm{sum}}(\Omega)$ with $1/p + 1/p' = 1$. Now, we can proceed as in [5, Chapter III.2] since the trace space of $W^{1,p'}_{\mathrm{sum}}(\Omega)$ is $W^{1-1/p',p'}(\partial\Omega)$.

Lemma 2.1. *Let $1/p + 1/p' = 1$. Then*

$$L^p_{\mathrm{ul}\sigma}(G) = \left\{ f \in L^p_{\mathrm{uloc}}(G) : \int\limits_G f\nabla\varphi \mathrm{d}x = 0 \textit{ for all } \varphi \in W^{1,p'}_{\mathrm{sum}}(G) \right\}. \tag{3}$$

Proof. This easily follows from (2). $\square$

Next, we characterize all $\pi \in \hat{W}^{1,p}_{\mathrm{uloc}}(G)$ satisfying $\nabla\pi \in L^p_{\mathrm{ul}\sigma}(G)$. We start with the case $G = \mathbb{R}^n$.

Lemma 2.2. *Let $\pi \in \hat{W}^{1,p}_{\mathrm{uloc}}(\mathbb{R}^n)$ satisfy $\nabla\pi \in L^p_{\mathrm{ul}\sigma}(\mathbb{R}^n)$. Then $\nabla\pi = K$ for some $K \in \mathbb{C}^n$.*

Proof. We only prove the assertion for $n \geq 3$. The case $n = 2$ follows similarly. Let $\alpha, \beta \in \mathbb{N}_0^n$ and $\varphi \in C_c^\infty(\mathbb{R}^n)$. We set $\Psi = E * \partial^\alpha\varphi$, where E denotes the fundamental solution of the Laplace equation. Then, an explicit calculation for $x \notin \mathrm{supp}\ \varphi$ yields

$$\left|\partial^\beta\Psi(x)\right| = \left|\left((\partial^{\alpha+\beta}E) * \varphi\right)(x)\right| \leq \frac{C(\varphi)}{\mathrm{dist}\ (x, \mathrm{supp}\ \varphi)^{n-2+|\alpha|+|\beta|}}.$$

Moreover, $\Psi \in C^\infty(\mathbb{R}^n)$ and $\Delta\Psi = \partial^\alpha\varphi$.

Since $\nabla\pi \in L^p_{\mathrm{uloc}}(\Omega)$ is harmonic, we have $\nabla\pi \in L^\infty(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$. Hence, $|\pi(x) - \pi(0)| \leq \|\nabla\pi\|_{L^\infty(\mathbb{R}^n)}|x|$, $x \in \mathbb{R}^n$. Therefore, integration by parts yields

$$0 = \int\limits_{\mathbb{R}^n} \nabla\pi\nabla\Psi \mathrm{d}x = -\int\limits_{\mathbb{R}^n} \pi\Delta\Psi \mathrm{d}x = -\int\limits_{\mathbb{R}^n} \pi\partial^\alpha\varphi \mathrm{d}x = \int\limits_{\mathbb{R}^n} \partial^\alpha\pi\varphi \mathrm{d}x$$

provided $|\alpha|$ is large enough. Since $\pi \in \hat{W}^{1,p}_{\mathrm{uloc}}(\mathbb{R}^n)$ by assumption, $\nabla\pi = K$ for some $K \in \mathbb{C}^n$. □

In particular, it follows from the previous lemma that $K \in L^p_{\mathrm{ul}\sigma}(\mathbb{R}^n)$. Hence, $L^p_\sigma(\mathbb{R}^n) \subsetneq L^p_{\mathrm{ul}\sigma}(\mathbb{R}^n)$.

Lemma 2.3. *Let $\pi \in \hat{W}^{1,p}_{\mathrm{uloc}}(\Omega)$ satisfy $\nabla\pi \in L^p_{\mathrm{ul}\sigma}(\Omega)$. Then $\pi = p_K + Kx$ for some $K \in \mathbb{C}^n$ and $p_K \in \hat{W}^{1,p}(\Omega)$, where p_K is uniquely determined. In particular, if $\pi \in \hat{W}^{1,p}(\Omega)$ then $\nabla\pi \equiv 0$.*

Proof. Let $\tilde{\pi}$ denote a smooth extension of π to $\mathbb{R}^n$. Then

$$\int\limits_{\mathbb{R}^n} \nabla\tilde{\pi}\nabla\Psi \mathrm{d}x = \int\limits_{\Omega} \nabla\pi\nabla\Psi \mathrm{d}x + \int\limits_{\Omega^c} \nabla\tilde{\pi}\nabla\Psi \mathrm{d}x = \int\limits_{\Omega^c} f\nabla\Psi \mathrm{d}x, \quad \Psi \in C_c^\infty(\mathbb{R}^n),$$

where $f = \nabla\tilde{\pi}|_{\Omega^c}$. Then the solution $\hat{\pi}$ of $\Delta\hat{\pi} = \mathrm{div}\ f$ in $\mathbb{R}^n$ satisfies $\hat{\pi} \in \hat{W}^{1,p}(\mathbb{R}^n)$. Since

$$\int\limits_{\mathbb{R}^n} \nabla(\tilde{\pi} - \hat{\pi})\nabla\Psi \mathrm{d}x = 0, \quad \Psi \in C_c^\infty(\mathbb{R}^n),$$

and $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{1,p}_{\mathrm{sum}}(\mathbb{R}^n)$, by Lemma 2.2, there exists $K \in \mathbb{C}^n$ with $\nabla(\tilde{\pi} - \hat{\pi}) = K$. Hence, $\nabla\pi = \nabla\hat{\pi}|_\Omega + K$. □

3. The Stokes operator in L^p_{uloc} spaces in exterior domains

In this section we present our main results for the Stokes operator in locally uniform spaces in exterior domains. We define $\Sigma_\theta := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \theta\}$. Here and in the following, we always assume $\theta \in (0, \pi)$.

Theorem 3.1. *Fix $\gamma > 0$ and let $\lambda \in \Sigma_\theta$ with $|\lambda| \geq \gamma$. Then, for $f \in L^p_{\mathrm{ul}\sigma}(\Omega)$ there exists $u \in W^{2,p}_{\mathrm{uloc}}(\Omega) \cap L^p_{\mathrm{ul}\sigma}(\Omega)$ and $p \in \hat{W}^{1,p}(\Omega)$ satisfying (1) with $G = \Omega$. Moreover, there exists $C > 0$, independent of u, p, f and λ, such that*

$$\lambda \|u\|_{L^p_{\mathrm{uloc}}(\Omega)} + \|u\|_{W^{2,p}_{\mathrm{uloc}}(\Omega)} + \|\nabla p\|_{L^p(\Omega)} \leq C \|f\|_{L^p_{\mathrm{ul}\sigma}(\Omega)}. \tag{4}$$

Proof. Let $\tilde{f}$ denote the extension of f by 0. By [2, Proposition 2.1 and Theorem 2.1] there exists a solution u_1 to

$$\lambda u_1 - \Delta u_1 = \tilde{f}, \quad \text{in } \mathbb{R}^n,$$

satisfying

$$\|u_1\|_{W^{2,p}_{\mathrm{uloc}}(\mathbb{R}^n)} + |\lambda| \|u_1\|_{L^p_{\mathrm{uloc}}(\mathbb{R}^n)} \leq C_1 \|\tilde{f}\|_{L^p_{\mathrm{uloc}}(\mathbb{R}^n)} = C_1 \|f\|_{L^p_{\mathrm{uloc}}(\Omega)}, \tag{5}$$

where $C_1 > 0$ is independent of f. Furthermore, we have div $u_1 = 0$. However, the boundary conditions are not fulfilled since u_1 is a solution in the whole space only.

Since Ω^c is compact, $u_1|_{\Omega^c} \in W^{2,p}(\Omega)$. Let E denote a strong 2-extension operator for Ω^c (see [1, Thm. 5.22]) and set $u_2 = Eu_1$. We then have $u_2 = u_1$ in Ω^c, and there exist $C_2, C_3 > 0$, independent of u_1, such that

$$\|u_2\|_{W^{s,p}(\mathbb{R}^n)} \leq C_2 \|u_1\|_{W^{s,p}(\Omega^c)} \leq C_2 C_3 \|u_1\|_{W^{s,p}_{\mathrm{uloc}}(\mathbb{R}^n)}, \quad s = 0, 1, 2. \tag{6}$$

By [4, Thm. 2.1], there exists $u_3 \in W^{2,p}(\Omega)$, $p_3 \in \hat{W}^{1,p}(\Omega)$ such that

$$\begin{aligned} \lambda u_3 - \Delta u_3 + \nabla p_3 &= \lambda u_2 - \Delta u_2, && \text{in } \Omega, \\ \text{div } u_3 &= \text{div } u_2, && \text{in } \Omega, \\ u_3 &= 0, && \text{on } \Omega. \end{aligned}$$

Moreover, it follows from (5), (6) and [4, Thm. 2.1] that

$$\begin{aligned} |\lambda| \|u_3\|_{L^p(\Omega)} + \|\nabla^2 u_3\|_{L^p(\Omega)} + \|\nabla p\|_{L^p(\Omega)} &\leq C_4 \left(\|u_2\|_{W^{2,p}(\Omega)} + |\lambda| \|u_2\|_{L^p(\mathbb{R}^n)} \right) \\ &\leq C_1 C_2 C_3 C_4 \|f\|_{L^p_{\mathrm{uloc}}(\Omega)}, \end{aligned}$$

where C_4 is independent of u_2 but it may depend on γ. Finally, we set $u := u_1 - u_2 + u_3$ and $p := p_3$. Then (u, p) satisfies (4) and

$$\begin{aligned} \lambda u - \Delta u + \nabla p &= f \text{ in } \Omega, \\ \text{div } u &= 0 \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

The proof is complete. □

Next, we investigate uniqueness of solutions to (1). Again, we start with the case $G = \mathbb{R}^n$.

Lemma 3.2. *Let $p \in (1,\infty)$, $\lambda \in \Sigma_\theta \cup \{0\}$. Assume that $u \in W^{2,p}_{\mathrm{uloc}}(\mathbb{R}^n)$ and $\pi \in \hat{W}^{1,p}_{\mathrm{uloc}}(\mathbb{R}^n)$ satisfy (1) with $f \equiv 0$ and $G = \mathbb{R}^n$. Then $\pi = \lambda K x$ and $u = K$ for some $K \in \mathbb{C}^n$.*

Proof. Multiplying (1) by $\nabla\Psi$, where $\Psi \in W^{1,p'}_{\mathrm{sum}}(\mathbb{R}^n)$, and integrating by parts, we obtain

$$\int\limits_{\mathbb{R}^n} \nabla\pi\nabla\Psi \mathrm{d}x = 0.$$

Hence, by Lemma 2.2, $\nabla\pi = K$ for some $K \in \mathbb{C}^n$. Obviously, $\tilde{u} := K/\lambda$ and $\pi = Kx$ is a solution of (1) for $\lambda \neq 0$. Since the solution is unique by [2, Proposition 2.1] the lemma follows for $\lambda \neq 0$. The case $\lambda = 0$ follows by standard arguments using the fact that ∇u is harmonic. □

Lemma 3.3. *Let $p \in (1,\infty)$, $\lambda \in \Sigma_\theta$ and let $u \in W^{2,p}_{\mathrm{uloc}}(\Omega)$ and $\pi \in \hat{W}^{1,p}_{\mathrm{uloc}}(\Omega)$ satisfy (1) with $f = 0$ and $G = \Omega$. Then $u = u_K + K$ and $\pi = \pi_K + \lambda K x$ with some $K \in \mathbb{C}^n$, $u_K \in W^{2,p}(\Omega)$ and $\pi_K \in \hat{W}^{1,p}(\Omega)$. In particular, if $\pi \in \hat{W}^{1,p}(\Omega)$, then $u = 0$, $\nabla\pi = 0$.*

Proof. We follow the ideas of the proof of [9, Theorem 1.2]. Let $\tilde{u}$, $\tilde{\pi}$ be a (smooth) extension to $\mathbb{R}^n$. Then $\tilde{u}$ and $\tilde{\pi}$ solve

$$\begin{aligned} \lambda\tilde{u} - \Delta\tilde{u} + \nabla\tilde{\pi} &= \tilde{f}, \quad \text{in } \mathbb{R}^n \\ \operatorname{div} \tilde{u} &= \tilde{g}, \quad \text{in } \mathbb{R}^n \end{aligned}$$

where $\tilde{g} := \operatorname{div} \tilde{u}$ and $\tilde{f} = \lambda\tilde{u} - \Delta\tilde{u} + \nabla\tilde{\pi}$. Note that $\tilde{g}$ and $\tilde{f}$ are compactly supported. Hence, $\tilde{g} \in W^{1,p}(\mathbb{R}^n)$ and $\tilde{f} \in L^p(\Omega)$. Taking divergence, we obtain

$$\Delta\tilde{\pi} = \operatorname{div} \tilde{f} - \lambda\tilde{g} - \Delta\tilde{g} = \operatorname{div} \tilde{f} - \lambda \operatorname{div} \tilde{u} - \Delta\tilde{g}. \tag{7}$$

We set $\hat{\pi} = E * (\operatorname{div} \tilde{f} - \lambda \operatorname{div} \tilde{u}) + \tilde{g}$, where E denotes the fundamental solution of the Laplace equation. It then follows that $\hat{\pi} \in \hat{W}^{1,p}(\mathbb{R}^n)$. Moreover, $\hat{\pi}$ satisfies (7). Hence,

$$\hat{u} := (\lambda - \Delta)^{-1}(\tilde{f} - \nabla\hat{\pi}) \in W^{2,p}(\mathbb{R}^n) \cap L^p_\sigma(\mathbb{R}^n)$$

and $\hat{\pi}$ satisfies (1) with $G = \mathbb{R}^n$ and $f = 0$. Therefore, Lemma 3.2 yields $\hat{u} - \tilde{u} = K$ and $\hat{\pi} - \tilde{\pi} = \lambda K x$ for some $K \in \mathbb{R}^n$. In particular, $u = K - \hat{u}$ and $\pi = \hat{\pi} - \lambda K x$. If $\pi \in \hat{W}^{1,p}(\Omega)$, then K must be zero so that $u \in W^{2,p}(\Omega)$ and $\pi \in \hat{W}^{1,p}(\Omega)$. By uniqueness results in $L^p(\Omega)$ (see [6], [4]), we have $u = 0$ and $\nabla\pi = 0$. □

Our existence and uniqueness result yields the analyticity of the Stokes semigroup in locally uniform L^p spaces. Let $R(\lambda)f$ denote the solution u of (1) in Theorem 3.1. The estimate (4) implies that $R(\lambda)$ is a bounded linear operator from $L^p_{\mathrm{ul}\sigma}(\Omega)$ to $W^{2,p}_{\mathrm{uloc}}(\Omega)$ for $\lambda \in \Sigma = \mathbb{C}\backslash(-\infty, 0]$. We define a closed linear operator in $L^p_{\mathrm{ul}\sigma}(\Omega)$ by

$$A := \lambda I - R(\lambda)^{-1}$$

whose domain equals the range of $R(\lambda)$ where $\lambda \in \Sigma$. We call this operator the Stokes operator in $L^p_{\mathrm{ul}\sigma}(\Omega)$. Apparently, the definition depends on λ. However, we easily obtain from (1) the 'resolvent identity'

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu) = (\mu - \lambda)R(\mu)R(\lambda)$$

by observing that the difference $w = R(\lambda)f - R(\mu)f$ solves

$$\begin{aligned} (\lambda - \Delta)w + \nabla q &= (\mu - \lambda)R(\mu)f && \text{in } G \\ \operatorname{div} w &= 0 && \text{in } G \\ w &= 0 && \text{on } \partial G \end{aligned}$$

with some $q \in \hat{W}^{1,p}(\Omega)$. The resolvent identity implies that the definition of the operator A is independent of $\lambda \in \Sigma$. Now, Theorem 3.1 yields the analyticity of the semigroup generated by A.

Theorem 3.4. *The operator $-A$ generates an analytic semigroup e^{-tA} in $L^p_{\mathrm{ul}\sigma}(\Omega)$.*

Remark 3.5. *The estimate* (4) *in Theorem* 3.1 *is not enough to claim that e^{-tA} is a bounded analytic semigroup since* (4) *is not uniform near $\lambda = 0$. Moreover, e^{-tA} is not expected to be a C_0-semigroup since the domain is not dense in $L^p_{\mathrm{ul}\sigma}(\Omega)$ and it is not C_0 even for $G = \mathbb{R}^n$.*

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Generation of Analytic Semigroups and Domain Characterization for Degenerate Elliptic Operators with Unbounded Coefficients Arising in Financial Mathematics. Part II

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Abstract. This paper is devoted to study the generation of analytic semigroup for a family of degenerate elliptic operators (with unbounded coefficients) which includes well-known operators arising in mathematical finance. The generation property is proved by assuming some compensation conditions among the coefficients and applying a suitable modification of the techniques developed in [16]. Using the results proved in [11] concerning the generation in the space $L^2(\mathbb{R}^d)$, we prove the generation results in $L^p(\mathbb{R}^d)$ for $p \in [1, +\infty]$. These results have several consequences in connection with the financial applications [3, 11].

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1. Introduction

In this paper we study the generation of analytic semigroups in $L^p(\mathbb{R}^d)$, with p in $[1, +\infty]$ for a family of degenerate elliptic operators with unbounded coefficients.

These results can be employed to obtain existence, uniqueness and regularity estimates for the solutions of the associated (linear or semilinear) parabolic problems, through the well-known theory of analytic semigroups (e.g., [12]). This has been done in [3] for the so-called "no-arbitrage" operators arising in pricing contingent claims. We consider the following differential operator in $\mathbb{R}^d$

$$\mathcal{A}(x, D) = \sum_{i,j=1}^{d} \psi_i(x)\psi_j(x)a_{i,j}(x)D_{i,j} + \sum_{i=1}^{d} b_i(x)D_i - \gamma^2(x)$$

(denoted simply by $\mathcal{A}$ in the following) where the weights $\psi_i : \mathbb{R}^d \mapsto \mathbb{R}$, $i = 1, \dots, d$, are differentiable sublinear functions vanishing in not more than a negligible set Z, the matrix $\{a_{i,j}\}_{i,j=1,\dots,d}$ is bounded and uniformly elliptic, the coefficients $b_i : \mathbb{R}^d \mapsto \mathbb{R}$, $i = 1, \dots, d$, are measurable functions and the function $\gamma : \mathbb{R}^d \mapsto \mathbb{R}$ is differentiable and locally square integrable with its first derivatives. The main difficulty to overcome here is the need of managing both the possible unboundedness of all the coefficients and the presence of zero's for those of the second-order terms. In general these operators do not generate analytic nor strongly continuous semigroups (for instance the Ornstein-Uhlenbeck operator in one dimension, where $\psi = 1$, $b(x) = x$, $\gamma = 0$). However, we prove that choosing suitable compensation conditions on the coefficients this become possible.

In [11] we considered the operator $\mathcal{A}$ defined in the whole space $\mathbb{R}^d$ and we proved the generation of analytic semigroup in the space $L^2(\mathbb{R}^d)$, by an application of Hilbert space techniques. This was possible thanks to some preliminary a priori estimates, which are established by an appropriate choice of some compensation conditions among the coefficients of the operator. Then we obtained a characterization of the domain of the operator in $L^2(\mathbb{R}^d)$ by a localization procedure which was adapted to the growth rate of the weights ψ's at infinity and close to the negligible set Z of all zeros of the ψ's.

The aim of this paper is to pass from the $L^2(\mathbb{R}^d)$ case to the $L^p(\mathbb{R}^d)$ one, when $1 \le p \le +\infty$, by using a suitable modification of the Stewart's method [15, 16, 17]. Of course the fitting localization procedure become more complicated here, since it now depends also on the growth rate of the zero-order coefficient γ.

A first result of our semigroup generation analysis is the existence of solutions of the no-arbitrage pricing problems, which is a central topic in the modern mathematical finance. However, a general existence result can be obtained via the probabilistic approach. So the main motivation to study these generation problems is based on the question of regularity of solutions. This also in order to apply suitable numerical methods.

The paper is organized as follows. In Section 2 we introduce the notation and recall some results about the generation of analytic semigroup in the spaces $L^2(\mathbb{R}^d)$ proved in [11] . In Section 3 we prove the generation of analytic semigroup and we obtain the domain characterization in the spaces $L^p(\mathbb{R}^d)$, $2 < p < \infty$ and $L^\infty(\mathbb{R}^d)$. This result implies the generation of analytic semigroup in the spaces $L^p(\mathbb{R}^d)$ for $1 \le p < 2$ using duality techniques.

2. Preliminary material and notation

Let Ω be an open subset of the d-dimensional Euclidean space $\mathbb{R}^d$. We denote by $C^\infty(\Omega)$ the linear space of all infinitely differentiable complex-valued functions on Ω, and we write $C_c^\infty(\Omega)$ for the linear submanifold of $C^\infty(\Omega)$ of all functions with compact support in Ω.
We denote by $W^{n,p}(\Omega)$ the usual *Sobolev space* (see, e.g., [1]), defined as the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W^{n,p}(\Omega)} \equiv \sum_{|\alpha|\le n} \Big(\int_\Omega |D^\alpha u(x)|^p \, dx \Big)^{1/p}$$

writing $L^p(\Omega)$ [resp. $H^n(\Omega)$] rather than $W^{0,p}(\Omega)$ [resp. $W^{n,2}(\Omega)$], and using the shorthands $W^{n,p}$ and L^p for $W^{n,p}(\mathbb{R}^d)$ and $L^p(\mathbb{R}^d)$, respectively.
We denote by $W^{n,p}_{\mathrm{loc}}$ [resp. L^p_{loc}, H^n_{loc}] the linear space of all measurable complex-valued functions on $\mathbb{R}^d$ belonging to $W^{n,p}(\Omega)$ [resp. $L^p(\Omega)$, $H^n(\Omega)$] for every open subset Ω of $\mathbb{R}^d$ having compact closure, and, for any fixed real-valued function $\xi \in W^{n,p}_{\mathrm{loc}}$, we define the *weighted Sobolev space* $W^{n,p}_\xi$ as the completion of $C_c^\infty(\mathbb{R}^d)$ with respect to the *weighted norm*

$$\|u\|_{W^{n,p}_\xi} \equiv \|\xi u\|_{W^{n,p}}.$$

It is well known that $W^{n,p}_\xi$ can also be defined as the space of all measurable functions u such that $\xi u \in W^{n,p}$. Similarly, for any choice of the functions α, β_i, $i = 1, \dots, d$, $\delta_{i,j}$, $i, j = 1, \dots, d$ belonging to L^p_{loc}, with $\operatorname{essinf} |\alpha| > 0$, we introduce the weighted Sobolev spaces $W^{1,p}_{(\alpha,\beta)}$ and $W^{2,p}_{(\alpha,\beta,\delta)}$ defined as the completion of $C_c^\infty(\mathbb{R}^d)$ with respect to the *weighted norm*

$$\|u\|_{W^{1,p}_{(\alpha,\beta)}} \equiv \|\alpha u\|_{L^p} + \sum_{i=1}^d \|\beta_i D_i u\|_{L^p}$$

and

$$\|u\|_{W^{2,p}_{(\alpha,\beta,\delta)}} \equiv \|\alpha u\|_{L^p} + \sum_{i=1}^d \|\beta_i D_i u\|_{L^p} + \sum_{i,j=1}^d \|\delta_{i,j} D_{i,j} u\|_{L^p}$$

respectively and we introduce also the spaces $W^{1,p}_{\xi,(\alpha,\beta)}$ [resp. $W^{2,p}_{\xi,(\alpha,\beta,\delta)}$] of all measurable functions u such that $\xi u \in W^{1,p}_{(\alpha,\beta)}$ [resp. $\xi u \in W^{2,p}_{(\alpha,\beta,\delta)}$], endowed with the norms

$$\|u\|_{W^{1,p}_{\xi,(\alpha,\beta)}} \equiv \|\xi u\|_{W^{1,p}_{(\alpha,\beta)}} \quad [\text{resp. } \|u\|_{W^{2,p}_{\xi,(\alpha,\beta,\delta)}} \equiv \|\xi u\|_{W^{2,p}_{(\alpha,\beta,\delta)}}].$$

Lastly we denote with L^p_ξ the space $W^{0,p}_\xi$.

Let us now consider the formal second-order differential operator

$$\mathcal{A}u \equiv \sum_{i,j=1}^d \psi_i(x)\psi_j(x) a_{i,j}(x) D_{i,j} u + \sum_{i=1}^d b_i(x) D_i u - \gamma^2(x) u. \tag{1}$$

Assumption 2.1.

1. For all $i, j = 1, \ldots, d$, the coefficients $a_{i,j}(x)$ are bounded differentiable real-valued functions on $\mathbb{R}^d$ such that $a_{i,j}(x) = a_{j,i}(x)$, and satisfying the strong ellipticity condition
$$\mathrm{Re} \sum_{i,j=1}^{d} a_{i,j}(x) z_i \overline{z}_j \geq E\, |z|^2 \quad \forall z \in \mathbb{C}^d,$$
for a suitable ellipticity modulus $E > 0$ independent of $x \in \mathbb{R}^d$;
2. for every $i = 1, \ldots, d$, the coefficients $b_i(x)$ are measurable real-valued functions on $\mathbb{R}^d$, while $\gamma(x)$ is a real-valued function in L^2_{loc} with $\mathrm{essinf}(\gamma) \geq 1$;[1]
3. for all $i = 1, \ldots, d$ the coefficients $\psi_i(x)$ are differentiable, and we have
$$\begin{aligned} |b_i(x)| &\leq B_1 E^{1/2} \eta_{1,i}(x)\, |\psi_i(x)|\, \gamma(x) \quad \forall x \in \mathbb{R}^d, \\ |D_j(\psi_i(x)\psi_j(x)a_{i,j}(x))| &\leq B_2 E^{1/2} \eta_{2,i,j}(x)\, |\psi_i(x)|\, \gamma(x) \quad \forall x \in \mathbb{R}^d, \end{aligned} \tag{2}$$
for suitable constants B_1 and B_2 such that $B_1 + B_2 < 2$ and measurable positive functions $\eta_{1,i}(x)$ and $\eta_{2,i,j}(x)$ satisfying
$$\sum_{i=1}^{d} \eta^2_{1,i}(x) = d \sum_{i,j=1}^{d} \eta^2_{2,i,j}(x) = 1.$$

Assumption 2.1 allows us to reduce the analysis of the nonvariational case to the analysis of the variational one. Indeed, introducing the sesquilinear form $a(\cdot,\cdot)$ associated to the operator $\mathcal{A}$, given by
$$a(u, v) \equiv \widehat{a}(u, v) - \int_{\mathbb{R}^d} \sum_{i,j=1}^{d} D_j(\psi_i(x)\psi_j(x)a_{i,j}(x)) D_i u(x) \overline{v}(x) dx,$$
for all $u \in H^{1,p}_{(\gamma,\psi)}$ $1 < p < \infty$ and $v \in H^{1,q}_{(\gamma,\psi)}$ where q is the conjugate of p , and writing
$$D(\mathcal{A}_p) \equiv \left\{ u \in H^{1,p}_{(\gamma,\psi)} : \exists K(u) > 0 \text{ s.t. } |a(u, \varphi)| \leq K(u) \|\varphi\|_q \ \forall \varphi \in C^\infty_c(\mathbb{R}^d) \right\},$$
one can study the realization $\mathcal{A}_p : D(\mathcal{A}_p) \to L^p$ of $\mathcal{A}$ by considering for each $\lambda \in \mathbb{C}$ such that $Re\lambda > 0$, the equation
$$(\lambda - \mathcal{A}_p)u = f. \tag{3}$$

In [11] the following results are proved:

Theorem 2.2. *Under Assumption* 2.1*, the operator $\mathcal{A}_2 : D(\mathcal{A}_2) \to L^2$ generates an analytic semigroup on L^2.*

[1] This condition could be replaced by the seemingly more general essinf $\gamma > 0$, provided to employ a standard normalization procedure.

Moreover,

Corollary 2.3. *Under Assumption* 2.1*, for every solution* $u \in D(\mathcal{A}_2)$ *of* (3)*, we have*

$$|\lambda|^{1/2} \|\gamma u\|_{L^2} \leq K' \|f\|_{L^2} \quad and \quad |\lambda|^{1/2} \|\psi_i D_i u\|_{L^2} \leq K'' \|f\|_{L^2},$$

for suitably chosen $K', K'' > 0$ *independent of* λ.

In order to obtain suitable estimates for the first-order derivatives we need the following assumption:

Assumption 2.4. Under 1. and 2. of Assumption 2.1, suppose in addition that γ is continuously differentiable and that, for all $i, j = 1, \ldots, d$ and $x \in \mathbb{R}^d$, we have

$$\begin{aligned} |b_i(x)| &\leq B_1 E^{1/2} \eta_{1,i}(x) \, |\psi_i(x)| \, \gamma(x), \\ |D_j(\psi_i(x)\psi_j(x)a_{i,j}(x))| &\leq B_2 E^{1/2} \eta_{2,i,j}(x) \, |\psi_i(x)| \, \gamma(x), \\ 2 \, |\psi_j(x) D_j \gamma(x) a_{i,j}(x)| &\leq B_3 E^{1/2} \eta_{3,i,j}(x) \gamma^2(x), \end{aligned}$$

for suitable constants B_1, B_2 and B_3 such that $B_1 + B_2 + B_3 < 2$ and suitable measurable functions $\eta_{1,i}(x)$, $\eta_{2,i,j}(x)$ and $\eta_{3,i,j}(x)$ on $\mathbb{R}^d$ satisfying $\sum_{i=1}^d \eta_{1,i}(x) = d \sum_{i,j=1}^d \eta_{2,i,j}^2(x) = d \sum_{i,j=1}^d \eta_{3,i,j}^2(x) = 1$.

We have

Theorem 2.5. *Under Assumption* 2.4*, both* $\gamma^2 u$ *and* $\psi_i \gamma D_i u$ *belong to* L^2*, for every* $i = 1, \ldots, d$*. More precisely,* u *belongs to* $H^1_{(\gamma^2, \gamma\psi)}$*, and*

$$\|u\|_{H^1_{(\gamma^2,\gamma\psi)}} \leq K \|f\|_{L^2}$$

holds true for a suitable $K > 0$*. In particular, for every* $i = 1, \ldots, d$*, also* $b_i D_i u$ *belongs to* L^2*, and we have*

$$\sum_{i=1}^d \|b_i D_i u\|_{L^2} \leq d^2 B_1 E^{1/2} \sum_{i=1}^d \|\psi_i \gamma D_i u\|_{L^2}.$$

Aiming to show that for all $i, j = 1, \ldots, d$ the single summand $\psi_i(x)\psi_j(x)D_{i,j}u(x)$ belongs to L^2, we need to strengthen our hypotheses on the coefficients $\psi_i(x)$'s. Therefore, having in mind our examples, we will assume then the negligibility of the set

$$Z \equiv \left\{ x \in \mathbb{R}^d : \psi_i(x) = 0, \text{ for some } i = 1, \ldots, d \right\},$$

of all zeros of the $\psi_i(x)$'s, and the existence of a suitable countable covering of $\mathbb{R}^d - Z$ which allows us to perform a localization procedure. Such a covering will be made by rectangles of the type

$$R(x_0, r\psi) \equiv \left\{ x \in \mathbb{R}^d : |x_i - x_i^{(0)}| \leq r|\psi_i(x_0)|, \ i = 1, \ldots, d \right\},$$

for $x_0 \equiv (x_1^{(0)}, \ldots, x_d^{(0)}) \in \mathbb{R}^d - Z$ and $r > 0$.

Assumption 2.6. Under Assumption 2.4, suppose in addition that

(i) for every $i = 1, \ldots, d$, the differentiable function $\psi_i(x)$ belongs to H^1_{loc} and the set Z is negligible;

(ii) there exist real numbers $r_1 > 0$ and $L > 0$ such that for every $0 < r \leq r_1$ we can find a countable set $N_r \subset \mathbb{R}^d - Z$ such that

  (a) the family $\mathcal{F}_1 \equiv \{R(x, r\psi)\}_{x \in N_r}$ is a covering of $\mathbb{R}^d - Z$;

  (b) each rectangle of the family $\mathcal{F}_2 \equiv \{R(x, 2r\psi)\}_{x \in N_r}$ does not contain any element of Z and has a nonempty intersection with at most a fixed number n_0 of other rectangles of $\mathcal{F}_2$ itself;

  (c) we have

$$\frac{1}{L} \leq \min_{i=1,\ldots,d} \inf_{x \in R(x_0, 2r\psi)} \frac{|\psi_i(x)|}{|\psi_i(x_0)|} \leq \max_{i=1,\ldots,d} \sup_{x \in R(x_0, 2r\psi)} \frac{|\psi_i(x)|}{|\psi_i(x_0)|} \leq L,$$

  for each $x_0 \in N_r$.

Remark 2.7. Assumption 2.6 is convenient for proving the characterization result. However, it is not easy to check it for given operators. In [11] it is proved that 2.6 is verified under a more treatable assumption, befitted with examples coming from financial mathematics. More precisely, it was shown that Assumption 2.8 below implies 2.6.

Assumption 2.8. Under Assumption 2.4, suppose in addition that

(i) Part (i) of Assumption 2.6 holds true;

(ii) there exist $r_0 > 0$ (small), $R_0 > 0$ (large), and $\alpha > 0$ such that for every $x \in \left\{x \in \mathbb{R}^d : \mathrm{dist}(x, Z) < r_0 \text{ or } \mathrm{dist}(x, 0) > R_0\right\} \equiv \mathcal{D}(r_0, R_0)$ and every $i = 1, \ldots, d$ we have

$$|D_j \psi_i(x)| \leq \alpha;$$

(iii) for every $i = 1, \ldots, d$ the function $\psi_i(x)$ depends only on the variable x_i.

Theorem 2.9. *Under Assumption* 2.6*, the functions*

$$\psi_i(x)\psi_j(x)D_{i,j}u(x)$$

belong to L^2 for all $i, j = 1, \ldots, d$. More precisely, we have $u \in H^2_{(\gamma^2, \gamma\psi, \psi^2)}$ and the estimate

$$|\lambda| \|u\|_{L^2} + |\lambda|^{1/2} \|u\|_{H^1_{(\gamma,\psi)}} + \|u\|_{H^2_{(\gamma^2,\gamma\psi,\psi^2)}} \leq K \|f\|_{L^2}$$

holds true for a suitable $K > 0$.

Such results can be extended to the case of weighted Sobolev spaces:

Theorem 2.10. *Assume that Assumption* 2.4 *still holds true when replacing the first-order term of the operator $\mathcal{A}$ with*

$$\sum_{i=1}^{d} b_i D_i + \sum_{i,j=1}^{d} \psi_i \psi_j a_{i,j} \Big(\frac{D_i \xi}{\xi} D_j + \frac{D_j \xi}{\xi} D_i \Big)$$

and the zero-order term with

$$-\gamma^2 + \sum_{i,j=1}^{d} \psi_i \psi_j a_{i,j} \Big(\frac{D_{i,j}\xi}{\xi} + 2\frac{D_i\xi D_j\xi}{\xi^2} \Big) + \sum_{i=1}^{d} b_i \frac{D_i\xi}{\xi};$$

then the operator $\mathcal{A}$ has a realization $\mathcal{A}_{2,\xi} : D(\mathcal{A}_{2,\xi}) \to L^2_\xi$ which generates an analytic semigroup on L^2_ξ. Moreover, for each $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\ \lambda > 0$*, the resolvent equation $\lambda u - \mathcal{A}_{2,\xi} u = f$ has, for every $f \in L^2$, a unique solution $u \in D(\mathcal{A}_{2,\xi})$, which satisfies the estimate*

$$|\lambda| \|u\|_{L^2_\xi} + |\lambda|^{1/2} \|u\|_{H^{1,2}_{\xi,(\gamma,\psi)}} + \|u\|_{H^{2,2}_{\xi,(\gamma^2,\gamma\psi,\psi^2)}} \le C \|f\|_{L^2_\xi},$$

for a suitable constant $C > 0$. In particular we have $D(\mathcal{A}_{2,\xi}) = H^2_{\xi,(\gamma^2,\gamma\psi,\psi^2)}$.

Remark 2.11. By using the Korn's argument it is possible to pass from generation results in the case of differentiable coefficients to similar result in the case of continuous coefficients. However, in such a general setting, it is impossible to find a general approach leading to this kind of results. This can be done in particular cases with different procedures.

3. Generation of analytic semigroups on $L^p(\mathbb{R}^d)$

In order to prove the L^p estimates we need an additional assumption. Actually we need the existence of a suitable countable covering of $\mathbb{R}^d - Z$ which allows us to perform a localization procedure. Let $r_{\gamma(x_0)}$ be the minimum between r and $\gamma(x_0)^{-1}$. Let

$$R(x_0, r_{\psi,\gamma}) \equiv \Big\{ x \in \mathbb{R}^d : |x_i - x_i^{(0)}| \le r_{\gamma(x_0)} |\psi_i(x_0)|,\ i = 1, \dots, d \Big\},$$

for $x_0 \equiv (x_1^{(0)}, \dots, x_d^{(0)}) \in \mathbb{R}^d - Z$ and $r > 0$.

Assumption 3.1. Under Assumption 2.4, suppose in addition that

(i) Part (i) of Assumption 2.6 holds true;

(ii) There exist real numbers $r_1 > 0$ and $L > 0$ such that for every $0 < r \le r_1$ we can find a countable set $N_r \subset \mathbb{R}^d - Z$ such that

  (a) the family $\mathcal{F}_1 \equiv \{R(x, r_{\psi,\gamma})\}_{x \in N_r}$ is a covering of $\mathbb{R}^d - Z$;

  (b) each rectangle of the family $\mathcal{F}_2 \equiv \{R(x, 2r_{\psi,\gamma})\}_{x \in N_r}$ does not contain any element of Z and has a nonempty intersection with at most a fixed number n_0 of other rectangles of $\mathcal{F}_2$ itself;

  (c) we have

$$\frac{1}{L} \le \inf_{x \in R(x_0, 2r_{\psi,\gamma})} \frac{|\gamma(x)|}{|\gamma(x_0)|} \le \sup_{x \in R(x_0, 2r_{\psi,\gamma})} \frac{|\gamma(x)|}{|\gamma(x_0)|} \le L,$$

  for each $x_0 \in N_r$.

Remark 3.2. As done in Remark 2.7, it is possible to find stronger conditions that imply Assumption 3.1 and that are usually satisfied by the classical problems arising from financial mathematics.

Lemma 3.3. *Under Assumption* 3.1*, assume to have proved that for a* $p \geq 2$ *the solution* u *of the resolvent equation* (3)*, related to some* $\lambda \in \mathbb{C}$ *such that* $\operatorname{Re}\lambda > 0$ *belongs to* $W^{2,p}_{(\gamma^2,\gamma\psi,\psi^2)}$ *and satisfies the estimate*

$$|\lambda|\|u\|_{L^p} + |\lambda|^{1/2}\|u\|_{W^{1,p}_{(\gamma,\psi)}} + \|u\|_{W^{2,p}_{(\gamma^2,\gamma\psi,\psi^2)}} \leq C'\|f\|_{L^p} \tag{4}$$

for some $C' > 0$ *and for each* $f \in L^p$.

Let $q \in (p, p^*)$ *and let* $f \in L^q$*. Assume that* $u \in W^{1,p}_{(\gamma^2,\gamma\psi)}$ *is a solution of* (3)*. Then* u *satisfies the estimate*

$$|\lambda|\|u\|_{L^q} + |\lambda|^{1/2}\|u\|_{W^{1,q}_{(\gamma,\psi)}} + \|u\|_{W^{1,q}_{(\gamma^2,\gamma\psi)}} \leq C\|f\|_{L^q}, \tag{5}$$

for every $\lambda \in \mathbb{C}$ *whose real part is greater than a suitable fixed positive real number* ω *and for a suitable* $C > 0$ *independent of* λ*. Moreover if* $p > d$ *we may choose* $q = \infty$*.*

Proof. For each $x_0 \equiv (x_1^0, \ldots, x_d^0) \in \mathbb{R}^d - Z$, we also consider the change of variables $T_{x_0,\psi} : \mathbb{R}^d \mapsto \mathbb{R}^d$ defined by

$$T_{x_0,\psi}(x) \stackrel{def}{=} ((x_1 - x_1^0)/\left|\psi_1(x_0)\right|, \ldots, (x_d - x_d^0)/\left|\psi_d(x_0)\right|),$$

of inverse

$$T^{-1}_{x_0,\psi}(x) = (x_1^0 + \left|\psi_1(x_0)\right| x_1, \ldots, x_d^0 + \left|\psi_d(x_0)\right| x_d).$$

Furthermore, for every $r > 0$ we denote by $B(x_0, r)$ the d-dimensional ball centered at 0 with radius r and we write $B(x_0, r\psi)$ for the d-dimensional ellipsoid centered at x_0 with semiaxes $r\left|\psi_1(x_0)\right|, \ldots, r\left|\psi_d(x_0)\right|$. Clearly

$$T_{x_0,\psi}(B(x_0, r\psi)) = B(0, r) \quad \text{and} \quad T^{-1}_{x_0,\psi}(B(0, r)) = B(x_0, r\psi).$$

Consider the change of variables

$$\widetilde{u}(x) \stackrel{def}{=} (u \circ T^{-1}_{x_0,\psi})(x),$$

and let $\theta(x)$ be any smooth cut-off function such that

$$\begin{cases} \theta(t) = 1 & \text{if } t \in [0,1] \\ \theta(t) = 0 & \text{if } t \in [2, +\infty[\end{cases},$$

for each $0 < r \leq r_0$ we can define a cut-off function on $\mathbb{R}^d$ by setting

$$\theta_r(x) \stackrel{def}{=} \theta\left(\frac{|x|}{r}\right),$$

and we can consider the function

$$v(x) = \theta_r(x)\widetilde{u}(x).$$

Clearly $v \in L^p$ satisfies the following equation

$$\begin{aligned}
&\sum_{i,j=1}^{d} \frac{\widetilde{\psi}_i(x)\widetilde{\psi}_j(x)}{\psi_i(x_0)\psi_j(x_0)}\widetilde{a}_{i,j}(x)D_{i,j}v(x) + \sum_{i=1}^{d}\frac{\widetilde{b}_i(x)}{\psi_i(x_0)}D_i v(x) - \widetilde{\gamma}^2 v(x) - \lambda v(x) \\
&= \theta_r(x)\widetilde{f}(x) + \sum_{i=1}^{d}\frac{\widetilde{b}_i(x)}{\psi_i(x_0)}D_i\theta_r(x)\widetilde{u}(x) \\
&+ \sum_{i,j=1}^{d} \frac{\widetilde{\psi}_i(x)\widetilde{\psi}_j(x)}{\psi_i(x_0)\psi_j(x_0)}\widetilde{a}_{i,j}(x)[\widetilde{u}(x)D_{i,j}\theta_r(x) + D_i\theta_r(x)D_j\widetilde{u}(x) + D_j\theta_r(x)D_i\widetilde{u}(x)],
\end{aligned}$$

whose right side $\widetilde{h}$ satisfies

$$\widetilde{h} \in L^p(B(0,2r)) \quad \text{and} \quad \widetilde{h} = 0 \text{ on } \partial B(0,2r).$$

On the other hand, by the assumption of the lemma the solution $v \in L^p$ satisfies the estimate

$$|\lambda|\|v\|_{L^p} + |\lambda|^{1/2}\|v\|_{W^{1,p}_{(\gamma,\psi)}} + \|v\|_{W^{2,p}_{(\gamma,\psi\gamma,\psi^2)}} \le C_1\|\widetilde{h}\|_{L^p},$$

for a suitable $C > 0$ independent of λ.
Furthermore we can also prove that

$$\begin{aligned}
\|\widetilde{h}\|_{L^p} \quad &\le \quad C_2\Bigg[\|\widetilde{f}\|_{L^p(B(0,2r))} + \frac{1}{r^2}\|\widetilde{u}\|_{L^p(B(0,2r))} + \frac{1}{r}\|D\widetilde{u}\|_{L^p(B(0,2r))} \\
&\quad + \frac{1}{r}\|\widetilde{\gamma}\widetilde{u}\|_{L^p(B(0,2r))}\Bigg],
\end{aligned}$$

for a suitable $C_2 > 0$ where the last estimate comes from equation (2).
Combining the above estimates, we obtain

$$\begin{aligned}
&|\lambda|\|\widetilde{u}\|_{L^p(B(0,r))} + |\lambda|^{1/2}\left[\gamma(x_0)\|\widetilde{u}\|_{L^p(B(0,r))} + \|D\widetilde{u}\|_{L^p(B(0,r))}\right] \\
&+\gamma^2(x_0)\|\widetilde{u}\|_{L^p(B(0,r))} + \gamma(x_0)\|D\widetilde{u}\|_{L^p(B(0,r))} + \|D^2\widetilde{u}\|_{L^p(B(0,r))} \\
&\le \quad C\Bigg[\|\widetilde{f}\|_{L^p(B(0,2r))} + \frac{1}{r^2}\|\widetilde{u}\|_{L^p(B(0,2r))} + \frac{1}{r}\|D\widetilde{u}\|_{L^p(B(0,2r))} \\
&\qquad + \frac{1}{r}\gamma(x_0)\|\widetilde{u}\|_{L^p(B(0,2r))}\Bigg],
\end{aligned} \tag{6}$$

where we suppose r small enough in order that Assumption 3.1 holds. Now, if we take $q \in (p, p^*)$ and $\delta(q) = d/q - d/p + 1$, for every $\varepsilon > 0$ there exists (see [13, p. 66]) $C(\varepsilon) > 0$ such that

$$\|\widetilde{u}\|_{L^q(B(0,r))} \le \varepsilon r^{\delta(q)}\|D\widetilde{u}\|_{L^p(B(0,r))} + C(\varepsilon)r^{\delta(q)-1}\|\widetilde{u}\|_{L^p(B(0,r))}, \tag{7}$$

and

$$\|D\widetilde{u}\|_{L^q(B(0,r))} \le \varepsilon r^{\delta(q)}\|D^2\widetilde{u}\|_{L^p(B(0,r))} + C(\varepsilon)r^{\delta(q)-2}\|\widetilde{u}\|_{L^p(B(0,r))}. \tag{8}$$

Combining estimates (7) and (8) we have

$$\begin{aligned}&\frac{1}{r^2}\|\widetilde{u}\|_{L^q(B(0,r))}+\frac{1}{r}\|D\widetilde{u}\|_{L^q(B(0,r))}\\ &\leq \frac{1}{r^2}\left[\varepsilon r^{\delta(q)}\|D\widetilde{u}\|_{L^p(B(0,r))}+C(\varepsilon)r^{\delta(q)-1}\|\widetilde{u}\|_{L^p(B(0,r))}\right]\\ &\quad+\frac{1}{r}\left[\varepsilon r^{\delta(q)}\|D^2\widetilde{u}\|_{L^p(B(0,r))}+C(\varepsilon)r^{\delta(q)-2}\|\widetilde{u}\|_{L^p(B(0,r))}\right],\end{aligned}$$

and, rearranging the terms,

$$\begin{aligned}&\frac{1}{r^2}\|\widetilde{u}\|_{L^q(B(0,r))}+\frac{1}{r}\|D\widetilde{u}\|_{L^q(B(0,r))}\\ &\leq C(\varepsilon)r^{\delta(q)-3}\|\widetilde{u}\|_{L^p(B(0,r))}+\varepsilon r^{\delta(q)-2}\|D\widetilde{u}\|_{L^p(B(0,r))}+\varepsilon r^{\delta(q)-1}\|D^2\widetilde{u}\|_{L^p(B(0,r))}.\end{aligned}$$

Taking into account (6), the above estimate implies

$$\begin{aligned}&\frac{1}{r^2}\|\widetilde{u}\|_{L^q(B(0,r))}+\frac{1}{r}\|D\widetilde{u}\|_{L^q(B(0,r))}\\ &\leq\left[C(\varepsilon)r^{\delta(q)-3}(|\lambda|+\gamma(x_0)^2)^{-1}+\varepsilon r^{\delta(q)-2}(|\lambda|^{\frac12}+\gamma(x_0))^{-1}+\varepsilon r^{\delta(q)-1}\right]\\ &\left[\|\widetilde{f}\|_{L^p(B(0,2r))}+\frac{1}{r^2}\|\widetilde{u}\|_{L^p(B(0,2r))}+\frac{1}{r}\|D\widetilde{u}\|_{L^p(B(0,2r))}+\frac{1}{r}\gamma(x_0)\|\widetilde{u}\|_{L^p(B(0,2r))}\right].\end{aligned}$$

By the Hölder inequality we get

$$\begin{aligned}&\frac{1}{r^2}\|\widetilde{u}\|_{L^q(B(0,r))}+\frac{1}{r}\|D\widetilde{u}\|_{L^q(B(0,r))}\\ &\leq\left[C(\varepsilon)r^{-2}(|\lambda|+\gamma(x_0)^2)^{-1}+\varepsilon r^{-1}(|\lambda|^{\frac12}+\gamma(x_0))^{-1}+\varepsilon\right]\\ &\left[\|\widetilde{f}\|_{L^q(B(0,2r))}+\frac{1}{r^2}\|\widetilde{u}\|_{L^q(B(0,2r))}+\frac{1}{r}\|D\widetilde{u}\|_{L^q(B(0,2r))}+\frac{1}{r}\gamma(x_0)\|\widetilde{u}\|_{L^q(B(0,2r))}\right].\end{aligned}$$

Finally, if we take:

- $\varepsilon > 0$ a small number to be chosen later
- $r_0 = (|\lambda|^{\frac12}+\gamma(x_0))^{-1}$
- $r = \alpha r_0$, where α is a number to be chosen later,

then we obtain

$$\begin{aligned}&\frac{1}{\alpha^2}(|\lambda|^{\frac12}+\gamma(x_0))^2\|\widetilde{u}\|_{L^q(B(0,r_0))}+\frac{1}{\alpha}(|\lambda|^{\frac12}+\gamma(x_0))\|D\widetilde{u}\|_{L^q(B(0,r_0))}\\ &\leq\left[C(\varepsilon)\frac{1}{\alpha^2}+\varepsilon\frac{1}{\alpha}+\varepsilon\right]\left[\|\widetilde{f}\|_{L^q(B(0,2r_0))}+\frac{1}{\alpha^2}(|\lambda|^{\frac12}+\gamma(x_0))^2\|\widetilde{u}\|_{L^q(B(0,2r_0))}\right.\\ &\left.+\frac{1}{\alpha}(|\lambda|^{\frac12}+\gamma(x_0))\|D\widetilde{u}\|_{L^q(B(0,2r_0))}+\frac{1}{\alpha}(|\lambda|^{\frac12}+\gamma(x_0))\gamma(x_0)\|\widetilde{u}\|_{L^q(B(0,2r_0))}\right] \quad (9)\end{aligned}$$

So, if $q < \infty$, by changing variable back, summing up to the covering and using Assumption 3.1, we have

$$\begin{aligned} &\frac{1}{\alpha^2}(|\lambda|\|u\|_{L^q} + \|\gamma^2 u\|_{L^q} + \frac{1}{\alpha}(|\lambda|^{1/2}\|Du\|_{L^q} + \|\gamma Du\|_{L^q}) \\ &\leq \; 2L^4 n_0 \left[C(\varepsilon)\frac{1}{\alpha^2} + \varepsilon\frac{1}{\alpha} + \varepsilon\right]\left[\frac{1}{\alpha^2}(|\lambda|\|u\|_{L^q} + \|\gamma^2 u\|_{L^q}) \right. \\ &\quad \left. + \frac{1}{\alpha}(|\lambda|^{1/2}\|\gamma u\|_{L^q} + \|\gamma^2 u\|_{L^q}) + \frac{1}{\alpha}(|\lambda|^{1/2}\|Du\|_{L^q} + \|\gamma Du\|_{L^q}) + \|f\|_{L^q}\right]. \end{aligned}$$

The statement follows from the above estimate choosing $\varepsilon = \frac{1}{8L^4 n_0}$ and $\alpha = 4C(\varepsilon)L^4 n_0$. If $q = \infty$ the argument is easier. Actually by changing variable back and by localizing around the points where $u(x)$, $\gamma^2(x)u(x)$, $Du(x)$, $\psi(x)Du(x)$ and $\gamma(x)\psi(x)Du(x)$ attain the maximum, the result follows directly from (9) without using a covering argument. $\square$

Remark 3.4. Assume $f \in L^\infty$. Arguing as in [15] from the proof of the previous lemma it is possible to get an estimate for the second derivatives. Actually, starting from equation (6), by the Hölder inequality and choosing a suitable $r = \alpha(|\lambda|^{\frac{1}{2}} + \gamma(x_0))^{-1}$, one gets that for each $q > d$:

$$\sup_{x_0 \in \mathbb{R}^d} (|\lambda|^{\frac{1}{2}} + \gamma(x_0))^{\frac{d}{q}} \|\psi^2 D^2 u\|_{L^q(B(x_0, r_0\psi))} \leq C\|f\|_{L^\infty}. \tag{10}$$

Remark 3.5. Estimate (4) implies the uniqueness of the solution of (3) in $W^{1,q}_{\gamma^2,\gamma\psi}$.

Lemma 3.6. *Under the assumptions of Lemma* 3.3 *we have that* $u \in W^{2,q}_{\gamma^2,\gamma\psi,\psi^2}$ *and*

$$\|\psi^2 D^2 u\|_{L^q} \leq C\|f\|_{L^q}.$$

Proof. Using the notation of the previous lemma we have that the function $\widetilde{u}(x)$ satisfies the following equation in $B(x_0, 2r)$

$$\sum_{i,j=1}^{d} \frac{\widetilde{\psi}_i(x)\widetilde{\psi}_j(x)}{\psi_i(x_0)\psi_j(x_0)}\widetilde{a}_{i,j}(x)D_{i,j}\widetilde{u}(x)) = -\sum_{i=1}^{d}\frac{\widetilde{b}_i(x)}{\psi_i(x_0)}D_i\widetilde{u}(x) - \widetilde{\gamma}^2\widetilde{u}(x) + \widetilde{f}(x) = \widetilde{h}(x).$$

Noting that the second-order differential operator

$$\widetilde{u}(x) \mapsto \sum_{i,j=1}^{d} \frac{\widetilde{\psi}_i(x)\widetilde{\psi}_j(x)}{\psi_i(x_0)\psi_j(x_0)}\widetilde{a}_{i,j}(x)D_{i,j}\widetilde{u}(x)$$

is a strongly elliptic operator in $B(0, 2r)$ and, thanks to known regularity results (see [6] and also [9, Theor. 17.2, p. 67], [10, 8.3, p. 173]) it follows that $\widetilde{u}(x) \in W^{2,q}(B(0, r))$ and

$$\| D^2\widetilde{u}\|_{L^q(B(x_0,2r)} \leq C\|\widetilde{h}\|_{L^q(B(x_0,2r)}.$$

The statement follows changing variable back, summing up to the covering and applying the results of Lemma 3.3. $\square$

Theorem 3.7. *Assume that* 3.1 *holds. Then for each* $2 \leq q \leq \infty$ *the operator* $\mathcal{A}_q : D(\mathcal{A}_q) \to L^q$ *generates an analytic semigroup on* L^p. *Moreover if* $q < \infty$ *for every solution* $u \in D(\mathcal{A}_q)$ *of* (3), *we have*

$$|\lambda|\|u\|_{L^q} + |\lambda|^{\frac{1}{2}}\|u\|_{W^{1,q}_{(\gamma,\psi)}} + \|u\|_{W^{2,q}_{(\gamma^2,\gamma\psi,\psi^2)}} \leq C\|f\|_{L^q} \tag{11}$$

for every $\lambda \in \mathbb{C}$ *whose real part is greater than a suitable fixed positive real number* ω *and for a suitable* $C > 0$ *independent of* λ.
If $q = \infty$ *then for every solution* $u \in D(\mathcal{A}_\infty)$ *of* (3), *we have*

$$|\lambda|\|u\|_{L^\infty} + |\lambda|^{\frac{1}{2}}\|u\|_{W^{1,\infty}_{(\gamma,\psi)}} + \|u\|_{W^{1,\infty}_{(\gamma^2,\gamma\psi)}} \leq C\|f\|_{L^\infty} \tag{12}$$

for every $\lambda \in \mathbb{C}$ *whose real part is greater than a suitable fixed positive real number* ω *and for a suitable* $C > 0$ *independent of* λ.

Proof. This Theorem is true for $p = 2$. Let $q \in (2, 2^*]$ where 2^* is the Sobolev exponent. Let u be a solution of (3) and assume that $f \in L^2 \cap L^q$. By applying Lemmata 3.3 and 3.6 we have that (11) holds.
If $f \in L^q$, we consider a sequence of functions $f_n \in L^2 \cap L^q$ converging to f in L^q. Then the sequence of associated solutions u_n is a Cauchy sequence in $W^{2,q}_{(\gamma^2,\gamma\psi,\psi^2)}$ converging to a function u that is a solution of (3). Moreover this solution is unique by Remark 3.5. Therefore the result is proved for any $q \in (2, 2^*]$.
If $q \in (2^*, (2^*)^*)$ one can prove the result iterating the previous argument. After a finite number of steps the statement follows. □

Using duality techniques one may prove a generation result in L^p with $1 \leq p < 2$.

Theorem 3.8. *Assume* 3.1 *holds. Let* $1 \leq p < 2$, *then the operator* $\widehat{A}_p : D(\mathcal{A}_p) \to L^p$ *generates an analytic semigroup on* L^p.

Proof. The statement follows if we show that a solution of (3) for $1 \leq p < 2$ exists and is unique, and moreover

$$|\lambda|\|u\|_{L^p} \leq C\|f\|_{L^p} \tag{13}$$

for every $\lambda \in \mathbb{C}$ whose real part is greater than a suitable fixed positive real number ω and for a suitable $C > 0$ independent of λ.
Assume first $f \in L^p \cap L^2$ so one has the existence of a solution of (3). Let $\widehat{\mathcal{A}}$ the operator in variational form

$$\widehat{\mathcal{A}}u \equiv \sum_{i,j=1}^{d} D_j(\psi_i(x)\psi_j(x)a_{i,j}(x)D_i u) + \sum_{i=1}^{d} b_i(x)D_i u - \gamma^2(x)u,$$

and let $\widehat{\mathcal{A}}^*$ its adjoint.
Define the function space $H = \{g \in L^{p'} : \|g\|_{L^{p'}} = 1\}$ where p' is the conjugate of p. Then

$$\|u\|_{L^p} = \sup_{g \in H} \int ugdx.$$

For each $g \in H$, let v_g be the solution of the equation

$$(\widehat{\mathcal{A}}^* - \lambda)v_g = g. \tag{14}$$

By Theorem 3.7

$$|\lambda| \|v_g\|_{L^{p'}} \le C\|g\|_{L^{p'}} \le C.$$

Therefore

$$\begin{aligned} \|u\|_{L^p} &= \sup_{g\in H} \int u(\widehat{\mathcal{A}}^* - \lambda)v_g dx = \sup_{g\in H} \int (\widehat{\mathcal{A}} - \lambda)uv_g dx = \sup_{g\in H} \int f v_g dx \\ &\le C|\lambda|^{-1} \|f\|_{L^p}. \end{aligned}$$

Note that the duality argument implies directly the uniqueness of a variational solution of (3). We have only to prove the existence in the general case. If $f \in L^p$ we can find a sequence of function $f_n \in L^p \cap L^2$ converging to $f \in L^p$. By the previous estimate we have that the solutions u_n are a Cauchy sequence in L^p. Using the regularity of the coefficients, it is not difficult to prove that the functions u_n converge to the solution u of (3) that satisfies estimate (13). □

Remark 3.9. If one assumes more regular coefficients, one may characterize the domain also in the case $p < 2$. Precisely for every solution $u \in D(\mathcal{A}_p)$ of (3) we have

$$|\lambda| \|u\|_{L^p} + |\lambda|^{\frac{1}{2}} \|u\|_{W^{1,p}_{(\gamma,\psi)}} + \|u\|_{W^{2,p}_{(\gamma^2,\gamma\psi,\psi^2)}} \le C\|f\|_{L^p} \tag{15}$$

or

$$|\lambda| \|u\|_{L^1} + |\lambda|^{\frac{1}{2}} \|u\|_{W^{1,1}_{(\gamma,\psi)}} + \|u\|_{W^{1,1}_{(\gamma^2,\gamma\psi)}} \le C\|f\|_{L^1} \tag{16}$$

according to wether $1 < p < 2$ or $p = 1$, for every $\lambda \in \mathbb{C}$ whose real part is greater than a suitable fixed positive real number ω and for a suitable $C > 0$ independent of λ.

Briefly let $f \in L^p \cap L^2$ (the general case follows as before from standard density arguments). So for every g in H there exists a solution v_g of (14) and by Theorem 3.7 we have

$$|\lambda| \|v_g\|_{L^{p'}} + |\lambda|^{\frac{1}{2}} \|v_g\|_{W^{1,p'}_{(\gamma,\psi)}} + \|v_g\|_{W^{2,p'}_{(\gamma^2,\gamma\psi,\psi^2)}} \le C \tag{17}$$

if $1 < p < 2$ or

$$|\lambda| \|v_g\|_{L^\infty} + |\lambda|^{\frac{1}{2}} \|v_g\|_{W^{1,\infty}_{(\gamma,\psi)}} + \|v_g\|_{W^{1,\infty}_{(\gamma^2,\gamma\psi)}} \le C$$

if $p = 1$.

Now (15) easily follows from (17) and estimates

$$\begin{aligned} \|u\|_{L^p} &\le \|v_g\|_{L^{p'}} \|f\|_{L^p} \\ \|u\|_{W^{1,p'}_{(\gamma,\psi)}} &\le \|v_g\|_{W^{1,p'}_{(\gamma,\psi)}} \|f\|_{L^p} \\ \|u\|_{W^{2,p}_{(\gamma^2,\gamma\psi,\psi^2)}} &\le \|v_g\|_{W^{2,p'}_{(\gamma^2,\gamma\psi,\psi^2)}} \|f\|_{L^p}. \end{aligned}$$

Similarly for (16).

Example. Consider the PDE for the price of a European contingent claim in the multifactor case, under the so-called no-arbitrage assumption

$$D_t v + \mathcal{A}v = 0,$$

where

$$\mathcal{A}v = \frac{1}{2}\mathrm{Tr}\ ((\sigma \mathrm{diag} x)\,(D_{i,j}v)\,(\sigma \mathrm{diag} x)^*) + (r-\rho)(1-\epsilon)\sum_{i=1}^{d} x_i D_i v - rv,$$

with terminal condition $v(x,T) = g(x)$ (see, e.g., [18]). Here (diag$x$) is the diagonal matrix with the components of $x \equiv (x_1, \ldots, x_d)$ on the main diagonal, r is the interest rate of a reference riskless asset in the market, σ is a given d-order matrix such that, writing σ^* for the transpose of σ, the matrix $\sigma^*\sigma$ is positive definite, $\rho \equiv \rho(x,t)$ is the dividend rate and $\epsilon \equiv \epsilon(x,t)$ is the tax rate on dividends. The solution $v \equiv v(x,t)$ represents the no-arbitrage price of a contingent claim having payoff $g \equiv g(x)$ at the expiration time T. In the case $d = 1$, $\rho = 0$, $\epsilon = 0$ and $g(x) = (x - E)^+$, where E is the maturity price of the option, we obtain the well-known Black and Scholes equation described in [4]. Also multifactor models, such as the ones appearing in [8], options on futures contracts, and swaps can be treated in our framework ([3, 18]), along with the example below.

Example. We consider here the structure model of interest rate derivatives. For the so-called affine single-term structure model the interest rate is modeled by the stochastic process $(X_t)_{t\geq 0}$ satisfying the differential equation

$$dX_t = (\alpha_1(t) + \alpha_2(t)X_t)\,dt + (\beta_1(t) + \beta_2(t)X_t)\,dW_t. \tag{18}$$

Suitably choosing the coefficients $\alpha_1(t)$, $\alpha_2(t)$, $\beta_1(t)$ and $\beta_2(t)$, different term-structure models can be obtained. In particular two models fitting our framework can be obtained by choosing

1. $\alpha_1 = \alpha_2 = \beta_1 = 0$ [7]
2. $\beta_1 = 0$ [5].

The price of a zero-coupon bond maturing at date T is the solution of the Cauchy problem

$$D_t v + \mathcal{A}v = 0,$$

with the end terminal condition $v(x,T) = 1$, where

$$\mathcal{A}v = \frac{1}{2}(\beta_1 + \beta_2 x)^2 D_{x,x}v + (\alpha_1 + \alpha_2 x)D_x v.$$

We remark in addition that our results allow us to treat also multifactor models with time-dependent coefficients (see [8, 3]), and also semilinear perturbations of the above equations.

Example. The following equation, coming from nonlinear filtering, is considered in [2, 14]:

$$D_t = D_{x,x} + xD_x v - x^2 v, \quad t > 0, x \in \mathbb{R}$$

with initial condition $v(0, x) = g(x)$. It can be easily checked that the second-order operator defined by the right-hand side of the above equation satisfies our assumptions.

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Numerical Approximation of Generalized Functions: Aliasing, the Gibbs Phenomenon and a Numerical Uncertainty Principle

Patrick Guidotti

To the memory of Günter Lumer

Abstract. A general recipe for high-order approximation of generalized functions is introduced which is based on the use of L_2-orthonormal bases consisting of C^∞-functions and the appropriate choice of a discrete quadrature rule. Particular attention is paid to maintaining the distinction between point-wise functions (that is, which can be evaluated point-wise) and linear functionals defined on spaces of smooth functions (that is, distributions). It turns out that "best" point-wise approximation and "best" distributional approximation cannot be achieved simultaneously. This entails the validity of a kind of "numerical uncertainty principle": The local value of a function and its action as a linear functional on test functions cannot be known at the same time with high accuracy, in general.

In spite of this, high-order accurate point-wise approximations can be obtained in special cases from a high accuracy distributional approximation when more information is available concerning the function which is to be approximated. A few special cases with application to PDEs are considered in detail.

Keywords. Generalized functions, approximation, Gibbs phenomenon.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be open. Starting with the spaces

$$\mathcal{D}(\Omega) = \mathrm{C}_c^\infty(\Omega) = \{\varphi \in \mathrm{C}^\infty(\Omega) \,|\, \mathrm{supp}(\varphi) \text{ is compact}\} \text{ and } \mathrm{C}^\infty(\Omega)$$

of test functions, generalized functions are introduced as

$$\mathcal{D}'(\Omega) = \mathcal{L}\big(\mathcal{D}(\Omega), \mathbb{K}\big) = \{u : \mathcal{D}(\Omega) \to \mathbb{K} \,|\, u \text{ is linear and continuous}\}$$
$$\mathcal{E}'(\Omega) = \mathcal{L}\big(\mathrm{C}^\infty(\Omega), \mathbb{K}\big) = \{u : \mathrm{C}^\infty(\Omega) \to \mathbb{K} \,|\, u \text{ is linear and continuous}\}$$

continuous linear functionals on the former. If $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathrm{L}_2(\Omega)$, then one has

$$[\varphi = \sum_{n\in\mathbb{N}} \varphi_n e_n \mapsto (\varphi_n)_{n\in\mathbb{N}}] \in \mathcal{GL}\big(\mathrm{L}_2(\Omega), l_2(\mathbb{N})\big)$$

is an isometry. In particular one has Bessel's equality

$$\|\varphi\|_2 = \big(\sum_{n\in\mathbb{N}} \varphi_n^2\big)^{1/2} \tag{1.1}$$

and Parseval's identity

$$\int_\Omega \varphi\psi\, dx = \sum_{n\in\mathbb{N}} \varphi_n \psi_n\,. \tag{1.2}$$

Unfortunately, even for $\varphi \in \mathcal{D}(\Omega)$ the "Fourier series"

$$\sum_{n=1}^{m} \varphi_n e_n \underset{m\to\infty}{\longrightarrow} \varphi \tag{1.3}$$

merely converges in the $\mathrm{L}_2(\Omega)$-topology, in general. Whenever it converges in the topology of $\mathrm{C}^\infty(\Omega)$ for all $\varphi \in \mathcal{D}(\Omega)$, then one also has Parseval's identity

$$\langle u, \varphi\rangle = \langle u, \sum_{n=1}^{m} \varphi_n e_n\rangle = \sum_{n=1}^{m} \varphi_n \langle u, e_n\rangle =: \sum_{n=1}^{m} \varphi_n u_n \tag{1.4}$$

for pairs $(u, \varphi) \in \mathcal{E}'(\Omega) \times \mathcal{D}(\Omega)$.

Remarks 1.1. **(a)** The fact that the basis functions are in most cases fully supported in the domain Ω entails that the convergence in (1.3) can only occur in $\mathrm{C}^\infty(\Omega)$ even though $\varphi \in \mathcal{D}(\Omega)$. For the same reason only compactly supported distributions $u \in \mathcal{E}'(\Omega)$ can be approximated by their coefficient series

$$\sum_{n=1}^{m} u_n e_n\,.$$

(b) In numerical analysis one is often confronted with the fact the approximations which are high order in the interior of the domain deteriorate as the boundary is approached. This is related to the previous remark.

The approximation procedure introduced in Section 3 for $u \in \mathcal{E}'(\Omega)$ is obtained bearing (1.4) in mind and is based on the choice of a convergent quadrature rule (x^m, q^m) given by

$$x^m = (x_j^m)_{j=1,\dots,g(m)}\,,\ q^m = (q_j^m)_{j=1,\dots,g(m)}\,.$$

In particular

$$q^m \cdot \varphi^m = \sum_{j=1}^{g(m)} \varphi_j^m q_j^m \underset{m\to\infty}{\longrightarrow} \int_\Omega \varphi(x)\, dx\,,\ \varphi \in \mathcal{D}(\Omega)$$

for $\varphi^m := \big(\varphi(x_j^m)\big)_{j=1,\dots,g(m)}$. Given $(e_n)_{n\in\mathbb{N}}$ it is desirable to choose the quadrature rule in such a way that the underlying orthogonality structure is preserved, that is,

$$e_j^m \cdot e_k^m = \delta_{jk}\,,\ 1 \le j,k \le g(m)$$

and $g(m) = m$. In the case of classical Jacobi polynomials this would correspond to working with Gauß quadrature.

Consider now the problem of approximating a general element $u \in \mathcal{E}'(\Omega)$. Only the information loss associated to analytically exact projections is considered here. For smooth or piecewise smooth functions one can choose to project either in *physical space* with $P^{\mathcal{P}}$ or *Fourier space* with $P^{\mathcal{F}}$. The projections are defined through

$$P^{\mathcal{P}}u := u^m = \big(u(x_j^m)\big)_{j=1,\dots,m} \tag{1.5}$$

$$P^{\mathcal{F}}u := \sum_{n=1}^{m} u_n e_n^m = \Big(\sum_{n=1}^{m} u_n e_n(x_j^m)\Big)_{j=1,\dots,m}\,. \tag{1.6}$$

The two projections have ranges of the same dimension. In the case of general distributions only the second projection can be used. A particularly interesting situation is that of piecewise smooth functions where one has the choice of using either projection. If one chooses $P^{\mathcal{P}}$ then one is confronted with the problem of aliasing, whereas the Gibbs phenomenon imposes limitations on the use of $P^{\mathcal{F}}$. The two well-known effects are dual to each other. The Gibbs phenomenon, or the appearance of oscillations at points of non-smoothness, is in reality the manifestation of the fact that the approximation defined by $P^{\mathcal{F}}$ is spectrally accurate in the sense of distributions (as will be shown in Section 4).

In any case it cannot be expected to obtain approximations which are of a high point-wise and distributional degree of accuracy simultaneously. The dual effects of aliasing and the Gibbs phenomenon make this an impossible goal to achieve. This is what is labeled *numerical uncertainty principle* in this paper.

The rest of the paper is organized as follows. In Section 2 a brief review of the basic concepts and facts about Schwartz' theory of distributions is given. Section 3 is devoted to introducing the fundamental concept of approximation family for a distribution. It will play a central role in the rest of the paper. A few simple concrete examples will be given. In Section 4 the important question of convergence of general approximation families is addressed. The rest of the paper is devoted to applications and examples. Section 5.2 deals with the Gibbs phenomenon and its resolution. In Section 6 a simple illustrative PDE example is considered.

The topic of this paper is clearly of interest but it does not seem to have received much attention in the literature, at least not directly and not in this form. Some references are given throughout the paper pointing to research papers in specific areas where some of the ideas presented here are at least implicitly present. The author is not aware of any relevant references specifically dealing with the issues considered here. A helpful reference might be the book by Devore [3] about constructive approximation.

2. Basics of the theory of distributions

A very brief summary of the main basic concepts and results of Schwartz' theory of distributions is needed to set the stage. The reader familiar with the theory can, however, skip this section and move on to the its numerical implications presented in the rest of the paper. A central role in the theory of generalized functions is played by the underlying duality structure. One of its advantages is that it allows to define and carry out many important (linear) operations for distributions at the level of test functions. At the discrete level it might seems pedantic and superfluous to want to keep a distinction between classical functions and distributions. It, however, turns out that many seemingly bad point-wise approximations are indeed good distributional ones. Here only a sketchy and incomplete overview of the theory of distributions is given. The interested reader is therefore referred to the literature [9, 8, 4, 12] for more in depth treatment. For Ω be a bounded open subset of $\mathbb{R}^n$ with $n \in \mathbb{N}$, the space of infinitely many times differentiable functions of compact support $\mathcal{D}(\Omega)$ is defined as

$$\mathcal{D}(\Omega) = \big\{ u \in \mathrm{C}^\infty(\Omega) \,\big|\, \operatorname{supp}(u) = \overline{[u \neq 0]} \subset\subset \Omega \big\}$$

where "$\subset\subset$" means "compactly contained". To be able to generate the appropriate associated class of distributions this space needs to be endowed with the inductive limit topology obtained by means of the following family of locally convex spaces

$$\mathcal{D}_{\Omega'}(\Omega) = \Big(\big\{ u \in \mathcal{D}(\Omega) \,\big|\, \operatorname{supp}(u) \subset \Omega' \subset\subset \Omega \big\}\,,\ \mathcal{P} = \big\{ p_{\Omega',m} \,|\, m \in \mathbb{N} \big\} \Big)$$

where $\mathcal{P}$ is the separating family of semi-norms defined through

$$p_{\Omega',m}(u) = \sup_{x \in \Omega'\,,\ |\alpha| \leq m} |\partial^\alpha u\,(x)|\,,\ m \in \mathbb{N}$$

and choosing the smallest topology which makes the following inclusions continuous

$$\mathcal{D}_{\Omega'}(\Omega) \hookrightarrow \mathcal{D}(\Omega)\,,\ \Omega' \subset\subset \Omega\,.$$

The reader can find the details of this topological construction in [4]. The corresponding space of distributions (generalized functions) is then obtained as the topological dual of $\mathcal{D}(\Omega)$, that is,

$$\mathcal{D}'(\Omega) = \mathcal{D}(\Omega)'\,.$$

It is endowed with its weak* topology. Since the main focus of this paper is on finite discrete approximations particular interest lies in the convergence of sequences and series. In the chosen topologies their convergence is equivalent to "point-wise"convergence. Indeed, let $(u_n)_{n\in\mathbb{N}}$ and $(\varphi_n)_{n\in\mathbb{N}}$ be sequences in $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$, respectively. Then the first converges to a limit $u_\infty \in \mathcal{D}'(\Omega)$ if and only if

$$\langle u_n, \varphi \rangle := u_n(\varphi) \to u_\infty(\varphi) = \langle u_\infty, \varphi \rangle \text{ for each } \varphi \in \mathcal{D}(\Omega)$$

whereas the second converges towards a limit $\varphi_\infty \in \mathcal{D}(\Omega)$ if and only if there exists $\Omega' \subset\subset \Omega$ such that $\operatorname{supp}(\varphi_n)\,,\ \operatorname{supp}(\varphi_\infty) \subset \Omega'$ for $n \in \mathbb{N}$ and

$$\langle u, \varphi_n \rangle \to \langle u, \varphi_\infty \rangle \text{ for each } u \in \mathcal{D}'(\Omega)$$

Given any $f \in \mathrm{L}_{1,\mathrm{loc}}(\Omega)$ in the space of locally integrable functions, a distribution u_f can be defined by

$$\langle u_f, \varphi \rangle := \int_\Omega f(x)\varphi(x)\,dx\,,\ \varphi \in \mathcal{D}(\Omega)\,.$$

In other words, locally integrable functions naturally act on test functions. Any distribution of this form is called *regular*. The duality between distributions and test functions and, in particular, the *integral duality* between regular distributions and test functions is exploited in the definition of all standard operations for distributions and will soon play an essential role in understanding *"discrete distributions"*. Another space of test functions, $\mathcal{E}(\Omega) = \mathrm{C}^\infty(\Omega)$ is endowed with its natural locally convex topology generated by the separating family

$$\bigl\{ p_{\Omega',m} \bigm| \Omega' \subset\subset \Omega\,,\ m \in \mathbb{N} \bigr\}\,.$$

Its dual $\mathcal{E}'(\Omega)$ of $\mathcal{E}(\Omega)$ can also naturally be viewed as a space of distributions. It actually is the space of distributions of compact support, where the support of a distribution u is defined as the closure of the complement of

$$\bigl\{ x \in \Omega \bigm| \text{there exists a neighborhood } \Omega' \text{ of } x \text{ s.t. } \langle u, \varphi \rangle = 0\,,\ \varphi \in \mathcal{D}(\Omega') \bigr\}$$

It turns out that all compactly supported distributions are of finite order. The order of a distribution is defined as follows. Any $u \in \mathcal{D}'(\Omega)$ is said to be of order $m \in \mathbb{N}$ if and only if for any given $\Omega' \subset\subset \Omega$ there is a constant $C = C(\Omega') > 0$ such that

$$|\langle u, \varphi \rangle| \le C\, p_{m,\Omega'}(\varphi)\,,\ \varphi \in \mathcal{D}_{\Omega'}(\Omega)\,.$$

In particular, the compactly supported distributions $\partial^\alpha \delta_y \in \mathcal{E}'(\Omega)$, $\alpha \in \mathbb{N}^n$ and $y \in \Omega$, defined by

$$\langle \partial^\alpha \delta_y, \varphi \rangle = (-1)^{|\alpha|}(\partial^\alpha \varphi)(y)\,,\ \varphi \in \mathcal{E}(\Omega) \tag{2.1}$$

are of finite order $|\alpha|$. The space of finite regularity test functions is given by

$$\mathcal{D}^m(\Omega) = \bigl\{ \varphi \in \mathrm{C}^m(\Omega) \bigm| \operatorname{supp}(\varphi) \subset\subset \Omega \bigr\} \tag{2.2}$$

in a way similar to $\mathcal{D}(\Omega)$, by duality one obtains the space $\mathcal{D}^m(\Omega)'$ of distributions of order at most $m \in \mathbb{N}$. Correspondingly, $\mathcal{E}^m(\Omega) = \mathrm{C}^m(\Omega)$ gives rise to the space $\mathcal{E}^m(\Omega)'$ of compactly supported distribution of order at most $m \in \mathbb{N}$.
It is important to point out that

$$\mathcal{D}(\Omega) \overset{d}{\hookrightarrow} \mathcal{E}(\Omega) \overset{d}{\hookrightarrow} \mathrm{L}_{1,loc}(\Omega) \overset{d}{\hookrightarrow} \mathcal{D}'(\Omega)\,, \tag{2.3}$$

where the "d" indicates density of the inclusion. The Hilbert space $\mathrm{L}_2(\Omega)$ and orthonormal bases on it are also basic ingredients of the approach presented here. It will be convenient to work with bases of smooth functions. They usually do not have compact support, however. Just think of all eigenfunction bases associated to boundary value problems. Given an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ for $\mathrm{L}_2(\Omega)$ with

$$e_n \in \mathrm{C}^\infty(\overline{\Omega})\,,$$

$(u_k)_{k\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ and a distribution $u \in \mathcal{E}'(\Omega)$ of compact support, it follows that

$$\sum_{k=1}^{m} u_k e_k \underset{m\to\infty}{\longrightarrow} u \text{ in } \mathcal{D}'(\Omega)$$

if and only if

$$\langle \sum_{k=1}^{m} u_k e_k, \varphi\rangle = \sum_{k=1}^{m} u_k \varphi_k \underset{m\to\infty}{\longrightarrow} \langle u, \varphi\rangle\,,\ \varphi \in \mathcal{D}(\Omega)\,,$$

where

$$\varphi_k = \int_\Omega e_k(x)\varphi(x)\,dx\,.$$

The following simple but important relation

$$\langle u, \varphi\rangle = \big\langle u, \sum_{k=1}^{\infty} \langle e_k, \varphi\rangle\, e_k\big\rangle = \sum_{k=1}^{\infty} \langle u, e_k\rangle\langle e_k, \varphi\rangle = \big\langle \sum_{k=1}^{\infty} \langle u, e_k\rangle\, e_k\,, \varphi\big\rangle\,,$$
$$u \in \mathcal{E}'(\Omega)\,,\ \varphi \in \mathcal{D}(\Omega) \tag{2.4}$$

shows that the basis coefficients of u can be computed by

$$u_k = \langle u, e_k\rangle\,,\ k \in \mathbb{N} \tag{2.5}$$

whenever convergence takes place. Unfortunately nothing can be said about the latter in general. In view of the duality construction used to introduce distributions, however, it is always the case that if either the series

$$\sum_{k=1}^{\infty} \langle \varphi, e_k\rangle\, e_k \tag{2.6}$$

converges in $\mathcal{E}(\Omega)$ or

$$\sum_{k=1}^{\infty} \langle u, e_k\rangle\, e_k \tag{2.7}$$

converges in $\mathcal{D}'(\Omega)$ then the validity of (2.4) is assured.

Remark 2.1. Formula (2.4) shows in particular that

$$\langle u, \varphi\rangle = \sum_{k=1}^{\infty} \langle u, e_k\rangle\langle e_k, \varphi\rangle = \sum_{k=1}^{\infty} u_k\varphi_k\,,\ u \in \mathcal{E}'(\Omega)\,,\ \varphi \in \mathcal{D}(\Omega)\,, \tag{2.8}$$

whenever there is convergence. Assuming that all functions and distributions considered be real-valued and switching to the case where $u, \varphi \in \mathrm{L}_2(\Omega)$, (2.8) becomes Parseval's identity. It is therefore legitimate to call it *generalized Parseval's identity.*

Remarks 2.2. (a) The requirement that (2.6)–(2.7) be convergent in the given topologies is essentially a "smoothness and boundary behavior assumption" on u

and φ with respect to the basis $(e_n)_{n\in\mathbb{N}}$. If no further information about properties of the basis are available, it is, however, not possible to measure this regularity in classical function spaces. Fortunately, many basis of practical interest consist of eigenfunctions of some operator. In that case more can be said about the convergence of (2.6)–(2.7) and consequently of (2.8). More details are found in Subsection 4.

(b) So far only the real-valued case is considered. The complex-valued case can clearly be covered with only minor modifications caused by the incongruence of the sesquilinearity of the scalar product and the bilinearity of the duality pairing.

Turning to a simple concrete example, let $y \in \Omega$ and $\alpha \in \mathbb{N}^n$ and consider the series

$$\partial^\alpha \delta_y = (-1)^{|\alpha|} \sum_{k\in\mathbb{Z}} (\partial^\alpha \bar{e}_k)(y)\, e_k\,. \tag{2.9}$$

It will provide with a smooth approximation

$$\big[(-1)^{|\alpha|} \sum_{|k|\leq m} (\partial^\alpha \bar{e}_k)(y)\, e_k\big]_{m\in\mathbb{N}} \tag{2.10}$$

to $\partial^\alpha \delta$ whenever the series converges in the sense of distributions. This is a very natural way of taking advantage of the duality built-in in the distributional framework without completely giving up the benefits of orthogonal expansions which are restricted to Hilbert spaces. A particular case of the above is given by Fourier series of periodic functions. Any test function of the periodicity cube $B_n = [-\pi,\pi]^n$ can be viewed as a periodic function. It can therefore be developed in a Fourier series

$$\varphi = \frac{1}{(2\pi)^{n/2}} \sum_{k\in\mathbb{Z}^n} \hat{\varphi}_k \exp(ik\cdot x) \tag{2.11}$$

with Fourier coefficients given by

$$\hat{\varphi}_k = \frac{1}{(2\pi)^{n/2}} \int_{B_n} \varphi(x) \exp(-ik\cdot x)\, dx\,. \tag{2.12}$$

In this case one has

Lemma 2.3. *Let $\varphi \in \mathcal{D}(B_n)$. Then its Fourier series expansion* (2.11) *converges in the topology of $\mathcal{E}(B_n)$. For any distribution $u \in \mathcal{E}'(B_n)$ its Fourier coefficient u_k is defined to be*

$$u_k = \frac{1}{(2\pi)^{n/2}} \langle u, \exp(-ik\cdot x)\rangle\,. \tag{2.13}$$

Then

$$\frac{1}{(2\pi)^n} \sum_{k=-m}^{m} u_k \exp(ik\cdot x) \to u\ (m\to\infty)\ \text{in}\ \mathcal{D}'(B_n)\,. \tag{2.14}$$

Proof. The proof follows from the general convergence result of Theorem 4.1. □

Example 1. Consider the distribution $\partial^\alpha\delta$ for $\alpha \in \mathbb{N}^n$. Then (2.13) gives

$$(\partial^\alpha\delta)_k = (-ik)^\alpha/(2\pi)^{n/2}$$

and

$$\frac{1}{(2\pi)^n}\sum_{k=-m}^{m}(-ik)^\alpha \exp(ik\cdot x) \underset{m\to\infty}{\longrightarrow} \partial^\alpha\delta \text{ in } \mathcal{D}'(B_n)\,. \tag{2.15}$$

Remarks 2.4. (a) More in general if a distribution is of finite order, then convergence in a weaker topology than that of $\mathcal{E}(\Omega)$ suffices to obtain convergence of its basis development in a stronger topology than that of the space $\mathcal{E}'(\Omega)$. This point will be raised again in Section 4.
(b) Specializing to $u = \delta$ and $n = 1$, the one-dimensional Dirac distribution supported in the origin, one sees that its approximating Fourier sum coincides with the classical Dirichlet kernel

$$D_m(x) = \frac{1}{2\pi}\sum_{k=-m}^{m}\exp(ik\cdot x)\,. \tag{2.16}$$

These very simple facts and examples turn out to be very important when approaching the problem of discretization. The problem of discretizing a (generalized) function is in fact not independent of the problem of discretizing the underlying duality structure between test functions and distributions. For this reason the discretization of functions and that of the duality pairing have to be related to each other in order to produce optimal results.

3. Approximating families

Next the discrete version of the above concepts is considered. Vectors will appear instead of distributions but the distinction between function and distribution in the duality sense will not be given up. This might seem a minor point since every finite-dimensional space is naturally isomorphic to its dual but it is not. It determines, among other things, the way in which the information content of a given vector of finite length has to be read. The main concept introduced in this section is that of *approximation family* for a distribution. It is meant to faithfully reproduce the continuous analytical structure at the finite-dimensional, discrete level. To realize a discrete duality pairing quadrature rules are used. Assume that a family of discretization points has been chosen

$$x^m = (x_1^m, \dots, x_{g(m)}^m)\,,\ m \in \mathbb{N} \tag{3.1}$$

where the strictly increasing function $g : \mathbb{N} \to \mathbb{N}$ with $g(m) \geq m$ counts the total number of grid points. A *discrete quadrature rule* for Ω with respect to a family of discretization points $(x^m)_{m\in\mathbb{N}}$ is a family of vectors

$$q^m = (q_1^m, \dots, q_{g(m)}^m) \tag{3.2}$$

such that

$$\sum_{k=1}^{g(m)} q_k^m \varphi(x_k^m) \underset{m\to\infty}{\longrightarrow} \int_\Omega \varphi(x)\, dx\,,\ \varphi \in \mathcal{E}(\Omega)\,. \tag{3.3}$$

The quadrature rules for Ω shall be denoted by $\big(x^m\,, q^m\big)_{m\in\mathbb{N}}$ or simply by $(q^m)_{m\in\mathbb{N}}$ if it is clear which discretization points have been fixed. The following definition plays a central and fundamental role.

Definition 3.1. Let $u \in \mathcal{D}'(\Omega)\ \big[\mathcal{E}'(\Omega)\big]$ be a given distribution. Then

$$[u^m, x^m, q^m]_{m\in\mathbb{N}}$$

is called a *discretization family* (in the sense of distributions) for u iff

(i) $u^m \in \mathbb{R}^{g(m)}\,,\ m \in \mathbb{N}\,.$ (3.4)

(ii) $\big(x^m\,,\ q^m\big)_{m\in\mathbb{N}}$ is quadrature rule for $\Omega\,.$ (3.5)

(iii) $u^m \cdot \varphi^m := \sum_{k=1}^{g(m)} u_k^m \varphi(x_k^m) q_k^m \underset{m\to\infty}{\longrightarrow} \langle u, \varphi\rangle$ for each $\varphi \in \mathcal{D}(\Omega)\ \big[\mathcal{E}(\Omega)\big]\,.$ (3.6)

If it is assumed that the approximated distribution is regular, that is, if $u \in \mathrm{L}_{1,\mathrm{loc}}(\Omega)$, then the above definition entails that

$$u^m \cdot \varphi^m \to \int_\Omega u(x)\varphi(x)\, dx \text{ for each } \varphi \in \mathcal{D}(\Omega)\ . \tag{3.7}$$

It follows that the definition is the discrete version of the classical concept of weak convergence for sequences of functions.

Remarks 3.2. (a) It should be pointed out that by modifying the definition of the approximating sequence as follows

$$\tilde{u}^m = (q_1^m u_1^m, \dots, q_{g(m)}^m u_{g(m)}^m)$$

the discrete duality pairing could be normalized to be the Euclidean scalar product. This will always be done whenever dealing with concrete examples. For abstract calculations, however, it is preferable to have the quadrature rule appear explicitly in the formulæ.

(b) Choosing $u^m = (1, \dots, 1)$, $m \in \mathbb{N}$, one can think of some distributions (read measures) as being approximated by quadrature rules. Or better still, one could view quadrature rules as discretizations families of measures in the sense of distributions. In the above definition quadrature rules are obviously encoded in the choice of duality pairing.

A few prototypical examples are considered next.

Example 2. Let $y \in (0,1)$ and $\mathbf{1}^m = (\underbrace{1, \dots, 1}_{m \text{ times}})$ and consider the family

$$[\delta_y^m, x^m, q^m] = \Big[2\sum_{k=0}^{m} \sin(k\pi y)\sin(k\pi x^m)\,,\ x^m = (k/m)_{k=1,\dots,m-1}\,,\ \frac{1}{m}\mathbf{1}^{m-1}\Big]_{m\in\mathbb{N}}$$

It approximates the Dirac distribution δ_y supported at $y \in (0,1)$ with respect to the trapezoidal rule of quadrature.

Proof. For any given test function $\varphi \in \mathcal{D}(0,1)$ the duality pairing is given by

$$\delta_y^m \cdot \varphi^m = \sum_{j=1}^m \Bigl(2 \sum_{k=1}^m \sin(k\pi y) \sin(k\pi x_j^m)\Bigr) \varphi(x_j^m) \frac{1}{m}$$
$$= \sum_{k=1}^m \sqrt{2} \sin(k\pi y) \Bigl(\frac{1}{m} \sum_{j=1}^m \sqrt{2} \sin(k\pi x_j^m) \varphi(x_j^m)\Bigr) .$$

The inner sum in the second line appears to be a trapezoidal rule discretization of

$$\sqrt{2} \int_0^1 \sin(k\pi x) \varphi(x)\, dx = \langle \varphi, \sqrt{2} \sin(k\pi \cdot) \rangle = \varphi_k$$

and it is therefore obtained that

$$\delta^m \cdot \varphi^m \longrightarrow \sum_{k=1}^\infty \varphi_k \sin(k\pi y) = \varphi(y)\ (m \to \infty) \tag{3.8}$$

which proves the claim. The convergence of the trapezoidal rule, which has implicitly been used, is elementary and omitted. □

Remark 3.3. In this case

$$e_k(x) := \sqrt{2} \sin(k\pi x)\, ,\ x \in (0,1)\, ,\ k \in \mathbb{N} \tag{3.9}$$

build an orthonormal basis of $\mathrm{L}_2(0,1)$ with $e_k \in \mathrm{C}^\infty[0,1]\, ,\ k \in \mathbb{N}$. It is very important that this basis consists of solutions (eigenfunctions) of

$$-\partial_{xx} u = \lambda u\, ,\ u(0) = u(1) = 0\, .$$

In more general situations, like in Example 5, this will provide the means for proving convergence.

This simple but important example can easily be extended to any dimension.

Example 3. Let $y \in \Omega = (0,1)^n$, $\alpha \in \mathbb{N}^n$ and

$$\mathbf{x}^m = \otimes_{k=1}^n x^m\, ,\ (\delta_y^\alpha)^m = \otimes_{k=1}^n (\delta_{y_k}^\alpha)^m \tag{3.10}$$

for

$$(\delta_{y_k}^\alpha)^m = 2 \sum_{j=0}^m (-j\pi)^{|\alpha_k|} \bigl(\partial^{\alpha_k} \sin\bigr)(j\pi y_k) \sin(j\pi x_k^m)\, ,\ k = 1, \dots, n\, .$$

Then $(\delta_y^\alpha)^m$ is a discretization family for $\delta_y^\alpha = \partial^\alpha \delta_y \in \mathcal{E}'(0,1)$.

In a periodic context this construction essentially produces the Dirichlet kernel D_m evaluated at the grid points.

Example 4. Let $\Omega = (-\pi, \pi)$ and let

$$x^m = \frac{\pi}{m}\big(-m, -m+1, \dots, m-1, m\big)$$

be the tuple of $2m+1$ equidistant discretization points. Fix the trapezoidal rule

$$q^m = \big(\frac{1}{2}, \underbrace{1, \dots, 1}_{2m-1\text{-times}}, \frac{1}{2}\big)$$

as the associated quadrature rule. Then the following modified Dirichlet kernel

$$\delta^m = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{m-1} \cos(kx^m) + \frac{1}{2\pi}\cos(m\,x^m) =$$
$$\frac{1}{2\pi}\frac{\sin\big((m-\frac{1}{2})x^m\big)}{\sin(\frac{x^m}{2})} + \frac{1}{2\pi}\cos(m\,x^m) = D_{m-1}(x^m) + \frac{1}{2\pi}\cos(m\,x^m) \quad (3.11)$$

defines a discretization family for $\delta \in \mathcal{E}'(-\pi, \pi)$.

Proof. The proof is identical to that of Example 2. □

Remarks 3.4. (a) In Example 4 the so-called alternating point trapezoidal rule given by the family of weights

$$q^m = (1, 0, 2, 0, \dots, 2, 0, 1)$$

with $2m+1$ equidistant discretization points as above could have been chosen.
(b) The approximating family obtained in Example 2 is closely related to that of Example 4. In fact

$$2\sum_{k=0}^{m} \sin(k\pi y)\sin(k\pi x^m) = \pi D_m(\pi(x-y)) - \pi D_m(\pi(x+y))\,. \quad (3.12)$$

(c) Disregarding the role played by the quadrature rule in the construction of the a discrete approximation for the Dirac distribution in the above example, it would seem natural to use the Dirichlet kernel (2.16) to produce a discretization family for δ. Even though this would be a viable discretization, it would unfortunately converge more slowly than (3.11).

Lastly a genuinely higher-dimensional example is considered.

Example 5. Let $\Omega = \mathbb{B}^2$ be the open unit circle parameterized by spherical coordinates and

$$e_{m,n}(r, \theta) = \frac{1}{c_{mn}} J_m(r\sqrt{\lambda_{mn}})e^{im\,\theta}\,,\ r \in [0,1]\,,\ \theta \in [-\pi, \pi)$$

be the orthonormal basis of $\mathrm{L}_2(\Omega)$ given by the eigenfunctions of the Dirichlet problem

$$-\triangle u = \lambda u \text{ in } \Omega\,,\ u = 0 \text{ on } \partial\Omega\,.$$

to eigenvalues λ_{mn}, $n \in \mathbb{N}$ given by the positive zeros of the Bessel function J_m, $m \in \mathbb{N}$. The constants c_{mn} are chosen so that $\|e_{mn}\|_{\mathrm{L}_2(\Omega)} = 1$.

Discretization points are

$$r_j^M = j/M\,,\ 0 \le j \le M \text{ and } \theta_j^N = \pi(-1 + j/N)\,,\ 0 \le j \le 2N\,,\ M, N \in \mathbb{N}\,.$$

For quadrature, the trapezoidal rule is used in both variables, that is,

$$q_r^M = \frac{1}{M}(r_0^M, r_1^M, \dots, r_M^M)\,,\ q_\theta^N = \frac{\pi}{N}\mathbf{1}^{2N+1}\,,\ M, N \in \mathbb{N}\,.$$

Then the family

$$\sum_{m=0}^{M}\sum_{n=-N}^{N} \frac{1}{c_{mn}^2} J_m(r_0\sqrt{\lambda_{mn}}) J_m(r^M\sqrt{\lambda_{mn}}) e^{im(\theta^N - \theta_0)} \tag{3.13}$$

defines a discretization family for $\delta_{(r_0,\theta_0)}$ for any $(r_0, \theta_0) \in \Omega$. Let now Γ be a closed smooth curve completely contained in Ω. Denote by δ_Γ the line integral distribution defined through

$$\langle \delta_\Gamma, \varphi \rangle = \int_\Gamma \varphi(x)\, d\sigma_\Gamma(x)\,,\ \varphi \in \mathrm{C}^\infty(\Omega)\,.$$

where σ_Γ is the surface measure. Then

$$\sum_{m=0}^{M}\sum_{n=-N}^{N} \frac{I_{mn}}{c_{mn}} J_m(r^M\sqrt{\lambda_{mn}}) e^{im\,\theta^N} \tag{3.14}$$

with

$$I_{mn} = \frac{1}{c_{mn}} \int_\Gamma J_m(r\sqrt{\lambda_{mn}}) e^{-im\,\theta}\, d\sigma_\Gamma(r, \theta)\,. \tag{3.15}$$

defines an approximation family for δ_Γ.

Proof. As for the Dirac distribution the claim would be a direct consequence of (2.4)–(2.5) and (2.9) combined with the know fact that $\big(e_{m,n}\big)_{m,n\in\mathbb{N}}$ is indeed an orthonormal basis of $\mathrm{L}_2(\Omega)$ and with the convergence of the chosen quadrature rule if

$$\sum_{m=0}^{M}\sum_{n=-N}^{N} \varphi_{mn} e_{m,n}$$

converged at least in $\mathrm{C}(\Omega)$ as $M, N \to \infty$ for any $\varphi \in \mathcal{E}(\Omega)$. This will be considered in a more general setting in Section 4. In the case of the line integral distribution the proof is similar and uses the complex version of (2.4)–(2.5). □

Remarks 3.5. (a) In the last example (3.15) can not be evaluated analytically. It can, however, be easily approximated numerically with high order of accuracy.

(b) Wavelet bases can of course also be used. One of their main purpose is, however, to provide localized basis functions. They also usually have finite degree of smoothness. Unless C^∞ wavelets are used, these two facts concur in making it impossible to achieve spectrality (in the sense of distributions) of any approximation method based on their use. Their main advantage lies of course in their multi-scale resolution properties.

4. Convergence

Definition 3.1 of approximation family given in the previous section leads to useful discrete approximations provided (2.8) does indeed converge for the chosen basis $(e_n)_{n\in\mathbb{N}}$. A criterion for their convergence is therefore derived which entails convergence of general discretization families. The standard basic tool is to trade smoothness for convergence.

Looking at Examples 2–5 given in Section 3 it is recognized that they all share a specific structure which yields the desired convergence. One starts with some operator pair $(\mathcal{A}, \mathcal{B})$ where

$$\mathcal{A} = \sum_{|\alpha|\leq m} a_\alpha \partial^\alpha \,,\ a_\alpha \in \mathbb{R}^M \text{ and } \mathcal{B} = \sum_{|\alpha|\leq k} b_\alpha \partial^\alpha \,,\ b_\alpha \in \mathbb{R}^M \tag{4.1}$$

are a vector of differential operators on Ω of order $m \geq 1$ and a vector of boundary differential operators on $\partial\Omega$ of order $k < m$, respectively. It is always possible to introduce

$$L : \operatorname{dom} L \subset \mathrm{L}_2(\Omega) \to \mathrm{L}_2(\Omega)^M \tag{4.2}$$

where

$$\operatorname{dom}(L) = \mathrm{H}^m_{\mathcal{B}}(\Omega) := \big\{ u \in \mathrm{H}^m(\Omega) \,\big|\, \mathcal{B}u = 0 \big\} \text{ and } Lu = \mathcal{A}u \,,\ u \in \operatorname{dom}(L) \,, \tag{4.3}$$

with the understanding that the homogeneous boundary condition has to be imposed in the sense of traces whenever it makes sense and has to considered empty otherwise. Then, if L is closed and densely defined, the self-adjoint operator

$$A = L'L : \operatorname{dom}(A) \subset \mathrm{L}_2(\Omega) \to \mathrm{L}_2(\Omega) \tag{4.4}$$

can be defined. If the latter turns out to be invertible, it has a compact resolvent by the compact embedding

$$\operatorname{dom}(A) \subset \mathrm{H}^{2m}(\Omega) \hookrightarrow \mathrm{L}_2(\Omega) \,.$$

It is therefore possible to introduce an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ on $\mathrm{L}_2(\Omega)$ consisting of the necessarily smooth eigenfunctions of A to the ordered family of positive eigenvalues $(\lambda_n)_n \in \mathbb{N}$. Then this basis has all the needed properties.

Theorem 4.1. *Assume that $(\mathcal{A}, \mathcal{B})$ is such that the operator L defined through (4.2)–(4.3) be closed and densely defined. Let the self-adjoint operator A given in (4.4) have finite-dimensional kernel, then*

$$\sum_{k\in\mathbb{N}} \langle e_k, \varphi \rangle e_k \overset{k\to\infty}{\longrightarrow} \varphi \text{ in } \mathcal{E}(\Omega) \tag{4.5}$$

for any $\varphi \in \mathcal{D}(\Omega)$ and

$$\sum_{k\in\mathbb{N}} \langle u, e_k \rangle e_k \overset{k\to\infty}{\longrightarrow} u \text{ in } \mathcal{D}'(\Omega) \tag{4.6}$$

for any $u \in \mathcal{E}'(\Omega)$.

Proof. Factoring out the finite-dimensional kernel, if necessary, it can be assumed that A is invertible and therefore has compact inverse by the Rellich-Kondratev embedding theorem, [11, Proposition 4.4]. The spectral theory of compact operators implies that $(e_k)_{k\in\mathbb{N}}$ is indeed a orthonormal basis of $\mathrm{L}_2(\Omega)$. Series (4.5) therefore converges in $\mathrm{L}_2(\Omega)$ at least. Now

$$\varphi \in \mathrm{dom}(A^m)\,,\ m \in \mathbb{N}\,,$$

since $\mathcal{D}(\Omega) \subset \mathrm{dom}(A)$ and $A\varphi \in \mathcal{D}(\Omega)$ for any $\varphi \in \mathcal{D}(\Omega)$. The closure of A implies that

$$A^m \sum_{k\in\mathbb{N}} \langle e_k, \varphi\rangle e_k = \sum_{k\in\mathbb{N}} \langle e_k, \varphi\rangle A^m e_k = \sum_{k\in\mathbb{N}} \langle \lambda_k^m e_k, \varphi\rangle e_k = \sum_{k\in\mathbb{N}} \langle e_k, A^m\varphi\rangle e_k$$

which yields $\mathrm{H}^m(\Omega)$ convergence for the series since $\|A^m \cdot\|_{\mathrm{L}_2(\Omega)}$ is an equivalent norm on $\mathrm{H}^m(\Omega)$ and $A^m\varphi \in \mathrm{L}_2(\Omega)$. Notice that factoring the kernel has no impact on the convergence of the series in any way, since it is assumed to be finite dimensional. Sobolev embedding theorem (cf. [11, Prop. 1.5])

$$\mathrm{H}^s(\Omega) \hookrightarrow \mathrm{C}^{s-n/2}(\overline{\Omega})\,,\ s > 0 \text{ with } s - n/2 > 0$$

then implies convergence in $\mathrm{C}^s(\Omega)$ for every $s > 0$ which implies the first assertion. The second convergence claim follows from (2.4). □

Example 6. In Examples 2–3 and 5 the basis functions are eigenfunctions of the Laplacian on $\mathrm{L}_2(\Omega)$ with Dirichlet boundary conditions for $\Omega = (0,1)\,,\ (0,1)^n$ and $\mathbb{B}^2$, respectively.

Example 7. Taking the operator $L = \nabla$ with domain of definition

$$\mathrm{dom}(L) = \mathrm{H}^1_p\big((-\pi,\pi)^n\big)$$

where the subscript "p" stands for periodic on the periodicity box $(-\pi,\pi)^n$, one has that

$$L' = -\,\mathrm{div}$$

with same domain of definition, and therefore

$$A = -\,\mathrm{div}(\nabla\cdot)$$

with domain $\mathrm{dom}(A) = H^2_p\big((-\pi,\pi)^n\big)$. This is the case of Fourier series, Example 4.

This, together with the proofs given there, shows that the families of Examples 2–4 are indeed discretization families in the sense of definition 3.1. As to the general situation, assume that an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of $\mathrm{L}_2(\Omega)$ is given. Let $u \in \mathcal{E}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$ be given. Choose a family of discretization points $(x^m)_{m\in\mathbb{N}}$ and an associated convergent quadrature rule $(q^m)_{m\in\mathbb{N}}$. Let $\bar{u}^m$ and $\bar{\varphi}^m$

be approximations to the exact coefficient vector u^m and to the exact grid point values φ^m, respectively. Then

$$\begin{aligned}\langle u,\varphi\rangle - \bar{u}^m\cdot\bar{\varphi}^m &= \sum_{k=m+1}^{\infty} u_k\varphi_k + \sum_{k=1}^{m} u_k\varphi_k - u^m\cdot\varphi^m \\ &\quad + (u^m-\bar{u}^m)\cdot\bar{\varphi}^m + u^m\cdot(\varphi^m-\bar{\varphi}^m) \\ &= \sum_{k=m+1}^{\infty} u_k\varphi_k + \sum_{k=1}^{m} u_k(\varphi_k-\hat{\varphi}_k^m) + (u^m-\bar{u}^m)\cdot\bar{\varphi}^m + u^m\cdot(\varphi^m-\bar{\varphi}^m) \qquad (4.7)\end{aligned}$$

where

$$\hat{\varphi}_k^m = e_k^m\cdot\varphi^m = \sum_{k=1}^{m} e_k(x_k^m)\varphi(x_k^m)q_k^m \,.$$

Disregarding the numerical error in the approximation of the coefficients, the error depends on the order of the distribution u and on the difference between the exact and discrete basis coefficients of the test function φ. The first determines the polynomial growth rate of the u_k's, the latter is super-algebraically convergent for any test function φ. More precisely, since the distribution u is of finite order, one has that

$$|u_k| \le c(u)k^p\,,\ k\in\mathbb{N}$$

for some positive constants $c(u)$ and some order $p\in\mathbb{N}$. As for the test function, for every $P\ge 0$ a constant $c(P,\varphi)\ge 0$ can be found such that

$$|\varphi_k| \le c(P,\varphi)k^{-P}\,,\ k\in\mathbb{N}\,.$$

It follows that

$$\Big|\sum_{k=m+1}^{\infty} u_k\varphi_k\Big| \le c(P',u,\varphi)\frac{1}{m^{P'}}$$

for any $P'\ge 0$. As for the error $|\varphi_k-\hat{\varphi}_k|$ incurred in the computation of the discrete basis coefficient, it is determined by the effect of aliasing and can be bound as follows. It can be easily seen from

$$\varphi = \sum_{k=1}^{\infty}\varphi_k e_k$$

that the discrete basis coefficient differs from the exact one in the amount

$$|\varphi_j-\hat{\varphi}_j| = \Big|\sum_{k=m+1}^{\infty}\varphi_k e_k^m\cdot e_j^m\Big|\,.$$

This follows from the choice of the discrete duality pairing which preserves orthogonality of the first m basis functions. Again, since φ is a test function and since $e_k^m\cdot e_j^m = O(1)$, it follows

$$|\varphi_j-\hat{\varphi}_j| \le c(P,\varphi)\frac{1}{m^P}$$

for any $P\ge 0$ and suitable constant $c(P,\varphi)\ge 0$. Therefore one obtains

Theorem 4.2. *Let $u \in \mathcal{E}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$. Then for any $P \geq 0$, a constant $c = c(P, u, \varphi)$ can be found such that*

$$|\langle u, \varphi \rangle - u^m \cdot \varphi^m| \leq c(P, u, \varphi) \frac{1}{m^P} \tag{4.8}$$

Remark 4.3. Approximations for test functions are best generated and handled in physical space. This is due to the fact that there is no easy way to tell when a rapidly convergent basis expansion will have compact support. On the other hand, special distributions often allow for an easy computation of their exact coefficients in the basis expansion. This implies that, in many cases of interest, the errors neglected in (4.7) can be made very small as well.

5. Interpolation

The general procedure described above applied to special distributions leads to recovering many interesting applications. One of them is interpolation. It can be naturally viewed as the approximation of (derivatives of the) Dirac distributions. In this case the above procedure delivers an "optimal" interpolation functional. It also turns out that a high order (in the sense of distributions) approximation of a (possibly only piecewise) smooth function has to carry non vanishing oscillations at the right scale. Put differently, even if singularities occur at non-grid points, their locations and intensity are captured via oscillations which degrade the point-wise quality of the approximation. However, if the oscillations are not viewed point-wise but rather in the sense of distributions, they are seen contain high-order information concerning the approximated object. In the case of piecewise regular functions this allows for developing "recovery" methods which produce genuinely point-wise approximation. Even though it is not formulated in the general abstract terms of this paper, this is precisely what a series of methods proposed in the literature implicitly exploit (see for instance [6, 5]).

In general, however, the oscillatory behavior of high-order distributional approximation causes a limitation in the point-wise accuracy that can be achieved without giving up non-local out-of-grid information concerning the limit. Even worse, the high-order implicit information encoded in the oscillations can, in general, not be made explicit. This is precisely what has earlier been labeled as *numerical uncertainty principle.*

5.1. Location and intensity of singularities

So far it has been observed that distributions give rise to oscillations when they are approximated. Next the question is investigated of how to find out which distribution (if any) is represented by some given set of data with oscillatory behavior. To answer this question, the concept of φ-sweep, or test function sweep, is introduced. Consider the test functions given by

$$\varphi_{x_0,\alpha,\beta}(x) = \begin{cases} \exp\bigl(-\alpha \frac{\psi^2(x,x_0,\beta)}{1-\psi^2(x,x_0,\beta)}\bigr)\,, & |x - x_0| \leq \beta \\ 0\,, & |x - x_0| > \beta \end{cases} \tag{5.1}$$

for

$$\psi(x, x_0, \beta) = \frac{x - x_0}{\beta}\,. \tag{5.2}$$

and $\alpha,\ \beta > 0$. To distinguish the continuous test function from its discrete counterpart, the latter is denoted by $\varphi^m_{x_0,\alpha,\beta}$.

Definition 5.1. If u^m is a vector of discrete values approximating some distribution $u \in \mathcal{D}'(\Omega)$ with $\mathrm{supp}(u) \subset\subset \Omega$, the function defined by

$$S\big(u^m, \varphi^m_{\cdot,\alpha,\beta}\big)(x) = u^m \cdot \varphi^m_{x,\alpha,\beta}\,,\ x \in \Omega\,, \tag{5.3}$$

is called a $\varphi^m_{\alpha,\beta}$-sweep of u^m.

Assume u^m is the manifestation of a Dirac distribution $c_0\delta_{x_0}$ supported at $x_0 \in \Omega$ and with intensity $c_0 \in \mathbb{R}$, for instance. By applying a φ-sweep to it, an approximation to $\varphi_{x_0,\alpha,\beta}$ would be produced since

$$\langle c_0\delta_{x_0}, \varphi_{x,\alpha,\beta}\rangle = c_0\,\varphi_{\cdot,\alpha,\beta}(x_0) = c_0\,\varphi_{x_0,\alpha,\beta}(x)\,,\ x \in \Omega\,,$$

by definition of φ-sweep. Consequently the location of the peak would point to the location of the singularity and the height of the peak to its intensity. For a one-dimensional illustration, take

$$m = 64\,,\ u^m = \delta^m_{\pi/6} - 0.5\delta^m_{1/\sqrt{2}} \qquad (\pi/6 \approx 0.5236\,,\ 1/\sqrt{2} \approx 0.7071)$$

and compute its φ-sweep

$$S\big(u^m, \varphi^m_{\cdot,\alpha,\beta}\big)$$

with $\alpha = 10$, $\beta = 0.1$ at $N = 256$ equidistant grid points. Figure 1 shows the result. The peaks are located at

$$x_1 = 0.5273 \text{ and } x_2 = 0.7109$$

and their heights are

$$p_1 \approx 0.9996 \text{ and } p_2 \approx 0.4999\,,$$

respectively. The example shows that, given oscillations due to Dirac singularities on a certain grid, it is possible to approximately locate their support with the help of test function sweeps, even if it is located outside the grid.

Remarks 5.2. (a) The above simple example has illustrative purposes only. The approximate location of the singularities is taken as the grid point at which the maximum is reached. By interpolation or, equivalently, by using finer sweeps better results can be achieved (up to a limit depending on m).

(b) If one is given some random oscillatory data without further information it wont be possible to use sweeps to find out anything, in general. Fortunately, in many cases, the information available is not confined to the data set itself.

(c) In general, how would can it be checked if the singularities observed are of δ-type, and not worse: say of δ'-type, for instance? A rough diagnosis can be given by looking at order of the discretization family, that is, at the power p for which $u^m = O(m^p)$. It roughly points to the order of the distributions.

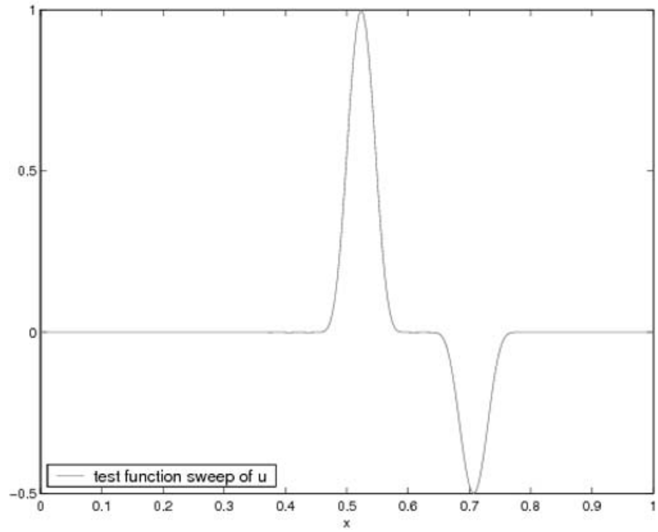

FIGURE 1. Locating singularities with a φ-sweep.

It is now possible to go back to the interpolation problem for discontinuous functions.

5.2. A remark about Gibbs phenomenon

Gibbs phenomenon is a widely observed phenomenon with which everybody working in numerical analysis or applied mathematics is very familiar with. It is observed whenever a discontinuous function is approximated by its Fourier series. The approximating series converges (albeit poorly) away from the jump discontinuity but keeps oscillating around it and does therefore not converge point-wise to its limit (due to persistent overshooting in the standard example). An historical perspective on the topic which raises and answers questions about the correct attribution of the discovery of this phenomenon can be found in [6]. This paper also contains a selection of many useful references from the vast literature about this problem.

Consider the simple sawtooth function

$$s_0(x) = \begin{cases} -(x/\pi + 1)/2\,, & x \in [-\pi, 0] \\ (1 - x/\pi)/2\,, & x \in (0, \pi] \end{cases}$$

defined on the interval $[-\pi, \pi]$. This is one of the standard one-dimensional example used to illustrate Gibbs phenomenon. The function s_0 is to be considered as a 2π-periodic function. It obviously has a jump discontinuity at $x = 0$. If one tries to approximate s_0 by its Fourier series

$$\sum_{n\in\mathbb{Z}^*} \frac{1}{\pi n}\exp(inx)\,,\ x\in[-\pi,\pi)\,,$$

then one can observe "experimentally" that the oscillations around the discontinuity point will concentrate but never disappear or attenuate no matter how many terms in the series are taken into the approximating sum. The series of course converges in $\mathrm{L}_2(-\pi,\pi)$ but this is not quite good enough for many purposes. It is actually more advantageous to think of s_0 as a distribution than it is to consider it a function. It can in fact be argued that the most interesting convergence which is taking place is the one in the sense of distributions.
Now, if the $\mathrm{L}_{2,p}(0,1)$ convergence is not good enough, why would one be interested in an even weaker convergence?
Well, this should be clear in light of the analysis presented in the previous sections.

It can be argued that Gibbs phenomenon, at the discrete level, is essentially an interpolation phenomenon. Truncating a Fourier series one always obtains an approximation with finite-dimensional information content. Now, if one plots the truncated Fourier series as a function of a continuous variable one is in a sense implicitly interpolating the value of the approximation at infinitely many points. In the presence of a singularity this automatically leads to oscillations. However, at the discrete level, it is more natural to evaluate the truncated Fourier series at a finite number of points only. If the points are chosen appropriately

$$\big[x_j^m \mapsto \delta_{m,x_j^m}\cdot z^m\,,\ j=1,\dots,m\big]\,,$$

then one obtains an approximating sequence which converges (albeit poorly) to z. Oscillations are still observed but are now decaying. In fact even for the Dirac distribution itself, one has "point-wise" convergence in the sense that

$$\frac{1}{2\pi m}D_{m,x_{j^m}}(x_j)=\delta_{jj^m}+O(\frac{1}{m})\,,\ j=-m,\dots,m\,,$$

assuming that its support always sits on a grid point. This again indicates that persistent oscillations are manifestation of the fact that of out-of-grid information needs to be approximated or accounted for (read *interpolated*). But even when oscillations occur, their exact behavior carries the information needed to determine the location of the singularity. The information has, however, to be read in the sense of distributions. As for the rest, the oscillatory approximation is as good any point-wise converging approximation. In fact, since finite-dimensional approximations are used, the space of test functions can be spanned by finitely many "functions" and the distributional information content of an oscillating approximation and that of a point-wise approximation do indeed have to coincide. It just needs to be extracted in the proper way. In many concrete applications it is not known where exactly a singularity might be located and the occurrence of persistent oscillations will be the rule rather than the exception. The point-wise convergence observed in that case, however, is so poor that the local method is always preferred.

Dirichlet's kernel (2.16) is nothing but the manifestation of a very general way of producing discretization families for the Dirac distribution. See (2.9) and Example 3, in particular. A δ^m-sweep (defined similarly to a test function sweep) by means of Dirichlet's kernel for a function ψ can be rephrased in terms of a convolution with it in view of the periodic structure. Or, in other words,

$$D_{m,x_0}(x) = D_m(x - x_0)\,,\ x \in [-\pi, \pi]\,.$$

In terms of the distributional framework, one sees that

$$S\big(D_{m,x_0}, \psi_m\big) = \sum_{k=-m}^{m} \hat{\psi}_m^k \exp(ikx_0)\,,\ x_0 \in (-\pi, \pi)$$

for the approximate discrete Fourier coefficients (computed w.r.t. the trapezoidal rule)

$$\hat{\psi}_m^k = \psi_m \cdot \exp(-ikx^m)$$

in this particular case. This means that oscillations observed whenever using a δ-sweep to represent or interpolate a discontinuous function are in this particular case what is known as Gibbs phenomenon. By using localized biased δ-sweeps obtained by multiplication with both test functions like (5.1) and with one-sided functions like

$$\begin{aligned} h^-_{x_0,\gamma}(x) &= \big(1 - \tanh(\gamma\,(x - x_0))\big)\,, \\ h^+_{x_0,\gamma}(x) &= 1 - h_{x_0,\gamma}(x)\,,\ x \in [-\pi, \pi]\,,\ \gamma > 0 \end{aligned}$$

it is possible to take advantage of the oscillations and get rid of them at the same time. To be able to effectively use one-sided localization one needs to estimate the location of the singularity. This can be done by using left- and right-sided (local) δ-sweeps and observing their difference. It is large where left and right limit do not coincide. The results obtained by approximating

$$z(x) := \begin{cases} \pi + x\,, & x \leq \frac{1}{\sqrt{2}} \\ x - \pi\,, & x > \frac{1}{\sqrt{2}} \end{cases}$$

are depicted in Figure 2. $n = 129$ equidistant discretization points are used for $(-\pi, \pi)$ in order to perform sweeps at $N = 257$ points (midpoints of the original intervals are added). The choice of the discontinuity point makes sure it is never a grid point. As to the other parameters

$$\alpha = \frac{10}{\pi}\,,\ \beta = \frac{\pi}{5}\,,\ \gamma = 100\,.$$

A global δ-sweep produces z^b_{128} whereas a local δ-sweep with one-sided bias within distance $\pi/5$ of the singularity yields z^g_{128} from the discrete information contained in

$$z_{64} = \left(z\left(\frac{j}{64}\pi\right)\right)_{j=-64,\dots,64}\,.$$

The definition of test function is adapted to the periodic case in that (5.1) is still used but (5.2) is modified to

$$\psi(x, x_0, \beta) = \Big[\min\Big(\max(x, x_0) - \min(x, x_0), 2\pi - \max(x, x_0) + \min(x, x_0)\Big)\Big]/\beta$$

which is the periodic distance function.

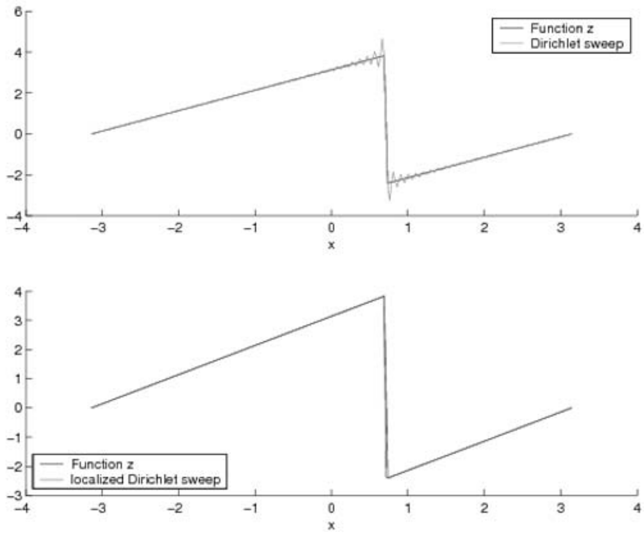

FIGURE 2. "Good and bad" approximations z^g_{128} and z^b_{128} of z.

Both approximations are equivalent in the sense of distributions, that is, in the sense that z^b_{128} and z^g_{128} both determine perfectly valid and equivalent discretization families for the distribution z.

It is also observed in [6] that Gibbs phenomenon is observed when developing smooth functions with finitely many jump discontinuities in the more general framework of Gegenbauer polynomials. It follows from distributional perspective adopted here that oscillations are to be expected whenever approximating distributions by means of bases of smooth (oscillatory) functions and that oscillations, at the discrete level at least, are the nonlocal way in which local singular behavior manifests itself.

6. Oscillations in the wave equations

Finally an example is presented where a distribution or its approximation is not the starting point but rather where it appears as the solution of an equation. The example will demonstrate that the oscillations observed when solving the wave equation for discontinuous data are an approximation to the exact solution provided they are interpreted in the sense of distributions. A standard test problem in this context is provided by the propagation of box waves. Here the natural problem of propagating Dirac impulses is chosen. In other words, the fundamental solution (Riemann function) of the wave operator will be computed numerically.

Consider the one-dimensional wave equation

$$\begin{cases} u_{tt} - u_{xx} = f(t,x)\,, & t \in \mathbb{R}\,,\ x \in (0,1) \\ u(t,0) = u(t,1) = 0\,, & t \in \mathbb{R} \\ u(0,x) = u_0(x)\,,\ u_t(0,x) = u_1(x)\,, & x \in [0,1]\,. \end{cases} \tag{6.1}$$

To compute the fundamental solution one needs to solve (6.1) for

$$\big(f, u_0, u_1\big) = \big(0, 0, \delta_y\big)\,,\ y \in (0,1)\,. \tag{6.2}$$

Taking the analytical view point first, let

$$A : \mathrm{dom}(A) \subset \mathrm{L}_2(0,1) \longrightarrow \mathrm{L}_2(0,1)$$

be the operator defined through

$$\mathrm{dom}(A) = \mathrm{H}^2(0,1) \cap \mathrm{H}^1_0(0,1)\,,\ Au = -u_{xx}\,,\ u \in \mathrm{dom}(A)$$

A is a positive definite self-adjoint operator with compact resolvent and therefore admits a calculus (cf. [10]). With it one can define

$$A^{-1/2}\sin(tA) \in \mathcal{L}\big(\mathrm{L}_2(\Omega), \mathrm{L}_2(\Omega)\big) \text{ for every } t \in \mathbb{R} \tag{6.3}$$

by

$$A^{-1/2}\sin(tA)u = \sum_{k=1}^{\infty} \sin(kt)\langle u, e_k\rangle e_k$$

for $e_k = \sin(k\pi\cdot)$. Unfortunately the Dirac distribution $\delta_y \notin \mathrm{L}_2(\Omega)$ and the functional calculus approach breaks down. The distributional framework, however, comes to rescue and one has

$$A^{-1/2}\sin(tA)\delta_y = \sum_{k=1}^{\infty} \frac{1}{k}\sin(kt)\langle \delta_y, e_k\rangle e_k = \sum_{k=1}^{\infty} \frac{1}{k}\sin(kt)\sin(k\pi y)\sin(k\pi x) \tag{6.4}$$

converges in the sense of distributions and represents the Riemann function.
When numerically solving the above equation by any consistent scheme, one expects to obtain some discrete approximation to (6.4) and, consequently, oscillations. Finding a numerically acceptable solution therefore becomes the problem of filtering the information out in the spirit of Subsection 5.1. Since the propagation u_t of the initial Dirac impulse $u_t(0) = \delta_y$ is of interest, the problem is reformulated into the wave equation with initial condition

$$u(0,x) = \delta_y\,,\ u_t(0,x) = 0\,. \tag{6.5}$$

The analytical expression for the solution then reads

$$\sum_{k=1}^{\infty} \cos(k\pi t)\sin(k\pi y)\sin(k\pi x)$$
$$= \sum_{k=1}^{\infty} \frac{1}{2}\big[\sin(k\pi(y-t)) + \sin(k\pi(y+t))\big]\sin(k\pi x)\,. \quad (6.6)$$

The discretization is chosen as in Example 2, that is, a discretization with n equidistant points x_j^n, $j = 1, \ldots, n$ is used. In addition the spatial differential operator is discretized by means of a centered second-order finite difference approximation to $-\partial_{xx}$. The same is done for ∂_{tt}. Then the scheme consists in marching implicitly in time. More specifically, if

$$u^m = (0, u_2, \ldots, u_{n-1}, 0)$$

is the solution vector at time mdt, then

$$u^{m+1} = \big(1 + \frac{dt^2}{dx^2}A^n\big)^{-1}\big(2u^m - u^{m-1}\big) \quad (6.7)$$

where A^n is defined by

$$A(j,k) = \begin{cases} 2\,, & |j-k| = 0\,,\ 1 < j,k < n\,, \\ -1\,, & |j-k| = 1\,,\ 1 < j,k < n\,, \\ 1\,, & j = k = 1\,,\ j = k = n \\ 0\,, & \text{otherwise}\,. \end{cases}$$

As to the initial conditions it is chosen

$$u^0 = e_{j_0^m}/dx\,,\ u^1 = 0\,,\ j_0^m = 13, 33\,,\ m = 32, 64\,.$$

The solution is computed up to time $t = 0.13$ as to make sure that distributions which live on a grid point are avoided. To ensure convergence a small time step $dt = t/10^4$. Figure 3 show the solutions u_n^{10000} obtained for $n = 32, 64$. The functions $S_{n,100}$, $n = 32, 64$ obtained by using a $\varphi_{10,0.1}$-sweep at $N = 100$ equidistant points are depicted in Figure 4. The peaks are located at

$$x_{32}^- = 0.36\,,\ x_{32}^+ = 0.62 \text{ and } x_{64}^- = 0.37\,,\ x_{64}^+ = 0.63$$

and the maxima are

$$M_{32}^- = 0.4952\,,\ M_{32}^+ = 0.4954 \text{ and } M_{64}^- = 0.4911\,,\ M_{64}^+ = 0.4979\,,$$

respectively.

