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Abstract Harmonic Analysis of Continuous Wavelet Transforms

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Preface

This volume discusses a construction situated at the intersection of two different mathematical fields: Abstract harmonic analysis, understood as the theory of group representations and their decomposition into irreducibles on the one hand, and wavelet (and related) transforms on the other. In a sense the volume reexamines one of the roots of wavelet analysis: The paper [60] by Grossmann, Morlet and Paul may be considered as one of the initial sources of wavelet theory, yet it deals with a unitary representation of the affine group, citing results on discrete series representations of nonunimodular groups due to Duflo and Moore. It was also observed in [60] that the discrete series setting provided a unified approach to wavelet as well as other related transforms, such as the windowed Fourier transform.

We consider generalizations of these transforms, based on a representation-theoretic construction. The construction of continuous and discrete wavelet transforms, and their many relatives which have been studied in the past twenty years, involves the following steps: Pick a suitable basic element (the *wavelet*) in a Hilbert space, and construct a system of vectors from it by the action of certain prescribed operators on the basic element, with the aim of expanding arbitrary elements of the Hilbert space in this system. The associated *wavelet transform* is the map which assigns each element of the Hilbert space its expansion coefficients, i.e. the family of scalar products with all elements of the system. A *wavelet inversion formula* allows the reconstruction of an element from its expansion coefficients.

Continuous wavelet transforms, as studied in the current volume, are obtained through the action of a group via a unitary representation. Wavelet inversion is achieved by integration against the left Haar measure of the group. The key questions that are treated –and solved to a large extent– by means of abstract harmonic analysis are: Which representations can be used? Which vectors can serve as wavelets?

The representation-theoretic formulation focusses on one aspect of wavelet theory, the inversion formula, with the aim of developing general criteria and providing a more complete understanding. Many other aspects that have made

wavelets such a popular tool, such as discretization with fast algorithms and the many ensuing connections and applications to signal and image processing, or, on the more theoretical side, the use of wavelets for the characterization of large classes of function spaces such as Besov spaces, are lost when we move on to the more general context which is considered here. One of the reasons for this is that these aspects often depend on a specific *realization* of a representation, whereas abstract harmonic analysis does not differentiate between unitarily equivalent representations.

In view of these shortcomings there is a certain need to justify the use of techniques such as direct integrals, entailing a fair amount of technical detail, for the solution of problems which in concrete settings are often amenable to more direct approaches. Several reasons could be given: First of all, the inversion formula is a crucial aspect of wavelet and Gabor analysis. Analogous formulae have been – and are being – constructed for a wide variety of settings, some with, some without a group-theoretic background. The techniques developed in the current volume provide a systematic, unified and powerful approach which for type I groups yields a complete description of the possible choices of representations and vectors. As the discussion in Chapter 5 shows, many of the existing criteria for wavelets in higher dimensions, but also for Gabor systems, are covered by the approach.

Secondly, Plancherel theory provides an attractive theoretical context which allows the unified treatment of related problems. In this respect, my prime example is the discretization and sampling of continuous transforms. The analogy to real Fourier analysis suggests to look for nonabelian versions of Shannon's sampling theorem, and the discussion of the Heisenberg group in Chapter 6 shows that this intuition can be made to work at least in special cases. The proofs for the results of Chapter 6 rely on a combination of direct integral theory and the theory of Weyl-Heisenberg frames. Thus the connection between wavelet transforms and the Plancherel formula can serve as a source of new problems, techniques and results in representation theory.

The third reason is that the connection between the initial problem of characterizing wavelet transforms on one side and the Plancherel formula on the other is beneficial also for the development and understanding of Plancherel theory. Despite the close connection, the answers to the above key questions require more than the straightforward application of known results. It was necessary to prove new results in Plancherel theory, most notably a precise description of the scope of the pointwise inversion formula. In the nonunimodular case, the Plancherel formula is obscured by the *formal dimension operators*, a family of unbounded operators needed to make the formula work. As we will see, these operators are intimately related to *admissibility conditions* characterizing the possible wavelets, and the fact that the operators are unbounded has rather surprising consequences for the existence of such vectors. Hence, the drawback of having to deal with unbounded operators, incurring the necessity to check domains, turns into an asset.

Finally the study of admissibility conditions and wavelet-type inversion formulae offers an excellent opportunity for getting acquainted with the Plancherel formula for locally compact groups. My own experience may serve as an illustration to this remark. The main part of the current is concerned with the question how Plancherel theory can be employed to derive admissibility criteria. This way of putting it suggests a fixed hierarchy: First comes the general theory, and the concrete problem is solved by applying it. However, for me a full understanding of the Plancherel formula on the one hand, and of its relations to admissibility criteria on the other, developed concurrently rather than consecutively. The exposition tries to reproduce this to some extent. Thus the volume can be read as a problem-driven – and reasonably self-contained– introduction to the Plancherel formula.

As the volume connects two different fields, it is intended to be open to researchers from both of them. The emphasis is clearly on representation theory. The role of group theory in constructing the continuous wavelet transform or the windowed Fourier transform is a standard issue found in many introductory texts on wavelets or time-frequency analysis, and the text is intended to be accessible to anyone with an interest in these aspects. Naturally more sophisticated techniques are required as the text progresses, but these are explained and motivated in the light of the initial problems, which are existence and characterization of admissible vectors. Also, a number of well-known examples, such as the windowed Fourier transform or wavelet transforms constructed from semidirect products, keep reappearing to provide illustration to the general results. Specifically the Heisenberg group will occur in various roles.

A further group of potential readers are mathematical physicists with an interest in generalized coherent states and their construction via group representations. In a sense the current volume may be regarded as a complement to the book by Ali, Antoine and Gazeau [1]: Both texts consider generalizations to the discrete series case. [1] replaces the square-integrability requirement by a weaker condition, but mostly stays within the realm of irreducible representations, whereas the current volume investigates the irreducibility condition. Note however that we do not comment on the relevance of the results presented here to mathematical physics, simply for lack of competence.

In any case it is only assumed that the reader knows the basics of locally compact groups and their representation theory. The exposition is largely self-contained, though for known results usually only references are given. The somewhat introductory Chapter 2 can be understood using only basic notions from group theory, with the addition of a few results from functional and Fourier analysis which are also explained in the text. The more sophisticated tools, such as direct integrals, the Plancherel formula or the Mackey machine, are introduced in the text, though mostly by citation and somewhat concisely. In order to accomodate readers of varying backgrounds, I have marked some of the sections and subsections according to their relation to the core material of the text. The core material is the study of admissibility conditions, dis-

cretization and sampling of the transforms. Sections and subsections with the superscript * contain predominantly technical results and arguments which are indispensable for a rigorous proof, but not necessarily for an understanding and assessment of results belonging to the core material. Sections and subsections marked with a superscript ** contain results which may be considered diversions, and usually require more facts from representation theory than we can present in the current volume. The marks are intended to provide some orientation and should not be taken too literally; it goes without saying that distinctions of this kind are subjective.

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Contents

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Introduction

1.1 The Point of Departure

In one of the papers initiating the study of the continuous wavelet transform on the real line, Grossmann, Morlet and Paul [60] considered systems $(\psi_{b,a})_{b,a\in\mathbb{R}\times\mathbb{R}'}$ arising from a single function $\psi \in \mathrm{L}^2(\mathbb{R})$ via

$$\psi_{b,a}(x) = |a|^{-1/2}\psi\left(\frac{x-b}{a}\right) \quad .$$

They showed that every function ψ fulfilling the **admissibility condition**

$$\int_{\mathbb{R}'} \frac{|\widehat{\psi}(\omega)|^2}{|\omega|} d\omega = 1 \quad , \tag{1.1}$$

where $\mathbb{R}' = \mathbb{R} \setminus \{0\}$, gives rise to an **inversion formula**

$$f = \int_{\mathbb{R}} \int_{\mathbb{R}'} \langle f, \psi_{b,a}\rangle \psi_{b,a} \frac{da}{|a|^2} db \quad , \tag{1.2}$$

to be read in the weak sense. An equivalent formulation of this fact is that the **wavelet transform**

$$f \mapsto V_\psi f \quad , \quad V_\psi f(b,a) = \langle f, \psi_{b,a}\rangle$$

is an isometry $\mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R}\times\mathbb{R}', db\frac{da}{|a|^2})$. As a matter of fact, the inversion formula was already known to Calderón [27], and its proof is a more or less elementary exercise in Fourier analysis.

However, the admissibility condition as well as the choice of the measure used in the reconstruction appear to be somewhat obscure until read in group-theoretic terms. The relation to groups was pointed out in [60] –and in fact earlier in [16]–, where it was noted that $\psi_{b,a} = \pi(b,a)\psi$, for a certain representation π of the affine group G of the real line. Moreover, (1.1) and (1.2)

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have natural group-theoretic interpretations as well. For instance, the measure used for reconstruction is just the left Haar measure on G.

Hence, the wavelet transform is seen to be a special instance of the following construction: Given a (strongly continuous, unitary) representation $(\pi, \mathcal{H}_\pi)$ of a locally compact group G and a vector $\eta \in \mathcal{H}_\pi$, we define the **coefficient operator**

$$V_\eta : \mathcal{H}_\pi \ni \varphi \mapsto V_\eta\varphi \in C_b(G) \quad , \quad V_\eta\varphi(x) = \langle \varphi, \pi(x)\eta \rangle \quad .$$

Here $C_b(G)$ denotes the space of bounded continuous functions on G.

We are however mainly interested in inversion formulae, hence we consider V_η as an operator $\mathcal{H}_\pi \to \mathrm{L}^2(G)$, with the obvious domain $\mathrm{dom}(V_\eta) = \{\varphi \in \mathcal{H}_\pi : V_\eta\varphi \in \mathrm{L}^2(G)\}$. We call η **admissible** whenever $V_\eta : \mathcal{H} \to \mathrm{L}^2(G)$ is an isometric embedding, and in this case V_η is called **(generalized) wavelet transform**. While the definition itself is rather simple, the problem of identifying admissible vectors is highly nontrivial, and the question whether these vectors exist for a given representation does not have a simple general answer. It is the main purpose of this book to develop in a systematical fashion criteria to deal with both problems.

As pointed out in [60], the construction principle for wavelet transforms had also been studied in mathematical physics, where admissible vectors η are called **fiducial vectors**, systems of the type $\{\pi(x)\eta : x \in G\}$ **coherent state systems**, and the corresponding inversion formulae **resolutions of the identity**; see [1, 73] for more details and references.

Here the earliest and most prominent examples were the original coherent states obtained by time-frequency shifts of the Gaussian, which were studied in quantum optics [114]. Perelomov [97] discussed the existence of resolutions of the identity in more generality, restricting attention to irreducible representations of unimodular groups. In this setting **discrete series** representations, i.e., irreducible subrepresentations of the regular representation λ_G of G turned out to be the right choice. Here every nonzero vector is admissible up to normalization. Moreover, Perelomov devised a construction which gives rise to resolutions of the identity for a large class of irreducible representations which were not in the discrete series. The idea behind this construction was to replace the group as integration domain by a well-chosen quotient, i.e., to construct isometries $\mathcal{H}_\pi \hookrightarrow \mathrm{L}^2(G/H)$ for a suitable closed subgroup H. In all of these constructions, irreducibility was essential: Only the well-definedness and a suitable intertwining property needed to be proved, and Schur's lemma would provide for the isometry property.

While we already remarked that [60] was not the first source to comment on the role of the affine group in constructing inversion formulae, suitably general criteria for nonunimodular groups were missing up to this point. Grossmann, Morlet and Paul showed how to use the orthogonality relations, established for these groups by Duflo and Moore [38], for the characterization of admissible vectors. More precisely, Duflo and Moore proved the existence of a uniquely

defined unbounded selfadjoint operator C_π associated to a discrete series representation such that a vector η is admissible iff it is contained in the domain of C_π, with $\|C_\pi \eta\| = 1$. A second look at the admissibility condition (1.1) shows that in the case of the wavelet transform on $\mathrm{L}^2(\mathbb{R})$ this operator is given on the Plancherel transform side by multiplication with $|\omega|^{-1/2}$. This framework allowed to construct analogous transforms in a variety of settings, which was to become an active area of research in the subsequent years; a by no means complete list of references is [93, 22, 25, 48, 68, 49, 50, 51, 83, 7, 8]. See also [1] and the references therein.

However, it soon became apparent that admissible vectors exist outside the discrete series setting. In 1992, Mallat and Zhong [92] constructed a transform related to the original continuous wavelet transform, called the **dyadic wavelet transform**. Starting from a function $\psi \in \mathrm{L}^2(\mathbb{R})$ satisfying the **dyadic admissibility condition**

$$\sum_{n\in\mathbb{Z}} |\widehat{\psi}(2^n\omega)|^2 = 1 \quad , \quad \text{for almost every } \omega \in \mathbb{R} \tag{1.3}$$

one obtains the (weak-sense) inversion formula

$$f = \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}} \langle f, \psi_{b,2^n}\rangle \psi_{b,2^n} 2^{-n} db \quad , \tag{1.4}$$

or equivalently, an isometric **dyadic wavelet transform** $\mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R} \times \mathbb{Z}, db2^{-n}dn)$, where dn denotes counting measure. Clearly the representation behind this transform is just the restriction of the above representation π to the closed subgroup $H = \{(b, 2^n) : b \in \mathbb{R}, n \in \mathbb{Z}\}$ of G, and the measure underlying the dyadic inversion formula is the left Haar measure of that subgroup. However, in one respect the new transform is fundamentally different: The restriction of π to H is no longer irreducible, in fact, it does not even contain irreducible subrepresentations (see Example 2.36 for details). Therefore (1.3) and (1.4), for all the apparent similarity to (1.1) and (1.2), cannot be treated in the same discrete series framework.

The example by Mallat and Zhong, together with results due to Klauder, Isham and Streater [67, 74], was the starting point for the work presented in this book. In each of these papers, a more or less straightforward construction led to admissibility conditions – similar to (1.1) and (1.3) – for representations which could not be dealt with by means of the usual discrete series arguments. The initial motivation was to understand these examples under a representation-theoretic perspective, with a view to providing a general strategy for the systematic construction of wavelet transforms.

The book departs from a few basic realizations: Any wavelet transform V_η is a unitary equivalence between π and a subrepresentation of λ_G, the left regular representation of G on $\mathrm{L}^2(G)$. Hence, the **Plancherel decomposition** of the latter into a direct integral of irreducible representations should

play a central role in the study of admissible vectors, as it allows to analyze invariant subspaces and intertwining operators.

A first hint towards direct integrals had been given by the representations in [67, 74], which were constructed as direct integrals of irreducible representations. However, the particular choice of the underlying measure was not motivated, and it was unclear to what extent these constructions and the associated admissibility conditions could be generalized to other groups. Properly read, the paper by Carey [29] on reproducing kernel subspaces of $\mathrm{L}^2(G)$ can be seen as a first source discussing the role of Plancherel measure in this context.

1.2 Overview of the Book

The contents of the remaining chapters may be roughly summarized as follows:

2. Introduction to the group-theoretic approach to the construction of continuous wavelet transforms. Embedding the discussion into $\mathrm{L}^2(G)$. Formulation of a list of tasks to be solved for general groups. Solution of these problems for the toy example $G = \mathbb{R}$.
3. Introduction to the Plancherel transform for type I groups, and to the necessary representation-theoretic machinery.
4. Plancherel inversion and admissibility conditions for type I groups. Existence and characterization of admissible vectors for this setting.
5. Examples of admissibility conditions in concrete settings, in particular for quasiregular representations.
6. Sampling theory on the Heisenberg group.

Chapter 2 is concerned with the collection of basic notions and results, concerning coefficient operators, inversion formulae and their relation to convolution and the regular representations. In this chapter we formulate the problems which we intend to address (with varying degrees of generality) in the subsequent chapters. We consider existence and characterization of inversion formulae, the associated reproducing kernel subspaces of $\mathrm{L}^2(G)$ and their properties, and the connection to discretization of the continuous transforms and sampling theorems on the group. Support properties of the arising coefficient functions are also an issue. Section 2.7 is crucial for the following parts: It discusses the solution of the previously formulated list of problems for the special case $G = \mathbb{R}$. It turns out that the questions mostly translate to elementary problems in real Fourier analysis.

Chapter 3 provides the "Fourier transform side" for locally compact groups of type I. The Fourier transform of such groups is obtained by integrating functions against irreducible representations. The challenge for Plancherel theory is to construct from this a unitary operator from $\mathrm{L}^2(G)$ onto a suitable direct integral space. This problem may be seen as analogous to the case of the reals, where the tasks consists in showing that the Fourier transform defined on $\mathrm{L}^1(\mathbb{R})$ induces a unitary operator $\mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R})$. However, for

arbitrary locally compact groups the right hand side first needs to be constructed, which involves a fair amount of technique. The exposition starts from a representation-theoretic discussion of the toy example, and during the exposition to follow we refer repeatedly to this initial example.

Chapter 4 contains a complete solution of the existence and characterization of admissible vectors, at least for type I groups and up to unitary equivalence. The technique is a suitable adaptation of the Fourier arguments used for the toy example. It relies on a pointwise Plancherel inversion formula, which in this generality has not been previously established. In the course of argument we derive new results concerning the Fourier algebra and Fourier inversion on type I locally compact groups, as well as an L^2-version of the convolution theorem, which allows a precise description of L^2-convolution operators, including domains, on the Plancherel transform side 4.18. We comment on an interpretation of the support properties obtained in Chapter 2 in connection with the so-called "qualitative uncertainty principle". Using existence and uniqueness properties of direct integral decompositions, we then describe a general procedure how to establish the existence and criteria for admissible vectors (Remark 4.30). We also show that these criteria in effect characterize the Plancherel measure, at least for unimodular groups. Section 4.5 shows how the Plancherel transform view allows a unified treatment of wavelet and Wigner transforms associated to nilpotent Lie groups.

Chapter 5 shows how to put the representation-theoretic machinery developed in the previous chapters to work on a much-studied class of concrete representations, thereby considerably generalizing the existing results and providing additional theoretic background. We discuss semidirect products of the type $\mathbb{R}^k \rtimes H$, with suitable matrix groups H. These constructions have received considerable attention in the past. However, the representation-theoretic results derived in the previous chapters allow to study generalizations, e.g. groups of the sort $N \rtimes H$, where N is a homogeneous Lie group and H is a one-parameter group of dilations on N. The discussion of the Zak-transform in the context of Weyl-Heisenberg frames gives further evidence for the scope of the general representation-theoretic approach.

The final chapter contains a discussion of sampling theorems on the Heisenberg group $\mathbb{H}$. We obtain a complete characterization of the closed leftinvariant subspaces of $L^2(\mathbb{H})$ possessing a sampling expansion with respect to a lattice. Crucial tools for the proof of these results are provided by the theory of Weyl-Heisenberg frames.

1.3 Preliminaries

In this section we recall the basic notions of representation theory, as far as they are needed in the following chapter. For results from representation theory, the books by Folland [45] and Dixmier [35] will serve as standard references.

The most important standing assumptions are that *all locally compact groups in this book are assumed to be Hausdorff and second countable* and *all Hilbert spaces in this book are assumed to be separable.*

Hilbert Spaces and Operators

Given a Hilbert space $\mathcal{H}$, the space of bounded operators on it is denoted by $\mathcal{B}(\mathcal{H})$, and the operator norm by $\|\cdot\|_\infty$. $\mathcal{U}(\mathcal{H})$ denotes the group of unitary operators on $\mathcal{H}$. Besides the norm topology, there exist several topologies of interest on $\mathcal{B}(\mathcal{H})$. Here we mention the **strong operator topology** as the coarsest topology making all mappings of the form

$$\mathcal{B}(\mathcal{H}) \ni T \mapsto T\eta \in \mathcal{H} \quad ,$$

with $\eta \in \mathcal{H}$ arbitrary, continuous, and the **weak operator topology**, which is the coarsest topology for which all coefficient mappings

$$\mathcal{B}(\mathcal{H}) \ni T \mapsto \langle \varphi, T\eta \rangle \in \mathbb{C},$$

with $\varphi, \eta \in \mathcal{H}$ arbitrary, are continuous. Furthermore, let the **ultraweak topology** denote the coarsest topology for which all mappings

$$\mathcal{B}(\mathcal{H}) \ni T \mapsto \sum_{n\in\mathbb{N}} \langle \varphi_n, T\eta_n \rangle$$

are continuous. Here $(\eta_n)_{n\in\mathbb{N}}$ and $(\varphi_n)_{n\in\mathbb{N}}$ range over all families fulfilling

$$\sum_{n\in\mathbb{N}} \|\eta_n\|^2 < \infty \quad , \quad \sum_{n\in\mathbb{N}} \|\varphi_n\|^2 < \infty \quad .$$

We use the abbreviations **ONB** and **ONS** for orthonormal bases and orthonormal systems, respectively. $\dim(\mathcal{H})$ denotes the Hilbert space dimension, i.e., the cardinality of an arbitrary ONB of $\mathcal{H}$. Another abbreviation is the word **projection**, which in this book always refers to selfadjoint projection operators on a Hilbert space. For separable Hilbert spaces, the Hilbert space dimension is in $\mathbb{N} \cup \{\infty\}$, where the latter denotes the countably infinite cardinal. The standard index set of cardinality m (wherever needed) is $I_m = \{1, \ldots, m\}$, where $I_\infty = \mathbb{N}$, and the standard Hilbert space of dimension m is $\ell^2(I_m)$.

If $(\mathcal{H}_i)_{i\in I}$ is a family of Hilbert spaces, then $\bigoplus_{i\in I} \mathcal{H}_i$ is the space of vectors $(\varphi_i)_{i\in I}$ in the cartesian product fulfilling in addition

$$\|(\varphi_i)_{i\in I}\|^2 := \sum_{i\in I} \|\varphi_i\|^2 < \infty \quad .$$

The norm thus defined on $\bigoplus_{i\in I} \mathcal{H}_i$ is a Hilbert space norm, and $\bigoplus_{i\in I} \mathcal{H}_i$ is complete with respect to the norm. If the $\mathcal{H}_i$ are orthogonal subspaces of a common Hilbert space $\mathcal{H}$, $\bigoplus_{i\in I} \mathcal{H}_i$ is canonically identified with the closed subspace generated by the union of the $\mathcal{H}_i$.

If T is a densely defined operator on $\mathcal{H}$ which has a bounded extension, we denote the extension by $[T]$.

Unitary Representations

A **unitary, strongly continuous representation**, or simply **representation**, of a locally compact group G is a group homomorphism $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ that is continuous, when the right hand side is endowed with the strong operator topology. Since weak and strong operator topology coincide on $\mathcal{U}(\mathcal{H}_\pi)$, the continuity requirement is equivalent to the condition that all coefficient functions of the type

$$G \ni x \mapsto \langle \varphi, \pi(x)\eta \rangle \in \mathbb{C},$$

are continuous.

Given representations σ, π, and operator $T : \mathcal{H}_\sigma \to \mathcal{H}_\pi$ is called **intertwining operator**, if $T\sigma(x) = \pi(x)T$ holds, for all $x \in G$. We write $\sigma \simeq \pi$ if σ and π are **unitarily equivalent**, which means that there is a *unitary* intertwining operator $U : \mathcal{H}_\sigma \to \mathcal{H}_\pi$. It is elementary to check that this defines an equivalence relation between representations. For any subset $\mathcal{K} \subset \mathcal{H}_\pi$ we let

$$\pi(G)\mathcal{K} = \{\pi(x)\eta : x \in G, \eta \in \mathcal{K}\} \quad .$$

A subspace of $\mathcal{K} \subset \mathcal{H}_\pi$ is called **invariant** if $\pi(G)\mathcal{K} \subset \mathcal{K}$. Orthogonal complements of invariant subspaces are invariant also. Restriction of a representation to invariant subspaces gives rise to **subrepresentations**. We write $\sigma < \pi$ if σ is unitarily equivalent to a subrepresentation of π. σ and π are called **disjoint** if there is no nonzero intertwining operator in either direction. A vector $\eta \in \mathcal{H}_\pi$ is called **cyclic** if $\pi(G)\eta$ spans a dense subspace of $\mathcal{H}_\pi$. A **cyclic representation** is a representation having a cyclic vector. All representations of interest to us are cyclic. In particular our standing assumption that G is second countable implies that all representations occurring in the book are realized on separable Hilbert spaces. π is called **irreducible** if every nonzero vector is cyclic, or equivalently, if the only closed invariant subspaces of $\mathcal{H}_\pi$ are $\{0\}$ and $\mathcal{H}_\pi$. Given a family $(\pi_i)_{i\in I}$, the direct sum $\pi = \bigoplus_{i\in I} \pi_i$ acts on $\bigoplus_{i\in I} \mathcal{H}_{\pi_i}$ via

$$\pi(x)\,(\varphi_i)_{i\in I} = (\pi_i(x)\varphi_i)_{i\in I} \quad .$$

The main result in connection with irreducible representations is Schur's lemma characterizing irreducibility in terms of intertwining operators. See [45, 3.5] for a proof.

Lemma 1.1. *If π_1, π_2 are irreducible representations, then the space of intertwining operators between π_1 and π_2 has dimension 1 or 0, depending on $\pi_1 \simeq \pi_2$ or not.*
In other words, π_1 and π_2 are either equivalent or disjoint.

Using the spectral theorem the following generalization can be shown. The proof can be found in [66, 1.2.15].

Lemma 1.2. *Let π_1, π_2 be representations of G, and let $T : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ be a closed intertwining operator, defined on a dense subspace $\mathcal{D} \subset \mathcal{H}_{\pi_1}$. Then $\overline{\mathrm{Im}T}$ and $(\mathrm{ker}T)^{\perp}$ are invariant subspaces and π_1, restricted to $(\mathrm{ker}T)^{\perp}$, is unitarily equivalent to the restriction of π_2 to $\overline{\mathrm{Im}T}$).*
If, moreover, π_1 is irreducible, T is a multiple of an isometry.

Given G, the **unitary dual** $\widehat{G}$ denotes the equivalence classes of irreducible representations of G. Whenever this is convenient, we assume the existence of a fixed choice of representatives of $\widehat{G}$, taking recourse to Schur's lemma to identify arbitrary irreducible representations with one of the representatives by means of the essentially unique intertwining operator.

We next describe the contragredient $\overline{\pi}$ of a representation π. For this purpose we define two involutions on $\mathcal{B}(\mathcal{H}_\pi)$, which are closely related to taking adjoints. For this purpose let $T \in \mathcal{B}(\mathcal{H}_\pi)$. If $(e_i)_{i\in I}$ is any orthonormal basis, we may define two linear operators T^t and $\overline{T}$ by prescribing

$$\langle T^t e_i, e_j\rangle = \langle T e_j, e_i\rangle \quad , \quad \langle \overline{T} e_i, e_j\rangle = \overline{\langle T e_i, e_j\rangle} \quad .$$

It is straightforward to check that these definitions do not depend on the choice of basis, and that $T^* = \overline{T}^t$, as we expect from finitedimensional matrix calculus. Additionally, the relations $\overline{T^t} = \overline{T}^t = T^*$ and $(ST)^t = T^t S^t$, $\overline{ST} = \overline{S}\,\overline{T}$ are easily verified.

Now, given a representation $(\pi, \mathcal{H}_\pi)$, the **(standard realization of the) contragredient representation** $\overline{\pi}$ acts on $\mathcal{H}_\pi$ by $\overline{\pi}(x) = \overline{\pi(x)}$. In general, $\overline{\pi} \not\simeq \pi$.

Commuting Algebras

The study of the commuting algebra, i.e., the bounded operators intertwining a representation with itself, is a central tool of representation theory. In this book, the **commutant** of a subset $M \subset \mathcal{B}(\mathcal{H})$, is denoted by M', and it is given by

$$M' = \{T \in \mathcal{B}(\mathcal{H}) : TS = ST \quad , \quad \forall S \in M\} \quad .$$

It is a **von Neumann algebra**, i.e. a subalgebra of $\mathcal{B}(\mathcal{H})$ which is closed under taking adjoints, contains the identity operator, and is closed with respect to the strong operator topology. The von Neumann density theorem [36, Theorem I.3.2, Corollary 1.3.1] states for selfadjoint subalgebras $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, that closedness in any of the above topologies on $\mathcal{B}(\mathcal{H})$ is equivalent to $\mathcal{A} = \mathcal{A}''$.

There are two von Neumann algebras associated to any representation π, the **commuting algebra** of π, which is the algebra $\pi(G)'$ of bounded operators intertwining π with itself, and the **bicommutant** $\pi(G)''$, which is the von Neumann algebra generated by $\pi(G)$. Since $\mathrm{span}(\pi(G))$ is a selfadjoint algebra, the von Neumann density theorem entails that it is dense in $\pi(G)''$ with respect to any of the above topologies. Invariant subspaces are conveniently discussed in terms of $\pi(G)'$, since a closed subspace $\mathcal{K}$ is invariant under π iff the projection onto $\mathcal{K}$ is contained in $\pi(G)'$.

Von Neumann algebras are closely related to the spectral theorem for selfadjoint operators, in the following way: Let $\mathcal{A}$ be a von Neumann algebra, and let T be a bounded selfadjoint operator. If S is an arbitrary bounded operator, it is well-known that S commutes with T iff S commutes with all spectral projections of T. Applying this to $S \in \mathcal{A}'$, the fact that $\mathcal{A} = \mathcal{A}''$ yields the following observation.

Theorem 1.3. *Let $\mathcal{A}$ is a von Neumann algebra on $\mathcal{H}$ and $T = T^* \in \mathcal{B}(\mathcal{H})$. Then $T \in \mathcal{A}$ iff all spectral projections of T are in $\mathcal{A}$.*

A useful consequence is that von Neumann algebras are closed under the functional calculus of selfadjoint operators, as described in [101, VII.7].

Corollary 1.4. *Let $\mathcal{A}$ is a von Neumann algebra on $\mathcal{H}$ and $T = T^* \in \mathcal{A}$ selfadjoint. Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function which is bounded on the spectrum of T. Then $f(T) \in \mathcal{A}$.*

Proof. Every spectral projection of $f(T)$ is a spectral projection of T. Hence the previous theorem yields the statement.

For more details concerning the spectral theorem we refer the reader to [101, Chapter VII]. The relevance of the spectral theorem for the representation theory of the reals is sketched in Section 2.7.

Tensor Products

The tensor product notation is particularly suited to treating direct sums of equivalent representations. Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces. The **Hilbert space tensor product** $\mathcal{H} \otimes \mathcal{K}$ is defined as the space of bounded linear operators $T : \mathcal{K} \to \mathcal{H}$ satisfying

$$\|T\|^2_{\mathcal{H}\otimes\mathcal{K}} := \sum_{j\in J} \|Te_j\|^2 < \infty \quad .$$

Here $(e_j)_{j\in J}$ is an ONB of $\mathcal{K}$. The Parseval equality can be employed to show that the norm is independent of the choice of basis, making $\mathcal{H} \otimes \mathcal{K}$ a Hilbert space with scalar product

$$\langle S, T\rangle = \sum_{j\in J} \langle Se_j, Te_j\rangle \quad .$$

Of particular interest are the operators of rank one. We define the **elementary tensor** $\varphi \otimes \eta$ as the rank one operator $\mathcal{K} \to \mathcal{H}$ defined by $\mathcal{K} \ni z \mapsto \langle z, \eta\rangle\varphi$. The scalar product of two rank one operators can be computed as

$$\langle \eta \otimes \varphi, \eta' \otimes \varphi'\rangle_{\mathcal{H}\otimes\mathcal{K}} = \langle \eta, \eta'\rangle_{\mathcal{H}} \langle \varphi', \varphi\rangle_{\mathcal{K}} \quad .$$

Note that our definition differs from the one in [45] in that our tensor product consists of *linear* operators as opposed to *conjugate-linear* in [45]. As a consequence, our elementary tensors are only conjugate-linear in the $\mathcal{K}$ variable, as witnessed by the change of order in the scalar product. However, the arguments in [45] are easily adapted to our notation. For computations in $\mathcal{H} \otimes \mathcal{K}$, it is useful to observe that ONB's $(\eta_i)_{i\in I} \subset \mathcal{H}$ and $(\varphi_j)_{j\in J} \subset \mathcal{K}$ yield an ONB $(\eta_i \otimes \varphi_j)_{i\in I, j\in J}$ of $\mathcal{H} \otimes \mathcal{K}$ [45, 7.14]. By collecting terms in the expansion with respect to the ONB, one obtains that each $T \in \mathcal{H} \otimes \mathcal{K}$ can be written as

$$T = \sum_{j\in J} a_j \otimes \varphi_j = \sum_{i\in I} \eta_i \otimes b_i \quad , \tag{1.5}$$

where the a_j and b_i are computed by $a_j = T\varphi_j$ and $b_i = T^*\eta_i$, yielding

$$T = \sum_{j\in J} (T\varphi_j) \otimes \varphi_j = \sum_{i\in I} \eta_i \otimes (T^*\eta_i) \quad , \tag{1.6}$$

as well as

$$\|T\|_2^2 = \sum_{j\in J} \|a_j\|^2 = \sum_{i\in I} \|b_i\|^2 \quad .$$

By polarization of this equation we find that given a second operator $S = \sum_{j\in J} c_j \otimes \varphi_j$, the scalar product can also be computed via

$$\langle T, S\rangle = \sum_{j\in J} \langle a_j, c_j\rangle \quad .$$

Operators $T \in \mathcal{B}(\mathcal{H})$, $S \in \mathcal{B}(\mathcal{K})$ act on elements on $\mathcal{H}\otimes\mathcal{K}$ by multiplication. On elementary tensors, this action reads as

$$(T \otimes S)(\eta \otimes \varphi) = T \circ (\eta \otimes \varphi) \circ S = (T\eta) \otimes (S^*\varphi) \quad ,$$

which will be denoted by $T \otimes S \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$. Keep in mind that this tensor is also only sesquilinear. Given two representations π, σ, the **tensor product representation** $\pi \otimes \overline{\sigma}$ is the representation of the direct product $G \times G$ acting on $\mathcal{H}_\pi \otimes \mathcal{H}_\sigma$ via $\pi \otimes \overline{\sigma}(x, y) = \pi(x) \otimes \sigma(y)^*$. On elementary tensors this action is given by

$$(\pi \otimes \overline{\sigma}(x, y))(\eta \otimes \varphi) = (\pi(x)\eta) \otimes (\sigma(x)\varphi) \quad .$$

Observe that the sesquilinearity of our tensor product notation entails that the restriction of $\pi \otimes \overline{\sigma}$ to $\{1\} \otimes G$ is indeed equivalent to $\dim(\mathcal{H}_\pi) \cdot \overline{\sigma}$, where $\overline{\sigma}$ is the contragredient of σ.

One can use the tensor product notation to define a compact realization of the multiple of a fixed representation. Given such a representation σ, the standard realization of $\pi = m \cdot \sigma$ acts on $\mathcal{H}_\pi = \mathcal{H}_\sigma \otimes \ell^2(I_m)$ by

$$\pi(x) = \sigma(x) \otimes \mathrm{Id}_{\ell^2(I_m)} \quad .$$

The advantage of this realization lies in compact formulae for the associated von Neumann algebras, if σ is irreducible:

$$\pi(G)' = 1 \otimes \mathcal{B}(\ell^2(I_m)) \quad , \tag{1.7}$$

which is understood as the algebra of all operators of the form $\mathrm{Id}_{\mathcal{H}_\sigma} \otimes T$, and

$$\pi(G)'' = \mathcal{B}(\mathcal{H}_\sigma) \otimes 1 \quad , \tag{1.8}$$

with analogous definitions. The follow for instance by [105, Theorem 2.8.1].

Trace Class and Hilbert-Schmidt Operators

Given a bounded positive operator T on a separable Hilbert space $\mathcal{H}$, T it is called **trace class operator** if its **trace class norm**

$$\|T\|_1 = \mathrm{trace}(T) = \sum_{i \in I} \langle T\eta_i, \eta_i \rangle < \infty \quad ,$$

where $(\eta_i)_{i \in I}$ is an ONB of $\mathcal{H}$. $\|T\|_1$ can be shown to be independent of the choice of ONB. An arbitrary bounded operator T is a **trace class operator** iff $|T|$ is of trace class. This defines the Banach space $\mathcal{B}_1(\mathcal{H})$ of trace clase operators. The trace

$$\mathrm{trace}(T) = \sum_{n \in \mathbb{N}} \langle T\eta_i, \eta_i \rangle$$

is a linear functional on $\mathcal{B}_1(\mathcal{H})$, and again independent of the choice of ONB. A useful property of the trace is that $\mathrm{trace}(TS) = \mathrm{trace}(ST)$, for all $T \in \mathcal{B}_1(\mathcal{H})$ and $S \in \mathcal{B}_\infty(\mathcal{H})$.

More generally, we may define for arbitrary $1 \leq p < \infty$ the **Schatten-von Neumann space of order** p as the space $\mathcal{B}_p(\mathcal{H})$ of operators T such that $|T|^p$ is trace class, endowed with the norm

$$\|T\|_p = \|(T^*T)^{p/2}\|_1^{1/p} \quad .$$

Again $\mathcal{B}_p(\mathcal{H})$ is a Banach space with respect to $\|\cdot\|_p$. An operator T is in $\mathcal{B}_p(\mathcal{H})$ iff $|T|$ has a discrete p-summable spectrum (counting multiplicities). This also entails that $\mathcal{B}_p(\mathcal{H}) \subset \mathcal{B}_r(\mathcal{H})$, for $p \leq r$, and that these spaces are contained in the space of compact operators on $\mathcal{H}$. Moreover, it entails that $\|\cdot\|_\infty \leq \|\cdot\|_p$.

As a further interesting property, $\mathcal{B}_p(\mathcal{H})$ is a twosided ideal in $\mathcal{B}(\mathcal{H})$, satisfying

$$\|ATB\|_p \leq \|A\|_\infty \|T\|_p \|B\|_\infty \quad .$$

We will exclusively be concerned with $p = 1$ and $p = 2$. Elements of the latter space are called **Hilbert-Schmidt operators**) . $\mathcal{B}_2(\mathcal{H})$ is a Hilbert space, with scalar product

$$\langle S, T\rangle = \mathrm{trace}(ST^*) = \mathrm{trace}(T^*S) \quad .$$

In fact, as the formula

$$\mathrm{trace}(T^*T) = \sum_{i\in I} \|T\eta_i\|^2 = \|T\|^2_{\mathcal{H}\otimes\mathcal{H}}$$

shows, $\mathcal{B}_2(\mathcal{H}) = \mathcal{H}\otimes\mathcal{H}$. In particular, all facts involving the role of rank-one operators and elementary tensors presented in the previous section hold for $\mathcal{B}_2(\mathcal{H})$.

Measure Spaces

In this book integration, either on a locally compact group or its dual, is ubiquitous. Borel spaces provide the natural context for our purposes, and we give a sketch of the basic notions and results. For a more detailed exposition, confer the chapters dedicated to the subject in [15, 17, 94].

Let us quickly recall some definitions connected to measure spaces. A **Borel space** is a set X equipped with a σ-algebra $\mathcal{B}$, i.e. a set of subsets of X (containing the set X itself) which is closed under taking complements and countable unions. $\mathcal{B}$ is also called **Borel structure**. Elements of $\mathcal{B}$ are called **measurable** or **Borel**. A σ-algebra **separates points**, if it contains the singletons. Arbitrary subsets A of a Borel space $(X, \mathcal{B})$, measurable or not, inherit a Borel structure by declaring the intersections $A \cap B$, $B \in \mathcal{B}$, as the measurable sets in A.

In most cases we will not explicitly mention the σ-algebra, since it is usually provided by the context. For a locally compact group, it is generated by the open sets. For countable sets, the power set will be the usual Borel structure. A **measure space** is a Borel space with a (σ-additive) measure μ on the σ-algebra. **ν-nullsets** are sets A with $\nu(A) = 0$, whereas **conull sets** are complements of nullsets.

If μ and ν are measures on the same space, μ is **ν-absolutely continuous** if every ν-nullset is a μ-nullset as well. We assume all measures to be σ-finite. In particular, the Radon-Nikodym Theorem holds [104, 6.10]. Hence absolute continuity of measures is expressable in terms of densities.

Measurable mappings between Borel spaces are defined by the property that the preimages of measurable sets are measurable. A bijective mapping $\phi : X \to Y$ between Borel spaces is a **Borel isomorphism** iff ϕ and ϕ^{-1} is measurable. A mapping $X \to Y$ is **μ-measurable** iff it is measurable outside a μ-nullset. For complex-valued functions f given on any measure space, we let $\mathrm{supp}(f) = f^{-1}(\mathbb{C} \setminus 0)$. Inclusion properties between supports are understood to hold only up to sets of measure zero, which is reasonable if one deals with L^p-functions. Given a Borel set A, we let $\mathbf{1}_A$ denote its indicator function.

Given a measurable mapping $\Phi : X \to Y$ between Borel spaces and a measure μ on X, the image measure $\Phi^*(\mu)$ on Y is defined as $\Phi^*(\mu)(A) =$

$\mu(\Phi^{-1}(A))$. A measure ν on Y is a **pseudo-image** of μ under Φ if ν is equivalent to $\Phi^*(\tilde{\mu})$, and $\tilde{\mu}$ is a finite measure on X which is equivalent to μ. $\tilde{\mu}$ exists if μ is σ-finite. Clearly two pseudo-images of the same measure are equivalent.

Let us now turn to locally compact groups G and G-spaces. A **G-space** is a set X with a **an action of G on X**, i.e., a mapping $G \times X \to X$, $(g, x) \mapsto g.x$, fulfilling $e.x = x$ and $g.(h.x) = (gh).x$. A **Borel G-space** is a G-space with the additional property that G and X carry Borel structures which make the action measurable; here $G \times X$ is endowed with the product Borel structure. If X is a G-space, the **orbits** $G.x = \{g.x : g \in G\}$ yield a partition of X, and the set of orbits or **orbit space** is denoted X/G for the orbit space. This notation is also applied to invariant subsets: If $A \subset X$ is **G-invariant**, i.e. $G.A = A$, then A/G is the space of orbits in A, canonically embedded in X/G. If X is a Borel space, the **quotient Borel structure** on X is defined by declaring all subsets $A \subset X/G$ as Borel for which the corresponding invariant subset of X is Borel. It is the coarsest Borel structure for which the quotient map $X \to X/G$ is measurable.

For $x \in X$ the **stabilizer of** x is given by $G_x = \{g \in G : g.x = x\}$. The canonical map $G \ni g \mapsto g.x$ induces a bijection $G/G_x \to G.x$.

2

Wavelet Transforms and Group Representations

In this chapter we present the representation-theoretic approach to continuous wavelet transforms. Only basic representation theory and functional analysis (including the spectral theorem) are required. The main purpose is to clarify the role of the regular representation, and to develop some related notions, such as selfadjoint convolution idempotents, which are then used for the formulation of the problems which the book addresses in the sequel. Most of the results in this chapter may be considered well-known, or are more or less straightforward extensions of known results, with the exception of the last two sections: The notion of sampling space and the related results presented in Section 2.6 are apparently new. Section 2.7 contains the discussion of an example which is crucial for the following: It motivates the use of Fourier analysis and thus serves as a blueprint for the arguments in the following chapters.

2.1 Haar Measure and the Regular Representation

Given a second countable locally compact group G, we denote by μ_G a **left Haar measure** on G, i.e. a Radon measure on the Borel σ-algebra of G which is invariant under left translations: $\mu_G(xE) = \mu_G(E)$. Since G is σ-compact, any Radon measure ν on G is inner and outer regular, i.e., for all Borel sets $A \subset G$ and $\epsilon > 0$ there exist sets $C \subset A \subset V$ with C compact, V open such that $\nu(V \setminus C) < \epsilon$.

One of the pillars of representation theory of locally compact groups is the fact that Haar measure always exists and is unique up to normalization. We use a simple dx to denote integration against μ_G, and $|A| = \mu_G(A)$ for Borel subsets $A \subset G$. An associated rightinvariant measure, the so called **right Haar measure** is obtained by letting $\mu_{G,r}(A) = |A^{-1}|$. The modular function $\Delta_G : G \to \mathbb{R}^+$ measures the rightinvariance of the left Haar measure. It is given by $\Delta_G(x) = \frac{|Ex|}{|E|}$, for an arbitrary Borel set E of finite positive measure. Using the fact that μ_G is unique up to normalization, one can show

H. Führ: LNM 1863, pp. 15–58, 2005.

that Δ_G is a well-defined continuous homomorphism, and independent of the choice of E. The homomorphism property entails that Δ_G is either trivial or unbounded: $\Delta_G(G)$ is a subgroup of the multiplicative $\mathbb{R}^+$, and all nontrivial subgroup of the latter are unbounded. Δ_G can also be viewed as a Radon-Nikodym derivative, namely

$$\Delta_G = \frac{d\mu_G}{d\mu_{G,r}} \quad ,$$

see [45, Proposition 2.31]. Hence the following formula, which will be used repeatedly [45, (2.32)]:

$$\int_G f(x)dx = \int_G f(x^{-1})\Delta_G(x^{-1})dx \quad . \tag{2.1}$$

G is called **unimodular** if $\Delta_G \equiv 1$, which is the case iff μ_G is rightinvariant also.

We will frequently use invariant and quasi-invariant measures on quotient spaces. If $H < G$ is a closed subgroup, we let $G/H = \{xH : x \in G\}$, which is a Hausdorff locally compact topological space. G acts on this space by $y.(xH) = yxH$, and the question of invariance of measures on G/H arises naturally. Given any measure ν on G/H let ν_g be the measure given by $\nu_g(A) = \nu(gA)$. Then ν is called **invariant** if $\nu_g = \nu$ for all $g \in G$, and **quasi-invariant** if ν_g and ν are equivalent. The following lemma collects the basic results concerning quasi-invariant measures on quotients.

Lemma 2.1. *Let G be a locally compact group, and $H < G$.*

(a) There exists a quasi-invariant Radon measure on G/H. All quasi-invariant Radon measures on G/H are equivalent.
(b) There exists an invariant Radon measure on G/H iff Δ_H is the restriction of Δ_G to H.
(c) If there exists an invariant Radon measure $\mu_{G/H}$ on G/H, it is unique up to normalization. After picking Haar measures on G and H, the normalization of $\mu_{G/H}$ can be chosen such as to ensure **Weil's integral formula**

$$\int_G f(x)dx = \int_{G/H}\int_H f(xh)dhd\mu_{G/H}(xH) \quad . \tag{2.2}$$

The invariance property of Haar measure implies that the left translation action of the group on itself gives rise to a unitary representation on $\mathrm{L}^2(G)$. The result is the **regular representation** defined next.

Definition 2.2. *Let G be a locally compact group. The* **left (resp. right) regular representation** *λ_G (ϱ_G) acts on $\mathrm{L}^2(G)$ by*

$$(\lambda_G(x)f)(y) = f(x^{-1}y) \quad \textit{resp.} \ (\varrho_G(x)f)(y) = \Delta_G(x)^{1/2} f(yx)$$

The **two-sided representation** *of the product group $G \times G$ is defined as*

$$(\lambda_G \times \varrho_G)(x,y) = \lambda_G(x)\varrho_G(y) \quad .$$

λ_G-invariant subspaces are called **leftinvariant**.

The convolution of two functions f, g on G is defined as the integral

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)dy \quad . \tag{2.3}$$

This is well-defined, with absolute convergence for almost every $x \in G$, whenever $f, g \in \mathrm{L}^1(G)$. But L^2-functions can be convolved also, if we employ a certain involution.

Definition 2.3. *Given any function f on G, define $f^*(x) = \overline{f(x^{-1})}$.*

Remark 2.4. If f is p-integrable with respect to left Haar measure, then f^* is p-integrable with respect to right Haar measure, and vice versa. In general, f^* will not be in $\mathrm{L}^p(G)$ if f is. Notable exceptions are given by the (trivial) case that G is unimodular, or more generally, that f is supported in a set on which Δ_G^{-1} is bounded.

The mapping $f \mapsto f^*$ is obviously a conjugate-linear involution. With respect to convolution, the involution turns out to be an antihomomorphism:

$$\begin{aligned}
(g * f)^*(x) &= \overline{\int_G g(y)f(y^{-1}x^{-1})dy} \\
&= \int_G g^*(y^{-1})f^*(xy)dy \\
&= \int_G f^*(y)g^*((x^{-1}y)^{-1})dy \\
&= \int_G f^*(x)g^*(y^{-1}x)dy \\
&= f^* * g^*(x) \quad .
\end{aligned}$$

Note that our definition differs from the notation in [45, 35]. Our choice is motivated by proposition 2.19 below which clarifies the connection between the involution and taking adjoints of coefficient operators.

The following simple observation relates convolution to coefficient functions:

Proposition 2.5. *For $f, g \in \mathrm{L}^2(G)$,*

$$(g * f^*)(x) = \int_G g(y)\overline{f(x^{-1}y)}dy = \langle g, \lambda_G(x)f\rangle \quad , \tag{2.4}$$

in particular the convolution integral $g * f^*$ converges absolutely for every x, yielding a continuous function which vanishes at infinity.

Proof. Equation (2.4) is self-explanatory, and it yields pointwise absolute convergence of the convolution product. Continuity follows from the continuity of the regular representation. Recall that a function f on G vanishes at infinity if for every $\epsilon > 0$ there exists a compact set $C \subset G$ such that $|f| < \epsilon$ outside of C. If f and g are compactly supported, it is clear that $g * f^*$ also has compact support, hence vanishes at infinity. For arbitrary L^2-functions f and g pick sequences $f_n \to f$ and $g_n \to g$ with $f_n, g_n \in C_c(G)$. Then the Cauchy-Schwarz inequality implies $g_m * f_n^* \to g * f^*$ uniformly, as $m, n \to \infty$. But then the limit vanishes at infinity also.

The von Neumann algebras generated by the regular representation are the *left and right group von Neumann algebras.*

Definition 2.6. *Let G be a locally compact group. The von Neumann algebras generated by the left and right regular representations are*

$$VN_l(G) = \lambda_G(G)'' \text{ and } VN_r(G) = \varrho_G(G)'' \quad .$$

$VN_l(G)$ and $VN_r(G)$ obviously commute; in fact $VN_l(G)' = VN_r(G)$. If the group is abelian, $VN_l(G) = VN_r(G) =: VN(G)$.

The equality $VN_l(G)' = VN_r(G)$ is a surprisingly deep result, known as the commutation theorem. For a proof, see [109].

2.2 Coherent States and Resolutions of the Identity

In this section we present a general notion of coherent state systems. Basically, the setup discussed in this section yields a formalization for the expansion of Hilbert space elements with respect to certain systems of vectors. The blueprint for this type of expansions is provided by ONB's: If $\eta = (\eta_i)_{i \in I}$ is an ONB of a Hilbert space $\mathcal{H}$, it is well-known that the **coefficient operator**

$$V_\eta : \mathcal{H} \ni \varphi \mapsto (\langle \varphi, \eta_i \rangle)_{i \in I} \in \ell^2(I) \tag{2.5}$$

is unitary, and that every $\varphi \in \mathcal{H}$ may be written as

$$\varphi = \sum_{i \in I} \langle \varphi, \eta_i \rangle \eta_i \quad . \tag{2.6}$$

The generalization discussed here consists in replacing I by a **measure space** $(X, \mathcal{B}, \mu)$, and summation by integration. In the following sections we will mostly specialize to the case $X = G$, a locally compact group, endowed with left Haar measure. However, in connection with sampling we will also need to discuss tight frames (obtained by taking a discrete space with counting measure), which is why have chosen to base the discussion on a slightly more abstract level.

Definition 2.7. *Let $\mathcal{H}$ be a Hilbert space. Let $\eta = (\eta_x)_{x\in X}$ denote a family of vectors, indexed by the elements of a measure space $(X, \mathcal{B}, \mu)$.*

(a) If for all $\varphi \in \mathcal{H}$, the **coefficient function**

$$V_\eta\varphi : X \ni x \mapsto \langle \varphi, \eta_x\rangle$$

is μ-measurable, we call η a **coherent state system**.

(b) Let $(\eta_x)_{x\in X}$ be a coherent state system, and define

$$\mathrm{dom}(V_\eta) := \{\varphi \in \mathcal{H} : V_\eta\varphi \in \mathrm{L}^2(X,\mu)\} \quad ,$$

which may be trivial. Denote by $V_\eta : \mathcal{H} \to \mathrm{L}^2(X,\mu)$ the (possibly unbounded) **coefficient operator** *or* **analysis operator** *with domain $\mathcal{D}_\eta$.*

(c) The coherent state system $(\eta_x)_{x\in X}$ is called **admissible** *if the associated coefficient operator $V_\eta : \varphi \mapsto V_\eta\varphi$ is an isometry, with* $\mathrm{dom}(V_\eta) = \mathcal{H}$.

It would be more precise to speak of μ-admissibility, since obviously the property depends on the measure. However, we treat the measure space $(X, \mathcal{B}, \mu)$ as given; it will either be a locally compact group with left Haar measure, or a discrete set with counting measure.

We next collect a few basic functional-analytic properties of coherent state systems. The following observation will frequently allow density arguments in connection with coefficient operators:

Proposition 2.8. *For any coherent state system $(\eta_x)_{x\in X}$, the associated coefficient operator is a closed operator.*

Proof. Let $\varphi_n \to \varphi$, where $\varphi_n \in \mathrm{dom}(V_\eta)$. Assume in addition that $V_\eta\varphi_n \to F$ in $\mathrm{L}^2(X,\mu)$. After passing to a suitable subsequence we may assume in addition pointwise almost everywhere convergence. Now the Cauchy-Schwarz-inequality entails

$$|V_\eta\varphi_n(x) - \langle\varphi, \eta_x\rangle| = |\langle\varphi_n - \varphi, \eta_x\rangle| \le \|\varphi_n - \varphi\|\,\|\eta_x\| \to 0 \quad ,$$

hence $F = V_\eta\varphi$ a.e., in particular the right hand side is in $\mathrm{L}^2(X,\mu)$.

Next we want to describe adjoint operators. For this purpose weak integrals will be needed.

Definition 2.9. *Let $(\eta_x)_{x\in X}$ be a coherent state system. If the right-hand side of*

$$\varphi \mapsto \int_X \langle\varphi, \eta_x\rangle d\mu(x)$$

converges absolutely for all φ, and defines a continuous linear functional on $\mathcal{H}$, we let the element of $\mathcal{H}$ corresponding to the functional by the Fischer-Riesz theorem be denoted by the **weak integral**

$$\int_X \eta_x d\mu(x) \quad .$$

Hence we obtain the following defining relation for $\int_X \eta_x d\mu(x)$*:*

$$\left\langle \varphi, \int_X \eta_x d\mu(x) \right\rangle = \int_X \langle \varphi, \eta_x \rangle d\mu(x) \tag{2.7}$$

For a family of operators $(T_x)_{x\in X}$ *we define the weak operator integral* $\int_X T_x dx$ *pointwise as*

$$\left(\int_X T_x dx \right)(\varphi) = \int_X T_x(\varphi)\ dx \quad ,$$

whenever the right-hand sides converges weakly for every φ*.*

Proposition 2.10. *Let* $(\eta_x)_{x\in X}$ *be a coherent state system. The associated coefficient operator* V_η *is bounded on* $\mathcal{H}$ *iff* $\mathrm{dom}(V_\eta) = \mathcal{H}$*. In that case, its adjoint operator is the* **synthesis operator***, given pointwise by the weak integral*

$$V_\eta^*(F) = \int_X F(x)\eta_x d\mu(x) \quad . \tag{2.8}$$

Proof. The first statement follows from the closed graph theorem and the previous proposition. For (2.8) we compute

$$\begin{aligned} \langle V_\eta \varphi, F \rangle &= \int_X \langle \varphi, \eta_x \rangle \overline{F(x)} d\mu(x) = \int_X \langle \varphi, F(x)\eta_x \rangle d\mu(x) \\ &= \left\langle \varphi, \int_X F(x)\eta_x d\mu(X) \right\rangle \quad . \end{aligned}$$

We will next apply the proposition to admissible coherent state systems. Note that for such systems η the isometry property entails that $V_\eta^* V_\eta$ is the identity operator on $\mathcal{H}$, and $V_\eta V_\eta^*$ is the projection onto the range of V_η. The first formula, the inversion formula, can then be read as a (usually continuous and redundant) expansion of a given vector in terms of the coherent state system. An alternative way of describing this property, commonly used in mathematical physics, expresses the identity operator as the (usually continuous) superposition of rank-one operators. In order to present this formulation, we use the **bracket notation** for rank-one operators:

$$|\eta\rangle\langle\psi| : \varphi \mapsto \langle \psi|\varphi\rangle \eta \quad . \tag{2.9}$$

Note the attempt to reconcile mathematics and physics notation by letting $\langle \eta|\varphi\rangle = \langle \varphi, \eta\rangle$. In particular, the bracket (2.9) is linear in η and antilinear in ψ. Outside the following proposition, we will however favor the tensor product notation $\eta \otimes \psi$ over the bracket notation.

Proposition 2.11. *If $(\eta_x)_{x\in X}$ is an admissible coherent state system, then for every $\varphi \in \mathcal{H}$, the following (weak-sense) reconstruction formula (or* **coherent state expansion***) holds:*

$$\varphi = \int_X \langle \eta_x | \varphi \rangle \eta_x d\mu(x) \quad . \tag{2.10}$$

Equivalently, we obtain the **resolution of the identity** *as a weak operator integral*

$$\int_X |\eta_x\rangle\langle\eta_x| d\mu(x) = \mathrm{Id}_{\mathcal{H}} \quad . \tag{2.11}$$

Proof. Recall that by the defining relation (2.7) the right hand side of (2.10) denotes the Hilbert space element $\psi \in \mathcal{H}$ satisfying for all $z \in \mathcal{H}$ the equation

$$\langle \psi, z \rangle = \int_X \langle \varphi, \eta_x \rangle \langle \eta_x, z \rangle d\mu(x) \quad .$$

But the right-hand side of this equation is just $\langle V_\eta \varphi, V_\eta z \rangle_{\mathrm{L}^2(X)} = \langle \varphi, z \rangle$, by the isometry property of V_η. Hence $\psi = \varphi$. Equation (2.11) is just a rephrasing of (2.10).

As a special case of (2.10) we retrieve (2.6) (with a somewhat weaker sense of convergence), observing that by (2.5) ONB's are admissible coherent state systems. Next we identify the ranges of coefficient mappings.

Proposition 2.12. *Let $(\eta_x)_{x\in X}$ be an admissible coherent state system. Then the image space $\widetilde{\mathcal{K}} = V_\eta(\mathcal{H}) \subset \mathrm{L}^2(X, \mu)$ is a reproducing kernel Hilbert space, i.e., the projection P onto $\mathcal{K}$ is given by*

$$PF(x) = \int_X F(y) \langle \eta_y, \eta_x \rangle d\mu(y) \quad .$$

Proof. Note that the integral converges absolutely since $V_\eta(\eta_y) \in \mathrm{L}^2(X)$. If we assume that V_η is an isometry, then $P = V_\eta V_\eta^*$. Plugging in (2.8) gives the desired equation:

$$\begin{aligned} V_\eta V_\eta^* F(x) &= \langle V_\eta^* F, \eta_x \rangle \\ &= \int_X F(y) \langle \eta_y, \eta_x \rangle d\mu(y) \quad . \end{aligned}$$

2.3 Continuous Wavelet Transforms and the Regular Representation

We now introduce the particular class of coherent state expansions associated to group representations which this book studies in detail. We first exhibit the close relation to the regular representation of the group. After that we investigate the functional-analytic basics of the coefficient operators in this setting, i.e., domains and adjoints.