This shows that the location, type and intensity of the singularity can be computed approximately by means of the oscillatory finite difference approximations to (6.5) to obtain

$$u(t,\cdot) \approx M_n^- \delta_{x_n^-(t)} + M_n^+ \delta_{x_n^+(t)}\,. \quad (6.8)$$

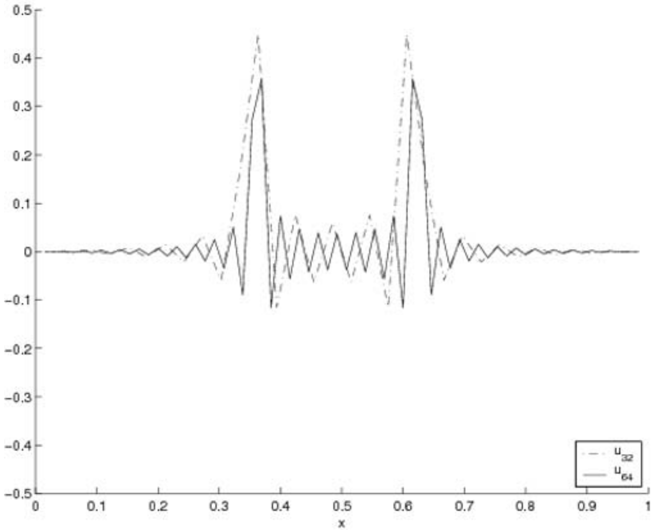

FIGURE 3. The finite difference solutions u_n, $n = 32, 64$.

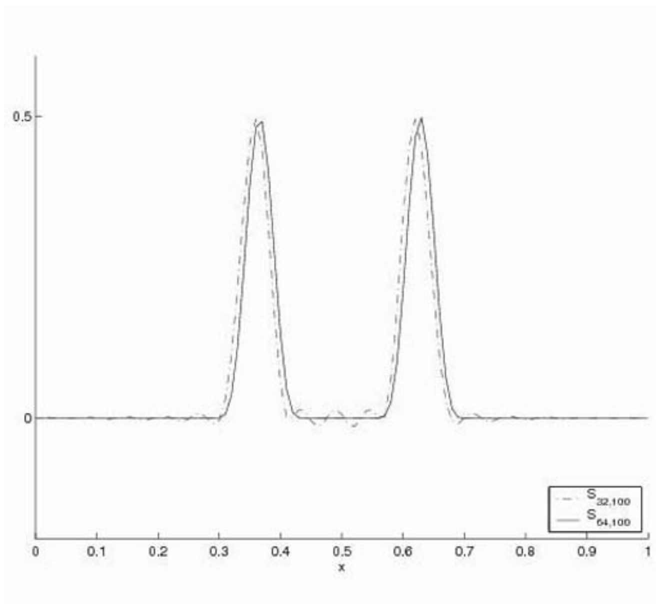

FIGURE 4. Test function sweeps $S_{n,100}$ of u_n^m, $n = 32, 64$, $m = 10^4$.

Remark 6.1. In this example it would have also been possible to work in the space

$$\mathrm{H}^{-1}(0,1) = \big(\mathrm{H}_0^1(0,1)\big)'$$

in order to look for the mild solution (cf. [7] for a definition) of the weak formulation of (6.1) with (6.5) as a function $u : \mathbb{R} \to \mathrm{H}^{-1}(0,1)$ satisfying

$$\begin{cases} \langle u_{tt}, \varphi\rangle + a(u,\varphi) = 0\,, & \varphi \in \mathrm{H}_0^1(0,1)\,, \\ u(0) = \delta_y\,,\ u_t(0) = 0\,, & t = 0 \end{cases}$$

where $a : \mathrm{H}_0^1(0,1) \times \mathrm{H}_0^1(0,1) \to \mathbb{R}$ is defined through

$$a(u,\varphi) = \int_0^1 u_x(\xi)\varphi_x(\xi)\,d\xi\,.$$

The above discussion then shows that the numerical scheme automatically produces the mild weak solution of (6.5) and that it needs to be interpreted as such. A convergence analysis would reveal that the formal order of the scheme employed is indeed realized if understood in the sense of distributions.
It has been observed (cf. [1, 2] and the overview and references in the second paper) that the accuracy of "weak" solutions obtained by finite element methods can be improved by post-processing techniques. This is possible since the solution has a higher degree of accuracy in negative Sobolev norms. That is a further manifestation of the general point made in this paper albeit in the context of normed spaces and can be understood and explained by the framework proposed here.

7. Conclusion

In this paper it is shown how the theory of distribution can naturally be linked to and used in the discrete world. Actually, even more is true. The use of the natural duality structure existing between distributions and test functions in the discrete world paves the way to a systematic understanding of numerical techniques and the treatment of problems in which distributions are natural key players. It turns out that a central role in this approach is played by smooth orthonormal bases (with respect to $\mathrm{L}_2(\Omega)$) which can be used to effectively approximate any compactly supported distribution. The discretization obtained can be viewed as a kind of *abstract spectral method*. Bases consisting of functions with only a finite degree of smoothness m only allow for approximating distributions of finite order at most m. In particular, bases consisting of merely $\mathrm{L}_2(\Omega)$ functions can only approximate $\mathrm{L}_2(\Omega)$ functions. Many methods have been developed which can be found in the literature (a small part of which is cited here) which take implicit advantage of the structure made explicit in this paper.

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No Radial Symmetries in the Arrhenius–Semenov Thermal Explosion Equation

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Abstract. Nonlinear evolution equations in the theory of exothermic chemical reactions lead to semilinear parabolic and elliptic boundary value problems with exponential nonlinearities. In contrast to a commonly employed (Frank–Kamenetskii) approximation, which permits similarity variables for the asymptotic analysis of solution behavior near thermal runaway, we show that the more correct (Arrhenius–Semenov) equation permits no radial symmetries. We also establish that a more general class of thermal nonlinearities also possess no symmetries.

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Keywords. Nonlinear evolution equations; Bifurcation; Symmetries; Arrhenius–Semenov equation; explosion criticality.

1. Introduction

In a series of papers [1–8] about 20 years ago, the first named author (K. G.) and his Ph.D. students (B. Eaton and E. Ash) investigated the number of solutions, and how to count them, for the Arrhenius–Semenov thermal explosion equation

$$-\Delta u = \lambda e^{\frac{u}{1+\epsilon u}}, \qquad \epsilon > 0, \quad \lambda > 0. \tag{1.1}$$

Here Δ is the Laplacian operator $\sum \partial^2 u/\partial x_i^2$ in one, two, or three dimensions. The parameter ϵ is essentially the reciprocal of the activation energy for the material. Thus large ϵ corresponds to relatively low activation energy and not much tendency on the part of the material to spontaneously ignite. On the other hand, small ϵ implies a material prone to spontaneous ignition.

A more commonly studied equation in the combustion and mathematical literature is the Frank–Kamenetskii thermal explosion equation

$$-\Delta u = \lambda e^u, \qquad \lambda > 0. \tag{1.2}$$

As can be seen, (1.2) is a simplification of (1.1) obtained by taking $\epsilon = 0$ in (1.1). Physically, this amounts to assuming an infinite activation energy for the material. The equation (1.2) carries many other names from its appearance in different contexts, most notably, the equation of Bratu, the equation of Poisson–Boltzman, the equation of Chandrasekhar. Both (1.1) and (1.2) are the steady-state equations of the time-dependent evolution equations

$$\frac{\partial u}{\partial t} - \Delta u = \text{RHS of (1.1) or (1.2)} \tag{1.3}$$

describing the evolution of a positive solution $u(x,t)$ from a positive initial profile $u(x,0)$, along with various given boundary conditions.

For more background as to the physical issues, the analytical issues, the numerical issues, and the historical issues (e.g., names attached to these equations), we refer the reader to our survey [4]. Those who study these explosion equations may be seen to separate into two communities, which for convenience here we will simply call the elliptic and the parabolic equation communities. We are in the former and our goal is to understand the multiple solutions of (1.1) or (1.2). These steady solutions then govern the possible limiting behaviors as time increases of the parabolic equations (1.3). Equation (1.2) has no solution for λ beyond some critical value λ^*. This led us to take the position that the 'artificial' assumption of infinite activation energy that simplifies (1.1) to (1.2) should be avoided. Accordingly, we concentrate our investigations on the Arrhenius–Semenov equation (1.1). Under appropriate boundary conditions, for (1.1) solutions exist for all $\lambda > 0$. However for λ in critical bands, there can be multiple solutions. This situation pertains whether one is speaking of either the Equations (1.1) or (1.2).

A rather good concise history and description of these explosion equations from the chemical-mathematical point of view may be found in the 1983 paper of Boddington, Feng, and Gray [9]. As for the chemists, our original interests [1–4] were to see how well we could refine and tune mathematical numerical techniques to determine as precisely as possible the critical ignition parameters λ_c and ϵ_c for the true Arrhenius rate law equation (1.1) for the slab, cylinder, and sphere geometries. The critical $\lambda_c(\epsilon)$ is that value of the exothermicity parameter λ for which the smallest positive solution of (1.1) reaches an infinite change of temperature u with respect to λ. Such λ_c is then taken to be the critical ignition parameter for given ϵ. The largest ϵ for which such ignition occurs is called ϵ_c and is of substantial physical importance. As Tables 3 and 6 of [9] show, our numerical methods produced competitive answers: $\epsilon_c(slab) = 0.245780$, $\epsilon_c(cylinder) = 0.242106$, $\epsilon_c(sphere) = 0.238797$.

There exists a wide literature in the chemistry, engineering, and mathematical communities about such equations and their criticalities, and we have no intention of providing any full overview here. However, let us note a few key mathematical results. Consider just the slab, cylinder, sphere geometries, respectively the unit 1-, 2-, and 3-dimensional spheres, along with Dirichlet boundary conditions $u = 0$ at $||x|| = 1$. Consider the general partial differential equation boundary value problem

$$\begin{cases} -\Delta u = f(u) \quad \text{in} \quad ||x|| < 1 \\ \quad u = 0 \quad \text{at} \quad ||x|| = 1 \end{cases} \tag{1.4}$$

Then Gidas, Ni, and Nirenberg [10] showed that any positive solution u is necessarily spherically symmetric. It was assumed that f be C^1 and u be $C^2(\overline{\Omega})$. Earlier Keller and Cohen [11] and Sattinger [12] had shown that if also $f(u) > 0$, $f'(u) > 0$, and $f(u)$ is unbounded; then there exists a critical λ^* for which no solutions to (1.4) exist for $\lambda > \lambda^*$. On the other hand, should $f(u)$ be bounded for $u > 0$, then solutions to (1.4) exist for all $\lambda > 0$. See Lacey [13] and Amann [14] for further related early mathematical basic results.

Possibly because it is a simpler equation, there followed a substantial number of mathematical investigations of the Frank–Kamenetskii equation (1.2) and its solution 'blow-up'. Also, we may comment that the terminology 'blow-up' is more dramatic that the terminology 'thermal runaway' that was used originally by the chemists and scientists studying the Arrhenius–Semenov equation (1.1). For the thermal runaway point of view we refer the reader especially to the fine early studies [15, 16] of the Arrhenius kinetics, which employed sophisticated inner and outer expansions, and analytical studies of the dynamics of the passage of solutions through criticalities to the higher temperature stable solutions.

The bifurcation diagram for the Arrhenius–Semenov equation (1.1) is quite interesting. See for example [4] or [6]. For small activation energy (large ϵ), one has unique solutions and a one-to-one relationship between $||u||$ = maximum temperature over the region Ω, and the (material exothermicity) bifurcation parameter $0 < \lambda < \infty$. However, as ϵ decreases (higher activation energies), the bifurcation diagram (see, e.g., Figure 1 of [4]) becomes more S-like, and at the critical ϵ_c, one reaches 'thermal runaway' in the sense that there is a critical $\lambda_c(\epsilon_c)$ for which $d||u||/d\lambda = \infty$. For any higher activation energies (smaller ϵ), one has multiple solutions to the problem, most notably, a stable lower branch (small $||u||$) and a stable higher branch (large $||u||$), with a number of unstable branches in between. These intermediate solutions carry physical meanings such as extinction and reignition, snapback thyristor, other, as one passes through the turning points. See for example [7, 8] and the papers [15, 16, 17, 18] for more details.

To conclude this introduction, our interests [5–8] became more analytical, even as we found the 34 numerical solutions to (1.1) at $\epsilon = 0.01$ that we reported in the paper [6]. We also at that time had carried out a local Lie group analysis of (1.1) to determine the radial symmetries it may have. Such symmetries if present would provide a reduction for its solution. It surprised us that they are not present, except in the Frank–Kamenetskii simplified equation (1.2). We vaguely [6, p. 559, bottom] mentioned such results and promised a paper to appear. However, our interests at that time went elsewhere and we never got around to publishing that interesting analysis and finding. Therefore I take this opportunity to present those results here.

2. No symmetries in the Arrhenius–Semenov equation

For the local transformation group techniques which we will use without further comment in the following, see [19–23] and the bibliographies cited therein. Such Lie-group based methods are now well known, although they are sometimes extremely tedious. Indeed, symbolic manipulation computer schemes are sometimes used to automate the work. As we did this analysis in the late 1980's, our principal guide was the classic book Bluman and Cole [21]. For (1.1), by the result of Gidas, Ni, Nirenberg [10], for the classical spherical domains we need only consider the ordinary differential equation

$$y'' + \frac{m}{x}y' + \exp(y/(1+\epsilon y)) = 0,$$

i.e., the radial part of (1.1). Here $m = n-1$ where n is the space dimension. From the theory of local Lie groups [20,21] we see that the second-order differential equation $\Omega(x,y,y',y'') = 0$ is invariant under the group generated by $U = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}$ iff $U'' = 0$ where $U'' = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \eta'\frac{\partial}{\partial(y')} + \eta''\frac{\partial}{\partial(y'')}$, $\eta' = \frac{d\eta}{dx}\eta - y'\frac{d}{dx}\xi$, and $\eta'' = \frac{d}{dx} - y''\frac{d}{dx}\xi$. Our no-symmetries result for (1.1) is the following.

Theorem 2.1. *The Arrhenius–Semenov differential equation*

$$\Omega(x,y,y'y,'') \equiv y'' + \frac{m}{x}y' + \exp\left(\frac{y}{1+\epsilon y}\right) = 0, \quad \epsilon > 0,$$

has only the symmetry group generated by $\xi(x,y) = e$, $\eta(x,y) = 0$, if $m = 0$, $e \in R$, and $\xi(x,y) = 0$, $\eta(x,y) = 0$, if $m \neq 0$.

Proof. We seek group invariant solutions for

$$\Omega(x,y,y',y'') \equiv y'' + \frac{m}{x}y' + \exp\left(\frac{y}{1+\epsilon y}\right) = 0.$$

Let

$$\omega(x,y,y') = -\left(\frac{m}{x}y' + \exp\left(\frac{y}{1+\epsilon y}\right)\right).$$

The condition $U''\Omega = 0$ implies

$$-\xi\frac{\partial\omega}{\partial x} - \eta\frac{\partial\omega}{\partial y} - \left\{\eta_x + (\eta_y - \xi_x)y' - \xi_y(y')^2\right\}\frac{\partial\omega}{\partial(y')} + \eta_{xx}$$

$$+(2\eta_{xy} - \xi_{xx})y' + (\eta_{xx} - 2\xi_{xy})(y')^2 - \xi_{yy}(y')^3 + \{\eta_y - 2\xi_x - 3\xi_y y'\}\omega(x,y,y') = 0$$

for all x, y, y'. Now

$$\frac{\partial\omega}{\partial x} = \frac{my'}{x^2}, \quad \frac{\partial\omega}{\partial y} = -\exp\left\{\frac{y}{1+\epsilon y}\right\}\frac{1}{(1+\epsilon y)^2}, \quad \text{and} \quad \frac{\partial\omega}{\partial(y')} = -\frac{m}{x}.$$

Let

$$f(y) = \exp\left\{\frac{y}{1+\epsilon y}\right\}\frac{1}{(1+\epsilon y)^2}$$

and

$$g(y) = \exp\left\{\frac{y}{1+\epsilon y}\right\}.$$

Hence we must have

$$\left[\eta f(y) + \eta_x\frac{m}{x} + \eta_{xx} - (\eta_y - 2\xi_x)\right] + \left[-\xi\frac{m}{x^2} + 2\eta_{xy} - \xi_{xx} + 3\xi_y g(y)\right] y'$$
$$+ \left[-\frac{2m}{x}\xi_y + \eta_{yy} - 2\xi_{xy}\right](y')^2 + [-\xi_{yy}](y')^3 = 0$$

for all x, y, y'. Thus we are led to the four equations which must be satisfied simultaneously:

$$-\xi_{yy} = 0 \tag{2.1}$$

$$\frac{2m}{x}\xi_y + \eta_{yy} - 2\xi_{xy} = 0 \tag{2.2}$$

$$-\xi\frac{m}{x^2} + \frac{m}{x}\xi_x + 2\eta_{xy} - \xi_{xx} + 3\xi_y g(y) = 0 \tag{2.3}$$

$$\eta f(y) + \eta_x\frac{m}{x} + \eta_{xx} - (\eta_y - 2\xi_x)g(y) = 0. \tag{2.4}$$

From (2.1) we get

$$\xi(x,y) = a(x)y + b(y)$$

for some functions a and b. From (2.2)

$$\eta_{yy} = 2\left[a'(x) - \frac{m}{x}a(x)\right] \equiv \alpha(x)$$

for some function α. Hence

$$\eta(x,y) = \frac{1}{2}\alpha(x)y^2 + \beta(x)y + \gamma(x)$$

for some functions β and γ. Inserting the above expressions for ξ and η into (2.4) and collecting similar terms involving powers of y and $g(y)$ yields the following equations. From the y^0 terms,

$$\frac{m}{x}\gamma'(x) + \gamma''(x) = 0, \tag{2.5}$$

y terms,

$$2\epsilon\left(\frac{m}{x}\gamma'(x) + \gamma''(x)\right) + \left(\frac{m}{x}\beta'(x) + \beta''(x)\right) = 0, \tag{2.6}$$

y^2 terms,

$$\epsilon^2\left(\frac{m}{x}\gamma'(x) + \gamma''(x)\right) + 2\epsilon\left(\frac{m}{x}\beta'(x) + \beta''(x)\right) + \epsilon\frac{m}{2x}\alpha'(x) + \frac{1}{2}\alpha''(x) = 0, \tag{2.7}$$

y^3 terms,

$$\epsilon^2\left(\frac{m}{x}\beta'(x) + \beta''(x)\right) + \epsilon\left(\frac{m}{x}\alpha'(x) + \alpha''(x)\right) = 0, \tag{2.8}$$

y^4 terms,

$$\frac{\epsilon^2}{2}\left(\frac{m}{x}\alpha'(x)+\alpha''(x)\right)=0, \tag{2.9}$$

$g(y)$ terms,

$$\gamma(x)+2b'(x)-\beta(x)=0, \tag{2.10}$$

$yg(y)$ terms,

$$\beta(x)+(2a'(x)-\alpha(x))+2\epsilon(2b'(x)-\beta(x))=0, \tag{2.11}$$

$y^2g(y)$ terms,

$$\frac{1}{2}\alpha(x)+\epsilon^2(2b'(x)-\beta(x))+2\epsilon(2a'(x)-\alpha(x))=0, \tag{2.12}$$

and $y^3g(y)$ terms,

$$\epsilon_2(2a'(x)-\alpha(x))=0. \tag{2.13}$$

We now treat the case $m \neq 1$. From equations (2.5), (2.6), and (2.7) we obtain

$$\gamma(x)=c_\gamma x^{1-m}+d_\gamma, \quad \beta(x)=c_\beta x^{1-m}+d_\beta, \text{ and } \alpha(x)=c_\alpha x^{1-m}+d_\alpha$$

for some constants $c_\alpha, c_\beta, c_\gamma, d_\alpha, d_\beta, d_\gamma \in \mathbf{R}$. Equations (2.8) and (2.9) are also satisfied. From Equation (2.13),

$$a'(x)=\frac{1}{2}\alpha(x)=\frac{c_\alpha}{2}x^{1-m}+\frac{d_\alpha}{2}.$$

Hence

$$\alpha(x)=\begin{cases} c_a x^{2-m}+d_a x+e_a, & \text{if} \quad m\neq 2\\ c_a\log x+d_a x+e_a, & \text{if} \quad m=1\end{cases}$$

where

$$c_a=\begin{cases} \frac{c_\alpha}{2(2-m)}, & \text{if} \quad m\neq 2\\ \frac{c_\alpha}{2}, & \text{if} \quad m=2\end{cases}$$

$$d_a=\frac{d_\alpha}{2}, \quad e_a\in R.$$

From (2.12) we get

$$b'(x)=\frac{1}{2}\beta(x)-\frac{1}{4\epsilon^2}\alpha(x).$$

Hence

$$b(x)=\begin{cases} \frac{1}{4\epsilon^2}\left[\frac{2\epsilon^2 c_\beta-c_\alpha}{2-m}x^{2-m}+(2\epsilon^2 d_\beta-d_\alpha)x+e_b\right] & \text{if} \quad m\neq 2\\ \frac{1}{4\epsilon^2}\left[\frac{2\epsilon^2 c_\beta-c_\alpha}{2-m}\log x+(2\epsilon^2 d_\beta-d_\alpha)x+e_b\right] & \text{if} \quad m=2.\end{cases}$$

From (2.11),

$$b'(x)=\left(\frac{2\epsilon-1}{4\epsilon}\right)\beta(x)$$

and

$$\alpha(x)=\epsilon\beta(x),$$

and from equation (2.10)

$$\gamma(x) = \frac{1}{2\epsilon}\beta(x).$$

Recapitulating, from Equation (2.4) the following must be satisfied.

$$\begin{aligned}
\alpha(x) &= cx^{1-m} + d, \quad c, d \in R \\
\beta(x) &= \tfrac{1}{\epsilon}\alpha(x) \\
\gamma(x) &= \tfrac{1}{2\epsilon^2}\alpha(x)
\end{aligned}$$

$$a(x) = \begin{cases} \frac{c}{2(2-m)}x^{m-2} + \frac{d}{2}x + e_a, & m \neq 2, \quad e_a \in R \\ \frac{c}{2}\log x + \frac{d}{2}x + e_a, & m = 2, \quad e_a \in R \end{cases}$$

$$b(x) = \begin{cases} \frac{1}{4\epsilon^2}\left[\frac{2\epsilon-1}{2-m}cx^{2-m} + (2\epsilon-1)x + e_b\right], & m \neq 2, \quad e_b \in R \\ \frac{1}{4\epsilon^2}\left[(2\epsilon-1)c\log x + (2\epsilon-1)x + e_b\right], & m = 2, \quad e_b \in R \end{cases}$$

$$b'(x) = \frac{2\epsilon - 1}{4\epsilon}\beta(x).$$

From equation (2.3) we have

$$\xi_y = a(x) = 0.$$

This further implies $\alpha = \beta = \gamma = \beta' = 0$. Hence if $m \neq 0, 1$ we have $\xi = 0$, $\eta = 0$, and if $m = 0$ we have $\xi(x, y) = e_b$ and $\eta(x, y) = 0$.

Now we restrict ourself to the case $m = 1$. In this case,

$$\alpha(x) = c_\alpha \log x + d_\alpha, \ \beta(x) = c_\beta \log x + d_\beta, \text{ and } \gamma(x) = c_\gamma \log x + d_\gamma.$$

Equations (2.10) and (2.11) yield as before $\beta(x) = \frac{1}{\epsilon}\alpha(x)$ and $\gamma(x) = \frac{1}{2\epsilon^2}\alpha(x)$. Equation (2.13) gives

$$a(x) = \frac{1}{2}c_\alpha(\log x - 1)x + \frac{d_\alpha}{2} + e_a.$$

Again from (2.3) $a(x) = 0$, and therefore $\alpha = \beta = \gamma = \beta' = 0$. Equation (2.3) further implies that $b(x) = 0$.

Therefore in any case, the *only* symmetry groups for the Arrhenius–Semenov equation are given by $\xi(x, y) = e_b$, $\eta(x, y) = 0$, if $m = 0$, and $\xi(x, y) = 0$, $\eta(x, y) = 0$, if $m \neq 0$. $\square$

Next, we consider a more general nonlinearity.

Theorem 2.2. *The nonlinear differential equation*

$$\Omega(x, y, y', y'') \equiv y'' + \frac{m}{x}y' + f(y) = 0$$

has group invariant solutions only if $f(y) = ce^{ky}$, $c, k \in R$. *Hence it is necessary for* f *to be of the Frank–Kamenetskii type for group invariant solutions to exist.*

Proof. We seek the general form of f which insures group invariant solutions to

$$y'' + \frac{m}{x}y' + f(y) = 0.$$

Here we have

$$\Omega(x, y, y', y'') \equiv y'' - \omega(x, y, y')$$

where

$$\omega(x, y, y') = -\left(\frac{m}{x}y' + f(y)\right).$$

The condition

$$U''\Omega = 0 \quad \text{when} \quad \Omega = 0$$

gives

$$-\xi\frac{\partial\omega}{\partial x} - \eta\frac{\partial\omega}{\partial y} - \{\eta_x + (\eta_y - \xi_x)y' - \xi_y(y')^2\}\frac{\partial\omega}{\partial(y')}$$
$$+\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})(y')^2 - \xi_{yy}(y')^3 + \{\eta_y - 2\xi_x - 3\xi_y y\}\omega = 0$$

for all x, y, y', and

$$\frac{\partial\omega}{\partial x} = \frac{m}{x^2}y', \quad \frac{\partial\omega}{\partial y} = -f'(y), \quad \frac{\partial\omega}{\partial(y')} = -\frac{m}{x}.$$

Collecting like terms involving powers of y', we are led to consider the following equations.

$$\xi_{yy} = 0 \tag{2.14}$$

$$-\frac{m}{x}\xi_y + \eta_{yy} - 2\xi_{xy} + 3\frac{m}{x}\xi_y = 0 \tag{2.15}$$

$$-\xi\frac{m}{x^2} + \frac{m}{x}\xi_x + 2\eta_{xy} - \xi_{xx} + 3\xi_y f(y) = 0 \tag{2.16}$$

$$\eta f'(y) + \frac{m}{x}\eta_x + \eta_{xx} - f(y)(\eta_y - 2\xi_x) = 0 \tag{2.17}$$

for all x, y. These equations resemble (2.1) through (2.4) but here we are considering a more general nonlinearity.

As before, (2.14) gives

$$\xi(x, y) = a(x)y + b(x)$$

for some functions a and b. Equation (2.15) gives

$$\eta_{yy} = 2\left(a'(x) - \frac{m}{x}a(x)\right).$$

Let

$$\alpha(x) = a'(x) - \frac{m}{x}a(x).$$

Then

$$\eta(x, y) = \alpha(x)y^2 + \beta(x)y + \gamma(x)$$

for some functions β and γ. From Equation (2.16) we have if

$$\xi_y = a(x) \neq 0$$

that

$$f(y) = \frac{1}{3\xi_y}\left[\xi\frac{m}{x^2} - \frac{m}{x}\xi_x + \xi_{xx} - 2\eta_{xy}\right].$$

Thus

$$f'(y) = \frac{1}{3\xi_y}\left[\xi_y\frac{m}{x^2} - \frac{m}{x}\xi_{xy} + \xi_{xxy} - 2\eta_{xyy}\right].$$

Since

$$\xi_{xy} = a'(x), \quad \xi_{xxy} = a''(x),$$

and

$$\eta_{xyy} = 2\alpha'(x) = 2a''(x) - 2\frac{d}{dx}\left(\frac{m}{x}a(x)\right)$$

we have

$$f'(y) = -\frac{1}{3a(x)}\left[a''(x) - \frac{m}{x}a'(x) + \frac{m}{x^2}a(x)\right]$$

which is a function of x. Therefore $f'(y) = k$ for some constant k. If $\xi_y \neq 0$, then $f(y) = ky$ and $a(x)$ must satisfy the equation

$$a''(x) - \frac{m}{x}a'(x) + \left(3k - \frac{m}{x^2}\right)a(x) = 0.$$

This linear case ($\xi_y \neq 0$) $f(y) = ky$ does not interest us here. However, the second-order differential equation for $a(x)$ just given, along with a comparable equation for $b(x)$, could be pursued to obtain $\xi(x, y)$, should there be interest.

In the case that $\xi_y = a(x) = 0$, Equation (2.16) becomes

$$-\frac{m}{x^2}\xi + \frac{m}{x}\xi_x + 2\eta_{xy} - \xi_{xx} = 0 \tag{2.16$'$}$$

where $\xi(x, y) = b(x)$ and $\eta(x, y) = \beta(x)y + \gamma(x)$.

Equation (2.17) implies

$$(\beta y + \gamma)f'(y) + \frac{m}{x}(\beta' y + \gamma) + \beta'' y + \gamma'' - f(y)(\beta - 2b') = 0. \tag{2.17$'$}$$

Differentiating (2.17′) twice with respect to y gives

$$\beta f''(y) + (\beta y + \gamma)f^{(3)}(y) + 2f''(y)b' = 0.$$

If we suppose $\beta \neq 0$ then

$$f''(y) + yf^{(3)}(y) + A(x)f^{(3)}(y) + B(x)f''(y) = 0 \tag{2.18}$$

where

$$A(x) = \frac{\gamma(x)}{\beta(x)} \quad \text{and} \quad B(x) = \frac{2b'(x)}{\beta(x)}.$$

Differentiating both sides of (2.18) with respect to x yields

$$A'(x)f^{(3)}(y) + B'(x)f''(y) = 0$$

or

$$\frac{f^{(3)}(y)}{f^{(2)}(y)} = -\frac{B'(x)}{A'(x)} = k$$

for some constant k. Hence $f(y) = k_1 e^{ky} + k_2 y + k_3$, $k_0, k_1, k_2 \in R$. Substituting this expression for $f(y)$ back into Equation (2.18) yields

$$k^2 k_1[1 + ky + kA(x) + B(x)] = 0.$$

Hence if $k_1 \neq 0$ then $k = 0$ and $f(y)$ is linear.

Now assuming $\beta(x) = 0$, and therefore that

$$\eta(x, y) = \gamma(x), \quad \xi(x, y) = b(x),$$

from Equation (2.17)

$$\eta f'(y) + \frac{m}{x}\eta_{xx} - 2f(y) = 0.$$

Differentiating both sides of (2.17) with respect to y gives

$$\eta f''(y) + 2f'(y)\xi_x = 0$$

or

$$\frac{f''(y)}{f'(y)} = -\frac{2\xi_x}{\eta} = k$$

for some constant k. Therefore $f(y) = k_1 e^{ky} + k_2$. Equation (2.16) gives

$$b''(x) - \frac{m}{x}b'(x) + \frac{m}{x^2}b(x) = 0.$$

Equation (2.17) gives

$$\gamma''(x) + \frac{m}{x}\gamma'(x) - k_2 k\gamma(x) = 0.$$

By considerations as above, the above equations for b and γ yield the following conclusions. Of course we have assumed that $f(y)$ possesses the needed regularity as we have differentiated it in the above analysis.

If $m = 0$, the one space dimensional case, then either

$$f(y) = k_1 e^{ky} + k, \quad \xi(x, y) = b, \quad \eta(x, y) = 0, \quad \text{for} \quad b \in R,$$

or

$$f(y) = k_1 e^{ky}, \quad \xi(x, y) = ax + b, \quad \eta(x, y) = -\frac{2a}{k} \quad \text{for} \quad a, b \in R.$$

If $m = 1$, the two space dimensional case, then

$$f(y) = k_1 e^{ky},\ \xi(x, y) = ax\log x + bx,\ \eta(x, y) = -\frac{2}{k}(a\log x + (a + b)),\ a, b \in R.$$

If $m > 1$, the three and higher space dimensional case, then

$$f(y) = k_1 e^{ky}, \quad \xi(x, y) = ax, \quad \eta(x, y) = -\frac{2a}{k}, \quad a \in R. \qquad \square$$

3. Conclusions and discussion

In Section 1 we reviewed the important Arrhenius–Semenov equation of thermal explosion theory. In Section 2 we proved the previously unknown result that the Arrhenius–Semenov equation (1.1) possesses no radial symmetries. We also established that among a more general class of thermal nonlinearities, the only one which possesses symmetries is the Frank–Kamenetskii equation (1.2). To our knowledge, that is also a previously unknown fact.

In proving their result that positive solutions must be spherically symmetric, for a class of equations that includes both (1.1) and (1.2), Gidas, Ni and Nirenberg [10] employed the maximum principle for elliptic partial differential equations, plus an interesting device of 'moving parallel planes'. They did not do a Lie group analysis as we have done in the present paper. However, their result that positive solutions on spherical domains must be spherically symmetric, does indeed constitute a group symmetry, i.e., solution invariance under $SO(n)$. Thus within the context of [10], especially in view of our Theorem 2.2 which looks at general nonlinearities f, our results may be viewed as extending and complementing [10]. That is, [10] established a symmetry in the angular variable. We have established the nonsymmetry for the Arrhenius–Semenov equation, and the symmetries for the Frank–Kamenetskii equation, in the radial variable.

For space dimensions 2 and higher ($m > 0$) we showed that the infinitesimal generator $U = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$ (and its prolongations) admitted no radial symmetries for the Arrhenius–Semenov equation. In one space dimension ($m = 0$) from Theorem 2.1 we do get the spatial translation invariance of solutions. This fact can be seen without a Lie group analysis just from the one-dimensional Laplacian $y''(x)$. That operator also enjoys (see [22]) an $SO(2)$ symmetry in x and y, but from our results we see that the nonlinear term precludes such for the Arrhenius–Semenov equation.

Our Theorem 2.2 does establish the infinitesimal generators for the radial symmetries of the Frank–Kamenetskii equation. These infinitesimal symmetries may be exponentiated by the standard methods (e.g., see [22]) to determine the corresponding symmetry group. We do not do the details here. Let us briefly comment. For $m = 0$ we get the trivial spatial translation invariance mentioned above, plus a scale and translation invariance in x coupled with a translation of the dependent variable y. The $m > 1$ (space dimension 3 or higher) is similar. However, the space dimension 2 ($m = 1$) has a more complicated symmetry group, reflecting the nature of the fundamental singularity for the Laplacian in 2 dimensions.

One could perform a similar local transformation group analysis of the time dependent equation (1.3) with the Arrhenius–Semenov nonlinearity. We do not know if that has been done. In the literature usually one finds the Frank–Kamenetskii approximation imposed (see [24] and citations therein) and then use of auxiliary variables such as $\tau = -\ln(T - t)$, $y = x(T - t)^{-1/2}$ permits an asymptotic analysis near the 'blow-up' time T. In both cases (1.1) and (1.2), the solution narrows to a 'hot spot', see our δ-function-like solution profiles which we computed in [6]. However, the Arrhenius–Semenov equation enables these hot spots to eventually spread out as one passes through criticality. Thus in our opinion the asymptotics of the fold points on the Arrhenius–Semenov bifurcation curve are more interesting than the approximation 'blow-up'. These Arrhenius–Semenov fold dynamics have been studied in the work of Kapila [15,16], the later work of Winters and Cliffe [17] and Van de Velde and Ward [18], and also in the chemistry literature, see, e.g., Boddington, Gray, and Kay [25]. When these asymptotics are combined with initial value profiles, one finds [26,27] that one needs to start with an initial profile u_0 which has moment properties resembling a Gaussian, i.e., the tendency to concentrate heat should already be in the initial distribution. Also [27] the initial maximum temperature should be small enough that one does not evolve into the unstable steady solution profiles. There is also interesting travelling wave behavior [28] that has been detected.

Our result that the Arrhenius–Semenov equation (1.1) possesses no radial symmetries raises a number of questions that could be suitable for further investigation. We just mention a few here. Analytic solutions for the simpler equation (1.2) are known in $n = 1, 2$ dimensions (see [4]). What is the general relation of our result to integrability? Does it fit into a larger class of equations for which no symmetries and no integrability are related? What about the implications of our result for other combustion nonlinearities that one finds in use, e.g., $e^{-1/u}$, $u^{1/2}e^{-1/u}$, and other boundary conditions such as $\partial u/\partial n + \beta u = 0$? It should be noted that we did not impose any boundary conditions during our analysis. It must be admitted (in our opinion) that, generally speaking, Lie Group methods for determining the group symmetries of a differential equation, do not naturally extend to accommodate boundary conditions.

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Mild Well-posedness of Abstract Differential Equations

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Abstract. We obtain spectral conditions that characterize mild well-posed inhomogeneous differential equations in a general Banach space X. L^p periodic solutions of first and second-order equations are considered. The results are expressed in terms of operator-valued Fourier multipliers. Our approach provides a unified framework for various notions of strong and mild solutions. Applications to semilinear equations of second order in Hilbert spaces are given.

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1. Introduction

Operator-valued Fourier multipliers and their applications to differential equations have received much attention recently. Among the many papers on the subject, we mention Arendt-Bu [5], Weis [18] and Denk-Hieber-Prüss [10]. Mild solutions of abstract differential equations are of great importance and are connected to operator semigroups and cosine functions for first and second-order problems respectively (see, e.g., the monograph [3]). It was discovered recently that for strong solutions of the first-order problem, well-posedness did not require that the operator involved be the generator of a semigroup. In Arendt-Bu [5], a very simple and elegant characterization of strong well-posedness was established for periodic solutions. However the problem of characterizing mild well-posedness was left open, except when the operator A generates C_0-continuous semigroup. See the remark after Proposition 3.4. in [5].

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The main objective of this paper is to establish a characterization of mild well-posedness for periodic solutions of differential equations of first and second order. We work with a different definition of mild solution and show that, for the first-order Cauchy problem, it coincides with the one adopted by Arendt and Bu [5] in case A generates a strongly continuous semigroup. Actually, the definition of mild solutions that we adopt is inspired by Staffans [15] where he worked with a first-order equation in Hilbert space.

Let A be a closed and densely defined operator in a Banach space X. We consider the inhomogeneous problem with periodic boundary conditions

$$P_{\mathrm{per}}(f)\left\{\begin{array}{lcl} u'(t) &=& Au(t)+f(t), \quad t\in[0,2\pi],\\ u(0) &=& u(2\pi), \end{array}\right.$$

where $f \in L^p((0,2\pi);X)$, $1 \le p < \infty$. A *strong* L^p-solution of $P_{\mathrm{per}}(f)$ is a function $u \in W^{1,p}((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ such that $P_{\mathrm{per}}(f)$ is satisfied t-a.e. Assuming that X is a UMD space, Arendt and Bu [5] (see also Arendt [2]) have characterized strong L^p-well-posedness of the periodic problem $P_{\mathrm{per}}(f)$ in terms of the R-boundedness of the set $\{k(ikI-A)^{-1} : k \in \mathbb{Z}\}$.

Let $1 \le p < \infty$. We will prove that $P_{\mathrm{per}}(f)$ is $(W^{1,p}, L^p)$ mildly well posed (see Definition 3.1) if and only if $i\mathbb{Z} \subset \rho(A)$ and $((ikI-A)^{-1})_{k\in\mathbb{Z}}$ is an L^p-multiplier. In the case of the Cauchy problem of second order, we introduce two new notions of mild solutions and this allows us to distinguish between having $((-k^2I-A)^{-1})_{k\in\mathbb{Z}}$ and $(ik(-k^2I-A)^{-1})_{k\in\mathbb{Z}}$ as L^p-multipliers. The latter gives a more transparent description of the concept of C^1-mild solution of the second-order problem (see [8], [13] and [14]).

The interest in using Fourier multipliers comes from the fact that sufficient conditions for operator-valued Fourier multipliers have been established recently (see [5], [10], [18] and [12]).

The paper is organized as follows. In Section 2, we give some preliminaries on operator-valued Fourier multipliers and strong well-posedness of $P_{\mathrm{per}}(f)$. In Section 3, we establish a characterization of mild well-posedness of $P_{\mathrm{per}}(f)$ and its connection to strongly continuous semigroups. Section 4 is concerned with the second-order problem. There, we present two notions of mild well-posedness and characterize them through Fourier multipliers. Furthermore, we examine the situation when A is the generator of a strongly continuous cosine function on X. In Section 5, we present a unified approach to mild well-posedness for the first- and second-order problems in UMD Banach spaces using Hardy-Sobolev spaces. Finally, in Section 6, an application to semilinear equations in Hilbert spaces is considered.

2. Preliminaries

Let X be a Banach space. We denote by $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators on X. If Y is another Banach space, we write $\mathcal{L}(X,Y)$ for the space of bounded linear operators from X to Y. By $\rho(A)$ we denote the resolvent set of the operator A, and we write $R(\lambda, A) = (\lambda - A)^{-1}$ when $\lambda \in \rho(A)$.

For $f \in L^1((0, 2\pi); X)$ denote by

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} f(t) dt$$

the kth Fourier coefficient of f, where $k \in \mathbb{Z}$. The Fourier coefficients determine the function f; i.e., $\hat{f}(k) = 0$ for all $k \in \mathbb{Z}$ if and only if $f(t) = 0$ a.e.

We shall frequently identify the spaces of (vector or operator-valued) functions defined on $[0, 2\pi]$ with their periodic extensions to $\mathbb{R}$. Thus, throughout, we consider the space $L^p_{2\pi}(\mathbb{R}; X)$ (which is also denoted by $L^p((0, 2\pi); X)$, $1 \le p \le \infty$) of all 2π-periodic Bochner measurable X-valued functions f such that the restriction of f to $[0, 2\pi]$ is p-integrable (essentially bounded if $p = \infty$).

We recall the notion of operator-valued Fourier multiplier.

Definition 2.1. Let $1 \le p < \infty$. A sequence $(M_k)_{k \in \mathbb{Z}} \subset \mathcal{L}(X)$ is an L^p-multiplier if, for each $f \in L^p((0, 2\pi); X)$ there exists a function $g \in L^p((0, 2\pi); X)$ such that

$$M_k \hat{f}(k) = \hat{g}(k), \quad k \in \mathbb{Z}.$$

If a sequence $(M_k)_{k \in \mathbb{Z}} \subset \mathcal{B}(X)$ is an L^p-multiplier, then there exists a unique bounded operator $\mathcal{M} : L^p((0, 2\pi); X) \to L^p((0, 2\pi); X)$ such that

$$\widehat{(\mathcal{M}f)}(k) = M_k \hat{f}(k),$$

for all $k \in \mathbb{Z}$ and all $f \in L^p((0, 2\pi); X)$.

Recall that a family $\mathbf{T} \subset \mathcal{L}(X, Y)$ is called R-bounded if there is a constant $C \ge 0$ such that

$$|| \sum_{j=1}^{n} r_j \otimes T_j x_j ||_{L^p(0,1;Y)} \le C || \sum_{j=1}^{n} r_j \otimes x_j ||_{L^p(0,1;X)} \tag{2.1}$$

for all $T_1, \ldots, T_n \in \mathbf{T}$, $x_1, \ldots, x_n \in X$ and $n \in \mathbb{N}$, for some $p \in [1, \infty)$. More information on R-boundedness and its relationship to L^p multipliers can be found in the references [5], [10], [18]. If X is isomorphic to a Hilbert space, then, R-boundedness in $\mathcal{L}(X)$ is equivalent to boundedness. On the other hand, in any Banach space, R-boundedness is a necessary condition for L^p multipliers (see [5, Proposition 1.11, Proposition 1.13 and Proposition 1.17]).

We say that problem $P_{\text{per}}(f)$ is *strongly L^p-well posed* if for each $f \in L^p((0, 2\pi); X)$ there exists a unique strong L^p-solution of $P_{\text{per}}(f)$.

In [5, Theorem 2.3] the following remarkable result was established: if X is a UMD space and $1 < p < \infty$ then the following assertions are equivalent:

(i) $P_{\text{per}}(f)$ is strongly L^p-well posed.
(ii) $i\mathbb{Z} \subset \rho(A)$ and $(kR(ik, A))_{k \in \mathbb{Z}}$ is an L^p-multiplier.
(iii) $i\mathbb{Z} \subset \rho(A)$ and $(kR(ik, A))_{k \in \mathbb{Z}}$ is R-bounded.

The equivalence (i)$\Leftrightarrow$(ii) is valid in any Banach space and for $p = 1$ as well.

The concept of mild solution studied in [5, Section 3] is the following. Let $f \in L^1((0,2\pi);X)$. A function $u \in C([0,2\pi];X)$ is called a *mild solution* of the problem $P_{\text{per}}(f)$ if $u(0)=u(2\pi)$ and

$$\begin{cases} \displaystyle\int_0^t u(s)ds \in D(A), \quad \text{and} \\ \displaystyle u(t)-u(0) = A\int_0^t u(s)ds + \int_0^t f(s)ds \end{cases} \tag{2.2}$$

for all $t \in [0,2\pi]$. It is clear that every strong L^p-solution is a mild solution.

We say that problem $P_{\text{per}}(f)$ is L^p *mildly well posed* if for each $f \in L^p((0,2\pi);X)$ there exists a unique mild solution of $P_{\text{per}}(f)$.

Now recall from [5, Proposition 3.4] that if $\overline{D(A)} = X$, and the problem $P_{\text{per}}(f)$ is L^p mildly well posed then we have that $i\mathbb{Z} \subset \rho(A)$ and $(R(ik,A))_{k\in\mathbb{Z}}$ is an L^p-multiplier. In the following section we will use the above condition to characterize mild well-posedness of $P_{\text{per}}(f)$, adopting a different notion of mild solution.

3. Mild-well-posedness and L^p-multipliers

Let A be a closed operator in X with domain $D(A)$ and $1 \le p < \infty$. Define the operator $\mathcal{A}$ on $L^p((0,2\pi);X)$ by $D(\mathcal{A}) = W^{1,p}((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ and

$$\mathcal{A}u = u' - Au.$$

Here $W^{1,p}((0,2\pi);X)$ is the vector-valued Sobolev space. When considering the space $L^p((0,2\pi);D(A))$, we equip $D(A)$ with the graph norm. We now define the notion of mild solution that we will use.

Definition 3.1. We say that the problem $P_{\text{per}}(f)$ is $(W^{1,p},L^p)$ mildly well posed if there exists a linear operator $\mathcal{B}$ that maps $L^p((0,2\pi);X)$ continuously into itself as well as $W^{1,p}((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ into itself and which satisfies

$$\mathcal{A}\mathcal{B}u = \mathcal{B}\mathcal{A}u = u$$

for all $u \in W^{1,p}((0,2\pi);X) \cap L^p((0,2\pi);D(A))$. In this case the function $\mathcal{B}f$ is called the $(W^{1,p},L^p)$ mild solution of $P_{\text{per}}(f)$ and $\mathcal{B}$ the solution operator.

Clearly, the solution operator $\mathcal{B}$ above is unique, if it exists at all. The above notion of well-posedness is suggested by the paper Staffans [15] in case where $p=2$ and X is a Hilbert space.

Our first main result in this paper characterizes $(W^{1,p},L^p)$ mildly well-posedness in terms of operator-valued L^p-multipliers in Banach spaces.

Theorem 3.2. *Let A be closed linear operator and assume $\overline{D(A)} = X$. Let $1 \le p < \infty$. Then the following assertions are equivalent:*

(i) *$P_{\text{per}}(f)$ is $(W^{1,p},L^p)$ mildly well posed.*
(ii) *$i\mathbb{Z} \subset \rho(A)$ and $(R(ik,A))_{k\in\mathbb{Z}}$ is an L^p-multiplier.*

Proof. (ii) $\to$ (i). Let $\mathcal{B}$ be the operator which maps $f \in L^p((0,2\pi);X)$ into the function $u \in L^p((0,2\pi);X)$ whose kth Fourier coefficient is $R(ik,A)\hat{f}(k)$, i.e.,

$$\widehat{(\mathcal{B}f)}(k) = R(ik,A)\hat{f}(k) = \hat{u}(k), \tag{3.1}$$

for all $k \in \mathbb{Z}$ and all $f \in L^p((0,2\pi);X)$. By the remark following Definition 2.1, $\mathcal{B}$ is a bounded linear operator on $L^p((0,2\pi);X)$. Let $g \in W^{1,p}((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ and set $h = \mathcal{B}g$. Then,

$$ik\hat{h}(k) = R(ik,A)ik\hat{g}(k) = R(ik,A)\hat{g'}(k), \tag{3.2}$$

for all $k \in \mathbb{Z}$. Since $g' \in L^p((0,2\pi);X)$, there exists $w \in L^p((0,2\pi);X)$ such that

$$\hat{w}(k) = R(ik,A)\hat{g'}(k) \tag{3.3}$$

for all $k \in \mathbb{Z}$. Hence from (3.2), (3.3) and [5, Lemma 2.1] we obtain $h \in W^{1,p}((0,2\pi);X)$. Note that $\hat{h}(k) \in D(A)$, $k \in \mathbb{Z}$ since $\hat{h}(k) = R(ik,A)\hat{g}(k)$. From (3.2), it follows that

$$A\hat{h}(k) = AR(ik,A)\hat{g}(k) = R(ik,A)\hat{g'}(k) - \hat{g}(k) \tag{3.4}$$

for all $k \in \mathbb{Z}$. Hence from (3.3), [5, Lemma 3.1] and the closedness of A, we conclude that $h(t) \in D(A)$ and $Ah(t) = w(t) - g(t)$ for almost all $t \in [0,2\pi]$. We have proved that $\mathcal{B}$ that maps $W^{1,p}((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ into itself. Continuity of $\mathcal{B}$ follows from the Closed Graph Theorem since the space $W^{1,p}((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ embeds continuously into $L^p((0,2\pi);X)$.

Finally, for $u \in W^{1,p}((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ we have

$$\widehat{(\mathcal{A}u)}(k) = (ikI - A)\hat{u}(k), \tag{3.5}$$

for all $k \in \mathbb{Z}$. Hence from (3.1) and [5, Lemma 3.1] we obtain $\mathcal{A}\mathcal{B}u = \mathcal{B}\mathcal{A}u = u$.

(i) $\to$ (ii). Let $x \in X$ and $x_n \in D(A)$ such that $x_n \to x$. Fix $k \in \mathbb{Z}$ and let $f_n(t) = e^{ikt}x_n$ for all $n \in \mathbb{N}$ and $f_0(t) = e^{ikt}x$. Note that $\hat{f}_n(k) = x_n$ and $\hat{f}_n(j) = 0$ for $j \neq k$. Clearly $f_n \to f_0$ in the L^p-norm. Let $u_n = \mathcal{B}f_n$. Then we have

$$ik\hat{u}_n(k) - A\hat{u}_n(k) = \widehat{(\mathcal{A}u_n)}(k) = \widehat{(\mathcal{A}\mathcal{B}f_n)}(k) = \hat{f}_n(k) = x_n.$$

Since $\mathcal{B}$ is bounded on $L^p((0,2\pi);X)$, $u_n \to u_0 := \mathcal{B}f_0$ in the L^p-norm, we conclude that $\hat{u}_n(k) \to \hat{u}_0(k)$, and

$$ik\hat{u}_0(k) - A\hat{u}_0(k) = x.$$

Hence, for all $k \in \mathbb{Z}$, $(ikI - A)$ is surjective.

Let $x \in D(A)$ be such that $(ikI - A)x = 0$, for $k \in \mathbb{Z}$ fixed. Define $u(t) = e^{ikt}x$. Then, clearly, $u \in W^{1,p}((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ and $u'(t) - Au(t) = \mathcal{A}u = 0$. Hence

$$u = \mathcal{B}\mathcal{A}u = 0,$$

and therefore $x = 0$. Since A is closed, we have proved that $i\mathbb{Z} \subset \rho(A)$.

Next we show that $(R(ik,A))_{k \in \mathbb{Z}}$ is an L^p-multiplier. Let $f \in L^p((0,2\pi);X)$. We observe that since $\overline{D(A)} = X$ and $1 \le p < \infty$, the space $W^{1,p}((0,2\pi);X) \cap$

$L^p((0,2\pi);D(A))$ is dense in $L^p((0,2\pi);X)$. Hence there exists a sequence $f_n \in W^{1,p}((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ such that $f_n \to f$ in the L^p-norm. Define

$$g_n = \mathcal{B} f_n, \quad n \in \mathbb{N}.$$

Then $g_n \in W^{1,p}((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ and

$$g_n' - A g_n = \mathcal{A} g_n = \mathcal{A}\mathcal{B} f_n = f_n, \quad n \in \mathbb{N}.$$

Taking the Fourier coefficients, and using the fact that $i\mathbb{Z} \subset \rho(A)$, we obtain from the above

$$\hat{g}_n(k) = (ikI - A)^{-1} \hat{f}_n(k) \tag{3.6}$$

for all $k \in \mathbb{Z}$. Next, we note that $\{g_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p((0,2\pi);X)$. By continuity of $\mathcal{B}$, there exists $g \in L^p((0,2\pi);X)$ such that $g_n \to g$ in the L^p-norm. From this and using Hölder's inequality we deduce that $\hat{g}_n(k) \to \hat{g}(k)$ and, analogously, $\hat{f}_n(k) \to \hat{f}(k)$. Therefore we conclude from (3.6) that $\hat{g}(k) = (ikI - A)^{-1}\hat{f}(k)$, for all $k \in \mathbb{Z}$. The claim is proved. □

When X is a Hilbert space, the result was obtained by Staffans for $p = 2$. Even in this case, he could not obtain the full range $1 \le p < \infty$ since his proof relied on Plancherel's theorem which is only valid when $X = H$ is a Hilbert space and $p = 2$.

Indeed, in the case of a Hilbert space, and for $1 < p < \infty$, a sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(H)$ is an L^p-multiplier if

$$\sup_{k\in\mathbb{Z}} (\|M_k\| + \|k(M_{k+1} - M_k)\|) < \infty. \tag{3.7}$$

However, if in addition $p = 2$, then as a consequence of Plancherel's theorem,

$$\sup_{k\in\mathbb{Z}} (\|M_k\|) < \infty \tag{3.8}$$

is a necessary and sufficient condition for $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(H)$ to be a multiplier.

In a general Banach space, even finite-dimensional, this is no longer the case. In [5, Theorem 1.3] (see also [2]), it is shown that for UMD spaces, R-boundedness of the sequences $(M_k)_{k\in\mathbb{Z}}$ and $(k(M_{k+1} - M_k))_{k\in\mathbb{Z}}$ is sufficient for $(M_k)_{k\in\mathbb{Z}}$ to be an L^p-multiplier for $1 < p < \infty$. In the case of Hilbert spaces, the sufficiency of condition (3.7) is much older (see, e.g., [6, Theorem 6.1.6, p. 135]). It is known that in a Banach space X, if condition (3.7) always implies that $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$ is an L^p multiplier for $1 < p < \infty$, then X is isomorphic to a Hilbert space (see [5, Section 1]).

If follows from the proof that the concept of mild solution considered here is related to the one studied by Da Prato and Grisvard in [9]. In that paper, they call *strict solutions* ("solutions strictes") what we call strong solutions and they term *strong solutions* ("solutions fortes") what corresponds to our $(W^{1,p}, L^p)$ mild solutions. In a sense, the present concept of mild solutions seems more natural. They appear as strong limits (in L^p) of strong solutions. Such solutions are important in the analysis of nonlinear problems.

It should also be noted that the solution $u(\cdot)$ in Theorem 3.2 depends continuously on the function f. Specifically, there exists a positive constant C such that

$$\|u\|_{L^p((0,2\pi);X)} \leq C\|f\|_{L^p((0,2\pi);X)},\ f \in L^p((0,2\pi);X). \tag{3.9}$$

This is clear from the proof and is otherwise a consequence of Definition 3.1.

As direct consequence of [5, Proposition 3.4] we obtain the following result.

Corollary 3.3. *Let X be a Banach space and assume $\overline{D(A)} = X$. If $P_{\text{per}}(f)$ is L^p mildly well posed then $P_{\text{per}}(f)$ is $(W^{1,p}, L^p)$ mildly well posed.*

Using Theorem 3.2 and [5, Theorem 3.6] we obtain the following consequence in case A generates a C_0-semigroup.

Corollary 3.4. *Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X and let $1 \leq p < \infty$. Then the following are equivalent.*

(i) *$P_{\text{per}}(f)$ is L^p mildly well posed.*
(ii) *$P_{\text{per}}(f)$ is $(W^{1,p}, L^p)$ mildly well posed.*
(iii) *$i\mathbb{Z} \subset \rho(A)$ and $(R(ik,A))_{k\in\mathbb{Z}}$ is an L^p-multiplier.*
(iv) *$1 \in \rho(T(2\pi))$.*

It is important to note that the mild solutions provided by this corollary are continuous.

Remark 3.5. We observe that according [5, Proposition 1.11] condition (ii) in Theorem 3.2 implies that

(iii) $i\mathbb{Z} \subset \rho(A)$ and $(R(ik,A))_{k\in\mathbb{Z}}$ is R-bounded.

However the converse is false. This was proved in [5, Example 3.7]

4. Mild solutions for second-order equations

This section is concerned with second-order inhomogeneous problems of the form

$$P^2_{\text{per}}(f) \begin{cases} u''(t) &= Au(t) + f(t), \qquad 0 \leq t \leq 2\pi \\ u(0) &= u(2\pi), \\ u'(0) &= u'(2\pi), \end{cases} \tag{4.1}$$

in the space $L^p_{2\pi}(\mathbb{R};X), 1 \leq p < \infty$.

A *strong* L^p-solution of $P^2_{\text{per}}(f)$ is a function

$$u \in W^{2,p}((0,2\pi);X) \cap L^p((0,2\pi);D(A))$$

such that $P^2_{\text{per}}(f)$ is satisfied t-a.e.

We say that problem $P^2_{\text{per}}(f)$ is *strongly L^p well posed* if for each $f \in L^p((0,2\pi);X)$ there exists a unique strong L^p-solution of $P^2_{\text{per}}(f)$.

We define the operator $\mathcal{A}$ on $L^p((0,2\pi);X)$ by $D(\mathcal{A}) = W^{2,p}((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ and

$$\mathcal{A}u = u'' - Au \quad \text{for} \quad u \in D(\mathcal{A}).$$

Mild solutions of second-order problems have been studied in the paper [13] (see also [8] and [14]). There, two notions of mild solutions where considered. These notions, roughly speaking, correspond to integrating equation (4.1) once and twice respectively. Here, we introduce two new notions of mild solutions for (4.1) and establish characterizations which differentiate between the corresponding well-posedness in terms of Fourier multipliers even in case of Hilbert spaces. We show that when A generates a strongly continuous cosine function, then the notions of mild solutions introduced here coincide with those studied in [13].

Definition 4.1. We say that the problem $P^2_{\text{per}}(f)$ is $(W^{2,p}, L^p)$ mildly well posed if there exists a linear operator $\mathcal{B}$ that maps $L^p((0,2\pi);X)$ continuously into itself as well as $W^{2,p}((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ into itself and which satisfies

$$\mathcal{A}\mathcal{B}u = \mathcal{B}\mathcal{A}u = u$$

for all $u \in W^{2,p}((0,2\pi);X) \cap L^p((0,2\pi);D(A))$. In this case the function $\mathcal{B}f$ is called the mild solution of order 2 (or $(W^{2,p}, L^p)$ mild solution) of $P^2_{\text{per}}(f)$ and $\mathcal{B}$ the solution operator.

Proceeding as in the previous section, one obtains the following analog of Theorem 3.2.

Theorem 4.2. *Let A be closed and assume $\overline{D(A)} = X$. Let $1 \leq p < \infty$. Then the following assertions are equivalent:*

(i) *$P^2_{\text{per}}(f)$ is $(W^{2,p}, L^p)$ mildly well posed.*

(ii) *$\{-k^2,\, k \in \mathbb{Z}\} \subset \rho(A)$ and $(R(-k^2, A))_{k\in\mathbb{Z}}$ is an L^p-multiplier.*

Proof. (ii) $\to$ (i). For each $f \in L^p((0,2\pi);X)$, let $\mathcal{B}$ be the operator which maps f into the function $u \in L^p((0,2\pi);X)$ whose k^{th} Fourier coefficient is $R(-k^2, A)\hat{f}(k)$, i.e.,

$$\widehat{(\mathcal{B}f)}(k) = R(-k^2, A)\hat{f}(k) = \hat{u}(k), \tag{4.2}$$

for all $k \in \mathbb{Z}$ and all $f \in L^p((0,2\pi);X)$. Clearly, $\mathcal{B}$ is a bounded linear operator on $L^p((0,2\pi);X)$. Let $g \in W^{2,p}((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ and set $h = \mathcal{B}g$. Then,

$$-\, k^2\hat{h}(k) = R(-k^2, A)(ik)^2\hat{g}(k) = R(-k^2, A)\hat{g''}(k), \tag{4.3}$$

for all $k \in \mathbb{Z}$. Since $g'' \in L^p((0,2\pi);X)$, there exists $w \in L^p((0,2\pi);X)$ such that

$$\hat{w}(k) = R(-k^2, A)\hat{g''}(k) \tag{4.4}$$

for all $k \in \mathbb{Z}$. Hence from (4.2), (4.3) and [5, Lemma 2.1] we obtain

$$h \in W^{2,p}((0,2\pi);X).$$

Since $h = \mathcal{B}g$, from (4.2) and (4.3) it follows that

$$\begin{aligned} A\hat{h}(k) = AR(-k^2, A)\hat{g}(k) &= -k^2R(-k^2, A)\hat{g}(k) - \hat{g}(k) \\ &= R(-k^2, A)\hat{g''}(k) - \hat{g}(k) \end{aligned} \tag{4.5}$$

for all $k \in \mathbb{Z}$. Hence from (4.4), [5, Lemma 3.1] and the closedness of A, we conclude that $h(t) \in D(A)$ and $Ah(t) = w(t) - g(t)$ for almost all $t \in [0, 2\pi]$. We have proved that $\mathcal{B}$ that maps $W^{2,p}((0, 2\pi); X) \cap L^p((0, 2\pi); D(A))$ into itself. Continuity of $\mathcal{B}$ follows from the Closed Graph Theorem since the space $W^{2,p}((0, 2\pi); X) \cap L^p((0, 2\pi); D(A))$ embeds continuously into $L^p((0, 2\pi); X)$.

Finally, for $u \in W^{2,p}((0, 2\pi); X) \cap L^p((0, 2\pi); D(A))$ we have

$$\widehat{(\mathcal{A}u)}(k) = (-k^2 I - A)\hat{u}(k), \tag{4.6}$$

for all $k \in \mathbb{Z}$. Hence from (4.2) and [5, Lemma 3.1] we obtain $\mathcal{A}\mathcal{B}u = \mathcal{B}\mathcal{A}u = u$.

(i) $\to$ (ii). We shall only give a sketch of the proof since it is analogous to the proof of the corresponding implication in Theorem 3.2. For $x \in X, k \in \mathbb{Z}$, fixed, we let $x_n \to x$ where $x_n \in D(A)$, $n \in \mathbb{N} \cup \{0\}$. Set $f_n(t) = e^{ikt}x_n$ and $f_0(t) = e^{ikt}x$. One first establishes that $\{-k^2,\ k \in \mathbb{Z}\} \subset \rho(A)$ and then using an approximation procedure, one proves that $(R(-k^2, A))_{k\in\mathbb{Z}}$ is an L^p-multiplier. Note that $\widehat{\mathcal{B}f}(k) = R(-k^2, A)\hat{f}(k)$, $k \in \mathbb{Z}$. $\square$

Suppose A generates a strongly continuous cosine function $C(t)$ and denote by $S(t)$ the associate sine function. In what follows, we shall make use of the set

$$E = \{x \in X : t \to C(t)x \text{ is once continuously differentiable }\},$$

which under the norm $||x||_E = ||x|| + \sup_{0\leq t\leq 1} ||AS(t)x||$ is a Banach space (cf. [11] and [3, Section 3.14]).

Observe that if $(x, y) \in D(A) \times E$ and f is continuously differentiable on $[0,\ 2\pi]$, then the formula

$$u(t) = C(t)x + S(t)y + \int_0^t S(t-s)f(s)ds, \tag{4.7}$$

defines a strong (classical) solution of the differential equation in (4.1) (see, e.g., Travis and Webb [16, Proposition 2.4] or [3, Chapter 3] or [11]).

Using [13, Theorem 4.6] one immediately has the following corollary to Theorem 4.2.

Corollary 4.3. *Let A be the generator of a strongly continuous cosine function $C(t)$ and denote by $S(t)$ the associated sine function. For $1 \leq p < \infty$ the following are equivalent:*

(i) *$P^2_{\mathrm{per}}(f)$ is $(W^{2,p}, L^p)$ mildly well posed.*
(ii) *For any $f \in L^p_{2\pi}(\mathbb{R}; X)$ there exists a unique $(x, y) \in X \times X$ such that u given by (4.7) is differentiable at $t = 0$ and 2π-periodic, i.e., $u(0) = u(2\pi)$ and $u'(0) = u'(2\pi)$.*
(iii) *$S(2\pi) \in \mathcal{B}(X, E)$ is invertible.*

In the context of Hilbert spaces, using [13, Cor. 4.7] we have the following.

Corollary 4.4. *Let H be a Hilbert space and let A be the generator of a strongly continuous cosine family $C(t)$. For $1 \leq p < \infty$ the following are equivalent:*

(i) *$P^2_{\text{per}}(f)$ is $(W^{2,p}, L^p)$ mildly well posed.*
(ii) *$\{-k^2 : k \in \mathbb{Z}\} \subseteq \rho(A)$ and $\sup_{k\in\mathbb{Z}} ||R(-k^2; A)|| < \infty$.*

We introduce the following definition of mild solution to equation (4.1).

Definition 4.5. We say that the problem $P^2_{\text{per}}(f)$ is $(W^{2,p}, W^{1,p})$ mildly well posed if there exists a linear operator $\mathcal{B}$ that maps $L^p((0, 2\pi); X)$ continuously into itself with range in $W^{1,p}((0, 2\pi); X)$, as well as $W^{2,p}((0, 2\pi); X) \cap L^p((0, 2\pi); D(A))$ into itself and which satisfies

$$\mathcal{A}\mathcal{B}u = \mathcal{B}\mathcal{A}u = u$$

for all $u \in W^{1,p}((0, 2\pi); X) \cap L^p((0, 2\pi); D(A))$. In this case the function $\mathcal{B}f$ is called the mild solution of order 1 (or $(W^{2,p}, W^{1,p})$ mild solution) of $P^2_{\text{per}}(f)$ and $\mathcal{B}$ the solution operator.

Observe that this new notion of mild solutions is stronger than the previous one, namely the $(W^{2,p}, L^p)$ mild solution. This will be apparent in what follows.

When X and Y are Banach spaces, we write $X \hookrightarrow Y$ to indicate that X is continuously embedded into Y. The assertions contained in the following lemma are well known.

Lemma 4.6. *Let X, Y and Z be Banach spaces such that $Y \hookrightarrow Z$. Then the following hold:*

(i) *If the linear operator $T : X \longrightarrow Y$ is continuous, then $T : X \longrightarrow Z$ is continuous.*
(ii) *If the linear operator $T : X \longrightarrow Z$ is continuous and $T(X) \subset Y$, then $T : X \longrightarrow Y$ is continuous.*

Proof. (i) follows by direct verification while (ii) is an immediate consequence of the Closed Graph Theorem. □

In view of the lemma, in Definition 4.5, we can instead require that the solution operator $\mathcal{B}$ map $L^p((0, 2\pi); X)$ into $W^{1,p}((0, 2\pi); X)$ continuously. One obtains the following result which, together with the above theorems, recognizes the multipliers establishing the differences between strong solutions, mild solutions of order one and mild solutions of order two.

Theorem 4.7. *Let A be closed and assume $\overline{D(A)} = X$. Let $1 < p < \infty$. Then the following assertions are equivalent:*

(i) *$P^2_{\text{per}}(f)$ is $(W^{2,p}, W^{1,p})$ mildly well posed.*
(ii) *$\{-k^2, k \in \mathbb{Z}\} \subset \rho(A)$ and $(ikR(-k^2, A))_{k\in\mathbb{Z}}$ is an L^p-multiplier.*

Proof. (ii) → (i). Observe that if $(ikR(-k^2, A))_{k\in\mathbb{Z}}$ is an L^p-multiplier, then so is $(R(-k^2, A))_{k\in\mathbb{Z}}$. Again from the proof of Theorem 4.2, we have that from (ii) it follows that we can construct a solution operator $\mathcal{B}$. It remains to show that $\mathcal{B}$ maps $L^p((0, 2\pi); X)$ into $W^{1,p}((0, 2\pi); X)$. Since $(ikR(-k^2, A))_{k\in\mathbb{Z}}$ is an L^p-multiplier, for any $f \in L^p((0, 2\pi); X)$, we can find a function $w \in L^p((0, 2\pi); X)$ such that $ikR(-k^2, A)\hat{f}(k) = \hat{w}(k)$, $k \in \mathbb{Z}$. Recall that $\widehat{\mathcal{B}f}(k) = R(-k^2, A)\hat{f}(k)$, $k \in \mathbb{Z}$. Hence, $ik\widehat{\mathcal{B}f}(k) = \hat{w}(k)$, $k \in \mathbb{Z}$. Application of [3, Lemma 2.2] yields that $\mathcal{B}f \in W^{1,p}((0, 2\pi); X)$.
(i) → (ii). From the definition of well-posedness and Theorem 4.2, we see that (i) implies that $\{-k^2,\ k \in \mathbb{Z}\} \subset \rho(A)$ and $(R(-k^2, A))_{k\in\mathbb{Z}}$ is an L^p-multiplier. We have to show that $(ikR(-k^2, A))_{k\in\mathbb{Z}}$ is an L^p-multiplier. Let $f \in L^p((0, 2\pi); X)$.

Since $\mathcal{B}$ maps $L^p((0, 2\pi); X)$ into $W^{1,p}((0, 2\pi); X)$ and there exists $g \in W^{1,p}((0, 2\pi); X)$ such that $\widehat{\mathcal{B}f}(k) = \hat{g}(k) = R(-k^2, A)\hat{f}(k)$, $k \in \mathbb{Z}$, it follows from [5, Lemma 2.1], Definition 2.1 and the relation $\hat{g'}(k) = ik\hat{g}(k) = ikR(-k^2, A)\hat{f}(k)$, $k \in \mathbb{Z}$ that $(ikR(-k^2, A))_{k\in\mathbb{Z}}$ is an L^p-multiplier. □

Remark 4.8. Observe that we have the following string of implications

$$\begin{aligned}\text{Strongly } L^p \text{ well-posed} &\Longrightarrow (W^{2,p}, W^{1,p}) \text{ mildly well-posed}\\ &\Longrightarrow (W^{2,p}, L^p) \text{ mildly well-posed.}\end{aligned}$$

Finally, from [13, Theorem 5.3 and Corollary 5.4] we obtain the following corollaries.

Corollary 4.9. *Let A be the generator of a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ and let $1 \le p < \infty$. Then the following assertions are equivalent:*
(i) *$P^2_{\mathrm{per}}(f)$ is $(W^{2,p}, W^{1,p})$ mildly well posed.*
(ii) *For any $f \in L^p(0, 2\pi; X)$ there exists a unique $(x, y) \in E \times X$ such that u given by (4.7) is of class C^1 and 2π-periodic, i.e., $u(0) = u(2\pi)$ and $u'(0) = u'(2\pi)$.*
(iii) *$I - C(2\pi) \in \mathcal{B}(X; X)$ is invertible.*

In the context of Hilbert spaces, we have:

Corollary 4.10. *Let H be a Hilbert space and A the generator of a strongly continuous cosine family $C(t)$ and let $1 \le p < \infty$. Then the following are equivalent:*
(i) *$P^2_{\mathrm{per}}(f)$ is $(W^{2,p}, W^{1,p})$ mildly well posed.*
(ii) *$\{-k^2 : k \in \mathbb{Z}\} \subseteq \rho(A)$ and $\sup_{k\in\mathbb{Z}} ||kR(-k^2; A)|| < \infty$.*

Of course these two assertions are equivalent to assertion (ii) of the previous corollary.

5. Fractional differentiation and well-posedness

Let us consider the first-order problem $P_{\mathrm{per}}(f)$. We note from an examination of the proof of Theorem 3.2 that if in the definition of well-posedness (Definition 3.1) we further require that $\mathcal{B}$ map $L^p((0, 2\pi); X)$ into $W^{1,p}((0, 2\pi); X)$, one can show that this is equivalent to say that $\{ik\}_{k\in\mathbb{Z}} \subset \rho(A)$ and $(ikR(ik, A))_{k\in\mathbb{Z}}$ is an L^p multiplier. This shows that strong well-posedness (see Section 1 and [5, Theorem 2.3]) fits well into our framework. More precisely we have:

Theorem 5.1. *Let A be a closed and densely defined linear operator on X and let $1 \le p < \infty$. The following assertions are equivalent:*

(i) *Problem $P_{\mathrm{per}}(f)$ is $(W^{1,p}, L^p)$ mildly well posed and the solution operator $\mathcal{B}$ maps $L^p((0, 2\pi); X)$ continuously into itself with range in $W^{1,p}((0, 2\pi); X)$*

(ii) *$\{ik\}_{k\in\mathbb{Z}} \subset \rho(A)$ and $(ikR(ik, A))_{k\in\mathbb{Z}}$ is an L^p-multiplier.*

Remark 5.2. Observe that if $P_{\mathrm{per}}(f)$ is strongly L^p well posed then condition (i) is satisfied. The converse is valid in UMD spaces by [5, Theorem 2.3].

Likewise, for the second-order problem $P^2_{\mathrm{per}}(f)$, we have the following theorem (Compare [5, Theorem 6.1] and [13, Theorem 2.1 (with $\alpha = 0$)]). See Section 4 to recall the definition of strongly L^p well-posedness for problem $P^2_{\mathrm{per}}(f)$.

Theorem 5.3. *Let A be a closed and densely defined linear operator on X and let $1 \le p < \infty$. The following assertions are equivalent:*

(i) *Problem $P^2_{\mathrm{per}}(f)$ is $(W^{2,p}, L^p)$ mildly well posed and the solution operator $\mathcal{B}$ maps $L^p((0, 2\pi); X)$ continuously into itself with range in $W^{2,p}((0, 2\pi); X)$.*

(ii) *$\{-k^2\}_{k\in\mathbb{Z}} \subset \rho(A)$ and $(k^2R(-k^2, A))_{k\in\mathbb{Z}}$ is an L^p multiplier.*

Remark 5.4. Note that if $P^2_{\mathrm{per}}(f)$ is strongly L^p well posed then condition (i) is satisfied. The converse is valid in UMD spaces, which follows by the proof of [5, Theorem 6.1] (see also [13, Theorem 2.11]).

In UMD spaces, if $1 < p < \infty$, the multiplier conditions (ii) in Theorem 5.1 and Theorem 5.3 are equivalent respectively to the R-boundedness of $(ikR(ik, A))_{k\in\mathbb{Z}}$ and $(k^2R(-k^2, A))_{k\in\mathbb{Z}}$ (see [5]).

Comparing with [5] and [13], the difference is that here, we require the domain of A to be dense in X (see however Remark 5.8 below). And here we employ different proofs.

The above suggests that one can consider a one parameter family of concepts of well-posedness. In what follows, we shall restrict ourselves to the case of UMD spaces. So, let X be a UMD space. For $1 < p < \infty$ and $0 \le \alpha$, we define the space $H^{\alpha,p}((0, 2\pi); X)$ as: $H^{\alpha,p}((0, 2\pi); X) = \{f \in L^p((0, 2\pi); X), \exists g \in L^p((0, 2\pi); X)$ such that $\hat{g}(k) = |k|^\alpha \hat{f}(k),\ k \in \mathbb{Z}\}$.

In the case of Hilbert spaces, this situation was studied by O. Staffans [15]. We note due to the UMD property (more precisely the continuity of the Hilbert transform on $L^p(0, 2\pi); X)$, we have

$$W^{m,p}((0, 2\pi); X) = H^{m,p}((0, 2\pi); X), \quad \text{for } 1 < p < \infty \ \text{ and } \ m \in \mathbb{N} \cup \{0\} \quad (5.1)$$

(see for example [17, Chapter III], [1] and for the relationship with intermediate spaces, see [7, Chapter IV, especially Section 4.4, p.272]). Now we give the definition of (α, p) well-posedness for $P_{\mathrm{per}}(f)$.

Definition 5.5. We say that the problem $P_{\mathrm{per}}(f)$ is (α, p) mildly well posed if there exists a linear operator $\mathcal{B}$ that maps $L^p((0,2\pi);X)$ continuously into itself with range in $H^{\alpha,p}((0,\,2\pi);\,X)$, as well as $W^{1,p}((0,2\pi);X)\cap L^p((0,2\pi);D(A))$ into itself and which satisfies

$$\mathcal{A}\mathcal{B}u = \mathcal{B}\mathcal{A}u = u$$

for all $u \in W^{1,p}((0,2\pi);X) \cap L^p((0,2\pi);D(A))$.

Then we have the following.

Theorem 5.6. *Let X be a UMD space and $0 \le \alpha \le 1$. Let A be closed linear operator and assume $\overline{D(A)} = X$ and $1 \le p < \infty$. Then the following assertions are equivalent:*

(i) *$P_{\mathrm{per}}(f)$ is (α, p) mildly well posed.*
(ii) *$i\mathbb{Z} \subset \rho(A)$ and $(|k|^{\alpha} R(ik, A))_{k\in\mathbb{Z}}$ is an L^p-multiplier.*

Proof. The proof is a modification of the proof of Theorem 3.2 and we omit it. □

In a similar manner, we can deal with the second-order problem $P^2_{\mathrm{per}}(f)$. For the definition of (α, p) *mild well-posedness*, we now modify Definition 4.5 (or Definition 4.1 for that matter) to require that $\mathcal{B}$ map $L^p((0,\,2\pi);X)$ into $H^{2\alpha,p}((0,\,2\pi);X)$ for $0 \le \alpha \le 1$.

The result is the following theorem.

Theorem 5.7. *Let X be a UMD space and $0 \le \alpha \le 1$. Let A be closed linear operator and assume $\overline{D(A)} = X$ and $1 \le p < \infty$. Then the following assertions are equivalent:*

(i) *$P^2_{\mathrm{per}}(f)$ is (α, p) mildly well posed.*
(ii) *$\{-k^2\}_{k\in\mathbb{Z}} \subset \rho(A)$ and $(|k|^{2\alpha} R(-k^2, A))_{k\in\mathbb{Z}}$ is an L^p-multiplier.*

In UMD spaces, the case $\alpha = 1$ and $1 < p < \infty$ in Theorem 5.6 is Theorem 5.1. The reason is the continuity of the Hilbert transform on $L^p((0,\,2\pi);X)$. Clearly, Theorem 5.7 with $\alpha = 1$ corresponds to Theorem 5.3. On the other hand, if $\alpha = 1/2$ in Theorem 5.7, then we see that Theorem 5.7 corresponds to Theorem 4.7.

Corresponding results may be stated in general Banach spaces. However, given the recently proved theorems on operator-valued L^p multipliers in UMD spaces (see [5], [10] and [18]), it seems reasonable to single out this family of spaces. Observe that the spaces $H^{\alpha,p}((0,\,2\pi);\,X)$ were used in [5, Section 4] in conjunction with Sobolev embedding theorems to obtain continuity of mild solutions, the latter however being defined differently than ours.

For $\alpha = 0$, the first-order problem admits continuous mild solutions if we assume that A generates a strongly continuous semigroup. In the case of the second-order problem, continuous mild solutions are obtained under the condition that A be the generator of a strongly continuous cosine function. We refer to [5] and [13] respectively.

Remark 5.8. Suppose X is a reflexive Banach space. Let A be a closed linear operator with domain and range in X. Then, as is well known, if $\{(\lambda_n)\} \subset \rho(A)$, $\lim_{n\to\infty} |\lambda_n| = \infty$ and $(\lambda_n R(\lambda_n, A))$ is bounded, then $\overline{D(A)} = X$. Therefore, when the condition (ii) in Theorem 5.1 or Theorem 5.3 is satisfied in a reflexive Banach space (in particular a UMD space), the closed operator A is automatically densely defined.

In order to justify the reasonableness of the restriction on α (i.e., $0 \le \alpha \le 1$) in the previous theorems, we establish the following proposition. It is probably well known but we do not have a ready reference.

Proposition 5.9. *Let X be a Banach space ($X \neq \{0\}$) and $A : D(A) \subset X \to X$ be a closed linear operator. Suppose that $(\lambda_n)_{n\in\mathbb{N}} \subset \rho(A)$ and $\lim_{n\to\infty} |\lambda_n| = \infty$. Then for every $\varepsilon > 0$, $(|\lambda_n|^{1+\varepsilon} R(\lambda_n, A))$ is unbounded.*

Proof. Suppose to the contrary that $(|\lambda_n|^{1+\varepsilon} R(\lambda_n, A))$ is bounded, that is, there exists $M > 0$ such that $|\lambda_n|^{1+\varepsilon}\|R(\lambda_n, A))\| \le M$, $n \in \mathbb{N}$. Let $x \in D(A)$. Then there exist $\mu \in \rho(A)$ and $y \in X$ such that $x = R(\mu, A)y$. Clearly we may assume that $|\mu| < |\lambda_n|$, $n \in \mathbb{N}$.

Using the resolvent equation, we have

$$\begin{aligned} |\lambda_n|^{1+\varepsilon} R(\lambda_n, A)x &= |\lambda_n|^{1+\varepsilon} R(\lambda_n, A)R(\mu, A)y \\ &= \frac{|\lambda_n|^{1+\varepsilon}}{\mu - \lambda_n} (R(\lambda_n, A)y - R(\mu, A)y) \end{aligned}$$

It follows that $\frac{|\lambda_n|^{1+\varepsilon}}{|\mu-\lambda_n|}\|R(\lambda_n, A)y - R(\mu, A)y\| \le M\|x\|$ and thus

$$\frac{|\lambda_n|^{1+\varepsilon}}{|\mu - \lambda_n|}\|R(\mu, A)y\| \le M\|x\| + \frac{|\lambda_n|^{1+\varepsilon}}{|\mu - \lambda_n|}\|R(\lambda_n, A)y\| \le M\left(\|x\| + \frac{\|y\|}{|\mu - \lambda_n|}\right)$$

for all $n \in \mathbb{N}$. Obviously, since $\lim_{n\to\infty} |\lambda_n| = \infty$, this is only possible if $R(\mu, A)y = 0$ and thus $y = 0$, that is, $x = 0$. $\square$

In the light of this proposition, we see that the range of the parameter α in the last two theorems is the right one. Moreover, in view of the fact that every Fourier multiplier is bounded, we can say that the condition (ii) of Theorem 5.1 or Theorem 5.3 are the strongest possible.

6. Application to semi-linear equations in Hilbert spaces

Let X be a Hilbert space and denote by

$$Z := W^{2,p}((0, 2\pi); X) \cap L^p((0, 2\pi); D(A)).$$

In this section we consider the semilinear problem of second order

$$u''(t) = Au(t) + f(t, u(t)), \quad t \in [0, 2\pi], \tag{6.1}$$

where f is a continuous mapping of $L^p((0, 2\pi); X)$ into itself.

We say that a closed linear operator A belongs to the *class* $\mathcal{K}^2(X)$ if

$$\{-k^2 : k \in \mathbb{Z}\} \subseteq \rho(A) \text{ and } \sup_{k\in\mathbb{Z}} ||k^2 R(-k^2; A)|| < \infty. \tag{6.2}$$

Define the Nemytskii's superposition operator $\mathcal{N} : Z \to L^p((0, 2\pi); X)$ by $\mathcal{N}(v)(t) = f(t, v(t))$ and the bounded linear operator

$$\mathcal{B} := \mathcal{A}^{-1} : L^p((0, 2\pi); X) \to Z$$

by $\mathcal{B}(g) = u$ where u is the unique solution of the linear problem

$$u''(t) = Au(t) + g(t).$$

Then, in order to obtain *strong solutions for (*6.1*)*, i.e., $u \in Z$ such that (6.1) is satisfied, we have to show that the operator $\mathcal{H} : Z \to Z$ defined by $\mathcal{H} = \mathcal{B}\mathcal{N}$ has a fixed point.

For example, if we assume that $\mathcal{B}$ is a compact operator, and we suppose that for some $M > 0$,

$$\sup_{\|u\|\le M} \|\mathcal{N}(u)\|_{L^p((0,2\pi);X)} \le M/\|\mathcal{B}\|, \tag{6.3}$$

then one may apply Schauder's fixed point theorem to $\mathcal{H}$ in the ball

$$\{u \in L^p((0, 2\pi); X) : \|u\| \le M\}$$

to get existence of a strong solution for (6.1). This way one obtains the existence of solutions on $[0, 2\pi]$. More precisely, by applying the preceding argument, one proves the following result in Hilbert spaces.

Theorem 6.1. *Let H be a Hilbert space, and suppose $A \in \mathcal{K}^2(H)$. Assume that the closed unit ball of $D(A)$ is compact in H. Let f be given such that* (6.3) *is satisfied. Then the equation* (6.1) *has a strong solution, with $\|u\|_{L^2((0,2\pi);H)} \le M$.*

Proof. Since $A \in \mathcal{K}^2(H)$, for each $K \in \mathbb{Z}$ we can define operators $\mathcal{B}_K : L^2((0, 2\pi); H) \to L^2((0, 2\pi); H)$ by

$$(\mathcal{B}_K g)(t) = \sum_{k=-K}^{K} R(-k^2, A)\hat{g}(k)e^{ikt}. \tag{6.4}$$

Since the closed unit ball of $D(A)$ is compact in H, for each K, the operator $\mathcal{B}_K$ is a finite sum of compact operators, hence compact. Now, because of (6.2), as $K \to \infty$, $\mathcal{B}_K$ converges in norm to $\mathcal{B}$, so $\mathcal{B}$ is compact. The conclusion of the theorem is achieved by applying Schauder's fixed point theorem to the equation $u = \mathcal{B}f(u)$ in $\{u \in Z : \|u\| \le M\}$. □

Remark 6.2. Note that if $P^2_{\text{per}}(f)$ is strongly L^p well posed then $A \in \mathcal{K}^2(H)$. Indeed, we have by (*ii*) in Theorem 5.3 that $\{-k^2 : k \in \mathbb{Z}\} \subseteq \rho(A)$. Moreover, by Remark 5.4 and the comments following Definition 2.1, we know that

$$\sup_{k\in\mathbb{Z}} \|k^2 R(-k^2, A)\| < \infty. \tag{6.5}$$

On the other hand, we also obtain $A \in \mathcal{K}^2(H)$ under the weaker condition (i) in Theorem 5.3.

We can also obtain mild solutions for the semilinear problem (6.1) by relying instead on Corollary 4.10. Here, we take

$$Z = W^{1,p}((0, 2\pi); X).$$

We say that a closed linear operator A belongs to the *class* $\mathcal{K}^1(X)$ if A is the generator of a strongly continuous cosine family $C(t)$ on X and satisfies

$$\{-k^2 : k \in \mathbb{Z}\} \subseteq \rho(A) \text{ and } \sup_{k \in \mathbb{Z}} ||kR(-k^2; A)|| < \infty.$$

If A belongs to the class $\mathcal{K}^1(X)$ then, by Corollary 4.10, there exists a bounded linear operator

$$\mathcal{B} : L^p((0, 2\pi); X) \to Z.$$

We say that $u \in Z$ is a $(W^{2,p}, W^{1,p})$ *mild solution for* (6.1) if u is a fixed point of the equation

$$u = \mathcal{B} f(u)$$

With the same arguments as above, we arrive at:

Theorem 6.3. *Let H be a Hilbert space, and $A \in \mathcal{K}^1(H)$. Assume that the unit ball of $D(A)$ is compact in H. Let f be given such that* (6.3) *is satisfied. Then equation* (6.1) *has a $(W^{2,p}, W^{1,p})$ mild solution, with $\|u\|_{L^2((0,2\pi);H)} \leq M$.*

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Backward Uniqueness in Linear Thermoelasticity with Time and Space Variable Coefficients

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Abstract. Backward uniqueness for thermoelastic plates and thermoelastic waves with time- and space-dependent coefficients is established. While this result has been proved recently, in the case of *time-independent* coefficients, it is new for the case of time-dependent coefficients. The proof relies on a combination of energy and Carleman's estimates, hence it is very different from the one given in [LRT], which is based on complex analysis methods. These latter methods are not applicable to nonlinear models and to models with time-dependent coefficients. Our results have consequences for several nonlinear models of thermoelasticity.

1. Introduction

In a recent paper [I] Isakov has proved observability from the boundary of part of the temperature and the displacement field from the boundary. Backward uniqueness complements this result to full observability from the boundary provided the time is sufficiently large. This has been shown in [LRT] for *time-independent* coefficients. It is the purpose of this note to provide an alternative proof based on Carleman inequalities, which is applicable to *time-dependent coefficients* and which requires very little regularity. We consider this result to be of interest in its own right. It also has consequences for nonlinear thermoelasticity, since backward uniqueness for nonlinear problems follows from backward uniqueness for linear variable coefficient problems.

The equations of linear thermoelasticity consist of a coupled system of hyperbolic and parabolic equations. For both of them backward uniqueness is well known: The hyperbolic equation is well posed backward in time whereas Carleman estimates imply backward uniqueness for the parabolic problem.

We use this observation in order to prove backward uniqueness for the system of thermoelasticity. This will be accomplished by establishing appropriate independent Carleman inequalities for the parabolic and the hyperbolic component. Large parameters appearing in the Carleman estimates allow to "decouple" the system.

2. Main result

We consider equations of thermoelasticity on a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$ with sufficiently smooth boundary Γ. We shall consider two classes of models: one arising in the modeling of thermoelastic plates, and the other one describing thermoelastic waves.

2.1. Thermoelastic plates

Thermoelastic plate can be modeled by the following coupled heat and plate equations [LL, L2, L1]:

$$\begin{aligned} \mathcal{M}_\gamma w_{tt} + \mathcal{A}(x,t,\partial)^2 w - \mathcal{B}(x,t,\partial)\theta + F(x,t,w) &= 0 \qquad \text{in } Q = \Omega \times (0,T); \\ \theta_t - \mathcal{C}(x,t,\partial)\theta + \mathcal{B}(x,t,\partial) w_t &= 0 \qquad \text{in } Q; \\ w(0) = w_0, \quad w_t(0) = w_1, \quad \theta(0) &= \theta_0, \quad \text{in } \Omega. \end{aligned} \tag{1}$$

Here $\gamma > 0$ is a parameter accounting for rotational forces, and it is assumed to be small. $\mathcal{M}_\gamma$ is a stiffness operator and is defined by

$$\mathcal{M}_\gamma \equiv I - \gamma\Delta$$

$\mathcal{A}(x,t,\partial), \mathcal{B}(x,t,\partial), \mathcal{C}(x,t,\partial)$ denote second-order, strongly elliptic operator with sufficiently smooth coefficients which are time- and space-dependent.

Assumption 2.1. *More specifically we shall assume*

$$\mathcal{A}(x,t,\partial)u = -\partial_i(a^{ij}\partial_j u)$$

where the coefficients $a^{ij}(x,t)$ are symmetric and uniformly Lipschitz continuous. We assume ellipticity in the form

$$\lambda^{-1}|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq \lambda|\xi|^2$$

and some time regularity

$$|(\partial_t a^{ij})\xi_i\xi_j| \leq \mu|\xi|^2.$$

We assume that the coefficients (b^{ij}) and (c^{ij}) of the second-order divergence form operators $\mathcal{B}(x,t,\partial)$ and ,$\mathcal{C}(x,t,\partial)$ are symmetric, elliptic

$$\lambda^{-1}|\xi|^2 \leq b^{ij}\xi_i\xi_j \leq \lambda|\xi|^2, \qquad \lambda^{-1}|\xi|^2 \leq c^{ij}\xi_i\xi_j \leq \lambda|\xi|^2,$$

Lipschitz continuous

$$|(\partial_t b^{ij})\xi_i\xi_j| + |(\partial_x b^{ij})\xi_i\xi_j| \leq \mu|\xi|^2, \qquad |(\partial_t c^{ij})\xi_i\xi_j| + |(\partial_x c^{ij})\xi_i\xi_j| \leq \mu|\xi|^2$$

and satisfy

$$|(\partial_t\partial_x b^{ij})\xi_i\xi_j| \leq \mu|\xi|^2, \qquad |(\partial_t\partial_x c^{ij})\xi_i\xi_j| \leq \mu|\xi|^2.$$

These coefficients (a^{ij}) describe the elastic response, the coefficients (c^{ij}) the heat transfer, and the coefficients (b^{ij}) describe generation of heat through deformations, and the body forces due to temperature changes. Lower-order terms do not alter the arguments in any significant way.

Assumption 2.2. *The forcing term $F(x, t, w) \in L_\infty(0, T, H^{-1}(\Omega))$ is assumed linear in w and satisfies:*

$$\|F(x, t, w)\|_{H^{-1}(\Omega)} \leq M_0 \|w\|_{H^2(\Omega)}, \;\; x \in \Omega, t \in (0, T)$$

for some positive constant M_0 which is uniform in $w \in H^2(\Omega)$, and $x, t \in Q$.

A typical example is the following:

$$\sum_{i=1}^{n} \sum_{j,k=1}^{n} \partial_i d^{ijk}(x, t) \partial_{jk}^2$$

with bounded and measurable coefficients d^{ijk}.

We complement the system by adding boundary conditions.

Assumption 2.3. *We assume that the temperature θ satisfies either Dirichlet* (*fixed temperature*) *of natural* (*insulated*) *boundary conditions.*

With (1) we associate two types of canonical boundary conditions:

The **hinged plate**

$$w = \mathcal{A}(x, t, \partial)w = \theta = 0, \quad \text{on} \;\; \Sigma \equiv \Gamma \times (0, T) \tag{2}$$

and the **clamped plate**

$$w = \frac{\partial}{\partial \nu_{\mathcal{A}}} w = \theta = 0, \quad \text{on} \;\; \Sigma. \tag{3}$$

where $\frac{\partial}{\partial \nu_{\mathcal{A}}}$ stands for conormal derivative.

The main goal of this note is to prove *backward uniqueness* for the thermoelastic model presented above. By *backward uniqueness property*, we mean: Suppose that (w, θ) satisfies (1) and $(w(., t), w_t(., T), \theta(., T)) = (0, 0, 0)$. Then $(w(t), w_t(t), \theta(t)) \equiv 0$, for all $0 \leq t \leq T$.

It is well known that the above property holds true if $\gamma = 0$ and time-independent coefficients. This holds because the model not accounting for rotational inertia (i.e., $\gamma = 0$) defines an analytic semigroup [LR, LT, LT1], for which backward uniqueness obviously holds.

The situation is more complicated in the "hyperbolic" case (when $\gamma > 0$) and results and techniques depend on the model. The hinged plate with time-independent coefficients is the simplest case. Then the problem is spectral, i.e., it admits a Riesz basis spanned by the eigenfunctions of the underlying operator and the conclusion on backward uniqueness is straightforward. This simple argument fails both for other boundary conditions and for time-dependent coefficients.

Backward uniqueness has been shown for the hinged, clamped and free plate with time-independent coefficients in [LRT] using Phragmen Lindeloff Theorem combined with several (static) trace estimates. This method is not applicable to *time-dependent* problems.

Our goal is to prove backward uniqueness for time-dependent models.

Theorem 2.4. *Suppose that the Assumptions* 2.1*,* 2.3 *and* 2.2 *hold, that*

$$w \in C([0,T];H^2(\Omega)) \cap C^1([0,T];H^1(\Omega))), \qquad \textit{and } \theta \in L_2([0,T];H^1(\Omega))$$

satisfy (1) *equipped with either hinged or clamped boundary conditions. If, moreover,* $w(T) = w_t(T) = \theta(T) = 0$ *then* $(w,\theta) \equiv 0$.

Remark 2.5. Result and conclusion are true even when lower-order perturbations are added to the differential operators $\mathcal{A}(x,t,\partial), \mathcal{B}(x,t,\partial), \mathcal{C}(x,t,\partial)$.

Remark 2.6. Theorem 2.4 can be applied to some nonlinear models such as von Karman thermoelastic plates or semilinear problems: We apply it to the difference of two solutions and conclude that they are the same if their Cauchy data coincide at one time T.

Remark 2.7. The result stated above applies to hinged or clamped boundary conditions. It remains true if we have Neumann boundary conditions for the thermal variable. However, there are other boundary conditions coupling thermal and mechanical variables which do not fit the framework presented below. While the backward uniqueness property has been established for such model in the static case, the situation of time-dependent coefficients is not completely understood.

2.2. Thermoelastic waves

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz boundary. Denote

$$\|u\| \equiv \|u\|_{[L_2(\Omega)]^n}, \quad \text{and} \quad (u,v) \equiv (u,v)_{[L_2(\Omega)]^n}.$$

In what follows C denotes a generic constant, different in different occurrences. Critical dependence of the constants on the parameters involved will be appropriately emphasized.

We consider the following abstract version of a system of thermoelasticity describing thermoelastic waves in a bounded domain Ω:

$$\begin{aligned} w_{tt} - \mathcal{A}(x,t,\partial)w + E(x,t,\partial)\theta = 0 \\ \theta_t - \mathcal{C}(x,t,\partial)\theta + F(x,t,\partial)w_t = 0 \end{aligned} \tag{4}$$

where the operator $\mathcal{C}(x,t,\partial)$ has the form $-\partial_i c^{ij}\partial_j$ with the (time- and space-dependent) uniformly Lipschitz continuous coefficients. We assume that the coefficients c^{ij} satisfy conditions listed in Assumption 2.1. For simplicity we assume that both w and θ satisfy homogeneous Dirichlet boundary conditions:

$$w = 0 \quad \text{and} \quad \theta = 0 \quad \text{on } \partial\Omega.$$

The operator $\mathcal{A}(x,t,\partial)$ describes the elastic properties. We assume it to be positive and self-adjoint on $[L_2(\Omega)]^n$. In addition we impose the following assumption.

Assumption 2.8.

1. *There exist positive constants* λ, μ *such that*
$$\lambda^{-1}\|\nabla u\|^2 \leq (\mathcal{A}(x,t,\partial)u,u),$$
$$|(\mathcal{A}(x,t,\partial)u,v)| + \Big|\Big(\frac{d}{dt}\mathcal{A}(x,t,\sigma)u,v\Big)\Big| \leq \mu\|\nabla u\|\|\nabla v\|. \tag{5}$$
for all w *and* θ *vanishing at the boundary and all* $(x,t) \in Q$.
2. *The linear operators* $F(x,t,\partial)$ *and* $E(x,t,\partial)$ *are assumed to have the following mapping properties: there exists a positive constant* M_0 *such that*
$$\|E(x,t,\partial)\theta\| \leq M_0\|\theta\|_{H_0^1(\Omega)}, \qquad \|F(x,t,\partial)u\|_{H^{-1}(\Omega)} \leq M_0\|u\|. \tag{6}$$
for all w *and* θ *vanishing at the boundary and all* $(x,t) \in Q$.

Theorem 2.9. *Suppose that the coefficients of system* (4) *satisfy Assumption* 2.8 *e, and assume that*
$$w \in C([0,T],[H^1(\Omega)]^n) \cap C^1([0,T],[L_2(\Omega)]^n), \;\; \theta \in L_2([0,T],H^1(\Omega))$$
satisfies (4) *in a weak sense as well as homogeneous Dirichlet boundary conditions. If* $w(T) = w_t(T) = \theta(T) = 0$ *then* $(w,\theta) \equiv 0$.

Remark 2.10. In classical applications to thermoelastic waves in an isotropic material one has
$$[\mathcal{A}(x,t,\partial)u]^j = -\sum_{i=1}^n \partial_i\left(\lambda(\sum_{k=1}^n \epsilon_{kk}(u))\delta_{ij} + 2\nu\epsilon_{ij}(u)\right),$$
$$F(x,t,\partial) = f^j(x,t)\partial_j, \quad E(x,t,\partial) = e_j(x,t)\partial_j,$$
where λ and ν are the Lame coefficients and ϵ_{ij} is strain tensor given by $\epsilon_{ij} = 1/2(\partial_{x_i}u^j + \partial_{x_j}u^i)$. The arguments work also for other boundary conditions.

3. The energy estimates

Our strategy is to derive Carleman inequalities for solutions compactly supported on $[0,T]$ of the following non-homogeneous plate problem
$$\begin{aligned} \mathcal{M}_\gamma w_{tt} + \mathcal{A}(x,t,\partial)^2 w - \mathcal{B}(x,t,\partial)\theta + F(x,t,w) &= f \text{ in } Q; \\ \theta_t - \mathcal{C}(x,t,\partial)\theta + \mathcal{B}(x,t,\partial)w_t &= g \text{ in } Q, \end{aligned} \tag{7}$$
and the wave problem
$$\begin{aligned} w_{tt} + \mathcal{A}(x,t,\partial)w + E(x,t,\partial)\theta &= f \text{ in } Q; \\ \theta_t - \mathcal{C}(x,t,\partial)\theta + F(x,t\partial)w_t &= g \text{ in } Q, \end{aligned} \tag{8}$$
where f, g are prescribed forcing terms supported on $(0,T)$. Here and in the sequel we suppress the x dependence in the notation whenever this is convenient. In the first step we shall derive energy estimates for the plate equation and for the wave equation. Energy estimates backward in time trivially imply strong Carleman inequalities.

3.1. Energy estimates for the plate equation

We introduce the following energy functional

$$E_w(t) \equiv \int_\Omega (\mathcal{M}_\gamma w_t(t), w_t(t)) + |\mathcal{A}(x,t,\partial)w(t)|^2\, dx.$$

Lemma 3.1. *Let w and θ solution to* (7) *with homogeneous Dirichlet boundary conditions for θ and either clamped or hinged boundary conditions for w. Suppose that both are supported in $(0,T)$. Then the following inequality holds.*

$$E_w(t) \le C\int_t^\infty [\|\theta(s)\|^2 + E_w(s) + \|f(s)\|^2_{H^{-1}(\Omega)} + \|g(s)\|^2_{H^{-1}(\Omega)}]ds + C\|\theta(t)\|^2_{H^{-1}(\Omega)} \tag{9}$$

where the constant C is an intrinsic constant depending only on Ω and the constants M_0, λ, μ introduced in Assumption 2.1*, Assumption* 2.2.

Proof. In the sequel we make formal manipulations. It is not hard to see that they can be justified. We note that we can use the second equation in (7) and the boundary condition for θ to write

$$\theta = \mathcal{C}^{-1}[-\theta_t + \mathcal{B}w_t + g]$$

where $-\mathcal{C}$, $-\mathcal{B}$ stand for (unbounded) operators, defined on $L_2(\Omega)$ and corresponding to $\mathcal{C}(x,t,\partial), \mathcal{B}(x,t,\partial)$ with zero Dirichlet boundary conditions. Similarly we denote by A the operator corresponding to $\mathcal{A}(x,t,\partial)^2$ and equipped with either hinged or clamped boundary conditions. M denotes $\mathcal{M}_\gamma$ equipped with zero Dirichlet data on the boundary.

With the above notation we represent dynamics of thermoelastic plate via the following abstract equation:

$$Mw_{tt} + Aw + \mathcal{B}\mathcal{C}^{-1}\mathcal{B}w_t - \mathcal{B}\mathcal{C}^{-1}\theta_t + F(w) = f - \mathcal{B}\mathcal{C}^{-1}g \tag{10}$$

For notational simplicity we assume that $T = \infty$.

Recalling Assumption 2.1 and Assumption 2.2, we obtain for all $t \in [0,T]$

$$(F(w), w_t(t)) \le \|w_t(t)\|_{H_0^1(\Omega)}\|F(w)\|_{H^{-1}(\Omega)} \le C[E(t) + \|w(t)\|^2_{H^2(\Omega)}] \le CE(t)$$

where generic constant C depends on M_0, λ, μ and Ω.

Standard energy arguments applied to (10), followed by integration from t to ∞, give

$$E_w(t) + \int_t^\infty (\mathcal{B}\mathcal{C}^{-1}\theta_t, w_t)ds \le C\Big[\int_t^\infty [E_w(s) + \|f(s)\|^2_{H^{-1}(\Omega)} + \|g(s)\|^2_{H^{-1}(\Omega)}]ds\Big],$$

and

$$\left|\int_t^\infty [(\mathcal{B}\mathcal{C}^{-1}\theta_t, w_t) + (\mathcal{B}\mathcal{C}^{-1}\theta, w_{tt})]ds + ((\mathcal{B}\mathcal{C}^{-1}\theta(t), w_t(t))\right| \le C\int_t^\infty \|w_t\|\|\theta\|ds.$$

where we used that

$$\frac{d}{dt}(\mathcal{B}\mathcal{C}^{-1}) = \mathcal{B}_t\mathcal{C}^{-1} - \mathcal{B}\mathcal{C}^{-1}\mathcal{C}_t\mathcal{C}^{-1}$$

is bounded in L^2. This is the point where the Lipschitz continuity (with respect to x) of the coefficients enters along with the boundedness of the coefficients b_{xt}^{ij} and c_{xt}^{ij} asserted by Assumption 2.1.

Since $\mathcal{C} : H_0^1(\Omega) \to H^{-1}(\Omega)$ is invertible we can estimate (for all $t \in R$)

$$\begin{aligned}\left|((\mathcal{B}\mathcal{C}^{-1}\theta(t), w_t(t))\right| &\le C\|\theta(t)\|_{H^{-1}(\Omega)}\|w_t(t)\|_{H_0^1(\Omega)}\\ &\le \frac{C}{\varepsilon}\|\theta(t0\|^2_{H^{-1}(\Omega)} + \varepsilon\|w_t(t)\|^2_{H_0^1(\Omega)},\end{aligned}$$

and we arrive at (with a different small ε)

$$\begin{aligned}E_w(t) \le \frac{C}{\varepsilon)}\int_t^\infty &[E_w(s) + \|\theta(s)\|^2_{L_2(\Omega)} + \|f(s)\|^2_{H^{-1}(\Omega)} + \|g(s)\|^2_{H^{-1}(\Omega)}]ds\\ &+ \epsilon\int_t^\infty \|w_{tt}(s)\|^2_{L_2(\Omega)}ds + \frac{C}{\varepsilon}\|\theta(t)\|^2_{H_0^{-1}(\Omega)}\end{aligned} \tag{11}$$

where the constant C is generic and depends only on M_0, μ, λ and Ω. We need to estimate the L_2 norm of w_{tt}. In order to accomplish this we first write

$$w_{tt} = -M^{-1}[Aw + \mathcal{B}\theta + F(w) - f].$$

As above, $\mathcal{B}M^{-1}$ is bounded on L^2, and thus the same is true for its adjoint $M^{-1}\mathcal{B}$. We obtain for all $t \in [0, T]$

$$\begin{aligned}\|w_{tt}(t)\|_{L_2(\Omega)} \le C[&\|\theta(t)\|_{L_2(\Omega)} + \|M^{-1}f(t)\|_{L_2(\Omega)}\\ &+ \|M^{-1}Aw(t)\|_{L_2(\Omega)} + \|M^{-1}F(w)\|_{L_2(\Omega)}].\end{aligned} \tag{12}$$

The main difficulty of the proof is to provide a good bound for the third term. For hinged boundary conditions this estimate is straightforward and follows from the estimate

$$\|M^{-1}Aw(t)\|_{L_2(\Omega)} \le C\|w(t)\|_{H^2(\Omega)\cap H_0^1(\Omega)} \le cE_w^{\frac{1}{2}}(t) \tag{13}$$

which, in turn, is a consequence of the boundedness of

$$A : H^2 \cap H_0^1 \to (H^2 \cap H_0^1)^*,$$

and

$$M^{-1} : L^2 \to H^2 \cap H_0^1,$$

and hence by duality

$$M^{-1} : (H^2 \cap H_0^1)^* \to L^2.$$

Thus, in the hinged case

$$\|w_{tt}(t)\|_{L_2(\Omega)} \le C[\|\theta(t)\|_{L_2(\Omega)} + E_w^{1/2}(t) + \|M^{-1}f(t)\|_{L_2(\Omega)} + \|M^{-1}F(w)(t)\|_{L_2(\Omega)}], \tag{14}$$

and we complete the proof by

$$\|M^{-\frac{1}{2}}F(w)(t)\|_{L_2(\Omega)} \le C\|w(t)\|_{H^2(\Omega)}.$$

with a generic constant C depending only on the data of the problem ($M_0, \lambda, \mu, \Omega$).

The clamped case is more delicate because there is no natural embedding of $(H^2)^*$ into $(H^2 \cap H_0^1)^*$. In particular, (13) does not make sense, and it is even false on $C_0^\infty(\Omega)$.

Instead, in the clamped case we obtain the following more involved estimate:

Proposition 3.2. *Suppose that w and θ satisfy* (7)*, that θ is zero at the boundary and that the clamped boundary conditions hold. If moreover w is supported compactly in* $(0,\infty)$ *then for all $t \in [0,T]$*

$$\int_t^\infty \|M^{-1}Aw(s)\|^2_{L_2(\Omega)} ds \le C \int_t^\infty [E_w(s) + \|\theta(s)\|^2_{L_2(\Omega)} + \|f(s)\|^2_{H^{-1}(\Omega)}]\, ds \\ +\; C[E(t) + \|\theta(t)\|^2_{H^{-1}(\Omega)}].$$

where the constant C depends only on $M_0, \lambda, \mu, \Omega$.

A priori it is not clear that the left-hand side is well defined. It suffices to verify a uniform a priori estimate for smooth solutions to a smooth problem. We omit the technical aspects.

Proof. By duality it suffices to provide good bounds for $(M^{-1}Aw, z) = (Aw, M^{-1}z)$. We compute by using Green's formula

$$\begin{aligned}(\mathcal{A}(x,t,\partial)^2 w, M^{-1}z) &\le C\|w\|_{H^2(\Omega)}\|M^{-1}z\|_{H^2(\Omega)} \\ &\quad + \|\mathcal{A}(x,t,\partial)w|_\Gamma\|_{H^{-1/2}(\Gamma)}\|\frac{\partial}{\partial\nu}M^{-1}z|_\Gamma\|_{H^{1/2}(\Gamma)} \\ &\le C[\|w\|_{H^2(\Omega)} + \|\mathcal{A}(x,t,\partial)w|_\Gamma\|_{H^{-1/2}(\Gamma)}]\|z\|_{L_2(\Omega)}\end{aligned} \tag{15}$$

where in the last step we used trace theorem. On the other hand, by applying multiplier techniques on the problem at hand one obtains the following trace estimate:

$$\int_t^\infty \|\mathcal{A}(x,t,\partial)w|_\Gamma\|^2_{L_2(\Gamma)} ds \le C \int_t^\infty [E_w(s) + \|\theta(s)\|^2_{L_2(\Omega)} + \|f(s)\|^2_{H^{-1}(\Omega)}]\, ds \\ +\; C[E(t) + \|\theta(t)\|^2_{H^{-1}(\Omega)}]. \tag{16}$$

where the constant C depends only on $M_0, \lambda, \mu, \Omega$. Indeed, the above estimate is obtained by multiplying the original plate equation by $h(x)\nabla w$ with the vector field $h(x)$ parallel to the normal on the boundary. The technical details are very similar to these given in [LRT]. The argument does not depend on time independence of the coefficients, under the regularity of the coefficients stated in the theorem. We do not repeat the proof. Combining (15) and (16) yields the desired estimate in Proposition 3.2. □

We are now ready to prove the counter part of (14) for the clamped plate. From (12) and the estimate in Proposition 3.2 we obtain

$$\int_t^\infty \|w_{tt}(s)\|^2_{L_2(\Omega)} ds \le C \int_t^\infty [\|\theta(s)\|^2_{L_2(\Omega)} + E_w(s) + \|f(s)\|^2_{H^{-1}(\Omega)}] ds \\ + C[E_w(t) + \|\theta(t)\|^2_{H^{-1}(\Omega)}]. \tag{17}$$

In our final step we combine estimate in (11) with the one obtained in (17) (or (14) in the hinged case). This gives

$$E_w(t) \leq \frac{C}{\epsilon}\int_t^\infty [E_w(s) + \|\theta(s)\|^2_{L_2(\Omega)} + \|f(s)\|^2_{H^{-1}(\Omega)} + \|g(s)\|^2_{H^{-1}(\Omega)}]\, ds$$
$$+ \epsilon[E_w(t) + \|\theta(t)\|^2_{H^{-1}(\Omega)}]$$

Taking ϵ sufficiently small yields the desired result in Lemma 3.1. □

3.2. Energy estimate for the wave equation

Let

$$Pu \equiv u_{tt} - \mathcal{A}(x,t,\partial)u$$

We define energy function by

$$E_u(t) = \int_\Omega [\|\sqrt{\mathcal{A}(x,t,\partial)}u\|^2 + \|u_t\|^2]dt$$

The following energy estimate is standard:

Lemma 3.3. *Let $u \in C(\mathbb{R};[H^1_0(\Omega)]^n) \cap C^1(\mathbb{R};[L_2(\Omega)]^n)$. Then, for $0 \leq s,t < \infty$,*

$$E_u(t) \leq CE_u(s) + \left|\int_s^t \|Pu\|d\tau\right|^2. \tag{18}$$

a where the constant C depends only on λ, ν, M_0 -constants introduced in Assumption 2.8 and also on Ω.

4. Carleman estimates for parabolic equations

We recall the assumptions on the coefficients c^{ij}: they are measurable, uniformly elliptic and uniformly Lipschitz continuous in time. More precisely we assume

$$\lambda^{-1}|\xi|^2 \leq c^{ij}\xi_i\xi_j \leq \lambda|\xi|^2,$$
$$|(\partial_t c^{ij})\xi_i\xi_j| \leq \mu|\xi|^2,$$

and study the operator

$$Pu = u_t - \partial_i(c^{ij}\partial_j u).$$

Let $h(t) = \tau e^{\kappa t}$ with $\kappa \gg \mu$.

Lemma 4.1. *Let*

$$u \in L^2([0,T];H^1(\Omega)); u_t \in L^2([0,T];H^{-1}(\Omega))$$

satisfy $u(T) = u(0) = 0$. Then, there exists a constant $c > 0$, depending only on λ, μ, Ω, T, such that

$$\begin{aligned}\tau^{1/2}\|e^h u\|_{L^2([0,T];H^{-1}(\Omega))} + \|e^h u\|_{L^2([0,T]\times\Omega)} + \tau^{-1/2}\|e^h\nabla u\|_{L^2([0,T]\times\Omega)} \\ \leq c\|e^h Pu\|_{L^2([0,T];H^{-1}(\Omega))},\end{aligned} \tag{19}$$

and also

$$\tau^{1/2}\|e^h u\|_{L^2([0,T]\times\Omega)} + \|e^h\nabla u\|_{L^2([0,T]\times\Omega)} \leq c\|e^h Pu\|_{L^2([0,T]\times\Omega)}. \tag{20}$$

Proof. We extend u by zero to all $t \in \mathbb{R}$, and we prove first that for κ large enough (depending on λ, μ, Ω one has:

$$\sqrt{\tau}\|e^h u\|_{L^2(\Omega\times\mathbb{R})} \le c\|e^h Pu\|_{L^2(\Omega\times\mathbb{R})}. \tag{21}$$

for some constant $c > 0$ depending on λ, μ, Ω, T. Introducing the notation $v \equiv e^h u$ we obtain

$$P_\tau v := Pv - h'v = e^h Pu. \tag{22}$$

and (21) is equivalent to

$$\sqrt{\tau}\|v\|_{L^2(\mathbb{R}\times\Omega)} \le c\|P_\tau v\|_{L^2(\mathbb{R}\times\Omega)}. \tag{23}$$

We expand

$$\begin{aligned}\|P_\tau v\|^2_{L^2(\mathbb{R}\times\Omega)} &= \|v_t\|^2_{L^2(\mathbb{R}\times\Omega)} + \|(\mathcal{C}-h')v\|^2_{L^2(\mathbb{R}\times\Omega)} + 2\int_{\mathbb{R}}(v_t,(\mathcal{C}-h')v)ds \\ &\qquad\qquad \text{since the support of } v \in [0,T] \\ &= \|v_t\|^2_{L^2(\mathbb{R}\times\Omega)} + \|(\mathcal{C}-h')v\|^2_{L^2(\mathbb{R}\times\Omega)} + \int_{\mathbb{R}}([\mathcal{C}-h',\partial_t]v,v)ds,\end{aligned} \tag{24}$$

where the commutator $[A,B] \equiv AB - BA$.

Direct calculations give

$$[\mathcal{C}-h',\partial_t] = h'' - \partial_i(\partial_t c^{ij})\partial_j; \tag{25}$$

where we recall, $h'' = \kappa h' \ge C_T\kappa^2\tau$ on $[0,T]$. Equality in (24), after accounting for non-negativity of the first two terms on the right side of (24), and (25) imply:

$$\begin{aligned}\kappa\|(h')^{\frac{1}{2}}v\|^2_{L^2(\mathbb{R}\times\Omega)} &\le ([\mathcal{C}-h',\partial_t]v,v) + \mu\|\nabla v\|^2_{L^2(\mathbb{R}\times\Omega)}. \\ &\le \|P_\tau v\|^2_{L^2(\mathbb{R}\times\Omega)} + \mu\|\nabla v\|^2_{L^2(\mathbb{R}\times\Omega)}.\end{aligned} \tag{26}$$

Energy estimate applied to (22) and followed by (26) gives

$$\begin{aligned}\lambda^{-1}\|\nabla v\|^2_{L^2(\mathbb{R}\times\Omega)} &\le \int_{\mathbb{R}}[(h'v,v) + (P_\tau v,v)]dt \\ \le \kappa^{-1}(\|P_\tau v\|^2_{L^2(\mathbb{R}\times\Omega)} + \mu\|\nabla v\|^2_{L^2(\mathbb{R}\times\Omega)}) &+ \epsilon\|v\|^2_{L^2(\mathbb{R}\times\Omega)} + \frac{4}{\epsilon}\|P_\tau v\|^2_{L^2(\mathbb{R}\times\Omega)}\end{aligned} \tag{27}$$

Selecting large κ and small ϵ so that $\lambda^{-1} - \kappa^{-1}\nu - \epsilon C_\Omega > 1/2\lambda^{-1}$ leads to

$$\|\nabla v\|^2_{L^2(\mathbb{R}\times\Omega)} \le C\|P_\tau v\|^2_{L^2(\mathbb{R}\times\Omega)} \tag{28}$$

where κ is large enough and positive constant C depends on λ, μ, Ω.

Estimate in (28) when combined with (26) leads to (23), hence (21) has been proved. (28) and (21) imply the final conclusion in (20).

The proof of (19) proceeds along similar lines. We first establish (for κ large enough)

$$\sqrt{\tau}\|v\|_{L^2(\mathbb{R},H^{-1}(\Omega))} \le c\|P_\tau v\|_{L^2(\mathbb{R};H^{-1}(\Omega))}, \tag{29}$$

where κ is large enough and positive constant c depends only on λ, μ, T, Ω.

The map $\mathcal{C}(t) : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isomorphism and we denote its inverse again by $\mathcal{C}(t)^{-1}$. Since $C(t)$ is uniformly elliptic with the bounds independent on t, we can afford using an abbreviated notation $\mathcal{C}$. Then, expanding as in (24)

$$\begin{aligned}\int_{\mathbb{R}} \|P_\tau v\|^2_{H^{-1}(\Omega)} dt &\sim \int_{\mathbb{R}} (\mathcal{C}^{-1} P_\tau v, P_\tau v) dt \\ = \quad &\int_{\mathbb{R}} [(\mathcal{C}^{-1}(\mathcal{C}v - h'v), \mathcal{C}v - h'v) + (\mathcal{C}^{-1} v_t, v_t) + ([v - \mathcal{C}^{-1} h', \partial_t] v, v)] dt.\end{aligned}$$

The commutator is

$$\mathcal{C}^{-1} h'' - \mathcal{C}^{-1} (\frac{d}{dt} \mathcal{C}) \mathcal{C}^{-1} h',$$

where, again, the first term yields a good positive term

$$(\mathcal{C}^{-1} h'' v, v) \geq \kappa (\mathcal{C}^{-1} h' v, v),$$

and the second one can be estimated by

$$\left| (\mathcal{C}^{-1} (\frac{d}{dt} \mathcal{C}) \mathcal{C}^{-1} h' v, v) \right| \leq \mu \|(h')^{\frac{1}{2}} v\|^2_{H^{-1}(\Omega)}.$$

Hence

$$\kappa \int_{\mathbb{R}} (\mathcal{C}^{-1} h' v, v) dt \leq C_\lambda \int_{\mathbb{R}} [\|P_\tau v\|^2_{H^{-1}(\Omega)} + \mu \|(h')^{\frac{1}{2}} v\|^2_{H^{-1}(\Omega)}] dt. \tag{30}$$

Selecting κ large enough so $\frac{1}{2}\kappa - C_\Omega \mu > 0$ leads to

$$\kappa \int_{\mathbb{R}} (\mathcal{C}^{-1} h' v, v) dt \leq C_{\lambda,\mu} \int_{\mathbb{R}} \|P_\tau v\|^2_{H^{-1}(\Omega)} dt \tag{31}$$

Recalling $kh' > C_T \kappa^2 \tau$, we obtain inequality (29) from (31).

In the second step we apply energy estimate to (22), which give

$$(Pv - h'v, v) = (e^h Pu, v) \Rightarrow$$

$$\frac{1}{2} \frac{d}{dt} |v|^2_{L_2(\Omega)} + \lambda^{-1} |\mathcal{C}^{1/2} v|^2_{L_2(\Omega)} \leq |e^h Pu|_{H^{-1}(\Omega)} |\mathcal{C}^{1/2} v|_{L_2(\Omega)} + C_\lambda |h'v|^2_{H^{-1}(\Omega)}.$$

Hence

$$\lambda^{-1} \int_R \|\mathcal{C}^{1/2} v\|^2_{L_2(\Omega)} dt \leq \|e^h Pu\|^2_{L_2(R;H^{-1}(\Omega))} + C_\lambda \kappa \tau \|(h')^{\frac{1}{2}} v\|^2_{L_2(\mathbb{R};H^{-1}(\Omega))} \tag{32}$$

Combining this inequality with (25) gives

$$\|\nabla v\|_{L^2(\mathbb{R}\times\Omega)} \leq c\tau^{1/2} \|e^h Pu\|_{L^2(\mathbb{R};H^{-1}(\Omega))}, \tag{33}$$

where the positive constant Cdepends only on μ, λ, Ω, T. By interpolation

$$\|v\|^2_{L^2(\mathbb{R}\times\Omega)} \leq \|v\|_{L_2(\mathbb{R};H^{-1}(\Omega))} \|\nabla v\|_{L^2(\mathbb{R}\times\Omega)} \leq c \|e^h Pu\|^2_{L^2(\mathbb{R};H^{-1}(\Omega))},$$

where we have used once more (29). The first inequality (19) in Lemma 4.1 follows by combining the above inequality with (29) and (33). $\square$

Remark 4.2. It is obvious that the same arguments work for systems as well as for different boundary conditions.

5. Carleman estimates for thermoelastic system

5.1. Thermoelastic plates

Lemma 5.1. *There exists $\kappa \gg 1$ (depending on $\lambda, \mu, M_0, \Omega$) such that for all $\tau \geq 1$ and for all solutions $w \in C(\mathbb{R}; H^2(\Omega)) \cap C^1(\mathbb{R}; H^1(\Omega)), \theta \in L_2(\mathbb{R}; H^1(\Omega))$ satisfying* (7) *with one of the previous choices of boundary conditions and which are compactly supported in* $(0, \infty)$ *we have*

$$\int_{\mathbb{R}} E_w(t) e^{2h(t)} dt \leq \frac{1}{\tau} \int_{\mathbb{R}} \|e^{h(t)} f(t)\|^2_{H^{-1}(\Omega)} + \|e^{h(t)} g(t)\|^2_{H^{-1}(\Omega)}\, dt. \tag{34}$$

Proof. We multiply both sides of the inequality in Lemma 3.1 by $e^{2h(t)}$ and integrate over $\mathbb{R}$. This gives by Fubini's Theorem after accounting for $h' \geq \kappa\tau$

$$\begin{aligned}\int_{\mathbb{R}} E_w(t) e^{2h(t)} dt \leq & \int_{\mathbb{R}} \int_t^{\infty} [e^{2h(t)-2h(s)} [E_w(s) e^{2h(s)} + \|\theta(s) e^{h(s)}\|^2_{L_2(\Omega)} \\ & + \|f(s) e^{h(s)}\|^2_{H^{-1}(\Omega)} + \|g(s) e^{h(s)}\|^2_{H^{-1}(\Omega)}] ds dt \\ & + C \int_{\mathbb{R}} \|\theta(s) e^{h(s)}\|^2_{H^{-1}(\Omega)} ds \\ \leq & \frac{C}{\kappa\tau} \int_{\mathbb{R}} [E_w(s) e^{2h(s)} + \|\theta(s) e^{h(s)}\|^2_{L_2(\Omega)} \\ & + \|f(s) e^{h(s)}\|^2_{H^{-1}(\Omega)} + \|g(s) e^{h(s)}\|^2_{H^{-1}(\Omega)}] ds \\ & + C \int_{\mathbb{R}} \|\theta(s) e^{h(s)}\|^2_{H^{-1}(\Omega)} ds,\end{aligned} \tag{35}$$

where the constant $C > 0$ depends only on $\lambda, \mu, M_0, \Omega$. The heat component is estimated by the Carleman estimate in Lemma 4.1 applied with $u = \theta$ and $Pu = -\mathcal{B}(x, t, \partial) w_t + g$:

$$\begin{aligned}&\sqrt{\tau} \|e^h \theta\|_{L_2(\mathbb{R}; (H^{-1}(\Omega))} + \|e^h \theta\|_{L_2(\mathbb{R}, L_2(\Omega))} \\ &\quad \leq C[\|e^h g\|_{L_2(\mathbb{R}; H^{-1}(\Omega))} + \|e^h \mathcal{B} w_t\|_{L_2(\mathbb{R}: (H^{-1}(\Omega))}] \\ &\quad \leq C[\|e^h g\|_{L_2(\mathbb{R}; H^{-1}(\Omega))} + \|e^h w_t\|_{L_2(\mathbb{R}, H^1(\Omega))}] \\ &\quad \leq C[\|e^h g\|_{L_2(\mathbb{R}, H^{-1}(\Omega))} + |e^{2h} E_w(.)|^{1/2}_{L_2(\mathbb{R})}].\end{aligned} \tag{36}$$

Combining (35) with (36) we obtain

$$\int_{\mathbb{R}} E_w(t) e^{2h(t)} dt \leq \frac{C}{\kappa\tau} \int_R [E_w(s) e^{2h(s)} + \|f(s) e^{h(s)}\|^2_{H^{-1}(\Omega)} + \|g(s) e^{h(s)}\|^2_{H^{-1}(\Omega)}] ds, \tag{37}$$

where the constant $C > 0$ depends only on $\lambda, \mu.M_0, \Omega$. Taking κ large yields the result stated in Lemma 5.1. ☐

5.2. Thermoelastic waves

The Carleman estimate below follows for thermoelastic waves in the same way as for the thermoelastic plate. We state the Carleman estimate for completeness.

Lemma 5.2. *There exists* $\kappa \gg 1$ (*depending only on* $\lambda, \mu, M_0, \Omega$) *such that for all* $\tau \geq 1$ *and all*

$$w \in C(\mathbb{R}; [H^1(\Omega)]^n) \cap C^1(\mathbb{R}, [L_2(\Omega)]^n), \qquad \theta \in L_2(\mathbb{R}, H^1(\Omega))$$

satisfying (8) *with Dirichlet boundary conditions and compactly supported in* $(0, \infty)$ *we have*

$$\tau^2 \int_{\mathbb{R}} e^{2h(t)} E_w(t) dt \leq C \int_{\mathbb{R}} [\tau \| e^{h(t)} g(t) \|^2_{H^{-1}(\Omega)} + \| e^{h(t)} f(t) \|^2] dt \tag{38}$$

$$\begin{aligned} \int_{\mathbb{R}} [\tau \| e^{h(t)} \theta(t) \|^2_{H^{-1}(\Omega)} + \| e^{h(t)} \theta(t) \|^2_{L_2(\Omega)} + \tau^{-1} \| e^{h(t)} \theta(t) \|^2_{H^1(\Omega)}] \, dt \\ \leq C \int_{\mathbb{R}} [\| e^{h(t)} g(t) \|^2_{H^{-1}(\Omega)} + \tau^{-2} \| e^{h(t)} f(t) \|^2] \, dt, \end{aligned} \tag{39}$$

where the constant $C > 0$ *depends only on* $\lambda, \mu, M_0, \Omega$.

Proof. From (18) we obtain

$$E_u(t) \leq \left(\int_t^\infty \| Pu \| ds \right)^2$$

hence

$$\begin{aligned} \int_{\mathbb{R}} e^{2h} E_u dt &\leq C \left\| \int_t^\infty e^{h(t) - h(s)} \| e^{h(s)} Pu \| ds \right\|^2_{L^2(\mathbb{R})} \\ &\leq C \left\| \int_t^\infty e^{\kappa \tau (t - s)} \| e^{h(s)} Pu \| ds \right\|^2_{L^2(\mathbb{R})} \leq \frac{C}{(\kappa \tau)^2} \| e^h Pu \|^2_{L^2(\Omega \times \mathbb{R})}, \end{aligned}$$

where we used $h' \geq \kappa \tau$ and Young's inequality for convolutions in the last step. The constant C depends on $\lambda, \mu, M_0, \Omega$. Thus

$$\kappa^2 \tau^2 \int_{\mathbb{R}} e^{2h(t)} E_w(t) dt \leq C \int_{\mathbb{R}} [\| e^{h(t)} \theta(t)) \|^2_{H^1(|\Omega)} + \| e^{h(t)} f(t) \|^2] dt \tag{40}$$

and from (4.1)

$$\begin{aligned} \int_{\mathbb{R}} [\tau \| e^{h(t)} \theta(t) \|^2_{H^{-1}(\Omega)} + \| e^{h(t)} \theta(t) \|^2 + \tau^{-1} \| e^{h(t)} \theta(t) \|^2_{H^1(\Omega)}] \, dt \\ \leq C \int_{\mathbb{R}} [\| e^{h(t)} g(t) \|^2_{H^{-1}(\Omega)} + \| e^{h(t)} w_t(t) \|^2] dt. \end{aligned} \tag{41}$$

Combining the two estimates yields

$$\begin{aligned} \int_{\mathbb{R}} \tau^2 \| e^{h(t)} w_t \|^2 dt &\leq \tau^2 \int_{\mathbb{R}} [e^{2h(t)} E_w(t) dt \\ &\leq C \int_{\mathbb{R}} [\tau \| e^{h(t)} w_t \|^2 + \| e^{h(t)} f(t) \|^2 + \tau \| e^{h(t)} g(t) \|^2_{H^{-1}(\Omega)}] dt. \end{aligned} \tag{42}$$

Taking κ large gives the estimate in the first inequality of the Lemma. The second inequality follows by combining (41) and (42). □

6. Completion of the proof

We suppose that w and θ vanish of $t \geq T$. We may and do assume that $T = 1$. Let χ be a smooth cutoff function, which is 1 in $[-1/2, 3/2]$, with support in $(0, \infty)$. Let $\bar{w} = \chi w$ and $\bar{\theta} = \chi\theta$.

6.1. Proof of Theorem 2.4

With the above notation, the estimate in Lemma 5.1 holds with w replaced by $\bar{w}$ and with

$$f = [\mathcal{M}_\gamma, \chi]w = \chi_t w_t + \chi_{tt} w - \gamma\chi_t \Delta w_t - \gamma\chi_{tt}\Delta w$$
$$g = [D_t, \chi]\theta + [\mathcal{B}(x,t,\partial)D_t, \chi]w = \chi_t\theta + \chi_t\mathcal{B}(x,t,\partial)w,$$

where both f and g are supported on, say, $[0, 1/2]$. In addition

$$\|f(t)\|_{H^{-1}(\Omega)} \leq C\left(\|w_t(t)\|_{H^1(\Omega)} + \|w(t)\|_{H^1(\Omega)}\right)$$
$$\|g(t)\|_{H^{-1}(\Omega)} \leq C\left(\|\theta(t)\|_{H^{-1}(\Omega)} + \|w(t)\|_{H^1(\Omega)}\right).$$

Hence

$$\begin{aligned}\|e^h f\|^2_{L_2(\mathbb{R};H^{-1}(\Omega))} + \|e^h g\|^2_{L_2(\mathbb{R};H^{-1}(\Omega))} &\leq C\int_0^{1/2} [\|e^h f\|^2_{H^{-1}(\Omega)} + \|e^h g\|^2_{H^{-1}(\Omega)}]dt \\ &\leq Ce^{h(1/2)}[E_w(0) + \|\theta(0)\|^2_{L_2(\Omega)}]. \end{aligned} \tag{43}$$

where the constant C depends on Ω. Now uniqueness follows by the standard arguments. From Lemma 5.1 and (43) we obtain

$$\int_{1/2}^{\infty} e^{2h} E_w(t)dt \leq \frac{C}{\tau} e^{h(1/2)}[E_w(0) + \|\theta(0)\|^2_{L_2(\Omega)}],$$

where the constant $C > 0$ depends only on $\lambda, \mu, M_0, \Omega$. Taking $\tau \to \infty$ implies that $w \equiv 0$ on $(1/2, 1)$, hence, from the equation $\theta \equiv 0$ on $(1/2, 1)$. Thus vanishing at $T = 1$ implies vanishing in $[1/2, 1]$. This conclusion applies to any time shift and we obtain the desired statement.

6.2. Proof of Theorem 2.9

Theorem 2.9 follows in the same way from Lemma 5.2 applied to $\bar{w} = \chi w$ and $\bar{\theta} = \chi\theta$ with $f \equiv \chi_t w_t + \chi_{tt} w$ and $g \equiv \chi_t\theta$ gives

$$\tau\int_{\mathbb{R}} e^{2h(t)} E_{\bar{w}}(t)dt \leq C\int_{\mathbb{R}} [||e^{h(t)}\theta(t)||^2 + \tau^{-1}\|e^{h(t)} w_t(t)\|^2 + \tau^1\|e^{h(t)} w(t)\|^2]dt, \tag{44}$$

and similarly we obtain from Lemma 4.1

$$\begin{aligned}&\int_{\mathbb{R}} \tau\|e^{h(t)}\bar{\theta}(t)\|^2_{H^{-1}(\Omega)} + \|e^{h(t)}\bar{\theta}(t)\|^2 dt \\ &\qquad \leq C\tau^{-2}\int_{\mathbb{R}} [|e^{h(t)}\theta(t)|^2_{H^{-1}(\Omega)} + \|e^{h(t)} w_t\|^2 + \|e^{h(t)} w\|^2]dt.\end{aligned} \tag{45}$$

where the constant $C > 0$ depends only on $\lambda, \mu.M_0.|omega$. Proceeding as before and taking τ large in (44) and (45) yields the conclusion $\bar{w} \equiv 0$ and $\bar{\theta} \equiv 0$, as desired.

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Measure and Integral: New Foundations after One Hundred Years

Heinz König

Dedicated to the Memory of GÜNTER LUMER

Abstract. The present article wants to describe the main ideas and developments in the theory of measure and integral in the course and at the end of the first century of its existence.

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Keywords. Traditional abstract measures and Radon measures, construction of measures after Carathéodory and Daniell-Stone, inner regularity, construction via inner premeasures, finite and infinite products, the inner representation theorem, projective limits and stochastic processes.

The theory of measure and integral had been created by BOREL and LEBESGUE around 1900 as the *concrete* theory of the Lebesgue measure on $\mathbb{R}$. The decisive point was a small collection of entirely new and powerful theorems: the theorems of type Beppo Levi-Fatou-Lebesgue, Fubini-Tonelli, $\ldots$. The theory soon became a kind of foundation of mathematical analysis.

Likewise the theory soon turned into an *abstract* one. This is the usual fate of mathematical theories, but in the case of measure and integral a powerful impact came from the fact that the whole of mathematical analysis, like the whole of mathematics, went through a continuous chain of vivid abstractions all over the 20th century: Each new step of abstraction required its specific class of measures, in order that those powerful theorems could be put into action. Examples of first rank were the *locally compact topological groups* (HAAR 1933, VON NEUMANN 1934/36, ANDRÉ WEIL 1940) and the *mathematical theory of probability* (WIENER 1923, KOLMOGOROV 1933, DOOB 1953).

It so happened that for measure and integral the process of abstraction involved a *particular twofold task*: It is, in order to develop the theory for some abstract frame, the task to discover on the one hand the adequate *concepts* and *classes* of measures, and on the other hand the adequate *procedures* which lead to produce these measures from basic data of preconceived nature. As a rule these are hard problems, but decisive for the success of the enterprise.

In the course of the 20th century thus two comprehensive abstract theories of measure and integral came into existence: the *traditional abstract theory*, as presented for example in the famous 1950 textbook of Halmos [4], and the theory of *Radon measures* on Hausdorff topological spaces, developed in particular in the 1952–69 treatise of Bourbaki [1]. For all their power and splendour, both theories came to show some *essential weaknesses* with respect to the above particular tasks. We shall attempt to describe these weaknesses in Sections 1 and 2.

The time of release then came with the end of the 20th century. The second part of this article will describe the systematization due to the present author, based on ideas which date back to 1968–70. Another development of different nature is the monumental treatise 2000–2003 of Fremlin [3]. Its basic aim is the comprehensive description of measure and integral in both the abstract and topological theories, rather than their unification under new concepts like the present *premeasures*. Even so it is plain that there are overlaps in facts and spirit, in particular in the emphasis on *inner regular* and *nonsequential* procedures.

1. The two abstract theories of the 20th century

The Traditional Abstract Theory. The basic notion is that of a *measure* $\alpha : \mathfrak{A} \to [0, \infty]$, understood to be defined on a σ algebra $\mathfrak{A}$ of subsets of a nonvoid set X. The fundamental weakness of the theory is its total limitation to *sequential procedures*, inclusive of its neglect of *regularity*: In the main parts of the textbooks, devoted to the abstract situation, there is, aside from the ubiquitous σ:=*sequential* upward/downward continuity (almost always in the unfortunate guise of *countable additivity*), no τ:=*nonsequential* upward/downward continuity (defined via directed set systems) and no outer/inner *regularity*. Examples of instant consequences of this impoverishment are the lack of uniqueness results, for instance for finite products of measures and for Daniell-Stone representation, and the smallness of domains of certain fundamental constructions, for instance for finite and infinite products.

Then in the back of the textbooks there are specific chapters where X is assumed to be a Hausdorff topological space, with its usual set systems $\mathrm{Op}(X)$ and $\mathrm{Cl}(X) \supset \mathrm{Comp}(X)$ and its Borel σ algebra $\mathrm{Bor}(X)$. Here one finds, for the Borel measures $\alpha : \mathrm{Bor}(X) \to [0, \infty]$ and related ones, the concepts which were absent so far, but this time restricted in specific manner to $\mathrm{Op}(X)$ and $\mathrm{Comp}(X)$, as it had been most common in the concrete case of the Borel-Lebesgue measure $\lambda : \mathrm{Bor}(\mathbb{R}) \to [0, \infty]$:

λ outer regular $\mathrm{Op}(\mathbb{R})$ $\quad$ λ inner regular $\mathrm{Comp}(\mathbb{R})$,
$\lambda|\mathrm{Op}(\mathbb{R})$ upward τ continuous $\quad$ $\lambda|\mathrm{Comp}(\mathbb{R})$ downward τ continuous.

In the course of time it became clear that *inner* regularity is much more important than outer one, due to the predominant rôle of compactness in topology. This was the point of departure for the opponent *Radon measure* theory.

THE RADON MEASURE THEORY. Here one assumes X to be a Hausdorff topological space. A measure $\alpha : \mathfrak{A} \to [0,\infty]$ on X is called *Radon* iff $\mathfrak{A} \supset \mathrm{Comp}(X)$ such that $\alpha|\mathrm{Comp}(X) < \infty$ (in part of the literature fortified to local finiteness) and such that α is *inner regular* $\mathrm{Comp}(X)$. One deduces that $\alpha|\mathrm{Comp}(X)$ is downward τ continuous. A simple extension procedure permits to assume that $\mathfrak{A} \supset \mathrm{Bor}(X)$. One then proves that $\alpha|\mathrm{Op}(X)$ is upward τ continuous. But as a rule α is not outer regular $\mathrm{Op}(X)$.

The most common particular cases are: 1) λ is a Radon measure. 2) When X is compact then not all finite Borel measures must be Radon. 3) When X is Polish then all locally finite Borel measures are Radon.

The present definition of Radon measures is not the one in BOURBAKI [1], but the two definitions are equivalent (up to local finiteness). The explanation is the credo of Bourbaki that the theory of measure and integral must be based on *integrals* and not on *set functions*. But of course *set functions are the bones* in the body of measure and integral, and hence an essential part of the basic labour is predestined to produce the fundamental set functions from whatever had been declared to be the basic entities. In his treatise Bourbaki was able to develop his conception in the frame of *locally compact* spaces X: we call this the *initial version* of the Radon measure theory. But in his last chapter, where X is an arbitrary Hausdorff space, Bourbaki seemed to have made peace with the basic rôle of set functions. The final end of the conception then came with the 1973 treatise of LAURENT SCHWARTZ [14] – which does not mean that all authors of textbooks have realized this fact.

We list a few achievements of the Radon measure theory: 1) Existence and uniqueness of finite products. 2) Existence and uniqueness in the Riesz representation theorem. 3) The notion of *support* for Radon measures. 4) The existence of (countable or uncountable) decompositions of measure spaces based on compact subsets, the so-called *concassages*.

It follows that success in these points is due to *inner regularity* – of course in strict connection with *topological compactness*. For the traditional abstract theory this is a serious hint and challenge. On the other side, for Radon measures the strict attachment to topological compactness can be a severe obstacle. An important instance are the infinite products. We shall see that here neither theory will be satisfactory.

INFINITE PRODUCTS AND PROJECTIVE LIMITS. Let T be an infinite index set and $(Y_t)_{t\in T}$ be a family of nonvoid sets with product set $X = \prod_{t\in T} Y_t$.

The *traditional abstract* theory assumes a family of probability (=:prob) measure spaces $(Y_t, \mathfrak{B}_t, \beta_t)_{t\in T}$ (defined to mean that $\beta_t(Y_t) = 1$). In X one forms the product σ algebra $\mathfrak{A}$, defined to be generated by the product sets

$$A = \prod_{t \in T} B_t \ \text{ with } B_t \in \mathfrak{B}_t \text{ and } B_t = Y_t \text{ for almost all } t \in T,$$

where as usual *almost all* means *all except finitely many.* Then there exists a unique *product measure* $\alpha : \mathfrak{A} \to [0, \infty]$, in the sense that

$$\alpha(A) = \prod_{t \in T} \beta_t(B_t) \ \text{ for the above } A = \prod_{t \in T} B_t.$$

However, in the case of an uncountable T this result has the basic defect that its domain $\mathfrak{A}$ can be much *too small*: its members $A \in \mathfrak{A}$ are all *countably determined* in the intuitive sense. Thus let for example $T = [0, \infty[$ and $Y_t = \mathbb{R}$ for $t \in T$, so that the members of $X = \mathbb{R}^T$ are the *paths* $x : T = [0, \infty[\to \mathbb{R}$. Then the subset $\mathrm{C}(T, \mathbb{R}) \subset X$ of the *continuous* paths is not countably determined, and even worse, any countably determined $A \subset \mathrm{C}(T, \mathbb{R})$ must be $A = \varnothing$.

On the other side the *Radon measure* theory starts from a family of Hausdorff topological spaces $(Y_t)_{t \in T}$, with the product topology on X. One assumes $\mathfrak{B}_t = \mathrm{Bor}(Y_t)$ and Borel-Radon prob measures β_t for $t \in T$. Then the previous $\mathfrak{A}$ satisfies $\mathfrak{A} \subset \mathrm{Bor}(X)$, and hence the desired result would be that the previous product measure $\alpha : \mathfrak{A} \to [0, \infty[$ has an extension to a Radon measure $\beta : \mathrm{Bor}(X) \to [0, \infty[$. However, this is far from true: It is quite obvious that such an extension does not exist when T is uncountable and $\beta_t | \mathrm{Comp}(Y_t) < 1$ for all $t \in T$. The reason is the *smallness* of the compact sets in X.

After this we turn to the context of *projective limits.* Let $I = I(T)$ consist of the nonvoid finite subsets $p, q, \ldots$ of T. For $p \in I$ we form the product set $Y_p = \prod_{t \in p} Y_t$ and the canonical projection $H_p : X \to Y_p$, and for $p \subset q$ in I the canonical projection $H_{pq} : Y_q \to Y_p$. Let us assume the *traditional abstract* situation: From $(\mathfrak{B}_t)_{t \in T}$ as above we form the family $(\mathfrak{B}_p)_{p \in I}$ of the product σ algebras $\mathfrak{B}_p$ in Y_p, and then from $(\beta_t)_{t \in T}$ as above the family $(\beta_p)_{p \in I}$ of the product measures β_p on $\mathfrak{B}_p$. Then on the one hand the family $(\beta_p)_{p \in I}$ is *consistent* in the sense that

$$(\leftrightarrow) \quad \beta_p(B) = \beta_q(H_{pq}^{-1}(B)) \ \forall\, B \in \mathfrak{B}_p \quad \text{for all } p \subset q \text{ in } I,$$

and on the other hand the above characterization of the product measure $\alpha : \mathfrak{A} \to [0, \infty[$ of $(\beta_t)_{t \in T}$ can be written

$$(\Leftrightarrow) \quad \beta_p(B) = \alpha(H_p^{-1}(B)) \ \forall B \in \mathfrak{B}_p \quad \text{for all } p \in I,$$

that is in terms of the family $(\beta_p)_{p \in I}$. All this evokes a natural *variant* of the previous product formation: From a prescribed family of prob measures $(\beta_p)_{p \in I}$, assumed to be consistent $(\leftrightarrow)$, one is asked to produce a prob measure $\alpha : \mathfrak{A} \to [0, \infty[$ which satisfies $(\Leftrightarrow)$. Then α must be unique and is called the *projective limit* of the family $(\beta_p)_{p \in I}$.

It is clear first of all from $(\leftrightarrow)$ $(\Leftrightarrow)$ that each prob measure $\alpha : \mathfrak{A} \to [0, \infty[$ is the projective limit of a unique consistent family $(\beta_p)_{p \in I}$, much in contrast to the previous product formation which furnishes but a small portion of these α. But of course the essential point is to determine those consistent families $(\beta_p)_{p \in I}$

which produce prob measures $\alpha : \mathfrak{A} \to [0, \infty[$, that is those which via ($\Leftrightarrow$) come from these α. Let us call them *solvable*. It is known that not all consistent families $(\beta_p)_{p \in I}$ are solvable; it seems that some kind of *compactness* is involved. For the moment we quote the famous positive result due to KOLMOGOROV [6]: *If Y_t is a Polish topological space and $\mathfrak{B}_t = \mathrm{Bor}(Y_t)\ \forall\ t \in T$ then all consistent families $(\beta_p)_{p \in I}$ are solvable.* A comprehensive answer will be presented at the end of this article; it will at the same time be able to overcome the barrier of countably determined $A \subset X$.

The present context of projective limits is the basis of the traditional theory of *stochastic processes*. Here one assumes that $Y_t = Y$ and $\mathfrak{B}_t = \mathfrak{B}$ independent of $t \in T$. A *stochastic process for T and $(Y, \mathfrak{B})$* can be defined to be a prob measure $\alpha : \mathfrak{A} \to [0, \infty[$ of the above kind, so that it is equivalent to be a solvable consistent family $(\beta_p)_{p \in I}$ (the usual definition looks quite different, it is in the guise of so-called *versions* of α). After this definition the members $A \in \mathfrak{A}$ are those sets of paths $x : T \to Y$ in the path space $X = Y^T$ which the stochastic process α is *able to measure*. Thus the fact that all $A \in \mathfrak{A}$ are countably determined can lead to misfortune in the case that T is uncountable. A specific problem are those subsets of the path space which support the *essential features* of a stochastic process α and could be named the *essential* sets for α; they can a priori be far from obvious. The most prominent example is the stochastic process of *Brownian motion* = the *Wiener measure* α, with $T = [0, \infty[$ and $Y = \mathbb{R}$ (in one dimension). Here the prime candidate for an essential set is $\mathrm{C}(T, \mathbb{R}) \subset X$. It must be noted that the idea for this candidate came from experimental observations *outside of mathematics*; the mathematical side has to admit that the set is not countably determined and hence not in $\mathfrak{A}$. In its more than 50 years the traditional theory of stochastic processes has not been able to produce an adequate notion of essentials sets.

INSERTION: SET-THEORETICAL COMPACTNESS. We recall the set-theoretical notions of compactness initiated in MARCZEWSKI [12]. These notions are weaker and more flexible than topological compactness, and in our new development all aspects of compactness will be based on them. Let X be a nonvoid set. A lattice $\mathfrak{S}$ in X with $\varnothing \in \mathfrak{S}$ is called *σ/τ compact* iff each nonvoid *countable/arbitrary* subsystem $\mathfrak{M} \subset \mathfrak{S}$ which is downward directed with intersection $\varnothing$, in symbols $\mathfrak{M} \downarrow \varnothing$, satisfies $\varnothing \in \mathfrak{M}$.

We list some immediate properties:

1) If X is a Hausdorff topological space then $\mathrm{Comp}(X)$ is τ compact.
2) If $\mathfrak{S}$ is σ/τ compact then $\mathfrak{S} \cup \{X\}$ is σ/τ compact as well.
3) If X is a *non-compact* Hausdorff space then the τ compact lattice $\mathrm{Comp}(X) \cup \{X\}$ does not come via 1) from any Hausdorff topology on X.

We use the occasion to introduce some further notations. For a nonvoid set system $\mathfrak{M}$ in X we define $\mathfrak{M}^{\star} \subset \mathfrak{M}^{\sigma} \subset \mathfrak{M}^{\tau}$ to consist of the *unions* of the nonvoid *finite/countable/arbitrary* subsystems of $\mathfrak{M}$, and $\mathfrak{M}_{\star} \subset \mathfrak{M}_{\sigma} \subset \mathfrak{M}_{\tau}$ to consist of the respective *intersections*. Likewise for a nonvoid function system $M \subset \overline{\mathbb{R}}^X$

we define $M^\star \subset M^\sigma \subset M^\tau$ to consist of the *pointwise suprema* of the nonvoid *finite/countable/arbitrary* subsystems of M, and $M_\star \subset M_\sigma \subset M_\tau$ to consist of the respective *infima*.

In conclusion we want to introduce the *shorthand notation* $\bullet = \star\sigma\tau$, to mean that $\bullet$ can in a fixed context be read as one and the same of the symbols $\star/\sigma/\tau$ or of the words *finite/countable/arbitrary*, like *variables* are in common use all over mathematics.

2. The generation of measures in the two previous theories

THE TRADITIONAL ABSTRACT THEORY: CARATHÉODORY 1914. In the traditional abstract theory the method of CARATHÉODORY [2] is the most fundamental source of nontrivial measures. Let X be a nonvoid set. The basic idea is to form for a set function $\Theta : \mathfrak{P}(X) \to [0,\infty]$ with $\Theta(\varnothing) = 0$ the set system

$$\mathfrak{C}(\Theta) := \{A \subset X : \Theta(M) = \Theta(M \cap A) + \Theta(M \cap A')\ \forall\, M \subset X\},$$

the members of which are called *measurable* Θ. One proves that $\Theta|\mathfrak{C}(\Theta)$ is a content on an algebra. On the other side one defines for a set function $\varphi : \mathfrak{S} \to [0,\infty]$ on a set system $\mathfrak{S}$ in X with $\varnothing \in \mathfrak{S}$ and $\varphi(\varnothing) = 0$ the so-called *outer measure* $\varphi^\circ : \mathfrak{P}(X) \to [0,\infty]$ to be

$$\varphi^\circ(A) = \inf\{\sum_{l=1}^{\infty} \varphi(S_l) : (S_l)_l \text{ in } \mathfrak{S} \text{ with } A \subset \bigcup_{l=1}^{\infty} S_l\},$$

which is a familiar formation since Borel and Lebesgue. These two ideas of Carathéodory then furnish the theorem: *Assume that* $\mathfrak{S}$ *is a ring. If* φ *is a content and upward* σ *continuous, then* $\varphi^\circ|\mathfrak{C}(\varphi^\circ)$ *is a measure and an extension of* φ. Thus a set function on a ring can be extended to a measure iff it is a content and upward σ continuous.

For all its power the above theorem has been under quite some criticism. In the traditional frame the attacks are towards the formation $\mathfrak{C}(\cdot)$, as an unmotivated and artificial one, while as a rule no doubt falls upon the outer measure formation. However, we shall see that the opposite is true: There are in fact serious weaknesses around the theorem, but it is the particular form of $\varphi \mapsto \varphi^\circ$ which must be blamed for them, whereas the formation $\mathfrak{C}(\cdot)$ remains the decisive methodical idea and even improves when put into the adequate context. We formulate the main deficiencies of the Carathéodory theorem as follows:

1) The formation φ° and hence the measure $\varphi^\circ|\mathfrak{C}(\varphi^\circ)$ are *outer regular* $\mathfrak{S}^\sigma$ by their very definition: *first* $\uparrow$ *then* $\downarrow$. It is mysterious how an *inner* regular counterpart could look.
2) The sequential character of φ° implies that *sequential continuity* carries over from φ to $\varphi^\circ|\mathfrak{C}(\varphi^\circ)$. It is mysterious how a *nonsequential* counterpart could look. Both times the *sum* in the definition of φ° is a crucial obstacle.

3) The domains $\mathfrak{S}$ of basic data $\varphi : \mathfrak{S} \to [0, \infty]$ are as a rule not *rings* but at most *lattices*. This becomes even more obvious when regularity and nonsequential continuity enter the scene (while the so-called *semirings* will not be in demand). But the proof of the Carathéodory theorem suffers a total breakdown when one attempts to pass from rings to lattices $\mathfrak{S}$.

With respect to 3) the present author produced a certain relief in an analysis course 1969/70: Instead of φ° he defined for an *isotone* set function $\varphi : \mathfrak{S} \to [0, \infty]$ on a set system $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ and $\varphi(\varnothing) = 0$ the formation $\varphi^\sigma : \mathfrak{P}(X) \to [0, \infty]$ to be

$$\varphi^\sigma(A) = \inf\{ \lim_{l\to\infty} \varphi(S_l) : (S_l)_l \text{ in } \mathfrak{S} \text{ isotone with } A \subset \bigcup_{l=1}^{\infty} S_l \}.$$

It is obvious that $\varphi^\sigma = \varphi^\circ$ when φ is a content on a ring $\mathfrak{S}$, so that the Carathéodory theorem persists when formulated with φ^σ instead of φ°. But for φ^σ the same proof furnishes a much more comprehensive theorem: Let us define a set function $\varphi : \mathfrak{S} \to [0, \infty]$ on a lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ to be a *content* iff it is isotone with $\varphi(\varnothing) = 0$ and *modular* in the sense that

$$\varphi(A \cup B) + \varphi(A \cap B) = \varphi(A) + \varphi(B) \quad \text{for all } A, B \in \mathfrak{S},$$

which is the old notion when $\mathfrak{S}$ is a ring. Then for φ^σ the Carathéodory theorem carries over from rings to the class of lattices $\mathfrak{S}$ with the condition

$$B \setminus A \in \mathfrak{S}^\sigma \quad \text{for all pairs } A \subset B \text{ in } \mathfrak{S}.$$

Note for example that this condition is fulfilled for the lattices $\mathrm{Cl}(X)$ and $\mathrm{Comp}(X)$ in a metric space X! Yet the author saw no trace of the extended theorem in the traditional abstract theory.

Much later then he realized that the formation φ^σ is superior to φ° with respect to the other defects 1) 2) as well. This will be one of the two decisive points in the present enterprise. The author dares say that the world of measure and integral in the 20th century would have been another one if Carathéodory in 1914 had conducted his ideas with φ^σ instead of φ°.

Both Theories: Positive Linear Functionals. In the *traditional abstract* theory another fundamental source of nontrivial measures are the positive linear functionals via the Daniell-Stone theorem. Later the *initial version of the Radon measure* theory adopted and adapted the idea. The common set-up is as follows: On the nonvoid set X one assumes a vector space of real-valued functions $F \subset \mathbb{R}^X$ which is a lattice under the pointwise max and min operations $\vee\wedge$ and *Stonean*, defined to mean that $f \in F \Rightarrow f \wedge t \in F$ for $0 < t < \infty$. One considers the linear functionals $J : F \to \mathbb{R}$ which are *isotone* (=: *positive*). The *traditional abstract* theory assumes J to be σ *continuous*, to mean that the pointwise convergence $f_n \downarrow 0$ implies that $J(f_n) \downarrow 0$, or that each *countable* nonvoid $M \subset F$ which is downward directed with pointwise infimum 0, in symbols $M \downarrow 0$, satisfies $\inf\limits_{f\in M} J(f) = 0$. The *initial Radon measure* theory assumes X to be a *locally compact* Hausdorff topological space and $F = \mathrm{CK}(X, \mathbb{R})$ to consist of the continuous real-valued functions with compact support. The *Dini* theorem then implies that the $J : F \to \mathbb{R}$ are τ *continuous*, to mean that an *arbitrary* nonvoid $M \subset F$ such that $M \downarrow 0$ in the above sense satisfies $\inf_{f \in M} J(f) = 0$. Bourbaki in fact *defines* these J to be the *Radon measures* on X. We think that our discussion should combine the two cases, that means assume the functional J to be $\bullet$ *continuous* for some $\bullet = \sigma\tau$. The procedure then runs as follows: the fundamental point is that it is of *outer regular* character like the previous Carathéodory 1914 procedure.

One defines the *outer envelope* $J^\bullet : \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}$ to be

$$J^\bullet(f) = \inf\{\sup_{u \in M} J(u) : M \subset F \text{ nonvoid } \bullet \text{ with } M \uparrow \geqq f\},$$

where $M \uparrow \geqq f$ means that M is upward directed with $\sup_{u \in M} u \geqq f$; thus one has *first* $\uparrow$ *then* $\downarrow$ as before. As a descendant of $J^\bullet$ one forms the *inner envelope* $J_\bullet : \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}$ to be $J_\bullet(f) = -J^\bullet(-f)$ or

$$J_\bullet(f) = \sup\{\inf_{u \in M} J(u) : M \subset F \text{ nonvoid } \bullet \text{ with } M \downarrow \leqq f\},$$

where $M \downarrow \leqq f$ means that M is downward directed with $\inf_{u \in M} u \leqq f$. One notes that $J^\bullet \geqq J_\bullet$ and $J^\bullet|F = J_\bullet|F = J$, and

$$(\circ) \quad \text{for } f \in \overline{\mathbb{R}}^X : \inf_{u \in F} J^\bullet(|f-u|) = 0 \iff J^\bullet(f) = J_\bullet(f) \in \mathbb{R}.$$

One defines $f \in \overline{\mathbb{R}}^X$ to be $\bullet$ *integrable* J iff it fulfils the two equivalent properties in $(\circ)$. Then one passes from J to a *set function*: One forms the set system $\mathfrak{a} := \{A \subset X : \chi_A \text{ is } \bullet \text{ integrable } J\}$, which turns out to be a lattice, and its *transporter* $\mathfrak{A} := \{A \subset X : A \cap M \in \mathfrak{a}\ \forall\, M \in \mathfrak{a}\} \supset \mathfrak{a}$, and on $\mathfrak{A}$ one defines $\beta(A) = J^\bullet(\chi_A)$ (we suppress the mark $\bullet = \sigma\tau$ for $\mathfrak{a}, \mathfrak{A}$ and β). In both cases $\bullet = \sigma\tau$ one has the result: *The set function $\beta : \mathfrak{A} \to [0, \infty]$ is a measure. A function $f \in \overline{\mathbb{R}}^X$ is $\bullet$ integrable J iff it is measurable $\mathfrak{A}$ and integrable β, and then $J^\bullet(f) = \int f d\beta$.* Moreover one proves that the set system

$$\mathfrak{N} := \{[f > t] : f \in F \text{ and } 0 < t < \infty\}$$

satisfies $\mathfrak{N}^\bullet \subset \mathfrak{A}$, and that $J^\bullet(\chi_.)$ and hence β are *outer regular* $\mathfrak{N}^\bullet$.

In the *traditional abstract* theory one has $\bullet = \sigma$. Thus the measure $\beta : \mathfrak{A} \to [0, \infty]$ furnishes the usual *Daniell-Stone* theorem and is *outer regular* $\mathfrak{N}^\sigma$. As a rule the traditional abstract theory is content with this result, as it is content with the result from the Carathéodory method, even though both results are *outer regular* and not *inner regular* measures. Both times the ideas of the traditional theory do not suffice to provide the construction of an *inner regular* measure with respect to an appropriate set system. In particular it is not clear whether in place of $J^\bullet$ the *inner* envelope $J_\bullet$ could be used: note that in the left of $(\circ)$ one cannot simply replace the *subadditive* $J^\bullet$ with the *superadditive* $J_\bullet$!

In the *initial Radon measure* theory $J : F = \mathrm{CK}(X, \mathbb{R}) \to \mathbb{R}$ one has $\bullet = \tau$ and $\mathfrak{N}^\tau = \mathrm{Op}(X)$, so that the measure β is *outer regular* $\mathrm{Op}(X) \subset \mathrm{Bor}(X) \subset \mathfrak{A}$.

As a rule β is *not Radon.* Thus there is an even more severe contrast to the aim, which is to produce a true *Radon measure.* There are several textbooks which are content with the result as it is, and thus formulate the *Riesz representation* theorem with β in place of a true Radon measure. Not so Bourbaki: *Faute de mieux* one continued to utilize the *outer* envelope J^τ as the basic construction, but then went on to put a *second* one on top of it, named the *essential* construction: As before Bourbaki applied the left side of ($\circ$), but in place of J^τ to its so-called *essential upper integral* $J^\tau_\circ : [0,\infty]^X \to [0,\infty]$ defined to be

$$J^\tau_\circ(f) = \sup\{J^\tau(f\chi_K) : K \in \mathrm{Comp}(X)\}.$$

The expression improves somewhat when one notes that it can be written

$$J^\tau_\circ(f) = \sup\{J^\tau(u) : 0 \leqq u \leqq f \text{ with } J^\tau(u) < \infty\}.$$

Thus $J^\tau_\circ(f) = J^\tau(f)$ when $J^\tau(f) < \infty$, and in particular $J^\tau_\circ(f) = J(f)$ for $f \geqq 0$ in F. This formation $J^\tau_\circ$ happens to work for the present purpose. Since it remains subadditive one can follow the former procedure. One comes back to the former $\mathfrak{A}$, where this time one defines $\alpha(A) = J^\tau_\circ(\chi_A)$, to obtain the result which follows: *The set function $\alpha : \mathfrak{A} \to [0,\infty]$ is a Radon measure. A function $f \in \overline{\mathbb{R}}^X$ is τ integrable with respect to $J^\tau_\circ$* (=: essentially τ integrable J) *iff it is measurable* $\mathfrak{A}$ *and integrable α, and in case $f \geqq 0$ then $J^\tau_\circ(f) = \int f d\alpha$.* It follows that the map $J \mapsto \alpha$ is one-to-one to all Borel-Radon measures $\alpha : \mathrm{Bor}(X) \to [0,\infty]$. This is the true *Riesz representation* theorem: it identifies the present ad-hoc Radon measures with the *true Borel-Radon measures.*

All the above is restricted to *locally compact* spaces X. In the final chapter of Bourbaki [1] the development continues with the *definition and construction* of his ad-hoc Radon measures on *arbitrary* Hausdorff spaces X and their identification with the *true Borel-Radon measures.*

Summary. The overall picture at the end of the 20th century shows that the foundations of measure and integral are in conflictful condition. One knows from both old concrete facts and the Radon measure theory that *regularity*, above all *inner regularity*, and *nonsequential continuity* are fundamental and indispensable concepts and tools. But we have said that the textbooks in the unspoilt traditional abstract fields pass over these concepts in complete silence. However, in an unbelievable contrast, the two central methods which serve to produce measures from basic data are such that the resultant measures are all equipped with a natural *outer regular* structure. Thus *regularity exists in silent omnipresence* – in form of *outer* regularity.

But that is about all what the two abstract theories of the 20th century have to offer: In the traditional abstract theory the method of Carathéodory 1914 shows no hint at all how to produce inner regularity nor nonsequential continuity. In the representation theory for positive linear functionals it is, in order to produce inner regular outcomes, far from appropriate to continue with the weapons of the outer arsenal – and this must in fact be repaired at once and wherever with that unfortunate *essential construction*. All that will look even less natural when in our final section compared with the new conception. The subsequent Radon measure theories on arbitrary Hausdorff topological spaces X in Bourbaki 1969 and Schwartz 1973 improved the access to inner regular set functions; but after all the exposition of Bourbaki is based on the former one, and Schwartz insisted that a Radon measure be tied to an outer regular companion. Above all the development remained restricted to the topological context. In the *abstract* context there were a few lines of research with an emphasis on inner regularity, in particular the somewhat isolated area around the *compact* and *perfect* measures, for example in [3, Sections 342 and 451]. But on the whole the fundamental relevance of inner regularity and nonsequential continuity had been left without adequate structure.

3. The origin of the new systematization

The turn to the release started with two *natural ideas*: The first idea is to consider in a Hausdorff topological space X those set functions $\varphi : \mathrm{Comp}(X) \to [0, \infty[$ that can be extended to Radon measures, and to *characterize* these set functions. Of course one can assume that φ is isotone with $\varphi(\varnothing) = 0$. Then the second idea is to *extend* this characterization to the *abstract situation* of a lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ in a nonvoid set X, that is to characterize those isotone set functions $\varphi : \mathfrak{S} \to [0, \infty[$ with $\varphi(\varnothing) = 0$ that can be extended to measures which are *inner regular* $\mathfrak{S}$. It is immediate that the respective extensions $\alpha : \mathfrak{A} \to [0, \infty]$ are *unique*: If one defines the *crude inner envelope* $\varphi_\star : \mathfrak{P}(X) \to [0, \infty]$ of φ to be

$$\varphi_\star(A) = \sup\{\varphi(S) : S \in \mathfrak{S} \text{ with } S \subset A\},$$

then each such $\alpha : \mathfrak{A} \to [0, \infty]$ must be $\alpha = \varphi_\star | \mathfrak{A}$.

As for the first idea, we note three theorems in the literature which characterize those isotone set functions $\varphi : \mathrm{Comp}(X) \to [0, \infty[$ with $\varphi(\varnothing) = 0$ that can be extended to Radon measures, henceforth called the *Radon premeasures*.

Choquet 1953: *φ is a* locally bounded Radon premeasure *iff it is modular and* continuous from above: *for any $A \in \mathrm{Comp}(X)$ and $\varepsilon > 0$ there exists an open $U \supset A$ such that all compact $K \subset U$ fulfil $\varphi(K) < \varphi(A) + \varepsilon$.* Note that the last condition implies that φ is *downward τ continuous*.

Bourbaki 1969: *Assume that φ is* locally bounded (which in fact can be dispensed with)*. Then φ is a* Radon premeasure *iff it is modular and downward τ continuous.*

Kisyński 1968: *φ is a* Radon premeasure *iff*

$$\varphi(B) = \varphi(A) + \varphi_\star(B \setminus A) \quad \textit{for all } A \subset B \textit{ in } \mathrm{Comp}(X).$$

As for the second idea, the three theorems are of different kind. The Choquet condition, where besides $\mathrm{Comp}(X)$ also $\mathrm{Op}(X)$ comes in, is so close to the topological context that the natural attempts at extension lead back to that context. The Bourbaki condition breaks down even for certain bounded φ on certain lattices $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ which fulfil $\mathfrak{S} = \mathfrak{S}_\tau$ and are τ compact, like the former $\mathfrak{S} = \mathrm{Comp}(X)$ (for an example see [8, Remark 3.3]). The miraculous event is the Kisyński [5] theorem: It was not recorded in Bourbaki 1969 and Schwartz 1973. But in no time Topsøe [15, 16] realized that this theorem is capable of an abstract extension. His basic achievement is for both $\bullet = \sigma\tau$:

Topsøe 1970: *Let $\varphi : \mathfrak{S} \to [0, \infty[$ on the lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ be isotone with $\varphi(\varnothing) = 0$. Consider the properties*

1) *φ can be extended to a measure which is* inner regular $\mathfrak{S}$*, and φ is downward $\bullet$ continuous.*
2) *$\varphi(B) = \varphi(A) + \varphi_\star(B \setminus A)$ for all $A \subset B$ in $\mathfrak{S}$, and φ is downward $\bullet$ continuous at $\varnothing$.*

Then 1)$\Rightarrow$2) *is obvious, and* 2)$\Rightarrow$1) *holds true when $\mathfrak{S} = \mathfrak{S}_\bullet$.*

However, without $\mathfrak{S} = \mathfrak{S}_\bullet$ the implication 2)$\Rightarrow$1) becomes false. What remains true is the implication 2)$\Rightarrow$1$\bullet$) *φ can be extended to a measure of domain $\supset \mathfrak{S}_\bullet$ which is inner regular $\mathfrak{S}_\bullet$, and φ is downward $\bullet$ continuous.* But this time the converse 1$\bullet$)$\Rightarrow$2) becomes false without $\mathfrak{S} = \mathfrak{S}_\bullet$. Thus it appears that beyond $\mathfrak{S} = \mathfrak{S}_\bullet$ the formulation of 2) in terms of the *crude* inner envelope $\varphi_\star$ of φ ceases to be adequate and prevents an equivalence assertion. As another evidence we invoke the *outer* situation of Carathéodory 1914, which did *not* use the obvious *crude* outer counterpart $\varphi^\star$ of $\varphi_\star$, but the more subtle φ° or the later φ^σ.

All this underlines that *new envelopes* are required as the fundamental tools for systematization – while the basic ideas of Kisyński and Topsøe must remain in force. These new envelopes and the subsequent systematization are the contribution of the present author around the end of the 20th century.

4. The new theory

The new systematization is structured in order to meet both of the particular tasks formulated in the introduction. Its first and central aim is to produce certain *distinguished classes of measures* from certain particular classes of basic data, in the spirit that can be expected from what has been said so far. The foundational part consists of an *inner* and an *outer* theory, which are parallel in almost all essentials and have been developed in parallel at the outset [7]. But it soon became clear that the *inner* version is the superior one in the most decisive places. Therefore in the present article the explicit description will be restricted to the inner theory. The development will be almost uniform in the three columns $\bullet = \star\sigma\tau$, thanks to an adequate formulation of the basic notions.

In the sequel we assume that $\mathfrak{S}$ is a lattice with $\varnothing \in \mathfrak{S}$ in a nonvoid set X and that $\varphi : \mathfrak{S} \to [0, \infty[$ is an isotone set function with $\varphi(\varnothing) = 0$. The basic definitions are as follows: We define an *inner $\bullet$ extension* of φ to be a content $\alpha : \mathfrak{A} \to [0, \infty]$ on a ring $\mathfrak{A}$ which is an extension of φ, and is such that even $\mathfrak{S} \subset \mathfrak{S}_\bullet \subset \mathfrak{A}$ with

α is *inner regular* $\mathfrak{S}_\bullet$ and

$\alpha|\mathfrak{S}_\bullet$ is *downward $\bullet$ continuous* (void for $\bullet = \star$).

We define φ to be an *inner $\bullet$ premeasure* iff it possesses inner $\bullet$ extensions.

The subsequent *inner extension theorem* characterizes those φ which are inner $\bullet$ premeasures, and then *describes* all inner $\bullet$ extensions of φ. The decisive weapons are the new *inner* $\bullet$ *envelopes* $\varphi_\bullet : \mathfrak{P}(X) \to [0,\infty]$ announced above and defined to be

$$\varphi_\bullet(A) = \sup\{\inf_{M\in\mathfrak{M}} \varphi(M) : \mathfrak{M} \subset \mathfrak{S} \text{ nonvoid} \bullet \text{ with } \mathfrak{M} \downarrow\subset A\},$$

where $\mathfrak{M} \downarrow\subset A$ means that $\mathfrak{M}$ is downward directed with $\bigcap_{M\in\mathfrak{M}} M \subset A$; thus one has *first* $\downarrow$ *then* $\uparrow$. It follows that $\varphi_\bullet$ is *inner regular* $\mathfrak{S}_\bullet$. For $A \in \mathfrak{S}$ we have $\varphi(A) \leqq \varphi_\bullet(A)$, and $\varphi(A) = \varphi_\bullet(A)$ iff φ is downward $\bullet$ continuous at A. Furthermore $\varphi_\star \leqq \varphi_\sigma \leqq \varphi_\tau$, and $\varphi_\star$ is the previous *crude* inner envelope, while φ_σ can be defined via *sequences* like the previous *outer* counterpart. We also need the *satellites* $\varphi_\bullet^B : \mathfrak{P}(X) \to [0,\infty]$ with $B \subset X$, defined via limitation to those $\mathfrak{M} \subset \mathfrak{S}$ as above which consist of subsets $M \subset B$.

Inner Extension Theorem: *Let $\varphi : \mathfrak{S} \to [0,\infty[$ be isotone with $\varphi(\varnothing) = 0$. Then the following are equivalent.*

0) *φ is an inner $\bullet$ premeasure.*
1) *$\varphi(B) \leqq \varphi(A) + \varphi_\bullet(B \setminus A)$ for all $A \subset B$ in $\mathfrak{S}$, and φ is supermodular and downward $\bullet$ continuous.*
2) *$\varphi(B) \leqq \varphi(A) + \varphi_\bullet^B(B \setminus A)$ for all $A \subset B$ in $\mathfrak{S}$, and φ is supermodular and downward $\bullet$ continuous at $\varnothing$.*
3) *The set function $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ is an extension of φ.*

In this case $\Phi := \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ is an inner $\bullet$ extension of φ; it is a complete *content, and a* measure *when $\bullet = \sigma\tau$. All inner $\bullet$ extensions of φ are restrictions of Φ.*

The prominent rôle of $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ as the *unique maximal* inner $\bullet$ extension of φ emphasizes the *fundamental nature* of the formation $\mathfrak{C}(\cdot)$ due to Carathéodory. It appears that not until the present new theory this formation has achieved its adequate position. There is no such position in the traditional abstract theory!

The inner extension theorem has several important addenda. First of all the Localization Principle: *If $A \subset X$ fulfils $A \cap S \in \mathfrak{C}(\varphi_\bullet)$ for all $S \in \mathfrak{S}$ then* $A \in \mathfrak{C}(\varphi_\bullet)$. Also note that $\mathfrak{S} \subset \mathfrak{S}_\bullet \subset \mathfrak{C}(\varphi_\bullet)$, and in particular in case $\bullet = \tau$ that $\mathfrak{S}_\tau$ can be of an *immense* size.

An important special case for $\bullet = \sigma\tau$ is that $\mathfrak{S}$ is $\bullet$ *compact*. In this case the above set functions φ are all downward $\bullet$ continuous at $\varnothing$. Thus the equivalent condition 2) in the inner extension theorem becomes much simpler.

The most natural example is the *topological* case: Let X be a Hausdorff topological space and $\mathfrak{S} = \mathrm{Comp}(X)$. For each $\bullet = \star\sigma\tau$ then φ is an inner $\bullet$ premeasure iff it is a *Radon premeasure*, and in this case the envelopes $\varphi_\bullet$ and hence the measures $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ are the same for $\bullet = \star\sigma\tau$. Thus the common Φ is the *unique maximal* Radon measure extension of φ.

A brief word on the parallel new *outer* theory $\bullet = \star\sigma\tau$ and on the connections between the two theories: The outer theory starts with $\varphi : \mathfrak{S} \to [0,\infty]$, but this deviation finds its natural explanation in the extended version of the two theories developed in [7] – in that extended version they are even *identical*! The outer theory is based on the *outer* $\bullet$ *envelopes* $\varphi^{\bullet} : \mathfrak{S} \to [0, \infty]$ of φ, of which $\varphi^{\star}$ is the obvious *crude* outer one and φ^{σ} the previous 1969/70 variant of the Carathéodory formation φ°. The resultant *outer extension theorem* corresponds to the present inner one in the essentials, except that it has of course no extra condition with the senseless *upward* $\bullet$ *continuity at* $\varnothing$ and hence no satellites, but in return a certain *safety barrier* at ∞ in case $\bullet = \tau$. The case $\bullet = \sigma$ contains the result of Carathéodory 1914 and its 1969/70 extension, but goes far beyond.

In conclusion we want to recall the two *decisive ideas* which combine to form the basis of the present new theories: The first one is the idea of Kisyński and Topsøe how to express the existence of *inner regular* extensions for set functions defined on *lattices*. The second one is the 1969/70 idea to pass from the Carathéodory formation φ° to its variant φ^{σ}. It is remarkable that these two ideas came up in the same small period of time before 1970. Much later then the present author returned to the context and noticed that, in contrast to φ°, the formation φ^{σ} has an obvious *inner* counterpart φ_{σ}, and that the two of them have obvious *nonsequential* counterparts φ^{τ} and φ_{τ}, defined via directed set systems. What then remained was the *systematization*, to start off with an adequate formulation of the basic notions in order to arrive at the *necessary and sufficient conditions* in our inner and outer extension theorems.

5. The further development in a few examples

In the last few years the present author was pleased to note that the inner and outer extension theorems – and in particular the nature of their basic concepts – opened the road for an extensive development in measure and integration and beyond, the results of which are not more complicated and at times even simpler to formulate, but can be much more powerful and comprehensive than the earlier ones. In particular the author thinks it is for the first time that an *abstract* theory of measure and integral contains the respective *topological* theory as an *explicit special case*. He developed a number of topics in [7] and in subsequent papers. All this has been summarized in the survey articles [9, 11]. The present section wants to offer a few examples, related to the points of criticism in Sections 1 and 2.

The Choquet Integral. We shall need the notion of the *integral* due to Choquet 1953/54. Our version will be adapted to our situation of *two parallel* theories. Let $\mathfrak{S}$ be a lattice with $\varnothing \in \mathfrak{S}$ in the nonvoid set X. We define the function classes $\mathrm{Inn}(\mathfrak{S})$ and $\mathrm{Out}(\mathfrak{S})$ to consist of the functions $f \in [0, \infty]^X$ with $[f \geqq t] \in \mathfrak{S}$ and $[f > t] \in \mathfrak{S}$ respectively for $0 < t < \infty$. Then for $\varphi : \mathfrak{S} \to [0, \infty]$ isotone with $\varphi(\varnothing) = 0$ the *Choquet integral* $⨍ f d\varphi \in [0, \infty]$ is defined to be

$$= \int_{0\leftarrow}^{\to\infty} \varphi([f \geqq t])dt \text{ for } f \in \mathrm{Inn}(\mathfrak{S}) \quad \text{and} \quad = \int_{0\leftarrow}^{\to\infty} \varphi([f > t])dt \text{ for } f \in \mathrm{Out}(\mathfrak{S}),$$

both times as an improper Riemann integral of a decreasing function $\geqq 0$. One verifies that for $f \in \mathrm{Inn}(\mathfrak{S}) \cap \mathrm{Out}(\mathfrak{S})$ the two second members are equal. In particular $⨍\chi_A d\varphi = \varphi(A)$ for $A \in \mathfrak{S}$. When $\mathfrak{S}$ is a σ algebra then $\mathrm{Inn}(\mathfrak{S}) = \mathrm{Out}(\mathfrak{S})$ consists of the usual $f \in [0,\infty]^X$ *measurable* $\mathfrak{S}$, and when moreover φ is a measure then $⨍ f d\varphi$ is the usual integral $\int f d\varphi$. This notion of an integral is so natural and simple that one could wonder why it did not become the foundation for all of integration theory. But the basic hardship with the Choquet integral is that it is a priori obscure whether and when it is *additive*. For this context we refer to [10].

Positive Linear Functionals. Our first point is the representation of positive linear functionals as discussed in Section 2. Let as before $F \subset \mathbb{R}^X$ be a Stonean vector lattice and $J : F \to \mathbb{R}$ be a positive linear functional, assumed to be $\bullet$ continuous for some $\bullet = \sigma\tau$. Besides $J^\bullet$ and $J_\bullet$ we form the *crude* envelopes

$J^\star : J^\star(f) = \inf\{J(u) : u \in F \text{ with } u \geqq f\}$,
$J_\star : J_\star(f) = \sup\{J(u) : u \in F \text{ with } u \leqq f\}$;

and besides $\mathfrak{N}$ we form the set system

$$\mathfrak{M} := \{[f \geqq t] : f \in F \text{ and } 0 < t < \infty\},$$

both of which are lattices with $\varnothing$. Note that $F^+ := \{f \in F : f \geqq 0\} \subset \mathrm{Inn}(\mathfrak{M}) \cap \mathrm{Out}(\mathfrak{N})$.

The basic trouble in Section 2 was with *inner regular* representations. We restricted ourselves to the particular *initial Radon measure* situation with $\bullet = \tau$ and described the route via $J_\circ^\tau$ due to Bourbaki. Now in the new systematization the *inner extension theorem* produces the answer which follows, in the *full situation* and in striking contrast to the former one.

Inner Representation Theorem: *There is a unique inner $\bullet$ premeasure $\varphi : \mathfrak{M} \to [0,\infty[$ which represents J in the sense that $J(f) = ⨍ f d\varphi$ for all $f \in F^+$. This is $\varphi = J^\star(\chi_\cdot)|\mathfrak{M}$. It even fulfils $J_\bullet(f) = ⨍ f d\varphi_\bullet$ for all $f \in [0,\infty]^X$, and hence $J_\bullet(\chi_\cdot) = \varphi_\bullet = \Phi_\star$.* It follows that $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ represents J in the sense that all $f \in F$ are integrable Φ with $J(f) = \int f d\Phi$.

In the particular *initial Radon measure* case one has $\mathfrak{M}_\tau = \mathrm{Comp}(X)$, so that Φ is the unique maximal Radon measure which represents J. One proves that in fact $\Phi = \alpha$. In this context a final word on the old formation $J_\circ^\tau$: One proves that $J_\circ^\tau(f) = ⨍ f d\Phi^\star$ for all $f \in [0,\infty]^X$, and hence $J_\circ^\tau(\chi_\cdot) = \Phi^\star$. This shows that the formation is of *hybrid type*: From its definition it is of *inner* type, but its properties are more like those of an *outer* formation. For example, as a rule $J_\circ^\tau(\chi_\cdot) = \Phi^\star = \alpha^\star$ is far from inner regular $\mathrm{Comp}(X)$. Therefore $J_\circ^\tau$ has no place in the new systematization.

The new *outer procedure* is parallel to the new inner one. But it is of course closer to the old procedure, which after all has been of *outer* character: it is in terms of $\mathfrak{N}$ and ends up at the former β.

On the whole it seems clear that we are arrived at the adequate method of representation. The two new representation theorems are in [9] in much more comprehensive versions than described above: thus their domains are subsets of

$[0,\infty[^X$ and $[0,\infty]^X$, assumed to be positive-homogeneous with 0 and *Stonean* lattices in the appropriate sense, but need not even be *stable under addition.* The final theorems are the precise counterparts of the earlier inner and outer extension theorems for set functions. In particular the inner theorem furnishes a wide extension of the *Riesz representation* theorem to the class of *all* Hausdorff topological spaces X. We note that all these results have substantial predecessors in Pollard-Topsøe [13] and Topsøe [17].

Finite Products. It is well known and has been noted in Section 1 that the two abstract theories of the 20th century are quite different in their treatment of *finite products* of measures: thus the *Radon product measure* of two Radon measures is out of reach of the traditional abstract theory. Our next point is to show that with the new systematization the situation becomes totally different. We note at once that this point – like the final one – is a domain of the *inner* theory: there is no full outer counterpart.

We fix nonvoid sets X and Y. For nonvoid set systems $\mathfrak{S}$ in X and $\mathfrak{T}$ in Y we have the usual *product* set system $\mathfrak{S}\times\mathfrak{T} := \{S\times T : S\in\mathfrak{S} \text{ and } T\in\mathfrak{T}\}$ in $X\times Y$. For *lattices* $\mathfrak{S}$ and $\mathfrak{T}$ with $\varnothing$ then $\mathfrak{R} := (\mathfrak{S}\times\mathfrak{T})^\star$ is a lattice with $\varnothing$ as well (and the same for rings and algebras). Now let $\varphi:\mathfrak{S}\to[0,\infty]$ and $\psi:\mathfrak{T}\to[0,\infty]$ be isotone set functions with $\varphi(\varnothing)=\psi(\varnothing)=0$. One proves for $E\in\mathfrak{R}$ that the function $x\mapsto\psi(E(x))$, where $E(x):=\{y\in Y:(x,y)\in E\}\in\mathfrak{T}$ is the vertical section of E at $x\in X$, is in $\operatorname{Inn}(\mathfrak{S})\cap\operatorname{Out}(\mathfrak{S})$. We define the *product set function*

$$\vartheta=\varphi\times\psi:\mathfrak{R}\to[0,\infty] \quad \text{to be} \quad \vartheta(E)=⨍\psi(E(\cdot))d\varphi.$$

It follows that ϑ is isotone with $\vartheta(\varnothing)=0$ and fulfils $\vartheta(S\times T)=\varphi(S)\psi(T)$ for $S\in\mathfrak{S}$ and $T\in\mathfrak{T}$ (with $0\infty=0$ as usual). Also ϑ inherits from φ and ψ the properties to be *modular*, to be *finite*, and to be *finite and downward* $\bullet$ *continuous.* The fundamental fact is the

Product Theorem: *Assume that* $\varphi:\mathfrak{S}\to[0,\infty[$ *and* $\psi:\mathfrak{T}\to[0,\infty[$ *are inner* $\bullet$ *premeasures* ($\bullet=\star\sigma\tau$). *Then* $\vartheta=\varphi\times\psi:\mathfrak{R}\to[0,\infty[$ *is an inner* $\bullet$ *premeasure as well, and* $\Theta:=\vartheta_\bullet|\mathfrak{C}(\vartheta_\bullet)$ *is an extension of the product* $\Phi\times\Psi$ *of* $\Phi=\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ *and* $\Psi=\psi_\bullet|\mathfrak{C}(\psi_\bullet)$.

If in particular X and Y are Hausdorff topological spaces with $\mathfrak{S}=\operatorname{Comp}(X)$ and $\mathfrak{T}=\operatorname{Comp}(Y)$ then one notes that $\mathfrak{R}_\tau=\operatorname{Comp}(X\times Y)$. Thus if φ and ψ are Radon premeasures on X and Y with $\vartheta=\varphi\times\psi$, then $\pi:=\vartheta_\tau|\mathfrak{R}_\tau$ is a Radon premeasure on $X\times Y$ and fulfils $\pi_\tau=\pi_\star=(\vartheta_\tau|\mathfrak{R}_\tau)_\star=\vartheta_\tau$, so that $\Theta=\vartheta_\tau|\mathfrak{C}(\vartheta_\tau)=\pi_\tau|\mathfrak{C}(\pi_\tau)$ is an extension of $\Phi\times\Psi$ which is *maximal Radon* on $X\times Y$.

Projective Limits. Our final point is the context of projective limits as discussed in Section 1. The aim is a comprehensive *projective limit theorem* in terms of the new *inner theory.* As before we fix an infinite index set T and a family $(Y_t)_{t\in T}$ of nonvoid sets with product set X, and we recall the index set $I=I(T)$ and the family $(Y_p)_{p\in I}$ of partial product sets, with the projections $H_p:X\to Y_p$ and $H_{pq}:Y_q\to Y_p$ for $p\subset q$ in I.

This time now we assume, in the spirit of the new inner systematization, a family $(\mathfrak{K}_t)_{t\in T}$ of lattices $\mathfrak{K}_t$ in Y_t, such that $\mathfrak{K}_t$ contains the finite subsets of Y_t and is $\bullet$ *compact* for some fixed $\bullet = \sigma\tau$. We form the family $(\mathfrak{K}_p)_{p\in I}$ of partial product lattices $\mathfrak{K}_p = \{\prod_{t\in p} K_t : K_t \in \mathfrak{K}_t\}^\star$ in Y_p, which retain these properties. The decisive formation is

$$\mathfrak{S} := \{\prod_{t\in T} S_t : S_t \in \mathfrak{K}_t \cup \{Y_t\} \text{ with } S_t = Y_t \text{ for almost all } t \in T\}^\star,$$

which is a lattice in X with $\varnothing, X \in \mathfrak{S}$ and likewise $\bullet$ *compact.* Our theorem then reads as follows.

Projective Limit Theorem: *There is a one-to-one correspondence between*

the inner $\bullet$ prob *premeasures* $\varphi : \mathfrak{S} \to [0,\infty[$ (i.e. $\varphi(X) = 1$) *and*

the families $(\varphi_p)_{p\in I}$ *of inner* $\bullet$ prob *premeasures* $\varphi_p : \mathfrak{K}_p \to [0,\infty[$ *which are* projective *in the sense that*

$$(\leftrightarrow) \quad \varphi_p(B) = (\varphi_q)_\bullet\big(H_{pq}^{-1}(B)\big)\ \forall B \in \mathfrak{K}_p \quad \textit{for all } p \subset q \textit{ in } I.$$

The correspondence reads

$$(\Leftrightarrow) \quad \varphi_p(B) = \varphi\big(H_p^{-1}(B)\big)\ \forall B \in \mathfrak{K}_p \quad \textit{for all } p \in I.$$

One even has for all $B \subset Y_p$ *and* $p \in I$

$$(\varphi_p)_\bullet(B) = \varphi_\bullet\big(H_p^{-1}(B)\big) \quad \textit{and} \quad B \in \mathfrak{C}((\varphi_p)_\bullet) \Leftrightarrow H_p^{-1}(B) \in \mathfrak{C}(\varphi_\bullet).$$

Moreover one has for $A \in \mathfrak{S}_\bullet$

$$H_p(A) \in \mathfrak{C}((\varphi_p)_\bullet)\ \forall\, p \in I \quad \textit{and} \quad \Phi(A) = \inf_{p\in I} \Phi_p(H_p(A)).$$

In the context of *stochastic processes* one assumes that $Y_t = Y$ and $\mathfrak{K}_t = \mathfrak{K}$ independent of $t \in T$. In the spirit of the new inner systematization a *stochastic process* for T and $(Y, \mathfrak{K})$ can be defined to be an inner τ prob premeasure $\varphi : \mathfrak{S} \to [0,\infty[$. It is fundamental to take $\bullet = \tau$: Then $\Phi = \varphi_\tau|\mathfrak{C}(\varphi_\tau)$ is a measure on the path space $X = Y^T$ with an *immense domain* $\mathfrak{S} \subset \mathfrak{S}_\tau \subset \mathfrak{C}(\varphi_\tau)$, which in case of an uncountable T reaches far beyond the frame of *countably determined* subsets of $X = Y^T$. This situation opens the chance for an adequate definition: we define the *essential sets* for the stochastic process $\varphi : \mathfrak{S} \to [0,\infty[$ to be those subsets $E \in \mathfrak{C}(\varphi_\tau)$ which have full measure $\Phi(E) = 1$. At the same time we obtain the τ continuities which are in the $\bullet = \tau$ version of our inner systematization.

In case that Y is a *Polish topological space* with $\mathfrak{B} = \mathrm{Bor}(Y)$ and $\mathfrak{K} = \mathrm{Comp}(Y)$ one proves that there is a one-to-one correspondence between

the traditional stochastic processes $\alpha : \mathfrak{A} \to [0,\infty[$ for T and $(Y, \mathfrak{B})$ and

the new stochastic processes $\varphi : \mathfrak{S} \to [0,\infty[$ for T and $(Y, \mathfrak{K})$.

The correspondence rests upon $\mathfrak{S} \subset \mathfrak{A} \subset \mathfrak{C}(\varphi_\tau)$ and reads $\varphi = \alpha|\mathfrak{S}$ and $\alpha = \Phi|\mathfrak{A}$. Moreover $\varphi_\tau = (\alpha^\star|\mathfrak{S}_\tau)_\star$. In the example of the *Brownian motion = Wiener measure* with $T = [0,\infty[$ and $Y = \mathbb{R}$ one proves that $\mathrm{C}(T,\mathbb{R}) \in \mathfrak{C}(\varphi_\tau)$ with $\Phi(\mathrm{C}(T,\mathbb{R})) = 1$. Thus $\mathrm{C}(T,\mathbb{R})$ is in fact an *essential set* for this stochastic process. The present development has been summarized in [11]. It should be compared with the previous ones, for example in Fremlin [3, Chapter 45].

In conclusion we want to specialize the new projective limit theorem to the case of *infinite products*. Assume that $(\vartheta_t)_{t\in T}$ is a family of inner $\bullet$ prob premeasures $\vartheta_t : \mathfrak{K}_t \to [0,\infty[$. For $p \in I$ we define the inner $\bullet$ prob premeasure $\varphi_p : \mathfrak{K}_p \to [0,\infty[$ to be the *product* of the finite family $(\vartheta_t)_{t\in p}$ under the obvious extension of the product formation in the last example to any finite number of factors. One verifies that $(\varphi_p)_{p\in I}$ is a *projective family* in the sense of the present theorem. Thus it produces an inner $\bullet$ prob premeasure $\varphi : \mathfrak{S} \to [0,\infty[$. Then $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ has the obvious position of the natural *infinite product* of the family $(\Theta_t)_{t\in T}$ of the prob measures $\Theta_t = (\vartheta_t)_\bullet|\mathfrak{C}((\vartheta_t)_\bullet)$. This formation is far more comprehensive than the former one for Radon prob measures. It makes clear that in the present context the adequate notion of *compactness* is not the *topological* but the *set-theoretical* $\bullet$ one.

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Post-Widder Inversion for Laplace Transforms of Hyperfunctions

Peer Christian Kunstmann

To the memory of Prof. Günter Lumer

Abstract. We prove a Post-Widder inversion formula for the Laplace transform of hyperfunctions with compact support in $[0,\infty)$. We observe that any hyperfunction with support in $[0,\infty)$ has Laplace transforms which are analytic on the right half-plane $\mathbb{C}_+$, and we extend the Post-Widder inversion formula to suitably bounded representatives of arbitrary hyperfunctions with support in $[0,\infty)$.

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Introduction

Ever since Heaviside introduced the method of operational calculus at the end of the 19th century there have been attempts to base it on rigorous mathematical grounds. This was one reason for the development of Laplace transform theory (cf. G. Doetsch's books). But also L. Schwartz refers back to Heaviside in the preface to the first edition (1950) of his book on distribution theory.

A lack of Laplace transform methods is that, in order to be Laplace transformable, a function or distribution has to be exponentially bounded in some sense. This has been overcome by Prof. Lumer and Frank Neubrander (cf. [9]) who introduced an asymptotic Laplace transform for L^1_{loc}-functions, based on earlier work of C. Vignaux from 1939.

A few years earlier, H. Komatsu developed a Laplace transform theory for hyperfunctions (cf. [3], [4], [5]). The (asymptotic) Laplace transform of (generalized and hyper-) functions has been our favorite topic in discussions with Prof. Lumer whenever we met, and I always enjoyed the talks he gave on his research in that field and admired his deep insight into the subject.

It was shown in [10] that, after suitably modifying the definition from [9], the asymptotic Laplace transform coincides for L^1_{loc}-functions with the Laplace transform of hyperfunctions in the sense of Komatsu. This is preserved when extending asymptotic Laplace transform to hyperfunctions (cf. [10, Sect. 4]).

In the present paper we study Post-Widder type inversion formulae for the Laplace transform of hyperfunctions. The possibility of such a formula relies on the property that, for any hyperfunction, there is always a Laplace transform that is analytic on the right half-plane $\mathbb{C}_+$. Although this does not seem to be stated explicitly anywhere, it follows immediately from Komatsu's argument to prove surjectivity of the natural map $\mathscr{B}^{\exp}([a,\infty)) \to \mathscr{B}([a,\infty)$ (cf. [4], [10, Lem. 1.1], we reproduce the argument in Section 2 below). Nevertheless, things are always conceptually easier for hyperfunctions with compact support, and for this case we formulate and prove the formula in Section 1. In Section 2 we revise general hyperfunctions with support in $[0,\infty)$, indicating and discussing the possibility of "half-space theories" of (asymptotic) Laplace transforms. In the final Section 3 we give an extension of our Post-Widder inversion formula to the general case.

We restrict ourselves to the scalar-valued case here, but the results can easily be generalized to hyperfunctions with values in an arbitrary Banach space.

Throughout the paper we write, for any $G \subset \mathbb{C}$, $f : G \to \mathbb{C}$ and $\varepsilon > 0$: $G_\varepsilon := \{z \in \mathbb{C} : d(z,G) < \varepsilon\}$ and $\|f\|_{\infty,G} := \sup\{|f(z)| : z \in G\}$. For $\omega \in (0,\pi)$ we denote by $\Sigma(\omega)$ the open sector $\Sigma(\omega) := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \omega\}$ where the argument is taken with values in $(-\pi,\pi]$.

1. Hyperfunctions with compact support

In this section we recall the basics for hyperfunctions with compact support in $\mathbb{R}$, and we prove the Post-Widder inversion formula for hyperfunctions with compact support in $[0,\infty)$ (Theorem 1.1 and Corollary 1.3).

For a compact subset K of $\mathbb{R}$ we denote by $\mathscr{A}'(K)$ the set of linear functionals T on the set of entire functions $\mathscr{A}(\mathbb{C})$ such that, for all open and bounded supersets $U \subset \mathbb{C}$ of K, there is a constant C_U such that, for all $\varphi \in \mathscr{A}(\mathbb{C})$,

$$|T(\varphi)| \le C_U \|\varphi\|_{\infty,U}.$$

It is clearly sufficient to consider $U = K_\varepsilon$ in the definition, and $\mathscr{A}'(K)$ is a Fréchet space for the best constants $C_{K_{1/n}}$, $n \in \mathbb{N}$, as seminorms. By continuity, a $T \in \mathscr{A}'(K)$ can be extended to any function φ analytic in a complex neighborhood of K (cf. [2, 9.1.2]), and T is uniquely determined by it values on polynomials.

Let $T \in \mathscr{A}'(K)$ and $U, V \subset \mathbb{C}$ be open and bounded supersets of K, bounded by finitely many closed piecewise C^1-curves and such that $V \subset \overline{V} \subset U$. Then we have, orienting ∂U in the usual way, for any φ analytic in a neighborhood of $\overline{U}$ by Cauchy's formula

$$\varphi = \frac{1}{2\pi i} \int_{\partial U} \frac{\varphi(z)}{z - \cdot}\, dz \quad \text{on } V.$$

This leads, for such φ, to the representation

$$T(\varphi) = \frac{1}{2\pi i} \int_{\partial U} T\Big(\frac{1}{z - \cdot}\Big) \varphi(z)\, dz. \tag{1.1}$$

Here, the analytic function $\mathscr{C}T : \mathbb{C} \setminus K, z \mapsto T(\frac{1}{z-\cdot})$ is called the *Cauchy transform of* T. The representation (1.1) implies the estimate

$$|\mathscr{C}T(z)| \leq \frac{C_V}{d(z,V)} \quad \text{for } z \notin \overline{V},$$

in particular $\mathscr{C}T$ is analytic at ∞ and $\mathscr{C}T(\infty) = 0$.

On the other hand, if U is a complex neighborhood of K and the function F is analytic on $U \setminus K$ then

$$T(\varphi) = \frac{1}{2\pi i} \int_{\partial V} F(z)\varphi(z)\, dz \tag{1.2}$$

defines an element of $\mathscr{A}'(K)$ where the open neighborhood V is chosen such that $V \subset \overline{V} \subset U$, V is bounded by finitely many closed piecewise C^1-curves, and φ is analytic on a neighborhood of $\overline{V}$.

Writing $\mathscr{O}(V)$ for the set of analytic functions on the open subset $V \subset \mathbb{C}$, we thus have

$$\mathscr{A}'(K) \simeq \mathscr{O}(\mathbb{C} \setminus K)/\mathscr{O}(\mathbb{C}) \simeq \mathscr{O}(U \setminus K)/\mathscr{O}(U)$$

for any open complex neighborhood U of the compact set K.

Laplace transform. For $K \subset \mathbb{R}$ compact and $T \in \mathscr{A}'(K)$, the Laplace transform $\mathscr{L}T$ of T is given by

$$\mathscr{L}T(\lambda) := T(e^{-\lambda\cdot}) = \frac{1}{2\pi i} \int_{\partial V} \mathscr{C}T(z) e^{-\lambda z}\, dz, \quad \lambda \in \mathbb{C}, \tag{1.3}$$

where V is an open and bounded complex neighborhood of K, bounded by finitely many piecewise C^1-curves. For $K = [a,b] \subset [0,\infty)$, the case we shall contrate on, we have, for each $\varepsilon > 0$, estimates

$$|\mathscr{L}T(\lambda)| \leq \begin{cases} C_\varepsilon \exp(-a\mathrm{Re}\,\lambda + \varepsilon|\lambda|) & , \quad \mathrm{Re}\,\lambda \geq 0 \\ C_\varepsilon \exp(-b\mathrm{Re}\,\lambda + \varepsilon|\lambda|) & , \quad \mathrm{Re}\,\lambda < 0 \end{cases} \tag{1.4}$$

which characterize Laplace transforms of elements in $\mathscr{A}'([a,b])$ (cf. [2, Sect. 9.1]). Another characterization can be found in [8].

The following is our first version of Post-Widder inversion.

Theorem 1.1. *Let $[a,b] \subset (0,\infty)$ be compact and $T \in \mathscr{A}'([a,b])$. Then for all $\theta \in (0,\pi/2)$ there is a constant C_θ such that, for all $n \in \mathbb{N}_0$,*

$$\sup_{\lambda>0} |\lambda^{n+1} \frac{(-1)^n}{n!} (\mathscr{L}T)^{(n)}(\lambda)| \leq \frac{C_\theta}{(\cos\theta)^{n+1}}. \tag{1.5}$$

For any $v, w \in \mathbb{C} \setminus [0,\infty)$ we have, as $n \to \infty$,

$$\int_0^\infty \Big(\frac{n}{t}\Big)^{n+1} \frac{(-1)^n}{n!} (\mathscr{L}T)^{(n)}\Big(\frac{n}{t}\Big) \Big(\frac{1}{w-t} - \frac{1}{v-t}\Big)\, dt \longrightarrow \mathscr{C}T(w) - \mathscr{C}T(v). \tag{1.6}$$

Let us recall (cf. [1]) that Post-Widder inversion asserts

$$\Big(\frac{n}{t}\Big)^{n+1}\frac{(-1)^n}{n!}(\mathscr{L}g)^{(n)}\Big(\frac{n}{t}\Big)\to g(t)\quad \text{as } n\to\infty, \tag{1.7}$$

if $g\in L^\infty(0,\infty)$ is continuous at $t>0$. Theorem 1.1 is proved via the following more general result.

Lemma 1.2. *Under the assumptions of Theorem* 1.1 *let* φ *be analytic and bounded on some sector* $\Sigma(\omega)$*, where* $\omega\in(0,\pi/2)$*, and such that*

$$r\mapsto\varphi(re^{i\theta})\in L^1[0,\infty),\quad |\theta|<\omega. \tag{1.8}$$

Then

$$\int_0^\infty \Big(\frac{n}{t}\Big)^{n+1}\frac{(-1)^n}{n!}(\mathscr{L}T)^{(n)}\Big(\frac{n}{t}\Big)\varphi(t)\,dt\longrightarrow T(\varphi)\quad \textit{as } n\to\infty. \tag{1.9}$$

Clearly, the assumption (1.8) holds for $\varphi=(w-\cdot)^{-1}-(v-\cdot)^{-1}$, $v,w\in\mathbb{C}\setminus[0,\infty)$. Hence it suffices to prove the lemma.

Proof. We write $F:=\mathscr{C}T$ and $G:=\mathscr{L}T$. Moreover, we let

$$G_n(\lambda):=\lambda^{n+1}\frac{(-1)^n}{n!}(\mathscr{L}T)^{(n)}(\lambda)\quad\text{for } \lambda>0 \text{ and } n\in\mathbb{N}_0.$$

First we deform the contour in the integral (1.3) to $\Gamma(\theta):=\partial\Sigma(\theta)$ where $\theta\in(0,\omega)$. Then we have, for any $n\in\mathbb{N}_0$ and $\lambda>0$,

$$G_n(\lambda)=\lambda^{n+1}\frac{(-1)^n}{n!}G^{(n)}(\lambda)=\frac{1}{2\pi i}\int_{\Gamma(\theta)}\lambda^{n+1}F(z)\frac{z^n}{n!}e^{-\lambda z}\,dz,$$

and for $\Gamma_\pm:=\{z\in\Gamma(\theta):\arg z=\pm\theta\}$ we obtain

$$\frac{1}{2\pi}\int_{\Gamma_\pm}\lambda^{n+1}|F(z)|\frac{|z|^n}{n!}e^{-\lambda Re\, z}\,|dz|\le\frac{1}{\pi}\|F\|_{\infty,\Gamma(\theta)}(\cos\theta)^{-(n+1)},$$

where we wrote $z=re^{\pm i\theta}$ and substituted $r=\frac{s}{\lambda\cos\theta}$.

This implies that the following integral is absolutely convergent and that we may apply Fubini to obtain

$$\begin{aligned}\int_0^\infty G_n\Big(\frac{n}{t}\Big)\,\varphi(t)\,dt&=\int_0^\infty\Big(\frac{n}{t}\Big)^{n+1}\frac{(-1)^n}{n!}G^{(n)}\Big(\frac{n}{t}\Big)\varphi(t)\,dt\\&=\frac{1}{2\pi i}\int_{\Gamma(\theta)}F(z)\int_0^\infty\Big(\frac{n}{t}\Big)^{n+1}\frac{z^n}{n!}e^{-nz/t}\varphi(t)\,dt\,dz\\&=\frac{1}{2\pi i}\int_{\Gamma(\theta)}F(z)\int_0^{e^{-i\arg z}\infty}\Big(\frac{n}{s}\Big)^{n+1}\frac{1}{n!}e^{-n/s}\varphi(sz)\,ds\,dz,\end{aligned}$$

where we used the substitution $t=sz$. Now we replace $\int_0^{e^{\mp i\theta}\infty}\dots ds$ by $\int_0^\infty\dots ds$ by Cauchy's theorem. Indeed, for $\gamma_R(u):=Re^{iu}$, $\pm u\in(0,\theta)$ and $n\in\mathbb{N}$, we have

$$\Big|\int_{\gamma_R}\dots du\Big|\le C_nR^{-n}\theta\|\varphi\|_{\infty,\Sigma(\theta)}\to 0\quad\text{as } R\to\infty.$$

Thus we are lead to

$$\int_0^\infty G_n\Big(\frac{n}{t}\Big)\,\varphi(t)\,dt = \frac{1}{2\pi i}\int_{\Gamma(\theta)} F(z)\int_0^\infty \Big(\frac{n}{s}\Big)^{n+1}\frac{1}{n!}e^{-n/s}\varphi(sz)\,ds\,dz,$$

and another Fubini argument and the substitution $s = 1/r$ yield

$$= \int_0^\infty \frac{n^{n+1}}{n!}\,r^n\,e^{-nr}\,\frac{1}{2\pi i}\int_{\Gamma(\theta)} F(z)\,\frac{1}{r}\varphi\Big(\frac{z}{r}\Big)\,dz\,dr. \tag{1.10}$$

Letting $g_\varphi(r) := \frac{1}{2\pi i}\int_{\Gamma(\theta)} F(z)\,\frac{1}{r}\varphi(\frac{z}{r})\,dz$ we have

$$|g_\varphi(r)| \le \frac{1}{2\pi}\|F\|_{\infty,\Gamma(\theta)}\|\varphi\|_{L^1(\Gamma(\theta))}, \quad r > 0,$$

i.e., $g_\varphi \in L^\infty(0,\infty)$, and the expression (1.10) equals

$$= n^{n+1}\frac{(-1)^n}{n!}(\mathscr{L}g_\varphi)^{(n)}(n).$$

Since it can easily be shown that g_φ is continuous at $t = 1$, the Post-Widder inversion formula (1.7) asserts convergence, as $n \to \infty$, to

$$g_\varphi(1) = \frac{1}{2\pi i}\int_{\Gamma(\theta)} F(z)\,\varphi(z)\,dz = T(\varphi).$$

This ends the proof. □

The problem with $a = 0$ in Theorem 1.1 is that (1.5) need not hold (cf. the estimates (1.4)). For $T \in \mathscr{A}'([0,b])$ and $\varepsilon > 0$, the translation $T_\varepsilon : \varphi \mapsto T_\varepsilon(\varphi) := T(\varphi(\varepsilon + \cdot)$, belongs to $\mathscr{A}'([\varepsilon, b+\varepsilon])$ and satisfies $\mathscr{L}T_\varepsilon(\lambda) = e^{-\varepsilon\lambda}\mathscr{L}T(\lambda)$ and $\mathscr{C}T_\varepsilon(z) = \mathscr{C}T(z-\varepsilon)$. Thus we have the following.

Corollary 1.3. *Let $T \in \mathscr{A}'([0,b])$. Then we have for any $\varepsilon > 0$ and $v, w \in \mathbb{C}\backslash[0,\infty)$, as $n \to \infty$,*

$$\int_0^\infty \Big(\frac{n}{t}\Big)^{n+1}\frac{(-1)^n}{n!}(e^{-\varepsilon\cdot}\mathscr{L}T)^{(n)}\Big(\frac{n}{t}\Big)\Big(\frac{1}{w-t} - \frac{1}{v-t}\Big)\,dt \longrightarrow \mathscr{C}T(w-\varepsilon) - \mathscr{C}T(v-\varepsilon).$$

2. Hyperfunctions on $[0,\infty)$

In this section we recall definitions and properties of hyperfunctions supported in $[0,\infty)$ and we sketch the possibility of "half-space theories" for their Laplace transforms. We also refer to [2, Sect. 9.2] and [7].

For any open subset $Q \subset \mathbb{R}$ the set $\mathscr{B}(Q)$ of hyperfunctions on U is defined as $\mathscr{O}(U \setminus Q)/\mathscr{O}(U)$ where U is an open complex neighborhood of Q such that $U \cap \mathbb{R} = Q$. The definition does not depend on U. It is clear that there are canonical restriction mappings $\mathscr{B}(Q) \to \mathscr{B}(Q')$ if $Q \supset Q'$. Actually, these restriction mappings are surjective and this fact is referred to as *the flabbiness of hyperfunctions.* Given an open subset $Q \subset \mathbb{R}$ and $T = [F] \in \mathscr{B}(Q)$ the *support* $\operatorname{supp} T$ of T is the complement (in Q) of all points $t \in Q$ such that F is analytic in a neighborhood of t.

For $a \in [0,\infty)$, the set $\mathscr{B}([a,\infty))$ is defined as the set of hyperfunctions with support in $[a,\infty)$, i.e., we have

$$\mathscr{B}([a,\infty)) \simeq \mathscr{O}(\mathbb{C}\setminus[a,\infty))/\mathscr{O}(\mathbb{C}) \simeq \mathscr{O}(U\setminus[a,\infty))/\mathscr{O}(U), \tag{2.1}$$

where U is an arbitrary open complex neighborhood of $[a,\infty)$.

The idea behind Komatsu's Laplace transform of hyperfunctions is to replace arbitrary analytic functions in this representation by analytic functions that are exponentially bounded in a suitable sense. The key observation is that coclasses in (2.1) contain such exponentially bounded representatives. The argument from [4] actually yields representatives which are bounded in a certain sense.

We introduce the following notation: For any $a \in [0,\infty)$ we let $\mathscr{O}_a^b$ denote the set of functions that are analytic on $\mathbb{C}\setminus[a,\infty)$ and bounded outside any set $\Sigma(\theta)_\varepsilon$ for $\theta\in(0,\pi/2)$ and $\varepsilon>0$. We also let $\mathscr{O}_\infty^b := \bigcap_{a>0}\mathscr{O}_a^b$ denote the set of all entire functions that are bounded on any set $\mathbb{C}\setminus(a+\Sigma(\theta))$ where $a>0$, $\theta\in(0,\pi/2)$.

Theorem 2.1. *Let $T\in\mathscr{B}([0,\infty))$, $\varepsilon>0$ and U be a complex neighborhood of $[0,\infty)$. Then there exists a representing function $F\in\mathscr{O}(\mathbb{C}\setminus[0,\infty))$ such that $|F(z)|\le\varepsilon$ for $z\in\mathbb{C}\setminus U$. In particular we have $\mathscr{B}([0,\infty))\simeq\mathscr{O}_0^b/\mathscr{O}_\infty^b$.*

We start with the following decomposition result.

Lemma 2.2. *Any $T\in\mathscr{B}([0,\infty))$ can be written as $T=\sum_{j=1}^\infty T_j$ where, for each $j\in\mathbb{N}$, $T_j\in\mathscr{A}'([j-1,j])$, and the sum is understood locally, i.e., restricted to bounded open intervals where it is actually a finite sum.*

Proof. The lemma follows if, for $b>a\ge0$, we can write $S\in\mathscr{B}([a,\infty))$ as $S=S_0+S_1$ with $S_0\in\mathscr{A}'([a,b])$ and $S_1\in\mathscr{B}([b,\infty))$. To this end we define $\widetilde{S}_0\in\mathscr{B}(\mathbb{R}\setminus\{b\})$ to be equal to S on $(-\infty,b)$ and equal to 0 on (b,∞). Then we extend $\widetilde{S}_0$ to S_0 by flabbiness and let $S_1:=S-S_0$. □

Proof of Theorem 2.1. We find a sequence $(\alpha(j))_{j\in\mathbb{N}}$ with $\bigcup_j[j-1,j]_{\alpha(j)}\subset U$. Using Lemma 2.2 we write $T=\sum_j T_j$. Now we replace subsequently the summands T_j by summands $\widetilde{T}_j\in\mathscr{A}'([j-1,j])$ whose Cauchy transforms $\widetilde{F}_j$ are bounded by $2^{-j}\varepsilon$ outside $[j-1,j]_{\alpha(j)}$. Then the series $F(z):=\sum_{j=1}^\infty\widetilde{F}_j(z)$ converges locally uniformly for $z\in\mathbb{C}\setminus[0,\infty)$, and F represents T. We start with T_1 and approximate $F_1:=\mathscr{C}T_1$ outside $[0,1]_{\alpha(1)}$ with a polynomial P_1 (without constant term) in $(z-1)^{-1}$ such that $|F_1(z)-P_1((z-1)^{-1}|\le\varepsilon/2$ for $z\in\mathbb{C}\setminus[0,1]_{\alpha(1)}$. This is possible by Runge's theorem. The function $z\mapsto P_1((z-1)^{-1})$ is the Cauchy transform of a Schwartz distribution S_1 with support in $\{1\}$, and we let $\widetilde{T}_1:=T_1-S_1$. Observe that $\widetilde{T}_1=T_1$ on $[0,1)$. In the next step we let $F_2:=\mathscr{C}(T_2+S_1)$ and find a polynomial P_2 such that $|F_2(z)-P_2((z-2)^{-1})|\le 2^{-2}\varepsilon$ for $z\in\mathbb{C}\setminus[1,2]_{\alpha(2)}$. Again, $z\mapsto P_2((z-2)^{-2})$ is the Cauchy transform of a Schwartz distribution S_2 with support in $\{2\}$, and we let $\widetilde{T}_2:=T_2+S_1-S_2$. Observe that $\widetilde{T}_1+\widetilde{T}_2=T_1+T_2$ on $[0,2)$. We apply this procedure successively to any index $j>2$, and this proves the first assertion of the theorem.

We find representatives in $\mathscr{O}_0^b$ if $U:=\bigcup_j[j-1,j]_{2^{-j}}$. □

For any $F \in \mathscr{O}_0^b$ we denote by $[F]$ the induced hyperfunction in $\mathscr{B}([0,\infty))$.

Laplace transform. For any $F \in \mathscr{O}_0^b$ we define the Laplace transform $\mathscr{L}F$ by

$$\mathscr{L}F(\lambda) = \frac{1}{2\pi i}\int_{\partial(\Sigma(\theta)\cup B(0,\varepsilon))} F(z)e^{-\lambda z}\,dz \quad \text{for } \lambda \in \mathbb{C}_+. \tag{2.2}$$

This definition is independent of $\varepsilon > 0$ and $\theta \in (0,\pi/2)$ if $\operatorname{Re}\lambda > \tan\theta|\operatorname{Im}\lambda|$. Observe also that, for a hyperfunction $T \in \mathscr{A}'([a,b])$ with compact $[a,b] \subset [0,\infty)$, the Laplace transform $\mathscr{L}T$ in the sense of Section 1 coincides with $\mathscr{L}(\mathscr{C}T)$ (by an argument similar to the one used in the proof of Lemma 1.2). If $F \in \mathscr{O}_a^b$ for some $a > 0$ we may take $\varepsilon = 0$. For $\lambda \in \mathbb{C}_+$ and $\theta \in (0,\pi/2)$ with $\operatorname{Re}\lambda > \tan\theta|\operatorname{Im}\lambda|$ we then easily see

$$|\mathscr{L}F(\lambda)| \le C_\theta(\operatorname{Re}\lambda - \tan\theta|\operatorname{Im}\lambda|)^{-1}.$$

For $T = [F]$, the function $\mathscr{L}F$ is a Laplace transform of T in the sense of Komatsu ([5]) and it is an asymptotic Laplace transform of T in the sense of Lumer/Neubrander ([10]). Observe that $\mathscr{L}F$ is analytic on $\mathbb{C}_+$.

Remark 2.3. Of course, $\mathscr{O}_0^b$ is too small to be closed under multiplication with polynomials. This can be remedied by considering $\mathscr{O}_0^p := \{p\cdot F : F \in \mathscr{O}_0^b, p \text{ polynomial}\}$ instead. Observe that (2.2) makes sense for $F \in \mathscr{O}_0^p$ and that $\mathscr{L}F$ is analytic on $\mathbb{C}_+$ for such F. Moreover, one has $\mathscr{L}(-zF)(\lambda) = (\mathscr{L}F)'(\lambda)$ for $F \in \mathscr{O}_0^p$ (cf. [10] for a discussion of this property). This would yield a Laplace transform theory similar to the one for tempered Schwartz distributions with support in $[0,\infty)$.

If one is also interested in having $\mathscr{L}(e^{az}F)(\lambda) = \mathscr{L}F(\lambda - a)$ for $a > 0$ then one should consider $\mathscr{O}_0^e := \bigcup_{a>0} e^{a(\cdot)}\cdot\mathscr{O}_0^p$. For each $F \in \mathscr{O}_0^e$ the Laplace transform $\mathscr{L}F$ exists on a half-plane $\{\operatorname{Re}\lambda > a\}$ for some $a > 0$. This would yield a Laplace transform theory that parallels the one for exponentially bounded distributions with support in $[0,\infty)$.

A reason for requiring "post-sectors" in [3, 4, 5, 9, 10] (see [9] for the definition) and not half-planes as domains of analyticity is, of course, that otherwise Ouchi's characterization for the existence of hyperfunction fundamental solutions to abstract Cauchy problems (cf. [5]) could not be derived *within the theory.* Actually, this already happens for distributional fundamental solutions (cf. Chazarain's characterization quoted in [5, 6] which involves "logarithmic regions"). This may be a hint that hyperfunction fundamental solutions have a bit more structure than general hyperfunctions (cf. [6] where it is shown that this is the case for distributional fundamental solutions compared to general distributions).

The question we evoke here is the "consistency" of Laplace transforms $\mathscr{L}F$ with Laplace transforms of $T = [F]$ obtained by other means, e.g., by the classical Laplace transform for exponentially bounded functions on $[0,\infty)$ or by the method in [6] for distributions in $\mathcal{W}'$ (cf. [6] for the definition). In any case, Laplace transforming hyperfunctions requires to consider them as linear functionals (cf., e.g., Lemma 3.2 below) which in turn seems to be impossible without having to deal with coclasses, in the sense that a single hyperfunction gives rise to a bunch of functionals.

3. Post-Widder inversion for general hyperfunctions

In this section we comment on Post-Widder inversion for general hyperfunctions $T \in \mathscr{B}([a,\infty))$ or rather, on Post-Widder inversion for their representations $F \in \mathscr{O}^b_a$. Since the case $a = 0$ may again be treated by translation (cf. Corollary 1.3) we study only the case $a > 0$.

Theorem 3.1. *Let $a > 0$ and $T = [F] \in \mathscr{B}([a,\infty))$ where $F \in \mathscr{O}^b_a$. Then for all $\theta \in (0, \pi/2)$ there is a constant C_θ such that, for all $n \in \mathbb{N}_0$,*

$$\sup_{\lambda>0} |\lambda^{n+1} \frac{(-1)^n}{n!} (\mathscr{L}F)^{(n)}(\lambda)| \le \frac{C_\theta}{(\cos\theta)^{n+1}}. \tag{3.1}$$

For any $v, w \in \mathbb{C} \setminus [0,\infty)$ we have, as $n \to \infty$,

$$\int_0^\infty \Big(\frac{n}{t}\Big)^{n+1} \frac{(-1)^n}{n!} (\mathscr{L}F)^{(n)}\Big(\frac{n}{t}\Big)\Big(\frac{1}{w-t} - \frac{1}{v-t}\Big)\, dt \longrightarrow F(w) - F(v). \tag{3.2}$$

In particular, we can use (3.2) *to reconstruct T from $\mathscr{L}F$.*

The proof of Lemma 1.2 also proves the following.

Lemma 3.2. *Let $a > 0$ and $F \in \mathscr{O}^b_a$. Then*

$$S(\varphi) := \frac{1}{2\pi i} \int_{\Gamma(\theta_0)} F(z)\, \varphi(z)\, dz \tag{3.3}$$

is well defined for any bounded $\varphi \in \mathscr{O}(\Sigma(\omega))$ with $\omega \in (0, \pi/2)$ satisfying (1.8) *and independent of $\theta_0 \in (0, \omega)$. For such φ we have*

$$\int_0^\infty \Big(\frac{n}{t}\Big)^{n+1} \frac{(-1)^n}{n!} (\mathscr{L}F)^{(n)}\Big(\frac{n}{t}\Big)\varphi(t)\, dt \longrightarrow S(\varphi) \quad \text{as } n \to \infty. \tag{3.4}$$

Proof of Theorem 3.1. Again, the function $\varphi = (w - \cdot)^{-1} - (v - \cdot)^{-1}$ satisfies the assumptions of Lemma 3.2 and it rests to prove $S(\varphi) = F(w) - F(v)$. We consider (3.3) for $0 < \theta_0 < \min\{|\arg w|, |\arg v|\}$ and, for $R > \max\{|v|, |w|\}$ large, the contour γ_R given by $\gamma_R(\sigma) := Re^{-i\sigma}$, $\sigma \in [\theta, 2\pi - \theta]$. We observe $\sup_{z\in\gamma_R} |\varphi(z)| = O(R^{-2})$ as $R \to \infty$. Hence

$$\int_{\gamma_R} |F(z)||\varphi(z)|\, dz = O(R \cdot R^{-2}) = O(R^{-1}) \quad \text{as } R \to \infty.$$

Denoting $\Gamma_R := \{z \in \Gamma(\theta_0) : |z| \le R\}$, oriented in the same way as $\Gamma(\theta_0)$, we have by Cauchy's theorem

$$F(w) - F(v) = \frac{1}{2\pi i} \int_{\Gamma_R - \gamma_R} F(z)\varphi(z)\, dz \longrightarrow S(\varphi) \quad \text{as } R \to \infty,$$

which ends the proof. □

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On a Class of Elliptic Operators with Unbounded Time- and Space-dependent Coefficients in $\mathbb{R}^N$

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In memory of Günter Lumer

Abstract. We prove optimal Schauder estimates for classical solutions of the nonhomogeneous Cauchy problem associated with a class of elliptic operators with unbounded coefficients depending both on time and space variables. We deal both with the case when the coefficients of the elliptic operator are continuous and the case when they are merely measurable in the pair (t, x). In both the cases we assume that they are Hölder continuous in x, uniformly with respect to t.

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1. Introduction

In the last years the interest towards elliptic operators $\mathcal{L}$ with unbounded coefficients has grown considerably due to their numerous applications in many fields of sciences (mainly mathematical finance). Most of the literature is concerned with the autonomous case in which the coefficients of the operator $\mathcal{L}$ depend only on space variables. The study of such operators goes back to the pioneering papers by Azencott and Itô (see [1, 9] and also [21]) who proved that, under very weak assumptions on the smoothness of the coefficients and assuming growth conditions

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only on the potential term (i.e., it should be bounded from above), the homogeneous Cauchy problem

$$\begin{cases} D_t u(t,x) = \mathcal{L}u(t,x), & t > 0, \quad x \in \mathbb{R}^N, \\ u(0,x) = f(x), & \qquad\quad x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

admits, for any bounded and continuous function f, (at least) one classical solution (i.e., a function u which is (i) bounded and continuous in $[0,T]\times\mathbb{R}^N$, for any $T > 0$, (ii) once continuously differentiable with respect to time and twice continuously differentiable with respect to space variables in $\mathbb{R}_+ \times \mathbb{R}^N$, (iii) solves the Cauchy problem (1.1)).

The most famous example of an elliptic operator with unbounded coefficients is the Ornstein-Uhlenbeck operator, given by

$$\mathcal{L}u(x) = \mathrm{Tr}(QD^2u(x)) + \langle Ax, Du(x)\rangle, \qquad x \in \mathbb{R}^N, \tag{1.2}$$

where Q and A are $N \times N$ matrices with Q (strictly) positive definite (see, e.g., [5, 6, 8, 13, 17, 20, 22, 23] and the monograph [4]) In such a situation, a representation formula for the (unique) classical solution to the Cauchy problem (1.1) is available. More precisely,

$$u(t,x) := (T(t)f)(x) = \frac{1}{(4\pi)^{N/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle Q_t^{-1}y,y\rangle} f(y - e^{tA}x)dy,$$

for any $t > 0$ and any $x \in \mathbb{R}^N$, where

$$Q_t = \int_0^t e^{sA}Qe^{sA^*}ds, \qquad t > 0.$$

The family $\{T(t)\}$ defines a semigroup of bounded operators in $C_b(\mathbb{R}^N)$ (the space of bounded and continuous functions in $\mathbb{R}^N$). Such a semigroup is neither strongly continuous nor analytic in $C_b(\mathbb{R}^N)$ and in $BUC(\mathbb{R}^N)$ (the subspace of $C_b(\mathbb{R}^N)$ of all uniformly continuous functions). Nonetheless it exhibits smoothing properties that are typical of semigroups associated with elliptic operators with bounded coefficients. For instance, for any $f \in C_b(\mathbb{R}^N)$ and any $t > 0$, the function $T(t)f$ belongs to $C^\infty(\mathbb{R}^N)$ and all its derivatives are bounded. Moreover, for any $h, k > 0$, with $0 \le h \le k$, and any $f \in C_b^h(\mathbb{R}^N)$ there exists a positive constant $C_{h,k}$ such that

$$\|T(t)f\|_{C_b^k(\mathbb{R}^N)} \le C_{h,k}t^{-\frac{k-h}{2}}\|f\|_{C_b^h(\mathbb{R}^N)}, \qquad t \in]0,1]. \tag{1.3}$$

Such estimates have been used as the keystone to prove optimal Schauder estimates for the classical solution to the nonhomogeneous Cauchy problem

$$\begin{cases} D_t u(t,x) = \mathcal{L}u(t,x) + g(t,x), & t \in [0,T], \quad x \in \mathbb{R}^N, \\ u(0,x) = f(x), & \qquad\qquad x \in \mathbb{R}^N, \end{cases} \tag{1.4}$$

when f, g are bounded and smooth enough functions. To be more precise, it is well known that, if $f \in C_b^{2+\theta}(\mathbb{R}^N)$ (for some $\theta \in]0,1[$) and $g \in C_b([0,T] \times \mathbb{R}^N)$ is such that $g(t,\cdot)$ is in $C_b^\theta(\mathbb{R}^N)$ for any $t \in [0,T]$, with $\sup_{t\in[0,T]} \|g(t,\cdot)\|_{C_b^\theta(\mathbb{R}^N)} < +\infty$, then problem (1.4) admits a unique classical solution u, Moreover, $u(t,\cdot) \in$

$C_b^{2+\theta}(\mathbb{R}^N)$ for any $t \in [0,T]$ and the sup of the $C_b^{2+\theta}$-norms (when t runs in $[0,T]$) can be estimated in terms of the data. We refer the reader to [6] for the proof of the previous results.

Uniform estimates similar to (1.3) and, consequently, optimal Schauder estimates for the solutions of the Cauchy problem (1.4) have been recently proved in [3, 18] for a rather general class of elliptic operators with unbounded x-dependent coefficients. In such a situation, a different approach based on the classical Bernstein method (see [2]) has been applied, since no representation formulas for the solution to (1.1) are available.

Aim of this paper is to extend these results to a class of elliptic operators with unbounded coefficients, depending also on time. More precisely, we consider the following class of elliptic operators

$$\mathcal{L}u(t,x) = \mathrm{Tr}(Q(t,x)D^2u(t,x)) + \langle A(t)x + b(t,x), Du(t,x)\rangle, \qquad t>0,\ \ x \in \mathbb{R}^N, \tag{1.5}$$

under suitable assumptions on the function matrices Q and A and on the vector function b, that will be made clear in Section 2. Note, that in the case when Q and A are constant and $b \equiv 0$, the operator $\mathcal{L}$ reduces to the Ornstein-Uhlenbeck operator in (1.2), whereas in the case when the coefficients Q and b of $\mathcal{L}$ in (1.5) are independent of x and periodic in t, the operator $\mathcal{L}$ has been extensively studied in [7], where the authors proved that one can associate a backward evolution family $\{P(t,s)\}$ with the Cauchy problem

$$\begin{cases} D_s u(s,x) + \mathcal{L}u(s,x) = 0, & s \in]-\infty, t[, \\ u(t,t) = f \in C_b(\mathbb{R}^N). \end{cases}$$

Moreover, they studied the asymptotic behaviour of the function $P(t,s)f$ both when s tends to $-\infty$ (and t is fixed) and when t tends to $+\infty$ (and s is fixed).

The main results we prove in this paper are collected in the following two theorems. The first one deals with the case when the coefficients of the operator $\mathcal{L}$ are continuous in the pair (t,x).

Theorem 1.1. *Let Hypotheses* 2.1 *in Section* 2 *be satisfied and fix* $\theta \in]0,1[$. *Further, suppose that* $f \in C_b^{2+\theta}(\mathbb{R}^N)$ *and* $g \in C_b([0,T]\times\mathbb{R}^N)$ *is such that* $g(t,\cdot) \in C_b^\theta(\mathbb{R}^N)$ *for any* $t \in [0,T]$ *and*

$$\sup_{t\in[0,T]} \|g(t,\cdot)\|_{C_b^\theta(\mathbb{R}^N)} < +\infty.$$

Then, the Cauchy problem (1.4) *admits a unique classical solution* u. *For any* $t>0$, *the function* $u(t,\cdot)$ *belongs to* $C_b^{2+\theta}(\mathbb{R}^N)$ *and there exists a positive constant* C, *depending only on the ellipticity constant and the sup-norms of the coefficients of the operator* $\mathcal{L}$, *such that*

$$\sup_{t\in[0,T]} \|u(t,\cdot)\|_{C_b^{2+\theta}(\mathbb{R}^N)} \le C\bigg(\|f\|_{C_b^{2+\theta}(\mathbb{R}^N)} + \sup_{t\in[0,T]} \|g(t,\cdot)\|_{C_b^\theta(\mathbb{R}^N)}\bigg). \tag{1.6}$$

Our results extend [10, 11, 12] where similar results have been proved for elliptic operators with bounded coefficients.

The latter result that we prove is concerned with the case when the coefficients of the operator $\mathcal{L}$ are only measurable with respect to the pair (t,x) and they are Hölder continuous with respect to the space variables. Before stating it, we give the definition of solution to problem (1.4) adapted to our situation.

Definition 1.2. Suppose that the coefficients of the operator $\mathcal{L}$ are everywhere defined, bounded and measurable in $[0,T]\times\mathbb{R}^N$ for some $T>0$. Further, let $f\in C^2_b(\mathbb{R}^N)$ and $g\in C_b([0,T]\times\mathbb{R}^N)$. A function $u:[0,T]\times\mathbb{R}^N\to\mathbb{R}$ is called a solution to (1.4) if the following conditions are satisfied:

(i) the function u is Lipschitz continuous in $[0,T]\times\overline{B}(R)$ for any $R>0$, its first- and second-order space derivatives are bounded and continuous functions in $[0,T]\times\mathbb{R}^N$;
(ii) $u(0,x)=f(x)$ for any $x\in\mathbb{R}^N$;
(iii) there exists a set $F\subset[0,T]\times\mathbb{R}^N$, with negligible complement, such that $D_tu(t,x)=\mathcal{L}u(t,x)+g(t,x)$ for any $(t,x)\in F$. Moreover, for any $x\in\mathbb{R}^N$, the set $F(x)=\{t\in[0,T]:(t,x)\in F\}$ is measurable with measure T.

Theorem 1.3. *Let Hypotheses* 2.2 *in Section* 2 *be satisfied and fix* $\theta\in]0,1[$. *Further, suppose that* $f\in C^{2+\theta}_b(\mathbb{R}^N)$ *and* g *is a bounded and measurable function, everywhere defined in* $[0,T]\times\mathbb{R}^N$, *such that* $g(t,\cdot)\in C^\theta_b(\mathbb{R}^N)$ *for any* $t\in[0,T]$ *and*

$$\sup_{t\in[0,T]}\|g(t,\cdot)\|_{C^\theta_b(\mathbb{R}^N)}<+\infty.$$

Then, the Cauchy problem (1.4) *admits a unique solution* u *according to Definition* 1.2. *For any* $t>0$, *the function* $u(t,\cdot)$ *belongs to* $C^{2+\theta}_b(\mathbb{R}^N)$ *and there exists a positive constant* C *such that* (1.6) *holds true.*

Theorem 1.3 extends to the case of unbounded coefficients the results in [14]. We quote the monograph [19] for related results for elliptic operators with discontinuous coefficients.

We stress that the proof of this theorem essentially relies on the maximum principle in Proposition 2.3 and the results in Theorem 1.1, which allow us to apply a compactness argument to prove the existence and the uniqueness of the solution of the Cauchy problem (1.4) in the sense of Definition 1.2. Note that the techniques in Section 5 apply also to the situation considered in [14] and allow to obtain the same results in a simpler and elegant way from those in [10, 11, 12].

The paper is structured as follows. First, in Section 2, we state the main assumptions on the coefficients that we need in this paper and prove a maximum principle. Next, in Sections 3 and 4, we consider the case when the coefficients of $\mathcal{L}$ are continuous with respect to (t,x). First, in Section 3, we prove Theorem 1.1 in the case when the coefficients Q and b are independent of x. For this purpose, we prove that we can associate an evolution family $\{P(t,s)\}$ with the operator $\mathcal{L}$. We also estimate the sup-norm of the space derivatives (up to the third-order) of the function $P(t,s)f$ when f belongs to several spaces of (Hölder) continuous functions. These estimates will provide us with a fundamental tool to prove

Theorem 1.1 via an interpolation argument. Then, in Section 4, we consider the operator $\mathcal{L}$ in its general form. Using the classical method of continuity, we prove Theorem 1.1 in its generality. In Section 5, we turn our attention to the case when the coefficients of the operator $\mathcal{L}$ are measurable with respect to (t, x) and satisfy Hypotheses 2.2. Using an approximation argument and Theorem 1.1, we prove Theorem 1.3.

Notations

Given an open set $\Omega \subset \mathbb{R}^N$, a smooth function $u : \Omega \to \mathbb{R}$ and a multi-index $\alpha = (\alpha_1, \ldots, \alpha_N)$, we denote by $D^\alpha u$ the derivative $\frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1}, \ldots \partial x_N^{\alpha_N}}$, where $|\alpha|$ denotes the length of the vector α. When $k = 1, 2$ we use the notation $D_i u$ and $D_{ij} u$ to denote the derivatives $\frac{\partial u}{\partial x_i}$ and $\frac{\partial^2 u}{\partial x_i x_j}$, respectively. For a general $k \in \mathbb{N}$ and $x \in \Omega$, we denote by $D^k u(x)$ the vector of all the kth-order derivatives of u at x. Moreover, we set $\|D^k u\|_\infty = \sup_{x\in\Omega} |D^\alpha u(x)|$, where $|D^\alpha u(x)|$ denotes the Euclidean norm of the vector $D^\alpha u(x)$.

By $C_b(\mathbb{R}^N)$ we denote the space of all bounded and continuous functions $u : \mathbb{R}^N \to \mathbb{R}$, and we endow it with the sup-norm. For any $k > 0$, we denote by $C_b^k(\mathbb{R}^N)$ the subspace of $C_b(\mathbb{R}^N)$ of functions u which are continuously differentiable in $\mathbb{R}^N$ up to the $[k]$th-order with all the derivatives which are bounded and, those of maximal order, $(k - [k])$-Hölder continuous in $\mathbb{R}^N$. Here, $[k]$ denotes the integer part of k. We endow the space $C_b^k(\mathbb{R}^N)$ with the norm

$$\|u\|_{C_b^k(\mathbb{R}^N)} = \sum_{|\alpha|\le[k]} \|D^\alpha u\|_{C_b(\mathbb{R}^N)} + \sum_{|\alpha|=[k]} [D^\alpha u]_{C_b^{k-[k]}(\mathbb{R}^N)},$$

where $[\,\cdot\,]_{C_b^{k-[k]}(\mathbb{R}^N)}$ denotes the $(k - [k])$-Hölder seminorm in $\mathbb{R}^N$.

For any $\theta \in]0, 3[$ and any $T > 0$, we denote by $B^{0,\theta}([0, T]\times\mathbb{R}^N)$ the space of all bounded and measurable functions $f : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ such that $f(t, \cdot) \in C_b^\theta(\mathbb{R}^N)$ for any $t \in [0, T]$ and the sup of the C_b^θ-norms of $f(t, \cdot)$, when t runs in $[0, T]$, is finite. We norm $B^{0,\theta}([0, T] \times \mathbb{R}^N)$ by setting

$$\|f\|_{B^{0,\theta}([0,T]\times\mathbb{R}^N)} := \sup_{t\in[0,T]} \|f(t, \cdot)\|_{C_b^\theta(\mathbb{R}^N)}.$$

Similarly, by $C^{0,\theta}([0, T]\times\mathbb{R}^N)$ we denote the subset of $B^{0,\theta}([0, T]\times\mathbb{R}^N)$ of functions f which are continuous in $[0, T] \times \mathbb{R}^N$ together with their space derivatives up to the $[\theta]$-order. We endow $C^{0,\theta}([0, T] \times \mathbb{R}^N)$ with the norm of $B^{0,\theta}([0, T] \times \mathbb{R}^N)$. Further, by $C_b([0, T] \times \mathbb{R}^N)$ we denote the set of all bounded and continuous functions $f : [0, T] \times \mathbb{R}^N \to \mathbb{R}$, endowed with the sup-norm. Finally, if $a, b \in \mathbb{R}$ and $a < b$, we denote by $\mathrm{Lip}([a, b] \times \mathbb{R}^N)$ the set of all functions $f : [a, b] \times \mathbb{R}^N \to \mathbb{R}$ which are Lipschitz continuous in $[a, b] \times \mathbb{R}^N$, but not therein necessarily bounded.

By $B(R)$ we denote the open ball in $\mathbb{R}^N$ with centre at the origin and radius R, and by $\overline{B}(R)$ its closure. If A is a measurable set in $\mathbb{R}^N$, we denote by χ_A the characteristic function of the set A. Given two Banach spaces X and Y, we denote by $L(X,Y)$ the set of all bounded linear operators from X to Y. When $X=Y$, we simply write $L(X)$.

For any $N\times N$ matrix $Q=(q_{ij})$, we denote, respectively, by Q^* and $\mathrm{Tr}(Q)$ the transpose matrix and the trace of Q. Moreover, we denote by $\|Q\|$ its Euclidean norm, i.e., $\|Q\|^2=\sum_{i,j=1}^N |q_{ij}|^2$, and, when the entries of Q depend on y in some set F and are bounded, we set $\|Q\|_\infty=\sup_{y\in F}\|Q(y)\|$.

Finally, by $\langle x,y\rangle$ we denote the Euclidean inner product of the vectors $x,y\in\mathbb{R}^N$.

2. Main assumptions and preliminaries

In this section we list the assumptions on the coefficients of the operator $\mathcal{L}$ in (1.5) and we prove a maximum principle that we need in what follows.

2.1. Hypotheses

Throughout this paper we assume that either of Hypotheses 2.1 or 2.2 are satisfied.

Hypotheses 2.1.

(i) $q_{ij}=q_{ji}\in C^{0,\theta}([0,T]\times\mathbb{R}^N)$ for some $\theta\in]0,1[$ and any $i,j=1,\dots,N$, and

$$\sum_{i,j=1}^N q_{ij}(t,x)\xi_i\xi_j\ge\nu|\xi|^2,\qquad t\in[0,T],\ \ x,\xi\in\mathbb{R}^N,\tag{2.1}$$

for some positive constant ν;

(ii) $A=(a_{ij})$ with $a_{ij}\in C([0,T])$ for any $i,j=1,\dots,N$;

(iii) $b=(b_j)$ with $b_j\in C^{0,\theta}([0,T]\times\mathbb{R}^N)$ for any $j=1,\dots,N$.

Hypotheses 2.2.

(i) $q_{ij}=q_{ji}\in B^{0,\theta}([0,T]\times\mathbb{R}^N)$ for some $\theta\in]0,1[$ and any $i,j=1,\dots,N$, and condition (2.1) is satisfied by any $t\in\mathcal{D}$ and any $x,\xi\in\mathbb{R}^N$, where $[0,T]\setminus\mathcal{D}$ is a negligible set;

(ii) $A=(a_{ij})$ with a_{ij} everywhere defined, bounded and measurable in $[0,T]$ for any $i,j=1,\dots,N$;

(iii) $b=(b_j)$ with $b_j\in B^{0,\theta}([0,T]\times\mathbb{R}^N)$ for any $j=1,\dots,N$.

2.2. A maximum principle

Here, we state a maximum principle for elliptic operators with unbounded coefficients that we need in what follows. Its proof can be found, e.g., in [18] in the case of elliptic operators with continuous coefficients independent of t. Since we need to extend it to the case when the coefficients are t-dependent and satisfy Hypotheses 2.2, we go into details.

Proposition 2.3. *Let $\mathcal{L}$ be the elliptic operator defined by*

$$\mathcal{L}u(t,x)=\sum_{i,j=1}^N q_{ij}(t,x)D_{ij}u(t,x)+\sum_{j=1}^N b_j(t,x)D_ju(t,x),$$

where the coefficients q_{ij} *and* b_j *are measurable, bounded and everywhere defined in* $]0,T[\times\mathbb{R}^N$ $(i,j=1,\dots,N)$*, and condition* (2.1) *is satisfied. Further, assume that there exists a smooth* (*Lyapunov*) *function* $\varphi:[0,T]\times\mathbb{R}^N\to\mathbb{R}$ *such that*

$$\lim_{|x|\to+\infty}\inf_{t\in[0,T]}\varphi(t,x)=+\infty \tag{2.2}$$

and

$$\sum_{i,j=1}^N q_{ij}(t,x)D_{ij}\varphi(t,x)+\sum_{j=1}^N b_j(t,x)D_j\varphi(t,x)-D_t\varphi(t,x)-\lambda\varphi(t,x)\le C, \tag{2.3}$$

for any $t\in[0,T]$*, any* $x\in\mathbb{R}^N$ *and some positive constants* λ *and* C*. Let* $u:[0,T]\times\mathbb{R}^N\to\mathbb{R}$ *be a bounded and continuous function such that*

(i) *u admits continuous first- and second-order space derivatives in* $]0,T]\times\mathbb{R}^N$*;*
(ii) $u\in\mathrm{Lip}(K)$ *for any compact set* $K\subset]0,T]\times\mathbb{R}^N$ *and is differentiable with respect to time in* $\mathcal{C}\times\mathbb{R}^N$*, where* $\mathcal{C}\subset]0,T]$ *has a negligible complement;*
(iii) $D_tu=\mathcal{L}u+g$ *in* $\mathcal{C}\times\mathbb{R}^N$ *and* $u(0,\cdot)=f$*, where* $f\in C_b(\mathbb{R}^N)$ *and* g *is everywhere defined, bounded and measurable in* $]0,T]\times\mathbb{R}^N$*.*

If $g\le 0$ *then* $u\le\sup f^+$*, where* $f^+(x)=\max\{f(x),0\}$*. Similarly, if* $g\ge 0$*, then* $u\ge\inf f^-$*, where* $f^-(x)=\min\{f(x),0\}$*. In particular, the Cauchy problem* (1.4) *admits at most one solution in the sense of Definition* 1.2*.*

Proof. Without loss of generality we can limit ourselves to proving the first statement of the proof. Indeed, the second one will follow easily by replacing u with $-u$. Similarly, we can assume that $C\le 0$ in (2.3) and φ is everywhere positive in $[0,T]\times\mathbb{R}^N$: it suffices to replace φ with $\varphi+M$ for a suitable positive constant M.

Let u be a solution to problem (1.4) and, for any $n\in\mathbb{N}$, let us introduce the function $v_n:[0,T]\times\mathbb{R}^N\to\mathbb{R}$ defined by

$$v_n(t,x)=e^{-\lambda t}u(t,x)-\frac{1}{n}\varphi(t,x),\qquad (t,x)\in[0,T]\times\mathbb{R}^N.$$

By (2.2),

$$\lim_{|x|\to+\infty}\inf_{t\in[0,T]}v_n(t,x)=-\infty. \tag{2.4}$$

Moreover, the function v_n is a solution to the problem

$$\begin{cases} D_tv_n(t,x)-\mathcal{L}v_n(t,x)+\lambda v_n(t,x)\le e^{\lambda t}g(t,x), & t\in\mathcal{C},\quad x\in\mathbb{R}^N,\\ v_n(0,x)\le f(x), & x\in\mathbb{R}^N.\end{cases} \tag{2.5}$$

Since $v_n\in W^{1,\infty}_{\rm loc}(]0,T[\times B(R))$ for any $R>0$, from property (i) in the statement of the proposition, v_n belongs to the parabolic Sobolev space $W^{1,2}_{N+1}(]0,T[\times B(R))$ for any $R>0$, and it is a solution to (2.5) in the sense of distributions. From the Nazarov-Ural'tseva maximum principle (see [24, Theorem 1]) we deduce that

$$v_n(t,x)\le\sup_{x\in\overline{B}(R)}f^+(x)+\sup_{(t,x)\in]0,T[\times\partial B(R)}v_n^+(t,x),$$

for any $(t,x) \in [0,T] \times \overline{B}(R)$ and any $R > 0$. Thanks to (2.4), we can determine a positive number R_n such that $\sup_{(t,x)\in]0,T[\times \partial B(R)} v_n^+(t,x) = 0$, for any $R \geq R_n$. It follows that

$$v_n(t,x) \leq \sup_{x\in\mathbb{R}^N} f^+(x), \qquad (t,x) \in [0,T] \times \mathbb{R}^N.$$

Now, the assertion follows letting n go to $+\infty$. □

3. The case of continuous coefficients independent of the space variables

In this section we consider the case when the coefficients Q and b of the operator $\mathcal{L}$ are independent of x. Therefore, here $\mathcal{L}$ is given (on smooth functions u) by

$$\mathcal{L}u(t,x) = \sum_{i,j=1}^{N} q_{ij}(t) D_{ij} u(t,x) + \sum_{i,j=1}^{N} (a_{ij} x_j + b_j(t)) D_i u(t,x), \tag{3.1}$$

for any $t \in [0,T]$ and any $x \in \mathbb{R}^N$.

As a first step, we prove that, for any $s \in [0,T[$ and any $\varphi \in C_b(\mathbb{R}^N)$, the Cauchy problem

$$\begin{cases} D_t u(t,x) = \mathcal{L}u(t,x), & t \in]s,T], \quad x \in \mathbb{R}^N, \\ u(s,x) = \varphi(x), & \qquad\qquad\quad x \in \mathbb{R}^N, \end{cases} \tag{3.2}$$

admits a unique classical solution u and we find out its representation formula. For this purpose, we introduce the evolution family $\{U(t,s)\}$ defined as follows:

$$\begin{cases} D_t U(t,s) = U(t,s) A(t), & t \in]s,T], \\ U(s,s) = I. \end{cases}$$

Here, I denotes the identity matrix. Further, we introduce the families of matrices $\{Q(s,t)\}$ and functions $\{B(t,s)\}$ defined by

$$Q(s,t) = \int_s^t U(t,r) Q(r) U(t,r)^* dr, \qquad B(t,s) = \int_s^t U(r,s) b(r) dr,$$

for any $0 \leq s \leq t \leq T$. As it is immediately seen, $U(t,s)$ is defined for any $s,t \in [0,T]$. Moreover, $U(s,r)U(t,s) = U(t,r)$ for any $0 \leq r \leq s \leq t \leq T$. Therefore, $U(t,s)$ is invertible for any $0 \leq s \leq t \leq T$ and $U(t,s)^{-1} = U(s,t)$. As a byproduct, it follows that the matrix $Q(s,t)$ is strictly positive definite for any $s < t$. Finally,

$$\|U(t,s)\| \leq e^{\|A\|_\infty T}, \qquad s,t \in [0,T]. \tag{3.3}$$

We can now prove the following result.

Theorem 3.1. *Assume that Hypotheses* 2.1(i)–(iii) *are satisfied with the coefficients of the operator $\mathcal{L}$ being independent of x. Then, the Cauchy problem* (3.2) *admits* *a unique classical solution $u : [s,T] \times \mathbb{R}^N \to \mathbb{R}$ for any $f \in C_b(\mathbb{R}^N)$ and any $s \in [0,T[$. The function u is given by the following representation formula:*

$$u(t,x) = \frac{1}{(4\pi)^{N/2}(\det Q(s,t))^{1/2}} \int_{\mathbb{R}^N} e^{-\frac{1}{4}\langle Q(s,t)^{-1}y,y\rangle} f(y - U(t,s)x - B(t,s))dy, \tag{3.4}$$

for any $(t,x) \in [0,T] \times \mathbb{R}^N$.

Proof. A straightforward computation shows that the function u in (3.4) is a classical solution of problem (3.2) (see also the proof of Theorem 3.2). The uniqueness part of the theorem follows from Proposition 2.3. Indeed, the function $\varphi : \mathbb{R}^N \to \mathbb{R}$, defined by $\varphi(x) = 1+|x|^2$ for any $x \in \mathbb{R}^N$, is a Lyapunov function for the operator $\mathcal{L}$ since

$$\begin{aligned}
\mathcal{L}\varphi(t,x) &= 2\mathrm{Tr}(Q(t)) + 2\langle A(t)x + b(t), x\rangle \\
&\leq 2\sqrt{N}\|Q\|_\infty + 2\|A\|_\infty |x|^2 + \|b\|_\infty |x| \\
&\leq \lambda\varphi(x),
\end{aligned}$$

for any $t > 0$ and any $x \in \mathbb{R}^N$, where $\lambda = \max\{2\sqrt{N}\|Q\|_\infty + \|b\|_\infty^2, 1 + 2\|A\|_\infty\}$. □

Now, for any $s,t \in [0,T]$, with $s < t$, and any $f \in C_b(\mathbb{R}^N)$, we set $P(t,s)f = u(t,\cdot)$, where u is the solution to problem (3.2) provided by Theorem 3.1. By Proposition 2.3, $P(t,s)$ is a bounded linear operator satisfying

$$\|P(t,s)\|_{L(C_b(\mathbb{R}^N))} \leq 1, \qquad 0 \leq s < t \leq T. \tag{3.5}$$

Moreover, $\{P(t,s)\}$ is an evolution family. To see it, it suffices to observe that, for any $0 \leq r < s < t \leq T$, the functions $v(t,x) = (P(t,s)P(s,r)f)(x)$ and $w(t,x) = (P(t,r)f)(x)$ turn out to be solutions to the differential equation $D_t u(t,x) = \mathcal{L}u(t,x)$ for any $t \in]s,T]$ and any $x \in \mathbb{R}^N$, and they satisfy the same initial condition $v(s,\cdot) = w(s,\cdot) = P(s,r)f$.

3.1. Estimates of the derivatives of the evolution family $\{P(t,s)\}$

In this subsection we prove some uniform estimates for the space derivatives of the function $P(t,s)f$, when f belongs to several spaces of (Hölder-) continuous functions. Such estimates will be the keystone to prove, in Subsection 3.2, Theorem 1.1 in the case when Q and b are independent of x. We are going to show that, for any $0 \leq h \leq k \leq 3$, there exist positive constants $C_{h,k}$ such that

$$\|D^k P(t,s)f\|_\infty \leq C_{h,k}(t-s)^{-\frac{k-h}{2}} \|f\|_{C_b^h(\mathbb{R}^N)}, \qquad 0 \leq s < t \leq T. \tag{3.6}$$

The main effort consists in proving estimate (3.6) when h and k are integers. Indeed, the general case then will follow by a classical interpolation argument.

Theorem 3.2. *For any $h,k = 0,\ldots,3$, with $h \leq k$, there exists a positive constant $C_{h,k}$ such that* (3.6) *holds true.*

Proof. Throughout the proof we denote by C_j ($j \in \mathbb{N}$) positive constants, independent of s and t.

As a first step, we estimate the norm of $Q(s,t)^{-1/2}$. For this purpose, we observe that, since $U(t,s)^{-1} = U(s,t)$ and $\|U(t,s)^*\| = \|U(t,s)\|$ for any $0 \le s \le t \le T$, from (3.3) we deduce that $|U(t,s)^*\xi| \ge C_1|\xi|$, for any $\xi \in \mathbb{R}^N$ and any $s,t \in [0,T]$. Hence, taking (2.1) into account, we obtain

$$\langle Q(t,s)\xi,\xi\rangle = \int_s^t \langle U(t,r)Q(r)U(t,r)^*\xi,\xi\rangle dr \ge \nu \int_s^t |U(t,r)^*\xi|^2 dr \ge C_2(t-s)|\xi|^2,$$

for any $\xi \in \mathbb{R}^N$, so that

$$\|Q(t,s)^{-\frac{1}{2}}\| \le \sqrt{C_2}(t-s)^{-\frac{1}{2}}, \qquad 0 \le s < t \le T. \tag{3.7}$$

Let us now prove (3.6) with $h = 0$ and $k = 1$. For this purpose, we rewrite $P(t,s)f$ as follows:

$$\begin{aligned}(P(t,s)f)(x) \;=\; & \frac{1}{(4\pi)^{N/2}(\det Q(s,t))^{1/2}} \\ & \times \int_{\mathbb{R}^N} e^{-\frac{1}{4}\langle Q(s,t)^{-1}(y+U(t,s)x),y+U(t,s)x\rangle} f(y-B(t,s))dy,\end{aligned}$$

for any $(t,x) \in]s,T] \times \mathbb{R}^N$. By the dominated convergence theorem, we deduce that u is differentiable with respect to the space variables in $]s,T] \times \mathbb{R}^N$ and

$$\begin{aligned}(D_jP(t,s)f)(x) \;=\; & -\frac{1}{2^{N+1}\pi^{N/2}} \int_{\mathbb{R}^N} \langle Q(s,t)^{-1/2}y, U(t,s)_j\rangle e^{-\frac{1}{4}|y|^2} \\ & \times f(Q(t,s)^{1/2}y - U(t,s)x - B(t,s))dy,\end{aligned}$$

for any $j = 1,\dots,N$, where $U(t,s)_j$ denotes the jth column of the matrix $U(t,s)$. Hence, by (3.3) and (3.7), we get

$$\begin{aligned}\|DP(t,s)f\|_\infty \;\le\; & \frac{1}{2^{N+1}\pi^{N/2}}\|Q(s,t)^{-1/2}\|\,\|U(t,s)\|\,\|f\|_\infty \int_{\mathbb{R}^N} |y|e^{-\frac{1}{4}|y|^2}dy \\ \le\; & C_3(t-s)^{-\frac{1}{2}}\|f\|_\infty,\end{aligned} \tag{3.8}$$

that is (3.6).