Definition 2.13. *Let $(\pi, \mathcal{H}_\pi)$ denote a strongly continuous unitary representation of the locally compact group G. In the following, we endow G with left Haar measure. Associate to $\eta \in \mathcal{H}_\pi$ the orbit $(\eta_x)_{x\in G} = (\pi(x)\eta)_{x\in G}$. This is clearly a coherent state system in the sense of Definition 2.7(a), in particular the coefficient operators V_η can be defined according to 2.7(b).*

(a) η is called **admissible** *iff $(\pi(x)\eta)_{x\in G}$ is admissible.*
(b) If η is admissible, then $V_\eta : \mathcal{H}_\pi \hookrightarrow \mathrm{L}^2(G)$ is called **(generalized) continuous wavelet transform**.
(c) η is called **a bounded vector** *if $V_\eta : \mathcal{H}_\pi \to \mathrm{L}^2(G)$ is bounded on $\mathcal{H}_\pi$.*

We note in passing that η is cyclic iff V_η, this time viewed as an operator $\mathcal{H}_\pi \to C_b(G)$, is injective: Indeed, $V_\eta\varphi = 0$ iff $\varphi \perp \pi(G)\eta$, and that is equivalent to the fact that φ is orthogonal to the subspace spanned by $\pi(G)\eta$.

A straightforward but important consequence of the definitions is that

$$V_\eta(\pi(x)\varphi)(y) = \langle \pi(x)\varphi, \pi(y)\eta\rangle = \langle \varphi, \pi(x^{-1}y)\eta\rangle = (V_\eta\varphi)(x^{-1}y) \quad , \qquad (2.12)$$

i.e., coefficient operators intertwine π with the action by left translations on the argument. The same calculation shows that $\mathrm{dom}(V_\eta)$ is invariant under π.

Our next aim is to shift the focus from general representations of G to subrepresentations of λ_G. For this purpose the following simple proposition concerning the action of the commuting algebra on admissible (resp. bounded, cyclic) vectors is useful.

Proposition 2.14. *Let $(\pi, \mathcal{H}_\pi)$ be a representation of G and $\eta \in \mathcal{H}_\pi$. If $T \in \pi(G)'$, then*

$$V_{T\eta} = V_\eta \circ T^* \quad . \qquad (2.13)$$

In particular, suppose that $\mathcal{K}$ is an invariant closed subspace of $\mathcal{H}_\pi$, with projection $P_\mathcal{K}$. If $\eta \in \mathcal{H}_\pi$ is admissible (resp. bounded or cyclic) for $(\pi, \mathcal{H}_\pi)$, then $P_\mathcal{K}\eta$ has the same property for $(\pi|_\mathcal{K}, \mathcal{K})$.

Proof. $V_{T\eta}\varphi(x) = \langle \varphi, \pi(x)T\eta\rangle = \langle T^*\varphi, \pi(x)\eta\rangle$ shows (2.13), in particular the natural domain of $V_\eta \circ T^*$ coincides with $\mathrm{dom}(V_{T\eta})$. As a consequence $V_{P_\mathcal{K}\eta}$ is the restriction of V_η to $\mathcal{K}$. The remaining statements are immediate from this: The restriction of an isometry (resp. bounded or injective operator) has the same property.

The following rather obvious fact, which follows from similar arguments, will be used repeatedly.

Corollary 2.15. *Let T be a unitary operator intertwining the representations π and σ. Then $\eta \in \mathcal{H}_\pi$ is admissible (cyclic, bounded) iff $T\eta$ has the same property.*

We will next exhibit the central role of the regular representation for wavelet transforms. In view of the intertwining property (2.12), the remaining problems have more to do with functional analysis. The chief tool for this is the generalization of Schur's lemma given in 1.2.

Proposition 2.16. *(a) If π has a cyclic vector η for which V_η is densely defined, there exists an isometric intertwining operator $T : \mathcal{H}_\pi \hookrightarrow \mathrm{L}^2(G)$. Hence $\pi < \lambda_G$.*
(b) If $\varphi \in \mathcal{H}_\pi$ is such that $V_\varphi : \mathcal{H}_\pi \to \mathrm{L}^2(G)$ is a topological embedding, there exists an admissible vector $\eta \in \mathcal{H}_\pi$.
(c) Suppose that η is admissible and define $\mathcal{H} = V_\eta(\mathcal{H}_\pi)$. Then $\mathcal{H} \subset \mathrm{L}^2(G)$ is a closed, leftinvariant subspace, and the projection onto $\mathcal{H}$ is given by right convolution with $V_\eta\eta$.

Proof. For part (a) note that by assumption V_η is densely defined, and it intertwines π and λ_G on its domain, by (2.12). Hence Lemma 1.2 applies. Since η is cyclic, $\ker V_\eta = 0$, yielding $\pi < \lambda_G$.

For (b) define $U = V_\eta^* V_\eta$ and $\eta = U^{-1/2}\varphi$. Note that by assumption U is a selfadjoint bounded operator with bounded inverse, hence $U^{-1/2}$ is bounded also. Moreover, $U \in \pi(G)'$, hence 1.4 implies $U^{-1/2} \in \pi(G)'$.

Then by (2.13), $V_\eta^* V_\eta = U^{-1/2} U U^{-1/2} = \mathrm{Id}_{\mathcal{H}_\pi}$. The statements in (c) are obvious; for the calculation of the projection confer Proposition 2.12.

The proposition shows that up to unitary equivalence all representations of interest are subrepresentations of the left regular representation. In this setting, wavelet transforms are right convolution operators. We next want to discuss adjoint operators in this setting. Before we do this, we need to insert a small lemma.

Lemma 2.17. *Let a be a measurable bounded function, $b \in \mathrm{L}^2(G)$ such that for all $g \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G)$,*

$$\int_G a(x)\overline{g(x)}dx = \langle b, g\rangle \quad .$$

Then $g = f$ almost everywhere.

Proof. Assuming that a and b differ on a Borel set M of positive, finite measure, we find a measurable function g supported on M, with modulus 1 and such that $g(x)\overline{(b(x) - a(x))} > 0$ on M. But then $g \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G)$ yields the desired contradiction.

Remark 2.18. The nontrivial aspect of this lemma is that its proof is not just a density argument. Initially it is not even clear whether a is square-integrable. For this type of argument, replacing $\mathrm{L}^1(G) \cap \mathrm{L}^2(G)$ by some dense subspace generally does not work, as the following example shows: Consider the constant function $a(x) = 1$ on G and the subspace $\mathcal{H} = \{g \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G) : \int_G g(x)dx = 0\} \subset \mathrm{L}^2(G)$. $\mathcal{H}$ is dense if G is noncompact, and for all $g \in \mathcal{H}$,

$$\int_{\mathbb{R}} g(x)\overline{a(x)}dx = 0 \quad ,$$

with absolute convergence, but of course $a \neq 0 \in \mathrm{L}^2(\mathbb{R})$.

One of the reasons we single this argument out is that we will meet it again in connection with the Plancherel Inversion Theorem 4.15.

Proposition 2.19. *Suppose that* $f \in \mathrm{L}^2(G)$.

(a) $V_f : \mathrm{L}^2(G) \to \mathrm{L}^2(G)$ *is a closed operator with domain*

$$\mathrm{dom}(V_f) = \{g \in \mathrm{L}^2(G) : g * f^* \in \mathrm{L}^2(G)\} \quad ,$$

and acts by $V_f g = g * f^*$. *The subspace* $\mathrm{dom}(V_f)$ *is invariant under left translations.*

(b) If $f\Delta^{-1/2} \in \mathrm{L}^1(G)$, *then* f *is a bounded vector, with* $\|V_f\| \leq \|f\Delta^{-1/2}\|_1$. *This holds in particular when* f *has compact support.*

(c) If $f^* \in \mathrm{L}^2(G)$ *then* $\mathrm{L}^1(G) \cap \mathrm{L}^2(G) \subset \mathrm{dom}(V_f)$.

(d) Suppose that $f^* \in \mathrm{L}^2(G)$. *Then* $V_f^* \subset V_{f^*}$. *If one of the operators is bounded, so is the other, and they coincide.*

Proof. The first part of (a) was shown in Proposition 2.8. $V_f g = g * f^*$ was observed in equation (2.4). (b) and (c) are nonabelian versions of Young's inequality. We prove (b) along the lines of [45, Proposition 2.39], the proof of part (c) is similar (and can be found in [45]). We write

$$\begin{aligned} g * f^*(x) &= \int_G g(y)\overline{f(x^{-1}y)}dy \\ &= \int_G g(xy)\overline{f(y)}dy \\ &= \int_G (R_y g)(x)\overline{f(y)}dy \quad , \end{aligned}$$

where $(R_y g)(x) = g(xy)$. An application of the generalized Minkowski inequality then yields

$$\begin{aligned} \|g * f^*\|_2 &\leq \int_G \|R_y g\|_2 |f(y)|dy = \int_G \|g\|_2 \Delta_G(y)^{-1/2}|f(y)|dy \\ &= \|g\|_2 \|f\Delta_G^{-1/2}\|_1 \quad . \end{aligned}$$

For the computation of the adjoint operator in (d), let $g \in \mathrm{dom}(V_f^*)$. For all $h \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G) \subset \mathrm{dom}(V_f)$, we note that

$$\int_G \int_G |h(x)\overline{f(y^{-1}x)g(y)}|dydx \leq \int_G |h(x)|\|f^*\|_2\|g\|_2 dx < \infty \quad ,$$

hence we may apply Fubini's theorem to compute

$$\begin{aligned}\langle h, V_f^* g\rangle &= \langle V_f h, g\rangle \\ &= \int_G \int_G h(x)\overline{f(y^{-1}x)}dx\overline{g(y)}dy \\ &= \int_G h(x)\overline{\int_G g(y)f(y^{-1}x)dydx} \\ &= \int_G h(x)\overline{\int_G g(y)\overline{f^*(x^{-1}y)}dydx} \\ &= \int_G h(x)\overline{V_{f^*}g(x)}dx\end{aligned}$$

Note that $V_{f^*}g$ here denotes the coefficient function as an element of $C_b(G)$; we have yet to establish that $g \in \mathrm{dom}(V_{f^*}$. Here Lemma 2.17 applies to prove $V_{f^*}g = V_f^* g \in \mathrm{L}^2(G)$ and thus $V_f^* \subset V_{f^*}$. Assuming that V_f is bounded, it follows that $V_{f^*} \supset V_f^*$ is everywhere defined and closed, hence bounded. Conversely, V_f^* being contained in a bounded operator clearly implies that V_f^* is bounded.

Remark 2.20. Part (c) of the proposition implies that V_f is densely defined for arbitrary $f \in \mathrm{L}^2(G)$, when G is unimodular. This need not be true in the nonunimodular case, see example 2.29 below.

We note the following existence theorem for bounded cyclic vectors.

Theorem 2.21. *There exists a bounded cyclic vector for λ_G. Hence, an arbitrary representation π has a bounded cyclic vector iff $\pi < \lambda_G$.*

Proof. Losert and Rindler [84] proved for arbitrary locally compact groups the following statement: There exists $f \in C_c(G)$ which is a cyclic vector for λ_G iff G is first countable. Thus second countable groups have a cyclic vector $f \in C_c(G)$. But then 2.19 (b) entails that V_f is bounded on $\mathrm{L}^2(G)$, i.e. f is a bounded cyclic vector for $\mathrm{L}^2(G)$. Propositions 2.14 and 2.16 (a) yield the second statement.

Remark 2.22. When dealing with subrepresentations $\pi_1 < \pi_2$ and a vector $\eta \in \mathcal{H}_{\pi_1} \subset \mathcal{H}_{\pi_2}$, the notation V_η is somewhat ambiguous. Nonetheless, we refrain from introducing extra notation, since no serious confusion can occur: Denoting $V_\eta^{\pi_i}$ for the operator on $\mathcal{H}_{\pi_i}$ $(i = 1, 2)$, we find that $V_\eta^{\pi_2} = V_\eta^{\pi_1}$ on $\mathcal{H}_{\pi_1}$, and $V_\eta^{\pi_2} = 0$ on $\mathcal{H}_{\pi_1}^\perp$. Hence $V_\eta^{\pi_2}$ is just the trivial extension of $V_\eta^{\pi_1}$.

We close the section with a first short discussion of direct sum representations.

Proposition 2.23. *Let $\pi = \bigoplus_{i\in I} \pi_i$, and $\eta \in \mathcal{H}$. Let P_i denote the projection onto $\mathcal{H}_{\pi_i}$, and $\eta_i = P_i\eta$. Then the following are equivalent:*

(a) η is admissible.

(b) $\eta_i := P_i\eta$ *is admissible for* π_i*, for all* $i \in I$*, and* $\mathrm{Im}(V_{\eta_i}) \perp \mathrm{Im}(V_{\eta_j})$*, for all* $i \neq j$*.*

Proof. For $(a) \Rightarrow (b)$, the admissibility of η_i is due to Proposition 2.14. Moreover, if V_η is isometric, then it respects scalar products; in particular, the pairwise orthogonal subspaces $(P_i(\mathcal{H}))_{i\in I}$ have orthogonal images. But since $V_\eta \circ P_i = V_{\eta_i}$, this is precisely the second condition. The converse direction is similar.

One way of ensuring the pairwise orthogonality of image spaces in part (b) of the proposition is to choose the representations π_i as pairwise disjoint:

Lemma 2.24. *Let* π_1 *and* π_2 *be disjoint representations, and* $\eta_i \in \mathcal{H}_{\pi_i}$ *be bounded vectors* $(i = 1, 2)$*. Then* $V_{\eta_1}(\mathcal{H}_{\pi_1}) \perp V_{\eta_2}(\mathcal{H}_{\pi_2})$ *in* $\mathrm{L}^2(G)$*.*

Proof. $V_{\eta_2}^* V_{\eta_1} : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ is an intertwining operator, hence zero. Therefore, for all $\varphi_1 \in \mathcal{H}_{\pi_1}$ and $\varphi_2 \in \mathcal{H}_{\pi_2}$,

$$0 = \langle V_{\eta_2}^* V_{\eta_1} \varphi_1, \varphi_2 \rangle = \langle V_{\eta_1} \varphi_1, V_{\eta_2} \varphi_2 \rangle \quad ,$$

which is the desired orthogonality relation.

2.4 Discrete Series Representations

The major part of this book is concerned with the following two questions:

- Which representations π have admissible vectors?
- How can the admissible vectors be characterized?

For *irreducible* representations (such as the above mentioned examples), these questions have been answered by Grossmann, Morlet and Paul [60]; the key results can already be found in [38]. Irreducible subrepresentations of λ_G are called **discrete series representations**. The complete characterization of admissible vectors is contained in the following theorem. We will not present a full proof here, since the theorem is a special case of the more general results proved later on. However, some of the aspects of more general phenomena encountered later on can be studied here in a somewhat simpler setting, and we will focus on these.

Theorem 2.25. *Let* π *be an irreducible representation of* G*.*

(a) π *has admissible vectors iff* $\pi < \lambda_G$*.*
(b) A nonzero $\eta \in \mathcal{H}_\pi$ *is admissible (up to normalization) if* $V_\eta\eta \in \mathrm{L}^2(G)$*, or equivalently, if* $V_\eta\varphi \in \mathrm{L}^2(G)$*, for some nonzero* $\varphi \in \mathcal{H}_\pi$*.*

(c) There exists a unique, densely defined positive operator C_π with densely defined inverse, such that

$$\eta \in \mathcal{H}_\pi \text{ is admissible } \iff \eta \in \mathrm{dom}(C_\pi), \text{ with } \|C_\pi \eta\| = 1 \quad . \tag{2.14}$$

This condition follows from the **orthogonality relation**

$$\langle C_\pi \eta', C_\pi \eta \rangle \langle \varphi, \varphi' \rangle = \langle V_\eta \varphi, V_{\eta'} \varphi' \rangle \quad , \tag{2.15}$$

which holds for all $\varphi, \varphi' \in \mathcal{H}_\pi$ and $\eta, \eta' \in \mathrm{dom}(C_\pi)$. Conversely, $V_\psi \varphi \notin \mathrm{L}^2(G)$ whenever $\psi \notin \mathrm{dom}(C_\pi)$ and $0 \neq \varphi \in \mathcal{H}_\pi$.

(c) $C_\pi = c_\pi \times \mathrm{Id}_{\mathcal{H}_\pi}$ for a suitable $c_\pi > 0$ iff G is unimodular, or equivalently, if every nonzero vector is admissible up to normalization.

(d) Up to normalization, C_π is uniquely characterized by the **semi-invariance relation**

$$\pi(x) C_\pi \pi(x)^* = \Delta_G(x)^{1/2} C_\pi \quad . \tag{2.16}$$

The normalization of C_π is fixed by (2.15).

Proof. The "only-if" part of (a) is noted in Proposition 2.16 (a). For the converse direction assume $\pi < \lambda_G$, w.l.o.g. π acts by left translation on a closed subspace of $\mathrm{L}^2(G)$. Then projecting any $\eta \in C_c(G)$ into $\mathcal{H}_\pi$ yields a bounded vector, by 2.19(b) and 2.14. Since $C_c(G)$ is dense in $\mathrm{L}^2(G)$, we thus obtain a nonzero bounded vector η. Since π is irreducible, it follows that V_η is isometric up to a constant (by Lemma 1.2), hence we have found the admissible vector.

For the proof of part (b) note that the following chain of implications is trivial:

$$\begin{aligned} \eta \text{ is admissible up to normalization } &\Rightarrow V_\eta \eta \in \mathrm{L}^2(G) \\ &\Rightarrow (\exists \varphi \in \mathcal{H}_\pi \setminus \{0\} \ : \ V_\eta \varphi \in \mathrm{L}^2(G)) \quad . \end{aligned}$$

For the converse direction, assume $V_\eta \varphi \in \mathrm{L}^2(G)$ for a nonzero φ. Then $\mathrm{dom}(V_\eta)$ is nonzero and invariant, hence it is dense by irreducibility of π. But then Lemma 1.2 applies to yield that V_η is isometric up to a constant. Since $V_\eta \eta \neq 0$, the constant is nonzero, and thus η is admissible up to normalization.

The construction of the operators C_π requires additional tools from functional analysis. The basic idea is the following: Fix a normalized vector $\varphi \in \mathcal{H}_\pi$ and consider the positive definite sesquilinear form

$$B_\varphi : (\eta, \eta') \mapsto \langle V_{\eta'} \varphi, V_\eta \varphi \rangle \quad ,$$

which is the right hand side of (2.15) for the special case that $\varphi = \varphi'$. The domain of this form is $\mathcal{D} \times \mathcal{D}$, where $\mathcal{D}$ is the space of vectors η which are admissible up to normalization. Note that $\mathcal{D}$ is dense, being nonzero and invariant.

Recalling from linear algebra the representation theorem establishing a close connection between quadratic forms and symmetric matrices, we are looking for a positive selfadjoint operator A such that

$$B_\varphi(\eta, \eta') = \langle A\eta, \eta' \rangle \quad ,$$

and then letting $C_\pi = A^{1/2}$ should do the trick. Here we are in the situation that the domain is only a dense subset. We intend to use the representation theorem [101, Theorem VIII.6], and for this we need to show that B_φ is **closed**. This amounts to checking the following condition, for every sequence $(\eta_n)_{n\in\mathbb{N}}$ and $\eta \in \mathcal{H}_\pi$ such that $\eta_n \to \eta$: If

$$B_\varphi(\eta_n - \eta_m, \eta_n - \eta_m) \to 0 \quad , \quad \text{as } n, m \to \infty \tag{2.17}$$

then $\eta \in \mathcal{D}$ and $B_\varphi(\eta_n - \eta, \eta_n - \eta) \to 0$. It turns out that this is precisely the argument from the proof of Proposition 2.8: Observing that

$$B_\varphi(\eta - \eta', \eta - \eta') = \|V_{\eta-\eta'}\varphi\|_2^2 = \|V_\eta\varphi - V_{\eta'}\varphi\|_2^2 \quad ,$$

we see that condition (2.17) is equivalent to saying that $(V_{\eta_n}\varphi)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{L}^2(G)$. Hence after passing to a suitable subsequence we find that $V_{\eta_n}\varphi \to F \in \mathrm{L}^2(G)$, both in L^2 and pointwise almost everywhere. On the other hand, $\eta_n \to \eta$ entails $V_{\eta_n}\varphi \to V_\eta\varphi$ uniformly, by Cauchy-Schwarz. Hence $F = V_\eta$, and $\eta \in \mathcal{D}$ by part (a). Therefore we obtain the operator A, and letting $C_\pi = A^{1/2}$ yields

$$\langle V_{\eta'}\varphi, V_\eta\varphi \rangle = \langle C_\pi\eta, C_\pi\eta' \rangle \tag{2.18}$$

The first step for deriving the general orthogonality relations consists in observing that B_φ (and consequently C_π) is independent of the choice of normed vector φ: Fixing an arbitrary admissible η, the fact that V_η is the multiple of an isometry yields for all normed φ

$$B_\varphi(\eta, \eta) = \|V_\eta\varphi\|_2^2 = c_\eta\|\varphi\|^2$$

where c_η is a constant independent of φ. By polarization this implies that B_φ is independent of φ. Hence we obtain for arbitrary $\varphi \in \mathcal{H}$ and admissible vectors η, η'

$$\langle V_{\eta'}\varphi, V_\eta\varphi \rangle = \|\varphi\|^2 \langle C_\pi\eta, C_\pi\eta \rangle \quad .$$

Polarization with respect to φ yields (2.15).
Part (c) follows from (d), for (d) we refer to [38].

We note that (2.16) entails that C_π is unbounded in the nonunimodular case, since the operator norm on $\mathcal{B}(\mathcal{H}_\pi)$ is invariant under conjugation with unitaries. The operators C_π are called *Duflo-Moore* operators. More details on these operators can be found in Section 3.8. The proof given here basically follows the argument in [60]. The main reason we have reproduced it in part is

to demonstrate the close connection between the admissibility condition and the construction of the operators: The admissibility criterion (2.14) implies the orthogonality criterion (2.15) by polarization, and the latter was used to define C_π. Let us also point out the crucial role of irreducibility, which particularly implies that the space of admissible vectors (up to normalization) is dense in $\mathcal{H}_\pi$.

Remark 2.26. Note that the Duflo-Moore operators C_π studied here relate to the **formal dimension operators** K_π in [38] as $K_\pi^{-1/2} = C_\pi$. The terminology "formal dimension operator" is best understood by considering compact groups: Let π be an irreducible representation of a compact group G. Since coordinate functions are bounded, it is obvious that π is square-integrable. G is unimodular, thus C_π is scalar. Now the Schur orthogonality relations for compact groups [45, 5.8] yield for a normalized vector φ that

$$\|V_\varphi \varphi\|_2^2 = d_\pi^{-1} \|\varphi\|^2$$

where $d_\pi = \dim(\mathcal{H}_\pi)$. Thus $C_\pi = d_\pi^{-1/2} \cdot \mathcal{H}_\pi$, and the formal dimension operator $K_\pi = C_\pi^{-2}$ is multiplication with the Hilbert space dimension of $\mathcal{H}_\pi$.

The theorem of Grossmann, Morlet and Paul provides a rich reservoir of cases. In fact the large majority of papers dealing with the construction of wavelet transforms refers to this result. We give a small sample which contains the most popular examples.

Example 2.27. **Windowed Fourier transform.** Consider the **reduced Heisenberg group**, given as the set $\mathbb{H}_r = \mathbb{R}^2 \times \mathbb{T}$, with the group law

$$(p, q, z)(p', q', z') = (p + p', q + q', zz' \mathrm{e}^{\pi \mathrm{i}(pq' - qp')}) \ .$$

Haar measure here is given by $dpdqdz$, where dz is the rotation-invariant measure on the torus, normalized to one. G is unimodular. It acts on $\mathrm{L}^2(\mathbb{R})$ via the **Schrödinger representation** given by

$$(\pi(p, q, z)f)(x) = z \mathrm{e}^{2\pi \mathrm{i} q(x + p/2)} f(x + p) \quad . \tag{2.19}$$

Straightforward calculation allows to establish that

$$\begin{aligned}
\|V_\eta g\|_2^2 &= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{T}} \left| \int_{\mathbb{R}} g(x) \overline{z} \mathrm{e}^{-2\pi \mathrm{i} q(x+p/2)} \overline{\eta(x+p)} dx \right|^2 dz dq dp \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} g(x) \mathrm{e}^{-2\pi \mathrm{i} q(x+p/2)} \overline{\eta(x+p)} dx \right|^2 dq dp \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} g(x) \mathrm{e}^{-2\pi \mathrm{i} q x} \overline{\eta(x+p)} dx \right|^2 dq dp \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} |\widehat{H_p}(q)|^2 dp dq \quad ,
\end{aligned}$$

where $H_p(x) = g(x)\overline{\eta(x+p)}$, which for fixed $p \in \mathbb{R}$ is an integrable function. An application of Fubini's and Plancherel's theorem for the reals yields

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} |\widehat{H_p}(q)|^2 dpdq &= \int_{\mathbb{R}}\int_{\mathbb{R}} |g(x)\overline{\eta(x+p)}|^2 dxdp \\ &= \|\eta\|_2^2 \|g\|_2^2 \quad .\end{aligned}$$

This relation implies first of all that π is irreducible: V_η is injective for every nonzero η, i.e., η is cyclic. Moreover, every $\eta \in \mathrm{L}^2(\mathbb{R})$ is admissible up to normalization; more precisely, iff $\|\eta\| = 1$. This is what we are to expect by Theorem 2.25: G is unimodular, hence the formal dimension operator is a scalar multiple of the identity. In addition, we have established by elementary calculation that the scalar equals one.

Since the torus acts by multiplication, we have $|V_f(p,q,z)| = |V_f(p,q,1)|$, for all $z \in \mathbb{T}$. Hence the map $\mathcal{W}_f : g \mapsto (V_f g)|_{\mathbb{R}^2 \times \{1\}}$ is isometric as well. $\mathcal{W}_f$ is the **windowed Fourier transform** associated to the window f.

Hence we have derived for all $f \in \mathrm{L}^2(\mathbb{R})$ with $\|f\| = 1$ the transform

$$\mathcal{W}_f g(p,q) = \int_{\mathbb{R}} g(x) \mathrm{e}^{2\pi \mathrm{i} q(x+p/2)} \overline{f(x+p)}\, dx \quad ,$$

with inversion formula

$$g(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{W}_f g(p,q)\ \mathrm{e}^{-2\pi \mathrm{i} q(x+p/2)} \overline{f(x+p)}\ dpdq \quad .$$

Note that this inversion is to be understood in the weak sense and usually does not hold pointwise.

Example 2.28. **1-D CWT.** This is the original "continuous wavelet transform" introduced in [60]. It is based on the $ax + b$ group, the semidirect product $\mathbb{R} \rtimes \mathbb{R}'$. As a set G is given as $G = \mathbb{R} \times \mathbb{R}'$, with group law

$$(b,a)(b',a') = (b + ab', aa') \quad .$$

The left Haar measure is $db|a|^{-2}da$, which is distinct from the right Haar measure $db|a|^{-1}da$. Wavelets arise from the **quasi-regular representation** π acting on $\mathrm{L}^2(\mathbb{R})$ via

$$(\pi(b,a)f)(x) = |a|^{-1/2} f\left(\frac{x-b}{a}\right) \quad .$$

Again, computing L^2-norms of wavelet coefficients turns out to be an exercise in real Fourier analysis. First observe that on the Fourier transform side π acts as

$$(\pi(b,a)f)^\wedge(\omega) = |a|^{1/2} \mathrm{e}^{-2\pi \mathrm{i} \omega b} \widehat{f}(a\omega)$$

Hence, using the Plancherel theorem for the reals we can compute

$$\begin{aligned}
\|V_\eta g\|_2^2 &= \int_G |\langle g, \pi(b,a)\eta\rangle|^2 \, d\mu_G(b,a) \\
&= \int_G |\langle \hat{g}, (\pi(b,a)\eta)^\wedge\rangle|^2 \, d\mu_G(b,a) \\
&= \int_{\mathbb{R}'} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \hat{f}(\gamma)|a|^{1/2} \mathrm{e}^{2\pi \mathrm{i}\gamma b} \overline{\hat{\eta}}(a\gamma) d\gamma \right|^2 |a|^{-2} dbda \\
&= \int_{\mathbb{R}'} \int_{\mathbb{R}} \left| \int_{\widehat{\mathbb{R}^k}} \hat{f}(\gamma) \mathrm{e}^{2\pi \mathrm{i}\gamma b} \overline{\hat{\eta}}(a\gamma) d\gamma \right|^2 |a|^{-1} dbda \\
&= \int_{\mathbb{R}'} \int_{\mathbb{R}} \left| \widehat{\phi_a}(-b) \right|^2 |a|^{-1} dbda,
\end{aligned}$$

where $\phi_a(\gamma) = \hat{g}(\gamma)\overline{\hat{\eta}}(a\gamma)$. The Plancherel theorem allows thus to continue

$$\begin{aligned}
\int_{\mathbb{R}'} \int_{\mathbb{R}} \left| \widehat{\phi_a}(-b) \right|^2 |a|^{-1} dbda &= \int_{\mathbb{R}'} \int_{\mathbb{R}} \left| \hat{g}(\gamma)\overline{\hat{\eta}}(a\gamma) \right|^2 |a|^{-1} dbda \\
&= \int_{\mathbb{R}} |\hat{g}(\gamma)|^2 \left(\int_{\mathbb{R}'} |\hat{\eta}(a\gamma)|^2 |a|^{-1} da \right) d\gamma \\
&= \left(\int_{\mathbb{R}} |\hat{g}(\gamma)|^2 d\gamma \right) \cdot \left(\int_{\mathbb{R}'} |\hat{\eta}(a\gamma)|^2 |a|^{-1} da \right) \\
&= c_\eta^2 \|g\|^2 \quad ,
\end{aligned}$$

where we used the fact that the measure $a^{-1}da$ is Haar measure of the multiplicative group $\mathbb{R}'$. Hence we have derived

$$\|V_\eta g\|_2^2 = c_\eta^2 \|g\|^2 \quad , \tag{2.20}$$

where

$$c_\eta^2 = \int_{\mathbb{R}} \frac{|\widehat{\eta}(\omega)|^2}{|\omega|} d\omega \quad . \tag{2.21}$$

Note that our calculations also include the case $c_\eta = \infty$, where (2.20) means that $V_\eta f \notin \mathrm{L}^2(G)$. For this additional observation we need the following extended version of the Plancherel theorem:

$$\forall h \in \mathrm{L}^1(\mathbb{R}) \quad : \quad \Big(h \in \mathrm{L}^2(\mathbb{R}) \Longleftrightarrow \widehat{h} \in \mathrm{L}^2(\mathbb{R}) \Big) \quad . \tag{2.22}$$

Now "$\Longrightarrow$" is due to Plancherel's theorem, but the other direction is not. In order to show it, let $g \in \mathrm{L}^2(\mathbb{R})$ denote the inverse Plancherel transform of $\widehat{h}$, we have to show $g = h$. But this follows from the injectivity of the Fourier transform on the space of tempered distributions, since restriction to $\mathrm{L}^1(G)$ resp. $\mathrm{L}^2(G)$ yields the Fourier- resp. Plancherel transform.

As in the case of the windowed Fourier transform, (2.20) implies that the representation is irreducible. This time, the admissibility condition reads as:

$$\eta \in \mathrm{L}^2(\mathbb{R}) \ \text{ is admissible } \Leftrightarrow c_\eta = 1 \quad . \tag{2.23}$$

Comparing our findings to Theorem 2.25, we see that we have a discrete series representation of a nonunimodular group. Accordingly, the admissibility condition is more restrictive, requiring not just the right normalization. As a matter of fact, it is straightforward to check the semi-invariance relation (2.16) to show that the Duflo-Moore operator is given by

$$(C_\pi f)^\wedge(\omega) = |\omega|^{-1/2}\widehat{f}(\omega) \quad ,$$

as (2.21) suggests.

Example 2.29. As observed in Remark (2.20) above, V_f need not be densely defined for arbitrary $f \in \mathrm{L}^2(G)$, when G is nonunimodular. Here we construct such an example for the case that G is the $ax+b$-group. For this purpose consider the quasi-regular representation π from Example 2.28. Pick a $\psi \in \mathrm{L}^2(\mathbb{R})$ which is not in the domain of the Duflo-Moore operator, and an admissible vector η. Defining $f = V_\eta\psi$ and $\mathcal{H} = V_\eta(\mathrm{L}^2(\mathbb{R})) \subset \mathrm{L}^2(G)$, we see that $V_f g = 0$ for $g \in \mathcal{H}^\perp$, whereas for $g = V_\eta\varphi \in \mathcal{H}$,

$$V_f g(x) = \langle V_\eta\varphi, \lambda_G(x)V_\eta\psi\rangle = \langle\varphi, \pi(x)\psi\rangle = V_\psi\phi(x) \quad ,$$

and the latter function is not in $\mathrm{L}^2(G)$ by 2.25 (b) and the choice of ψ. Hence $\mathrm{dom}(V_f) = \mathcal{H}^\perp$, and $V_f = 0$ on this domain.

Example 2.30. **2-D CWT.** This construction was first introduced by Murenzi [93], as a natural generalization of the continuous transform in one dimension. We consider the **similitude group** $G = \mathbb{R}^2 \rtimes (SO(2) \times \mathbb{R}^+)$. Hence G is the set $\mathbb{R}^2 \times SO(2) \times \mathbb{R}^+$ with the group law

$$(x, h, r)(x', h', r') = (x + rhx', hh', rr') \quad .$$

The group can be identified with the subgroup of the full affine group of the plane generated the translations, the rotations and the dilations. It thus acts naturally on $\mathbb{R}^2$, which gives rise to the quasi-regular representation π acting on $\mathrm{L}^2(\mathbb{R}^2)$ via

$$(\pi(x, h, r)f)(y) = |r|^{-1} f(r^{-1}h^{-1}(y - x)) \quad .$$

An adaptation of the argument for the 1D CWT yields

$$\|V_\eta f\|_2^2 = c_\eta^2 \|f\|_2 \quad , \tag{2.24}$$

where this time

$$c_\eta^2 = \int_{\mathbb{R}^2} \frac{|\widehat{f}(\omega)|^2}{|\omega|^2} d\omega \quad .$$

Therefore the admissibility condition reads

$$\eta \text{ is admissible } \iff \int_{\mathbb{R}^2} \frac{|\widehat{f}(\omega)|^2}{|\omega|^2} d\omega = 1 \quad .$$

As in the case of the 1D-CWT, we obtain from the norm equality that π is again irreducible. The Duflo-Moore operator is computed as

$$(C_\pi f)^\wedge(\omega) = |\omega|^{-1} \widehat{f}(\omega) \quad .$$

Next let us consider direct sums of discrete series representations. The following theorem describes how far the discrete series arguments carry. Recall that Proposition 2.23 gives criteria for direct sum representations, and the orthogonality relations for the discrete series case allow to derive admissibility criteria for multiplicities greater than one.

Theorem 2.31. *Let $\pi = \bigoplus_{i\in I} \pi_i$, where each π_i is a discrete series representation. Denote by P_i the projection onto the representation space $\mathcal{H}_{\pi_i}$, and by C_{π_i} the associated Duflo-Moore operators. Since the π_i are irreducible, there exist (up to normalization) unique intertwining operators $S_{i,j} : \mathcal{H}_{\pi_i} \to \mathcal{H}_{\pi_j}$. Then the following are equivalent:*

(a) η is admissible.
(b) $\eta_i \in \mathrm{dom}(C_{\pi_i})$, with $\|C_{\pi_i}\eta_i\| = 1$. Moreover, for all i, j with $\pi_i \simeq \pi_j$,

$$\langle C_{\pi_j} S_{i,j} \eta_i, C_{\pi_j} \eta_j \rangle = 0 \tag{2.25}$$

Proof. We apply Proposition 2.23. By Theorem 2.25(c), $\|C_{\pi_i}\eta_i\| = 1$ is the admissibility condition on η_i. Moreover, the orthogonality relation (2.15) shows that whenever $\pi_i \simeq \pi_j$,

$$\mathrm{Im}(V_{\eta_i}) \perp \mathrm{Im}(V_{\eta_i}) \Longleftrightarrow \langle C_{\pi_j} S_{i,j} \eta_i, C_{\pi_j} \eta_j \rangle = 0 \quad .$$

Since any two irreducible representations are either equivalent or disjoint, Lemma 2.24 yields $\mathrm{Im}(V_{\eta_i}) \perp \mathrm{Im}(V_{\eta_i})$ for arbitrary vectors η_i and η_j, whenever $\pi_i \not\simeq \pi_j$. Hence the proof is finished.

The following remark is a preliminary version of one of the main results contained in this book: The existence criterion for admissible vectors given in Theorem 4.22. Here we only consider the case of direct sums of discrete series representations. Some of the phenomena encountered in the general case can be already examined in this simpler setting, in particular the striking difference between unimodular and nonunimodular groups and the role of the formal dimension operators in this context.

Remark 2.32. Let $\pi = \bigoplus_{i\in I} \pi_i$, where each π_i is a discrete series representation. We associate a *multiplicity function* $m_\pi : \widehat{G} \to \mathbb{N}_0 \bigcup \{\infty\}$ to π, by letting

$$m_\pi(\sigma) = |\{i \in I : \sigma \simeq \pi_i\}| \quad ,$$

where $|\cdot|$ denotes cardinality. m_π simply counts the multiplicity with which the representation π is contained in π. Note that $\mathcal{H}_\pi$ is assumed to be separable, hence the cardinalities are countably infinite at most, and only countably many σ have a nonzero multiplicity. Fix unitary intertwining operators $T_i : \mathcal{H}_{\pi_i} \to \mathcal{H}_\sigma$, for the unique $\sigma \in \widehat{G}$ with $\sigma \simeq \pi_i$. The uniqueness property of the Duflo-Moore operators entails that $C_\sigma = T_i C_{\pi_i} T_i^*$.

Using the operators T_i, the admissibility conditions from Theorem 2.31 can also be written as

$$\langle C_\sigma T_i \eta_i, C_\sigma T_j \eta_j \rangle = 0 \quad (\pi_i \simeq \pi_j) \tag{2.26}$$

$$\|C_{\pi_i} \eta_i\| = 1 \quad (\forall i \in I) \tag{2.27}$$

Both relations can be used to derive necessary conditions for the multiplicity function: (2.26) clearly implies that

$$m_\pi(\sigma) \leq \dim(\mathcal{H}_\sigma) \quad . \tag{2.28}$$

Moreover, the orthogonal decomposition of $\mathcal{H}_\pi$ yields in particular that

$$\infty > \|\eta\|^2 = \sum_{i \in I} \|P_i \eta\|^2 = \sum_{i \in I} \|\eta_i\|^2 \quad . \tag{2.29}$$

Now assume that G is unimodular. Then $C_\sigma = c_\sigma \times \mathrm{Id}_{\mathcal{H}_\sigma}$, with positive scalars c_σ. Hence (2.27) entails the necessary condition

$$\sum_{\sigma \in \widehat{G}} m_\pi(\sigma) c_\sigma^{-2} < \infty \quad . \tag{2.30}$$

Conversely, it is easily seen that vectors fulfilling (2.26) and (2.27) exist once (2.28) and (2.30) hold, therefore we have found a characterization of direct sums of discrete series representations with admissible vectors. Note that (2.30) implies $m_\pi(\sigma) < \infty$, which can be seen as a sharpening of (2.28).

In the nonunimodular case the situation is much less transparent. However, it turns out that the restrictions actually vanish! To begin with, $\dim(\mathcal{H}_\sigma) = \infty$ follows from the existence of an unbounded operator C_σ on $\mathcal{H}_\sigma$. In addition, while (2.29) still holds, implying in particular that (at least for I infinite) the norms of the η_i become arbitrarily small, it is no contradiction to (2.27). Here the fact that the C_{π_i} are unbounded makes it conceivable that there exist vectors that actually fulfill both conditions. Note that we still need to ensure (2.26), which requires more knowledge of the formal dimension operators than we have currently at our disposal. In any case the existence of an unbounded operator on $\mathcal{H}_\sigma$ entails $\dim(\mathcal{H}_\sigma) = \infty$, i.e., (2.28) holds trivially.

We will next study the space $\mathrm{L}^2_\pi(G)$ spanned by all coefficient functions associated to a fixed discrete series representation π. Most of the following is due to Duflo and Moore. The results can be seen as precursors of the Plancherel formula, or more precisely, as the contribution of the discrete series to the Plancherel formula. They also provide further insight into the role of Hilbert-Schmidt operators and the quasi-invariance relation (2.16).

Theorem 2.33. *Let π be a discrete series representation, and define*

$$\mathrm{L}^2_\pi(G) = \overline{\mathrm{span}\{\mathcal{H} : \lambda_G|_{\mathcal{H}} \simeq \pi\}} \quad .$$

(a) $\mathrm{L}^2_\pi(G) = \overline{\mathrm{span}\{V_\eta\varphi : \varphi, \eta \in \mathcal{H}_\pi \text{ such that } V_\eta\varphi \in \mathrm{L}^2(G)\}}$.
(b) $\mathrm{L}^2_\pi(G)$ *is* $\lambda_G \times \varrho_G$*-invariant, with* $\lambda_G \times \varrho_G|_{\mathrm{L}^2_\pi(G)} \simeq \pi \otimes \overline{\pi}$. *In particular,* $\lambda_G \times \varrho_G|_{\mathrm{L}^2_\pi(G)}$ *is irreducible.*
(c) Let $(\eta_i)_{i\in I}$ *denote an ONB of* $\mathcal{H}_\pi$ *contained in* $\mathrm{dom}(C_\pi^{-1})$. *Then* $\left(V_{C_\pi^{-1}\eta_i}\eta_j\right)_{i,j\in I}$ *is an ONB of* $\mathrm{L}^2_\pi(G)$.
(d) If σ *is another discrete series representation with* $\sigma \not\simeq \pi$, *then* $\mathrm{L}^2_\sigma(G) \perp \mathrm{L}^2_\pi(G)$.

Proof. For part (a) we let $\mathcal{H}_0 = \mathrm{span}\{V_\eta\varphi : \varphi, \eta \in \mathcal{H}_\pi, \eta \text{ is admissible}\}$. Then for every $f = V_\eta\varphi \in \mathcal{H}_0$, the leftinvariant space spanned by f is just the image of $\mathcal{H}_\pi$ under V_η, which is a unitary equivalence. Hence $\mathcal{H}_0 \subset \mathrm{L}^2_\pi(G)$, which extends to the closure of $\mathcal{H}_0$.

For the other direction we argue indirectly. Assume that there exists $g \in \mathrm{L}^2(G)$ such that the restriction of λ_G to the leftinvariant subspace generated by g is equivalent to π, yet g is not contained in the closed span of $\mathcal{H}_0$. W.l.o.g. we may assume that $g \perp \mathcal{H}_0$. Observe that $\mathcal{H}_0$ is rightinvariant also, since

$$V_{\pi(x)\eta}\varphi(y) = \langle \varphi, \pi(yx)\eta\rangle = \Delta_G(x)^{-1/2}\left(\rho_G(x)V_\eta\varphi\right)(y) \quad .$$

Let Q denote the projection onto the leftinvariant space generated by g. Pick $h \in C_c(G)$ such that $Qh \neq 0$. Then we have

$$V_{Qh}g = V_h g = g * h^* \in \mathrm{L}^2(G)$$

by choice of h and 2.19 (b). Moreover, $V_{Qh}g$ is nonzero since Qh and g are nonzero and π is irreducible. By definition of $\mathcal{H}_0$ we have $V_{Qh}g \in cal H_0$. On the other hand, rightinvariance of $\mathcal{H}_0$ yields that if $g \perp \mathcal{H}_0$, then $g * h^* \perp \mathcal{H}_0$, and we have the desired contradiction.

For part (b), consider the mapping

$$T : \varphi \otimes \eta \mapsto V_{C_\pi^{-1}\eta}\varphi \quad ,$$

defined for all elementary tensors $\varphi \otimes \eta$ satisfying $\eta \in \mathrm{dom}(C_\pi^{-1})$. Since C_π^{-1} is densely defined, these tensors span a dense subspace of $\mathcal{H}_\pi \otimes \mathcal{H}_\pi$. Moreover, by the orthogonality relation (2.15), T is isometric. Hence there exists a unique linear isometry, also denoted by $T : \mathcal{H}_\pi \otimes \mathcal{H}_\pi \to \mathrm{L}^2_\pi(G)$. By part (a), it has dense image, hence T is in fact unitary. We will next show that T is an intertwining operator. For this purpose observe that (2.16) gives rise to

$$\pi(x)C_\pi^{-1}\pi(x)^* = \Delta_G(x)^{-1/2}C_\pi^{-1} \quad .$$

Then we compute

$$\begin{aligned} T(\varphi \otimes \pi(x)\eta)(y) &= \langle \varphi, \pi(y) C_\pi^{-1} \pi(x)\eta \rangle \\ &= \langle \varphi, \Delta_G^{1/2} \pi(y)\pi(x) C_\pi^{-1} \eta \rangle \\ &= \Delta_G(x)^{1/2} T(\varphi \otimes \pi)(yx) \\ &= \left(\varrho(T(\varphi \otimes \pi)\right)(y) \quad . \end{aligned}$$

This shows that T intertwines $1 \otimes \overline{\pi}$ with ϱ_G; the left half is obvious.

Now part (c) is obtained by applying T to the ONB $(\eta_i \otimes \eta_j)_{i,j \in I}$. Part (d) follows from (a) and 2.24.

Observe that an ONB as in part (c) of the theorem always exists, since $\mathrm{dom}(C_\pi^{-1})$ is dense; simply apply Gram-Schmidt orthonormalization.

We will next show the announced contribution of π to the Plancherel formula. For this purpose we need the following definition: For $f \in \mathrm{L}^1(G)$ and a representation π, let

$$\pi(f) = \int_G f(x)\pi(x)dx \quad ,$$

where convergence is in the weak sense, which for $f \in \mathrm{L}^1(G)$ is guaranteed. This construction will be seen to yield the **operator-valued Fourier transform**, which is discussed in more detail in Chapter 3. We postpone a more complete discussion of the Fourier transform to that chapter, and only show the following result.

Theorem 2.34. *Let π be a discrete series representation. Denote by P_π the projection onto $\mathrm{L}^2_\pi(G)$. Then, for all $f \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G)$, $\pi(f)C_\pi^{-1}$ extends to a Hilbert-Schmidt operator, with*

$$\|\pi(f)C_\pi^{-1}\| = \|P_\pi(f)\| \quad .$$

Proof. Let an ONB $(\eta_i)_{i \in I} \subset \mathrm{dom}(C_\pi^{-1})$ of $\mathcal{H}_\pi$ be given. Then by part (c) of the previous theorem, we can compute the norm of $P_\pi(f)$ as

$$\begin{aligned} \|P_\pi(f)\|^2 &= \sum_{i,j \in I} \left| \langle f, V_{C_\pi^{-1}\eta_j} \eta_i \rangle \right|^2 \\ &= \sum_{i,j \in I} \left| \int_G f(x) \overline{\langle \eta_i, \pi(x) C_\pi^{-1} \eta_j \rangle} dx \right|^2 \\ &= \sum_{i,j \in I} |\langle \pi(f) C_\pi^{-1} \eta_j, \eta_i \rangle|^2 \quad , \end{aligned}$$

where the last equation used the definition of the weak operator integral. But the last term is just the Hilbert-Schmidt norm of $\pi(f)C_\pi^{-1}$.

Let us now give a few examples for which the discrete series approach cannot work. Clearly, if the underlying group is compact, then every irreducible

representation is in the discrete series: Wavelet coefficients are bounded functions and the Haar measure is finite, hence every wavelet coefficient is trivially in L^2. At the other end of the scale we have the reals: Every irreducible representation is a character, i.e. a group homomorphism $\mathbb{R} \to \mathbb{T}$. Matrix coefficients are constant multiples of that character, hence never square-integrable. The following theorem extends this observation to a larger class. The result is probably folklore, though I am not aware of a reference.

Theorem 2.35. *Let G be a* **SIN-group**, *i.e., every neighborhood of unity contains a conjugation-invariant neighborhood. If G has a discrete series representation, then G is compact.*
In particular, if G is discrete and has a discrete series representation, then G is finite. If G is abelian and has a discrete series representation, then G is compact.

Proof. Note that SIN-groups are unimodular: For any conjugation-invariant neighborhood U of unity, and any $x \in G$ we have $|Ux| = |x^{-1}Ux| = |U|$.

Now let π be a discrete series representation. The first step consists in showing that $\dim(\mathcal{H}_\pi)$ is finite. For this purpose pick a conjugation-invariant neighborhood of unity such that $\pi(\mathbf{1}_U) \neq 0$. The existence of such a neighborhood is seen as follows: Since the characteristic functions of a neighborhood base at unity span a dense subspace of $\mathrm{L}^1(G)$, we would otherwise obtain $\pi(f) = 0$ for all $f \in \mathrm{L}^1(G)$. This would contradict [35, 13.3.1], hence U exists.

We next show that $\pi(\mathbf{1}_U)$ is an intertwining operator. Using conjugation-invariance of U and rightinvariance of Haar measure, we find

$$\begin{aligned}
\langle \phi, \pi(\mathbf{1}_U)\,\pi(y)\eta\rangle &= \int_G \mathbf{1}_U(x)\langle \phi, \pi(x)\pi(y)\eta\rangle d\mu_G(x) \\
&= \int_U \langle \phi, \pi(xy)\eta\rangle d\mu_G(x) \\
&= \int_{Uy} \langle \phi, \pi(x)\eta\rangle d\mu_G(x) \\
&= \int_{yU} \langle \phi, \pi(x)\eta\rangle d\mu_G(x) \\
&= \int_U \langle \phi, \pi(yx)\eta\rangle d\mu_G(x) \\
&= \langle \phi, \pi(y)\pi(\mathbf{1}_U)\,\eta\rangle.
\end{aligned}$$

Hence, by Schur's lemma, $\pi(\mathbf{1}_U)$ is a scalar, which is nonzero by choice of $\mathbf{1}_U$. On the other hand, $\pi(\mathbf{1}_U)$ is Hilbert-Schmidt. Hence $\dim(\mathcal{H}_\pi) < \infty$.

Now assume that G is not compact. Since G is σ-compact and locally compact, there is a sequence $(C_n)_{n\in\mathbb{N}}$ of compact sets in G with the property that $A \subset G$ is compact iff there exists $n \in \mathbb{N}$ such that $A \subset C_n$. Pick a sequence $(x_n)_{n\in\mathbb{N}} \subset G$ with $x_n \in G \setminus C_n$. Then for every compact set $A \subset G$ there exists $n_A \in \mathbb{N}$ with $x_k \notin C$ for all $k \geq n_A$, and this property is inherited

by subsequences. Since $\dim(\mathcal{H}_\pi) < \infty$, we may assume that $\mathcal{H}_\pi = \mathbb{C}^n$ and $\pi(G) \subset U(n)$. Since $U(n)$ is compact, passing to a subsequence allows the assumption that $\pi(x_n) \to S \in U(n)$. Picking any unit vector η, we thus arrive at

$$V_\eta(S\eta)(x_n) = \langle S\eta, \pi(x_n)\eta\rangle \to \|S\eta\|^2 = 1 \quad .$$

On the other hand, $V_\eta(S\eta)$ vanishes at infinity, by 2.19, which yields the desired contradiction.

Note that the somewhat complicated choice of the sequence x_n is only made to avoid that any subsequence is relatively compact.

Let us close with an example that cannot be covered by the results in this section.

Example 2.36. **Dyadic wavelet transform.** This construction was first considered by Mallat and Zhang [92], though without referring to any group structure. We consider the group $H = \mathbb{R} \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts by powers of 2. Hence H is the subgroup of the $ax+b$-group generated by $\mathbb{R} \times \{0\}$ and $(0, 2)$; let's call it the $2^k x + b$-group. We are interested in admissible vectors for the restriction of the quasiregular representation from Example 2.28 to H. An easy adaptation of the calculations there yields

$$\|V_\eta f\|_2^2 = \int_{\mathbb{R}} |\widehat{f}(\omega)|^2 \Phi_\eta(\omega) d\omega \quad , \tag{2.31}$$

where the function Φ_η is given by

$$\Phi_\eta(\omega) = \sum_{n\in\mathbb{Z}} |\widehat{\eta}(2^n\omega)|^2 \quad .$$

For the proof of a more general result we refer the reader to Theorem 5.8 below.

Unlike the previous examples, this representation is not irreducible: Consider a function f such that $\widehat{f}$ is supported in $[1, 1.5]$, and η with $\mathrm{supp}(\widehat{\eta}) \subset [1.5, 2]$. Then $\Phi_\eta = 0$ on the support of f, and thus (2.31) implies $V_\eta f = 0$.

On the other hand, (2.31) yields the admissibility criterion

$$\eta \text{ is admissible} \Leftrightarrow \Phi_\eta \equiv 1 \quad ,$$

and it is easy to construct such functions, say $\widehat{\eta} = \mathbf{1}_{[-2,-1]} + \mathbf{1}_{[1,2]}$. Hence we have found a representation which is not covered by the discrete series case. As a matter of fact, π does not contain irreducible subrepresentation: Suppose that $\mathcal{H} \subset \mathrm{L}^2(\mathbb{R})$ is an irreducible subspace. Since π has admissible vectors, the subrepresentation also does, by Proposition 2.14. Let $\eta \in \mathcal{H}$ be admissible. Then $\pi(G)\eta$ spans $\mathcal{H}$, therefore relation (2.31) yields that

$$\begin{aligned} \mathcal{H}^\perp &= \{\varphi \in \mathrm{L}^2(\mathbb{R}) : V_\eta\varphi = 0\} \\ &= \{\varphi \in \mathrm{L}^2(\mathbb{R}) : |\mathrm{supp}(\widehat{\varphi}) \cap \mathrm{supp}(\Phi_\eta|) = 0\} \quad . \end{aligned}$$

But the orthogonal complement of the latter space is easily computed, yielding

$$\mathcal{H} = \{\varphi \in \mathrm{L}^2(\mathbb{R}) : \mathrm{supp}(\widehat{\varphi}) \subset \mathrm{supp}(\Phi_\eta)\}$$

Recall that supp denotes the measure-theoretic support, and inclusion is understood up to sets of measure zero. Now it is easy to construct two nonzero vectors ξ_1 and ξ_2 such that Φ_{ξ_1} and Φ_{ξ_2} have disjoint supports, both contained in $\mathrm{supp}(\Phi_\eta)$. To see this observe that $\Phi_\eta(2^n\omega) = \Phi_\eta(\omega)$ implies that $\mathrm{supp}(\Phi_\eta) \subset \cup_{n\in\mathbb{Z}} 2^n A$, where $A = [1, 2[\cap \mathrm{supp}(\Phi_\eta)$. In particular, A has positive measure. Hence, if we pick $B_1, B_2 \subset A$ disjoint with positive measure, and let $\widehat{\xi_i} = \mathbf{1}_{B_i}$, we obtain two nonzero functions such that $\mathrm{supp}(\Phi_{\xi_1})$ and $\mathrm{supp}(\Phi_{\xi_2})$ are disjoint. But then (2.31) implies $V_{\xi_1}\xi_2 = 0$, in particular ξ_1 is not cyclic for $\mathcal{H}$.

Thus π has no irreducible subrepresentation, in particular Theorem 2.31 has significance either.

2.5 Selfadjoint Convolution Idempotents and Support Properties

We now continue the discussion of the subspaces of $\mathrm{L}^2(G)$ which arise as image spaces of wavelet transforms. The following notion describes the associated reproducing kernels. After proving this observation, we will draw several consequences from the properties of the reproducing kernel spaces. In particular, we study support properties of wavelet transforms, as well as the existence of admissible vectors for λ_G.

Definition 2.37. *$S \in \mathrm{L}^2(G)$ is called* **(right selfadjoint) convolution idempotent** *if $S = S * S^* = S^*$.*

Convolution idempotents in $\mathrm{L}^1(G)$ have been studied for instance in [59], and generally the existence of such idempotents is a strong restriction on the group. By contrast, we will see that L^2-convolution idempotents exist in abundance. But first the connection between convolution idempotents and generalized wavelet transforms.

Proposition 2.38. *(a) Let $S \in \mathrm{L}^2(G)$ be a convolution idempotent, and denote by $\mathcal{H}$ the closed leftinvariant subspace generated by S, i.e., $\mathcal{H} = \overline{\mathrm{span}(\lambda_G(G)S)}$. Then the projection onto $\mathcal{H}$ is given by right convolution with S, i.e. $\mathcal{H} = \mathrm{L}^2(G) * S = \{g * S : g \in \mathrm{L}^2(G)$. Moreover, if T is another convolution idempotent in $\mathcal{H}$ with $\mathcal{H} = \mathrm{L}^2(G) * T$, then $T = S$.*

(b) S is a selfadjoint convolution idempotent iff there exists a representation π and an admissible $\eta \in \mathcal{H}_\pi$ such that $S = V_\eta\eta$. Consequently, the image spaces of continuous wavelet transforms are precisely the spaces of the form $\mathrm{L}^2(G) * S$.

Proof. For (a) observe that clearly $f = f * S$ holds for all $f \in \mathrm{span}(\lambda_G(G)S)$, as well as $f * S = 0$ for all $f \perp \mathcal{H}$. Hence on a dense subspace $V_S = P_{\mathcal{H}}$, the latter being the projection onto $\mathcal{H}$. Since V_S is closed, the result follows. The uniqueness statement follows from $T = T * S$ and $T = T^*$, hence

$$T = S^* * T^* = S * T = S \quad .$$

The "if"-part of (b) is due to 2.16 (c). For the other direction let π be the restriction of λ_G to $\mathcal{H} = \mathrm{L}^2(G) * S$. Then $f = f * S = V_S f$ for $f \in \mathcal{H}$ shows that the inclusion map is a continuous wavelet transform. The last statement is obvious by now.

The next property will be relevant for sampling theorems, allowing to conclude uniform convergence from L^2-convergence. Note that this observation holds for a larger class of reproducing kernel Hilbert spaces.

Proposition 2.39. *Let $S \in \mathrm{L}^2(G)$ be a selfadjoint convolution idempotent, then for all $f \in \mathcal{H} = \mathrm{L}^2(G) * S$ we have $\|f\|_\infty \leq \|f\|_2 \|S\|_2$.*

Proof. This follows from the Cauchy-Schwarz inequality:

$$|f(x)| = |(f * S^*)(x)| = |\langle f, \lambda_G(x)S\rangle| \leq \|f\|_2 \|S\|_2 \quad .$$

The following proposition gives rise to a somewhat subtle distinction between unimodular and nonunimodular groups: In the unimodular case, any invariant subspace of $\mathrm{L}^2(G)$ which has admissible vectors possesses one in the form of a convolution idempotent. This will not be the case for nonunimodular groups, as will be clarified in Remark 2.43 below.

Proposition 2.40. *Suppose that $\mathcal{H} \subset \mathrm{L}^2(G)$ is closed and leftinvariant. Assume that $\mathcal{H}$ has an admissible vector η with $\eta^* \in \mathrm{L}^2(G)$. Then there exists a right convolution idempotent $S \in \mathcal{H}$ such that $\mathcal{H} = \mathrm{L}^2(G) * S$. In particular, in such a case $\mathcal{H} \subset C_0(G)$.*

Proof. Suppose that an admissible vector $\eta \in \mathcal{H}$ exists, then the projection onto $\mathcal{H}$ is given by $V_{\eta^*} V_\eta$. Since V_η is bounded, 2.19 (d) implies that $S = \eta^* * \eta = V_\eta^* \eta^* \in \mathcal{H}$. Hence, using associativity of convolution, $f = (f * \eta^*) * \eta = f * S$, for all $f \in \mathcal{H}$, whereas $f * (\eta^* * \eta) = 0$, for $f \perp \mathcal{H}$. Therefore $\mathcal{H} = \mathrm{L}^2(G) * S$, and S is the desired selfadjoint convolution idempotent.