Now, we assume that $f \in C_b^1(\mathbb{R}^N)$ and observe that

$$\begin{aligned}(DP(t,s)f)(x) \;=\; & -\frac{1}{(4\pi)^{N/2}(\det Q(s,t))^{1/2}} \\ & \times \int_{\mathbb{R}^N} e^{-\frac{1}{4}\langle Q(s,t)^{-1}y,y\rangle} U(t,s)^* Df(y-U(t,s)x-B(t,s))dy \\ =\; & -U(t,s)^*(P(t,s)Df)(x),\end{aligned}$$

for any $(t,x) \in]s,T] \times \mathbb{R}^N$. Therefore, taking (3.3) into account, we get

$$\|DP(t,s)f\|_\infty \le \|U(t,s)^*\|\,\|Df\|_\infty \le e^{\|A\|_\infty T}\|Df\|_\infty, \qquad 0 \le s < t \le T,$$

that is, (3.6) with $h = k = 1$. Now, to get (3.6) in the remaining cases it suffices to recall that $\{P(t,s)\}$ is an evolution family. For instance, suppose that $h = 0$, $k = 2$ and fix $s, t \in [0,T]$ with $s < t$. Then, setting $r = (t+s)/2$, we get

$$D_{ij}P(t,s)f = D_{ij}P(t,r)P(r,s)f = \sum_{h=1}^{N} U(t,r)^*_{jh} D_i \left(P(t,r)D_h P(r,s)f\right),$$

for any $i, j = 1, \ldots, N$. Hence, using (3.3) and, twice, (3.8), yields

$$\|D_{ij}P(t,s)f\|_\infty \le C_4(t-s)^{-\frac{1}{2}}\|D_j P(r,s)f\|_\infty \le C_5(t-s)^{-1}\|f\|_\infty,$$

and (3.6) follows also in this case. The remaining cases can be treated likewise. □

Corollary 3.3. *Estimate* (3.6) *holds true for any* $h, k \in [0,3]$ *such that* $h \le k$.

Proof. As it has been already claimed, the proof follows by an interpolation argument. Two are the main ingredients needed to make the interpolation argument work. The first one is the well-known set equality $(C_b^h(\mathbb{R}^N), C_b^k(\mathbb{R}^N))_{\theta,\infty} = C_b^{h+\theta(k-h)}(\mathbb{R}^N)$, with equivalence of the corresponding norms, which holds for any $h, k \ge 0$ and any $\theta \in]0,1[$ (see, e.g., [15, Corollary 1.2.18]). The latter one is the fact that, given a quadruplet of Banach spaces Y_i, X_i $(i = 1,2)$ with Y_i continuously embedded in X_i $(i = 1,2)$, any linear operator T which belongs to $L(X_1, X_2)$ and to $L(Y_1, Y_2)$ is bounded also from $(X_1, Y_1)_{\theta,\infty}$ to $(X_2, Y_2)_{\theta,\infty}$, for any $\theta \in]0,1[$, and there exists a positive constant C such that

$$\|T\|_{L((X_1,Y_1)_{\theta,\infty},(X_2,Y_2)_{\theta,\infty})} \le C\|T\|^{1-\theta}_{L(X_1,X_2)}\|T\|^{\theta}_{L(Y_1,Y_2)}.$$

See, e.g., [15, Proposition 1.2.6]. □

Remark 3.4. Repeating the same arguments as in the proofs of Theorem 3.2 and Corollary 3.3, one can show that estimate (3.6) actually holds true for any $h, k \ge 0$ such that $h \le k$.

3.2. The nonhomogeneous case

In this section, we prove Theorem 1.1 in the case when $\mathcal{L}$ is given by (3.1). The candidate to be the solution to such a problem is the function u given by the pointwise variation-of-constants formula, i.e.,

$$u(t,x) = (P(t,0)f)(x) + \int_0^t (P(t,r)g(r,\cdot))(x)dr := (P(t,0)f)(x) + v(t,x), \quad (3.9)$$

for any $t \in [0,T]$ and any $x \in \mathbb{R}^N$. Note that the function v is well defined for any $(t,x) \in [0,T] \times \mathbb{R}^N$ and it is therein bounded and continuous. This can be easily seen from (3.4), performing the natural change of variable $z = Q(t,s)^{1/2}y$ in the integral term.

In view of the results in Theorem 3.1 and in Subsection 3.1, to prove that the function u is, actually, a solution to the Cauchy problem (1.4), we need to study the smoothness of the function v. For this purpose, we will take advantage of the following lemma, whose proof can be found in [16, Section 3].

Lemma 3.5. *Let $\theta, \alpha \in]0,1[$ be such that $\theta < \alpha$ and let $I \subset \mathbb{R}$ be an interval. Further, let $\varphi : I \to C_b^\theta(\mathbb{R}^N)$ be any function with the following properties:*

(i) *the function $t \mapsto (\varphi(t))(x)$ is measurable for any $x \in \mathbb{R}^N$;*
(ii) *$\|\varphi(t)\|_{C_b^\alpha(\mathbb{R}^N)} \le c(t)$ (resp. $\|\varphi(t)\|_{C_b^{2+\alpha}(\mathbb{R}^N)} \le c(t)$) for any $t \in I$ and some function $c \in L^1(I)$.*

Then, the function $\psi : \mathbb{R}^N \to \mathbb{R}$, defined by

$$\psi(x) = \int_I (\varphi(t)(x))dt, \qquad x \in \mathbb{R}^N,$$

belongs to $C_b^\alpha(\mathbb{R}^N)$ (resp. $C_b^{2+\alpha}(\mathbb{R}^N)$) and there exists a positive constant C, independent of φ and c, such that

$$\|\psi\|_{C_b^\alpha(\mathbb{R}^N)} \le C\|c\|_{L^1(I)}, \qquad (\textit{resp. } \|\psi\|_{C_b^{2+\alpha}(\mathbb{R}^N)} \le C\|c\|_{L^1(I)}).$$

The main properties of the function v defined in (3.9) are contained in the following proposition.

Proposition 3.6. *Fix $\theta \in]0,1[$. For any $g \in C^{0,\theta}([0,T]\times\mathbb{R}^N)$, the function v belongs to $C^{0,2+\theta}([0,T]\times\mathbb{R}^N)$. Moreover, there exists a positive constant C, independent of g, such that*

$$\|v\|_{C^{0,2+\theta}([0,T]\times\mathbb{R}^N)} \le C\|g\|_{C^{0,\theta}([0,T]\times\mathbb{R}^N)}. \tag{3.10}$$

Finally, v is continuously differentiable with respect to t in $[0,T]\times\mathbb{R}^N$ and solves the Cauchy problem (1.4) with $f \equiv 0$.

Proof. To begin with, let us show that the function v belongs to $C^{0,2+\theta}([0,T]\times\mathbb{R}^N)$ and satisfies (3.10). Since $C_b^{2+\theta}(\mathbb{R}^N) = (C_b^\alpha(\mathbb{R}^N), C_b^{2+\alpha}(\mathbb{R}^N))_{1-(\alpha-\theta)/2,\infty}$ with equivalence of the corresponding norms, for any $\alpha \in]\theta,1[$, to prove that $v(t,\cdot) \in C_b^{2+\theta}(\mathbb{R}^N)$ for any $t \in [0,T]$, as well as estimate (3.10), it suffices to show that, for any $\xi \in]0,1[$ and any $t \in [0,T]$, there exist two functions $a_\xi(t,\cdot) \in C_b^\alpha(\mathbb{R}^N)$ and $b_\xi(t,\cdot) \in C_b^{2+\alpha}(\mathbb{R}^N)$ such that $v(t,\cdot) = a_\xi(t,\cdot) + b_\xi(t,\cdot)$ and

$$\|a_\xi(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^N)} + \xi\|b_\xi(t,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^N)} \le C_1\xi^{1-(\alpha-\theta)/2}\|g\|_{C^{0,\theta}([0,T]\times\mathbb{R}^N)}, \tag{3.11}$$

for some positive constant C_1, independent of ξ, t and g. For this purpose, we set

$$a_\xi(t,\cdot) = \begin{cases} \displaystyle\int_{t-\xi}^t (P(t,r)g(r,\cdot))(x)dr, & \xi \in [0,t[, \\ \displaystyle\int_0^t (P(t,r)g(r,\cdot))(x)dr, & \text{otherwise}, \end{cases}$$

and

$$b_\xi(t,\cdot) = \begin{cases} \displaystyle\int_0^{t-\xi} (P(t,r)g(r,\cdot))(x)dr, & \xi \in [0,t[, \\ 0, & \text{otherwise}. \end{cases}$$

By Theorem 3.2, we can apply Lemma 3.5 to the functions $a_\xi(t,\cdot)$ and $b_\xi(t,\cdot)$, showing that they belong to $C_b^\alpha(\mathbb{R}^N)$ and $C_b^{2+\alpha}(\mathbb{R}^N)$, respectively, for any $\alpha\in]\theta,1[$, and that (3.11) holds.

With a similar argument, one can also show that the function v belongs to $C_b([0,T]\times\mathbb{R}^N)$. Hence, in particular, $v\in C([0,T];C(\overline{B}(R)))$ for any $R>0$. Recalling that $C^2(\overline{B}(R))$ belongs to the class $J_{2/(2+\theta)}$ between $C(\overline{B}(R))$ and $C^{2+\theta}(\overline{B}(R))$, it follows that

$$\begin{aligned}\|u(t,\cdot)-u(s,\cdot)\|_{C^2(\overline{B}(R))} &\le C_2\|u(t,\cdot)-u(s,\cdot)\|_{C^{2+\theta}(\overline{B}(R))}^{\theta/(2+\theta)}\|u(t,\cdot)-u(s,\cdot)\|_{C(\overline{B}(R))}^{2/(2+\theta)}\\ &\le C_3\|g\|_{C^{0,\theta}([0,T]\times\mathbb{R}^N)}^{\theta/(2+\theta)}\|u(t,\cdot)-u(s,\cdot)\|_{C(\overline{B}(R))}^{2/(2+\theta)},\end{aligned}\tag{3.12}$$

for any $s,t\in[0,T]$ and some positive constants C_2 and C_3, independent of s and t. Since the last side of (3.12) vanishes as $|t-s|$ tends to 0, it follows that all the space derivatives of u up to the second-order are continuous in $[0,T]\times\mathbb{R}^N$.

To conclude the proof, we just need to show that the function v is differentiable with respect to time in $[0,T]\times\mathbb{R}^N$ and $D_tv=\mathcal{L}v+g$. For this purpose, we show that v is differentiable from the right in $[0,T[\times\mathbb{R}^N$ and the right derivatives D_t^+v equals $\mathcal{L}v+g$ in $[0,T[\times\mathbb{R}^N$. Then, [25, p. 239] will allow us to conclude. As a straightforward computation shows, for any $(t,x)\in[0,T[\times\mathbb{R}^N$ and any $h>0$ sufficiently small, it holds that

$$\begin{aligned}\frac{v(t+h,x)-v(t,x)}{h} &= \frac{1}{h}\int_0^t((P(t+h,r)-P(t,r))\,g(r,\cdot))(x)dr\\ &\quad+\frac{1}{h}\int_t^{t+h}(P(t+h,r)g(r,\cdot))(x)dr\\ &= I_h^{(1)}(t,x)+I_h^{(2)}(t,x).\end{aligned}\tag{3.13}$$

Let us observe that $D_t^+(P(t,r)g(r,\cdot))(x)=(\mathcal{L}P(t,r)g(r,\cdot))(x)$ for any $r,t\in]0,T[$, with $r<t$, and any $x\in\mathbb{R}^N$. Moreover, by (3.5) and (3.10) we know that

$$\sup_{0\le r<s\le T}(s-r)^{1-\frac{\theta}{2}}|(\mathcal{L}(P(s,r)g(r,\cdot)))(x)|\le C_4\|g\|_{C^{0,\theta}([0,T]\times\mathbb{R}^N)},$$

for some positive constant $C_4=C_4(x)$. Hence, we can take the limit as h tends to 0^+ in the first integral term in (3.13) obtaining

$$\lim_{h\to0^+}I_h^{(1)}(t,x)=\int_0^t(\mathcal{L}(P(t,s)g(s,\cdot)))(x)ds=\mathcal{L}v(t,x).$$

Since the function $(t,s)\mapsto(P(t,s)g(s,\cdot))(x)$ is continuous in $D=\{(t,s)\in[0,T]\times[0,T]: s\le t\}$ (this can be easily checked by a straightforward change of variables in the representation formula (3.4)), we easily deduce that $I_h^{(2)}(t,x)$ tends to $g(t,x)$ as h tends to 0^+. Therefore, v is differentiable (from the right) with respect to t at the point $(t,x)\in[0,T[\times\mathbb{R}^N$ and $D_t^+v(t,x)=\mathcal{L}v(t,x)+g(t,x)$. This completes the proof. □

Corollary 3.7. *Assume that $\mathcal{L}$ is given by* (3.1) *with the coefficients satisfying Hypotheses* 2.1*. Then, Theorem* 1.1 *holds true. In particular, for any f and g as in the statement of Theorem* 1.1*, the unique classical solution to* (1.4) *is given by* (3.9)*.*

Proof. From Theorem 3.1 and Proposition 3.6, we know that the function u in (3.9) is a classical solution to problem (1.4) and satisfies (1.6). The uniqueness of the solution to problem (1.4) follows from Proposition 2.3. □

4. The general case (when the coefficients are continuous in (t, x))

In this section we assume that the coefficients Q and b of the operator $\mathcal{L}$ depend also on the space variables. First, in Subsection 4.1 we prove that, if u is a solution to the Cauchy problem

$$\begin{cases} D_t u(t,x) = \displaystyle\sum_{i,j=1}^N q_{ij}(t,x) D_{ij} u(t,x) + \sum_{i,j=1}^N a_{ij}(t) x_j D_i u(t,x) \\ \qquad\qquad + \displaystyle\sum_{j=1}^N b_j(t,x) D_j u(t,x) + g(t,x), & t \in [0,T],\ x \in \mathbb{R}^N, \\ u(0,x) = f(x), & x \in \mathbb{R}^N, \end{cases} \tag{4.1}$$

corresponding to $f \in C_b^{2+\theta}(\mathbb{R}^N)$ and $g \in C^{0,\theta}([0,T] \times \mathbb{R}^N)$, then there exists a positive constant C, depending only on ν (see (2.1)) and the sup-norm of the coefficients of the operator $\mathcal{L}$, such that

$$\|u\|_{C^{0,2+\theta}([0,T]\times\mathbb{R}^N)} \le C \left(\|f\|_{C_b^{2+\theta}(\mathbb{R}^N)} + \|g\|_{C^{0,\theta}([0,T]\times\mathbb{R}^N)} \right). \tag{4.2}$$

Such an *a priori* estimate will be the keystone to apply the classical method of continuity and prove Theorem 1.1.

4.1. Proof of (4.2)

Theorem 4.1. *Let $u \in C^{0,2+\theta}([0,T] \times \mathbb{R}^N)$ be a classical solution to problem* (4.1)*. Then, there exists a positive constant C, depending only on the ellipticity constant ν and on the sup-norm of the coefficients of the operator $\mathcal{L}$, such that estimate* (4.2) *holds true.*

Proof. The proof can be obtained adapting the proof of [14, Theorem 4.1]. For the reader's convenience we go into details. We split the proof into two steps. In the first one, we prove that there exists a positive constant $\hat{C}$ such that

$$\|u\|_{C^{0,2+\theta}([0,T^*]\times\mathbb{R}^N)} \le \hat{C} \left(\|u\|_{C_b([0,T^*]\times\mathbb{R}^N)} + \|f\|_{C_b^{2+\theta}(\mathbb{R}^N)} + \|g\|_{C^{0,\theta}([0,T^*]\times\mathbb{R}^N)} \right), \tag{4.3}$$

for any $T^* \in]0,T]$. Then, in Step 2, using (4.3) we will conclude the proof.

Step 1. To prove (4.3) we use a localization argument freezing the diffusion and (part of) the drift coefficients of the operator $\mathcal{L}$. For this purpose, let us fix $T^* \in$

$]0,T[$. With any $\delta>0$ and any $x_0\in\mathbb{R}^N$, we introduce the function ψ_{δ,x_0} defined by

$$\psi_{\delta,x_0}(x)=\psi((x-x_0)/\delta),\qquad x\in\mathbb{R}^N,$$

where $\psi\in C^3(\mathbb{R}^N)$ is any function such that $\chi_{B(1/2)}\le\psi\le\chi_{B(1)}$. As it is easily seen, the function $v_{\delta,x_0}=u\psi_{\delta,x_0}$ solves the Cauchy problem

$$\begin{cases} D_t w_{\delta,x_0}(t,x)=\mathcal{L}_{x_0}v_{\delta,x_0}(t,x)+g(t,x)\psi_{\delta,x_0}(t,x)-h_{\delta,x_0}(t,x)+k_{\delta,x_0}(t,x), \\ \hfill t\in]0,T^*],\quad x\in\mathbb{R}^N, \\ v_{\delta,x_0}(0,x)=f(x)\psi_{\delta,x_0}(x), \hfill x\in\mathbb{R}^N, \end{cases} \tag{4.4}$$

where

$$\mathcal{L}_{x_0}v_{\delta,x_0}(t,x)=\mathrm{Tr}(Q(t,x_0)D^2v_{\delta,x_0}(t,x))+\langle A(t)x+b(t,x_0),Dv_{\delta,x_0}(t,x)\rangle,$$

$$h_{\delta,x_0}(t,x)=2\langle Q(t,x)Du(t,x),D\psi_{\delta,x_0}(t,x)\rangle+u(t,x)\mathcal{L}\psi_{\delta,x_0}(t,x)$$

and

$$\begin{aligned} k_{\delta,x_0}(t,x) \;=\; &\sum_{i,j=1}^N\left(q_{ij}(t,x)-q_{ij}(t,x_0)\right)D^2_{ij}v_{\delta,x_0}(t,x) \\ &+\sum_{j=1}^N\left(b_j(t,x)-b_j(t,x_0)\right)D_jv_{\delta,x_0}(t,x), \end{aligned} \tag{4.5}$$

for any $(t,x)\in]0,T^*]\times\mathbb{R}^N$. To simplify the notation, in the rest of the proof of this step we drop out the dependence on δ and x_0 of the functions we deal with.

As a straightforward computation shows, the functions $f\psi$ and h belong to $C^{2+\theta}_b(\mathbb{R}^N)$ and $C^{0,\theta}([0,T^*]\times\mathbb{R}^N)$, respectively, and there exists a positive constant $C_1(\delta)$ such that

$$\begin{aligned} &\|f\psi\|_{C^{2+\theta}_b(\mathbb{R}^N)}+\|g\psi\|_{C^{0,\theta}([0,T^*]\times\mathbb{R}^N)}+\|h\|_{C^{0,\theta}([0,T^*]\times\mathbb{R}^N)} \\ &\le\; C_1(\delta)\left(\|f\|_{C^{2+\theta}_b(\mathbb{R}^N)}+\|g\|_{C^{0,\theta}([0,T^*]\times\mathbb{R}^N)}+\|u\|_{C^{0,1+\theta}([0,T^*]\times\mathbb{R}^N)}\right). \end{aligned} \tag{4.6}$$

Note that the constant $C_1(\delta)$ may blow up as δ goes to $+\infty$.

Let us now estimate the function k. For this purpose, we observe that

$$\begin{aligned} |\left(q_{ij}(t,x)-q_{ij}(t,x_0)\right)D^2_{ij}v(t,x)| \;&\le\; [q_{ij}(t,\cdot)]_{C^\theta_b(\mathbb{R}^N)}|x-x_0|^\theta\|D^2_{ij}v\|_\infty \\ &\le\; [q_{ij}(t,\cdot)]_{C^\theta_b(\mathbb{R}^N)}\delta^\theta\|D^2_{ij}v\|_\infty, \end{aligned} \tag{4.7}$$

for any $(t,x)\in[0,T^*]\times\mathbb{R}^N$ and any $i,j=1,\dots,N$. Similarly, for any $t\in[0,T^*]$ and any $x_1,x_2\in x_0+\overline{B}(\delta)$, one has

$$\begin{aligned} &|\left(q_{ij}(t,x_2)-q_{ij}(t,x_0)\right)D^2_{ij}v(t,x_2)-\left(q_{ij}(t,x_1)-q_{ij}(t,x_0)\right)D^2_{ij}v(t,x_1)| \\ &\quad\le|q_{ij}(t,x_2)-q_{ij}(t,x_1)||D^2_{ij}v(t,x_2)| \\ &\qquad+|q_{ij}(t,x_1)-q_{ij}(t,x_0)||D^2_{ij}v(t,x_2)-D^2_{ij}v(t,x_1)|. \end{aligned} \tag{4.8}$$

Since $D^2_{ij}v(t,\cdot)$ vanishes on $x_0+\partial B(\delta)$ $(i,j=1,\ldots,N)$, we easily deduce that

$$\begin{aligned}|D^2_{ij}v(t,x_2)| &= |D^2_{ij}v(t,x_2)-D^2_{ij}v(t,x^*)|\\ &\le [D^2_{ij}v(t,\cdot)]_{C^\theta_b(\mathbb{R}^N)}|x_2-x^*|^\theta\\ &\le [D^2_{ij}v(t,\cdot)]_{C^\theta_b(\mathbb{R}^N)}\delta^\theta, \end{aligned}\tag{4.9}$$

where $x^*=x_0+\delta(x-x_0)|x-x_0|^{-1}$. Estimates (4.8) and (4.9) now give

$$\begin{aligned}&|\left(q_{ij}(t,x_2)-q_{ij}(t,x_0)\right)D^2_{ij}v(t,x_2)-\left(q_{ij}(t,x_1)-q_{ij}(t,x_0)\right)D^2_{ij}v(t,x_1)|\\ &\quad\le 2\delta^\theta\|q_{ij}\|_{C^{0,\theta}([0,T]\times\mathbb{R}^N)}[D^2_{ij}v(t,\cdot)]_{C^\theta_b(\mathbb{R}^N)}|x_2-x_1|^\theta,\end{aligned}\tag{4.10}$$

for any $t\in[0,T^*]$ and any $x_1,x_2\in x_0+\overline{B}(\delta)$. Estimates (4.7) and (4.10) now give

$$\begin{aligned}&\sup_{t\in[0,T^*]}\|\left(q_{ij}(t,\cdot)-q_{ij}(t,x_0)\right)D^2_{ij}v(t,\cdot)\|_{C^\theta(x_0+\overline{B}(\delta))}\\ &\qquad\le 2\delta^\theta\|q_{ij}\|_{C^{0,\theta}([0,T^*]\times\mathbb{R}^N)}\|D^2_{ij}v\|_{C^{0,\theta}([0,T^*]\times\mathbb{R}^N)}.\end{aligned}\tag{4.11}$$

Since v identically vanishes outside the ball $x_0+B(\delta)$, it follows that the function $(t,x)\mapsto(q_{ij}(t,x)-q_{ij}(t,x_0))D^2_{ij}v(t,x)$ belongs to $C^{0,\theta}([0,T^*]\times\mathbb{R}^N)$ and (4.11) holds true with the ball $x_0+\overline{B}(\delta)$ being replaced by $\mathbb{R}^N$.
Similarly, we can estimate the terms in the second sum of (4.5). We finally obtain that k belongs to $C^{0,\theta}([0,T^*]\times\mathbb{R}^N)$ and

$$\|k\|_{C^{0,\theta}([0,T^*]\times\mathbb{R}^N)}\le C_2\delta^\theta\|v\|_{C^{0,2+\theta}([0,T^*]\times\mathbb{R}^N)},\tag{4.12}$$

where C_2 is a positive constant, independent of δ, T^* and v. Hence, from (4.4), (4.6), (4.12) and Corollary 3.7, we deduce that there exists a positive constant C_3, independent of δ and T^*, such that

$$\begin{aligned}\|v\|_{C^{0,2+\theta}([0,T^*]\times\mathbb{R}^N)} \le\ & C_3\Big(C_1(\delta)\|g\|_{C^{0,\theta}([0,T^*]\times\mathbb{R}^N)}+C_1(\delta)\|u\|_{C^{0,1+\theta}([0,T^*]\times\mathbb{R}^N)}\\ &+C_1(\delta)\|f\|_{C^{2+\theta}_b(\mathbb{R}^N)}+C_2\delta^\theta\|v\|_{C^{0,2+\theta}([0,T^*]\times\mathbb{R}^N)}\Big).\end{aligned}\tag{4.13}$$

Now, fixing δ sufficiently small, we can get rid of the term $\|v\|_{C^{0,2+\theta}([0,T^*]\times\mathbb{R}^N)}$ in the right-hand side of (4.13). Recalling that $v\equiv u$ in $[0,T]\times x_0+B(\delta/2)$ and covering $\mathbb{R}^N$ with balls of radius $\delta/2$, we obtain

$$\begin{aligned}&\|u\|_{C^{0,2+\theta}([0,T^*]\times\mathbb{R}^N)}\\ &\le C_4(\delta)\left(\|u\|_{C^{0,1+\theta}([0,T^*]\times\mathbb{R}^N)}+\|f\|_{C^{2+\theta}_b(\mathbb{R}^N)}+\|g\|_{C^{0,\theta}([0,T^*]\times\mathbb{R}^N)}\right),\end{aligned}$$

for some positive constant $C_4(\delta)$, independent of x_0. Recalling that, for any $\varepsilon>0$, there exists a positive constant $C_5(\varepsilon)$ such that

$$\|\varrho\|_{C^{1+\theta}_b(\mathbb{R}^N)}\le\varepsilon\|\varrho\|_{C^{2+\theta}_b(\mathbb{R}^N)}+C_5(\varepsilon)\|\varrho\|_{C_b(\mathbb{R}^N)},$$

for any $\varrho\in C^{2+\theta}_b(\mathbb{R}^N)$, estimate (4.3) now easily follows.

Step 2. We now conclude the proof. Again, we freeze part of the coefficients at x_0. Namely, we rewrite problem (4.1) as follows:

$$\begin{cases} D_t u(t,x) = \mathcal{L}_{x_0} u(t,x) + g(t,x) + m(t,x), & t \in [0,T], \quad x \in \mathbb{R}^N, \\ u(0,x) = f(x), & x \in \mathbb{R}^N, \end{cases}$$

where m is given by the right-hand side of (4.5), with the function v_{δ,x_0} being replaced by u. Taking (4.3) into account, it is easy to show that the function m belongs to $C^{0,\theta}([0,T] \times \mathbb{R}^N)$ and there exists a positive constant C_6, independent of x_0, s, u, f and g, such that

$$\|m\|_{C^{0,\theta}([0,s]\times\mathbb{R}^N)} \leq C_6 \left(\|u\|_{C_b([0,s]\times\mathbb{R}^N)} + \|f\|_{C_b^{2+\theta}(\mathbb{R}^N)} + \|g\|_{C^{0,\theta}([0,s]\times\mathbb{R}^N)} \right),$$

for any $s \in]0,T[$. Now, by the proof of Corollary 3.7, we know that u can be written as follows:

$$u(t,x) = (P(t,0)f)(x) + \int_0^t \{P(t,s)(g(s,\cdot) + m(s,\cdot))\}(x)ds,$$

for any $(t,x) \in [0,T^*] \times \mathbb{R}^N$, where $\{P(t,s)\}$ is the evolution family associated with the operator $\mathcal{L}_{x_0}$. By estimate (3.5), we know that

$$\begin{aligned} |u(t,x)| \ \leq \ & \|f\|_{C_b(\mathbb{R}^N)} + C_6 T \left(\|f\|_{C_b^{2+\theta}(\mathbb{R}^N)} + \|g\|_{C^{0,\theta}([0,T]\times\mathbb{R}^N)} \right) \\ & + C_6 \int_0^t \|u\|_{C_b([0,s]\times\mathbb{R}^N)} ds, \end{aligned}$$

for any $(t,x) \in [0,T] \in \mathbb{R}^N$ or, better,

$$\begin{aligned} \|u\|_{C_b([0,t]\times\mathbb{R}^N)} \ \leq \ & \|f\|_{C_b(\mathbb{R}^N)} + C_6 T \left(\|f\|_{C_b^{2+\theta}(\mathbb{R}^N)} + \|g\|_{C^{0,\theta}([0,T]\times\mathbb{R}^N)} \right) \\ & + C_6 \int_0^t \|u\|_{C([0,s]\times\mathbb{R}^N)} ds, \end{aligned}$$

for any $t \in [0,T]$. Now, the Gronwall lemma applies and gives

$$\|u\|_{C_b([0,T]\times\mathbb{R}^N)} \leq C_7 \left(\|f\|_{C_b^{2+\theta}(\mathbb{R}^N)} + \|g\|_{C^{0,\theta}([0,T]\times\mathbb{R}^N)} \right), \tag{4.14}$$

for some positive constant C_7, depending on T, but being independent of u, f and g. Replacing (4.14) into (4.3), estimate (4.2) easily follows. This completes the proof. □

4.2. The method of continuity: proof of Theorem 1.1

In this subsection we prove Theorem 1.1 in the general case, taking advantage of the classical method of continuity.

Proof of Theorem 1.1. For any $\lambda \in [0,1]$, let $\mathcal{L}_\lambda$ be the differential operator defined by

$$\mathcal{L}_\lambda = \lambda \mathcal{L} + (1-\lambda)(\Delta + \langle A(t)x, D\rangle) := \mathrm{Tr}(Q_\lambda(t,x)D^2) + \langle A(t)x + b_\lambda(t,x), D\rangle.$$

As it is immediately seen, the operator $\mathcal{L}_\lambda$ is uniformly elliptic and $\langle Q_\lambda(t,x)\xi,\xi\rangle \geq \min\{\nu,1\}|\xi|^2$, for any $t\in[0,T]$ and any $x,\xi\in\mathbb{R}^N$. Moreover, the coefficients Q_λ and b_λ belong to $C^{0,\theta}([0,T]\times\mathbb{R}^N)$ with norms that can be estimated from above by positive constants independent of λ.

We now denote by $\mathcal{F}$ the set of $\lambda\in[0,1]$ such that the problem

$$\begin{cases} D_t u(t,x) = \mathcal{L}_\lambda u(t,x) + g(t,x), & t\in[0,T], \quad x\in\mathbb{R}^N, \\ u(0,x) = f(x), & x\in\mathbb{R}^N, \end{cases} \tag{4.15}$$

admits a unique classical solution $u_\lambda\in C^{0,2+\theta}([0,T]\times\mathbb{R}^N)$ for any $f\in C_b^{2+\theta}(\mathbb{R}^N)$ and any $g\in C^{0,\theta}([0,T]\times\mathbb{R}^N)$. From Theorem 4.1 it follows that, if $\lambda\in\mathcal{F}$, then there exists a positive constant C_1, independent of λ, f and g, such that the solution u_λ to problem (4.15) satisfies

$$\|u_\lambda\|_{C^{0,2+\theta}([0,T]\times\mathbb{R}^N)} \leq C_1\left(\|f\|_{C_b^{2+\theta}(\mathbb{R}^N)} + \|g\|_{C^{0,\theta}([0,T]\times\mathbb{R}^N)}\right). \tag{4.16}$$

We are going to prove that $\mathcal{F}$ is both open and closed in $[0,1]$. This will be enough to conclude that $\mathcal{F}=[0,1]$, and Theorem 1.1 will follow. Let us first show that $\mathcal{F}$ is closed. For this purpose, we fix $\lambda\in\overline{\mathcal{F}}$. Then, there exists a sequence $\{\lambda_n\}\subset\mathcal{F}$ converging to λ as n tends to $+\infty$. From (4.16), we immediately deduce that the sequence $\{u_{\lambda_n}\}$ is bounded $C^{0,2+\theta}([0,T]\times\mathbb{R}^N)$. Moreover, from the differential equation in (4.15), we get

$$\|u_{\lambda_n}\|_{\mathrm{Lip}([0,T]\times\overline{B}(R))} \leq C_2\left(\|f\|_{C_b^{2+\theta}(\mathbb{R}^N)} + \|g\|_{C^{0,\theta}([0,T]\times\mathbb{R}^N)}\right), \tag{4.17}$$

for any $R>0$ and some positive constant C_2, depending on R, but being independent of n. Hence, $u_{\lambda_n}\in\mathrm{Lip}([0,T];C(\overline{B}(R)))$ and it is bounded in $[0,T]$ with values in $C^{2+\theta}(\overline{B}(R))$. Arguing as in the proof of (3.12), we deduce that $u_n(\cdot,x)$, $D_iu_{\lambda_n}(\cdot,x)$ and $D_{ij}u_{\lambda_n}(\cdot,x)$ belong to $C^{\theta/(2+\theta)}([0,T])$ for any $x\in\overline{B}(R)$. Moreover, (4.16) (with λ_n instead of λ) and (4.17) yield

$$\begin{aligned} &\sup_{x\in\overline{B}(R)}\left(\|u_{\lambda_n}(\cdot,x)\|_{C^{\theta/(2+\theta)}([0,T])} + \|D_iu_{\lambda_n}(\cdot,x)\|_{C^{\theta/(2+\theta)}([0,T])}\right.\\ &\qquad\qquad \left.+\|D_{ij}u_{\lambda_n}(\cdot,x)\|_{C^{\theta/(2+\theta)}([0,T])}\right)\\ &\leq C_3\left(\|f\|_{C_b^{2+\theta}(\mathbb{R}^N)} + \|g\|_{C^{0,\theta}([0,T]\times\mathbb{R}^N)}\right), \end{aligned} \tag{4.18}$$

for any $i,j=1,\dots,N$ and some positive constant C_3, independent of f and g. Combining estimates (4.16) and (4.18), it follows that the restriction to $[0,T]\times\overline{B}(R)$ of the function u_{λ_n} and its space derivatives up to the second-order satisfy the assumptions of the Ascoli-Arzelà theorem. By the arbitrariness of $R>0$, we infer that, up to a subsequence, the functions u_{λ_n}, Du_{λ_n} and $D^2u_{\lambda_n}$ converge locally uniformly in $[0,T]\times\mathbb{R}^N$ to u, Du and D^2u, respectively, for some function $u:[0,T]\times\mathbb{R}^N\to\mathbb{R}$. Moreover, $u\in C^{0,2+\theta}([0,T]\times\mathbb{R}^N)$. The previous arguments also show that $D_tu_{\lambda_n}$ converges locally uniformly in $[0,T]\times\mathbb{R}^N$. Hence, u is differentiable with respect to time in $[0,T]$ and satisfies (4.15). This implies that $\lambda\in\mathcal{F}$, so that $\mathcal{F}$ is closed.

To conclude the proof, we show that $\mathcal{F}$ is open in $[0,1]$. So, we fix $\lambda_0 \in \mathcal{F}$ and prove that there exists a neighborhood of λ_0 contained in $\mathcal{F}$. For this purpose, we observe that a function u_λ is a solution to problem (4.15), for some $\lambda \in [0,1]$, if and only if it solves the Cauchy problem

$$\begin{cases} D_t u(t,x) = \mathcal{L}_{\lambda_0} u(t,x) + g(t,x) + h_{\lambda,\lambda_0}(t,x,u), & t \in [0,T], \quad x \in \mathbb{R}^N, \\ u(0,x) = f(x), & x \in \mathbb{R}^N, \end{cases}$$

where

$$h_{\lambda,\lambda_0}(t,x,u) = (\lambda - \lambda_0)\left\{\mathrm{Tr}((Q(t,x) - I)D^2 u(t,x)) + \langle b(t,x), Du(t,x)\rangle\right\},$$

for any $t \in [0,T]$ and any $x \in \mathbb{R}^N$. As it is immediately seen, for any u in $C^{0,2+\theta}([0,T] \times \mathbb{R}^N)$, the function $h(\cdot,u)$ belongs to $C^{0,\theta}([0,T] \times \mathbb{R}^N)$ and there exists a positive constant C_4, independent of λ and λ_0, such that

$$\|h_{\lambda,\lambda_0}(\cdot,u)\|_{C^{0,\theta}([0,T]\times\mathbb{R}^N)} \le C_4|\lambda - \lambda_0|\|u\|_{C^{0,2+\theta}([0,T]\times\mathbb{R}^N)}. \tag{4.19}$$

According to Corollary 3.7, any classical solution to problem (4.15) is a fixed point of the operator $\Gamma \in L(C^{0,2+\theta}([0,T] \times \mathbb{R}^N))$ defined by

$$(\Gamma(u))(t,x) = (P(t,0)f)(x) + \int_0^t (P(t,s)(g(s,\cdot) + h_{\lambda,\lambda_0}(s,\cdot,u)))(x)ds,$$

for any $(t,x) \in [0,T] \times \mathbb{R}^N$. Here, $\{P(t,s)\}$ is the evolution family associated with the operator $\mathcal{L}_{x_0}$. Taking advantage of the estimates (1.6) and (4.19), we infer that

$$\|\Gamma(u_2) - \Gamma(u_1)\|_{C^{0,2+\theta}([0,T]\times\mathbb{R}^N)} \le C_5|\lambda - \lambda_0|\,\|u_2 - u_1\|_{C^{0,2+\theta}([0,T]\times\mathbb{R}^N)},$$

for some positive constant C_5, independent of λ, λ_0, u_1 and u_2. It turns out that Γ is a contraction provided that $C_5|\lambda - \lambda_0| < 1$. Therefore, for such λ's the Cauchy problem (4.15) is (uniquely) solvable with a classical solution u which belongs to $C^{0,2+\theta}([0,T] \times \mathbb{R}^N)$. This means that λ_0 is an interior point of $\mathcal{F}$ and, hence, $\mathcal{F}$ is open in $[0,1]$. This completes the proof. $\square$

5. The case when the coefficients are only measurable with respect to (t,x)

In this section, we extend the results in the previous sections to the case when the coefficients of the operator $\mathcal{L}$ are only measurable with respect to $(t,x) \in [0,T] \times \mathbb{R}^N$ and are θ-Hölder continuous with respect to the space variables (see Hypotheses 2.2). For this purpose, we begin by a lemma that we need in the proof of Theorem 1.3.

Lemma 5.1. *Under Hypotheses 2.2, there exist three sequences $Q^{(n)} = (q_{ij}^{(n)})$, $A^{(n)} = (a_{ij}^{(n)})$ and $B^{(n)} = (b_j^{(n)})$ such that $q_{ij}^{(n)}$, $b_j^{(n)}$ belong to $C^{0,\theta}([0,T] \times \mathbb{R}^N)$ and $a_{ij}^{(n)}$ belongs to $C([0,T])$ for any $i,j = 1,\ldots,N$ and any $n \in \mathbb{N}$. Moreover:*

(i) *there exists a measurable set $\mathcal{E} \subset [0,T]$, whose complement is negligible, such that, for any $i,j = 1,\ldots,N$, $q_{ij}^{(n)}(t,x)$, $a_{ij}^{(n)}(t)$ and $b_{ij}^{(n)}(t,x)$ converge pointwise, respectively to $q_{ij}(t,x)$, $a_{ij}(t)$ and $b_j(t,x)$, as n tends to $+\infty$, for any $(t,x) \in \mathcal{E} \times \mathbb{R}^N$;*

(ii) $\|a_{ij}^{(n)}\|_{C([0,T])} \le \|a_{ij}\|_{L^\infty(]0,T[)}$, $\|q_{ij}^{(n)}\|_{C^{0,\theta}([0,T]\times\mathbb{R}^N)} \le \|q_{ij}\|_{B^{0,\theta}([0,T]\times\mathbb{R}^N)}$ *and* $\|b_j^{(n)}\|_{C^{0,\theta}([0,T]\times\mathbb{R}^N)} \le \|b_j\|_{B^{0,\theta}([0,T]\times\mathbb{R}^N)}$ *for any $i,j = 1,\ldots,N$;*

(iii) *there exists a positive constant ν_0 such that $\langle Q^{(n)}(t,x)\xi,\xi\rangle \ge \nu_0|\xi|^2$ for any $t \in [0,T]$, any $x,\xi \in \mathbb{R}^N$ and any $n \in \mathbb{N}$.*

Proof. We limit ourselves to proving the statements related to the matrix function Q, the other ones being similar and even simpler. We define the matrix $Q^{(n)}(t,x)$ at any $(t,x) \in [0,T]\times\mathbb{R}^N$, by setting

$$q_{ij}^{(n)}(t,x) = \left(\frac{n}{4\pi}\right)^{\frac{1}{2}} \int_0^T q_{ij}(s,x)\exp\left(-\frac{n}{4}|t-s|^2\right)ds, \qquad i,j = 1,\ldots,N,\ \ n \in \mathbb{N}.$$

Note that the functions $q_{ij}^{(n)}$ are well defined in $[0,T]\times\mathbb{R}^N$ for any $i,j = 1,\ldots,N$ and any $n \in \mathbb{N}$. Indeed, if $\psi \in B^{0,\theta}([0,T]\times\mathbb{R}^N)$, then, by the Fubini theorem, the function $\psi(\cdot,x)$ is measurable for any $x \in \hat{\mathcal{E}}$, where $\mathbb{R}^N \setminus \hat{\mathcal{E}}$ is negligible. Actually, $\hat{\mathcal{E}} = \mathbb{R}^N$. Indeed, for any $x \in \mathbb{R}^N$, we can find out a sequence $\{x_n\} \subset \hat{\mathcal{E}}$ converging to x as n tends to $+\infty$. Since $\psi(t,\cdot) \in C_b^\theta(\mathbb{R}^N)$, then $\psi(t,x_n)$ tends to $\psi(t,x)$ as n tends to $+\infty$, for any $t \in [0,T]$, so that $\psi(\cdot,x)$ is measurable.

Now, a straightforward computation shows that $Q^{(n)}$ belongs to $C^{0,\theta}([0,T]\times\mathbb{R}^N)$ and the $C^{0,\theta}$-norms of the coefficients satisfy condition (ii) in the statement of the lemma. Moreover, taking Hypotheses 2.2(i) into account, we get

$$\begin{aligned}
\langle Q^{(n)}(t,x)\xi,\xi\rangle &\ge \nu|\xi|^2\left(\frac{n}{4\pi}\right)^{\frac{1}{2}}\int_0^T e^{-\frac{n}{4}|t-s|^2}ds \\
&= \nu|\xi|^2\left(\frac{n}{4\pi}\right)^{\frac{1}{2}}\left\{\int_0^t e^{-\frac{n}{4}s^2}ds + \int_0^{T-t} e^{-\frac{n}{4}s^2}ds\right\} \\
&\ge \nu|\xi|^2\left(\frac{n}{4\pi}\right)^{\frac{1}{2}}\int_0^{T/2} e^{-\frac{n}{4}s^2}ds \\
&\ge \nu|\xi|^2\frac{1}{2\sqrt{\pi}}\int_0^{T/2} e^{-\frac{1}{4}s^2}ds,
\end{aligned}$$

for any $\xi \in \mathbb{R}^N$ and any $(t,x) \in [0,T]\times\mathbb{R}^N$. Further, since $q_{ij}^{(n)}(\cdot,x)$ converges to $q_{ij}(\cdot,x)$ in $L^p(]0,T[)$ for any $p \in [1,+\infty[$, any $x \in \mathbb{R}^N$ and any $i,j = 1,\ldots,N$, we can find out an increasing sequence $\{n_k(x)\} \subset \mathbb{N}$ such that the subsequence $q_{ij}^{(n_k(x))}(t,x)$ converges to $q_{ij}(t,x)$ as n tends to $+\infty$ a.e. in $[0,T]$.

By a classical diagonal procedure, we can determine an increasing sequence $\{n_k\} \subset \mathbb{N}$ and a measurable set $\mathcal{E} \subset [0,T]$, whose complement is negligible in

$[0,T]$, such that

$$\lim_{k\to+\infty} q_{ij}^{(n_k)}(t,x) = q_{ij}(t,x), \qquad (t,x) \in \mathcal{E}\times\mathbb{Q}^N,\ \ i,j=1,\dots,N. \tag{5.1}$$

Let us now show that we can extend (5.1) to any $(t,x)\in\mathcal{E}\times\mathbb{R}^N$. For this purpose, we fix $(t,x)\in\mathcal{E}\times\mathbb{R}^N$ and denote by $\{x_m\}$ any sequence in $\mathbb{Q}^N$ converging to x as m tends to $+\infty$. Since $q_{ij}^{(n_k)}\in C^{0,\theta}([0,T]\times\mathbb{R}^N)$ and $q_{ij}\in B^{0,\theta}([0,T]\times\mathbb{R}^N)$ for any $k\in\mathbb{N}$ and any $i,j=1,\dots,N$, taking condition (ii) into account, we can write

$$\begin{aligned}
&|q_{ij}^{(n_k)}(t,x) - q_{ij}(t,x)|\\
&\qquad\le |q_{ij}^{(n_k)}(t,x_m) - q_{ij}^{(n_k)}(t,x)| + |q_{ij}^{(n_k)}(t,x_m) - q_{ij}(t,x_m)|\\
&\qquad\quad + |q_{ij}(t,x_m) - q_{ij}(t,x)|\\
&\qquad\le 2\|q_{ij}\|_{B^{0,\theta}([0,T]\times\mathbb{R}^N)}|x-x_m|^\theta + |q_{ij}^{(n_k)}(t,x_m) - q_{ij}(t,x_m)|,
\end{aligned} \tag{5.2}$$

for any $k,m\in\mathbb{N}$. Taking, first, the limsup when k goes to $+\infty$ in the first- and last-side of (5.2), and letting, then, m go to $+\infty$, (5.1) follows, for any $x\in\mathbb{R}^N$. $\square$

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. The uniqueness of a solution to problem (4.1), in the sense of Definition 1.2, follows immediately from Proposition 2.3. So, we just need to show the existence part of the theorem. For this purpose, for any $n\in\mathbb{N}$ we introduce the operator $\mathcal{L}^{(n)}$ defined by

$$\mathcal{L}^{(u)}u(t,x) = \mathrm{Tr}(Q^{(n)}(t,x)D^2u(t,x)) + \langle A^{(n)}(t)x + b^{(n)}(t,x), Du(t,x)\rangle,$$

for any $(t,x)\in[0,T]\times\mathbb{R}^N$, where $Q^{(n)}$, $A^{(n)}$ and $b^{(n)}$ are given by Lemma 5.1. Further, let f and g be as in the statement of the theorem. Using the same arguments as in the proof of the quoted lemma, we approximate the function g with a sequence $\{g_n\}$ of functions in $C^{0,\theta}([0,T]\times\mathbb{R}^N)$, such that their $C^{0,\theta}([0,T]\times\mathbb{R}^N)$-norms can be estimated from above by $\|g\|_{B^{0,\theta}([0,T]\times\mathbb{R}^N)}$, for any $n\in\mathbb{N}$. From Theorem 1.1, we know that the problem

$$\begin{cases} D_t u(t,x) = \mathcal{L}^{(n)}u(t,x) + g^{(n)}(t,x), & t\in[0,T],\quad x\in\mathbb{R}^N,\\ u(0,x) = f(x), & x\in\mathbb{R}^N,\end{cases}$$

admits a unique classical solution u_n which belongs to $C^{0,2+\theta}([0,T]\times\mathbb{R}^N)$. Moreover, there exists a constant C_1 (which can be taken independent of n due to Lemma 5.1) such that

$$\begin{aligned}
\|u_n\|_{C^{0,2+\theta}([0,T]\times\mathbb{R}^N)} &\le C_1\left(\|f\|_{C_b^{2+\theta}(\mathbb{R}^N)} + \|g^{(n)}\|_{C^{0,\theta}([0,T]\times\mathbb{R}^N)}\right)\\
&\le C_1\left(\|f\|_{C_b^{2+\theta}(\mathbb{R}^N)} + \|g\|_{B^{0,\theta}([0,T]\times\mathbb{R}^N)}\right).
\end{aligned}$$

Arguing as in the proof of Theorem 1.1, we can now show that the functions u_n, D_iu_n and $D_{ij}u_n$ ($i,j=1,\dots,N$, $n\in\mathbb{N}$) are equibounded and equicontinuous in $[0,T]\times\overline{B}(R)$ for any $R>0$. Moreover, any u_n is Lipschitz continuous in

$[0,T]\times B(R)$, with Lipschitz norm that can be estimated from above by a positive constant, independent of n. By the arbitrariness of $R>0$ and invoking the Ascoli-Arzelà theorem, we infer that there exists an increasing sequence $\{n_k\}\subset\mathbb{N}$ such that u_{n_k} and its first- and second-order space derivatives converge, locally uniformly in $[0,T]\times\mathbb{R}^N$, to a function $u\in C^{0,2+\theta}([0,T]\times\mathbb{R}^N)$ which belongs also to $\mathrm{Lip}([0,T]\times\overline{B}(R))$ for any $R>0$. Hence, u is differentiable a.e. in $[0,T]\times\mathbb{R}^N$ with respect to t. To prove that the function $u(\cdot,x)$ is differentiable a.e. in $[0,T]$ for any arbitrarily fixed $x\in\mathbb{R}^N$, we observe that

$$\begin{aligned} u_{n_k}(t,x) &= f(x)+\int_0^t D_t u_{n_k}(s,x)ds \\ &= f(x)+\int_0^t\left(\mathcal{L}^{(n_k)}u_{n_k}(s,x)+g^{(n_k)}(s,x)\right)ds, \end{aligned} \tag{5.3}$$

for any $(t,x)\in[0,T]\times\mathbb{R}^N$ and any $n\in\mathbb{N}$. By Lemma 5.1, we deduce that there exists a measurable set $\mathcal{E}\subset[0,T]$, whose complement is negligible in $[0,T]$, such that

$$\lim_{k\to+\infty}\{\mathcal{L}^{(n_k)}u_{n_k}(t,x)+g^{(n_k)}(t,x)\}=\mathcal{L}u(t,x)+g(t,x),\qquad (t,x)\in\mathcal{E}\times\mathbb{R}^N.$$

Moreover, the sup-norm of $\mathcal{L}^{(n_k)}u_{n_k}$, when (t,x) runs in $[0,T]\times\overline{B}(R)$ and $R>0$ is arbitrarily fixed, is bounded from above by a positive constant independent of n. Hence, we can take the limit as n tends to $+\infty$ in (5.3), getting

$$u(t,x)=f(x)+\int_0^t(\mathcal{L}u(s,x)+g(s,x))\,ds,\qquad (t,x)\in[0,T]\times\mathbb{R}^N. \tag{5.4}$$

Since the function $\mathcal{L}u+g$ is locally bounded in $[0,T]\times\mathbb{R}^N$, it follows that, for any $x\in\mathbb{R}^N$, there exists a set $\mathcal{G}(x)$ with measure T such that $u(\cdot,x)$ is differentiable in $\mathcal{G}(x)$ and $D_tu(t,x)=\mathcal{L}u(t,x)+g(t,x)$ for any $t\in\mathcal{G}(x)$. Let now $\mathcal{G}=\bigcap_{x\in\mathbb{Q}^N}\mathcal{G}(x)$. Of course, $\mathcal{G}$ is measurable with measure T. We are going to prove that, for any $x\in\mathbb{R}^N$, the function $u(\cdot,x)$ is differentiable in $\mathcal{G}$ and $D_tu=\mathcal{L}u+g$ in $\mathcal{G}\times\mathbb{R}^N$. For this purpose, we denote by w the integral function in (5.4). Moreover, we fix $(t,x)\in\mathcal{G}\times\mathbb{R}^N$ and let $\{x_n\}\subset\mathbb{Q}^N\cap B(|x|+1)$ be any sequence converging to x as n tends to $+\infty$. For notational convenience we denote by $\Delta_h w(t,x)$ the incremental ratio (with respect to the time variable) of the function w at the point (t,x). Then, there exists a positive constant C_2, independent of h, n and t, such that

$$\begin{aligned} &|(\Delta_h w)(t,x)-(\Delta_h w)(t,x_n)| \\ &\quad=\left|\frac{1}{h}\int_t^{t+h}(\mathcal{L}u(s,x)+g(s,x)-\mathcal{L}u(s,x_n)-g(s,x_n))\,ds\right| \\ &\quad\le C_2|x-x_n|^\theta. \end{aligned}$$

Here, we have used the fact that, for any $t\in[0,T]$, the function $(\mathcal{L}u)(t,\cdot)+g(t,\cdot)$ is θ-Hölder continuous in $B(|x|+1)$, with Hölder norm being independent of t.

Hence,

$$\begin{aligned}
&|(\Delta_h w)(t,x) - \mathcal{L}u(t,x) - g(t,x)| \\
&\qquad \leq |(\Delta_h w)(t,x) - (\Delta_h w)(t,x_n)| + |(\Delta_h w)(t,x_n) - \mathcal{L}w(t,x_n) - g(t,x_n)| \\
&\qquad\quad + |\mathcal{L}w(t,x) + g(t,x) - \mathcal{L}w(t,x_n) - g(t,x_n)| \\
&\qquad \leq |(\Delta_h w)(t,x_n) - \mathcal{L}w(t,x_n) - g(t,x_n)| + C_3|x - x_n|^\theta,
\end{aligned} \tag{5.5}$$

for some positive constant C_3, independent of h and n. Taking the limsup when h tends to 0 in (5.5) and, then, letting n go to $+\infty$, we deduce that $w(\cdot,x)$ is differentiable with respect to time at t and $D_t u(t,x) = \mathcal{L}u(t,x) + g(t,x)$.

Finally, we observe that (5.4) also implies that $u(0,\cdot) \equiv f$. Hence, u is a solution to problem (4.1) in the sense of Definition 1.2. □

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Time-dependent Nonlinear Perturbations of Analytic Semigroups

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Abstract. This paper is concerned with time-dependent relatively continuous perturbations of analytic semigroups and applications to convective reaction-diffusion systems. A general class of time-dependent semilinear evolution equations of the form $u_t = (A + B(t))u(t),\ t \in (s, \tau);\ u(s) = v \in D(s)$ is introduced in a general Banach space X. Here A is the generator of an analytic semigroup in X and $B(t)$ is a possibly nonlinear operator from a subset of the domain of a fractional power $(-A)^\alpha$ into X and $D(t) = D(B(t)) \subset D((-A)^\alpha)$. This type of semilinear evolution equations admit only local and mild solutions in general. In order to restrict the growth of mild solutions and formulate a Lipschitz conditions in a local sense for $B(t)$, a lower semicontinuous functional $\varphi : D((-A)^\alpha) \to [0, +\infty]$ is introduced and the growth condition of $u(\cdot)$ is formulated in terms of the nonnegative function $\varphi(u(\cdot))$ and the nonlinear operator $B(t)$ is assumed to be Lipschitz continuous on $D_\rho(t) \equiv \{v \in D(t) : \varphi(v) \le \rho\}$ for $\rho > 0$. The main objective is to establish a generation theorem for a nonlinear evolution operator which provides mild solutions to the semilinear evolution equation under the assumption that a consistent discrete scheme exists under a growth condition with respect to φ as well as closedness condition for the noncylindrical domain $\bigcup(\{t\} \times D_\rho(t))$. Moreover, a characterization theorem for the existence of such evolution operator is established in terms of the existence of φ-bounded discrete schemes. Our generation theorem can be applied to a variety of semilinear convective reaction-diffusion systems. We here make an attempt to apply our result to a mathematical model which describes a complex physiological phenomena of bone remodeling.

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Keywords. Nonlinear perturbation, analytic semigroup, evolution operator, subtangential type condition, generation theorem, characterization of evolution operators, local multiple Laplace transform, mathematical model of bone remodeling.

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1. Introduction

Of concern in this paper are the semilinear evolution equations in a Banach space $(X, |\cdot|)$ of the form

$$\text{(SE)} \qquad u'(t) = (A + B(t))u(t), \quad s < t < \tau,$$

under the initial conditions

$$\text{(IC)} \qquad u(s) = v \in D(s) \subset Y, \quad 0 \leq s < \tau.$$

Here A is assumed to be the generator of an analytic semigroup $\{T(t) : t \geq 0\}$ in X, Y is a Banach space which is contained in X and has a stronger norm defined through a fractional power of $-A$, and $\{B(t) : t \geq 0\}$ is a one-parameter family of nonlinear operators from subsets of Y. The set $D(s)$ stands for the domain of the nonlinear operator $B(s)$. The objective of the present paper is to discuss the time-dependent nonlinear perturbations of analytic semigroups in the case where $D(A)$ is not necessarily dense in X and Y is supposed to be a subspace of the domain of a fixed fractional power $(-A)^\alpha$ equipped with its graph norm and $B(t)$ is locally Lipschitz continuous from Y with the stronger norm into the original space X. Accordingly, $A + B(t)$ are understood to be time-dependent relatively continuous perturbations of the analytic semigroup $\{T(t) : t \geq 0\}$ in X. Generation and characterization of evolution operators providing mild solutions to the semilinear problem (SE)-(IC) are discussed.

The importance of semilinear problems of the type (SE)-(IC) has constantly been recognized in various branches of mathematical sciences. In fact, many of mathematical models describing nonlinear convective diffusion phenomena are formulated as semilinear evolution problems for reaction-diffusion systems, quasilinear transport-diffusion systems and so on.

In this paper we introduce a general class of time-dependent locally Lipschitz continuous perturbations of analytic semigroups in X and discuss the necessary and sufficient condition for the family $\{A+B(t)\}$ of semilinear operators to provide the mild solutions to the problem (SE)-(IC) in a global sense. This is the most significant feature of our argument. Those necessary and sufficient conditions are formulated in terms of existence of approximate difference scheme to (SE)-(IC). By means of those approximate difference schemes, approximate solutions to (SE)-(IC) are constructed. In consequence, generation and characterization of the associated evolution operators is discussed. Namely, under the assumption that an evolution operator provides mild solutions to (SE)-(IC), the existence of those approximate difference scheme can be derived.

So far, semilinear problems of the type (SE)-(IC) have been studied by many authors. First, let D be a closed subset of X and $D(t) \equiv D \cap Y$ in (IC). Lightbourne and Martin discussed in [11] the construction of mild solutions to (SE) under the subtangential condition

$$\text{(s.1)} \qquad \liminf_{h \downarrow 0} h^{-1} d(v + hB(t)v, D) = 0 \quad \text{for } t \in [0, \tau) \text{ and } v \in D \cap Y$$

and the assumption that D is invariant under $T(t)$ in the sense that $T(t)[D] \subset D$. H. Amann then considered in [2] the time-dependent case

$$\text{(SP)} \qquad \begin{cases} u'(t) = (A(t) + B(t))u(t), \quad 0 < t < \tau, \\ u(0) = v \in D(0) \end{cases}$$

and treated the existence of mild solutions to (SP) under (s.1) and the assumption that $U(t,s)[D] \subset D$, where $U(t,s)$ denotes the evolution operator generated by $\{A(t)\}$. Secondly, we consider the case where $D(t)$ be a closed subset of the Banach space $[D((-A(t))^\alpha)]$ equipped with the graph norm in (IC). In this setting J. Prüss showed in [27] that for the existence of local mild solutions to (SP) it is necessary and sufficient that the generalized subtangential condition below holds :

(s.2) There is a constant $\eta > 0$ such that for every $t \in [0,\tau)$, $v \in D(t)$ and $\varepsilon > 0$, there exist $h > 0$ and $w_h \in D(t+h)$ and

$$z_h = U(t+h,t)v + \int_t^{t+h} U(t+h,\xi)B(t)v\, d\xi - w_h$$

satisfies $|z_h| \le \varepsilon h$ and $|(-A(t+h))^\alpha z_h| \le \varepsilon h^\eta$.

Thirdly, it is also natural to treat the case in which D be a closed subset of X and put $D(t) = D \cap D((-A(t))^\alpha)$ for $t \in [0,\tau]$. Under the condition of flow invariance $U(t,s)[D] \subset D$, Chen showed in [4] that Pavel's subtangential condition

$$\text{(s.3)} \qquad \lim_{h \downarrow 0} h^{-1} d(U(t+h,t)v + hB(t)v, D) = 0$$

gives a necessary and sufficient condition for the existence of local mild solutions to (SP).

On the other hand, Oharu and Pazy [22], Oharu [21] and Oharu and Tebbs [24] considered the time-independent case $B(t) \equiv B$ and treated necessary and sufficient conditions on A and B for the existence of mild solutions to (SE) in a global sense. This problem is important from both theoretical and practical points of view. They interpret this problem as a characterization problem of a nonlinear semigroup which provides mild solutions to (SE) satisfying an appropriate growth condition and discuss the characterization of such nonlinear semigroup in terms of the range condition of the form

$$\text{(R)} \qquad R(I - \lambda(A+B)) \ni y \quad \text{for } y \in D(B), \lambda \in (0, \lambda(y)) \text{ and some } \lambda(y) > 0$$

under the convexity condition on $D(B)$ and $\varphi(\cdot)$.

The arguments in [22] contain three features: First, a lower semicontinuous functional $\varphi : Y \to [0,+\infty]$ is employed to define a local Lipschitz continuity of B and restrict the growth of solutions to (SE) by means of nonnegative function $\varphi(u(\cdot))$. Secondly, the semilinear operator $A + B$ is assumed to satisfy the range condition (R). We here employ the same point of view as in [22] and interpret the semilinear problem (SE)-(IC) as the generation problem of a nonlinear evolution operator which gives the mild solutions.

As mentioned in the Abstract, we employ a general functional $\varphi : Y \to [0, +\infty]$ to define the local Lipschitz continuity of the operators $B(t)$ and restrict the growth of mild solutions to (SE) as well as the time-dependence of $B(t)$. In this paper we employ the growth condition of the form

$$\text{(G)} \qquad \varphi(u(t)) \leq \Psi(\tau, 0; \varphi(v))$$

for $t \in [0, \tau]$ and $v \in D(0)$. Here the function Ψ is determined in various ways in accordance with the properties of the semilinear operators $A + B(t)$. In fact, (G) is nothing else but a boundedness condition for the solutions of the semilinear problem (SE)-(IC), and so Ψ is often found through *a priori* estimates for the solutions.

First, we establish a characterization theorem for such evolution operators under much weaker assumptions than those imposed in [22]. We here show that the existence of approximate scheme is a necessary and sufficient condition for the existence of an evolution operator associated with the problem (SE)-(IC). With this result, we investigate the uniqueness and the regularity of the mild solutions of the semilinear problem (SE)-(IC).

In this paper we apply two basic fundamental concepts. One of the important tools of deriving our main results is the so-called measure of noncompactness. In [22] it is shown with the aid of measure of noncompactness that the existence of a nonlinear semigroup associated with (SE) implies the range condition for $A + B$. Here we also use the measure of noncompactness to establish a generation theorem for a nonlinear evolution operator which provides mild solutions to (SE). Another important tool is discrete local multiple Laplace transform which is used for the proof of characterization theorem.

This paper is organized as follows: In Sections 2 and 3, we summarize the definition and properties of analytic semigroups as well as their fractional powers. Fundamental notions such as mild solution and nonlinear evolution operators are introduced and basic assumptions are imposed in Section 4. Under these assumptions the main results of this paper are also stated in this section. Section 5 deals with the uniqueness and regularity of mild solutions to the problem (SE)-(IC). Section 6 is devoted to the proof of the generation theorem for nonlinear evolution operators providing mild solutions to the problem (SE)-(IC). In Section 7 we discuss discrete local multiple Laplace transforms. Applying these results to Section 8, we demonstrate that the existence of an evolution operator associated with (SE)-(IC) implies the existence of approximate difference schemes. Accordingly, necessary and sufficient conditions for the global existence of mild solutions to the semilinear problem are obtained. Finally, in Section 9, the approximation-solvability of a mathematical model for bone-remodeling phenomena is verified by applying our result. The solvability of this model has not been known so far.

2. A linear theory

In this section we outline the definition and the characterization theorem of analytic semigroups.

Definition 2.1. Let $\mathscr{T} \equiv \{T(t) : t \geq 0\}$ be a semigroup in X. Let $0 < \theta < \frac{\pi}{2}$. We say that $\mathscr{T}$ is an *analytic semigroup* in the sector $\Sigma_\theta = \{t \in \mathbb{C} : |\arg t| < \theta\}$ if it satisfies the following conditions:

(T1) $\mathscr{T}$ can be continued analytically to Σ_θ.

(T2) For each $0 < \varepsilon < \theta$,

$$\lim_{\substack{t \to 0 \\ |\arg t| \leq \theta - \varepsilon}} T(t)x = x \quad \text{for } x \in \overline{D(A)}.$$

The theorem below gives the characterization of such analytic semigroups.

Theorem 2.2. *Let A be a closed linear operator in X. Then A is the generator of an analytic semigroup satisfying*

(C) *For each $0 < \varepsilon < \theta$ there is a constant $M_\varepsilon > 0$ such that*

$$\left|e^{-\omega t} T(t)\right| \leq M_\varepsilon \quad \textit{for some } \omega \in \mathbb{R} \textit{ and } t \in \overline{\Sigma_{\theta - \varepsilon}}$$

if and only if A satisfies the following two conditions:

(A1) $\rho(A) \supset \{\lambda \in \mathbb{C} : |\arg(\lambda - \omega)| < \frac{\pi}{2} + \theta\}$.

(A2) *For each $0 < \varepsilon < \theta$ there is a constant $M_\varepsilon > 0$ such that*

$$|(\lambda - A)^{-1}| \leq \frac{M_\varepsilon}{|\lambda - \omega|} \qquad \textit{for } \lambda \textit{ with } |\arg(\lambda - \omega)| \leq \frac{\pi}{2} + \theta - \varepsilon.$$

Remark 2.3. $\mathscr{T}$ is defined by

$$T(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda - A)^{-1} \, d\lambda \quad \text{for } t \in \Sigma_\theta, \tag{2.1}$$

where Γ is a smooth curve in Σ_θ running from $\infty e^{-i(\theta - \varepsilon)}$ to $\infty e^{i(\theta - \varepsilon)}$ in the complex plane. $\mathscr{T}$ is not of class (C_0) but has properties similar to those in the case of $\overline{D(A)} = X$. For details we refer to Sinestrari [28]. The limit $\lim_{t \to 0} T(t)x$ exists if and only if $x \in \overline{D(A)}$. In this case we have $\lim_{t \to 0} T(t)x = x$. By Da Prato and Sinestrari [5, Theorem 10.2], we have

$$(\lambda - A)^{-1} x = \int_0^\infty e^{-\lambda t} T(t) x \, dt \quad \text{for } \lambda > 0 \text{ and } x \in X. \tag{2.2}$$

It should be noted here that A is the generator of the integrated semigroup $\{W(t) = \int_0^t T(s) \, ds : t \geq 0\}$. For details on integrated semigroups and related topics we refer to [3, 5, 9, 12, 20, 29, 30, 31].

Exponential formula

Finally, we demonstrate that the analytic semigroup $\mathscr{T}$ is obtained by the so-called exponential formula. Without loss of generality, we may assume that $\omega = 0$. Let $\tau > 0$. For each $\varepsilon \in (0,1)$ we choose a sequence $\{t_j^\varepsilon\}_{j=0}^{N(\varepsilon)}$ in the time interval $[0,\tau]$ satisfying

$$\begin{aligned} &0 = t_0^\varepsilon < t_1^\varepsilon < \cdots < t_{N(\varepsilon)}^\varepsilon \leq \tau, \\ &\max\{t_j^\varepsilon - t_{j-1}^\varepsilon : 1 \leq j \leq N(\varepsilon)\} < \varepsilon, \\ &\lim_{\varepsilon \downarrow 0} t_{N(\varepsilon)}^\varepsilon = \tau. \end{aligned} \tag{2.3}$$

Set $h_j^\varepsilon = t_j^\varepsilon - t_{j-1}^\varepsilon$, $(1 \leq j \leq N(\varepsilon))$. We then define an operator-valued function $J_\varepsilon(\cdot)$ over $[0, t_{N(\varepsilon)}^\varepsilon]$ by

$$J_\varepsilon(t) = \begin{cases} I & \text{for } \ t = 0, \\ \displaystyle\prod_{j=1}^k (I - h_j^\varepsilon A)^{-1} & \text{for } t \in (t_{k-1}^\varepsilon, t_k^\varepsilon] \ \text{ and } \ 1 \leq k \leq N(\varepsilon). \end{cases}$$

Theorem 2.4. *Let A be a closed linear operator in X and $\tau > 0$. Suppose that A satisfies* (A1) *and* (A2). *Then the analytic semigroup $\mathscr{T} \equiv \{T(t) : t > 0\}$ is represented as*

$$T(t)x = \lim_{\varepsilon \downarrow 0} J_\varepsilon(t)x \quad \text{for } x \in X \text{ and } t \in (0,\tau], \tag{2.4}$$

and for each $\delta \in (0,\tau)$ the convergence is uniform on $[\delta,\tau]$. In particular we have

$$T(t)x = \lim_{n\to\infty} \left(I - \frac{t}{n}A\right)^{-n} x \quad \text{for } x \in X \text{ and } t \in (0,\tau].$$

Proof. Since the integral representation (2.1) of $T(t)$ does not depend on the choice of Γ, we take the following integral path:

$$\begin{aligned} &\Gamma = \Gamma^3 \cup \Gamma^2 \cup \Gamma^1, \quad \Gamma^1 : \lambda = re^{i\phi}, \quad \Gamma^2 : \lambda = e^{i\psi} \quad (|\psi| \leq \phi), \\ &\Gamma^3 : \lambda = re^{-i\phi}, \quad \phi = \frac{\pi}{2} + \theta - \varepsilon, \quad 1 \leq r < \infty. \end{aligned}$$

We first show that

$$\prod_{j=1}^k (I - h_j^\varepsilon A)^{-1} = \frac{1}{2\pi i}\int_\Gamma \prod_{j=1}^k (1 - h_j^\varepsilon \lambda)^{-1}(\lambda - A)^{-1}\, d\lambda \tag{2.5}$$

holds for $t \in (t_{n-1}^\varepsilon, t_n^\varepsilon]$. The elementary inequality $(1+a)^{-1}(1+b)^{-1} \leq (1+a+b)^{-1}$ for $a, b > 0$ and the fact that $\cos\phi < 0$ imply the inequality

$$\prod_{j=1}^k (1 - h_j^\varepsilon r\cos\phi)^{-1} \leq \left\{1 - (\sum_{j=1}^k h_j^\varepsilon) r\cos\phi\right\}^{-1}. \tag{2.6}$$

Using this inequality we obtain

$$\begin{aligned}
&\int_{\Gamma^1} \left| \prod_{j=1}^{k} (1 - h_j^{\varepsilon}\lambda)^{-1}(\lambda - A)^{-1} \right| dr \\
&\quad \leq \int_1^{\infty} \prod_{j=1}^{k} \left\{(1 - h_j^{\varepsilon} r\cos\phi)^2 + (h_j^{\varepsilon} r \sin\phi)^2\right\}^{-1/2} \frac{M_\varepsilon}{r}\, dr \\
&\quad \leq \int_1^{\infty} \left[\prod_{j=1}^{k} (1 - h_j^{\varepsilon} r\cos\phi)^{-1}\right] \frac{M_\varepsilon}{r}\, dr \\
&\quad \leq \int_1^{\infty} \left\{1 - (\sum_{j-1}^{k} h_j^{\varepsilon}) r\cos\phi\right\}^{-1} \frac{M_\varepsilon}{r}\, dr < +\infty.
\end{aligned}$$

Similarly, we see that

$$\int_{\Gamma^3} \left| \prod_{j=1}^{k} (1 - h_j^{\varepsilon}\lambda)^{-1}(\lambda - A)^{-1} \right| dr < +\infty,$$

and that

$$\int_{\Gamma^2} \left| \prod_{j=1}^{k} (1 - h_j^{\varepsilon}\lambda)^{-1}(\lambda - A)^{-1} \right| d\psi \leq \int_{-\phi}^{\phi} M_\varepsilon \prod_{j=1}^{k} (1 - h_j^{\varepsilon})^{-1}\, d\psi < +\infty.$$

Hence the integral on the right-hand side of (2.5) makes sense. We then take a real number $R > \max\{h_j^{-1} : 1 \leq j \leq n\}$ and set

$$\Gamma_R \equiv \Gamma_R^3 \cup \Gamma^2 \cup \Gamma_R^1, \quad \Gamma_R^1 : \lambda = re^{i\phi}, \quad \Gamma_R^3 : \lambda = re^{-i\phi} \quad (1 \leq r \leq R)$$

$$C_R : \lambda = Re^{i\psi} \quad (|\psi| \leq \phi).$$

It follows from the residual theorem that

$$\begin{aligned}
&\frac{1}{2\pi i} \int_{\Gamma_R} \prod_{j=1}^{k} (1 - h_j^{\varepsilon}\lambda)^{-1}(\lambda - A)^{-1}\, d\lambda \\
&\quad = \frac{1}{2\pi i} \int_{C_R} \prod_{j=1}^{k} (1 - h_j^{\varepsilon}\lambda)^{-1}(\lambda - A)^{-1}\, d\lambda + \prod_{j=1}^{k} (I - h_j^{\varepsilon} A)^{-1}.
\end{aligned}$$

It is also seen that

$$\begin{aligned}
\int_{C_R} \left| \prod_{j=1}^{k} (1 - h_j^{\varepsilon}\lambda)^{-1}(\lambda - A)^{-1} \right| d\psi &\leq \int_{-\phi}^{\phi} M \prod_{j=1}^{k} (1 - h_j^{\varepsilon} R)^{-1}\, d\psi \\
&\longrightarrow 0 \qquad \text{as } R \to \infty.
\end{aligned}$$

Thus we have asserted that (2.5) holds. Since (2.6) holds for $\lambda \in \Gamma_1 \cup \Gamma_3$, (2.4) follows by taking the limit as $k \to \infty$ in (2.5).

We next show that the convergence is uniform with respect to $t \in [\delta, \tau]$ for $\delta \in (0,\tau)$. Take $\delta \in (0,\tau)$ and fix it. Let $\{t_j^\varepsilon\}_{j=0}^{N(\varepsilon)} \subset [0, 2\tau]$ be the sequence in (2.3) replaced τ by 2τ. Then there exists $\varepsilon_0 \equiv \varepsilon_0(\delta,\tau) \in (0,1)$ such that, to each $\varepsilon \in (0,\varepsilon_0]$ and each $t \in [\delta,\tau]$, there corresponds $n \geq 2$ satisfying $t \in (t_{n-1}^\varepsilon, t_n^\varepsilon]$. Assume $\delta \in (t_{k-1}^\varepsilon, t_k^\varepsilon]$ and set

$$J'_\varepsilon(\delta)x = \prod_{j=1}^{k-1}(I - h_j^\varepsilon A)^{-1}x, \quad \text{and} \quad K = \{J'_\varepsilon(\delta)x : \varepsilon \in (0,\varepsilon_0]\} \cup \{T(\delta)x\}.$$

Then it is easily seen that K is a compact subset of $D(A)$. Since the restriction of $\mathscr{T}$ on $\overline{D(A)}$ is a C_0-semigroup on $\overline{D(A)}$, we have

$$\lim_{\varepsilon\downarrow 0} \prod_{j=k}^{n}(I - h_j^\varepsilon A)^{-1}y = T(t-\delta)y \quad \text{for } t \in [\delta,\tau] \text{ and } y \in K \tag{2.7}$$

and the convergence is uniform. By (2.2) we see that there exists $M > 0$ satisfying

$$\left|\prod_{j=k}^{n}(I - h_j^\varepsilon A)^{-1}\right| \leq M.$$

Take arbitrary $\eta > 0$. Then it follows from (2.4) and (2.7) that there exists $\varepsilon_1 \in (0,\varepsilon_0)$ satisfying

$$\left|T(\delta)x - \prod_{j=1}^{k-1}(I - h_j^\varepsilon A)^{-1}x\right| \leq \frac{\eta}{2M} \tag{2.8}$$

and

$$\left|\prod_{j=k}^{n}(I - h_j^\varepsilon A)^{-1}T(\delta)x - T(t-\delta)T(\delta)x\right| \leq \frac{\eta}{2} \tag{2.9}$$

for $\varepsilon \in (0,\varepsilon_1)$ and $t \in [\delta,\tau]$. (2.8) and (2.9) together imply

$$\begin{aligned}\left|\prod_{j=1}^{n}(I - h_j^\varepsilon A)^{-1}x - T(t)x\right| &\leq \left|\prod_{j=k}^{n}(I - h_j^\varepsilon A)^{-1}T(\delta)x - T(t-\delta)T(\delta)x\right| \\ &\quad + \left|\prod_{j=k}^{n}(I - h_j^\varepsilon A)^{-1}\right|\left|T(\delta)x - \prod_{j=1}^{k-1}(I - h_j^\varepsilon A)^{-1}x\right| \\ &\leq \eta.\end{aligned}$$

for $t \in [\delta,\tau]$. This completes the proof. □

3. Fractional powers of non-densely defined closed linear operators

In this section we outline the properties of the fractional powers of non-densely defined closed linear operators in X.

Let $\omega \in (0, \pi/2)$. Following Pazy [25], we make the two assumptions below:

(A1)$'$ A is a closed linear operator in X for which

$$\rho(A) \supset \Sigma^+ = \{X : 0 \le |\arg \lambda| \le \pi - \omega\} \cup V.$$

where V is a neighborhood in $\mathbb{C}$ of zero.

(A2)$'$ There exists a constant $M > 0$ such that

$$|(\lambda - A)^{-1}| \le \frac{M}{1 + |\lambda|} \quad \text{for } \lambda \in \Sigma^+.$$

Note that A is the generator of an analytic semigroup $\mathscr{T} \equiv \{T(t) : t \ge 0\}$ and $0 \in \rho(A)$. As in the case of $\overline{D(A)} = X$ ([13, Lemma 2.1.6 and Chapter 2, Section 2.2]), a fractional power $(-A)^{-\alpha}$ is defined by

$$(-A)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t)\, dt. \tag{3.1}$$

The integral converges in the uniform topology for every $\alpha > 0$. Since $(-A)^{-\alpha}$ is injective, a fractional power of $-A$ is defined by

$$D((-A)^\alpha) = \text{Ran }((-A)^{-\alpha}),$$

$$(-A)^\alpha = ((-A)^{-\alpha})^{-1} \quad \text{for } \alpha > 0 \text{ and } (-A)^0 = I.$$

For $0 < \alpha < 1$ and $x \in D(A)$, the fractional power $(-A)^\alpha x$ has an integral representation

$$(-A)^\alpha x = \frac{-\sin \pi\alpha}{\pi} \int_0^\infty t^{\alpha-1} A(tI - A)^{-1} x\, dt. \tag{3.2}$$

The proof is exactly same as in [25, Theorem 6.9]. For details we refer to [13, Chapter 2].

We need the following two lemmas for the subsequent arguments.

Lemma 3.1. *Assume* (A1)$'$ *and* (A2)$'$. *Then the following are valid:*

(a) $\overline{D((-A)^\alpha)} = \overline{D(A)}$ *for every* $\alpha > 0$.

(b) $\overline{D(A)}^{|\cdot|_\alpha} = \{x \in D((-A)^\alpha) : (-A)^\alpha x \in \overline{D(A)}\}$,

where $|\cdot|_\alpha$ *is the graph norm of* $(-A)^\alpha$.

Proof. We first show that (a) holds. Since $T(t) : X \to \overline{D(A)}$ by (2.1) for $t > 0$, we obtain from (3.1) that

$$D((-A)^\alpha) = R((-A)^{-\alpha}) \subset \overline{D(A)},$$

and so that $\overline{D((-A)^\alpha)} \subset \overline{D(A)}$. We denote by A_0 the part of A in $\overline{D(A)}$. Then A_0 is the infinitesimal generator of an analytic semigroup on $\overline{D(A)}$ and we have $\overline{D(A_0)} = \overline{D(A)}$. Since $\overline{D(A_0^n)} = \overline{D(A)}$ for $n \geq 1$ and $\overline{D(A_0^n)} \subset \overline{D(A^n)} \subset \overline{D(A)}$, we obtain $\overline{D(A^n)} = \overline{D(A)}$ for $n \geq 1$. Since $D((-A)^\alpha) \supset D(A^n)$ for $\alpha \leq n$, it is seen that assertion (a) holds. Using the fact that

$$\lim_{\lambda \downarrow 0} (I - \lambda A)^{-1} x = x \quad \text{for } x \in \overline{D(A)},$$

for $y \in \{x \in D((-A)^\alpha) : (-A)^\alpha x \in \overline{D(A)}\}$, we obtain

$$\lim_{\lambda \downarrow 0} (-A)^\alpha (I - \lambda A)^{-1} y = \lim_{\lambda \downarrow 0} (I - \lambda A)^{-1} (-A)^\alpha y = (-A)^\alpha y.$$

This implies that $\overline{D(A)}^{|\cdot|_\alpha} \supset \{x \in D((-A)^\alpha) : (-A)^\alpha x \in \overline{D(A)}\}$. Conversely, let $x \in D(A)$. Then we have by (3.2) $(-A)^\alpha x \in \overline{D(A)}$. This implies that $\overline{D(A)}^{|\cdot|_\alpha} \subset \{x \in D((-A)^\alpha) : (-A)^\alpha x \in \overline{D(A)}\}$. □

Lemma 3.2. (Moments Inequality) *Let $0 \leq \alpha < \beta < \gamma \leq 1$. Then there exists a constant $C_{\alpha,\beta,\gamma} > 0$ such that for every $x \in D((-A)^\gamma)$ and $\rho > 0$, we have*

$$|(-A)^\beta x| \leq C_{\alpha,\beta,\gamma}(\rho^{\beta-\alpha}|(-A)^\alpha x| + \rho^{\beta-\gamma}|(-A)^\gamma x|)$$

and

$$|(-A)^\beta x| \leq 2C_{\alpha,\beta,\gamma}|(-A)^\alpha x|^{(\gamma-\beta)/(\gamma-\alpha)}|(-A)^\gamma x|^{(\beta-\alpha)/(\gamma-\alpha)}.$$

The proof is same as in [6, Theorem 5.34].

4. Nonlinear perturbations of analytic semigroups

We consider the semilinear problem in a Banach space $(X, |\cdot|)$ of the form

$$u'(t) = (A + B(t))u(t), \quad s < t < \tau, \tag{SE}$$

$$u(s) = v \in D(s), \quad s \in [0, \tau). \tag{IC}$$

Here A is the generator of an analytic semigroup $\mathscr{T} \equiv \{T(t) : t \geq 0\}$ which satisfies condition (C) stated in Section 2 and Y is a subspace of the domain $D((-A)^\alpha)$ of the fractional power of A equipped with the graph norm. Throughout this paper we fix the time interval $[0, \tau]$, the exponent $\alpha \in (0, 1)$ and $p \in [1, \infty]$. Let $\mathscr{D} = \{D(t) : t \in [0, \tau]\}$ be a family of nonempty subsets of Y. For each $t \in [0, \tau]$, $B(t)$ is assumed to be a possibly nonlinear operator in X such that the domain $D(t) \equiv D(B(t))$ is a subset of Y. To restrict the time-dependence of the nonlinear operators $B(t)$ we introduce the following family of nonnegative functions defined on all of $[0, \tau]^2$. $f \in L^p((0, \tau) \times (0, \tau))$ belongs to $\mathcal{F}_p$ if and only if there exists a Banach space $(Z, \|\cdot\|_Z)$, $g \in L^p(0, \tau; Z)$ and a nondecreasing function $\psi : [0, \infty) \to [0, \infty)$ such that $\lim_{r \to 0} \psi(r) = 0$ and f is represented as

$f(s,t) = \|g(s)-g(t)\|+\psi(|t-s|)$. It is easily seen that $f \in \mathcal{F}_p$ satisfies the following properties:

$$f(s,t) \geq 0, \quad f(s,t) = f(t,s), \quad f(s,s) = 0,$$
$$f(s,t) \leq f(r,s) + f(r,t), \quad f(\cdot,s) \in L^p(0,\tau),$$
$$\lim_{h\downarrow 0} h^{-1} \int_s^{s+h} f(\xi,s)^p \, d\xi = 0 \quad \text{for almost all } s \in [0,\tau).$$

In this section we investigate the semilinear problem (SE)-(IC) from the same point of view as in Oharu and Pazy [22].

We begin by making assumptions on the operators A and $B(t)$ in the Banach space $(X, |\cdot|)$. Since the semilinear operator $A + B(t)$ can be represented as $A + B(t) = (A - \omega I) + (B(t) + \omega I)$, we may assume that the analytic semigroup $\mathscr{T}$ satisfies the following condition:

(H1) There exist $M \geq 1$ and $\omega > 0$ such that

$$|T(t)| \leq Me^{-\omega t} \quad \text{for } t > 0.$$

We introduce a new Banach space $(Y, \|\cdot\|)$ defined by

$$\|v\| = |(-A)^\alpha v| \quad \text{for } v \in D((-A)^\alpha), \qquad Y = \overline{D(A)}^{\|\cdot\|}.$$

By Theorem 3.1, Y may be represented as

$$Y = \{v \in D((-A)^\alpha) : (-A)^\alpha v \in \overline{D(A)}\}.$$

To restrict the growth of solutions of (SE) and Lipschitz continuity in a local sense of the nonlinear operator $B(t)$, we employ a nonnegative lower semicontinuous functional φ on the Banach space $(Y, \|\cdot\|)$. That is, we impose the following condition:

(H2) For each $t \in [0,\tau]$ the domain of $B(t)$ coincides with $D(t)$ and $D(t) \subset D(\varphi) \equiv \{y \in Y : \varphi(y) < \infty\}$.

(H3) For each $\rho > 0$ there exist $\omega_\rho \geq 0$ and $f_\rho \in \mathscr{F}_p$ such that

$$|B(s)v - B(t)w| \leq \omega_\rho \|v - w\| + f_\rho(s,t)$$

for s, $t \in [0,\tau]$, $v \in D_\rho(s)$ and $w \in D_\rho(t)$.

Because of the localized condition (H3) on $B(t)$ the semilinear problem (SE)-(IC) may admit only local solutions, and it is necessary to impose an appropriate growth condition to construct global solutions. In this paper we employ a function $\Psi : [0,\tau]^2 \times [0,\infty) \to [0,\infty)$ such that for each $s \in [0,\tau]$, $t \in [s,\tau]$ and each $\rho \geq 0$, $\Psi(t,s;\cdot)$ and $\Psi(\cdot,s;\rho)$ are nondecreasing, respectively. Using this function, we impose the growth condition of the form

$$\text{(G)} \qquad \varphi(u(t)) \leq \Psi(\tau,s;\varphi(v)) \quad \text{for } t \in [s,\tau].$$

As mentioned in the introduction, the function Ψ is often constructed through *a priori* estimates for the solutions to the semilinear problems (SE)-(IC). On the other hand, one of the features of our generation argument is that appropriate discrete solutions to a discretization in time of the semilinear problem are first constructed and then boundedness of the discrete solutions with respect to an appropriate lower semicontinuous functional φ is investigated, and so that bounds of those discrete solutions would suggest a concrete form of such function Ψ.

For the semilinear problem (SE)-(IC) we introduce the following notion of generalized solution.

Definition 4.1. Let $0 \le s < \tau$. A Y-valued function $u(\cdot)$ on $[s,\tau]$ is said to be a *mild solution* of the problem (SE)-(IC), if $u(\cdot) \in C([s,\tau];X) \cap L^p(s,\tau;Y)$, $u(t) \in D(t)$ and satisfies

$$u(t) = T(t-s)v + \int_s^t T(t-\xi)B(\xi)u(\xi)\,d\xi \quad \text{for } t \in [s,\tau],$$

where the integral is taken in the Banach space $(X, |\cdot|)$ and in the sense of Bochner.

We next introduce the notion of nonlinear evolution operator associated with the semilinear problem (SE)-(IC).

Definition 4.2. A two-parameter family $\mathscr{U} \equiv \{U(t,s) : 0 \le s \le t \le \tau\}$ of nonlinear operators from Y into itself is called a nonlinear evolution operator in Y, if it satisfies the two properties below :

(E1) $U(t,s) : D(s) \to D(t), \quad U(r,r)v = v,$
$U(t,r)v = U(t,s)U(s,r)v \quad \text{for } 0 \le r \le s \le t \le \tau$ and $v \in D(r)$.

(E2) For each $s \in [0,\tau)$ and $v \in D(s)$, $U(\cdot,s)v \in C([s,\tau];X) \cap L^p(s,\tau;Y)$.

A nonlinear evolution operator $\mathscr{U}$ is said to be *locally equi-Lipschitz continuous* on $\mathscr{D}$, if it satisfies the following condition:

(E3) For each $\rho > 0$ there exists a constant $M(\rho) \ge 1$ such that
$\|U(t,s)v - U(t,s)w\| \le M(\rho)\|v - w\|$
for $s \in [0,\tau),\ t \in [s,\tau]$ and $v,\ w \in D_\rho(s)$.

The main result of this paper is to investigate the relationship between the two statements below :

(I) There exists a locally equi-Lipschitz continuous evolution operator $\mathscr{U}$ in Y such that, for $s \in [0,\tau)$, $t \in [s,\tau]$ and $v \in D(s)$,

$$U(t,s)v = T(t-s)v + \int_s^t T(t-\xi)B(\xi)U(\xi,s)v\,d\xi, \tag{I.a}$$

$$\varphi(U(t,s)v) \le \Psi(\tau,s;\varphi(v)). \tag{I.b}$$

(II) For each $\varepsilon > 0$, $s \in [0,\tau)$ and $v \in D(s)$ there exists a partition $\Delta = \{s = t_0 < t_1 < \cdots < t_{N+1} = \tau\}$ and finite sequences $\{\hat{t}_i\}_{i=1}^N$ in $[0,\tau]$, $\{v_i\}_{i=0}^N$ in X,

$\{u_i\}_{i=1}^N$ in Y and $\{z_i\}_{i=1}^N$ in X satisfying $\max_{1\le i\le N+1}\{t_i - t_{i-1}\} < \varepsilon$, $\hat{t}_i \in [t_{i-1}, t_i]$, $v_0 = v$, $v_i \in D(A)$, $u_i \in D(\hat{t}_i)$, $\sum_{i=1}^N (t_i - t_{i-1})\|u_i - v_i\| < \varepsilon$, $\sum_{i=1}^N (t_i - t_{i-1})|z_i| < \varepsilon$ and the following four conditions:

(II.a) $v_i - (t_i - t_{i-1})(Av_i + B(\hat{t}_i)u_i) - v_{i-i} = (t_i - t_{i-1})z_i, \quad \text{for } i = 1, 2, \ldots, N.$

(II.b) $$\varphi(u_i) \le \Psi(\tau, s; \varphi(v)) + \varepsilon \ (\equiv R_\varepsilon) \quad \text{for } i = 1, 2, \ldots, N.$$

(II.c) $$\sum_{i=1}^N \int_{t_{i-1}}^{t_i} f_{R_\varepsilon}(\hat{t}_i, \xi)\, d\xi < \varepsilon.$$

where f_{R_ε} is a function in $\mathscr{F}_p$ specified in (H3) replaced r by R_ε.

(II.d) If $(\hat{t}_i, u_i) \to (t, u)$ in $[0, \tau] \times Y$ as $\varepsilon \to 0$, then $u \in D_{R_0}(t)$.

Condition (I) states that given initial-data s and $v \in D(s)$ there exists a mild solution on $[s, \tau]$ to (SE)-(IC). Accordingly, the implication (II) $\Rightarrow$ (I) may be called the generation theorem.

We here make two remarks.

Remark 4.3. First, for each $s \in [0, \tau)$ the semilinear operator $A + B(s)$ is the infinitesimal generator of the evolution operator $U(t, s)$ in the sense that

$$\lim_{h\downarrow 0} h^{-1}[U(s+h, s)v - v] = (A + B(s))v \quad \text{for} \quad v \in D(A) \cap D(s).$$

In fact, let $\rho > 0$ and $s \in [0, \tau)$. Under (I.b), condition (H3) implies that

$$B(\cdot)U(\cdot, s)v \in L^p(s, \tau; X) \quad \text{for} \quad v \in D_\rho(s).$$

Secondly, suppose that $v \in D(A) \cap D_\rho(s)$, and that

$$\lim_{h\downarrow 0} h^{-1} \int_0^h f_R(\xi + s, s)\, d\xi = 0,$$

for some $R > \Psi(\tau, s; \rho)$. Then we see from (I) that

$$\lim_{h\downarrow 0} h^{-1}[U(s+h, s)v - v] = (A + B(s))v.$$

Indeed, for $v \in D(A) \cap D_\rho(s)$,

$$U(s+h, s)v \to v \quad \text{in } Y \quad \text{as } h \downarrow 0, \quad \text{and}$$
$$|B(s+\xi)U(s+\xi, s)v - B(s)v| \le \omega_R\|U(s+\xi, s)v - v\| + f_R(s+\xi, s).$$

Therefore we see with the aid of the given properties of f_R that

$$\begin{aligned} h^{-1}[U(s+h, s)v - v] &= h^{-1}[T(h)v - v] + h^{-1}\int_0^h T(h-\xi)B(s)v\, d\xi \\ &\quad + h^{-1}\int_0^h T(h-\xi)[B(s+\xi)U(s+\xi, s)v - B(s)v]\, d\xi \end{aligned}$$

converges to $(A + B(s))v$ in X as $h \downarrow 0$.

Remark 4.4. In many applications to semilinear evolution equations, what is called 'semi-implicit' schemes (SI) as mentioned below play an important role to construct discrete approximate solutions.

(SI) For each $\bar{\varepsilon} > 0$, $s \in [0, \tau)$ and $v \in D(s)$ there exist sequences $\{t_k\}_{k=0}^N$ in $[s, \tau]$, $\{v_k\}_{k=0}^N$ in Y such that $v_0 = v$, $t_0 = s$, $t_N = \tau$, $0 < t_k - t_{k-1} < \bar{\varepsilon}$, $v_k \in D(A) \cap D(t_k)$, $k = 1, 2, \ldots, N$ and

$$v_k = (I - h_k A)^{-1}(v_{k-1} + h_k B(t_{k-1})v_{k-1}),$$

$$\varphi(v_k) \leq \Psi(\tau, s; \varphi(v_0)) + \bar{\varepsilon},$$

where $h_k = t_k - t_{k-1}$, $i = 1, 2, \ldots, N$.

Now the statement (II) may be regarded as a generalization of (SI). In fact, it is shown that (SI) implies (II) under some additional conditions. Suppose that (H1) and (H2) hold. Assume further that (H3) holds with $\mathscr{F}_p$ replaced by $\mathscr{F}_C = \{f \in C([0, \tau]^2) : f(s, t) = f(t, s),\ f(t, t) = 0 \text{ for } s, t \in [0, \tau]\}$ and that (H4) holds:

(H4) If $t_i \in [0, \tau]$, $x_i \in D(A) \cap D(t_i)$, $t_i \uparrow t$, $x_i \to x$ in Y as $i \to \infty$, then $x \in D(t)$.

Then (SI) implies (II). Actually, let $\varepsilon > 0$, $s \in [0, \tau)$, $v \in D(s)$ and set $R = \Psi(\tau, s; \varphi(v)) + \varepsilon$. One then finds a large number ν so that $M_\alpha \omega_R \Gamma(1-\alpha)\nu^{\alpha-1} < 1/2$. Let $\gamma \in (\alpha, 1)$ and $\hat{R} = 2(M+1)e^{\nu\tau}\|v\| + 2M_\alpha e^{\nu\tau}(|f_R|_\infty + |B(s)v|)(1-\alpha)^{-1}\tau^{1-\alpha}$. We then choose an $\bar{\varepsilon} \in (0, \varepsilon)$ small enough to satisfy

$$\sup_{h \in [0, \bar{\varepsilon}]} \|(I - hA)^{-1}v - v\| < \varepsilon/(2M\tau), \quad f_R(t, \xi) \leq \varepsilon/\tau \quad \text{for } |t - \xi| \leq \bar{\varepsilon} \quad \text{and}$$

$$(\omega_R \hat{R} + |B(s)v| + |f_R|_\infty)(M_\alpha + M_{\gamma-\alpha}M_\gamma(1-\gamma)^{-1})\tau^{1-\gamma}\bar{\varepsilon} < \varepsilon/(2\tau).$$

Then there exist sequences $\{t_k\}_{k=0}^N$ in $[s, \tau]$ and $\{v_k\}_{k=0}^N$ in Y satisfying (SI). Set $u_k = v_{k-1}$, $\hat{t}_k = t_{k-1}$ and $z_k = 0$ for $k = 1, 2, \ldots, N$. We have only to prove that $\sum_{k=1}^N (t_k - t_{k-1})\|u_k - v_k\| = \sum_{k=1}^N h_k \|v_k - v_{k-1}\| < \varepsilon$. To this end, we first verify that $\{v_k\}_{k=1}^N$ is bounded in Y. Since $\{v_k\}$ satisfies the identity

$$v_k = \prod_{j=1}^k (I - h_j A)^{-1} v + \sum_{j=1}^k h_j \prod_{i=j}^k (I - h_i A)^{-1} B(t_{j-1})v_{j-1} \tag{4.1}$$

for $k = 1, 2, \ldots, N$, we have

$$\begin{aligned} e^{-\nu t_k}\|v_k - v\| &\leq (M+1)\|v\| \\ &\quad + M_\alpha \omega_R \sum_{j=1}^k h_j e^{-\nu(t_k - t_{j-1})}(t_k - t_{j-1})^{-\alpha} e^{-\nu t_{j-1}}\|v_{j-1} - v\| \\ &\quad + M_\alpha \sum_{j=1}^k h_j (t_k - t_{j-1})^{-\alpha}(|f_R|_\infty + |B(s)v|) \end{aligned}$$

$$
\begin{aligned}
&\le (M+1)\|v\| \\
&\quad + M_\alpha \omega_R \int_s^{t_k} e^{-\nu(t_k-\xi)}(t_k-\xi)^{-\alpha}\, d\xi \sup_{1\le j\le N+1} e^{-\nu t_{j-1}}\|v_{j-1}-v\| \\
&\quad + M_\alpha(|f_R|_\infty + |B(s)v|)\int_s^{t_k}(t_k-\xi)^{-\alpha}\, d\xi \\
&\le (M+1)\|v\| + M_\alpha(|f_R|_\infty + |B(s)v|)(1-\alpha)^{-1}\tau^{1-\alpha} \\
&\quad + M_\alpha\omega_R\Gamma(1-\alpha)\nu^{\alpha-1}\sup_{0\le j\le N} e^{-\nu t_j}\|v_j - v\|.
\end{aligned}
$$

This implies that $\sup_{1\le k\le N}\|v_k - v\| \le \hat{R}$. and thus $\{v_k\}_{k=1}^N$ is bounded in Y.

We now apply this fact and use (4.1) to get

$$
\begin{aligned}
\|v_k - v_{k-1}\| &\le M\|(I-h_kA)^{-1}v - v\| + h_k\|(I-h_kA)^{-1}B(t_{k-1})v_{k-1}\| \\
&\quad + \left\|((I-h_kA)^{-1}-I)\sum_{j=1}^{k-1} h_j \prod_{i=j}^{k-1}(I-h_iA)^{-1}B(t_{j-1})v_{j-1}\right\| \\
&\le \varepsilon/(2\tau) + h_k^{1-\alpha}M_\alpha(\omega_R\|v_{k-1}-v\| + f_R(t_{k-1},s) + |B(s)v|) \\
&\quad + M_{\gamma-\alpha}M_\gamma h_k^{\gamma-\alpha}\sum_{j=1}^{k-1} h_j(t_{k-1}-t_{j-1})^{-\gamma}(\omega_R\|v_{j-1}-v\| \\
&\quad + f_R(t_{j-1},s) + |B(s)v|) \\
&\le \varepsilon/(2\tau) + \bar{\varepsilon}^{1-\alpha}M_\alpha(\omega_R\hat{R} + |B(s)v| + |f_R|_\infty) \\
&\quad + M_{\gamma-\alpha}M_\gamma\bar{\varepsilon}^{\gamma-\alpha}(\omega_R\hat{R} + |B(s)v| + |f_R|_\infty)\int_s^{t_{k-1}}(t_{k-1}-\xi)^{-\gamma}\, d\xi \\
&< \varepsilon/\tau.
\end{aligned}
$$

Hence $\sum_{k=1}^N h_k\|v_k - v_{k-1}\| < \varepsilon$ and the implication (SI) $\Rightarrow$ (II) is verified.

Finally, we need the following theorem in Section 8 and the proof is similar to [23].

Theorem 4.5. *Let A and $B(t)$ satisfy conditions* (H1), (H2) *and* (H3). *Let $\mathscr{U} \equiv \{U(t,s) : 0 \le s \le t \le \tau\}$ be a nonlinear evolution operator in Y such that $B(\cdot)U(\cdot,s)v \in L^p(s,\tau;X)$ for each $v \in D(s)$. Then the following are equivalent:*

(a) *For $s \in [0,\tau)$, $v \in D(s)$ and $t \in [s,\tau]$,*

$$U(t,s)v = T(t-s)v + \int_0^t T(t-s)B(\xi)U(\xi,s)v\, d\xi.$$

(b) *For* $s \in [0,\tau)$, $v \in D(s)$, $\int_s^t U(\xi,s)v\,d\xi \in D(A)$ *and*

$$U(t,s)v = v + A\int_s^t U(\xi,s)v\,d\xi + \int_s^t B(\xi)U(\xi,s)v\,d\xi \quad \textit{for } t \in [s,\tau].$$

We are now in a position to state our main result.

Theorem 4.6. (**Characterization Theorem**) *Under conditions* (H1) *through* (H3), (I) *and* (II) *are equivalent.*

The implication (II) $\Rightarrow$ (I) is called a generation theorem, while the implication (I) $\Rightarrow$ (II) may be called a characterization theorem. The above theorem extends Theorem 1.1 in Oharu-Pazy [22] to the time-dependent case under considerably weaker conditions. The precise proof of Theorems 4.6 is given in Section 6 through Section 8.

5. Uniqueness and regularity of mild solutions

In this section we discuss the continuous dependence of the mild solutions on initial-data in the Banach space Y. We also investigate the regularity in the original Banach space X of mild solutions. Throughout this section we assume that A and $B(t)$ satisfy conditions (H1), (H2) and (H3).

Before starting the uniqueness and regularity arguments, we prepare a useful inequality which are often applied in the subsequent discussions. The following statements and its proof is given in Henry [8, Lemma 7.1.1].

Lemma 5.1. (Henry's inequality) *Suppose* a, $b \geq 0$, $\alpha < 1$. *Suppose further that* $u(t)$ *is nonnegative and locally integrable on* $[0,\tau)$ *and satisfies*

$$u(t) \leq a + b\int_0^t (t-s)^{-\alpha}u(s)\,ds \quad \textit{for } t \in [0,\tau).$$

Then, we have

$$u(t) \leq aE_{1-\alpha}(\theta t) \quad \textit{for } t \in [0,\tau),$$

where $E_{1-\alpha}(t) = \sum_{n=0}^{\infty} t^{n(1-\alpha)}/\Gamma(n(1-\alpha)+1)$ *and* $\theta = (b\Gamma(1-\alpha))^{1/(1-\alpha)}$.

Now our uniqueness argument is given as follows:

Theorem 5.2. *Let* $s \in [0,\tau)$ *and let* v, $\hat{v}$ *be two initial values given in* $D(s)$. *Suppose that* $u(\cdot)$ *and* $\hat{u}(\cdot)$ *are the corresponding mild solutions to* (SE)-(IC) *and that*

$$u(t), \hat{u}(t) \in D_R(t) \quad \textit{for } t \in [s,\tau] \textit{ and some } R > 0.$$

Then we have

$$\|u(t) - \hat{u}(t)\| \leq ME_{1-\alpha}(\theta_\alpha(t-s))\|v - \hat{v}\| \quad \text{for } t \in [s,\tau],$$

where $\theta_\alpha = (M_\alpha\omega_R\Gamma(1-\alpha))^{1/(1-\alpha)}$.

Proof. Since $u(\cdot)$ and $\hat{u}(\cdot)$ satisfy the integral equations in Definition 4.1, we have

$$\begin{aligned}\|u(t)-\hat{u}(t)\| &\le M\|v-\hat{v}\| + M_\alpha \int_0^t (t-s)^{-\alpha}|B(s)u(s)-B(s)\hat{u}(s)|\,ds\\ &\le M\|v-\hat{v}\| + M_\alpha\omega_R \int_0^t (t-s)^{-\alpha}\|u(s)-\hat{u}(s)\|\,ds.\end{aligned}$$

By applying Henry's inequality, we obtain the desired estimate. □

Let $\beta \in (0,1)$ and let V be a Banach space. In order to state the regularity of mild solutions we introduce the following two function spaces:

$$C^\beta([a,b];V) \equiv \left\{ f \in C([a,b];V) : \sup\left\{ \frac{|f(s)-f(t)|}{|s-t|^\beta} : s,t \in [a,b],\ s \ne t \right\} < \infty \right\},$$

$$C^\beta_{\ell oc}(0,\infty;V) \equiv \{ f \in C((0,\infty);V) : f \in \bigcap_{b>a>0} C^\beta([a,b];V)\}.$$

Our regularity theorem may be stated as follows:

Theorem 5.3. (**Regularity Theorem**) *Let $s \in [0,\tau)$, $v \in D(s)$ and let $u(\cdot)$ be the associated mild solution of the problem* (SE)-(IC) *satisfying the growth condition* (G)*. Then we have:*

(a) *If $\alpha \in (0, 1-p^{-1})$, then*
$$u(\cdot) \in C([s,\tau];Y) \cap C^\delta_{loc}((s,\tau];D((-A)^\gamma))$$
for $\gamma \in [\alpha, 1-p^{-1})$ and $\delta \in (0, 1-p^{-1}-\gamma)$.

(b) *If $\alpha \in [1-p^{-1}, 1)$, then*
$$u(\cdot) \in C^\delta([s,\tau];D((-A)^\gamma)) \quad \textit{for } \gamma \in [0,1-p^{-1}) \textit{ and } \delta \in (0,1-p^{-1}-\gamma).$$

Proof. Let q be the conjugate exponent to p and set $g(\xi) = B(\xi)u(\xi)$. We first show that $u(\cdot) \in C([s,\tau];Y)$. Let $t \in [s,\tau)$ and $h \in [0,\tau-t]$. Using the identity

$$\begin{aligned}u(t+h)-u(t) = T(t-s)(T(h)v-v) &+ \int_s^t T(t-\xi)(T(h)g(\xi)-g(\xi))\,d\xi\\ &+ \int_t^{t+h} T(t+h-\xi)g(\xi)\,d\xi,\end{aligned}$$

we have

$$\begin{aligned}\|u(t+h)-u(t)\| &\le M\|T(h)v-v\| + M_\alpha \int_s^t (t-\xi)^{-\alpha}|T(h)g(\xi)-g(\xi)|\,d\xi\\ &\quad + M_\alpha \int_t^{t+h} (t+h-\xi)^{-\alpha}|g(\xi)|\,d\xi\\ &\le M\|T(h)v-v\| + M_\alpha(1-q\alpha)^{-1/q}\tau^{1/q-\alpha}|T(h)g(\cdot)-g(\cdot)|_{L^p}\\ &\quad + M_\alpha(1-q\alpha)^{-1/q}h^{1/q-\alpha}|g|_{L^p}\end{aligned}$$

This implies that $u(\cdot)$ is right continuous on $[s, \tau)$ in Y. The left continuity of $u(\cdot)$ is proved in a similar way.

Let $0 \le s < \tau$, $v \in D(s)$. Take any positive number μ such that $0 < 2\mu < \tau - s$. Let f_R be a function given in $\mathscr{F}_p$ by (H3) and let $\gamma \in [\alpha, 1-p^{-1})$. Note that $q\gamma < 1$. Let $\delta \in (0, 1-p^{-1}-\gamma)$, $t \in [s+\mu, \tau)$ and $h \in (0, \tau - t]$. Then it is easily seen that $u(\cdot) \in L^\infty_{loc}(s, \tau; D((-A)^{\gamma+\delta}))$. This implies the estimate

$$|(T(h)-I)u(t)|_{D((-A)^\gamma)} \le C_{\gamma,\delta} h^\delta |u(t)|_{D((-A)^{\gamma+\delta})} \le N C_{\gamma,\delta} h^\delta,$$

where $N = \sup\{|u(\xi)|_{D((-A)^{\gamma+\delta})} : s+\mu \le \xi \le \tau\}$. Using the relation

$$u(t+h) - u(t) = (T(h)-I)u(t) + \int_0^h T(h-\xi) g(\xi + t)\, d\xi.$$

and Hölder's inequality, we obtain that

$$\begin{aligned}\left|(-A)^\gamma \int_0^h T(h-\xi) g(\xi+t)\, d\xi\right| &\le M_\gamma \int_0^h (h-\xi)^{-\gamma} |g(\xi+t)|\, d\xi \\ &\le M_\gamma |g|_{L^p(0,\tau;X)} (1-q\gamma)^{-1/q} h^{1/q-\gamma}.\end{aligned}$$

Therefore, we have

$$|u(t+h) - u(t)|_{D((-A)^\gamma)} \le C' h^\delta$$

and this shows that (a) holds. Assertion (b) is proved in a way similar to the proof of (a). □

For further regularity results on mild solutions to (SE)-(IC) we refer to [1, 13].

6. Generation of nonlinear evolution operator $\mathscr{U}$ in Y

In this section we give proof of the implication (II) ⇒ (I) of Theorem 4.6. Throughout this section we assume conditions (H1), (H2), (H3) and (II). The proof is divided into four steps. We notice that the proof is similar to that in [15].

Step 1. Let $n \ge 1$, $s \in [0, \tau)$ and choose an initial-value $v \in D(s)$. Let $R = \Psi(\tau, s; \varphi(v)) + 1$ and f_R a function given in (H3). We then choose a real number $\nu > 0$ so that $8M_\alpha(1+\omega_R)\Gamma(1-\alpha)\nu^{\alpha-1} < 1$. Then condition (II) implies that for each positive integer n there exists a partition $\Delta_n = \{s = t_0^n < t_1^n < \cdots < t_{N(n)+1}^n = \tau\}$ and finite sequences $\{\hat{t}_k^n : k = 1, \ldots, N(n)\}$ in $[0, \tau]$, $\{v_k^n : k = 0, \ldots, N(n)\}$ in Y, $\{u_k^n : k = 1, \ldots, N(n)\}$ in X and $\{z_k^n : k = 1, \ldots, N(n)\}$ in X such that, for $k = 1, 2, \ldots, N(n)$, $t_k^n - t_{k-1}^n$, $\tau_0 - t_{N(n)}^n \le 1/n$, $\hat{t}_k^n \in [t_{k-1}^n, t_k^n]$, $v_k^n \in D(A)$, $u_k^n \in D(\hat{t}_k^n)$,

$$\sum_{k=1}^{N(n)} (t_k^n - t_{k-1}^n) \|v_k^n - u_k^n\| \le 1/n,$$

$$\sum_{k=1}^{N(n)} (t_k^n - t_{k-1}^n) |z_k^n| \le 1/n,$$

$$v_k^n - v_{k-1}^n - (t_k^n - t_{k-1}^n)(Av_k^n + B(\hat{t}_k^n)u_k^n) = (t_k^n - t_{k-1}^n)z_k^n, \quad v_0^n = v, \tag{6.1}$$

$$\varphi(u_k^n) \le \Psi(\tau, s; \varphi(v)) + 1/n, \tag{6.2}$$

$$\sum_{k=1}^{N(n)} \int_{t_{k-1}^n}^{t_k^n} f_R(\hat{t}_k^n, \xi)\, d\xi < 1/n. \tag{6.3}$$

By induction we have

$$v_k^n = \prod_{j=1}^{k}(I - h_j^n A)^{-1} v + \sum_{j=1}^{k} h_j^n \prod_{i=j}^{k}(I - h_i^n A)^{-1} B(\hat{t}_j^n)u_j^n + \sum_{j=1}^{k} h_j^n \prod_{i=j}^{k}(I - h_i^n A)^{-1} z_j^n, \tag{6.4}$$

for $k = 1, 2, \ldots, N(n)$, where $h_k^n = t_k^n - t_{k-1}^n$.

For simplicity in notation, we introduce the following eight step functions:

$$\sigma_n(t) = \begin{cases} s & \text{for } t \in \{s\} \cup \ (t_{N(n)}^n, \tau], \\ \hat{t}_k^n & \text{for } t \in (t_{k-1}^n, t_k^n], \quad 1 \le k \le N(n), \end{cases}$$

$$v_n(t) = \begin{cases} v & \text{for } t \in \{s\} \cup \ (t_{N(n)}^n, \tau], \\ v_k^n & \text{for } t \in (t_{k-1}^n, t_k^n], \quad 1 \le k \le N(n), \end{cases}$$

$$u_n(t) = \begin{cases} v & \text{for } t \in \{s\} \cup \ (t_{N(n)}^n, \tau], \\ u_k^n & \text{for } t \in (t_{k-1}^n, t_k^n], \quad 1 \le k \le N(n), \end{cases}$$

$$z_n(t) = \begin{cases} 0 & \text{for } t \in \{s\} \cup \ (t_{N(n)}^n, \tau], \\ z_k^n & \text{for } t \in (t_{k-1}^n, t_k^n], \quad 1 \le k \le N(n), \end{cases}$$

$$J_n(t) = \begin{cases} I & \text{for } t \in \{s\} \cup \ (t_{N(n)}^n, \tau], \\ \displaystyle\prod_{j=1}^{k}(I - h_j^n A)^{-1} & \text{for } t \in (t_{k-1}^n, t_k^n], \quad 1 \le k \le N(n), \end{cases}$$

$$J_n(t,r) = \begin{cases} \displaystyle\prod_{i=j}^{k}(I - h_i^n A)^{-1} & \text{for } t \ge r, t \in (t_{k-1}^n, t_k^n], \text{ and } r \in [t_{j-1}^n, t_j^n), \\ & 1 \le j \le k \le N(n), \\ 0 & \text{otherwise}, \end{cases}$$

$$Q_n(t) = \begin{cases} s & \text{for } t \in \{0\} \bigcup \ (t_{N(n)}^n, \tau], \\ \displaystyle\int_0^{t_k^n} J(t_k^n, s)B(\sigma_n(s))u_n(s)\, ds & \text{for } t \in (t_{k-1}^n, t_k^n], \quad 1 \le k \le N(n), \end{cases}$$

$$\Theta_n(t) = \begin{cases} s & \text{for } t \in \{0\} \bigcup \ (t_{N(n)}^n, \tau], \\ \displaystyle\int_0^{t_k^n} J(t_k^n, s)z_n(s)\, ds & \text{for } t \in (t_{k-1}^n, t_k^n], \quad 1 \le k \le N(n). \end{cases}$$

Using these functions, we may rewrite (6.2) and (6.4) in the following forms:

$$v_n(t) = J_n(t)v + Q_n(t) + \Theta_n(t), \tag{6.5}$$

$$\varphi(u_n(t)) \leq \Psi(\tau, s; \varphi(v)) + n^{-1} \leq R, \tag{6.6}$$

$$u_n(t) \in D(\sigma_n(t)), \quad \text{for } t \in [s, \tau]. \tag{6.7}$$

Also, (6.3) implies

$$\int_s^\tau f_R(\sigma_n(\xi), \xi)\, d\xi \leq 1/n. \tag{6.8}$$

In order to estimate terms on the right-hand side of (6.5), we need the two lemmas below.

Lemma 6.1. (G. Nakamura and S. Oharu [19]) *Let $\beta \in (0,1)$, $1 \leq j \leq k \leq N(n)$ and $\xi \in [t_{j-1}^n, t_j^n)$. Then we have*

$$|(-A)^\beta J_n(t_k^n, \xi)| \leq M_\beta (t_k^n - t_{j-1}^n)^{-\beta} \leq M_\beta (t_k^n - \xi)^{-\beta}$$

for some constant $M_\beta > 0$.

Lemma 6.2. *Let $\beta \in [0,1)$, $\nu > 0$ and $\mathfrak{u}(\cdot) \in L^1(s, \tau; X)$. Then we have*

$$\sum_{k=1}^{N(n)} \int_{t_{k-1}^n}^{t_k^n} e^{-\nu t}\, dt \int_s^{t_k^n} |(-A)^\beta J_n(t_k^n, \xi) u(\xi)|\, d\xi$$
$$\leq M_\beta \Gamma(1-\beta) \nu^{\beta-1} e^{\nu|\Delta_n|} \int_s^{t_{N(n)}} e^{-\nu\xi} |u(\xi)|\, d\xi,$$

where $|\Delta_n| = \max\{|t_k^n - t_{k-1}^n| : k = 1, 2, \ldots, N(n)\}$.

The proof of Lemma 6.2 is given at the end of this section. First we demonstrate that $\{u_n(\cdot) : n \geq 1\}$, $\{Q_n(\cdot) : n \geq 1\}$ and $\{\Theta_n(\cdot) : n \geq 1\}$ are all bounded in the space $L^1(s, \tau; Y)$ endowed with the norm defined by

$$\|v\|_1 = \int_s^\tau e^{-\nu t} \|v(t)\|\, dt.$$

If $t \in (t_{k-1}^n, t_k^n]$, $k = 1, 2, \ldots, N(n)$, then it follows from (6.5) that

$$e^{-\nu t}\|v_n(t) - v\| \leq M\|v\| + e^{-\nu t} \int_s^{t_k^n} |(-A)^\alpha J_n(t_k^n, \xi)[B(\sigma_n(\xi))u_n(\xi) - B(s)v]|\, d\xi$$
$$+ e^{-\nu t} \int_s^{t_k^n} |(-A)^\alpha J(t_k^n, \xi)[B(s)v + z_n(\xi)]|\, d\xi. \tag{6.9}$$

Noting that

$$\int_s^\tau e^{-\nu t}\|v_n(t) - v\|\, dt = \sum_{k=1}^{N(\tau)} \int_{t_{k-1}^n}^{t_k^n} e^{-\nu t} \|u_n(t) - v\|\, dt,$$

we have

$$\int_s^\tau e^{-\nu t}\|v_n(t)-v\|\,dt \le M(\tau-s)\|v\|$$
$$+\sum_{k=1}^{N(n)}\int_{t_{k-1}^n}^{t_k^n} e^{-\nu t}\,dt\left(\int_s^{t_k^n} |(-A)^\alpha J_n(t_k^n,\xi)[B(\sigma_n(\xi))u_n(\xi)-B(s)v]|\,d\xi\right)$$
$$+\sum_{k=1}^{N(n)}\int_{t_{k-1}^n}^{t_k^n} e^{-\nu t}\left(\int_s^{t_k^n} |(-A)^\alpha J(t_k^n,\xi)[B(s)v+z_n(\xi)]|\,d\xi\right). \tag{6.10}$$

By Lemmas 6.1 and 6.2, we have

$$\sum_{k=1}^{N(n)}\int_{t_{k-1}^n}^{t_k^n} e^{-\nu t}\int_s^{t_k^n} |(-A)^\alpha J_n(t_k^n,\xi)[B(\sigma_n(\xi))u_n(\xi)-B(s)v]|\,d\xi$$
$$\le M_\alpha\Gamma(1-\alpha)\nu^{\alpha-1}e^{\nu/n}\int_s^\tau e^{-\nu\xi}|B(\sigma_n(\xi))u_n(\xi)-B(s)v|\,d\xi$$
$$\le M_\alpha\Gamma(1-\alpha)\nu^{\alpha-1}e^{\nu/n}\Big[\omega_R\int_s^\tau e^{-\nu\xi}\|u_n(\xi)-v\|\,d\xi$$
$$+\int_s^\tau (f_R(\sigma_n(\xi),\xi)+f_R(\xi,s))\,d\xi\Big]$$

and

$$\sum_{k=1}^{N(n)}\int_{t_{k-1}^n}^{t_k^n} e^{-\nu t}\,dt\int_s^{t_k^n} |(-A)^\alpha J_n(t_k^n,\xi)[B(s)v+z_n(\xi)]|\,d\xi$$
$$\le M_\alpha\Gamma(1-\alpha)\nu^{\alpha-1}e^{\nu/n}((\tau-s)|B(s)v|+1).$$

Combining these estimates with (6.10) and noting that $4M_\alpha\Gamma(1-\alpha)(1+\omega_R)\nu^{\alpha-1} < 1$, we obtain

$$\int_s^\tau e^{-\nu t}\|v_n(t)-v\|\,dt \le K+\frac{1}{2}\int_s^\tau e^{-\nu t}\|u_n(t)-v\|\,dt$$

for $n \ge \nu/\log 2$, where $K = M(\tau-s)\|v\|+(\tau-s)|B(s)v|+1+\int_s^\tau f_R(\xi,s)\,d\xi$. Hence we have

$$\int_s^\tau e^{-\nu t}\|u_n(t)-v\|\,dt \le 2(K+1). \tag{6.11}$$

This shows that $\{u_n(\cdot) : n \ge 1\}$ is bounded in $L^1(s,\tau;Y)$. The boundedness of $\{v_n(\cdot) : n \ge 1\}$ and $\{Q_n(\cdot) : n \ge 1\}$ follows from that of $\{u_n(\cdot) : n \ge 1\}$. It is easily shown that $\{\Theta_n(\cdot) : n \ge 1\}$ is also bounded in $L^1(s,\tau;Y)$.

Step 2. Here we demonstrate that $\{u_n(\cdot) : n \ge 1\}$ is precompact in $L^1(s,\tau;Y)$. For this purpose we use the what is called ball measure of noncompactness $\beta(\cdot)$.

For a bounded subset V of a Banach space E we define the ball measure of non-compactness $\beta(V)$ by

$$\beta(V) = \inf\{r > 0 : V \text{ admits a finite covering by balls of radius } r\}.$$

We need the following typical properties of $\beta(\cdot)$ below:

Lemma 6.3. *Let V and W be bounded subsets of E. Then the following are valid:*

(a) $\beta(V + W) \le \beta(V) + \beta(W)$.

(b) $\beta(\mu W) = \mu\beta(W)$ *for* $\mu > 0$.

(c) $\beta(V) = 0$ *if and only if V is precompact in E.*

(d) $\beta(W) = 0$ *implies* $\beta(V \cup W) = \beta(V)$.

We refer to [14] for the proof of this lemma and for further properties of β. In particular, the fourth property refers to the notion of measure on noncompactness.

Let ℓ be a natural number satisfying $\ell \ge \nu/\log 2$. Set $V_\ell = \{u_n(\cdot) : n \ge \ell\}$ and $W_\ell = \{Q_n(\cdot) : n \ge \ell\}$. We denote by $S(r; y)$ a ball of radius r with center y. Let $\eta > 0$ and set $\delta = \beta(V) + \eta$. By the definition of $\beta(\cdot)$ there exist $y_1(\cdot)$, $y_2(\cdot)$, $\dots$, $y_m(\cdot)$ in $L^1(s, \tau; Y)$ such that

$$V \subset \bigcup_{i=1}^{m} S(\delta; y_i(\cdot)). \tag{6.12}$$

By choosing $u_i(\cdot) \in V$ with $\|u_i(\cdot) - y_i(\cdot)\|_1 < \delta$, we may rewrite (6.12) as

$$V \subset \bigcup_{i=1}^{m} S(2\delta; u_i(\cdot)),$$

where $\|w(\cdot)\|_1$ denotes the norm of $w \in L^1(0, \tau; Y)$. For each i we define a step function $Q_n^i(\cdot)$ on $[0, \tau]$ by

$$Q_n^i(t) = \begin{cases} 0 & \text{for } t \in \{s\} \bigcup (t_{N(n)}^n, \tau] \\ \displaystyle\int_s^{t_k^n} J_n(t_k^n, \xi) B(\sigma_i(\xi)) u_i(\xi)\, d\xi & \text{for } t \in (t_{k-1}^n, t_k^n],\ k = 1, 2, \dots, N(n). \end{cases}$$

We set $W_\ell^i = \{Q_n^i(\cdot) : n \ge \ell\}$ for $i = 1, 2, \dots, m$ and show that each W_ℓ^i is precompact in $L^1(s, \tau; Y)$. It should be noted that for each compact subset $K \subset X$, $J_n(t, \xi)v$ converges uniformly to $T(t - \xi)v$ in X with respect to ξ, $t \in [s, \tau]$ with $\xi \le t$ and $v \in K$. This implies that for each i

$$Q_n^i(\cdot) \longrightarrow \int_s^{\cdot} T(\cdot - \xi) B(\sigma_i(\xi)) u_i(\xi)\, d\xi \quad \text{in } L^1(s, \tau; X) \text{ as } n \to \infty. \tag{6.13}$$

Let $\gamma \in (\alpha, 1)$. Since the sequence $(-A)^\gamma Q_n^i(\cdot)$ is bounded in $L^1(s, \tau; X)$ by Lemma 6.1, the convergence (6.13), the moments inequality

$$|(-A)^\alpha w| \le C(\alpha, \gamma)|w|^{1-\alpha/\gamma}|(-A)^\gamma w|^{\alpha/\gamma} \quad \text{for } w \in D((-A)^\gamma)$$

and Hölder's inequality together implies that

$$Q_n^i(\cdot) \longrightarrow \int_s^{\cdot} T(\cdot - \xi)B(\sigma_i(\xi))u_i(\xi)\, d\xi \quad \text{in } L^1(s,\tau;Y).$$

This means that each W_ℓ^i is precompact in $L^1(s,\tau;Y)$. Therefore there exist a finite number of elements $\zeta_{i,j}(\cdot) \in L^1(s,\tau;Y)$, $j = 1, 2, \ldots, \nu(i)$, such that

$$W_\ell^i \subset \bigcup_{j=1}^{\nu(i)} S(\eta; \zeta_{i,j}(\cdot)). \tag{6.14}$$

Set $\kappa = 2M_\alpha \Gamma(1-\alpha)\nu^{\alpha-1}(\omega_R \delta + 2/\ell) + \eta$. Then

$$W_\ell \subset \bigcup_{i=1}^{m} \bigcup_{j=1}^{\nu(i)} S(\kappa; \zeta_{i,j}(\cdot)) \tag{6.15}$$

holds. In fact, it follow from (6.12) and (6.14) that for each $u_n(\cdot) \in V_\ell$ there exist $u_i(\cdot)$ and $\zeta_{i,j}(\cdot)$ such that $u_n(\cdot) \in S(2\delta; u_i(\cdot))$ and $Q_l^i(\cdot) \in S(\eta; \zeta_{i,j}(\cdot))$. In view of this fact, Lemmas 6.1 and 6.2, we obtain

$$\begin{aligned} &\int_s^\tau e^{-\nu t}\|Q_n(t) - \zeta_{i,j}(t)\|\, dt \\ &\qquad \le \int_s^\tau e^{-\nu t}\|Q_n(t) - Q_n^i(t)\|\, dt + \int_s^\tau e^{-\nu t}\|Q_n^i(t) - \zeta_{i,j}(t)\|\, dt \\ &\qquad \le 2M_\alpha \Gamma(1-\alpha)\nu^{\alpha-1}(\omega_R\delta + 2/\ell) + \eta. \end{aligned} \tag{6.16}$$

This shows that (6.15) is valid. Letting $\eta \downarrow 0$ in (6.16), we have

$$\beta(W_\ell) \le 2M_\alpha\Gamma(1-\alpha)\nu^{\alpha-1}(\omega_R \beta(V_\ell) + 2/\ell) \le \beta(V_\ell)/2 + 1/\ell. \tag{6.17}$$

Since $J_n(t)v \to T(t-s)v$ and $\Theta_n(t) \to 0$ in $L^1(s,\tau;Y)$ as $n \to \infty$, the sets $\{J_n(\cdot)v : n \ge 1\}$ and $\{\Theta_n(\cdot) : n \ge 1\}$ are precompact in $L^1(s,\tau;Y)$. So, using the inclusion

$$V_\ell \subset \{J_n(\cdot)v : n \ge \ell\} + W_\ell + \{\Theta_n(\cdot) : n \ge \ell\} + \{u_n - v_n : n \ge \ell\}$$

and Lemma 6.3 (a), we infer that $\beta(V_\ell) \le \beta(W_\ell) + 1/\ell$. This together with (6.17) implies

$$\beta(V_\ell) \le \frac{3}{\ell}.$$

Since $\beta(V_\ell) = \beta(V_1)$ for all $\ell \ge 1$ by Lemma 6.3 (d), we see that $\beta(V_1) \le 0$. Therefore, the application of Lemma 6.3 (c) implies that V is precompact in $L^1(s,\tau;Y)$.