We use the proposition to prove the following result due to Rieffel [102].

Proposition 2.41. *Let G be a unimodular group and $\mathcal{H} \subset \mathrm{L}^2(G)$ a closed, leftinvariant subspace. Then $\mathcal{H}$ contains a nonzero selfadjoint convolution idempotent.*

Proof. We start by choosing a nonzero bounded vector $\phi \in \mathcal{H}$: Pick $\phi_0 \in C_c(G)$ with nontrivial projection $P_{\mathcal{H}}\phi$ in $\mathcal{H}$. Then ϕ_0 is a bounded vector by 2.19(b), and Proposition 2.14 implies that $\phi = P_{\mathcal{H}}\phi_0$ is bounded as well. Pick a nonzero spectral projection Q of the selfadjoint operator $U = V_\phi^* V_\phi$ corresponding to a subset in $\mathbb{R}^+$ bounded away from zero. Then U restricted to $\mathcal{K} := Q(\mathcal{H})$ is a topological mapping. It follows that as a mapping $\mathcal{K} \to V_\phi(\mathcal{K})$ the operator $V_\phi = V_{Q\phi}$ is topological. Now Proposition 2.16 (b) ensures the existence of an admissible vector η in $\mathcal{K}$, and then Proposition 2.40 entails that $\mathcal{K}$ is generated by a convolution idempotent. (Note that the last step was the only instance where we used that G is unimodular; otherwise we have no way of checking $\eta^* \in \mathrm{L}^2(G)$.)

Our next aim is to decide for which unimodular groups G the regular representation itself allows admissible vectors. Note that in view of Proposition 2.14, the existence of admissible vectors for λ_G provides admissible vectors for all subrepresentations as well. Hence for these groups the necessary condition $\pi < \lambda_G$ is also sufficient, which yields a complete answer at least to the existence part of our problem. For unimodular groups, this approach turns out to be too bold, except for the somewhat trivial case of discrete groups. The following theorem first appeared in [53], with a somewhat sketchy proof.

Theorem 2.42. *Let G be unimodular. Then λ_G has an admissible vector iff G is discrete.*

Proof. First, if G is discrete, then the indicator function of $\{e_G\}$, where e_G is the neutral element of G, is admissible: The associated wavelet transform is the identity operator. Now assume that λ_G has an admissible vector, then by Proposition 2.40 $\mathrm{L}^2(G)$ consists of bounded continuous functions, in particular $\mathrm{L}^2(G) \subset \mathrm{L}^\infty(G)$. In order to show that this implies discreteness of G, we first show that for G nondiscrete there exist measurable sets of arbitrarily small, positive measure. For suppose otherwise, i.e.,

$$\epsilon := \inf\{|A| : A \subset G \text{ Borel }, |A| > 0\} > 0 \quad .$$

Then the infimum is actually attained: For $n \in \mathbb{N}$ there exists U_n such that $|U_n| < \epsilon + 1/n$. Using regularity, we find $V_n \supset U_n$ open with $|V_n \setminus U_n| < 1/n$. Pick $x_n \in V_n$, then $x_n^{-1} V_n$ is an open neighborhood of unity in G. It follows that letting for arbitrary $N \in \mathbb{N}$,

$$W_N = \bigcap_{n=1}^{N} x_n^{-1} V_n$$

defines a decreasing series of open neighborhoods of unity satisfying

$$\epsilon \leq |W_N| \leq |V_N| < \epsilon + 2/N \quad .$$

But then $U = \bigcap_{n\in\mathbb{N}} U_n$ has measure ϵ.

Next pick $C \subset U \subset V$, C compact, V open with $\mu(V \setminus C) < \epsilon$. C and V exist by regularity of Haar measure. Then $V \setminus C$ is open and has zero measure (by minimality of ϵ), hence $U = V$ is open. If U contains two distinct points, they can be divided by two disjoint open sets contained in U, which contradicts the minimality of $\mu(U)$. Hence U is an open singleton, and G is discrete, contrary to our assumptions.

Now suppose G is nondiscrete. Pick a sequence of Borel sets $U_n \subset G$ with the $0 < |U_n| < n^{-6}$, and define $f = \sum_{n\in\mathbb{N}} n \cdot \mathbf{1}_{U_n}$. Since $\|n\mathbf{1}_{U_n}\|_2 \leq n^{-2}$, the sum converges in $\mathrm{L}^2(G)$, but clearly the limit is not in $\mathrm{L}^\infty(G)$. Hence $\mathrm{L}^2(G) \not\subset \mathrm{L}^\infty(G)$.

Remark 2.43. We note that if G is nonunimodular and type I, Theorem 4.22 provides the existence of an admissible vector for λ_G, even though G is obviously nondiscrete. Hence there is a sharp contrast between the unimodular and the nonunimodular setting.

The admissible vector for λ_G also shows that the assumption $\eta^* \in \mathrm{L}^2(G)$ in Proposition 2.40 cannot be dispensed with: If η is admissible for λ_G and such that $\eta^* \in \mathrm{L}^2(G)$, the Proposition implies $\mathrm{L}^2(G) \subset C_0(G)$. But then the proof of 2.42 can be adapted to show that G is discrete, which contradicts the fact that G is nonunimodular.

As one application of the connection between wavelets and convolution idempotents, we want to prove that wavelet coefficients have noncompact supports, at least for a large class of groups. We will see later on that in the abelian setting these results are related to the qualitative uncertainty property, stating that any L^2-function having support of finite Lebesgue measure both in time and frequency domains must be zero. The analog of that theorem for nonabelian groups will be given in Corollary 4.28 below.

But now let us show the result concerning the supports of wavelet coefficients. It was established by Wilczok both for the one-dimensional continuous wavelet transform and for the windowed Fourier transform [115], but the reasoning can be extended to a much larger class of groups. This was done by Arnal and Ludwig [14], who showed the unimodular version of 2.45, by an adaptation of the proof by Amrein and Berthier [5]. The only contribution of the author is realizing that with minor adjustments the proof goes through in the general case as well. First a small lemma is needed. Recall for this lemma that the **connected component** of a locally compact group is by definition the connected component of the unit element. It is a closed subgroup.

Lemma 2.44. *Let G be a locally compact group with noncompact connected component. Let $V \subset W$ be two measurable subsets of G such that $0 \neq |V|, |W| < \infty$. Then, whenever $|V| > \epsilon > 0$, there exists $x \in G$ such that*

$$|V| - 2\epsilon < |xV \cap W| < |V| - \epsilon \quad . \tag{2.32}$$

Proof. The function $\phi : x \mapsto |xV \cap W| = \langle \mathbf{1}_W, \lambda_G(x)\mathbf{1}_V \rangle$ vanishes at infinity, being a matrix coefficient associated to two L^2-functions (confer Proposition 2.5). Hence, if G_0 is the connected component of G, $\phi(G_0) \subset \mathbb{R}^+$ is a

connected set, and since G_0 is noncompact, the closure of $\phi(G_0)$ contains 0. On the other hand, $\phi(e) = |V|$, hence $\phi(G)$ contains the half-open interval $]0, |V|]$. Hence there exists $x \in G_0$ such that $|V| - 2\epsilon < \phi(x) < |V| - \epsilon$, which is (2.32).

Theorem 2.45. *Let G be a locally compact group with noncompact connected component. Let $f \in \mathrm{L}^2(G)$ and suppose that there exists $S \in \mathrm{L}^2(G)$ such that $f = f * S$ and $S^* \in \mathrm{L}^2(G)$. If f is supported in a set of finite Haar measure, then $f = 0$.*

In particular, if $\eta \in \mathcal{H}_\pi$ is admissible for the representation π, and $V_\eta\varphi$ is supported in a set of finite Haar measure, for some $\varphi \in \mathcal{H}_\pi$, then $\varphi = 0$.

Proof. Suppose that $f \neq 0$ fulfills $f = f * S$, and in addition $C = f^{-1}(\mathbb{C} \setminus \{0\})$ has finite Haar measure. We pick $x_0 = e$, and apply the last lemma to recursively pick $x_1, x_2, \ldots \in G$ satisfying

$$|C| - \frac{1}{2^{k-1}} < |x_k C \cap C_{k-1}| < |C| - \frac{1}{2^k}, \quad k \in \mathbb{N}, \tag{2.33}$$

where $C_{k-1} = \bigcup_{i=0}^{k-1} x_i C \supset C$. Then, if we define $C_\infty = \bigcup_{i\in\mathbb{N}} C_i$, we find that $|C_\infty| < \infty$. Indeed, by (2.33), we have

$$\begin{aligned} |C_{k+1}| &= |C_k \cup x_{k+1}C| = |C_k| + |x_{k+1}C \setminus C_k| \\ &= |C_k| + |x_{k+1}C| - |C_{k+1} \cap x_k C_k| \leq |C_k| + \frac{1}{2^k} \quad , \end{aligned}$$

which entails the desired finiteness.

Now define $\varphi = \mathbf{1}_{C_\infty}$, and consider the operator $K : g \mapsto \varphi \cdot (g * S)$. Writing

$$K(g)(x) = \varphi(x) \int_G g(y) S(y^{-1}x) dy = \int_G g(y)\varphi(x) S(y^{-1}x) dy$$

shows that K is an integral operator with kernel $(x, y) \mapsto \varphi(x)S(y^{-1}x)$. Since

$$\int_G \int_G |\varphi(x)S(y^{-1}x)|^2 dy dx = \mu_G(C_\infty)\|S^*\|_2^2 < \infty \quad ,$$

K is a Hilbert-Schmidt operator [101, VI.23], hence compact.

On the other hand, $(\lambda_G(x_k)f) * S = \lambda_G(x_k)f$ and $\mathrm{supp}(\lambda_G(x_k)f) = x_k C \subset C_\infty$ show that $\lambda_G(x_k)f$ is an eigenvector of K for the eigenvalue 1. In addition,

$$\left| \mathrm{supp}(\lambda_G(x_k)f) \setminus \bigcup_{i=0}^{k-1} \mathrm{supp}(\lambda_G(x_i)f) \right| > 0$$

by the lower inequality of (2.33), hence the $\lambda_G(x_k)f$ are linearly independent. But this means that the eigenspace of K for the eigenvalue 1 is infinite-dimensional, which contradicts the compactness.

The condition concerning the connected component is clearly necessary: If G is a compact group, the supports of the wavelet transforms are trivially of finite measure, yet there are many nontrivial convolution idempotents in $\mathrm{L}^2(G)$, arising for instance from irreducible representations.

Using Theorem 2.45 and Proposition 2.40, we can formulate the following sharpening of Theorem 2.42:

Corollary 2.46. *Let G be a locally compact unimodular group with noncompact connected component. Suppose that $\mathcal{H} \subset \mathrm{L}^2(G)$ is closed and leftinvariant. If $\mathcal{H}$ contains a nonzero function whose support has finite Haar measure, there is no admissible vector for $\mathcal{H}$.*

This concludes the discussion of the relations between continuous wavelet transforms and λ_G. Let us summarize the main results:

- A necessary condition for π to have admissible vectors is that $\pi < \lambda_G$. For nondiscrete unimodular groups, it is not sufficient.
- Embedding π into λ_G and making suitable identifications, we may assume that $\mathcal{H}_\pi = \mathrm{L}^2(G) * S$, with S a selfadjoint convolution idempotent.
- Admissible vectors in $\mathcal{H}_\pi$ are those η for which $f \mapsto f * \eta^*$ defines an isometry on $\mathcal{H}_\pi$. For $\mathcal{H}_\pi = \mathrm{L}^2(G) * S$, these vectors are characterized by $\eta^* * \eta = S$.

Therefore, in order to give a complete classification of representations with admissible vectors, we are faced with the following list of tasks:

T1 Give a concrete description of the closed, leftinvariant subspaces of $\mathrm{L}^2(G)$. In terms of the commuting algebra: Characterize the projections in $\mathrm{VN}_r(G)$.
T2 Given a leftinvariant subspace $\mathcal{H}$, give **admissibility criteria**, i.e. criteria for a right convolution operators $g \mapsto g * f^*$, with $f \in \mathcal{H}$, to be isometric.
T3 Characterize the subspaces $\mathcal{H}$ for which the admissibility conditions can be fulfilled. Equivalently, characterize the right convolution idempotents S.
T4 Given a concrete representation π, decide whether $\pi < \lambda_G$; if yes, make the criteria for **T1** - **T3** explicit.

Remark 2.47. Item **T4** accounts for the fact that the discussion of the problem in $\mathrm{L}^2(G)$, while it makes perfect sense from a representation-theoretic point of view, limits the scope of the characterizations for concrete cases, where the realization of the representation is usually **not** given by left action on some suitable subspace of $\mathrm{L}^2(G)$. Indeed, in the case of the original wavelets arising from the $ax + b$-group, the focus of interest is on the action of that group on the real line by affine transformations, and the corresponding quasiregular representation. First finding an appropriate embedding into $\mathrm{L}^2(G)$ hence

turns out to be a serious obstacle which must be overcome before the results presented here can be applied. However, for type I groups direct integral decompositions provide a systematic way of translating questions of containment of representations to the problem of absolute continuity of measures on the dual, and Chapter 5 contains a large class of examples to which this scheme is applicable.

2.6 Discretized Transforms and Sampling

In this section we want to embed the discretization problem into the L^2-setting, in a way which is complementary to the treatment of continuous transforms. In effect, we will only be able to do this in a satisfactory manner for unimodular groups.

Definition 2.48. *A family $(\eta_x)_{x\in X}$ of vectors in a Hilbert space $\mathcal{H}$ is called a* **frame** *if the associated coefficient operator is a topological embedding into $\ell^2(X)$, i.e., if there exist constants $0 < A \leq B$ (called* **frame constants***) such that*

$$A \sum_{x\in X} |\langle \phi, \eta_x \rangle|^2 \leq \|\phi\|^2 \leq B \sum_{x\in X} |\langle \phi, \eta_x \rangle|^2 \quad .$$

A frame is **tight** *if $A = B$, and* **normalized tight** *if $A = B = 1$.*

In the terminology established in Section 1.1., a normalized tight frame is an admissible coherent state system based on a discrete space X with counting measure. We next formalize the notion of discretization.

Definition 2.49. *Let π be a representation and $\eta \in \mathcal{H}_\pi$ an admissible vector. Given a discrete subset $\Gamma \subset G$, the* **associated discretization of** V_η *is the coefficient operator $V_{\eta,\Gamma} : \mathcal{H}_\pi \to \ell^2(\Gamma)$ associated to the coherent state system $(\pi(\Gamma)\eta)$.*

Remark 2.50. (1) By 2.11 a discretization of V_η gives rise to the discrete reconstruction formula

$$f = \frac{1}{c_\eta} \sum_{\gamma\in\Gamma} V_\eta f(\gamma)\ \pi(\gamma)\eta \quad ,$$

which may be viewed as a Riemann sum version of the continuous reconstruction formula (2.10).
(2) Not all frames of the form $\pi(\Gamma)\eta$ arise as discretizations of continuous transforms, i.e., η need not be admissible. For instance, there exist frames associated to representations which are only square-integrable on a suitable quotient of the group [9]; these representations do not even possess admissible vectors in the sense discussed here.

On the other hand, the admissibility of functions giving rise to wavelet frames has been established in various settings, e.g., [33, 48, 8], which seems

to indicate that under certain topological conditions on the sampling set there is a strong connection between discrete and continuous transforms. See also Proposition 2.60 for the case that the sampling set is a lattice.
(3) We deal with discretization in a rather restrictive way, since only isometries are admitted. By now there exists extensive literature concerning the construction of wavelet frames and related constructions such as Gabor frames, see the monograph [28] and the references therein. We have refrained from discussing the discretization problem in full depth, since our focus is on Plancherel theory and its possible uses in connection with discretization.

The same remark applies to the structure of the sampling set: As the example of multiresolution ONB's of $\mathrm{L}^2(\mathbb{R})$ shows, the sampling set need not be a subgroup, i.e. it is not required to be regular. However, we will mostly concentrate on **regular sampling**, i.e., the sampling set will be a subgroup. It is obvious that the scope of purely group-theoretic techniques for dealing with irregular sampling will be limited, although examples like multiresolution ONB's are intriguing. A possible approach to obtain more general group-theoretic results, even in the irregular sampling case, could consist in adapting the techniques developed in [43] for certain discrete series representations (so-called *integrable representations*) to a more general setting.

Clearly discretization is closely connected to sampling the continuous transform. Hence the following notion arises quite naturally:

Definition 2.51. *Let G be a locally compact group, $\Gamma \subset G$. Let $\mathcal{H} \subset \mathrm{L}^2(G)$ be a leftinvariant closed subspace of $\mathrm{L}^2(G)$ consisting of continuous functions. We call $\mathcal{H}$ a* **sampling space (with respect to Γ)** *if it has the following two properties:*

(S_1) *There exists a constant $c_{\mathcal{H}} > 0$, such that for all $f \in \mathcal{H}$,*

$$\sum_{\gamma \in \Gamma} |f(\gamma)|^2 = c_{\mathcal{H}} \|f\|_2^2 \quad .$$

In other words, the restriction mapping $R_\Gamma : \mathcal{H} \ni f \mapsto (f|_\Gamma) \in \ell^2(\Gamma)$ is a scalar multiple of an isometry.

(S_2) *There exists $S \in \mathcal{H}$ such that every $f \in \mathcal{H}$ has the expansion*

$$f(x) = \sum_{\gamma \in \Gamma} f(\gamma) S(\gamma^{-1} x) \quad , \tag{2.34}$$

with convergence both in L^2 and uniformly.

The function S from condition (S_2) is called **sinc-type function**. *Furthermore, we say that a sampling space* **has the interpolation property** *if R_Γ maps onto all of $\ell^2(\Gamma)$, i.e. any element in $\ell^2(\Gamma)$ can be interpolated by a function in $\mathcal{H}$.*

It will become apparent below that the Heisenberg group allows a variety of sampling spaces associated to lattices, but none that has the interpolation property.

The definition is modelled after the following, prominent example:

*Example 2.52 (***Whittaker, Shannon, Kotel'nikov).** Let $G = \mathbb{R}$, $\Gamma = \mathbb{Z}$ and

$$\mathcal{H} = \{f \in \mathrm{L}^2(\mathbb{R}) : \mathrm{supp}(\widehat{f}) \subset [-0.5, 0.5]\} \quad .$$

Then $\mathcal{H}$ is a sampling subspace with the interpolation property, with associated sinc-type function

$$S(x) = \mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \quad .$$

A short proof of this fact, which uses the notions developed here, can be found in Remark 2.55 (1) below.

Our further discussion requires some basic and widely known facts about tight frames.

Proposition 2.53. *Let $(\eta_i)_{i\in I} \subset \mathcal{H}$ be a tight frame with frame constant c.*

(a) If $\mathcal{H}' \subset \mathcal{H}$ is a closed subspace and $P : \mathcal{H} \to \mathcal{H}'$ is the projection onto $\mathcal{H}'$, then $(P\eta_i)_{i\in I}$ is a tight frame of $\mathcal{H}'$ with frame constant c.
(b) Suppose that $c = 1$. Then $(\eta_i)_{i\in I}$ is an ONB iff $\|\eta_i\| = 1$ for all $i \in I$.
(c) If $\|\eta_i\| = \|\eta_j\|$, for all $i, j \in I$, then $\|\eta_i\|^2 \leq c$.
(d) $(\eta_i)_{i\in I}$ is an orthonormal basis iff $c = 1$ and the coefficient operator is onto.

Proof. Part (a) follows from the fact that on $\mathcal{H}'$ the coefficient operator associated to $(P\eta_i)_{i\in I}$ coincides with the coefficient operator associated to $(\eta_i)_{i\in I}$. The "only-if"-part of (b) is clear. The "if"-part follows from

$$1 = \|\eta_i\|^2 = \sum_{i\in I} |\langle \eta_i, \eta_j\rangle|^2 = 1 + \sum_{i\neq j} |\langle \eta_i, \eta_j\rangle|^2 \quad ,$$

whence $\langle \eta_i, \eta_j\rangle$ vanishes for $i \neq j$. Part (c) follows from a similar argument. The "only if" part of (d) is obvious. For the converse let $\delta_i \in \ell^2(I)$ be the Kronecker-delta at i, and let $T : \mathcal{H} \to \ell^2(I)$ denote the coefficient operator. Then $\langle T^*\delta_i, \varphi\rangle = \langle \delta_i, T\varphi\rangle = \langle \eta_i, \varphi\rangle$ for all $\varphi \in H$ implies $T^*\delta_i = \eta_i$, or $T\eta_i = \delta_i$ (T is by assumption unitary), which is the desired orthonormality relation.

The following proposition notes an elementary connection between sampling and discretization.

Proposition 2.54. *Let $\eta \in \mathcal{H}_\pi$ be admissible, and such that $\pi(\Gamma)\eta$ a tight frame with frame constant c_η. Then $\mathcal{H} = V_\eta(\mathcal{H}_\pi)$ is a sampling space, and $S = \frac{1}{c_\eta} V_\eta\eta$ is the associated sinc-type function for $\mathcal{H}$.*

Proof. Clearly $V_\eta(\mathcal{H}_\pi)$ consists of continuous functions. Using the isometry property of V_η together with the tight frame property of $\pi(\Gamma)\eta$, we obtain for all $f = V_\eta\phi \in \mathcal{H}$

$$\begin{aligned} f = V_\eta\phi &= V_\eta \left(\frac{1}{c_\eta} \sum_{\gamma\in\Gamma} \langle \phi, \pi(\gamma)\eta\rangle \pi(\gamma)\eta \right) \\ &= \sum_{\gamma\in\Gamma} \frac{1}{c_\eta} V_\eta\phi(\gamma) V_\eta(\pi(\gamma)\eta) = \sum_{\gamma\in\Gamma} f(\gamma) S(\gamma^{-1}\cdot) \quad , \end{aligned}$$

with convergence in $\|\cdot\|_2$. Uniform convergence follows from this by Proposition 2.39, since $V_\eta(\mathcal{H}_\pi) = \mathrm{L}^2(G) * V_\eta\eta$.

Remark 2.55. (1) The original sampling theorem in 2.52 can be seen to fit into this setting. If we pick η to be the sinc-function, we find that $V_\eta : \mathcal{H} \to \mathrm{L}^2(\mathbb{R})$ is just the inclusion map, hence η is admissible. Moreover, the Fourier transform of $(\lambda_{\mathbb{R}}(n)\eta)_{n\in\mathbb{Z}}$ yields precisely the Fourier basis of $\mathrm{L}^2([-1/2, 1/2])$. Hence Proposition 2.54 applies.
(2) The proposition shows that various results on the relation between discrete wavelet or Weyl-Heisenberg systems and continuous ones give rise to sampling theorems: For the wavelet case, the underlying group is the $ax + b$-group. A result by Daubechies [33] ensures that every wavelet giving rise to a tight frame is in fact an admissible vector (up to normalization), hence we are precisely in the setting of the proposition. Similarly for discrete Weyl-Heisenberg system, where the underlying group is the reduced Heisenberg group we encountered in Example 2.27. Here admissibility of the window function is trivial. Again the expansion coefficients are sampled values of the windowed Fourier transform, which is the underlying continuous wavelet transform.

The following theorem serves various purposes. First of all it shows that, at least for a unimodular groups, the definition of a sampling space is redundant: Property (S_2) follows from (S_1). Moreover it shows that every sampling space can be obtained from the construction in Proposition 2.54, hence the construction of sampling subspaces and the discretization problem are (in a somewhat abstract sense) equivalent.

Theorem 2.56. *Assume that G is unimodular. Let $\mathcal{H} \subset \mathrm{L}^2(G)$ be a leftinvariant closed space consisting of continuous functions, and assume that it has property (S_1). Then $\mathcal{H}$ is a sampling subspace. More precisely, there exists a unique selfadjoint convolution idempotent S, such that $\frac{1}{c_\mathcal{H}} S$ is the associated sinc-type function, and in addition $\mathcal{H} = \mathrm{L}^2(G) * S$. In particular,*

$$\forall f \in \mathcal{H} \ , \ \forall \gamma \in \Gamma \quad : \quad f(\gamma) = \langle f, \lambda_G(\gamma) S \rangle \quad ,$$

and thus $\lambda_G(\Gamma)S$ is a tight frame for $\mathcal{H}$. $\mathcal{H}$ has arbitrary interpolation iff $\lambda_G(\Gamma)\frac{1}{\sqrt{c_\mathcal{H}}} S$ is an ONB of $\mathcal{H}$.

Proof. Define $S_\gamma = R_\Gamma^*(\delta_\gamma)$, where $\delta_\gamma \in \ell^2(\Gamma)$ is the Kronecker delta at γ. Then $\frac{1}{c_\eta} R_\Gamma^* R_\Gamma = \mathrm{Id}_\mathcal{H}$ shows that

$$f = \sum_{\gamma \in \Gamma} f(\gamma) \frac{1}{c_\mathcal{H}} S_\gamma \quad , \tag{2.35}$$

with convergence in the norm. The orthogonal projection $P : \ell^2(\Gamma) \to R_\Gamma(\mathcal{H})$ is given by $P = \frac{1}{c_\mathcal{H}} R_\Gamma R_\Gamma^*$. Moreover, we compute

$$f(\gamma) = \langle R_\Gamma f, \delta_\gamma \rangle = \langle f, R_\Gamma^* \delta_\gamma \rangle = \langle f, S_\gamma \rangle \quad . \tag{2.36}$$

Next use Zorn's lemma to pick a maximal family $(\mathcal{H}_i)_{i \in I}$ of nontrivial pairwise orthogonal closed subspaces of the form $\mathcal{H}_i = \mathrm{L}^2(G) * S_i$, where the S_i are selfadjoint convolution idempotents in $\mathrm{L}^2(G)$. Then Proposition 2.41 implies that $\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i$; this is the only place where we need that G is unimodular. Since right convolution with S_i is the orthogonal projection onto $\mathcal{H}_i$, equation (2.36) implies for all $f \in \mathcal{H}_i$

$$\langle f, S_\gamma * S_i \rangle = \langle f * S_i, S_\gamma \rangle = \langle f, S_\gamma \rangle = f(\gamma) = \langle f, \lambda_G(\gamma) S_i \rangle \quad .$$

Here the first equality used 2.19(d) and $S_i = S_i^*$, and the second one used $f = f * S_i$. As a consequence, $S_\gamma * S_i = \lambda_G(\gamma) S_i$. For all $\gamma \in \Gamma$,

$$S_\gamma = \sum_{i \in I} S_\gamma * S_i = \sum_{i \in I} \lambda_G(\gamma) S_i \quad , \tag{2.37}$$

with unconditionally converging sums. Since $\lambda_G(\gamma)$ is unitary, we can thus define

$$S = \sum_{i \in I} S_i$$

and conclude from (2.37) that

$$S_\gamma = \lambda(\gamma) S \quad .$$

Moreover, $S_i = S_i^*$ for all $i \in I$ implies $S = S^*$. Finally, for all $f \in \mathcal{H}$,

$$(f * S^*)(x) = \langle f, \lambda_G(x) S \rangle = \langle f, \sum_{i \in I} \lambda_G(x) S_i \rangle = \left(\sum_{i \in I} f * S_i \right)(x) = f(x) \quad .$$

Hence $\mathcal{H} = \mathrm{L}^2(G) * S$, and uniqueness of S was noted in Proposition 2.38. Now (2.35) and (2.37) shows that for $\frac{1}{c_\mathcal{H}} S$ to be the associated sinc-type function, only the uniform convergence of the sampling expansion remains to be shown, which follows from the normconvergence by Proposition 2.39. The statement concerning the tight frame property of $\lambda_G(\Gamma) S$ is now obvious. The last statement follows from Proposition 2.53 (d).

We next collect some additional observations which concerning regular sampling.

Definition 2.57. *A discrete subgroup $\Gamma < G$ is called a* **lattice** *if the quotient G/Γ carries a finite invariant measure. If a lattice exists, G is unimodular. If $A \subset G$ is any Borel transversal mod Γ, which exists by 3.4, we let*

$$\mathrm{covol}(\Gamma) = |A| \quad ,$$

which is independent of the choice of A.

The well-definedness of $\mathrm{covol}(\Gamma)$ is immediate from Weil's integral formula (2.2). The existence of a lattice implies that G is unimodular.

Proposition 2.58. *Let $\Gamma < G$, and suppose that there exists a frame of the form $\pi(\Gamma)\varphi$, with $\varphi \in \mathcal{H}_\pi$. Then there exist $\eta \in \mathcal{H}_\pi$ such that $\pi(\Gamma)\eta$ is a tight frame.*

Proof. First note that up to normalization the tight frame property is precisely admissibility for the restriction of π to Γ. Hence the statement is immediate from 2.16 (b).

Proposition 2.59. *Let G be unimodular and $\Gamma < G$ a discrete subgroup . Assume that $\mathcal{H} \subset \mathrm{L}^2(G)$ is a sampling subspace for Γ. Then Γ is a lattice, with* $\mathrm{covol}(\Gamma) = \frac{1}{c_{\mathcal{H}}}$.

Proof. If $f \in \mathcal{H}$ is any nonzero vector, and A is any measurable transversal, we compute

$$\begin{aligned} \|f\|^2 &= \int_A \sum_{\gamma\in\Gamma} |f(x\gamma)|^2 d\mu_G(x) = \int_A c_{\mathcal{H}} \|\lambda_G(x^{-1})f\|^2 d\mu_G(x) \\ &= \|f\|^2 c_{\mathcal{H}}\ \mathrm{covol}(\Gamma) \quad . \end{aligned}$$

The following general observation was pointed out to the author by K. Gröchenig:

Proposition 2.60. *Let $\Gamma < G$ be a lattice and assume that $\pi(\Gamma)\eta$ is a normalized tight frame. Then $\frac{1}{\mathrm{covol}(\Gamma)}\eta$ is admissible, i.e., the frame is a discretization of a continuous wavelet transform.*

Proof. For arbitrary $\phi \in \mathcal{H}_\pi$ and any measurable transversal A mod Γ

$$\begin{aligned} \|V_\eta\phi\|_2^2 &= \int_A \sum_{\gamma\in\Gamma} |\langle \phi, \pi(x\gamma)\eta\rangle|^2 dx \\ &= \int_A \sum_{\gamma\in\Gamma} |\langle \pi(x)^*\phi, \pi(\gamma)\eta\rangle|^2 dx \\ &= \int_A \|\lambda_G(x^{-1})\phi\|^2 dx \\ &= |A|\|\phi\|^2 \quad . \end{aligned}$$

The next proposition gives a representation-theoretic criterion for sampling spaces with the interpolation property.

Proposition 2.61. *Let $\Gamma < G$ be a lattice. There exists a sampling space $\mathcal{H} \subset \mathrm{L}^2(G)$ with interpolation property with respect to Γ iff there exists a representation π of G such that $\pi|_\Gamma \simeq \lambda_\Gamma$.*

Proof. For the "only-if" part pick $\pi = \lambda_G|_{\mathcal{H}}$. For the "if"-part, if $T : \mathcal{H} \to \ell^2(\Gamma)$ is a unitary equivalence, then $\eta = T^{-1}(\delta_0)$ is such that $\pi(\Gamma)\eta$ is an ONB of $\mathcal{H}$. Now the previous proposition implies that $V_\eta : \mathcal{H}_\pi \to \mathrm{L}^2(G)$ is an isometric embedding, and $\mathcal{H} = V_\eta(\mathcal{H}_\pi)$ has the interpolation property.

At least for unimodular groups, Theorem 2.56 implies that the discretization problem fits quite well into the framework developed for the continuous transforms. In particular the construction of sampling spaces and the discretization of continuous transforms are equivalent problems. We thus find one more task for our list:

T5 Characterize those convolution idempotents $S \in \mathrm{L}^2(G)$ such that in addition $\lambda_G(\Gamma)S$ is a tight frame of $\mathcal{H} = \mathrm{L}^2(G) * S$. For the interpolation property, decide which of these frames are in fact ONB's.

2.7 The Toy Example

In this section, we solve **T1** through **T5** for the group $G = \mathbb{R}$. This example will provide orientation for the further development, since in this setting the solutions turn out to be fairly simple exercises in real Fourier analysis; maybe with the exception of **T4**, which requires more sophisticated arguments.

As we saw in Section 2.3 , every representation of interest can be realized on some translationinvariant subspace on $\mathrm{L}^2(\mathbb{R})$. Moreover, in this setting wavelet transforms are convolution operators, hence it is quite natural to expect that the convolution theorem plays a role. Usually the convolution theorem is given on L^1, however for our purposes the following L^2-version will be more useful:

Theorem 2.62. *Let $f, g \in \mathrm{L}^2(\mathbb{R})$. Then $f * g^* \in \mathrm{L}^2(\mathbb{R})$ iff $\widehat{f}\overline{\widehat{g}} \in \mathrm{L}^2(\widehat{\mathbb{R}})$. In that case, $(f * g^*)^\wedge = \widehat{f}\overline{\widehat{g}}$.*

Proof. The computation

$$\begin{aligned}(f * g^*)(x) &= \langle f, \lambda_{\mathbb{R}}(x) g \rangle \\ &= \langle \widehat{f}, \mathrm{e}^{-2\pi \mathrm{i} x \cdot} \widehat{g} \rangle \\ &= \int_{\mathbb{R}} \widehat{f}(\omega) \overline{\widehat{g}(\omega)} \mathrm{e}^{2\pi \mathrm{i} x \omega} d\omega\end{aligned}$$

shows that the convolution theorem boils down to the "extended Plancherel formula" (2.22) proved in Example 2.28.

Solution to T1

Given a measurable subset $U \subset \widehat{\mathbb{R}}$, define

$$\mathcal{H}_U = \{f \in \mathrm{L}^2(\mathbb{R}) : \mathrm{supp}(\widehat{f}) \subset U\} \quad ,$$

where as usual inclusion is understood up to sets of measure zero. Then it is straightforward to show that $\mathcal{H}_U$ is a closed, translationinvariant subspace of $\mathrm{L}^2(\mathbb{R})$. The following theorem, which is essentially [104, 9.16], shows that this construction of translationinvariant subspaces is exhaustive.

Theorem 2.63. *The mapping*

$$\{U \subset \widehat{\mathbb{R}} \text{ measurable }\}/ \text{ nullsets} \longrightarrow \{\mathcal{H} \subset \mathrm{L}^2(G) \text{ closed, leftinvariant}\} \quad ,$$
$$U \mapsto \mathcal{H}_U$$

is a bijection.

Solution to T2

Theorem 2.64. *Let $f \in \mathcal{H}_U$, for some measurable $U \subset \widehat{\mathbb{R}}$.*

$$f \text{ is admissible} \Longleftrightarrow |\widehat{f}| = 1 \quad \text{a.e. on } U \quad , \tag{2.38}$$
$$f \text{ is a bounded vector} \Longleftrightarrow \widehat{f} \in \mathrm{L}^\infty(U) \quad , \tag{2.39}$$
$$f \text{ is cyclic} \Longleftrightarrow \widehat{f} \neq 0 \quad (\text{almost everywhere on } U) \quad . \tag{2.40}$$

Proof. Given any $f \in \mathrm{L}^2(\mathbb{R})$, denote by $M_{\widehat{f}}$ the multiplication operator with $\widehat{f}$, with the natural domain $\{g \in \mathrm{L}^2(\mathbb{R}) : \widehat{f}g \in \mathrm{L}^2(\mathbb{R})\}$. Then the L^2-convolution theorem implies that V_f and $M_{\widehat{f}}$ are conjugate under the Plancherel transform, including the domains. Now the equivalences follow immediately.

Solution to T3

Theorem 2.65. *$\mathcal{H}_U \subset \mathrm{L}^2(\mathbb{R})$ has admissible vectors iff $|U| < \infty$. $S \in \mathrm{L}^2(\mathbb{R})$ is a convolution idempotent iff $S = S_U := \mathbf{1}_U^\vee$, for $U \subset \widehat{\mathbb{R}}$ with $|U| < \infty$.*

Proof. Any admissible vector $f \in \mathcal{H}_U$ has to fulfill $|\widehat{f}| = 1$ on U, and of course $\widehat{f} \in \mathrm{L}^2(U)$. Thus follows the first condition. The characterization of convolution idempotents is immediate from the convolution theorem and $\widehat{f^*} = \overline{\widehat{f}}$.

Remark 2.66. The arguments for **T1** through **T3** generalize directly to locally compact abelian groups G. Simply replace $\widehat{\mathbb{R}}$ by the character group $\widehat{G}$ and Lebesgue measure by Haar measure on that group. This applies in particular to the cases $G = \mathbb{T}, \mathbb{Z}$.

Support Properties and the Qualitative Uncertainty Principle

Combining Theorems 2.65 and 2.45 yields the *qualitative uncertainty principle* over the reals:

Corollary 2.67. *If* $f \in \mathrm{L}^2(\mathbb{R})$ *fulfills* $|\mathrm{supp}(f)| < \infty$ *and* $|\mathrm{supp}(\widehat{f})| < \infty$, *then* $f = 0$.

Proof. $|\mathrm{supp}(\widehat{f})| < \infty$ implies $f = f * S = V_S f$ for a suitable convolution idempotent in $\mathrm{L}^2(\mathbb{R})$. Hence 2.45 applies.

Solution to T4

The arguments in this subsection were developed together with Keith Taylor. Suppose that $(\pi, \mathcal{H}_\pi)$ is an arbitrary representation of $\mathbb{R}$. A detailed description of such representation is obtainable by a combination of Stone's theorem and the spectral theorem for (possibly unbounded) operators. More precisely, Stone's theorem [101, VIII.8] implies the existence of an *infinitesimal generator*, i.e., a densely defined selfadjoint operator A on $\mathcal{H}_\pi$, such that

$$\pi(t) = \mathrm{e}^{-2\pi \mathrm{i} t A} \quad .$$

In order to understand this formula, we need to recall the spectral theorem [101, Chapter VIII]. Let Π denote the *spectral measure* of A. Then Π is a map from the Borel σ-algebra of $\mathbb{R}$ to the set of orthogonal projections on $\mathcal{H}$ mapping disjoint sets to projections with orthogonal ranges, satisfying $\Pi(A \cap B) = \Pi(A) \circ \Pi(B)$ as well as $\Pi(\mathbb{R}) = \mathrm{Id}_{\mathcal{H}_\pi}$. Π assigns to each pair of vectors $x, y \in \mathrm{dom}(A)$ a complex measure $\Pi_{x,y}$ on $\mathbb{R}$ by letting $\Pi_{x,y}(E) = \langle \Pi(E)x, y\rangle$. The spectral measure describes A via

$$\langle Ax, y\rangle = \int_{\mathbb{R}}^{\oplus} s d\Pi_{x,y}(s) \quad .$$

Th shorthand for this formula we use

$$A = \int_{\mathbb{R}} s d\Pi(s) \quad .$$

The spectral theorem can be viewed as a diagonalization of the selfadjoint operator. In particular, exponentiating amounts to exponentiating the diagonal elements, hence

$$\langle \mathrm{e}^{-2\pi \mathrm{i} t A} x, y\rangle = \int_{\mathbb{R}}^{\oplus} \mathrm{e}^{-2\pi \mathrm{i} s t} d\Pi_{x,y}(s) \quad ,$$

defines the unitary operator $\mathrm{e}^{-2\pi \mathrm{i} t A}$.

We want to decide in terms of the spectral measure whether admissible vectors exist. Recall that admissible vectors are in particular cyclic. The following lemma translates cyclicity into a property of the spectral measure, for

which the construction of the representation π takes a somewhat more concrete form. The result is well-known, see for instance [95, Chapter I, Proposition 7.3]. For the proof of the Lemma we need one more ingredient, namely the *Fourier-Stieltjes* transform on $\mathbb{R}$. For this purpose let $M(\mathbb{R})$ denote the space of complex (finite) measures on $\mathbb{R}$. For every $\mu \in M(\mathbb{R})$, the exponentials are absolutely integrable with respect to $|\mu|$, hence the Fourier-Stieltjes-transform $\widehat{\mu} : \mathbb{R} \to \mathbb{C}$ of μ, given by

$$\widehat{\mu}(\omega) = \int_{\mathbb{R}} \mathrm{e}^{-2\pi \mathrm{i}\omega x} d\mu(x)$$

is well-defined. The crucial property of the Fourier-Stieltjes transform is that it is injective on $M(\mathbb{R})$ [45, 4.17]. As a consequence, to be used repeatedly in the next proof, we conclude for all positive $\nu \in M(\mathbb{R})$ that the exponentials are total in $\mathrm{L}^2(\mathbb{R}, \nu)$: If $0 \neq f \in \mathrm{L}^2(\mathbb{R}, \nu) \subset \mathrm{L}^1(\mathbb{R}, \nu)$, then $f\nu \in M(\mathbb{R})$, and thus the Fourier-Stieltjes transform of $f\nu$ is nonzero. But the Fourier-Stieltjes transform of $f\nu$ is just the family of scalar products of f with the exponentials, taken in $\mathrm{L}^2(\mathbb{R}, \nu)$.

Lemma 2.68. *The following are equivalent:*

(i) π *is cyclic.*

(ii) There exists a positive, finite Borel-measure μ *on* $\mathbb{R}$ *and a unitary map* $T : \mathcal{H}_\pi \to \mathrm{L}^2(\mathbb{R}, \mu)$ *such that, for all* $B \in \mathcal{B}(\mathbb{R})$*, we have*

$$\Pi(B) = T^{-1} P_B T \quad , \tag{2.41}$$

where $P_B : \mathrm{L}^2(\mathbb{R}, \mu) \to \mathrm{L}^2(\mathbb{R}, \mu)$ *denotes multiplication with* $\mathbf{1}_B$*, as well as*

$$\pi(t) = T^{-1} M_t T \quad , \tag{2.42}$$

where M_t *is multiplication with* $\mathrm{e}^{-2\pi \mathrm{i} t \cdot}$*.*

The measure μ *is unique up to equivalence.*

Proof. "$(i) \Rightarrow (ii)$": Let η be a cyclic vector, and define $\mu = \Pi_{\eta,\eta}$. Since μ is finite, all the characters $\mathrm{e}^{-2\pi \mathrm{i} t \cdot}$ are in $\mathrm{L}^2(\mathbb{R}, \mu)$, and the equality

$$\langle \pi(t)\eta, \pi(s)\eta \rangle = \int_{\mathbb{R}} \mathrm{e}^{-2\pi \mathrm{i}(t-s)\omega} d\Pi_{\eta,\eta} = \langle \mathrm{e}^{-2\pi \mathrm{i} t \cdot}, \mathrm{e}^{-2\pi \mathrm{i} s \cdot} \rangle_{\mathrm{L}^2(\mathbb{R},\mu)}$$

implies that the mapping $\pi(t)\eta \mapsto \mathrm{e}^{-2\pi \mathrm{i} t \cdot}$ may be extended linearly to an isometry $T : \mathcal{H}_\pi \to \mathrm{L}^2(\mathbb{R}, \mu)$; here we used that $\pi(\mathbb{R})\eta$ spans a dense subspace of $\mathcal{H}_\pi$. It is onto, since $T(\mathcal{H}_\pi$ is closed and contains the exponentials, which are total in $\mathrm{L}^2(\mathbb{R}, \mu)$.

For equation (2.41), we compute

$$\begin{aligned}\langle T\Pi(B)\pi(s)\eta, \mathrm{e}^{-2\pi \mathrm{i} t\cdot}\rangle &= \langle \Pi(B)\pi(s)\eta, \pi(t)\eta\rangle \\ &= \int_B \mathrm{e}^{-2\pi \mathrm{i}(s-t)\lambda} d\Pi_{\eta,\eta}(\lambda) \\ &= \int_{\mathbb{R}} \mathbf{1}_B(\lambda)\mathrm{e}^{-2\pi \mathrm{i} s\lambda}\mathrm{e}^{-\mathrm{i}t\lambda} d\mu(\lambda) \\ &= \langle P_B T(\pi(s)\eta), \mathrm{e}^{-2\pi \mathrm{i} t\cdot}\rangle \quad ,\end{aligned}$$

and thus $T\Pi(B) = P_B T$: The exponentials are total in $\mathrm{L}^2(\mathbb{R}, \mu)$, and η is cyclic. Equation (2.42) is immediate on $\pi(\mathbb{R})\eta$, and extends to all of $\mathcal{H}$ in the same way.

"$(ii) \Rightarrow (i)$": The totality of the exponentials in $\mathrm{L}^2(\mathbb{R}, \mu)$ implies that the constant function $\mathbf{1}$ is a cyclic vector with respect to the representation $t \mapsto M_t$. Hence $\eta = T^{-1}(\mathbf{1})$ is cyclic for π.

For the uniqueness result we observe that (2.41) clearly implies $\mu(B) = 0$ iff $\Pi(B) = 0$.

Now we can characterize arbitrary representations with admissible vectors. The argument is obtained by sharpening the proof of the lemma.

Theorem 2.69. *π has admissible vectors iff it is cyclic, and in addition the real-valued measure μ associated to its spectral measure by the previous lemma is absolutely continuous with respect to Lebesgue measure, with support in a set of finite Lebesgue measure.*

Proof. Suppose that η is an admissible vector for π. The proof consists essentially in repeating the construction proving the previous lemma and seeing that for admissible vectors η the measure μ is as desired. For this purpose we calculate

$$\begin{aligned}V_\eta\phi(t) &= \langle \phi, \pi(t)\eta\rangle \\ &= \langle \phi, \mathrm{e}^{-2\pi \mathrm{i} t A}\eta\rangle \\ &= \int_{\mathbb{R}} \mathrm{e}^{2\pi \mathrm{i} t\omega} d\Pi_{\eta,\phi}(\omega) \quad ,\end{aligned}$$

which exhibits $V_\eta\phi$ as the Fourier-Stieltjes transform of the measure $\Pi_{\eta,\phi}$. On the other hand, $V_\eta\phi$ is an L^2-function, hence $\Pi_{\eta,\phi}$ turns out to be absolutely continuous with respect to Lebesgue-measure λ. We let

$$T_\eta(\phi) = \frac{d\Pi_{\phi,\eta}}{d\lambda} \quad . \tag{2.43}$$

This sets up an isometry between $\mathcal{H}$ and some subspace of $\mathrm{L}^2(\mathbb{R})$. Let us next show that the projection onto $T(\mathcal{H})$ is given by restriction to an appropriately chosen subset Σ. For this purpose we compute $T_\eta(\Pi(B)\eta)$, for an arbitrary measurable subset B. On the one hand,

$$\langle \Pi(B)\eta, \eta\rangle = \int_{\mathbb{R}} d\mu_{\Pi(B)\eta,\eta}(\lambda) = \int_B d\Pi_{\eta,\eta}(\lambda) = \int_B T(\eta)(\lambda)d\lambda \quad .$$

On the other hand, the isometry property of T_η gives that

$$\begin{aligned}\langle \Pi(B)\eta, \eta\rangle &= \langle T_\eta(\Pi(B)\eta), T_\eta(\eta)\rangle = \int_{\mathbb{R}} T_\eta(\Pi(B))(\lambda)\overline{T_\eta(\eta)(\lambda)}d\lambda \\ &= \int_B T_\eta(\eta)(\lambda)\overline{T_\eta(\eta)(\lambda)}d\lambda \quad .\end{aligned}$$

Since this holds for all subsets B, we obtain that $T_\eta(\eta)(\lambda) = T_\eta(\eta)(\lambda)\overline{T_\eta(\eta)(\lambda)}$ a.e., which entails $T_\eta(\eta)(\lambda) \in \{0,1\}$. On the other hand, $T_\eta(\eta)$ is square-integrable, hence it is the characteristic function of a set Σ of finite Lebesgue measure. To show that $T_\eta(\mathcal{H}) = \mathrm{L}^2(\Sigma, dx)$, we note that $T_\eta(\Pi(B)\eta) = \mathbf{1}_B$, and the characteristic functions span a dense subspace of L^2.

Replacing $\Pi(B)\eta$ by $\mu(B)\phi$ in the above argument gives $T_\eta(\Pi(B)\phi) = \mathbf{1}_B T_\eta(\phi)$. Similarly we obtain $T_\eta(\pi(t)\phi) = \mathrm{e}^{-2\pi \mathrm{i} t\cdot} T_\eta(\phi)$, and we have shown the "only-if"-direction.

For the other direction, we construct an admissible vector for the equivalent representation acting on $\mathrm{L}^2(\Sigma)$, by picking $\eta = \mathbf{1}_\Sigma$. Then for every $\phi \in \mathrm{L}^2(\Sigma) \subset \mathrm{L}^1(\Sigma)$

$$V_\eta\phi(t) = \int_\Sigma \mathrm{e}^{2\pi \mathrm{i} t\lambda}\phi(\lambda)d\lambda = \widehat{\phi}(-t) \quad ,$$

which immediately implies $\|V_\eta\phi\|_{\mathrm{L}^2} = \|\phi\|$.

Remark 2.70. Given a representation π with admissible vector, we have now found two different ways to arrive at an equivalent representation $\widehat{\pi_U}$ acting on $\mathrm{L}^2(U, dx) \subset \mathrm{L}^2(\mathbb{R})$ for some measurable $U \subset \mathbb{R}$ by

$$\left(\widehat{\pi}(t)f\right)(\omega) = \mathrm{e}^{-2\pi \mathrm{i} t\omega} f(\omega) \quad .$$

The first one consists in embedding $\mathcal{H}_\pi$ in $\mathrm{L}^2(G)$ via V_η, and then applying Theorem 2.63 to see that $\pi \simeq \widehat{\pi}_U$ for a suitable U.

A shortcut is described by the mapping T_η constructed in the proof of 2.69. In fact, it is not hard to see that we have the following commutative diagram

$$\begin{array}{ccc} & \overset{V_\eta}{\longrightarrow} & \\ \mathcal{H}_\pi & & \mathcal{H}_U \\ & {\scriptstyle T_\eta}\searrow & \downarrow \mathcal{F} \\ & & \mathrm{L}^2(U, dx) \end{array} \quad .$$

Indeed, observing that $T_\eta(\varphi) \in \mathrm{L}^2(\Sigma) \subset \mathrm{L}^1(\Sigma)$, we can apply the Fourier inversion formula to (2.43), obtaining

$$\begin{aligned} T(\varphi)^\vee(s) &= \int_{\mathbb{R}} \mathrm{e}^{2\pi \mathrm{i} st} T(\varphi)(t)dt = \int_{\mathbb{R}} \mathrm{e}^{2\pi \mathrm{i} st} d\Pi_{\varphi,\eta}(t) = \langle \varphi, \mathrm{e}^{-2\pi \mathrm{i} sA}\eta\rangle \\ &= V_\eta\varphi(s) \quad .\end{aligned}$$

Note also that the admissibility condition obtained in 2.64 coincides with the admissibility condition which is derived in the proof of Theorem 2.69. This is owed to the fact that both are just the admissibility conditions for the representation $\widehat{\pi_U}$, translated to the respective settings by the associated intertwining operators.

Solution to T5

Here we focus on regular sampling, i.e., $\Gamma = \alpha\mathbb{Z}$ is assumed to be a lattice, with $\alpha > 0$. The following theorem can be seen as a refinement of Shannon's sampling theorem. It can be regarded as folklore. Similar results for arbitrary locally compact groups were obtained for instance by Kluvánek [76].

Theorem 2.71. *Let $\mathcal{H}_U$ with associated idempotent $S = \mathbf{1}_U^\vee$. The following are equivalent:*

(a) $\lambda_\mathbb{R}(\Gamma)S$ is a tight frame, i.e., $\mathcal{H}_U$ is a sampling space. The constant $c_\mathcal{H}$ associated to $\mathcal{H}_U$ is $1/\alpha$.
(a') There exists $f \in \mathcal{H}_U$ such that $\lambda_\mathbb{R}(\Gamma)f$ is a frame.
(a") There exists $f \in \mathcal{H}_U$ such that $\lambda_\mathbb{R}(\Gamma)f$ is total.
(b) $|U \cap \frac{1}{\alpha}k + U| = 0$, for all $0 \neq k \in \mathbb{Z}$

Regarding the interpolation property, we have the following equivalent conditions:

(i) $\mathcal{H}_U$ is a sampling space with interpolation property.
(ii) $|U \cap \frac{k}{\alpha} + U| = 0$, for all $0 \neq k \in \mathbb{Z}$, and $|U| = \frac{1}{\alpha}$.

Proof. (a) ⇒ (a') ⇒ (a") is obvious. Now assume that (a") holds. If $\lambda_\mathbb{R}(\Gamma)f$ is total in $\mathcal{H}_U$ then f is a cyclic vector for the translation action of $\lambda_\mathbb{R}$ on $\mathcal{H}_U$. Hence $\widehat{f} \neq 0$ almost everywhere on U, by condition (2.40). Suppose that (b) is violated, i.e., there exists $A \subset U$ measurable with $|A| > 0$ and $\frac{k}{\alpha} + A \subset U$, for a suitable nonzero k. Possibly after passing to a smaller set A we may assume that $|\widehat{f}(\omega + \alpha k)| \geq \epsilon$ for some fixed $\epsilon > 0$ and all $\omega \in A$. Then letting

$$\widehat{g}(\omega) = \begin{cases} 1 & \text{for } \omega \in A \\ -\frac{\widehat{f}(\omega - k/\alpha)}{\widehat{f}(\omega)} & \text{for } \omega \in \frac{k}{\alpha} + A \end{cases}$$

defines an L^2-function supported on $A \cup \frac{k}{\alpha} + A$ satisfying

$$\widehat{f}(\omega)\widehat{g}(\omega) = -\widehat{f}(\omega + \frac{k}{\alpha})\widehat{g}(\omega + \frac{k}{\alpha})$$

on A. Given $\ell \in \mathbb{Z}$, the $1/\alpha$-periodicity of $\exp(2\pi\mathrm{i}\alpha\ell\cdot)$ then implies that

$$\langle g, \lambda_\mathbb{R}(\alpha\ell)f\rangle = \langle \widehat{g}, \mathrm{e}^{-2\pi\mathrm{i}\alpha\ell\cdot}\widehat{f}\rangle = 0 \quad .$$

Hence $\lambda_\mathbb{R}(\alpha\mathbb{Z})f$ is not complete, which gives the desired contradiction.

Now assume (b). Pick a Borel set $V \supset U$ fulfilling (b) and such that in addition, $|B| = 1/\alpha$. In other words, V is a measurable transversal containing U. Then the $1/\alpha$-periodicity of the exponentials and the fact that the exponentials (suitably normalized) form an ONB of $\mathrm{L}^2([0, 1/\alpha])$ implies that they also form an ONB of $\mathrm{L}^2(V)$. Then the restriction of the exponentials to U are the image of an ONB under a projection operator, i.e., still a normalized tight frame with frame constant $1/\alpha$, by 2.53(a). Pulling this back to $\mathcal{H}_U$ gives the desired statement. Moreover, we have also shown (ii) $\Rightarrow$ (i).

For (i) $\Rightarrow$ (ii) we note that the first condition in (ii) follows by (a) $\Rightarrow$ (b), whereas the second one follows from the requirement that $\alpha^{1/2}\|S\|_2 = 1$.

We close the section with the observation that not every continuous transform can be regularly sampled to give a discrete transform.

Example 2.72. There exists a space $\mathcal{H}_U$ which has admissible vectors but does not admit frames of the form $\lambda_{\mathbb{R}}(\alpha\mathbb{Z})f$. For this purpose, pick $U \subset \widehat{\mathbb{R}}$ open, dense and of finite measure, say a union of suitably small open balls around the rationals. Then there exist admissible vectors by Theorem 2.63, but since for $t \in \mathbb{R}$ arbitrary $U \cap t + U$ is open and nonempty, condition (b) of Theorem 2.71 is always violated.

Question: Does there exist a discrete set $\Gamma \subset \mathbb{R}$ and $\eta \in \mathcal{H}_U$ such that $\lambda_{\mathbb{R}}(\Gamma)\eta$ is a frame?

3

The Plancherel Transform for Locally Compact Groups

This chapter provides, for general locally compact groups, the Fourier side that has proved so useful for the treatment of the toy example. We start out by taking a second look at the toy example, this time with a more representation-theoretic slant.

3.1 A Direct Integral View of the Toy Example

This section contains a first, nonrigorous introduction to direct integrals. The purpose is twofold: We use the arguments given in Section 2.7 as an illustration of the direct integral view, and thus obtain an overview of the results needed to extend the reasoning to other groups.

Abstract harmonic analysis as used in this book can be described as the endeavor to completely understand the representation theory of a given group in terms of its irreducible representations. The idea behind this approach is that irreducible representations should serve as building blocks, and that a technique of uniquely decomposing arbitrary into those building blocks would allow to compare different representations, to characterize intertwining operators etc.

For illustration purposes let us first consider a finite group G. Then any given representation π of G can be decomposed into a direct sum of irreducibles,

$$\pi \simeq \bigoplus_{i \in I} \sigma_i \quad . \tag{3.1}$$

If we define the associated *multiplicity function* on $\widehat{G}$ by

$$m_\pi(\sigma) = |\{i \in I : \sigma_i \simeq \sigma\}|$$

as in Remark 2.32, we can rewrite the decomposition as

$$\pi \simeq \bigoplus_{\sigma \in \widehat{G}} m_\pi(\sigma) \cdot \sigma \quad . \tag{3.2}$$

H. Führ: LNM 1863, pp. 59–103, 2005.

One of the advantages of (3.2) over (3.1) is that the invariant subspaces corresponding to $m_\pi(\sigma) \cdot \sigma$ in (3.2) are unique, by contrast to the subspaces carrying the irreducible subrepresentations σ_i in (3.1). Now the significance of (3.2) is that, given a second representation $\widetilde{\pi}$ with associated multiplicity function $m_{\widetilde{\pi}}$ it can be shown that

$$\pi < \widetilde{\pi} \Leftrightarrow \forall \sigma \in \widehat{G} : m_\pi(\sigma) \leq m_{\widetilde{\pi}}(\sigma) \quad . \tag{3.3}$$

In other words, we are able to break the problem of deciding equivalence or containment of arbitrary representations down into classifying and counting the irreducible components.

Thus the basic task in representation theory can be stated in form of the following two questions:

1. *Which* subrepresentations occur?
2. For those irreducible that do occur, what is their *multiplicity*?

While the multiplicity function contains the answers to both questions, the advantage of this formulation is that it generalizes to more general, in particular noncompact, groups. This will become clear if we switch attention to the case $G = \mathbb{R}$. The first thing to realize is that direct *sums* of irreducible representations no longer suffice to describe all representations of $\mathbb{R}$. The simplest example is $\lambda_\mathbb{R}$: We have already observed before, in Theorem 2.35, that noncompact abelian groups do not have a discrete series. Therefore $\lambda_\mathbb{R}$ cannot have irreducible subrepresentations; though in a sense made more transparent below they occur *infinitesimally* in the decomposition.