Step 3. Since V is precompact in $L^1(s,\tau;Y)$, we can choose a convergent subsequence $\{u_{n_l}(\cdot)\}_{l=1}^\infty$. Let $u_{n_l}(\cdot)$ converge to $u(\cdot)$ in $L^1(s,\tau;Y)$ as $l \to \infty$. In this step we show that $u(\cdot)$ is a local mild solution to (SE)-(IC). Without loss of generality we may assume that $u_n(\cdot)$ converges to $u(\cdot)$. It follows from (6.6), (6.7) and (II.d) that

$$\begin{aligned} &\varphi(u(t)) \le \Psi(\tau, s; \varphi(v)), \\ &u(t) \in D(t) \qquad \text{for a.e. } t \in [s,\tau]. \end{aligned} \tag{6.18}$$

Since

$$|B(\sigma_n(\xi))u_n(\xi) - B(\xi)u(\xi)| \le \omega_R \|u_n(\xi) - u(\xi)\| + f_R(\sigma_n(\xi), \xi)$$

for a.e. $\xi \in [s,\tau]$, $B(\sigma_n(\cdot))u_n(\cdot)$ converges to $B(\cdot)u(\cdot)$ in $L^1(s,\tau;X)$ as $n \to \infty$. We now define a step function $\widehat{Q}_n(\cdot)$ over $[0,\tau]$ by

$$\widehat{Q}_n(t) = \begin{cases} 0 & \text{for } t \in \{s\} \bigcup (t^n_{N(n)}, \tau] \\ \displaystyle\int_s^{t^n_k} J_n(t^n_k, \xi) B(\xi)u(\xi)\, d\xi & \text{for } t \in (t^n_{k-1}, t^n_k], \quad k = 1,2,\ldots,N(n). \end{cases}$$

Then we infer from Lemma 6.1 that

$$\begin{aligned} &\left\| Q_n(\cdot) - \int_s^{\cdot} T(\cdot - \xi)B(\xi)u(\xi)\, d\xi \right\|_1 \\ &\qquad \le \|Q_n(\cdot) - \widehat{Q}_n(\cdot)\|_1 + \left\| \widehat{Q}_n(\cdot) - \int_s^{\cdot} T(\cdot - \xi)B(\xi)u(\xi)\, d\xi \right\|_1 \\ &\qquad \le 2M_\alpha \Gamma(1-\alpha)\nu^{\alpha-1} \int_s^\tau e^{-\nu\xi} |B(\sigma_n(\xi))u_n(\xi) - B(\xi)u(\xi)|\, d\xi \\ &\qquad\quad + \left\| \widehat{Q}_n(\cdot) - \int_s^{\cdot} T(\cdot - \xi)B(\xi)u(\xi)\, d\xi \right\|_1 \end{aligned}$$

for $n \ge \nu/\log 2$. Since the dominated convergence theorem implies

$$\widehat{Q}_n(\cdot) \longrightarrow \int_s^{\cdot} T(\cdot - \xi)B(\xi)u(\xi)\, d\xi \quad \text{in } L^1(s,\tau;Y) \text{ as } n \to \infty,$$

it follows that $Q_n(\cdot)$ converges to $\int_s^{\cdot} T(\cdot - \xi)B(\xi)u(\xi)\, d\xi$ in $L^1(s,\tau;Y)$. Hence (6.5) implies the identity

$$u(t) = T(t-s)v + \int_s^t T(t-\xi)B(\xi)u(\xi)\, d\xi \quad \text{for a.e. } t \in [s,\tau]. \tag{6.19}$$

We next show that $u(\cdot) \in L^p(s,\tau;Y)$. By (6.19) we have

$$\|u(t) - v\| \le C(t) + M_\alpha \omega_R \int_s^t (t-\xi)^{-\alpha} \|u(\xi) - v\|\, d\xi \tag{6.20}$$

for a.e. $t \in [s,\tau]$, where

$$C(t) = (M+1)\|v\| + M_\alpha (1-\alpha)^{-1} \tau^{1-\alpha} |B(s)v| + M_\alpha \int_s^t (t-\xi)^{-\alpha} f_R(\xi, s)\, d\xi.$$

By Henry's inequality, Lemma 5.1, we obtain that

$$\|u(t)-v\| \le C(t) + \theta_\alpha \int_s^t E'_{1-\alpha}(\theta_\alpha(t-\xi))C(\xi)\,d\xi \quad \text{for } t \in [s,\tau],$$

where θ_α is a positive constant given in Theorem 5.2. This implies that $u(\cdot) \in L^p(s,\tau;Y)$. Since $B(\cdot)u(\cdot) \in L^p(s,\tau;X)$, it is easily seen that $u(\cdot)$ is continuous in X. Thus, we have shown that $u(\cdot)$ is a mild solution to (SE)-(IC).

Step 4. From the lower semicontinuity of $\varphi(\cdot)$ it is seen that a mild solution $u(\cdot)$ constructed in the previous steps satisfies the growth condition (G). Furthermore, Theorem 5.2 asserts that $u(\cdot)$ is uniquely determined by the initial value v.

For each $s \in [0,\tau)$ and $v \in D(s)$ we write $u(\cdot;s,v)$ for the unique mild solution to (SE)-(IC). Hence we may define a two parameter family of nonlinear operators $\mathscr{U} = \{U(t,s) : 0 \le s \le t \le \tau\}$ in Y by setting

$$U(t,s)v = u(t;s,v) \quad \text{for } t \in [s,\tau] \text{ and } v \in D(s).$$

In view of Theorem 5.2 and the construction of $u(\cdot;s,v)$, it is verified that $\mathscr{U}$ forms a locally equi-Lipschitz continuous evolution operator in Y satisfying (I.a) and (I.b). Thus the proof of the generation part of our main theorem, Theorem 4.6, is complete.

Finally, we close this section by giving:

Proof of Lemma 6.2. Applying Lemma 6.1, we have

$$\begin{aligned}
&\sum_{k=1}^{N(n)} \int_{t_{k-1}^n}^{t_k^n} e^{-\nu t}\,dt \int_s^{t_k^n} |(-A)^\beta J_n(t_k^n,\xi)u(\xi)|\,d\xi \\
&\quad\le \sum_{k=1}^{N(n)} \int_{t_{k-1}^n}^{t_k^n} e^{-\nu t}\,dt \sum_{j=1}^{k} \int_{t_{j-1}^n}^{t_j^n} M_\beta (t_k^n - t_{j-1}^n)^{-\beta}|u(\xi)|\,d\xi \\
&\quad= M_\beta \sum_{j=1}^{N(n)} \int_{t_{j-1}^n}^{t_j^n} |u(\xi)|\,d\xi \sum_{k=j}^{N(n)} \int_{t_{k-1}^n}^{t_k^n} (t_k^n - t_{j-1}^n)^{-\beta} e^{-\nu t}\,dt \\
&\quad\le M_\beta \sum_{j=1}^{N(n)} \int_{t_{j-1}^n}^{t_j^n} |u(\xi)|\,d\xi \int_{t_{j-1}^n}^{t_{N(n)}^n} (t - t_{j-1}^n)^{-\beta} e^{-\nu t}\,dt \\
&\quad\le M_\beta \Gamma(1-\beta)\nu^{\beta-1} e^{\nu|\Delta_n|} \int_s^{t_{N(n)}^n} e^{-\nu\xi}|u(\xi)|\,d\xi. \qquad \square
\end{aligned}$$

7. Discrete local multiple Laplace transforms

As discussed in [19], the concept of local multiple Laplace transform is considerably useful to discuss products of resolvents of semilinear operators. In order to verify the implication from (I) to (II) in Theorem 4.6. We need various technical estimates for the local multiple Laplace transforms which are organized in this section.

For each $i \geq 1$ we consider the set $V_i(t) = \{(s_1, \dots, s_i) \in \mathbb{R}_+^i : \sum_{j=1}^i s_i \leq t\}$ and $E_i(\xi_1, \dots, \xi_i; s_1, \dots, s_i) = \exp(-s_1/\xi_1)\exp((1/\xi_1 - 1/\xi_2)s_2) \cdots \exp((1/\xi_{i-1} - 1/\xi_i)s_i)$, and then define $F_i : [0, \infty) \times (0, \infty)^i \to [0, \infty)$ by

$$\begin{aligned} &F_i(t, \xi_1, \xi_2, \dots, \xi_i) \\ &\quad = \frac{1}{\prod_{j=1}^i \xi_j} \int \cdots \int_{V_i(t)} e^{-\sum_{j=1}^i s_j/\xi_j} \, ds_1 \cdots ds_i \\ &\quad = \frac{1}{\prod_{j=1}^i \xi_j} \int_0^t \int_0^{s_1} \cdots \int_0^{s_{i-1}} E_i(\xi_1, \xi_2, \dots, \xi_i; s_1, s_2, \dots, s_i) \, ds_i \cdots ds_1. \end{aligned}$$

To investigate the maximum and minimum of F_i under the condition that $\sum_{j=1}^i \xi_i =$ constant, we need the following lemma:

Lemma 7.1. *Let $t > 0$ and $\lambda > 0$. Let g be a continuous function over $[0, +\infty)$. For each $\xi_2 > \xi_1 > 0$ satisfying $\xi_1 + \xi_2 = \lambda$ we set*

$$h(t, \xi_1, \xi_2) = \frac{1}{\xi_1 \xi_2} \int_0^t \int_0^s e^{-(t-s)/\xi_1} e^{-(s-r)/\xi_2} g(r) \, dr \, ds \tag{7.1}$$

$$= \frac{1}{\xi_2 - \xi_1} \int_0^t (e^{-(t-s)/\xi_2} - e^{-(t-s)/\xi_1}) g(s) \, ds. \tag{7.2}$$

Then for any $\varepsilon \in (0, \xi_1)$

$$h(t, \lambda/2, \lambda/2) \leq h(t, \xi_1, \xi_2) \leq h(t, \xi_1 - \varepsilon, \xi_2 + \varepsilon) \leq \int_0^t e^{-(t-r)/\lambda} g(r) \, dr. \tag{7.3}$$

Proof. Noting the relation $\xi_2 = \lambda - \xi_1$ and differentiating both sides of (7.1) with respect to ξ_1, we have

$$\begin{aligned} h_{\xi_1}(t, \xi_1, \xi_2) = &-\frac{1}{\xi_1} h + \frac{1}{\xi_1^3 \xi_2} \int_0^t \int_0^s (t-s) e^{-(t-s)/\xi_1} e^{-(s-r)/\xi_2} g(r) \, dr \, ds \\ &+ \frac{1}{\xi_2} h - \frac{1}{\xi_1 \xi_2^3} \int_0^t \int_0^s (s-r) e^{-(t-s)/\xi_1} e^{-(s-r)/\xi_2} g(r) \, dr \, ds. \end{aligned} \tag{7.4}$$

We show that $h_{\xi_1} < 0$. By integration by parts and using the identity

$$\int_0^s e^{-(s-r)/\xi_2} g(r) \, dr = \xi_1 \xi_2 h_s + \xi_2 h, \tag{7.5}$$

the second term of the above identity is calculated as follows:

$$\begin{aligned}
&\frac{1}{\xi_1^3\xi_2}\int_0^t\int_0^s (t-s)e^{-(t-s)/\xi_1}e^{-(s-r)/\xi_2}g(r)\,dr\,ds \\
&\quad = \frac{1}{\xi_1^3\xi_2}\int_0^t (t-s)e^{-(t-s)/\xi_1}(\xi_1\xi_2 h_s + \xi_2 h)\,ds \\
&\quad = \frac{1}{\xi_1^2}\int_0^t (t-s)e^{-(t-s)/\xi_1}h_s\,ds + \frac{1}{\xi_1^3}\int_0^t (t-s)e^{-(t-s)/\xi_1}h(s)\,ds \\
&\quad = \frac{1}{\xi_1^2}\int_0^t e^{-(t-s)/\xi_1}h(s)\,ds \\
&\quad = \frac{1}{\xi_1}h - h_t + \int_0^t e^{-(t-s)/\xi_1}h_{ss}(s)\,ds.
\end{aligned}$$

Similarly, the forth term of (7.4) is written as below:

$$\begin{aligned}
&-\frac{1}{\xi_1\xi_2^3}\int_0^t\int_0^s (s-r)e^{-(t-s)/\xi_1}e^{-(s-r)/\xi_2}g(r)\,dr\,ds \\
&\quad = -\frac{1}{\xi_1\xi_2^3}\int_0^t\int_0^s e^{-(t-s)/\xi_1}e^{-(s-r)/\xi_2}\int_0^r e^{-(r-\eta)/\xi_2}g(\eta)\,d\eta\,dr\,ds \\
&\quad = -\frac{1}{\xi_1\xi_2^3}\int_0^t\int_0^s e^{-(t-s)/\xi_1}e^{-(s-r)/\xi_2}(\xi_1\xi_2 h_\eta + \xi_2 h)\,d\eta\,dr\,ds \\
&\quad = -\frac{\xi_1}{\xi_2(\xi_2-\xi_1)}\int_0^t (e^{-(t-s)/\xi_2} - e^{-(t-s)/\xi_1})h_s\,dr\,ds \\
&\qquad - \frac{1}{\xi_2(\xi_2-\xi_1)}\int_0^t (e^{-(t-s)/\xi_2} - e^{-(t-s)/\xi_1})h\,dr\,ds \\
&\quad = -\frac{1}{\xi_2^2}\int_0^t e^{-(t-s)/\xi_2}h\,ds \\
&\quad = -\frac{1}{\xi_2}h + h_t - \int_0^t e^{-(t-s)/\xi_2}h_{ss}\,ds.
\end{aligned}$$

Here we have used the identity (7.2) replaced g by h and h_η. Thus we obtain

$$h_{\xi_1}(t,\xi_1,\xi_2) = \int_0^t (e^{-(t-s)/\xi_1} - e^{-(t-s)/\xi_2})h_{ss}\,ds < 0.$$

Hence the first and the second inequalities of (7.3) are verified. Taking the limit as $\xi_1 \to 0$ in (7.2) gives the third inequality of (7.3). □

Since $F_i(\tau, \xi_1, \xi_2 \ldots, \xi_i)$ is invariant under the permutation of ξ_j, the following two lemmas are consequences of successive applications of the previous lemma.

Lemma 7.2. *Let $\tau > 0$, $\lambda > 0$, $i \geq 1$ and let $\xi_j > 0$, $j = 1, 2, \ldots, i$. Assume that $\sum_{j=1}^{i} \xi_j = \lambda$. Then*

$$F_i(\tau, \xi_1, \xi_2, \ldots, \xi_i) \geq F_i(\tau, \lambda/i, \ldots, \lambda/i) = \frac{1}{(i-1)!}\left(\frac{i}{\lambda}\right)^i \int_0^\tau s^{i-1} e^{-is/\lambda}\, ds. \quad (7.6)$$

Lemma 7.3. *Let $\tau > 0$, $\lambda > 0$, $i \geq 1$ and let $\eta \in (0, 1)$. Take a natural number l satisfying $(l-1)\eta < \lambda \leq l\eta$. Then we have*

$$F_i(\tau, \xi_1, \xi_2, \ldots, \xi_i) \leq F_l(\tau, \lambda - (l-1)\eta, \eta, \ldots, \eta)$$

for ξ_j satisfying $\xi_j \in (0, \eta]$, $j = 1, 2, \ldots, i$, and $\sum_{j=1}^{i} \xi_j = \lambda$.

Let $N \geq 2$. Let $\Delta = \{0 = \sigma_0 < \sigma_1 < \cdots < \sigma_{N+1} = 1\}$ be a partition of $[0, 1]$ and set $\xi_i = \sigma_i - \sigma_{i-1}$. We introduce two families of functions by

$$G_i(t) = \frac{1}{\xi_{i+1}} \int_0^t e^{-(t-s)/\xi_{i+1}} G_{i-1}(s)\, ds, \quad G_0(t) = \xi_1^{-1} e^{-t/\xi_1}, \quad (7.7)$$

for $t \in [0, 1]$ and $i = 1, 2, \ldots, N$, and

$$a_i(t) = \int_0^t G_{i-1}(s)\, ds = F_i(t, \xi_1, \xi_2, \ldots, \xi_i) \quad (7.8)$$

for $t \geq 0$ and $i = 1, 2, \ldots, N+1$.

Lemma 7.4. *Let $n \geq 1$. Then*

$$\frac{n^n}{(n-1)!} \int_0^1 s^{n-1} e^{-ns}\, ds > 1/2. \quad (7.9)$$

Proof. Since $n^{2(n-i)} \geq \prod_{j=i+1}^{i} (2n-j)j$ for $0 \leq i \leq n-1$, we see that

$$\sum_{i=0}^{n-1} \frac{n^i}{i!} \leq \sum_{i=0}^{n-1} \frac{n^{2n-i-1}}{(2n-i-1)!} = \sum_{i=n}^{2n-1} \frac{n^i}{i!} < \sum_{i=n}^{\infty} \frac{n^i}{i!}.$$

This together with the representation

$$\frac{n^n}{(n-1)!} \int_0^1 s^{n-1} e^{-ns}\, ds = 1 - e^{-n} \sum_{i=0}^{n-1} n^i / i!$$

implies the desired result. □

The following three lemmas show some technical but useful properties of $a_i(t)$ and $G_i(t)$.

Lemma 7.5.

(i) *For $t > 0$ and $i = 1, 2, \ldots, N$, $a_i(t) > a_{i+1}(t)$.*

(ii) *For $i = 1, 2, \ldots, N+1$, $\lim_{t \to \infty} a_i(t) = 1$ and $a_i(1) > 1/2$.*

(iii) *$\sum_{i=1}^{N+1} \xi_i G_{i-1}(t) \leq 1$ for $t \in [0, 1]$ and $\int_0^1 G_{i-1}(t)\, dt < 1$.*

Proof. Integrating (7.8) by parts, we have

$$a_{i+1}(t) = -\xi_{i+1}G_i(t) + a_i(t) < a_i(t). \tag{7.10}$$

The assertion $a_i(1) > 1/2$ follows from (7.10), Lemmas 7.1 and 7.4. It follows from direct calculations that $\lim_{t\to\infty} a_i(t) = 1$. Next we verify assertion (iii). Set $Q_N(t) = \sum_{i=1}^{N+1} \xi_i G_{i-1}(t)$. Then we see that

$$Q_N(t)' = -G_N(t) \leq 0.$$

This together with $Q_N(0) = 1$ implies that $Q_N(t) \leq 1$ for $t \in [0,1]$. Finally, it follows from Lemma 7.5 that

$$\int_0^1 G_{i-1}(t)\,dt = a_i(1) \leq a_1(1) < 1.$$

The proof is now complete. □

Lemma 7.6. *Let $\delta \in (0,1)$ and $\eta \in (0,\delta/2)$. Then, for each partition Δ of $[0,1]$ satisfying $|\Delta| \leq \eta$, it holds that*

$$a_i(\sigma_i - \delta) \leq \sqrt{\frac{1}{2\pi\eta}}(1-\delta)^{-1/\delta}\left((1-\delta)e^{\delta}\right)^{1/\eta} \quad \textit{for } \sigma_i \in (\delta, 1].$$

Proof. Let l be a natural number satisfying $(l-1)\eta < \sigma_i \leq l\eta$. By Lemma 7.3 we see that

$$\begin{aligned}
a_i(\sigma_i - \delta) &\leq F_l(\sigma_i-\delta, \sigma_i-(l-1)\eta, \eta, \ldots, \eta)\\
&= \frac{1}{(l-2)!\eta^{l-1}(\sigma_i-(l-1)\eta)}\int_0^{\sigma_i-\delta}\int_0^t s^{l-2}e^{-(t-s)/(\sigma_i-(l-1)\eta)}e^{-s/\eta}\,ds\,dt\\
&= -\frac{1}{(l-2)!\eta^{l-1}}e^{-(\sigma_i-\delta)/(\sigma_i-(l-1)\eta)}\int_0^{\sigma_i-\delta} s^{l-2}e^{s(l\eta-\sigma_i)/(\eta(\sigma_i-(l-1)\eta)}\,ds\\
&\quad + \frac{1}{(l-2)!\eta^{l-1}}\int_0^{\sigma_i-\delta} s^{l-2}e^{-s/\eta}\,ds\\
&\leq \frac{1}{(l-2)!\eta^{l-1}}\int_0^{\sigma_i-\delta} s^{l-2}e^{-s/\eta}\,ds.
\end{aligned}$$

Since $s \mapsto s^{l-2}e^{-s/\eta}$ is monotone increasing over the interval $[0, \sigma_i - \delta]$, and since the function $x \mapsto x^{l-1}e^{-x}$ is monotone decreasing over $[l-1, \infty)$, we have

$$\begin{aligned}
&\frac{1}{(l-2)!\eta^{l-1}}\int_0^{\sigma_i-\delta} s^{l-2}e^{-s/\eta}\,ds \leq \frac{1}{(l-2)!\eta^{l-1}}(\sigma_i-\delta)^{l-1}e^{-(\sigma_i-\delta)/\eta}\\
&= \frac{1}{(l-2)!}\left(\frac{\sigma_i}{\eta}\right)^{l-1} e^{-\sigma_i/\eta}\left(1-\delta/\sigma_i\right)^{l-1}e^{\delta/\eta}\\
&\leq \frac{(l-1)^{l-1}}{(l-2)!}e^{-(l-1)}\left(1-\delta/\sigma_i\right)^{l-1}e^{\delta/\eta} \leq \sqrt{\frac{l-1}{2\pi}}\left(1-\delta/\sigma_i\right)^{l-1}e^{\delta/\eta}.
\end{aligned}$$

Here we have used the Stirling's formula to show the last inequality. Since $x \mapsto (1-\delta/x)^x$ is monotone increasing over $(\delta, 1]$ and $l/\sigma_i \geq 1/\eta$, we have

$$\begin{aligned}\sqrt{\frac{l-1}{2\pi}}\,(1-\delta/\sigma_i)^{l-1}\,e^{\delta/\eta} &\leq \sqrt{\frac{1}{2\pi\eta}}\,((1-\delta/\sigma_i)^{\sigma_i})^{(l-1)/\sigma_i}\,e^{\delta/\eta} \\ &\leq \sqrt{\frac{1}{2\pi\eta}}(1-\delta)^{-1/\delta}\left((1-\delta)e^{\delta}\right)^{1/\eta}. \qquad \square\end{aligned}$$

Lemma 7.7. *Let $\delta \in (0,1)$ and $\eta \in (0, \delta + (1-\delta)\log(1-\delta)]$. Then, for any partition $\Delta = \{0 = \sigma_0 < \sigma_1 < \cdots < \sigma_{N+1} = 1\}$ of $[0,1]$ satisfying $|\Delta| \leq \eta$, it holds that*

$$1 - a_i(\sigma_i + \delta) \leq \sqrt{\frac{1-\delta}{2\pi}}\frac{1}{\sqrt{\eta}}\left((1-\delta)^{\delta-1}e^{-\delta}\right)^{1/\eta} \quad \textit{for } \sigma_i \in (0, 1-\delta].$$

Proof. Let $\delta \in (0,1)$, $\eta \in (0, \delta + (1-\delta)\log(1-\delta)]$ and let Δ be a partition of $[0,1]$ satisfying $|\Delta| \leq \eta$. Let $\sigma_i \in (0, 1-\delta)$. Since $s \mapsto se^{-s/\sigma_i}$ is decreasing for $s > \sigma_i$, we have, by Lemma 7.2 and Stirling's formula,

$$\begin{aligned}1 - a_i(\sigma_i+\delta) &\leq 1 - \frac{1}{(i-1)!}\left(\frac{i}{\sigma_i}\right)^i \int_0^{\sigma_i+\delta} s^{i-1}e^{-is/\sigma_i}\,ds \\ &= \frac{1}{(i-1)!}\left(\frac{i}{\sigma_i}\right)^i \int_{\sigma_i+\delta}^{\infty} s^{i-1}e^{-is/\sigma_i}\,ds \\ &\leq \frac{1}{(i-1)!}\left(\frac{i}{\sigma_i}\right)^i ((\sigma_i+\delta)e^{-(\sigma_i+\delta)/\sigma_i})^{i-1}\int_{\sigma_i+\delta}^{\infty} e^{-s/\sigma_i}\,ds \\ &\leq \sqrt{\frac{i}{2\pi}}(1+\delta/\sigma_i)^{i-1}e^{-i\delta/\sigma_i} \\ &\leq \sqrt{\frac{1-\delta}{2\pi}}\sqrt{\frac{i}{\sigma_i}}((1+\delta/\sigma_i)^{\sigma_i}e^{-\delta})^{i/\sigma_i}.\end{aligned}$$

Since $\xi \mapsto (1+\delta/\xi)^{\xi}$ is increasing for $\xi \in (0, 1-\delta]$, $(1-\delta)^{\delta-1}e^{-\delta} < 1$, $i/\sigma_i \geq 1/\eta$ and since $x \mapsto \sqrt{x}((1-\delta)^{\delta-1}e^{-\delta}))^x$ is decreasing for $x \geq 1/(\delta + (1-\delta)\log(1-\delta))$, we conclude that

$$1 - a_i(\sigma_i+\delta) \leq \sqrt{\frac{1-\delta}{2\pi}}\frac{1}{\sqrt{\eta}}((1-\delta)^{\delta-1}e^{-\delta})^{1/\eta}. \qquad \square$$

Now we introduce the notion of discrete *local multiple Laplace transforms* and investigate some properties of them.

Let $(Z, \|\cdot\|_Z)$ be a Banach space and $v : [0,1] \to Z$. Let $\Delta = \{0 = \sigma_0 < \sigma_1 < \cdots < \sigma_{N+1} = 1\}$ be a partition of $[0,1]$. Then we define a discrete local multiple Laplace transform of v by

$$\hat{v}_\Delta(\sigma_i) = \frac{1}{a_i(1)}\int_0^1 G_{i-1}(t)v(t)\,dt \quad \text{for } i = 1, 2, \ldots, N+1.$$

Lemma 7.8. *Let $(Z, \|\cdot\|_Z)$ be a Banach space and let $v \in C([0,1];Z)$. Then, to each $\varepsilon > 0$, there corresponds $\eta > 0$ such that for any partition $\Delta = \{0 = \sigma_0 < \sigma_1 < \cdots < \sigma_{N+1} = 1\}$ satisfying $|\Delta| < \eta$, we have*

$$\|\hat{v}_\Delta(\sigma_i) - v(\sigma_i)\| < \varepsilon/2, \quad \textit{for } i = 1, 2, \ldots, N+1.$$

Proof. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $|v(t) - v(\sigma_i)| < \varepsilon/12$ for $t \in [\sigma_i - \delta, \sigma_i + \delta] \cap [0,1]$ and $i = 1, 2, \ldots, N+1$. Choose a positive number η such that $\eta < \min\{\delta + (1-\delta)\log(1-\delta), \delta/2\}$ and

$$\eta^{-1/2} \max\{(1-\delta)^{\delta-1} e^{-\delta}, (1-\delta)e^{\delta}\}^{1/\eta} < \frac{(1-\delta)^{1/\delta}\varepsilon}{24(1 + \|v\|_\infty)},$$

where $\|f\|_\infty = \max\{\|f(t)\|_Z : t \in [0,1]\}$. By Lemmas 7.6 and 7.7, we obtain

$$\begin{aligned}
&\|\hat{v}_\Delta(\sigma_i) - v(\sigma_i)\| \\
&\quad \le 2\int_0^{(\sigma_i-\delta)\vee 0} G_{i-1}(t)\|v(t) - v(\sigma_i)\|_Z\,dt + 2\int_{(\sigma_i+\delta)\wedge 1}^1 G_{i-1}(t)\|v(t) - v(\sigma_i)\|_Z\,dt \\
&\qquad + 2\int_{(\sigma_i-\delta)\vee 0}^{(\sigma_i+\delta)\wedge 1} G_{i-1}(t)\|v(t) - v(\sigma_i)\|_Z\,dt \\
&\quad \le 4\|v\|_\infty \left(\int_0^{(\sigma_i-\delta)\vee 0} G_{i-1}(t)\,dt + \int_{(\sigma_i+\delta)\wedge 1}^1 G_{i-1}(t)\,dt \right) + \varepsilon/6 \\
&\quad \le 4\|v\|_\infty (a_i((\sigma_i-\delta)\vee 0) + a_i(1) - a_i((\sigma_i+\delta)\wedge 1)) + \varepsilon/6 \\
&\quad < \varepsilon/2,
\end{aligned}$$

where $s \vee t = \max\{s,t\}$ and $s \wedge t = \min\{s,t\}$. Thus the proof is obtained. □

Lemma 7.9. *Let $(Z, \|\cdot\|_Z)$ be a Banach space, $g \in L^1(0,1;Z)$, and let $\varepsilon > 0$. Then there exists a partition $\Delta = \{0 = \sigma_0 < \sigma_1 < \cdots < \sigma_{N+1} = 1\}$ of $[0,1]$ such that $|\Delta| < \varepsilon$ and*

$$\sum_{i=1}^N \xi_i \|\hat{g}_\Delta(\sigma_i) - g(\sigma_i)\|_Z + \xi_{N+1}\|\hat{g}_\Delta(1) - g(\sigma_N)\|_Z < \varepsilon,$$

where $\xi_i = \sigma_i - \sigma_{i-1}$ and $i = 1, 2, \ldots, N+1$.

Proof. Let $g \in L^1(0,1)$ and $\varepsilon > 0$. Then there exists a continuous function v over $[0,1]$ such that $\int_0^1 \|g(t) - v(t)\|_Z\,dt < \varepsilon/16$. Let η_0 be a positive number such that $\|v(s) - v(t)\|_Z < \varepsilon/16$ for s, $t \in [0,1]$ with $|s-t| < \eta_0$. Let η be a positive number specified as in Lemma 7.8. By [10, Lemma 3.3.1] there exists a partition $\Delta : 0 = \sigma_0 < \sigma_1 < \cdots < \sigma_{N+1} = 1$ satisfying $|\Delta| \le \min\{\eta_0, \eta, \varepsilon\}$ and

$$\sum_{i=1}^N \int_{\sigma_{i-1}}^{\sigma_i} \|g(s) - g(\sigma_i)\|_Z\,ds + \int_{\sigma_N}^1 \|g(s) - g(\sigma_N)\|_Z\,ds < \varepsilon/16. \tag{7.11}$$

Then we have

$$
\begin{aligned}
&\sum_{i=1}^{N} \xi_i \|\hat{g}_\Delta(\sigma_i) - g(\sigma_i)\|_Z + \xi_{N+1}\|\hat{g}_\Delta(1) - g(\sigma_N)\|_Z \\
&\quad \le \sum_{i=1}^{N} \frac{1}{a_i(1)} \int_{\sigma_{i-1}}^{\sigma_i} \int_0^1 G_{i-1}(t)\|g(t) - g(\sigma_i)\|_Z \, dt\, ds \\
&\qquad + \frac{\xi_{N+1}}{a_{N+1}(1)} \int_0^1 G_N(t)\|g(t) - g(\sigma_N)\|_Z \, dt \\
&\quad \le \sum_{i=1}^{N+1} \frac{1}{a_i(1)} \int_{\sigma_{i-1}}^{\sigma_i} \int_0^1 G_{i-1}(t)\|v(t) - v(s)\|_Z \, dt\, ds + 4\int_0^1 \|v(t) - g(t)\|_Z \, dt \\
&\qquad + 2\sum_{i=1}^{N} \int_{\sigma_{i-1}}^{\sigma_i} \|g(s) - g(\sigma_i)\|_Z \, ds + 2\int_{\sigma_N}^{1} \|g(s) - g(\sigma_N)\|_Z \, ds \\
&\quad \le \sum_{i=1}^{N+1} \frac{1}{a_i(1)} \int_{\sigma_{i-1}}^{\sigma_i} \int_0^1 G_{i-1}(t)\|v(t) - v(\sigma_i)\|_Z \, dt\, ds + \varepsilon/2 < \varepsilon.
\end{aligned}
$$

Here we have used Lemma 7.5 (iii). Thus, the proof is complete. □

8. Characterization of nonlinearly perturbed analytic semigroups

In this section we give the proof of implication (I) ⇒ (II) in Theorem 4.6. Once this is done, then we will have our characterization theorem. Throughout this section we assume (H1) through (H3) and (I).

Let $\varepsilon > 0$, $s \in [0, \tau)$, $v \in D(s)$ and set $R = \Psi(\tau, s; \varphi(v))$. Let $\mathscr{U}$ be the evolution operator satisfying (I), ω_R a positive constant and $f_R \in \mathcal{F}_p$ a function specified in (H3) replaced r by R. We then set $u(t) = U(t, s)v$ and $C = \sup_{s \le t \le \tau} |u(t)|$. We also choose $\sigma_* \in (0, 1)$ so that σ_* is close enough to 1 and

$$|u(\tau) - u(s + (\tau - s)\sigma)| < \varepsilon/16 \quad \text{for } \sigma \in [\sigma_*, 1].$$

Let η_0 be a positive number such that $8C\sqrt{2/\pi}x^{-1/2}(\sigma_*^{-\sigma_*}e^{\sigma_*-1})^{1/x} < \varepsilon$ for $x \in (0, \eta_0]$. Note that $\sigma_*^{-\sigma_*}e^{\sigma_*-1} < 1$. Finally, let η be a positive number specified in Lemma 7.8 replaced Z, $v(t)$ and ε by Y, $u(s + (\tau - s)t)$ and $\varepsilon/8(1 + \omega_R\tau + \tau)$, respectively. Then by Lemma 7.9 replaced ε by $\min\{\eta_0, \eta, 1 - \sigma_*, \varepsilon/4(\tau - s)\}$ and (7.11), there exists a partition $\{0 = \sigma_0 < \sigma_1 < \cdots < \sigma_{N+1} = 1\}$ of $[0, 1]$ such that $\max\{\sigma_i - \sigma_{i-1}\} < \min\{\eta_0, \eta, 1 - \sigma_*, \varepsilon/4(\tau - s)\}$ and

$$\sum_{i=1}^{N} \int_{\sigma_{i-1}}^{\sigma_i} f_R(s + (\tau - s)t, s + (\tau - s)\sigma_i)\, dt < \frac{\varepsilon}{\tau - s}, \tag{8.1}$$

$$\sum_{i=1}^{N} \frac{\xi_i}{a_i(1)} \int_0^1 G_{i-1}(t) f_R(s + (\tau - s)\sigma_i, s + (\tau - s)t)\, dt < \frac{\varepsilon}{4(\tau - s)}. \tag{8.2}$$

Set $t_i = s+(\tau-s)\sigma_i$, $i = 0,1,\ldots,N+1$. Then $\Delta = \{s = t_0 < t_1 < \cdots < t_{N+1} = \tau\}$ is a partition of $[s,\tau]$. Set $h_i = t_i - t_{i-1}$ for $i = 1,2,\ldots,N+1$. We define sequences $\{v_i : 0 \le i \le N\}$ and $\{B_i v : 1 \le i \le N\}$ in X, and a sequences $\{u_i : 1 \le i \le N\}$ in Y by $v_0 = v$, $u_i = u(t_i)$ and

$$v_i = a_i(1)^{-1}\int_0^1 G_{i-1}(t)u(s+(\tau-s)t)\,dt,$$
$$B_i v = a_i(1)^{-1}\int_0^1 G_{i-1}(t)B(s+(\tau-s)t)u(s+(\tau-s)t)\,dt,$$

where $a_i(t)$ and $G_{i-1}(t)$ are the functions defined by (7.7) and (7.8) with $\sigma_i = t_i/(\tau-s)$ and $\xi_i = (t_i - t_{i-1})/(\tau-s)$ for $i = 1,2,\ldots,N$. The integral is taken in X in the sense of Bochner.

The next result is crucial for the subsequent discussions.

Lemma 8.1. *The sequences $\{v_i\}$, $\{B_i v\}$ and $\{u_i\}$ have the following properties:*

(i) *$v_i \in D(A)$ and $(I - h_i A)v_i$ can be written as*

$$v_{i-1} + h_i B_i v + h_i((\tau-s)a_i(1))^{-1}G_{i-1}(1)(v_{i-1} - u(\tau)) \quad \textit{for } i = 1,2,\ldots,N.$$

(ii) *$u_i \in D(t_i)$ and $\varphi(u_i) \le \Psi(\tau,s;\varphi(v))$ for $i = 1,2,\ldots,N$.*

(iii) *For $i = 1,2,\ldots,N$*

$$\|v_i - u_i\| < \varepsilon/\tau \quad \textit{and} \quad \sum_{i=1}^N h_i|B_i v - B(t_i)u_i| < \varepsilon/2.$$

Proof. Let $s \in [0,\tau)$ and $v \in D(s)$ and let $u(\cdot)$ be a mild solution to (SE)-(IC)-(G). Assertion (ii) is obvious. We next prove assertion (i). To this end, we employ a function $p_i(t)$ defined by $p_i(t) = (a_i(1))^{-1}a_i(t)$. By Theorem 4.5 (b), it holds that

$$\begin{aligned} A\int_0^{(\tau-s)t} u(s+r)\,dr &= u(s+(\tau-s)t) - v \\ &\quad - \int_0^{(\tau-s)t} B(s+r)u(s+r)\,dr \quad \text{for } t \in [0,1]. \end{aligned} \tag{8.3}$$

Multiplying both sides of the above identity by $(a_i(1))^{-1}G_{i-1}(t)$ and integrating the resultant equation over $[0,1]$, we have

$$\begin{aligned} &(a_i(1))^{-1}\int_0^1\left(G_{i-1}(t)A\int_0^{(\tau-s)t}u(s+r)\,dr\right)dt \\ &= v_i - v - (a_i(1))^{-1}\int_0^1 G_{i-1}(t)\int_0^{(\tau-s)t}B(s+r)u(s+r)\,dr\,dt \\ &= v_i - v + (\tau-s)\int_0^1 p_i(t)B(s+(\tau-s)t)u(s+(\tau-s)t)\,dt - \int_0^{\tau-s}B(s+r)u(s+r)\,d\sigma. \end{aligned}$$

In view of (8.3) the last term on the right-hand side is replaced by

$$u(\tau) - v - A\int_s^\tau u(r)\,dr.$$

On the other hand, the left-hand side can be written as

$$A\left(-(\tau-s)\int_0^1 p_i(t)u(s+(\tau-s)t)\,dt + \int_s^\tau u(r)\,dr\right)$$

and thus we obtain

$$\begin{aligned}
&-(\tau-s)A\int_0^1 p_i(t)u(s+(\tau-s)t)\,dt \\
&\qquad = v_i - u(\tau) + (\tau-s)\int_0^1 p_i(t)B(s+(\tau-s)t)u(s+(\tau-s)t)\,dt
\end{aligned} \tag{8.4}$$

for $i = 1, 2, \ldots, N$. Assume that $i \geq 2$. Using the relation

$$p_i(t) = -\xi_i a_i(1)^{-1}G_{i-1}(t) + (a_{i-1}(1)/a_i(1))p_{i-1}(t),$$

we have, by (8.4),

$$\begin{aligned}
h_iAv_i &= v_i - u(\tau) - h_iB_iv + \frac{a_{i-1}(1)}{a_i(1)}(\tau-s)A\int_0^1 p_{i-1}(t)u(s+(\tau-s)t)\,dt \\
&\quad + \frac{a_{i-1}(1)}{a_i(1)}(\tau-s)\int_0^1 p_{i-1}(t)B(s+(\tau-s)t)u(s+(\tau-s)t)\,dt \\
&= v_i - u(\tau) - h_iB_iv + \frac{a_{i-1}(1)}{a_i(1)}(u(\tau) - v_{i-1}).
\end{aligned}$$

This together with (7.10) implies assertion (i) for $i \geq 2$. The case where $i = 1$ is proved in a similar way. Finally, assertion (iii) follows from Lemmas 7.8, 7.9 and (8.2). □

The implication (I) ⇒ (II) in our Characterization Theorem, Theorem 4.6, is a direct consequence of the theorem below:

Theorem 8.2. *Let $s \in [0,\tau)$, $\varepsilon > 0$ and let $v \in D(s)$. Then there exists a partition $\Delta : s = t_0 < t_1 < \cdots < t_{N+1} = \tau$, finite sequences $\{v_i\}_{i=0}^{N+1}$, $\{u_i\}_{i=1}^{N}$ in Y and $\{z_i\}_{i=1}^{N}$ in X such that $v_0 = v$, $v_i \in D(A)$, $u_i \in D(t_i)$,*

$$\begin{aligned}
&\|v_i - u_i\| < \varepsilon/\tau, \quad \sum_{i=1}^N (t_i - t_{i-1})|z_i| < \varepsilon, \quad \sum_{i=1}^N \int_{t_{i-1}}^{t_i} f_R(t_i,\xi)\,d\xi < \varepsilon, \\
&v_i - (t_i - t_{i-1})(Av_i + B(t_i)u_i) = v_{i-1} + (t_i - t_{i-1})z_i, \\
&\varphi(u_i) \leq \Psi(\tau, s; \varphi(v)), \quad \textit{for } i = 1, 2, \ldots, N,
\end{aligned}$$

and such that if $(t_i, u_i) \to (t, u)$ as $\varepsilon \to 0$, then $u \in D(t)$.

Proof. Let $s \in [0, \tau)$, $\varepsilon > 0$ and let $v \in D(s)$. Let Δ, $\{v_i\}$ and $\{u_i\}$ be a partition and sequences employed in Lemma 8.1. Set $z_i = B_i v - B(t_i)u_i + ((\tau - s)a_i(1))^{-1}G_{i-1}(1)(v_{i-1} - u(\tau))$. In view of Lemma 8.1 and (8.1), we have only to show that

$$\frac{1}{(\tau - s)a_i(1)} \sum_{i=1}^{N} h_i G_{i-1}(1)|v_{i-1} - u(\tau)| \\ \leq \frac{2}{\tau - s} \sum_{i=i}^{N} h_i G_{i-1}(1)|v_{i-1} - u(\tau)| < \varepsilon/2. \quad (8.5)$$

By Lemmas 7.5 and 7.8 we have

$$\frac{2}{\tau - s} \sum_{i=i_*+1}^{N+1} h_i G_{i-1}(1)|v_{i-1} - u(\tau)| < \varepsilon/4, \quad (8.6)$$

where i_* is the minimum integer of the set $\{i : \sigma_* \leq \sigma_i < 1\}$. Since the function $x \mapsto (e/x)^x$ is monotone increasing on $(0, 1]$, we have, by (7.10) and Lemma 7.7,

$$\begin{aligned} \frac{2}{\tau - s} \sum_{i=1}^{i_*} h_i G_{i-1}(1)|v_{i-1} - u(\tau)| &\leq 4C \left(\xi_1 G_0(1) + \sum_{i=2}^{i_*} (a_{i-1}(1) - a_i(1)) \right) \\ &= 4C(1 - a_{i_*}(1)) \\ &\leq 2C\sqrt{2/\pi}\eta^{-1/2}(\sigma_{i_*}^{-\sigma_{i_*}} e^{\sigma_{i_*} - 1})^{1/\eta} \quad (8.7) \\ &\leq 2C\sqrt{2/\pi}\eta^{-1/2}(\sigma_*^{-\sigma_*} e^{\sigma_* - 1})^{1/\eta} \\ &< \varepsilon/4. \end{aligned}$$

Combining (8.6) with (8.7), we obtain (8.5). The proof is now complete. □

9. Applications to convective reaction-diffusion systems

As an application of our characterization theorem, Theorem 4.6, we here deal with the existence and uniqueness of mild solutions to a convective reaction-diffusion system describing the bone remodeling phenomena. The mathematical model takes the following form :

$$\text{(RDS)} \quad \begin{cases} u_t = d_1 \Delta u - \alpha_1 \mathbb{E} \cdot \nabla u + \gamma w u - \beta v u - c_1 u, & (t, x) \in (0, \tau) \times \Omega, \\ v_t = d_2 \Delta v - \alpha_2 \mathbb{E} \cdot \nabla v + a_2 \nabla u \cdot \nabla v + \varepsilon_2 u v - c_2 v, \\ w_t = d_3 \Delta w + \alpha_3 \mathbb{E} \cdot \nabla w - a_3 \nabla u \cdot \nabla w - \varepsilon_3 u w + c_3 w \end{cases}$$

under the initial condition

$$\text{(IC)} \qquad u(0, x) = u_0(x) \geq 0, \quad v(0, x) = v_0(x) \geq 0, \quad w(0, x) = w_0(x) \geq 0$$

and the boundary condition

$$\text{(BC)} \qquad \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0.$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, $\tau > 0$ and ν denotes the outward unit normal to $\partial\Omega$. Coefficients d_i, α_i, c_i, a_j, ε_j, $i = 1, 2, 3$, $j = 2, 3$, β and γ are all positive constants. We assume that $\mathbb{E} \in C([0, \tau]; (L^\infty(\Omega))^n)$.

In this model, $u = u(t, x)$ represents the concentration of calcium at $(t, x) \in [0, \tau] \times \Omega$. $v = v(t, x)$ and $w = w(t, x)$ stand for the cell densities of osteoblasts and osteoclasts, respectively. $\mathbb{E}$ represents an electric filed generated by stress-strain distribution through the bone. Advection effects along the negative and positive directions of physical and chemical stimulation $\mathbb{E}$ are denoted by the convection terms $-\alpha_1 \mathbb{E} \cdot \nabla u$, $-\alpha_2 \mathbb{E} \cdot \nabla v$ and $\alpha_3 \mathbb{E} \cdot \nabla w$. The terms $a_2 \nabla v \cdot \nabla u$ and $a_3 \nabla w \cdot \nabla u$ describe the advection effects on osteoblasts and osteoclasts along the gradient of the concentration of u. This model was first introduced in [17] and [18] as a mathematical model which describes a complex physiological phenomena of bone metabolism. In [16] an attempt was made to show the solvability of the model, although this section contains a complete version of the proof.

Now we rewrite the nonlinear problem (RDS)-(BC)-(IC) as the abstract Cauchy problem in the product Banach space $(L^\infty(\Omega))^3$. First, we introduce a Banach space $X = (L^\infty(\Omega))^3$ equipped with the norm $|\mathbf{v}|_X = \max_{k=1,2,3} |v_k|_\infty$ for $\mathbf{v} = (v_1, v_2, v_3) \in X$. By $L^\infty_+(\Omega)$ we denote the set of all nonnegative elements in $(L^\infty(\Omega))^3$ and set $X_+ = (L^\infty_+(\Omega))^3$. Let $\omega_0 > 0$. A linear operator $\boldsymbol{A}$ in X is defined by

$$D(\boldsymbol{A}) = \{\mathbf{v} = (v_k) : v_k \in W^{2,p}(\Omega) \text{ for } p > n \text{ and } \partial v_k/\partial\nu = 0,\ k = 1, 2, 3\},$$

$$\boldsymbol{A}\mathbf{v} = (A_1 v_1, A_2 v_2, A_3 v_3) \quad \text{for } \mathbf{v} \in D(\boldsymbol{A}),$$

$$A_1 v_1 = d_1 \Delta v_1 - \omega_0 v_1, \quad A_2 v_2 = d_2 \Delta v_2 - \omega_0 v_2, \quad A_3 v_3 = d_3 \Delta v_3 - \omega_0 v_3.$$

Then it is known that each A_i generates an analytic semigroup $T_i(t)$ on $L^\infty(\Omega)$ satisfying

$$|T_i(t)|_\infty \le e^{-\omega_0 t} \quad \text{for } t \ge 0, \tag{9.1}$$

and hence $\boldsymbol{A}$ generates an analytic semigroup $\boldsymbol{T}(t) = (T_1(t), T_2(t), T_3(t))$ on X satisfying

$$|\boldsymbol{T}(t)|_X \le e^{-\omega_0 t} \quad \text{for } t \ge 0. \tag{9.2}$$

For details we refer to [13, Corollaries 3.1.21 and 3.1.24].

Choose a real number $\theta \in (1/2, 1)$. We then introduce another Banach space $(Y, \|\cdot\|_Y)$ defined by

$$Y = \{\mathbf{v} \in D((-\boldsymbol{A})^\theta) : (-\boldsymbol{A})^\theta \mathbf{v} \in \overline{D(\boldsymbol{A})}\},$$

$$\|\mathbf{v}\|_Y = |(-\boldsymbol{A})^\theta \mathbf{v}|_X \quad \text{for } \mathbf{v} \in Y.$$

We here remark that $(-\boldsymbol{A})^\theta$ is represented as

$$(-\boldsymbol{A})^\theta \mathbf{v} = ((-A_1)^\theta v_1, (-A_2)^\theta v_2, (-A_3)^\theta v_3) \quad \text{for } \mathbf{v} \in Y.$$

It is well known that there exists a constant $M_\theta \geq 1$ satisfying

$$|v|_\infty \leq M_\theta |(-A_i)^\theta v|_\infty, \tag{9.3}$$

$$|(I - \lambda A_i)^{-1} v - v|_\infty \leq M_\theta \lambda^\theta |(-A_i)^\theta v|_\infty \quad \text{for } v \in D((-A_i)^\theta),\ i = 1, 2, 3, \tag{9.4}$$

$$|(-A_i)^\theta (I - \lambda A_i)^{-1} v|_\infty \leq M_\theta \lambda^{-\theta} |v|_\infty \quad \text{for } \lambda > 0 \text{ and } v \in L^\infty(\Omega), \tag{9.5}$$

Y is embedded into $(C^1(\overline{\Omega}))^3$, and that there exists a constant c_0 such that

$$\|\mathbf{v}\|_{(C^1(\overline{\Omega}))^3} \leq c_0 \|\mathbf{v}\|_Y \quad \text{for } \mathbf{v} \in Y. \tag{9.6}$$

Next, for each $t \in [0, \tau]$, we define a nonlinear operator $\boldsymbol{B}(t)$ by

$$D(t) := D(\boldsymbol{B}(t)) = Y, \quad \boldsymbol{B}(t) = \boldsymbol{B}_c(t) + \boldsymbol{B}_r, \tag{9.7}$$

$$\boldsymbol{B}_c(t)\mathbf{v} = (B_{c,1}(t)\mathbf{v}, B_{c,2}(t)\mathbf{v}, B_{c,3}(t)\mathbf{v}), \quad \boldsymbol{B}_r \mathbf{v} = (B_{r,1}\mathbf{v}, B_{r,2}\mathbf{v}, B_{r,3}\mathbf{v}),$$

$$B_{c,1}(t)\mathbf{v} = -\alpha_1 \mathbb{E}(t) \cdot \nabla u, \quad B_{c,2}(t)\mathbf{v} = -\alpha_2 \mathbb{E}(t) \cdot \nabla v + a_2 \nabla u \cdot \nabla v,$$

$$B_{c,3}(t)\mathbf{v} = \alpha_3 \mathbb{E}(t) \cdot \nabla w - a_3 \nabla u \cdot \nabla w,$$

$$B_{r,1}\mathbf{v} = \gamma w u - \beta u v + (\omega_0 - c_1) u, \quad B_{r,2}\mathbf{v} = \varepsilon_2 u v + (\omega_0 - c_2) v,$$

$$B_{r,3}\mathbf{v} = -\varepsilon_3 u w + (\omega_0 + c_3) w \quad \text{for } \mathbf{v} = (u, v, w) \in Y.$$

We also employ a continuous functional φ on Y defined by

$$\varphi(\mathbf{v}) = \|\mathbf{v}\|_Y \quad \text{for } \mathbf{v} \in Y. \tag{9.8}$$

Lemma 9.1. *For each $\lambda > 0$, we have*

$$|(I - \lambda A_i)^{-1} v|_\infty \leq (1 + \lambda \omega_0)^{-1} |v|_\infty \quad \textit{for } v \in L^\infty(\Omega) \textit{ and } i = 1, 2, 3.$$

For the proof of Lemma 9.1 we employ 0-1 measures defined as follows:

Definition 9.2. Let (S, Σ, μ) be a measure space. A measure μ on Σ is said to be 0-1 measure, if either $\mu(E) = 1$ or $\mu(E^c) = 1$.

Let $\mathscr{M}$ be the class of all Lebesgue measurable subsets of Ω and $ba(\Omega)$ the set of all finitely additive bounded measures on $\mathscr{M}$ which vanish on sets of Lebesgue measure zero. Let $\mu \in ba(\Omega)$ be a 0-1 measure. We set $\mathscr{M}(\mu) = \{E \in \mathscr{M} : \mu(E) = 1\}$. Then it is known that there exists a unique point $a \in \overline{\Omega}$ such that for any neighborhood U of a, $U \cap \Omega \in \mathscr{M}(\mu)$, $\bigcap \overline{\mathscr{M}(\mu)} = \{a\}$ and $\bigcap \mathscr{M}(\mu) = \emptyset$, where $\overline{\mathscr{M}(\mu)} = \{\overline{E} : E \in \mathscr{M}(\mu)\}$. (See [26, Theorem 3.1.3].) The singleton set $\{a\}$ of the uniquely determined point a is called the essential support of μ.

We here state some crucial properties of 0-1 measures in $ba(\Omega)$:

Lemma 9.3. *Let $\mu \in ba(\Omega)$ be a* 0-1 *measure with the essential support at $a \in \overline{\Omega}$. By $\langle f, w\rangle$ we denote the value of $w \in L^\infty(\Omega)^*$ at $f \in L^\infty(\Omega)$. Then the following are valid:*

(a) $ba(\Omega) = L^\infty(\Omega)^*$.
(b) $\|\mu\| = 1$ *and* $\langle f, \mu\rangle = f(a)$ *for* $f \in C(\overline{\Omega})$.
(c) $\langle fg, \mu\rangle = \langle f, \mu\rangle\langle g, \mu\rangle$ *for* $f,\ g \in L^\infty(\Omega)$.

For details we refer to [26, 32].

The next lemma is used in the proofs of Lemmas 9.1 and 9.7.

Lemma 9.4. *Let $u \in W^{2,p}(\Omega)$ for some $p > N$ and $\Delta u \in L^\infty(\Omega)$. Assume that u has nonnegative maximum at some $a \in \overline{\Omega}$. Assume also either* (i) *$a \in \Omega$, or* (ii) *$a \in \partial\Omega$, $u(a) > u(x)$ for $x \in \Omega$ and $(\partial u/\partial \nu)(a) = 0$. Then there exists a* 0-1 *measure $\mu \in ba(\Omega)$ such that μ has the essential support at a and $\langle \Delta u, \mu\rangle \le 0$.*

Proof. The arguments in the following are the combination of those of [7, Lemma 3.4] and [26, Lemma 4.3.2]. By $B(x; r)$ we denote a ball with center at x and radius r. Let $r > 0$ and set $E_r = \{x \in B(a; r) \cap \Omega : \Delta u \le 0\}$. First we show that $\mathfrak{m}(E_r) > 0$ for sufficiently small r, where $\mathfrak{m}$ denotes the Lebesgue measure. Assume to the contrary that $\Delta u > 0$ a.e. in $B(a; r) \cap \Omega$. In the case of (i), we choose $r_0 > 0$ small enough to satisfy $B(a; r_0) \subset \Omega$. Then, by strong maximum principle ([7, Theorem 9.6]), u must be a constant on $B(a; r)$ for $r \in (0, r_0)$. This is a contradiction. Hence $\mathfrak{m}(E_r) > 0$ for $r \in (0, r_0)$. In the case of (ii), the proof is given as follows. Since $\partial\Omega$ is smooth, there exists a ball $B(y; R) \subset B(a; r) \cap \Omega$ with $a \in \partial B(y; R)$. For $\rho \in (0, R)$ we define an auxiliary function v by $v(x) = e^{-\delta|x-y|^2} - e^{-\delta R^2}$, where $\delta = 1/(2\rho^2)$. Then we see that

$$\Delta v(x) = e^{-\delta|x-y|^2}\{4\delta^2|x-y|^2 - 2n\delta\}$$

holds on $D := B(y; R) \setminus B(y; \rho)$. Hence we have $\Delta v \ge 0$ on D provided that δ is chosen large enough. Since $u(x) - u(a) < 0$ on $\partial B(y; \rho)$, there is a constant $\eta > 0$ for which $u(x) - u(a) + \eta v(x) \le 0$ on $\partial B(y; \rho)$. This inequality also holds on $\partial B(y; R)$. Note that $v(x) = 0$ on the boundary. Thus we have $\Delta(u(x) - u(a) + \eta v(x)) \ge 0$ in D, and $u(x) - u(a) + \eta v(x) \le 0$ on ∂D. The weak maximum principle ([7, Theorem 9.1]) implies that $u(x) - u(a) + \eta v(x) \le 0$ on D. Taking the normal derivative at a, we obtain $(\partial u/\partial \nu)(a) \ge -\eta(\partial v/\partial \nu)(a) = -\eta v'(R) > 0$. This contradicts the assumption that $(\partial u/\partial \nu)(a) = 0$. Therefore we have shown that $\mathfrak{m}(E_r) > 0$ in both cases.

Let $\mathscr{E} = \{E_r : r \in (0, r_0)\}$. Then for any finite subsets $\mathscr{F}$ of $\mathscr{E}$, the Lebesgue measure of the intersection of all elements in $\mathscr{F}$ is positive. Hence, by [32, Theorem 4.1], there exists a 0-1 measure μ with the property that $\mu(E) = 1$ for any $E \in \mathscr{E}$. It follows from the definition of $\mathscr{E}$ that the essential support of μ is $\{a\}$. Thus we have

$$\langle \Delta u, \mu\rangle = \int_\Omega \Delta u \, d\mu = \int_{E_r} \Delta u \, d\mu \le 0.$$

The proof is now complete. $\square$

Proof of Lemma 9.1. Since the resolvent $(I - \lambda A_i)^{-1}$ exists for sufficiently large λ, we show that $A_i - \omega_0$ is dissipative in $L^\infty(\Omega)$. Let $v \in D(A_i)$. Without loss of generality, we may assume that $|v|_\infty = v(a) \geq 0$. Let μ be the 0-1 measure obtained by Lemma 9.4. Let $\lambda > 0$ and set $f = v - \lambda A_i v$. Then, we have

$$\begin{aligned}\langle v, \mu\rangle &= \lambda\langle (d_i\Delta - \omega_0)v, \mu\rangle + \langle f, \mu\rangle \\ &\leq -\lambda\omega_0|v|_\infty + |f|_\infty.\end{aligned}$$

Since $\langle v, \mu\rangle = v(a) = |v|_\infty$, this means that $A_i - \omega_0$ is dissipative. □

The next lemma directly follows from the definitions of $\boldsymbol{B}_r$ and $\boldsymbol{B}_c(t)$.

Lemma 9.5.

(i) *There exists a nondecreasing function* $L_r : [0, \infty) \to [0, \infty)$ *such that*

$$|\boldsymbol{B}_r\mathbf{v}|_X \leq L_r(|\mathbf{v}|_X)(1 + \|\mathbf{v}\|_X) \quad \textit{for } \mathbf{v} \in Y.$$

(ii) *For* $\mathbf{v} = (u, v, w) \in Y \cap X_+$ *it holds that*

$$\begin{aligned}|B_{r,1}\mathbf{v}|_\infty &\leq (\gamma + \omega_0)|w|_\infty|u|_\infty, \\ |B_{r,2}\mathbf{v}|_\infty &\leq (\varepsilon_2 + \omega_0)|v|_\infty|u|_\infty, \\ |B_{r,3}\mathbf{v}|_\infty &\leq (c_3 + \omega_0)|w|_\infty.\end{aligned}$$

(iii) *There exists a positive constant* C_c *such that*

$$\begin{aligned}|B_{c,1}(t)\mathbf{v}|_\infty &\leq C_c|(-A_1)^\theta u|_\infty, \\ |B_{c,2}(t)\mathbf{v}|_\infty &\leq C_c(1 + |(-A_1)^\theta u|_\infty)|(-A_2)^\theta v|_\infty, \\ |B_{c,3}(t)\mathbf{v}|_\infty &\leq C_c(1 + |(-A_1)^\theta u|_\infty)|(-A_3)^\theta w|_\infty.\end{aligned}$$

Therefore we have

$$|\boldsymbol{B}_c(t)\mathbf{v}|_X \leq C_c(1 + \|\mathbf{v}\|_Y)\|\mathbf{v}\|_Y \quad \textit{for } t \in [0, \tau] \textit{ and } \mathbf{v} = (u, v, w) \in Y. \tag{9.9}$$

For each $\rho > 0$ and $t \in [0, \tau]$ we set $D_\rho(t) = \{\mathbf{v} \in Y : \varphi(\mathbf{v}) \leq \rho\}$. We need the following four lemmas:

Lemma 9.6. (Local Lipschitz condition) *For each* $\rho > 0$ *there exists a constant* $\omega_{B,\rho} \geq 0$ *and a nondecreasing function* $g_\rho : [0, \tau] \to [0, \infty)$ *such that* $\lim_{r\downarrow 0} g_\rho(r) = g_\rho(0) = 0$ *and*

$$|\boldsymbol{B}_r\mathbf{v} - \boldsymbol{B}_r\hat{\mathbf{v}}|_X \leq \omega_{B,\rho}|\mathbf{v} - \hat{\mathbf{v}}|_X, \tag{9.10}$$

$$|\boldsymbol{B}_c(s)\mathbf{v} - \boldsymbol{B}_c(t)\hat{\mathbf{v}}|_X \leq \omega_{B,\rho}\|\mathbf{v} - \hat{\mathbf{v}}\|_Y + g_\rho(|t - s|) \tag{9.11}$$

for s, $t \in [0, \tau]$, $\mathbf{v} \in D_\rho(s)$ *and* $\hat{\mathbf{v}} \in D_\rho(t)$.

Proof. Let $s,\ t \in [0,\tau]$ and $\rho > 0$. Let $\mathbf{v} = (u,v,w) \in D_\rho(s)$ and $\hat{\mathbf{v}} = (\hat{u},\hat{v},\hat{w}) \in D_\rho(t)$. Then by (9.3) and (9.6) we have

$$|B_{r,1}\mathbf{v} - B_{r,1}\hat{\mathbf{v}}|_\infty \le (2(\beta+\gamma)M_\theta\rho + c_1 + \omega_0)|\mathbf{v} - \hat{\mathbf{v}}|_X,$$

$$|B_{r,2}\mathbf{v} - B_{r,2}\hat{\mathbf{v}}|_\infty \le (2\varepsilon_2 M_\theta\rho + c_2 + \omega_0)|\mathbf{v} - \hat{\mathbf{v}}|_X,$$

$$|B_{r,3}\mathbf{v} - B_{r,3}\hat{\mathbf{v}}|_\infty \le (2\varepsilon_3 M_\theta\rho + c_3 + \omega_0)|\mathbf{v} - \hat{\mathbf{v}}|_X,$$

and

$$|B_{c,1}(s)\mathbf{v} - B_{c,1}(t)\hat{\mathbf{v}}|_\infty \le \alpha_1\|\mathbb{E}\|_\infty c_0\|\mathbf{v} - \hat{\mathbf{v}}\|_Y + g_\rho(|t-s|),$$

$$|B_{c,2}(s)\mathbf{v} - B_{c,2}(t)\hat{\mathbf{v}}|_\infty \le (\alpha_2\|\mathbb{E}\|_\infty + 2a_2c_0\rho)c_0\|\mathbf{v} - \hat{\mathbf{v}}\|_Y + g_\rho(|t-s|),$$

$$|B_{c,3}(s)\mathbf{v} - B_{c,3}(t)\hat{\mathbf{v}}|_\infty \le (\alpha_3\|\mathbb{E}\|_\infty + 2a_3c_0\rho)c_0\|\mathbf{v} - \hat{\mathbf{v}}\|_Y + g_\rho(|t-s|),$$

where $\|\mathbb{E}\|_\infty = \max\{|\mathbb{E}(t)|_\infty : t \in [0,\tau]\}$ and

$$g_\rho(r) = \rho\max\{\alpha_1,\alpha_2,\alpha_3\}c_0\max\{|\mathbb{E}(s) - \mathbb{E}(t)|_\infty : s,\ t \in [0,\tau],\ |s-t| \le r\}.$$

These inequalities together imply the desired estimates. □

Lemma 9.7. (Subtangential condition and positivity preserving property)

(i) *For each $R > 0$ and $\tilde{\varepsilon} > 0$ there exists $\lambda_0 = \lambda_0(R,\tilde{\varepsilon}) \in (0,\tilde{\varepsilon})$ such that to each $s \in [0,\tau)$, $\mathbf{v}_0 = (u_0,v_0,w_0) \in D_R(s)$ and $\lambda \in (0,\lambda_0] \cap (0,\tau - s]$ there corresponds $\mathbf{v}_\lambda \in D(\boldsymbol{A})$ satisfying*

$$\mathbf{v}_\lambda - \lambda(\boldsymbol{A} + \boldsymbol{B}_c(s+\lambda))\mathbf{v}_\lambda = \mathbf{v}_0 + \lambda\boldsymbol{B}_r\mathbf{v}_0. \tag{9.12}$$

(ii) *If $\mathbf{v}_0 \in D_R(s) \cap X_+$, then $\mathbf{v}_\lambda$ obtained above belongs to $D(\boldsymbol{A}) \cap X_+$.*

Proof. Let $R > 0$ and $\tilde{\varepsilon} > 0$. Choose $\lambda_0 \in (0,\tilde{\varepsilon})$ small enough to satisfy

$$\lambda_0^{1-\theta}M_\theta(L_r(M_\theta R)(1 + M_\theta R) + C_c(2+R)R) < 1,$$

$$M_\theta\omega_{B,R+1}\lambda_0^{1-\theta} < 1/2 \quad \text{and} \quad \lambda_0((\beta+\varepsilon_3)M_\theta R + c_1 + c_2) < 1.$$

Let $s \in [0,\tau)$, $\mathbf{v}_0 \in D_R(s)$ and $\lambda \in (0,\lambda_0] \cap (0,\tau - s]$. We then introduce a closed subset $W \subset D(s+\lambda) = Y$ and a mapping $F : W \to Y$ defined by

$$W = \{\mathbf{w} : \|\mathbf{w}\|_Y \le R+1\} \quad \text{and}$$

$$F\mathbf{w} = (I - \lambda\boldsymbol{A})^{-1}(\mathbf{v}_0 + \lambda\boldsymbol{B}_r\mathbf{v}_0 + \lambda\boldsymbol{B}_c(s+\lambda)\mathbf{w}) \quad \text{for } \mathbf{w} \in W.$$

For $\mathbf{w}$, $\hat{\mathbf{w}} \in W$ we apply (9.3), (9.5), Lemmas 9.5 and 9.6 to get

$$\begin{aligned}\|F\mathbf{w}\|_Y &\le \|\mathbf{v}_0\|_Y + M_\theta\lambda^{1-\theta}(L_r(|\mathbf{v}_0|_X)(1 + |\mathbf{v}_0|_X) + C_c(1 + \|\mathbf{w}\|_Y)\|\mathbf{w}\|_Y)\\ &\le \|\mathbf{v}_0\|_Y + M_\theta\lambda_0^{1-\theta}(L_r(M_\theta R)(1 + M_\theta R) + C_c(2+R)R) < R+1\end{aligned}$$

and

$$\|F\mathbf{w} - F\hat{\mathbf{w}}\|_Y \le M_\theta\omega_{B,R+1}\lambda^{1-\theta}\|\mathbf{w} - \hat{\mathbf{w}}\|_Y < (1/2)\|\mathbf{w} - \hat{\mathbf{w}}\|_Y.$$

By the contracting mapping theorem, one finds $\mathbf{v}_\lambda \in W$ satisfying $\mathbf{v}_\lambda = F\mathbf{v}_\lambda$. This shows assertion (i).

Next we prove assertion (ii) by contradiction. Let $s \in [0,\tau)$, $\mathbf{v}_0 = (u_0, v_0, w_0) \in D_R(s) \cap X_+$, $\lambda \in (0, \lambda_0]$ and let $\mathbf{v}_\lambda = (u_\lambda, v_\lambda, w_\lambda) \in D(\boldsymbol{A})$ satisfy (9.12). Suppose to the contrary that $\mathbf{v}_\lambda \notin X_+$. We consider the case where u_λ is not nonnegative. Then, $-u_\lambda$ attains its positive maximum at $a \in \overline{\Omega}$. Since u_λ satisfies (BC), it follows from Lemma 9.4 that there exists a 0-1 measure μ such that $\langle \Delta u_\lambda, \mu \rangle \geq 0$. Noting that $u_\lambda \in C^1(\overline{\Omega})$, $\nabla u_\lambda(a) = 0$, and the identity

$$u_\lambda - \lambda(d_1 \Delta - \omega_0) u_\lambda - \lambda \alpha_1 \mathbb{E}(s + \lambda) \cdot \nabla u_\lambda = (1 + \lambda(\omega_0 + \gamma w_0 - \beta v_0 - c_1)) u_0,$$

we have

$$(1 + \lambda \omega_0) u_\lambda(a) \geq (1 - \lambda(\beta |v_0|_\infty + c_1)) u_0(a) \geq 0.$$

This contradicts $u_\lambda(a) < 0$. Similarly, the other cases lead to a contradiction. Thus assertion (ii) is proved. □

Lemma 9.8. (A priori estimate with respect to X norm)
Let $s \in [0, \tau)$ and $\mathbf{v}_0 = (u_0, v_0, w_0) \in D(s) \cap X_+$. Assume that there exist sequences $\{t_i\}_{i=0}^N$ in $[s, \tau]$ and $\{\mathbf{v}_i\}_{i=1}^N$ in $D(\boldsymbol{A}) \cap X_+$ satisfying $t_0 = s$ and

$$\mathbf{v}_i - h_i(\boldsymbol{A} + \boldsymbol{B}_c(t_i))\mathbf{v}_i = \mathbf{v}_{i-1} + h_i \boldsymbol{B}_r \mathbf{v}_{i-1}, \ i = 1, 2, \ldots, N, \tag{9.13}$$

where $h_i = t_i - t_{i-1}$, $i = 1, 2, \ldots, N$. Then it holds that

$$|\mathbf{v}_i|_X \leq C(\tau, |\mathbf{v}_0|_X) \equiv \max\{\exp(\gamma\tau|\mathbf{v}_0|_X \exp(c_3\tau))|\mathbf{v}_0|_X, \exp(c_3\tau)|\mathbf{v}_0|_X, \kappa\}$$

for $i = 1, 2, \ldots, N$, where $\kappa = \exp(\varepsilon_2 \tau |\mathbf{v}_0|_X \exp(\gamma\tau|\mathbf{v}_0|_X \exp(c_3\tau)))|\mathbf{v}_0|_X$.

Proof. Let $s \in [0, \tau)$ and $\mathbf{v}_0 = (u_0, v_0, w_0) \in D(s) \cap X_+$. Let $\{t_i\}_{i=0}^N$ in $[s, \tau]$ and $\{\mathbf{v}_i\}_{i=1}^N$ in $D(\boldsymbol{A}) \cap X_+$ satisfy (9.13). We first show that $|w_i|_\infty \leq e^{c_3\tau}|w_0|_\infty$, $i = 1, 2, \ldots, N$. Let w_i attain its nonnegative maximum at $a \in \overline{\Omega}$ and let μ be a 0-1 measure specified in Lemma 9.4 with u replaced by w_i. Noting that $\nabla w_i(a) = 0$ and

$$w_i - h_i(d_3 \Delta - \omega_0) w_i - h_i(\alpha_3 \mathbb{E}(t_i) \cdot \nabla w_i - a_3 \nabla u_i \cdot \nabla w_i) = w_{i-1} + h_i B_{r,3} \mathbf{v}_{i-1},$$

Lemma 9.5 implies

$$(1 + h_i \omega_0) w_i(a) \leq |w_{i-1}|_\infty + h_i(c_3 + \omega_0)|w_{i-1}|_\infty.$$

Since $w_i(a) = |w_i|_\infty$, the above inequality implies

$$|w_i|_X \leq e^{c_3\tau}|w_0|_\infty, \ i = 1, 2, \ldots, N.$$

Similarly, we have

$$\begin{aligned}(1 + h_i\omega_0)|u_i|_\infty &\leq |u_{i-1}|_\infty + h_i(\gamma + \omega_0)|w_{i-1}|_\infty |u_{i-1}|_\infty \\ &\leq |u_{i-1}|_\infty + h_i(\gamma + \omega_0) e^{c_3\tau} |w_0|_\infty |u_{i-1}|_\infty\end{aligned}$$

and

$$(1 + h_i\omega_0)|v_i|_\infty \leq |v_{i-1}|_\infty + h_i(\varepsilon_2 + \omega_0)|u_{i-1}|_\infty |v_{i-1}|_\infty \quad \text{for } i = 1, 2, \ldots, N.$$

These inequalities imply that

$$\begin{aligned}&|u_i|_\infty \leq \exp(\gamma\tau|w_0|_\infty \exp(c_3\tau))|u_0|_\infty \\ &|v_i|_\infty \leq \exp(\varepsilon_2\tau|u_0|_\infty \exp(\gamma\tau|w_0|_\infty \exp(c_3\tau)))|v_0|_\infty.\end{aligned}$$ □

Lemma 9.9. (A priori estimate with respect to Y norm)
There exists a function $\Psi : [0,\tau]^2 \times [0,\infty) \to [0,\infty)$ *having the following two properties:*

(i) *For each* $s \in [0,\tau]$, $t \in [s,\tau]$ *and* $r \geq 0$, $\Psi(t,s;\cdot)$ *and* $\Psi(\cdot,s;r)$ *are monotone nondecreasing.*

(ii) *For all sequences* $\{t_i\}_{i=0}^N$ *in* $[s,\tau]$ *and* $\{\mathbf{v}_i\}_{i=0}^N$ *in* X_+ *satisfying* $t_0 = s$, $\mathbf{v}_0 \in D_\rho(s)$, $\mathbf{v}_i \in D(\boldsymbol{A})$, $|\Delta| \leq ((1-\theta)/(2M_\theta C_c))^{1/(1-\theta)}$ *and*

$$\mathbf{v}_i - h_i(\boldsymbol{A} + \boldsymbol{B}_c(t_i))\mathbf{v}_i = \mathbf{v}_{i-1} + h_i \boldsymbol{B}_r \mathbf{v}_{i-1},\ i = 1,2,\ldots,N, \tag{9.14}$$

where $h_i = t_i - t_{i-1}$ *and* $|\Delta| = \max\{h_i : 1 \leq i \leq N\}$, *the following estimate holds;*

$$\|\mathbf{v}_i\|_Y \leq \Psi(\tau,s;\rho), \quad \textit{for } i = 1,2,\ldots,N.$$

Proof. Let $s \in [0,\tau)$, $\rho > 0$ and $\mathbf{v}_0 \in D_\rho(s) \cap X_+$. Let $\{t_i\}_{i=0}^N$ in $[s,\tau]$ and $\{\mathbf{v}_i\}_{i=1}^N$ in $D(\boldsymbol{A}) \cap X_+$ satisfy (9.14). By (9.14), we see that

$$\mathbf{v}_i = \prod_{j=1}^{i}(I - h_j\boldsymbol{A})^{-1}\mathbf{v}_0 + \sum_{j=1}^{i} h_j \prod_{k=j}^{i}(I - h_k\boldsymbol{A})^{-1}(\boldsymbol{B}_c(t_j)\mathbf{v}_j + \boldsymbol{B}_r\mathbf{v}_{j-1}) \tag{9.15}$$

holds for $i = 1,2,\ldots,N$. First we give the estimate for $|(-A_1)^\theta u_i|_\infty$. By (9.5), (9.15), Lemmas 6.1 and 9.5, we have

$$\begin{aligned}\|(-A_1)^\theta u_i\|_\infty \leq |(-A_1)^\theta u_0|_\infty + M_\theta \sum_{j=1}^{i} h_j (t_i - t_{j-1})^{-\theta} C_c |(-A_1)^\theta u_j|_\infty \\ + M_\theta \sum_{j=1}^{i} h_j (t_i - t_{j-1})^{-\theta}(\gamma + \omega_0)|w_{j-1}|_\infty |u_{j-1}|_\infty,\ i = 1,2,\ldots,N,\end{aligned} \tag{9.16}$$

We define

$$\begin{aligned}&q(s) = |(-A_1)^\theta u_0|_\infty \quad \text{and}\\ &q(t) = \max\{|(-A_1)^\theta u_0|_\infty, |(-A_1)^\theta u_1|_\infty, \ldots, |(-A_1)^\theta u_i|_\infty\}\end{aligned}$$

for $t \in (t_{i-1}, t_i]$ and $i = 1,2,\ldots,N$. By (9.16), we have for $t \in (t_{i-1}, t_i]$, $i = 1,2,\ldots,N$,

$$\begin{aligned}q(t) &\leq |(-A_1)^\theta u_0|_\infty + M_\theta C_c \int_s^{t_i} (t_i - \xi)^{-\theta} q(\xi)\,d\xi + \widetilde{C}(\tau,\rho) \int_s^{t_i} (t_i - \xi)^{-\theta}\,d\xi \\ &\leq K_\rho + M_\theta C_c \int_s^{t_i} (t_i - \xi)^{-\theta} q(\xi)\,d\xi \\ &\leq K_\rho + M_\theta C_c \int_{t_{i-1}}^{t_i} (t_i - \xi)^{-\theta} q(\xi)\,d\xi + M_\theta C_c \int_s^{t_{i-1}} (t_i - \xi)^{-\theta} q(\xi)\,d\xi \\ &\leq K_\rho + M_\theta C_c (1-\theta)^{-1} |\Delta|^{1-\theta} q(t) + M_\theta C_c \int_s^{t} (t - \xi)^{-\theta} q(\xi)\,d\xi,\end{aligned}$$

where $K_\rho = \rho+(1-\theta)^{-1}\tau^{1-\theta}\widetilde{C}(\tau,\rho)$, $\widetilde{C}(\tau,\rho) = (\gamma+\varepsilon_2+c_3+\omega_0)M_\theta C(\tau, M_\theta\rho)^2$ and $C(\tau, M_\theta\rho)$ is a constant specified in Lemma 9.8. Since $M_\theta C_c(1-\theta)^{-1}|\Delta|^{1-\theta} \le 1/2$, we obtain

$$q(t) \le 2K_\rho + 2M_\theta C_c \int_s^t (t-\xi)^{-\theta} q(\xi)\, d\xi \quad \text{for } t \in [s, t_N].$$

Applying Henry's inequality, Lemma 5.1, we conclude that

$$q(t) \le 2K_\rho E_{1-\theta}(\delta_\theta(t-s)) \quad \text{for } t \in [s, t_N],$$

where $\delta_\theta = (2M_\theta C_c \Gamma(1-\theta))^{1/(1-\theta)}$. This implies that

$$|(-A_1)^\theta u_i|_\infty \le 2K_\rho E_{1-\theta}(\delta_\theta(\tau - s)) \quad \text{for } i = 1, 2, \ldots, N.$$

Next we show the estimate for $|(-A_2)^\theta v_i|_\infty$. By (9.15) we have

$$\begin{aligned}
|(-A_2)^\theta v_i|_\infty &\le |(-A_2)^\theta v_0|_\infty + \widetilde{C}(\tau,\rho) \sum_{j=1}^{i} h_j (t_i - t_{j-1})^{-\theta} \\
&\quad + M_\theta \sum_{j=1}^{i} h_j (t_i - t_{j-1})^{-\theta} C_c (1 + |(-A_1)^\theta u_j|_\infty) |(-A_2)^\theta v_j|_\infty \\
&\le \rho + \widetilde{C}(\tau,\rho) \sum_{j=1}^{i} h_j (t_i - t_{j-1})^{-\theta} \\
&\quad + M_\theta C_c (1 + 2K_\rho E_{1-\theta}(\delta_\theta \tau)) \sum_{j=1}^{i} h_j (t_i - t_{j-1})^{-\theta} |(-A_2)^\theta v_j|_\infty
\end{aligned}$$

for $i = 1, 2, \ldots, N$. In a way similar to the estimate for $|(-A_1)^\theta u_i|_\infty$, we obtain

$$|(-A_2)^\theta v_i|_\infty \le 2K_\rho E_{1-\theta}(\hat{\delta}_\theta(\tau - s)) \quad \text{for } i = 1, 2, \ldots, N,$$

where $\hat{\delta}_\theta = (2M_\theta C_c(1+2K_\rho E_{1-\theta}(\delta_\theta\tau))\Gamma(1-\theta))^{1/(1-\theta)}$. Finally $|(-A_3)^\theta w_i|$ satisfies the same estimate as $|(-A_2)^\theta v_i|$. Since $\delta_\theta < \hat{\delta}_\theta$, assertion (ii) holds for the function

$$\Psi(t, s; \rho) = 2K_\rho E_{1-\theta}(\hat{\delta}_\theta(t-s)). \tag{9.17}$$

Accordingly, a growth condition in this model is given by the above function Ψ. □

Theorem 9.10. *For each* $\mathbf{u}_0 \in Y \cap X_+$ *there exists a unique mild solution* $\mathbf{u}(\cdot) \in X_+$ *to* (RDS)-(BC)-(IC) *satisfying*

$$|\mathbf{u}(t)|_X \le C(\tau, |\mathbf{u}_0|_X), \quad \text{for } t \in [0, \tau].$$

$$\|\mathbf{u}(t)\|_Y \le \Psi(\tau, 0; \|\mathbf{u}_0\|_Y) \quad \text{for } t \in [0, \tau],$$

where Ψ *is a function determined by* (9.17).

Proof. In (9.2), (9.7), (9.8) and Lemma 9.6, we have checked that assumptions (H1) through (H3) in Theorem 4.6 hold. If we could check assumption (II) in Theorem 4.6, then the desired result would be obtained by Theorem 4.6 and Lemma 9.8. To this end, let $\varepsilon > 0$, $s \in [0, \tau)$ and $\mathbf{v}_0 \in D(s) \cap X_+$, and set $\rho = \varphi(\mathbf{v}_0)$ and $R = \Psi(\tau, s; \rho) + \varepsilon$. Let λ_* be the maximum of the numbers satisfying $\lambda \in (0, 1)$ and $\lambda^\theta \omega_{B,R}(M_\theta R + C_c R(1 + R) + L_r(M_\theta R)(1 + M_\theta R)) < \varepsilon$. Let g_R be a function specified in Lemma 9.6. Since $\lim_{r\downarrow 0} g_R(r) = 0$, there exists $\eta > 0$ such that $|g_R(r)| < \varepsilon/\tau$ for $r < \eta$. Set $\tilde{\varepsilon} = \min\{\varepsilon, \eta, ((1 - \theta)/(2M_\theta C_c))^{1/(1-\theta)}, \lambda_*\}$. Then, it follows from Lemmas 9.7 and 9.9 that there exist $\lambda_0 = \lambda_0(R, \tilde{\varepsilon}) \in (0, \tilde{\varepsilon})$ and $\mathbf{v}_1 \in D(\boldsymbol{A}) \cap X_+$ such that

$$\mathbf{v}_1 - \lambda_1(\boldsymbol{A} + \boldsymbol{B}(s + \lambda_1))\mathbf{v}_1 = \mathbf{v}_0 + \lambda_1 \mathbf{z}_1 \quad \text{and} \quad \varphi(\mathbf{v}_1) \le R,$$

where $\lambda_1 = \min\{\lambda_0, \tau - s\}$ and $\mathbf{z}_1 = \boldsymbol{B}_r\mathbf{v}_0 - \boldsymbol{B}_r\mathbf{v}_1$. From Lemmas 9.5, 9.6, (9.3) and (9.4), we infer that

$$\begin{aligned} |\mathbf{z}_1|_X &\le \omega_{B,R}|\mathbf{v}_0 - \mathbf{v}_1|_X \\ &\le \omega_{B,R}(|(I - \lambda_1\boldsymbol{A})^{-1}\mathbf{v}_0 - \mathbf{v}_0|_X + \lambda_1|\boldsymbol{B}_r\mathbf{v}_0|_X) + \lambda_1|\boldsymbol{B}_c(t_1)\mathbf{v}_1|_X) \\ &\le \lambda_1^\theta \omega_{B,R}(M_\theta R + C_c R(1 + R) + L_r(M_\theta R)(1 + M_\theta R)) < \varepsilon. \end{aligned}$$

Letting $t_1 = s + \lambda_1$ and using Lemmas 9.5, 9.6, 9.7, 9.9, (9.3) and (9.4) again, we can find $\mathbf{v}_2 \in D(\boldsymbol{A}) \cap X_+$ satisfying

$$\mathbf{v}_2 - \lambda_2(\boldsymbol{A} + \boldsymbol{B}(t_1 + \lambda_2))\mathbf{v}_2 = \mathbf{v}_1 + \lambda_2\mathbf{z}_2 \quad \text{and} \quad \varphi(\mathbf{v}_2) \le R.$$

Here $\lambda_2 = \min\{\lambda_0, \tau - t_1\}$, $\mathbf{z}_2 = \boldsymbol{B}_r\mathbf{v}_1 - \boldsymbol{B}_r\mathbf{v}_2$ and $|\mathbf{z}_2|_X < \varepsilon$. Repeating the above arguments finite times, we obtain the sequences $\{t_i\}_{i=0}^{N+1}$ in $[s, \tau]$, $\{\mathbf{v}_i\}_{i=0}^N$ in $Y \cap X_+$ and $\{\mathbf{z}_i\}_{i=1}^N$ in X satisfying assumption (II) with $\mathbf{u}_i = \mathbf{v}_i$, $\mathbf{z}_i = \boldsymbol{B}_r\mathbf{v}_{i-1} - \boldsymbol{B}_r\mathbf{v}_i$ and $f_R = g_R$. The proof is now complete. □

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A Variational Approach to Strongly Damped Wave Equations

Delio Mugnolo

Dedicated to the inspiring memory of Günter Lumer

Abstract. We discuss a Hilbert space method that allows to prove analytical well-posedness of a class of linear strongly damped wave equations. The main technical tool is a perturbation lemma for sesquilinear forms, which seems to be new. In most common linear cases we can furthermore apply a recent result due to Crouzeix–Haase, thus extending several known results and obtaining optimal analyticity angle.

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1. Introduction

Of concern of this note are complete second-order abstract Cauchy problems of the form

$$\begin{cases} \ddot{u}(t) + Au(t) + B\dot{u}(t) = 0, & t \geq 0, \\ u(0) = u_{10}, \qquad \dot{u}(0) = u_{20}, \end{cases} \tag{1.1}$$

where the elastic operator A is in the literature usually assumed to be a self-adjoint, strictly positive definite operator on a Hilbert space H. It is known that such elastic systems exhibit good properties whenever B is a multiplication operator: e.g., they are forward as well as backward solvable, they admit energy decay estimates if B is dissipative, or else blow-up estimates if B is accretive, see, e.g., [20, 21, 23] and references therein.

It is interesting to note that, in particular, the standard model of an electrical transmission line by means of the telegraph equation fits this framework, the case of B negative multiplication operator corresponding to *viscous* damping.

In [8], Chen–Russell proposed a family of different, strongly (or *structural*) damping effects: theoretical arguments and empirical studies motivated them to consider damping operators that are unbounded on H, cf. references in [8]. For the sake of simplicity, they mostly investigated the special cases of $B = A$ and $B = 2\rho A^{\frac{1}{2}}$. However, they also pointed out that the crucial property is the so-called *frequency response* estimate

$$\|\lambda R(i\lambda, \mathbf{A})\| \le M, \qquad \lambda \in \mathbb{R},$$

satisfied by the resolvent operator of $\mathbf{A}$, where

$$\mathbf{A} := \begin{pmatrix} 0 & -I \\ A & B \end{pmatrix} \tag{1.2}$$

is the reduction matrix associated with (1.1). Thus, following Chen–Russell the issue becomes to find conditions on A, B ensuring that $\mathbf{A}$ (or rather its closure) generates an analytic semigroup in the candidate phase space $\mathbf{H} := D(A^{\frac{1}{2}}) \times H$.

Ever since, several authors including Dautray–Lions, Chen–Triggiani, Xiao–Liang, and Chill–Srivastava have further investigated these kind of parabolic systems, significantly extending the results of Chen–Russell. Chen–Triggiani still imposed the assumption that the damping effect is at most as strong as the elastic one, i.e., that

$$B = \rho A^{\alpha}, \qquad \text{for } \alpha \in [0,1] \text{ and } \rho \in (0,\infty), \tag{1.3}$$

and then showed, by methods based on spectral analysis, that the semigroup generated by the closure (of a suitable part) of $-\mathbf{A}$ is analytic if and only if $\alpha \in [\frac{1}{2}, 1]$, cf. [9, Thm. 1.1]. Successively, Xiao–Liang have proved similar results in the slightly more general case where $B = f(A)$ for a suitable class of functions f , cf. [30, Thm. 6.4.2]. Similar, less sharp results have also been obtained in [15, § 6.3] by a technique based on the theory of operator matrices. We observe that strongly damped wave equations are also of interest in the framework of control theory, see, e.g., [7, 22], and references therein. Energy decay estimates have also been extensively investigated, see, e.g., [19, 5].

More recently, Chill–Srivastava have discussed L^p-maximal regularity properties for the solution to

$$\begin{cases} \ddot{u}(t) + Au(t) + B\dot{u}(t) = f(t), & t \in [0,T], \\ u(0) = 0, \qquad \dot{u}(0) = 0. \end{cases} \tag{1.4}$$

While they are not directly interested in the analyticity of the semigroup generated by $-\mathbf{A}$, their results in some sense extend those of [9, 30]: if (1.3) holds and A is a sectorial operator on an L^q-space, $q \in (1,\infty)$, and under further technical assumptions, it turns out that (1.4) has maximal L^p-regularity if $\alpha \in (\frac{1}{2}, 1]$, cf. [11, Thm. 4.1]. Observe that, in particular, if (1.4) has L^p-maximal regularity, then a differentiable semigroup governs (1.1) on a certain phase space, cf. [11, Cor. 2.5].

The case of $D(B) \subset D(A)$ has been treated less frequently, see, e.g., [28, 26]; moreover, most authors have not discussed analyticity properties. In [25, Thm. 6.2], we have showed that if B generates a cosine operator function with phase space $V \times H$, and if A is bounded from V to H, then (1.1) is governed by an analytic semigroup of angle $\frac{\pi}{2}$. In many relevant cases this amounts to saying that $A = \rho B^\alpha$, for $\alpha \leq \frac{1}{2}$ and $\rho \in \mathbb{C}$.

Aim of this paper is to discuss (1.1) under assumptions on B that complement, or perhaps interpolate, those of the above mentioned papers. In fact, we will assume B to be *at least* as unbounded as A. The quoted results suggest that $\alpha = \frac{1}{2}$ is a critical exponent, whenever (1.3) holds. In fact, we will show that the exponent $\alpha = 1$ is critical, too. More precisely if $\alpha = 1$, then the leading term in (1.1) is not A anymore, but B. In fact, we show that (1.1) is governed by an analytic semigroup under quite weak boundedness assumptions on A, whenever B is associated with a closed, H-elliptic form. In particular, we show that no closedness or spectral conditions on A are necessary. Our method is based on the introduction of a suitable weak formulation of (1.1), and then on the application of the theory of sesquilinear forms on complex Hilbert spaces. We refer to [27, 2] for comprehensive treatments of this mature theory that goes back to Kato and Lions, and to [14] for a similar, slightly less general approach to damped wave equations due to Dautray–Lions.

In Section 2 we introduce our general framework and show a first well-posedness result for (1.1). To this aim we prove a perturbation lemma for sesquilinear forms that may be of independent interest. We also obtain a first estimate on the angle of analyticity. In Section 3 we impose slightly stronger conditions and, by means of a recent result due to Crouzeix–Haase, we find sufficient condition in order that the semigroup is analytic of angle $\frac{\pi}{2}$: this includes the relevant case of self-adjoint damping operator B. Some applications to semilinear problems are also considered.

Remark added in revision: The anonymous referee has informed us that a variational approach to linear damped wave equations has also been pursued in [14, § XVIII.5.1], see also [14, § XVIII.6]. Indeed, Dautray–Lions' methods are quite similar to those presented in Section 2 below, and they also consider the neutral equation $C\ddot{u}(t) + Au(t) + B\dot{u}(t) = 0$, $t \geq 0$, even in the nonautonomous case, where $D(B) \subset D(A)$. Though, the assumptions in [14, § XVIII.5.1] are restricted to the case of A, B differential operators whose principal part is self-adjoint and (in the case of B) also strictly positive definite, and no angle of analyticity is proved there. However, their main result [14, Thm. XVIII.1] is admittedly very close to Corollary 3.2 below.

2. First well-posedness results

Let V, H be complex Hilbert spaces such that V is continuously and densely imbedded in H. Let $a : V \times V \to \mathbb{C}$, $b : V \times V \to \mathbb{C}$ be sesquilinear forms[1].

[1] One can often think of sesquilinear forms in terms of physical quantities. In fact, if a, b are symmetric, $2a(u,u), 2b(u,u)$ merely represent the energy functionals associated with the elastic and damping operators A, B that appear in (1.1), respectively.

More precisely, we recall that the *operator associated with a* is by definition given by

$$\begin{array}{rcl} D(A) & := & \{f \in V : \exists h \in H \text{ s.t. } a(f,g) = (h \mid g)_H \ \forall g \in V\}, \\ Af & := & h, \end{array}$$

and likewise for the operator associated with b.

The following perturbation lemma seems to be of independent interest. It is the form equivalent of a well-known perturbation result for operators due to Desch–Schappacher. In the following we denote by H_α any interpolation space between V and H that verifies the interpolation inequality

$$\|f\|_{H_\alpha} \leq M_\alpha \|f\|_V^\alpha \|f\|_H^{1-\alpha}, \qquad f \in V. \tag{2.1}$$

Lemma 2.1. *Let $a : V \times V \to \mathbb{C}$ be a sesquilinear mapping. Let $\alpha \in [0,1)$ such that $a_1 : V \times H_\alpha \to \mathbb{C}$ and $a_2 : H_\alpha \times V \to \mathbb{C}$ be continuous sesquilinear mappings. Then a is H-elliptic if and only if $a + a_1 + a_2 : V \times V \to \mathbb{C}$ is H-elliptic.*

Proof. Let a be H-elliptic and let

$$|a_1(f,g)| \leq M\|f\|_V \|g\|_{H_\alpha} \qquad \text{and} \qquad |a_2(g,f)| \leq M\|g\|_V \|f\|_{H_\alpha},$$

for some constant $M > 0$ and for all $f \in V$, $g \in H_\alpha$, so that by (2.1) we can estimate both $|a_1(f,f)|$ and $|a_2(f,f)|$ by $MM_\alpha \|f\|_V^{1+\alpha} \|f\|_H^{1-\alpha}$.

By Young's inequality one has for all $\alpha \in [0,1)$ and all $x, y > 0$ that

$$xy \leq \frac{1+\alpha}{2} x^{\frac{2}{1+\alpha}} + \frac{1-\alpha}{2} y^{\frac{2}{1-\alpha}}.$$

Thus, for all $\epsilon > 0$ letting $x = (\sqrt{\epsilon}\|f\|_V)^{1+\alpha}$ and $y = (\frac{1}{\sqrt{\epsilon}}\|f\|_V)^{1-\alpha}$ one obtains

$$\|f\|_V^{1+\alpha} \|f\|_H^{1-\alpha} \leq \frac{1+\alpha}{2} \epsilon \|f\|_V^2 + \frac{1-\alpha}{2\epsilon} \|f\|_H^2, \qquad f \in V.$$

Accordingly, for all $\epsilon > 0$ there exists $M(\epsilon) > 0$ such that

$$-\epsilon \|f\|_V^2 + M(\epsilon) \|f\|_H^2 \leq a_1(f,f) + a_2(f,f), \qquad f \in V.$$

By assumption a is H-elliptic, i.e., $\mathrm{Re} a(f,f) \geq \alpha \|f\|_V^2 - \omega \|f\|_H^2$ for some $\alpha > 0$ and $\omega \in \mathbb{R}$. Thus, that for $\epsilon = \alpha/2$

$$\begin{array}{rcl} \mathrm{Re}(a + a_1 + a_2)(f,f) & = & \mathrm{Re} a(f,f) + a_1(f,f) + a_2(f,f) \\ & \geq & \alpha \|f\|_V^2 - \omega \|f\|_H^2 - \epsilon \|f\|_V^2 - M(\epsilon) \|f\|_H^2 \\ & \geq & \dfrac{\alpha}{2} \|f\|_V^2 - (\omega + M(\epsilon)) \|f\|_H^2, \end{array}$$

for all $f \in V$. This completes the proof. ☐

With the aim of discussing the abstract damped wave equation (1.1) we introduce $\mathbf{V} := V \times V$ as well as the candidate energy space $\mathbf{H} := V \times H$. Observe that $\mathbf{V}$ is continuously and densely imbedded into $\mathbf{H}$ and that both $\mathbf{V}$ and $\mathbf{H}$ have a canonical Hilbert space structure. Define

$$\mathbf{a}(\mathbf{u}, \mathbf{v}) := -(u_2 \mid v_1)_V + a(u_1, v_2) + b(u_2, v_2), \tag{2.2}$$

where we have considered

$$\mathbf{u} = (u_1, u_2)^\top, \mathbf{v} = (v_1, v_2)^\top \in \mathbf{V},$$

i.e., $\mathbf{a}$ is a sesquilinear form with domain $\mathbf{V}$. Observe that $\mathbf{a}$ is in general not symmetric.

Lemma 2.2. *The following assertions hold.*

1) *The form* $\mathbf{a}$ *is continuous with respect to* $\mathbf{V}$ *if and only if* a, b *are continuous with respect to* V*.*
2) *The form* $\mathbf{a}$ *is* $\mathbf{H}$*-elliptic if and only if* b *is* H*-elliptic.*
3) *Let* $\mathrm{Re}a(u,v) = \mathrm{Re}(u \mid v)_V$ *for all* $u, v \in V$*. If* b *is accretive, then* $\mathbf{a}$ *is accretive.*
4) *If if* $\mathbf{a}$ *is accretive, then* b *is accretive.*

Observe that, as a direct consequence of the sesquilinearity of a, $\mathrm{Re}a(u,v) = \mathrm{Re}(u \mid v)_V$ for all $u, v \in V$ if and only if $\mathrm{Re}a(u,v) \geq \mathrm{Re}(u \mid v)_V$ for all $u, v \in V$.

Proof. 1) Let $\mathbf{a}$ be continuous. Then for some constant $M_{\mathbf{a}} > 0$ and all $u, v \in V$ one has

$$|b(u,v)| = |\mathbf{a}(\mathbf{u},\mathbf{v})| \leq M_{\mathbf{a}} \|\mathbf{u}\|_{\mathbf{V}} \|\mathbf{v}\|_{\mathbf{V}} = M_{\mathbf{a}} \|u\|_V \|v\|_V,$$

where we have set $\mathbf{u} := (0,u)^\top$ and $\mathbf{v} := (0,v)^\top$. Similarly, setting $\mathbf{u} := (u,0)^\top$ and $\mathbf{v} := (0,v)^\top$ we obtain that

$$|a(u,v)| = |\mathbf{a}(\mathbf{u},\mathbf{v})| \leq M_{\mathbf{a}} \|\mathbf{u}\|_{\mathbf{V}} \|\mathbf{v}\|_{\mathbf{V}} = M_{\mathbf{a}} \|u\|_V \|v\|_V.$$

Let now a, b be continuous, i.e., assume that for some $M_a, M_b \geq 0$ there holds

$$|a(u,v)| \leq M_a \|u\|_V \|v\|_V, \qquad u, v \in V,$$

as well as

$$|b(u,v)| \leq M_b \|u\|_V \|v\|_V, \qquad u \in V.$$

A tedious computation then shows that

$$\begin{aligned} |\mathbf{a}(\mathbf{u},\mathbf{v})|^2 \quad \leq \quad & \|u_2\|_V^2 \|v_1\|_V^2 + M_a^2 \|u_1\|_V^2 \|v_2\|_V^2 + M_b^2 \|u_2\|_V^2 \|v_2\|_V^2 \\ & +2M_a \|u_1\|_V \|u_2\|_V \|v_1\|_V \|v_2\|_V + 2M_b \|u_2\|_V^2 \|v_1\|_V \|v_2\|_V \\ & +2M_a M_b \|u_1\|_V \|u_2\|_V \|v_2\|_V^2 \\ \leq \quad & M_{\mathbf{a}}^2 (\|u_1\|_V^2 + \|u_2\|_V^2)(\|v_1\|_V^2 + \|v_2\|_V^2), \end{aligned}$$

i.e., $|\mathbf{a}(\mathbf{u},\mathbf{v})| \leq M_{\mathbf{a}} \|\mathbf{u}\|_{\mathbf{V}} \|\mathbf{v}\|_{\mathbf{V}}$, where

$$M_{\mathbf{a}}^2 := \frac{M_a}{2} + M_a M_b + \max\left\{M_a^2, 1, M_b^2\right\}. \tag{2.3}$$

2) To begin with, consider the form $\mathbf{a}_0 : \mathbf{V} \times \mathbf{V} \to \mathbb{C}$ defined by

$$\mathbf{a}_0(\mathbf{u},\mathbf{v}) := b(u_2, v_2).$$

A direct computation shows that $\mathbf{a}_0$ is $\mathbf{H}$-elliptic if and only if b is H-elliptic. Similarly, define the continuous sesquilinear mappings $\mathbf{a}_1 : \mathbf{H} \times \mathbf{V} \to \mathbb{C}$ and $\mathbf{a}_2 : \mathbf{V} \times \mathbf{H} \to \mathbb{C}$ by

$$\mathbf{a}_1(\mathbf{u},\mathbf{v}) := -(u_2 \mid v_1)_V \qquad \text{and} \qquad \mathbf{a}_2(\mathbf{u},\mathbf{v}) := a(u_1, v_2).$$

By Lemma 2.1 we conclude that $\mathbf{a} = \mathbf{a}_0 + \mathbf{a}_1 + \mathbf{a}_2$ is $\mathbf{H}$-elliptic if and only if $\mathbf{a}_0$ is $\mathbf{H}$-elliptic if and only if b is H-elliptic.

3) If b is accretive and $\mathrm{Re} a(u,v) = \mathrm{Re}(u \mid v)_V$ for all $u, v \in V$, then

$$\mathrm{Re}\mathbf{a}(\mathbf{u},\mathbf{u}) = \mathrm{Re} b(u_2,u_2) \geq 0, \qquad \mathbf{u} = (u_1,u_2)^\top \in \mathbf{V},$$

i.e., $\mathbf{a}$ is accretive.

4) Conversely, if $\mathbf{a}$ is accretive, we obtain that for all $u \in V$

$$\mathrm{Re} b(u,u) = \mathrm{Re}\mathbf{a}(\mathbf{u},\mathbf{u}) \geq 0,$$

where we have set $\mathbf{u} := (0,u)$. □

By [27, Prop. 1.51 and Thm. 1.52] we can now state the following.

Theorem 2.3. *Let a, b be continuous. Let further b be H-elliptic. Then the operator associated with $\mathbf{a}$ is closed. It generates a C_0-semigroup $(e^{-t\mathbf{a}})_{t\geq 0}$ on $\mathbf{H}$ which is analytic of angle $\frac{\pi}{2} - \arctan M$, where $M_{\mathbf{a}}$ is defined as in (2.3). The semigroup $(e^{-t\mathbf{a}})_{t\geq 0}$ is contractive if b is accretive and $\mathrm{Re} a(u,v) = \mathrm{Re}(v \mid u)_V$ for all $u, v \in V$.*

We emphasize that in the above theorem we are assuming a neither to be H-elliptic, nor to be (quasi-)accretive. In other words, the operator A associated with a need not be closed or (quasi-)dissipative. Thus, in the limiting case of A bounded from $D(B)$ to H, where B is the operator associated with b, Theorem 2.3 extends the well-posedness results of [9, 30, 11]. In this sense, we say that the leading term in (1.1) is not the elastic, but rather the damping one.

Remark 2.4. 1) Let $V \neq \{0\}$. The form $\mathbf{a}$ is self-adjoint if and only if b is self-adjoint and $a(\cdot,\cdot) = -(\cdot \mid \cdot)_V$. Let in fact $\mathbf{u} := (u,0)^\top$ and $\mathbf{v} := (0,v)^\top$, with $u, v \in V$, $v \neq 0 \neq u$. Then, one has

$$\mathbf{a}(\mathbf{u},\mathbf{v}) = a(u,v) \qquad \text{and} \qquad \mathbf{a}(\mathbf{v},\mathbf{u}) = -(v \mid u)_V.$$

On the other hand, if $\mathbf{u} := (0,u)^\top$ and $\mathbf{v} := (0,v)^\top$, with $u, v \in V$, $v \neq 0 \neq u$, then

$$\mathbf{a}(\mathbf{u},\mathbf{v}) = b(u,v) \qquad \text{and} \qquad \mathbf{a}(\mathbf{v},\mathbf{u}) = b(v,u).$$

To prove the converse implication, it suffices to observe that if b is self-adjoint and $a(\cdot,\cdot) = -(\cdot \mid \cdot)_V$, then

$$\mathbf{a}(\mathbf{u},\mathbf{v}) = b(u_2,v_2) = \overline{b(v_2,u_2)} = \overline{\mathbf{a}(\mathbf{v},\mathbf{u})}.$$

2) The form $\mathbf{a}$ is not coercive, unless $V = \{0\}$. Let in fact $\mathbf{u} := (u,0)^\top$, with $0 \neq u \in V$. Then one has

$$\mathrm{Re}\mathbf{a}(\mathbf{u},\mathbf{u}) = 0 < \|\mathbf{u}\|^2_{\mathbf{V}}.$$

This shows that there exists no $\epsilon > 0$ such that the estimate $\|e^{-t\mathbf{a}}\| \leq e^{-\epsilon t}$ holds for all $t \geq 0$. This should be compared with the exponential stability result in [9, Thm. 1.1].

3) In the relevant case of dim $\mathbf{V} = \infty$ the imbedding of $\mathbf{V}$ in $\mathbf{H}$ is not compact. Thus if Theorem 2.3 applies, then $(e^{-t\mathbf{a}})_{t\geq 0}$ is not compact.

4) An advantage of dealing with sesquilinear forms instead of operators is the flexibility of this theory. Let us briefly discuss the case of time-dependent damped wave equations. Consider families $(a_t)_{t\in[0,T]}$ and $(b_t)_{t\in[0,T]}$ of sesquilinear forms with joint (time-independent) dense domain V. Assume them to be equicontinuous. Let furthermore the mappings $t \mapsto a_t(u,v)$ and $t \mapsto b_t(u,v)$ be measurable for all $u,v \in V$. If finally $(b_t)_{t\geq 0}$ is equi-H-elliptic, i.e.,

$$\mathrm{Re} b_t(u,u) + \omega\|u\|_H^2 \geq \alpha\|u\|_V^2, \qquad u \in V,\ t \geq 0,$$

for some $\omega \in \mathbb{R}, \alpha > 0$, then it is easy to see that the family of sesquilinear forms $(\mathbf{a}_t)_{t\geq 0}$ defined by

$$\mathbf{a}_t(\mathbf{u},\mathbf{v}) := -(u_2 \mid v_1)_V + a_t(u_1,v_2) + b_t(u_2,v_2),$$

fits the framework presented in [29, Chapt. 3], and we conclude that the nonautonomous abstract Cauchy problem associated with $(\mathbf{a}_t)_{t\geq 0}$ is well posed in a suitably weak sense. We refer to [29] for details.

In order to interpret Theorem 2.3 as a well-posedness result for (1.1), we still have to determine the operator $(\mathbf{A}, D(\mathbf{A}))$ associated with $\mathbf{a}$, which by definition is

$$\begin{aligned} D(\mathbf{A}) &:= \{\mathbf{u} \in \mathbf{V} : \exists \mathbf{z} \in \mathbf{H} \text{ s.t. } \mathbf{a}(\mathbf{u},\mathbf{v}) = (\mathbf{z} \mid \mathbf{v})_{\mathbf{V}} \text{ for all } \mathbf{v} \in \mathbf{V}\}, \\ \mathbf{Au} &:= \mathbf{z}. \end{aligned}$$

In fact, the expression "$Au + B\dot{u}$" in (1.1) is in general purely formal, as the solution u to (1.1) need not satisfy $u \in C(\mathbb{R}_+, D(A)) \cap C^1(\mathbb{R}_+, D(B))$. However, in our framework a direct computation shows that the following holds.

Proposition 2.5. *The operator* $\mathbf{A}$ *on* $\mathbf{V}$ *associated with the form* $\mathbf{a}$ *is given by*

$$\begin{aligned} D(\mathbf{A}) &= \{\mathbf{u} \in \mathbf{V} : \exists w \in H \ \textit{s.t.}\ a(u_1,v) + b(u_2,v) = (w \mid v)_H \ \textit{for all}\ v \in V\}, \\ \mathbf{Au} &= (u_2, w)^\top. \end{aligned}$$

In the remainder of this section we assume V, H to be function spaces over a measure space (X,μ). The following is a direct consequence of the above proposition and should be compared with the results of [10].

Corollary 2.6. *Let* $\rho \in H$ *such that* $\rho u \in V$ *and* $a(u,v) = b(\rho u, v)$ *for all* $u,v \in V$. *Then*

$$\begin{aligned} D(\mathbf{A}) &= \{\mathbf{u} \in \mathbf{V} : \exists w \in H \ \textit{s.t.}\ b(\rho u_1 + u_2, v) = (w \mid v)_H \ \textit{for all}\ v \in V\} \\ &= \{\mathbf{u} \in \mathbf{V} : \rho u_1 + u_2 \in D(B)\}, \\ \mathbf{Au} &= (u_2, B(\rho u_1 + u_2))^\top. \end{aligned}$$

where B *denotes the operator associated with* b.

While throughout the paper we consider complex Hilbert spaces, it is of interest for applications to ensure that solutions to (1.1) are in fact real whenever the initial data are real. In the following we denote the closed convex subsets $V_\mathbb{R}$ and $H_\mathbb{R}$ defined by the real-valued functions belonging to V and H, respectively.

Proposition 2.7. *Let a, b be continuous and b be H-elliptic. Assume further that* $\mathrm{Re}u \in V$ $a(\mathrm{Re}u, \mathrm{Im}u), (\mathrm{Re}u \mid \mathrm{Im}u)_V \in \mathbb{R}$ *for all* $u \in V$. *Then* $(e^{-t\mathbf{a}})_{t\geq 0}$ *is real* (*i.e., it leaves invariant* $V_\mathbb{R} \times H_\mathbb{R}$) *if and only if the semigroup associated with b is real* (*i.e., it leaves invariant* $H_\mathbb{R}$).

Proof. Without loss of generality we can assume both b and $\mathbf{a}$ to be accretive, since reality of a semigroup is invariant under rescaling. Let the semigroup associated with b be real. Then by [27, Prop. 2.5] one has $\mathrm{Re}u \in V$ for all $u \in V$ and $b(\mathrm{Re}u, \mathrm{Im}u) \in \mathbb{R}$. Thus, for an arbitrary $\mathbf{u} = (u_1, u_2)^\top \in \mathbf{V}$, one has $\mathrm{Re}\mathbf{u} = (\mathrm{Re}\mathbf{u}_1, \mathrm{Re}\mathbf{u}_2)^\top \in \mathbf{V}$ and moreover

$$\mathbf{a}(\mathrm{Re}\mathbf{u}, \mathrm{Im}\mathbf{u}) = -(\mathrm{Re}u_2 \mid \mathrm{Im}u_1)_V + a(\mathrm{Re}u_1, \mathrm{Im}u_2) + b(\mathrm{Re}u_2, \mathrm{Im}u_2) \in \mathbb{R}.$$

Since the projection $\mathbf{P}$ of $\mathbf{H}$ onto $\mathbf{H}_\mathbb{R}$ is given by

$$\mathbf{P}\mathbf{u} = (\mathrm{Re}u_1, \mathrm{Re}u_2), \qquad \mathbf{u} = (u_1, u_2)^\top \in \mathbf{H},$$

the claim follows by [27, Thm. 2.2]. Conversely, let $(e^{-t\mathbf{a}})_{t\geq 0}$ be real and let $u \in V$. Set $\mathbf{u} := (0, u)^\top \in \mathbf{V}$. Then, $\mathrm{Re}\mathbf{u} = (0, \mathrm{Re}u) \in \mathbf{V}$ and $b(\mathrm{Re}u, \mathrm{Im}u) = \mathbf{a}(\mathrm{Re}\mathbf{u}, \mathrm{Im}\mathbf{u}) \in \mathbb{R}$. □

3. Interpolation spaces and nonlinear problems

In Theorem 2.3 we have shown that if a, b are continuous and b is H-elliptic, the form $\mathbf{a}$ is associated with an analytic semigroup on $\mathbf{H}$. We can sharpen this result under the additional assumption that for some constant $M_b > 0$

$$|\mathrm{Im}b(u, u)| \leq M_b \|u\|_H \|u\|_V, \qquad u \in V. \tag{3.1}$$

Theorem 3.1. *If* (3.1) *holds, then the operator* $\mathbf{A}$ *associated with* $\mathbf{a}$ *generates a cosine operator function on* $\mathbf{H}$. *Moreover, the form domain* $\mathbf{V}$ *is isometric to the fractional power domain* $D(\lambda + \mathbf{A})^{\frac{1}{2}}$, *for* $\lambda > 0$ *large enough.*

Proof. We first show that $|\mathrm{Im}\mathbf{a}(\mathbf{u}, \mathbf{u})| \leq M_\mathbf{a} \|\mathbf{u}\|_\mathbf{V} \|\mathbf{u}\|_\mathbf{H}$ for some constant $M_\mathbf{a}$ and all $\mathbf{u} \in \mathbf{V}$. Let to this aim $\mathbf{u} = (u_1, u_2)^\top \in \mathbf{V}$. Since $|a(u, v)| \leq M_a \|u\|_V \|v\|_V$ for some $M_a > 0$ and all $u, v \in V$, there holds

$$\begin{aligned}
|\mathrm{Im}\mathbf{a}(\mathbf{u}, \mathbf{u})|^2 \leq\ & (1 + M_a^2)\|u_1\|_V^2 \|u_2\|_V^2 + M_b^2 \|u_2\|_H^2 \|u_2\|_V^2 \\
& \quad + 2M_a \|u_1\|_V^2 \|u_2\|_V^2 + 2M_b(1 + M_a)\|u_1\|_V \|u_2\|_H \|u_2\|_V^2 \\
\leq\ & \left((1 + M_a)^2 \|u_1\|_V^2 + M_b^2 \|u_2\|_H^2\right) \|u_2\|_V^2 \\
& \quad + M_b(1 + M_a) \left(\|u_1\|_V^2 + \|u_2\|_H^2\right) \|u_2\|_V^2 \\
\leq\ & (1 + M_a + M_b)^2 \left(\|u_1\|_V^2 + \|u_2\|_H^2\right) \|u_2\|_V^2 \\
\leq\ & (1 + M_a + M_b)^2 \|\mathbf{u}\|_\mathbf{H}^2 \|\mathbf{u}\|_\mathbf{V}^2.
\end{aligned}$$

This shows in particular that the numerical range of $\mathbf{a}$ is contained in a parabola (see [17, p. 204]) and thus, applying a result due to Crouzeix [13], we promptly obtain that $\mathbf{A}$ generates a cosine operator function on $\mathbf{H}$.

Moreover, by Haase's converse of Crouzeix's theorem (see [1, § 5.6.6]) there exists an equivalent scalar product $((\cdot \mid \cdot))_{\mathbf{H}}$ on $\mathbf{H}$ and $\lambda > 0$ such that the numerical range of $\mathbf{a}_\lambda := \mathbf{a} + \lambda((\cdot \mid \cdot))_{\mathbf{H}}$ lies in a parabola. Now it follows by a result due to McIntosh (see again [1, § 5.6.6]) that $\mathbf{A}$ has the square root property. This concludes the proof. ☐

The following result should be compared with [14, Thm. XVIII.5.1].

Corollary 3.2. *Let $B = B_0 + B_1$, where B_0 is a self-adjoint and strictly positive definite operator. Assume A to be bounded from $D(B_0^{\frac{1}{2}})$ to $D(B_0^{-\frac{1}{2}})$ and B_1 to be bounded from $D(B_0^{\frac{1}{2}})$ to H. Then problem* (1.1) *is governed by an analytic semigroup of angle $\frac{\pi}{2}$ on $D(B_0^{\frac{1}{2}}) \times H$.*

In particular, (1.1) *admits a unique mild solution for all initial data $u_{10} \in D(B_0^{\frac{1}{2}})$ and $u_{20} \in H$. If $A = \rho B$ for $\rho \in \mathbb{C}$, then* (1.1) *admits a unique classical solution for all $u_{10}, u_{20} \in D(B_0^{\frac{1}{2}})$ such that $\rho u_{10} + u_{20} \in D(B)$.*

Proof. Let $b_0 : D(B_0) \times D(B_0) \to \mathbb{C}$ the coercive, symmetric sesquilinear form associated with B_0. In particular, B_0 has the square root property (cf. [1, § 5.5.1]) and therefore the form norm of b_0 is isomorphic to $D(B_0^{\frac{1}{2}})$. Since now for the sesquilinear form b associated with B holds

$$|\mathrm{Im} b(u,u)| = |\mathrm{Im}(B_0 u \mid u)_H + \mathrm{Im}(B_1 u|u)_H| \le \|B_1 u\|_H \|u\|_H \le M\|u\|_{D(B_0)}\|u\|_H$$

for some constant $M > 0$, one sees that (3.1) is satisfied. After defining by a the sesquilinear form associated with A, Theorem 3.1 can be applied. Since every cosine operator function generator also generates an analytic semigroup of angle $\frac{\pi}{2}$ (see [3, Thm. 3.14.17]), the claim holds. ☐

Example 3.3. For an open bounded domain $\Omega \subset \mathbb{R}^n$ with C^2-boundary $\partial\Omega$ consider the complete second-order problem

$$\begin{cases} \ddot{u}(t,x) &= \nabla \cdot \Big(\alpha(x)\nabla u(t,x) + \beta(x)\nabla \dot{u}(t,x)\Big), & t \ge 0,\ x \in \Omega, \\ u(t,z) &= \dot{u}(t,z) = 0, & t \ge 0,\ z \in \partial\Omega, \\ u(0,x) &= u_{10}(x), & x \in \Omega, \\ \dot{u}(0,x) &= u_{20}(x), & x \in \Omega, \end{cases}$$

where $\alpha, \beta \in C^1(\overline{\Omega})$ such that $0 < \beta(x)$ for all $x \in \overline{\Omega}$.

Let $B = -\nabla \cdot (\beta\nabla)$ and $A = -\nabla \cdot (\alpha\nabla)$ on $H := L^2(\Omega)$, and accordingly introduce the forms

$$b(f,g) := \int_\Omega \beta(x)\nabla f(x)\overline{\nabla g(x)} \qquad \text{and} \qquad a(f,g) := \int_\Omega \alpha(x)\nabla f(x)\overline{\nabla g(x)}.$$

Then $D(B^{\frac{1}{2}}) = H_0^1(\Omega)$ and by Corollary 3.2 and Corollary 2.6 one concludes that the operator

$$\begin{aligned} D(\mathbf{A}) &= \left\{(u_1,u_2)^\top \in (H_0^1(\Omega))^2 : \alpha\nabla u_1 + \beta\nabla u_2 \in H_0^1(\Omega)\right\}, \\ \mathbf{A}\mathbf{u} &= \left(u_2, \nabla\left(\alpha\nabla u_1 + \beta\nabla u_2\right)\right)^\top. \end{aligned}$$

generates on $H^1_0(\Omega) \times L^2(\Omega)$ an analytic semigroup of angle $\frac{\pi}{2}$. This semigroup is contractive if $\alpha \equiv 1$ (and more generally also whenever $\alpha > 0$, up to considering weighted phase space). It yields the solutions to the above problem, which are real valued whenever $u_{10} \in H^1_0(\Omega)$ and $u_{20} \in L^2(\Omega)$ are real valued.

The analytical well-posedness of the above problem has been shown in [9] only in the case of α strictly positive, whereas we allow for α to be a complex-valued function.

We can now exploit the technique developed in [24, Chapt. 7] for semilinear parabolic problems, which heavily relies on interpolation theory. In order to avoid technicalities, we consider in the remainder of this section the special case of $A = \rho B$ for some $\rho \in \mathbb{C}$. This case is relevant in many concrete contexts, e.g., whenever investigating semilinear strongly damped equations like the Klein–Gordon one, see, e.g., [18, 4, 16, 6]. As an example of a possible application, we formulate the following, which is a direct consequence of [24, Thm. 7.1.3 and 7.1.10]. More refined results, also yielding global well-posedness, can be obtained by applying further tools from [24, § 7.2].

Corollary 3.4. *Let B satisfy the assumptions of Corollary 3.2. Assume $G : [0,T] \times D(B^{\frac{1}{2}}) \times D(B^{\frac{1}{2}}) \to H$ to be a continuous mapping that is locally Hölder continuous with respect to the first variable and locally Lipschitz continuous with respect to the second and third ones. Then for small initial data $u_{10}, u_{20} \in D(B^{\frac{1}{2}})$*

$$\begin{cases} \ddot{u}(t) + B(\rho u + \dot{u})(t) = G(t, u_1(t), \dot{u_2}(t)), & t \in [0,T], \\ u(0) = u_{10}, \qquad \dot{u}(0) = u_{20}, \end{cases}$$

has a unique classical solution, locally in time.

Theorem 3.1 also allows to apply the theory developed in [12] for quasilinear parabolic problems, where determining interpolation spaces is a crucial step, too. A prototypical result is the following, which can be compared with [11, Thm. 5.1].

Corollary 3.5. *Let D be a subspace of H with $D \hookrightarrow V$. Let the mapping*

$$B : V \times V \to \mathcal{L}(\{(u,v)^\top \in V \times V : \rho u + v \in D\}, H)$$

be well defined and locally Lipschitz continuous. Let $u_{10}, u_{20} \in V$ and assume the operator $B(u_{10}, u_{20})$ to satisfy the assumptions of Corollary 3.2 with $D(B(u_{10}, u_{20})^{\frac{1}{2}}) = V$. Then for all $f \in L^2(\mathbb{R}_+, H)$ and all $g \in \mathrm{Lip}(\mathbb{R}_+ \times V, H)$ there exists $\tau > 0$ such that the problem

$$\begin{cases} \ddot{u}(t) + B(u(t), \dot{u}(t))(\rho u(t) + \dot{u}(t)) = f(t) + g(t, u(t)), & t \in (0,\tau), \\ u(0) = u_{10}, \qquad \dot{u}(0) = u_{20}, \end{cases}$$

has a solution $u \in H^2((0,\tau), H) \cap H^1((0,\tau), V)$ with $\rho u + \dot{u} \in L^2((0,\tau), D)$.

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Exponential and Polynomial Stability Estimates for the Wave Equation and Maxwell's System with Memory Boundary Conditions

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To Günter Lumer

Abstract. We give exponential and polynomial stability results for the wave equation with variable coefficients in a bounded domain of $\mathbb{R}^n$, subject to a Dirichlet boundary condition on one part of the boundary and boundary conditions of memory type on the other part of the boundary. Moreover, analogous stability results are given for a system of Maxwell's equations in heterogeneous media subject to dissipative boundary conditions with memory.

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1. Introduction

In this paper we consider the wave equation with variable coefficients with Dirichlet boundary condition on one part of the boundary and a dissipative boundary condition of memory type on the other part of the boundary.

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with a smooth boundary Γ. We assume that Γ is divided into two closed and disjoint parts Γ_0 and Γ_1, i.e., $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\Gamma_0 \cap \Gamma_1 = \emptyset$. Moreover we assume that the measure of Γ_0 is positive.

Consider the problem

$$u_{tt} + \mathcal{A}u = 0 \quad \text{in} \quad \Omega \times (0, +\infty) \tag{1.1}$$

$$u = 0 \quad \text{on} \quad \Gamma_0 \times (0, +\infty) \tag{1.2}$$

$$\frac{\partial u}{\partial \nu_{\mathcal{A}}}(t) + \int_0^t k(t-s) u_t(s) ds + b u_t(t) = 0 \quad \text{on} \quad \Gamma_1 \times (0, +\infty) \tag{1.3}$$

$$u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{in} \quad \Omega \tag{1.4}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}u = -\operatorname{div}(A\nabla u) \tag{1.5}$$

when A is a symmetric matrix

$$A(x) = (a_{ij}(x))_{1\le i,j\le n} \tag{1.6}$$

with coefficients $a_{ij} \in C^1(\overline{\Omega})$ and satisfying

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ge \alpha \sum_{i=1}^{n} \xi_i^2, \quad \forall x \in \Omega, \quad \forall \xi \in \mathbb{R}^n, \tag{1.7}$$

for some constant $\alpha > 0$.

In condition (1.3), $k : [0, +\infty) \to \mathbb{R}$ is a function of class C^2, b is a positive constant and $\frac{\partial u}{\partial \nu_{\mathcal{A}}}$ is the co-normal derivative

$$\frac{\partial u}{\partial \nu_{\mathcal{A}}} = \langle A\nabla u, \nu\rangle, \tag{1.8}$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^n$ and $\nu(x)$ denotes the outward unit normal vector to the point $x \in \Gamma$. In the sequel we will also use $v \cdot w$ to denote the usual inner product between two vectors v, w.

The integral boundary condition (1.3) describes the memory effect which can be caused, for instance, by the interaction with another viscoelastic element. This boundary condition is quite general and covers a fairly large variety of physical configurations. We refer to [22] for some discussions about this model.

Frictional dissipative boundary condition (i.e., the case $k = 0$ in (1.3)) for the wave equation was studied by many authors, see [10, 11, 12] and their references. On the contrary for boundary condition with memory, only a few number of papers exists [1, 2, 3, 6, 9, 24, 25, 26]. In these papers, the authors consider the wave equation with constant coefficients and prove the decay of the energy by combining the multiplier method with the use of a suitable Lyapounov functional or integral inequalities.

Here we extend the previous results to the case of variable coefficients by using the approach from differential geometry initiated in [27] and by introducing suitable Lyapounov functionals.

Moreover, we consider in a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary Γ, the homogeneous Maxwell's system

$$D' - \operatorname{curl}(\mu B) = 0 \quad \text{in} \quad \Omega \times (0, +\infty) \tag{1.9}$$

$$B' + \operatorname{curl}(\lambda D) = 0 \quad \text{in} \quad \Omega \times (0, +\infty) \tag{1.10}$$

$$\operatorname{div} D = \operatorname{div} B = 0 \quad \text{in} \quad \Omega \times (0, +\infty) \tag{1.11}$$

$$D(0) = D_0 \quad \text{and} \quad B(0) = B_0 \quad \text{in} \quad \Omega \tag{1.12}$$

$$\lambda\mu D_\tau(t) = k_0 B(t) \times \nu + \int_0^t k(s)B(t-s) \times \nu ds \quad \text{on } \Gamma \times (0, +\infty) \tag{1.13}$$

where D, B are three-dimensional vector-valued functions of t, $x = (x_1, x_2, x_3)$; $\mu = \mu(x)$ and $\lambda = \lambda(x)$ are scalar functions in $C^2(\overline{\Omega})$ bounded from below by a positive constant, i.e.,

$$\lambda(x) \geq \lambda_0 > 0, \quad \mu(x) \geq \mu_0 > 0, \quad \forall x \in \overline{\Omega}; \tag{1.14}$$

D_0, B_0 are the initial data in a suitable space. As before, in the boundary condition (1.13), ν denotes the outward unit normal vector to the boundary Γ and $k : [0, +\infty) \to \mathbb{R}$ is a positive function of class C^2. Moreover, k_0 is a positive constant and D_τ denotes the tangential component of the vector field D, that is

$$D_\tau = \nu \times (D \times \nu).$$

The integral boundary condition (1.13) describes the memory effect and means that the boundary is a medium with a high but finite electric conductivity [5, 8, 15].