The correct generalization of the decomposition theory of finite groups, which includes the real case, thus turns out to be provided by **direct integrals**, which we sketch next. Suppose we are given a family (or bundle) of (separable) Hilbert space $(\mathcal{H}_\omega)_{\omega \in \mathbb{R}}$. On each $\mathcal{H}_\omega$, we let $\mathbb{R}$ act unitarily by the character π_ω given by

$$\pi_\omega(s)\eta = \mathrm{e}^{-2\pi \mathrm{i} \omega s}\eta \quad .$$

Hence, $\pi_\omega \simeq m_\pi(\omega) \cdot \chi_\omega$, where $m_\pi(\omega) = \dim(\mathcal{H}_\omega)$. We define **vector fields** as cross-sections to the bundle, i.e. mappings $f : \mathbb{R} \to \bigcup_{\omega \in \mathbb{R}} \mathcal{H}_\omega$ satisfying $f(\omega) \in \mathcal{H}_\omega$. The idea is to let the group act on vector fields f by acting on each fibre $f(\omega)$ by the corresponding representation π_ω. The means of synthesizing the different representation spaces into one Hilbert space is now obtained by introducing a positive measure ν_π on $\mathbb{R}$, defining the **direct integral space**

$$\mathcal{H}_\pi = \int_\mathbb{R}^\oplus \mathcal{H}_\omega d\nu_\pi(\omega)$$

as the set of all vector fields satisfying

$$\|f\|_2^2 := \int_\mathbb{R} \|f(\omega)\|^2 d\nu_\pi(\omega) < \infty \quad .$$

All measure-theoretic detail necessary to make the construction rigorous will be given in the following sections. The only purpose of this section is to provide motivation, and we use direct integrals in a purely formal fashion. The scalar product of two elements $f, g \in \mathcal{H}_\pi$ is given by

$$\langle f, g \rangle = \int_{\mathbb{R}} \langle f(\omega), g(\omega) \rangle_{\mathcal{H}_\omega} \, d\nu_\pi(\omega) \quad .$$

The representation $\pi = \int_{\mathbb{R}}^{\oplus} \pi_\omega d\nu_\pi(\omega)$ acts on $\mathcal{H}_\pi$ by

$$(\pi(t)f)(\omega) = \pi_\omega(t) f(\omega) = \mathrm{e}^{-2\pi \mathrm{i} \omega t} f(\omega) \quad .$$

Since each π_ω is unitary, it is intuitively clear that π is unitary. The measure ν prescribes *which* irreducible representations occur (and *how*), whereas m_π prescribes the multiplicities. Now the central statements of direct integral theory for $\mathbb{R}$ (extending to *groups of type I*) are:

1. Every representation is unitarily equivalent to a direct integral representation, and the associated pair (ν_π, m_π) is unique up to measure-theoretic technicalities: ν is unique up to equivalence, and m_π is unique up to ν_π-nullsets.
2. $\pi < \widetilde{\pi}$ iff ν_π is $\nu_{\widetilde{\pi}}$-absolutely continuous, and $m_\pi \leq m_{\widetilde{\pi}}$, ν_π-almost everywhere.

Let us now reexamine the Fourier arguments used in Section 2.7 in terms of direct integrals.

Containment in the Regular Representation

Straightforward calculation yields that the Plancherel transform on $\mathbb{R}$ has the following intertwining property:

$$(\lambda_{\mathbb{R}}(x)f)^\wedge(\omega) = \chi_\omega(x) \widehat{f}(\omega) \tag{3.4}$$

We consider $\widehat{f}$ as a vector field $(\widehat{f}(\omega))_{\omega \in \mathbb{R}}$, where each $\widehat{f}(\omega) = \mathcal{H}_\omega = \mathbb{C}$. Then the square-integrability property of $\widehat{f}$ yields

$$\widehat{f} \in \int_{\mathbb{R}}^{\oplus} \mathcal{H}_\omega d\omega \quad .$$

Hence the Plancherel formula yields that the Fourier transform is a unitary equivalence intertwining the regular representation of $\mathbb{R}$ with a direct integral representation $\widehat{\pi}$, where $\nu_{\widehat{\pi}}$ is Lebesgue measure and the multiplicity function $\nu_{\widehat{\pi}} \equiv 1$.

We first apply this observation to Theorem 2.69. If π is an arbitrary representation of $\mathbb{R}$, we can write

$$\pi \simeq \int_{\mathbb{R}}^{\oplus} m_\pi(\omega) \cdot \chi_\omega \, d\nu_\pi(\omega) \quad .$$

Direct integrals provide a refinement of the argument in the proof of Theorem 2.69 using the spectral theorem: For a Borel set $B \subset \mathbb{R}$ let $\Pi(B)$ be the projection operator on $\mathcal{H}_\pi$ given by

$$\Pi(B)f(\omega) = \mathbf{1}_B(\omega)f(\omega) \quad .$$

Then we can, at least formally, define an operator A via the relations

$$\langle Ax, y\rangle = \int_{\mathbb{R}} \omega d\Pi_{x,y}(\omega)$$

with $\Pi_{x,y}(B) = \langle \Pi(B)x, y\rangle$. This definition entails $\pi(t) = \mathrm{e}^{-2\pi i t A}$. Hence we have retrieved from the direct integral all the data used in the proof of 2.69. Lemma 2.68 can now be understood as a characterization of cyclic representations in terms of the multiplicity function: A representation of $\mathbb{R}$ is cyclic iff $m_\pi \equiv 1$, at least ν_π-almost everywhere. Moreover 2.69 becomes an instance of the general procedure for checking containment: After we have established that the measure underlying the decomposition of $\lambda_{\mathbb{R}}$ is Lebesgue measure, with multiplicity function $\equiv 1$, every subrepresentation necessarily arises from a Lebesgue-absolutely continuous measure, with multiplicity ≤ 1.

A second instance of this argument applies to Theorem 2.71. Let us consider a representation π_U of $\mathbb{R}$ acting on a space $\mathcal{H}_U$, and its restriction to $\Gamma = \alpha\mathbb{Z}$. We are looking for tight frames of the form $\pi(\Gamma)\eta$, i.e. admissible vectors $\eta \in \mathcal{H}_U$ for the restriction $\pi_U|_\Gamma$.

First we apply similar reasoning to $\mathbb{Z}$ instead of $\mathbb{R}$ to obtain that the Fourier transform on $\ell^2(\mathbb{Z})$ yields a decomposition

$$\lambda_{\mathbb{Z}} \simeq \int_{[0,1]}^{\oplus} \chi_\omega \, d\omega \tag{3.5}$$

where $\chi_\omega(n) = \mathrm{e}^{-2\pi i n\omega}$ denotes obviously an element of $\widehat{\mathbb{Z}}$. Hence the decomposition of $\lambda_{\mathbb{Z}}$ is based on Lebesgue measure on the interval, and multiplicity $\equiv 1$.

Next consider the representation π_0 of $\mathbb{Z}$ defined by $\pi_0(k) = \pi_U(\alpha k)$. Again we need to compute a decomposition into a direct integral of irreducible representations over $\mathbb{Z}$. For this purpose we start out by writing

$$\mathrm{L}^2(\mathbb{R}) \simeq \int_{[0,1/\alpha[}^{\oplus} \ell^2(\mathbb{Z}) d\omega \quad . \tag{3.6}$$

This can be seen by mentally rearranging functions $f \in \mathrm{L}^2(\mathbb{R})$ as families of sequences $((F_\omega(k))_{k\in\mathbb{Z}})_{\omega\in[0,1/\alpha]}$, where

$$F_\omega(k) = f(\omega + k/\alpha) \quad .$$

The map $f \to F$ then implements (3.6). Now let $I_\omega = \{k \in \mathbb{Z} : \omega + k/\alpha \in U\}$, and

$$m_{\pi_0}(\omega) = \sum_{k\in\mathbb{Z}} \mathbf{1}_U(\omega + k/\alpha) = \dim(\ell^2(I_\omega)) \quad .$$

From the definition of I_ω we obtain immediately that the restriction of the operator yielding (3.6) to $\mathrm{L}^2(U)$ provides

$$\mathrm{L}^2(U) \simeq \int_{[0,1/\alpha[}^{\oplus} \ell^2(I_\omega) d\omega \quad . \tag{3.7}$$

Finally, the fact that the restrictions of χ_ω and $\chi_{\omega+t}$ to Γ are equivalent iff t is an integer multiple of $1/\alpha$ yields that (3.7) effects a decomposition into irreducible representations,

$$\pi_0 \simeq \int_{[0,1/\alpha]}^{\oplus} m_{\pi_0}(\omega) \cdot \chi_\omega d\omega \tag{3.8}$$

Now checking $\pi_0 < \lambda_\mathbb{Z}$ amounts to comparing (3.5) with (3.8), yielding the necessary condition

$$m_{\pi_0}(\omega) \leq 1 \quad \text{, for almost every } \omega \in [0, 1/\alpha[\quad ,$$

which is condition (b) from Theorem 2.71. It is also sufficient: Since $\mathbb{Z}$ is discrete, $\lambda_\mathbb{Z}$ has an admissible vector, and thus every subrepresentation does, by 2.14.

Invariant Subspaces and the Decomposition of the Commuting Algebra

The characterization of invariant subspaces given in Theorem 2.63 also has a representation-theoretic background. Since invariant subspaces correspond to projection in $\pi(G)'$, it is natural to extend the decomposition theory to commuting algebras. For this purpose let us take a look at direct sum decompositions, i.e. consider again

$$\pi = \bigoplus_{i\in I} \sigma_i$$

with irreducible σ_i. Write an operator T in the commuting algebra as a block matrix operator $(T_{i,j})_{i,j\in I}$, where $T_{i,j} : \mathcal{H}_{\sigma_i} \to \mathcal{H}_{\sigma_j}$ is the restriction of $P_j \circ T$ to $\mathcal{H}_i$, with P_j denoting the projection onto $\mathcal{H}_{\sigma_j}$. Clearly $T_{i,j}$ is an intertwining operator, therefore $T_{i,j} = 0$ whenever $\sigma_i \not\simeq \sigma_j$, by Schur's lemma. Hence, rewriting again the decomposition of π as

$$\pi = \bigoplus_{\sigma\in\widehat{G}} m_\pi(\sigma) \cdot \sigma \tag{3.9}$$

we observe that every intertwining operator T has block diagonal structure $(T_\sigma)_{\sigma\in\widehat{G}}$, where each T_σ intertwines $m_\pi(\sigma)\cdot\sigma$ with itself. (As a matter of fact, it is this property of the decomposition (3.9) that guarantees the containment criterion (3.3).) Now the correct generalization of the decomposition of intertwining operators to the case $G = \mathbb{R}$ is

3. If $\pi = \int_{\mathbb{R}}^{\oplus} m_\pi(\omega)\cdot\chi_\omega \, d\nu_\pi(\omega)$ is a direct integral of irreducibles, then every intertwining operator T of π has the form

$$(Tf)(\omega) = T_\omega f(\omega) \quad ,$$

where $T_\omega : \mathcal{H}_\omega \to \mathcal{H}_\omega$ are suitably chosen operators. Here $\mathcal{H}_\omega$ denotes the representation space of $m_\pi(\omega)\cdot\chi_\omega$.

An application of this fact to the decomposition of the regular representation yields that every operator commuting with the regular representation decomposes on the Fourier side into operators $T_\omega : \mathbb{C} \to \mathbb{C}$, i.e., functions. Since operator multiplication translates to pointwise multiplication, and taking adjoints to complex conjugation, we find in particular that projections correspond to functions with values in $\{0, 1\}$. Hence the projection onto an invariant subspace is given by multiplication with the characteristic function of a suitable Borel set, which is precisely the statement of Theorem 2.63.

Wavelet Transform, Admissible Vectors and Plancherel Inversion

Finally let us take a closer look at the admissibility conditions. For $U \subset \widehat{\mathbb{R}}$ of finite measure, let $\widehat{\pi}_U$ denote the representation acting on $\mathrm{L}^2(U)$ via

$$(\widehat{\pi}(x)F)(\omega) = \mathrm{e}^{-2\pi\mathrm{i}\omega x}F(\omega) \quad ,$$

then obviously

$$\widehat{\pi}_U = \int_U^{\oplus} \chi_\omega d\omega$$

and the representation is conjugate under the Plancherel transform to the restriction of $\lambda_{\mathbb{R}}$ to $\mathcal{H}_U$. Now, picking $\eta \in \mathrm{L}^2(U)$ we obtain

$$V_\eta F(x) = \int_U F(\omega)\eta(\omega)\mathrm{e}^{2\pi\mathrm{i}\omega x} dx \quad ,$$

hence V_η is pointwise multiplication with η, followed by Plancherel inversion. Since the latter is unitary, V_η is an isometry iff $|\eta| = \mathbf{1}_U$; and similar properties of η such as cyclicity, boundedness etc. are just as easily read off.

Summary

Our discussion of the toy example used the following representation-theoretic facts:

1. Representations of $\mathbb{R}$ (and $\mathbb{Z}$) can be decomposed uniquely into irreducibles; containment of representations can be checked by comparing measures and multiplicities.
2. The commuting algebras of the representations are decomposed as well.
3. The Plancherel transform on $\mathrm{L}^2(\mathbb{R})$ yields the decomposition of $\lambda_{\mathbb{R}}$ into irreducible representations. Given a realization of the representation on the Plancherel transform side, admissibility conditions and wavelet transforms are closely related to pointwise Plancherel inversion.

For the remainder of this chapter we will be concerned with deriving and constructing the analogous results and objects for general type I groups. The chief object of interest will be the Plancherel transform. The aim is to construct a unitary operator

$$\mathcal{P} : \mathrm{L}^2(G) \to \int_{\widehat{G}}^{\oplus} \mathcal{B}_2(\mathcal{H}_\sigma)\, d\nu_G(\sigma) \quad , \tag{3.10}$$

having properties analogous to 1. through 3. from above. The following sections are dedicated to elucidating the various objects occurring on the right hand side, and the necessary assumptions to guarantee their existence: Section 3.2 contains the basic notions concerning Borel spaces. In particular, regularity properties of such spaces are crucial to make direct integral theory work. We discuss direct integrals, their construction, as well as existence and uniqueness of the decomposition of unitary representations into irreducible representations, in Section 3.3. Section 3.5 then presents the Plancherel theorem for unimodular groups. Nonunimodular groups require a different treatment; the construction needs to be modified by use of a family of unbounded operators, the **Duflo-Moore-operators**. These operators are intimately related to admissibility conditions, thus a detailed study is indispensable. This requires an excursion into Mackey's theory (Section 3.6), as well as a section containing somewhat technical results concerning operator-valued integral kernels (Section 3.7). Section 3.8 finally presents the Plancherel formula for nonunimodular groups. We include a proof of the isometry part of the Plancherel formula which illustrates the role of Mackey's theory in establishing the Plancherel formula for the group.

The definitions and results presented in the course of this chapter are not intended as a self-contained introduction to direct integral decompositions and the Plancherel transform, but rather as a guided tour of key results. For a more complete exposition we refer the reader to the monographs [35, 45, 82]. However, we intend to give sufficient details and references to guarantee a rigourous use of the results in the following chapters.

3.2 Regularity Properties of Borel Spaces*

Regularity of Borel spaces is an important issue in direct integral theory. Here the central notion is that of a standard Borel space:

Definition 3.1. *A Borel space X is called*

1. **countably separated** *if there exists a countable family $(A_n)_{n\in\mathbb{N}}$ of measurable sets such that for all $x \in X$: $\{x\} = \bigcup\{A_n : x \in A_n\}$.*
2. **standard** *if there exists a Borel isomorphism $X \to Y$, where Y is a complete separable metric space, endowed with the Borel σ-algebra generated by the topology.*
3. **analytic** *if it is countably separated, and in addition there exists a surjective Borel map $Y \to X$, where Y is a standard Borel space.*

That standardness is a rather strong property of Borel spaces can be read off a famous classification result due to Kuratowski: Every standard Borel space is Borel isomorphic either to a countable (discrete) space or to $[0, 1]$ endowed with the usual structure [94, Theorem 2.14]. Furthermore we obtain the following useful result [94, Theorem 2.13].

Theorem 3.2. *Let $\phi : X \to Y$ be an injective measurable mapping, where X is standard and Y is analytic. Then $\phi(X)$ is a Borel subset of Y. In particular, if Y is standard as well and ϕ is bijective, it is a Borel isomorphism.*

As an immediate consequence we see that a subset of a standard Borel space is standard itself iff it is Borel in the larger space.

We will say that a **measure μ is countably separated** (or standard) if there exists a measurable subset $X' \subset X$ with $\mu(X \setminus X') = 0$ which is countably separated (standard).

An interesting and important question is the behavior of Borel structures when passing to quotients. If $\mathcal{R} \subset X \times X$ is an equivalence relation, the **quotient Borel structure** on the quotient space $X/\mathcal{R}$ is the finest σ-algebra making the projection map $X \to X/\mathcal{R}$ measurable. Hence a subset of $X/\mathcal{R}$ is in the quotient σ-algebra iff its preimage under the projection map is in the σ-algebra on X. The key problem here, which is closely related to the type I condition discussed in more detail below, is the fact that $X/\mathcal{R}$ need not be standard even when X is.

Given a locally compact group G and $H < G$, G/H become Borel spaces in the usual way by taking the σ-algebra generated by the open sets. If the group is second countable, it is in fact completely metrizable and thus standard; the same holds for the quotients. As a consequence of these facts and Theorem 3.2 we note that under mild conditions on the given group action, quotients and orbits can be identified measure-theoretically. This fact will be used repeatedly throughout the text to identify quotients and orbit in a measure-theoretic sense.

Proposition 3.3. *Let X be an analytic Borel G-space. Then for all $x \in X$ the canonical map $G \ni g \mapsto g.x$ induces a Borel isomorphism between G/G_x and $G.x$. Hence the latter is a Borel subset of X.*

We will sometimes need measurable cross-sections and transversals. Denoting by $q_H : G \to G/H$ the quotient map, a **cross-section** $\alpha : G/H \to G$ is a right inverse to q, i.e., a map fulfilling $\alpha(xH)H = xH$, for all $x \in G$. A **transversal (mod H)** is a set $A \subset G$ meeting each right coset xH in precisely one point. There is a natural bijection between cross-sections and transversals: Every transversal A induces a cross-section α by letting $\alpha(xH) = y \in A$ with $yH = xH$. Conversely, $\alpha(G/H) \subset G$ is clearly a transversal, if α is a cross-section. The close connection extends to measurability properties: The cross-section is a Borel map iff the associated transversal is a Borel set; this is seen by applying 3.2 to the cross-section. We have the following existence result:

Lemma 3.4. *Let G be second countable and $H < G$ closed. Then there exists measurable cross-sections and transversals of G/H.*

Proof. See e.g. [86, Lemma 1.1].

3.3 Direct Integrals

In this section we discuss the construction of direct integrals of Hilbert spaces and von Neumann algebras.

3.3.1 Direct Integrals of Hilbert Spaces

We use the definition of direct integrals in [45]. Given a family $(\mathcal{H}_x)_{x\in X}$ of separable Hilbert spaces, the **fibre spaces** indexed by elements of a Borel space X, a **vector field** is cross-section, i.e. a mapping $\eta : X \to \bigcup_{x\in X} \mathcal{H}_x$ satisfying $\eta(x) \in \mathcal{H}_x$ for all $x \in X$. Direct integral spaces are spaces of vector fields which are square-integrable. Thus we need to have a notion of measurability. Not surprisingly, measurable vector fields are defined in terms of coordinates with respect to certain bases (or at least total systems) of the fibre spaces. More precisely, a **measurable structure** on $(\mathcal{H}_x)_{x\in X}$ is a family of vector fields $(\mathrm{e}^n_x)_{x\in X}$ $(n \in \mathbb{N})$ having the properties

- For all m, n, the mapping $x \mapsto \langle e^m_x, e^n_x\rangle$ is Borel.
- For all $x \in X$, the family $(e^n_x)_{n\in\mathbb{N}}$ is total in $\mathcal{H}_x$.

If a measurable structure exists, $(\mathcal{H}_x)_{x\in X}$ is a **measurable field of Hilbert spaces**. **Vector fields** $(\eta_x)_{x\in X}$ are defined to be **measurable** if all mappings $x \mapsto \langle \eta_x, e^m_x\rangle$ are Borel. The following Proposition [45, Proposition 7.27], shows that if measurable structures exist, they can come in the form of measurable fields of ONB's. The following proposition rests among other things on the observation that Gram-Schmidt orthonormalization does not affect measurability.

Proposition 3.5. *Let $(\mathcal{H}_x)_{x\in X}$ be a measurable field of Hilbert spaces. Then the mapping $d : x \mapsto \dim(\mathcal{H}_x)$ is a measurable mapping. Moreover, there exists an equivalent* **measurable family of ONB's**, *i.e., a family $(e_x^n)_{x\in X}$ such that for all $x \in X$, the vectors $(e_x^n)_{n=1,\dots d(x)}$ are an ONB of $\mathcal{H}_x$, and in addition, $((e_x^n)_{x\in X})_{n\in\mathbb{N}}$ has the same measurable vector fields as the original measurable structure.*

Given a measure ν on X, a **ν-measurable field of Hilbert spaces** is a field of Hilbert spaces with the property that there exists a ν-nullset Y such that $(\mathcal{H}_x)_{x\in X\setminus Y}$ is a measurable field of Hilbert spaces based on $X \setminus Y$.

Now, given a Borel measure ν and a ν-measurable field $(\mathcal{H}_x)_{x\in X}$ of Hilbert space, the **direct integral space** $\mathcal{H} = \int_X^\oplus \mathcal{H}_x d\nu(X)$ is given as the space of measurable vector fields $(\eta_x)_{x\in X}$ satisfying

$$\int_X \|\eta_x\|^2 d\nu(x) < \infty \quad ,$$

where, as usual, ν-almost everywhere agreeing fields are identified. $\mathcal{H}$ is a Hilbert space with scalar product

$$\langle (\eta_x)_{x\in X}, (\varphi_x)_{x\in X}\rangle = \int_X \langle \eta_x, \varphi_x\rangle d\nu(x) \quad .$$

While the space of measurable vector fields depends on the particular measurable structure, the direct integral space is, up to a canonical unitary equivalence, unique. Also, changes on a ν-nullset clearly do not affect the construction.

A particular case of direct integrals, also arising in the context of induced representations, is of a *constant* field of Hilbert spaces, i.e. we consider

$$\int_X^\oplus \mathcal{H}_x d\nu(x) \quad , \quad \mathcal{H}_x = \mathcal{H} \quad .$$

Clearly the simplest realization of this is obtained by taking a fixed ONB of $\mathcal{H}$ to induce a measurable structure. Thus we see that

$$\int_X^\oplus \mathcal{H}_x d\nu(x) = \mathrm{L}^2(X, \nu; \mathcal{H}) \quad ,$$

the space of all measurable, square-integrable vector fields on X with values in $\mathcal{H}$. Here $\mathcal{H}$ is endowed with the Borel structure induced by the weak topology, which for separable Hilbert spaces coincides with the Borel structure induced by the norm topology. For later use we note the following result which shows that $\mathrm{L}^2(X, \nu; \mathcal{H})$ is closely related to the tensor product $\mathrm{L}^2(X, \nu) \otimes \mathcal{H}$.

Lemma 3.6. *If $(\phi_j)_{j\in J}$ is an ONB of $\mathrm{L}^2(X, \nu)$ and $(e_i)_{i\in I}$ an ONB of $\mathcal{H}$, then $(\phi_j e_i)_{i\in I, j\in J}$ constitutes an ONB of $\mathrm{L}^2(X, \nu; \mathcal{H})$*

Proof. It is clear that the system is an ONS. Hence it remains to show that it is total. For this purpose let $F \in \mathrm{L}^2(X, \nu; \mathcal{H})$ be nonzero. Since

$$\|F(x)\|^2 = \sum_{i \in I} |\langle F(x), e_i \rangle|^2$$

does not vanish identically, it follows that at least one of the functions

$$f_i(x) = \langle F(x), e_i \rangle$$

is nonzero. Then the completeness of the ϕ_j implies that at least for one $j \in J$

$$0 \neq \langle f_i, \phi_j \rangle = \langle \phi_j e_i, F \rangle \quad .$$

For use with the Plancherel formula, we will also need direct integrals of operator spaces. Given a measurable field $(\mathcal{H}_x)_{x \in X}$ of Hilbert spaces, the direct integral space

$$\int_X^{\oplus} \mathcal{B}_2(\mathcal{H}_x) d\nu(x)$$

can be conveniently constructed from the direct integral of the $\mathcal{H}_x$. E.g., a measurable structure $(e_x^n)_{n \in I_x}$ for $\int_X^{\oplus} \mathcal{H}_x dx$ yields a measurable structure $(e_x^n \otimes e_x^m)_{m,n \in I_x}$ for $\int_X^{\oplus} \mathcal{B}_2(\mathcal{H}_x) dx$. Using (1.5), any field T of Hilbert-Schmidt operators can be written as

$$T_x = \left(\sum_{n \in I_x} \varphi_x^n \otimes e_x^n \right) \quad ,$$

where the $(e_x^n)_{n \in I_x}$ are the ONB of $\mathcal{H}_x$ provided by the measurable structure, and it is immediate to check that T is measurable iff $(\varphi_x^n)_{n \in \mathbb{N}}$ is a measurable vector field, for each $n \in \mathbb{N}$. These criteria may be applied to check the following result which will be used below.

Lemma 3.7. *Let a measurable field $(\mathcal{H}_x)_{x \in X}$ of Hilbert spaces be given, and let $(A_x)_{x \in X}$ be a measurable field of Hilbert-Schmidt operators. Let $A_x = U_x |A_x|$ be the polar decomposition of A_x. Then $(U_x)_{x \in X}$ and $(|A_x|)_{x \in X}$ are measurable as well.*

3.3.2 Direct Integrals of von Neumann Algebras

Measurable operator fields are families of (not necessarily bounded) operators $T_x : \mathcal{H}_x \to \mathcal{H}_x$ such that $x \mapsto \langle \eta_x, T_x \varphi_x \rangle$ is measurable for all measurable vector fields η, φ with $\varphi_x \in \mathrm{dom}(T_x)$. For a given measure ν, ν-measurable operator fields are defined in the obvious sense. In order to check measurability, it is sufficient to plug in elements of the measurable structure for η, and for φ if a measurable structure with elements in $\mathrm{dom}(T_x)$ exists.

Given a measurable field of bounded operators T_x such that the operator norms are essentially bounded, then

$$\left(\int_X^{\oplus} T_x d\nu(x)\right)(\eta) = (T_x \eta_x)_{x\in X}$$

defines a bounded operator $\int_X^{\oplus} T_x d\nu(x)$ on $\int_X^{\oplus} \mathcal{H}_x d\nu(x)$. Operators of this type are called **decomposable**; if almost all T_x are scalar, the operator is called **diagonalizable**.

In connection with commuting algebras, the decomposition of von Neumann algebras into direct integrals will be crucial. In the following let $\mathcal{H} = \int_X^{\oplus} \mathcal{H}_x d\nu(x)$ and a family of von Neumann algebras $(\mathcal{A}_x)_{x\in X}$ with $\mathcal{A}_x \subset \mathcal{B}(\mathcal{H}_x)$ be given. As might be expected, there is a measure-theoretic condition involved [36, II.3.2 Definition 1]:

Definition 3.8. $(\mathcal{A}_x)_{x\in X}$ *is* **measurable** *if there exists a countable family of measurable operator fields* $((T_x^n)_{x\in X})_{n\in\mathbb{N}}$ *such that, for* ν*-almost every* $x \in X$, $(T_x^n)_{n\in\mathbb{N}}$ *generates* $\mathcal{A}_x$.

Obvious examples of measurable fields of von Neumann algebras are $\mathcal{A}_x = \mathcal{B}(\mathcal{H}_x)$, or $\mathcal{A}_x = \mathbb{C} \cdot \mathrm{Id}_{\mathcal{H}_x}$.

A direct integral of a measurable field of von Neumann algebras $(\mathcal{A}_x)_{x\in X}$ is constructed in the expected manner, i.e.,

$$\int_X^{\oplus} \mathcal{A}_x d\nu(x)$$

consists of all ν-measurable operator fields $(T_x)_{x\in X}$, essentially bounded in the operator norm, such that $T_x \in \mathcal{A}_x$ (ν-a.e.). Conversely, given a von Neumann algebra

$$\mathcal{A} \subset \mathcal{B}(\int_X^{\oplus} \mathcal{H}_x d\nu(x)) \quad ,$$

we say that $\mathcal{A}$ **decomposes** if

$$\mathcal{A} = \int_X^{\oplus} \mathcal{A}_x d\nu(x)$$

for some measurable field of von Neumann algebras. Obviously the algebras of decomposable resp. diagonalizable operators are decomposable von Neumann algebras. In fact, it is obvious from the definition that these are the largest resp. smallest decomposable von Neumann algebras. Note that not every von Neumann algebra consisting of decomposable operators is decomposable; the former condition only means that $\mathcal{A} \subset \int_X^{\oplus} \mathcal{B}(\mathcal{H}_x) d\mu(x)$.

In all cases of interest for us, taking commutants behaves in the expected way ([36, II.3.3 Theorem 4])

Theorem 3.9. *Suppose that X is standard. If*

$$\mathcal{A} = \int_X^{\oplus} \mathcal{A}_x d\nu(x)$$

then

$$\mathcal{A}' = \int_X^{\oplus} \mathcal{A}'_x d\nu(x) \quad .$$

3.4 Direct Integral Decomposition

Now let $\mathcal{H} = \int_X^{\oplus} \mathcal{H}_x d\nu(x)$. For a group G, let a family $(\sigma_x)_{x\in X}$ of unitary representations, with σ_x acting on $\mathcal{H}_x$, be given. The family is called **measurable field of representations** if for all $g \in G$, $(\sigma_x(g))_{x\in X}$ is measurable. The **direct integral representation**

$$\left(\int_X^{\oplus} \sigma_x d\nu(x)\right)(g) = \int_X^{\oplus} \sigma_x(g) d\nu(x)$$

is a unitary representation of G, which is again well-defined and unique up to unitary equivalence.

Recall that our principal aim is direct integral *decomposition*, meaning that we are looking for ways of establishing a unitary equivalence between a given representation and a direct integral, hopefully with useful uniqueness properties. Also, we would like to be able to use the dual $\widehat{G}$ as base space X. As will be seen below, the general case requires that we need to put up with a usually larger (and less transparent) carrier space for direct integrals, the **quasi-dual** $\check{G}$.

Our main source for this section is [35]. Results given there for C^*-algebras are cited here for the group case, without further comment. The justification for this is given by [35, 13.3.5,13.9]. One further piece of vocabulary that is useful for checking the citations from [35] is the word "postliminal" which keeps reappearing in [35]. Recall that we always assume the underlying group to be second countable. Here "postliminal" is synonymous to "type I" [35, 13.9.4], and the latter will be defined in the next subsection.

3.4.1 The Dual and Quasi-Dual of a Locally Compact Group*

Definition 3.10. *Let G be a locally compact group. A* **factor representation** *of G is a representation π fulfilling $\pi(G)' \cap \pi(G)'' = \mathbb{C} \cdot \mathrm{Id}_{\mathcal{H}_\pi}$.*

The suitable equivalence relation between factor representations is quasi-equivalence, as can be seen by the following statement [35, Corollary 5.3.6].

Proposition 3.11. *Let π, σ be factor representations. Then either $\pi \approx \sigma$ or π and σ are disjoint.*

The equivalence classes of factor representations modulo $\approx$ yield the *quasi-dual* $\check{G}$ of G. Clearly, irreducible representations are factor representations. The following proposition notes that there is in fact a canonical embedding $\widehat{G} \hookrightarrow \check{G}$. The first statement follows from [35, 5.3.4], the second statement is [35, 5.3.3]

Proposition 3.12. *Let $\pi = m \cdot \sigma$ with $\sigma \in \widehat{G}$. Then π is a factor representation. Moreover, $\pi \approx n \cdot \sigma'$, for $\sigma' \in \widehat{G}$, iff $\sigma \simeq \sigma'$.*

Factor representations corresponding to elements of $\widehat{G}$ are also called factors **of type I**. In view of this terminology (which originates from the theory of von Neumann algebras), the following definition is natural. However, the full scope of this notion will only become clear later on.

Definition 3.13. *G is called* **type I** *if $\widehat{G} = \check{G}$.*

Remark 3.14. Since the type I property of the underlying group will be a standing assumption below, a few remarks regarding the type I condition are in order.

The class of type I groups contains the abelian and the compact groups. In the connected Lie group case, the exponential solvable (in particular the nilpotent) groups, confer [23, Chap. VI, 2.11], as well as the semisimple ones [63], are known to be type I. Other groups obtained from these classes via group extensions can be dealt with using the Mackey machine; see in particular Theorem 3.39(c). This result allows to check for instance that many of the semidirect product groups arising in mathematical physics, such as the Poincaré and Euclidean motion groups, are type I.

By contrast, a discrete group G is of type I only occurs if it contains a normal abelian subgroup of finite index [111]. Thus the following hardly ever applies to discrete groups, with the lucky exception of the group considered in Section 5.5.

For integration on $\check{G}$ and $\widehat{G}$ we need Borel structures on these spaces. For this purpose let us fix Hilbert spaces $\mathcal{H}_n$ ($n \in \mathbb{N} \cup \{\infty\}$) of Hilbert space dimensions n. Now let

$$Rep_n(G) = \{\pi : \pi \text{ representation of } G \text{ on } \mathcal{H}_n\}$$

and endow it with the smallest σ-algebra for which all the coefficient maps

$$\pi \mapsto \langle \eta, \pi(x)\xi \rangle \quad ,$$

with $x \in G$, $\eta, \xi \in \mathcal{H}_n$, are measurable. Moreover, let

$$Rep(G) = \bigcup_{n \in \mathbb{N} \cup \{\infty\}} Rep_n$$

and endow it with the direct sum structure

$$A \subset Rep(G) \text{ measurable } \Leftrightarrow \forall n \in \mathbb{N} \cup \{\infty\} : A \cap Rep_n \subset Rep_n \text{ measurable } .$$

Let $Fac(G) \subset Rep(G)$ denote the subset of factor representations. Since G is separable by assumption, every factor representation can be realized up to unitary equivalence on some $\mathcal{H}_n$, thus we obtain a canonical bijection

$$Fac(G)/\approx \to \check{G} \ ,$$

which serves to endow $\check{G}$ with the quotient structure, the **Mackey Borel structure**. $\widehat{G} \subset \check{G}$ is endowed with the relative structure. Given that this construction goes through several steps, it is conceivable that measurability questions can become quite tedious. At least we know that the Borel structure separates points, i.e., singletons in $\check{G}$ are Borel [35, 7.2.4]. Moreover, $\widehat{G}$ is a Borel set in $\check{G}$ [35, 7.3.6].

The next lemma shows sheds some more light on the Mackey Borel structure. Its purpose is to identify an arbitrary direct integral, based on some standard Borel space, as a direct integral against a standard Borel measure on $\check{G}$. Statements of this type are often used implicitly, however we have felt the need to clarify this point. This is the purpose of the next statement.

Lemma 3.15. *Assume that $(\sigma_x)_{x\in X}$ is a measurable field of factor representations of the group G, based on the standard Borel space Y. Then $\Phi : X \ni x \mapsto [\sigma_x]_\approx$, where $[\sigma_x]_\approx$ denotes the quasi-equivalence class, is a Borel map into $\check{G}$, endowed with the Mackey Borel structure.*

Proof. In order to relate X to $\check{G}$, let $n(x) = \dim(\mathcal{H}_x)$. Using the measurable field of ONB's $(\varphi_x^n)_{x\in X, n\in\mathbb{N}}$ and fixed ONB's $(\psi_n^i)_{i=1,\dots,n}$ on the $\mathcal{H}_n$ for the $\mathcal{H}_x$, we can set up a field of unitary equivalences $U_x : \mathcal{H}_x \to \mathcal{H}_{n(x)}$, by letting

$$U_x\varphi = \sum_{i\in I_{n(x)}} \langle \varphi, \varphi_x^i\rangle \psi_n^i \ .$$

This yields a mapping $\widetilde{\Phi} : X \to Fac(G)$. The measurability of the ONB's implies that the sets $X_n = \{x \in X : \dim(\mathcal{H}_x) = x\}$ are a measurable partition of X, and it is enough to show that $\widetilde{\Phi}$ is measurable on these. For this property we need to check measurability of the mappings

$$x \mapsto \langle \eta, \widetilde{\Phi}(x)(g)\xi\rangle = \langle U_x^*\eta, \sigma_x(g)U_x^*\xi\rangle \ , \tag{3.11}$$

for all $g \in G$ and $\eta, \xi \in \mathcal{H}_n$. Now it is straightforward to verify that

$$x \mapsto U_x^*\eta \ , \quad x \mapsto U_x^*\xi$$

are measurable vector fields, hence the measurability requirement on the σ_x yields (3.11). Thus $\widetilde{\Phi}$ is measurable.

Φ is obtained by concatenating $\widetilde{\Phi}$ with the quotient map $Fac(G) \to \check{G}$, which is measurable by the choice of Borel structure on $\check{G}$. □

3.4.2 Central Decompositions*

Unfortunately no useful decomposition theory, valid for *all* representations of *all* locally compact groups, can be based on the unitary dual. In this section we will see that $\check{G}$ provides a certain substitute for this. This substitute has at least two shortcomings: The first is the empirical observation that in the cases where $\widehat{G}$ and $\check{G}$ in fact differ, the latter is virtually impossible to compute. Note that already the computation of $\widehat{G}$ for large classes of groups G (e.g., for group extensions, nilpotent or exponential Lie groups) marks milestones in harmonic analysis of the past fifty years. By contrast, to my knowledge there does not seem to be a single group G, not of type I, for which $\check{G}$ is actually known explicitly.

The second drawback is the fact that the uniqueness property of the decomposition has to be taken with caution. A representation π may be decomposable into factors in several essentially different ways. A uniqueness result is only available if we stipulate in addition that the center of the commuting algebra, $\pi(G)' \cap \pi(G)''$, is decomposed as well. This **central decomposition** is the subject of the next theorem.

Theorem 3.16. *Let π be a representation of G.*

(a) **Existence.**
There exist
- *a standard measure ν_π on $\check{G}$,*
- *a ν_π-measurable field of representations $(\sigma_\vartheta)_{\vartheta\in\check{G}}$ with $\sigma \in \vartheta$,*
- *a unitary operator $T : \mathcal{H}_\pi \to \int_{\check{G}}^{\oplus} \mathcal{H}_{\sigma_\vartheta} d\nu_\pi(\vartheta)$*

such that T gives rise to the following equivalences

$$\pi \simeq \int_{\check{G}}^{\oplus} \sigma_\vartheta d\nu_\pi(\vartheta) \tag{3.12}$$

$$\pi(G)' \cap \pi(G)'' \simeq \int_{\check{G}}^{\oplus} \mathbb{C} \cdot \mathrm{Id}_{\mathcal{H}_{\pi_\vartheta}} d\nu_\pi(\vartheta) \quad . \tag{3.13}$$

(b) **Uniqueness.**
Let $\widetilde{\nu}$ be a standard measure on $\check{G}$ and $(\widetilde{\sigma}_\vartheta)_{\vartheta\in\check{G}}$ a measurable field of representatives, effecting decompositions analogous to (3.12) and (3.13), then $\widetilde{\nu}$ is equivalent to ν_π, and there exists a ν_π-almost everywhere defined measurable field of unitary equivalences between σ_ϑ and $\widetilde{\sigma}_\vartheta$.

(c) The operator T from part (a) in addition yields

$$\pi(G)'' \simeq \int_{\check{G}}^{\oplus} \sigma_\vartheta(G)'' d\nu_\pi(\vartheta) \tag{3.14}$$

$$\pi(G)' \simeq \int_{\check{G}}^{\oplus} \sigma_\vartheta(G)' d\nu_\pi(\vartheta) \tag{3.15}$$

Proof. Parts (a) and (b) are [35, 8.4.2]. For (3.14) observe that since G is separable, $\pi(G)''$ is generated, as a von Neumann algebra, by a countable set, and $\pi(G)'' \cap \pi(G)'$ consists of the diagonal operators by (3.13). Then [35, A.88] applies to show (3.14). The statement about the commutant follows from this and Theorem 3.9.

Note that instead of (3.13), one could also require (3.14) to characterize the central decomposition. Indeed, (3.15) follows again, and taking intersections on both sides, observing that $\sigma_\vartheta(G)' \cap \sigma_\vartheta(G)'' = \mathbb{C} \cdot \mathrm{Id}_{\mathcal{H}_{\pi_\vartheta}}$, yields (3.13).

Let us shortly comment on the practical use of the central decomposition. The main problem with this decomposition is that it does not work backwards: If we have a direct integral representation

$$\pi = \int_{\check{G}}^{\oplus} \sigma_\vartheta d\nu(\vartheta) \quad ,$$

the right hand need not be the central decomposition from Theorem 3.16. For this it remains to check that $\pi(G)' \cap \pi(G)''$ coincides with the diagonal operators, which may be rather hard to prove even when it is true. Hence the uniqueness statement is not as easily applicable as one might have hoped. For instance, it can be shown that every representation can be decomposed into a direct integral of irreducible representations (e.g., [45, 7.38]). However, if we have started with a factor representation π (not of type I), it coincides with its central decomposition, with measure concentrated on $\{[\pi]_\approx\} \subset \check{G} \setminus \widehat{G}$, whereas the decomposition into irreducibles is supported in $\widehat{G}$. Thus we have obtained two disjoint supports, and an example of a decomposition which is not central.

Nonetheless, the decomposition of the commuting algebras allows to formulate the following result relating quasicontainment and absolute continuity [35, 8.4.4,8.4.5].

Theorem 3.17. *Let π_1, π_2 be representations, and let ν_{π_1}, ν_{π_2} be the measures underlying the respective central decompositions. Then π_1 is quasi-equivalent to a subrepresentation of π_2 iff ν_{π_1} is absolutely continuous with respect to ν_{π_2}. In particular $\pi_1 \approx \pi_2$ iff ν_{π_1} and ν_{π_2} are equivalent.*

This yields the following characterization of disjointness. Recall that two measures on the same space are called disjoint if every set is a nullset with respect to at least one of them.

Corollary 3.18. *Let π_1, π_2 be representations, and let ν_{π_1}, ν_{π_2} be the measures underlying the respective central decompositions. Then π_1 and π_2 are disjoint iff ν_{π_1} and ν_{π_2} are disjoint measures.*

3.4.3 Type I Representations and Their Decompositions

From the discussion so far one might expect that a decent decomposition theory is obtained by considering those representations whose central decom-

position is supported on $\widehat{G}$. This intuition actually works, and the representations which fall in this category can be characterized in a way which does not directly refer to the central decomposition.

Definition 3.19. *A representation π is called* **multiplicity-free** *if $\pi(G)'$ is commutative. π is called* **type I** *if it is quasi-equivalent to a multiplicity-free representation.*

Multiplicity-free representations owe their name to the following characterization:

Proposition 3.20. *Given a representation π, the following are equivalent:*

(a) π is multiplicity-free.
(b) There exists a standard measure ν_π on $\widehat{G}$ such that

$$\pi \simeq \int_{\widehat{G}}^{\oplus} \sigma \, d\nu_G(\sigma)$$

as well as

$$\pi(G)' \simeq \int_{\widehat{G}}^{\oplus} \mathbb{C} \cdot \mathrm{Id}_{\mathcal{H}_\sigma} \, d\nu_G(\sigma) \quad .$$

In other words, the central decomposition of π is supported on $\widehat{G}$, and each $\sigma \in \widehat{G}$ occurs with multiplicity (at most) one.

A combination of Proposition 3.20 and Theorem 3.17 yields the following criterion.

Theorem 3.21. *Let π be a representation, and ν_π the associated measure of the central decomposition. Then π is type I iff $\nu_\pi(\check{G} \setminus \widehat{G}) = 0$.*

As a corollary, we find that we can split each representation into a well-behaved and a pathological part.

Corollary 3.22. *Let π be a representation. Then $\mathcal{H}_\pi = \mathcal{H}_\pi^I \oplus \mathcal{H}_\pi^{I\perp}$, where the corresponding restrictions π^I and $\pi^{I\perp}$ are disjoint representations, and every invariant subspace carrying a type I subrepresentation is contained in $\mathcal{H}_\pi^I$.*

Proof. Note that the characteristic functions of the measurable sets $\widehat{G}$ and $\check{G} \setminus \widehat{G}$ give rise to projection operators in the commuting algebra, acting as pointwise multiplication on the central decomposition side. This yields the two invariant subspaces. The disjointness statement follows from 3.17.

For type I groups, the decomposition theory takes the expected form. The following theorem relates the type I property to the regularity of the dual.

Theorem 3.23. *For a second countable group G the following are equivalent:*

(a) G is type I.
(b) Every representation of G is type I.

(c) $\widehat{G}$ *is a standard Borel space.*

Proof. $(a) \Rightarrow (b)$ follows from Theorem 3.21, since $\nu(\check{G} \setminus \widehat{G}) = 0$ is trivial by assumption. The converse is clear: If G is not type I, there exist factors which are not of type I. For $(a) \Leftrightarrow (c)$, confer [45, Theorem 7.6].

Theorem 3.24. *Let* π *be a representation of the type I group* G*.*

(a) There exists a standard measure ν_π *on* $\widehat{G}$ *and a multiplicity function* $m_\pi : \widehat{G} \to \mathbb{N}_0 \cup \{\infty\}$ *such that*

$$\pi \simeq \int_{\widehat{G}}^{\oplus} m_\pi(\sigma) \cdot \sigma \; d\nu_\pi(\sigma) \quad . \tag{3.16}$$

(b) The unitary equivalence effecting (3.16) also induces a decomposition of the von Neumann algebras associated to π*, yielding*

$$\pi(G)' \simeq \int_{\widehat{G}}^{\oplus} 1 \otimes \mathcal{B}(\ell^2(I_{m_\pi(\sigma)})) \; d\nu_\pi(\sigma)$$

and

$$\pi(G)'' \simeq \int_{\widehat{G}}^{\oplus} \mathcal{B}(\mathcal{H}_\sigma) \otimes 1 \; d\nu_\pi(\sigma) \quad .$$

Next to the uniqueness statement. The formulation of the following theorem may seem excessively general, involving arbitrary Borel spaces. However, it implies that it is enough to realize that a certain direct integral decomposition, not necessarily based on a subset of the dual, is the unique direct integral decomposition into irreducibles, after a suitable identification of the underlying measure space. The fact that the identification comes in the form of a Borel isomorphism onto a suitable subset of $\widehat{G}$ is part of the statement. Moreover, it shows that *any* decomposition of a representation of a type I group into a direct integral of irreducible representations is in fact the central decomposition, unlike the case we sketched at the end of the previous subsection.

Theorem 3.25. *Let* G *be a type I group. Let* X *be a standard Borel space with measure* ν*, and* $(\sigma_x)_{x \in X}$ *a measurable field of irreducible representations of* G *satisfying* $\sigma_x \not\simeq \sigma_y$ *for* $x \neq y$*. Let* $m : X \to \mathbb{N}_0 \cup \{\infty\}$ *be measurable, and define*

$$\pi = \int_X^{\oplus} m(x) \cdot \sigma_x d\nu(x) \quad .$$

Assume that π *has the central decomposition*

$$\pi \simeq \int_{\widehat{G}}^{\oplus} m_\pi(\sigma) \cdot \sigma \; d\nu_\pi(\sigma) \quad . \tag{3.17}$$

Then the injective mapping $\Phi : x \mapsto [\sigma_x]_\simeq$ is a Borel isomorphism from X onto a Borel subset of $\widehat{G}$, ν_π is equivalent to the image measure of ν under Φ, and $m = m_\pi \circ \Phi$ (nu_π- almost everywhere). Now the uniqueness statement of [35, 8.6.6] yields the rest.

Proof. By assumption, $\Phi : X \hookrightarrow \widehat{G}$ is an injection. Moreover, since the field of representations is measurable, Φ is measurable also, by 3.15. Since it is an injective mapping between the standard spaces X and $\widehat{G}$, 3.3 that it is a Borel isomorphism onto a subspace of $\widehat{G}$, and the image measure is a standard measure on $\widehat{G}$.

As a consequence of the uniqueness theorem, we obtain the containment criterion.

Theorem 3.26. *Let π, ϱ be two type I representations, and let ν_π, m_π, ν_ϱ, m_ϱ denote the associated measures and multiplicity functions. Then $\pi < \varrho$ iff ν_π is ν_ϱ-absolutely continuous, with $m_\pi \leq m_\varrho$, ν_π-almost everywhere.*

Proof. The "if"-part is straightforward: If $m_\pi(\sigma) \leq m_\varrho(\sigma)$, then there is a canonical embedding $\ell^2(I_{m_\pi(\sigma)}) \subset \ell^2(I_{m_\varrho(\sigma)})$. This can be used to construct a canonical isometric intertwining operator

$$T_\sigma : \mathcal{H}_\sigma \otimes \ell^2(I_{m_\pi(\sigma)}) \hookrightarrow \mathcal{H}_\sigma \otimes \ell^2(I_{m_\varrho(\sigma)}) \quad .$$

It is elementary to check that this is a measurable field of intertwining operators.

For the "only-if" part we identify ϱ with its central decomposition, i.e

$$\varrho = \int_{\widehat{G}}^{\oplus} m_\varrho(\sigma) \cdot \sigma \; d\nu_\varrho(\sigma) \quad .$$

Then π is equivalent to a restriction of ϱ to some invariant subspace $\mathcal{K}$. The projection onto $\mathcal{K}$ is in $\varrho(G)'$, thus given by a field $(1 \otimes P_\sigma)_{\sigma \in \widehat{G}}$. This establishes that

$$\pi \simeq \int_{\widehat{G}}^{\oplus} m_\pi(\sigma) \cdot \sigma \; d\nu_\pi(\sigma) \quad ,$$

where $m_\pi(\sigma) = \operatorname{rank}(P_\sigma) \leq m_\varrho(\sigma)$. Moreover, every intertwining operator on $\mathcal{K}$ is decomposed (it intertwines ϱ), thus we have obtained the central decomposition, by the remark following Theorem 3.16.

We observe that the *irreducible subrepresentations* of a representations correspond to the *atoms* of the measure underlying the decomposition of π into irreducibles.

Corollary 3.27. *Let G be a type I group, and π a representation with $\pi \simeq \int_{\widehat{G}}^{\oplus} m_\pi(\sigma) \cdot \sigma \; d\nu_\pi(\sigma)$. Then $\sigma_0 \in \widehat{G}$ is contained in π iff $\nu_\pi(\{\sigma_0\}) > 0$.*

3.4.4 Measure Decompositions and Direct Integrals

The direct integral decompositions obtained in the later chapters are not so much *constructed* but *recognized* as direct integrals. By this we mean that the initial representation space $\mathcal{H}_\pi$ is mapped into a suitable L^2-space, and then the latter is seen to be a direct integral by use of a measure decomposition. This is a geometrically intuitive way of viewing direct integrals, already used by Mackey [87]. We have already encountered a particularly simple example in the discussion in Section 3.1: The unitary equivalence (3.7) was obtained by identifying the function $f \in \mathrm{L}^2(U)$ with the family of sequences $((f(\omega + k/\alpha))_{k\in\mathbb{Z}})_{\omega\in[0,1/\alpha[}$, yielding a unitary equivalence

$$\mathrm{L}^2(U) \simeq \int_{[0,1/\alpha[}^{\oplus} \ell^2(I_\omega)\, d\omega \quad .$$

The identification is unitary because of the simple equality

$$\int_U |f(\omega)|^2 d\omega = \int_{[0,1/\alpha[} \sum_{k\in I_\omega} |f(\omega + k/\alpha)|^2 d\omega$$

which we interpret by an integral first over the set $\omega + I_\alpha$, and then an integral over $[0, 1/\alpha[$. But this is nothing but a **measure decomposition** (or **disintegration**) of Lebesgue measure on U, along the partition given by the sets $\{\omega + k/\alpha : k \in I_\omega\}$, where ω through $[0, 1/\alpha[$.

Before we show that this procedure works generally, let us give criteria for the existence of measure decompositions. Again the regularity conditions for Borel spaces prove useful.

Proposition 3.28. *Let $(X, \mathcal{B}, \mu)$ be a standard measure space. Let $\mathcal{R}$ be an equivalence relation on X, such that $X/\mathcal{R}$ is a countably separated measure space. Then there exists a Borel measure $\overline{\mu}$ on $X/\mathcal{R}$ and on each equivalence class $[x]$ a measure $\nu_{[x]}$, such that for all Borel sets $B \subset X$, the mapping*

$$\phi_B : X/\mathcal{R} \ni [x] \mapsto \nu_A([x] \cap B)$$

is $\overline{\mu}$-measurable on $X/\mathcal{R}$ and fulfills

$$\int_{X/\mathcal{R}} \phi_B([x]) d\overline{\mu}([x]) = \mu(B) \quad .$$

The measure class of $\overline{\mu}$ is unique, and given a fixed choice of $\overline{\mu}$, the $\nu_{[x]}$ are uniquely defined ($\overline{\mu}$-almost everywhere).

Proof. See [40, Lemma 4.4]

The following proposition tells how to read certain representations acting on an L^2-space with measure decomposition as direct integrals. The representation-theoretic part was also used in Section 3.1.

Proposition 3.29. *Let π be a representation of G acting on $\mathcal{H}_\pi = \mathrm{L}^2(X, d\mu; \mathcal{H}_0)$, with (X, μ) standard. Suppose that there exists an equivalence relation $\mathcal{R}$ on X such that $X/\mathcal{R}$ is standard, and let $(\nu_{[x]})_{[x]\in X/\mathcal{R}}$ and $\overline{\mu}$ be a measure decomposition as in Proposition 3.28.*

(a) Let $f \in \mathrm{L}^2(X, d\mu; \mathcal{H}_0)$, then, for $\overline{\mu}$-almost $[x] \in X/\mathcal{R}$, $f|_{[x]} \in \mathrm{L}^2([x], \nu_{[x]}; \mathcal{H}_0)$. This gives rise to a unitary equivalence

$$\mathrm{L}^2(X, d\mu; \mathcal{H}_0) \simeq \int_{X/\mathcal{R}}^{\oplus} \mathrm{L}^2([x], d\nu_{[x]}; \mathcal{H}_0) d\overline{\mu}([x]) \quad .$$

(b) Assume moreover that there exist unitary representations $(\sigma_{[x]})_{[x]\in X/\mathcal{R}}$, with $\sigma_{[x]}$ acting on $\mathrm{L}^2([x], \nu_{[x]}; \mathcal{H}_0)$, such that for all $f \in \mathcal{H}_\pi$ and $\overline{\mu}$-almost all $[x] \in X/G$, we have

$$(\pi(y)f)|_{[x]} = \pi_{[x]}(y)(f|_{[x]}) \quad .$$

Then the operator S from (a) implements a unitary equivalence

$$\pi \simeq \int_{X/\mathcal{R}}^{\oplus} \sigma_{[x]} d\overline{\mu}[x] \quad .$$

Proof. This is a special case of [40, Lemma 4.5]. As a matter of fact, only the equivalence in part (b) is given there, but a closer look at the proof shows that the intertwining operator is obtained by the construction in (a).

3.5 The Plancherel Transform for Unimodular Groups

Plancherel theory describes the decomposition of the regular representation into a direct integral of irreducible representations. We already know that such a decomposition exists if we only assume that G (or, more generally, λ_G) is type I. In this connection, we give a formal definition of the term "Plancherel measure".

Definition 3.30. *Let G be a second countable locally compact group with type I regular representation. A standard measure ν_G on $\widehat{G}$ is called* **Plancherel measure** *if it yields the central decomposition*

$$\lambda_G \simeq \int_{\widehat{G}}^{\oplus} m(\sigma) \cdot \sigma \; d\nu_G(\sigma) \quad .$$

It follows from the results in Section 3.3 that Plancherel measures exist, and that all Plancherel measures are equivalent. If Plancherel theory is only understood in the sense of providing *some* deomposition of the regular representation, the answers given in section 3.3 are thus wholly sufficient. However, the special feature of the Plancherel theorem below is that it gives a concrete

description of the intertwining operator implementing the decomposition, as well as the fibre spaces in the direct integral decomposition. In addition, for unimodular groups it will be possible to single out a particularly useful choice of Plancherel measure from the equivalence class.

The key turns out to be provided by the **operator-valued Fourier transform**: As in the case of the reals, the construction of the Plancherel transform for a type I *unimodular* group proceeds by first defining a Fourier transform on $\mathrm{L}^1(G)$ and extending it (with suitable modifications) to $\mathrm{L}^2(G)$. The **operator-valued Fourier transform** (or **group Fourier transform**) on G maps each $f \in \mathrm{L}^1(G)$ to the family $\mathcal{F}(f) = (\sigma(f))_{\sigma \in \widehat{G}}$ of operators, where each $\sigma(f)$ is defined by the weak operator integral

$$\sigma(f) := \int_G f(x)\sigma(x)dx \quad . \tag{3.18}$$

This defines a field of bounded operators, in fact, a simple estimate establishes

$$\|\sigma(f)\|_\infty \leq \|f\|_1 \quad . \tag{3.19}$$

Indeed,

$$\begin{aligned} |\langle \sigma(f)\eta, \varphi\rangle| &= \left| \int_G f(x)\langle \sigma(x)\eta, \varphi\rangle dx \right| \\ &\leq \int_G |f(x)|\ |\langle \sigma(x)\eta, \varphi\rangle|\ dx \\ &\leq \|f\|_1 \|\eta\|\ \|\varphi\| \quad . \end{aligned}$$

Note that the definition of $\mathcal{F}$ is entirely analogous to the definition of the Fourier transform over the reals, by the usual parametrization of $\widehat{\mathbb{R}}$.

Another simple computation shows the convolution theorem on $\mathrm{L}^1(G)$, namely the fact that convolution becomes operator multiplication on the Fourier side:

$$\sigma(f * g) = \sigma(f) \circ \sigma(g) \quad . \tag{3.20}$$

This is justified by

$$\begin{aligned} \langle \sigma(f * g)\eta, \varphi\rangle &= \int_G (fastg)(x)\langle \sigma(x)\eta, \varphi\rangle\ dx \\ &= \int_G \int_G f(y)g(y^{-1}x)\langle \sigma(x)\eta, \varphi\rangle\ dy\ dx \\ &= \int_G \int_G f(y) \int_G g(y^{-1}x)\langle \sigma(x)\eta, \varphi\rangle\ dx\ dy \\ &= \int_G \int_G f(y) \int_G g(x)\langle \sigma(y)\sigma(x)\eta, \varphi\rangle\ dx\ dy \\ &= \int_G \int_G f(y) \int_G g(x)\langle \sigma(x)\eta, \sigma(y)^*\varphi\rangle\ dx\ dy \end{aligned}$$

$$= \int_G f(y)\langle \sigma(g)\eta, \sigma(y)^*\varphi\rangle \; dy$$
$$= \langle \sigma(f)(\sigma(g)\eta), \varphi\rangle \quad .$$

A similar computation establishes for $f \in \mathrm{L}^1(G)$ and $g(x) = f(y^{-1}xz)$

$$\sigma(g) = \sigma(y)\sigma(f)\sigma(z)^* \quad , \tag{3.21}$$

if the group is unimodular.

While in the real case there is a natural Hilbert space to contain the images of $\mathrm{L}^2(\mathbb{R}) \cap \mathrm{L}^1(\mathbb{R})$ under the Fourier transform, namely $\mathrm{L}^2(\mathbb{R})$, in the nonabelian case this Hilbert space needs to be constructed. We have spent the previous sections mainly with providing the elements of this construction: The Fourier transform $(\sigma(f))_{\sigma\in\widehat{G}}$ forms a field of bounded operators, indexed by elements of $\widehat{G}$. Furthermore, this field is measurable, as follows from the definition of the Mackey Borel structure on $\widehat{G}$, and it intertwines the action of the twosided regular representation with the canonical action from left and right. It is thus reasonable to expect $\mathcal{H}$ to be the direct integral over the measure space $(\widehat{G}, \nu_G)$, with ν_G the measure underlying the central decomposition of λ_G. The natural choice for the fibres is given by the Hilbert-Schmidt operators $\mathcal{H}_\sigma \to \mathcal{H}_\sigma$.

The following Plancherel theorem contains the rigorous formulation of these facts; see [35, 18.8] for a proof.

Theorem 3.31. *Let G be a unimodular type I group. There exists a Plancherel measure ν_G on $\widehat{G}$ with the following properties:*

(a) For all $f \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G)$, $\sigma(f) \in \mathcal{B}_2(\mathcal{H}_\sigma)$, for ν_G-almost all $\sigma \in \widehat{G}$, and

$$\|f\|_2^2 = \int_{\widehat{G}} \|\sigma(f)\|_2^2 \; d\nu_G(\sigma) \quad . \tag{3.22}$$

(b) $\mathcal{F}$ extends uniquely to a unitary operator

$$\mathcal{P} : \mathrm{L}^2(G) \to \int_{\widehat{G}}^{\oplus} \mathcal{B}_2(\mathcal{H}_\sigma) \; d\nu_G(\sigma) \quad , \tag{3.23}$$

the **Plancherel transform** *of G.*

(c) $\mathcal{P}$ implements the following unitary equivalences

$$\lambda_G \simeq \int_{\widehat{G}}^{\oplus} \sigma \otimes 1 \; d\nu_G(\sigma) \tag{3.24}$$

$$\varrho_G \simeq \int_{\widehat{G}}^{\oplus} 1 \otimes \overline{\sigma} \; d\nu_G(\sigma) \tag{3.25}$$

$$VN_l(G) \simeq \int_{\widehat{G}}^{\oplus} \mathcal{B}(\mathcal{H}_\sigma) \otimes 1 \; d\nu_G(\sigma) \tag{3.26}$$

$$VN_r(G) \simeq \int_{\widehat{G}}^{\oplus} 1 \otimes \mathcal{B}(\mathcal{H}_\sigma) \; d\nu_G(\sigma) \tag{3.27}$$

(d) ν_G satisfies the **inversion formula**

$$f(x) = \int_{\widehat{G}} \operatorname{trace}(\widehat{f}(\sigma)\sigma(x)^*)\ d\nu_G(\sigma) \quad , \tag{3.28}$$

for all f in a suitable dense $\mathcal{J} \subset \mathrm{L}^2(G)$. Either of (3.22) and (3.28) fixes ν_G uniquely.

We call the choice of ν_G fulfilling (3.22) and (3.28) **the canonical choice of Plancherel measure** for G. One obvious requirement for the space $\mathcal{J}$ is that the integral in (3.28) converges in a suitable sense. A possible choice is the space spanned by $(\mathrm{L}^1(G) \cap \mathrm{L}^2(G)) * (\mathrm{L}^1(G) \cap \mathrm{L}^2(G))$ [45, p. 234]. A substantial part of Chapter 4 will be concerned with the proof of Theorem 4.15 containing a precise description of the scope of (3.28), and a version for nonunimodular groups.

Remark 3.32. In the following, we use $\widehat{\ }$ to denote the Plancherel transform, which should not be confused with the Fourier transform. The direct integral Hilbert space on the right hand side of (3.23) is denoted by $\mathcal{B}_2^{\oplus}$. The scalar product of two elements $A_i \in \mathcal{B}_2^{\oplus}$, $i = 1, 2$, consisting of the measurable fields $(A_i(\sigma) \in \mathcal{B}_2(\mathcal{H}_\sigma))_{\sigma \in \widehat{G}}$, is thus given by

$$\langle A_1, A_2 \rangle_{\mathcal{B}_2^{\oplus}} = \int_{\widehat{G}} \operatorname{trace}\ (A_1(\sigma)A_2(\sigma)^*)\ d\nu_G(\sigma).$$

Given an element $T \in VN_l(G)$, the operator field corresponding to it according to (3.26) is denoted by $(\widehat{T}_\sigma \otimes 1)_{\sigma \in \widehat{G}}$, and we write $T \simeq (\widehat{T}_\sigma \otimes 1)_{\sigma \in \widehat{G}}$. We define the analogous notation $T \simeq (1 \otimes \widehat{T}_\sigma)_{\sigma \in \widehat{G}}$ for $T \in VN_r(G)$.

Remark 3.33. Since we use the toy example $G = \mathbb{R}$ as orientation, it is instructive to see how the reals fit into the general scheme. This can be seen using the following dictionary:

unitary dual	$\longleftrightarrow$	$\widehat{\mathbb{R}} \simeq \mathbb{R}$, via $\mathbb{R} \ni \omega \mapsto \chi_\omega := \mathrm{e}^{-2\pi \mathrm{i} \omega \cdot}$
Fourier transform	$\longleftrightarrow$	$\mathcal{F}(f)(\omega) = \int_{\mathbb{R}} f(x)\chi_\omega(x)dx$
C_σ	$\longleftrightarrow$	$\mathrm{Id}_{\mathbb{C}}$
Plancherel measure	$\longleftrightarrow$	$d\omega$
$\mathcal{B}_2^{\oplus}$	$\longleftrightarrow$	$\mathrm{L}^2(\widehat{\mathbb{R}}) \cong \int_{\widehat{\mathbb{R}}}^{\oplus} \mathbb{C}\ d\omega$
projection field $(\widehat{P}_\sigma)_{\sigma \in \widehat{G}}$	$\longleftrightarrow$	$U \subset \widehat{\mathbb{R}}$

Most of the table is self-explanatory, maybe except for the last two lines. Since the representation spaces of the characters are one-dimensional, the corresponding Hilbert-Schmidt-spaces (or tensor products) are again conveniently identified with $\mathbb{C}$. Measurable complex-valued functions on $\mathbb{R}$ can thus be viewed as measurable fields of operators $\mathbb{C} \to \mathbb{C}$. This allows to view $\mathrm{L}^2(\widehat{\mathbb{R}})$ as direct integral of one-dimensional operator spaces, and to view the characteristic function $\mathbf{1}_U$ as a field of projections

$$\widehat{P}_\sigma = \begin{cases} \mathrm{Id}_{\mathbb{C}} & \text{for } \sigma \in U \\ 0 & \text{otherwise} \end{cases} .$$

In particular, Theorem 2.63 is now easily recognised as a special case of Corollary 4.17.

Finally, we observe that inserting

$$\mathrm{trace}(\widehat{f}(\omega)\chi_\omega(x)^*) = \widehat{f}(\omega)\overline{\chi_\omega(x)} = \widehat{f}(\omega)\mathrm{e}^{2\pi \mathrm{i}\omega x}$$

into the inversion formula (3.28) yields the well-known Fourier inversion formula

$$f(x) = \int_{\mathbb{R}} \widehat{f}(\omega)\mathrm{e}^{2\pi\mathrm{i}\omega x} d\omega \quad .$$

Note that this inversion formula holds for all $f \in \mathrm{L}^2(\mathbb{R})$ with $\widehat{f} \in \mathrm{L}^1(\mathbb{R})$, which is the natural domain for inversion formulae. The analog of this space for nonabelian groups is described in Chapter 4.

The following remark explains how the discrete series representations fit into the Plancherel picture.

Remark 3.34. By 3.27 it is clear that $\pi \in \widehat{G}$ is in the discrete series iff $\nu_G(\{\pi\}) > 0$. Let $\eta, \varphi, \eta', \varphi' \in \mathcal{H}_\pi$ and consider the rank-one operators $A_\pi = \varphi \otimes \eta$ and $B_\pi = \varphi' \otimes \eta'$. Since $\{\pi\}$ is Borel, we can extend these trivially to measurable operator fields $(A_\sigma)_{\sigma\in\widehat{G}} \in \mathcal{B}_2^\oplus$ and $(B_\sigma)_{\sigma\in\widehat{G}}$. Let $a, b \in \mathrm{L}^2(G)$ denote the respective preimages under the Plancherel transform. Assuming that the inversion formula (3.28) is applicable, we find that

$$a(x) = \int_{\widehat{G}} \mathrm{trace}(A_\sigma \sigma(x)^*)\, d\nu_G(\sigma) = \nu_G(\{\pi\})\langle \varphi, \pi(x)\eta\rangle = \nu_G(\{\pi\})V_\eta\varphi(x) \quad ,$$

and likewise for b. Note that here the absolute convergence of the integral is trivially guaranteed; this fact allows us to justify the use of Plancherel inversion by referring to Theorem 4.15 below. Since Plancherel transform is unitary, we obtain

$$\begin{aligned} \langle V_\eta\varphi, V_{\eta'}\varphi'\rangle &= \nu_G(\{\pi\})^{-2}\langle a, a'\rangle \\ &= \nu_G(\{\pi\})^{-2}\int_{\widehat{G}} \mathrm{trace}(A_\sigma B_\sigma^*) d\nu_G(\sigma) \\ &= \nu_G(\{\pi\})^{-2}\langle \varphi, \varphi'\rangle\langle \eta', \eta\rangle \quad . \end{aligned}$$

A comparison of this equation with the orthogonality relation (2.15), observing in addition that $C_\pi = c_\pi \cdot \mathrm{Id}_{\mathcal{H}_\pi}$, yields that

$$c_\pi = \nu_G(\{\pi\})^{-1} \quad .$$

In particular, computing the constant c_π, which is the only step necessary for establishing admissibility conditions, is equivalent to computing the canonical Plancherel measure of the singleton set $\{\pi\}$.

3.6 The Mackey Machine*

In this book the Mackey machine is the chief technical tool for the computation of duals and Plancherel formulae. It serves two purposes: On the one hand we need Mackey's theory to obtain an explicit description of the Plancherel measure of a nonunimodular group, using the unimodular normal subgroup $N = \mathrm{Ker}(\Delta)$. Moreover, the examples given in Chapter 5 all rely on computation of Plancherel measure for group extensions, which is based on Mackey's theory. We therefore include a short survey, which will also allow to fix some notation. For a more complete exposition we refer the reader to [45].

Let G be a locally compact group, and $N \lhd G$ closed. Then G operates on N by conjugation, i.e.

$$x.n = xnx^{-1}$$

is a left action of G on N by automorphisms. This gives rise to a left action, the **dual action** of G on $\widehat{N}$, if we let $(x.\sigma)$ denote the representation acting on $\mathcal{H}_\sigma$ by

$$(x.\sigma)(n) = \sigma(x^{-1}.n) = \sigma(x^{-1}nx) \quad .$$

Clearly N acts trivially on $\widehat{N}$, since $\sigma(n)$ acts as intertwining operator between σ and $n.\sigma$, if $n \in N$. Hence we obtain an action of the factor group $H = G/N$ on $\widehat{G}$, which is also called the dual action. Given $\gamma \in \widehat{N}$, the associated stabilizer in G is denoted by G_γ. The **little fixed group** H_γ denotes the quotient G_γ/N. In the case that G is a semidirect product $G = N \rtimes H$, the H_γ can be identified with subgroups of H. The associated dual orbit is denoted by $\mathcal{O}(\gamma)$. The dual action respects the Mackey Borel structure, more precisely, $\widehat{N}$ is a Borel G-space. In particular, whenever N is type I, $\widehat{N}$ is standard, and then 3.3 implies that $\mathcal{O}(\gamma) \cong G/G_\gamma \cong H/H_\gamma$ as Borel spaces. This identification will be used later on frequently in the discussion of quasi-invariant measures on the dual orbits.

The discussion somewhat simplifies if G is in fact a **semidirect product** $G = N \rtimes H$. This is the case if H can be identified with a closed subgroup of G, i.e., there exists a cross-section $G/N \to G$ which is a continuous group homomorphism. Then we can write elements of G as pairs (n, h), with group law $(n, h)(n', h') = (n\alpha_h(n'), hh')$. Here α_h is the automorphism of N obtained by conjugating with h. In this setting the little fixed group H_γ is also a subgroup of G, and $G_\gamma = N \rtimes H_\gamma$.

The following regularity condition on the dual orbit space turns out to be central for the complete description of the dual of a group extension.

Definition 3.35. *Let $N \lhd G$ and $H = G/N$. N is called* **regularly embedded** *if the quotient space $\widehat{N}/H$ is countably separated.*

We can now state the objectives of Mackey's theory. Given a group extension $N \lhd G$ fulfilling certain additional assumptions, we want to compute $\widehat{G}$ from the following data:

- the dual orbit space $\widehat{N}/H$
- for each $\gamma \in \widehat{N}$, the associated little fixed group H_γ and its dual $\widehat{H_\gamma}$

The procedure which allows to construct all elements of $\widehat{G}$ from these ingredients is **induction**. There exist several realizations of induced representations; for our purposes the most useful one uses cross-sections. For a proof, see for instance [50, Proposition 2.6.3]. A version for right cosets instead of left ones may be found in [75].

Proposition 3.36. *Let G be a locally compact group and $H < G$. Suppose that ϱ is a representation of H. Pick a quasi-invariant measure μ on G/H, and let*

$$\mathcal{H}_\pi := \mathrm{L}^2(G/H, \mu; \mathcal{H}_\sigma) \quad .$$

Given $x \in G$, define the measure μ_x by $\mu_x(E) = \mu(Ex)$. Let $\alpha : G/H \to G$ denote a Borel-measurable cross-section. Letting

$$(\pi(x)f)(yH) := \sqrt{\frac{d\mu_{x^{-1}}}{d\mu}(yH)}\sigma\left(\alpha(yH)^{-1}x\alpha(x^{-1}yH)\right) f(x^{-1}yH), \quad (3.29)$$

defines a unitary representation π with $\pi \simeq \mathrm{Ind}_H^G \sigma$.

We will not give Mackey's theorem in its most general form, since this would necessitate to discuss projective representations. The following definition formulates the technical condition which allows to avoid this.

Definition 3.37. *Let $N \lhd G$, and $\sigma \in \widehat{N}$. If σ can be extended to a unitary representation of G_σ on $\mathcal{H}_\sigma$, it is said to have a* **trivial Mackey obstruction**.

Remark 3.38. Let us explain the notion of Mackey obstruction some more. For simplicity assume that $G = N \rtimes H$, and let $\sigma \in \widehat{N}$. Then for every $h \in H_\sigma$ there exists a unitary operator $T(h) : \mathcal{H}_\sigma \to \mathcal{H}_\sigma$ intertwining σ and $h.\sigma$. Hence $\sigma(\alpha_h(n)) = T(h)\sigma(n)T(h)^*$. By Schur's lemma, $T(h)$ is unique up to an element in $\mathbb{T}$. Now the Mackey obstruction is trivial precisely if we can choose the mapping $h \mapsto T(h)$ to be a homomorphism.

For the purposes of Chapter 5, we introduce an auxiliary notion. Given a semidirect product $G = N \rtimes H$, we say that the Mackey obstruction of $\sigma \in \widehat{N}$ is **particularly trivial** if $(n, h) \mapsto \sigma(n)$ is a representation of $G_\gamma = N \rtimes H_\gamma$. Let us mention two cases where the obstruction is particularly trivial: If N is abelian, then $T(h)$ and $\sigma(n)$ commute, as linear operators on one-dimensional space. Hence we may as well choose $T(h) = 1$. The other, even more obvious, case is $N = G_\gamma$. These two cases cover all examples in Chapter 5.

The following theorem shows how the induction procedure allows to compute the duals of group extensions with regularly embedded normal subgroup.

Theorem 3.39. [Mackey]
Let $N \lhd G$, and assume that all $\sigma \in \widehat{N}$ have trivial Mackey obstruction.

(a) Given $\gamma \in \widehat{N}$, let $\widetilde{\gamma}$ denote a fixed extension of N to G_γ. Given $\varrho \in \widehat{H_\gamma}$, let

$$(\gamma \times \varrho)(y) := \widetilde{\gamma}(y)\varrho(yN) \quad .$$

This defines a representation of G_γ which satisfies $\mathrm{Ind}_{G_\gamma}^{G} \gamma \times \varrho \in \widehat{G}$.

(b) Let $N \lhd G$ be regularly imbedded, with N is type I. Pick a measurable transversal $\Sigma \subset \widehat{N}$ of $\widehat{N}/H$. Then the Mackey correspondence

$$\{(\gamma, \varrho) : \gamma \in \Sigma, \varrho \in \widehat{H_\gamma}\} \to \widehat{G} \tag{3.30}$$

$$(\gamma, \varrho) \mapsto \mathrm{Ind}_{G_\gamma}^{G} \gamma \times \varrho \tag{3.31}$$

is a bijection.

(c) Suppose that N is regularly embedded and type I. Then G is type I iff all the H_γ are.

We remark that N need not be regularly embedded for G to be type I, as witnessed by a counterexample in [17].

Besides allowing to derive all of $\widehat{G}$ by brute force calculation, Mackey's theory also provides a geometric intuition which can be exploited for various purposes. More precisely, the **Mackey correspondence** allows to view $\widehat{G}$ as a *fibred set*

$$\widehat{G} = \bigcup_{\mathcal{O}(\sigma) \in \widehat{N}/H} \{\mathrm{Ind}_{G_\sigma}^{G} \sigma \times \varrho : \varrho \in \widehat{H_\sigma}\} \equiv \bigcup_{\mathcal{O}(\sigma) \in \widehat{N}/H} \{\sigma\} \times \widehat{H_\sigma}$$

with *base space* $\widehat{N}/H$ and *fibre* $\{\mathcal{O}(\sigma)\} \times \widehat{H_\sigma} \equiv \widehat{H_\sigma}$ associated to $\mathcal{O}(\sigma) \in \widehat{G}/H$. This view allows an educated guess concerning the Plancherel measure of G: All fibres carry respective Plancherel measure, which can be "glued together" using a pseudoimage of ν_G under the projection map $\widehat{N} \to \widehat{N}/H$. The following theorem shows that this construction works. Note that it is somewhat more general than the version of Mackey's theorem given above, since 3.40 only requires good behavior of the dual action on a conull set. An even more general version can be found in [75, I, Theorem 10.2]. The procedure for obtaining the correct normalization of the canonical Plancherel measure for the case that G is unimodular follows from [82, Ch. III,Theorem 6].

Theorem 3.40. [Kleppner/Lipsman]
Let G and $N \lhd G$ be type I. Suppose that N is unimodular and regularly embedded, and that there is a G-invariant Borel conull subset $\Sigma \subset N$ such that all $\sigma \in \Sigma$ have trivial Mackey obstruction. Then a ν_G-conull subset of $\widehat{G}$ is given by the set

$$\Omega = \bigcup_{\mathcal{O}(\sigma) \in \Sigma/H} \{\mathrm{Ind}_{G_\sigma}^{G} \sigma \times \varrho : \varrho \in \widehat{H_\sigma}\} \equiv \bigcup_{\mathcal{O}(\sigma) \in \Sigma/H} \{\sigma\} \times \widehat{H_\sigma} \quad .$$

The class of Plancherel measure is obtained by the following procedure: Pick a pseudo-image $\overline{\nu_N}$ *of* ν_N *on* Σ/H*. Then there exists, for almost all* $\mathcal{O}(\gamma) \in \Sigma/H$*, a normalization of Plancherel measure* ν_{H_γ} *(which is necessarily unique in the unimodular case) such that*

$$d\nu_G = d\nu_{H_\gamma} d\overline{\nu_N} \quad . \tag{3.32}$$

For G *unimodular, the correct normalizations are obtained as follows:*

- *There exists a unique family of measures* $\mu_{\mathcal{O}(\gamma)}$ *on the* G*-orbits in* Σ *such that*
$$d\mu_{\mathcal{O}(\gamma)} d\overline{\nu_N}(\mathcal{O}(\gamma)) = d\nu_N(\gamma) \quad .$$
- *Since* G *is unimodular, the* $\mu_{\mathcal{O}(\gamma)}$ *are* H*-invariant, at least* ν_N*-almost everywhere. Hence there exists a unique choice of Haar measure* μ_{H_γ} *such that Weil's formula*
$$d\mu_N d\mu_{H_\gamma} d\mu_{\mathcal{O}(\gamma)} = d\mu_G$$
holds.
- *Normalizing* ν_{H_γ} *according to the choice of* μ_{H_γ} *in the previous step ensures (3.32).*

Since discrete series correspond to the atoms with respect to Plancherel measure, the following result follows directly from the fibred measure theorem of Kleppner and Lipsman:

Corollary 3.41. *Let* G *and* $N \lhd G$ *be as in Theorem 3.40. Then* π *is a discrete series representation iff* $\pi = \mathrm{Ind}_{G_\sigma}^G \sigma \times \varrho$*, where* $\sigma \in \widehat{N}$ *is such that* $\nu_N(\mathcal{O}(\sigma)) > 0$ *and* ϱ *is in the discrete series of* H_σ*.*

For the sake of illustration, let us discuss the theorems for the groups from Section 2.4.

Remark 3.42. Consider the group $\mathbb{H}_r = \mathbb{R}^2 \times \mathbb{T}$ as in Example 2.27, with the group law

$$(p, q, z)(p', q', z') = (p + p', q + q', zz' \mathrm{e}^{\pi \mathrm{i}(q'p - qp')}) \ ,$$

and inversion $(p, q, z)^{-1} = (-p, -q, \overline{z})$. Then $N = \{0\} \times \mathbb{R} \times \mathbb{T}$ is an abelian normal subgroup, and the dual group is parameterized by $\mathbb{R} \times \mathbb{Z}$, via $\chi_{(\omega,n)}(0, q, z) = z^n \mathrm{e}^{-2\pi \mathrm{i} \omega q}$. Letting $H = \mathbb{R} \times \{0\} \times \{1\}$, we see that $\mathbb{H}_r = N \rtimes H$, and the conjugation action of H on N is given by

$$(p, 0, 1)(0, q, z)(-p, 0, 1) = (p, 0, 1)(-p, q, z\mathrm{e}^{\pi \mathrm{i} qp}) = (0, q, z\mathrm{e}^{2\pi \mathrm{i} pq}) \quad .$$

The dual action is then computed as

$$\begin{aligned} [(p, q, z).\chi_{(\omega,n)}](q_0, z_0) &= [(p, 0, 1).\chi_{(\omega,n)}](q_0, z_0) = (\omega, n)(q_0, z_0 \mathrm{e}^{2\pi \mathrm{i} pq}) \\ &= z_0^n \mathrm{e}^{2\pi \mathrm{i} npq_0} \mathrm{e}^{-2\pi \mathrm{i} \omega q_0} = \chi_{(\omega - np, n)}(q_0, z_0) \quad . \end{aligned}$$

Therefore, the dual orbits are given by

$$\mathcal{O}_{(\omega,0)} = \{(\omega,0)\}\ ,\ \omega \in \mathbb{R}$$
$$\mathcal{O}_n = \mathbb{R} \times \{n\}\ ,\ n \in \mathbb{Z} \setminus \{0\}\quad .$$

The little fixed groups associated to the orbits are H in the first case, and trivial in the second case. Thus Mackey's Theorem yields

$$\widehat{\mathbb{H}_r} = \{\sigma_{r,s} : r,s \in \mathbb{R}\} \bigcup \{\pi_k : k \in \mathbb{Z} \setminus \{0\}\}\quad . \tag{3.33}$$

Here $\sigma_{r,s}(p,q,t) = \mathrm{e}^{-2\pi \mathrm{i}(rp+sq)}$, the characters of the quotient group $G/(\{0\} \times \{0\} \times \mathbb{T}$, and $\pi_k = \mathrm{Ind}_N^{\mathbb{H}_r} \chi_{0,k}$. Identifying $\mathbb{R} \times \{n\}$ with $\mathbb{R}$, and observing once again that the representation of to be induced acts on $\mathbb{C}$, we realize the induced representations on $\mathrm{L}^2(\mathbb{R})$, using the cross-sections $\alpha : \mathbb{R} \to \mathbb{H}_r$ given by $\alpha(\omega) = (-\omega/n, 0, 1)$. We note that Lebesgue measure is in fact invariant under the shift action in formula (3.29), hence the Radon-Nikodym derivative equals one. Computing

$$\alpha(\omega)^{-1}(p,q,z)\alpha((p,q,z)^{-1}\omega) = (\omega/n,0,1)(p,q,z)\alpha(\omega+np)$$
$$= (\omega/n,0,1)(p,q,z)(-\omega/n-p,0,1) = (0,q,z\mathrm{e}^{2\pi \mathrm{i} q(\omega/k+p/2)})$$

Plugging this into (3.29) yields

$$\pi_k(p,q,z)f(\omega) = \chi_{(0,k)}(0,q,z\mathrm{e}^{2\pi \mathrm{i} q(p/2+\omega/k)})f(\omega+kp) = z^k \mathrm{e}^{2\pi \mathrm{i} q(\omega+pk/2)}\quad .$$

Comparing this to Equation (2.19), we see that π_{-1} is identical to the representation π considered in Example 2.27. In order to establish that the representation is in the discrete series (which is of course known by the discussion in 2.27), we next compute Plancherel measure, or at least its measure class. The Plancherel measure on $\widehat{N} \equiv \mathbb{R} \times \mathbb{Z}$ is $d\omega dn$, where $d\omega$ is Lebesgue measure, and dn counting measure. It is obvious that taking Lebesgue measure on the continuous part of $\widehat{G}$ (i.e., on the $\{\sigma_{r,s} : r,s \in \mathbb{R}\}$) and counting measure on the complement is a pseudoimage of this. This yields the measure class; in particular, $\nu_G(\{\pi_{-1}\}) > 0$ establishes once again that the representation from Example 2.27 is in the discrete series. Moreover, our discussion has provided the additional discrete series representations $\{\pi_k : k \neq 0, -1\}$.

Remark 3.43. There are different possible ways of viewing the reduced reduced Heisenberg group as a group extension. However, not every possibility allows to apply Theorem 3.39. For instance, consider the central subgroup $Z = \{0\} \times \{0\} \times \mathbb{T}$. Then clearly $Z \lhd G$, with $G/Z \cong \mathbb{R}^2$. Since Z is central, the conjugation action of G on Z is trivial, and so is the dual action. It follows that $G_\gamma = G$ for all $\gamma \in \widehat{Z}$. Assuming that all Mackey obstructions were trivial, this would imply that all irreducible representations are extensions of elements of $\widehat{Z}$. Thus they would act in onedimensional spaces, i.e. all elements of $\widehat{G}$ would be homomorphisms $G \to \mathbb{T}$. Since $\widehat{G}$ separates points, this would imply that G is abelian, which it is not.

Remark 3.44. Let G be the $ax+b$-group, $G = \mathbb{R} \rtimes \mathbb{R}'$ from Example 2.28. A more detailed study of the representations of this group, including the Plancherel formula, may be found in the papers by Khalil [71] and Eymard and Terp [42]. These papers may also be considered as precursors to the results of Kleppner and Lipsman, and Duflo and Moore, on the Plancherel formula for general nonunimodular groups.

The arguments given here for the $ax+b$-group will be applied to general semidirect products in Chapter 5. The role of the normal subgroup N is played by the subset $\{(b,1) : b \in \mathbb{R}\}$, whereas the quotient group is isomorphic to the subgroup $\{(0,a) : a \in \mathbb{R}'\} \cong \mathbb{R}'$. Using

$$(0,a)(b,1)(0,a^{-1}) = (ab,1)$$

we find that

$$(a.\chi_\omega)(b) = \chi_\omega(a^{-1}b) = \mathrm{e}^{-2\pi \mathrm{i} a^{-1} b\omega} = \chi_{a^{-1}\omega}(b) \quad . \tag{3.34}$$

Thus we have two dual orbits in $\widehat{N} \equiv \mathbb{R}$, namely $\{0\}$ and $\mathbb{R}'$, with associated fixed groups G and N, respectively. The trivial representation of N, which corresponds to the orbit $\{0\}$, clearly extends to the trivial representation of G. Hence the trivial orbit contributes the representations

$$\varrho_t(b,a) = a^{it}\mathrm{sign}(a)^\epsilon \quad (t \in \mathbb{R}^+, \epsilon \in \{0,1\}) \quad ,$$

which is just the dual of the quotient group $G/N \equiv \mathbb{R}^*$.

We pick $\chi_1 \in \widehat{N}$ as a representative of the other orbit. It is clear from (3.34) that the little fixed group is trivial, thus the orbit contributes precisely one more representation

$$\sigma = \mathrm{Ind}_N^G \chi_1 \quad .$$

Noting that χ_1 has $\mathbb{C}$ as representation space, we realize π on the space $\mathrm{L}^2(\mathbb{R}, dx)$, using the cross-section $\alpha : \mathbb{R}' \to G$ given by $\alpha(\omega) = (0, \omega^{-1})$. With a view to evaluating (3.29) we compute

$$\alpha(\omega)^{-1}(b,a)\alpha((b,a)^{-1},\omega) = (0,\omega)(b,a)(0,a^{-1}\omega^{-1}) = (\omega b, 0) \quad .$$

Plugging this into (3.29) yields

$$[\sigma(b,a)f](\omega) = |a|^{1/2}\mathrm{e}^{-2\pi \mathrm{i} b\omega} f(a\omega) \quad .$$

Note that the factor $|a|^{1/2}$ is the Radon-Nikodym derivative term in (3.29), as it serves to making $\sigma(b,a)$ unitary.

A comparison to Example 2.28 shows that σ, being the only infinite-dimensional irreducible representation of G, must be equivalent to the irreducible representation π which we studied in 2.28. A straightforward computation shows that

$$(\pi(b,a)f)(\omega) = \mathrm{e}^{-2\pi \mathrm{i}\omega b}|a|^{1/2} f(a\omega) = \left(\sigma(b,a)\widehat{f}\right)(\omega) \quad ,$$

which means that the Plancherel transform on $\mathrm{L}^2(\mathbb{R})$ intertwines σ and π. This is also our reason for realizing $\mathrm{Ind}_N^G \chi_1$ on $\mathrm{L}^2(\mathbb{R}, dx)$, instead of choosing the measure $|x|^{-1}dx$, which is invariant under the dual action. The induced representation would look slightly simpler, but the unitary equivalence to the representation from 2.28 is less easy to see.

We next turn to the Plancherel measure, which is a rather easy exercise: ν_N is supported on the conull orbit $\mathbb{R}'$, which contributes the representation σ. Now Theorem 3.40 yields that, since the other orbit $\{0\}$ has measure zero, the Plancherel measure of G is supported in $\{\sigma\}$. In other words, the

$$\lambda_G \simeq \infty \cdot \sigma \quad ,$$

and the Plancherel theorem implies that

$$f \mapsto \sigma(f) C_\sigma$$

defined on $\mathrm{L}^1(G) \cap \mathrm{L}^2(G)$, extends to a unitary operator $\mathrm{L}^2(G) \to \mathcal{B}_2(\mathcal{H}_\sigma)$.

The similitude group considered in Example 2.30 can be treated much in the same way. Again the dual action has two orbits, $\{0\}$ and its complement, and on the latter, the action is free. As for the 1D case, this implies that $\lambda_G \simeq \infty \cdot \sigma$, for a single irreducible representation σ.

The following remark dealing with the dyadic wavelet transform can be seen as a precursor to the discussion in Chapter 5, in particular the first three sections therein.

Remark 3.45. Finally let us consider the $2^k x + b$-group underlying the dyadic wavelet transform. Since this is a subgroup of the $ax + b$-group containing $\mathbb{R}$, we obtain the same dual action as in Remark 3.44, which yields the following orbit space

$$\{\{2^k x : k \in \mathbb{Z}\} \quad : \quad x \in \pm[1, 2[\cup \{0\} \cup\} \equiv \pm[1, 2[\cup \{0\}$$

The associated little fixed groups are trivial for orbits different from $\{0\}$, whereas the little fixed group of the latter is obviously $\mathbb{Z}$. Thus we find that

$$\widehat{G} = \{\chi_\gamma : \gamma \in [0, 1[\} \cup \{\sigma_\omega : \omega \in \pm[1, 2[\} \quad ,$$

with $\chi_\gamma(b, k) = \mathrm{e}^{-2\pi \mathrm{i} \gamma k}$, and $\sigma_\omega = \mathrm{Ind}_N^G \omega$. Proceeding as in the previous example, we realize σ_ω on $\ell^2(\mathbb{Z})$, yielding

$$\sigma_\omega(b, k) f(n) = \mathrm{e}^{2\pi \mathrm{i} \omega n} f(n - k) \quad . \tag{3.35}$$

Let us describe the Plancherel measure. The group being nonunimodular, only the measure class will be of interest. We may safely ignore the representations χ_γ, since they correspond to the orbit $\{0\}$ which has zero measure. For the complement, the fact that the dual action is free there implies that it remains

to compute a pseudoimage of Plancherel measure on $\widehat{\mathbb{R}}$ under the quotient map associated to the dual action. Clearly Lebesgue measure on

$$\pm[1,2[\equiv \{\sigma_\omega : \omega \in \pm[1,2[\}$$

serves this purpose. As a first simple observation we note that this measure does not have atoms, and hence G has no discrete series representations.

Let us now consider the quasi-regular representation π from Example 2.36. We already know that π has admissible vectors, and we want to express this in terms of Plancherel measure. For this purpose we compute the action of the representation $\widehat{\pi}$ obtained by conjugating π with the Fourier transform. In the last remark, we have already computed this action for the full affine group, obtaining

$$\left(\widehat{\pi}(b,k)f\right)(\omega) = |2|^{k/2}\mathrm{e}^{-2\pi\mathrm{i}\omega k} f(2^k\omega) \quad . \tag{3.36}$$

In order to relate this to the Plancherel decomposition, we associate to each $\xi \in \pm[1/2,1[$ the Hilbert space $\mathcal{H}_\xi = \ell^2(2^{\mathbb{Z}}\xi, 2^k dk)$. That is, elements of $\ell^2(2^{\mathbb{Z}}\xi, 2^k dk)$ are sequences indexed by $\{2^k\xi : k \in \mathbb{Z}\}$ which are square-summable w.r.t. (weighted) counting measure. The motivation for this definition is that if we parameterize $\mathbb{R}'$ by $\pm[1,2[\times\mathbb{Z} \ni (\xi,k) \mapsto 2^k\xi$. Lebesgue measure is then given by $d\xi 2^{k-1}dk$, with $d\xi$ Lebesgue measure on the intervals, and dk counting measure. Hence any element of $\mathrm{L}^2(\mathbb{R})$ corresponds uniquely to a family $(f|_{2^{\mathbb{Z}}\xi})_{\xi\in\pm[1,2[}$, yielding a unitary equivalence

$$\mathrm{L}^2(\mathbb{R}) \simeq \int_{\pm[1,2[} \mathcal{H}_\xi d\xi \quad .$$

As a matter of fact, this argument is yet another instance of the general procedure of viewing a measure decomposition as direct integral decomposition of the associated L^2-spaces, confer Subsection 3.4.4. The relevance of this particular decomposition is that a second look at (3.36) convinces us that we also have a decomposition of representations,

$$\widehat{\pi} \simeq \int_{\pm[1,2[} \pi_\xi d\xi$$

where the action of π_ξ on $\mathcal{H}_\xi$ is

$$\left(\pi_\xi(b,k)f\right)(\omega) = |2|^{k/2}\mathrm{e}^{-2\pi\mathrm{i}\omega k} f(2^k\omega) \quad . \tag{3.37}$$

Now the comparison to (3.35) allows to check immediately that the operator $T_\xi : \ell^2(\mathbb{Z}) \to \mathcal{H}_\xi$, defined by

$$\left(T_\xi(f)\right)(2^k\xi) = 2^{k/2} f(k)$$

intertwines σ_ξ with π_ξ. Thus we arrive at

$$\pi \simeq \int_{\pm[1,2[}^{\oplus} \sigma_\xi d\xi \quad ,$$

and the right-hand side is a multiplicity-free direct integral representation with Plancherel measure as underlying measure. Thus we have established $\pi \approx \lambda_G$. In particular, π has no irreducible subrepresentations either, a fact we have already derived by other means in Example 2.36.

3.7 Operator-Valued Integral Kernels*

The Plancherel formula for nonunimodular groups is obtained by applying the results of the previous section to the normal subgroup $N = \mathrm{Ker}(\Delta_G)$. This involves explicit calculation using induced representations, and operator-valued integral kernels arise naturally in this context.

It is a well-known fact that for a given measure space (X, μ) the Hilbert-Schmidt operators on $\mathrm{L}^2(X)$ can be conveniently identified with integral kernels in $\mathrm{L}^2(X \times X, \mu \otimes \mu)$. In this section we discuss the extension of this observation to vector-valued L^2-spaces, mostly with the aim of computing a trace formula. Abstract computation –ignoring possible unpleasant linearity-vs.-sesquilinearity issues– allows to figure out what to expect:

$$\begin{aligned}\mathcal{B}_2(\mathrm{L}^2(X;\mathcal{H})) &\simeq (\mathrm{L}^2(X) \otimes \mathcal{H}) \otimes (\mathrm{L}^2(X) \otimes \mathcal{H}) \simeq \mathrm{L}^2(X \times X) \otimes \mathcal{B}_2(\mathcal{H}) \\ &\simeq \mathrm{L}^2(X \times X; \mathcal{B}_2(\mathcal{H})) \quad ;\end{aligned}$$

hence replacing the scalar-valued kernels by Hilbert-Schmidt-valued ones should do the trick.

In this section we give a proof of this correspondence, and then give a trace formula which is our chief motivation for using operator-valued kernels. For the proof of the trace formula Bochner integration is a useful tool. Following [117], we call a function $f : X \to B$, where X is a measure space and B a Banach space, **strongly measurable** if it is the pointwise limit (in the norm) of measurable finite-valued functions $X \to B$. Here measurability for finite-valued functions simply means that all preimages are measurable. For separable Banach spaces, this turns out to be equivalent to **weak measurability** [117, V.4 Theorem], which is defined by the requirement that $\phi \circ f : X \to \mathbb{C}$ is measurable, for all $\phi \in B^*$. For the cases under consideration, this measurability condition is straightforward to check and will not be addressed explicitly. By [117, V.5], it is possible to construct a **Bochner integral** assigning every strongly measurable f such that $x \mapsto \|f(x)\|_B$ is integrable, a unique value $\int_X f(x) d\mu(x)$. The following lemma makes the identification $\mathcal{B}_2(\mathrm{L}^2(X, d\mu; \mathcal{H})) \simeq \mathrm{L}^2(X \times X, d\mu; \mathcal{B}_2(\mathcal{H}))$ explicit.

Lemma 3.46. *Let X be a measure space, with measure μ, and consider the mapping $T : \mathrm{L}^2(X \times X; \mathcal{B}_2(\mathcal{H})) \to \mathcal{B}_2(\mathrm{L}^2(H; \mathcal{H}))$, $F \mapsto T_F$ given by*

$$(T_F)(g)(x) = \int_H F(x,\xi)g(\xi)d\mu(\xi) \quad . \tag{3.38}$$

Here the right-hand side converges for every $x \in H$ satisfying the condition $F(x,\cdot) \in \mathrm{L}^2(X, d\mu; \mathcal{B}_2(\mathcal{H}))$, as an $\mathcal{H}$-valued Bochner integral. T is a unitary map, fulfilling $(T_F)^* = T_{F^*}$, where $F^*(\xi,\xi') = (F(\xi',\xi))^*$.

Proof. We compute

$$\begin{aligned}\int_X \|F(x,\xi)g(\xi)\|d\mu(\xi) &\leq \int_H \|F(x,\xi)\|_\infty \|g(\xi)\|d\mu(\xi) \\ &\leq \int_X \|F(x,\xi)\|_2 \|g(\xi)\|d\mu(\xi) \\ &\leq \|F(x,\cdot)\|_2 \|g\|_2 \quad ,\end{aligned}$$

where the last inequality is due to Cauchy-Schwarz. Fubini's theorem implies that $F(x,\cdot) \in \mathrm{L}^2(X, \mathcal{B}_2(\mathcal{H}))$ for almost all x. For these x we have thus proved convergence of the right hand side of (3.38), while

$$\int_H \|(T_F g)(x)\|_2^2 d\mu(x) \leq \int_H \|F(x,\cdot)\|_2^2 \|g\|_2^2 d\mu(x) \leq \|F\|_2^2 \|g\|^2$$

entails $\|T_F\|_\infty \leq \|F\|_2$. Hence $T_{F_n} \to T_F$ in $\|\cdot\|_\infty$ whenever $F_n \to F$ in $\mathrm{L}^2(X \times X; \mathcal{B}_2(\mathcal{H}))$.

Next pick ONB's $(e_i)_{i\in I}$ of $\mathcal{H}$ and $(\varphi_j)_{j\in J}$ of $\mathrm{L}^2(X, d\mu)$. Then $(\phi_i e_j)_{i\in I, j\in J}$ is an ONB of $\mathrm{L}^2(X, d\mu; \mathcal{H})$ by 3.6, and then $((\varphi_k e_\ell) \otimes (\varphi_m e_n))_{k,\ell,m,n}$ is an ONB of $\mathcal{B}_2(\mathrm{L}^2(X, d\mu; \mathcal{H}))$. It is easily seen that the preimage under K of these operators is given by the operator-valued functions

$$\varphi_k \overline{\varphi_m}\ e_\ell \otimes e_k \quad : \quad (\xi,\xi') \mapsto \varphi_k(\xi)\overline{\varphi_m(\xi')} e_\ell \otimes e_k \quad ,$$

but the latter are an ONB of $\mathrm{L}^2(X \times X, d\mu; \mathcal{B}_2(\mathcal{H}))$.

An arbitrary $F \in \mathrm{L}^2(X \times X; \mathcal{B}_2(\mathcal{H}))$ can be expanded in this ONB. Then the partial sums are mapped under T to a convergent sequence in $\mathcal{B}_2(\mathrm{L}^2(X, d\mu; \mathcal{H}))$, while on the other hand, they converge to T_F in $\|\cdot\|_\infty$. This shows $T_F \in \mathcal{B}_2(\mathrm{L}^2(X, d\mu; \mathcal{H}))$ and $\|T_F\|_{\mathcal{B}_2} = \|F\|_2$. The statement concerning adjoints is immediate.

The next lemma studies how operator multiplication and taking traces relates to the integral kernels. Since the result is of a strictly auxiliary nature, we do not consider the problem in full generality. In particular, we restrict attention to the case where $X = H$, a locally compact group, and $\mu = \mu_H$.

Note that $\mathcal{B}_1 \subset \mathcal{B}_2$ implies that each trace class operator on $\mathrm{L}^2(H; \mathcal{H})$ also has a unique Hilbert-Schmidt kernel. In the case of multiplication, the behavior is as expected, i.e., the formula for the kernel of $T_F \circ T_G$ turns out to be a continuous analogue of matrix multiplication. On the other hand, the first guess for a trace formula,

$$\operatorname{trace}(T_A) = \int_H \operatorname{trace}(A(\xi,\xi))d\xi$$

is clearly problematic: Unless H is discrete, the diagonal in $H \times H$ is a set of measure zero. Hence if the operator field $(A(\xi,\xi'))_{\xi,\xi'\in H}$ is just viewed as an element of $\mathrm{L}^2(H \times H; \mathcal{B}_2(\mathcal{H}))$, the formula does not even make sense. The following lemma shows however that the integral kernels of trace class operators can be chosen in such a way that not only the trace of T_A, but also the traces of related operators can be computed by integrating the traces on certain subsets $A \subset H \times H$ of product measure zero.

Lemma 3.47. *Let H be a locally compact group. Let $F, G \in \mathrm{L}^2(H\times H, \mathcal{B}_2(\mathcal{H}))$, and define*

$$A(\xi,\xi'') = \int_H F(\xi,\xi')G(\xi',\xi'')d\xi' \qquad (3.39)$$
$$\text{whenever } F(\xi,\cdot), G(\cdot,\xi'') \in \mathrm{L}^2(H, \mathcal{B}_2(\mathcal{H}))$$

(a) The set $\{(\xi,\xi'') : F(\xi,\cdot), G(\cdot,\xi'') \in \mathrm{L}^2(H, \mathcal{B}_2(\mathcal{H}))\} \subset H\times H$ has a complement of measure zero. On this set the right hand side of (3.39) converges as a $\mathcal{B}_1(\mathcal{H})$-valued Bochner integral. In particular, $A(\xi,\xi)$ is well-defined for almost every $\xi \in H$.

(b) A as defined in (3.39) fulfills $A \in \mathrm{L}^2(H \times H; \mathcal{B}_2(\mathcal{H}))$ and $T_A = T_F \circ T_G$.

(c) Let A be given by (3.39). Let $\gamma \in H$, and $(U_{\xi''})_{\xi''\in H}$ a measurable field of unitary operators on $\mathcal{H}$. Then the operator kernel

$$B : (\xi,\xi'') \mapsto A(\xi,\gamma^{-1}\xi'')U_{\xi''}$$

is in $\mathrm{L}^2(H \times H, \mathcal{B}_2(\mathcal{H}))$, with T_B trace class. The trace is computed as

$$\operatorname{trace}(T_B) = \int_H \operatorname{trace}\left(A(\xi,\gamma^{-1}\xi)U_\xi\right)\, d\xi \quad . \qquad (3.40)$$

We moreover obtain the inequality

$$\int_H \|A(\xi,\gamma^{-1}\xi)U_\xi\|_1 d\xi \le \|F\|_2\|G\|_2 \qquad (3.41)$$

Proof. Clearly, $F(\xi,\xi')G(\xi',\xi'') \in \mathcal{B}_1(\mathcal{H})$. Now the calculation

$$\begin{aligned}\int_H \|F(\xi,\xi')G(\xi',\xi'')\|_1 d\xi' &\le \int_H \|F(\xi,\xi')\|_2\|G(\xi',\xi'')\|_2 d\xi' \\ &\le \|F(\xi,\cdot)\|_2\|G(\cdot,\xi'')\|_2\end{aligned}$$

establishes the convergence statement of (a). That the sets on which this convergence holds are conull (both in $H \times H$ and on the diagonal) follows from Fubini's Theorem.

For part (b) we observe that Bochner convergence of (3.39) in $\mathcal{B}_1(\mathcal{H})$ entails Bochner convergence in $\mathcal{B}_2(\mathcal{H})$, hence

$$\begin{aligned}
&\int_H \int_H \|A(\xi,\xi'')\|_2^2 d\xi d\xi'' \leq \\
&\leq \int_H \int_H \left(\int_H \|F(\xi,\xi')\|_2 \|G(\xi',\xi'')\|_2 d\xi' \right)^2 d\xi d\xi'' \\
&\leq \int_H \int_H \left(\int_H \|F(\xi,x)\|_2^2 dx \right) \left(\int_H \|G(y,\xi'')\|_2^2 dy \right) d\xi d\xi'' \\
&= \|F\|_2^2 \|G\|_2^2 \quad .
\end{aligned}$$

Hence $A \in \mathrm{L}^2(H \times H, \mathcal{B}_2(\mathcal{H}))$, and $T_A = T_F \circ T_G$ is verified by a straightforward calculation.

For the proof of part (c) we first observe that

$$B(\xi,\xi'') = \int_H F(\xi,\xi') K(\xi',\xi'') d\xi' \quad ,$$

where

$$K : H \times H \ni (\xi',\xi'') \mapsto G(\xi',\gamma^{-1}\xi'') U_{\xi''} \quad .$$

Leftinvariance of the measure and the fact that the $U_{\xi''}$ are unitary ensure that $K \in \mathrm{L}^2(H \times H, \mathcal{B}_2(\mathcal{H}))$. Now the same argument as for (a) yields that

$$B(\xi,\xi) = \int_H F(\xi,\xi') G(\xi',\gamma^{-1}\xi) U_\xi \; d\xi' \quad , \tag{3.42}$$

with Bochner convergence in $\mathcal{B}_1(\mathcal{H})$ whenever $\|F(\xi,\cdot)\|_2, \|G(\cdot,\gamma^{-1}\xi)\|_2 < \infty$, i.e., for almost every ξ. More precisely, we obtain for these ξ that

$$\|B(\xi,\xi)\|_1 \leq \|F(\xi,\cdot)\|_2 \|G(\cdot,\gamma^{-1}\xi)\|_2 \quad .$$

Integration over ξ yields

$$\int_H \|B(\xi,\xi)\|_1 d\xi \leq \left(\int \|F(\xi,\cdot)\|_2^2 d\xi \right)^{1/2} \left(\int \|G(\cdot,\gamma^{-1}\xi)\|_2^2 d\xi \right)^{1/2} = \|F\|_2 \|G\|_2 \ ,$$

i.e., (3.41).

For the proof of (3.40), observe that $F \mapsto T_F$ is unitary. Then the definition of the scalar product on $\mathrm{L}^2(H \times H, \mathcal{B}_2(\mathcal{H}))$ entails

$$\begin{aligned}
\mathrm{trace}(T_B) &= \mathrm{trace}(T_F \circ T_K) = \langle T_F, T_K^* \rangle = \langle T_F, T_{K^*} \rangle \\
&= \int_H \int_H \mathrm{trace}(F(\xi,\xi') \left((K^*)(\xi,\xi')\right)^*) \; d\xi d\xi' \\
&= \int_H \int_H \mathrm{trace}\left(F(\xi,\xi') G(\xi',\gamma^{-1}\xi) U_\xi\right) d\xi' d\xi \quad ,
\end{aligned} \tag{3.43}$$

where we used the definition of the adjoint kernel K^* in the previous lemma to compute

$$(K^*)(\xi,\gamma^{-1}\xi') = U_{\xi'}^* G(\xi',\gamma^{-1}\xi)^* \quad .$$

Now the Bochner convergence of (3.42) together with the fact that the trace is a continuous functional on $\mathcal{B}_1(\mathcal{H})$ allow to pull the trace out of the inner integral of (3.43), by [117, V.5, Corollary 2], thus leaving us with the desired relation.

3.8 The Plancherel Formula for Nonunimodular Groups

In this section we give an outline of the construction of the Plancherel transform of a nonunimodular type I group, as it was derived by Duflo and Moore [38, Paragraph 6]. Similar results were obtained by Tatsuuma [110], as well as Kleppner and Lipsman [75].

3.8.1 The Plancherel Theorem

The following theorem summarizes both the unimodular and the nonunimodular cases.

Theorem 3.48. *Let G be a locally compact group. Assume that G and $N = \mathrm{Ker}(\Delta_G)$ are type I groups, with N regularly embedded. Then there exist*

- *a Plancherel measure ν_G on $\widehat{G}$,*
- *a measurable field $(C_\sigma)_{\sigma\in\widehat{G}}$ of densely defined selfadjoint positive operators with densely defined inverses,*

such that the following statements hold:

(a) For all $f \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G)$ and the closure of the operator $\sigma(f)C_\sigma^{-1}$ is in $\mathcal{B}_2(\mathcal{H}_\sigma)$, and

$$\|f\|_2^2 = \int_{\widehat{G}} \|[\sigma(f)C_\sigma^{-1}]\|_2^2 \, d\nu_G(\sigma) \quad . \tag{3.44}$$

(b) $\mathcal{F}$ extends uniquely to a unitary operator

$$\mathcal{P} : \mathrm{L}^2(G) \to \mathcal{B}_2^{\oplus} = \int_{\widehat{G}}^{\oplus} \mathcal{B}_2(\mathcal{H}_\sigma) \, d\nu_G(\sigma) \quad , \tag{3.45}$$

the **Plancherel transform** *of G.*

(c) $\mathcal{P}$ implements the following unitary equivalences

$$\lambda_G \simeq \int_{\widehat{G}}^{\oplus} \sigma \otimes 1 \, d\nu_G(\sigma) \tag{3.46}$$

$$\varrho_G \simeq \int_{\widehat{G}}^{\oplus} 1 \otimes \overline{\sigma} \, d\nu_G(\sigma) \tag{3.47}$$

$$VN_l(G) \simeq \int_{\widehat{G}}^{\oplus} \mathcal{B}(\mathcal{H}_\sigma) \otimes 1 \, d\nu_G(\sigma) \tag{3.48}$$

$$VN_r(G) \simeq \int_{\widehat{G}}^{\oplus} 1 \otimes \mathcal{B}(\mathcal{H}_\sigma) \, d\nu_G(\sigma) \tag{3.49}$$

(d) ν_G *and the operator field* $(C_\sigma)_{\sigma\in\widehat{G}}$ *can be chosen to satisfy the* **inversion formula**

$$f(x) = \int_{\widehat{G}} \operatorname{trace}([\sigma(f)C_\sigma^{-2}]\sigma(x)^*)\, d\nu_G(\sigma) \quad , \tag{3.50}$$

for all f *in a suitable dense* $\mathcal{D} \subset \mathrm{L}^2(G)$*. The inversion formula converges absolutely in the sense that* ν_G*-almost every* $\sigma(f)C_\sigma^{-2}$ *extends to a trace-class operator, and the integral over the trace-class norms is finite.*

(e) The pair $(\nu_G, (C_\sigma)_{\sigma\in\widehat{G}}$ *is essentially unique: The* **semi-invariance relation**

$$\sigma(x)C_\sigma\sigma(x)^* = \Delta_G(x)^{1/2}C_\sigma \tag{3.51}$$

fixes each C_σ *uniquely up to multiplication by a scalar, and once these are fixed, so is* ν_G*. Conversely, one can fix* ν_G *(which is a priori only unique up to equivalence) and thereby determine the* C_σ *uniquely.*

(f) G *is unimodular if and only if for* ν_G*-almost all* σ*,* C_σ *is a multiple of the identity* $\mathrm{Id}_{\mathcal{H}_\sigma}$ *on* $\mathcal{H}_\sigma$*. In this case the requirement* $C_\sigma = \mathrm{Id}_{\mathcal{H}_\sigma}$*, yields that* ν_G *is the canonical Plancherel measure.*
If G *is nonunimodular,* C_σ *is an unbounded operator for (*ν_G*-almost all)* $\sigma \in \widehat{G}$*.*

Let us clarify the relation between the normalizations of ν_G and the C_σ: Assume that $\widetilde{\nu_G}$, $\left(\widetilde{C_\sigma}\right)_{\sigma\in\widehat{G}}$ also fulfill the properties of the theorem. In particular, since (3.44) holds with $\widetilde{\nu_G}$ and $\widetilde{C_\sigma}$ in the place of ν_G and $\widetilde{C_\sigma}$, one deduces

$$\widetilde{C_\sigma} = \left(\frac{d\nu_G}{d\widetilde{\nu_G}}(\sigma)\right)^{-1/2} C_\sigma \quad .$$

Let us repeat the observation, already made in Section 2.4 for the discrete series case, that if the group is nonunimodular, relation (3.51) necessarily entails that C_σ is unbounded.

Remark 3.49. The inversion formula (3.50) generalizes (3.28). It was shown in [38] to hold for the space of Bruhat functions $\mathcal{D}$ introduced in [26]. This space is defined as

$$\mathcal{D}(G) = \bigcup \{C_c^\infty(G/K) : K \subset G \text{ compact such that } G/K \text{ is a Lie group } \} \quad ,$$

where $C_c^\infty(G/K)$ is the space of arbitrarily smooth functions on G/K with compact support, canonically embedded into $C_c(G)$. Theorem 4.15 below contains an extension of this formula to the natural maximal domain of this formula, which will turn out to be crucial for establishing admissibility criteria.

We adopt the notations fixed in Remark 3.32 for unimodular groups for the general case. In particular, the distinction between the Fourier and Plancherel transform, and the fact that we reserve the notation $\widehat{\cdot}$ for the latter, becomes all the more important due to the fact that the two now differ even for functions in $\mathrm{L}^1(G) \cap \mathrm{L}^2(G)$.

3.8.2 Construction Details*

The results presented in this subsection are indispensable for our purposes, as they provide concrete realizations of the representations entering the Plancherel formula, along with a detailed description of the Duflo-Moore operators and their domains. Both are needed to prove the results in Chapter 4.

We already remarked that the key technique for the construction of the Plancherel measure of a nonunimodular group G is to apply Theorem 3.40 to the group extension

$$1 \to N \to G \to H \to 1 \quad ,$$

where $N = \mathrm{Ker}(\Delta_G)$ and $H = G/N$. This approach provides in particular an explicit description of the Duflo-Moore operators, which will be crucial for the construction of admissible vectors. While the measure-theoretic details are somewhat tedious, there is a quite intuitive scheme behind these results, based on the double role of the factor group H: On the one hand the identification, via cosets, of $G \equiv N \times H$ gives rise to a measure decomposition of μ_G in terms of μ_N, μ_H and Δ_G. On the other hand, it turns out that the operation of H on $\widehat{N}$ is free ν_N-almost everywhere, giving rise to a Borel isomorphism $\widehat{N} \equiv V \times H$. Now V can be suitably identified with a subset of $\widehat{G}$, via the induction map, and the measure decomposition of ν_N along V and H provides a measure of V, which turns out to be the Plancherel measure of G. The connection between the various objects will be provided by the operator-valued kernel calculus.

The following proposition spells out the measure-theoretic technicalities. We will not provide a full proof that the construction actually yields the Plancherel measure, however we will show that the transform constructed from the C_σ and ν_G is indeed an isometry $\mathrm{L}^1(G) \cap \mathrm{L}^2(G) \to \mathcal{B}_2^{\oplus}$. The direct computation showing this provides a nice illustration of the interplay between the measure decompositions on G and $\widehat{N}$.

Proposition 3.50. *Let G be nonunimodular and type I, with $N = \mathrm{Ker}(\Delta_G)$ type I and regularly embedded. Then there exists an H-invariant Borel subset $U \subset \widehat{N}$ with the following properties:*

(i) $\nu_N(\widehat{N} \setminus U) = 0$. The quotient space U/H is a standard Borel space.

(ii) For every $\sigma \in U$, $\mathrm{Ind}_N^G \sigma \in \widehat{G}$. Moreover, H operates freely on the orbit $\mathcal{O}(\sigma)$.

(iii) The mapping $(\mathcal{O}(\varrho)) \mapsto \mathrm{Ind}_N^G \varrho$ is a Borel isomorphism $U/H \to V = \mathrm{Ind}(U)$, where V is a standard Borel subset of $\widehat{G}$. Moreover, there exists a Borel cross section $\tau : V \to U$, such that $H \times V \ni (H, \sigma) \mapsto H\tau(\sigma) \in U$ is a Borel isomorphism. In particular, both U and $\tau(V)$ are standard.

(iv) Let the function $\psi : U \to \mathbb{R}^+$ be defined by pulling the mapping $H \times V \ni (\gamma, \sigma) \mapsto \Delta_G(H)^{-1/2}$ back to U via τ. Then ψ is measurable.

(v) Plancherel measure on $V \cong U/H$ is obtained by a measure disintegration along H-orbits: If each orbit $\mathcal{O}(\tau(\sigma))$ is endowed with the image $\mu_{\mathcal{O}(\tau(\sigma))}$ of Haar measure under the (bijective) projection map $H \ni \gamma \mapsto \gamma.\tau(\sigma)$, there exists a unique measure ν_G on V such that

$$d\nu_N(\varrho) = \psi(\varrho)^{-2} d\mu_{\mathcal{O}(\tau(\sigma))}(\varrho) d\nu_G(\sigma) \quad ; \tag{3.52}$$

the measure ν_G thus obtained is the Plancherel measure of G.