As for the scalar wave equation, frictional dissipative boundary condition for Maxwell's system (i.e., the case $k = 0$ in (1.13)) was studied by many authors, see [4, 7, 10, 11, 16, 17, 20, 21]. On the contrary, in the case of boundary conditions with memory, we know only a few number of papers [5, 8, 15]. In these papers, the authors consider Maxwell's equations with constant coefficients and prove the exponential decay of the energy by combining the multiplier method with the use of Pazy's theorem.

Here we extend the previous results to the case of variable coefficients by using the multiplier method and by introducing a suitable Lyapounov functional. Under appropriate assumptions on k, we also prove the polynomial decay of the energy.

The paper is organized as follows.

In Section 2 we give exponential and polynomial stability results for the wave equation while in Section 3 we consider the stabilization of Maxwell's equations. Finally, in Section 4 we give some examples where our assumptions are illustrated.

2. The wave equation

To obtain our stability estimates we assume that the kernel $k(\cdot)$ in the boundary condition (1.3) satisfies one of the following sets of assumptions:

$$k(t) \geq 0, \quad k'(t) \leq -\gamma_0 k(t), \quad k''(t) \geq -\gamma_1 k'(t) \tag{2.1}$$

for positive constants γ_0, γ_1; or

$$k(t) \geq 0, \quad k'(t) \leq -\gamma_0 [k(t)]^{1+\frac{1}{p}}, \quad k''(t) \geq \gamma_1 [-k'(t)]^{1+\frac{1}{p+1}} \tag{2.2}$$

for positive constants γ_0, γ_1 and for some $p > 1$.

Note that assumptions (2.1) imply that the function k and $-k'$ are exponentially decaying to 0; while assumptions (2.2) imply that k and $-k'$ are polynomially decaying to 0 as $1/(1+t)^p$ and $1/(1+t)^{p+1}$ respectively.

These assumptions are relatively standard, see [9, 25]. Note that in [1, 3, 6], the asymptotic behaviour of k is a little bit weaker but other conditions at 0 and on u_0 are imposed.

With our assumptions (2.1) or (2.2), problem (1.1)–(1.4) can be formulated as an evolutionary integral equation of variational type [22, 23]. Therefore the results from [22, 23] allow to state the following results, where we recall that

$$H^1_{\Gamma_0}(\Omega) := \{u \in H^1(\Omega) : u = 0 \quad \text{on} \quad \Gamma_0\}.$$

Theorem 2.1. *Let the above assumptions on k be satisfied. Then for all initial data $(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$, there exists a unique weak solution $u \in C^1(\mathbb{R}^+; L^2(\Omega)) \cap C(\mathbb{R}^+; H^1_{\Gamma_0}(\Omega))$ of (1.1)–(1.4). If furthermore $(u_0, u_1) \in (H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)) \times H^1_{\Gamma_0}(\Omega)$ satisfies the compatibility condition*

$$\frac{\partial u_0}{\partial \nu_{\mathcal{A}}} + b u_1 = 0 \quad \text{on} \quad \Gamma_1,$$

then the weak solution u of (1.1)–(1.4) has the regularity

$$u \in C^2(\mathbb{R}^+; L^2(\Omega)) \cap C^1(\mathbb{R}^+; H^1_{\Gamma_0}(\Omega)) \cap C(\mathbb{R}^+; H^2(\Omega)).$$

Remark 2.2. In [3, 25, 26], instead of (1.3) the authors consider the boundary condition

$$u(t) + \int_0^t g(t-s) \frac{\partial u}{\partial \nu}(s) ds = 0 \quad \text{on} \quad \Gamma_1 \times (0, +\infty), \tag{2.3}$$

for some function g. But in these papers it was shown that this boundary condition is equivalent to (1.3) under the assumption $u_0 = 0$ on Γ_1, where k is the resolvent kernel of g. Therefore our results below also hold for such a boundary condition. Note further that any strong solution of problem (1.1), (1.2), (2.3) and (1.4) satisfies $u(0) = 0$ on Γ_1 and therefore the assumption $u_0 = 0$ on Γ_1 is necessary to have strong solutions to that system.

We define the energy of the problem (1.1)–(1.4) by

$$\begin{aligned} E(t) := \frac{1}{2} \int_\Omega \left\{ u_t^2 + \langle A \nabla u, \nabla u \rangle \right\} dx + \frac{1}{2} \int_{\Gamma_1} k(t)[u(t) - u(0)]^2 d\Gamma \\ - \frac{1}{2} \int_{\Gamma_1} \int_0^t k'(t-s)[u(t) - u(s)]^2 ds d\Gamma. \end{aligned} \tag{2.4}$$

This is a "good" definition of the energy in order to obtain stability results. Indeed, a direct computation shows that the energy defined by (2.4) is decreasing. For any regular solution of the problem (1.1)–(1.4) we have

$$\begin{aligned} E'(t) = \frac{1}{2} \int_{\Gamma_1} k'(t)[u(t) - u(0)]^2 d\Gamma - b \int_{\Gamma_1} u_t^2 d\Gamma \\ - \frac{1}{2} \int_{\Gamma_1} \int_0^t k''(t-s)[u(t) - u(s)]^2 ds d\Gamma \le 0. \end{aligned} \tag{2.5}$$

Now, we want to give sufficient conditions on Ω and on the operator $\mathcal{A}$ in order to guarantee the exponential decay of the energy under the assumption (2.1) and the polynomial decay under assumption (2.2).

According to [27] (see also [13]) let us introduce the Riemannian metric generated by the spatial operator. Let

$$G(x) = (g_{ij}(x))_{1\leq i,j\leq n} = A^{-1}(x).$$

For any $x \in \mathbb{R}^n$ define the inner product and the norm on the tangent space $\mathbb{R}^n_x = \mathbb{R}^n$ by

$$g(X,Y) = \langle X,Y\rangle_g = \sum_{i,j=1}^n g_{ij}(x)\alpha_i\beta_j, \quad \forall\ X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i},\ Y = \sum_{i=1}^n \beta_i \frac{\partial}{\partial x_i} \in \mathbb{R}^n_x; \tag{2.6}$$

$$|X|_g = \langle X,X\rangle_g^{\frac{1}{2}}, \quad \forall\ \ X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i} \in \mathbb{R}^n_x. \tag{2.7}$$

Denote the Levi-Civita connection in the Riemannian metric g by $\mathcal{D}$. Let

$$H = \sum_{i=1}^n h_i \frac{\partial}{\partial x_i}, \quad X = \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i}$$

be vector fields on $(\mathbb{R}^n, g)$. The covariant differential $\mathcal{D}H$ of H determines a bilinear form on $\mathbb{R}^n_x \times \mathbb{R}^n_x$, for any $x \in \mathbb{R}^n$, defined by

$$\mathcal{D}H(Y,X) = \langle \mathcal{D}_X H, Y\rangle_g \quad \forall X,Y \in \mathbb{R}^n_x$$

where $\mathcal{D}_X H$ is the covariant derivative of H with respect to X

$$\mathcal{D}_X H = \sum_{k=1}^n \mathcal{D}_X \left(h_k \frac{\partial}{\partial x_k}\right) = \sum_{k=1}^n X \cdot \nabla h_k \frac{\partial}{\partial x_k} + \sum_{k,j=1}^n h_k \xi_j \mathcal{D}_{\partial/\partial x_j}\left(\frac{\partial}{\partial x_k}\right), \tag{2.8}$$

where

$$\mathcal{D}_{\partial/\partial x_j}\left(\frac{\partial}{\partial x_k}\right) = \sum_{l=1}^n \Gamma^l_{jk} \frac{\partial}{\partial x_l}, \tag{2.9}$$

Γ^l_{jk} being the Christoffel symbols of the connection $\mathcal{D}$,

$$\Gamma^l_{jk} = \frac{1}{2}\sum_{p=1}^n a_{lp}\left(\frac{\partial g_{kp}}{\partial x_j} + \frac{\partial g_{jp}}{\partial x_k} - \frac{\partial g_{jk}}{\partial x_p}\right). \tag{2.10}$$

Then, by (2.9) and (2.10)

$$\mathcal{D}_X H = \sum_{l=1}^n \left(X \cdot \nabla h_l + \sum_{k,i=1}^n h_k \xi_i \Gamma^l_{ik}\right)\frac{\partial}{\partial x_l}. \tag{2.11}$$

Therefore, by (2.11)

$$\mathcal{D}H(X,X) = \langle \mathcal{D}_X H, X\rangle_g = \sum_{i,j=1}^{n}\left(\sum_{l=1}^{n}\frac{\partial h_l}{\partial x_i}g_{lj} + \sum_{k,l=1}^{n} h_k g_{lj}\Gamma_{ik}^{l}\right)\xi_i\xi_j\,. \tag{2.12}$$

See [27] for more details.

Assumptions on $\mathcal{A}$. As in [27] we assume that there exists a C^1 vector field H in the Riemannian metric $(\mathbb{R}^n, g)$ such that

$$\langle \mathcal{D}_X H, X\rangle_g \geq a_0|X|_g^2, \quad \forall x\in\overline{\Omega}, \quad \forall X\in\mathbb{R}_x^n, \tag{2.13}$$

for some positive constant a_0.

Moreover we assume that

$$\sup_{\Omega}\ \operatorname{div} H < \inf_{\Omega}\ \operatorname{div} H + 2a_0, \tag{2.14}$$

and

$$H\cdot\nu \leq 0, \quad \text{on } \Gamma_0 \quad \text{and} \quad H\cdot\nu\geq\delta \quad \text{on } \Gamma_1, \tag{2.15}$$

for a constant $\delta > 0$.

Remark 2.3. Assumption (2.13) has been first introduced by Yao in [27] to extend to the case of variable coefficients the standard identity with multiplier. Obviously it holds in the case of constant coefficients taking as H the standard multiplier $m(x) = x - x_0$. Note that in this case also (2.14) is verified. We refer to [27, 13] for examples of function H verifying this assumption in the non constant case. We also refer to Section 4 for examples verifying the assumptions (2.13) and (2.14).

Remark 2.4. Observe that assumption (2.13) is verified if there exists a function v of class C^2 strictly convex with respect to the metric g, that is a function v such that

$$\mathcal{D}^2 v(X,X) = \langle \mathcal{D}_X(\nabla_g v), X\rangle_g \geq a_0|X|_g^2, \quad \forall x\in\overline{\Omega}, \quad \forall X\in\mathbb{R}_x^n.$$

In that case (2.13) holds with $H = \nabla_g v$, see [13].

Consider the standard energy

$$\mathcal{E}(t) := \int_{\Omega}\left\{u_t^2 + \langle A\nabla u, \nabla u\rangle\right\}dx\,, \tag{2.16}$$

and define

$$Mu = 2(H\cdot\nabla u) + \theta u\,, \tag{2.17}$$

where H is defined in assumption (2.13) and θ is a constant such that

$$\sup_{\Omega}\ \operatorname{div} H - 2a_0 < \theta < \inf_{\Omega}\operatorname{div}\ H. \tag{2.18}$$

From assumption (2.14) it follows that such a constant θ exists. We can give the following estimate. Its proof is based on Green's formula, the assumptions (1.7) and (2.13), Young's inequality and Poincaré's theorem.

Proposition 2.5. Assume that k satisfies (2.1). Then, for any regular solution of the problem (1.1)–(1.4), we have

$$\begin{aligned}\frac{d}{dt}\left\{\int_\Omega u_t M u dx\right\} \leq -c_0\mathcal{E}(t) + C\left\{\int_{\Gamma_1} u_t^2 d\Gamma + \int_{\Gamma_1} k(t)[u(t)-u(0)]^2 d\Gamma \right. \\ \left. - \int_{\Gamma_1}\int_0^t k'(t-s)[u(t)-u(s)]^2 ds d\Gamma\right\},\end{aligned} \tag{2.19}$$

for suitable positive constants c_0, C.

By introducing the Lyapounov functional

$$\tilde{E}(t) := E(t) + \hat{\gamma}\int_\Omega u_t(t) M u(t) dx\,, \tag{2.20}$$

where $\hat{\gamma}$ is a positive constant sufficiently small, and using (2.19), we can deduce the exponential stability result for problem (1.1)–(1.4). See [18] for the full details.

Theorem 2.6. *Assume that* $\Gamma = \Gamma_0 \cup \Gamma_1$ *with* Γ_0, Γ_1 *closed sets with* $\Gamma_0 \cap \Gamma_1 = \emptyset$*, that the matrix* A *satisfies* (1.7) *and that* $b > 0$*. Furthermore, assume that there exists a* C^1 *vector field* H *verifying* (2.13), (2.14), (2.15)*. If the memory kernel* k *satisfies* (2.1)*, then there exist two positive constants* C_1, C_2 *such that for any regular solution of problem* (1.1)–(1.4)*,*

$$E(t) \leq C_1 E(0) e^{-C_2 t}, \quad \forall t > 0. \tag{2.21}$$

Assuming that the function k in the boundary condition (1.3) satisfies the assumptions (2.2) we can give a polynomial stability result. In this case, instead of Proposition 2.5, we use the following estimate.

Proposition 2.7. Assume that k satisfies (2.2). Then, any regular solution of the problem (1.1)–(1.4) satisfies

$$\begin{aligned}\frac{d}{dt}\left\{\int_\Omega u_t M u dx\right\} \leq -c_0\mathcal{E}(t) + C\left\{\int_{\Gamma_1} u_t^2 d\Gamma + \int_{\Gamma_1} [k(t)]^{1+\frac{1}{p}}[u(t)-u(0)]^2 d\Gamma \right. \\ \left. + \int_{\Gamma_1}\int_0^t [-k'(t-s)]^{1+\frac{1}{p+1}}[u(t)-u(s)]^2 ds d\Gamma\right\},\end{aligned} \tag{2.22}$$

for suitable positive constants c_0, C.

Using (2.22) and the Lyapounov functional (2.20) we obtain the polynomial stability estimate. See [18] for the proof of such a result.

Theorem 2.8. *Assume* $\Gamma = \Gamma_0 \cup \Gamma_1$ *with* Γ_0, Γ_1 *closed sets with* $\Gamma_0 \cap \Gamma_1 = \emptyset$*, that the matrix* A *satisfies* (1.7) *and that* $b > 0$*. Furthermore, assume that there exists a* C^1 *vector field* H *verifying* (2.13), (2.14), (2.15) *and that the memory kernel* k *satisfies* (2.2)*. Let* r *be a real number such that*

$$\frac{1}{p+1} < r < 1\,.$$

Then, for any regular solution u of problem (1.1)–(1.4)*, we have*

$$E(t) \leq \frac{C}{(1+t)^{(1-r)(p+1)}}, \quad \forall t > 0\,, \tag{2.23}$$

for a suitable positive constant C depending on $E(0)$.

Remark 2.9. If $H(x) \cdot \nu(x) = 0$ for some $x \in \Gamma_1$, then a term like

$$\int_0^T \int_{\Gamma_1} |\nabla_\tau u|^2 d\Gamma dt,$$

where ∇_τ denotes the tangential gradient, remains in the estimate of boundary terms. For the standard boundary condition

$$\frac{\partial u}{\partial \nu} + u_t = 0\,,$$

this term is usually eliminated (see, e.g., [13]) using micro-local analysis and a compactness argument. This compactness argument cannot be used here since we need an estimate independent of T.

Remark 2.10. Note that everything still holds if the function k depends also on the variable space, that is

$$k := k(x,t)\,, \quad k : \Gamma_1 \times [0,\infty) \to \mathbb{R}\,, \quad k \in C^2(\Gamma_1 \times [0,\infty))\,.$$

The same remark applies to the following section.

3. Maxwell's equations

In this section, we consider the system (1.9)–(1.13) and give exponential and polynomial stability results. We assume that the function $k(\cdot)$ in the boundary condition (1.13) satisfies the following assumptions:

$$k'(t) \leq 0, \quad k''(t) \geq 0. \tag{3.1}$$

With our assumptions (3.1), problem (1.9)–(1.13) can be formulated as a standard evolution equation $\dot{u} + Au = 0, u(0) = u_0$ [15] with an appropriate operator A, which generates a strongly continuous semigroup (proved using the same techniques as the ones from Theorem 2.1 of [15]). This allows to obtain the following results:

Theorem 3.1. *Let assumption* (3.1) *be satisfied. Then, for all initial data $D_0, B_0 \in L^2(\Omega)^3$ such that $\operatorname{div} D_0 = \operatorname{div} B_0 = 0$ in Ω, $\operatorname{curl}(\mu B_0), \operatorname{curl}(\lambda D_0) \in L^2(\Omega)^3$ and satisfying the compatibility condition*

$$\lambda\mu D_{0\tau}(t) = k_0 B_0 \times \nu \quad \text{on} \quad \Gamma,$$

there exists a unique solution (D, B) of (1.9)–(1.13) *with the regularity $D, B \in C^1(\mathbb{R}^+; L^2(\Omega)^3)$, $\operatorname{curl}(\mu B) \in C(\mathbb{R}^+; L^2(\Omega)^3)$, $\operatorname{curl}(\lambda D) \in C(\mathbb{R}^+; L^2(\Omega)^3)$.*

We define the energy of our system (1.9)–(1.13) by

$$E(t) := \frac{1}{2}\int_\Omega \left\{\lambda|D(t)|^2 + \mu|B(t)|^2\right\} dx + \frac{1}{2}\int_\Gamma k(t)\left|\int_0^t B(t-\tau)\times\nu d\tau\right|^2 d\Gamma$$
$$- \frac{1}{2}\int_\Gamma\int_0^t k'(s)\left|\int_0^s B(t-\tau)\times\nu d\tau\right|^2 dsd\Gamma. \tag{3.2}$$

Note that our definition is different from the one in [8, 15] and is inspired from the definition of the energy for the wave equation with memory boundary conditions.

For any regular solution of the Maxwell's system (1.9)–(1.13) the energy is decreasing since

$$\begin{aligned} E'(t) = -k_0\int_\Gamma |B(t)\times\nu|^2 d\Gamma + \frac{1}{2}\int_\Gamma k'(t)\Big|\int_0^t B(t-\tau)\times\nu d\tau\Big|^2 d\Gamma \\ - \frac{1}{2}\int_\Gamma\int_0^t k''(s)\Big|\int_0^s B(t-\tau)\times\nu d\tau\Big|^2 dsd\Gamma \le 0. \end{aligned} \tag{3.3}$$

We assume that there exists a C^1 vector field q such that

$$\lambda\,\mathrm{div}\, q|\xi|^2 - 2\lambda\sum_{i,k=1}^{3}\frac{\partial q_i}{\partial x_k}\xi_i\xi_k - q\cdot\nabla\lambda|\xi|^2 \ge \rho\lambda|\xi|^2\,, \quad \forall x\in\Omega, \quad \forall\xi\in\mathbb{R}^3\,, \tag{3.4}$$

and

$$\mu\,\mathrm{div}\, q|\xi|^2 - 2\mu\sum_{i,k=1}^{3}\frac{\partial q_i}{\partial x_k}\xi_i\xi_k - q\cdot\nabla\mu|\xi|^2 \ge \rho\mu|\xi|^2\,, \quad \forall x\in\Omega, \quad \forall\xi\in\mathbb{R}^3\,. \tag{3.5}$$

We further assume that $q\cdot\nu > 0$ on Γ. Therefore, for a suitable positive constant δ,

$$q\cdot\nu \ge \delta \quad \text{on } \Gamma\,. \tag{3.6}$$

Moreover, in order to have an exponential stability estimate we assume that the integral kernel k satisfies assumption (2.1).

Now we recall a standard identity with multiplier. For a proof see Lemma 2.2 of [21] that extended to variable λ, μ a previous identity by Komornik [10].

Proposition 3.2. Let (D, B) be a regular solution of problem (1.9)–(1.12) and let q be a C^1 vector field. Then the following identity holds:

$$\begin{aligned} 2\int_\Omega (B\times D)'q dx = &-\int_\Omega (\mathrm{div}\, q)(\lambda|D|^2 + \mu|B|^2)dx \\ &+ 2\int_\Omega\sum_{i,k=1}^{3}\frac{\partial q_i}{\partial x_k}(\lambda D_iD_k + \mu B_iB_k)dx + \int_\Omega\{(q\cdot\nabla\lambda)|D|^2 + (q\cdot\nabla\mu)|B|^2\}dx \\ &+ \int_\Gamma\{(\lambda|D|^2 + \mu|B|^2)(q\cdot\nu) - 2\mu(q\cdot B)(\nu\cdot B) - 2\lambda(q\cdot D)(\nu\cdot D)\}d\Gamma. \end{aligned} \tag{3.7}$$

Remark 3.3. Note that identity (3.7) holds without assuming any particular boundary condition.

Using the boundary condition (1.13), Young's and Cauchy-Schwarz's inequalities and the assumption (2.1), we can estimate the boundary terms in (3.7) as follows.

Proposition 3.4. Assume that (2.1) holds. Let (D, B) be a solution of problem $(1.9) - (1.13)$ and let q be a C^1 vector field verifying (3.4), (3.5), (3.6). Then,

$$\begin{aligned}\int_\Gamma \{(\lambda|D|^2 + \mu|B|^2)(q\cdot\nu) - 2\mu(q\cdot B)(\nu\cdot B) - 2\lambda(q\cdot D)(\nu\cdot D)\}d\Gamma \\ \leq C\left\{k_0\int_\Gamma |B(t)\times\nu|^2 d\Gamma + \int_\Gamma k(t)\left|\int_0^t B(t-\tau)\times\nu d\tau\right|^2 d\Gamma\right. \\ \left. - \int_\Gamma\int_0^t k'(s)\left|\int_0^s B(t-\tau)\times\nu d\tau\right|^2 dsd\Gamma\right\},\end{aligned} \tag{3.8}$$

for a suitable positive constant C.

By the estimate (3.8) and the assumptions (3.4), (3.5) on q, (3.7) gives

$$\begin{aligned}2\int_\Omega (B\times D)' q dx \leq -\rho\int_\Omega(\lambda|D|^2+\mu|B|^2)dx \\ +C\left\{k_0\int_\Gamma |B(t)\times\nu|^2 d\Gamma + \int_\Gamma k(t)\left|\int_0^t B(t-\tau)\times\nu d\tau\right|^2 d\Gamma\right. \\ \left. - \int_\Gamma\int_0^t k'(s)\left|\int_0^s B(t-\tau)\times\nu d\tau\right|^2 dsd\Gamma\right\}.\end{aligned} \tag{3.9}$$

Now, we can give the exponential stability result. We need to introduce the Lyapounov functional

$$\tilde{E}(t) := E(t) + 2\hat{\gamma}\int_\Omega (B\times D)\cdot q dx, \tag{3.10}$$

where $\hat{\gamma}$ is a positive constant chosen sufficiently small and q a C^1 vector field verifying (3.4), (3.5) and (3.6). To obtain the stability estimate we use the derivative of the energy (3.3), inequality (3.9) and the new functional (3.10). A complete proof of such a result is given in [19].

Theorem 3.5. *Assume that* (3.4)*,* (3.5)*,* (3.6) *and* (2.1) *hold. Then, for any regular solution* (D, B) *of problem* (1.9)–(1.13)*,*

$$E(t) \leq C_1 E(0) e^{-C_2 t}, \quad \forall t > 0, \tag{3.11}$$

for suitable positive constants C_1, C_2*.*

In order to have a polynomial stability estimate we assume that the function k in the boundary condition (1.13) satisfies (2.2).

Instead of Proposition 3.4 we have, in this case, the following result.

Proposition 3.6. Assume that (2.2) holds. Let (D, B) be a solution of problem (1.9) – (1.13) and let q be a C^1 vector field verifying (3.4), (3.5), (3.6). Then,

$$\begin{aligned}
&\int_\Gamma \{(\lambda|D|^2 + \mu|B|^2)(q\cdot\nu) - 2\mu(q\cdot B)(\nu\cdot B) - 2\lambda(q\cdot D)(\nu\cdot D)\}d\Gamma \\
&\quad \le C\left\{k_0\int_\Gamma |B(t)\times\nu|^2 d\Gamma + \int_\Gamma [k(t)]^{1+\frac{1}{p}}\left|\int_0^t B(t-\tau)\times\nu d\tau\right|^2 d\Gamma\right. \\
&\qquad \left. + \int_\Gamma\int_0^t [-k'(s)]^{1+\frac{1}{p+1}}\left|\int_0^s B(t-\tau)\times\nu d\tau\right|^2 ds d\Gamma\right\},
\end{aligned} \tag{3.12}$$

for a suitable positive constant C.

By the estimate (3.12) and the assumptions (3.4), (3.5) on q, (3.7) gives

$$\begin{aligned}
2\int_\Omega (B\times D)' q dx &\le -\rho\int_\Omega (\lambda|D|^2 + \mu|B|^2)dx \\
&\quad + C\left\{k_0\int_\Gamma |B(t)\times\nu|^2 d\Gamma + \int_\Gamma [k(t)]^{1+\frac{1}{p}}\left|\int_0^t B(t-\tau)\times\nu d\tau\right|^2 d\Gamma\right. \\
&\quad \left. + \int_\Gamma\int_0^t [-k'(s)]^{1+\frac{1}{p+1}}\left|\int_0^s B(t-\tau)\times\nu d\tau\right|^2 ds d\Gamma\right\}.
\end{aligned} \tag{3.13}$$

We are ready to give the stabilization result. To prove the polynomial stabilization we use the derivative of the energy (3.3), the Lyapounov functional (3.10) and the above estimate (3.13). For the details we refer to [19].

Theorem 3.7. *Assume that* (3.4), (3.5), (3.6) *and* (2.2) *hold. Let* (D, B) *be a regular solution of problem* (1.9)–(1.13) *and let* r *be such that*

$$\frac{1}{p+1} < r < \frac{p}{p+1}.$$

Then,

$$E(t) \le \frac{C}{(1+t)^{(1-r)(p+1)-1}}, \quad \forall t > 0\,, \tag{3.14}$$

for a suitable positive constant C.

4. Examples

We end up with some examples that illustrate the geometric assumption (2.13) and the assumption (2.14) (Subsection 4.1) and some examples that illustrate the assumptions (3.4) and (3.5) (Subsection 4.2).

4.1. Examples for the wave equation

Example (1). We take A such that $a_{ij}(x) = a(x)\delta_{ij}$, for some smooth function a satisfying $a \geq \alpha > 0$ in Ω. Then clearly $g_{ij}(x) = a(x)^{-1}\delta_{ij}$, and direct calculations yield

$$\Gamma^l_{jk} = -\frac{1}{2a}\left(\frac{\partial a}{\partial x_j}\delta_{kl} + \frac{\partial a}{\partial x_k}\delta_{jl} - \frac{\partial a}{\partial x_l}\delta_{jk}\right).$$

We now choose $H(x) = a(x)(x - x_0)$ for some $x_0 \in \mathbb{R}^n$ and again standard calculations give

$$\mathcal{D}H(X,X) = a^{-1}\sum_{i,j=1}^{n}\left(a\delta_{ij} + \frac{\partial a}{\partial x_i}(x_j - x_{0j}) - \frac{1}{2}(x - x_0)\cdot\nabla a\delta_{ij}\right)\xi_i\xi_j\,.$$

Therefore the assumption (2.13) is equivalent to

$$a|X|^2 + X\cdot\nabla aX\cdot(x - x_0) - \frac{1}{2}|X|^2(x - x_0)\cdot\nabla a \geq a_0|X|^2, \forall x \in \Omega, X \in \mathbb{R}^n, \quad (4.1)$$

where $|X|$ means the Euclidean norm of X. Since Cauchy-Schwarz's inequality leads to

$$\left|X\cdot\nabla aX\cdot(x - x_0) - \frac{1}{2}|X|^2(x - x_0)\cdot\nabla a\right| \leq \frac{3}{2}\|x - x_0\|_\infty\|\nabla a\|_\infty|X|^2,$$

where $\|w\|_\infty = \sup_{x\in\Omega}|w(x)|$, the estimate (4.1) will be true if

$$a - \frac{3}{2}\|x - x_0\|_\infty\|\nabla a\|_\infty \geq a_0 \text{ in } \Omega,$$

or equivalently if

$$\frac{3}{2}\|x - x_0\|_\infty\|\nabla a\|_\infty < \inf_{x\in\Omega} a(x), \quad (4.2)$$

and in that case we may take

$$a_0 = \inf_{x\in\Omega} a(x) - \frac{3}{2}\|x - x_0\|_\infty\|\nabla a\|_\infty.$$

Since

$$\operatorname{div} H(x) = na(x) + \nabla a(x)\cdot(x - x_0),$$

the condition (2.14) is equivalent to

$$\sup_{x\in\Omega}(na(x) + \nabla a(x)\cdot(x - x_0)) < \inf_{x\in\Omega}(na(x) + \nabla a(x)\cdot(x - x_0)) + 2a_0. \quad (4.3)$$

Again using Cauchy-Schwarz's inequality, this estimate is satisfied if

$$n\|a\|_\infty + 2\|x - x_0\|_\infty\|\nabla a\|_\infty < n\inf_{x\in\Omega} a(x) + 2a_0.$$

Taking the above value of a_0, we deduce that (2.14) holds if

$$n\|a\|_\infty + 5\|x - x_0\|_\infty\|\nabla a\|_\infty < (n + 2)\inf_{x\in\Omega} a(x). \quad (4.4)$$

Roughly speaking the conditions (4.2) and (4.4) hold together if a does not vary too much in Ω.

Example (2). We keep the setting of example 1 with $a(x) = 1 + |x|^2$ and $H(x) = a(x)x$. Then from the above considerations (2.13) reduces to (see (4.1))

$$|X|^2(1 + |x|^2) + 2(X \cdot x)^2 - |X|^2|x|^2 \geq a_0|X|^2, \qquad \forall x \in \Omega, X \in \mathbb{R}^n,$$

that is $a_0|X|^2 \leq |X|^2 + 2(X \cdot x)^2$, which always holds by choosing $a_0 = 1$.

Concerning (2.14), it is equivalent to (with the choice $a_0 = 1$)

$$(n + 2)r_{\max}^2 < (n + 2)r_{\min}^2 + 2.$$

where for brevity we write

$$r_{\min} = \inf_{x\in\Omega} |x|, r_{\max} = \sup_{x\in\Omega} |x|.$$

In other words, (2.14) is equivalent to

$$r_{\max}^2 - r_{\min}^2 < 2/(n + 2).$$

For instance it holds if $\Omega = B(0, r_{\max}) \setminus B(0, r_{\min})$ with the above constraint between $r_{\min}$ and $r_{\max}$. In that case $\Gamma_0 = S(0, r_{\min})$ and $\Gamma_1 = S(0, r_{\max})$.

Example (3). As in Example 4.1 of [27] we take

$$a_{ij}(x) = (1 + |x|^2)^2\delta_{ij}.$$

Here contrary to [27] we simply take $H(x) = (1 + |x|^2)^2 x$. As before (2.13) is equivalent to

$$|X|^2(1 + |x|^2) + 4(X \cdot x)^2 - 2|X|^2|x|^2 \geq \frac{a_0}{1 + |x|^2}|X|^2, \qquad x \in \Omega, X \in \mathbb{R}^n. \quad (4.5)$$

This inequality will be satisfied if

$$|X|^2(1 + |x|^2) - 2|X|^2|x|^2 \geq \frac{a_0}{1 + |x|^2}|X|^2, \qquad x \in \Omega, X \in \mathbb{R}^n,$$

or equivalently

$$1 - |x|^2 \geq \frac{a_0}{1 + |x|^2}, \qquad x \in \Omega.$$

If $\bar{\Omega} \subset B(0, 1) = \{x \in \mathbb{R}^n : |x| < 1\}$, then the above condition holds with $a_0 = 1 - r_{\max}^4$. Since $\operatorname{div} H(x) = (1 + |x|^2)(n + (4 + n)|x|^2)$, the condition (2.14) is equivalent to

$$(1 + r_{\max}^2)(n + (4 + n)r_{\max}^2) < (1 + r_{\min}^2)(n + (4 + n)r_{\min}^2) + 2 - 2r_{\max}^4,$$

which is satisfied if $r_{\max}$ is not too far from $r_{\min}$.

Example (4). We take the matrix $A(x)$ as a perturbation of a symmetric positive definite matrix A. More precisely we take

$$A(x) = A - R(x),$$

and assume that the perturbation R is small in the sense that there exists $r \in (0, 1)$ such that

$$\sup_{x\in\Omega} \|A^{-1}\|_2\|R(x)\|_2 \leq r, \max_{k=1,\ldots,n} \sup_{x\in\Omega} \|\frac{\partial R}{\partial x_k}\|_2 \leq r, \quad (4.6)$$

where $\|\cdot\|_2$ means the Euclidean matrix norm. This condition implies that

$$\|A^{-1}R(x)\|_2 \leq \|A^{-1}\|_2\|R(x)\|_2 < 1,$$

and using a Neumann series, the matrix $A(x)$ is invertible and its inverse matrix is given by

$$G(x) = A(x)^{-1} = \sum_{k=0}^{\infty}(A^{-1}R(x))^k A^{-1}. \tag{4.7}$$

From this expression, we see that

$$\frac{\partial G}{\partial x_k} = \sum_{k=1}^{\infty} k(A^{-1}R(x))^{k-1}A^{-1}\frac{\partial R}{\partial x_k}A^{-1},$$

and therefore by the assumptions (4.6), we deduce that

$$\|\frac{\partial G}{\partial x_k}\|_2 \leq \frac{r}{(1-r)^2}\|A^{-1}\|_2^2. \tag{4.8}$$

Now we choose

$$H(x) = A(x)(x - x_0),$$

and find

$$\frac{\partial h_i}{\partial x_k} = a_{ik}(x) + \sum_{j=1}^{n}\frac{\partial r_{ij}}{\partial x_k}(x_j - x_{0j}).$$

From this identity and the estimate (4.8), we deduce that

$$\mathcal{D}H(X,X) = |X|^2 + (R_1(x)X,X), \forall x \in \Omega, X \in \mathbb{R}^n,$$

where the matrix function $R_1(x)$ satisfies

$$\|R_1(x)\|_2 \leq Cr, \forall x \in \Omega,$$

for some $C > 0$. Therefore for r small enough, the assumption (2.13) will be satisfied.

Similarly as

$$\operatorname{div} H(x) = \text{ tr } A + r_2(x),$$

with

$$|r_2(x)| \leq Cr, \ \forall x \in \Omega,$$

for some $C > 0$, the condition (2.14) holds if r is small enough.

4.2. Examples for Maxwell's equations

Example (5). If we take λ and μ satisfying for any $x \in \Omega$,

$$\lambda(x) - \nabla\lambda(x)\cdot(x - x_0) \geq c\lambda(x), \tag{4.9}$$

$$\mu(x) - \nabla\mu(x)\cdot(x - x_0) \geq c\mu(x), \tag{4.10}$$

for a given point $x_0 \in \Omega$ and a positive constant c, then (3.4) and (3.5) hold for the standard multiplier $q(x) := x - x_0$. If the domain Ω is strictly star-shaped with respect to x_0, then (3.6) also holds. In particular (4.9) and (4.10) hold if

$$\nabla\lambda(x)\cdot(x - x_0) \leq 0, \ \nabla\mu(x)\cdot(x - x_0) \leq 0, \quad \forall x \in \Omega.$$

Example (6). If $\lambda \equiv \mu$ and $q := \lambda^2(x)(x - x_0)$, then (3.4) and (3.5) become

$$\frac{1}{2}\nabla\lambda^2 \cdot (x - x_0)|\xi|^2 + \lambda^2|\xi|^2 - 2(\nabla\lambda^2 \cdot \xi)((x - x_0) \cdot \xi) \geq \rho|\xi|^2, \quad \forall x \in \Omega, \quad \forall \xi \in \mathbb{R}^3.$$

In particular, this is verified if

$$\|x - x_0\|_\infty \|\nabla\lambda^2\|_\infty < \frac{2}{5}(\inf_\Omega \lambda)^2.$$

Example (7). If $\lambda \equiv \mu \equiv 1 + |x|^2$ and if we take $q := x$, then (3.4) and (3.5) are equivalent to

$$(1 + |x|^2)|\xi|^2 - x \cdot \nabla(1 + |x|^2)|\xi|^2 \geq \rho(1 + |x|^2)|\xi|^2,$$

that is

$$(1 - |x|^2)|\xi|^2 \geq \rho(1 + |x|^2)|\xi|^2.$$

Consequently if $\overline{\Omega} \subset \{\, x \in \mathbb{R}^3 \;:\; |x| < 1 \,\}$, then there exists $\rho > 0$ such that this condition is satisfied.

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Maximal Regularity for Degenerate Evolution Equations with an Exponential Weight Function

Jan Prüss and Gieri Simonett

Dedicated to the memory of Günter Lumer

Abstract. In this contribution we consider degenerate evolution equations on the real line that have the distinguished feature that they contain an exponential weight function in front of the time derivative.

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1. Introduction

In this contribution we consider degenerate evolution equations on the real line that have the distinguished feature that they contain an exponential weight function. More precisely, we consider evolution equations of the type

$$e^{sx}\partial_t u + h(\partial_x)u = f, \quad u(0) = u_0, \tag{1.1}$$

where $s > 0$ is a fixed number, $x \in \mathbb{R}$ and $u = u(t,x)$. Here $h(\partial_x)$ is a pseudo-differential operator whose symbol $h = h(i\xi)$ is meromorphic in a vertical strip around the imaginary axis and satisfies appropriate growth conditions.

Our interest is motivated by problems that arise from elliptic or parabolic equations on angles and wedges, and by free boundary problems with moving contact lines. To describe the class of symbols we have in mind, let us consider the case of dynamic boundary conditions. It can be shown that the boundary symbol for the Laplace equation $\Delta u = 0$ on an angle $G = \{(r\cos\phi, r\sin\phi);\ r > 0\ \phi \in (0,\alpha)\}$ in $\mathbb{R}^2$ with Dirichlet condition $u = 0$ on $\phi = \alpha$ and dynamic boundary condition $\partial_t u + \partial_\nu u = g$ on $\phi = 0$ is given by

$$\partial_t e^x + \psi_0(-(\partial_x + \beta)^2), \quad \psi_0(z) = \sqrt{z}\coth(\alpha\sqrt{z}), \quad z \in \mathbb{C}.$$

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Here $\beta \in \mathbb{R}$ is a parameter that will ultimately determine the weight function corresponding to the angle α. Similarly, if one considers the one-phase Mullins-Sekerka problem in two dimensions with boundary intersection and prescribed contact angle $\alpha \in (0, \pi]$, one is led to the boundary symbol

$$\partial_t e^{3x} - \psi_1(-(\partial_x + \beta)^2)(\partial_x + \beta + 1)(\partial_x + \beta + 2),$$

where this time $\psi_1(z) = \sqrt{z}\tanh(\alpha\sqrt{z})$. The free boundary problem for the stationary Stokes equations with boundary contact and prescribed contact angle in two dimensions leads to

$$\partial_t e^x + \psi(\partial_x + \beta),$$

where

$$\psi(z) = (1+z)\frac{\cos(2\alpha z) - \cos(2\alpha)}{\sin(2\alpha z) + z\sin(2\alpha)},$$

in the slip case and

$$\psi(z) = \frac{(1+z)}{4}\frac{\sin(2\alpha z) - z\sin(2\alpha)}{z^2\sin^2(\alpha) - \cos^2(\alpha z)}$$

in the non-slip case. This motivates the study of equations of the type (1.1) and its parametric form

$$\nu e^{sx}u + h(\partial_x)u = f, \tag{1.2}$$

where $s > 0$, $\nu \in \mathbb{C}$.

It is our goal to identify function spaces such that the operators in (1.1) and (1.2) become topological isomorphisms between these spaces, i.e., to obtain optimal solvability results. We will do this in the framework of L_p-spaces. Our main tools are recent results on sums of sectorial operators, their $\mathcal{H}^\infty$-calculi, and $\mathcal{R}$-boundedness of associated operator families, see for instance [1, 2, 3, 4, 6, 7, 9].

Once this goal is achieved, one can go on to study symbols of higher-dimensional or time-dependent problems. The symbols for the Mullins-Sekerka problem in higher dimensions, for the Stefan problem with surface tension, and for the free boundary of the non-steady Stokes problem will be the subject of future work.

The case where h is a second-order polynomial is studied in detail in [8], and an application to a parabolic evolution equation in a wedge domain with dynamic boundary conditions is given.

Observe that equations (1.1) and (1.2) are highly degenerate, due to the presence of the exponentials. Therefore they are not directly accessible by standard methods for pseudo-differential operators. Moreover, the basic ingredients of these symbols, namely e^x and ∂_x, do not commute. Still, there is a close relation between these operators. In fact, e^{sx} is an eigenfunction of ∂_x with eigenvalue s, or to put it in a different way, the commutator between e^{sx} and ∂_x is se^{sx}. It is this relation we base our approach on. It allows us to apply abstract results on sums of non-commuting operators.

The plan for this paper is as follows. In Section 2 we introduce the symbol class $\mathcal{M}^r_{a,b}$. Our first main result, Theorem 2.5, states that parametric symbols of the form (1.2) lead to sectorial operators in $L_p(\mathbb{R})$ which admit a bounded $\mathcal{H}^\infty$-calculus. This result is used in Section 3 to show that problem (1.1) generates a bounded, strongly continuous, analytic semigroup on $L_p(\mathbb{R})$ for every symbol $h \in \mathcal{M}^r_{a,b}$, see Theorem 3.1 We can also show that the degenerate evolution equation (1.1) enjoys L_p-maximal regularity, provided h is replaced by $\omega_0 + h$ with an appropriate nonnegative number ω_0, see Proposition 3.2. We pose the open question whether or not ω_0 can in fact be chosen to be zero, and we answer this question in the affirmative in case that $p = 2$. Finally, in Section 5 we study some of the functions introduced above, and we characterize values of β so that the associated symbol h_β belongs to the symbol class $\mathcal{M}^r_{a,b}$.

In order to keep this paper short, we refer to [2, 7] for the definitions and for background material on sectorial operators, their $\mathcal{H}^\infty$-calculus, and the concept of $\mathcal{R}$-boundedness. For the reader's convenience, we will include a recent result on an $\mathcal{H}^\infty$-calculus for the sum of non-commuting operators. For this, we consider two sectorial operators A and B and we assume that A and B satisfy the Labbas-Terreni commutator condition [5], which reads as follows.

$$\begin{cases} 0 \in \rho(A). \text{ There are constants } c > 0,\ \ 0 \le \alpha < \beta < 1, \\ \psi_A > \phi_A,\ \psi_B > \phi_B,\ \psi_A + \psi_B < \pi, \\ \text{ such that for all } \lambda \in \Sigma_{\pi-\psi_A},\ \mu \in \Sigma_{\pi-\psi_B} \\ \|A(\lambda + A)^{-1}[A^{-1}(\mu + B)^{-1} - (\mu + B)^{-1}A^{-1}]\| \le c/(1 + |\lambda|)^{1-\alpha}|\mu|^{1+\beta}. \end{cases} \tag{1.3}$$

Assuming this condition we have the following generalization of a result by Kalton-Weis [3] on sums of operators to the non-commuting case, see [7].

Theorem 1.1. *Suppose $A \in \mathcal{H}^\infty(X)$, $B \in \mathcal{R}S(X)$ and suppose that (1.3) holds for some angles $\psi_A > \phi^\infty_A$, $\psi_B > \phi^R_B$ with $\psi_A + \psi_B < \pi$.*
Then there is a number $\omega_0 \ge 0$ such that $\omega_0 + A + B$ is invertible and sectorial with angle $\phi_{\omega_0+A+B} \le \max\{\psi_A, \psi_B\}$. Moreover, if in addition $B \in \mathcal{R}H^\infty(X)$ and $\psi_B > \phi^{R\infty}_B$, then $\omega_0 + A + B \in \mathcal{H}^\infty(X)$ and $\phi^\infty_{\omega_0+A+B} \le \max\{\psi_A, \psi_B\}$.

2. Parametric symbols

In this section we consider the parametric problem

$$\nu e^{sx} u + h(\partial_x) u = f, \tag{2.1}$$

where $f \in L_p(\mathbb{R})$ for $1 < p < \infty$, $\nu \in \Sigma_\theta$, $s \in \mathbb{R}$, $s \neq 0$, and $h(\partial_x)$ is a pseudo-differential operator whose symbol h belongs to the class $\mathcal{M}^r_{a,b}$ defined below. We study the unique solvability of (2.1) in $L_p(\mathbb{R})$ with optimal regularity. This means that we are looking for a unique solution u of (2.1) such that $e^{sx}u \in L_p(\mathbb{R})$ and $u \in H^r_p(\mathbb{R})$, where $r \in \mathbb{R}$ denotes the order of the symbol $h(z)$. It is an important objective to obtain estimates for the solutions that are uniform in $\nu \in \Sigma_\theta$.

We introduce now the class of symbols. For this purpose we consider the vertical strip

$$S_{(a,b)} = \{z \in \mathbb{C} : \ a < \operatorname{Re} z < b\} \quad \text{where} \quad 0 \in (a,b).$$

Definition 2.1. Let $r \geq 1$ be a fixed number.
Then h is said to belong to the class $\mathcal{M}^r_{a,b}$ if

(i) $h(z)$ is a meromorphic function defined on the strip $S_{(a,b)}$,
(ii) $h(z)/|z|^r \to 1$ as $|z| \to \infty$, $z \in S_{(a,b)}$,
(iii) there are constants $C, N > 0$ such that

$$|zh'(z)| \leq C(1 + |z|^r), \quad z \in S_{(a,b)}, \quad |z| \geq N,$$

(iv) h has no poles on the line $i\mathbb{R}$,
(v) there exists a number $c_0 > 0$ such that $\operatorname{Re} h(i\xi) \geq c_0$ for all $\xi \in \mathbb{R}$.

Remark 2.2. The following properties are easy consequences of Definition 2.1.

(a) Suppose h satisfies (i)–(ii) in Definition 2.1. Then h has only finitely many poles in $S_{(a,b)}$.
(b) Suppose h satisfies (i)–(ii) and (iv)–(v) in Definition 2.1. Let

$$\theta_h := \sup\{|\arg h(i\xi)| : \xi \in \mathbb{R}\}.$$

Then $\theta_h < \pi/2$.

In the next proposition, we study some mapping properties of $h(\partial_x)$ and we derive an expression for the commutator $[e^{sx}, h(\partial_x)]$.

Proposition 2.3. *Let $r > 0$ and $1 < p < \infty$. Suppose $0, -s \in (a,b)$ and suppose that*

(i) *$g : S_{(a,b)} \to \mathbb{C}$ is meromorphic,*
(ii) *there are positive constants C and N such that*

$$|g(z)| + |zg'(z)| \leq C(1 + |z|^r), \quad z \in S_{(a,b)}, \ |z| \geq N,$$

(iii) *g has no poles on the lines $i\mathbb{R}$ and $i\mathbb{R} - s$.*

Let $g(\partial_x)$ and $g(\partial_x - s)$ be the pseudo-differential operators defined by

$$g(\partial_x)u := \mathcal{F}^{-1}(g(i\xi)\mathcal{F}u), \quad g(\partial_x - s)u := \mathcal{F}^{-1}(g(i\xi - s)\mathcal{F}u), \quad u \in \mathcal{S}(\mathbb{R}),$$

respectively, where $\mathcal{F}$ denotes the Fourier transform, and $\mathcal{S}(\mathbb{R})$ is the Schwartz space of rapidly decaying functions. Then

(a) *the operators $g(\partial_x)$ and $g(\partial_x - s)$ are well defined and*

$$g(\partial_x), \ g(\partial_x - s) \in \mathcal{B}(H^r_p(\mathbb{R}), L_p(\mathbb{R})).$$

(b) *For any function $v \in H^r_p(\mathbb{R})$ such that $e^{sx}v \in H^r_p(\mathbb{R})$ we have the identity*

$$e^{sx}g(\partial_x)v(x) = g(\partial_x - s)e^{sx}v(x) + e^{sx}\sum_k \int_{\mathbb{R}} p_k(x-y)e^{z_k(x-y)}v(y)dy,$$

for $x \in \mathbb{R}$, where z_k denote the finitely many poles with order m_k of g in the strip $S_{(-s,0)}$ and $p_k(x)$ are polynomials of order $m_k - 1$.

Proof. (a) Let m_σ be defined by $m_\sigma(\xi) = h(i\xi - \sigma)/(1 + |\xi|^2)^{r/2}$ for $\xi \in \mathbb{R}$ and $\sigma = 0, s$. It is not difficult to see that m_σ satisfies $\sup_{\xi\in\mathbb{R}}(|m_\sigma(\xi)| + |\xi m'_\sigma(\xi)|) < \infty$, and the assertion follows from Mikhlin's multiplier theorem.

(b) Let $v \in \mathcal{D}(\mathbb{R})$ be a test function. Then by definition of the pseudo-differential operator $g(\partial_x)$ we have

$$g(\partial_x)v(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi} g(i\xi)\mathcal{F}v(\xi)d\xi, \quad x \in \mathbb{R}.$$

Note that by assumption (ii) there are only finitely many poles z_k in the strip $S_{(-s,0)}$. Multiplying with e^{sx} and applying the residue theorem yields

$$\begin{aligned} e^{sx}g(\partial_x)v(x) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{(s+i\xi)x} g(i\xi)\mathcal{F}v(\xi)d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x} g(i\xi - s)\mathcal{F}v(\xi + is)d\xi + e^{sx}\sum_k \mathrm{Res}[e^{zx}g(z)\mathcal{F}v(-iz)]_{z=z_k} \\ &= g(\partial_x - s)e^{sx}v(x) + e^{sx}\sum_k \int_{\mathbb{R}} e^{z_k(x-y)} p_k(x-y)v(y)dy, \end{aligned}$$

where the $p_k(x)$ are polynomials of order $m_k - 1$ corresponding to the order m_k of the pole of $g(z)$ at $z = z_k$. The assertion now follows from an approximation argument. □

Next we state a result on kernel bounds for $h(\partial_x)^{-1}$ which is also of independent interest.

Proposition 2.4. *Suppose $r \geq 1$ and*

(i) *$h : S_{(-d,d)} \to \mathbb{C}$ is holomorphic for some $d > 0$,*
(ii) *there are positive constants c, C such that*

$$|h(z)| \geq c(|z|^r + 1) \quad \text{and} \quad |h(z)| + |zh'(z)| \leq C(1 + |z|^r), \; z \in S_{(-d,d)}.$$

Then

(a) *the operator $h(\partial_x)$ is well defined and*

$$h(\partial_x) \in \mathit{Isom}(H^r_p(\mathbb{R}), L_p(\mathbb{R})).$$

(b) *$h(\partial_x)^{-1}$ is a convolution operator with kernel k, where $e^{\delta|\cdot|}k \in L_1(\mathbb{R})$ for some $\delta > 0$.*

Proof. (a) Mikhlin's theorem implies that $h(\partial_x)$ is a well-defined invertible operator with domain $H^r_p(\mathbb{R})$.

(b) The kernel of $h(\partial_x)^{-1}$ is given by the inverse Fourier transform of $h(i\xi)^{-1}$, i.e.,

$$k(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x}\frac{d\xi}{h(i\xi)}, \quad x \in \mathbb{R}.$$

Shifting the path of integration by $2\delta < d$ to the left or to the right, we obtain by Cauchy's theorem

$$e^{\pm 2\delta x} k(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} \frac{d\xi}{h(i\xi \mp 2\delta)}, \quad x \in \mathbb{R}.$$

Plancherel's theorem then yields $e^{2\delta|x|}k \in L_2(\mathbb{R})$. Using that $e^{\delta|x|}k = e^{-\delta|x|}e^{2\delta|x|}k$ we obtain from Hölder's inequality that $e^{\delta|x|}k \in L_1(\mathbb{R})$. $\square$

We will now state our main result for problem (2.1). Before doing so, we introduce the following spaces

$$\begin{aligned} X_0 &:= L_p(\mathbb{R}), \\ X_1 &:= H_p^r(\mathbb{R}) \cap \{v \in L_p(\mathbb{R}) \ : e^{sx} v \in L_p(\mathbb{R})). \end{aligned} \tag{2.2}$$

Theorem 2.5. *Let $1 < p < \infty$, $r \geq 1$, and $a, b \in \mathbb{R}$ with $0, -s \in (a, b)$. Suppose the symbol h belongs to the class $\mathcal{M}_{a,b}^r$ and let θ_h be as in Remark 2.2. Then*

(a) $(\nu e^{sx} + h(\partial_x)) \in \mathit{Isom}\,(X_1, X_0)$ *for each* $\nu \in \Sigma_{\pi-\theta_h}$.

(b) *For each $\theta > \theta_h$ there is a positive number M_θ such that*

$$\|(\nu e^{sx} + h(\partial_x))^{-1}\|_{\mathcal{B}(L_p, H_p^r)} + \|\nu e^{sx}(\nu e^{sx} + h(\partial_x))^{-1}\|_{\mathcal{B}(L_p)} \leq M_\theta, \tag{2.3}$$

for every $\nu \in \Sigma_{\pi-\theta}$.

(c) $(\nu e^{sx} + h(\partial_x)) \in \mathcal{H}^\infty(L_p(\mathbb{R}))$ *for each* $\nu \in \Sigma_{\pi-\theta_h}$.

Proof. (1) Let $\theta > \theta_h$ be fixed and choose $\nu \in \Sigma_{\pi-\theta}$. Let A be the operator in $X_0 = L_p(\mathbb{R})$ defined by means of $(Au)(x) = \nu e^{sx} u(x),\ x \in \mathbb{R}$, for

$$u \in D(A) = \{u \in L_p(\mathbb{R}) : \ e^{sx} u \in L_p(\mathbb{R})\}.$$

A is a multiplication operator, hence it is sectorial and admits a bounded $\mathcal{H}^\infty$-calculus with angle $\phi_A^\infty = \phi_A = |\arg \nu| \leq \pi - \theta$. Next we introduce the operator B in X_0 given by

$$Bu = h(\partial_x)u, \quad u \in D(B) = H_p^r(\mathbb{R}).$$

As in the proofs of Proposition 2.3 and Proposition 2.4 we obtain from Mikhlin's theorem that B is well defined, invertible, sectorial, and admits a bounded $\mathcal{H}^\infty$-calculus with angle $\phi_B^\infty = \theta_h$.

We would now like to apply Theorem 1.1 to the sum $A + B$. For this purpose we have to check the commutator condition (1.3). In order to do so, it turns out to be convenient to first remove the poles of h in the strip $\bar{S}_{(-s,0)}$, decomposing h as $h = h_1 + h_2$, where h_1 is holomorphic in $S_{(-s-\varepsilon,\varepsilon)}$ and h_2 is rational and bounded at infinity. By adding a sufficiently large constant to h_1 (and subtracting it off from h_2) we can assume that $\operatorname{Re} h_1(i\xi - \sigma) \geq c_0 > 0$ for all $\sigma \in [0, s]$, and also that $\theta_{h_1} \leq \theta_h$. Therefore, the operators $h_1(\partial_x)$ and $h_1(\partial_x - s)$ have the same properties as B. In particular, the parabolicity condition $\phi_A^\infty + \phi_{h_1(\partial_x)}^\infty \leq \pi - \theta + \theta_h < \pi$ is satisfied. For $\eta > 0$ fixed we obtain from Proposition 2.3(b), with $g = (\mu + h_1)^{-1}$ and $(a, b) = (-s - \varepsilon, \varepsilon)$, that

$$\begin{aligned}(\eta + A)(\lambda + \eta + A)^{-1}&[(\eta + A)^{-1}, (\mu + h_1(\partial_x))^{-1}]\\ &= (\lambda + \eta + A)^{-1}[(\mu + h_1(\partial_x))^{-1}, A](\eta + A)^{-1}\\ &= -(\lambda + \eta + A)^{-1}(\mu + h_1(\partial_x - s))^{-1}[h_1(\partial_x) - h_1(\partial_x - s)]\\ &\qquad\qquad \cdot (\mu + h_1(\partial_x))^{-1}A(\eta + A)^{-1}.\end{aligned}$$

Since $|h_1(i\xi) - h_1(i\xi - s)| \sim |\xi|^{r-1}$ we see that the function m defined by

$$m(\xi) := \frac{h_1(i\xi) - h_1(i\xi - s)}{(1 + \xi^2)^{(r-\delta)/2}}$$

belongs to $L_2(\mathbb{R})$, and also that $m' \in L_2(\mathbb{R})$ for each $\delta \in (0, 1/2)$. This implies that m is the Fourier transform of an L_1-function and it follows that

$$(h_1(\partial_x) - h_1(\partial_x - s)) \in \mathcal{B}(H_p^{r-\delta}(\mathbb{R}), L_p(\mathbb{R})).$$

Hence we obtain the estimate

$$\begin{aligned}&\|(\eta + A)(\lambda + \eta + A)^{-1}[(\eta + A)^{-1}, (\mu + h_1(\partial_x))^{-1}]\|\\ &\le C(|\lambda| + \eta)^{-1}|\mu|^{-1}\|h_1(\partial_x) - h_1(\partial_x - s)\|_{\mathcal{B}(H_p^{r-\delta}, L_p)}\|(\mu + h_1(\partial_x))^{-1}\|_{\mathcal{B}(L_p, H_p^{r-\delta})}\\ &\le C_\eta(1 + |\lambda|)^{-1}|\mu|^{-(1+\delta/r)},\end{aligned}$$

and (1.3) holds with $\alpha = 0$, $\beta = \delta/r$, and $\psi_A > \phi_A$, $\psi_B > \phi_B$ such that $\psi_A + \psi_B < \pi$. Thus by Theorem 1.1 and [7, Remark 2.1] there is a sufficiently large ω_0 such that $\omega_0 + A + h_1(\partial_x)$ is invertible, sectorial, and belongs to $\mathcal{H}^\infty(X_0)$ with angle less than $\max\{\psi_A, \psi_B\}$. Since $h_2(\partial_x)$ is bounded, the same results hold for

$$\omega_0 + A + B = \omega_0 + A + h_1(\partial_x) + h_2(\partial_x),$$

possibly at the expense of enlarging ω_0. This implies in particular that $A + B$ with domain

$$D(A + B) = D(A) \cap D(B) = X_1$$

is closed.

In the remaining part of the proof we want to remove ω_0 by means of a Fredholm type argument. Suppose we know that $\omega + A + B$ is injective and has closed range for all $\omega \in [0, \omega_0]$. Then these operators are semi-Fredholm, hence their index is well defined and constant, by the well-known result on the continuity of the Fredholm index. Now, for $\omega = \omega_0$ this index is zero since $\omega + A + B$ is bijective as proved above. Then it must be zero for all $\omega \in [0, \omega_0]$, hence the operators $\omega + A + B$ must also be surjective since they are injective. We can then conclude from [2, Proposition 2.7] that $A + B$ is sectorial and admits a bounded $\mathcal{H}^\infty$-calculus as well.

(2) Let us first consider the easiest case $p = 2$. Suppose $u \in D(A) \cap D(B)$ satisfies

$$\nu e^{sx}u + \omega u + h(\partial_x)u = f.$$

Taking the inner product with u in $L_2(\mathbb{R})$ yields

$$\nu\|e^{sx/2}u\|_2^2 + \omega\|u\|_2^2 + (h(\partial_x)u|u) = (f|u).$$

By means of Plancherel's theorem we have

$$(h(\partial_x)u|u) = (\mathcal{F}(h(\partial_x)u)|\mathcal{F}u) = (h(i\xi)\mathcal{F}u|\mathcal{F}u),$$

and by taking real parts we obtain

$$c_0\|u\|_2^2 \leq \mathrm{Re}\,(h(\partial_x)u|u) \leq \|f\|_2\|u\|_2,$$

provided $\mathrm{Re}\,\nu \geq 0$. This implies the a priori bound

$$\|u\|_2 \leq c_0^{-1}\|f\|_2,$$

which is independent of $\omega \geq 0$ and $\mathrm{Re}\,\nu \geq 0$, i.e., injectivity and closed range of $\omega + A + B$ follow. In the case of a general angle $\theta > \theta_h$ we set $\rho = \tan\theta_h$. Then

$$|\mathrm{Im}\,h(i\xi)| \leq \rho\mathrm{Re}\,h(i\xi), \quad \xi \in \mathbb{R}.$$

Taking real parts we get this time

$$\mathrm{Re}\,\nu\|e^{sx/2}u\|_2^2 + \omega\|u\|_2^2 + \int_{\mathbb{R}} \mathrm{Re}\,h(i\xi)|\mathcal{F}u|^2\,d\xi \leq \|f\|_2\|u\|_2,$$

and taking imaginary parts we obtain

$$|\mathrm{Im}\,\nu|\,\|e^{sx/2}u\|_2^2 - \int_{\mathbb{R}} |\mathrm{Im}\,h(i\xi)||\mathcal{F}u|^2\,d\xi \leq \|f\|_2\|u\|_2.$$

Thus

$$(|\mathrm{Im}\nu| + (\varepsilon+\rho)\mathrm{Re}\nu)\|e^{sx/2}u\|_2^2 + \int_{\mathbb{R}} \big((\varepsilon+\rho)\mathrm{Re}h(i\xi) - |\mathrm{Im}h(i\xi)|\big)|\mathcal{F}u|^2 d\xi \leq c\|f\|_2\|u\|_2.$$

For $|\mathrm{Im}\,\nu| + (\varepsilon+\rho)\mathrm{Re}\,\nu \geq 0$ we may now conclude that

$$\|u\|_2 \leq (1+\rho+\varepsilon)/c_0\varepsilon)\|f\|_2.$$

The assumptions $|\arg\nu| \leq \pi - \theta$ and $\theta > \theta_h$ allow for such a choice of $\varepsilon > 0$. Hence in any case we have shown that $\omega + A + B$ is injective and has closed range for all $\omega \geq 0$, which completes the proof of the theorem for $p = 2$.

(3) We next prove injectivity for all $p \in (1,\infty)$. Suppose $u \in X_1$ satisfies

$$\nu e^{sx}u + \omega u + h(\partial_x)u = 0.$$

Then $u, e^{sx}u \in L_p(\mathbb{R})$ implies that $e^{\sigma x}u \in L_p(\mathbb{R})$ for all $\sigma \in [0,s]$. But this gives

$$e^{-\varepsilon x}u = -e^{-\varepsilon x}(\omega + h(\partial_x))^{-1}\nu e^{sx}u = -(\omega + h(\partial_x+\varepsilon))^{-1}\nu e^{(s-\varepsilon)x}u,$$

where $\varepsilon > 0$ is so small that $\mathrm{Re}\,h(i\xi+\sigma) \geq c_0/2$ for all $\xi \in \mathbb{R}$ and $0 \leq \sigma \leq \varepsilon$. It follows that $e^{\sigma x}u \in L_p(\mathbb{R})$ for all $\sigma \in [-\varepsilon, s]$. Using the Sobolev embedding $H_p^r(\mathbb{R}) \hookrightarrow C_0(\mathbb{R})$ and Hölder's inequality we get

$$\int_{\mathbb{R}} |u|^2\,dx \leq \|u\|_\infty\Big(\int_{\mathbb{R}} e^{-\varepsilon p'|x|}\,dx\Big)^{1/p'}\Big(\int_{\mathbb{R}} e^{\varepsilon p|x|}|u|^p\,dx\Big)^{1/p}$$

and we conclude that $u \in L_2(\mathbb{R})$. Uniqueness in $L_2(\mathbb{R})$ now implies $u = 0$, i.e., $\omega + A + B$ is injective in $L_p(\mathbb{R})$ for all $\omega \geq 0$.

(4) Closedness of the ranges is more involved for $p \neq 2$ since we cannot refer to Parseval's theorem. Moreover, B will in general not be accretive in $L_p(\mathbb{R})$. So assume to the contrary that $R(\omega + A + B)$ is not closed in $L_p(\mathbb{R})$, for some $\omega \geq 0$. Then there is a sequence $(u_n) \subset D(A) \cap D(B)$ with

$$\|u_n\|_p = 1 \text{ and } f_n := (\omega + A + B)u_n \to 0 \text{ in } L_p(\mathbb{R}) \text{ as } n \to \infty.$$

Since $\omega_0 + A + B$ is invertible by step (1) this implies that u_n is bounded in $H_p^r(\mathbb{R})$ and $e^{sx}u$ is bounded in $L_p(\mathbb{R})$. By reflexivity of these spaces there exists a function $u \in H_p^r(\mathbb{R}) \cap L_p(\mathbb{R}, e^{psx}dx)$ and a subsequence (w.l.o.g. the full sequence) such that

$$u_n \rightharpoonup u \ \text{ in } H_p^r(\mathbb{R}), \quad Bu_n \rightharpoonup Bu \ \text{ in } L_p(\mathbb{R}) \quad \text{and } e^{sx}u_n \rightharpoonup e^{sx}u \ \text{ in } L_p(\mathbb{R}).$$

The function u then satisfies $\nu e^{sx}u + \omega u + h(\partial_x)u = 0$. Hence $u = 0$ by the uniqueness result proved in the previous step.

We want to show $u_n \to 0$ in $L_p(\mathbb{R})$ which gives a contradiction to $\|u_n\|_p = 1$. To achieve this we use the embedding $H_p^r(\mathbb{R}) \hookrightarrow BUC^\alpha(\mathbb{R})$ for $\alpha = r - 1/p > 0$. Since u_n converges weakly to 0 in $L_p(\mathbb{R})$ and is relatively compact in $C(\mathbb{R})$ w.r.t the topology of uniform convergence on compact sets by the Arzela-Ascoli theorem, we may conclude that $u_n \to 0$ locally uniformly. Let $a \in \mathbb{R}$ be a fixed number. Then given any $\varepsilon > 0$ there exists numbers $b > a$ and $k \in \mathbb{N}$ such that for any $n \geq k$

$$\begin{aligned}\int_a^\infty |u_n|^p dx &\leq e^{-sbp} \int_b^\infty |u_n e^{sx}|^p dx + \int_a^b |u_n|^p dx \\ &\leq e^{-sbp} \sup_n |u_n e^{sx}|_p^p + (b-a) \sup\{|u_n(x)|^p : \ x \in [a,b], \ n \geq k\} \leq \varepsilon.\end{aligned}$$

Hence $\int_a^\infty |u_n|^p dx \to 0$ as $n \to \infty$ for each $a \in \mathbb{R}$.

We will now apply Proposition 2.4 to $\omega + h(z)$ and we find that its inverse has a kernel k such that $e^{\delta|x|}k \in L_1(\mathbb{R})$ for $\delta > 0$ sufficiently small. This yields

$$u_n = (\omega + h(\partial_x))^{-1}(f_n - \nu e^{sx}u_n) = k * f_n - k * \nu e^{sx}u_n =: k * f_n - v_n.$$

Observe that $e^{-\delta x}v_n = (e^{-\delta x}k) * (\nu e^{(s-\delta)x}u_n)$. Since $e^{(s-\delta)x}u_n$ is uniformly bounded in $L_p(\mathbb{R})$ with respect to n and $e^{-\delta x}k \in L_1(\mathbb{R})$ we conclude that $e^{-\delta x}v_n$ is also uniformly bounded in $L_p(\mathbb{R})$. Let $\varepsilon > 0$ be given. Then we can find numbers $a \in \mathbb{R}$ and $k \in \mathbb{N}$ such that

$$\begin{aligned}(\int_{-\infty}^a |u_n|^p dx)^{1/p} &\leq \|k\|_1 \|f_n\|_p + e^{\delta a}(\int_{-\infty}^a |e^{-\delta x}v_n|^p dx)^{1/p} \\ &\leq \|k\|_1 \|f_n\|_p + e^{\delta a}\|e^{-\delta x}v_n\|_p \leq \varepsilon\end{aligned}$$

whenever $n \geq k$. (This can be done by choosing first a sufficiently negative and then k sufficiently large.) Hence $u_n \to 0$ in $L_p(\mathbb{R})$, and so the range of $\omega + A + B$ must be closed for each $\omega \geq 0$.

(5) Finally we prove the estimate (2.3) by a scaling argument. Let τ_a denote the translation group on $L_p(\mathbb{R})$, i.e., $(\tau_a v)(x) = v(x+a)$, and observe that $h(\partial_x)$ commutes with this group. Then with $a = \frac{1}{s}\ln|\nu|$ and $\vartheta = \arg\nu$ we have

$$\nu e^{sx}u(x) + h(\partial_x)u(x) = f(x), \quad x \in \mathbb{R},$$

if and only if

$$e^{i\vartheta}e^{sx}\tau_{-a}u + h(\partial_x)\tau_{-a}u = \tau_{-a}f.$$

Setting $T_\vartheta = e^{i\vartheta}e^{sx}(e^{i\vartheta}e^{sx} + h(\partial_x))^{-1}$ this gives the representation

$$\nu e^{sx}u = \nu e^{sx}(\nu e^{sx} + h(\partial_x))^{-1}f = \tau_a T_\vartheta \tau_{-a} f.$$

The family $\{T_\vartheta\}_{\vartheta\in[-\theta,\theta]} \subset \mathcal{B}(L_p(\mathbb{R}))$ is continuous in ϑ, hence uniformly bounded. Since the translations are isometries on $L_p(\mathbb{R})$ we obtain the estimate

$$\|\nu e^{sx}(\nu e^{sx} + h(\partial_x))^{-1}\|_{\mathcal{B}(L_p(\mathbb{R}))} \le \sup_{|\vartheta|\le\theta} \|T_\vartheta\|_{\mathcal{B}(L_p(\mathbb{R}))} < \infty. \tag{2.4}$$

This proves estimate (2.3) since $h(\partial_x) \in \mathcal{B}(H_p^r(\mathbb{R}), L_p(\mathbb{R}))$ is an isomorphism. $\square$

3. The evolution equation

By means of the transformation $v(x) = e^{sx}u(x)$, (2.1) is equivalent to the parametric problem

$$\nu v + h(\partial_x)e^{-sx}v = f. \tag{3.1}$$

We thus consider the new operator C on $X = L_p(\mathbb{R})$ given by

$$Cv = h(\partial_x)(e^{-sx}v), \quad v \in \mathsf{D}(C) = \{v \in L_p(\mathbb{R}) : \ e^{-sx}v \in H_p^r(\mathbb{R})\}. \tag{3.2}$$

We have the following result.

Theorem 3.1. *Let the assumptions of Theorem* 2.5 *be satisfied. Then C is sectorial with $\phi_C = \theta_h < \pi/2$. Hence $-C$ is the generator of a bounded analytic C_0-semigroup on X.*

Proof. It is clear that C is densely defined, since $\mathcal{D}(\mathbb{R}) \subset \mathsf{D}(C)$. Observing that

$$\nu(\nu + C)^{-1} = \nu e^{sx}(\nu e^{sx} + h(\partial_x))^{-1}, \quad \nu \in \Sigma_{\pi-\theta_h}, \tag{3.3}$$

it follows from Theorem 2.5(b) that C is sectorial with angle θ_h. This shows that $-C$ is the generator of a bounded analytic C_0-semigroup in $X = L_p(\mathbb{R})$. The ergodic theorem $X = N(C) \oplus \overline{R(C)}$ shows also that the range of C is dense in X since obviously C is injective. $\square$

We pose the question whether the Cauchy problem

$$\dot{v} + Cv = f, \quad v(0) = 0, \tag{3.4}$$

has maximal L_p-regularity. This is not clear from Theorem 2.5, but the first step of its proof shows that (3.4) has in fact maximal L_p-regularity if C is replaced by $C_\omega = (\omega + h(\partial_x))e^{-sx}$ with $\omega \ge \omega_0$, where ω_0 is an appropriate nonnegative number. It is an interesting open question whether ω_0 can be chosen to be 0.

Due to the transformation $u(t,x) = e^{-sx}v(t,x)$ is is clear that every solution of the Cauchy problem (3.4) is also a solution of the following degenerate Cauchy-problem

$$\begin{aligned} e^{sx}\partial_t u(t,x) + h(\partial_x)u(t,x) = f(t,x), \quad t > 0,\ x \in \mathbb{R}, \\ u(0,x) = 0. \end{aligned} \tag{3.5}$$

Thanks to Theorem 3.1 we know that problem (3.4) admits a unique solution v for an appropriate function f, and hence, problem (3.5) also admits a unique solution (whose regularity properties can be deduced from the regularity properties of v via the transformation $u = e^{-sx}v$).

It is an open problem whether or not (3.5) has maximal regularity. In that direction, we can only prove the following weaker result.

Proposition 3.2. *Let the assumptions of Theorem* 2.5 *be satisfied. Then there exists a non-negative number* ω_0 *such that*

$$\begin{aligned} e^{sx}\partial_t u(t,x) + \omega u(t,x) + h(\partial_x)u(t,x) = f(t,x), \quad t > 0,\ x \in \mathbb{R}, \\ u(0,x) = 0, \end{aligned} \tag{3.6}$$

admits a unique solution u *with maximal* L_p*-regularity for every* $\omega \geq \omega_0$*. That is, for each* $f \in L_p(J \times \mathbb{R})$*, problem* (3.6) *admits a unique solution* $u \in L_p(J, H_p^r(\mathbb{R}))$ *such that* $e^{sx}\partial_t u \in L_p(J \times \mathbb{R})$ *where* $J = (0,T)$*. There is a constant* $M = M_\omega > 0$*, independent of* f*, such that*

$$\|e^{sx}\partial_t u\|_{L_p(J\times\mathbb{R})} + \|u\|_{L_p(J,H_p^r(\mathbb{R}))} \leq M\|f\|_{L_p(J\times\mathbb{R})}.$$

Moreover, the operator $L = \partial_t e^{sx} + \omega + h(\partial_x)$ *admits a bounded* $\mathcal{H}^\infty$*-calculus on* $L_p(J \times \mathbb{R})$ *for* $\omega \geq \omega_0$*.*

Proof. Repeating step (1) of the proof of Theorem 2.5 in $L_p(J\times\mathbb{R}) = L_p(J, L_p(\mathbb{R}))$ with A replaced by $A = \partial_t e^{sx}$, we obtain a number ω_0 such that the operator

$$\omega_0 + \partial_t e^{sx} + h(\partial_x),$$

with natural domain, is invertible and admits a bounded $\mathcal{H}^\infty$-calculus. Propositions 1.3.(iv) and 2.7 in [2] imply that this is also true for any $\omega \geq \omega_0$. □

On the other hand, we do obtain maximal L_p-regularity for problem (3.5) in case that $X = L_2(\mathbb{R})$. This is the statement of the next theorem.

Theorem 3.3. *Let the assumptions of Theorem* 2.5 *be satisfied. Then for each* $f \in L_p(J, L_2(\mathbb{R}))$*, problem* (3.6) *admits a unique solution* $u \in L_p(J, H_2^r(\mathbb{R}))$ *such that* $e^{sx}\partial_t u \in L_p(J, L_2(\mathbb{R}))$*. There is a constant* $M > 0$*, independent of* f*, such that*

$$\|e^{sx}\partial_t u\|_{L_p(J,L_2(\mathbb{R}))} + \|u\|_{L_p(J,H_2^r(\mathbb{R}))} \leq M\|f\|_{L_p(J,L_2(\mathbb{R}))}.$$

The operator $L = \partial_t e^{sx} + h(\partial_x)$ *admits a bounded* $\mathcal{H}^\infty$*-calculus on* $L_p(J, L_2(\mathbb{R}))$*.*

Proof. Let $X = L_2(\mathbb{R})$. According to Theorem 3.1 we know that the operator C is sectorial with $\phi_C = \theta_h$. Since X is a Hilbert space, we have that C is, in addition, also $\mathcal{R}$-sectorial with $\phi_C^R = \theta_h$, see for instance [2, Remark 3.2.(3)]. This implies that the Cauchy problem (3.4) has maximal L_p-regularity, see for instance [2, Theorem 4.4]. Since $\omega + h(z)$ satisfies the same assumptions as $h(z)$ for each $\omega \geq 0$ we deduce that the Cauchy problem (3.4) with C replaced by C_ω also has maximal L_p-regularity. That is, for each $f \in L_p(J, X)$, with $X = L_2(\mathbb{R})$, there is a unique solution $v \in H_p^1(J, X)$ of (3.4), and there is a positive constant $M = M(\omega)$ independent of f such that

$$\|\dot{v}\|_{L_p(J,X)} + \|C_\omega v\|_{L_p(J,X)} \leq M\|f\|_{L_p(J,X)}, \quad f \in L_p(J, X).$$

Going to (3.6) via the transformation $u = e^{-sx}v$ yields a unique solution of (3.6) and the estimate

$$\|e^{sx}\partial_t u\|_{L_p(J,X)} + \|(\omega + h(\partial_x))u\|_{L_p(J,X)} \leq M\|f\|_{L_p(J,X)}, \quad f \in L_p(J, X).$$

Since $\omega + h(\partial_x) \in \mathcal{B}(H_p^r(\mathbb{R}), L_p(\mathbb{R}))$ is an isomorphism for each $\omega \geq 0$, this yields invertibility of the operators $\omega + \partial_t e^{sx} + h(\partial_x)$ on $L_p(J, L_2(\mathbb{R}))$ with natural domain, for each $\omega \geq 0$. As in Theorem 3.1 we obtain that there is a number $\omega_0 \geq 0$ such that $\omega_0 + \partial_t e^{sx} + h(\partial_x)$ admits a bounded $\mathcal{H}^\infty$-calculus on $L_p(J, L_2(\mathbb{R}))$. Using [2, Proposition 2.7] we conclude that $\partial_t e^{sx} + h(\partial_x)$ is invertible, sectorial and admits a bounded $\mathcal{H}^\infty$-calculus on $L_p(J, L_2(\mathbb{R}))$, and this completes the proof. □

4. Examples

In this section we discuss some of the examples introduced in Section 1.

(i) We first consider the symbol of the Laplace equation in an angle $G = \{(r\cos\phi, r\sin\phi) : r > 0,\ \phi \in (0, \alpha)\}$ with homogeneous Dirichlet condition on $\phi = \alpha$ and dynamic boundary condition $\partial_t u + \partial_\nu u = g$ on $\phi = 0$. Then we obtain a problem of the form (1.1) with $s = 1$ and

$$h_\beta(z) = \psi_0(-(z+\beta)^2) \quad \text{with} \quad \psi_0(\zeta) = \sqrt{\zeta}\coth(\alpha\sqrt{\zeta}).$$

Since the function $\coth(\zeta)$ is odd, ψ_0 is a meromorphic function on $\mathbb{C}$ with poles in $\{\zeta_k = -r_k^2 = -k^2(\pi/\alpha)^2 : k \in \mathbb{N}\}$. Since $\coth\zeta \to 1$ for $|\zeta| \to \infty$, $|\arg(\zeta)| \leq \theta < \pi/2$, it is easy to see that $h_\beta(z)/|z| \to 1$ as $|z| \to \infty$, in any strip $S_{(a,b)}$. In particular $r = 1$ and h satisfies (i)–(iii) of Definition 2.1 in $S_{(a,b)}$ for all $a < b$. Next we determine the values of β which are admissible. The parabola $P_\beta = \{-(i\xi+\beta)^2 : \xi \in \mathbb{R}\}$ passes through a pole of ψ_0 if and only if $|\beta| = r_k$ for some $k \in \mathbb{N}$. Therefore Definition 2.1(iv) is satisfied if and only if $|\beta| \neq r_k$ for all $k \in \mathbb{N}$. To check Definition 2.1(v), we compute the real part of $h_\beta(i\xi)$, to the result

$$\operatorname{Re} h_\beta(i\xi) = \frac{|\xi|\sinh(2\alpha|\xi|) + \beta\sin(2\alpha\beta)}{\cosh(2\alpha|\xi|) - \cos(2\alpha\beta)}.$$

This shows that the real part of $h_\beta(i\xi)$ is strictly positive for all values of $\xi \in \mathbb{R}$ if and only if $\operatorname{Re} h_\beta(0) > 0$, which in turn is equivalent to $|\beta|\cot(\alpha|\beta|) > 0$. This yields the range

$$|\beta| \in [0, \pi/2\alpha) \bigcup_{k\geq 1} (k\pi/\alpha, (k+1/2)\pi/\alpha).$$

(ii) If in (i) we change the Dirichlet condition on $\phi = \alpha$ into a Neumann condition then the function h becomes

$$h_\beta(z) = \psi_1(-(z+\beta)^2) \quad \text{with} \quad \psi_1(\zeta) = \sqrt{\zeta}\tanh(\alpha\sqrt{\zeta}).$$

Here we have again a meromorphic function, $s = r = 1$, but the poles are this time in $\{\zeta_k = -s_k^2 = -(2k+1)^2(\pi/2\alpha)^2 : k \in \mathbb{N}_0\}$. The admissible values of β then are $|\beta| \neq s_k$ for $k \in \mathbb{N}_0$. For the real part of $h_\beta(i\xi)$ we get

$$\operatorname{Re} h_\beta(i\xi) = \frac{|\xi|\sinh(2\alpha|\xi|) - \beta\sin(2\alpha\beta)}{\cosh(2\alpha|\xi|) + \cos(2\alpha\beta)}.$$

Thus the real part of $h_\beta(i\xi)$ is in this case strictly positive for all $\xi \in \mathbb{R}$ if and only if $\operatorname{Re} h_\beta(0) > 0$ which in turn is equivalent to $|\beta|\tan(\alpha|\beta|) < 0$. This yields the range $|\beta| \in \cup_{k\geq 1}(k\pi/\alpha, (k+1/2)\pi/\alpha)$.

(iii) We next discuss the symbol of the two-dimensional Mullins-Sekerka problem

$$h_\beta(z) = -\psi_1(-(z+\beta)^2)(z+\beta+1)(z+\beta+2),$$

with ψ_1 as in (ii), where we restrict attention to the physically relevant range $\alpha \in (0, \pi)$. Here again h is meromorphic and we have $s = r = 3$. The poles are the same as in (ii), and for the real part of $h_\beta(i\xi)$ we get the more complicated expression

$$\operatorname{Re} h_\beta(i\xi) = \frac{|\xi|\sinh(2\alpha|\xi|)(\xi^2 - 3\beta(\beta+2) - 2) + (\beta+1)\sin(2\alpha\beta)(\beta(\beta+2) - 3\xi^2)}{2(\sinh^2(\alpha\xi) + \cos^2(\alpha\beta))}.$$

For $\beta > 0$ we set $\xi_0^2 = \beta(\beta+2)/3$ to see that $\xi_0^2 - 3\beta(\beta+2) - 2 < 0$, hence $\operatorname{Re} h_\beta(i\xi_0) < 0$. If $\beta = 0$, then we also have $\operatorname{Re} h_\beta(i\xi) < 0$ for ξ sufficiently small. Thus nonnegative values of β are not admissible, and neither are small negative values of β. On the other hand, if $\beta \leq -2$ then the same choice of ξ_0 shows $\operatorname{Re} h_\beta(i\xi_0) \leq 0$, so that such values of β do also not meet (iv) of Definition 2.1. This shows that the admissible values of β are contained in the interval $(-2, 0)$. Next we look at $h_\beta(0)$ which is

$$h_\beta(0) = |\beta|\tan(\alpha|\beta|)(\beta+1)(\beta+2).$$

There are two distinguished cases, namely $-2 < \beta < -1$ and $-1 < \beta < 0$, as $h_\beta(0) = 0$ for $\beta = -1$. If $-1 < \beta < 0$ we always have the window $-\pi/2\alpha < \beta < 0$. Restricting attention to this range, a sufficient condition for $\operatorname{Re} h_\beta(i\xi) \geq c_0 > 0$ is

$$\max\{-1, -\pi/2\alpha\} < \beta < -1 + 1/\sqrt{3}.$$

In fact, we then have $\sin(2\alpha|\beta|) > 0$ as well as $3|\beta|(\beta+2) - 2 > 0$, which implies $\operatorname{Re} h_\beta(i\xi) > 0$ for all $\xi \in \mathbb{R}$. On the other hand, if ξ^2 is such that the coefficient of $\sinh(2\alpha|\xi|)|\xi|$ is negative, i.e., if $\xi^2 - 3\beta(\beta+2) - 2 < 0$, then we may estimate

$$\begin{aligned}
&\sin(2\alpha\beta)(\beta+1)(-3\xi^2 + \beta(\beta+2)) + \sinh(2\alpha|\xi|)|\xi|(\xi^2 - 3\beta(\beta+2) - 2)\\
&\le 2\alpha|\beta|(1-|\beta|)\big[3\xi^2 + |\beta|(2-|\beta|)\big] + 2\alpha\xi^2\big[\xi^2 + 3|\beta|(2-|\beta|) - 2\big]\\
&= 2\alpha\big[\xi^4 - (2 + 6|\beta|^2 - 9|\beta|)\xi^2 + |\beta|^2(1-|\beta|)(2-|\beta|)\big].
\end{aligned}$$

The last line becomes negative for some value of $\xi^2 > 0$ if and only if

$$|\beta|^2(1-|\beta|)(2-|\beta|) < (1 + 3|\beta|^2 - 9|\beta|/2)^2,$$

which shows that the range $-0.195 \le \beta < 0$ is forbidden. Computations with a computer algebra system suggest that there is an increasing function $\beta^*(\alpha)$ such that the range of well-posedness is given by $-1 < \beta < \beta^*(\alpha)$, and $-0.32 < \beta^*(\alpha) < -0.195$.

(iv) Finally we discuss the symbol of the stationary Stokes problem with boundary contact and prescribed contact angle in the slip case in two dimensions. This symbol reads as

$$h_\beta(z) = \psi(z+\beta) \quad \text{with} \quad \psi(\zeta) = (1+\zeta)\frac{\cos(2\alpha\zeta) - \cos(2\alpha)}{\sin(2\alpha\zeta) + \zeta\sin(2\alpha)}.$$

This symbol is much more complex than those discussed before, and we do not intend to present a complete discussion here. Obviously, $\beta = 0$ leads to a first order pole, hence neither of the intervals $[-\delta, 0]$ and $[0, \delta]$ are admissible. We want to concentrate on a neighborhood of $\beta = 1$. Computing the real part of $\psi(1 + i\xi)$ leads to the expression

$$\operatorname{Re}\psi(1+i\xi) = \frac{(\cosh(2\alpha\xi) - 1)(\xi\sinh(2\alpha\xi) + \xi^2\sin(4\alpha)/2)}{\sin^2(2\alpha)(\cosh(2\alpha\xi)+1)^2 + (\cos(2\alpha)\sinh(2\alpha\xi) + \xi\sin(2\alpha))^2}.$$

This representation of $\operatorname{Re}\psi(1+i\xi)$ shows that it is strictly positive except at $\xi = 0$. Thus $\beta = 1$ is not admissible. We expand the symbol at $(\beta, \xi) = (1, 0)$ to the result

$$h_\beta(i\xi) = 2\alpha(1 - \beta - i\xi) + o(|\beta - 1| + |\xi|).$$

This shows by means of a compactness argument that $\operatorname{Re} h_\beta(i\xi)$ is bounded below for $\xi \in \mathbb{R}$ when β is restricted to an interval $(\beta^*(\alpha), 1)$ with $\beta^*(\alpha) < 1$. This range of β is admissible, i.e., for such numbers β the conditions (iv) and (v) of Definition 2.1 are satisfied.

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An Analysis of Asian options

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Dedicated to the memory of Günter Lumer

Abstract. The objective of this paper is to provide an analytic theory for pricing of Asian options of European type. We present a partial differential equation describing the fair price process of an Asian option. This appears as

$$(\partial_t - A - x \cdot \nabla_y)u = 0$$

and the associated payoff function as the end value. Here the operator A is the d-dimensional Black-Scholes operator, and $B = x \cdot \nabla_y$ represents the path dependence in terms of the price averaging in Asian options. The main result will be to prove, that a solution of this partial differential equation exists, is unique, and depends continuously on the data in appropriate function spaces, i.e., that the problem is well posed. On our way we are going to employ semigroup methods, in particular the *Lumer-Phillips theorem*.

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1. Introduction

Let $J := [0, T]$ and $(\Omega, \mathcal{F}, P)$ a complete probability space with filtration $\{\mathcal{F}_t\}_{t\in J}$; thus

$$\mathcal{F}_0 = \{\emptyset, \Omega\} \cup \mathcal{N} \subset \mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}_T \subset \mathcal{F} \qquad s, t \in J, \quad s < t,$$

and $\mathcal{N} = \{N \in \mathcal{F} : P(N) = 0\}$. We consider a market containing $d + 1$ assets, the 0th being riskless, the remaining d being risky with prices $S_t^i : \Omega \to \mathbb{R}$, $i = 0, \dots, d$, in time $t \in J$. Let $\{S_t\} = \{(S_t^0, \dots, S_t^d)^T\} \in L_2(J \times \Omega; \mathbb{R}^{d+1})$ be the $\{\mathcal{F}_t\}$-adapted vectorial price process. We assume as usual that the asset price processes S_t are driven by stochastic differential equations

$$\begin{cases} \mathrm{d}S_t^0 = r(t)S_t^0\mathrm{d}t, & S_0^0 = 1 \\ \mathrm{d}S_t^i = \mu^i(t)S_t^i\mathrm{d}t + \sum_{j=1}^m \sigma_j^i(t)S_t^i\mathrm{d}B_t^j, & S_0^i > 0 \text{ given} \end{cases} \tag{1.1}$$

where $i = 1, \dots, d$ and $t \in J$.

Here $r \in L_\infty(J)$ is the deterministic rate of interest of the riskless asset at time t, $\mu^i \in L_\infty(J \times \Omega)$ the growth rate of asset i, $\sigma_j^i \in L_\infty(J)$ the variances, also known as volatilities of the market which is assumed to be complete, i.e., the matrix (σ_j^i) is surjective. The process $\{B_t\}_{t\in J}$ denotes a m-dimensional Brownian motion with independent components B_t^j.

Henceforth $\mathcal{F}_t$ is assumed to be the complete σ-algebra, which is induced by the history of the Brownian motion $\{B_s\}_{s<t}$. Moreover we assume that $\{\mu^i(t)\}$ is an adapted process, i.e., $\mu^i(t)$ is $\mathcal{F}_t$-adapted. We define

$$\mu(t) := (\mu^1(t), \dots, \mu^d(t))^T$$

and

$$\sigma(t) := (\sigma_j^i(t))_{i,j}.$$

Our objective is to find a self-financing portfolio strategy θ_t and the fair price for an option at time t, if a function $g \in C((0,T] \times \mathbb{R}_+^d \times \mathbb{R}_+^d)$ is appointed, which replicates g, i.e.,

$$Z := g(T, S_T^*, I(S_T^*)), \quad \text{with} \quad S_t^* := \left(S_t^1, \dots, S_t^d\right)^T$$

and

$$I(S_t^*) := \left(I(S_t^1), \dots, I(S_t^d)\right)^T \quad \text{is defined by } I(S_t^k) := \int_0^t S_\tau^k \mathrm{d}\tau.$$

Subject to these conditions $Y_t := \theta_t \cdot S_t$ is the fair price at time t with end constraint $Y_T = Z$. Our candidates of interest for g are

$$g(t, S_t^*, I(S_t^*)) = \left[\frac{1}{t}I(S_t^k) - K\right]_+ \quad \text{(average price call on asset } k\text{)},$$

$$g(t, S_t^*, I(S_t^*)) = \left[S_t^k - \frac{1}{t}I(S_t^k)\right]_+ \quad \text{(average strike call on asset } k\text{)},$$

$$g(t, S_t^*, I(S_t^*)) = \left[K - \frac{1}{t}I(S_t^k)\right]_+ \quad \text{(average price put on asset } k\text{)},$$

$$g(t, S_t^*, I(S_t^*)) = \left[\frac{1}{t}I(S_t^k) - S_t^k\right]_+ \quad \text{(average strike put on asset } k\text{)},$$

with $k = 1, \dots, d$. We restrict our attention to the "European case" which means that an option can only be executed at expiration time T. Therefore it will suffice to evaluate the payoff function g in $t = T$ and we will write $g(T, x, y) =: g(x, y)$.

In the following section we will construct a $2d+1$-dimensional partial differential equation of elliptic-hyperbolic type, describing the fair price process of an Asian option. Then, in Section 3, the Lumer-Phillips theorem is employed to obtain well-posedness of the Euler transformed problem on the spaces

$$X_\infty := C_0(\mathbb{R}^d \times \mathbb{R}^d_+) := \left\{ u \in C(\mathbb{R}^d \times \mathbb{R}^d_+) : \lim_{|x|+|y|\to\infty} u(x,y) = 0 \right\}$$

endowed with norm $\|u\|_\infty = \sup_{x\in\mathbb{R}^d, y\in\mathbb{R}^d_+} \{|u(x,y)|\}$, and

$$X_p := L_p(\mathbb{R}^d \times \mathbb{R}^d_+), \quad 1 \le p < \infty,$$

equipped with their natural norm. We will make use of Yosida approximation to verify the generator property of a sum of noncommuting generators. Finally in Section 4 we present an appropriate scaling of the end conditions and we show that well-posedness is invariant under this scaling. Lastly the Lumer-Phillips theorem is the key to obtain the well-posedness results.

We introduce some notations. The dot between two vectors a and b denotes the inner product, i.e., $a \cdot b = \sum_{i=1}^d a_i b_i$. A double dot between two matrices A and B similarly denotes the double summation, i.e., $A : B = \sum_{i=1}^d \sum_{j=1}^d a_{ij} b_{ij}$. Moreover ∇_x^2 means $(\nabla_x^2)_{ij} = \partial_{x_i}\partial_{x_j}$ and we define $(xy)_i := x_i y_i$, $(xx\nabla_x^2)_{ij} := x_i x_j \partial_{x_i}\partial_{x_j}$ for $x, y \in \mathbb{R}^d$.

2. The Black-Scholes approach

The basic idea consists in the approach following the fundamental work of Black-Scholes [BS73]. We use the ansatz $Y_t = u(t, S_t^*, I(S_t^*))$ and try to determine the function $u(t,x,y)$ with $t \in J$, $x = (x^1, \dots, x^d) \in \mathbb{R}^d_+$, and $y = (y^1, \dots, y^d) \in \mathbb{R}^d_+$. We already know that u satisfies the end constraint $u(T, S_T^*, I(S_T^*)) = Y_T = Z = g(S_T^*, I(S_T^*))$, i.e.,

$$u(T,x,y) = g(x,y), \qquad x \in \mathbb{R}^d_+, \quad y \in \mathbb{R}^d_+. \tag{2.1}$$

Applying Itô's formula we have

$$\begin{aligned} \mathrm{d}Y_t &= \partial_t u \mathrm{d}t + \partial_x u \mathrm{d}S_t^* + \partial_y u \mathrm{d}I_t + \frac{1}{2}\partial_x^2 u \mathrm{d}S_t^* \mathrm{d}S_t^* + \partial_{xy}^2 u \mathrm{d}S_t^* \mathrm{d}I_t \\ &\quad + \frac{1}{2}\partial_y^2 u \mathrm{d}I_t \mathrm{d}I_t \\ &= \partial_t u \mathrm{d}t + \sum_{i=1}^d \partial_{x_i} u \mathrm{d}S_t^i + \sum_{i=1}^d \partial_{y_i} u \mathrm{d}I_t^i + \frac{1}{2}\sum_{i=1}^d \sum_{k=1}^d \partial_{x_i}\partial_{x_k} u \mathrm{d}S_t^i \mathrm{d}S_t^k \\ &\quad + \sum_{i=1}^d \sum_{k=1}^d \partial_{x_i}\partial_{y_k} u \mathrm{d}S_t^i \mathrm{d}I_t^k + \frac{1}{2}\sum_{i=1}^d \sum_{k=1}^d \partial_{y_i}\partial_{y_k} u \mathrm{d}I_t^i \mathrm{d}I_t^k. \end{aligned} \tag{2.2}$$

The relation $\mathrm{d}I_t^k = S_t^k \mathrm{d}t$ and (1.1) yield

$$\begin{aligned}
\mathrm{d}Y_t &= \partial_t u \mathrm{d}t + \sum_{i=1}^d \partial_{x_i} u \left[\mu_t^i S_t^i \mathrm{d}t + \sum_{j=1}^m \sigma_j^i S_t^i \mathrm{d}B_t^j\right] + \sum_{i=1}^d \partial_{y_i} u \left[S_t^i \mathrm{d}t\right] \\
&\quad + \frac{1}{2}\sum_{i=1}^d \sum_{k=1}^d \partial_{x_i}\partial_{x_k} u \left[\left(\mu_t^i S_t^i \mathrm{d}t + \sum_{j=1}^m \sigma_j^i S_t^i \mathrm{d}B_t^j\right)\left(\mu_t^k S_t^k \mathrm{d}t + \sum_{j=1}^m \sigma_j^k S_t^k \mathrm{d}B_t^j\right)\right] \\
&\quad + \sum_{i=1}^d \sum_{k=1}^d \partial_{x_i}\partial_{y_k} u \left[\left(\mu_t^i S_t^i \mathrm{d}t + \sum_{j=1}^m \sigma_j^i S_t^i \mathrm{d}B_t^j\right)\left(S_t^k \mathrm{d}t\right)\right] \\
&\quad + \frac{1}{2}\sum_{i=1}^d \sum_{k=1}^d \partial_{y_i}\partial_{y_k} u \left[\left(S_t^i \mathrm{d}t\right)\left(S_t^k \mathrm{d}t\right)\right].
\end{aligned}$$

With subject to the common conventions $(\mathrm{d}t)^2 = \mathrm{d}t \cdot \mathrm{d}B_t = 0$ and $(\mathrm{d}B_t)^2 = \mathrm{d}t$, the following equation follows

$$\begin{aligned}
\mathrm{d}Y_t = &\underbrace{\left[\partial_t u + \sum_{i=1}^d \mu_t^i S_t^i \partial_{x_i} u + \sum_{i=1}^d S_t^i \partial_{y_i} u + \frac{1}{2}\sum_{i=1}^d \sum_{k=1}^d \left(\sum_{j=1}^m \sigma_j^i \sigma_j^k\right) S_t^i S_t^k \partial_{x_i}\partial_{x_k} u\right] \mathrm{d}t}_{\text{deterministic part}} \\
&+ \underbrace{\sum_{j=1}^m \left(\sum_{i=1}^d \partial_{x_i} u \sigma_j^i S_t^i\right) \mathrm{d}B_t^j}_{\text{stochastic part}}. \qquad (2.3)
\end{aligned}$$

On the other hand $Y_t = \theta_t \cdot S_t = \theta_t^* \cdot S_t^* + \theta_t^0 \cdot S_t^0$, hence with the self-financing condition

$$\mathrm{d}\theta_t \cdot S_t + \mathrm{d}\theta_t \cdot \mathrm{d}S_t = 0 \qquad (2.4)$$

another representation of $\mathrm{d}Y_t$ arises as

$$\begin{aligned}
\mathrm{d}Y_t = \theta_t \cdot \mathrm{d}S_t &= \theta_t^0 r_t S_t^0 \mathrm{d}t + \sum_{i=1}^d \theta_t^i S_t^i \mu_t^i \mathrm{d}t + \sum_{i=1}^d \sum_{j=1}^m \theta_t^i S_t^i \sigma_j^i \mathrm{d}B_t^j \\
&= \underbrace{\left[\theta_t^0 r_t S_t^0 + \sum_{i=1}^d \theta_t^i S_t^i \mu_t^i\right]}_{\text{deterministic part}} \mathrm{d}t + \underbrace{\sum_{j=1}^m \sum_{i=1}^d \theta_t^i S_t^i \sigma_j^i}_{\text{stochastic part}} \mathrm{d}B_t^j.
\end{aligned} \qquad (2.5)$$

Thanks to the uniqueness of the Itô transform we are able to compare the deterministic and stochastic coefficients of both representations (2.3) and (2.5).

The comparison of the stochastic parts yields

$$\sigma_t^T \cdot (S_t^* \nabla_x u) = \sigma_t^T \cdot (\theta_t^* S_t^*)$$

and due to the injectivity of the matrix σ_t^T we obtain

$$\theta_t^* = \nabla_x u(t, S_t^*, I(S_t^*)). \tag{2.6}$$

The comparison of the deterministic coefficients provides

$$\partial_t u + (S_t^* \mu_t) \cdot \nabla_x u + S_t^* \cdot \nabla_y u + \frac{1}{2} \sigma_t \sigma_t^T : (S_t^* S_t^* \nabla_x^2 u) = \theta_t^0 r_t S_t^0 + \theta_t^* \cdot (S_t^* \mu_t). \tag{2.7}$$

After insertion of (2.6) the remaining equation reads as follows

$$\partial_t u + S_t^* \cdot \nabla_y u + \frac{1}{2} (\sigma_t \sigma_t^T) : (S_t^* S_t^* \nabla_x^2 u) = \theta_t^0 r_t S_t^0. \tag{2.8}$$

Since $u(t, S_t^*, I(S_t^*)) = Y_t = \theta_t^* \cdot S_t^* + \theta_t^0 S_t^0$ we have

$$\theta_t^0 S_t^0 = u - \theta_t^* \cdot S_t^* \overset{(2.6)}{=} u - S_t^* \cdot \nabla_x u$$

and therefore

$$\theta_t^0 = (S_t^0)^{-1} \left[u(t, S_t^*, I(S_t^*)) - S_t^* \cdot \nabla_x u(t, S_t^*, I(S_t^*)) \right]. \tag{2.9}$$

As a last step we have to insert (2.9) into (2.8) and resubstitute S_t^* with x and $I(S_t^*)$ with y, so that the following partial differential equation arises

$$\partial_t u + x \cdot \nabla_y u + \frac{1}{2} \underbrace{(\sigma_t \sigma_t^T)}_{=:a_t} : xx \nabla_x^2 u = r_t (u - x \cdot \nabla_x u), \qquad x, y \in \mathbb{R}_+^d.$$

Hence we derived a $2d+1$-dimensional partial differential equation with inverse time direction

$$\begin{cases} \partial_t u + \displaystyle\sum_{i=1}^d x_i \partial_{y_i} u + \frac{1}{2} \sum_{i=1}^d \sum_{k=1}^d a_{ik}(t) x_i x_k \partial_{x_i} \partial_{x_k} u = r(t) \left[u - \sum_{i=1}^d x_i \partial_{x_i} u \right], \\ \qquad\qquad\qquad\qquad\qquad\qquad u(T, x, y) = g(T, x, y), \end{cases} \tag{2.10}$$

where $x, y \in \mathbb{R}_+^d$ and $t \in J$.

For simplicity, we assume, that $a(t) \equiv a$ and $r(t) \equiv r$. By doing so we can write (2.10) as

$$\begin{cases} \partial_t u + \displaystyle\sum_{i=1}^d x_i \partial_{y_i} u + \frac{1}{2} \sum_{i=1}^d \sum_{k=1}^d a_{ik} x_i x_k \partial_{x_i} \partial_{x_k} u = r \left[u - \sum_{i=1}^d x_i \partial_{x_i} u \right], \\ \qquad\qquad\qquad\qquad\qquad\qquad u(T, x, y) = g(x, y), \end{cases} \tag{2.11}$$

with $x, y \in \mathbb{R}_+^d$ and $t \in [0, T]$.

This is the basic model for the pricing of an Asian option of European style. In short form it is written as

$$\begin{aligned} \partial_t u + x \cdot \nabla_y u + \frac{1}{2} a : xx \nabla_x^2 u &= r(u - x \cdot \nabla_x u), \qquad t \in [0, T),\ x, y \in \mathbb{R}_+^d, \\ u(T, x, y) &= g(x, y). \end{aligned}$$

In order to eliminate the strong degeneracy of the coefficients in (2.11) we are going to run an Euler transformation. Therefore we substitute $x_i \rightsquigarrow e^{\xi_i}$ and $u \rightsquigarrow v$ with

$$v(t,\xi,y) = u(t,x,y) = u(t,e^{\xi},y) = v(t,\log x,y),$$

where $e^{\xi} = (e^{\xi_1},\dots,e^{\xi_d})^T$ and $\log x = (\log x_1,\dots,\log x_d)^T$. Hence the first partial derivative of v with respect to ξ is

$$\partial_{\xi_i} v(t,\xi,y) = x_i \partial_{x_i} u(t,x,y) \tag{2.12}$$

and the relevant second partial derivation results as

$$\begin{aligned}\partial_{\xi_i}\partial_{\xi_j} v(t,\xi,y) &= (x_j\partial_{x_j})(x_i\partial_{x_i}u)(t,x,y)\\ &= \begin{cases} x_i x_j \partial_{x_i}\partial_{x_j} u(t,x,y) & : i \neq j \\ (x_i)^2 \partial_{x_i}^2 u(t,x,y) + x_i \partial_{x_i} u(t,x,y) & : i = j \end{cases}.\end{aligned} \tag{2.13}$$

Thus the differential equation in v appears as

$$\partial_t v + \sum_{i=1}^d e^{\xi_i}\partial_{y_i} v + \frac{1}{2}\sum_{i=1}^d\sum_{k=1}^d a_{ik}\partial_{\xi_i}\partial_{\xi_k} v = r\left[v - \sum_{i=1}^d \partial_{\xi_i} v\right] + \frac{1}{2}\sum_{i=1}^d a_{ii}\partial_{\xi_i} v.$$

Time direction can be inverted with a time reflection by substituting $t \rightsquigarrow T - t$ and $v \rightsquigarrow w$ with $w(t,\xi,y) = v(T-t,\xi,y)$. Thus the final problem can be written as

$$\begin{gathered}-\partial_t w + \sum_{i=1}^d e^{\xi_i}\partial_{y_i} w + \frac{1}{2}\sum_{i=1}^d\sum_{k=1}^d a_{ik}\partial_{\xi_i}\partial_{\xi_k} w = r\left[w - \sum_{i=1}^d \partial_{\xi_i} w\right] + \frac{1}{2}\sum_{i=1}^d a_{ii}\partial_{\xi_i} w,\\ w(0,\xi,y) = g\left(e^{\xi},y\right),\end{gathered}$$

with $\xi \in \mathbb{R}^d,\ y \in \mathbb{R}_+^d$ and $t \in [0,1]$. Introducing the vector b by $b_i := r - \frac{a_{ii}}{2}$ we obtain the following problem

$$\begin{cases} \partial_t w + rw - b\cdot\nabla_{\xi} w - \dfrac{1}{2} a : \nabla_{\xi}^2 w = e^{\xi}\cdot\nabla_y w,\ t\in(0,1],\ \xi\in\mathbb{R}^d,\ y\in\mathbb{R}_+^d, \\ \qquad w(0,\xi,y) = w_0(\xi,y) := g\left(e^{\xi},y\right). \end{cases} \tag{2.14}$$

It is this problem we will study mathematically, the inverse Euler transform is left to the reader.

3. Well-posedness of the problem

Our objective is to prove that the problem (2.14) is well posed in the spaces X_p, $1 < p \leq \infty$, introduced in Section 1, i.e., that its solution exists, is unique, and depends continuously on the data. We start providing two preliminary lemmata.

Lemma 3.1. *The family of operators* $\{T_B(t)\}_{t\geq 0} \subset \mathcal{B}(X_p)$, $1 \leq p \leq \infty$ *given by*

$$(T_B(t)u_0)(\xi, y) = u_0(\xi, y + te^{\xi}), \quad t \geq 0,\ \xi \in \mathbb{R}^d,\ y \in \mathbb{R}^d_+$$

defines a C_0*-semigroup of contractions in* X_p.

Proof. It is obvious that $\{T_B(t)\}_{t\geq 0}$ satisfies the semigroup property. The following equation assures the contractivity of semigroup $\{T_B(t)\}_{t\geq 0}$ for $1 \leq p < \infty$

$$\begin{aligned}\|T_B(t)u_0(\xi)\|_p^p &= \int\limits_{\mathbb{R}^d_+} \left|u_0\left(\xi, y + te^{\xi}\right)\right|^p \mathrm{d}y \\ &= \int\limits_{U\subset\mathbb{R}^d_+} |u_0(\xi, z)|^p \,\mathrm{d}z, \qquad (\text{with } z = y + te^{\xi}) \\ &\leq \|u_0(\xi)\|_p^p.\end{aligned}$$

Integrating over $\xi \in \mathbb{R}^d$ yields the claim. For $p = \infty$ we receive

$$\|T_B(t)u_0(\xi)\|_\infty = \sup_{y\in\mathbb{R}^d_+} \left|u_0\left(\xi, y + te^{\xi}\right)\right| \leq \sup_{y\in\mathbb{R}^d_+} |u_0(\xi, y)| = \|u_0(\xi)\|_\infty .$$

This implies that $\|T_B(t)u_0\|_p \leq \|u_0\|_p$ holds for $1 \leq p \leq \infty$.
The C_0 property for $1 \leq p < \infty$ follows directly from the fact that the space of test functions is dense in X_p. □

In the sequel we denote with B the generator of $\{T_B(t)\}_{t\geq 0}$; note that $B = e^{\xi} \cdot \nabla_y$ holds at least on $C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d_+) \subset D(B)$.

Lemma 3.2. *The operator* $A : D(A) \subset X_p \to X_p$, $1 < p \leq \infty$, *defined by*

$$Au := \frac{1}{2} a : \nabla_\xi^2 u + b \cdot \nabla_\xi u - ru$$

with domain

$$D(A) = \begin{cases} W_p^2\left(\mathbb{R}^d; L_p(\mathbb{R}^d_+)\right) & \text{for } 1 < p < \infty, \\ \left\{u \in C_0(\mathbb{R}^d) \cap \bigcap\limits_{q>1} W^2_{q,loc}(\mathbb{R}^d) : Au \in C_0(\mathbb{R}^d)\right\} & \text{for } p = \infty \end{cases}$$

generates an analytic C_0*-semigroup of contractions in* X_p.

Proof. Lunardi proved in [Lun95, Corollary 3.1.9] that A generates an analytic C_0-semigroup $\{T_A(t)\}_{t\geq 0}$. This semigroup is given by

$$(T_A(t)u_0)(\xi, y) = e^{-rt} \int\limits_{\mathbb{R}^d} u_0(\xi - \eta, y)\gamma_{tb,ta}(\eta)\mathrm{d}\eta, \tag{3.1}$$

where $\gamma_{\mu,\sigma}(\eta)$ is the Gaussian distribution, i.e.,

$$\gamma_{\mu,\sigma}(\eta) = \frac{1}{\sqrt{(2\pi)^d \det\sigma}} \exp\left\{-\frac{1}{2}(\sigma^{-1}(\eta - \mu) \mid \eta - \mu)\right\}.$$

Since $\gamma_{\mu,\sigma}(\eta) \geq 0$ and $\int_{\mathbb{R}^d} \gamma_{\mu,\sigma}(\eta)\mathrm{d}\eta = 1$ we obtain by Young's inequality

$$\|T_A(t)\|_p \leq e^{-rt}$$

for all $1 \leq p \leq \infty$. □

In the following $B_\lambda := B(I-\lambda B)^{-1}$ denotes the Yosida-approximation of the operator B. It is well known that $\lim_{\lambda\to 0^+} B_\lambda x = Bx$ for $x \in D(B)$ and also that if B is a generator of a C_0-semigroup of contractions then so is B_λ.

Proposition 3.1. *Let X be a reflexive Banach space. Let A and B be dissipative generators and suppose that the solution u_λ of*

$$\omega u - Au - B_\lambda u = f \tag{3.2}$$

satisfies

$$\sup_{\lambda\in(0,1)} |B_\lambda u_\lambda| < \infty \tag{3.3}$$

for a dense set of right-hand sides f. Then $\overline{A+B}$ is the generator of a C_0-semigroup of contractions in X.

Proof. We employ the theorem of Lumer and Phillips [LP61]. Obviously $A+B$ is dissipative. Due to the assumption that $\{B_\lambda u_\lambda\}$ is bounded it holds that $\{u_\lambda\}$ is bounded and hence $\{Au_\lambda\}$ is bounded as well. Since X is reflexive there is a sequence $(\lambda_n) \subset \mathbb{R}$ with $\lambda_n \to 0$ such that $u_n := u_{\lambda_n} \rightharpoonup u$, $Au_n \rightharpoonup v$, and $B_{\lambda_n} u_n \rightharpoonup w$. Because graph$(A)$ is closed and convex we have

$$\mathrm{graph}(A) \ni (u_n, Au_n) \rightharpoonup (u,v) \in \mathrm{graph}(A) \quad \text{in } X \times X \tag{3.4}$$

and thus $(u,v) = (u, Au)$ with $u \in D(A)$. Moreover, since $(I-\lambda_n B)^{-1}u_n \rightharpoonup u$ as well, we obtain also

$$B_{\lambda_n} u_n = B(I-\lambda_n B)^{-1} u_n \rightharpoonup w = Bu \tag{3.5}$$

with $u \in D(B)$. Summarizing, we have proven that u_n converges weakly to a solution $u \in D(A) \cap D(B)$ of

$$\omega u - Au - Bu = f \tag{3.6}$$

for a dense set of right-hand sides f; hence $\overline{R(w - Au - Bu)} = X$ and the theorem of Lumer-Phillips applies. □

Theorem 1. *Let the operator $L : D(L) \subset X_p \to X_p$, $1 < p \leq \infty$, be defined as*

$$Lw := \frac{1}{2} a : \nabla_\xi^2 w + b \cdot \nabla_\xi w - rw + e^\xi \partial_y w,$$

with domain $D(L) = D(A) \cap D(B)$ and consider the abstract Cauchy problem

$$\begin{cases} \partial_t w - \overline{L} w = 0 & \text{in} \quad (0,T] \times \mathbb{R}^d \times \mathbb{R}^d_+, \\ \qquad\quad w = w_0 & \text{on} \quad \{t=0\} \times \mathbb{R}^d \times \mathbb{R}^d_+. \end{cases} \tag{3.7}$$

Then problem (3.7) is well posed in X_p.

Proof. Lemma 3.2 resp. Lemma 3.1 the Lumer-Phillips theorem [LP61] imply that the operator A given by $Au = \frac{1}{2}a : \nabla_\xi^2 u + b \cdot \nabla_\xi u - ru$ and the operator B introduced after Lemma 3.1 are dissipative generators. We want to show that the range of $\omega + A + B$ is dense in X_p. So we consider the equation

$$\omega u_\lambda - Au_\lambda - B_\lambda u_\lambda = f \tag{3.8}$$

which admits a unique solution $u_\lambda \in D(A)$, $\lambda > 0$. Here we take $f \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d_+)$, $\operatorname{supp} f$ compact; this set is dense in X_p for each $1 \le p \le \infty$.
Since A and e^{ξ_j}, $j = 1, \ldots, d$, do not commute we obtain

$$\begin{aligned} &\omega(e^{\xi_j} u_\lambda) - A(e^{\xi_j} u_\lambda) - B_\lambda(e^{\xi_j} u_\lambda) \\ &\quad = e^{\xi_j} f - A(e^{\xi_j} u_\lambda) + e^{\xi_j} A u_\lambda \\ &\quad = e^{\xi_j} f - [A, e^{\xi_j}] u_\lambda, \quad j = 1, \ldots, d, \end{aligned} \tag{3.9}$$

where $[A, e^{\xi_j}]$ denotes the commutator of A and e^{ξ_j}. Employing the sum convention we have

$$\begin{aligned} [A, e^{\xi_j}]v &= \frac{a_{ik}}{2} \partial_i \partial_k (e^{\xi_j} v) + b_i \partial_i (e^{\xi_j} v) - e^{\xi_j} \frac{a_{ik}}{2} \partial_i \partial_k v - e^{\xi_j} b_i \partial_i v \\ &= \frac{a_{ik}}{2} \partial_i (e^{\xi_j} \partial_k v + \delta_{jk} e^{\xi_j} v) + e^{\xi_j} b_i (\partial_i v + \delta_{ji} v) - e^{\xi_j} \frac{a_{ik}}{2} \partial_i \partial_k v - e^{\xi_j} b_i \partial_i v \\ &= \frac{a_{ik}}{2} \left(\delta_{ji} e^{\xi_j} \partial_k v + \delta_{jk} e^{\xi_j} \partial_i v + \delta_{ji} \delta_{jk} e^{\xi_j} v \right) + \left(r - \frac{a_{jj}}{2} \right) e^{\xi_j} v \\ &= \frac{a_{jk}}{2} e^{\xi_j} \partial_k v + \frac{a_{ij}}{2} e^{\xi_j} \partial_i v + r e^{\xi_j} v \\ &= a_{jk} e^{\xi_j} \partial_k v + r e^{\xi_j} v \\ &= a_{jk} \partial_k (e^{\xi_j} v) - a_{jj} (e^{\xi_j} v) + r(e^{\xi_j} v), \end{aligned}$$

due to $b_j = r - a_{jj}/2$ and since (a_{ij}) is symmetric. Thus we obtain the equation

$$(\omega + r - a_{jj})(e^{\xi_j} u_\lambda) - A(e^{\xi_j} u_\lambda) - A_j(e^{\xi_j} u_\lambda) - B_\lambda(e^{\xi_j} u_\lambda) = e^{\xi_j} f \tag{3.10}$$

with $A_j v := -a_{jk} \partial_k v$. Because the operators A_j are also dissipative we choose

$$\omega > 2 \max\{a_{jj} - r : j = 1, \ldots, d\}$$

and it results

$$\|e^{\xi_j} u_\lambda\|_p \le \frac{2}{\omega} \|e^{\xi_j} f\|_p \quad \text{as well as} \quad \|e^{\xi_j} \partial_{y_k} u_\lambda\|_p \le \frac{2}{\omega} \|e^{\xi_j} \partial_{y_k} f\|_p. \tag{3.11}$$

This implies that

$$\|B_\lambda u_\lambda\|_p = \left\| \frac{1}{\lambda} \left(\frac{1}{\lambda} - B \right)^{-1} B u_\lambda \right\|_p \le c_1 \|B u_\lambda\|_p \le c$$

for $1 < p \le \infty$ and Proposition 3.1 applies for $1 < p < \infty$. For the case $p = \infty$ it follows that $\|Au_\lambda\|_\infty \le c$ and we obtain $\partial_{\xi_i} u_\lambda \in C_\xi^\alpha$ as well as $\partial_{\xi_i} \partial_{y_k} u_\lambda \in C_\xi^\alpha$ with $\alpha \in (0, 1)$. Thus there is a sequence $(\lambda_n) \subset \mathbb{R}_+$ with $\lim_{n \to \infty} \lambda_n = 0$, such that $\nabla_y u_{\lambda_n} \to \nabla_y u$ and $u_{\lambda_n} \to u$ as $n \to \infty$ uniformly on compact sets. In particular this means that for every ball $B_r(0)$ with radius $r \in \mathbb{N}$ we find a sequence $(\lambda_{r,n})_n \subset \mathbb{R}_+$ with $\lim_{n\to\infty} \lambda_{r,n} = 0$ such that $B_{\lambda_{r,n}} u_{\lambda_{r,n}} \to Bu$ and $Au_{\lambda_{r,n}} \to Au$ as $n \to \infty$ uniformly on $B_r(0)$. By a diagonal-sequence argument we obtain the existence of a sequence $(\lambda_k) \subset \mathbb{R}_+$ with $\lim_{k\to\infty} \lambda_k = 0$ such that $B_{\lambda_k} u_{\lambda_k} \to Bu$ as well as $Au_{\lambda_k} \to Au$ as $k \to \infty$ on an arbitrary compact set $K \subset \mathbb{R}^d \times \mathbb{R}^d_+$. This implies

$$\omega u + Au + Bu = f \tag{3.12}$$

also in the case $p = \infty$, and the theorem is proved. □

4. The call-put parity

Suppose that u_c is a solution of problem (2.11) with $g(x,y) = [\frac{1}{T} y_k - K]_+$ resp. $g(x,y) = [x_k - \frac{1}{T} y_k]_+$. Accordingly suppose that u_p is a solution of (2.11) with $g(x,y) = [K - \frac{1}{T} y_k]_+$ resp. $g(x,y) = [\frac{1}{T} y_k - x_k]_+$. Since problem (2.11) is linear, we obtain that $u_c - u_p$ is a solution for $g(x,y) = \frac{1}{T} y_k - K$ resp. $g(x,y) = x_k - \frac{1}{T} y_k$.

In this section we will present an appropriate scaling $\tilde{g}$ of the end conditions g, such that $\tilde{g} \in X_\infty$ holds and that the scaled problem is still well posed. Then the Lumer-Phillips theorem results that in particular the solutions of the scaled problem, hence the solutions of (2.11), are unique. Thus the following propositions provide the call-put-parities for the average price option resp. the average strike option.

Proposition 4.1. *$u(t,x,y) = e^{r(t-T)} \left[T^{-1} \left(y_k + r^{-1} x_k \left(e^{r(T-t)} - 1\right)\right) - K\right]$ is a solution of* (2.11) *with $g(x,y) = \frac{1}{T} y_k - K$.*

Proof. Obviously, the end condition with the postulated g holds. Thus it remains to prove, that u is a solution of the partial differential equation (2.11):

$$\begin{aligned}
&\partial_t u + \sum_{i=1}^d x_i \partial_{y_i} u + \frac{1}{2} \sum_{i,j=1}^d a_{ij} x_i x_j \partial_{x_i} \partial_{x_j} u \\
&\quad = e^{r(t-T)} \left[\frac{1}{T} (r y_k - x_k) - rK\right] + \frac{1}{T} e^{r(t-T)} x_k \\
&\quad = \frac{r}{T} e^{r(t-T)} y_k - \frac{1}{T} e^{r(t-T)} x_k - r e^{r(t-T)} K + \frac{1}{T} e^{r(t-T)} x_k \\
&\quad = \frac{r}{T} e^{r(t-T)} y_k + \frac{1}{T} x_k - \frac{1}{T} e^{r(t-T)} x_k - r e^{r(t-T)} K - \frac{1}{T} x_k + \frac{1}{T} e^{r(t-T)} x_k \\
&\quad = r \left[u - \sum_{i=1}^d x_i \partial_{x_i} u\right].
\end{aligned}$$

□

Proposition 4.2. *$u(t,x,y) = x_k - T^{-1} e^{r(t-T)} \left[y_k + x_k r^{-1} \left(e^{r(T-t)} - 1\right)\right]$ is a solution of* (2.11) *with $g(x,y) = x_k - \frac{1}{T} y_k$.*

Proof. The end condition with the postulated g holds obviously. Thus it remains to prove, that u is a solution of the partial differential equation (2.11):

$$\begin{aligned}
&\partial_t u + \sum_{i=1}^{d} x_i \partial_{y_i} u + \frac{1}{2} \sum_{i,j=1}^{d} a_{ij} x_i x_j \partial_{x_i} \partial_{x_j} u \\
&= \frac{1}{T} e^{r(t-T)} (x_k - r y_k) - x_k \frac{1}{T} e^{r(t-T)} \\
&= -\frac{r}{T} e^{r(t-T)} y_k + \frac{1}{T} e^{r(t-T)} x_k - \frac{1}{T} e^{r(t-T)} x_k \\
&= r x_k - \frac{r}{T} e^{r(t-T)} y_k - \frac{1}{T} x_k + \frac{1}{T} e^{r(t-T)} x_k - r x_k + \frac{1}{T} x_k - \frac{1}{T} e^{r(t-T)} x_k \\
&= r \left[u - \sum_{i=1}^{d} x_i \partial_{x_i} u \right].
\end{aligned}$$

□

Consider the scaling of the end condition

$$v_0 = \frac{u_0}{1 + |x|^2 + |y|^2}, \quad u_0 \in \left\{ [x_k - \tfrac{1}{T} y_k]_+ ; [K - \tfrac{1}{T} y_k]_+ \right\} \tag{4.1}$$

for $x, y \in \mathbb{R}_+^d$, where $|x|^2 = \sum_{i=1}^d x_i^2$. Hence we have $u = (1 + |x|^2 + |y|^2)v$ and we compute the relevant derivatives, i.e.,

$$\partial_t u = (1 + |x|^2 + |y|^2) \partial_t v \tag{4.2}$$

$$\partial_{x_i} u = (1 + |x|^2 + |y|^2) \partial_{x_i} v + 2 x_i v \tag{4.3}$$

$$\partial_{y_i} u = (1 + |x|^2 + |y|^2) \partial_{y_i} v + 2 y_i v \tag{4.4}$$

$$\partial_{x_j} \partial_{x_i} u = (1 + |x|^2 + |y|^2) \partial_{x_j} \partial_{x_i} v + 2 x_j \partial_{x_i} v + 2 x_i \partial_{x_j} v + \delta_{ij} 2v. \tag{4.5}$$

By means of sum convention and equation (2.11) we have

$$\begin{aligned}
&\partial_t v + x_i \partial_{y_i} v + \frac{1}{2} a_{ij} x_i x_j \partial_{x_i} \partial_{x_j} v - r (v - x_i \partial_{x_i} v) \\
&\qquad = -m(x,y)(2 a_{ij} x_i x_j^2 \partial_{x_i} + a_{ii} x_i^2 + r|x|^2 + 2 y_i) v,
\end{aligned} \tag{4.6}$$

with $m(x,y) := 1/(1 + |x|^2 + |y|^2)$. To run an Euler transform in x, i.e., $x_i \rightsquigarrow e^{x_i}$, $i = 1, \ldots, d$ and $v \rightsquigarrow w$ with $w(t, \xi, y) = v(t, e^{\xi}, y)$, we use the calculated derivatives (2.12) and (2.13) and receive

$$\begin{aligned}
&\partial_t w + e^{\xi_i} \partial_{y_i} w + b_i \partial_{\xi_i} w + \frac{1}{2} a_{ij} \partial_{\xi_i} \partial_{x_j} w - r w \\
&\qquad = -m(e^{\xi}, y)(2 a_{ij} e^{2\xi_j} \partial_{\xi_i} + a_{ii} e^{2\xi_i} + r|e^{\xi}|^2 + 2 y_i) v,
\end{aligned} \tag{4.7}$$

with vector b given by $b_i = r - \frac{a_{ii}}{2}$. As a last step we invert the time, i.e., $t \rightsquigarrow T-t$, and denote the inverted function again with w; hence

$$\partial_t w - e^{\xi_i}\partial_{y_i} w - b_i \partial_{\xi_i} w - \frac{1}{2} a_{ij} \partial_{\xi_i} \partial_{x_j} w + rw \\ = m(e^{\xi}, y)(2a_{ij} e^{2\xi_j} \partial_{\xi_i} + a_{ii} e^{2\xi_i} + r|e^{\xi}|^2 + 2y_i)w \quad (4.8)$$

holds. Let us introduce the operators G and H by

$$Gw := 2m(e^{\xi}, y) \sum_{i=1}^{d} \sum_{j=1}^{d} a_{ij} e^{2\xi_j} \partial_{\xi_i} w$$

$$Hw := m(e^{\xi}, y) \left(\sum_{i=1}^{d} a_{ii} e^{2\xi_i} + r|e^{\xi}|^2 + 2y_i \right)$$

and recall that the left-hand side of equation (4.8) precisely is $(\partial_t - L)u$ with $L = A + B$ as defined in Theorem 1. With an easy calculus we obtain that

$$A + G = \frac{1}{2} a : \nabla_{\xi}^2 + \widetilde{b}(\xi) \cdot \nabla_{\xi} - r, \quad (4.9)$$

with $\widetilde{b}_i(\xi) := b_i + 2m(e^{\xi}, y) \sum_{j=1}^{d} a_{ij} e^{2\xi_j}$, $i = 1, \ldots, d$, is the ω-dissipative generator of an analytic C_0-semigroup with $\omega \le \sup\{|\operatorname{div} \widetilde{b}(\xi)| : \xi \in \mathbb{R}^d\}$, provided the right-hand side of this inequality is finite. And indeed we have

$$|\operatorname{div} \widetilde{b}(\xi)| = \left| \sum_{i=1}^{d} \partial_{\xi_i} m(e^{\xi}, y) \sum_{j=1}^{d} a_{ij} e^{2\xi_j} \right| \\ \le 2 \left(\sum_{i=1}^{d} \sum_{j=1}^{d} |a_{ij}| \left| \frac{e^{2\xi_i} e^{2\xi_j}}{(1 + |e^{\xi}|^2)^2} \right| + \sum_{i=1}^{d} |a_{ii}| \left| \frac{e^{2\xi_i}}{1 + |e^{\xi}|^2} \right| \right) \le 4d^2 M_a,$$

where $M_a := \max\{|a_{ij}| : i, j = 1, \ldots, d\}$. Thus Theorem 1 applies for the shifted operator sum $(\eta + A + G)$ and B, η sufficiently large, i.e., the sum $A + G + B$ generates a C_0-semigroup. Since the operator H is linear and bounded the Bounded Perturbation Theorem (e.g., [EN00]) applies and thus $A + G + B + H$ generates a C_0-semigroup and we obtain the existence of a solution of the scaled problem (4.8) with initial value $w(0, \xi, y) = v_0(e^{\xi}, y)$.