(vi) If ν_G is constructed as in (v), and $\sigma = \mathrm{Ind}_N^G \tau(\sigma) \in U$ is realised on $\mathrm{L}^2(H, d\mu_H; \mathcal{H}_{\tau(\sigma)})$ via cross sections, then C_σ is given by multiplication with $\Delta_G^{-1/2}$:

$$(C_\sigma \eta)(\gamma) = \Delta_G(\gamma)^{-1/2} \eta(\gamma) \quad ,$$

and $\mathrm{dom}(C_\sigma)$ is the set of all $\eta \in \mathrm{L}^2(H, d\mu_\Gamma; \mathcal{H}_{\tau(\sigma)})$ for which this product is also in $\mathrm{L}^2(H, d\mu_\Gamma; \mathcal{H}_{\tau(\sigma)})$.

Proof. Part (i) is given by [38, Theorem 6, 1., 2.], using in addition that λ_G is type I. The choice of U also guarantees $\mathrm{Ind}(U) \subset \widehat{G}$, [38, Lemma 8] which is the first statement of (ii). The second statement of (ii) then follows by [38, Lemma 7]. That Ind induces a Borel isomorphism $U/H \to V$ is due to [38, Lemma 13], possibly after passing to a smaller set $U' \subset U$ while preserving all other properties. The existence of a Borel cross section and the resulting Borel isomorphism are observed in the proof of [38, Proposition 10]; again we might have to pass to a ν_N-negligibly smaller set. In particular, both U and $\tau(V)$ are standard, as Borel-isomorphic images of standard Borel spaces. Part (iv) is obvious. Parts (v) and (vi) are [38, Theorem 6, 3.].

A pleasant consequence of part (ii) is that the Mackey obstructions for all $\sigma_0 \in U$ are particularly trivial in the sense of 3.38.

Corollary 3.51. *If λ_G is type I and G is nonunimodular, then $\lambda_G \simeq \infty \cdot \lambda_G$.*

Proof. Since $\mathbb{R}^+$ has no nontrivial noncompact subgroups, H is infinite. Hence $\mathcal{H}_\sigma = \mathrm{L}^2(H, d\mu_H; \mathcal{H}_{\tau(\sigma)})$ is infinite-dimensional. Accordingly, the left operation of σ on $\mathcal{B}_2(\mathcal{H}_\sigma)$ has infinite multiplicity, for almost every $\sigma \in \widehat{G}$. Since λ_G is the direct integral of these representations, the analogous statement for λ_G follows.

In the following computations we refer to an explicit construction of the direct integral space $\mathcal{B}_2^\oplus$. This amounts to making several concrete choices.

- We pick a fixed measurable $\tau : V \to \widehat{N}$ and let $U_0 = \tau(V)$.
- We fix a measurable realization of U_0, which exists by [88, Theorem 10.2], and since U_0 is standard.
- For $\gamma \in H$ and $\sigma_0 \in U_0$, we realize $\gamma.\sigma_0$ on $\mathcal{H}_{\sigma_0}$ by $(\gamma.\sigma_0)(x) = \sigma_0(\alpha(\gamma)^{-1} x \alpha(\gamma))$.
- The induced representations $\mathrm{Ind}_N^G \sigma_0$ are realized on $\mathcal{H}_\sigma = \mathrm{L}^2(H, d\mu_H; \mathcal{H}_{\sigma_0})$ using the cross-section α; see Proposition 3.36 for construction details.

- The measurable structure on the $\mathcal{H}_\sigma$ is defined as follows: Given a measurable family of ONB's $(e^n_{\sigma_0})_{\sigma\in U_0}$, of the $\mathcal{H}_{\sigma_0}$, and given an ONB $(a_n)_{n\in\mathbb{N}}$ of $\mathrm{L}^2(H)$, $(a_n e^\sigma_k)_{n,k}$ is an ONB of $\mathcal{H}_\sigma = \mathrm{L}^2(H, d\mu_H; \mathcal{H}_{\tau(\sigma)}) \simeq \mathcal{H}_{\sigma_0} \otimes \mathrm{L}^2(H)$, except for the zero vectors belonging to the indices $k > \dim(\mathcal{H}_{\tau(\sigma)})$, and it is straightforward to check that it is a measurable structure on $(\mathcal{H}_\sigma)_{\sigma\in U}$. Moreover, it is easy to construct a measurable family of ONB's from this: First note that the sets $V_\ell := \{\sigma : \dim(\mathcal{H}_{\tau(\sigma)}) = \ell\}$, for $\ell \in \mathbb{N}\cup\{\infty\}$, are Borel sets [88, Theorem 8.7]. On each V_ℓ, pick a fixed bijection $s_\ell : \mathbb{N}\times\{1,\dots,\ell\} \to \mathbb{N}$ (where $\{1,\dots,\infty\} := \mathbb{N}$). Then letting $u^\sigma_{s_\ell(n,k)} := a_n e^\sigma_k$, for $\sigma \in V_\ell$, removes the zero vectors from our measurable structure $(a_n e^k_\sigma)_{\sigma\in U}$. Moreover, on each V_ℓ, the measurability is easily checked, and this is sufficient.
- We make the identification

$$H \times U_0 \equiv U \text{ via } H\times U_0 \ni (\gamma,\sigma_0) \mapsto \gamma.\sigma_0 \quad . \tag{3.53}$$

  In this parametrization,

$$\psi((\gamma,\sigma_0)) = \Delta_G^{-1/2}(\gamma)$$

  and

$$d\nu_N(\gamma,\sigma_0) = \Delta_G(\gamma) d\nu_G(\sigma) d\mu_H(\gamma) \quad , \tag{3.54}$$

  where σ and σ_0 are related by $\sigma_0 = \tau(\sigma)$.

The following lemma computes the Haar measure of G in terms of μ_N and μ_H, and fixes the normalization we use in the following.

Lemma 3.52. *Fix a measurable cross-section $\alpha : H \to G$. Then the mapping $N\times H \ni (g_0,\gamma) \mapsto g_0\alpha(\gamma) \in G$ is a Borel isomorphism. We use the notation $g = g_0\alpha(\gamma) \equiv (g_0,\gamma)$. Then*

$$d\mu_G(g_0,\gamma) = d\mu_N(g_0)\Delta_G(\gamma)d\mu_H(\gamma) \tag{3.55}$$

is a left Haar measure.

Proof. Fix $g = g_0\alpha(\gamma), g' = g'_0\alpha(\gamma') \in G$, then

$$gg' = g\ \alpha(\gamma)g'\alpha(\gamma)^{-1}\ \alpha(\gamma)\alpha(\gamma)'\alpha(\gamma\gamma')^{-1}\ \alpha(\gamma\gamma')$$

with $\alpha(\gamma)g'_0\alpha(\gamma)^{-1}, \alpha(\gamma)\alpha(\gamma)'\alpha(\gamma\gamma')^{-1} \in N$ (observing $N \lhd G$). Hence right translation on G corresponds to right translation in the variables g_0,γ, though not by g'_0,γ'. Now the rightinvariance of μ_N,μ_H entails that $d\mu_N(g_0)d\mu_H(\gamma)$ is a right Haar measure on G. But then $\Delta_G d\mu_N d\mu_H$ is a left Haar measure.

In order to check the Parseval equality of the Plancherel transform constructed from the C_σ and ν_G as in the proposition, we need a few technical lemmas. The first one computes the action of $\mathrm{Ind}_N^G(\sigma_0)$, for $\sigma_0 \in U$.

Lemma 3.53. *Let $\sigma_0 \in U_0$ and $\sigma = \mathrm{Ind}_N^G \sigma_0$. Define the cocycle $\Lambda : H \times H \to N$ by*

$$\Lambda(\gamma, \xi) = \alpha(\xi)^{-1}\alpha(\gamma)\alpha(\alpha(\gamma)^{-1}\xi) \quad .$$

If we realize σ on $\mathrm{L}^2(H, d\mu_\gamma; \mathcal{H}_\sigma)$ via the cross-section α, we obtain for $x = g_1\alpha(\gamma)$

$$(\sigma(x)f)(\xi) = (\sigma_0\xi)(g_1)\sigma_0(\Lambda(\gamma, \xi))f(\gamma^{-1}\xi) \quad . \tag{3.56}$$

Proof. Since the measure is invariant, formula (3.29) for induction via cross-sections yields

$$\begin{aligned}(\sigma(g_0, \gamma)f)(\xi) &= \sigma_0\left(\alpha(\xi)^{-1}g_0\alpha(\gamma)\alpha(\alpha(\gamma)^{-1}g_0^{-1}\xi)\right) \; f(\alpha(\gamma)^{-1}g_0^{-1}\xi) \\ &= \sigma_0\left(\alpha(\xi)^{-1}g_0\alpha(\xi)\Lambda(\gamma, \xi)\right) \; f(\gamma^{-1}\xi) \\ &= (\xi.\sigma_0)(g_0)\sigma_0(\Lambda(\gamma, \xi))f(\gamma^{-1}\xi) \quad ,\end{aligned}$$

where $\alpha(\gamma)^{-1}g_0^{-1}\xi = \alpha(\gamma)^{-1}\xi$ is due to $N \lhd G$.

The next step consists in showing how the integrated representation $\mathrm{Ind}_N^G \sigma_0$ acts via an operator-valued integral kernel, thus bringing the techniques from Section 3.7 into play.

Lemma 3.54. *Let $\sigma_0 \in U_0$ and $\sigma = \mathrm{Ind}_N^G \sigma_0$. For $g \in \mathrm{L}^1(G)$ let $g_\gamma := g(\cdot, \gamma)$. Then $\sigma(g) : \mathrm{L}^2(H, d\mu_H; \mathcal{H}_{\sigma_0}) \to \mathrm{L}^2(H, d\mu_H; \mathcal{H}_{\sigma_0})$ can be written as*

$$\sigma(g)f(\xi) = \int_H k(\xi, \gamma)f(\gamma)d\gamma \quad ,$$

where k is an operator valued integral kernel given by

$$k(\xi, \gamma) = (\xi.\sigma_0)(g_{\gamma^{-1}\xi}) \circ \sigma_0(\Lambda(\gamma^{-1}\xi, \xi)) \cdot \Delta_G(\gamma^{-1}\xi) \quad .$$

Proof. First note that by Fubini's theorem $g_\gamma \in \mathrm{L}^1(N)$, for almost every $\gamma \in H$, which justifies the use of $(\xi.\sigma_0)(g_{\gamma^{-1}\xi})$. The following formal calculations can be made rigorous by plugging them into scalar products, according to the definition of the weak operator integral. Using the previous lemma, we see that

$$\begin{aligned}(\sigma(g)f)(\xi) &= \int_H \int_N g(g_1, \gamma)(\xi\sigma_0)(g_1)\sigma_0(\Lambda(\gamma, \xi))f(\gamma^{-1}\xi)dg_1\Delta_G(\gamma)d\gamma \\ &= \int_\Gamma (\xi.\sigma_0)(g_\gamma)\sigma_0(\Lambda(\gamma, \xi))f(\gamma^{-1}\xi)\Delta_G(\gamma)d\gamma \\ &= \int_H (\xi.\sigma_0)(g_{\gamma^{-1}})\sigma_0(\Lambda(\gamma^{-1}, \xi))f(\gamma\xi)\Delta_G(\gamma^{-1})d\gamma \\ &= \int_H (\xi.\sigma_0)(g_{\gamma^{-1}\xi})\sigma_0(\Lambda(\gamma^{-1}\xi, \xi))\Delta_G(\gamma^{-1}\xi)f(\gamma)d\gamma \quad ,\end{aligned}$$

which is the desired formula.

Now we can prove that on $\mathrm{L}^1(G) \cap \mathrm{L}^2(G)$ the Plancherel transform preserves the L^2-norm. Recall that the operator $\sigma(g)C_\sigma^{-1}$ has the operator kernel

$$(\xi.\sigma_0)(g_{\gamma^{-1}\xi}) \circ \sigma_0(\Lambda(\gamma^{-1}\xi, \xi)) \cdot \Delta_G(\gamma^{-1}\xi) \cdot \Delta_G(\gamma)^{1/2} \quad ,$$

and therefore

$$\begin{aligned}
&\int_{\widehat{G}} \left\|\sigma(g)C_\sigma^{-1}\right\|_2^2 d\nu_G(\sigma) \\
&= \int_U \int_H \\
&\quad \int_H \left\|(\xi.\sigma_0)(g_{\gamma^{-1}\xi})\sigma_0(\Lambda(\gamma^{-1}\xi,\xi))\Delta_G(\gamma^{-1}\xi)\Delta_G(\gamma)^{1/2}\right\|_2^2 d\xi \\
&\quad d\gamma d\nu_G(\sigma) \\
&= \int_U \int_H \int_H \left\|(\gamma\xi.\sigma_0)(g_\xi)\Delta_G(\xi)\Delta_G(\gamma)^{1/2}\right\|_2^2 d\xi d\gamma d\nu_G(\sigma) && (3.57) \\
&= \int_H \left(\int_U \int_H \left\|(\gamma.\sigma_0)(g_\xi)\Delta_G(\xi)\Delta_G(\gamma\xi^{-1})^{1/2}\right\|_2^2 d\gamma d\nu_G(\sigma) \right) d\xi && (3.58) \\
&= \int_H \left(\int_U \int_H \|(\gamma.\sigma_0)(g_\xi)\|_2^2 \, \Delta_G(\gamma) d\gamma d\nu_G(\sigma) \right) \Delta_G(\xi) d\xi \\
&= \int_H \left(\int_{\widehat{G}} \|\varrho(g_\xi)\|_2^2 \, d\nu_N(\varrho) \right) \Delta_G(\xi) d\xi && (3.59) \\
&= \int_H \|g_\xi\|_2^2 \Delta_G(\xi) d\xi && (3.60) \\
&= \|g\|_2^2 \quad . && (3.61)
\end{aligned}$$

Here the various equalities are justified as follows: (3.57) is obtained by dropping the unitary operators $\sigma_0(\Lambda(\gamma^{-1}\xi, \xi))$, and a change of variables. (3.58) is Fubini's theorem, and again a change of variables. In line (3.59) we used the measure decomposition (3.54) of ν_G. (3.60) is an application of the Plancherel formula for N, while the last equality uses the measure decomposition (3.55).

4

Plancherel Inversion and Wavelet Transforms

This chapter fully exhibits the relationship between the continuous wavelet transforms discussed in Chapter 2 with the Plancherel formula. A first instance of the connection was discernible in Remark 3.34 dealing with discrete series representations of unimodular groups: Computing the constant c_π governing the admissibility condition for such representations π was found to be equivalent to computing the Plancherel measure of the set $\{\pi\}$, and the wavelet transform was found to be a particular case of inverse Plancherel transform.

These observations are systematically expanded in the course of this chapter. We thus start out by discussing Fourier inversion, first as a mapping between a direct integral space $\mathcal{B}_1^\oplus$ of trace class operators and the Fourier Algebra $A(G)$. We then prove a Plancherel inversion formula (Theorem 4.15), from which the L^2-convolution Theorem 4.18 follows immediately. The same argument as for the toy example then yields admissibility conditions from the convolution theorem (Theorem 4.20). We characterize when admissible vectors exist (Theorem 4.22). Remark 4.30 offers a strategy for the solution of **T4**, sketching a systematic approach to treat arbitrary representations via direct integral theory. In the unimodular case it can actually be shown that the scheme from 4.30 characterizes the canonical Plancherel measure. We also discuss briefly how the scheme relates to the type I assumption. The final section of the chapter is devoted to a short diversion treating Wigner functions associated to nilpotent Lie groups.

The standing assumptions throughout this chapter are: G and $N = \mathrm{Ker}(\Delta_G)$ are type I, with N regularly embedded.

4.1 Fourier Inversion and the Fourier Algebra*

The natural domain for Fourier and Plancherel inversion formulae is given by the Fourier algebra $A(G)$ and its counterpart $\mathcal{B}_1^\oplus$ on the Fourier side. In order to motivate the latter space, recall the situation over the reals: Formally, the inverse Plancherel transform of $f \in \mathrm{L}^2(\mathbb{R})$ is given as

H. Führ: LNM 1863, pp. 105–138, 2005.

$$f(x) = \int_{\widehat{\mathbb{R}}} \widehat{f}(\omega) \mathrm{e}^{2\pi \mathrm{i} x\omega} d\omega \quad ,$$

and it is a well-known fact in Fourier analysis that this equation holds rigourously pointwise almost everywhere whenever $\widehat{f} \in \mathrm{L}^1(\mathbb{R})$, which is the natural condition to ensure absolute convergence of the integral. The analogous formula for general locally compact groups will be

$$a(x) = \int_{\widehat{G}} \mathrm{trace}(A_\sigma \sigma(x)^*) d\nu_G(\sigma) \quad . \tag{4.1}$$

The operator fields A for which this formula makes sense constitute the Banach space $\mathcal{B}_1^\oplus$ defined in the next lemma. Its proof is standard and therefore omitted.

Lemma 4.1. *Let $\mathcal{B}_1^\oplus$ be the space of measurable fields $(B(\sigma))_{\sigma \in \widehat{G}}$ of trace class operators, for which the norm*

$$\|B\|_{\mathcal{B}_1^\oplus} := \int_{\widehat{G}} \|B(\sigma)\|_1 \, d\nu_G(\sigma)$$

is finite. Here we identify operator fields which agree ν_G-almost everywhere. Then $(\mathcal{B}_1^\oplus, \|\cdot\|_{\mathcal{B}_1^\oplus})$ is a Banach space.

Let us next define the space of functions arising as left-hand sides of (4.1, which is the Fourier algebra.

Definition 4.2. *The* **Fourier algebra** *of G is defined as*

$$A(G) := \mathrm{L}^2(G) * \mathrm{L}^2(G)^* = \{f * g^* : f, g \in \mathrm{L}^2(G)\} \quad .$$

Endowed with the norm

$$\|u\|_{A(G)} = \inf\{\|f\|_2 \|g\|_2 : u = f * g^*\}$$

$A(G)$ becomes a Banach space of C_0-functions, with $\|u\|_{A(G)} \geq \|u\|_\infty$. $A(G)$ is closed with respect to pointwise multiplication and conjugation, which makes $A(G)$ a Banach--algebra.*

Remark 4.3. (a) By definition, $A(G)$ is just the space of coefficient functions for λ_G and its subrepresentations. Hence it seems a natural object of study in connection with continuous wavelet transforms. However, neither the norm on $A(G)$ nor its algebra structure seem to be related in a canonical way to questions concerning admissible vectors and wavelet transforms. Hence the usefulness of $A(G)$ in this connection is dubious. For our purposes, focussing on the Plancherel transform and its inversion, the benefit of considering $A(G)$ mainly lies in clarifying the role of the von Neumann algebra $VN_l(G)$ in connection with inversion formulae, and in the notion of positivity which will be useful for convergence issues.

(b) Note that 4.2 is not the initial definition of $A(G)$ given in [41], but equivalent to the original definition because of [41, Théorème, p. 218]. Likewise, the norm given here is not the original definition in [41], but it coincides with it by [41, Lemme 2.14].
(c) If $u \in A(G)$, so are $u^*, \overline{u}$ and $\check{u}$, the latter defined as $\check{u}(x) = u(x^{-1})$; see [41, Proposition 3.8]. Moreover, we have

$$\|u^*\|_{A(G)} = \|\overline{u}\|_{A(G)} = \|\check{u}\|_{A(G)} \quad .$$

The following theorem was given in [81] for unimodular groups.

Theorem 4.4. *Let $A = (A_\sigma)_{\sigma \in \widehat{G}} \in \mathcal{B}_1^\oplus$. Then*

$$a(x) = \mathcal{F}^{-1}_{A(G)}(A)(x) = \int_{\widehat{G}} \operatorname{trace}(A_\sigma \sigma(x)^*) d\nu_G(\sigma)$$

defines a function $a \in A(G)$. It satisfies the **Parseval equality**

$$\int_G f(x)\overline{a(x)}dx = \int_{\widehat{G}} \operatorname{trace}(\sigma(f)A_\sigma^*) d\nu_G(\sigma) \quad , \tag{4.2}$$

for all $f \in \mathrm{L}^1(G)$. The linear operator $\mathcal{F}^{-1}_{A(G)} : \mathcal{B}_1^\oplus \to A(G)$ is onto.

Proof. Let $A_\sigma = U_\sigma |A_\sigma|$ be the polar decomposition of A_σ. Then $B_{1,\sigma} = U_\sigma |A_\sigma|^{1/2}$ and $B_{2,\sigma} = |A_\sigma|^{1/2}$ defines elements $B_1, B_2 \in \mathcal{B}_2^\oplus$, since

$$\begin{aligned} \|B_1\|^2_{\mathcal{B}_2^\oplus} &= \int_{\widehat{G}} \operatorname{trace}(B_{1,\sigma} B_{1,\sigma}^*) d\nu_G(\sigma) \\ &= \int_{\widehat{G}} \operatorname{trace}(U_\sigma^* U_\sigma |A_\sigma|) d\nu_G(\sigma) \\ &= \|A\|_{\mathcal{B}_1^\oplus} \quad , \end{aligned}$$

and similarly $\|B_2\|^2_{\mathcal{B}_2^\oplus} = \|A\|_{\mathcal{B}_1^\oplus}$. For measurability confer Lemma 3.7. Denoting by $b_1, b_2 \in \mathrm{L}^2(G)$ the respective preimages under the Plancherel transform, we find

$$\begin{aligned} a(x) &= \int_{\widehat{G}} \operatorname{trace}(A_\sigma \sigma(x)^*) d\nu_G(\sigma) \\ &= \int_{\widehat{G}} \operatorname{trace}(B_{1,\sigma} B_{2,\sigma} \sigma(x)^*) d\nu_G(\sigma) \\ &= \langle b_1, \lambda_G(x) b_2 \rangle \\ &= (b_1 * b_2^*)(x) \quad , \end{aligned}$$

hence $a \in A(G)$. If, conversely, $a = b_1 * b_2^* \in A(G)$, define $(B_{i,\sigma})_{\sigma \in \widehat{G}} = \widehat{b_i}$. If we then let $A_\sigma = B_{1,\sigma} B_{2,\sigma}^*$, the same calculation shows that $a = \mathcal{F}^{-1}_{A(G)}(A)$. Hence the mapping is onto.

Finally, let us show the Parseval equality. Let $A_\sigma = B_{1,\sigma}B_{2,\sigma}$ as above, and denote by h_1, h_2 the inverse Plancherel transform of $(B^*_{2,\sigma})_{\sigma\in\widehat{G}}$ and $(B_{1,\sigma})_{\sigma\in\widehat{G}}$, respectively. It follows that

$$\begin{aligned}
\int_{\widehat{G}} \mathrm{trace}(\sigma(f)A^*_\sigma)d\nu_G(\sigma) &= \int_{\widehat{G}} \mathrm{trace}(\sigma(f)B^*_{2,\sigma}B^*_{1,\sigma})d\nu_G(\sigma) \\
&= \int_{\widehat{G}} \mathrm{trace}((f*h_1)^\wedge(\sigma)B^*_{1,\sigma})d\nu_G(\sigma) \\
&= \int_G (f*h_1)(x)\overline{h_2(x)}dx \\
&= \int_G f(y)\int_G h_1(y^{-1}x)\overline{h_2(x)}dxdy \\
&= \int_G f(y)\overline{\int_G h_2(x)\overline{h_1(y^{-1}x)}dx}\ dy \\
&= \int_G f(y)\overline{\int_{\widehat{G}} \mathrm{trace}(B_{1,\sigma}B^*_{\sigma,2}\sigma(y)^*)d\nu_G(\sigma)}\ dy \\
&= \int_G f(y)\overline{a(y)}dy \quad .
\end{aligned}$$

Remark 4.5. Below we will show that the Fourier inversion formula maps $\mathcal{B}_1^\oplus$ isometrically onto $A(G)$. Now we are faced with the somewhat puzzling situation that on the one hand, Plancherel measure – which defines the norm on $\mathcal{B}_1^\oplus$ – is not uniquely given, whereas the norm on $A(G)$ is defined independently of a choice of Plancherel measure.

This apparent contradiction is easily resolved: Suppose that ν_1 and ν_2 are two different choices of Plancherel measure. Denote the corresponding spaces of integrable trace class fields by $\mathcal{B}_1^\oplus(\nu_i)$. Then there exists a canonical isometric isomorphism $T : \mathcal{B}_1^\oplus(\nu_1) \to \mathcal{B}_1^\oplus(\nu_2)$, given by pointwise multiplication with $\frac{d\nu_1}{d\nu_2}$. Moreover, if we denote the corresponding Fourier inversion operators by

$$\mathcal{F}_i^{-1} : \mathcal{B}_1^\oplus(\nu_i) \to A(G) \quad ,$$

an easy computation establishes that $\mathcal{F}_2^{-1} = \mathcal{F}_1^{-1} \circ T$.

It is similarly remarkable that the Duflo-Moore operators do not make a single appearance in this section.

A most useful feature of $A(G)$ is its close relationship to the left von Neumann algebra $VN_l(G)$, as witnessed by the next theorem [41, Théorème 3.10]:

Theorem 4.6. *Let $A(G)'$ denote the Banach space dual of $A(G)$. For all $T \in VN_l(G)$, there exists a unique linear functional $\varphi_T \in A(G)'$ such that,*

$$\varphi_T((f*g^*)^\vee) = \langle Tf, g\rangle \quad .$$

Here $\vee$ denotes the involution from Remark 4.3(c). The mapping $T \mapsto \varphi_T$ is an isometric isomorphism $VN_l(G) \to A(G)'$, which is bicontinuous if $VN_l(G)$ is equipped with the ultra-weak topology and $A(G)'$ with the weak-topology $\sigma(A(G)', A(G))$. Conversely, the ultra-weakly continuous linear forms on $VN_l(G)$ are precisely given by the mappings $T \mapsto \varphi_T(u)$, for fixed $u \in A(G)$.*

Definition 4.7. *The predual property of $A(G)$ allows to lift the action of $VN_l(G)$ on itself to an action on $A(G)$, via duality [41, 3.16]. More precisely, for $T \in VN_l(G)$, let $T \mapsto \check{T}$ denote the adjoint of the involution $a \mapsto \check{a}$. Given $u \in A(G)$, the functional*

$$VN_l(G) \ni S \mapsto \varphi_{\check{T}S}(u)$$

is ultraweakly continuous, hence corresponds to a unique element $Tu \in A(G)$.

For $u \in A(G) \cap \mathrm{L}^2(G)$ and $T \in VN_l(G)$ the notation Tu is ambiguous, but [41, Proposition 3.17] notes that the two possible meanings coincide, and that $u \mapsto Tu$ is in fact norm-continuous on $A(G)$. Hence an alternative way of defining the mapping $u \mapsto Tu$ could proceed by extending it by continuity from $A(G) \cap \mathrm{L}^2(G)$, which by [41, Proposition 3.4] is dense in $A(G)$.

Positivity, as defined next, will be useful in connection with convergence issues.

Definition 4.8. *A function u on G is called* **of positive type** *if for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in G$ the matrix $(u(x_i^{-1}x_j))_{i,j=1,\ldots,n}$ is positive semi-definite. In such a case we write $u \gg 0$. If $u_1, u_2 \gg 0$ with $u_1 - u_2 \gg 0$, we write $u_1 \gg u_2$.*

Remark 4.9. $u \gg 0$ implies $u(x) = \overline{u(x^{-1})}$, i.e. $u = u^*$. Moreover, it is obvious that $u \gg 0$ iff $\overline{u} \gg 0$. Particular examples of functions of positive type are

$$x \mapsto \langle \pi(x)\xi, \xi \rangle \quad , \quad x \mapsto \langle \xi, \pi(x)\xi \rangle \quad ,$$

where π is an arbitrary unitary representation, and $\xi \in \mathcal{H}_\pi$. For the first example, confer [35, 13.4.5], while the second is just the complex conjugate of the first. The first example is the typical way of referring to functions of positive type, while the second is more adapted to coefficient functions.

Taking $\pi = \lambda_G$, we find in particular for all $f \in \mathrm{L}^2(G)$ that $f * f^* \gg 0$. Conversely, [57, p.73, Théorème 17] states that for all $u \in A(G)$ with $u \gg 0$ there exists $f \in \mathrm{L}^2(G)$ such that $u = f * f^*$.

Next let us take a closer look at the duality between $VN_l(G)$ and $A(G)$ on the Fourier side. For this purpose we need an invariance property of Plancherel measure under the taking contragredients. The result is probably folklore; it is mentioned for instance in [81, Section 3]. We have not been able to locate a proof, though. If π, σ are two irreducible representations with $\pi \simeq \overline{\sigma}$, we

use the unique unitary intertwining operator $\mathcal{H}_\pi \to \mathcal{H}_\sigma$ to identify π with the standard realization of $\overline{\sigma}$. This needs to be kept in mind in the next two lemmas, where we do not explicitly distinguish between operators on $\mathcal{H}_\sigma$ and operators on $\mathcal{H}_{\overline{\sigma}}$, even though for a given measurable realization of $\widehat{G}$ the representative of $[\overline{\sigma}]$ will not necessarily be the standard realization of the representative of $[\sigma]$.

Lemma 4.10. *(a) If G is unimodular and ν_G is the canonical choice of Plancherel measure, then ν_G is invariant under the mapping $\sigma \mapsto \overline{\sigma}$.*

(b) If G is nonunimodular, and ν_G is constructed from the canonical Plancherel measure of $\mathrm{Ker}(\Delta_G)$ according to Proposition 3.50, then ν_G is invariant under the mapping $\sigma \mapsto \overline{\sigma}$.

Proof. For part (a) we let $\overline{\nu}$ denote the measure given by $\overline{\nu}(A) = \nu_G(\{\overline{\sigma} : \sigma \in A\})$. We first show that $\overline{\nu}$ is ν_G-absolutely continuous. For this purpose observe that $\lambda_G \times \varrho_G \simeq \int_{\widehat{G}} \sigma \otimes \overline{\sigma} d\nu_G(\sigma)$. Now the fact that taking contragredients commutes with taking direct integrals and tensor products shows that $\overline{\lambda_G} \simeq \varrho_G$. On the other hand, $\lambda_G \simeq \varrho_G$ via the involution $f \mapsto \Delta_G^{-1/2} f^*$. Hence $\lambda_G \simeq \overline{\lambda_G}$. Now $\pi = \int_{\widehat{G}}^{\oplus} \sigma d\nu_G(\sigma)$ is multiplicity-free and quasi-equivalent to λ_G, while $\overline{\pi} = \int_{\widehat{G}}^{\oplus} \sigma d\overline{\nu}(\sigma)$ is multiplicity-free and quasi-equivalent to $\varrho_G \simeq \lambda_G$. Hence $\pi \approx \overline{\pi}$, which by [35, 5.4.6] entails that $\pi \simeq \overline{\pi}$. But then ν_G and $\overline{\nu}$ are equivalent.

In addition, we obtain that the unique unitary operators $\mathcal{H}_\sigma \mapsto \mathcal{H}_{\overline{\sigma}}$ intertwining the realization of $\overline{\sigma}$ used in the Plancherel decomposition with the standard realization on $\mathcal{H}_\sigma$, constitute a measurable field of operators (outside a ν_G-nullset); confer [36, Theorem 4, p.238]. Hence the identification of operators on $\mathcal{H}_\sigma$ and $\mathcal{H}_{\overline{\sigma}}$ can be obtained by a measurable field of operators.

As a first consequence, we see that the operator

$$(A_\sigma)_{\sigma \in \widehat{G}} \mapsto (A_{\overline{\sigma}})_{\sigma \in \widehat{G}} \quad ,$$

defined for all $A \in \mathcal{B}_2^{\oplus}$ for which the right hand side is in $\mathcal{B}_2^{\oplus}$, is densely defined and closed: It factors into the unitary map

$$T : (A_\sigma)_{\sigma \in \widehat{G}} \mapsto \left(A_{\overline{\sigma}} \sqrt{\frac{d\overline{\nu}}{d\nu_G}}(\sigma) \right)_{\sigma \in \widehat{G}}$$

followed by the densely defined closed operator consisting in pointwise multiplication with $\sqrt{\frac{d\nu_G}{d\overline{\nu}}}$.

Next we recall that the canonical choice of Plancherel measure entails for $f \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G)$ that $\widehat{f}(\sigma) = \sigma(f)$. A simple computation establishes

$$T(\widehat{f})(\sigma) = \overline{\sigma}(f) = \overline{\sigma(\overline{f})} = \overline{\widehat{\overline{f}}(\sigma)} \quad . \tag{4.3}$$

Note that this entails in particular that $\mathcal{P}(\mathrm{L}^1(G) \cap \mathrm{L}^2(G)) \subset \mathrm{dom}(T)$. Since taking conjugates of L^2-functions and Hilbert-Schmidt operators are obviously isometric operations, we obtain that

$$(\widehat{f}(\sigma))_{\sigma\in\widehat{G}} \mapsto \left(\overline{\widehat{\overline{f}}(\sigma)}\right)_{\sigma\in\widehat{G}}$$

is a unitary operator on $\mathcal{B}_2^{\oplus}$, and T is a densely defined closed operator coinciding with it on a dense subspace. Hence T is unitary itself. But then the multiplication operator arising from the Radon-Nikodym derivative is unitary also, which entails $\frac{d\overline{\nu}}{d\nu_G} \equiv 1$.

This proves the statement concerning unimodular G. The statement for nonunimodular groups follows from this, the fact that $\mathrm{Ind}_N^G\overline{\sigma} \simeq \overline{\mathrm{Ind}_N^G\sigma}$ [86, Theorem 5.1], and the construction of ν_G from a measure decomposition of $\nu_{\mathrm{Ker}(\Delta_G)}$, as sketched in Section 3.8.

Lemma 4.11. *Let $A \in \mathcal{B}_1^{\oplus}$, and $a = \mathcal{F}_{A(G)}^{-1}(A)$. Let $T \simeq (\widehat{T}_\sigma \otimes 1)_{\sigma\in\widehat{G}} \in VN_l(G)$. Then*

$$\varphi_T(a) = \int_{\widehat{G}} \mathrm{trace}(\widehat{T}_\sigma A_{\overline{\sigma}}^t) d\nu_G(\sigma) \quad . \tag{4.4}$$

Proof. In view of 4.10 we may assume that the Plancherel measure is invariant under taking contragredients. Hence, if $(A_\sigma)_{\sigma\in\widehat{G}} \in \mathcal{B}_1^{\oplus}$, then $(A_{\overline{\sigma}}^t)_{\sigma\in\widehat{G}} \in \mathcal{B}_1^{\oplus}$ as well. In particular, for fixed A the right hand side of (4.4) is ultraweakly continuous as a function of T.

By Theorem 4.6, the mapping $T \mapsto \varphi_T(a)$ is ultraweakly continuous as well, hence it remains to check (4.4) for $T = \lambda_G(x)$, $x \in G$ (these operators span a dense subalgebra). But here we have by [41, 3.14 Remarque], that

$$\begin{aligned}
\varphi_T(a) &= a(x) \\
&= \int_{\widehat{G}} \mathrm{trace}(A_\sigma \sigma(x)^*) d\nu_G(\sigma) \\
&= \int_{\widehat{G}} \mathrm{trace}(A_{\overline{\sigma}}\overline{\sigma}(x)^*) d\nu_G(\sigma) \\
&= \int_{\widehat{G}} \mathrm{trace}(A_{\overline{\sigma}}^t \sigma(x)) d\nu_G(x) \\
&= \int_{\widehat{G}} \mathrm{trace}(\sigma(x) A_{\overline{\sigma}}^t) d\nu_G(x) \quad .
\end{aligned}$$

Here the penultimate equality used the relations $\overline{\sigma(x)^*} = \sigma(x)^t$ and $\mathrm{trace}(ST^t) = \mathrm{trace}(S^tT)$. Since $\lambda_G(x) \simeq (\sigma(x) \otimes 1)_{\sigma\in\widehat{G}}$, we are done.

Now we can establish the Fourier transform on $A(G)$. For unimodular groups, the theorem is [81, Theorem 3.1], for nonunimodular groups it is new.

Theorem 4.12. *(a) $\mathcal{F}_{A(G)}^{-1}$, as defined in Theorem 4.4, is one-to-one. The inverse operator*

$$\mathcal{F}_{A(G)} : A(G) \to \mathcal{B}_1^{\oplus}$$

is thus a bijection, in fact an isometry of Banach spaces.

(b) If $T \simeq \left(\widehat{T}_\sigma \otimes 1\right)_{\sigma\in\widehat{G}} \in VN_l(G)$ *and* $a \in A(G)$*, then*

$$\mathcal{F}_{A(G)}(Ta) = \left(\widehat{T}_\sigma \mathcal{F}_{A(G)}(a)(\sigma)\right)_{\sigma\in\widehat{G}} \quad .$$

(c) Let $a = \mathcal{F}^{-1}_{A(G)}(A)$*. Then*

$$a \gg 0 \Longleftrightarrow A_\sigma \geq 0 \quad (\nu_G - a.e.)$$

Proof. Define $a = \mathcal{F}^{-1}_{A(G)}(A)$, and let $A_\sigma = U_\sigma |A_\sigma|$ denote the polar decomposition. Define $B_{1,\sigma}$ and $B_{2,\sigma}$ as in the proof of 4.4. We recall that $\|B_i\|_{\mathcal{B}_2^\oplus} = \|A\|^{1/2}_{\mathcal{B}_1^\oplus}$, and $\mathcal{F}^{-1}_{A(G)}(A) = b_1 * b_2^*$. Then $a = b_1 * b_2^*$ implies that

$$\|f\|_{A(G)} \leq \|b_1\|_2 \|b_2\|_2 = \|A\|_{\mathcal{B}_1^\oplus} \quad .$$

For the converse direction define $T \in VN_l(G)$ as $\widehat{T}_\sigma = (U^*_{\overline{\sigma}})^t \otimes 1$. Then the previous lemma entails

$$\varphi_T(a) = \int_{\widehat{G}} \mathrm{trace}(U_{\overline{\sigma}} U^*_{\overline{\sigma}} |A_{\overline{\sigma}}|) d\nu_G(\sigma) = \|A\|_{\mathcal{B}_1^\oplus} \quad .$$

But $a \mapsto \varphi_\bullet(a)$ is an isometric embedding of $A(G)$ into $VN_l(G)'$, by 4.6. In addition, $\|T\|_\infty \leq 1$, being defined by a field of partial isometries. Hence $\|a\|_{A(G)} \geq \|A\|_{\mathcal{B}_1^\oplus}$, and $\mathcal{F}^{-1}_{A(G)}$ is shown to be isometric, in particular one-to-one. This closes the proof of (a).

For the proof of (b) fix $T \in VN_l(G)$, and denote the corresponding operator field on the Plancherel transform side by $(\widehat{T}_\sigma \otimes 1)_{\sigma\in\widehat{G}}$. Consider the mappings

1. $a \mapsto Ta$
2. $a \mapsto \mathcal{F}^{-1}_{A(G)} \left(\left(\widehat{T}_\sigma \mathcal{F}_{A(G)}(a)(\sigma) \right)_{\sigma\in\widehat{G}} \right)$

on $A(G)$. We claim that they coincide on the subspace $C_c(G) * C_c(G) \subset A(G)$, which is dense by [41, Proposition 3.4]. Indeed, let $a = f * g^*$ with $f, g \in C_c(G)$. The usual calculation shows that $\mathcal{F}_{A(G)}(a) = (\widehat{f}(\sigma)\widehat{g}(\sigma)^*)_{\sigma\in\widehat{G}}$. g is a bounded vector by 2.19, i.e., $V_g \in VN_r(G)$. Since $VN_l(G)$ and $VN_r(G)$ commute,

$$\begin{aligned} [T(a)](x) &= [T(V_g f)](x) = [V_g(T(f))](x) = [T(f) * g^*](x) \\ &= \int_{\widehat{G}} \mathrm{trace}\left(\widehat{T}_\sigma \widehat{f}(\sigma) \widehat{g}(\sigma)^* \sigma(x)^* \right) d\nu_G(\sigma) \quad , \end{aligned}$$

which shows the claim. Now mapping 1. is bounded with respect to $\|\cdot\|_{A(G)}$ by [41, Proposition 3.17]. On the other hand, since $\mathcal{F}_{A(G)}$ is isometric, we only need to show that $(\widehat{T}_\sigma \otimes 1)_{\sigma\in\widehat{G}}$ acts as a bounded operator on $\mathcal{B}_1^\oplus$, which follows trivially from the boundedness of T. Hence we have shown (b) on all of $A(G)$.

For (c) first assume $A_\sigma > 0$, ν_G-almost everywhere. Then taking $\widehat{g}_\sigma = A_\sigma^{1/2}$ defines $\widehat{g} \in \mathcal{B}_2^\oplus$, and the calculation

$$\mathcal{F}^{-1}_{A(G)}(A)(x) = \int_{\widehat{G}} \operatorname{trace}(\widehat{g}_\sigma \widehat{g}_\sigma^* \sigma(x)^*) d\nu_G(\sigma) = g * g^*(x)$$

shows that indeed $\mathcal{F}^{-1}_{A(G)}(A) \gg 0$. Conversely, if $a = \mathcal{F}^{-1}_{A(G)}(A) \gg 0$, then $a = g * g^*$, and a similar calculation establishes that $\mathcal{F}_{A(G)}(a) = (\widehat{g}(\sigma)\widehat{g}(\sigma)^*)_{\sigma \in \widehat{G}}$, which clearly is a field of positive operators.

The following lemma will be instrumental in establishing the Plancherel inversion formula.

Lemma 4.13. *Let G be unimodular. Suppose that $u_1, u_2 \in \mathrm{L}^2(G) \cap A(G)$, with $u_2 \gg u_1 \gg 0$, and that u_1 is a bounded vector. Then $\|u_2\|_2 \geq \|u_1\|_2$.*

Proof. It suffices to show that

$$\langle u_1, u_2 \rangle \geq \|u_1\|_2^2 \quad ,$$

since this implies

$$\|u_2\|_2^2 - \|u_1\|_2^2 \geq \|u_2\|_2^2 + \|u_1\|_2^2 - 2\langle u_1, u_2 \rangle = \|u_2 - u_1\|_2^2 \quad .$$

By assumption, there exists $\psi \in \mathrm{L}^2(G)$ such that $u_2 - u_1 = \psi * \psi$. Pick $(\varphi_n)_{n \in \mathbb{N}} \subset C_c(G)$ converging to ψ in $\|\cdot\|_2$. Then

$$\begin{aligned} |\langle \psi, u_1 * \psi \rangle - \langle \varphi_n, u_1 * \varphi_n \rangle| &\leq |\langle \psi - \varphi_n, u_1 * \psi \rangle| + |\langle \varphi_n, u_1 * (\varphi_n - \psi) \rangle| \\ &\leq \|\psi - \varphi_n\|_2 \|u_1 * \psi\|_2 + \|\varphi_n\|_2 \|u_1 * (\phi_n - \psi)\|_2 \to 0 \end{aligned}$$

using the boundedness of u_1. On the other hand, $\langle \varphi_n * \varphi_n^*, u_1 \rangle \geq 0$ by positivity of u_1 ([35, 13.4.4], observe that for unimodular groups our involution coincides with the one in [35]). Hence

$$\langle u_2 - u_1, u_1 \rangle = \lim_{n \to \infty} \langle \varphi_n * \varphi_n^*, u_1 \rangle \geq 0 \quad ,$$

which finishes the proof.

4.2 Plancherel Inversion*

In this section we discuss the L^2-functions which can be obtained by the Fourier inversion formula (4.1). Clearly these functions are precisely the intersection $A(G) \cap \mathrm{L}^2(G)$. However, the description on the Plancherel transform side is much less obvious. In view of the last section, any theorem describing pointwise Plancherel inversion via (4.1) will contain some statement on the relationship between the spaces $\mathcal{B}_1^\oplus$ and $\mathcal{B}_2^\oplus$ and the operators $\mathcal{P}$ and $\mathcal{F}_{A(G)}$.

A first conjecture, which turns out to be correct for unimodular groups, could be that $\mathcal{F}_{A(G)}^{-1}$ maps $\mathcal{B}_2^{\oplus} \cap \mathcal{B}_1^{\oplus}$ bijectively onto $\mathrm{L}^2(G) \cap A(G)$. This statement implies in particular that $\mathcal{F}_{A(G)}$ and $\mathcal{P}$ coincide on $\mathrm{L}^2(G) \cap A(G)$.

In the nonunimodular case however, this cannot be expected. Using the next lemma, it is possible to compute at least for $V_g f$, $f \in \mathrm{L}^2(G), g \in C_c(G)$, what the image under Plancherel transform is:

$$(V_g f)^\wedge(\sigma) = \widehat{f}(\sigma)\sigma(\Delta_G^{-1/2} g^*) = [\widehat{f}(\sigma)\widehat{g}(\sigma)^* C_\sigma] \quad .$$

Recalling that operator fields of the form $\left(\widehat{f}(\sigma)\widehat{g}(\sigma)\right)_{\sigma \in \widehat{G}}$ are typical elements of $\mathcal{B}_1^{\oplus}$, this calculation motivates the conjecture that $\mathrm{L}^2(G) \cap A(G)$ is the image under $\mathcal{F}_{A(G)}^{-1}$ of the space

$$\{(A_\sigma)_{\sigma \in \widehat{G}} \in \mathcal{B}_1^{\oplus} : ([A_\sigma C_\sigma])_{\sigma \in \widehat{G}} \in \mathcal{B}_2^{\oplus}\} \quad .$$

Theorem 4.15 below proves this conjecture. But first a few basic computations concerning the interplay between convolution with functions in $C_c(G)$ and the Duflo-Moore operators. Not all of them are needed, but we include them for completeness.

Lemma 4.14. *Let $f \in C_c(G)$.*

(i) For ν_G-almost every σ, we have $\widehat{f}(\sigma)^* = C_\sigma^{-1}\sigma(\Delta_G^{-1} f^*)$. In particular the right hand side is everywhere defined and bounded.

(ii) For ν_G-almost every σ, we have

$$[\sigma(f)C_\sigma^{-1}] = C_\sigma^{-1}\sigma(\Delta_G^{-1/2} f) \quad ,$$

in particular the right hand side is everywhere defined and bounded.

(iii) For all $g \in \mathrm{L}^2(G)$, we have

$$\widehat{(g * f)}(\sigma) = \widehat{g}(\sigma)\sigma(\Delta_G^{-1/2} f) \quad ,$$
$$\widehat{(f * g)}(\sigma) = \sigma(f) \ .$$

(iv) For all $g \in \mathrm{L}^2(G)$, we have

$$(\Delta_g^{-1/2} g^*)^\wedge(\sigma) = \widehat{g}(\sigma)^* \quad .$$

Proof. For part (i) we invoke [104, Theorem 13.2], to find that, since C_σ^{-1} is selfadjoint and $\sigma(f)$ is bounded, $(\sigma(f)C_\sigma^{-1})^* = C_\sigma^{-1}\sigma(f)^*$. Moreover, since $\widehat{f}(\sigma)$ is bounded, the right hand side of the last equation is everywhere defined. We conclude the proof of (i) by computing

$$\begin{aligned}
\langle \sigma(\Delta_G^{-1} f^*)\phi, \eta\rangle &= \int_G \langle \Delta_G^{-1}(x)\overline{f(x^{-1})}\sigma(x)\phi, \eta\rangle dx \\
&= \int_G \langle \phi, f(x^{-1})\sigma(x^{-1})\eta\rangle \Delta_G(x)^{-1} dx \\
&= \int_G \langle \phi, f(x)\sigma(x)\eta\rangle dx \\
&= \langle \phi, \sigma(f)\eta\rangle \quad .
\end{aligned}$$

For (ii) we first note that by (i), applied to $\Delta_G^{-1/2} f \in C_c(G)$, the right hand side is bounded and everywhere defined. Moreover, the left-hand side is bounded since $f \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G)$. It thus remains to show that the equality holds on the dense subspace $\mathrm{dom}(C_\sigma^{-1})$: For $\phi, \eta \in \mathrm{dom}(C_\sigma^{-1})$ the definition of the weak operator integral yields

$$\begin{aligned}
\langle \phi, \sigma(f) C_\sigma^{-1} \eta \rangle &= \int_G \langle \phi, \sigma(x) C_\sigma^{-1} \eta \rangle f(x) d\mu_G(x) \\
&= \int_G \langle \phi, \Delta_G(x)^{-1/2} C_\sigma^{-1} \sigma(x) \eta \rangle f(x) d\mu_G(x) \\
&= \int_G \langle C_\sigma^{-1} \phi, \sigma(x) \eta \rangle \Delta_G(x)^{-1/2} f(x) d\mu_G(x) \\
&= \langle C_\sigma^{-1} \phi, \sigma(\Delta_G^{-1/2} f) \eta \rangle \\
&= \langle \phi, C_\sigma^{-1} \sigma(\Delta_G^{-1/2} f) \eta \rangle \quad ,
\end{aligned}$$

where the second equality uses the semi-invariance relation (3.51), and the selfadjointness of C_σ^{-1} was used on various occasions. This shows (ii).

Part (iii) is then immediate from (i) and (ii), at least for $g \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G)$. It extends by continuity to all of $\mathrm{L}^2(G)$: The left-hand sides are continuous operators, being convolution operators with $f \in C_c(G)$ (see 2.19(b)), and the right hand sides are continuous because of inequality (3.19).

For part (iv), we first observe that both sides of the equation are unitary mappings, hence it is enough to check the equality on $C_c(G)$. Here (i) and (ii) give $\widehat{g}(\sigma)^* = C_\sigma^{-1} \sigma(\Delta_G^{-1} g^*) = \sigma(\Delta_G^{-1/2} g^*) C_\sigma^{-1} = (\Delta_G^{-1/2} g)\widehat{\ }(\sigma)$.

The following theorem is one of the central new results of this book. For unimodular groups, it is stated in [81, Corollary 2.4, Corollary 2.5], though we will point out below that the argument in [81] seems to contain a gap. In any case, the nonunimodular part is new; one direction was proved in [4].

Theorem 4.15. *Let* $(A_\sigma)_{\sigma \in \widehat{G}} \in \mathcal{B}_1^\oplus$*, and define*

$$a(x) = \int_{\widehat{G}} \mathrm{trace}(A_\sigma \sigma(x)^*) d\nu_G(\sigma) \quad . \tag{4.5}$$

Then $a \in \mathrm{L}^2(G)$ *iff* $([A_\sigma C_\sigma])_{\sigma \in \widehat{G}} \in \mathcal{B}_2^\oplus$*. In that case we have* $([A_\sigma C_\sigma])_{\sigma \in \widehat{G}} = \widehat{a}$*.*

Proof. First assume that $([A_\sigma C_\sigma])_{\sigma \in \widehat{G}} \in \mathcal{B}_2^\oplus$, and let b be the inverse image of that under the Plancherel transform. Then for any $g \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G)$,

$$\begin{aligned}\langle g, b\rangle &= \int_{\widehat{G}} \operatorname{trace}(\widehat{g}(\sigma)[A_\sigma C_\sigma]^*) d\nu_G(\sigma) \\ &= \int_{\widehat{G}} \operatorname{trace}([\sigma(g)C_\sigma^{-1}][A_\sigma C_\sigma]^*) d\nu_G(\sigma) \\ &= \int_{\widehat{G}} \operatorname{trace}(\sigma(g)A_\sigma^*) d\nu_G(\sigma) \\ &= \int_G g(x)\overline{a(x)} dx \quad ,\end{aligned}$$

where the last equation is due to (4.2). Now Lemma 2.17 yields $b = a$ almost everywhere, and thus $\widehat{a} = ([A_\sigma C_\sigma])_{\sigma\in\widehat{G}}$.

Let us now show the converse direction. Assume that $a \in \mathrm{L}^2(G)$. Let $A_\sigma = U_\sigma|A_\sigma|$ be the polar decomposition of A_σ, let $U \simeq (U_\sigma \otimes 1)_{\sigma\in\widehat{G}}$ the element in $VN_l(G)$ corresponding to the partial isometries, and let

$$h(x) = \int_{\widehat{G}} \operatorname{trace}(|A_\sigma|\sigma(x)^*) d\nu_G(\sigma) \quad .$$

By Theorem 4.12(b) we have that $h = U^*a$, where the right hand side denotes the action of $VN_l(G)$ on $A(G)$. But $a \in \mathrm{L}^2(G)$ then implies $h \in \mathrm{L}^2(G)$, by [41, 3.17]. Since $(A_\sigma)_{\sigma\in\widehat{G}} \in \mathcal{B}_2^\oplus$ iff $(|A_\sigma|)_{\sigma\in\widehat{G}} \in \mathcal{B}_2^\oplus$, we may thus assume w.l.o.g. that $A_\sigma \geq 0$.

Now assume G to be unimodular. Pick an increasing sequence of Borel sets $\Sigma_n \subset \widehat{G}$ with $\|A_\sigma\|_2 \leq n$ on Σ_n, $\nu_G(\Sigma_n) < \infty$ and $\bigcup_{n\in\mathbb{N}} \Sigma_n = \widehat{G}$ (up to a Plancherel nullset). Consider the fields $A^n = (\mathbf{1}_{\Sigma_n}(\sigma)A_\sigma)_{\sigma\in\widehat{G}} \in \mathcal{B}_2^\oplus$. Then the direction proved first shows that

$$a_n(x) = \int_{\widehat{G}} \operatorname{trace}(A_\sigma^n \sigma(x)^*) d\nu_G(\sigma)$$

defines a sequence in $\mathrm{L}^2(G)$, with $\widehat{a_n} = (A_\sigma^n)_{\sigma\in\widehat{G}}$. Observe that

$$(a - a_n)(x) = \int_{\widehat{G}\setminus\Sigma_n} \operatorname{trace}(A_\sigma \sigma(x)^*) d\nu_G(\sigma) \quad ,$$

whence Theorem 4.12 (c) yields $a \gg a_n$. Next we check that the a_n are bounded vectors: By construction, $\|A_\sigma^n\|_\infty \leq \|A_\sigma^n\|_2 \leq n$, hence for all $f \in \mathrm{L}^2(G)$

$$V_{a_n} f(x) = \int_{\widehat{G}} \operatorname{trace}(\widehat{f}(\sigma)A_\sigma^n \sigma^*(x)) d\nu_G(\sigma)$$

where $(\widehat{f}(\sigma)A_\sigma^n)_{\sigma\in\widehat{G}} \in \mathcal{B}_2^\oplus$. Hence the previously established direction yields $V_{a_n} f \in \mathrm{L}^2(G)$, i.e., a_n is bounded. Now we may apply Lemma 4.13, implying $\|a_n\|_2 \leq \|a\|_2$. But then $\|A^n\|_{\mathcal{B}_2^\oplus} \leq \|a\|_2$, and thus $A \in \mathcal{B}_2^\oplus$.

Hence it remains to prove $a \in \mathrm{L}^2(G) \Rightarrow ([A_\sigma C_\sigma])_{\sigma\in\widehat{G}} \in \mathcal{B}_2^\oplus$ for G nonunimodular. In the following we use the notations from Subsection 3.8.2. Recall

in particular, that $\sigma = \mathrm{Ind}_N^G \sigma_0$, and that A_σ is given by a Hilbert-Schmidt valued kernel $(A_\sigma(\xi,\xi'))_{\xi,\xi' \in H}$. Again writing $A_\sigma = F_\sigma G_\sigma$, with $F_\sigma, G_\sigma \in \mathcal{B}_2^\oplus$, we appeal to 3.47 (a) and find that the integral kernel of A_σ is computed from the kernels of F_σ, G_σ by

$$A_\sigma(\xi,\xi'') = \int_H F_\sigma(\xi,\xi') G_\sigma(\xi',\xi'') d\mu(\xi') \quad ,$$
$$\text{whenever } F_\sigma(\xi,\cdot), G_\sigma(\cdot,\xi'') \in \mathrm{L}^2(H;\mathcal{B}_2(\mathcal{H}_\sigma)) \quad .$$

Lemma 3.47 (a) states that the right hand side is a Bochner integral converging in $\mathcal{B}_1^\oplus(\mathcal{H}_{\sigma_0})$.

We next compute the integral kernel of $A_\sigma \sigma(g)^*$. Relation (3.56) implies for $g = g_0\alpha(\gamma)$ and $\sigma_0 = \tau(\sigma)$ that $\sigma(g)A_\sigma^*$ has the integral kernel

$$(\xi,\xi') \mapsto (\xi.\sigma_0)(g_0) \circ \sigma_0(\Lambda(\gamma,\xi)) \circ A_\sigma(\xi',\gamma^{-1}\xi)^* \quad .$$

Transposing yields the kernel

$$B_{\sigma,\gamma} : (\xi,\xi') \mapsto A_\sigma(\xi,\gamma^{-1}\xi') \circ \sigma_0(\Lambda(\gamma,\xi'))^* \circ (\xi'.\sigma_0)(g_0)^*$$

for $A_\sigma\sigma(g)^*$.

By Fubini's theorem, the mapping $a(\cdot,\gamma) : g_0 \mapsto a(g_0,\gamma)$ is in $\mathrm{L}^2(N)$, for almost all γ. We intend to apply the unimodular part of the theorem to these functions. Plugging in the kernel for $A_\sigma\sigma(g)^*$ and using the trace formula (3.40), we obtain

$$\begin{aligned} a(g_0,\gamma) &= \\ &= \int_{\widehat{G}} \mathrm{trace}(A_\sigma\sigma(x)^*)\ d\nu_G(\sigma) \\ &= \int_{\widehat{G}} \int_H \mathrm{trace}\left(A_\sigma(\xi,\gamma^{-1}\xi)\sigma_0(\Lambda(\gamma,\xi))(\xi'.\sigma_0)(g_0)^*\right) d\xi d\nu_G(\sigma) \\ &= \int_{\widehat{N}} \mathrm{trace}\left(A_\sigma(\xi,\gamma^{-1}\xi)\sigma_0(\Lambda(\gamma,\xi))\Delta_G(\xi)^{-1}\sigma(g_0)^*\right) d\nu_N(\xi,\sigma_0) \quad , \quad (4.6) \end{aligned}$$

where the last equation uses (3.54). Next we estimate

$$\begin{aligned} &\int_{\widehat{N}} \|A_\sigma(\xi,\gamma^{-1}\xi)\sigma_0(\Lambda(\gamma,\xi))\Delta_G(\xi)^{-1}\|_1 d\nu_N(\xi,\sigma_0) = \\ &= \int_{\widehat{G}} \int_H \|A_\sigma(\xi,\gamma^{-1}\xi)\|_1 d\xi d\nu_G(\sigma) \\ &\leq \int_{\widehat{G}} \|F_\sigma\|_2 \|G_\sigma\|_2 d\nu_G(x) \leq \|F\|_{\mathcal{B}_2^\oplus} \|G\|_{\mathcal{B}_2^\oplus} \quad , \end{aligned}$$

where the first inequality is due to (3.41), and the second is again Cauchy-Schwarz. Hence we see that

$$D^\gamma(\sigma_0,\xi) = A_\sigma(\xi,\gamma^{-1}\xi)\sigma_0(\Lambda(\gamma,\xi))\Delta_G(\xi)^{-1} \quad (\sigma_0 \in U_0, \xi \in H)$$

defines a measurable operator field $D^\gamma \in \mathcal{B}_1^\oplus(N)$, and that (4.6) is Fourier inversion applied to this field. Now for every γ with $a(\cdot,\gamma) \in \mathrm{L}^2(N)$ the unimodular part of the theorem implies that $D^\gamma \in \mathcal{B}_2^\oplus(N)$, with

$$\|a(\cdot,\gamma)\|_2^2 = \int_{\widehat{N}} \|D^\gamma(\sigma_0,\xi)\|_2^2 d\nu_N(\sigma_0,\xi) \quad .$$

But then Fubini's theorem and (3.55) imply

$$\begin{aligned}
\|a\|_2^2 &= \int_H \|a(\cdot,\gamma)\|_2^2 \Delta_G(\gamma) d\gamma \\
&= \int_H \int_{U_0} \int_H \|D^\gamma(\sigma_0,\xi)\|_2^2 \Delta_G(\xi) d\xi d\nu_G(\sigma_0) \Delta_G(\gamma) d\gamma \\
&= \int_H \int_{\widehat{G}} \int_H \|A_\sigma(\xi,\gamma^{-1}\xi)\sigma_0(\Lambda(\gamma,\xi))\|_2^2 \Delta_G(\gamma\xi^{-1}) d\gamma d\nu_G(\sigma) \\
&= \int_{\widehat{G}} \int_H \int_H \|A_\sigma(\xi,\gamma^{-1}\xi)\|_2^2 \Delta_G(\gamma\xi^{-1}) d\gamma d\xi d\nu_G(\sigma) \\
&= \int_{\widehat{G}} \int_H \int_H \|A_\sigma(\xi,\gamma)\|_2^2 \Delta_G(\gamma^{-1}) d\gamma d\xi d\nu_G(\sigma) \\
&= \int_{\widehat{G}} \int_H \int_H \|A_\sigma(\xi,\gamma)\Delta_G(\gamma)^{-1/2}\|_2^2 d\gamma d\xi d\nu_G(\sigma) \quad .
\end{aligned}$$

Hence we find in particular that for ν_G-almost every $\sigma \in \widehat{G}$ the operator with kernel

$$(\xi,\gamma) \mapsto A_\sigma(\xi,\gamma)\Delta_G(\gamma)^{-1/2}$$

is Hilbert-Schmidt. On the other hand, recalling that C_σ acts via the multiplication with $\Delta_G(\gamma)^{-1/2}$, we see that $A_\sigma C_\sigma$ coincides with this Hilbert-Schmidt operator on $\mathrm{dom}(C_\sigma)$. Hence $[A_\sigma C_\sigma]$ exists and is in $\mathcal{B}_2(\mathcal{H}_\sigma)$, and we conclude

$$\|a\|_2^2 = \int_{\widehat{G}} \|[A_\sigma C_\sigma]\|_2^2 d\nu_G(\sigma) \quad ,$$

which finishes the proof.

Remark 4.16. (a) The unimodular version of the Plancherel inversion theorem was shown in [81], and the proof of

$$([A_\sigma C_\sigma])_{\sigma\in\widehat{G}} \in \mathcal{B}_2^\oplus \Rightarrow a \in \mathrm{L}^2(G) \tag{4.7}$$

immediately carries over to the general setting. Note that the argument relies on Lemma 2.17.
This observation is crucial in connection with the proof for the converse direction

$$a \in \mathrm{L}^2(G) \Rightarrow ([A_\sigma C_\sigma])_{\sigma\in\widehat{G}} \in \mathcal{B}_2^\oplus \quad .$$

The argument given in [81] for unimodular groups uses (4.2) to establish

$$\int_G a(x)\overline{k(x)}dx = \int_{\widehat{G}} \mathrm{trace}(A_\sigma \widehat{k}(\sigma)^*)d\nu_G(\sigma) \quad ,$$

for all $k \in \mathrm{L}^1(G)\cap\mathrm{L}^2(G)$, and then concludes by density of $\mathcal{P}(\mathrm{L}^1(G)\cap\mathrm{L}^2(G))$ in $\mathcal{B}_2^\oplus$ that $\widehat{a} = A$. This is the mirror image of the argument for $([A_\sigma C_\sigma])_{\sigma\in\widehat{G}} \in \mathcal{B}_2^\oplus \Rightarrow a \in \mathrm{L}^2(G)$, with $\mathcal{P}(\mathrm{L}^1(G)\cap\mathrm{L}^2(G))$ replacing $\mathrm{L}^1(G)\cap\mathrm{L}^2(G)$. But in Remark 2.18 we saw that density alone is insufficient, and an analogue of 2.17 for $\mathcal{P}(\mathrm{L}^1(G)\cap\mathrm{L}^2(G))$ instead of $\mathrm{L}^1(G)\cap\mathrm{L}^2(G)$ does not seem easily available.

This is why the argument presented here is rather more complicated than the one given in [81]. Note in particular that we used the action of $VN_l(G)$ on $A(G)$ and the notion of positivity from Section 4.1. A substantially shorter argument could be provided if the following rather intuitive result from integration theory were true. Unfortunately I have not been able to prove it:
Let a sequence $(a_n)_{n\in\mathbb{N}} \subset \mathrm{L}^2(G)$ be given with $a_n \to a$ uniformly. Suppose that the a_n have orthogonal increments. Then $a \in \mathrm{L}^2(G)$, with $\|a\|_2 \geq \|a_n\|_2$ for all n.
Note however that this observation only allows to shorten the unimodular part of the proof.
(b) The implication (4.7), with a much more complicated proof, may be found in [4]. This direction allows to prove sufficient admissibility conditions, as shown in [53]. For necessity of these conditions however, the converse of (4.7) seems indispensable.

4.3 Admissibility Criteria

In this section we solve **T1** through **T3** for the general setting, and discuss the qualitative uncertainty property.

Since leftinvariant subspaces correspond to projections in $VN_r(G)$, we obtain an answer to **T1** as an immediate consequence of the Theorem 3.48 (c).

Corollary 4.17. *Let $\mathcal{H} \subset \mathrm{L}^2(G)$ be a closed leftinvariant subspace. The projection P onto $\mathcal{H}$ corresponds to a field of projections $P \simeq (1 \otimes \widehat{P}_\sigma)_{\sigma\in\widehat{G}}$, i.e.*

$$(Pf)^\wedge(\sigma) = \widehat{f}(\sigma) \circ \widehat{P}_\sigma \quad . \tag{4.8}$$

Next let us consider admissible vectors. The proof turns out to be largely analogous to the proof for the toy example, once the L^2-convolution theorem is established. But this is now a formality.

Theorem 4.18. *For $f, g \in \mathrm{L}^2(G)$ we have $V_g f \in \mathrm{L}^2(G)$ iff $([\widehat{f}(\sigma)\widehat{g}(\sigma)^* C_\sigma])_{\sigma\in\widehat{G}} \in \mathcal{B}_2^\oplus$. In this case, we have ν_G-almost everywhere*

$$(V_g f)^\wedge(\sigma) = [\widehat{f}(\sigma)\widehat{g}(\sigma)^* C_\sigma] \quad . \tag{4.9}$$

In terms of operator domains, this means that

$$\mathrm{dom}(V_g) = \{f \in \mathrm{L}^2(G) : ([\widehat{f}(\sigma)\widehat{g}(\sigma)^* C_\sigma])_{\sigma\in\widehat{G}} \in \mathcal{B}_2^\oplus\} \quad .$$

Proof. Apply Theorem 4.15 to $A_\sigma := \widehat{f}(\sigma)\widehat{g}(\sigma)^*$, and observe that the unitarity of Plancherel transform yields

$$(V_g f)(x) = \langle f, \lambda_G(x)g\rangle = \int_G \mathrm{tr}(A_\sigma \sigma(x)^*) d\nu_G(x) \quad .$$

Now we only need to note some technical facts concerning multiplication operators on tensor products, and the criteria for admissibility and boundedness can be derived.

Lemma 4.19. *Let $\mathcal{H}, \mathcal{K}_1, \mathcal{K}_2$ be Hilbert spaces. Given a densely defined operator $A : \mathcal{K}_2 \to \mathcal{K}_1$, consider the operator $\mathcal{A} : \mathcal{H} \otimes \mathcal{K}_1 \to \mathcal{H} \otimes \mathcal{K}_2$ defined by letting $T \mapsto [TA]$, for all $T \in \mathcal{H} \otimes \mathcal{K}_1$ for which a bounded extension from* $\mathrm{dom}(A)$ *exists and is in $\mathcal{H} \otimes \mathcal{K}_2$.*

(a) $\mathcal{A}$ is closed.
(b) $\mathcal{A}$ is bounded iff A has a bounded extension.
(c) Now assume $A = S^*C$, where $S \in \mathcal{H} \otimes \mathcal{K}_1$, and $C : \mathcal{H} \to \mathcal{H}$ is selfadjoint, with $\mathcal{K}_1, \mathcal{K}_2 \subset \mathcal{H}$. If $S = \sum_{j\in J} a_j \otimes \varphi_j$, for some ONB $(\varphi_j)_{j\in J}$ of $\mathcal{K}$, then

$$\mathcal{A} \text{ is isometric } \Leftrightarrow (a_j)_{j\in J} \subset \mathrm{dom}(C) \quad \text{and } (Ca_j)_{j\in J} \text{ is an ONS} \quad . \tag{4.10}$$

Proof. For part (a) we pick a basis $(\eta_i)_{i\in I}$ of $\mathcal{H}_1$ to write arbitrary elements of $\mathcal{H} \otimes \mathcal{K}_1$ as

$$S = \sum_{i\in I} \eta_i \otimes b_i \quad .$$

Then $\mathrm{dom}(\mathcal{A})$ consists of all such S for which $(b_i)_{i\in I} \subset \mathrm{dom}(A^*)$ with $\sum_{i\in I} \|A^* b_i\|^2 < \infty$, and then

$$\mathcal{A}S = \sum_{i\in I} \eta_i \otimes A^* b_i \quad .$$

Since A^* is closed, it is easy to conclude from this that $\mathcal{A}$ is closed.

Moreover observe that the fact

$$\|\mathcal{A}S\|^2 = \sum_{i\in I} \|A^* b_i\|^2 \quad ,$$

clearly entails that $\mathcal{A}$ is bounded iff A^* is bounded, which entails (b).

For part (c) observe that the same argument proving (b) yields that $\mathcal{A}$ is isometric iff $A^* = CS$ is isometric. It is straightforward to conclude that $(a_j)_{j\in J} \subset \mathrm{dom}(A)$, and

$$CS = \sum_{j\in J} (Ca_j) \otimes \varphi_j \quad .$$

Since an isometry is characterized by the fact that the ONB $(\varphi_j)_{j\in J}$ must be mapped onto an ONS, $(Ca_j)_{j\in J}$ is an ONS iff CS is isometric.

Compare the following theorem to Theorem 2.64. The sufficiency of the criteria may be found in [53]. For unimodular groups, the characterization of selfadjoint convolution operators on the Fourier side was proved by Carey [29].

Theorem 4.20. *Let $\mathcal{H} \subset \mathrm{L}^2(G)$ be a leftinvariant closed subspace, with orthogonal projection $P \simeq (1 \otimes \widehat{P}_\sigma)_{\sigma \in \widehat{G}}$. Then we have the following equivalences for $\eta \in \mathcal{H}$*

$$\eta \text{ is admissible} \iff [C_\sigma \widehat{\eta}(\sigma)] \text{ is an isometry on } \widehat{P}_\sigma(\mathcal{H}_\sigma) \quad , \tag{4.11}$$

$$\eta \text{ is bounded} \iff \sigma \mapsto \|[C_\sigma \widehat{\eta}(\sigma)]\|_\infty \text{ is essentially bounded} \quad , \tag{4.12}$$

$$\eta \text{ is cyclic} \iff \sigma \mapsto \|[C_\sigma \widehat{\eta}(\sigma)]\|_\infty \text{ is injective on } \widehat{P}_\sigma(\mathcal{H}_\sigma) \quad . \tag{4.13}$$

Bounded vectors fulfill $\|V_\eta\|_\infty = \operatorname{ess\,sup}_{\sigma \in \widehat{G}} \|C_\sigma \widehat{\eta}(\sigma)\|_\infty$. Moreover, $S \in \mathrm{L}^2(G)$ is a selfadjoint convolution idempotent iff $[C_\sigma \widehat{S}(\sigma)]$ is a projection, for ν_G-almost every σ.

Proof. The chief technical difficulty remaining for the proof of necessity is that the L^2-convolution theorem for $V_\eta f$ holds only pointwise a.e. on the Plancherel transform side, and in principle the conull set may depend on f. Hence we need some additional functional analysis to make the argument work.

Pick a countable total subset $\mathcal{S} \subset \mathrm{L}^2(G)$. Then $\{\widehat{f}(\sigma) : f \in \mathcal{S}\}$ is total in $\mathcal{H}_\sigma$, for almost every σ; otherwise one could construct (measurably) the projections onto the nontrivial complements and obtain a nontrivial complement in $\mathrm{L}^2(G)$. Suppose that η is a bounded vector, say $\|V_\eta\|_\infty \leq k$. Then we find by Theorem 4.18 that for all $f \in \mathcal{S}$

$$\int_{\widehat{G}}^{\oplus} \|[\widehat{f}(\sigma)\widehat{\eta}(\sigma)^* C_\sigma]\|_2^2 d\nu_G(\sigma) \leq k^2 \int_{\widehat{G}}^{\oplus} \|\widehat{f}(\sigma)\|_2^2 d\nu_G(\sigma) \quad .$$

Passing from $\widehat{f}$ to $(\widehat{f}(\sigma)\mathbf{1}_B(\sigma))_{\sigma \in \widehat{G}}$, for a Borel subset B, we see that we replace $\widehat{G}$ as integration domain on both sides by B. Since B can be arbitrary, this implies the inequality for the integrands, i.e.,

$$\|[\widehat{f}(\sigma)\widehat{g}(\sigma)^* C_\sigma]\|_2^2 \leq k^2 \|\widehat{f}(\sigma)\|_2^2 \quad .$$

Now the totality of $\{\widehat{f}(\sigma) : f \in \mathcal{S}\}$ shows that the inequality holds on a dense subspace of $\mathcal{H}_\sigma$, for almost all $\sigma \in \widehat{G}$. Then parts (a) and (b) of Lemma 4.19 apply to yield that $\widehat{\eta}(\sigma)^* C_\sigma$ is bounded, and norm $\leq k$ almost everywhere follows by density considerations.

The remaining necessary conditions in (4.12) and (4.11), as well as the norm equality

$$\|V_\eta\|_\infty = \operatorname*{ess\,sup}_{\sigma \in \widehat{G}} \|C_\sigma \widehat{\eta}(\sigma)\|_\infty \quad .$$

(4.13) follow similarly. The converse directions are immediate from Theorem 4.18. Since S is a selfadjoint convolution idempotent iff V_S is a projection operator, the last statement is immediate.

Remark 4.21. The differences between the unimodular and the nonunimodular cases can be exemplified by the following observation: To any $f \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G)$ we can associate at least three objects on the Fourier transform side: The Fourier transform $(\sigma(f))_{\sigma\in\widehat{G}}$, the Plancherel transform $(\widehat{f}(\sigma))_{\sigma\in\widehat{G}}$, and the decomposition of the operator $V_g \simeq (1 \otimes \widehat{T}_\sigma)_{\sigma\in\widehat{G}}$. In the unimodular case, all three objects are basically identical: $\widehat{f}(\sigma) = \sigma(f)$ and $\widehat{T}_\sigma = \sigma(f)^*$. In the nonunimodular case however, they all differ by suitable powers of the Duflo-Moore operator.

Now we can easily characterize the subrepresentations of λ_G with admissible vectors. The statement concerning unimodular groups is partly contained in [29, Theorem 2.10]; in this form the theorem appeared in [53]. A special case of the theorem for the reals was given in Theorem 2.65. Also, the discussion of direct sums of discrete series representations contained in Remark 2.32, in particular for unimodular groups, is a special case of this theorem.

Theorem 4.22. *Let $\mathcal{H} \subset \mathrm{L}^2(G)$ be a leftinvariant closed subspace, and let $P \simeq (1 \otimes \widehat{P}_\sigma)_{\sigma\in\widehat{G}}$ denote the projection onto $\mathcal{H}$. Then $\mathcal{H}$ has admissible vectors iff either*

- *G is unimodular, almost all $\widehat{P}_\sigma$ have finite rank and*

$$\nu_{\mathcal{H}} = \int_{\widehat{G}} \mathrm{rank}(\widehat{P}_\sigma) d\nu_G(\sigma) < \infty \quad . \tag{4.14}$$

  In this case every admissible vector $g \in \mathcal{H}$ fulfills $\|g\|^2 = \nu_{\mathcal{H}}$.
- *G is nonunimodular. In that case, there exist admissible vectors with arbitrarily small or big norm.*

Proof. Assume first that G is unimodular. Suppose g is an admissible vector for $\mathcal{H}$, and define $h = g^* * g$. Then we have $P = (V_g)^* \circ (V_g) = V_{g^*} \circ V_g = V_h$ and $h = V_{g^*} g^* \in \mathrm{L}^2(G)$ (note that V_{g^*} is a bounded operator on all of $\mathrm{L}^2(G)$). Applying Theorem 4.18 first to V_h and then to V_{g^*} yields

$$P_\sigma = \widehat{h}(\sigma)^* = \widehat{g}(\sigma)^* \widehat{g}(\sigma) \quad ,$$

ν_G-almost everywhere. Hence

$$\begin{aligned} \|g\|^2 &= \int_{\widehat{G}} \mathrm{tr}(\hat{g}(\sigma)\hat{g}(\sigma)^*) d\nu_G(\sigma) \\ &= \int_{\widehat{G}} \mathrm{tr}(\hat{g}(\sigma)^*\hat{g}(\sigma)) d\nu_G(\sigma) \\ &= \int_{\widehat{G}} \mathrm{rank}(P_\sigma) d\nu_G(\sigma) \quad . \end{aligned}$$

For the sufficiency of (4.14) we note that $(\widehat{P}_\sigma)_{\sigma\in\widehat{G}} \in \mathrm{B}_2^\oplus$, and we let g be the inverse Plancherel transform of that. Then Theorem 4.18 shows that $P = V_g$,

which means that V_g is the identity operator on $\mathcal{H}$, and g is admissible.
In the nonunimodular case, the statement follows from Theorem 4.23 and its corollary.

Theorem 4.23. *Let G be nonunimodular. Then there exists an operator field $(A_\sigma)_{\sigma\in\widehat{G}} \in \mathcal{B}_2^{\oplus}$ such that $[C_\sigma A_\sigma]$ is an isometry, for ν_G-almost every $\sigma \in \widehat{G}$. As a consequence, if $a \in \mathrm{L}^2(G)$ is the preimage of A under the Plancherel transform, a is admissible for λ_G.*

Proof. The proof uses the techniques and objects described in detail in Section 3.8.2, in particular the notions from Proposition 3.50. We first give the A_σ pointwise and postpone the questions of measurability and square-integrability. Pick $c > 1$ in such a way that $\{\gamma \in H : 1 \leq \Delta_G^{-1/2}(\gamma) < c\}$ has positive Haar measure, and define, for $n \in \mathbb{N}$, $S_n := \{\gamma \in H : c^n \leq \Delta_G^{-1/2}(\gamma) < c^{n+1}\}$. Let $(u_n^\sigma)_{n\in\mathbb{N}} \subset \mathcal{H}_\sigma = \mathrm{L}^2(H, d\mu_H; \mathcal{H}_{\tau(\sigma)})$ be an ONB. Moreover let $(v_n^\sigma)_{n\in\mathbb{N}} \subset \mathrm{L}^2(H, d\mu_H; \mathcal{H}_{\tau(\sigma)})$ be a sequence of unit vectors with $\mathrm{supp}(v_n^\sigma) \subset S_n$. Define the operator A_σ by

$$A_\sigma = \sum_{n\in\mathbb{N}} \|\Delta_G^{-1/2} v_n^\sigma\|^{-1} v_n^\sigma \otimes u_n^\sigma \quad .$$

This defines a Hilbert-Schmidt operator, since $\|\Delta_G^{-1/2} v_n^\sigma\| \geq c^n$.

On the other hand, the construction of the v_n^σ implies that A_σ maps $\mathcal{H}_\sigma$ into $\mathrm{dom}(C_\sigma)$. In fact, given any $f = \sum_{n\in\mathbb{N}} \langle f, u_n^\sigma\rangle u_n^\sigma$, the disjointness of the supports of the v_n^σ implies for all $h \in H$ that

$$\begin{aligned} A_\sigma f(h) &= \sum_{n\in\mathbb{N}} \|\Delta_G^{-1/2} v_n^\sigma\|^{-1} \langle f, u_n^\sigma\rangle v_n^\sigma(h) \\ &= \begin{cases} \|\Delta_G^{-1/2} v_n^\sigma\|^{-1} \langle f, u_n^\sigma\rangle v_n^\sigma(h) & : h \in S_n \text{ for some } n \\ 0 & : \text{ otherwise} \end{cases} \quad . \end{aligned}$$

Thus

$$\begin{aligned} \|C_\sigma A_\sigma f\|^2 &= \sum_{n\in\mathbb{N}} \int_{S_n} \|\Delta_G^{-1/2} v_n^\sigma\|^{-2} \left| \Delta_G^{-1/2}(h) \langle f, u_n^\sigma\rangle \; v_n^\sigma(h) \right\|^2 \\ &= \sum_{n\in\mathbb{N}} |\langle f, u_n^\sigma\rangle|^2 \; \|\Delta_G^{-1/2} v_n^\sigma\|^{-2} \int_{S_n} \|\Delta_G^{-1/2}(h) v_n^\sigma(h)\|^2 dh \\ &= \|f\|^2 \quad . \end{aligned}$$

Therefore $C_\sigma A_\sigma$ is an isometry.

Let us next address the measurability requirement. We have already constructed a measurable family $(u_n^\sigma)_{n\in\mathbb{N}}$ of ONB's, so we only have to ensure that the images $(\|\Delta_G^{-1/2} v_n^\sigma\|^{-1} v_n^\sigma)_{n\in\mathbb{N}}$ can be chosen measurably as well. Pick any family $(b_n)_{n\in\mathbb{N}} \subset \mathrm{L}^2(H)$ of unit vectors, such that b_n is supported in S_n.

Moreover, let $(\xi^\sigma)_{\sigma\in V}$ be a measurable field of unit vectors $\xi^\sigma \in \mathcal{H}_{\tau(\sigma)}$, and define $v_n^\sigma = b_n\xi^\sigma$. Then

$$\sigma \mapsto \langle \|\Delta_G^{-1/2}a_n\|^{-1}v_n^\sigma, a_n e_k^\sigma\rangle = \|\Delta_G^{-1/2}a_n\|^{-1}\langle b_n, a_n\rangle\langle \xi^\sigma, e_k^\sigma\rangle$$

is measurable by the choice of the ξ^σ.

Finally, let us provide for square-integrability. For this purpose we observe that we may assume the constant c picked above to be ≥ 2, and then $\|A_\sigma\|_2^2 < 2$. Moreover, if we shift the construction in the sense that $u_n^\sigma \mapsto \|\Delta_G^{-1/2}v_{n+k}^\sigma\|^{-1}v_{n+k}^\sigma$, for $k > 0$, we obtain $\|A_\sigma\|_2^2 < 2^{1-k}$, while preserving all the other properties of A_σ. With this in mind, we can easily modify the construction to obtain an element of $\mathcal{B}_2^\oplus$: Since G is separable, ν_G is σ-finite, i.e., $\hat{G} = \bigcup_{n\in\mathbb{N}} \Sigma_n$ with the Σ_n pairwise disjoint and $\nu_G(\Sigma_n) < \infty$. Shifting on Σ_n by $k_n \in N$ with $\nu_G(\Sigma_n)2^{-k_n} < 2^{-n}$ ensures square-integrability without destroying measurability. (The latter is obvious on Σ_n.) Hence we are done.

The shifting argument employed in the proof also provides the following fact.

Corollary 4.24. *The operator field $(A_\sigma)_{\sigma\in\hat{G}}$ from Theorem 4.23 can be constructed with arbitrarily small or big norm.*

The following corollary is obtained by combining Theorems 2.42 and 4.22. We expect that it has already been noted elsewhere; it can also be derived from the results in the paper by Arnal and Ludwig [14].

Corollary 4.25. *Let G be a nondiscrete unimodular group. Then*

$$\int_{\hat{G}} \dim(\mathcal{H}_\sigma) d\nu_G(\sigma) = \infty \quad .$$

For completeness, we note the following characterization of unimodularity, which follows from Theorems 2.42 and 4.22.

Corollary 4.26. *Let G be a nondiscrete group. Then λ_G has an admissible vector iff G is nonunimodular*

Theorem 4.22 and Corollary 3.51 imply the following remarkable property.

Corollary 4.27. *Suppose that G is a nonunimodular group. Let π be a representation of G, and assume that $\pi = \bigoplus_{n\in\mathbb{N}} \pi_n$. If each π_n has an admissible vector, so does π.*

The corollary can be applied to the results of Liu and Peng [83] to construct a continuous wavelet transform on the Heisenberg group $\mathbb{H}$. The authors considered a particular group extension $\mathbb{H} \rtimes \mathbb{R}$ and its action on $\mathrm{L}^2(\mathbb{H})$ for the construction of wavelet transforms. They established that this representation decomposes into an infinite direct sum of discrete series representations, and they gave admissibility conditions for each. Now the corollary provides the

existence of an admissible vector for the whole representation. Corollary 5.28 below shows that this type of construction in fact works for arbitrary homogeneous Lie groups.

We next give a generalization of the qualitative uncertainty principle to nonunimodular groups. For the unimodular version, confer [14].

Corollary 4.28. *Suppose that G has a noncompact connected component. For $f \in \mathrm{L}^2(G)$ let P_σ denote the projection onto $\left(\mathrm{Ker}(\widehat{f}(\sigma))\right)^\perp$, and assume that*

$$r(\sigma) = \|[P_\sigma C_\sigma^{-1}]\|_2^2$$

is well-defined and finite ν_G-almost everywhere. Note that in the unimodular case $r(\sigma) = \mathrm{rank}(\widehat{f}(\sigma))$. Suppose that f fulfills

(i) $|\mathrm{supp}(f)| < \infty$ *and*
(ii) $\int_{\widehat{G}} r(\sigma) d\nu_G(\sigma) < \infty$.

Then $f = 0$.

Proof. Consider the operators $\widehat{S}_\sigma = [C_\sigma^{-1} P_\sigma]$, by assumption $(\widehat{S}_\sigma)_{\sigma \in \widehat{G}} \in \mathcal{B}_2^\oplus$. If we let S denote the inverse Plancherel transform of that, the L^2-convolution Theorem entails that

$$(V_S f)^\wedge = \left([\widehat{f}(\sigma) P_\sigma C_\sigma^{-1} C_\sigma]\right)_{\sigma \in \widehat{G}} = \widehat{f} \quad ,$$

since by construction $\widehat{f}(\sigma) \circ P_\sigma = \widehat{f}(\sigma)$. In other words, $f = V_S f = f * S^*$. Moreover, since $[C_\sigma \widehat{S}_\sigma] = P_\sigma$ is a projection operator, $S = S^*$ by Theorem 4.20, and thus $f = f * S$. Now assumption (i) and Theorem 2.45 imply $f = 0$.

Remark 4.29. In the real case condition (ii) specializes to the well-known condition $|\mathrm{supp}(\widehat{f})| < \infty$; this is our excuse for calling Corollary 4.28 a qualitative uncertainty principle. In the unimodular case the analogy to the qualitative uncertainty property is still quite comprehensible: The Plancherel measure of the support of $\widehat{f}$ is simply weighted with the rank of $\widehat{f}(\sigma)$. If we let $\mathcal{H}$ denote the leftinvariant closed subspace generated by f, then $\int_{\widehat{G}} r(\sigma) d\nu_G(\sigma) = \nu_{\mathcal{H}}$, the latter being the constant encountered in Theorem 4.22.

For nonunimodular groups the quantity $\int_{\widehat{G}} r(\sigma) d\nu_G(\sigma)$ is more difficult to interpret. It is however independent of the choice of Plancherel measure: If we pass to an alternative pair $(\widetilde{\nu_G}, (\widetilde{C_\sigma})_{\sigma \in \widehat{G}})$ of Plancherel measure and associated Duflo-Moore-operators and define $\widetilde{r}$ using the $\widetilde{C_\sigma}$, the renormalizations of $\widetilde{\nu_G}$ and $\widetilde{C_\sigma}$ cancel to yield

$$\int_{\widehat{G}} \widetilde{r}(\sigma) d\widetilde{\nu_G}(\sigma) = \int_{\widehat{G}} r(\sigma) d\nu_G(\sigma) \quad .$$

Let us next consider **T4**, i.e., the question how to make the criteria derived for the regular representation applicable to arbitrary representations π. We intend to employ the existence and uniqueness of direct integral decompositions for this task: Decomposing π into irreducibles, we can check containment in λ_G by comparing the underlying measure to Plancherel measure, and comparing multiplicities (if necessary). Moreover, once containment is established in this way, the admissibility conditions for λ_G directly carry over to the direct integral decomposition of π, much in the way that we could directly establish admissibility criteria for representations of $\mathbb{R}$, as described in Remark 2.70. Using Lemma 4.19(c), we can in addition break the admissibility condition formulated for operator fields down to conditions involving rank-one operators. Thus we obtain orthogonality conditions generalizing the discrete series case from Theorem 2.31.

Remark 4.30. Given an arbitrary representation $(\pi, \mathcal{H}_\pi)$ of a type I group, the following steps need to be carried out for establishing admissibility conditions:

- Explicitly construct a unitary intertwining operator

$$T : \mathcal{H}_\pi \to \int_{\widehat{G}}^{\oplus} \mathcal{H}_\sigma^{m_\pi(\sigma)} \, d\widetilde{\nu}(\sigma)$$
$$\pi \simeq \int_{\widehat{G}}^{\oplus} m_\pi(\sigma) \cdot \sigma \;\; d\widetilde{\nu}(\sigma)$$

  where $\widetilde{\nu}$ is a suitable measure on $\widehat{G}$ and m is a multiplicity function.
- There exist admissible vectors if and only if the following questions are answered in the affirmative:
  - Is $\widetilde{\nu}$ ν_G-absolutely continuous?
  - Is $m_\pi(\sigma) \leq \dim(\mathcal{H}_\sigma)$, $\widetilde{\nu}$-almost everywhere?
  - If G is unimodular, does the relation

$$\int_{\widehat{G}} m_\pi(\sigma) d\nu_G(\sigma) < \infty$$

    hold?

  Note that if G is nonunimodular, the answer to the second question is "yes" by Corollary 3.51.
- Compute the Radon-Nikodym derivative $\frac{d\widetilde{\nu}}{d\nu_G}$.
  If G is nonunimodular, compute the Duflo-Moore operators C_σ, using e.g. the description in Proposition 3.50, or the semi-invariance relation (3.51). T maps vectors $\eta \in \mathcal{H}_\pi$ to measurable families $((T\eta)(\sigma, i))_{\sigma \in \widehat{G}, i=1,\ldots,m_\pi(\sigma)}$. Then η is admissible iff it fulfills the following orthonormality relations, for ν_G-almost every $\sigma \in \Sigma$:

$$\left(\frac{d\widetilde{\nu}}{d\nu_G}(\sigma)\right)^{1/2} \|C_\sigma(T\eta)(\sigma, i)\| = 1, \;\; \text{for } 1 \leq i \leq m_\pi(\sigma) \tag{4.15}$$

$$\langle C_\sigma(T\eta)(\sigma, i), C_\sigma(T\eta)(\sigma, j)\rangle = 0, \;\; \text{for } 1 \leq i < j \leq m_\pi(\sigma) \tag{4.16}$$

Note that conditions (4.15) and (4.16) implicitly contain that $(T\eta)(\sigma, i) \in \mathrm{dom}(C_\sigma)$. Clearly, for the admissibility conditions to be explicit, T, the C_σ and $\frac{d\widetilde{\nu}}{d\nu_G}$ need to be known explicitly. The next chapter is devoted to carrying this program out for a rather general setting.

In the simplest possible case, i.e, G unimodular, π multiplicity-free and $\widetilde{\nu} = \nu_G$, the admissibility condition reduces to

$$\|(T\eta)(\sigma)\| \equiv 1, \ \nu_G - \text{almost everywhere on } \Sigma \quad . \tag{4.17}$$

We close the section with a partial converse of the results in the last remark, at least for unimodular groups: Any admissibility condition which has a similar structure as the admissibility criterion (4.15) and (4.16) necessarily has to coincide with it. Also, the admissibility condition characterizes the canonical Plancherel measure. We expect this statement to hold for arbitrary groups, but have preferred only to deal with the unimodular case and to avoid the problems that arise from the Duflo-Moore operators. For certain representations however, an analogous result is derived in Theorem 5.23.

In any case the next theorem provides further evidence for the central thesis of this book, namely that computing admissible vectors is in a sense equivalent to computing Plancherel measure. The following subsection will show that this observation actually extends to the type I condition.

Theorem 4.31. *Let G be unimodular. Suppose that*

$$\pi = \int_X^\oplus m(x)\sigma_x \ d\nu(x) \tag{4.18}$$

is a direct integral representation, where X is a standard Borel space with σ-finite measure ν, $m : X \to \mathbb{N} \cup \{\infty\}$ is a Borel map and $(\sigma_x)_{x\in X}$ is a measurable field of representations of G. Assume that the admissibility criterion

$$\begin{array}{l} (\eta_x^i)_{x\in X, i=1,\ldots,m(x)} \text{ is admissible} \\ \iff (\eta_x^i)_{i=1,\ldots,m(x)} \text{ is an ONS for } \nu\text{-a.e.} x \in X \quad . \end{array} \tag{4.19}$$

holds and that there exist vectors η fulfilling it.

(a) There exists a ν-conull subset $X' \subset X$ such that $x \mapsto \sigma_x$ is a Borel embedding $X' \hookrightarrow \widehat{G}$.
(b) In this identification, ν is the restriction of ν_G to a suitable subset of $\widehat{G}$.

Proof. Let $\pi_0 = \int_X^\oplus \sigma_x d\nu(x)$, and $\mathcal{H}_{\pi_0}$ the associated representation space. We first prove that (4.19) entails

$$\|V_\eta \phi\|_2^2 = \int_X \|\eta_x\|^2 \ \|\phi_x\|^2 d\nu(x) \quad , \tag{4.20}$$

for all vector fields $\eta = (\eta_x)_{x\in X}, \phi = (\phi_x)_{x\in X} \in \mathcal{H}_{\pi_0}$, in the extended sense that $V_\eta \phi \notin \mathrm{L}^2(G)$ whenever the right hand side is infinite. For this purpose define the auxiliary vector field $\widetilde{\eta} \in \mathcal{H}_\pi$, by letting

$$\widetilde{\eta}_{x,1} = \|\eta_x\|^{-1} \eta_x \quad ,$$

and choosing $\widetilde{\eta}_{x,i}, i = 2, \ldots, m(x)$ orthonormal to each other and to η_x, whenever $\eta_x \neq 0$. For those x for which $\eta_x = 0$, we pick an arbitrary orthonormal system $\widetilde{\eta}_{x,i}, i = 1, \ldots, m(x)$ (arbitrary within the measurability requirement, that is). Note that the assumption that there exists a square-integrable vector field fulfilling the admissibility condition implies that $m(x) \leq \dim(\mathcal{H}_x)$, as well as

$$\int_X m(x) \ d\nu(x) < \infty \quad .$$

Hence $\widetilde{\eta}$ can be constructed and is square-integrable as well. Moreover it is admissible by assumption. Next define a vector field $\widetilde{\phi}$ by

$$\widetilde{\phi}_{x,i} = \begin{cases} \|\eta_x\| \phi_x & i = 1 \\ 0 & i = 2, \ldots, m(x) \end{cases} \quad ,$$

and assume for the moment that $\widetilde{\phi}$ is square-integrable, i.e., in $\mathcal{H}_\pi$. Then

$$\begin{aligned} V_{\widetilde{\eta}} \widetilde{\phi}(y) &= \int_X \sum_{i=1}^{m(x)} \langle \widetilde{\phi}_{x,i}, \sigma_x(y) \widetilde{\eta}_{x,i} \rangle d\nu(x) \\ &= \int_X \langle \phi_x, \sigma_x(y) \eta_x \rangle d\nu(x) \\ &= V_\eta \phi(y) \quad . \end{aligned}$$

Hence admissibility of $\widetilde{\eta}$ entails

$$\|V_\eta \phi\|_2^2 = \|\widetilde{\phi}\|^2 = \int_X \|\eta_x\|^2 \ \|\phi_x\|^2 d\nu(x) \quad .$$

This proves (4.20) for the case that the right-hand side is finite; the general case is easily obtained by plugging in restrictions of ϕ to suitable Borel subsets $B \subset X$.

Now part (a) follows from Theorem 4.32 below. For part (b) we note that by part (a) (4.18) is a decomposition into irreducibles, hence Theorem 3.25 yields that ν_G and ν are equivalent.

Moreover, we can conclude in the same way that (4.20) holds also with ν_G replacing ν, by the admissibility criterion (4.17). Plugging in η with $\|\eta_x\| = 1$ almost everywhere yields

$$\int_X \|\varphi_x\|^2 d\nu_G(x) = \int_X \|\varphi_x\|^2 d\nu(x)$$

for all vector fields φ. Hence $\nu = \nu_G$.

4.4 Admissibility Criteria and the Type I Condition**

In this section we comment on the relation of the type I property to the existence of admissibility conditions. A major motivation for tackling non-type I groups is provided by discrete G: Here λ_G is type I iff G itself is [70], and the latter is the case only if G is a finite extension of an abelian normal subgroup [111], i.e., very rarely.

The following theorem shows that for any direct integral representation π with associated admissibility conditions as in Remark 4.30, π is necessarily type I. Thus admissibility criteria outside the type I setting will have to be of a different nature.

Theorem 4.32. *Let G be unimodular, not necessarily of type I. Suppose that $\pi = \int_X^\oplus m(x)\sigma_x \, d\nu(x)$ is a direct integral representation, where X is a standard Borel space with σ-finite measure ν, $m : X \to \mathbb{N} \cup \{\infty\}$ is a Borel map and $(\sigma_x)_{x\in X}$ is a measurable field of representations of G. Assume that the admissibility criterion*

$$\begin{array}{l}(\eta_x^i)_{x\in X, i=1,\ldots,m(x)} \text{ is admissible} \\ \iff (\eta_x^i)_{i=1,\ldots,m(x)} \text{ is an ONS for } \nu\text{-a.e.} x \in X \quad .\end{array} \tag{4.21}$$

holds and that there exist vectors η fulfilling it. Then π is type I.

Proof. Just as in the proof of Theorem 4.31, let $\pi_0 = \int_X^\oplus \sigma_x d\nu(x)$, and $\mathcal{H}_{\pi_0}$ the associated representation space. Recall from the proof of 4.31 that

$$\|V_\eta \phi\|_2^2 = \int_X \|\eta_x\|^2 \, \|\phi_x\|^2 d\nu(x) \quad . \tag{4.22}$$

Observe that the proof of that equation did not rely on the type I property of π.

We are going to show that π_0 is multiplicity-free. Since $\pi \approx \pi_0$, the type I property of π then follows immediately. Let $\mathcal{K} \subset \mathcal{H}_{\pi_0}$ be an invariant subspace. π is cyclic by assumption, hence there exists a cyclic vector ψ for the subspace $\mathcal{K}$ as well. Then (4.22) allows to compute the orthogonal complement of $\mathcal{K}$ in $\mathcal{H}_{\pi_0}$ as

$$\begin{aligned}\mathcal{K}^\perp &= \{\phi : V_\psi \phi = 0\} \\ &= \{\phi : \|\phi_x\| \|\psi_x\| = 0, \nu - \text{ almost everywhere } \} \quad .\end{aligned}$$

But this entails that the projection onto $\mathcal{K}$ is given by pointwise multiplication with the characteristic function of $\{x \in X : \psi_x \neq 0\}$. It follows that all invariant projections commute, and thus the commuting algebra of π_0 is commutative.

Remark 4.33. If π is as in the theorem, then π is a subrepresentation of λ_G^I, the type I part of λ_G, and ν is absolutely continuous with respect to the measure ν_G on $\widehat{G}$ underlying the decomposition of λ_G^I into irreducibles.

We are not aware of a systematic treatment of admissibility conditions outside the type I setting. There exists a substitute for the Plancherel theorem, see e.g. [35, 18.7.7], which uses the theory of traces and their decomposition. Some use of this result can be made to formulate admissibility criteria in the general setting, see [55].

4.5 Wigner Functions Associated to Nilpotent Lie Groups**

In this section we sketch the construction of Wigner functions associated to certain representations of nilpotent Lie groups. A full understanding of the results requires knowledge of Kirillov's theory of coadjoint orbits and their use in constructing the unitary dual. We refrain from giving an introduction to this theory here, and refer the interested reader to [30, 72] for details. While the results only hold in a specific context, they serve as an example how representation theory can provide a unified view of phenomena connected to continuous wavelet transforms. Also, several ideas that have occurred in the discussion so far make their appearance in this section: The extension of orthogonality relations to Hilbert-Schmidt-operators, as used in the proof of 2.33, or the problem of decomposing operators over a direct integral space. The latter problem will be seen to connect nicely to Kirillov's orbit theory.

The Wigner transform is intended as a *symbol calculus* associating to an operator on a certain Hilbert space a function on a phase space, i.e., to an observable in the quantum mechanical sense an observable in the classical sense. The original Wigner transform is closely related to the Heisenberg group (see the next subsection). The authors of [24] demonstrated how one could replace the Heisenberg group by the affine group and thus arrive at a different symbol calculus. As a matter of fact, there are various constructions based on different choices of groups around, see for instance [11, 12, 96, 24, 2]. In the following we will be concerned only with [2] and a variation of the construction, presented in [4]. The appeal of the approach presented here derives from the way it highlights the role of the Plancherel transform in the construction. The relations between this construction and the various other definitions of symbol calculi to be found in the literature are not entirely clear to us, and we do not have a particular claim to originality.

The authors of [2] singled out two main ingredients of the construction: A discrete series representation of the underlying group, with the associated orthogonality relations, and a Euclidean Fourier transform on the group obtained from an identification with its Lie algebra via the exponential map. [4] then showed how this construction generalizes to cases where discrete series representations are not available. We present a discussion which is suited to simply connected, connected nilpotent Lie groups; extensions to exponential Lie groups are possible.

The Original Wigner Transform

The starting point of the construction is the Heisenberg group $\mathbb{H}$. As a set, $G = \mathbb{R}^3$, with the group law

$$(p,q,t)(p',q',t') = (p+p', q+q', t+t'+(q'p-qp')/2) \quad .$$

The comparison with Example 2.27 shows that the reduced Heisenberg group $\mathbb{H}_r$ is the quotient group of $\mathbb{H}$ by the discrete central subgroup $\{0\} \times \{0\} \times \{2\pi\mathbb{Z}\}$. For the time being, this difference does not really matter, since in this subsection we are dealing with the windowed Fourier transform and ignore the action of the center $\{0\} \times \{0\} \times \mathbb{R}$. However, for the connection to Kirillov's theory later on it is crucial to have a simply connected group.

Recall that

$$\mathcal{W}_f g(p,q) = \int_{\mathbb{R}} g(x)\overline{f(x+p)}\mathrm{e}^{-2\pi \mathrm{i} q(x+p/2)}dx \quad .$$

Now the fact that $\mathcal{W}_f$ is isometric iff $\|f\| = 1$ entails that

$$\|\mathcal{W}_f g\|_{\mathrm{L}^2(\mathbb{R}^2)} = \|f\|_2 \ \left\|\mathcal{W}_{\frac{f}{\|f\|}} g\right\| = \|f\|_2 \ \|g\|_2 \quad ,$$

for arbitrary $0 \neq f, g \in \mathrm{L}^2(\mathbb{R})$. Then polarization provides the **biorthogonality relation**

$$\langle \mathcal{W}_{f_1} g_1, \mathcal{W}_{f_2} g_2 \rangle_{\mathrm{L}^2(\mathbb{R}^2)} = \langle f_2, f_1 \rangle \langle g_1, g_2 \rangle \quad ; \tag{4.23}$$

similar arguments were used in connection with the orthogonality relations (2.15) for general discrete series representations. Since the right-hand side is nothing but the Hilbert-Schmidt scalar product of the two rank-one operators $g_1 \otimes f_1$ and $g_2 \otimes f_2$, we find that the mapping

$$\mathcal{W} : (f,g) \mapsto \mathcal{W}_f g$$

linearly extends to a unique isometry $\mathcal{W} : \mathcal{B}_2(\mathrm{L}^2(\mathbb{R})) \to \mathrm{L}^2(\mathbb{R}^2)$. As a matter of fact the map is onto, hence unitary. Now the Wigner transform, which we denote by $\mathfrak{W}$, is obtained by taking the usual scalar-valued Plancherel transform after $\mathcal{W}$, i.e., formally

$$\begin{aligned} \mathfrak{W}(g \otimes f)(p^*,q^*) &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathcal{W}_f g(p,q) \mathrm{e}^{-2\pi \mathrm{i}(p^* p + q^* q)} dpdq \\ &= \int_{\mathbb{R}} \mathrm{e}^{-2\pi \mathrm{i} p^* p} g\left(-q^* - \frac{p}{2}\right) \overline{f\left(-q^* + \frac{p}{2}\right)} dp \quad . \end{aligned} \tag{4.24}$$

Here the second equality is Fourier inversion.

Let us collect the main properties of the operator $\mathfrak{W} : \mathcal{B}_2(\mathrm{L}^2(\mathbb{R})) \to \mathrm{L}^2(\mathbb{R}^2)$:

- **Unitarity**, usually expressed in the **overlap condition** [2], or **Moyal's identity**
$$\langle \mathfrak{W}(g_1 \otimes f), \mathfrak{W}(g_2 \otimes f_2) \rangle = \langle g_1, g_2 \rangle \langle f_2, f_1 \rangle \quad .$$
- **Covariance:** If $A \in \mathcal{B}_2(\mathrm{L}^2(\mathbb{R}))$, then
$$\mathfrak{W}(\pi(p,q,t) A \pi(p,q,t)^*)(p',q') = \mathfrak{W}(A)(p'-p, q'-q) \quad .$$
This is verified by direct calculation for rank-one operators and extends by density to arbitrary Hilbert-Schmidt operators.
- **Reality:** If $A \in \mathcal{B}_2(\mathrm{L}^2(\mathbb{R}))$, then
$$\mathfrak{W}(A^*) = \overline{\mathfrak{W}(A)} \quad .$$
Again, this is easily seen for rank-one operators.

Note that our calculations in this subsection are somewhat ad hoc. In particular the group-theoretic significance is not clear, since we considered restrictions to the subset $\mathbb{R}^2 \times \{1\} \subset G$, which is not a subgroup. The construction of the Fourier-Wigner operator from (restrictions of) wavelet coefficients closely resembles Plancherel inversion. This observation will be the basis of the general approach for the construction of a transform with the above three properties, which works for arbitrary simply connected nilpotent groups.

Wigner Functions Associated to Nilpotent Lie Groups

Let N be a simply connected, connected nilpotent Lie group, with Lie algebra $\mathfrak{n}$. The basic facts concerning these groups, as used in the following, are contained in [30]. N is an **exponential** Lie group, i.e., the exponential mapping $\exp : \mathfrak{n} \to N$ is bijective. Moreover, the image measure under exp of Lebesgue measure on $\mathfrak{n}$ turns out to be left- and rightinvariant on N [30, Proposition 1.2.9]. Let $\mathrm{L}^2(\mathfrak{n})$ denote the L^2-space with respect to Lebesgue measure, then we obtain a unitary map $\mathrm{EXP}^* : \mathrm{L}^2(N) \ni f \mapsto f \circ \exp \in \mathrm{L}^2(\mathfrak{n})$. Finally, we let $\mathcal{P}_{\mathfrak{n}} : \mathrm{L}^2(\mathfrak{n}) \to \mathrm{L}^2(\mathfrak{n}^*)$ denote the usual Euclidean Plancherel transform, defined on $\mathrm{L}^2(\mathfrak{n}) \cap \mathrm{L}^1(\mathfrak{n})$ by
$$\mathcal{P}_{\mathfrak{n}}(f)(X^*) = \int_{\mathfrak{n}} f(X) \mathrm{e}^{-i\langle X, X^* \rangle} dX \quad ,$$
where dX is Lebesgue measure and $\langle X, X^* \rangle$ denotes the duality between $\mathfrak{n}$ and $\mathfrak{n}^*$. For a suitable normalization of Lebesgue measure on $\mathfrak{n}^*$ the map is unitary. Now we define the **global Wigner transform** $\mathfrak{W} : \mathcal{B}_2^{\oplus} \to \mathrm{L}^2(\mathfrak{n}^*)$ as
$$\mathfrak{W} = \mathcal{P}_{\mathfrak{n}} \circ \mathrm{EXP}^* \circ \mathcal{P}^{-1}$$

The construction entails the following theorem. We let Ad and Ad^* denote the adjoint resp. coadjoint actions of N on $\mathfrak{n}$ resp. $\mathfrak{n}^*$. Ad is obtained by differentiating the conjugation action of N on itself, which gives rise to a linear action of N on $\mathfrak{n}$. Ad^* is obtained by transposing this action.

Theorem 4.34. *(a)* $\mathfrak{W}$ *is unitary.*
(b) $\mathfrak{W}$ *is* **covariant***: For all* $(A_\sigma)_{\sigma\in\widehat{N}} \in \mathcal{B}_2^{\oplus}$*, for all* $x \in N$*,*

$$\mathfrak{W}((\sigma(x)A_\sigma\sigma(x)^*)_{\sigma\in\widehat{N}})(X^*) = \mathfrak{W}(\mathrm{Ad}^*(x)X^*) \quad \textit{almost everywhere.}$$

(c) $\mathfrak{W}$ *is* **real***: For all* $(A_\sigma)_{\sigma\in\widehat{N}} \in \mathcal{B}_2^{\oplus}$*,*

$$\mathfrak{W}((A_\sigma^*)_{\sigma\in\widehat{N}})(X^*) = \overline{\mathfrak{W}((A_\sigma)_{\sigma\in\widehat{N}})(X^*)} \quad .$$

Proof. Part (a) is obvious from the definition. For part (b) we first use the intertwining property of the Plancherel transform to see that, for $a = \mathcal{P}^{-1}((A_\sigma)_{\sigma\in\widehat{N}})$,

$$\mathcal{P}^{-1}((\sigma(x)A_\sigma\sigma(x)^*)_{\sigma\in\widehat{N}})(y) = a(x^{-1}yx) \quad .$$

Moreover, we have the fundamental relation (see [30])

$$\exp((\mathrm{Ad})(x)Y) = x\,\exp(Y)\,x^{-1} \quad ,$$

as well as

$$\begin{aligned}\mathcal{P}_{\mathfrak{n}}(f\circ\mathrm{Ad}(x))(X^*) &= \int_{\mathfrak{n}} f(\mathrm{Ad}(x)Y)\mathrm{e}^{i\langle Y,X^*\rangle}dX\\ &= \int_{\mathfrak{n}} f(Y)\mathrm{e}^{i\langle \mathrm{Ad}(x^{-1})Y,X^*\rangle}dX\\ &= \int_{\mathfrak{n}} f(Y)\mathrm{e}^{i\langle Y,\mathrm{Ad}^*(x)X^*\rangle}dX\\ &= \mathcal{P}_{\mathfrak{n}}(f)(\mathrm{Ad}^*(x)X^*) \quad .\end{aligned}$$

Combining these covariances gives (b). For part (*c*) observe that by Lemma 4.14 (iv)

$$\mathcal{P}_N(a^*) = (A_\sigma^*)_{\sigma\in\widehat{N}} \quad ,$$

and we recall that $a^*(x) = \overline{a(x^{-1})}$. Since $\exp(-X) = \exp(X)^{-1}$, EXP^* intertwines the involution on $\mathrm{L}^2(N)$ with the analogous involution on $\mathfrak{n}$, viewed as a vector group. But this implies (c).

While this result was pleasantly simple to prove, it is still not clear how to recover the original Wigner transform from it; after all the latter maps single Hilbert-Schmidt operators to functions in two variables, not fields of operators to functions in three variables. As will be seen below, the missing link is provided by the role of the coadjoint orbits in the computation of $\widehat{N}$ and μ_N. This brings up Kirillov's orbit method.

The central result of harmonic analysis on nilpotent Lie group is the existence of the **Kirillov correspondence**, which is a bijection

$$\kappa : \mathfrak{n}^*/\mathrm{Ad}^*(N) \to \widehat{N} \quad .$$

In fact it is a Borel isomorphism, if we endow the right hand side with the Mackey Borel structure and the left hand side with the quotient structure. We denote the inverse mapping by $\kappa^{-1}(\sigma) = \mathcal{O}_\sigma^*$. The orbit space is countably separated, i.e., N is type I.

Besides providing a scheme to compute $\widehat{N}$, or at least a parametrization of it, the coadjoint orbits also give access to the Plancherel measure: Since the orbit space is countably separated, there exists a measure decomposition

$$dX^* = d\mu_{\mathcal{O}}(X^*)d\nu(\mathcal{O})$$

of Lebesgue measure on $\mathfrak{n}^*$ (see Proposition 3.28), and it can be chosen in such a way that the image measure of ν under κ is precisely ν_N [72].

By Proposition 3.29 the measure decomposition gives rise to a direct integral decomposition of $\mathrm{L}^2(\mathfrak{n}^*)$, namely

$$\mathrm{L}^2(\mathfrak{n}^*, dX) \simeq \int_{\mathfrak{n}^*/\mathrm{Ad}^*(G)}^{\oplus} \mathrm{L}^2(\mathcal{O}, d\mu_{\mathcal{O}})\ d\nu(\mathcal{O}) \quad .$$

Now $\mathfrak{W}$ can be read as an operator between two direct integral spaces, (essentially) based on the same measure space, and the following definition is natural:

Definition 4.35. *The Wigner transform* $\mathfrak{W}$ **decomposes** *if there exists a conull,* $\mathrm{Ad}^*(N)$*-invariant Borel set* $C \subset \mathfrak{n}^*$*, and a field of operators*

$$\mathfrak{W}_\sigma : \mathcal{B}_2(\mathcal{H}_\sigma) \to \mathrm{L}^2(\mathcal{O}_\sigma^*, \mu_{\mathcal{O}_\sigma^*})$$

for all $\sigma \in \widehat{N}$ *with* $\mathcal{O}_\sigma^* \subset C$*, such that the following relation on* $\mathcal{B}_2^{\oplus}$ *holds:*

$$\mathfrak{W}((A_\sigma)_{\sigma\in\widehat{N}}) = (\mathfrak{W}_\sigma(A_\sigma))_{\sigma\in\widehat{N}} \quad . \tag{4.25}$$

If the operators $\mathfrak{W}_\sigma$ *exist, they are called* **local Wigner transforms**.

Note that by definition the set of $\sigma \in \widehat{G}$ for which $\mathfrak{W}_\sigma$ is not defined has measure zero, hence (4.25) is meaningful for $\mathcal{B}_2^{\oplus}$.

Remark 4.36. (1) In the next subsection we will see that the concrete Wigner transform given above is a local Wigner operator in the sense of the definition. (2) The motivation for local Wigner operators is to obtain a correspondence between single operators and functions, instead of families of operators and (families of) functions. The desire to have Wigner functions supported by single coadjoint orbits is motivated by the applications of Wigner functions in mathematical physics. The Wigner transform is intended as a *quantization procedure*, assigning each operator on a given Hilbert space its *symbol*, i.e., a function on phase space. Now the symplectic structure on coadjoint orbits provides a natural interpretation of these orbits as phase spaces of physical systems, whereas $\mathfrak{n}^*$, as a disjoint union of such phase spaces, does not readily

lend itself to such an interpretation. For more details on quantization and coadjoint orbits, we refer the reader to [62, 72].
(3) The construction can be extended to exponential groups. The extra cost consists mainly in having to deal with densities when passing from N to $\mathfrak{n}$, i.e, Haar measure of N need not coincide with Lebesgue measure. Also the group can be nonunimodular. Confer [2, 3] for examples.
(4) The question whether $\mathfrak{W}$ decomposes has the following representation-theoretic background: The proof of the covariance property in Theorem 4.34 is based on the fact that $\mathcal{P}_{\mathfrak{n}} \circ \mathrm{EXP}^*$ decomposes the conjugation representation

$$x \mapsto \lambda_G(x)\varrho_G(x)$$

with the unitary action on $\mathrm{L}^2(\mathfrak{n}^*)$ induced by Ad^*, whereas $\mathcal{P}$ intertwines it with the direct integral representation

$$\int_{\mathfrak{n}^*/\mathrm{Ad}^*(N)}^{\oplus} \pi_{\mathcal{O}} d\nu(\mathcal{O}) \quad , \tag{4.26}$$

where $\pi_{\mathcal{O}}$ is a representation acting on $\mathcal{B}_2(\mathcal{H}_{\kappa(\mathcal{O})})$ via

$$\pi_{\mathcal{O}}(x)A = \kappa(\mathcal{O})(x) \otimes \kappa(\mathcal{O})(x) \quad .$$

Since both $\mathrm{L}^2(\mathfrak{n}^*)$ and $\int_{\mathfrak{n}^*/\mathrm{Ad}^*(N)}^{\oplus} \mathcal{B}_2(\mathcal{H}_{\kappa(\mathcal{O})}) d\nu(\mathcal{O})$ can be viewed as direct integral spaces based on the same measure space, the decomposition statement is natural. It is nontrivial, however: The point is that the decomposition (4.26) is not easily related to the decomposition of the conjugation representation into irreducibles. In particular, $\pi_{\mathcal{O}}$ may contain representations which correspond to coadjoint orbits other than $\mathcal{O}$. Hence additional assumptions are necessary, as witnessed by the next theorem showing that the local Wigner operators exist only in very restrictive settings.

Theorem 4.37. *Let N be a simply connected, connected nilpotent Lie group. The Wigner transform on N decomposes iff there exists a conull $\mathrm{Ad}^*(N)$-invariant Borel subset $C \subset \mathfrak{n}^*$ such that every coadjoint orbit in C is an affine subspace.*

Proof. We first prove that $\mathfrak{W}$ decomposes iff there exists a conull $\mathrm{Ad}^*(N)$-invariant Borel subset $C \subset \mathfrak{n}^*$ such that for all $f \in \mathrm{L}^1(N) \cap \mathrm{L}^2(N)$, and all $\sigma \in \widehat{N}$ with $\mathcal{O}_\sigma^* \subset C$,

$$\|\sigma(f)\|_2^2 = \int_{\mathcal{O}_\sigma^*} |\mathcal{P}_{\mathfrak{n}}(f)(\omega)|^2 \, d\mu_{\mathcal{O}_\sigma^*}(\omega) \quad . \tag{4.27}$$

Indeed, supposing that $\mathfrak{W}$ decomposes, the right hand side is precisely $\|\mathfrak{W}_\sigma(\sigma(f))\|^2_{\mathrm{L}^2(\mathcal{O}_\sigma^*)}$, and unitarity of $\mathfrak{W}$ entails ν_N-almost everywhere that $\mathfrak{W}_\sigma$ is unitary. This proves the "only-if" part.