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Linearized Stability and Regularity for Nonlinear Age-dependent Population Models

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To the memory of Günter Lumer

Abstract. The paper is concerned with the general theory of nonlinear age-dependent population dynamics. We present (a) a principle of linearized stability and (b) a result on regularity of solutions to the general nonlinear model.

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1. Introduction

In this paper, we shall address two problems in the theory of age-dependent population dynamics as developed by Glenn F. Webb ([7] and references therein). In its abstract form, the general age-dependent population problem (ADP) is formulated for population densities $\phi \in L^1 := L^1(\mathbb{R}_+, \mathbb{R}^n)$ as follows: Let F be a mapping from L^1 to $\mathbb{R}^n$, let G be a mapping from L^1 to L^1, and let $\phi \in L^1$. For given $T > 0$, a function $u \in C([0,T], L^1)$ will be called a *solution of* (ADP) with initial age distribution ϕ provided it satisfies the following laws:

$$\lim_{h\to 0^+} \int_0^\infty \left\| h^{-1}(u(t+h)(a+h) - u(t)(a)) - G(u(t))(a) \right\| da = 0, \ 0 \le t < T \quad (1.1)$$

(the *balance law* of the population),

$$\lim_{h\to 0^+} h^{-1} \int_0^h \|u(t+h)(a) - F(u(t))\| da = 0, \qquad 0 \le t < T \qquad (1.2)$$

(the *birth law* of the population), and

$$u(0) = \phi \qquad (1.3)$$

(the *initial age distribution* of the population).

F is called the *birth function*, and G is called the *aging function* of the population. In the classical Gurtin-McCamy model ([4]), F and G have the forms

$$F(\phi) = \int_0^\infty \beta(a, P\phi)\phi(a)da, \text{ and } \quad G(\phi)(a) = -\mu(a, P\phi)\phi(a), \quad \phi \in L^1,\ a \in \mathbb{R}_+,$$

respectively, where the *fertility modulus* β, and the *mortality modulus* μ are non-negative functions of two variables, and $P(\phi) = \int_0^\infty \phi(a)da$.

In [7], problem (ADP) is considered under the following general assumptions on the birth and on the aging functions:

(A1) $F : L^1 \to \mathbb{R}^n$, and $G : L^1 \to L^1$ are Lipschitz on norm-balls of L^1.

(A2) $F(L^1_+) \subset \mathbb{R}^n_+$, and there is an increasing function $c : \mathbb{R}_+ \to \mathbb{R}_+$ such that if $r > 0$, and $\phi \in L^1_+$ with $\|\phi\| \leq r$, then $G(\phi) + c(r)\phi \in L^1_+$ (where $L^1_+ := L^1(\mathbb{R}_+, \mathbb{R}^n_+)$).

It is shown in [7, Thm. 2.4] that under these assumptions, given $\phi \in L^1_+$, there exists a unique solution u of (ADP) on a maximal interval of existence $[0, T_\phi)$, $0 < T_\phi \leq \infty$, with $u(t) \in L^1_+$ for all $t \in [0, T_\phi)$.

Notice that the notion of a solution to (ADP) as given by (1.1)–(1.3) looks like the notion of a kind of a "mild" solution to the evolution problem

$$\dot{u}(t) + u(t)_a - Gu(t) = 0, \ \ u(0) = \phi, \text{ and } u(t)(0) = Fu(t),\ t \geq 0. \tag{1.4}$$

This will be made more precise in Section 3, and will be the basis for our considerations here.

The object of this paper is the answer to the following problems in the above context of (ADP):

(Q1) Under which conditions on the mappings F and G does there hold a *principle of linearized stability* for (ADP)? More precisely: assume that, under the above conditions **(A1)** and **(A2)**, there exists an equilibrium solution $\phi_e \in L^1_+$ to (ADP), and that the functions F and G have a "Fréchet-" derivative $\tilde{F}[\phi_e] \in B(L^1, \mathbb{R}^n)$ and $\tilde{G}[\phi_e] \in B(L^1)$, respectively (to be made precise in the next section). Assume, moreover, that the corresponding linear (ADP), with F and G being replaced by $\tilde{F}[\phi_e]$ and $\tilde{G}[\phi_e]$, respectively, is exponentially asymptotically stable. Is it then true that the equilibrium ϕ_e is locally exponentially stable for the original (nonlinear) (ADP)?

(Q2) Under which (differentiability-) conditions on the mappings F and G are solutions to (ADP) *regular*; i.e., such that they are absolutely continuous and differentiable (with respect to age $a \in \mathbb{R}_+$) a.e. $a \in \mathbb{R}_+$ for t in their interval of existence, and, roughly speaking, classical solutions to (1.4)?

These problems have already been addressed by G.F. Webb [7] and J. Prüß [6] under global differentiability conditions on F and G, and, in the case of **(Q1)**, under additional assumptions on special forms of F and G. The essential point of our results will be a considerable weakening of the differentiability assumptions on F and G, as well as no further restriction on the forms of F and G.

The methods of proof of the corresponding results (Theorems 2.2 and 2.3 in Section 2) will be based on the nonlinear semigroup generated by the solutions to (ADP), and will be carried out in terms of the general theory of accretive operators and nonlinear semigroups (Section 3).

Notation and terminology. Throughout the paper, we shall denote the space $L^1(\mathbb{R}_+, \mathbb{R}^n)$ by L^1, and its positive cone $L^1(\mathbb{R}_+, \mathbb{R}^n_+)$ by L^1_+. For (real) Banach spaces X and Y, $B(X,Y)$ will denote the space of bounded linear operators from X to Y, and, for $X = Y$, we shall abbreviate this to $B(X)$. As for the norms, we shall indiscriminately use the symbol $\|\cdot\|$ for the norms of $\mathbb{R}^n$, of L^1, as well as of $B(X,Y)$.

Given a subset D of a real Banach space X, $cl\, D$ will denote its closure in X. Recall that a subset $C \subset X \times X$ is said to be *accretive* in X if for each $\lambda > 0$ and each pair $[x_i, y_i] \in C, i \in \{1,2\}$, we have $\|(x_1 + \lambda y_1) - (x_2 + \lambda y_2)\| \geq \|x_1 - x_2\|$, and ω-*accretive* in X for some $\omega \in \mathbb{R}$, if $(C + \omega I)$ is accretive. If, in addition, $R(I + \lambda C) = X$ for all $\lambda > 0$ with $\lambda\omega < 1$, then C is said to be $\omega - m$-accretive. If $C \subset X \times X$ is ω-accretive, then, for any $\lambda > 0$ with $\lambda\omega < 1$, $J^C_\lambda = (I + \lambda C)^{-1}$ denotes the resolvent of C. For all these notions and the general theory of accretive sets and evolution equations, the reader is referred to [1, 5].

2. Linearized stability and regularity for (ADP)

In this section, we formulate and discuss our results on linearized stability and on regularity of solutions for problem (ADP). The proofs will be given in Section 3 below.

The following notions of (relative) Fréchet-differentiability of a nonlinear map from X to Y, X and Y Banach spaces, will be basic for our considerations.

Definition 2.1.

1. A mapping $H : D(H) \subset X \to Y$ is said to be F-differentiable at $x \in D(H)$ (relative to its domain of definition $D(H)$) if the following holds: There exists $\tilde{H}[x] \in B(X,Y)$ such that, given any $\epsilon > 0$, there exists $\delta > 0$ such that, if $z \in D(H)$, and $\|z - x\| < \delta$, then
$$\left\| Hz - Hx - \tilde{H}[x](z - x) \right\| \leq \epsilon \|z - x\|.$$
2. A mapping $H : D(H) \subset X \to Y$ is said to be continuously F-differentiable on $D \subset D(H)$ if it is F-differentiable at each $x \in D$ (in the sense of 1. above), and if the map $\{x \to \tilde{H}[x]\}$ is continuous from D to $B(X,Y)$.

Notice that, in contrast to classical Fréchet-differentiability, the approximation is only required on a relative neighbourhood (with respect to $D(H)$) (as $D(H)$, in general, may not even contain any X-open subset).

Assumptions on F and G. In order to keep assumptions to a minimum, and since we are interested only in nonnegative solutions to (ADP), we shall, from now on, require F and G only to be defined on L^1_+, and to fulfill the following assumptions:

(A1*) $F : L^1_+ \to \mathbb{R}^n_+$, and $G : L^1_+ \to L^1$ are Lipschitz on norm-balls of L^1_+.

(A2*) There exists an increasing function $c : \mathbb{R}_+ \to \mathbb{R}_+$ such that if $r > 0$, and $\phi \in L^1_+$ with $\|\phi\| \leq r$, then $G(\phi) + c(r)\phi \in L^1_+$.

The following is our result on linearized stability for equilibria of (ADP).

Theorem 2.2. *Assume that there exists an equilibrium solution $\phi_e \in L^1_+$ to* (ADP) (*i.e., $u(t) \equiv \phi_e$ for all $t \geq 0$*)*. Assume, moreover, that both F and G have F-differentials $\tilde{F}[\phi_e] \in B(L^1, \mathbb{R}^n)$ and $\tilde{G}[\phi_e] \in B(L^1)$ at ϕ_e, respectively. If the zero solution to the linearized* (ADP)*, with F and G replaced by $\tilde{F}[\phi_e]$ and $\tilde{G}[\phi_e]$, respectively, is exponentially stable, then the equilibrium ϕ_e is locally exponentially stable for* (*the original nonlinear*) (ADP)*; i.e., there exist $\delta > 0$, $M \geq 1$, and $\alpha > 0$ such that for all $\phi \in L^1_+$, with $\|\phi - \phi_e\| < \delta$, the solution u_ϕ to* (ADP) *exists for all times* ($T_\phi = \infty$), *$u_\phi(t) \in L^1_+$ for all $t \geq 0$, and*

$$\|u_\phi(t) - \phi_e\| \leq M e^{-\alpha t}\|\phi - \phi_e\| \quad \textit{for all} \quad t \geq 0.$$

In order to formulate our result on regularity of solutions to (ADP) – and to keep this formulation short, we temporarily introduce the following (kind of) regularity subset $R_{(ADP)}$ of L^1_+ for the problem (ADP):

$$R_{(ADP)} := \{\phi \in L^1_+ \mid \phi \text{ is absolutely continuous on } \mathbb{R}_+, \phi' \in L^1, \text{ and } \phi(0) = F(\phi)\}.$$

Theorem 2.3. *Assume that F and G are continuously differentiable on L^1_+* (*in the sense of Definition* 2.1, 2., *above*)*. Let $\phi \in R_{(ADP)}$, and let u_ϕ denote the corresponding solution to* (ADP) *on* $[0, T_\phi)$*. Then the following hold:*

(i) $u_\phi(t) \in R_{(ADP)}$ *for all $t \in [0, T_\phi)$, the function $\{t \to u_\phi(t)\}$ is continuously differentiable from $[0, T_\phi)$ to L^1, and u_ϕ is a classical solution to* (1.4).

(ii) $$\left\|\frac{d}{dt}u_\phi(t)\right\| \leq e^{t\tilde{\omega}(t)}\|\phi' - G\phi\| \quad \textit{for all} \quad 0 \leq t \leq T < T_\phi,$$
with $\tilde{\omega}(t) := \sup_{0\leq s\leq t}\left\|\tilde{F}[u_\phi(s)]\right\| + \sup_{0\leq s\leq t}\left\|\tilde{G}[u_\phi(s)]\right\|$.

(iii) *The mapping* $\left\{t \to \dfrac{d}{dt}u_\phi(t)\right\}$ *is a mild solution to the linearized nonautonomous version of* (ADP)

$$(ADP)_{\text{lin}} \quad \begin{cases} \dot{v}(t) + v(t)_a - \tilde{G}[u_\phi(t)]v(t) = 0, & 0 \leq t \leq T < T_\phi, \\ v(0) = -\phi' + G(\phi), \\ v(t)(0) = \tilde{F}[u_\phi(t)]v(t), & 0 \leq t \leq T < T_\phi. \end{cases}$$

Remarks 2.4.

1. Assertion (iii) of Theorem 2.3 means that $\dot{u}_\phi(t) = U(t,0)(-\phi' + G(\phi)), 0 \leq t \leq T < T_\phi$, where $\{U(t,s) \mid 0 \leq s \leq t \leq T\}$ is the (linear) evolution system generated by $(ADP)_{lin}$ (for details, see the corresponding proof in Section 3 below).
2. Both of Theorems 2.2 and 2.3 have been proved in [7] (see [7, Thm. 4.13 and Thm. 2.10]; for Thm. 2.2, compare also [6]) under the following global stronger differentiability assumptions on F and G : both F and G are F-differentiable on L^1 (in the sense of Definition 2.1, 1., above), and the maps $\{\phi \to \tilde{F}[\phi]\}$ and $\{\phi \to \tilde{G}[\phi]\}$ are (not only continuous, but) Lipschitz on balls of L^1. In addition, both in [7] and [6], for the analogue of Theorem 2.2, the authors have assumed (various kinds of) special forms of F and G.
3. Aside from the weakening in the assumptions from local Lipschitz-continuity to simply continuity of the differential-maps in the above Theorems, the main point of Theorem 2.2 is the fact that we dispense with global differentiability properties of F and G altogether, and require F-differentiability of both F and G only at the equilibrium point ϕ_e. Moreover, we pose no extra conditions on a special form of F and G. (It should be noted, however, that both in [7] and [6], the authors have used the assumptions on special forms of F and G to (a) deduce concrete criteria for the linearized problem to be exponentially asymptotically stable, and (b) to complement the stability result by a result on instability of the equilibrium ϕ_e.)

3. Proofs of Theorems 2.2 and 2.3

Our methods of proof will entirely be based on the (nonlinear) semigroup approach to problem (ADP) as developed by G.F. Webb [7]. The following are the basic results we shall need. Throughout this section, we assume the general hypotheses **(A1*)** and **(A2*)** to be in place.

We associate with (ADP) the operator A in L^1 defined by

$$\begin{cases} D(A) = \{\phi \in L^1_+ \mid \phi \text{ absolutely continuous on } \mathbb{R}_+, \, \phi' \in L^1, \, \phi(0) = F(\phi)\}, \\ \quad A\phi := \phi' - G(\phi) \, , \phi \in D(A). \end{cases}$$

Then, if in addition, both F and G are globally Lipschitz on L^1_+, and we let $\omega = \|F\|_{Lip} + \|G\|_{Lip}$, we have from [7, Props. 3.8 and 3.9]:

(W1) $(A + \omega I)$ is accretive, $R(I + \lambda A) = L^1_+$ for all $\lambda > 0$ with $\lambda\omega < 1$, and $cl\, D(A) = L^1_+$. Finally, if for $\phi \in L^1_+$, u_ϕ denotes the corresponding solution to (ADP) (global in this case), then $u_\phi(t) = S(t)\phi$ for all $t \geq 0$, where $(S(t))_{t\geq 0}$ denotes the semigroup of operators on L^1_+ generated by $-A$ (in the sense of Crandall-Liggett). In this sense, u_ϕ is a mild solution to (1.4).

(An independent proof of the latter fact will be given in the Appendix below.)

Furthermore, this approach can be 'localized', i.e., adapted to the general locally Lipschitz-case (for F and G) and associated, possibly only local solutions by (the method of proof of) [7, Thm. 3.3]: If, under our general standing hypotheses **(A1*)** and **(A2*)**, u_ϕ is a local solution to (ADP) on $[0, T_\phi)$, and $0 < T < T_\phi$ is given, we can let, say, $r = \sup_{0\le t\le T} \|u_\phi(t)\| + 1$, and consider the operator A_r corresponding to A above with F and G being replaced by their radial truncations F_r and G_r to conclude that $u_\phi(t) = S_r(t)\phi$ for $0 \le t \le T$, with $(S_r(t))_{t\ge 0}$ the semigroup generated by $-A_r$. Here, for a mapping $H : D(H) \subset X \to Y$, the radial truncation H_r for $r > 0$ is defined by

$$H_r(x) = \begin{cases} Hx & \text{for } \|x\| \le r \\ H\left(r\dfrac{x}{\|x\|}\right) & \text{for } \|x\| \ge r. \end{cases}$$

Thus, in the following, for the proofs of Theorems 2.2 and 2.3 we shall arrange things for having both F and G globally Lipschitz, and use **(W1)**.

We start with a technical Lemma that will be the starting point for the proofs of Theorems 2.2 and 2.3.

Let $\phi = \phi_e$ (Theorem 2.2), or $\phi \in L^1_+$ arbitrary (Theorem 2.3). Then, under the assumptions of Theorems 2.2 and 2.3, respectively, together with the (nonlinear) operator A as above, for $0 < T < T_\phi$, we also consider the family of (linear) operators $\{\tilde{A}(t) \mid 0 \le t \le T\}$, defined by

$$\begin{cases} D(\tilde{A}(t)) = \{\psi \in L^1 \mid \psi \text{ absolutely continuous} \\ \qquad\qquad \text{on } \mathbb{R}_+,\ \psi' \in L^1, \text{ and } \psi(0) = \tilde{F}[S(t)\phi](\psi)\}, \\ \tilde{A}(t)\psi := \psi' - \tilde{G}[S(t)\phi](\psi)\ ,\psi \in D(\tilde{A}(t)). \end{cases}$$

We shall use in the following that, according to [7, Prop. 3.11], any one of these operators is $\tilde{\omega}_t - m$-accretive with domain dense in L^1, where

$$\tilde{\omega}_t = \left\|\tilde{F}[S(t)\phi]\right\| + \left\|\tilde{G}[S(t)\phi]\right\|.$$

Notation. For the remainder of this section, we shall use the following notational conventions for $t \in [0, T]$, and ϕ as above:

1. $\tilde{\omega}(t) := \sup_{0\le r\le t} \left\|\tilde{F}[S(r)\phi]\right\| + \sup_{0\le r\le t} \left\|\tilde{G}[S(r)\phi]\right\|$.

2. $J_\lambda = {J_\lambda}^A$, and, for $r > 0$, $\tilde{J}_\lambda(r) = J_\lambda^{\tilde{A}[r]}$, for any $\lambda > 0$ with $\lambda \max(\omega, \tilde{\omega}_r) < 1$.

Lemma 3.1. *Let ϕ, and $0 < T < T_\phi$ be as above, and choose $\lambda_0 > 0$ such that $\lambda_0\omega < 1$, and $\lambda_0\tilde{\omega}(T + \delta) < 1$, and $\lambda_0 < \delta$, where $\delta > 0$ is chosen such that $(T + \delta) < T_\phi$. Further, let $\tilde{\omega} = \tilde{\omega}(T + \delta)$. Then, for $\psi \in L^1_+$, and $n \in \mathbb{N}$ so large* *that* $t < n\lambda_0$, *and* $1 < n\delta$, *we have, for* $\lambda = \frac{t}{n}$,

$$(1-\lambda\tilde{\omega})^{(n+1)}\left\| J_\lambda{}^n\psi - J_\lambda{}^n\phi - \prod_{j=1}^{n}\tilde{J}_\lambda(j\lambda)(\psi-\phi)\right\| \leq \int_\lambda^{(n+1)\lambda}(1-\lambda\tilde{\omega})^{[\tau/\lambda]} \tag{3.1}$$
$$\times \quad \left\| F(J_\lambda{}^{[\tau/\lambda]}\psi) - F(J_\lambda{}^{[\tau/\lambda]}\phi) - \tilde{F}[J_\lambda{}^{[\tau/\lambda]}\phi](J_\lambda{}^{[\tau/\lambda]}\psi - J_\lambda{}^{[\tau/\lambda]}\phi)\right\| d\tau$$
$$+ \quad \int_\lambda^{(n+1)\lambda}(1-\lambda\tilde{\omega})^{[\tau/\lambda]}\left\| G(J_\lambda{}^{[\tau/\lambda]}\psi) - G(J_\lambda{}^{[\tau/\lambda]}\phi) - \tilde{G}[J_\lambda{}^{[\tau/\lambda]}\phi]\right.$$
$$\left.(J_\lambda{}^{[\tau/\lambda]}\psi - J_\lambda{}^{[\tau/\lambda]}\phi)\right\| d\tau + \int_\lambda^{(n+1)\lambda}(1-\lambda\tilde{\omega})^{[\tau/\lambda]}\left\| (\tilde{F}[J_\lambda{}^{[\tau/\lambda]}\phi]\right.$$
$$- \quad \left.\tilde{F}[S(\lambda[\tau/\lambda])\phi])(J_\lambda{}^{[\tau/\lambda]}\psi - J_\lambda{}^{[\tau/\lambda]}\phi)\right\| d\tau + \int_\lambda^{(n+1)\lambda}(1-\lambda\tilde{\omega})^{[\tau/\lambda]}$$
$$\times \quad \left\| (\tilde{G}[J_\lambda{}^{[\tau/\lambda]}\phi] - \tilde{G}[S(\lambda[\tau/\lambda])\phi])(J_\lambda{}^{[\tau/\lambda]}\psi - J_\lambda{}^{[\tau/\lambda]}\phi)\right\| d\tau,$$

(*where* $[\tau/\lambda]$ *denotes the largest integer less than or equal to* τ/λ).

Proof. The proof will be an easy induction argument. The basic inequality we need is the following: Let $\rho \in L^1_+$, and let $\rho_\lambda = J_\lambda\rho$, i.e., ρ_λ solves the differential equation $\rho_\lambda + \lambda\rho'_\lambda - \lambda G(\rho_\lambda) = \rho$, $\rho_\lambda(0) = F(\rho_\lambda)$. Thus,

$$\rho_\lambda(a) = \exp\left(-\frac{a}{\lambda}\right)\left\{ F(\rho_\lambda) + \int_0^a e^{\frac{b}{\lambda}}[(G(\rho_\lambda) + \frac{1}{\lambda}\rho)(b)]\, db\right\}, a \in \mathbb{R}_+.$$

Thus,

$$\|\rho_\lambda\| \leq \|\rho\| + \lambda(\|F(\rho_\lambda)\| + \|G(\rho_\lambda)\|) \tag{3.2}$$

In the following, for $\phi, \psi \in L^1_+$, $k \in \mathbb{N}$, and $\lambda > 0$ small enough, let

$$\begin{aligned}
a_k &= \left\| J_\lambda{}^k\psi - J_\lambda{}^k\phi - \prod_{j=1}^{k}\tilde{J}_\lambda(j\lambda)(\psi-\phi)\right\| \\
b_k &= \left\| F(J_\lambda{}^k\psi) - F(J_\lambda{}^k\phi) - \tilde{F}[J_\lambda{}^k\phi](J_\lambda{}^k\psi - J_\lambda{}^k\phi)\right\| \\
c_k &= \left\| G(J_\lambda{}^k\psi) - G(J_\lambda{}^k\phi) - \tilde{G}[J_\lambda{}^k\phi](J_\lambda{}^k\psi - J_\lambda{}^k\phi)\right\| \\
d_k &= \left\| (\tilde{F}[J_\lambda{}^k\phi] - \tilde{F}[S(k\lambda)\phi])(J_\lambda{}^k\psi - J_\lambda{}^k\phi)\right\|, \qquad \text{and} \\
e_k &= \left\| (\tilde{G}[J_\lambda{}^k\phi] - \tilde{G}[S(k\lambda)\phi])(J_\lambda{}^k\psi - J_\lambda{}^k\phi)\right\|
\end{aligned}$$

(Notice that, for $\phi = \phi_e$, $d_k = e_k = 0$.)

Then, for λ_0 and $0 < t < T$, and $\tilde{\omega}$ as in Lemma 3.1, and for $0 < \lambda < \lambda_0$, for all $k \in \{1, \ldots, [t/\lambda]\}$,

$$(1 - \lambda\tilde{\omega})^{(k+1)} a_k \leq \lambda \sum_{j=1}^{k} (1 - \lambda\tilde{\omega})^j (b_j + c_j + d_j + e_j). \tag{3.3}$$

Proof. For $k = 1$, we use (3.2) to conclude that

$$\begin{aligned}
a_1 &= \left\| J_\lambda\psi - J_\lambda\phi - \tilde{J}_\lambda(\lambda)(\psi - \phi) \right\| \\
&\leq \lambda \left\| F(J_\lambda\psi) - F(J_\lambda\phi) - \tilde{F}[S(\lambda)\phi]\tilde{J}_\lambda(\lambda)(\psi - \phi) \right\| \\
&+ \lambda \left\| G(J_\lambda\psi) - G(J_\lambda\phi) - \tilde{G}[S(\lambda)\phi]\tilde{J}_\lambda(\lambda)(\psi - \phi) \right\| \\
&\leq \lambda[b_1 + c_1 + d_1 + e_1 + (\left\| \tilde{F}[S(\lambda)\phi] \right\| + \left\| \tilde{G}[S(\lambda)\phi] \right\|)a_1]
\end{aligned}$$

Rearranging, and noting that $\left(\left\| \tilde{F}[S(\lambda)\phi] \right\| + \left\| \tilde{G}[S(\lambda)\phi] \right\| \right) \leq \tilde{\omega}$ proves the case of $k = 1$.

Similarly, for the step from k to $(k + 1)$, we conclude by means of (3.2) that

$$\begin{aligned}
a_{k+1} &= \left\| J_\lambda({J_\lambda}^k\psi) - J_\lambda({J_\lambda}^k\phi) - \tilde{J}_\lambda((k+1)\lambda) \prod_{j=1}^{k} \tilde{J}_\lambda(j\lambda)(\psi - \phi) \right\| \\
&\leq a_k + \lambda \left\| F({J_\lambda}^{k+1}\psi) - F({J_\lambda}^{k+1}\phi) - \tilde{F}[S((k+1)\lambda)\phi] \prod_{j=1}^{k+1} \tilde{J}_\lambda(j\lambda)(\psi - \phi) \right\| \\
&\quad + \lambda \left\| G({J_\lambda}^{k+1}\psi) - G({J_\lambda}^{k+1}\phi) - \tilde{G}[S((k+1)\lambda)\phi] \prod_{j=1}^{k+1} \tilde{J}_\lambda(j\lambda)(\psi - \phi) \right\| \\
&\leq a_k + \lambda[b_{k+1} + c_{k+1} + d_{k+1} + e_{k+1} + (\left\| \tilde{F}[S((k+1)\lambda)\phi] \right\| \\
&\quad + \left\| \tilde{G}[S((k+1)\lambda)\phi] \right\|)a_{k+1}]
\end{aligned}$$

Rearranging, and invoking the induction hypothesis completes the proof of (3.3).

The estimate (3.1) now follows from (3.3) by the particular choice of $k = n$ and $\lambda = \frac{t}{n}$ with n, t as in Lemma 3.1, and by replacing any of the summands to the right of estimate (3.3) by an appropriate integral, such as, for instance, for b_j:

$$\begin{aligned}
\lambda(1 - \lambda\tilde{\omega})^j\, b_j &= \int_{j\lambda}^{(j+1)\lambda} (1 - \lambda\tilde{\omega})^{[\tau/\lambda]} \\
&\times \left\| F({J_\lambda}^{[\tau/\lambda]}\psi) - F({J_\lambda}^{[\tau/\lambda]}\phi) - \tilde{F}[{J_\lambda}^{[\tau/\lambda]}\phi]({J_\lambda}^{[\tau/\lambda]}\psi - {J_\lambda}^{[\tau/\lambda]}\phi) \right\| d\tau.
\end{aligned}$$

This completes the proof of Lemma 3.1. □

Proof of Theorem 2.2. If $\phi_e \in L^1_+$ is an equilibrium for (ADP), then, according to [7, Prop. 4.1], $\phi_e \in D(A)$, and $A\phi_e = 0$. Let $r := (\|\phi_e\| + 1)$, and consider the operator A in L^1 as above, with F and G being replaced by their respective r-truncations. Then, as $J_\lambda \phi_e = \phi_e$, and $S(t)\phi_e = \phi_e$, $t \geq 0$, the corresponding (linear) operators $\tilde{A}(t)$ as above are constant, denoted by, say, $\tilde{A}$. Denoting the (linear) semigroup generated by $-\tilde{A}$ by $(\tilde{S}(t))_{t\geq 0}$, we thus read from Lemma 3.1 for this particular case that, for $\tilde{\omega} = (\left\|\tilde{F}[\phi_e]\right\| + \left\|\tilde{G}[\phi_e]\right\|)$, and for any given $t > 0$,

$$\begin{aligned} & e^{-\tilde{\omega}t}\left\|S(t)\psi - \phi_e - \tilde{S}(t)(\psi - \phi_e)\right\| && (3.4) \\ \leq & \int_0^t e^{-\tilde{\omega}\tau}\left\|F(S(\tau)\psi) - F\phi_e - \tilde{F}[\phi_e](S(\tau)\psi - \phi_e)\right\| d\tau \\ + & \int_0^t e^{-\tilde{\omega}\tau}\left\|G(S(\tau)\psi) - G\phi_e - \tilde{G}[\phi_e](S(\tau)\psi - \phi_e)\right\| d\tau. \end{aligned}$$

Given $\epsilon > 0$, choose a joint $\delta' > 0$ for $\tilde{F}[\phi_e]$ and $\tilde{G}[\phi_e]$ as in Definition 2.1, 1., and let $\delta := e^{-\omega t}\delta'$. Then, as $\|S(\tau)\psi - \phi_e\| \leq e^{\omega\tau}\|\psi - \phi_e\|$, and thus $\|S(\tau)\psi - \phi_e\| \leq \delta'$ for all $\tau \in [0, t]$ and all $\psi \in L^1_+$ with $\|\psi - \phi_e\| < \delta$, we read from (3.4) that

$$\left\|S(t)\psi - \phi_e - \tilde{S}(t)(\psi - \phi_e)\right\| \leq \epsilon M(t)\|\psi - \phi_e\|$$

for all $\psi \in L^1_+$ with $\|\psi - \phi_e\| < \delta$, and some positive constant $M(t)$. This shows that, given any $t > 0$, $\tilde{S}(t)$ is an F-differential of $S(t)$ in the sense of Definition 2.1, 1. As, by the assumptions of Theorem 2.2, the semigroup $(\tilde{S}(t))_{t\geq 0}$ is exponentially stable, a result of Desch and Schappacher [2, Prop. 2.1] (compare also proposition 2. of the Final remarks below) implies that there exist $0 < \delta < 1$, $M \geq 1$, and $\alpha > 0$ such that for all $\phi \in L^1_+$, with $\|\phi - \phi_e\| < \delta$,

$$\|S(t)\phi - \phi_e\| \leq Me^{-\alpha t}\|\phi - \phi_e\| \quad \text{for all} \quad t \geq 0.$$

Replacing now the δ in this assertion by $0 < \delta_1 < M^{-1}\delta$, then we also have that $\|S(t)\phi\| < (\|\phi_e\| + 1) = r$ for all $t \geq 0$, and for all $\phi \in L^1_+$, with $\|\phi - \phi_e\| < \delta_1$. Thus, for all such ϕ, both $F(S(t)\phi) = F_r((S(t)\phi)$ and $G(S(t)\phi) = G_r((S(t)\phi)$, so that $u_\phi(t) = S(t)\phi$ also is a global solution to (the un-truncated, original) (ADP). This completes the proof of Theorem 2.2. □

Proof of Theorem 2.3. Given $\phi \in R_{(ADP)}$ as in Theorem 2.3, let $0 < T < T_\phi$, and choose $\delta > 0$ such that $(T + \delta) < T_\phi$. Let $C(\phi)$ be the closed convex hull of the set $\{u_\phi(s) \mid 0 \leq s \leq (T + \delta)\}$. Let $\rho = (\sup\{\|\psi\| \mid \psi \in C(\phi)\} + 1)$ and consider the operator A in L^1 as above, with F and G being replaced by their respective ρ-truncations. Then, as explained at the beginning of this section, $u_\phi(s) = S(s)\phi$ for $0 \leq s \leq (T + \delta)$. Moreover, as will be needed later on, at points of $C(\phi)$, the F-derivatives of the ρ-truncations of F and G can be taken as those of the original maps F and G.

Step 1: We consider the family $\{\tilde{A}(s) \mid 0 \leq s \leq t\}$ of linear operators in L^1 as defined above. These operators are all densely defined, and $\tilde{\omega}(t) - m$-accretive (with $\tilde{\omega}(t)$ as defined in Theorem 2.3, (ii)). Moreover, using the definitions and the basic inequality (3.2), it is easy to check that, given $\lambda_0 > 0$, with $\lambda_0\tilde{\omega}(t) < 1$, there exists $C \geq 1$ such that

$$\begin{aligned}\left\|\tilde{J}_\lambda(r)\psi - \tilde{J}_\lambda(s)\psi\right\| \leq \lambda C(&\left\|\tilde{F}[S(r)\phi] - \tilde{F}[S(s)\phi]\right\| \\ + &\left\|\tilde{G}[S(r)\phi] - \tilde{G}[S(s)\phi]\right\|)\|\psi\|\end{aligned} \tag{3.5}$$

for all $0 \leq r, s \leq t$, $\psi \in L^1$, and all $0 < \lambda < \lambda_0$. Thus, as it is assumed that the maps $\{\psi \to \text{the F-differential at}\,\psi\}$ are continuous on L^1_+ for both $\tilde{F}$ and $\tilde{G}$, we conclude from [1, Thm. 2.1] that the family $\{\tilde{A}(s) \mid 0 \leq s \leq t\}$ generates an evolution family $\{U(s,r) \mid 0 \leq r \leq s \leq t\}$ of bounded linear operators on L^1, which is given by

$$U(s,r)\psi = \lim_{n\to\infty}\prod_{j=1}^{n}\tilde{J}_{(s-r)/n}(r + j(\frac{s-r}{n}))\psi \tag{3.6}$$

(for $0 \leq r < s \leq t$; while $U(r,r) = I$), and fulfills the estimate

$$\|U(s,r)\psi - U(s,r)\rho\| \leq e^{\tilde{\omega}(t)(s-r)}\|\psi - \rho\| \tag{3.7}$$

for all $0 \leq r \leq s \leq t$, and all $\psi, \rho \in L^1$. (Actually, in [1, Thm. 2.1], the assumption on the controlling function in estimate (3,5) is that it be a continuous function with values in X. But, in the course of the proof, it is only the modulus of continuity of the function that comes into play; so the continuous control of (3.5) works as well.)

From (3.6), by using continuity of the differential maps for $\tilde{F}$ and $\tilde{G}$, and invoking Lebesgue's Dominated Convergence Theorem, we conclude from (3.1) of Lemma 3.1 that

$$\begin{aligned}&e^{-\tilde{\omega}t}\|S(t)\psi - S(t)\phi - U(t,0)(\psi - \phi)\| \\ &\leq \int_0^t e^{-\tilde{\omega}\tau}\left\|F(S(\tau)\psi) - F(S(\tau)\phi) - \tilde{F}[S(\tau)\phi](S(\tau)\psi - S(\tau)\phi)\right\|d\tau \\ &+ \int_0^t e^{-\tilde{\omega}\tau}\left\|G(S(\tau)\psi) - G(S(\tau)\phi) - \tilde{G}[S(\tau)\phi](S(\tau)\psi - S(\tau)\phi)\right\|d\tau\end{aligned}$$

for all $\psi \in L^1_+$ (with $\tilde{\omega}$ as defined in Lemma 3.1).

If we specialize this estimate for $\psi = S(h)\phi$, with $h \in [0,\delta)$, we arrive at

$$\begin{aligned}&e^{-\tilde{\omega}t}\|S(t+h)\phi - S(t)\phi - U(t,0)(S(h)\phi - \phi)\| \\ &\leq \int_0^t e^{-\tilde{\omega}\tau}\left\|F(S(\tau+h)\phi) - F(S(\tau)\phi) - \tilde{F}[S(\tau)\phi](S(\tau+h)\phi - S(\tau)\phi)\right\|d\tau \\ &+ \int_0^t e^{-\tilde{\omega}\tau}\left\|G(S(\tau+h)\phi) - G(S(\tau)\phi) - \tilde{G}[S(\tau)\phi](S(\tau+h)\phi - S(\tau)\phi)\right\|d\tau.\end{aligned} \tag{3.8}$$

Step 2: In order to get a uniform estimate on the integrands on the right-hand side of (3.8), we use the following "convexity-trick":

Given a mapping $H : D(H) \subset X \to Y$, X and Y real Banach spaces, that is continuously F-differentiable on a convex subset $C \subset D(H)$ in the sense of Definition 2.1, 2., and given $x, y \in C$, define the map $R : [0,1] \to Y$ by

$$R(\alpha) := H(x + \alpha(y-x)) - \alpha\tilde{H}[x](y-x).$$

Then, $R'(\alpha) = (\tilde{H}[x+\alpha(y-x)] - \tilde{H}[x])(y-x)$ for all $0 < \alpha < 1$. By continuity of both R and the differential map,

$$R(1) - R(0) = \int_0^1 (\tilde{H}[x+\alpha(y-x)] - \tilde{H}[x])(y-x)d\alpha. \tag{3.9}$$

We now specialize this result to the maps F and G in the context of (3.8): For $h \in [0,\delta)$, and $\tau \in [0,t]$ as above, let

$$R(\alpha) := F(S(\tau)\phi+\alpha(S(\tau+h)\phi-S(\tau)\phi))-\alpha\tilde{F}[S(\tau)\phi](S(\tau+h)\phi-S(\tau)\phi), \alpha \in [0,1].$$

Then, from (3.9),

$$\begin{aligned} &\left\|F(S(\tau+h)\phi) - F(S(\tau)\phi) - \tilde{F}[S(\tau)\phi](S(\tau+h)\phi - S(\tau)\phi)\right\| \\ &\quad\leq \|S(\tau+h)\phi - S(\tau)\phi\| \int_0^1 \left\|\tilde{F}[S(\tau)\phi + \alpha(S(\tau+h)\phi - S(\tau)\phi)] - \tilde{F}[S(\tau)\phi]\right\| d\alpha. \end{aligned} \tag{3.10}$$

At this point, note that the set $C(\phi)$ is (convex and) compact. Hence, the (continuous) differential map $\{\psi \to \tilde{F}[\psi]\}$ is uniformly continuous on $C(\phi)$. Let $\epsilon > 0$, and choose $\delta_1 > 0$ such that, if $\psi, \rho \in C(\phi)$, with $\|\psi - \rho\| < \delta_1$, then $\left\|\tilde{F}[\psi] - \tilde{F}[\rho]\right\| < \epsilon e^{-\omega t}$. Now, choose $0 < \delta_2 < \delta$ (δ from the beginning of the proof), such that, for $h \in [0,\delta_2)$, $\|S(h)\phi - \phi\| < \delta_1 e^{-\omega t}$. Then,

$$(S(\tau)\phi + \alpha(S(\tau+h)\phi - S(\tau)\phi)), S(\tau)\phi \in C(\phi), \text{ and}$$

$$\|(S(\tau)\phi + \alpha(S(\tau+h)\phi - S(\tau)\phi) - S(\tau)\phi)\| = \alpha\|S(\tau+h)\phi - S(\tau)\phi\| < \delta_1,$$

and thus

$$\left\|\tilde{F}[S(\tau)\phi + \alpha(S(\tau+h)\phi - S(\tau)\phi)] - \tilde{F}[S(\tau)\phi]\right\| < \epsilon e^{-\omega t} \tag{3.11}$$

for all $0 \leq h \leq \delta_2$, and all $\alpha \in [0,1]$, and $\tau \in [0,t]$.
By (3.10), this yields

$$\begin{aligned} \left\|F(S(\tau+h)\phi) - F(S(\tau)\phi) - \tilde{F}[S(\tau)\phi](S(\tau+h)\phi - S(\tau)\phi)\right\| \\ \leq \quad \epsilon\|S(h)\phi - \phi\| \end{aligned} \tag{3.12}$$

for all $0 \leq h < \delta_2$, and all $\tau \in [0,t]$.

As the same reasoning also works for G, we can thus conclude from (3.8) that there exists $0 < \delta_3 < \delta$ such that

$$\|S(t+h)\phi - S(t)\phi - U(t,0)(S(h)\phi - \phi)\| \leq \epsilon M(t)\|S(h)\phi - \phi\| \tag{3.13}$$

for all $0 \leq h < \delta_3$, and some positive constant $M(t)$.

Step 3: In [7, Thm. 3.1] it has been shown that, under our hypotheses, the operator $-A$ is (not only the generator in the sense of Crandall-Ligget, but also) the infinitesimal generator of $S(t))_{t\geq 0}$, i.e.,

$$\begin{cases} D(A) = \{\psi \in L^1_+ \mid \lim\limits_{h\to 0^+} h^{-1}(S(h)\psi - \psi) \text{ exists}\}, \text{ and} \\ -A\psi = \lim\limits_{h\to 0^+} h^{-1}(S(h)\psi - \psi)\ , \psi \in D(A). \end{cases}$$

As, by assumption, $\phi \in D(A)$, we thus conclude from (3.13) that $u_\phi(t) = S(t)\phi \in D(A)$ for all $0 \leq t < T_\phi$. (3.13) as well yields that, for $0 < t < T_\phi$, $u_\phi = S(\cdot)\phi$ is actually differentiable (not just from the right), so that, altogether, u_ϕ in fact is a classical solution to (1.4).

Moreover, as $\lim_{h\to 0^+} h^{-1}(S(h)\phi - \phi) = -A\phi = (-\phi' + G\phi)$, we read from (3.13) that $\dot{u}_\phi(t) = U(t,0)(-\phi' + G\phi)$ for $0 \leq t < T_\phi$, so that it is a mild solution to the evolution equation

$$(ADP)_{lin} \quad \begin{cases} \dot{v}(t) + \tilde{A}(t)v(t) = 0, \ \ 0 \leq t \leq T < T_\phi, \\ v(0) = -\phi' + G(\phi), \\ v(t)(0) = \tilde{F}[u_\phi(t)]v(t), \ \ 0 \leq t \leq T < T_\phi. \end{cases}$$

Finally, the estimate of proposition (ii) of Theorem 2.3 now follows from (3.7). This completes the proof of Theorem 2.3. $\square$

Final remarks. 1. Notice that, aside from the concrete problem (ADP), for our results in a general framework, the space $\mathbb{R}^n$ could be replaced by just any real Banach lattice X. In this context, the corresponding result to Theorem 2.2 for the operator A with $G \equiv 0$ has been given in [3, Thm. 4.4] under the assumption that F be (globally defined and) Fréchet-differentiable (in the usual sense) at the equilibrium, and with recourse to the Desch-Schappacher result as well.

2. In fact, the latter result [2, Prop. 2.1] has been proved for a (nonlinear) semigroup $(S(t))_{t\geq 0}$ defined on an open subset C of a Banach space X, and for Fréchet-differentiability of $S(t)$ at the equilibrium in the classical sense. However, the proof as given in [2] works as well for just any subset $C \subset X$ with $S(t)C \subset C$, and F-differentiability of $S(t)$ at an equilibrium $x_e \in C$ in the relative C-sense as in Definition 2.1, 1., above. As, in addition, the paper [2] may not always be easily accessible, we briefly indicate the arguments of proof of [2, Prop. 2.1] in this general context:

Given a semigroup of nonlinear operators $S(t) : C \to C$, $C \subset X$ any subset of X, such that, for some $\omega \geq 0$, $\|S(t)x - S(t)y\| \leq e^{\omega t}\|x - y\|, x, y \in C$, and with F-differentials $\tilde{S}(t) \in B(X)$ at an equilibrium $x_e \in C$ (in the relative C-sense as in Definition 2.1, 1., above), assume there exist $\tilde{\omega} > 0$, and $\tilde{M} \geq 1$ such that $\left\|\tilde{S}(t)\right\| \leq \tilde{M}e^{-\tilde{\omega}t}$ for $t \geq 0$. Fix $k \in \mathbb{N}$ such that $\left\|\tilde{S}(k)\right\| \leq (1/4)$. Then, for $\epsilon = (1/4)$, there exists $\delta = \delta(\epsilon, k) > 0$ such that, for all $x \in C$ with $\|x - x_e\| < \delta$,

$$\|S(k)x - x_e\| \leq (1/4)\|x - x_e\| + \left\|\tilde{S}(k)(x - x_e)\right\| \leq (1/2)\|x - x_e\| < (1/2)\delta.$$

At this point, choose any $\alpha > 0$ such that $0 < \alpha < (ln2/k)$, and let $\kappa := (1/2)e^{\alpha k}$. Then $0 < \kappa < 1$, and $e^{\alpha k}\|S(k)x - x_e\| \leq \kappa\|x - x_e\|$ for all $x \in C$ with $\|x - x_e\| < \delta$. By induction on $n \in \mathbb{N}$, we actually get

$$e^{\alpha nk}\|S(nk)x - x_e\| \leq \kappa^n\|x - x_e\| \text{ for all } x \in C \text{ with } \|x - x_e\| < \delta, \text{ and } n \in \mathbb{N}.$$

At this point, let $\delta_1 := e^{-\omega k}\delta$, and let $x \in C$ with $\|x - x_e\| < \delta_1$. Given any $t \geq k$, let $t = n_t k + \gamma_t$, with $n_t \in \mathbb{N}$, and $0 \leq \gamma_t < k$. By noting that $\|S(\gamma_t)x - x_e\| \leq e^{\omega\gamma_t}\|x - x_e\| < \delta$, we get

$$e^{\alpha t}\|S(t)x - x_e\| = e^{\alpha\gamma_t}e^{\alpha n_t k}\|S(n_t k)S(\gamma_t)x - x_e\| \leq e^{(\alpha+\omega)k}\kappa^{n_t}\|x - x_e\|.$$

Also, for $0 \leq t \leq k$, we have $e^{\alpha t}\|S(t)x - x_e\| \leq e^{(\alpha+\omega)k}\|x - x_e\|$. Thus, letting $M := e^{(\alpha+\omega)k}$, we have $\|S(t)x - x_e\| \leq Me^{-\alpha t}\|x - x_e\|$ for all $t \geq 0$, and for all $x \in C$ with $\|x - x_e\| < \delta_1$. This completes the proof.

4. Appendix

The fact that the semigroup solutions $u_\phi = S(\cdot)\phi$, $\phi \in L^1_+$, of the semigroup $(S(t))_{t\geq 0}$ generated by the operator $(-A)$ as defined in the second paragraph of Section 3 above are solutions to problem (ADP) has been shown in [7] via an equivalent integral equation (equation (1.49) in [7, Section 1.4]). In order to keep the present paper self-contained, we here give a direct proof. We restrict ourselves to the case of globally (on L^1_+) Lipschitz continuous mappings F and G; the local case can be dealt with via the radial truncations of these mappings.

With the notations as in Section 3 above, given $\phi \in L^1_+$, $\lambda > 0$, and $k \in \mathbb{N} \cup \{0\}$, we have

$$J_\lambda^{k+1}\phi - J_\lambda^k\phi = \lambda G(J_\lambda^{k+1}\phi) - \lambda(J_\lambda^{k+1}\phi)'.$$

Given $0 \leq s < t$, and $0 < \lambda < t$, summing this equation from $k = [s/\lambda]$ to $k = ([t/\lambda] - 1)$ yields

$$\begin{aligned} J_\lambda^{[t/\lambda]}\phi - J_\lambda^{[s/\lambda]}\phi &= \lambda\sum_{k=s_\lambda}^{k=t_\lambda-1} G(J_\lambda^{k+1}\phi) - \lambda\sum_{k=s_\lambda}^{k=t_\lambda-1}(J_\lambda^{k+1}\phi)' \qquad (4.1)\\ &= \int_{(s_\lambda+1)\lambda}^{(t_\lambda+1)\lambda} G(J_\lambda^{[\tau/\lambda]})d\tau - \int_{(s_\lambda+1)\lambda}^{(t_\lambda+1)\lambda}(J_\lambda^{[\tau/\lambda]}\phi)'d\tau, \end{aligned}$$

with $s_\lambda = [s/\lambda]$, and $t_\lambda = [t/\lambda]$.

Letting $f_n = \int_{(s_{\lambda_n}+1)\lambda_n}^{(t_{\lambda_n}+1)\lambda_n}(J_{\lambda_n}^{[\tau/\lambda_n]}\phi)d\tau$, for any sequence $0 < \lambda_n \to 0$, the sequence $(f_n)_n$ converges in L^1 to $\int_s^t S(\tau)\phi d\tau$. From (4.1) we read that the sequence $(f'_n)_n$ is L^1-convergent as well, and, by closedness of the operator of differentiation in L^1, we actually get from (4.1) that

$$(\int_s^t S(\tau)\phi d\tau)' = \int_s^t G(S(\tau)\phi)d\tau - (S(t)\phi - S(s)\phi) \qquad (4.2)$$

for all $0 \leq s \leq t$. Moreover, as both $(f_n)_n$ and $(f_n)'_n$ are L^1-convergent, the sequence $(f_n)_n$ is actually uniformly equicontinuous on $\mathbb{R}^+$, and uniformly convergent on compacta, so that, altogether, all functions $\int_s^t S(\tau)\phi d\tau$, $0 \leq s \leq t$, are continuous, and, particularly for $a = 0$,

$$(\int_s^t S(\tau)\phi d\tau)(0) = \int_s^t F(S(\tau)\phi)d\tau. \tag{4.3}$$

At this point, letting $M(a,b,s,t) := \int_a^b(\int_s^t S(\tau)\phi d\tau)(\xi)d\xi$, with $0 \leq a \leq b$, and $0 \leq s \leq t$, and, for $h \geq 0$, $N(h) := M(a+h, b+h, s+h, t+h)$, (4.2) implies that $N'(h) = \int_{a+h}^{b+h}(\int_{s+h}^{t+h} G(S(\tau)\phi)d\tau)(\xi)d\xi$, so that

$$\begin{aligned}
\int_{a+h}^{b+h}&(\int_{s+h}^{t+h} S(\tau)\phi d\tau)(\xi)d\xi - \int_a^b(\int_s^t S(\tau)\phi d\tau)(\xi)d\xi \\
&= N(h) - N(0) \\
&= \int_0^h N'(\rho)d\rho \\
&= \int_0^h [\int_{a+\rho}^{b+\rho}(\int_{s+\rho}^{t+\rho} G(S(\tau)\phi)d\tau)(\xi)d\xi)]d\rho.
\end{aligned}$$

Thus,

$$(\int_s^t S(\tau+h)\phi d\tau)_h - \int_s^t S(\tau)\phi d\tau = \int_0^h (\int_s^t G(S(\tau+\rho)\phi)d\tau)_\rho d\rho \tag{4.4}$$

for all $0 \leq s \leq t$, and all $h \geq 0$, where the suffix h, respectively ρ, indicates the respective translation of the function by h, respectively ρ. This implies that $(S(t+h)\phi)_h - S(t)\phi = \int_0^h (G(S(t+\rho)\phi))_\rho d\rho$, and thus

$$L^1 - \lim_{h\to 0^+} \frac{1}{h}[(S(t+h)\phi)_h - S(t)\phi] = G(S(t)\phi).$$

This shows that $u_\phi = S(\cdot)\phi$ fulfills the balance law (1.1).

As for the birth law (1.2), notice that, by continuity of the functions on the left-hand side of equation (4.4), (4.4) evaluated at $a = 0$, in conjunction with (4.3) implies

$$\begin{aligned}
\frac{1}{\rho}(\int_t^{t+\rho} &S(\tau+\xi)\phi d\tau)(\xi) - \frac{1}{\rho}\int_t^{t+\rho} F(S(\tau)\phi)d\tau \\
&= \quad \int_0^\xi \frac{1}{\rho}(\int_t^{t+\rho} G(S(\tau+\eta)\phi)d\tau)(\eta)d\eta
\end{aligned} \tag{4.5}$$

for all $\rho > 0$, and all $t, \xi \geq 0$. Thus, for $h, \rho > 0$,

$$\begin{aligned}
&\frac{1}{h}\int_0^h \|(S(h)\phi)(\xi) - F(\phi)\| d\xi \qquad (4.6)\\
&\leq \quad \frac{1}{h}\int_0^h \left\| (S(h)\phi)(\xi) - \frac{1}{\rho}\left(\int_h^{h+\rho} S(\tau)\phi d\tau\right)(\xi) \right\| d\xi \\
&+ \quad \frac{1}{h}\int_0^h \left\| \frac{1}{\rho}\left(\int_{h-\xi}^{h-\xi+\rho} S(\tau+\xi)\phi d\tau\right)(\xi) - \frac{1}{\rho}\int_{h-\xi}^{h-\xi+\rho} F(S(\tau)\phi) d\tau \right\| d\xi \\
&+ \quad \frac{1}{h}\int_0^h \left\| \frac{1}{\rho}\int_h^{h+\rho} F(S(\tau-\xi)\phi) d\tau - F(\phi) \right\| d\xi
\end{aligned}$$

By (4.5), the second term on the right-hand side of (4.6) is equal to

$$\frac{1}{h}\int_0^h \left\| \int_0^\xi \frac{1}{\rho}\left(\int_h^{h+\rho} G(S(\tau+\eta-\xi)\phi) d\tau\right)(\eta) d\eta \right\| d\xi.$$

Notice that, as $\rho \to 0^+$, $\frac{1}{\rho}\int_{h+\eta}^{h+\eta+\rho} G(S(\tau-\xi)\phi)d\tau$ is L^1-convergent to $G(S(h+\eta-\xi)\phi)$ uniformly over $0 \leq \xi, \eta \leq h$. Thus, letting $\rho \to 0^+$ in (4.6),

$$\begin{aligned}
&\frac{1}{h}\int_0^h \|(S(h)\phi)(\xi) - F(\phi)\| d\xi \\
&\leq \quad \frac{1}{h}\int_0^h \left\| \int_0^\xi G(S(h+\eta-\xi)\phi)(\eta) d\eta \right\| d\xi + \frac{1}{h}\int_0^h \|F(S(h-\xi)\phi) - F(\phi)\| d\xi \\
&\leq \quad \frac{1}{h}\int_0^h \left(\int_0^u \|G(S(u)\phi)(v)\| dv\right) du + \sup_{0\leq\xi\leq h} \|F(S(h-\xi)\phi) - F(\phi)\|,
\end{aligned}$$

where, in the last step, an interchange of the order of integration was used. As, given $h_0 > 0$, the set $\{G(S(u)\phi) \mid 0 \leq u \leq h_0\}$ is L^1-relatively compact, and thus uniformly integrable, and as F is Lipschitz continuous, both terms on the right-hand side tend to 0 as $h \to 0^+$. Replacing ϕ by $S(t)\phi$, with $\phi \in L^1_+$, and $t \geq 0$, reveals that $u_\phi = S(\cdot)\phi$ fulfills the birth law (1.2) as well. This completes the proof.

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Space Almost Periodic Solutions of Reaction Diffusion Equations

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Abstract. We consider reaction diffusion equations of the form (*) $\partial_t u = \nu\Delta u + \zeta u + \mathcal{P}(u)$, $\mathcal{P}(u) = \sum_z^m a_k u^k$ and seek solutions on $\mathbb{R}^n$ which are almost periodic in the space variables x. Such solutions are constructed in the space $H^0(\mathbb{R}^n)$ of almost periodic functions $f(x)$ subject to (**) $f(x) = \sum f_k e^{i\Lambda_k x}$, $\sum |f_k| < \infty$, provided that the coefficients a_k in (*) are also in this class. Such solutions are obtained via an instable manifold construction, which yields solutions on $t \in (-\infty, 0]$ of slow exponential decay. An extension of the method to Fourier transforms of complex measures is outlined.

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0. Introduction

In what follows we are interested in solutions of reaction diffusion equations such as

$$\partial_t u = \nu\Delta u + \zeta u + \mathcal{P}(u), \quad \mathcal{P}(u) = \sum_2^m q_k u^k,\ \zeta > 0 \tag{0.1}$$

or related equations such as the Ginzburg-Landau equation, which are defined and bounded on R^n but which do not necessarily tend to zero as $|x| \to \infty$. Such solutions have been studied in different contexts and for various reasons; see eq. [2] and the references therein. A class of possible solutions is provided by functions which are space almost periodic (s.a.p) in one sense or the other. Among these, a class of functions which are s.a.p. in the sense below has turned out to be particularly suited, i.e.:

$$f(x) = \sum a_k e^{i\Lambda_k x},\ \sum |a_k| < \infty,\ \Lambda_k \in R^n,\ x \in R^n. \tag{0.2}$$

Functions of type (0.2) where used in [9] to construct almost periodic breather solutions on a half-line for the one-dimensional wave equation

$$\partial_t^2 u = \partial_x^2 u = \zeta u + \mathcal{P}(u), \quad \zeta > 0, x \in R_+. \tag{0.3}$$

The procedure in [9] was based on a stable manifold construction which turns out to be applicable in modified form to (0.1). That is, we use an instable manifold construction based on arguments put forward in [9] in order to construct solutions of (0.1) which are defined on $(-\infty, 0]$, which decay exponentially as $t \to -\infty$, whereby the exponential decay may be chosen to be arbitrarily slow in a sense to be made precise below. In some cases this result can be combined with the maximum principle or another result which guarantees global existence for $t \geq 0$ so as to get a solution defined for $t \in \mathbb{R}$. As side result one infers Ljapounov instability.

There exists quite a literature on almost periodic solutions of nonlinear ODE's and PDE's ([3],[8]). However the interest is mostly in solutions almost periodic in time; solutions almost periodic in the space variables have found less interest, at least as far as dissipative systems are concerned. In [6] however, quasiperiodic solutions of elliptic equations on a strip are investigated, a step which aims at the construction s.a.p. equilibrium solutions of the Bénard problem, a task yet unsolved.

1. Notation

$\mathbb{R}$, $\mathbb{C}$ denote real and complex numbers, with $\overline{x}$ the complex conjugate of $x \in \mathbb{C}$ and $|x|$ its absolute value; if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ then $x^2 = \Sigma x_j^2$.

$C_b^0(\mathbb{R}^n)$ is the space of bounded and uniformly continuous functions f provided with the supnorm $\|f\|_{C^0} = \|f\|_\infty = \sup_x |f(x)|$.

The nonlinearity $\mathcal{P}(u)$ in (0.1) is assumed to be of the form

$$\mathcal{P}(u) = \Sigma a_{\alpha\beta} u^\alpha \overline{u}^\beta, \quad 2 \leq \alpha + \beta \leq m \tag{1.1}$$

with coefficients $a_{\alpha\beta}$ subject to (0.2), but in order to simplify the presentation we work with polynomials $P(u)$, i.e.,

$$\mathcal{P}(u) = \Sigma a_\alpha u^\alpha, \quad 2 \leq \alpha \leq m, \tag{1.2}$$

with the a'_αs subject to (0.2). However, the estimates which will be proved for $\mathcal{P}(u)$ given by (1.2) hold without restriction for $\mathcal{P}(u)$ given by (1.1). Finally, S is the σ-algebra of Borel sets of $\mathbb{R}^n$. If X is a Banach space, $\| \; \|_X$ denotes the norm of X; in a fixed context we simply may write $\| \; \|$.

2. Spaces of almost periodic functions

As indicated, we interpret eq. (0.1) in the space of functions (0.2), to be discussed in more detail below. To this end we recall the Banach space of Bohr almost periodic functions $B(R^n)$ ([1], [13]). A function $f(x) \in B(R^n)$ has an associated Fourier series

$$f(x) \sim \Sigma a_k e^{i\Lambda_k x}, \text{ where } \Lambda_k = (\Lambda_k^1, \dots, \Lambda_k^n), \quad x = (x_1, \dots, x_n) \in R^n$$
$$\text{and } \Sigma \Lambda_k^j x_j = \Lambda_k x, \quad \Lambda_k^2 = \Sigma(\Lambda_k^j)^2, \quad x^2 = \Sigma x_j^2, \tag{2.1}$$

endowed with a number of properties.

Definition 1. Let $f \in B(\mathbb{R}^n)$ and $f \sim \Sigma a_k e^{i\Lambda_x x}$; for $s \geq 0$ we stipulate that $f \in H^s(\mathbb{R}^n)$ iff $\|f\|_s = \Sigma(1+\Lambda_k^2)^{\frac{s}{2}}|a_k| < \infty$.

Remarks

(1) If $f \in H^s(R^n)$ then f coincides with its formal Fourier series: $f = \Sigma a_k e^{i\Lambda_k x}$, $x \in \mathbb{R}^n$.

(2) $H^s(\mathbb{R}^n)$ is a Banach space under the norm $\| \ \|_s$; if $s = 0$ it is even a Banach algebra, i.e., if $f, g \in H^0(\mathbb{R}^n)$ then $\|fg\|_0 \leq \|f\|_0\|g\|_0$ as one easily verifies, ([9]).

(3) For n fixed we also set $H^s = H^s(\mathbb{R}^n)$; likewise we set $\| \ \| = \| \ \|_0$, i.e., $\|f\| = \Sigma|a_k|$ for $f \in H^0$.

(4) In order to put (0.1) into the functional frame of Definition 1, we specify the data in (0.1) more precisely. We assume

$$\nu = 1 + i\alpha, \text{ some } \alpha \in \mathbb{R}. \tag{2.2}$$

If $\alpha = 0$ and if $\mathcal{P}(u)$ is given by (1.2) with real coefficients a_α, then we have a reaction diffusion equation; if $\alpha \neq 0$ and with $\mathcal{P}(u)$ as in (1.1) then we have an equation of Landau-Ginzburg type (see [2] for an instance of physical interest). In order to treat (0.1) as an evolution equation on $H^0(\mathbb{R}^n)$ we first consider the Laplacian Δ:

Definition 2

(a) $\mathrm{dom}(\Delta) = \mathrm{H}^2(\mathbb{R}^n)$,

(b) if $f = \Sigma a_k e^{i\Lambda_k x} \in H^2(\mathbb{R}^n)$ then $\Delta f = -\Sigma a_k \Lambda_k^2 e^{i\Lambda_k x}$.

Lemma 2.1. $(1+i\alpha)\Delta$ is a holomorphic semigroup generator on $H^0(\mathbb{R}^n)$, i.e., there are $\mathcal{V} = \mathcal{V}(\alpha) \in (0, \frac{\pi}{2})$ and $C = C(\alpha) > 0$ as follows: if $\zeta = \mathrm{r}e^{\mathrm{i}\varphi}$, $\varphi \in [-\frac{\pi}{2} - \mathcal{V}, \frac{\pi}{2} + \mathcal{V}]$ and $r > 0$ then

$$\zeta \in \varrho((1+i\alpha)\Delta) \text{ and } \|((1+i\alpha)\Delta - \zeta)^{-1}\| \leq C|\zeta|^{-1}. \tag{2.3}$$

The proof of Lemma 2.1 is based on the next proposition whose elementary proof we omit:

Proposition 2.1. Given $\alpha \in R$, there are $\mathcal{V} \in (0, \frac{\pi}{2})$ and $\varepsilon > 0$ as follows: if $\varphi \in [-\frac{\pi}{2} - \mathcal{V}, \frac{\pi}{2} + \mathcal{V}]$, $r > 0$ and $\lambda \geq 0$ then

$$\tfrac{\varepsilon}{2}(|\zeta| + \lambda) \leq |(1+i\alpha)\lambda + \zeta|. \tag{2.4}$$

Proof of Lemma 2.1. With $a \in \mathbb{R}$ fixed, we let $\mathcal{V} = \mathcal{V}(\alpha)$, $\varepsilon = \varepsilon(\alpha)$ be given according to Prop. 2.1; we also set

$$S_{\mathcal{V}} = \{\mathrm{re}^{\mathrm{i}\varphi}/\mathrm{r} > 0 \quad \& \quad -\tfrac{\pi}{2} - \mathcal{V} \leq \varphi \leq \tfrac{\pi}{2} + \mathcal{V}\}. \tag{2.5}$$

Moreover we set $C = C(\alpha) = 2\varepsilon^{-1}$ and fix $\zeta = \mathrm{re}^{\mathrm{i}\varphi}$ in $S_{\mathcal{V}}$. Given $g = \Sigma g_k e^{i\Lambda_k x}$ in H^0, a solution f of

$$((1 + i\alpha)\Delta - \zeta)f = -g \tag{2.6}$$

is formally given by

$$f = \Sigma f_k e^{i\Lambda_k x}, \quad f_k = ((1 + i\alpha)\Lambda_k^2 + \zeta)^{-1} g_k. \tag{2.7}$$

In order to show that indeed $f \in H^2(R^n)$ we invoke Prop. 2.1 according to which

$$(1 + \Lambda_k^2)|(1 + i\alpha)\Lambda_k^2 + \zeta|^{-1} \leq C(1 + \Lambda_k^2)(\Lambda_k^2 + |\zeta|)^{-1} \leq C_1 \tag{2.8}$$

for some $C_1 = C_1(|\zeta|)$ independent of k whence

$$\|f\|_2 = \Sigma(1 + \Lambda_k^2)|(1 + i\alpha)\Lambda_k^2 + \zeta|^{-1}|g_k| \leq C_1 \Sigma|g_k| = C_1\|g\| \tag{2.9}$$

On the other hand we have again by Prop. 2.1:

$$\begin{aligned} \|f\| &= \Sigma|(1 + i\alpha)\Lambda_k^2 + \zeta|^{-1}\, |g|_k \leq C\Sigma(\Lambda_k^2 + |\zeta|)^{-1}|g_k| \\ &\leq C|\zeta|^{-1}\|g\|, \ \zeta \in S_{\mathcal{V}}. \end{aligned} \tag{2.10}$$

Moreover, a solution $f \in H^2(\mathbb{R}^n)$ of (2.6) is unique since its Fourier coefficients are given by (2.7). To sum up, the operator $(1 + i\alpha)\Delta - \zeta$ maps $H^2(\mathbb{R}^n)$ one-one onto $H^0(\mathbb{R}^n)$, has a bounded inverse by (2.10) and is thus closed. It also satisfies the resolvent estimate (2.10).

Since $\mathrm{dom}(\Delta) = \mathrm{H}^2(\mathbb{R}^n)$ is dense in $H^0(\mathbb{R}^n)$ it follows that $-(1 + i\alpha)\Delta$ is sectorial and thus $(1 + i\alpha)\Delta$ a holomorphic semigroup generator ([7], Theorem 2.5.1) □

Formal arguments suggest that the semigroup W_t, $t \geq 0$ generated by $(1 + i\alpha)\Delta$ acts on $f = \Sigma f_k e^{i\Lambda_k x}$ according to

$$W_t f = \Sigma e^{-(1+i\alpha)\Lambda_k^2 t} e^{i\Lambda_k x} f_k, \quad t \geq 0 \tag{2.11}$$

We first note

Proposition 2.2. W_t, $t \geq 0$ given by (2.11) is a contraction semigroup:

(a) $\|W_t\| \leq 1$,
(b) $W_t W_s = W_{t+s}$,
(c) $W_h \to I$ strongly as $h \downarrow 0$.

Proof. (a), (b) follow directly from (2.11). As to (c) we define

$$H_T(\mathbb{R}^n) \text{ is the space of finite sums } f = \sum e^{i\Lambda_t x} f_k, \quad 0 \leq k \leq N. \tag{2.12}$$

We also stipulate

$$\text{if } f = \sum f_k e^{i\Lambda_k x} \text{ then } f^N = \sum f_k e^{i\Lambda_k x}, \quad k \leq N. \tag{2.13}$$

In order to prove (c) we note:

$$\|W_h g - g\| \to 0 \text{ as } h \downarrow 0 \text{ for } g \in H_T(\mathbb{R}^n). \tag{2.14}$$

Now assume $f = \sum f_k e^{i\Lambda_k x} \in H^0(\mathbb{R}^n)$. Then, using (a), we have

$$\begin{aligned}\|W_h f - f\| &\le \|W_h f - W_h f^N\| + \|W_h f^N - f^N\| + \|f^N - f\| \\ &\le 2\|f - f^N\| + \|W_h f^N - f^N\|.\end{aligned} \tag{2.15}$$

Clause (c) now follows by a standard argument from (2.13), (2.14). □

Lemma 2.2. $(1 + i\alpha)\Delta$ generates W_t, $t \ge 0$.

Proof. Based on Definition 1 we set $L = (1 + i\alpha)\Delta$ and let $\widetilde{L}$ be the generator of W_t, $t \ge 0$. Straightforward computation shows:

$$\text{if } g \in H_T(\mathbb{R}^n) \text{ then } g \in \mathrm{dom}(L) \cap \mathrm{dom}(\widetilde{L}) \text{ and } Lg = \widetilde{L}g. \tag{2.16}$$

Next let $f \in \mathrm{dom}(L) = H^2(\mathbb{R}^n)$; with f^N as in (2.13) one has:

$$\|f - f^N\|_2 \to 0, \text{ whence } \|Lf - Lf^N\| \to 0 \text{ as } N \uparrow \infty. \tag{2.17}$$

Thus $\widetilde{L}f^N = Lf^N \to Lf$ and $f^N \to f$ in $H^0(\mathbb{R}^n)$. Since $\widetilde{L}$ is closed this entails

$$f \in \mathrm{dom}(\widetilde{L}) \text{ and } \widetilde{L}f = Lf, \text{ i.e., } L \subseteq \widetilde{L}. \tag{2.18}$$

Since both L, $\widetilde{L}$ are semigroup generators, (2.18) implies $L = \widetilde{L}$, proving the Lemma. □

Corollary. $(1+i\alpha)\Delta+\zeta$ is the generator of the holomorphic semigroup $V_t = e^{\zeta t}W_t$ which acts on $f = \sum f_k e^{i\Lambda_k x}$ according to

$$V_t f = \sum f_k e^{(-(1+i\alpha)\Lambda_k^2+\zeta)t} e^{i\Lambda_k x}, \quad t \ge 0. \tag{2.19}$$

3. Slow instable manifolds

As indicated in the introduction, our aim is to construct small solutions $y(t) \in H^0(\mathbb{R}^n)$, $t \le 0$ which are of slow exponential decay as $t \to -\infty$, a concept to be made precise later. To this end we first construct instable manifolds ([4], Theorem 5.1.3) along established lines. The only point in which our present construction differs from that in [4] is that here we have no spectral sets. Since such situations are treated in some detail in [11], our presentation will be brief. We recall that ζ in (0.1) is assumed to satisfy $\zeta > 0$. To start with we fix β with

$$0 < \beta < \zeta, \text{ with } \zeta > 0 \text{ as in (0.1)}, \tag{3.1}$$

and define an operator Q_β acting on $H^0(\mathbb{R}^n)$ via

$$Q_\beta f = \sum_{\beta \le -\Lambda_k^2+\zeta} f_k e^{i\Lambda_k x} \text{ where } f = \sum f_k e^{i\Lambda_k x}; \tag{3.2}$$

if no confusion arises we simply set $Q = Q_\beta$. Q is a bounded projection operator (i.e., $Q^2 = Q$) which commutes with V_t,whereby

$$QV_tf = V_tQf = \sum_{\beta\le-\Lambda_k^2+\zeta} f_k e^{-(1+i\alpha)\Lambda_k^2+\zeta)t} e^{i\Lambda_k x}. \tag{3.3}$$

Now while V_t, given by (2.19), is defined for $t \ge 0$ only, V_tQ admits an obvious extension to $(-\infty, 0)$ as follows from (3.3) and in fact satisfies an estimate

$$\|V_tQ\| = \|QV_t\| \le e^{\beta t}, \quad t \le 0, \tag{3.4}$$

as one reads off from (3.3). Likewise we infer from (3.3):

$$\|(1-Q)V_t\| = \|V_t(1-Q)\| \le e^{\beta t}, \quad t \ge 0. \tag{3.5}$$

Next we define the Banach space B_-^β via

$$\begin{gathered} y \in B_-^\beta \text{ iff } y \in C^0((-\infty, 0], H^0(\mathbb{R}^n)) \text{ and} \\ [y]_\beta = \sup_{t\le0} e^{-\beta t}\|y(t)\| < \infty; \end{gathered} \tag{3.6}$$

if no confusion arises we write $[\ ]$ instead of $[\ ]_\beta$. We then define a mapping F which associates with $\eta \in \mathrm{rg}(Q)$ and $y \in B_-^\beta$ the image $F(\eta, y) = \widetilde{y} \in B_-^\beta$ via

$$\widetilde{y}(t) = V_t\eta - \int_t^0 V_{t-s}QP(y(s))\,ds + \int_{-\infty}^t V_{t-s}(1-Q)P(y(s))\,ds, \quad t \le 0. \tag{3.7}$$

Mappings such as (3.7), related to instable resp. stable manifolds are discussed at various places, see, e.g., [4], Theorem 5.1.3, [12], Section 11.3, [9], Section 3. We refer to these texts but elaborate some points which will be needed later. To this end we list some estimates related to the nonlinearity $\mathcal{P}(u)$ in (1.2).

Let $y \in B_-^\beta$; since $H^0(\mathbb{R}^n)$ is a Banach algebra and since

$$\|y(t)\| \le e^{\beta t}[y]; \quad [y^n] \le [y]^n$$

we have that

$$\|P(y(t))\| \le [y]^2 e^{2\beta t}\phi([y]) \text{ where } \phi(z) = \sum_2^m \|a_\alpha\| z^{\alpha-2}. \tag{3.8}$$

Setting

$$F_1(t, y) = \int_t^0 V_{t-s}QP(y)\,ds, \quad F_2(t, y) = \int_{-\infty}^t V_{t-s}(1-Q)P(y)\,ds$$

we infer from (3.4), (3.5), (3.8):

$$\begin{aligned} \|F_2(t, y)\| &\le \int_{-\infty}^t e^{\beta(t-s)}e^{2\beta s}\,ds[y]^2\phi([y]) \\ &\le \frac{e^{2\beta t}}{\beta}[y]^2\phi([y]), \quad t \le 0. \end{aligned} \tag{3.9}$$

Likewise we obtain

$$\|F_1(t,y)\| \le \frac{e^{\beta t}}{\beta}[y]^2\phi([y]), \quad t \le 0. \tag{3.10}$$

On the other hand by repeating the arguments in either of [9], proof of Prop. 3.3 or [12], Section 11.3 one infers

$$F_j(t,y),\ j=1,2 \text{ are in } C^0((-\infty,0],H^0(\mathbb{R}^n)). \tag{3.11}$$

From (3.9)–(3.11) we thus get

Proposition 3.1. $\widetilde{y}$, given by (3.7) is in B_-^β and satisfies:

$$[\widetilde{y}] \le \|\eta\| + \tfrac{2}{\beta}[y]^2\phi([y]),\ (\phi \text{ as in } (3.8)) \qquad \bullet$$

In order to estimate $[F_j(\cdot,y_1)-F_j(\cdot,y_2)]$ for $y_1,y_2 \in B_-^\beta$ we invoke without proof the elementary

Proposition 3.2. With the polynomial $P(z)=\sum_2^m a_\alpha z^\alpha$, $a_\alpha \in H^0(\mathbb{R}^n)$, there is associated a polynomial $\widehat{\phi}(z)=\sum_0^{m-2} b_j z^j$, $b_j \ge 0$, as follows: if $u_1,u_2 \in H^0(\mathbb{R}^n)$ then

$$\|P(u_1)-P(u_2)\| \le \|u_1-u_2\| \max(\|u_1\|,\|u_2\|)\widehat{\phi}(\max(\|u_1\|,\|u_2\|)) \qquad \bullet$$

From Prop. 3.2 we infer that if $y_1,y_2 \in B_-^\beta$ then the following holds:

$$\|P(y_2(t))-P(y_1(t))\| \le [y_2-y_1]\max([y_1],[y_2])\widehat{\phi}(\max([y_1],[y_2]))e^{2\beta t}, \quad t \le 0. \tag{3.12}$$

Using (3.12) we can argue as in (3.9) so as to get the estimates:

$$[F_j(\cdot,y_1)-F_j(\cdot,y_2)] \le \tfrac{1}{\beta}[y_2-y_1]\max([y_1],[y_2])\widehat{\phi}(\max([y_1],[y_2])) \tag{3.13}$$

where $y_1,y_2 \in B_-^\beta$, $j=1,2$. From (3.7), (3.13) we thus get

Proposition 3.3. If $y_1,y_2 \in B_-^\beta$ then we have

$$[\widetilde{y}_2-\widetilde{y}_1] \le \tfrac{2}{\beta}[y_2-y_1]\max([y_1],[y_2])\widehat{\phi}(\max([y_1],[y_2]) \qquad \bullet$$

In order to construct a fixpoint of the mapping given by (3.7) we define spheres

$$\begin{aligned} S(\varepsilon) &= \{\eta/\eta \in \mathrm{rg}(Q)\&\|\eta\| \le \varepsilon\} \\ S_\beta(\varepsilon) &= \{y/y \in B_-^\beta\&[y] \le \varepsilon\}. \end{aligned} \tag{3.14}$$

Lemma 3.1. Set $\varepsilon_0 = \min\left(\frac{\beta}{4\phi(1)}, \frac{\beta}{4\widehat{\phi}(1)}, 1\right)$, let $\varepsilon \le \varepsilon_0$, fix $\eta \in S(\frac{\varepsilon}{2})$. Then the mapping which associates with $y \in S_\beta(\varepsilon)$ the image $\widetilde{y} \in B_-^\beta$ via (3.7) is a contraction of $S_\beta(\varepsilon)$ which has a fixpoint $y=\varphi(\eta) \in S_\beta(\varepsilon)$, i.e., which satisfies

$$y(t) = V_t\eta - \int_t^0 V_{t-s}QP(y)\,ds + \int_{-\infty}^t V_{t-s}(1-Q)P(y)\,ds \tag{3.15}$$

for $t \leq 0$, with $y = y(s)$. One has the estimate

$$[\varphi(\eta_1) - \varphi(\eta_2)] \leq 2\|\eta_1 - \eta_2\|, \quad \eta_1, \eta_2 \in S\left(\tfrac{\varepsilon}{2}\right) \tag{3.16}$$

•

Proof. With $\varepsilon \leq \varepsilon_0$, $\eta \in S(\frac{\varepsilon}{2})$, $y \in S_\beta(\varepsilon)$ assumed, we have by Prop. 3.1:

$$[\widetilde{y}] \leq \|\eta\| + \tfrac{2}{\beta}[y]^2\phi([y]) \leq \tfrac{\varepsilon}{2} + \left(\tfrac{2}{\beta}\phi(1)\varepsilon\right)\varepsilon \leq \varepsilon,$$

that is, the mapping in (3.7) maps $S_\beta(\varepsilon)$ into itself. Next with $\eta \in S(\frac{\varepsilon}{2})$, $y_1, y_2 \in S_\beta(\varepsilon)$ assumed, let $\widetilde{y}_j$ be the image of y_j via (3.7), $j = 1, 2$. By Prop. 3.3 we have

$$[\widetilde{y}_2 - \widetilde{y}_1] \leq \tfrac{2}{\beta}[y_1 - y_2]\left(\frac{\beta}{4\widehat{\phi}(1)}\right)\widehat{\phi}(1) \leq \tfrac{1}{2}[y_2 - y_1],$$

i.e., the mapping in (3.7) is a contraction of $S_\beta(\varepsilon)$. This entails that there is a unique fixpoint $y = \varphi(\eta)$ of the mapping in (3.7) restricted to $S_\beta(\varepsilon)$, which thus satisfies (3.15).

In order to prove (3.16) we fix $\eta_1, \eta_2 \in S(\frac{\varepsilon}{2})$ and let $y_j = \varphi(\eta_j)$ be the associated fixpoints in $S_\beta(\varepsilon)$. We also recall the expressions $F_j(t, y)$, $j = 1, 2$ in (3.9), (3.10). Equation (3.15), satisfied by y_1, y_2, is then written as follows:

$$y_j(t) = V_t\eta_j - F_1(t, y_j) + F_2(t, y_j), \quad t \leq 0,\ j = 1, 2$$

whence

$$y_2(t) - y_1(t) = V_t(\eta_2 - \eta_1) - (F_1(t, y_2) - F_1(t, y_1)) + (F_2(t, y_2) - F_2(t, y_1))$$

and thus

$$[y_2 - y_1] \leq \|\eta_2 - \eta_1\| + [F_1(\ , y_2) - F_1(\ , y_1)] + [F_2(\ , y_2) - F_2(\ , y_2)].$$

Recalling $\varepsilon \leq \varepsilon_0$ and $y_j \in S_\beta(\varepsilon)$, we invoke (3.13) so as to infer from the last inequality:

$$\begin{aligned} &\leq \|\eta_2 - \eta_1\| + \tfrac{2}{\beta}[y_2 - y_1]\tfrac{\beta}{4\widehat{\phi}(1)}\,\widehat{\phi}(1) \\ &\leq \|\eta_2 - \eta_1\| + \tfrac{1}{2}[y_2 - y_1]. \end{aligned} \tag{3.17}$$

From (3.17), clause (3.16) readily follows. □

Remarks. The fixpoint $y = \varphi(\eta)$ in Lemma 3.1 satisfies by construction equation (3.15). Standard computations show that y also satisfies:

$$y(t) = V_{t-\tau}y(\tau) + \int_\tau^t V_{t-s}P(y(s))\,ds, \quad \tau \leq t \leq 0. \tag{3.18}$$

From (3.18) one infers by classical arguments ([14], proof of Theorem 3.1, pg. 196) that $y(t)$ is even a solution of (0.1) in the usual sense:

$$\begin{aligned} &y(t) \in \mathrm{dom}(\Delta),\ t \leq 0, \quad y \in C^1((-\infty, 0], H^0(\mathbb{R}^n)) \text{ and} \\ &y_t(t) = ((1 + i\alpha)\Delta + \zeta)y(t) + P(y(t)),\ t \leq 0 \text{ pointwise.} \end{aligned} \tag{3.19}$$

Thus Lemma 3.1 yields a solution $y(t)$, $t \leq 0$ of (0.1) which decays exponentially as $t \to -\infty$. Our aim is to obtain an improvement of Lemma 3.1, i.e.,

Theorem 1. Let $0 < \beta < \mu < \zeta$. Then there is a solution $y \in B_-^\beta$ of (0.1) with $y \notin B_-^\mu$.

Remarks. A similar result was obtained in [10] in a Hilbert space setting for stable manifolds, but instable manifolds, which require a different treatment, were not considered. As a preparation we fix $0 < \beta < \gamma < \zeta$ and associate with γ a projection operator R via

$$Rf = \sum_{\gamma \leq -\Lambda_k^2 + \zeta} f_k e^{i\Lambda_k x}, \text{ where } f = \sum f_k e^{i\Lambda_k x}. \tag{3.20}$$

The projection Q given by (3.2), commutes with R, i.e., $QR = RQ = R$, and $Q - R$ is given by

$$(Q-R)f = \sum_{\beta \leq -\Lambda_k^2 + \zeta < \gamma} f_k e^{i\Lambda_k x}. \tag{3.21}$$

Moreover, one easily verifies the estimate

$$\|V_t(Q-R)\| \leq e^{\gamma t}, \quad t \geq 0. \tag{3.22}$$

Proposition 3.4. Let $\gamma < \mu$ and let $y \in B_-^\mu$ be a solution of (0.1). Then

$$(Q-R)y(t) = \int_{-\infty}^t V_{t-s}(Q-R)P(y(s))\,ds, \quad t \leq 0. \tag{3.23}$$

Proof. With $y \in B_-^\mu$ a solution of equation (0.1), it satisfies its integrated version (3.18). Multiplication of (3.18) with $Q - R$ yields

$$(Q-R)y(t) = V_{t-\tau}(Q-R)y(\tau) + \int_\tau^t V_{t-s}(Q-R)P(y(s))\,ds, \tag{3.24}$$

for $\tau \leq t \leq 0$. We claim that the integral

$$I = \int_{-\infty}^t V_{t-s}(Q-R)P(y(s))\,ds, \quad t \leq 0$$

exists absolutely. In fact, using (3.8) with $[\ ] = [\ ]_\mu$ and recalling (3.22) we get, since $\gamma < \mu$,

$$\begin{aligned} \|I\| &\leq \int_{-\infty}^t \|V_{t-s}(Q-R)\| e^{2\mu s} \phi([y]_\mu)[y]_\mu^2\,ds \\ &\leq e^{\gamma t} \int_{-\infty}^t e^{(2\mu-\gamma)s}\,ds\, \phi([y]_\mu)[y]_\mu^2 < \infty. \end{aligned} \tag{3.25}$$

On the other hand, since $\gamma < \mu$, and in view of (3.22), we have that

$$\|V_{t-\tau}(Q-R)y(\tau)\| \leq e^{\gamma(t-\tau)}[y]_\mu e^{\mu\tau} \to 0 \text{ as } \tau \to -\infty. \tag{3.26}$$

By (3.25), (3.26) we can pass to the limit $\tau \to -\infty$ in (3.24) so as to get (3.23) $\square$

In our last step we fix $0 < \beta < \gamma < \zeta$ and $\varepsilon \le \varepsilon_0$ with ε_0 given by Lemma 3.1, while Q, R are the projections associated with β, γ via (3.2), (3.20) resp. Let $\varphi(\ )$ be the mapping from $S(\frac{\varepsilon}{2})$ to $S_\beta(\varepsilon)$ provided by Lemma 3.1.

Our principal claim is

Lemma 3.2. Assume in addition $\gamma < 2\beta$. Then there is $C = C(\beta, \gamma)$ as follows: if $\eta \in S(\frac{\varepsilon}{2})$ and $\eta \in \mathrm{rg}(Q - R)$, if moreover $\varphi(\eta) = y$ is in B_-^μ for some $\gamma < \mu$, then

$$\|\eta\| \le C\|\eta\|^2. \tag{3.27}$$

Proof. By our assumptions, $\varphi(\eta) = y \in S_\beta(\varepsilon)$ is a solution of the integral equation (3.15), i.e.,

$$y(t) = V_t\eta - \int_t^0 V_{t-s}QP(y(s))\,ds + \int_{-\infty}^t V_{t-s}(1-Q)P(y(s))\,ds \quad t \le 0. \tag{3.28}$$

As pointed out in the remark following (3.17), y is then a solution to (0.1) on $(-\infty, 0]$. Since by assumption also $y \in B_-^\mu$ for some $\gamma < \mu$, we can apply Prop. 3.4 and infer

$$(Q - R)y(t) = \int_{-\infty}^t V_{t-s}(Q - R)P(y(s))\,ds, \quad t \le 0. \tag{3.29}$$

We now apply $(Q-R)$ to (3.28), using that $(Q-R)(1-Q) = 0$ and $(Q-R)\eta = \eta$, so as to get

$$(Q - R)y(t) = V_t\eta - \int_t^0 V_{t-s}(Q - R)P(y(s))\,ds, \quad t \le 0. \tag{3.30}$$

By combining (3.29), (3.30) and after setting $t = 0$ we get

$$\eta = \int_{-\infty}^0 V_{-s}(Q - R)P(y(s))\,ds. \tag{3.31}$$

We now apply (3.8) and (3.22) to (3.31) so as to get

$$\begin{aligned} \|\eta\| &\le \int_{-\infty}^0 e^{-\gamma s}[y]_\beta^2\phi([y]_\beta)e^{2\beta s}\,ds \\ &\le \int_{-\infty}^0 e^{(2\beta-\gamma)s}\,ds\,\phi(1)[y]_\beta^2. \end{aligned} \tag{3.32}$$

But $y = \varphi(\eta)$ and $\varphi(0) = 0$ whence $[y]_\beta = [\varphi(\eta)]_\beta \le 2\|\eta\|$ by (3.16), what together with (3.32) and $2\beta > \gamma$ implies

$$\|\eta\| \le 4(2\beta - \gamma)^{-1}\|\eta\|^2\phi(1),$$

what proves the lemma □

Proof of Theorem 1. For $0 < \beta < \zeta$ let $\beta < \mu$; pick γ such that $\beta < \gamma < \zeta$, $\gamma < \mu$ and $\gamma < 2\beta$. Fix $\varepsilon \le \varepsilon_0$ with ε_0 given by Lemma 3.1, and let $\varphi : S(\frac{\varepsilon}{2}) \to S_\beta(\varepsilon)$ be the mapping given by Lemma 3.1; let also Q, R be the projections related to β, γ via (3.2), (3.20) resp.

Finally, fix η such that

$$\eta \in \mathrm{rg}(Q-R),\ \eta \neq 0,\ \|\eta\| < \min(\tfrac{\varepsilon}{2}, C^{-1}) \tag{3.33}$$

with C as in Lemma 3.2. The assumptions of Lemma 3.2 are thus satisfied. It follows that $\varphi(\eta) = y$ is a solution of (0.1) in B_-^β, and that if also $y \in B_-^\mu$ then $\|\eta\| \leq C\|\eta\|^2$; since $\eta \neq 0$ this entails $C^{-1} \leq \|\eta\|$, contradicting (3.33) □

Example. We consider the Landau-Ginzburg equation

$$u_t = (1+i\alpha)\Delta u + u - (1-i\delta)u^2\overline{u}, \quad \alpha, \delta \in \mathbb{R} \tag{3.34}$$

([2]). Here the nonlinearity is of the form (1.1), but as stressed in Section 1, amenable to our estimates (Prop. 3.1–3.3) without restriction. We thus can apply any of Lemma 3.1, or Theorem 1 to (3.34). As a result we find for $\beta \in (0,1)$ small solutions $y \in B_-^\beta$, $y \neq 0$ of (3.34). By Theorem II.1 in [2] each of these solutions admits an extension to $[0,\infty)$ into a classical solution $y(t)$, $t \geq 0$ such that $\sup_x |y(x,t)| \leq K$, $t \geq 0$, (some K) provided that $n = 1,2$ or $n = 3$ and $\alpha, \delta > 0$. It can be shown that these extensions are Bohr almost periodic in $x \in \mathbb{R}^n$. The problem arises whether this extension remains in $H^0(\mathbb{R}^n)$ for $t \geq 0$ and satisfies $\sup_{t \leq T} \|y(t)\| < \infty$ for any $T > 0$ or whether there is $T < \infty$ such that $y(\cdot, t) \in H^0(\mathbb{R}^n)$ for $t < T$ but with $\limsup_{t \to T} \|y(t)\| = \infty$.

We do not know if such a loss of regularity is possible. The application of Lemma 3.1 or Theorem 1 to (3.34) at the same time shows that the trivial equilibrium $u = 0$ of (3.34) is Ljapounov unstable with respect to perturbations in $H^0(\mathbb{R}^n)$.

Remark. If $\nu = 1$ in (0.1), i.e., $\alpha = 0$ in (2.2), and if $\mathcal{P}(u)$ has the form (1.2) with coefficients $a_\alpha \in H^0(\mathbb{R}^n)$ real, then one is interested in real solutions of (0.1). Now it is routine to show that the construction in Section 3 can be carried out in the subspace $H_r^0(\mathbb{R}^n)$ of real elements of $H^0(\mathbb{R}^n)$. As a result we obtain real versions of Lemmas 3.1, 3.2 and Theorem 1, i.e., we obtain real solutions $y \in B_-^\beta$ of (0.1), endowed with all relevant properties.

4. Outlook

(I) The reasons why we have considered eq. (0.1) in the narrow space $H^0(\mathbb{R}^n)$ (Definition 1) are as follows. On the one hand it is easy to construct the projection operators Q, R ((3.2), (3.20)), indispensable for all manifold constructions, although the linearity $(1+i\alpha)\Delta + \zeta$ has continuous spectrum and no spectral gaps; see [9], [11] for different but comparable situations. On the other hand the nonlinearity $\mathcal{P}(u)$ ((1.1), (1.2)) maps $H^0(\mathbb{R}^n)$ smoothly into itself, thanks to the Banach algebra property of $H^0(\mathbb{R}^n)$. It is now natural to seek broader spaces in which our constructions can still be carried out. For reasons of space we have to refrain from a discussion of the various possibilities and content us with a discussion of just one case. First we observe that an element $f \in H^0(\mathbb{R}^n)$, i.e., subject to (0.2), may be viewed as the Fourier transform of a complex measure μ:

$$f(x) = \int e^{i\lambda x} d\mu(\lambda), \text{ where } \mu(E) = \sum_{\Lambda_k \in E} a_k, \quad E \in S. \tag{4.1}$$

This causes us to consider the space $\widehat{\mathcal{C}}(\mathbb{R}^n)$ of Fourier transforms of finite complex measures, i.e.:

$$f(x) = \int e^{ix\lambda}\, d\mu(\lambda), \quad \mu \text{ a finite complex measure.} \tag{4.2}$$

For simplicity we restrict our alternation to the Banach space $\widehat{\mathcal{C}}_r(\mathbb{R}^n)$ of Fourier transforms of measures μ of the form

$$\mu = \mu_d + \mu_a, \ \mu_d \text{ discrete, } \mu_a \text{ absolutely continuous.} \tag{4.3}$$

Fourier transforms $f = \widehat{\mu}$ of such measures are of the form

$$f(x) = \sum a_k e^{i\Lambda_k x} + \int_{\mathbb{R}^n} g(\lambda) e^{i\lambda x}\, d\lambda, \quad \sum |a_k| < \infty, \ g \in \mathcal{L}^1(\mathbb{R}^n). \tag{4.4}$$

A norm $\|f\|$ on $\widehat{\mathcal{C}}_r(\mathbb{R}^n)$ is then given by

$$\|f\| = \sum |a_k| + \int |g(\lambda)|\, d\lambda. \tag{4.5}$$

One easily verifies that the representation in (4.4) is unique and that $\widehat{\mathcal{C}}_r(\mathbb{R}^n)$ is Banach under the norm $\| \ \|$ in (4.5). Projection operators P_d, P_a are then defined via

$$P_d f = \sum a_k e^{i\Lambda_k x}, \quad P_a f = \int g(\lambda) e^{i\lambda x}\, d\lambda \tag{4.6}$$

which give rise to the direct sum representation

$$\widehat{\mathcal{C}}_r(\mathbb{R}^n) = P_d \widehat{\mathcal{C}}_r(\mathbb{R}^n) \oplus P_a \widehat{\mathcal{C}}_r(\mathbb{R}^n) = H^0(\mathbb{R}^n) \oplus P_a \widehat{\mathcal{C}}_r(\mathbb{R}^n). \tag{4.7}$$

Below, Fourier transforms $\int g(\lambda) e^{i\lambda x}\, d\lambda$ are denoted by $\widehat{g}$, $(g(\lambda))^\wedge$ or $\widehat{g}(x)$ according to the case. In order to extend the considerations in Section 2.3 to the present context we need

Definition 3. $f(x)$, given by (4.4), is in $\operatorname{dom}(\Delta)$ iff $\sum |a_k| \Lambda_k^2 + \int \lambda^2 |g(\lambda)|\, d\lambda < \infty$; in this case

$$-\Delta f = \sum \Lambda_k^2 a_k e^{i\Lambda_k x} + \int \lambda^2 g(\lambda) e^{i\lambda}\, dx.$$

By Definition 3 and (4.6) it follows that Δ commutes with P_d, P_a:

$$P_d \Delta \subseteq \Delta P_d, \quad P_a \Delta \subseteq \Delta P_a \tag{4.8}$$

what implies that the spaces $P_d \widehat{\mathcal{C}}_r(\mathbb{R}^n)$, $P_a \widehat{\mathcal{C}}_r(\mathbb{R}^n)$ are invariant under Δ. Moreover, $P_d \Delta$ coincides with Δ given by Definition 2, as a comparison shows. From these observations it follows that Lemma 2.1 remains valid when interpreted in the present context. In fact, Lemma 2.1 has already been proved for $P_d \Delta$ in Section 2.

For $P_a \Delta$ however, the lemma follows straightforwardly from the identity

$$((1 + i\alpha)\Delta - \zeta)^{-1} \widehat{g} = -(((1 + i\alpha)\lambda^2 + \zeta)^{-1} g)^\wedge, \quad g \in \mathcal{L}^1(\mathbb{R}^n), \quad \zeta \in S_{\mathcal{V}}$$

and from Prop. 2.1 by the same arguments put forward in the proof of Lemma 2.1, Section 2. Routine arguments, similar to those used in the proof of Lemma 2.2 then show that the semigroup generated by $(1+i\alpha)\Delta+\zeta$ (some $\zeta>0$) acts on f, given by (4.4), according to

$$V_t f=\sum a_k e^{(-(1+i\alpha)\Lambda_k^2+\zeta)t}e^{i\Lambda_k x}+\int e^{(-(1+i\alpha)\lambda^2+\zeta)t}g(\lambda)e^{i\lambda x}\,d\lambda. \tag{4.9}$$

V_t, given by (4.9), commutes with P_a, P_d and is the direct sum of semigroups P_aV_t, P_dV_t acting on $P_a\widehat{\mathcal{C}}_r(\mathbb{R}^n)$, $P_d\widehat{\mathcal{C}}_r(\mathbb{R}^n)$ respectively, with P_dV_t given by (2.13). Next we also need projections corresponding to Q, R in (3.2), (3.20) respectively. As to Q we stipulate:

$$Qf=\sum_{\beta\le-\Lambda_k^2+\zeta}a_k e^{i\Lambda_x}+\int x_{[0,\zeta-\beta]}(\lambda^2)g(\lambda)e^{i\lambda x}\,d\lambda, \tag{4.10}$$

with f given by (4.4) and χ_I the characteristic function of I. Assuming as in Section 3 that $0<\beta<\gamma<\zeta$, the projection R is defined accordingly, i.e., via (4.10) but with γ in place of β. It is then routine to show that the estimates (3.4), (3.5), (3.22) remain valid in the present context, i.e., with V_t and Q, R defined via (4.9), (4.10) resp.

While the above remarks settle the linear part of our considerations, we also have to take care of the nonlinearity $\mathcal{P}(u)$ ((1.1), (1.2)). This reduces to a discussion of the multiplication in $\widehat{\mathcal{C}}_r(\mathbb{R}^n)$.

The proposition below summarizes those parts of Theorems (19.15), (19.18), (19.20) in [5], 269–273, which are of relevance here. Below, $*$ denotes convolution: if $g,h\in\mathcal{L}^1(\mathbb{R}^n)$ then

$$\|g*h\|_{L^1}=\|g\|_{L^1}\|h\|_{L^1}\text{ and }(g*h)^\wedge=\widehat{g}\widehat{h}. \tag{4.11}$$

Proposition 4.1

(A) Let $T=\sum a_k e^{i\Lambda_k x}\in H^0(\mathbb{R}^n)$, $g\in\mathcal{L}^1(\mathbb{R}^n)$; then $\varphi(\ )=\sum a_k g(\cdot\ -\Lambda_k)$ is in $\mathcal{L}^1(\mathbb{R}^n)$, $\widehat{\varphi}=T\widehat{g}$ and

$$\|\widehat{\varphi}\|=\|\varphi\|_{L^1}\le\left(\sum|a_k|\right)\|g\|_{L^1}=\|T\|\,\|\widehat{g}\|.$$

(B) Let $T_j\in H^0(\mathbb{R}^n)$, $g_j\in\mathcal{L}^1(\mathbb{R}^n)$, i.e., $T_j+\widehat{g}_j\in\widehat{\mathcal{C}}_r(\mathbb{R}^n)$, $j=1,2$. Then

$$(T_1+\widehat{g}_1)(T_2+\widehat{g}_2)=T_1T_2+T_1\widehat{g}_2+T_2\widehat{g}_1+(g_1*g_2)^\wedge\in\widehat{\mathcal{C}}_r(\mathbb{R}^n)$$

and $\|(T_1+\widehat{g}_1)(T_2+\widehat{g}_2)\|\le\|T_1+\widehat{g}_1\|\,\|T_2+\widehat{g}_2\|$. •

Remarks. While (A) is proved by elementary estimates, (B) follows from (4.11) and (A) by straightforward computation. With estimates (3.4), (3.5), (3.22) valid in the present setting, and having established that $\widehat{\mathcal{C}}_r(\mathbb{R}^n)$ is a Banach algebra, it is now straightforward to show that Lemma 3.1 and Theorem 1 still hold when interpreted in $\widehat{\mathcal{C}}_r(\mathbb{R}^n)$; the proofs remain the same. We content us to restate Theorem

1 properly. To this end we redefine B_-^β according to

$$\begin{aligned} &y \in B_-^\beta \text{ iff } y \in C^0((-\infty,0],\ \widehat{\mathcal{C}}_r(\mathbb{R}^n)) \text{ and} \\ &[y]_\beta = \sup_{t\le 0} e^{-\beta t}\|y(t)\| < \infty \quad (\|\ \| \text{ the norm in (4.5)}). \end{aligned} \tag{4.12}$$

Theorem 2. If $0 < \beta < \mu < \zeta$ there is a solution $y \in B_-^\beta$ (via (4.12) of (0.1) such that $y \notin B_-^\mu$.

Remarks

(0) Theorem 2 remains valid if we restrict our considerations to the subspace $P_a\widehat{\mathcal{C}}_r(\mathbb{R}^n)$ of Fourier transforms of absolutely continuous measures which is invariant under all operations involved in the proof of Theorem 2.

(1) The so generalized Theorem 1 and Lemma 3.1 apply to example (3.34) and thus yield a larger class of solutions of (3.34), defined on all of $\mathbb{R}$.

(2) It can be shown that Lemma 3.1 and Theorem 1 hold in the space $\widehat{\mathcal{C}}(\mathbb{R}^n)$ of Fourier transforms (4.2) of complex, finite, σ-additive measures μ. While the proof is still along the lines of Sections 2, 3 a certain amount of measure theory as treated in [5], Chapters 3, 5 cannot be avoided; the details will be presented elsewhere.

(II) We conclude with an outline of the structure which eq. (0.1) assumes in the Banach algebra $\widehat{\mathcal{C}}_r(\mathbb{R}^n)$. For simplicity we let $\alpha = 0$ in (2.2) and let $\mathcal{P}(u)$ be a polynomial via (1.2), i.e., with coefficients in $H^0(\mathbb{R}^n)$. We then have the representation

$$P(u+v) = \mathcal{P}(u) + vQ(u,v), Q(0,0) = 0 \tag{4.13}$$

with $Q(u,v)$ a polynomial in u, v with coefficients in $H^0(\mathbb{R}^n)$. Now let $u(\ )$ be a solution cf. (0.1) on $I = [0,\tau)$:

$$\begin{aligned} &u(t) \in \mathrm{dom}(\Delta) \text{ for } t \in I, \quad \Delta u \in C^0(I, \widehat{\mathcal{C}}_r(\mathbb{R}^n)), \\ &u \in C^1([I, \widehat{\mathcal{C}}_r(\mathbb{R}^n)), \quad u(0) = u_0 \text{ and } u \text{ satisfies (0.1) pointwise.} \end{aligned} \tag{4.14}$$

Based on the direct sum property (4.7), the solution u in (4.14) admits the decomposition

$$u = T + \widehat{g} \text{ with } P_d u = T \in H^0(\mathbb{R}^n), \quad P_a u = \widehat{g}, \quad g \in \mathcal{L}^1(\mathbb{R}^n). \tag{4.15}$$

We insert (4.15) into (0.1) by taking care of (4.13) and of Prop. 4.1; as a result we obtain a coupled system equivalent to (0.1):

$$\partial_t T = (\Delta+\zeta)T + P(T), \qquad P_d u(0) = T(0), \tag{4.16a}$$

$$\partial_t \widehat{g} = (\Delta+\zeta)\widehat{g} + \widehat{g}Q(T,\widehat{g}), \quad P_a u(0) = \widehat{g}(0). \tag{4.16b}$$

From (4.16) we read off that the almost periodic part $P_d u = T$ of the solution u of (0.1) is dominating: if $[0,\tau)$, $\tau < \infty$ is the maximal interval of existence of (4.16a) for the initial condition $P_d u(0) = T(0)$, then the maximal interval of existence $[0,\tau')$ for the whole system (4.16a) + (4.16b) satisfies $\tau' \le \tau$ regardless how $P_a u(0) = \widehat{g}(0)$ is chosen. Likewise, if $\tau = \infty$ but $\limsup_{t\to\infty} \|T(t)\| = \infty$

then either $\tau' < \infty$ or else $\tau' = \infty$ and $\limsup_{t\to\infty} \|u(t)\| = \infty$ for the solution $u = T + \widehat{g}$ of (4.16a)+(4.16b), i.e., of (0.1), regardless how $P_a u(0) = \widehat{g}(0)$ is chosen. That is, the absolutely continuous part $P_a u(0)$ of the initial condition $u(0)$ has no stabilizing effect.

Appendix

We add a few remarks concerning the relations between our considerations and the space $S(\mathbb{R}^n)$ of Stepanov almost periodic functions. In order to define $S(\mathbb{R}^n)$, let $Q \subseteq R^n$ denote any n-cube of length L and assume $f \in \mathcal{L}^2_{\text{loc}}(\mathbb{R}^n)$. By definition, $f \in \mathcal{L}^2_b(\mathbb{R}^n)$ iff

$$\|f\|_{\mathcal{L}^2_b} = \sup_Q \|f\|_{\mathcal{L}^2(Q)} < \infty. \tag{A.1}$$

Likewise, $f \in H^m_b$ iff $f \in H^m_{\text{loc}}(\mathbb{R}^n)$ and

$$\|f\|_{H^m_b} = \sup_Q \|f\|_{H^m(Q)} < \infty. \tag{A.2}$$

$\mathcal{L}^2_b$ and H^m_b are Banach spaces endowed with norms (a1), (a2) respectively. It is easily seen that up to norm equivalence, $\mathcal{L}^2_b$ and H^m_b do not depend on L. As to $S(\mathbb{R}^n)$ we recall the space $H_T(\mathbb{R}^n)$ of finite trigonometric sums in (2.12) and stipulate

$$f \in S(\mathbb{R}^n) \text{ iff there is a sequence } T_N \in H_T(\mathbb{R}^n) \text{ with } \lim_N \|f - T_N\|_{\mathcal{L}^2_b} = 0. \tag{A.3}$$

Likewise, $f \in HS^m(\mathbb{R}^n)$ iff there is a sequence $T_N \in H_T(\mathbb{R}^n)$ with

$$\lim \|f - T_N\|_{H^m_b} = 0.$$

A function $f \in S(\mathbb{R}^n)$ has a series of properties, similar to those of the Bohr almost periodic functions ([15]). In particular, f has an associated Fourier series

$$f \simeq \Sigma f_k e^{i\Lambda_k x}.$$

However the relation between f and its Fourier series is less simple than in case of $f \in H^s(\mathbb{R}^n)$ (Definition 1).

It is now straightforward to interpret equation (0.1) (with $\nu > 0$ for simplicity) as an evolution equation in $S(\mathbb{R}^n)$. To this end we stipulate

$$\mathrm{dom}(\Delta) = \mathrm{HS}^2(\mathbb{R}^n), \quad \Delta \mathrm{f} = \Sigma\partial_{\mathrm{j}}^2\mathrm{f}, \quad \mathrm{f} \in \mathrm{dom}(\Delta). \tag{A.4}$$

It turns out that Δ so defined is the generator of a holomorphic semigroup, i.e., the heat semigroup W_t, $t \geq 0$, restricted to $S(\mathbb{R}^n)$. Fractional power spaces B^γ can then be defined in terms of $(1-\Delta)^\gamma$, what gives the possibility to define the polynomial nonlinearity $P(u)$ in (0.1) properly. This leads to an interpretation of (0.1) as an evolution equation in $S(\mathbb{R}^n)$ in the sense of [14], pg. 196. We note that the spaces $H^m(\mathbb{R}^n)$ in Definition 1 are continuously embedded in $HS^m(\mathbb{R}^n)$:

$$H^m(\mathbb{R}^n) \hookrightarrow HS^m(\mathbb{R}^n). \tag{A.5}$$

Since $\zeta > 0$ in (0.1) by assumption, one expects that $u = 0$, considered as a solution of (0.1) in the space $S(\mathbb{R}^n)$, is Ljapunov unstable. One might try to prove this instability by construction of an instable manifold with the aid of a projection operator Q, endowed with properties similar as Q given by (3.2). However, it is here where the difficulties appear; in fact we were not able to construct a suitable projection operator Q. A way out is to rely on Lemma 3.1, i.e., we fix $\eta \in S(\frac{\varepsilon}{2})$, $\eta \neq 0$ in the terminology of Lemma 3.1 and seek the solution $y \in S_\beta(\varepsilon)$, $y = \varphi(\eta)$ of (0.1) which satisfies (3.15). By the embedding (A.5) we have that $y(t)$, $t \leq 0$ is also a solution of (0.1) when considered as an equation in $S(\mathbb{R}^n)$. On the other hand, $Qy(0) = \eta \neq 0$ by (3.15) whence

$$\|y(0)\|_{\mathcal{L}_b^2} = d > 0 \tag{A.6}$$

as is easily seen. We now fix $\delta > 0$, $T_0 > 0$ arbitrarily; since $y \in B_-^\beta$ there is $T > T_0$ such that $\|y(-T)\|k < \delta$ where k is an embedding constant such that

$$\|y(t)\|_{\mathcal{L}_b^2} \leq k\|y(t)\|, \quad t \leq 0 \tag{A.7}$$

obtained from (A.5). Finally we set $y_0(t) = y(t-T)$, $t \in [0,T]$. We then have that y_0 is a solution of (0.1) in $S(\mathbb{R}^n)$ such that

$$\|y_0(0)\|_{\mathcal{L}_b^2} \leq k\|y(-T)\|, \quad t \leq 0$$

obtained from (A.5).

We then have that y_0 is a solution of (0.1) in $S(\mathbb{R}^n)$ such that

$$\|y_0(0)\|_{\mathcal{L}_b^2} \leq k\|y(-T)\| < \delta, \quad \|y_0(T)\|_{\mathcal{L}_b^2} = \|y(0)\|_{\mathcal{L}_b^2} = d.$$

Thus Ljapounov instability of $u = 0$ as a solution of (0.1) in $S(\mathbb{R}^n)$ follows.

While Ljapounov instability of (0.1) in $S(\mathbb{R}^n)$ thus follows from the considerations in Section 3, the situation is less favorable in case of solutions of slow exponential decay. In fact, if $y \in B_-^\beta$ is a solution of (0.1) in $H^0(\mathbb{R}^n)$ such that

$$\sup_{t \leq 0} e^{-\mu t}\|y(t)\| = \infty, \quad \text{some } \beta < \mu,$$

we cannot necessarily infer

$$\sup_{t \leq 0} e^{\mu t}\|y(t)\|_{\mathcal{L}_b^2} = \infty \tag{A.8}$$

since we have only (A.7) at disposal. An attempt to construct a manifold of solutions of (0.1) in $S(\mathbb{R}^n)$, exhibiting slow exponential decay in the sense of Lemma 3.2 resp. Theorem 1, fails since we cannot construct the necessary projection operators Q, R. We therefore have to content us with a partial result. In order to outline the construction of solutions of (0.1) in $S(\mathbb{R}^n)$ which exhibit slow exponential decay in some sense we assume that the coefficients a_k in (0.1) are constant and let (for simplicity) $n = 1$. Next we recall ε_0 in Lemma 3.1; we fix $\varepsilon \leq \varepsilon_0$ and $0 < \beta < \gamma < \zeta$ such that $\gamma < 2\beta$. We then pick $a \in \mathbb{C}$, $a \neq 0$ such that $|a| \leq \min(C^{-1}, \frac{\varepsilon}{2})$ (with $C = C(\beta,\gamma)$ as in Lemma 3.2) and set

$$\eta = \tfrac{1}{2}(ae^{i\Lambda x} + \overline{a}e^{-i\Lambda x}) \quad \text{where} \quad \beta \leq \zeta - \Lambda^2 < \gamma. \tag{A.9}$$

Thus $\|\eta\| = |a| \neq 0$ and $\eta \in S(\frac{\varepsilon}{2})$ what entails that $\varphi(\eta) = y \in S_\beta(\varepsilon)$ is a solution of (0.1) in $H^0(\mathbb{R})$ such that

$$y \in B_-^\beta \text{ and } y \notin B_-^\mu \text{ for } \gamma < \mu. \tag{A.10}$$

Now η in (A.9) is $\frac{2\pi}{\Lambda}$-periodic; invariance arguments show that the solution $y = \varphi(\eta)$ of (0.1) (in $H^0(\mathbb{R})$) is $\frac{2\pi}{\Lambda}$-periodic too. By looking at y as a solution of (0.1) in a $\frac{2\pi}{\Lambda}$-periodic Sobolev setting we find

$$y \in C^1((-\infty, 0)], H^1_{\mathrm{per}}((0, \tfrac{2\pi}{\Lambda})) \tag{A.11}$$

where $H^k_{\mathrm{per}}((0, \frac{2\pi}{\Lambda}))$ are the usual $\frac{2\pi}{\Lambda}$-periodic Sobolev spaces. Now let

$$y(t) = \Sigma a_k(t) e^{i\frac{2\pi}{\Lambda}kx}.$$

From (A.11) we then infer

$$\|y(t)\|^2_{H^1_{\mathrm{per}}} = \Sigma |a_k(t)|^2 (1 + (\tfrac{2\pi}{\Lambda})^2 k^2) < \infty$$

what in turn entails

$$\|y(t)\| \leq c \|y(t)\|_{H^1_{\mathrm{per}}} \quad (t \leq 0), \text{ some constant } c,$$

whence

$$\infty = \sup_{t \leq 0} e^{-\mu t} \|y(t)\| \leq \sup_{t \leq 0} e^{-\mu t} \|y(t)\|_{H^1_{\mathrm{per}}}.$$

In a last step one then infers from (A.14):

$$\sup_{t \leq 0} e^{-\mu t} \|y(t)\|_{H^1_b} = \infty.$$

Thus a solution y of (0.1) in $S(\mathbb{R}^n)$ has been found which exhibits slow exponential decay in the H^1_b-norm.

Whether this result can be improved so as to yield solutions of slow exponential decay in the $\mathcal{L}^2_b$-norm is open.

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On the Oseen Semigroup with Rotating Effect

Yoshihiro Shibata

Dedicated to the memory of Günter Lumer

Abstract. This paper is concerned with the generation of C_0 semigroup associated with the Oseen equation with rotating effect and its L_p-L_q decay estimate. The theorems presented in this paper give us one of the key steps in order to show a globally in time existence of solutions to the Navier-Stokes equations describing the motion of viscous incompressible fluid flow past a rotating rigid body.

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1. Introduction and main results

Let Ω be an exterior domain in the Euclidean 3-space $\mathbb{R}^3$ with $C^{1,1}$ boundary $\partial\Omega$. In this paper, we report the existence of solutions and their decay property of the initial boundary value problem:

$$\begin{aligned} &u_t + L_{k,a}u + \nabla\pi = 0, \quad \operatorname{div} u = 0 \quad \text{in } \Omega \times \mathbb{R}_+, \\ &u|_{\partial\Omega} = 0, \quad u|_{t=0} = f. \end{aligned} \tag{1.1}$$

Here, $\mathbb{R}_+ = (0,\infty)$ and $L_{k,a}u = -\Delta u + k\partial_3 u - (\omega \times x)\cdot\nabla u + \omega \times u$,
k is a real constant, $\omega = a\mathbf{e}_3$ ($\mathbf{e}_3 = (0,0,1)^T$ [1]), a a real constant,
$u = (u_1,u_2,u_3)^T$ unknown velocity field, π unknown pressure,
t the time variable, $x = (x_1,x_2,x_3)$ the space variable in $\mathbb{R}^3$,
$\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$, $\Delta = \sum_{j=1}^3 \partial_j^2$, $u_t = (\partial_t u_1, \partial_t u_2, \partial_t u_3)$,
$\Delta u = (\Delta u_1, \Delta u_2, \Delta u_3)$, $(\omega \times x)\cdot\nabla = a(x_1\partial_2 - x_2\partial_1)$,
and $\nabla\pi = (\partial_1\pi, \partial_2\pi, \partial_3\pi)$.

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[1] M^T denotes the transposed M.

The first equation of (1.1) can be written in the componentwise:

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} - \Delta u_1 + k\frac{\partial u_1}{\partial x_3} - a(x_1\partial_2 - x_2\partial_1)u_1 - u_2 + \frac{\partial \pi}{\partial x_1} = 0,\\
&\frac{\partial u_2}{\partial t} - \Delta u_2 + k\frac{\partial u_2}{\partial x_3} - a(x_1\partial_2 - x_2\partial_1)u_2 + u_1 + \frac{\partial \pi}{\partial x_2} = 0,\\
&\frac{\partial u_3}{\partial t} - \Delta u_3 + k\frac{\partial u_3}{\partial x_3} - a(x_1\partial_2 - x_2\partial_1)u_3 + \frac{\partial \pi}{\partial x_3} = 0.
\end{aligned}$$

The problem (1.1) is obtained as a linearized problem of the Navier-Stokes equations describing the motion of incompressible viscous fluid flow past a rotating rigid body $\mathcal{O} = \mathbb{R}^3 \setminus \Omega$ with axis of rotation $\omega = a\mathbf{e}_3 = a(0,0,1)^T$ $(a \neq 0)$ which is moving with velocity $k \neq 0$ in the direction of its axis of rotation.

To be more precise, let us consider the Navier-Stokes equations:

$$\begin{aligned}
v_t + v\cdot\nabla v - \Delta v + \nabla\pi = g &\quad \text{in } \Omega(t), &&t > 0,\\
\operatorname{div} v = 0 &\quad \text{in } \Omega(t), &&t > 0,\\
v(y,t) = \omega \times y &\quad \text{on } \partial\Omega(t), &&t > 0,\\
v(y,t) \to u_\infty \neq 0 &\quad \text{as } |y| \to \infty, &&t > 0
\end{aligned} \tag{1.2}$$

with an initial value $v(y,0) = v_0(y)$ in the time-dependent exterior domain

$$\Omega(t) = \mathcal{O}(at)\Omega,$$

where $\mathcal{O}(t)$ denotes the orthogonal matrix

$$\mathcal{O}(t) = \begin{pmatrix} \cos t & -\sin t & 0\\ \sin t & \cos t & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Then, introducing

$$x = \mathcal{O}(t)^T y, \;\; u(x,t) = \mathcal{O}(t)^T(v(y,t) - u_\infty), \;\; p(x,t) = \pi(y,t), \tag{1.3}$$

we see that (u,p) satisfies the modified Navier-Stokes equations:

$$\begin{aligned}
&u_t + u\cdot\nabla u - \Delta u + (\mathcal{O}(t)^T u_\infty)\cdot\nabla u\\
&\quad -(\omega\times x)\cdot\nabla u + \omega\times u + \nabla\pi = f &&\text{in } \Omega\times(0,\infty),\\
&\qquad\qquad \operatorname{div} u = 0 &&\text{in } \Omega\times(0,\infty),\\
&\qquad\qquad u(x,t) = \omega\times x - \mathcal{O}(t)u_\infty &&\text{on } \partial\Omega\times(0,\infty),\\
&\qquad\qquad u(x,t) \to 0 &&\text{as } |x|\to\infty, t > 0
\end{aligned} \tag{1.4}$$

with an initial data $u(x,0) = v_0(x)$. In this paper, we consider only the case where $u_\infty = k\mathbf{e}_3$, so that $\mathcal{O}(t)^T u_\infty = k\mathbf{e}_3$ for all $t > 0$. Therefore, (1.4) leads to the system:

$$
\begin{aligned}
&u_t + u\cdot\nabla u - \Delta u + k\partial_3 u && \\
&-(\omega\times x)\cdot\nabla u + \omega\times u + \nabla\pi = f && \text{in } \Omega\times(0,\infty),\\
&\operatorname{div} u = 0 && \text{in } \Omega\times(0,\infty),\\
&u(x,t) = \omega\times x - k\mathbf{e}_3 && \text{on } \partial\Omega\times(0,\infty),\\
&u(x,t)\to 0 && \text{as } |x|\to\infty,\ t>0
\end{aligned}
\tag{1.5}
$$

with an initial data $u(x,0) = v_0(x)$. Dropping the nonlinear term $u\cdot\nabla u$, the external force f and boundary force $\omega\times x - k\mathbf{e}_3$, we have (1.1).

The mathematical analysis of viscous flow past rotating obstacles started with [1], where weak non-stationary solutions have been constructed in an even more general setting allowing for time-dependent functions $\omega(t)$ and $u_\infty(t)$. And, Farwig [2] proved the L_q estimate of second derivatives of u and the first derivatives of p, where u and p solve the equations:

$$-\Delta u + k\partial_3 u - (\omega\times x)\cdot u + \omega\times u + \nabla p = f, \quad \operatorname{div} u = g \quad \text{in } \mathbb{R}^3.$$

The author could not find any other papers published in any mathematical journal concerning the evolution equation (1.5) with non-zero k and ω.

To formulate (1.1) in the semigroup setting, we eliminate the pressure term p, because p has no evolution. For this purpose, we introduce the Helmholtz decomposition. Let D be one of $\mathbb{R}^3$, Ω and $\Omega_R = \Omega\cap B_R$ ($B_R = \{x\in\mathbb{R}^3 \mid |x| < R\}$). Let R be a large number such that $B_{R-5}\supset\mathbb{R}^3\setminus\Omega$ and $1<q<\infty$. Set

$$
\begin{aligned}
&L_q(D)^3 = \{f = (f_1,f_2,f_3) \mid f_i\in L_q(D)(i=1,2,3)\},\ \ J_q(D) = \overline{C^\infty_{0,\sigma}(D)}^{L_q(D)},\\
&G_q(D) = \{\nabla\pi \mid \pi\in\hat W^1_q(D)\}, \quad C^\infty_{0,\sigma}(D) = \{u\in C^\infty_0(D)^3 \mid \operatorname{div} u = 0 \ \text{in } D\},\\
&\hat W^1_q(D) = \{\pi\in L_{q,\mathrm{loc}}(\overline{D}) \mid \nabla\pi\in L_q(D)^3, \int_{\Omega_R}\pi\,dx = 0\}.
\end{aligned}
$$

Then, we have

$$L_q(D)^3 = J_q(D)\oplus G_q(D), \quad \oplus : \text{direct sum}.$$

To obtain such decomposition, given $f\in L_q(D)^3$ we take π such that π is a weak solution to the Laplace equation:

$$\Delta\pi = \operatorname{div} f \quad \text{in } D, \quad \partial_\nu\pi = \nu\cdot f. \tag{1.6}$$

Here, $\nu = (\nu_1,\nu_2,\nu_3)$ is the unit outer normal to ∂D and $\partial_\nu = \nu\cdot\nabla$. In particular, we know that $J_q(D) = \{g\in L_q(D) \mid \operatorname{div} g = 0 \text{ in } D,\ \nu\cdot g|_{\partial D} = 0\}$. When $f = g + \nabla\pi$ with $g\in J_q(D)$ and $\pi\in\hat W^1_q(D)$, we set

$$P_D f = g, \ \ Q_D f = \pi. \tag{1.7}$$

In particular, P_D is a bounded linear operator from $L_q(D)^3$ onto $J_q(D)$ and $Q_D f\in\hat W^1_q(D)$.

Applying P_D to $L_{k,a}$, we define the operator $\mathcal{L}_D$ with domain $\mathcal{D}_q(D)$ as follows:

$$\mathcal{L}_D u = P_D L_{k,a} u = P_D(-\Delta u + k\partial_3 u - (\omega\times x)\cdot\nabla u + \omega\times u) \quad (u\in\mathcal{D}_q(D)),$$
$$\mathcal{D}_q(D) = \{u\in J_q(D)\cap W_q^2(D) \mid u|_{\partial D}=0,\ (\omega\times x)\cdot\nabla u\in L_q(D)\}. \tag{1.8}$$

Using these symbols, (1.1) is written as follows:

$$\begin{aligned} u_t + \mathcal{L}_\Omega u &= 0 \quad \text{in } J_q(\Omega) && \text{for } t>0,\\ u|_{t=0} &= f \quad u(t)\in\mathcal{D}_q(\Omega) && \text{for } t>0. \end{aligned} \tag{1.9}$$

Theorem 1.1. *Let $1<q<\infty$. Then, $\mathcal{L}_\Omega$ generates a C^0 semigroup $\{T(t)\}_{t\geq 0}$ on $J_q(\Omega)$.*

Remark 1.2. Theorem 1.1 was proved by Hishida [6] when $q=2$ and $k=0$ and by Geissert-Heck-Hieber [5] when $1<q<\infty$ and $k=0$. Our proof is different from previous results and based on some new consideration on the pressure terms

According to Kato's theory [8], we know that so-called L_q-L_r estimate of the Stokes semigroup plays an essential role to show the stability of stationary flow. In the following theorem, we state such L_q-L_r estimates of solutions to (1.1).

Theorem 1.3. *Let $1<q<\infty$, $k_0>0$ and $a_0>0$. Assume that $|a|\leq a_0$ and $|k|\leq k_0$. Then, there hold the following estimates for $f\in J_q(\Omega)$ and $t>0$:*

$$\begin{aligned} \|T(t)f\|_{L_r(\Omega)} &\leq C_{q,r} t^{-\frac32\left(\frac1q-\frac1r\right)}\|f\|_{L_q(\Omega)} && 1<q\leq r\leq\infty,\ q\neq\infty,\\ \|\nabla T(t)f\|_{L_r(\Omega)} &\leq C_{q,r} t^{-\frac12-\frac32\left(\frac1q-\frac1r\right)}\|f\|_{L_q(\Omega)} && 1<q\leq r\leq 3. \end{aligned}$$

Here, the constant $C_{q,r}$ depends on a_0 and k_0 but is independent of a, k, t and f whenever $|a|\leq a_0$ and $|k|\leq k_0$.

Remark 1.4. When $k=0$, Theorem 1.3 was proved by Hishida and Shibata [7].

2. Analysis in $\mathbb{R}^3$

If we consider the equation (1.1) in $\mathbb{R}^3$ with initial data $f\in J_q(\mathbb{R}^3)$, then the solution u is given by the following formula:

$$\begin{aligned} u(t) = S_{\mathbb{R}^3}(t)f &= (4\pi t)^{-3/2}\int_{\mathbb{R}^3}\exp\Big(-\frac{|\mathcal{O}(at)x-y-k\mathbf{e}_3t|^2}{4t}\Big)\mathcal{O}(at)^T P_{\mathbb{R}^3}f(y)\,dy\\ &= \mathcal{F}^{-1}\left[e^{-(|\xi|^2+ik\xi_3)t}\mathcal{O}(at)^T\mathcal{F}[P_{\mathbb{R}^3}f](\mathcal{O}(at)\xi)\right](x). \end{aligned}$$

Here $P_{\mathbb{R}^3}$ and $Q_{\mathbb{R}^3}$ are given by the following formulas:

$$P_{\mathbb{R}^3}f = \mathcal{F}^{-1}\left[\left(\delta_{jk}-\frac{\xi_j\xi_k}{|\xi|^2}\right)\hat f(\xi)\right](x), \quad Q_{\mathbb{R}^3}f = \mathcal{F}^{-1}\left[\frac{\xi\cdot\hat f(\xi)}{i|\xi|^2}\right](x)+c(f)$$

and $c(f)$ is a constant such that $\int_{\Omega_R} Q_{\mathbb{R}^3} f\,dx = 0$. To show Theorems 1.1 and 1.3, we consider the resolvent problem. The resolvent of the problem (1.1) in $\mathbb{R}^3$ is given by the Laplace transform of $S_{\mathbb{R}^3}(t)f$. Set

$$\begin{aligned}\mathcal{A}_{\mathbb{R}^3,a,k}(\lambda)f = \mathcal{A}(\lambda)f &= \int_0^\infty e^{-\lambda t} S_{\mathbb{R}^3}(t) P_{\mathbb{R}^3} f\,dt \\ &= \mathcal{F}^{-1}\Big[\int_0^\infty e^{-(\lambda+|\xi|^2+i\xi_3 k)t}\mathcal{O}(at)^T\widehat{P_{\mathbb{R}^3}f}(\mathcal{O}(at)\xi)\,dt\Big](x).\end{aligned}$$

If we restrict ourselves to the case where f has a compact support, we have rather plenty of information to prove Theorems 1.1 and 1.3 as follows:

Theorem 2.1. *Let $1 < q < \infty$, $k_0 > 0$ and $a_0 > 0$. Assume that $|k| \le k_0$ and $|a| \le a_0$.*
(1) Let $\gamma > 0$, $0 < \epsilon < \pi/2$ and $N \in \mathbb{N}$ with $N \ge 4$. Set

$$\begin{aligned}&\mathbb{C}_\gamma = \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \ge \gamma\},\ \ \mathbb{C}_+ = \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\},\\ &\Sigma_\epsilon = \{\lambda \in \mathbb{C}\setminus\{0\} \mid |\arg\lambda| \le \pi-\epsilon\},\\ &L_{q,R-1}(\mathbb{R}^3) = \{f \in L_q(\mathbb{R}^3)^3 \mid f(x) = 0 \ \text{for } x \notin B_{R-1}\},\\ &\mathcal{L}_R(\mathbb{R}^3) = \mathcal{L}(L_{q,R-1}(\mathbb{R}^3), W_q^2(\mathbb{R}^3)^3).\end{aligned}$$

Then, $\mathcal{A}_{\mathbb{R}^3,a,k}(\lambda) \in \mathrm{Anal}(\mathbb{C}_+, \mathcal{L}_R(\mathbb{R}^3))$ and there exist three operators:

$$\begin{aligned}&\mathcal{A}^N_{1,a}(\lambda),\ \ \tilde{\mathcal{A}}^N_{1,a}(\lambda) \in \mathrm{Anal}(\mathbb{C}\setminus(-\infty,0], \mathcal{L}_R(\mathbb{R}^3)),\\ &\mathcal{A}^N_{2,a}(\lambda) \in \mathrm{Anal}(\mathbb{C}_+, \mathcal{L}_R(\mathbb{R}^3))\end{aligned}$$

such that

$$\begin{aligned}&\mathcal{A}_{\mathbb{R}^3,a,k}(\lambda)f = \mathcal{A}^N_{1,a}(\lambda) + \mathcal{A}^N_{2,a}(\lambda),\\ &\mathcal{A}^N_{1,a}(\lambda) = (\lambda - \Delta_{\mathbb{R}^3} + k\partial_3)^{-1}P_{\mathbb{R}^3} + \tilde{\mathcal{A}}^N_{1,a}(\lambda),\\ &\|\partial_x^\beta \mathcal{A}^N_{1,a}(\lambda)f\|_{L_q(\mathbb{R}^3)} \le C|\lambda|^{-(1-(|\beta|/2))}\|f\|_{L_q(\mathbb{R}^3)} \qquad (\lambda \in \Sigma_\epsilon, |\lambda| \ge c_\epsilon > 0),\\ &\|\partial_x^\beta \tilde{\mathcal{A}}^N_{1,a}(\lambda)f\|_{L_q(\mathbb{R}^3)} \le C|\lambda|^{-((3/2)-(|\beta|/2))}\|f\|_{L_q(\mathbb{R}^3)} \qquad (\lambda \in \Sigma_\epsilon, |\lambda| \ge c_\epsilon > 0),\\ &\|\partial_x^\beta \mathcal{A}^N_{2,a}(\lambda)f\|_{L_q(\mathbb{R}^3)} \le C\,\gamma^{-1}|\lambda|^{-(N/2)+(|\beta|/2)}\|f\|_{L_q(\mathbb{R}^3)} \quad (\lambda \in \mathbb{C}_\gamma)\end{aligned}$$

for any $f \in L_{q,R-1}(\mathbb{R}^3)$ provided that $|\beta| \le 2$ and $|\lambda| \ge 1$, where the constants c_ϵ and C depend on ϵ, a_0 and k_0 but are independent of a and k and we have set

$$(\lambda - \Delta_{\mathbb{R}^3} + k\partial_3)^{-1}g = \mathcal{F}_\xi^{-1}[(\lambda + |\xi|^2 + ik\xi_3)^{-1}\hat{g}(\xi)](x).$$

Here and hereafter, $\mathcal{L}(X,Y)$ denotes the set of all bounded linear operators from X into Y and $\mathrm{Anal}(I,X)$ the set of all X-valued analytic functions defined on I.

Theorem 2.2. *Let $1 < q < \infty$, $k_0 > 0$, $a_0 > 0$, γ_0 and $K \ge 10a_0 + 2$. Assume that $|k| \le k_0$, $|a| \le a_0$ and $0 \le \gamma \le \gamma_0$. Set $\mathcal{L}_{R,\mathrm{comp}}(\mathbb{R}^3) = \mathcal{L}(L_{q,R-1}(\mathbb{R}^3), W_q^2(B_R)^3)$ and denote the operator norm of $\mathcal{L}_{R,\mathrm{comp}}(\mathbb{R}^3)$ by $[\,\cdot\,]_{\mathbb{R}^3,R}$.*

Then, $\mathcal{A}(\lambda) = \mathcal{A}_{\mathbb{R}^3,a,k}(\lambda) \in C(\overline{\mathbb{C}_+}, \mathcal{L}_{R,\mathrm{comp}}(\mathbb{R}^3))$ *and satisfies the conditions:*

$$
\begin{aligned}
&\sup_{|s|\le K} [\mathcal{A}(\gamma+is)]_{\mathbb{R}^3,R} \le C_{\gamma_0,a_0,K},\\
&\int_{-K}^{K} [(\partial_\lambda \mathcal{A})(\gamma+is)]_{\mathbb{R}^3,R}^{\frac{3}{2}}\,ds \le C_{\gamma_0,a_0,K},\\
&\sup_{0<|h|\le 1} |h|^{-1/2}\Big(\int_{-K}^{K} [\mathcal{A}(\gamma+i(s+h)) - \mathcal{A}(\gamma+is)]_{\mathbb{R}^3,R}^{3}\,ds\Big)^{\frac{1}{3}} \le C_{\gamma_0,a_0,K},\\
&\sup_{0<|h|\le 1} |h|^{-1/2}\int_{-K}^{K} [(\partial_\lambda \mathcal{A})(\gamma+i(s+h)) - (\partial_\lambda \mathcal{A})(\gamma+is)]_{\mathbb{R}^3,R}\,ds \le C_{\gamma_0,a_0,K},\\
&\|(\partial_\lambda^m \mathcal{A}_{\mathbb{R}^3,a})(\gamma+is)\|_{\mathcal{L}(L_{q,R-1}(\mathbb{R}^3), W_q^j(B_R)^3)} \le C_{\gamma_0,a_0,K}|s|^{-m-(1-(j/2))}
\end{aligned}
$$

for $m = 0,1,2,3$ *and* $j = 0,1,2$*;* $s \in \mathbb{R}$ *with* $|s| \ge K-2$*. Moreover, we have*

$$
\begin{aligned}
&\lim_{\gamma\to 0+}\sup_{s\in\mathbb{R}} [\mathcal{A}(\gamma+is) - \mathcal{A}(is)]_{\mathbb{R}^3,R} = 0,\\
&\lim_{\gamma\to 0+}\int_{-\infty}^{\infty} [(\partial_\lambda \mathcal{A})(\gamma+is) - (\partial_\lambda \mathcal{A})(is)]_{\mathbb{R}^3,R} = 0,\\
&\lim_{r\to\infty} r^{-1}\int_{r\le|x|\le 2r} |[\mathcal{A}(\lambda)f](x)|^2\,dx = 0 \quad (k\ne 0),\\
&\lim_{r\to\infty} r^{-2}\int_{r\le|x|\le 2r} |[\mathcal{A}(\lambda)f](x)|^2\,dx = 0 \quad (k = 0),\\
&\|\mathcal{A}(\lambda_1)f - \mathcal{A}(\lambda_2)f\|_{W_q^2(B_R)} \le C_{q,R}|\lambda_1-\lambda_2|^{1/4}\|f\|_{L_q(\mathbb{R}^3)}
\end{aligned}
$$

for any λ*,* λ_1 *and* $\lambda_2 \in \overline{\mathbb{C}_+} = \{\lambda\in\mathbb{C} \mid \mathrm{Re}\,\lambda \ge 0\}$ *provided that* $f \in L_{q,R-1}(\mathbb{R}^3)$*.*

3. Rough ideas of proofs of Theorems 1.1 and 1.3

To show Theorem 1.1, the following theorem is a key step.

Theorem 3.1. *Let* $1 < q < \infty$ *and set*

$$L_{q,R-1}(D) = \{f \in L_q(D)^3 \mid f(x) = 0 \ \textit{for}\ |x| > R-1\}$$

with $D = \Omega$ *or* $\mathbb{R}^3$*. For every* $f \in L_{q,R-1}(\Omega)$*, the problem* (1.1) *replacing* f *by* $P_\Omega f$ *admits a unique* (u,π) *having the following regularity properties:*

$$
\begin{aligned}
&u \in C^0([0,\infty), J_q(\Omega)) \cap C^1((0,\infty), L_q(\Omega)) \cap C^0((0,\infty), W_q^2(\Omega)),\\
&\pi \in C^0((0,\infty), \hat{W}_q^1(\Omega))
\end{aligned}
$$

and satisfying the following estimates:

$$\|u(t)\|_{L_q(\Omega)} + t^{1/2}\|\nabla u(t)\|_{L_q(\Omega)} + t(\|u_t(t)\|_{L_q(\Omega)} + \|u(t)\|_{W_q^2(\Omega)} + \|\nabla\pi(t)\|_{L_q(\Omega)}) \leq C_\gamma e^{\gamma t}\|f\|_{L_q(\Omega)},$$
$$t^{(1/2)(1+(1/q))}(\|u_t(t)\|_{L_q(\Omega_b)} + \|\pi(t)\|_{L_q(\Omega_b)}) \leq C_{\gamma,b}e^{\gamma t}\|f\|_{L_q(\Omega)}$$

for any $t > 0$. Here, $\Omega_b = B_b \cap \Omega$ $(b > R)$, $\gamma > 0$ is any real number, and C_γ and $C_{\gamma,b}$ are constants depending on a_0 and k_0 whenever $|a| \leq a_0$ and $|k| \leq k_0$ but are independent of a, k, t and f.

Moreover, if $f \in L_{q,R-1}(\Omega) \cap \mathcal{D}_q(\Omega)$,

$$u \in C^0([0,\infty), W_q^2(\Omega)) \cap C^1([0,\infty), L_q(\Omega)),$$
$$\|u(t)\|_{W_q^2(\Omega)} + \|u_t(t)\|_{L_q(\Omega)} \leq C_\gamma e^{\gamma t}\|f\|_{W_q^2(\Omega)}.$$

Now, we shall give a sketch of proof of Theorem 1.1 by using Theorem 3.1. Let us define the operator $S_\Omega(t)$ by the formula: $S_\Omega(t)f = u(t)$ for $f \in L_{q,R-1}(\Omega)$, where $u(t)$ is a vector of functions mentioned in Theorem 3.1.

Theorem 3.1 $\Rightarrow$ Theorem 1.1

First step: Given $f \in \mathcal{D}_q(\Omega)$, let $\tilde{f}$ be an element in $\mathcal{D}_q(\mathbb{R}^3)$ such that

$$\tilde{f} = f \quad \text{on } \Omega \text{ and } \|\tilde{f}\|_{\mathcal{D}_q(\mathbb{R}^3)} \leq C_q\|f\|_{\mathcal{D}_q(\Omega)}$$

where

$$\|f\|_{\mathcal{D}_q(D)} = \|f\|_{W_q^2(D)} + \|(\omega \times x)\cdot\nabla f\|_{L_q(D)}.$$

Let φ be a function in $C_0^\infty(\mathbb{R}^3)$ such that $\varphi(x) = 1$ for $|x| \leq R-2$ and $\varphi(x) = 0$ for $|x| \geq R-1$ and set

$$v(t) = (1-\varphi)S_{\mathbb{R}^3}(t)\tilde{f} + \mathbb{B}[(\nabla\varphi)\cdot S_{\mathbb{R}^3}(t)\tilde{f}].$$

Here, $\mathbb{B}$ denotes the Bogovskiĭ-Pileckas operator satisfying the estimates:

$$\|\mathbb{B}[(\nabla\varphi)\cdot v]\|_{W_q^j(\mathbb{R}^3)} \leq C\|v\|_{W_q^{j-1}(\mathrm{supp}\,(\nabla\varphi))}, \quad j = 1,2;$$
$$\|\mathbb{B}[(\nabla\varphi)\cdot \nabla v]\|_{W_q^j(\mathbb{R}^3)} \leq C\|v\|_{W_q^{j}(\mathrm{supp}\,(\nabla\varphi))}, \quad j = 0,1,2.$$

2nd step: To obtain the solution $u(t)$ of (1.1), we set $u(t) = v(t) + w(t)$, and then $w(t)$ and $\pi(t)$ should satisfy the equations:

$$w_t + L_{k,a}w + \nabla\pi = F, \quad \mathrm{div}\, w = 0 \quad \text{in } \Omega\times(0,\infty),$$
$$w|_{\partial\Omega} = 0, \quad w|_{t=0} = \varphi f - \mathbb{B}[(\nabla\varphi)\cdot f] = g.$$

Here,

$$F = -2(\nabla\varphi)\cdot\nabla S_{\mathbb{R}^3}(t)\tilde{f} - (\Delta\varphi)S_{\mathbb{R}^3}(t)\tilde{f} + k(\partial_3\varphi)S_{\mathbb{R}^3}(t)\tilde{f}$$
$$- ((\omega\times x)\cdot\nabla\varphi)S_{\mathbb{R}^3}(t)\tilde{f} + (\partial_t + L_{k,a})\mathbb{B}[(\nabla\varphi)\cdot S_{\mathbb{R}^3}(t)\tilde{f}].$$

Observe that

$$\begin{aligned}&\partial_t \mathbb{B}[(\nabla\varphi)\cdot S_{\mathbb{R}^3}(t)\tilde{f}] = \mathbb{B}[(\nabla\varphi)\cdot\partial_t S_{\mathbb{R}^3}(t)\tilde{f}]\\ &\quad = \mathbb{B}[(\nabla\varphi)\cdot\Delta S_{\mathbb{R}^3}(t)\tilde{f}] + \mathbb{B}[(\nabla\varphi)\cdot(-k\partial_3 + ((\omega\times x)\cdot\nabla - \omega\times)S_{\mathbb{R}^3}(t)\tilde{f})].\end{aligned}$$

Therefore, we have

$$\|F(t)\|_{W_q^2(\Omega)} \le C_\gamma t^{-1/2}e^{\gamma t}\|f\|_{\mathcal{D}_q(\Omega)}, \quad \|F(t)\|_{L_q(\Omega)} \le C_\gamma t^{-1/2}e^{\gamma t}\|f\|_{L_q(\Omega)}.$$

If we write

$$w(t) = S_\Omega(t)g + \int_0^t S_\Omega(t-s)F(s)\,ds,$$

then by Theorem 3.1 we have

$$\begin{aligned}&w(t)\in C^0([0,\infty), W_q^2(\Omega))\cap C^1([0,\infty), L_q(\Omega),\\ &\|w(t)\|_{W_q^2(\Omega)} + \|w_t(t)\|_{L_q(\Omega)} \le C_\gamma e^{\gamma t}\|f\|_{\mathcal{D}_q(\Omega)},\\ &\|w(t)\|_{L_q(\Omega)} \le C_\gamma e^{\gamma t}\|f\|_{L_q(\Omega)}.\end{aligned}$$

Therefore, we can construct a solution

$$u(t)\in C^0([0,\infty), W_q^2(\Omega))\cap C^1([0,\infty), L_q(\Omega))$$

which satisfies the estimate:

$$\begin{aligned}&\|u(t)\|_{W_q^2(\Omega)} + \|u_t(t)\|_{L_q(\Omega)} \le C_\gamma e^{\gamma t}\|f\|_{\mathcal{D}_q(\Omega)},\\ &\|u(t)\|_{L_q(\Omega)} \le C_\gamma e^{\gamma t}\|f\|_{L_q(\Omega)}.\end{aligned}$$

If we define $\{T(t)\}_{t\ge 0}$ by the formula: $T(t)f = u(t)$, then the uniqueness of solutions and the denseness of $\mathcal{D}_q(\Omega)$ in $J_q(\Omega)$ imply that $\{T(t)\}_{t\ge0}$ is a C^0 semigroup on $J_q(\Omega)$. Since we can show that the resolvent set of $\mathcal{L}_q$ contains the complex plane with positive real part, we see that the generator of $\{T(t)\}_{t\ge0}$ is $\mathcal{L}_q$, which completes the proof of Theorem 1.1.

An idea of Proof of Theorem 1.3

By Young's inequality, we see easily that

$$\|\nabla^j S_{\mathbb{R}^3}(t)f\|_{L_r(\mathbb{R}^3)} \le C_{q,r}t^{-\frac{j}{2}-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}\|f\|_{L_q(\mathbb{R}^3)}$$

for $1 < q \le r \le \infty$ with $q \ne \infty$ and $t > 0$. Combining this estimate with Theorem 3.1 and the following local energy decay theorem by cut-off technique, we have Theorem 1.3.

Theorem 3.2 (Local Energy Decay). *Let $1 < q < \infty$, $a_0 > 0$ and $k_0 > 0$. Then, we have*

$$\|\partial_t^j T(t)P_\Omega f\|_{W_q^2(\Omega_R)} \le C_q t^{-\frac{3}{2}}\|f\|_{L_q(\Omega)}, \quad t > 1$$

for any $f \in L_{q,R-1}(\Omega)$ and $j = 0, 1$. Here, C_q denotes a constant depending on a_0 and k_0 whenever $|a| \le a_0$ and $|k| \le k_0$ but are independent of a, k, t and f.

4. On some new treatment of the pressure term

Although there are several new ideas are necessary to prove Theorems 1.1 and 1.3, the most important idea is a new treatment of the pressure term, which will be discussed in what follows. To explain our idea, for the simplicity instead of (1.1) we shall consider the resolvent problem for the usual Stokes operator with non-slip boundary condition:

$$\lambda u - \Delta u + \nabla\theta = f, \quad \operatorname{div} u = 0 \quad \text{in } D, \quad u|_{\partial D} = 0 \tag{4.1}$$

where D is one of Ω or Ω_R. We know the following theorem (cf. [10], [3] and references therein) except for the additional estimate of θ.

Theorem 4.1. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$ and $\lambda_0 > 0$. For every $f \in J_q(D)$ and $\lambda \in \Sigma_\epsilon$ with $|\lambda| \geq \lambda_0$, the problem (4.1) admits a unique solution $(u, \theta) \in W_q^2(D)^3 \times \hat{W}_q^1(D)$ possessing the estimate:*

$$\begin{aligned} &|\lambda|\|u\|_{L_q(D)} + |\lambda|^{1/2}\|\nabla u\|_{L_q(D)} + \|\nabla^2 u\|_{L_q(D)} \\ &\qquad + |\lambda|^{(1/2)(1-(1/q))}(|\lambda|\|u\|_{W_q^{-1}(\Omega_R)} + \|\theta\|_{L_q(\Omega_R)}) \leq C_{\epsilon,\lambda_0}\|f\|_{L_q(D)}. \end{aligned}$$

Here, $\Sigma_\epsilon = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg\lambda| \leq \pi - \epsilon\}$.

To get the additional estimate for θ, we use the following propositions concerning the Laplace equation with Neumann boundary condition.

Proposition 4.2. *Let $1 < q < \infty$ and D be a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary ∂D. Let $f \in L_q(D)$ and assume that $\int_D f\,dx = 0$. Then, there exists a unique $u \in W_q^2(D)$ which satisfies:*

$$\begin{aligned} &\Delta u = f \quad \text{in } D, \quad \partial_\nu u|_{\partial D} = 0, \quad \int_D u\,dx = 0, \\ &\|u\|_{W_q^2(D)} \leq C_q\{\|f\|_{L_q(D)} + \|g\|_{W_q^1(D)}\}. \end{aligned}$$

Proposition 4.3. *Let $1 < q < \infty$ and assume that Ω is an exterior domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary $\partial\Omega$. Set*

$$\begin{aligned} \hat{W}_q^2(\Omega) &= \{u \in L_{q,\mathrm{loc}}(\overline{\Omega}) \mid \nabla u \in W_q^1(\Omega)\}, \\ \hat{L}_{q,R-1}(\Omega) &= \{f \in L_q(\Omega) \mid f(x) = 0 \quad \text{for } |x| \geq R-1, \ \int_\Omega f\,dx = 0\}. \end{aligned}$$

Then, for every $f \in \hat{L}_{q,R-1}(\Omega)$ there exists a $u \in \hat{W}_q^2(\Omega)$ which satisfies:

$$\begin{aligned} &\Delta u = f \quad \text{in } \Omega, \quad \partial_\nu u = 0, \\ &\|u\|_{L_q(\Omega\cap B_R)} + \sup_{|x|\geq R}|x|^{n-1}|u(x)| + \|\nabla u\|_{W_q^1(\Omega)} \leq C\|f\|_{L_q(\Omega)}. \end{aligned}$$

To estimate θ itself in Ω_R, we take any $\varphi \in C_0^\infty(\Omega_R)$ and set $\tilde\varphi = \varphi - |\Omega_R|^{-1}\int_{\Omega_R}\varphi\,dx$. Let ψ be a solution to $\Delta\psi = \tilde\varphi$ in Ω_R and $\partial_\nu\psi|_{\partial\Omega_R} = 0$, and then noting that $\int_{\Omega_R}\theta\,dx = 0$, we have

$$(\theta,\varphi)_{\Omega_R} = (\theta,\tilde\varphi)_{\Omega_R} = (\theta,\Delta\psi)_{\Omega_R} = -(\nabla\theta,\nabla\psi)_{\Omega_R} = (\lambda u - \Delta u - f, \nabla\psi)_{\Omega_R}$$
$$= -\lambda(\operatorname{div} u,\psi)_{\Omega_R} - (\partial_\nu u,\nabla\psi)_{\partial\Omega_R} + (\nabla u,\nabla^2\psi)_{\Omega_R} - (f,\nabla\psi)_{\Omega_R}.$$

Since $f \in J_q(\Omega)$, $(f,\nabla\psi)_D = 0$. Therefore, noting that $\operatorname{div} u = 0$ in Ω and using the interpolation inequality about the trace operator, we have

$$|(\theta,\varphi)_{\Omega_R} \leq C\{\|\partial_\nu u\|_{L_q(\partial\Omega_R)} + \|\nabla u\|_{L_q(D)}\}\|\psi\|_{W^2_{q'}(\Omega_R)}$$
$$\leq C\{\|\nabla^2 u\|^{1/q}_{L_q(\Omega_R)}\|\nabla u\|^{1-(1/q)}_{L_q(\Omega_R)} + \|\nabla u\|_{L_q(\Omega_R)}\}\|\varphi\|_{L_{q'}(\Omega)}$$

which implies that

$$\|\theta\|_{L_q(\Omega_R)} \leq C\{\|\nabla^2 u\|^{1/q}_{L_q(\Omega_R)}\|\nabla u\|^{1-(1/q)}_{L_q(\Omega_R)} + \|\nabla u\|_{L_q(\Omega_R)}\}.$$

Since $\|u\|_{W_q^{-1}(\Omega_R)}$ can be estimated by $\|(\theta,\nabla u, f)\|_{L_q(\Omega_R)}$, we have the required estimate for θ and λu.

Now, for large λ we shall construct the parametrix of the solutions to the equations:

$$\lambda u - \Delta u + \nabla\theta = f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0. \tag{4.2}$$

Let φ be a function in C_0^∞ such that $\varphi = 1$ for $|x| \leq R-2$ and $\varphi = 0$ for $|x| \geq R-1$. Given $f \in L_{q,R-1}(\Omega)$, $f_0(x) = f(x)$ for $x \in \Omega$ and $f_0(x) = 0$ for $x \in \mathbb{R}^3 \setminus \Omega$ and $f|_{\Omega_R}$ denotes the restriction of f to Ω_R. Set

$$\Phi_0(\lambda)f = (1-\varphi)R_0(\lambda)P_{\mathbb{R}^3}f_0 + \varphi R_{0,\Omega_R}(\lambda)P_{\Omega_R}f|_{\Omega_R}$$
$$+ \mathbb{B}[(\nabla\varphi)\cdot(R_0(\lambda)P_{\mathbb{R}^3}f_0 - R_{0,\Omega_R}(\lambda)P_{\Omega_R}f|_{\Omega_R})],$$
$$\Psi_0 f = (1-\varphi)Q_{\mathbb{R}^3}f_0 + \varphi(Q_{\Omega_R}f|_{\Omega_R} + p_{0,\Omega_R}(\lambda)P_{\Omega_R}f|_{\Omega_R}).$$

Here,

$$R_0(\lambda)g = \mathcal{F}^{-1}[(\lambda+|\xi|^2)^{-1}\hat g(\xi)], \text{ and } u = R_{0,\Omega_R}(\lambda)P_{\Omega_R}h \text{ and } \theta = p_{0,\Omega_R}(\lambda)P_{\Omega_R}h$$

solve the equations:

$$\lambda u - \Delta u + \nabla\theta = P_{\Omega_R}h, \ \operatorname{div} u = 0 \ \text{ in } \Omega_R, \ u|_{\partial\Omega} = 0$$

for $h \in L_q(\Omega_R)$. In particular, we have

$$\lambda u - \Delta u + \nabla(\theta + Q_{\Omega_R}h) = h \quad \text{in } \Omega_R.$$

By the Fourier multiplier theorem, we have

$$(\lambda I - \Delta)R_0(\lambda)g + \nabla Q_{\mathbb{R}^3}g = g, \quad \operatorname{div} R_0(\lambda)g = 0 \ \text{ in } \mathbb{R}^3, \tag{4.3}$$

$$|\lambda|\|R_0(\lambda)g\|_{L_q(\mathbb{R}^3)} + |\lambda|^{1/2}\|\nabla R_0(\lambda)g\|_{L_q(\mathbb{R}^3)} + \|\nabla^2 R_0(\lambda)g\|_{L_q(\mathbb{R}^3)}$$
$$+ \|\nabla Q_{\mathbb{R}^3}f\|_{L_q(\mathbb{R}^3)} \leq C_{q,\epsilon}\|g\|_{L_q(\mathbb{R}^3)} \tag{4.4}$$

for any $g \in L_q(\mathbb{R}^3)$ and $\lambda \in \Sigma_\epsilon$ $(0 < \epsilon < \pi/2)$. By Theorem 4.1 we have

$$\begin{aligned} (\lambda I - \Delta)R_{0,\Omega_R}(\lambda)f + \nabla(Q_{\Omega_R}f + p_{0,\Omega_R}(\lambda)f) = f \quad &\text{in } \Omega_R, \\ \operatorname{div} R^0_{0,\Omega_R}(\lambda)f = 0 \quad &\text{in } \Omega_R, \\ R_{0,\Omega_R}(\lambda)f = 0 \quad &\text{on } \partial\Omega_R, \end{aligned} \tag{4.5}$$

$$\begin{aligned} |\lambda|\|R_{0,\Omega_R}(\lambda)f\|_{L_q(\Omega_R)} + |\lambda|^{1/2}\|\nabla R_{0,\Omega_R}(\lambda)f\|_{L_q(\Omega_R)} + \|\nabla^2 R_{0,\Omega_R}(\lambda)f\|_{L_q(\Omega_R)} \\ +\|\nabla p_{0,\Omega_R}(\lambda)f\|_{L_q(\Omega_R)} + (1+|\lambda|)^{(1/2)(1-(1/q))}\|p_{0,\Omega_R}(\lambda)f\|_{L_q(\Omega_R)} \\ +\|Q_{\Omega_R}f\|_{W^1_q(\Omega_R)} \le C_{q,\epsilon}\|f\|_{L_q(\Omega_R)} \end{aligned} \tag{4.6}$$

for any $f \in L_q(\Omega_R)$ and $\lambda \in \Sigma_\epsilon \cup \{\lambda \in \mathbb{C} \mid |\lambda| \le \sigma_0\}$.

Combining (4.3), (4.4), (4.5) and (4.6), we have

$$\begin{aligned} &\lambda\Phi_0(\lambda)f - \Delta\Phi_0(\lambda)f + \nabla\Psi_0(\lambda)f = (I+T)f + S_0(\lambda)f, \\ &\operatorname{div} u = 0 \quad \text{in } \Omega, \quad \Phi_0(\lambda)f|_{\partial\Omega} = 0. \end{aligned}$$

Here, T is the operator from $L_{q,R-1}(\Omega)$ into $L_{q,R-1}(\Omega) \cap W^1_q(\Omega)$ given by the formula:

$$Tf = -(\nabla\varphi)(Q_{\mathbb{R}^3}f_0 - Q_{\Omega_R}f|_{\Omega_R}) - \mathbb{B}[(\nabla\varphi)\cdot\nabla(Q_{\mathbb{R}^3}f_0 - Q_{\Omega_R}f|_{\Omega_R})], \tag{4.7}$$

and $S_0(\lambda)$ is the operator from $L_{q,R-1}(\Omega)$ into $L_{q,R-1}(\Omega) \cap W^1_q(\Omega)$ such that

$$\|S_0(\lambda)f\|_{L_q(\Omega)} \le C|\lambda|^{-(1/2)(1-(1/q))}\|f\|_{L_q(\Omega)}. \tag{4.8}$$

In fact, we have such remainder terms from the following observation:

$$\begin{aligned} &\lambda\mathbb{B}[(\nabla\varphi)\cdot(R_0(\lambda)P_{\mathbb{R}^3}f_0 - R_{0,\Omega_R}(\lambda)P_{\Omega_R}f|_{\Omega_R}] \\ &\quad = \mathbb{B}[(\nabla\varphi)\cdot\Delta(R_0(\lambda)P_{\mathbb{R}^3}f_0 - R_{0,\Omega_R}(\lambda)P_{\Omega_R}f|_{\Omega_R})] \\ &\qquad - \mathbb{B}[(\nabla\varphi)\cdot\nabla(Q_{\mathbb{R}^3}f_0 - Q_{\Omega_R}f|_{\Omega_R})] \\ &\qquad + \mathbb{B}[(\nabla\varphi)\cdot\nabla p_{0,\Omega_R}(\lambda)f|_{\Omega_R}]. \end{aligned}$$

Lemma 4.4 (Key Lemma). *Let $1 < q < \infty$. Then, there exists the inverse operator $(I+T)^{-1} \in \mathcal{L}(L_{q,R-1}(\Omega))$.*

In view of (4.8) and the Key lemma, there exists a large λ_0 such that for any $\lambda \in \Sigma_\epsilon$ with $|\lambda| \ge \lambda_0$ we have the solution formula of (4.2) given by the formula:

$$(u,\theta) = (\Phi_0(\lambda), \Psi_0(\lambda))(I+T+S_0(\lambda))^{-1}f,$$

where

$$(I+T+S_0(\lambda))^{-1} = \Big\{\sum_{j=0}^{\infty}((I+T)^{-1}S_0(\lambda))^j\Big\}(I+T)^{-1}.$$

Especially, the representation formula of θ seems to be a new treatment of the pressure term, from which we can see that the pressure term is represented by $\delta(t)a_0(x) + t^{-((1/2)(1+1/q)}a_1(x)+$ continuous function up to $t=0$, where $\delta(t)$ is the Dirac delta function with respect to time variable t. This gives us some information about the initial layer. Moreover, $(I+T)^{-1}$ plays an essential role to construct the parametrix for (1.1).

A proof of the Key Lemma. By (4.4) and (4.6) we have

$$|\lambda|\|\Phi_0(\lambda)f\|_{L_q(\Omega)} + \|\Phi_0(\lambda)f\|_{W_q^2(\Omega)} + \|\varphi p_{0,\Omega_R} P_{\Omega_R} f|_{\Omega_R}\|_{W_q^1(\Omega)} + |\lambda|^{(1/2)(1-(1/q))}\|\varphi p_{0,\Omega_R} P_{\Omega_R} f|_{\Omega_R}\|_{L_q(\Omega)} \leq C_{q,\epsilon}\|f\|_{L_q(\Omega)} \quad (4.9)$$

for any $\lambda \in \Sigma_\epsilon$ with $|\lambda| \geq 1$. From (4.9) we see that there exist a sequence $\{\lambda_j\}_{j=1}^\infty$ and a $v \in L_q(\Omega)$ such that $\lim_{j\to\infty} \lambda_j = \infty$ and

$$\lim_{j\to\infty} \lambda_j \Phi_0(\lambda_j)f = v \quad \text{weakly in } L_q(\Omega). \quad (4.10)$$

On the other hand, by (4.9) we have

$$\|\Phi_0(\lambda_j)f\|_{L_q(\Omega)} \leq C|\lambda_j|^{-1}\|f\|_{L_q(\Omega)} \to 0, \quad \|\Phi_0(\lambda_j)f\|_{W_q^2(\Omega)} \leq C\|f\|_{L_q(\Omega)},$$
$$\|\varphi p_{0,\Omega_R} P_{\Omega_R} f|_{\Omega_R}\|_{L_q(\Omega)} \leq C|\lambda_j|^{-(1/2)(1-(1/q))}\|f\|_{L_q(\Omega)} \to 0,$$
$$\|\nabla(\varphi p_{0,\Omega_R} P_{\Omega_R} f|_{\Omega_R})\|_{L_q(\Omega)} \leq C\|f\|_{L_q(\Omega)}$$

as $j \to \infty$, and therefore we have

$$\lim_{j\to\infty} \Delta\Phi_0(\lambda_j)f = 0, \quad \lim_{j\to\infty} \nabla(\varphi p_{0,\Omega_R}(\lambda_j)f|_{\Omega_R}) = 0 \quad \text{weakly in } L_q(\Omega). \quad (4.11)$$

In particular, we have

$$\lim_{j\to\infty} \nabla\Psi_0(\lambda_j)f = \nabla((1-\varphi)Q_{\mathbb{R}^3} f_0 + \varphi\, Q_{\Omega_R} f|_{\Omega_R}) \quad \text{weakly in } L_q(\Omega). \quad (4.12)$$

Since $\lambda_j\Phi_0(\lambda_j)f \in J_q(\Omega)$, we see that $v \in J_q(\Omega)$. Therefore, letting $j \to \infty$ in the relation: $\lambda\Phi_0(\lambda_j)f - \Delta\Phi_0(\lambda_j)f + \nabla\Psi_0(\lambda_j)f = (I+T)f + S_0(\lambda)f$, by (4.11), (4.12) and (4.13) we have

$$v + \nabla((1-\varphi)Q_{\mathbb{R}^3} f_0 + \varphi Q_{\Omega_R} f|_{\Omega_R}) = (I+T)f \quad \text{in } \Omega. \quad (4.13)$$

Since $Tf \in W_q^1(\Omega)$ and $\operatorname{supp} Tf \subset D_{R-2,R-1}$ as follows from (4.7), T is a compact operator on $L_{q,R-1}(\Omega)$, and therefore to show the invertibility of $I+T$ it suffices to show the injectivity of $I+T$ on $L_{q,R-1}(\Omega)$. Let f be a vector of functions in $L_{q,R-1}(\Omega)$ such that $(I+T)f = 0$, which implies that

$$f = -Tf = 0 \quad \text{for } x \notin D_{R-2,R-1}. \quad (4.14)$$

By (4.13) we have By (4.13) we have

$$v + \nabla((1-\varphi)Q_{\mathbb{R}^3} f_0 + \varphi Q_{\Omega_R} f|_{\Omega_R}) = 0 \quad \text{in } \Omega \quad (4.15)$$

with some $v \in J_q(\Omega)$. Since the Helmholtz decomposition is unique, it follows from (4.15) that

$$\nabla((1-\varphi)Q_{\mathbb{R}^3} f_0 + \varphi Q_{\Omega_R} f|_{\Omega_R}) = 0,$$

which implies that

$$(1-\varphi)Q_{\mathbb{R}^3} f_0 + \varphi Q_{\Omega_R} f|_{\Omega_R} = c \quad (4.16)$$

with some constant c. Since $\varphi(x) = 1$ for $|x| \le R-2$ and $\varphi(x) = 0$ for $|x| \ge R-1$, by (4.16) we have

$$Q_{\mathbb{R}^3} f_0 = c \quad \text{in } |x| \ge R-1, \quad Q_{\Omega_R} f|_{\Omega_R} = c \quad \text{in } |x| \le R-2. \tag{4.17}$$

If we define $w(x)$ by the formula: $w(x) = (Q_{\Omega_R} f|_{\Omega_R})(x)$ for $x \in \Omega_R$ and $w(x) = c$ for $x \notin \Omega$, then by (1.6) with $D = \Omega_R$, (4.17) and (4.14) we see that $w \in W^1_q(B_R)$ and w is a weak solution to the equation:

$$\Delta w = \operatorname{div} f_0 \quad \text{in } B_R, \quad \partial_\nu w = 0 \quad \text{on } S_R, \tag{4.18}$$

because $\partial\Omega_R = \partial\Omega \cup S_R$.

On the other hand, by (1.6) with $D = \mathbb{R}^3$ we have

$$\Delta Q_{\mathbb{R}^3} f_0 = \operatorname{div} f_0 \quad \text{in } \mathbb{R}^3$$

which combined with (4.17) implies that $Q_{\mathbb{R}^3} f_0$ also satisfies (4.18) weakly. Therefore, by the uniqueness of weak solutions we have that $w - Q_{\mathbb{R}^3} f_0 = d$ with some constant d. In particular, $Q_{\Omega_R} f|_{\Omega_R} - Q_{\mathbb{R}^3} f_0 = d$ in Ω_R, which combined with the facts that $\int_{\Omega_R} Q_{\Omega_R} f|_{\Omega_R}\, dx = \int_{\Omega_R} Q_{\mathbb{R}^3} f_0\, dx = 0$ implies that $d = 0$. Namely, we have $Q_{\Omega_R} f|_{\Omega_R} - Q_{\mathbb{R}^3} f_0 = 0$ in Ω_R, which inserted into (4.7) implies that $Tf = 0$. Recalling that $f + Tf = 0$, we have $f = 0$, which implies the injectivity of $I + T$ on $L_{q,R}(\Omega)$. This completes the proof of the lemma.

5. The idea of proofs of Theorems 3.1 and 3.2

Now, let us consider the resolvent problem:

$$\lambda u - \Delta u + M_{k,a} u + \nabla \pi = f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0.$$

Here, $M_{k,a} u = k\partial_3 u - (\omega \times x) \cdot \nabla u + \omega \times u$. We construct the parametrix of the form:

$$\begin{aligned}
\Phi(\lambda) f &= (1-\varphi)\mathcal{A}_{\mathbb{R}^3,a,k}(\lambda) f_0 + \varphi R_{\Omega_R}(\lambda) P_{\Omega_R} f|_{\Omega_R} \\
&\quad + \mathbb{B}[(\nabla\varphi) \cdot (\mathcal{A}_{\mathbb{R}^3,a,k}(\lambda) f_0 - \varphi R_{\Omega_R}(\lambda) P_{\Omega_R} f|_{\Omega_R})], \\
\Psi(\lambda) f &= (1-\varphi) Q_{\mathbb{R}^3} f_0 + \varphi(Q_{\Omega_R} f|_{\Omega_R} + p_{\Omega_R}(\lambda) P_{\Omega_R} f|_{\Omega_R})
\end{aligned}$$

for $\lambda \in \mathbb{C}_+ = \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\}$. Here, $u = R_{\Omega_R}(\lambda) g$ and $\theta = p_{\Omega_R}(\lambda) g$ solve the equation:

$$\begin{aligned}
\lambda u - \Delta u + M_{k,a} u + \nabla\theta &= P_{\Omega_R} g \quad && \text{in } \Omega_R, \\
\operatorname{div} u &= 0 && \text{in } \Omega_R, \\
u|_{\partial\Omega_R} &= 0.
\end{aligned}$$

We see that

$$\begin{gathered}
(\lambda - \Delta + M_{k,a})\Phi(\lambda) f + \nabla\Psi(\lambda) f = (I+T) f + S(\lambda) f, \\
\operatorname{div} \Phi(\lambda) f = 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0.
\end{gathered}$$

Here T is the operator defined in (4.7) and $S(\lambda)$ is a linear operator from $L_{q,R-1}(\Omega)$ into $S_{q,R-1}(\Omega) \cap W^1_q(\Omega)$ such that $\|S(\lambda)f\|_{L_q(\Omega)} \leq C|\lambda|^{-(1/2)(1-(1/q))}\|f\|_{L_q(\Omega)}$ for $\lambda \in \mathbb{C}_+$ with $|\lambda| \geq 1$.

In fact, $S(\lambda)$ is defined as follows:

$$
\begin{aligned}
&S(\lambda)f = 2(\nabla\varphi) : \nabla(\mathcal{C}_a(\lambda)f) + (\Delta\varphi)\mathcal{C}_a(\lambda)f - k(\partial_3\varphi)\mathcal{C}_a(\lambda)f \\
&\quad + [(\omega \times x)\cdot\nabla\varphi]\mathcal{C}_a(\lambda)f + \mathbb{B}[(\nabla\varphi)\cdot\Delta\mathcal{C}_a(\lambda)f] - \mathbb{B}[(\nabla\varphi)\cdot M_{k,a}\mathcal{C}_a(\lambda)f] \\
&\quad + \mathbb{B}[(\nabla\varphi)\cdot\nabla\mathcal{B}_{\Omega_R,a}(\lambda)f|_{\Omega_R}] - \Delta\mathbb{B}[(\nabla\varphi)\cdot\mathcal{C}_a(\lambda)f] + M_{k,a}\mathbb{B}[(\nabla\varphi)\cdot\mathcal{C}_a(\lambda)f] \\
&\quad + (\nabla\varphi)\mathcal{B}_{\Omega_R,a}(\lambda)f|_{\Omega_R}, \\
&\mathcal{C}_a(\lambda)f = \mathcal{A}_{\mathbb{R}^3,a}(\lambda)f_0 - \mathcal{A}_{\Omega_R,a}(\lambda)f|_{\Omega_R}.
\end{aligned}
$$

Then, we have

$$
(I+T+S(\lambda))^{-1} = \Big(I + \sum_{j=1}^{\infty}((I+T)^{-1}S(\lambda))^j\Big)(I+T)^{-1} = I + T_1(\lambda) + T_2(\lambda).
$$

Moreover, by Theorem 2.1 we see that

$$
\begin{aligned}
&T_1(\lambda) \in \mathrm{Anal}(\tilde{\Sigma}_\epsilon, \mathcal{L}(L_{q,R-1}(\Omega))), \\
&\|T_1(\lambda)f\|_{L_q(\Omega)} \leq C|\lambda|^{-(1/2)(1-(1/q))}\|f\|_{L_q(\Omega)} \quad (\lambda \in \tilde{\Sigma}_\epsilon,\ |\lambda| \geq 1), \\
&T_2(\lambda) \in \mathrm{Anal}(\mathbb{C}_+, \mathcal{L}(L_{q,R-1}(\Omega))), \\
&\|T_2(\lambda)f\|_{L_q(\Omega)} \leq C_\gamma|\lambda|^{-3}\|f\|_{L_q(\Omega)} \qquad (5.1)
\end{aligned}
$$

for $\lambda \in \mathbb{C}_+$ with $|\lambda| \geq 1$ and $\mathrm{Re}\,\lambda \geq \gamma > 0$. Here, $0 < \epsilon < \pi/2$,

$$
\tilde{\Sigma}_\epsilon = \mathbb{C}_+ \cup \{\lambda \in \mathbb{C}\setminus\{0\} \mid |\arg\lambda| \leq \pi - \epsilon,\ \ |\lambda| \geq c(\epsilon)\}
$$

and $c(\epsilon)$ is some constant depending on ϵ, a_0 and k_0 whenever $|a| \leq a_0$ and $|k| \leq k_0$.

Set $(R(\lambda), \Xi(\lambda)) = (\Phi(\lambda), \Psi(\lambda))(I+T+S(\lambda))^{-1}$. Then, $(R(\lambda), \Xi(\lambda))$ is the solution operator to the equations:

$$
(\lambda - \Delta + M_{k,a})u = f,\ \ \mathrm{div}\, u = 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0.
$$

Moreover, by Theorem 2.1 and (5.1) we have

$$
\begin{aligned}
&R(\lambda) = R_1(\lambda) + R_2(\lambda), \quad \Xi(\lambda) = \Xi_0 + \Xi_1(\lambda) + \Xi_2(\lambda), \\
&R_1(\lambda) \in \mathrm{Anal}(\tilde{\Sigma}_\epsilon, \mathcal{L}(L_{q,R-1}(\Omega), W^2_q(\Omega)), \\
&R_2(\lambda) \in \mathrm{Anal}(\mathbb{C}_+, \mathcal{L}(L_{q,R-1}(\Omega), W^2_q(\Omega)), \\
&\Xi_0 \in \mathcal{L}(L_{q,R-1}(\Omega), \hat{W}^1_q(\Omega)), \\
&\Xi_1(\lambda) \in \mathrm{Anal}(\tilde{\Sigma}_\epsilon, \mathcal{L}(L_{q,R-1}(\Omega), \hat{W}^1_q(\Omega))), \\
&\Xi_2(\lambda) \in \mathrm{Anal}(\mathbb{C}_+, \mathcal{L}(L_{q,R-1}(\Omega), \hat{W}^1_q(\Omega))), \\
&\|\nabla^j R_1(\lambda)f\|_{L_q(\Omega)} \leq C|\lambda|^{-1+(j/2)}\|f\|_{L_q(\Omega)}, \\
&|\lambda|^{(1/2)(1-(1/q))}\|\Xi_1(\lambda)f\|_{L_q(\Omega_b)} + \|\nabla\Xi_1(\lambda)f\|_{L_q(\Omega)} \leq C_b\|f\|_{L_q(\Omega)}
\end{aligned}
$$

for $\lambda \in \tilde{\Sigma}_\epsilon$ with $|\lambda| \geq 1$,

$$\|R_2(\lambda)f\|_{W_q^2(\Omega)} \leq C_\gamma |\lambda|^{-3} \|f\|_{L_q(\Omega)},$$
$$\|\Xi_2(\lambda)f\|_{L_q(\Omega_b)} + \|\nabla \Xi_2(\lambda)f\|_{L_q(\Omega)} \leq C_b |\lambda|^{-3} \|f\|_{L_q(\Omega)}$$

for $\lambda \in \mathbb{C}_+$ with $|\lambda| \geq 1$ and $\operatorname{Re}\lambda \geq \gamma > 0$. Combining the resolvent estimate in the whole space and the above estimates by cut-off technique, we have

Theorem 5.1. *Let $1 < q < \infty$. Then, the resolvent set of $\mathcal{L}_q$ contains $\mathbb{C}_+$.*

For $f \in L_{q,R-1}(\Omega)$ we set

$$u(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} R_1(\lambda) f \, d\lambda + \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} R_2(\lambda) f \, d\lambda,$$
$$\theta(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\Xi_0 f + \Xi_1(\lambda) f) \, d\lambda + \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} \Xi_2(\lambda) f \, d\lambda,$$

where

$$\Gamma = \bigcup_{\pm} \{\sigma + s e^{\pm i\kappa} \mid s \geq 0\}$$

with large $\sigma > 0$ and $\pi/2 < \kappa < \pi$. Then, we can show Theorem 3.1. In view of Theorem 2.2, we can show the local energy decay (Theorem 3.2), by shifting the contour in the definition of $u(t)$ to the imaginary axis and using the following lemma:

Lemma 5.2. *Let $0 < \kappa < 1$. Let X and $\|\cdot\|_X$ be a Banach space and its norm, respectively. Let $f(s)$ be a function in $L_1(\mathbb{R}, X)$, which satisfies the condition:*

$$\sup_{0 < |h| \leq 1} |h|^{-\kappa} \int_{\mathbb{R}} \|f(s+h) - f(s)\|_X \, ds \leq C_\kappa M \tag{5.2}$$

for some $M > 0$. Set $g(t) = \int_{\mathbb{R}} e^{its} f(s) \, ds$. Then, we have

$$\|g(t)\|_X \leq |e^{-i} - 1|^{-1} M t^{-\kappa}$$

for any $t \geq 1$.

6. Remark on the stability theorem

Let us consider the problem (1.5) in the case where $f = f(x)$ and assume that the existence of solution $w(x)$ with some pressure term $\theta(x)$ of the stationary problem:

$$\begin{aligned} w \cdot \nabla w - \Delta w + k\partial_3 w & \\ -(\omega \times x) \cdot \nabla w + \omega \times w + \nabla \theta = f & \quad \text{in } \Omega, \\ \operatorname{div} w = 0 & \quad \text{in } \Omega, \\ w = \omega \times x - k\mathbf{e}_3 & \quad \text{on } \partial\Omega, \\ w(x) \to 0 & \quad \text{as } |x| \to \infty. \end{aligned} \tag{6.1}$$

In this case, we set $u(x,t) = w(z) + z(x,t)$ and $\pi(x,t) = \theta(x) + \kappa(x,t)$ in (1.5) and then we have the equations for z and κ as follows:

$$\begin{aligned} & z_t + z\cdot\nabla z + z\cdot\nabla w + w\cdot\nabla z - \Delta z + k\partial_3 z \\ & \qquad -(\omega\times x)\cdot\nabla z + \omega\times z + \nabla\kappa = 0 && \text{in } \Omega\times(0,\infty), \\ & \qquad\qquad\qquad\qquad\qquad \operatorname{div} z = 0 && \text{in } \Omega\times(0,\infty), \\ & \qquad\qquad\qquad\qquad\qquad z = \omega\times x - k\mathbf{e}_3 && \text{on } \partial\Omega\times(0,\infty), \\ & \qquad\qquad\qquad\qquad\qquad z(x)\to 0 && \text{as } |x|\to\infty \quad t>0 \end{aligned} \tag{6.2}$$

with an initial data $v_0(x) - w(x)$. Following Kato [8], instead of (6.2) we consider the integral equation:

$$z(t) = T(t)(v_0 - w) - \int_0^t T(t-s)[z(s)\cdot\nabla z(s) + z(s)\cdot\nabla w + w\cdot\nabla z(s)]\,ds. \tag{6.3}$$

Then, using Theorem 1.3 and employing the same argument as in Shibata [9], we have the following stability theorem.

Theorem 6.1. *Let σ be a small positive number and $3 < p < \infty$. Then, there exists a small positive number ϵ depending on σ and p such that if $v_0 - w \in J_3(\Omega)$ and*

$$\|w\|_{L_{3-\sigma}(\Omega)\cap L_{3+\sigma}(\Omega)} + \|\nabla w\|_{L_{(3/2)-\sigma}(\Omega)\cap L_{(3/2)+\sigma}(\Omega)} + \|v_0 - w\|_{L_3(\Omega)} \le \epsilon, \tag{6.4}$$

then problem (6.3) *admits a unique solution*

$$z(t) \in C([0,\infty), J_q(\Omega)) \cap C((0,\infty), L_p(\Omega)\cap W^1_3(\Omega))$$

such that

$$\begin{gathered} [z]_{3,0,t} + [z]_{p,\mu(p),t} + [\nabla z]_{3,1/2,t} \le \sqrt{\epsilon}, \\ \lim_{t\to 0+}(\|z(t) - (v_0 - w)\|_{L_3(\Omega)} + [z]_{p,\mu(p),t} + [\nabla z]_{3,1/2,t}) = 0. \end{gathered}$$

Here, we have put

$$[z]_{q,\rho,t} = \sup_{0<s<t} s^\rho \|z(\cdot,s)\|_{L_q(\Omega)}, \quad \mu(p) = \frac{3}{2}\left(\frac{1}{3} - \frac{1}{p}\right) = \frac{1}{2} - \frac{3}{2p}.$$

Remark 6.2. Recently, Galdi and Silvestre [4] proved the existence of stationary solutions, but the author has not yet checked their stability.

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Exact Controllability in $L_2(\Omega)$ of the Schrödinger Equation in a Riemannian Manifold with $L_2(\Sigma_1)$-Neumann Boundary Control

Roberto Triggiani

Abstract. We consider the Schrödinger equation, with H^1-level terms having variable coefficients in time and space, as defined on an open bounded connected set Ω of an n-dimensional complete Riemannian manifold. We show that it is exactly controllable on the state space $L_2(\Omega)$ on an arbitrarily small interval $[0, T]$, by means of Neumann boundary controls in the class $L_2(0, T; L_2(\Gamma_1))$, where $\Gamma_1 = \partial\Omega \setminus \Gamma_0$, and the equation is homogeneous on Γ_0, either in the Dirichlet or in the Neumann B.C. Different geometric conditions apply in the two cases. This result is a vast generalization over the literature.

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1. Introduction. Problem statement. Assumptions

Preliminary notation. Throughout this paper, M is a complete n-dimensional, Riemannian manifold of class C^3 with C^3-metric $g(\cdot\,,\,\cdot) = \langle \cdot\,,\,\cdot \rangle$ and squared norm $|X|^2 = g(X, X)$. We may, on occasion, append a subscript g : $\langle \cdot\,,\,\cdot \rangle_g$; $|\cdot|_g$. We shall denote it by (M, g). Let Ω be an *open, bounded, connected* set of M with smooth boundary (say, of class C^2) $\partial\Omega \equiv \Gamma = \overline{\Gamma_0 \cup \Gamma_1}$. Here, Γ_0 is the uncontrolled or unobserved part of Γ and Γ_1 is the controlled or observed part of Γ, both relatively open in Γ. We let ν denote the outward unit normal field along the boundary Γ. Further, we denote by ∇ the gradient, by D the Levi-Civita connection, by D^2 the Hessian, by $\Delta = \operatorname{div}(\nabla)$ the Laplace (Laplace-Beltrami) operator [Do.1, p. 55, p. 83, p. 141], [Le.1, p. 28, pp. 43–44, p. 54, p. 68].

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Model. In this paper, we consider the following mixed Schrödinger problem in the (complex-valued) unknown $w(t,x)$ defined on Q,

$$\begin{cases} iw_t + \Delta w = F(w) & \text{in } Q \equiv (0,T] \times \Omega; & (1.1\text{a}) \\ w(0,\ \cdot\ ) = w_0 & \text{in } \Omega; & (1.1\text{b}) \\ \text{either } w|_{\Sigma_0} \equiv 0, \text{ or else } \langle Dw, \nu\rangle|_{\Sigma_0} \equiv 0, & \text{in } \Sigma_0 = (0,T] \times \Gamma_0; & (1.1\text{c}) \\ \langle Dw, \nu\rangle|_{\Sigma_1} \equiv u, & \text{in } \Sigma_1 \equiv (0,T] \times \Gamma_1, & (1.1\text{d}) \end{cases}$$

with Neumann boundary control $L_2(0,T; L_2(\Gamma_1))$. In the case $w|_{\Sigma_0} \equiv 0$, we also assume $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$. In (1.1a), we have set

$$F(w) = -i\langle R(t,x), Dw\rangle + q_0(t,x)w, \tag{1.2a}$$

where the coefficients are subject to the following assumptions. First:

(A.1) q_0 is a complex-valued function on $[0,T] \times \Omega$ and $R(t,x)$ is a *real-valued* vector field on $\mathbb{R}_t \times M$ (structural property [R-S.1]) satisfying the following regularity hypotheses

$$q_0 \in L_\infty(Q), \quad R \in L_\infty(0,T,\mathcal{X}(M))[\text{He.1}], \tag{1.2b}$$

so that for the energy level term F, we have

$$|F(w)|^2 \le C_T\{|Dw|^2 + |w|^2\}, \qquad \forall\ (t,x) \in Q \text{ a.e.}, \tag{1.2c}$$

where $Dw = \nabla w$ for the scalar function w. Thus, $Dw \in \mathcal{X}(M) =$ the set of all C^2 complex-valued vector fields on M.

Next, recall that the covariant differential (a 2-0 tensor T_2^0) of $R \in \mathcal{X}(M)$ determines a bilinear form on $TM \times TM$, for each $x \in M$, defined by $DR(X,Y) = \langle D_X R, Y\rangle_g$. Then, we require that:

(A.2)

$$\begin{cases} |DR(X,Y)| = |\langle D_X R, Y\rangle| \le C|X|\,|Y|, \quad 0 \le t \le T, \\ \text{or } DR \in L_\infty(0,T;T_2^0); \end{cases} \tag{1.2d}$$

and moreover,

$$|Dq_0| \in L_\infty(Q). \tag{1.2e}$$

Remark 1.1. Henceforth, we shall focus on the case dim $M \ge 2$, and leave the case dim $M = 1$ as a more direct generalization of [L-T-Z.2, Appendix A, Theorem A.1], whereby a change of variables *eliminates the first-order term.* □

Remark 1.2. Modulo a first-order additive operator $(Df)(w)$, we have that $\Delta_g w$ models the principal part of a second-order elliptic operator with variable coefficients $a_{ij}(x)$ in space, defined on an n-dimensional Euclidean bounded domain. See Section 6.1 below. In this case, the Riemannian manifold is $(\mathbb{R}^n, g)$, where the Riemannian metric g is derived from an inversion on the symmetric matrix $\{a_{ij}(x)\}$. See [C-H.1], [Du.1]. □

Exact controllability of the w-problem on $L_2(\Omega)$. The goal of this paper is to establish the new property (see literature below) of exact controllability of the mixed w-problem in (1.1a-b-c), on the state space $L_2(\Omega)$, within the class of $L_2(0,T;L_2(\Gamma_1))$-boundary controls in the Neumann B.C., where $T>0$ is arbitrarily preassigned. To this end, we introduce two geometrical conditions, one for the Dirichlet homogeneous B.C. $w|_{\Sigma_0}\equiv 0$, the other for the Neumann homogeneous B.C. $\langle Dw,\nu\rangle|_{\Sigma_0}\equiv 0$ in (1.1c).

Geometric assumptions. In addition to the standing assumptions (A.1) = (1.2b), (A.2) = (1.2d–e) on the first-order operator F in (1.2a), the following hypotheses (A.3) and (A.4) are postulated throughout this paper. (See Section 6 for classes of examples.)

(A.3) Given the triple $\{\Omega,\Gamma_0,\Gamma_1\}$, $\partial\Omega=\overline{\Gamma_0\cup\Gamma_1}$, there exists a non-negative, real-valued function $d:\overline{\Omega}\Rightarrow\mathbb{R}^+$ of class C^3 that is strictly convex in the metric g. This means that the Hessian D^2d of d (a two-order tensor) satisfies $D^2d(X,\overline{X})>0$, $\forall\ x\in\overline{\Omega}$, $\forall\ X\in T_xM=$ the tangent space at x. By compactness of Ω, we can always achieve that: There exists a constant $\rho>0$ such that

$$D^2d(X,\overline{X})\equiv\langle D_X(Dd),\overline{X}\rangle_g\geq\rho|X|_g^2,\quad\forall\ x\in\Omega,\ \forall\ X\in T_xM;\tag{1.3}$$

(A.4) moreover,

(a) $\langle Dd,\nu\rangle\leq 0$ on Γ_0, in the case of the Dirichlet B.C. $w|_{\Sigma_0}\equiv 0$ in (1.1c), whereby $\overline{\Gamma}_0\cap\overline{\Gamma}_1=\emptyset$; (1.4)

(b) $\langle Dd,\nu\rangle=0$ on Γ_0, in the case of the Neumann B.C. $\langle Dw,\nu\rangle|_{\Sigma_0}\equiv 0$ in (1.1c); (1.5)

□

Remark 1.3. In the Euclidean case, the geometric requirement in (1.4) in the case of Dirichlet B.C. is standard on Γ_0 (but also the geometrical condition $\langle Dd,\nu\rangle\geq 0$ on Γ_1 was traditionally made, which, however, we dispense with here, as in [L-T-Z.1], [L-T-Z.2], [L-T.2], [Tr.2]. The stronger geometrical requirement (1.5) in the case of Neumann B.C. in (1.1c) was introduced in [Tr.1, Section 5]. Reference [L-T-Z.1, Appendices A–C] provides, by different mathematical techniques, *several general classes* of triples $\{\Omega,\Gamma_0,\Gamma_1\}$, $\overline{\Gamma_0\cup\Gamma_1}=\Gamma$, in the Euclidean setting $\mathbb{R}^n$, $n\geq 2$, where assumptions (A.3) = (1.3), as well as the geometrical requirement on (1.5) are satisfied for a suitably constructed strictly convex function d. Reference [T-Y.2, Appendix B] extends a general sufficient condition of [L-T-Z.1] to the Riemannian setting. These results are also reported in [G-L-L-T.1]. We also refer to [T-X.1, Section 10] for additional insight and illustrations, partly reported in Section 6.2 below. □

An a priori energy estimate at the L_2-level [L-T-Z.2]**.** Here below we shall consider smooth solutions of the Schrödinger equation (1.1a), with a forcing term f, that is

$$iz_t+\Delta z=F(z)+f\equiv -i\langle R(t,x),Dz\rangle+q_0(t,x)z+f,\tag{1.6}$$

satisfying, in addition,

$$\text{either } z|_{\Sigma_0} \equiv 0, \text{ in which case } \langle Dd, \nu\rangle \leq 0 \text{ on } \Gamma_0, \tag{1.7a}$$

$$\text{or else } \langle Dz, \nu\rangle \equiv 0, \text{ in which case } \langle Dd, \nu\rangle \equiv 0 \text{ on } \Gamma_0, \tag{1.7b}$$

where d is the strictly convex function of hypothesis (A.3). No other boundary conditions on Γ_1 are imposed on z. Indeed, the traces of z on $\Sigma_1 = (0,T] \times \Gamma_1$ will be explicitly contained in the following *a priori* estimate.

Theorem 1.1. *Assume that* (A.1) = (1.2b), (A.2) = (1.2d–e). *Let z be a solution of Eqn.* (1.6) *satisfying, in addition, either the Dirichlet case* (1.7a)*, or else the Neumann case* (1.7b)*. Let $T > 0$ be arbitrary. Finally, let $f \in L_2(0,T; L_2(\Omega))$. Then, the following inequality holds true: There exists a constant $C_T > 0$ such that*

$$\begin{aligned}
&\int_0^T \left[\|z\|^2_{L_2(\Omega)} + \|z_t\|^2_{H^{-2}(\Omega)}\right] dt + \|z(0)\|^2_{L_2(\Omega)} + \|z_t(0)\|^2_{H^{-2}(\Omega)} \\
&\quad \leq C_T \bigg\{ \|z\|^2_{L_2(\Sigma_1)} + \|\langle Dz, \nu\rangle|_{\Gamma_1}\|^2_{H_a^{-1}(\Sigma_1)} \\
&\qquad + \int_0^T \int_{\Gamma_1} |\langle Dz, \nu\rangle|_{\Gamma_1}|\,|z| d\Gamma_1 dt + \|z\|^2_{H^{-1}(Q)} + \|f\|^2_{L_2(Q)} \bigg\},
\end{aligned} \tag{1.8}$$

where $H_a^{-1}(\Sigma_1)$ is the dual space to the anisotropic space $H_a^1(\Sigma_1)$, with respect to the pivot space $L_2(\Sigma_1)$:

$$H_a^{-1}(\Sigma_1) = (H_a^1(\Sigma_1))';\ H_a^1(\Sigma_1) \equiv H^{\frac{1}{2}}(0,T; L_2(\Gamma_1)) \cap L_2(0,T,H^1(\Gamma_1)). \tag{1.9}$$

□

Remark 1.4. The natural energy level for the Schrödinger equation is the $H^1(\Omega)$-level, *not* the $L_2(\Omega)$-level. Indeed, the proof of the energy estimate (1.8) at the $L_2(\Omega)$-level for (1.6) requires a heavy use of pseudo-differential/micro-local analysis machinery [L-T-Z.2, Sect. 10], to shift the more natural $H^1(\Omega)$-level energy estimate to the $L_2(\Omega)$-level. As a matter of fact, the proof in [L-T-Z.2, Sect. 10] refers specifically to an Euclidean domain Ω in $\mathbb{R}^n$, $\partial\Omega = \overline{\Gamma_0 \cup \Gamma_1}$. It is based on partition of unity of Ω, flattening the boundary locally, and consequent analysis in the half-space, by taking, as a starting point, the *a-priori* energy estimate at the $H^1(\Omega)$-level from [Tr.2], in the Euclidean case. However, it was already noted in [L-T-Z.2, Remark 2.6.2], that by taking this time, as a starting point, the *a-priori* $H^1(\Omega)$-energy level estimate in the Riemannian case from [T-Y.1], the same proof works also in the case where Ω is an open, bounded, connected set Ω of an n-dimensional, Riemannian manifold M, as in the present paper. □

We can now state our main result.

Theorem 1.2. *With reference to the mixed problem* (1.1a-b-c-d)*, assume hypotheses* (A.3)*, as well as* (A.4a) = (1.4) *in the Dirichlet case or* (A.4b) = (1.5) *in the Neumann case. Let the coefficients of F satisfy assumptions* (A.1) = (1.2b), (A.2)

= (1.2d–e). *Then, the w-problem* (1.1a-b-c-d) *is exactly controllable in the following sense. Let $T > 0$ be arbitrary. Given $w_0 \in L_2(\Omega)$ [respectively, $w_1 \in L_2(\Omega)$], there exists a boundary control $u \in L_2(0, T; L_2(\Gamma_1))$ such that the corresponding solution of the w-problem* (1.1) *due to the data $\{w_0, u\}$ [respectively, due to the data $\{w_0 = 0, u\}$] satisfies $w(T) = 0$ [respectively, $w(T) = w_1$].*

As in the case for most of the exact controllability results for hyperbolic and Petrowski-type evolution equations in the literature, the proof of Theorem 1.2 is by duality: that is, it consists of establishing the equivalent continuous observability inequality [L-T.3], [Tr.2], [T-Y.1], [G-L-L-T.1]. (An exception is the direct work of W. Littman [Li-Ta.1], [H-L.1, H-L.2].) In our present case, establishment of the continuous observability inequality in Section 3 relies critically on the $L_2(\Omega)$-energy level estimate (1.9).

Literature. Theorem 1.2 is new even in the Euclidean case and with no H^1-level terms ($R(t,x) \equiv 0$). Theorem 1.2 is a generalization of [Ma.1] in various directions. First, [Ma.1] considers only the pure Schrödinger equation $iw_t + \Delta w \equiv 0$ on Q, in the Euclidean case. Variable coefficients (in space) of the principal part are not permitted, nor are H^1-energy level terms, by the multiplier methods used in [Ma.1]. The same multiplier is just one tool used also in [L-T.3], [L-T.5, Sect. 10.9, p. 1042] in the more demanding case of Dirichlet boundary control, particularly in the corresponding uniform stabilization problem, which requires, in addition, a non-trivial shift of topologies. Unlike the Neumann case in [Ma.1], the results of [L-T.3] in the Dirichlet case are topologically optimal. Second, in the aforementioned more restricted setting, [Ma.1] claims exact controllability with $L_2(0, T; L_2(\Gamma_1))$-Neumann control and $w|_{\Sigma_0} \equiv 0$ only, however, on the state space $H^1(\Omega)$, not in $L_2(\Omega)$ as in the present Theorem 1.2. Third, the case $\frac{\partial w}{\partial \nu} \equiv 0$ on Σ_0 in the Euclidean treatment of [Ma.1] is excluded. Subsequent references, beginning with [Ta.1] and continuing with [Tr.2], [Ta.2], [T-Y.1], [L-T-Z.2], and [T-X.1], have greatly generalized the original *pure* Schrödinger equation in the Euclidean domain of [Ma.1], [L-T.3]. However, in all cases, exact controllability results of (1.1) (with $w|_{\Sigma_0} \equiv 0$) with $L_2(\Sigma_1)$-Neumann controls are still given in the state space $H^1(\Omega)$, not $L_2(\Omega)$. Different approaches have been pursued, all sharing the goal of seeking preliminary Carleman-type estimates: a unifying pseudo-differential approach in [Ta.1, Ta.2] under a pseudo-convexity assumption; different Riemannian geometric approaches yielding Carleman-type estimates with lower-order terms [T-Y.1] (following the Euclidean case of [Tr.2], counterpart of [L-T.4] for second-order hyperbolic equations) or without lower-order terms [T-X.1] (following the Euclidean case of [L-T-Z.2]). Thus, [L-T-Z.2] and [T-X.1] yield new global uniqueness results over, say, [Is.1, Is.2]. Two of these are critically invoked in the proof of Section 3 (just above Eqn. (3.18) and Eqn. (3.22)). Specialization of the present Riemannian setting to capture the case of variable coefficients (in space) of the principal part over a Euclidean domain is deferred to Section 6.1 below. Paper [L-T-Z.2, Sect. 11] gives, in the linear Euclidean case, a companion uniform stabilization result of the conservative pure Schrödinger equation at the $L_2(\Omega)$-level, by means of a boundary feedback control (dissipation) in the Neumann B.C. (L_2 in time and space). The corresponding uniform stabilization result, this time with a nonlinear boundary dissipation in the Neumann B.C., in the same topological setting with state space $L_2(\Omega)$ is given in [L-T.6]. □

Remark 1.5. The above result, Theorem 1.2, complements [L-T.7, Sect. 8.2], which shows – at least for the pure Schrödinger equation on a half-space, in dimension $\geq$ 2, with Neumann $L_2(\Sigma_T)$-boundary control – that the *range* of the input-solution operator $\{w_0 = 0, u\} \to w(T)$, applied to all of $L_2(\Sigma_1)$, is *not* contained in $H^\epsilon(\Omega)$, $\forall\ \epsilon > 0$. In contrast, our result shows that the totality of solutions points $w(T)$ due to the class of $L_2(\Sigma_1)$-controls in the Neumann-boundary conditions fill all of $L_2(\Omega)$. □

Remark 1.6. The next task is to seek a global exact controllability result in the semilinear case, following the abstract strategy of [L-T.8], [Tr.3]. Moreover, the case of plate equations needs to be revisited. A general Riemannian treatment where, however, the Carleman estimates have lower-order terms is given in [L-T-Y.2]. □

2. The adjoint problem and the equivalent COI under the working assumption $\langle R, \nu \rangle \equiv 0$ on Γ_1 (resp. on Γ)

The goal of the present section is two-fold. First, we shall seek to establish the PDE system which is obtained by duality or transposition over the mixed control problem (1.1a–d). This is the φ-problem (2.8) below. To this end, we shall make a temporary working assumption, (A.5) = (2.4) below, to be later removed in Section 5. Second, we shall obtain the relevant Continuous Observability Inequality (COI) for the φ-problem (2.8) which is *equivalent* to the exact controllability property of the w-problem in (1.1), as spelled out in the statement of Theorem 1.2. We begin by setting, for short

$$\mathcal{A}w \equiv i\Delta w - \langle R(t,x), Dw \rangle - iq_0(t,x)w, \tag{2.1}$$

with $R(t,x)$ the *real-valued* vector field on $\mathbb{R}_t \times M$, as in assumption (1.2a), (A.1). With reference to problem (1.1), define the operator $A : L_2(\Omega \supset \mathcal{D}(A) \to L_2(\Omega)$ (depending on t), by

$$Aw \equiv \mathcal{A}w,\ \mathcal{D}(A) \equiv \{w \in H^2(\Omega) : w|_{\Gamma_0} = \langle Dw, \nu \rangle|_{\Gamma_1} \equiv 0\}, \tag{2.2a}$$

or else

$$\mathcal{D}(A) \equiv \{w \in H^2(\Omega) : \langle Dw, \nu \rangle|_{\Gamma} \equiv 0\}. \tag{2.2b}$$

A preliminary identity. For $w, \overline{\varphi} \in H^1(\Omega)$, we compute via the divergence (Green) formula [Do.1], [Le.1], [L-T-Y.1]:

$$\int_\Omega \langle R, Dw\rangle \overline{\varphi}\, d\Omega = \int_\Gamma w\overline{\varphi}\langle R, \nu\rangle d\Gamma - \int_\Omega w \text{ div}(\overline{\varphi} R) d\Omega \tag{2.3a}$$

$$(\text{if } w|_{\Gamma_0} = 0) = \int_{\Gamma_1} w\overline{\varphi}\langle R, \nu\rangle d\Gamma_1 - \int_\Omega w \text{ div}(\overline{\varphi} R) d\Omega. \tag{2.3b}$$

Accordingly, throughout this section, we shall impose the following *working assumption*:

(A.5)

$$\begin{cases} \text{either } \langle R(t,x), \nu\rangle \equiv 0 \text{ on } \Gamma, & \text{if } \langle Dw, \nu\rangle|_{\Gamma_0} \equiv 0, \quad (2.4a) \\ \text{or else } \langle R(t,x), \nu\rangle \equiv 0 \text{ on } \Gamma_1, & \text{if } w|_{\Gamma_0} \equiv 0. \quad (2.4b) \end{cases}$$

This assumption will facilitate the analysis in establishing Theorem 1.2 at first. Later on, in Section 5, we shall dispense with assumption (A.5) = (2.4), by means of a natural change of variable, as in [L-T-Z.2, Appendix A, Proposition A.4, Eqn. (A.18), p. 107], whereby the geometrical assumption (A.5) = (2.4) will be satisfied by the new variable and exact controllability in the original variable will be equivalent to exact controllability in the new variable.

Thus, for $w, \overline{\varphi} \in H^1(\Omega)$, under both assumption (A.5) = (2.4a) and (A.5) = (2.4b), we have from (2.3):

$$\int_\Omega \langle R, Dw\rangle \overline{\varphi}\, d\Omega = -\int_\Omega w \text{ div}(\overline{\varphi} R) d\Omega. \tag{2.5}$$

The adjoint operator A^* of A under (A.5) = (2.4). The $L_2(\Omega)$-adjoint of the operator A in (2.2), subject to either (A.5) = (2.4a) or (A.5) = (2.4b), is:

$$A^*\varphi = -i\Delta\varphi + \langle R(t,x), D\varphi\rangle + i\tilde{q}_0\varphi, \ \tilde{q}_0 = \overline{q}_0 - \text{ div } R \in L_\infty(Q), \tag{2.6a}$$

and either

$$\mathcal{D}(A^*) = \mathcal{D}(A) = \{\varphi \in H^2(\Omega) : \varphi|_{\Gamma_0} = \langle D\varphi, \nu\rangle|_{\Gamma_1} \equiv 0\}, \tag{2.6b}$$

in case (2.2a), or else

$$\mathcal{D}(A^*) = \mathcal{D}(A) = \{\varphi \in H^2(\Omega) : \langle D\varphi, \nu\rangle|_\Gamma \equiv 0\}, \tag{2.6b$'$}$$

in case (2.2b). A direct computation using either assumption (A.5) = (2.4a), or else (A.5) = (2.4b), hence identity (2.5) in both cases, yields in fact, starting from (2.1):

$$(Aw, \varphi)_{L_2(\Omega)} = \int_\Omega (Aw)\overline{\varphi}\, d\Omega = \int_\Omega w\overline{(A^*\varphi)} d\Omega = (w, A^*\varphi)_{L_2(\Omega)},$$
$$w, \varphi \in \mathcal{D}(A) = \mathcal{D}(A^*). \tag{2.7}$$

The problem adjoint to (1.1a–d). On the basis of the operator A^* in (2.6) (under (A.5) = (2.4)), we consider the problem

$$\varphi_t = -A^*\varphi,\ \varphi(T) = \varphi_0; \begin{cases} \varphi_t = i\Delta\varphi - \langle R, D\varphi\rangle - i\tilde{q}_0\varphi, & \text{in } Q; \quad (2.8a) \\ \varphi|_{t=T} = \varphi_0, & \text{in } \Omega; \quad (2.8b) \\ \text{either } \varphi|_{\Sigma_0} \equiv 0, \text{ or } \langle D\varphi, \nu\rangle|_{\Sigma_0} \equiv 0, & \text{in } \Sigma_0; \quad (2.8c) \\ \langle D\varphi, \nu\rangle|_{\Sigma_1} \equiv 0, & \text{in } \Sigma_1. \quad (2.8d) \end{cases}$$

When the I.C. $w_0 = 0$ in (1.1b), then the φ-problem (2.8a–d) is the adjoint to the control w-problem (1.1a–d). More precisely, we have:

Proposition 2.1. *With reference to problems* (1.1) *and* (2.8)*, assume* (A.1), (A.2)*, and* (A.5)*. The closed map*

$$\mathcal{L}_T :\ \{w_0 = 0, u\} \to \mathcal{L}_T u = w(T),\ \textit{from } L_2(\Sigma_1) \supset \mathcal{D}(\mathcal{L}_T) \textit{ to } L_2(\Omega), \tag{2.9}$$

and the map

$$\mathcal{L}_T^* :\ \varphi_0 \to \mathcal{L}_T^*\varphi_0 = -i\varphi(\ \cdot\ ;\varphi_0)|_{\Sigma_1} \textit{ from } L_2(\Omega) \supset \mathcal{D}(\mathcal{L}_T^*) \textit{ to } L_2(\Sigma_1), \tag{2.10}$$

are adjoint of each other: for $u \in \mathcal{D}(\mathcal{L}_T)$ and $\varphi_0 \in \mathcal{D}(\mathcal{L}_T^*)$,

$$\begin{aligned} (\mathcal{L}_T u, \varphi_0)_{L_2(\Omega)} &= (w(T), \varphi_0)_{L_2(\Omega)} = \int_\Omega w(T)\overline{\varphi}_0 d\Omega \qquad (2.11) \\ &= \int_0^T \int_{\Gamma_1} u\overline{(-i\varphi)} d\Sigma_1 = (u, -i\varphi)_{L_2(\Sigma_1)} = (u, \mathcal{L}_T^*\varphi_0)_{L_2(\Sigma_1)}. \end{aligned}$$

Proof. Multiply Eqn. (1.1a) by $\overline{\varphi}$ and integrate by parts over Q, invoking $w_0 = 0$ and the B.C. (1.1c–d) for w, and (2.8a–d) for φ. Details are straightforward using (2.5). $\square$

Duality between exact controllability of the w-problem (1.1) with $w_0 = 0$ and continuous observability of the φ-problem (2.8). Exact controllability of problem (1.1a–d) with $w_0 = 0$, as spelled out in the statement of Theorem 1.2, over the interval $[0, T]$ on the state space $L_2(\Omega)$, within the class of Neumann-boundary controls $L_2(0, T; L_2(\Gamma_1))$ means precisely that the (closed) map $\mathcal{L}_T$ in (2.9) satisfies

$$\mathcal{L}_T : L_2(0, T; L_2(\Gamma_1)) \supset \mathcal{D}(\mathcal{L}_T) \xrightarrow{\text{onto}} L_2(\Omega). \tag{2.12}$$

Equivalently then [T-L.1, p. 235], the adjoint operator $\mathcal{L}_T^*$ in (2.10) is bounded below: there exists a constant $c_T' > 0$ such that

$$\|\mathcal{L}_T^* z\|_{L_2(0,T;L_2(\Gamma_1))} \geq c_T'\|z\|_{L_2(\Omega)}, \quad z \in \mathcal{D}(\mathcal{L}_T^*). \tag{2.13}$$

Recalling (2.10) for $\mathcal{L}_T^*$, we obtain the Continuous Observability Inequality (COI) in terms of the adjoint φ-problem (2.8), under the working assumption (A.5).

Proposition 2.2. *Assume* (A.1), (A.2), (A.5)*. The exact controllability property of problem* (1.1a–d) *spelled out in the statement of Theorem* 1.2 (*in symbols: property*

(2.12)) *is equivalent to the following COI: There exists a constant* $c_T > 0$ *such that the solution of problem* (2.8) *satisfies*

$$\|\varphi_0\|^2_{L_2(\Omega)} \le c_T \int_0^T \int_{\Gamma_1} |\varphi(\,\cdot\,;\varphi_0)|^2 d\Sigma_1, \tag{2.14}$$

whenever the right-hand side of (2.14) *is finite.*

Remark 2.1. A similar qualification as the last line of Proposition 2.2 occurs in the COI corresponding to the exact controllability on $H^1(\Omega) \times L_2(\Omega)$ of the wave equation with $L_2(\Sigma)$-Neumann controls [L-T.1]. □

3. Proof of the COI (2.14) under (A.5)

The goal of this section is to establish the *Continuous Observability Inequality* (2.14) for the adjoint φ-problem (2.8a-b-c-d), under the working assumption (A.5).

Regularity. First, however, we need to establish the regularity of problem (2.8).

Theorem 3.1. *Let* $T > 0$ *be arbitrary. Assume* (A.1), (A.2) (*so that* $\tilde{q}_0 \in L_\infty(Q)$, *see* (2.6a)). *With reference to the* φ*-problem* (2.8a–d) *with* $\varphi_0 \in L_2(\Omega)$, *we have that the solution map*

$$\varphi_0 \in L_2(\Omega) \to \varphi \in C([0,T]; L_2(\Omega)) \tag{3.0}$$

is continuous.

The proof is given in Section 4.

Continuous Observability Inequality. We next establish inequality (2.14) at first under the working assumption (A.5). This will be removed in Section 5.

Theorem 3.2. *Let* $T > 0$ *be arbitrary. With reference to the* φ*-problem* (2.8a–d) *with* $\varphi_0 \in L_2(\Omega)$, *assume* (A.1), (A.2) *for the coefficients, namely* $DR \in L_\infty(0,T;T_2^0)$ *and* $\tilde{q}_0 \in L_\infty(Q)$. *Further, assume the geometrical conditions* (A.3) = (1.3), *as well as* (A.4a) = (1.4) *in the Dirichlet case* $\varphi|_{\Sigma_0} \equiv 0$ *in* (2.8c), *and* (A.4b) = (1.5) *in the Neumann case* $\langle D\varphi, \nu\rangle|_{\Sigma_0} \equiv 0$ *in* (2.8c). *Further, assume the working assumption* (A.5). *Then, the following estimate holds true: There exists a constant* $c_T > 0$, *independent of* φ_0, *such that*

$$\|\varphi_0\|^2_{L_2(\Omega)} \le c_T \int_0^T \int_{\Gamma_1} |\varphi|^2 d\Sigma_1, \tag{3.1}$$

whenever the right-hand side of (3.1) *is finite.*

Proof. Step 1. We shall first show the estimate

$$\int_0^T \int_{\Gamma_1} |\varphi|^2 d\Sigma_1 + \|\varphi\|^2_{H^{-1}(Q)} \ge \tilde{c}_T \|\varphi_0\|^2_{L_2(\Omega)}, \tag{3.2}$$

for $\tilde{c}_T > 0$ independent of φ_0, which is inequality (3.1) polluted by an interior lower-order term. The key inequality (3.2) is readily seen to be a direct application of estimate (1.8) (with $f \equiv 0$) of Theorem 1.2, after using the homogeneous Neumann B.C. in (2.8c).

Step 2. Naturally, for $c_T > 0$ independent of φ_0, (3.2) implies *a fortiori*

$$\int_0^T \int_{\Gamma_1} |\varphi|^2 d\Sigma_1 + \|\varphi\|^2_{L_\infty(0,T;H^{-1}(\Omega)} \geq c_T \|\varphi_0\|^2_{L_2(\Omega)}, \tag{3.3}$$

as the interior term in (3.3) dominates the interior term in (3.2).

Step 3. Next, we need to absorb the interior *l.o.t.* $\varphi \in L_\infty(0,T;H^{-1}(Q))$ by a compactness/uniqueness argument, as usual. The uniqueness part is the delicate point. Thus, we seek to establish the following result in order to complete the proof of Theorem 3.2.

Lemma 3.3. *Assume the hypotheses of Theorem* 3.2*, and let* φ *be a solution of problem* (2.8) *satisfying inequality* (3.3)*. Then, in fact,*

$$\|\varphi\|^2_{L_\infty(0,T;H^{-1}(\Omega))} \leq k_T \int_0^T \int_{\Gamma_1} |\varphi|^2 d\Sigma_1, \tag{3.4}$$

for a constant $k_T > 0$ *independent of* φ_0.

Proof of Lemma 3.3. Assume by contradiction, as usual, that there exists a sequence φ_n of solutions of problem (2.8) with $\varphi_n|_{t=T} = \varphi_{n,0}$ such that

$$\|\varphi_n\|_{L_\infty(0,T;H^{-1}(\Omega))} \equiv 1, \ n = 1,2,\ldots; \ \int_0^T \int_{\Gamma_1} |\varphi_n|^2 d\Sigma_1 \to 0, \text{ as } n \to \infty. \tag{3.5}$$

Under present assumptions, each φ_n satisfies inequality (3.3), so that by (3.5), we have $\|\varphi_{n,0}\|_{L_2(\Omega)} \leq$ const, for all n. Thus, there is a subsequence, still called $\varphi_{n,0}$ converging weakly in $L_2(\Omega)$ to some $\varphi_0 \in L_2(\Omega)$. Call ψ the solution of problem (2.8) with such initial condition φ_0: $\psi|_{t=0} = \varphi_0$. Thus, ψ solves the following problem [same as (2.8)]:

$$\begin{cases} i\psi_t + \Delta\psi = -i\langle R, D\psi\rangle + \tilde{q}_0\psi \equiv \tilde{F}(\psi) & \text{in } Q; \quad (3.6\text{a}) \\ \psi(T, \ \cdot \ ) = \varphi_0 & \text{in } \Omega; \quad (3.6\text{b}) \\ \text{either } \psi|_{\Sigma_0} \equiv 0, \text{ or else } \langle D\psi, \nu\rangle|_{\Sigma_0} \equiv 0, & \text{in } \Sigma_0; \quad (3.6\text{c}) \\ \langle D\psi, \nu\rangle|_{\Sigma_1} \equiv 0, & \text{in } \Sigma_1, \quad (3.6\text{d}) \end{cases}$$

corresponding to the I.C. $\varphi_0 \in L_2(\Omega)$ given by the above limit point. By Theorem 3.1, we have

$$\psi \in C([0,T]; L_2(\Omega)). \tag{3.7}$$

It then follows, as usual, that:

$$\begin{cases} \varphi_n \ \to \ \psi \ \text{ in } L_\infty(0,T;L_2(\Omega)) \ \text{ weak-star}; & (3.8\text{a}) \\ (\varphi_n)_t \ \to \ \psi_t \ \text{ in } L_\infty(0,T;H^{-2}(\Omega)) \ \text{ weak-star}, & (3.8\text{b}) \end{cases}$$

and hence φ_n and $(\varphi_n)_t$ are accordingly uniformly bounded:

$$\|\varphi_n\|_{L_\infty(0,T;L_2(\Omega))} + \|(\varphi_n)_t\|_{L_\infty(0,T;H^{-2}(\Omega))} \leq \text{const}, \ \forall\ n. \tag{3.9}$$

Since the injection $L_2(\Omega) \hookrightarrow H^{-1}(\Omega)$ is compact (Ω being bounded), it follows from (3.9) via Aubin-Simon [A.1], [Si.1] with $0 < T < \infty$, that the injection

$$W \hookrightarrow L_\infty(0,T;H^{-1}(\Omega)) \text{ is likewise compact,} \tag{3.10}$$

where W is the Banach space equipped with the norm on the LHS of (3.9).

Accordingly, by (3.10), there is a subsequence, still called φ_n, such that

$$\varphi_n \to \psi \text{ in } L_\infty(0,T;H^{-1}(\Omega)), \text{ strongly,} \tag{3.11}$$

so that by (3.5), (LHS), we obtain

$$\|\psi\|_{L_\infty(0,T;H^{-1}(\Omega))} = 1. \tag{3.12}$$

Moreover, from (3.5) (RHS), and (3.11), we have that ψ also satisfies

$$\psi|_{\Sigma_1} \equiv 0, \qquad \text{in } \Sigma_1. \tag{3.13}$$

Combine (3.6) with (3.13) and conclude that the limit function ψ in (3.11) satisfies either one or the other of the following two over-determined problems:

$$\text{either} \begin{cases} i\psi_t + \Delta\psi &= \tilde{F}(\psi) \\ \psi(T,\ \cdot\ ) &= \varphi_0 \\ \psi|_\Sigma \equiv 0; \langle D\psi, \nu\rangle|_{\Sigma_1} \equiv 0 \end{cases} ;\ \text{or} \begin{cases} i\psi_t + \Delta\psi &= \tilde{F}(\psi) & \text{in } Q; & (3.14\text{a}) \\ \psi(T,\ \cdot\ ) &= \varphi_0 & \text{in } \Omega; & (3.14\text{b}) \\ \psi|_{\Sigma_1} \equiv 0; \langle D\psi, \nu\rangle|_\Sigma \equiv 0 & & & (3.14\text{c}) \end{cases}$$

with $\varphi_0 \in L_2(\Omega)$, recalling $\tilde{F}(\psi)$ in (3.6a). In either case, we seek to establish that $\psi \equiv 0$ in Q, which is a contradiction with (3.12). Thus, the proof is complete, as soon as we establish that either one of the over-determined problems in (3.14) implies that $\psi \equiv 0$ in Q. This will be done next following ideas from [L-T.8, p. 133–134] (see also [Tr.3, p. 284]).

Assume at first *the problem on the LHS of* (3.14)*, with over-determined B.C.*: Homogeneous Dirichlet B.C. on all of $\Sigma = (0,T] \times \Gamma$, and homogeneous Neumann B.C. on $\Sigma_1 = (0,T] \times \Gamma_1$. Under present assumptions on the coefficients of (3.6a), and the geometry $\{\Omega, \Gamma_0, \Gamma_1\}$ as in (A.3), (A.4a), and only invoking the B.C. $\psi|_\Sigma \equiv 0$, we can then conclude with the *continuous observability inequality, polluted by interior l.o.t.*, as in [Tr.2, Proposition 2.4.1, Eqn. (2.4), p. 488] in the Euclidean case, and [T-Y.1, Eqn. (146), p. 657] in the Riemannian case; this holds true yielding

$$\|\varphi_0\|^2_{H^1(\Omega)} \leq \int_0^T \int_{\Gamma_1} |\langle D\psi, \nu\rangle|^2 d\Sigma_1 + \text{const}_\tau \|\psi\|^2_{C([0,T];L_2(\Omega))} \tag{3.15}$$

$$\text{(by (3.14c))} \qquad \leq \text{const}_\tau \|\psi\|^2_{C([0,T];L_2(\Omega))} < \infty, \tag{3.16}$$

invoking (3.7) on ψ and $\langle D\psi, \nu\rangle|_{\Sigma_1} = 0$ by (3.14c) (LHS). Thus, in effect, the over-determined B.C.'s in (3.14c) (LHS) imply that

$$\varphi_0 \in H^1(\Omega), \tag{3.17}$$

a boost in the regularity of the initial condition by one unit over the assumed $\varphi_0 \in L_2(\Omega)$. Armed with (3.17), and on the basis of the present assumptions on the coefficients of $\tilde{F}(\psi)$ in (3.6a) and the geometry $\{\Omega, \Gamma_0, \Gamma_1\}$ as in (A.3) = (1.3), (A.4a) = (1.4), and only invoking the B.C. $\psi|_\Sigma \equiv 0$, we can then conclude with the *continuous observability inequality* as in [L-T-Z.1, Thm. 2.3.1, Eqn. (2.3.2a)] in the Euclidean case and its generalization as [T-X.1, Thm. 8.2, Eqn. (8.8)] in the Riemannian case: This holds true and yields

$$\int_0^T \int_{\Gamma_1} |\langle D\psi, \nu\rangle|^2 d\Sigma_1 \geq k_{\varphi,\tau}\mathbb{E}(0) = k_{\varphi,\tau}\|\varphi_0\|^2_{H^1(\Omega)}, \tag{3.18}$$

as justified by (3.17). But, because of the over-determined B.C. in (3.14c) (LHS), $\langle D\psi, \nu\rangle|_{\Sigma_1} \equiv 0$, the left-hand side of (3.18) is zero.

Thus, we obtain that the I.C. φ_0 is zero: $\varphi_0 = 0$. But then it follows that for the solution of problem (3.6) with $\varphi_0 = 0$ we have $\psi \equiv 0$ in Q, as desired. Lemma 3.3 is proved at least in the case of problem (3.14a-b-c) (LHS).

Next, assume the *problem on the RHS of* (3.14)*, with over-determine B.C.*: Homogeneous Neumann B.C. on all of $\Sigma = (0, T] \times \Gamma$; and homogeneous Dirichlet B.C. on $\Sigma_1 = (0, T] \times \Gamma_1$. Under present assumptions on the coefficients of $\tilde{F}(w)$ in (3.6a) and the geometry $\{\Omega, \Gamma_0, \Gamma_1\}$ as in (A.3) = (1.3), (A.4b) = (1.5), and only invoking the B.C. $\langle D\psi, \nu\rangle|_\Sigma = 0$, we can then conclude *with the continuous observability inequality polluted by interior l.o.t.*, as in [Tr.2] in the Euclidean case, and [T-Y.1] in the Riemannian case: this holds true yielding again by (3.7) on ψ:

$$\|\varphi_0\|^2_{H^1(\Omega)} \leq C_\tau \left[\int_0^T \int_{\Gamma_1} |\overline{\psi}\psi_t\langle Dd, \nu\rangle| d\Sigma_1\right] + \text{const}_\tau \|\psi\|^2_{C([0,T];L_2(\Omega))} \tag{3.19}$$

$$(\text{by (3.14c)}) \leq \text{const}_\tau \|\psi\|^2_{C([0,T];L_2(\Omega))} < \infty. \tag{3.20}$$

As a matter of fact, the above inequality (3.19) is not *explicitly* written out in the latter two references [Tr.2] and [T-Y.1], as these did not consider explicitly exact controllability of problem (1.1) with $\langle Dw, \nu\rangle|_{\Sigma_0} \equiv 0$ in (1.1c), hence the ψ-problem at the RHS of (3.14). However, in this case of problem (3.14) (RHS), the validity of Eqn. (3.19) can immediately be deduced from the Carleman estimates (second version) of these two references, by simply applying the hypotheses at hand $(*)$: $\psi|_{\Sigma_1} \equiv 0$, $\langle D\psi, \nu\rangle|_\Sigma \equiv 0$ (via (3.14c)) and $\langle Dh, \nu\rangle|_{\Gamma_0} = 0$ (via (A.4b) = (1.5)). Indeed, the starting point is the Carleman estimate in [Tr.2, Thm. 2.1.2, Eqn. (2.1.12), p. 466] (Euclidean case) and [T-Y.1, Thm. 3.4, Eqn. (64), p. 641] (Riemannian case), where the boundary terms in [Tr.2, Eqn. (2.1.14), p. 466] and [T-Y.1, Eqn. (65), p. 641] yield, at first, $BT_1|_\Sigma = BT|_\Sigma$ since $\langle D\psi, \nu\rangle|_\Sigma \equiv 0$. Next, by [Tr.2, Eqn. (2.1.10), p. 465] and [T-Y.1, Eqn. (58), p. 640], one obtains

$$BT|_\Sigma = \frac{1}{2}\left|\int_{\Sigma_1} \overline{\psi}\psi_t e^{\tau\phi}\langle Dd, \nu\rangle d\Sigma_1\right|, \tag{3.21}$$

by using all of the three aforementioned properties $(*)$ at hand, whereby then, on Σ_1 where $\psi|_{\Sigma_1} \equiv 0$, we have: $|D\psi|^2 \equiv |\nabla_g\psi|^2 \equiv |\langle D\psi, \nu\rangle| \equiv 0$. Then, the quoted

Carleman estimates of these two references yield at once the desired inequality (3.19), by use of (3.21) with $C_\tau = \frac{1}{2}\sup_{\Sigma_1} e^{\tau\phi}$.

Thus, in effect, this time the over-determined B.C.'s in (3.14c) (RHS) imply again the boost $\varphi_0 \in H^1(\Omega)$ in (3.17) in regularity of the initial condition over the assumed $\varphi_0 \in L_2(\Omega)$. Armed with (3.17), and on the basis of the present assumptions on the coefficients of $\tilde{F}(\psi)$ in (3.6a) and the geometry $\{\Omega, \Gamma_0, \Gamma_1\}$, as in (A.3) = (1.3), (A.4b) = (1.5), and only invoking the B.C. $\langle D\psi, \nu\rangle|_\Sigma = 0$, we can then conclude with the continuous observability inequality as in [L-T-Z.2, Thm. 2.4.1, Eqn. (2.4.2), plus absorption of $\ell.o.t.$] in the Euclidean as and its generalization as in [T-X.1, Thm. 8.4, Eqn. (8.11)] in the Riemannian case; which is

$$\int_0^T \int_{\Gamma_1} |\psi_t|^2 d\Sigma_1 \geq c_T \mathbb{E}(0) = c_T \|\varphi_0\|^2_{H^1(\Omega)}, \tag{3.22}$$

as justified by (3.17) in our present case. But, because of the over-determined B.C. in (3.14c) (RHS): $\psi|_{\Sigma_1} \equiv 0$, hence $\psi_t|_{\Sigma_1} \equiv 0$, the left-hand side of (3.22) is zero. Thus, we obtain once again that $\varphi_0 = 0$, and hence $\psi \equiv 0$ in Q. Lemma 3.3 is proved also in the case of problem (3.14a-b-c) (RHS). □

The working assumption (A.5) = (2.4) will be removed, after the analysis of Section 4, by use of the change of variable (4.3).

4. Proof of Theorem 3.1

We return to problem (2.8a–d) rewritten for convenience as

$$\begin{cases} \varphi_t = i\Delta\varphi - \langle R, D\varphi\rangle - iq_0\varphi, & \text{in } Q \quad (4.0a) \\ \varphi(0, \,\cdot\,) = \varphi_0, & \text{in } \Omega; \quad (4.0b) \\ \text{either } \varphi|_{\Sigma_0} \equiv 0, \text{ or else } \langle D\varphi, \nu\rangle|_{\Sigma_0} \equiv 0 & \text{in } \Sigma_0; \quad (4.0c) \\ \langle D\varphi, \nu\rangle|_{\Sigma_1} \equiv 0 & \text{in } \Sigma_1, \quad (4.0d) \end{cases}$$

under assumption (A.1) for R and q_0: $R \in L_\infty(0, T; \mathcal{X}(M))$, $q_0 \in L_\infty(Q)$ [so that in the case of (2.8a), where actually the coefficient of φ is $\tilde{q}_0 = \overline{q}_0 - i \text{ div } R$, we have $\tilde{q}_0 \in L_\infty(Q)$, as required, by invoking (A.2) = (1.2d)]. In this section, we find more convenient to call $q_0 \in L_\infty(Q)$ the coefficient of φ in (4.0a).

We next list a few properties of problem (4.0a–d), which will play a critical role in this section. For definiteness, we shall explicitly focus on the case $\varphi|_{\Sigma_0} \equiv 0$ in (4.0c).

1. A preliminary identity (for R real). Identity (2.3b) for $w = \varphi \in H^1(\Omega)$, with $\varphi|_{\Sigma_0} = 0$ yields

$$\text{Re}\left\{\int_\Omega \langle R, D\varphi\rangle \overline{\varphi}\, d\Omega\right\} = \frac{1}{2}\left\{\int_{\Gamma_1} |\varphi|^2 \langle R, \nu\rangle d\Gamma_1 - \int_\Omega |\varphi|^2 \text{ div } R\, d\Omega\right\}. \tag{4.1}$$

In fact, by (2.3b), expanding its last term:

$$\int_{\Omega}\langle R, D\varphi\rangle\overline{\varphi}\,d\Omega = \int_{\Gamma_1}|\varphi|^2\langle R,\nu\rangle d\Gamma_1 - \int_{\Omega}|\varphi|^2 \text{ div } R\,d\Omega - \int_{\Omega}\langle R, D\overline{\varphi}\rangle\varphi\,d\Omega, \quad (4.2)$$

and (4.1) follows from (4.2), as desired, as $\langle R, D\overline{\varphi}\rangle\varphi = \overline{\langle R, D\varphi\rangle\overline{\varphi}}$.

2. Change of variables. [L-T-Z.2, Appendix A, Proposition A.4, and Proposition A.5]. Let dim $\Omega \geq 2$, let $p(t,x)$ be a real-valued scalar function, subject to the assumptions p_t, $|Dp|$, $\Delta p \in L_\infty(Q)$. Then, the change of variable

$$\chi(t,x) = e^{-\frac{i}{2}p(t,x)}\varphi(t,x) \quad (4.3)$$

has the following effects:

(a) It transforms the Schrödinger equation (4.0a) with $|R|$, $q_0 \in L_\infty(Q)$, into the new form

$$\chi_t = i\Delta\chi - \langle R(t,x) + Dp(t,x), D\chi\rangle - iq_1(t,x)\chi, \quad (4.4)$$

where

$$q_1 = \left[q_0 + \frac{1}{2}\,p_t - \frac{i}{2}\,\Delta p - \frac{1}{4}|Dp|^2 + \frac{1}{2}\langle R + Dp, Dp\rangle\right] \in L_\infty(Q), \quad (4.5)$$

still with purely real coefficient $\tilde{R} \equiv (R + Dp)$ of the energy level term $D\chi$, and still with $|\tilde{R}|$, q_1 being preserved in $L_\infty(Q)$;

(b) it transforms the boundary conditions (4.0c) (LHS), (4.0d) of Neumann type for φ into

$$\chi|_{\Sigma_0} \equiv 0; \qquad \langle D\chi,\nu\rangle + \left(\frac{i}{2}\,\langle Dp,\nu\rangle\right)\chi \equiv 0 \quad \text{on } \Sigma_1, \quad (4.6)$$

of Robin type for χ. Moreover, given the original real-valued vector field $R(t,x)$, it is always possible to select, in infinitely many ways, a smooth real function p such that

$$\inf_{\Gamma_1}\langle\tilde{R},\nu\rangle = \inf_{\Gamma_1}\left[\langle R,\nu\rangle + \langle Dp,\nu\rangle\right] \geq 0, \quad (4.7)$$

by means of the inverse trace theorem.

3. The case where the coefficients R and q_0 in (4.0a) are time-independent. Here, at first, we further assume that

(A.6)

$$\text{the coefficients } R \text{ and } q_0 \text{ are time-independent,} \quad (4.8)$$

recalling (4.0a) and (2.6a). We then have a more precise result:

Theorem 4.1. *Assume hypotheses* (A.1) = (1.2b), (A.2) = (1.2d–e), (A.6) *on the coefficients R, q_0 of* (4.0a)*, as well as*

(i) *preliminarily,*

$$\langle R,\nu\rangle \geq 0 \quad \textit{on } \Gamma_1. \quad (4.9)$$

Then, with reference to problem (4.0a–d)*, the map*

$$\varphi_0 \in L_2(\Omega) \to \varphi \in C([0,T]; L_2(\Omega))$$

defines – after a suitably large translation – a s.c. contraction semigroup on $L_2(\Omega)$*. More precisely, let*

$$Af = i\Delta f - \langle R, Df\rangle - iq_0 f, \quad f \in \mathcal{D}(A); \tag{4.10a}$$

$$\mathcal{D}(A) = \left\{ f \in H^2(\Omega) : \ f|_{\Gamma_0} = \langle Df, \nu\rangle|_{\Gamma_1} \equiv 0 \right\}. \tag{4.10b}$$

Let k^2 *be a sufficiently large constant, say*

$$k^2 > \frac{1}{2}|div\ R|_{L_\infty(\Omega)} + |q_0|_{L_\infty(\Omega)}. \tag{4.11}$$

Then: (i$_1$) *the operator* $(A - k^2 I) : L_2(\Omega) \supset \mathcal{D}(A) \to L_2(\Omega)$ *is maximal dissipative and, accordingly, it generates a s.c. contraction semigroup, so that*

$$\left\| e^{At} \right\|_{\mathcal{L}(L_2(\Omega))} \le 1 \cdot e^{k^2 t}, \quad t \ge 0. \tag{4.12}$$

(ii) *Assumption* (4.9) *is redundant: more precisely, under the sole assumptions* (A.1) = (1.2b), (A.2) = (1.2d–e) *on the coefficients* [*but without assumption* (4.9)]*, the operator* A *generates a s.c. semigroup* e^{At} *on* $L_2(\Omega)$*. Thus,* a fortiori*, Theorem* 3.1 *holds true for time-independent coefficients.* □

Proof (i). *Dissipativity.* For $f \in \mathcal{D}(A)$ in (4.10b), we compute by virtue of identity (4.1), where (,) denotes the $L_2(\Omega)$-inner product:

$$\mathrm{Re}(Af, f) = \mathrm{Re}\{i(\Delta f, f)\} - \mathrm{Re} \int_\Omega \langle R, Df\rangle \bar{f}\, d\Omega - \mathrm{Re}\{i(q_0 f, f)\} \tag{4.13}$$

$$(\text{by (4.1)}) = -\frac{1}{2}\int_{\Gamma_1} |f|^2 \langle R, \nu\rangle d\Gamma_1 + \frac{1}{2}\int_\Omega |f|^2 \mathrm{div}\ R\, d\Omega - \mathrm{Re}\{i(q_0 f, f)\}, \tag{4.14}$$

using also that $(\Delta f, f)$ is real. [If q_0 happens to be real, the last term in (4.14) would vanish.] Next, under assumption (4.9) and recalling (4.11), we then obtain from (4.14),

$$\mathrm{Re}((A - k^2 I)f, f) \le \left[\frac{1}{2}|\mathrm{div}\, R|_{L_\infty(\Omega)} + |q_0|_{L_\infty(\Omega)} - k^2\right] \|f\|^2_{L_2(\Omega)} \tag{4.15}$$

$$(\text{by (4.11)}) \qquad \le 0, \tag{4.16}$$

and dissipativity is proved.

Maximality. We must show the range condition: that for k^2 sufficiently large, for $\lambda > 0$ and $g \in L_2(\Omega)$, there exists an $f \in \mathcal{D}(A)$ such that $[\lambda I - (A - k^2)]f = g$. Via (4.10) for A, this yields a corresponding elliptic problem

$$\begin{cases} -i\Delta f + \langle R, Df\rangle + iq_0 f + (\lambda + k^2) f = g & \text{on } \Omega; & (4.17\text{a}) \\ f|_{\Gamma_0} \equiv 0; \qquad \langle Df, \nu\rangle|_{\Gamma_1} \equiv 0 & \text{on } \Gamma_i,\ i = 0, 1, & (4.17\text{b}) \end{cases}$$

and elliptic theory yields the desired conclusion. Part (i) is established.

(ii) The proof of Part (i) applies to the χ-system consisting of Eqn. (4.4) and the B.C. (4.6), via a suitable smooth function p for which the required estimate (4.7) holds true. Then, the property that $\chi_0 \to \chi$, defines a s.c. semigroup on $L_2(\Omega)$ transfers to the map $\varphi_0 \to \varphi$ via the change of variable (4.3) □

4. The case of time-varying coefficients: $|R|$, $q_0 \in L_\infty(Q)$. Here, we omit assumption (A.6) = (4.8) and prove likewise Theorem 3.1 in its full strength by energy methods.

Proof of Theorem 3.1. We multiply Eqn. (4.0a) by $\bar{\varphi}$ and integrate over Ω. With $\| \quad \|$ and $(\,\cdot\,,\,\cdot\,)$ the $L_2(\Omega)$-norm and inner product, and recalling identity (4.1), we obtain:

$$\frac{1}{2}\,\frac{d}{dt}\|\varphi\|^2 = \mathrm{Re}(\varphi_t,\varphi) = -\mathrm{Re}(\langle R, D\varphi\rangle,\varphi) - \mathrm{Re}\{i(q_0\varphi,\varphi)\} \tag{4.18}$$

$$\text{(by (4.1))} = -\frac{1}{2}\int_{\Gamma_1}|\varphi|^2\langle R,\nu\rangle\, d\Gamma_1 + \frac{1}{2}\int_\Omega |\varphi|^2 \mathrm{div}\; R\, d\Omega - \mathrm{Re}\{i(q_0\varphi,\varphi)\}. \tag{4.19}$$

Hence, integrating in time over $[0,t]$, $0 < t \le T_0 \le T$, we obtain

$$\begin{aligned} &\|\varphi(t)\|^2 + \int_0^t\int_{\Gamma_1}|\varphi|^2\langle R,\nu\rangle\, d\Gamma_1 d\tau \\ &\quad = \|\varphi(0)\|^2 + \int_0^t\int_\Omega |\varphi|^2\mathrm{div}\; R\, d\Omega\, dt - 2\;\mathrm{Re}\left\{i\int_0^t\int_\Omega q_0|\varphi|^2 d\Omega\, d\tau\right\} \qquad (4.20)\\ &\quad \le \|\varphi(0)\|^2 + T_0 C_{T,R,q_0}\|\varphi\|^2_{C([0,T_0];L_2(\Omega))}, \quad 0 \le t \le T_0, \qquad (4.21)\end{aligned}$$

where

$$\begin{aligned}\int_0^t\int_\Omega [|\mathrm{div}\; R| + 2|q_0|]d\Omega\, dt &\le T_0\left\{\|\mathrm{div}\; R\|_{L_\infty(Q_T)} + 2\|q_0\|_{L_\infty(Q_T)}\right\}\\ &\equiv T_0 C_{T,R,q_0}, \quad 0 \le t \le T_0, \end{aligned}\tag{4.22}$$

where $Q_T \equiv (0,T] \times \Omega$.

(i) Assume, at first, hypothesis (4.9) on R. Thus, for T_0 sufficiently small, whereby $[1 - T_0 C_{T,R,q_0}] > 0$, we obtain from (4.21) by taking the sup in t over $[0,T_0]$:

$$\|\varphi\|^2_{C([0,T_0];L_2(\Omega))} \le \frac{1}{1 - T_0 C_{T,R,q_0}}\;\|\varphi(0)\|^2. \tag{4.23}$$

In fact, one first obtains inequality (4.23) with the $C([0,T_0];L_2(\Omega))$-norm replaced by the $L_\infty(0,T;L_2(\Omega))$-norm; then one boosts the regularity from $L_\infty(0,T_0;L_2(\Omega))$ to $C([0,T_0];L_2(\Omega))$ by an approximation/density argument. We next repeat the argument starting from (4.19) and integrating now over $[T_0,2T_0]$. We obtain the counterpart of (4.23):

$$\|\varphi\|^2_{C([T_0,2T_0];L_2(\Omega))} \le \frac{1}{1 - T_0 C_{T,R,q_0}}\|\varphi(T_0)\|^2 \tag{4.24}$$

$$\text{(by (4.23)} \qquad = \left(\frac{1}{1 - T_0 C_{T,R,q_0}}\right)^2\|\varphi(0)\|^2. \tag{4.25}$$

After a finite number of steps we obtain

$$\|\varphi\|^2_{C([0,T];L_2(\Omega))} \le \mathrm{const}_{T,R,q_0}\|\varphi(0)\|^2, \tag{4.26}$$

and Theorem 3.1 is proved, at least under the additional assumption (4.9) on R.

(ii) Next, we remove assumption (4.9). Namely, in view of the change of variable (4.3) and of property (4.7) for a smooth real function p, we apply the first part (i) of the proof to the χ-problem (4.4), (4.6), and obtain the counterpart of (4.26):

$$\|\chi\|_{C([0,T];L_2(\Omega))} \leq \text{const}_T \|\chi(0)\|. \tag{4.27}$$

Next, inequality (4.27) is transferred to φ, whereby inequality (4.26) holds true again (with a different constant), by virtue of the change of variable (4.3). Thus, Theorem 3.1 is proved. □

5. Proof of Theorem 1.2: Removal of Assumption (A.5) = (2.4)

In this section we complete the proof of Theorem 1.2 in its full strength, by removing the working assumption (A.5) = (2.4). To this end, we perform in the original w-problem a change of variable as the one taken in (4.3) for the dual φ-problem (2.8a–d) or (4.0a–d); that is, we set

$$y(t,x) = e^{-\frac{i}{2}p(t,x)} w(t,x), \tag{5.1}$$

for a smooth real function $p(t,x)$. Then, the problem in y corresponding to the w-problem (1.1a–d) is

$$\begin{cases} y_t = i\Delta y - \langle R(t,x) + Dp(t,x), Dy\rangle - iq_1(t,x)y, & \text{in } Q & (5.2\text{a}) \\ \text{either } y|_{\Sigma_0} \equiv 0, \text{ or else } \left[\langle Dy,\nu\rangle + \frac{i}{2}\langle Dp,\nu\rangle y\right]_{\Sigma_0} \equiv 0. & \text{in } \Sigma_0; & (5.2\text{b}) \\ \langle Dy,\nu\rangle + \left(\frac{i}{2}\langle Dp,\nu\rangle\right) y = e^{-\frac{i}{2}p(t,x)} u & \text{in } \Sigma_1, & (5.2\text{c}) \end{cases}$$

where $\tilde{R}(t,x) \equiv R(t,x) + Dp(t,x)$ is a real-valued vector field on $\mathbb{R}_t \times M$, satisfying the same assumptions $\tilde{R} \in L_\infty(0,T;\mathcal{X}(M))$ and $D\tilde{R} \in L_\infty(0,T;T_2^0)$, as (A.1) = (1.2b) and (A.2) = (1.2d) for R. Similarly, q_1 – which is given by (4.5) – satisfies $q_1, |Dq_1| \in L_\infty(Q)$, as required by (1.2b), (1.2c). Moreover, given the original real-valued vector field $R(t,x)$ in (1.2a), it is always possible to select, in infinitely many ways, a smooth real function p such that

$$\langle \tilde{R},\nu\rangle|_\Sigma = [\langle R,\nu\rangle + \langle Dp,\nu\rangle]_\Gamma = 0. \tag{5.3}$$

Thus, to the y-problem (5.2), we can apply the same duality argument used in Section 3 with respect to the original w-problem in (1.1) (except for the noncritical fact that the B.C. (5.2c) for y is of Robin-type, while the B.C. (1.1d) for w is of Neumann-type. Accordingly, by Section 3, Theorem 3.2 is applied to the dual of the y-problem (5.2). We conclude that the y-problem (5.2) is exactly controllable on the state space $L_2(\Omega)$ by means of $L_2(0,T;L_2(\Gamma_1))$ controllers of the type $e^{-\frac{i}{2}p(t,x)}u(t,x)$. But then by (5.1), the w-problem (1.1a-b-c) is likewise exactly controllable on the state space $L_2(\Omega)$, by means of controllers of the type u in $L_2(0,T;L_2(\Gamma_1))$. Theorem 1.2 is fully proved.

6. Illustrations and examples

In this section, we provide classes of examples of strictly convex functions on the Riemannian manifold $\{M, g\}$; that is, satisfying assumption (A.3) = (1.3). Two main cases will be emphasized. Subsection 6.1 is devoted to the case arising from Schrödinger equations with variable coefficient (in space) principal part, which are defined on a Euclidean bounded domain $\Omega \subset \mathbb{R}^n$. Here, then, the Riemannian manifold is $\{\mathbb{R}^n, g\}$, where the metric g is derived from the coefficients $a_{ij}(x)$ of the basic differential operator; in fact, $[g_{ij}(x)] = [a_{ij}(x)]^{-1}$. For this case – a primary reason for studying Schrödinger equations on a Riemannian manifold – several classes of examples of strictly convex functions were already given explicitly in prior references [T-Y.1], [L-T-Y.1], [Y.1], [Y.2]. Subsection 6.2 considers instead additional genuine Riemannian manifolds $\{M, g\}$, $M \neq \mathbb{R}^n$, following [T-X.1, Sect. 10].

6.1. Variable coefficient Schrödinger equations defined on a bounded Euclidean domain

Schrödinger equations on a Euclidean domain. In this section, $\tilde{\Omega}$ is an open bounded domain in $\mathbb{R}^n$, with boundary $\partial\tilde{\Omega} = \tilde{\Gamma}$ of class, say, C^2. Let $x = [x_1, x_2, \ldots, x_n]$, and let

$$\mathcal{A}w = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial w}{\partial x_j}\right); \quad \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq a\sum_{i=1}^{n}\xi_i^2, \quad x \in \tilde{\Omega}, \tag{6.1.1}$$

be a second-order differential operator, with real coefficients $a_{ij} = a_{ji}$ of class C^1, see Remark 6.1.1 below, satisfying the uniform ellipticity condition for some positive constant $a > 0$. Thus, we can extend $a_{ij}(x)$ smoothly to all of $\mathbb{R}^n$, so that the matrices

$$A(x) = (a_{ij}(x)); \quad G(x) = [A(x)]^{-1} = (g_{ij}(x)), \; i, j = 1, \ldots, n, \; x \in \mathbb{R}^n \tag{6.1.2}$$

are positive definite on any $x \in \mathbb{R}^n$.

Let $\tilde{F}$ be a linear, first-order differential operator: $\tilde{F}(w) = \tilde{R}(t,x) \cdot \nabla_0 w + r(t,x)w$, satisfying

$$|\tilde{F}(w)| \leq C_T[|\nabla_0 w|^2 + |w|^2], \quad t, x \in \tilde{Q} = (0,T] \times \tilde{\Omega} \text{ a.e.}, \tag{6.1.3}$$

where ∇_0 is the gradient in $\mathbb{R}^n$ and " $\cdot$ " is the $\mathbb{R}^n$-inner product. The corresponding Schrödinger equation on $\tilde{\Omega}$ is

$$iw_t + \mathcal{A}w = \tilde{F}(w) \quad \text{in } \tilde{Q} \tag{6.1.4}$$

to be supplemented by the initial condition and by boundary conditions.

Riemannian metric. It is easily checked that $(\mathbb{R}^n, g)$ is a complete Riemannian manifold with the Riemannian metric $g = \sum_{i,j=1}^{n} g_{ij}dx_i dx_j$. (If $A(x) = I$, i.e., $\mathcal{A} = -\Delta$, then $G(x) = I$ and g is the Euclidean $\mathbb{R}^n$-metric.) One also has [Y.2, Eqn. (2.2.11), p. 393]

$$\mathcal{A}w = -\Delta_g w + (Df)(w); \quad f(x) = \frac{1}{2}\ln\, \det[a_{ij}(x)], \tag{6.1.5}$$

where Δ_g is the corresponding Laplace-Beltrami operator; that is, under the change of metric, from the original Euclidean metric to g, we have that the second-order elliptic operator (6.1.1) becomes Δ_g on $(\mathbb{R}^n, g)$, modulo a first-order term. Thus, Eqn. (6.1.4) is turned into Eqn. (1.1), where (1.2c) is satisfied.

Remark 6.1.1. Let the coefficients a_{ij} in (6.1.1) be of class C^1, as assumed. Then the entries g_{ij} in (6.1.2) are of class C^1 as well. Thus, the connection coefficients (Christoffel symbols) Γ^{ℓ}_{ik}, see [Do.1, p. 54], are of class C^0. The geodesic solutions to a corresponding second-order, nonlinear ordinary differential equation [Do.1, p. 62] are then of class C^2. Thus, the square of the distance function $\operatorname{dist}^2_g(x, x_0)$ is in C^2. Typically, but by no means always, the required strictly convex function is taken to be $\operatorname{dist}^2_g(x, x_0)$, under suitable assumptions on the sectional curvature. See below. We also notice that in our case, where the manifolds are complete, the geodesics exist globally. $\square$

Classes of examples of strictly convex functions $d(x)$ in the Riemannian metric g. Several classes of $(a_{ij}(x))$ yielding strictly convex functions $d(x)$ in the metric g induced by $(g_{ij}(x)) = (a_{ij}(x))^{-1}$ on all of $\mathbb{R}^n$ are given in [L-T-Y.1], [T-Y.1], [Y.1], [Y.2]. Thus, *any* (sufficiently smooth) open bounded domain may be taken. Often, Green-Wu's [G-W.1] theorem is invoked, see Theorem 6.2.2.2 below. This assortment of examples can also be derived from the more systematic treatment of Section 6.2 to follow.

6.2. The general Riemannian case $\{M, g\}$

Orientation. In this subsection, we briefly summarize the treatment in [T-X.1, Section 10.2], aimed at providing classes of examples of strictly convex functions (that is, satisfying (A.1)) on a general Riemannian manifold $\{M, g\}$. To this end, [T-X.1, Section 10.2] proceeds according to the following strategy:

(i) At first, one provides strictly convex functions $d(x)$ on three canonical Riemannian manifolds: the sphere (S^n, can); the Euclidean space $(\mathbb{R}^n, \text{can})$; the hyperbolic space (H^n, can), with canonical metrics. These are the typical manifolds with constant sectional curvature K: respectively, $K > 0$, $K = 0$, $K < 0$. In these three canonical cases, the required strictly convex function $d(x)$ is constructed as a composition $d(x) = h(\rho(x))$ of a suitable function $h(\,\cdot\,)$ and the underlying distance function $\rho(x) = \operatorname{dist}_g(x, x_0)$, $x_0 \in M$.

(ii) Subsequently, the Hessian Comparison Theorem is invoked and used to further enlarge the class of examples in point (i), by making a comparison with the three canonical cases.

Here we omit Step (1) and refer to [T-X.1, Section 10.2.1]. Instead, we quote its consequences.

The Hessian Comparison Theorem. Using the Hessian Comparison Theorem [S-Y.1, Thm. 1.1], we obtain the following results:

Proposition 6.2.1 (Square of distance function). *If all sectional curvatures of (M, g) have an upper bound K, then:*

(i) *if* $K > 0$*:* $d(x) = \frac{1}{2}\mathrm{dist}^2(x, x_0)$ *is a strictly convex function on* M *when* $\mathrm{dist}(x, x_0) < \frac{\pi}{2\sqrt{K}}$, $\forall\, x_0 \in M$*;*

(ii) *if* $K \leq 0$*:* $d(x) = \frac{1}{2}\mathrm{dist}^2(x, x_0)$ *is a strictly convex function on* M*, where* M *is simply connected.* □

Proposition 6.2.2. *Let* $h(t)$ *be defined by*

$$h(t) = \begin{cases} -\cos(\sqrt{K}t), & K > 0; \\ \dfrac{1}{2}\, t^2, & K = 0; \\ \cosh(\sqrt{-K}t), & K < 0. \end{cases}$$

If all sectional curvatures of (M, g) *have an upper bound* K*, then:*

(i) *if* $K > 0$*:* $d(x) = h(\rho(x))$ *is a strictly convex function on* M *when* $\rho(x) = \mathrm{dist}(x, x_0) < \frac{\pi}{2\sqrt{K}}$, $\forall\, x_0 \in M$*;*

(ii) *if* $K \leq 0$*:* $d(x) = h(\rho(x))$ *is a strictly convex function on* M*, where* M *is simply connected.* □

Remark 6.2.1. When $K \leq 0$, the reason why we require M to be simply connected is in order to eliminate examples such as a flat torus or H_n/G [Do.1]. Indeed, when M is simply connected and has $K \leq 0$, then M is homeomorphic to $\mathbb{R}^n$ by the exponential map $\exp_{x_0} : T_{x_0}M \equiv \mathbb{R}^n \to M$. □

Two general results of the existence of strictly convex functions. We report here two known results: one of recent origin and valid on 2-dimensional manifolds dim $n = 2$, and one very established and well-known.

Theorem 6.2.3 ([B-G-L.1] **Strictly convex function in 2-D case via curvature flows).** *Let* $\bar{\Omega}$ *be a two-dimensional, smooth, compact, Riemannian surface whose boundary* $\partial\Omega$ *has positive second fundamental form. Assume there are no closed geodesics in* Ω*, then there exists a* C^2 *strictly convex function in* $\bar{\Omega}$*.*

This theorem is proved in [B-G-L.1] by using a nonlinear parabolic equation which arises in a quite unrelated geometric problem of curve-shortening flows.

Theorem 6.2.4 (Theorem 1(a) in [G-W.1]**. Complete non-compact manifold with positive sectional curvature).** *If* M *is a complete, non-compact, Riemannian manifold of everywhere positive sectional curvature, then there exists a* C^∞ *Lipschitz continuous strictly convex exhaustion function on* M*.*

Here a function $f : M \to \mathbf{R}$ *is an exhaustion function if for every* $\lambda \in \mathbf{R}$*,* $f^{-1}((-\infty, \lambda])$ *is a compact subset of* M*.*

6.3. Examples of strictly convex functions $d(x)$ satisfying the geometrical condition (1.5): $\langle Dd, \nu\rangle|_{\Gamma_0} = \frac{\partial d}{\partial \nu}|_{\Gamma_0} = 0$

6.3.1. Geodesic flat boundary. Let (M, g) be a complete Riemannian manifold with metric g. Let Ω be an open, bounded, connected subset of M with smooth boundary (say, of class C^2) $\partial\Omega = \Gamma = \overline{\Gamma_0 \cup \Gamma_1}$, $\Gamma_0 \cap \Gamma_1 = \emptyset$.

Definition 6.3.1. *An open subset Γ_0 of the boundary Γ is called "locally geodesic flat" if for any two close points $x_1, x_2 \in \Gamma_0$, the unique geodesic $\gamma(t)$ joining x_1 and x_2 with respect to the metric g is contained in Γ_0.*

Now assume that Ω is chosen as to satisfy Proposition 6.2.1 or 6.2.2, and Γ_0 be a locally geodesic flat connected part of boundary $\partial\Omega$. In order to define the strictly convex function $d(x)$, we need only to choose a suitable point x_0. Choose $x_0 \in M \backslash \overline{\Omega}$ such that x_0 is on a locally geodesic flat hypersurface S of M which includes Γ_0 as a subset. Define first $\rho(x) = \operatorname{dist}(x, x_0)$ $\forall x \in \Omega$, and then define $d(x) = h(\rho(x))$ as in Proposition 6.2.1 or 6.2.2. We then know that $d(x)$ is a strictly convex function on Ω.

Next we check that $\frac{\partial d}{\partial \nu}|_{\Gamma_0} = 0$: Using a geodesic frame on a neighborhood of S, one has $\frac{\partial \rho(x)}{\partial \nu} = 0$, $\forall\, x \in S$. Hence

$$\frac{\partial d(x)}{\partial \nu} = \frac{\partial h(\rho(x))}{\partial \nu} = h'(\rho)\frac{\partial \rho(x)}{\partial \nu} = 0, \qquad \forall x \in \Gamma_0, \quad \text{as desired.}$$

Remark 6.3.1. In the Euclidean setting $\mathbb{R}^n$, we can choose Γ_0 to be an open subset of a hyperplane and define $d(x) = \|x - x_0\|^2$ with x_0 on the hyperplane but away from Γ_0 to get $\frac{\partial d}{\partial \nu} = 2(x - x_0) \cdot \nu = 0$ on Γ_0 [L-T-Z.1, p. 288]. The present concept of "geodesic flat boundary" is a natural generalization of the Euclidean case. □

6.3.2. Strictly convex function $d(x)$, as a perturbation of d_0 with $\frac{\partial d_0}{\partial \nu}|_{\Gamma_0} \le 0$. Let (M, g) be an n-dimensional Riemannian manifold with Levi-Civita connection D. Let Ω be an open, bounded, connected subset of M with smooth boundary (say, of class C^2) $\partial\Omega = \Gamma = \overline{\Gamma_0 \cup \Gamma_1}$, $\Gamma_0 \cap \Gamma_1 = \emptyset$. The portion Γ_0 of $\partial\Omega$ is defined as follows: Let $\ell : M \to \mathbf{R}$ be a function of class C^2. Then we define

$$\Gamma_0 = \{x \in \partial\Omega :\ \ell(x) = 0\} \textit{ with } D\ell = \operatorname{grad}_g \ell \neq 0, \textit{ on } \Gamma_0. \tag{6.3.1}$$

Theorem 6.3.2. ([T-Y.1, Theorem B.1]). *In above setting, assume that:*

(i) (*Convexity of ℓ near Γ_0*) $D^2\ell(X, X)(x) \ge 0, \quad \forall\, x \in \Gamma_0,\ \forall\, X \in T_xM;$

(ii) *there exists a function $d_0 : \bar{\Omega} \to \mathbf{R}$ of class C^2, such that*

(ii$_1$) $D^2 d_0(X, X)(x) \ge \rho_0 |X|_g^2$, *where* $\rho_0 > 0, \quad \forall\, x \in \Gamma_0, \quad \forall\, X \in T_xM;$

(ii$_2$) $$\left.\frac{\partial d_0}{\partial \nu}\right|_{\Gamma_0} = \langle Dd_0, \nu \rangle_g \le 0, \qquad \textit{on } \Gamma_0.$$

Then, there exists a function $d : M \to \mathbf{R}$ of class C^2 (which is explicitly constructed in a layer (collar) of Γ_0, the critical set), such that it satisfies the following two conditions:

(a) $\frac{\partial d}{\partial \nu}|_{\Gamma_0} = \langle Dd, \nu \rangle_g = 0, \qquad$ *on* Γ_0;

(b) $D^2 d(X, X)(x) \ge (\rho_0 - \epsilon)|X|_g^2, \quad \forall\, x \in \Gamma_0, \quad \forall\, X \in T_xM,$

where $\epsilon > 0$ is arbitrarily small.

The function $d(x)$ is explicitly constructed near Γ_0, within Ω, as a perturbation of the original function d_0 assumed in (ii) above. For details of proof, we refer to [T-Y.2, Appendix B].

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