For the "if-part" we construct $\mathfrak{W}_\sigma^{-1}$ as follows: Fix a coadjoint orbit $\mathcal{O}_\sigma^*$, for $\sigma \in C$. By [30, 3.1.4], $\mathcal{O}_\sigma^*$ is a closed submanifold of $\mathfrak{n}^*$. Pick a function $g \in C_c^\infty(\mathfrak{n}^*)$, then $\mathcal{P}_\mathfrak{n}^{-1}(g) \in \mathrm{L}^1(N) \cap \mathrm{L}^2(N)$. Moreover, the restriction of g to $\mathcal{O}_\sigma^*$ is in $C_c^\infty(\mathcal{O}_\sigma^*)$. We define the operator $G_\sigma \in \mathcal{B}_2(\mathcal{H}_\sigma)$

$$G_\sigma = \sigma\left(\mathcal{P}_\mathfrak{n}^{-1}(g)\right)$$

We claim that G_σ only depends on the restriction $g|_{\mathcal{O}_\sigma^*}$: Indeed, if $h \in C_c^\infty(\mathfrak{n}^*)$ fulfills $h|_{\mathcal{O}_\sigma^*} = g|_{\mathcal{O}_\sigma^*}$, then (4.27)yields

$$\|\sigma(\mathcal{P}_\mathfrak{n}^{-1}(f-h))\|_2^2 = 0 \quad .$$

Hence the operator $\mathfrak{V}_\sigma : g|_{\mathcal{O}_\sigma^*} \mapsto G_\sigma$ is well-defined. Moreover, the assumption yields $\|g|_{\mathcal{O}_\sigma^*}\|_2 = \|G_\sigma\|_2$. Since $\mathcal{O}_\sigma^*$ is a closed submanifold, the set of restrictions of C_c^∞-functions is precisely $C_c^\infty(\mathcal{O}_\sigma^*)$, which is dense in $\mathrm{L}^2(\mathcal{O}_\sigma^*)$. Hence $\mathfrak{V}_\sigma$ has a unique isometric extension $\mathrm{L}^2(\mathcal{O}_\sigma^*) \to \mathcal{B}_2(\mathcal{H}_\sigma)$. By construction, $\mathfrak{W}^{-1}$ thus coincides with the direct integral of the field $(\mathfrak{V}_\sigma)_{\sigma\in C}$, at least on $C_c^\infty(\mathfrak{n}^*)$. But this space is dense, hence we conclude that the $\mathfrak{V}_\sigma$ are unitary, and $\mathfrak{W}$ decomposes into the $\mathfrak{W}_\sigma = \mathfrak{V}_\sigma^{-1}$.

Hence the equivalence is established, and we can finish by appealing to [85, Theorem 1] which states that (4.27) holds iff $\mathcal{O}_\sigma^*$ is an affine subspace.

Remark 4.38. There appears to be a way of defining local Wigner transforms outside the flat orbit case. It consists in replacing the operator $\mathcal{P}_\mathfrak{n}$ by a different integral operator$\mathcal{P}_\mathfrak{n}^{\mathrm{ad}}$, the so-called **adapted Fourier transform** [10, 12, 13]. The transform is designed to ensure

$$\|\sigma(f)\|_2^2 = \int_{\mathcal{O}_\sigma^*} |\mathcal{P}_\mathfrak{n}^{\mathrm{ad}}(f)(\omega)|^2 \, d\mu_{\mathcal{O}_\sigma^*}(\omega) \quad , \tag{4.28}$$

and as in the proof of the theorem, this entails that the **adapted Wigner transform**

$$\mathfrak{W}^{\mathrm{ad}} = \mathcal{P}_\mathfrak{n}^{\mathrm{ad}} \circ \mathrm{EXP}^* \circ \mathcal{P}^{-1}$$

decomposes. However, we have not been able to check whether the adapted transform has the covariance property.

The Original Wigner Function Revisited

Let us now compute the global Wigner transform for the Heisenberg group, and show how to recover the original Wigner transform from it. The formal calculations below can be made rigourous, either by restricting to suitable functions for which the integrals exist pointwise, or by weak arguments, i.e., taking scalar products with L^2-functions and using Plancherel's theorem instead of Fourier inversion. We have refrained from doing either.

The Heisenberg Lie algebra of the group is $\mathfrak{h} = \mathbb{R}^3$, with Lie bracket

$$[(p,q,t),(p',q',t')] = (0,0,pq'-p'q) \quad .$$

The associated Lie group $\mathbb{H}$ can be thought of as $\mathfrak{h}$, endowed with the Campbell-Baker-Hausdorff formula, which yields the group structure

$$(p,q,t)(p',q',t') = (p+p', q+q', t+t'+(pq'-p'q)/2) \quad .$$

Note that taking for $\mathbb{H}$ the Lie algebra $\mathfrak{h}$ with Campbell-Baker Hausdorff formula amounts to taking the identity operator as exponential map. The adjoint representation of H on $\mathfrak{h}$ is computed as

$$\mathrm{Ad}(p,q,t)(p',q',t') = (p',q',t'+pq'-p'q) \quad ,$$

and if (p^*,q^*,t^*) are the coordinates with respect to the dual basis,

$$\mathrm{Ad}^*(p,q,t)(p^*,q^*,t^*) = (p^*+t^*p, q^*+t^*q, t^*) \quad .$$

It follows that the set of coadjoint orbits consists of the singletons $\{(p^*,q^*,0)\}$, with $(p^*,q^*)\in\mathbb{R}^2$, and the hyperplanes $\mathcal{O}^*_{t^*} = \mathbb{R}^2\times\{t^*\}$. The representation corresponding to $\mathcal{O}^*_{t^*}$ under the Kirillov map is the **Schrödinger representation** ϱ_{-t^*} (observe the sign change). Here ϱ_{t^*} acts on $\mathrm{L}^2(\mathbb{R})$ via

$$[\varrho_{t^*}(p,q,t)f](x) = \mathrm{e}^{2\pi\mathrm{i}(qx+t^*t+t^*pq/2)} f(x+t^*p) \quad .$$

Now the Plancherel measure is seen to be supported by the Schrödinger representations, where it is given by $d\nu_{\mathbb{H}}(\sigma_{t^*}) = |t^*|dt^*$ [45, Section 7.6], with dt^* denoting Lebesgue measure on the real line.

We have now collected all necessary ingredients for the computation of $\mathfrak{W}$. Note that we know already by Theorem 4.37 that $\mathfrak{W}$ decomposes, so we will also be interested in the components. Pick a measurable field of rank-one operators $R = (g_{t^*}\otimes f_{t^*})_{t^*\in\mathbb{R}'} \in \mathcal{B}_2^{\oplus}\cap\mathcal{B}_1^{\oplus}$. Then the Plancherel inversion formula yields for $r = \mathcal{P}^{-1}(R)$ that

$$\begin{aligned} r(p,q,t) &= \int_{\mathbb{R}'} \langle g_{t^*}, \varrho_{-t^*}(p,q,t) f_{t^*}\rangle |t^*| dt^* \\ &= \int_{\mathbb{R}'}\int_{\mathbb{R}} g_{t^*}(x)\overline{f_{t^*}(x-t^*p)}\mathrm{e}^{-2\pi\mathrm{i}(qx-t^*t-t^*pq/2)}|t^*|dt^* \end{aligned}$$

Plugging this into the Euclidean Plancherel transform yields formally

$$\begin{aligned} &\mathfrak{W}(R)(p^*,q^*,s^*) \\ &= \int_{\mathbb{R}^5} g_{t^*}(x)\overline{f_{t^*}(x-t^*p)}\mathrm{e}^{-2\pi\mathrm{i}(qx-t^*t-t^*pq/2+q^*q+p^*p+s^*t)}|t^*|dt^*dxdpdqdt \end{aligned}$$

Now the integration against $dxdq$ can be simplified by Fourier inversion to the substitution $x\mapsto -q^*+t^*p/2$, thus the integral becomes

$$\int_{\mathbb{R}^3} g_{t^*}(-q^* + t^*p/2)\overline{f_{t^*}(-q^* - t^*p/2)}\mathrm{e}^{-2\pi\mathrm{i}(-t^*t+p^*p+s^*t)}|t^*|dt^*dpdt$$
$$= \int_{\mathbb{R}} g_{s^*}(-q^* + s^*p/2)\overline{f_{s^*}(-q^* - s^*p/2)}\mathrm{e}^{-2\pi\mathrm{i}p^*p}|s^*|dp \quad .$$

Here another Fourier inversion allowed to discard the integral $dtdt*$, resulting in the substitution $t^* \mapsto s^*$. Now a closer look at the remaining term reveals that indeed the values of $\mathfrak{W}(R)$ on the coadjoint orbit $\mathbb{R}^2 \times \{s^*\}$ only depend on $g_{s^*} \otimes f_{s^*}$, and that the local Wigner operators are given by

$$[\mathfrak{W}_{s^*}(g \otimes f)](p^*, q^*, s^*) = |s|^* \int_{\mathbb{R}} g\left(-q^* + \frac{s^*p}{2}\right)\overline{f\left(-q^* - \frac{s^*p}{2}\right)}\mathrm{e}^{-2\pi\mathrm{i}p^*p}dp \quad .$$

Now a comparison with (4.24) yields that the original Wigner transform discussed above is just the local Wigner transform $\mathfrak{W}_1$.

5

Admissible Vectors for Group Extensions

In this chapter we discuss a class of examples which has received considerable attention in recent years. The aim consists in making the abstract admissibility conditions developed in Chapter 4 explicit, in particular the criteria from Remark 4.30. Recall that this requires computing the Plancherel measure as well as the direct integral decomposition of the representation at hand into irreducibles. We consider certain group extensions

$$1 \to N \to G \to H \to 1 \quad ,$$

using the techniques of Kleppner and Lipsman from Section 3.6 for the computation of ν_G. Of particular interest will be the case where $G = N \rtimes H$, and π is the *quasi-regular representation* $\pi = \mathrm{Ind}_H^G 1$. The case $N = \mathbb{R}^k$ and $H < \mathrm{GL}(k, \mathbb{R})$ has been studied by a number of authors, see the list below, and we start by discussing this setting by more or less basic arguments. In fact, Examples 2.28, 2.30 and 2.36 are all special cases of this setting. It turns out that the arguments dealing with these examples extend to the general case $G = \mathbb{R}^k \rtimes H$, yielding elementary admissibility conditions which avoid explicit reference to the Plancherel transform of G (Theorem 5.8). Having done that, we show in Theorem 5.23 how the concrete admissibility conditions relate to the scheme described in Remark 4.30. As a result we obtain a very concrete description of the various objects that enter the Plancherel formula for the group under consideration. The argument is based on a general result about the containment of a quasi-regular representation in the regular representation (Theorem 5.22), which allows to conclude the existence of admissible vectors for a wide range of settings.

The result can also be applied to cases where N is nonabelian; in particular when N is a homogeneous Lie group and H a one-parameter group of dilations acting on N (Corollary 5.28). Finally, we show how the Zak transform criteria for Weyl-Heisenberg frames with integer oversampling can be regarded as admissibility conditions with respect to a certain discrete group.

Thus the results of this chapter generalize and/or complement the findings of various authors:

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- Murenzi's [93] 2D continuous wavelet transform discussed in 2.30.
- The dyadic wavelet transform of Mallat and Zhong [92] considered in 2.36
- Bohnke [25] introduced $H = \mathbb{R}^+ \cdot \mathrm{SO}_0(1,1)$ and $H = \mathbb{R}^+ \cdot \mathrm{SO}_0(2,1)$. Again the representations were irreducible.
- General characterizations of dilation groups H giving rise to *discrete series* subrepresentations of the quasiregular representation were given in [22, 50].
- Klauder, Isham and Streater [67, 74] considered $H = \mathrm{SO}(k)$ and $H = \mathrm{SO}(k-1,1)$. As a matter of fact, the representations they consider are not explicitly defined as subrepresentations of π, but they are direct integrals which are immediately recognised as subrepresentations of the representation $\widehat{\pi}$ obtained by conjugating π with the Fourier transform on $\mathbb{R}^k$.
- Cyclic and one-parameter subgroups of $\mathrm{GL}(k,\mathbb{R})$ were considered by Gröchenig, Kaniuth and Taylor [59]. A general discussion of discrete dilation groups can be found in [51].
- The characterization of dilation groups allowing the existence of admissible vectors was addressed in full generality in [77, 113], as well as in [52], and the discussion in the first two sections follows the latter paper closely.
- Lemarié [78] established the existence of wavelet orthonormal bases on $\mathrm{L}^2(N)$, where N is a stratified Lie group. These bases arise from the action of a lattice in N and a discrete group of dilations, in an entirely analogous fashion to the multiresolution wavelets on $\mathbb{R}$. Our results provide a continuous analogue of these systems, for the larger class of homogeneous Lie groups.
- Liu and Peng [83] considered a semidirect product $\mathbb{H} \rtimes \mathbb{R}$, where $\mathbb{R}$ denotes a one-parameter group of dilations, and an associated quasi-regular representation π on $\mathrm{L}^2(\mathbb{H})$. They then showed that π splits into discrete series representations, and gave admissibility conditions for each. Our results yield admissible vectors for π itself.
- Daubechies [33] (among other authors) characterized tight Gabor frames for the case of rational oversampling, making use of the Zak transform. The discussion in Section 5.5 exhibits this criterion as yet another instance of the scheme from Remark 4.30.

The standing assumptions of this chapter are: G is a type I group, $N \lhd G$ is regularly embedded and type I. Whenever G is nonunimodular, we assume that the $\mathrm{Ker}(\Delta_G)$ has the same properties as N. The assumptions are chosen to allow to apply the results from the previous chapters. As the discussion of the concrete admissibility conditions shows, the conditions are somewhat more restrictive than seems necessary. However, for all concrete examples that have so far been considered in the literature, the standing assumptions can in fact be verified.

5.1 Quasiregular Representations and the Dual Orbit Space

For this and the next section let $H < \mathrm{GL}(k,\mathbb{R})$ be a closed subgroup, and $G = \mathbb{R}^k \rtimes H$. Elements of G are denoted by (x,h) with $x \in \mathbb{R}^k$ and $h \in H$; the group law is then given by $(x_0,h_0)(x_1,h_1) = (x_0 + h_0x_1, h_0h_1)$. Left Haar measure of G is given by $d\mu_G(x,h) = |\det(h)|^{-1}dxd\mu_H(h)$, and the modular function is computed as $\Delta_G(x,h) = \Delta_H(h)|\det(h)|^{-1}$. For simplicity we will sometimes write $\Delta_G(h)$ instead of $\Delta_G(0,h)$. The quasiregular representation π of G acts on $\mathrm{L}^2(\mathbb{R}^k)$ by

$$(\pi(x,h)f)(y) = |\det(h)|^{-1/2} f(h^{-1}(y-x)).$$

The closedness of H in $\mathrm{GL}(k,\mathbb{R})$ may seem a somewhat arbitrary condition (Lie subgroups might also work), but it is in fact not a real restriction, because of the following fact. It was observed in [49, Proposition 5] for the discrete series case, but the proof does not use irreducibility. We reproduce it for the sake of completeness.

Proposition 5.1. *Let H be a subgroup of $\mathrm{GL}(k,\mathbb{R})$, endowed with some locally compact group topology. Assume that the semidirect product $\mathbb{R}^k \rtimes H$ is a topological semidirect product, and that the quasiregular representation has a nontrivial subrepresentation with an admissible vector. Then H is a closed subgroup of $\mathrm{GL}(k,\mathbb{R})$, and the topology on H is the relative topology.*

Proof. Since all wavelet coefficients vanish at infinity by 2.19, there exists a nontrivial C_0 matrix coefficient V_fg for π. We may assume $V_fg(e) = 1$. Let W_fg denote the matrix coefficient of the quasiregular representation of $\mathbb{R}^k \rtimes \mathrm{GL}(k,\mathbb{R})$ corresponding to the same pair of functions f,g. Then V_fg is the restriction of W_fg to $\mathbb{R}^k \rtimes H$. Since W_fg is continuous, there exists a compact neighborhood U of 1 in $\mathrm{GL}(k,\mathbb{R})$ with $|W_fg| > 1/2$ on U. $U \cap H$ is $\mathcal{T}$-closed in H, since U is closed and $\mathcal{T}$ is finer than the relative topology. Since H is closed in $\mathbb{R}^k \rtimes H$, the restriction of V_fg to H vanishes at infinity. By choice of U, this implies that $U \cap H$ is contained in a $\mathcal{T}$-compact set, hence is $\mathcal{T}$-compact itself. But then the inclusion map from $U \cap H$ to H is a homeomorphism onto its image, being a continuous map from a compact space to a Hausdorff space. Hence $\mathcal{T}$ coincides on $U \cap H$ with the relative topology. Since $U \cap H$ is a neighborhood of 1 for both topologies, it follows that the neighborhood filters of both topologies at unity coincide. Since a group topology is uniquely determined by the neighborhoods at unity, the topologies themselves coincide. In particular the relative topology is locally compact, which means H is closed.

Mackey's theory directs our attention towards the dual action. As it turns out, the decomposition of the quasiregular representation is also closely related to the dual orbit space. Let us quickly recall the notions of Mackey's theory

for this particular setup. The dual group $\widehat{\mathbb{R}^k}$ is the character group of $\mathbb{R}^k$, identified with $\mathbb{R}^k$ itself. Denoting the usual scalar product on $\mathbb{R}^k$ by $(\omega, x) \mapsto \omega \cdot x$, the duality between $\widehat{\mathbb{R}^k}$ and $\mathbb{R}^k$ is given by

$$\langle \omega, x \rangle = \mathrm{e}^{-2\pi \mathrm{i} \omega \cdot x} \quad .$$

In this identification, the dual operation is given by

$$h.\omega = (h^t)^{-1}\omega \quad , \tag{5.1}$$

where the right-hand side denotes the product of a matrix with a column vector. $\widehat{\mathbb{R}^k}/H$ is the dual orbit space. For $\gamma \in \widehat{\mathbb{R}^k}$, H_γ denotes the stabilizer of γ in H; it is a closed subgroup of H.

For the discussion of subrepresentations of π, it is useful to introduce the representation $\widehat{\pi}$ obtained by conjugating π with the Fourier transform on $\mathbb{R}^k$. It is readily seen to operate on $\mathrm{L}^2(\widehat{\mathbb{R}^k})$ via

$$(\widehat{\pi}(x,h)\widehat{f})(\omega) = |\det(h)|^{1/2} \mathrm{e}^{-2\pi \mathrm{i} \omega \cdot x} f(h^{-1}.\omega) \quad . \tag{5.2}$$

The action of $\widehat{\pi}$ allows to identify subrepresentations in a simple way: Every invariant closed subspace $\mathcal{H} \subset \mathrm{L}^2(\mathbb{R}^k)$ is of the form

$$\mathcal{H} = \mathcal{H}_U = \{ g \in \mathrm{L}^2(\mathbb{R}^k) : \widehat{g} \text{ vanishes outside of } U \ \} \quad ,$$

where $U \subset \widehat{\mathbb{R}^k}$ is a measurable, H-invariant subset (see [48] for a detailed argument). We let π_U denote the subrepresentation acting on $\mathcal{H}_U$.

Remark 5.2. **The structure of the dual orbit space** As the structure of the dual orbit space is important for the decomposition of the quasi-regular representation and for the construction of admissible vectors on the one hand, but also for the computation of Plancherel measure on the other, let us take a closer look at its measure-theoretic structure. For our discussion, the following two sets will be central

$$\Omega_c = \{ \omega \in \widehat{\mathbb{R}^k} : H_\omega \text{ is compact } \} \ , \ \Omega_{rc} = \{ \omega \in \Omega_c : \mathcal{O}(\omega) \mid \text{ is locally closed } \} \ .$$

The set $\Omega_{rc} \subset \Omega_c$ consists of the "regular" orbits in Ω_c; i.e., it is the "well-behaved" part of Ω_c. Loosely speaking, Ω_c is the set we have to deal with, and Ω_{rc} is the set we can deal with. Put more precisely: While Theorem 5.8 below shows that subrepresentations with admissible vectors necessarily correspond to invariant subsets U of Ω_c, the existence result in Theorem 5.12 only considers subsets of the smaller set Ω_{rc}. However, this distinction is not due to a shortcoming of our approach: Remark 5.13 gives an example of a subset of Ω_c that does not allow admissible vectors for the corresponding subrepresentation.

The measure-theoretic properties of the two sets are summarized as follows: Ω_c can be shown to be measurable, see Corollary 5.6. But usually Ω_c is

not open, even when it is conull, as is illustrated by the example of $\mathrm{SL}(2,\mathbb{Z})$: It is easy to see that Ω_c consists of all the vectors (ω_1, ω_2) such that ω_1/ω_2 is irrational. This is a conull set with dense complement in $\widehat{\mathbb{R}^2}$.

By contrast, Ω_{rc} is always open, by Proposition 5.7. A pleasant consequence of this is that Glimm's Theorem [56] applies (since Ω_{rc} is locally compact), which entails a number of useful properties of the orbit space Ω_{rc}/H: It is a standard Borel space having a measurable cross section $\Omega_{rc}/H \to \Omega_{rc}$, and there exists a measurable transversal, i.e., a Borel subset $A \subset \Omega_{rc}$ meeting each orbit in precisely one point.

Unfortunately, the example of $\mathrm{SL}(2,\mathbb{Z})$ shows that Ω_{rc} can be empty even when Ω_c is conull: Ω_c contains no nonempty open set, since its complement is dense.

The rest of the section is devoted to proving the measurability of Ω_c and the openness of Ω_{rc}. The proof for the first result uses the subgroup space of H, as introduced by Fell [44].

Definition 5.3. *Let G be a locally compact group. The* **subgroup space of G** *is the set $K(G) := \{H < G : H \text{ is closed }\}$, endowed with the topology generated by the sets*

$$U(V_1, \ldots, V_n; C) := \{H \in K(G) : H \cap V_i \neq \emptyset, \forall 1 \leq i \leq n, H \cap C = \emptyset\},$$

where $V_1, \ldots, V_n$ denotes any finite family of open subsets of G and $C \subset G$ is compact.
With this topology $K(G)$ is a compact Hausdorff space.

The motivation for introducing the $K(G)$ to our discussion is the following:

Proposition 5.4. *Let X be a countably separated Borel space, and G a locally compact group acting measurably on X. Consider the* **stabilizer map** *$s : X \to K(G)$ defined by $s(x) = \{g \in G : g.x = x\}$. Then s is Borel.*

Proof. That s indeed maps into $K(G)$ is due to [17, Chapter I, Proposition 3.7], the measurability is [17, Chapter II, Proposition 2.3].

Proposition 5.5. *Let G be a σ-compact, locally compact group. Then $K_c(G) = \{H \in K(G) : H \text{ is compact }\}$ is a Borel subset of $K(G)$.*

Proof. We first construct a sequence of compact set $(C_n)_{n\in\mathbb{N}}$ such that $A \subset G$ is relatively compact iff $A \subset C_n$ for some $n \in \mathbb{N}$. For this purpose let $G = \bigcup_{n\in\mathbb{N}} K_n$ with compact K_n. Pick a compact neighborhood V of the unit elements, and define $C_n = V \cdot \bigcup_{i=1}^{n} C_i$. Then the open kernels of the C_n cover G. Hence, if $A \subset G$ is compact, the fact that the C_n increase implies that $A \subset C_n$ for some n. It follows that

$$\begin{aligned} K_c(G) &= \bigcup_{n\in\mathbb{N}} \{H \in K(G) : H \cap (G \setminus C_n) = \emptyset\} \\ &= \bigcup_{n\in\mathbb{N}} K(G) \setminus U(G \setminus C_n, \emptyset) \end{aligned}$$

is an F_σ-set.

Combining 5.4 and 5.5, we obtain the desired measurability.

Corollary 5.6. *Ω_c is a Borel subset of $\widehat{\mathbb{R}^k}$.*

The proof of the following proposition uses ideas from [77].

Proposition 5.7. *Ω_{rc} is open.*

Proof. Define the *ϵ-stabilizer*

$$H_\omega^\epsilon = \{h \in H : |h.\omega - \omega| \leq \epsilon\} \quad ,$$

where $|\cdot|$ denotes the euclidean norm on $\widehat{\mathbb{R}^k}$. If H_ω^ϵ is compact for some $\epsilon > 0$, then $B_\epsilon(\omega) \cap \mathcal{O}(\omega) = H_\omega^\epsilon.\omega$ is compact. Here $B_\epsilon(x)$ denotes the closed ϵ-ball around x. Hence the orbit $\mathcal{O}(\omega)$ is locally closed.

Conversely, assume that $B_\epsilon(\omega) \cap \mathcal{O}(\omega)$ is compact for some $\epsilon > 0$ and that H_ω is compact. There exists a measurable cross-section $\tau : \mathcal{O}(\omega) \to H$ which maps compact sets in $\mathcal{O}(\omega)$ to relatively compact sets in H. Hence $H_\omega^\epsilon \subset H_\omega \tau(B_\epsilon(\omega))$ is relatively compact and closed, hence compact.

In short, we have shown

$$\omega \in \Omega_{rc} \Longleftrightarrow \exists \epsilon > 0 : H_\omega^\epsilon \text{ is compact} \quad ,$$

and we are going to use this characterization to prove the openness of Ω_{rc}.

If the origin is in Ω_{rc}, then H is compact, and $\Omega_{rc} = \widehat{\mathbb{R}^k}$. In the other case, pick ω in Ω_{rc} and $\epsilon > 0$ with H_ω^ϵ compact. Since $\mathrm{GL}(k, \mathbb{R})$ acts transitively on $\widehat{\mathbb{R}^k} \setminus \{0\}$, we may (possibly after passing to a smaller ϵ) assume that there exists a continuous cross-section $\alpha : B_\epsilon(\omega) \to \mathrm{GL}(k, \mathbb{R})$ with relatively compact image, i.e., $\alpha(\gamma).\omega = \gamma$, for all $\gamma \in B_\epsilon(\omega)$, and $\alpha(B_\epsilon(\omega)) \subset U$, where U is a compact neighborhood of the identity in $\mathrm{GL}(k, \mathbb{R})$. We are going to show that $B_\epsilon(\omega) \subset \Omega_{rc}$. For this purpose let $\gamma \in B_\epsilon(\omega)$. Clearly it is enough to prove that

$$C := \{h \in H : h.\gamma \in B_\epsilon(\omega)\} = \{h \in \mathrm{GL}(k, \mathbb{R}) : h.\gamma \in B_\epsilon(\omega)\} \cap H$$

is relatively compact. By assumption,

$$H_\omega^\epsilon = \{h \in \mathrm{GL}(k, \mathbb{R}) : h.\omega \in B_\epsilon(\omega)\} \cap H$$

is compact. Hence

$$\begin{aligned} C &= \{h \in \mathrm{GL}(k, \mathbb{R}) : \omega\alpha(\gamma).h \in B_\epsilon(\omega)\} \cap H \\ &= \alpha(\gamma)^{-1}\{h \in \mathrm{GL}(k, \mathbb{R}) : h.\omega \in B_\epsilon(\omega)\} \cap H \\ &\subset U^{-1}(\{h \in \mathrm{GL}(k, \mathbb{R}) : h.\omega \in B_\epsilon(\omega)\} \cap H) \end{aligned}$$

i.e., C is contained in the product of two compact sets, and thus relatively compact. Note that we used here that H is a closed subgroup of $\mathrm{GL}(k, \mathbb{R})$, hence compactness in H is the same as compactness in $\mathrm{GL}(k, \mathbb{R})$.

5.2 Concrete Admissibility Conditions

We will now derive the admissibility condition for the quasiregular representation and its subrepresentations. The proof of the admissibility condition is fairly straightforward. It is only when we address the existence of functions fulfilling the condition that we are forced to use more involved arguments. The theorem was derived for certain concrete groups H in [92, 67, 74], and the arguments presented in this section are generalizations of those in [92, 67, 74]. The general version given here appears also in [77, 52]. Note that the admissibility condition also figures as a part of the definition of the notion of "projection generating function" in [59, Definition 2.1]. Thus the following theorem also answers a question raised in [59, Remark 2.6(b)]: There the authors observe that taking a projection generating function as wavelet gives rise to orthogonality relations among the wavelet coefficients which closely resemble those for irreducible square-integrable representations, even though the representation at hand is not irreducible. The explanation for this phenomenon is that the orthogonality relations in the discrete series case are particular instances of the orthogonality relations arising in connection with the Plancherel formula.

A comparison of the following theorem with Theorem 4.20 gives a first hint towards the connection between abstract and concrete admissibility conditions.

Theorem 5.8. *Let $(\pi_U, \mathcal{H}_U)$ be a subrepresentation of π corresponding to some invariant measurable subset U. Then*

$$g \in \mathcal{H}_U \text{ is bounded } \Leftrightarrow \int_H |\widehat{g}(h^{-1}.\omega)|^2 d\mu_H(h) \leq \text{constant} \tag{5.3}$$

$$g \in \mathcal{H}_U \text{ is cyclic } \Leftrightarrow \int_H |\widehat{g}(h^{-1}.\omega)|^2 d\mu_H(h) \neq 0 \tag{5.4}$$

$$g \in \mathcal{H}_U \text{ is admissible } \Leftrightarrow \int_H |\widehat{g}(h^{-1}.\omega)|^2 d\mu_H(h) = 1 \tag{5.5}$$

Here all right-hand sides are understood to hold almost everywhere. In particular, if π_U has a bounded cyclic vector, then $U \subset \Omega_c$ (up to a null set).

Proof. We start by explicitly calculating the L^2-norm of $V_g f$, for $f, g \in \mathcal{H}_U$. The following computations are generalizations of the argument used for the 1D continuous wavelet transform in Example 2.28, see also [22, 48, 113].

$$\begin{aligned}
\|V_g f\|^2_{\mathrm{L}^2(G)} &= \int_G |\langle f, \pi(x,h)g\rangle|^2 \, d\mu_G(x,h) \\
&= \int_G \left| \langle \widehat{f}, (\pi(x,h)g)^\wedge \rangle \right|^2 d\mu_G(x,h) \\
&= \int_G \left| \int_{\widehat{\mathbb{R}^k}} \widehat{f}(\omega) |\det(h)|^{1/2} \mathrm{e}^{2\pi \mathrm{i} \gamma x} \overline{\widehat{g}}(h^{-1}.\omega) d\omega \right|^2 d\mu_G(x,h) \\
&= \int_H \int_{\mathbb{R}^k} \left| \int_{\widehat{\mathbb{R}^k}} \widehat{f}(\omega) \mathrm{e}^{2\pi \mathrm{i} \omega x} \overline{\widehat{g}}(h^{-1}.\omega) d\omega \right|^2 d\lambda(x) d\mu_H(h) \\
&= \int_H \int_{\mathbb{R}^k} |\mathcal{F}(\phi_h)(-x)|^2 \, d\lambda(x) d\mu_H(h).
\end{aligned}$$

Here $\phi_h(\omega) = \widehat{f}(\omega)\overline{\widehat{g}}(h^{-1}.\omega)$, and $\mathcal{F}$ denotes the Fourier transform on $\mathrm{L}^1(\widehat{\mathbb{R}^k})$. An application of Plancherel's formula – or more precisely, the extension of 2.22 to $\mathbb{R}^k$ – to the last expression yields

$$\begin{aligned}
\int_H \int_{\widehat{\mathbb{R}^k}} |\phi_h(\omega)|^2 \, d\omega d\mu_H(h) &= \int_H \int_{\widehat{\mathbb{R}^k}} \left|\widehat{f}(\omega)\right|^2 \left|\widehat{g}(h^{-1}.\omega)\right|^2 d\omega d\mu_H(h) \\
&= \int_{\widehat{\mathbb{R}^k}} \left|\widehat{f}(\omega)\right|^2 \left(\int_H \left|\widehat{g}(h^{-1}.\omega)\right|^2 d\mu_H(h) \right) d\omega \ .
\end{aligned}$$

Now (5.3) through (5.5) are obvious. Moreover, it is easily seen that whenever the stabilizer H_ω is noncompact, we have

$$\int_H |\widehat{g}(h^{-1}.\omega)|^2 d\mu_H(h) \in \{0, \infty\},$$

(cf. also the proof of [48, Theorem 10]), hence $V_g f \in \mathrm{L}^2(G)$ entails that the pointwise product $\widehat{f}\overline{\widehat{g}}$ vanishes a.e. outside of Ω_c. In particular, a bounded cyclic vector vanishes almost everywhere outside of Ω_c, hence we obtain in such a case that $U \subset \Omega_c$ (up to a null set).

For the construction of admissible vectors we first decompose Lebesgue measure λ on Ω_{rc} into measures on the orbits and a measure on Ω_{rc}/H. Then we address the relationship of the measures on the orbits to the Haar measure of H.

Lemma 5.9. *(a) There exists a measure $\overline{\lambda}$ on Ω_{rc}/H and on each orbit $\mathcal{O}(\gamma)$ a measure $\beta_{\mathcal{O}(\gamma)}$ such that for every measurable $A \subset \Omega_{rc}$ the mapping*

$$\mathcal{O}(\gamma) \mapsto \int_{\mathcal{O}(\gamma)} \mathbf{1}_A(\omega) d\beta_{\mathcal{O}(\gamma)}(\omega)$$

is $\overline{\lambda}$-measurable, and in addition

$$\lambda(A) = \int_{\widehat{\mathbb{R}^k}/H} \int_{\mathcal{O}(\gamma)} \mathbf{1}_A(\omega) d\beta_{\mathcal{O}(\gamma)}(\omega) d\overline{\lambda}(\mathcal{O}(\gamma)).$$

(b) Let $(\overline{\lambda}, (\beta_{\mathcal{O}(\gamma)})_{\mathcal{O}(\gamma)\in\Omega_{rc}/H})$ *be as in (a). For* $\gamma \in \Omega_{rc}$ *define* $\mu_{\mathcal{O}(\gamma)}$ *as the image measure of* μ_H *under the projection map* $p_\gamma : h \mapsto h^{-1}.\gamma$. $\mu_{\mathcal{O}(\gamma)}$ *is a* σ*-finite measure, and its definition is independent of the choice of representative* γ. *Then, for almost all* $\gamma \in \Omega_{rc}$, *the* $\mu_{\mathcal{O}(\gamma)}$ *and* $\beta_{\mathcal{O}(\gamma)}$ *are equivalent, with globally Lebesgue-measurable Radon-Nikodym-derivatives: There exists an (essentially unique) Lebesgue-measurable function* $\kappa : \Omega_{rc} \to \mathbb{R}^+$ *such that for* $\omega \in \mathcal{O}(\gamma)$,

$$\frac{d\beta_{\mathcal{O}(\gamma)}}{d\mu_{\mathcal{O}(\gamma)}}(\omega) = \kappa(\omega) \quad .$$

(c) The function κ *fulfills the* **semi-invariance relation**

$$\kappa(h^{-1}.\omega) = \kappa(\omega)\Delta_G(h)^{-1} \quad . \tag{5.6}$$

In particular, κ *is* H*-invariant iff* G *is unimodular. In that case, we can in fact assume that* $\kappa = 1$ *almost everywhere. This choice determines the measure* $\overline{\lambda}$ *uniquely.*

Proof. Statement (a) is a special instance of Proposition 3.28.

In order to prove part (b), well-definedness and σ-finiteness of $\mu_{\mathcal{O}(\gamma)}$ follow from compactness of H_γ. The independence of the representative γ of the orbit follows from the fact that μ_H is leftinvariant. To compute the Radon-Nikodym derivative κ, we first introduce an auxiliary function $\ell : \Omega_{rc} \to \mathbb{R}_0^+$: Fix a Borel-measurable transversal $A \subset \Omega_{rc}$ of the H-orbits. Then the mapping $\tau : A \times H \to \Omega_{rc}$, $\tau(\omega, h) = h^{-1}.\omega$ is bijective and continuous, hence, since $A \times H$ is a standard Borel space, $\tau^{-1} : \Omega_{rc} \to A \times H$ is Borel as well, by [15, Theorem 3.3.2]. If we let $\tau^{-1}(\gamma)_H$ denote the H-valued coordinate of $\tau^{-1}(\gamma)$, then $\ell(\gamma) := \Delta_G(\tau^{-1}(\gamma)_H)$ is a Borel-measurable mapping. Since Δ_G is constant on every compact subgroup (in particular on all the little fixed groups of elements in Ω_{rc}), a straightforward calculation shows that ℓ satisfies the semi-invariance relation $\ell(h^{-1}.\omega) = \ell(\omega)\Delta_G(h)^{-1}$.

Next fix an orbit $\mathcal{O}(\gamma)$ and let us compare the measures $\beta_{\mathcal{O}(\gamma)}$ and $\ell\mu_{\mathcal{O}(\gamma)}$: Since

$$d\mu_{\mathcal{O}(\gamma)}(h^{-1}.\omega) = \Delta_H(h)d\mu_{\mathcal{O}(\gamma)}(\omega) \quad \text{and} \quad d\beta_{\mathcal{O}(\gamma)}(h^{-1}.\omega) = |\det(h)|d\beta_{\mathcal{O}(\gamma)}(\omega) \ ,$$

the definition of ℓ ensures that $\ell\mu_{\mathcal{O}(\gamma)}$ and $\beta_{\mathcal{O}(\gamma)}$ behave identically under the action of H. Moreover, they are σ-finite and quasi-invariant, hence equivalent. Since they have the same behaviour under the operation of H, the Radon-Nikodym derivative turns out to be a positive constant on the orbit. Summarizing, we find for $\omega \in \mathcal{O}(\gamma)$ that

$$\frac{d\beta_{\mathcal{O}(\gamma)}}{d\mu_{\mathcal{O}(\gamma)}}(\omega) = \ell(\omega)c_{\mathcal{O}(\gamma)} \quad ,$$

with $\ell, c_{\mathcal{O}(\gamma)} > 0$, and it remains to show that $c_{\mathcal{O}(\gamma)}$ depends measurably on the orbit.

For this purpose pick a relatively compact open neighborhood $B \subset H$ of the identity. Then $AB = \tau(A \times B) \subset \Omega_{rc}$ is Borel-measurable, as a continuous image of a standard space, hence $\mathbf{1}_{AB}$, the indicator function of AB, is a Borel-measurable function. Both

$$\phi_1 : \mathcal{O}(\gamma) \mapsto \int_{\mathcal{O}(\gamma)} \mathbf{1}_{AB}(\omega) d\beta_{\mathcal{O}(\gamma)}(\omega)$$

and

$$\phi_2 : \mathcal{O}(\gamma) \mapsto \int_{\mathcal{O}(\gamma)} \mathbf{1}_{AB}(\omega)\ell(\omega) d\mu_{\mathcal{O}(\gamma)}(\omega)$$

are measurable functions: The first one is by choice of the $\beta_{\mathcal{O}(\gamma)}$, see part (a). The second one is measurable by Fubini's theorem, applied to the mapping $(\omega, h) \mapsto \mathbf{1}_{AB}(h^{-1}.\omega)\ell(h^{-1}.\omega)$ on $\widehat{\mathbb{R}^k} \times H$ (recall the definition of $\mu_{\mathcal{O}(\gamma)}$).

In addition, both functions are finite and positive on Ω_{rc}. We have

$$\phi_2(\mathcal{O}(\gamma)) = \int_{\mathcal{O}(\gamma)} \mathbf{1}_{AB}(\omega)\ell(\omega) d\mu_{\mathcal{O}(\gamma)}(\omega) = \int_{p_\gamma^{-1}(AB)} \Delta_G(h) d\mu_H(h) \ ,$$

and $p_\gamma^{-1}(AB)$ is relatively compact and open, hence it has finite and positive Haar measure. Since in addition Δ_G is positive and bounded on $p_\gamma^{-1}(AB)$, we find $0 < \phi_2(\mathcal{O}(\gamma)) < \infty$. Hence

$$\phi_1(\mathcal{O}(\gamma)) = c_{\mathcal{O}(\gamma)}\phi_2(\mathcal{O}(\gamma))$$

can be solved for $c_{\mathcal{O}(\gamma)}$, which thus turns out to depend measurably upon $\mathcal{O}(\gamma)$. Hence

$$\kappa(\omega) = \frac{d\beta_{\mathcal{O}(\gamma)}}{d\mu_{\mathcal{O}(\gamma)}} = \ell(\omega)c_{\mathcal{O}(\gamma)}$$

is a Lebesgue-measurable function.

The remaining part (*c*) is simple to prove: The semi-invariance relation of ℓ entails the relation for κ. The normalization is easily obtained: If κ is constant on the orbits, it defines a measurable mapping $\overline{\kappa}$ on Ω_{rc}/H. If we replace each $\beta_{\mathcal{O}(\gamma)}$ by $\mu_{\mathcal{O}(\gamma)}$, we can make up for it by taking $\overline{\kappa}(\mathcal{O}(\gamma))d\overline{\lambda}(\mathcal{O}(\gamma))$ as the new measure on the orbit space. The new choice has the desired properties. The uniqueness of $\overline{\lambda}$ follows from the usual Radon-Nikodym arguments.

Remark 5.10. Let us for the next two sections fix a choice of $\overline{\lambda}$. Note that this also uniquely determines the function κ. In the unimodular case we take κ to be 1, which in turn determines $\overline{\lambda}$ uniquely.

As we shall see in Theorem 5.23, the choice of a pair $(\overline{\lambda}, \kappa)$ corresponds exactly to a choice of Plancherel measure and the associated family of Duflo-Moore operators, at least on a subset of $\widehat{G}$.

Before we turn to the construction of admissible vectors, we introduce some notation to help clarify the construction: To a function $\widehat{g}$ on U we associate

two auxiliary H-invariant functions $T_H(\widehat{g})$ and $S_H(\widehat{g})$ such that admissibility of g translates to a condition on $T_H(\widehat{g})$ and square-integrability to a condition on $S_H(\widehat{g})$.

Definition 5.11. *For a measurable function $\widehat{g}$ on Ω_{rc}, let $T_H(\widehat{g})$ denote the function*

$$T_H(\widehat{g})(\omega) := \left(\int_{\mathcal{O}(\omega)} |\widehat{g}(\gamma)|^2 d\mu_{\mathcal{O}(\omega)}(\gamma)\right)^{1/2}$$
$$= \left(\int_{\mathcal{O}(\omega)} |\kappa(\gamma)^{-1/2}\widehat{g}(\gamma)|^2 d\beta_{\mathcal{O}(\omega)}(\gamma)\right)^{1/2} .$$

$T_H(\widehat{g})$ is a measurable, H-invariant mapping $\Omega_{rc} \to \mathbb{R}_0^+ \cup \{\infty\}$. The admissibility condition can then be reformulated:

$$g \in \mathrm{L}^2(U) \text{ is admissible } \Leftrightarrow T_H(\widehat{g}) \equiv 1 \quad (\text{ a.e. on } U) \quad . \tag{5.7}$$

Similarly, weak admissibility is equivalent to the requirement that $T_H(\widehat{g}) \in \mathrm{L}^\infty(U)$ and $T_H(\widehat{g}) > 0$ almost everywhere. We can also define

$$S_H(\widehat{g})(\omega) := \left(\int_{\mathcal{O}(\omega)} |\widehat{g}(\gamma)|^2 d\beta_{\mathcal{O}(\omega)}(\gamma)\right)^{1/2} .$$

By our choice of measures, S_H and T_H coincide iff G is unimodular. Both $T_H(\widehat{g})$ and $S_H(\widehat{g})$ may (and will) be regarded as functions on the quotient space U/H. By the choice of the $\beta_{\mathcal{O}(\omega)}$,

$$\int_U |\widehat{g}(\omega)|^2 d\omega = \int_{U/H} |S_H(\widehat{g})(\mathcal{O}(\omega))|^2 d\overline{\lambda}(\mathcal{O}(\omega)) \quad , \tag{5.8}$$

so that $\widehat{g}$ is square-integrable iff $S_H(\widehat{g})$ is a square-integrable function on U/H.

Now we can address the existence of admissible vectors. The following theorem is essentially the same as [77, Theorem 1.8]. Again, a comparison of this theorem with the abstract version in Theorem 4.22 is instructive. The connection will be made explicit in Theorem 5.23 below.

Theorem 5.12. *Let $U \subset \Omega_{rc}$ be measurable and H-invariant. Then π_U has a bounded cyclic vector. It has an admissible vector iff either*

(i) G is unimodular and $\overline{\lambda}(U/H) < \infty$.
(ii)G is nonunimodular.

Note that the strategy for the construction of admissible vectors in the following proof is similar to the arguments in [77], but also to the construction in Theorem 4.23. It amounts to treating the admissibility condition – involving T_H – first, and then adjusting the construction to fulfill the square-integrability condition – involving S_H – as well.

Proof. Recall that by the last remark we have for each admissible vector g that $T_H(\widehat{g})$ is constant almost everywhere. At the same time, in the unimodular case it is square-integrable as a function on U/H, because of $S_H = T_H$. This shows the necessity of (i) in the unimodular case.

To prove the existence of admissible vectors, we first construct a function $\widehat{g}$ on U fulfilling the admissibility condition (5.7), and then modify the construction to provide for square-integrability.

For this purpose we recycle the sets $A \subset \Omega_{rc}$ and $B \subset H$ from the proof of Lemma 5.9. We already observed there that $\widehat{f} = \mathbf{1}_{AB}$ is Lebesgue-measurable, and that $T_H(\widehat{f})$ is positive and finite almost everywhere on U. Hence we may define $\widehat{g} = \widehat{f}/T_H(\widehat{f})$, which fulfills the admissibility criterion. In the unimodular case, equation (5.8) together with $S_H = T_H$ shows that $\widehat{g} \in \mathrm{L}^2(U)$.

In the nonunimodular case, we modify g as follows: For every $\gamma \in U$, the compactness of $p_\gamma^{-1}(AB)$ entails that Δ_G is bounded on that set. Thus $S_H(\widehat{g})$ is positive and finite almost everywhere. Since $\overline{\lambda}$ is σ-finite, we can write $U/H = \bigcup_{n\in\mathbb{N}} V_n$, with disjoint V_n of finite measure, such that in addition $S_H(\widehat{g})$ is bounded on each V_n (here we regard $S_H(\widehat{g})$ as a function on the quotient). In particular, $S_H(\widehat{g}) \cdot \mathbf{1}_{V_n}$ is square-integrable on U/H. Now let $U_n \subset U$ be the inverse image of V_n under the quotient map, and for $h_0 \in H$ and $n, k_n \in \mathbb{N}$, denote by

$$\widehat{g_n}(\omega) := \Delta_H(h_0)^{k_n/2}\, \widehat{f}_2(h^{-1}.\omega_0^{k_n}) \cdot \mathbf{1}_{U_n}(\omega) \quad .$$

Then the normalization ensures that $\widehat{g_n}$ has the following properties:

$$T_H(\widehat{g_n}) = \mathbf{1}_{U_n} \tag{5.9}$$

and

$$\begin{aligned} S_H(\widehat{g_n}) &= \Delta_H(h_0)^{k_n/2} |\det(h_0)|^{-k_n/2} S_H(\widehat{g}) \cdot \mathbf{1}_{U_n} \\ &= \Delta_G(h_0)^{k_n/2} S_H(\widehat{g}) \cdot \mathbf{1}_{U_n} \quad . \end{aligned} \tag{5.10}$$

Hence the following construction gives an admissible vector: Choose $h_0 \in H$ such that $\Delta_G(h_0) < 1/2$, pick $k_n \in \mathbb{N}$ satisfying

$$2^{-k_n} \| S_H(\widehat{g}) \cdot \mathbf{1}_{U_n} \|_2^2 < 2^{-n} \tag{5.11}$$

and let $\widehat{\widehat{g}}(\omega) := \Delta_H(h_0)^{k_n/2} \widehat{f}_2(h^{-1}.\omega_0^{k_n})$, for $\omega \in U_n$. Then (5.9) implies that $T_H(\widehat{\widehat{g}}) = 1$ a.e., whereas (5.10) and (5.11) ensure that $S_H(\widehat{\widehat{g}}) \in \mathrm{L}^2(U/H, \overline{\lambda})$.

A bounded cyclic vector for π_U –which is missing in the unimodular case– can be obtained by similar (somewhat simpler) methods.

Remark 5.13. In Theorem 5.12 we cannot replace Ω_{rc} by the bigger set Ω_c. To give a nonunimodular example, let $H = \{2^k h : k \in \mathbb{Z}, h \in \mathrm{SL}(2,\mathbb{Z})\}$, which is a discrete subgroup of $\mathrm{GL}(2,\mathbb{R})$. Whenever $(\gamma_1, \gamma_2) \in \widehat{\mathbb{R}^2}$ is such that γ_1/γ_2 is irrational, the stabilizer of (γ_1, γ_2) in H is finite. Hence the set Ω_c is a

conull subset in $\widehat{\mathbb{R}^2}$, whereas (as we already noted) Ω_{rc} is empty. H operates ergodically on $\widehat{\mathbb{R}^2}$ (already $\mathrm{SL}(2,\mathbb{Z})$ does, [118, 2.2.9]), and we can use this fact to show that π has no admissible vectors: For every $g \in \mathrm{L}^2(\mathbb{R}^2)$, the map

$$\mathbb{R}^2 \ni \gamma \mapsto \sum_{h\in H} |\hat{g}(\gamma h)|^2$$

is measurable and H-invariant, hence, by ergodicity, it is constant almost everywhere. By the same calculation as in the proof of the admissibility condition (5.5) we see that g is admissible iff the constant is finite.

Let $g \neq 0$. Pick $\epsilon > 0$ and a Borel set A on which $|\hat{g}|^2 > \epsilon$ holds. Then

$$\sum_{h\in H} |\hat{g}(\gamma h)|^2 \geq \epsilon \sum_{h\in H} |\mathbf{1}_A(\gamma h)| \quad .$$

Choose a sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint subsets of A satisfying $\lambda(A_n) > 0$. Then, for any fixed n, the set $B_n := \{\gamma \in U : \gamma H \cap A_n \neq \emptyset\}$ is H-invariant and contains A_n, hence, by ergodicity, it is a conull set. Hence the intersection B of all B_n is a conull set, and for every $\gamma \in B$ the set $\gamma H \cap A$ is infinite, the A_n being disjoint. But this implies

$$\sum_{h\in H} |\mathbf{1}_A(\gamma h)| = \infty$$

on B, and thus a forteriori

$$\sum_{h\in H} |\hat{g}(\gamma h)|^2 = \infty \quad ,$$

hence g is not admissible.

Let us now give a short summary of the steps which have to be carried out for the construction of wavelet transforms from semidirect products:

1. Compute the H-orbits in $\widehat{\mathbb{R}^k}$, possibly by giving a parametrization of $\widehat{\mathbb{R}^k}/H$.
2. Determine the set Ω_{rc}. If $\lambda(\Omega_{rc}) = 0$, stop.
3. Parametrize each orbit in Ω_{rc} and determine the image $\mu_{\mathcal{O}(\gamma)}$ of Haar measure under the projection map $h \mapsto h^{-1}.\gamma$.
4. Compute the measure decomposition $d\lambda(\gamma) = d\beta_{\mathcal{O}(\omega)}(\gamma) d\overline{\lambda}(\mathcal{O}(\omega))$.
5. Compute the Radon-Nikodym derivative κ.
6. The admissibility condition can then be formulated for subsets of Ω_{rc} just as in Theorem 5.8. Theorem 5.12 ensures the existence of admissible vectors.

In Remark 5.17 below we carry out the steps for the Poincaré group, which was already considered in [74]. Since the final step – the actual construction of admissible vectors – is missing, the description is somewhat incomplete. Clearly the construction given in the proof of Theorem 5.12 is not very practical, but it seems doubtful to us that a more explicit method is available which works in full generality. However, in many concrete cases where parameterizations of orbits and orbit spaces are possible, they can be given differentiably. Then computing the various measures and Radon-Nikodym derivatives reduces to computing the Jacobian determinants of those parameterizations. We expect that in such a setting the construction of admissible vectors should also be facilitated.

Further simplification can be obtained by the action of a matrix group in the normalizer of H, as we will see next. For the remainder of this section we focus on the case that G is unimodular. The main motivation for the following proposition is to show that certain subrepresentations of π do not have admissible vectors. In the light of Theorem 5.12, this amounts to proving that $\overline{\lambda}(U/H)$ is infinite, for the H-invariant set $U \subset \Omega_{rc}$ under consideration.

The argument proving the following proposition employs the action of the scalars on the orbit space Ω_{rc}/H. The group of scalars could be replaced by any group $A \subset \mathrm{GL}(k, \mathbb{R})$ which normalizes H. Symmetry arguments of this type could also simplify some of the steps 1. through 6. sketched above.

The multiplicative group $(\mathbb{R}', \cdot)$ operates on $\widehat{\mathbb{R}^k}/H$ by multiplication: If $a \in \mathbb{R}^+$ then $a \cdot (\mathcal{O}(\gamma)) = \mathcal{O}(a\gamma)$ is well-defined. Obviously Ω_{rc} is invariant, so that we obtain an operation on Ω_{rc}/H. The next proposition gives the behaviour of $\overline{\lambda}$ under this action. Obviously the fixed groups are constant along $\mathbb{R}'$-orbits, i.e., $H_{a\gamma} = H_\gamma$.

Proposition 5.14. *Assume that G is unimodular. Let the measures $\overline{\lambda}$ and $\mu_{\mathcal{O}(\gamma)}$ be as in Lemma 5.9. Assume that a constant choice of Haar measure on $H_{a\gamma} = H_\gamma$ was used to compute the $\mu_{\mathcal{O}.(a\gamma)}$ $(a \in \mathbb{R}')$. For $a \in \mathbb{R}'$ and $\gamma \in \widehat{\mathbb{R}^k}$ let $a^*(\mu_{\mathcal{O}(\gamma)})$ denote the image measure of $\mu_{\mathcal{O}(\gamma)}$ on $\mathcal{O}(a\gamma)$, i.e., for measurable $B \subset \mathcal{O}(a\gamma)$ let $a^*(\mu_{\mathcal{O}(\gamma)})(B) := \mu_{\mathcal{O}(\gamma)}(a^{-1}B)$. Moreover let the measure $\overline{\lambda}_a$ be given by $\overline{\lambda}_a(B) := \overline{\lambda}(aB)$ $(B \subset \widehat{\mathbb{R}^k}/H$ measurable$)$. Then on Ω_{rc}/H the following relations hold:*

$$\mu_{\mathcal{O}(a\gamma)} = a^*(\mu_{\mathcal{O}(\gamma)}),$$
$$\overline{\lambda}_a = |a|^n \overline{\lambda}.$$

Proof. The first equality is immediate from the definitions of $\mu_{\mathcal{O}(\gamma)}$ and $\mu_{\mathcal{O}(a\gamma)}$. For the second equation let us introduce the following notation: If $f : \Omega_{rc} \to \mathbb{R}$ is a positive, measurable function, let $q(f)$ denote the function on Ω_{rc}/H defined by

$$q(f)(\mathcal{O}(\gamma)) := \int_{\mathcal{O}(\gamma)} f(\omega) d\mu_{\mathcal{O}(\gamma)}(\omega).$$

Moreover let $f_a(\omega) := f(a^{-1}\omega)$, for all $\omega \in \Omega_{rc}$ and $a \in \mathbb{R}'$. From the first equation we obtain

$$\begin{aligned} q(f_a)(\mathcal{O}(\gamma)) &= \int_{\mathcal{O}(\gamma)} f(a^{-1}\omega) d\mu_{\mathcal{O}(\gamma)}(\omega) \\ &= \int_{\mathcal{O}(\gamma)a^{-1}} f(\omega) d\mu_{\mathcal{O}(a^{-1}\gamma)}(\omega) \\ &= q(f)(\mathcal{O}(a^{-1}\gamma)). \end{aligned}$$

Using this equation, we compute

$$\begin{aligned} \int_{\Omega_{rc}/H} q(f)(\mathcal{O}(\gamma)) d\overline{\lambda}(\mathcal{O}(\gamma)) &= a^{-n} \int_{\Omega_{rc}} f_a(\omega) d\lambda(\omega) \\ &= a^{-n} \int_{\Omega_{rc}/H} q(f_a)(\mathcal{O}(\gamma)) d\overline{\lambda}(\mathcal{O}(\gamma)) \\ &= a^{-n} \int_{\Omega_{rc}/H} q(f)(\mathcal{O}(a^{-1}\gamma)) d\overline{\lambda}(\mathcal{O}(\gamma)) \\ &= a^{-n} \int_{\Omega_{rc}/H} q(f)(\mathcal{O}(\gamma)) d\overline{\lambda}_a(\mathcal{O}(\gamma)) \end{aligned}$$

Using arguments similar to the one in the proof of Theorem 5.12, it is readily seen that for each measurable $A \subset \Omega_{rc}/H$ there exists a positive measurable f on Ω_{rc} with $q(f) = \mathbf{1}_A$. Hence we have shown the second equation.

As a first consequence we obtain that admissible vectors exist only for proper subsets of Ω_{rc}. This was already noted (in the special case where Ω_{rc} is conull in $\widehat{\mathbb{R}^k}$) in [77, Theorem 1.8].

Corollary 5.15. *Assume that G is unimodular, and that $U := \Omega_{rc}$ is not a nullset. Then the subrepresentation π_U does not have an admissible vector.*

Proof. By assumption we have $\overline{\lambda}(\Omega_{rc}/H) > 0$, and we need to show that $\overline{\lambda}(\Omega_{rc}/H) = \infty$. But for all $a \in \mathbb{R}'$, $a\Omega_{rc} = \Omega_{rc}$, and Proposition 5.14 yields $\overline{\lambda}(\Omega_{rc}/H) = \overline{\lambda}(a \cdot \Omega_{rc}/H) = |a|^{-k}\overline{\lambda}(\Omega_{rc}/H)$.

Recall from Theorem 2.25 (c) that, given a square integrable representation σ of a locally compact group G, every vector in $\mathcal{H}_\sigma$ is admissible iff σ is irreducible and G is unimodular. Hence discrete series representations of unimodular groups are particularly useful, having no restrictions at all on admissible vectors. But the following corollary excludes irreducible representations from our setting. The statement was proved first in [50] by a technique employing the Fell topology of the group.

Corollary 5.16. *Let G be unimodular. Then the quasiregular representation π does not contain any irreducible square-integrable subrepresentations.*

Proof. Assume the contrary and let π_U be an irreducible square integrable subrepresentation. Here U denotes the corresponding H-invariant subset of $\widehat{\mathbb{R}^k}$. Then, by [6, Theorem 1.1], U is (up to a null set) an orbit of positive measure, hence open by Sard's Theorem. In particular $U \subset \Omega_{rc}$, and $\overline{\lambda}(\{U\}) > 0$.

From the fact that $\mathcal{H}_U$ has admissible vectors we conclude that $\overline{\lambda}(\{U\}) < \infty$. On the other hand, an easy connectedness argument shows that for each $\gamma \in U$, the ray $\mathbb{R}^+\gamma$ is contained in the open orbit U. Hence the same argument which proved the previous corollary shows that $\overline{\lambda}(\{U\}) = \infty$, which yields the desired contradiction.

For illustration we now carry out the various steps for the Poincaré group, already considered in [74].

Remark 5.17. Admissibility conditions for the Poincaré groups. For $k \geq 3$ let L_k denote the Lorentz bilinear form on $\mathbb{R}^k$, i.e., $L_k(x,y) = -x_1y_1 + x_2y_2 + x_3y_3 + \ldots + x_ky_k$ and let $H = SO_0(k-1,1)$ denote the connected component of the linear group leaving L_k invariant. We exclude the case $k = 2$ for simplicity. The Plancherel measure was explicitly calculated in [75]. We want to determine the admissibility condition for the quasiregular representation associated with this group. For this purpose we need to compute the measure decomposition of Lebesgue-measure on $\widehat{\mathbb{R}^k}$, and we employ the symmetry arguments from 5.14 for this purpose.

1. The $SO_0(k-1,1)$-orbits in $\widehat{\mathbb{R}^k}$ can be parameterized as follows (see, e.g., [75, I, Example 2, Section 10])

$$\{0\}, \{\gamma \in \widehat{\mathbb{R}^k} : L_k(\gamma,\gamma) = 0, \gamma_1 > 0\}, \{\gamma \in \widehat{\mathbb{R}^k} : L_k(\gamma,\gamma) = 0, \gamma_1 < 0\},$$
$$\mathcal{O}_r^+ = \{\gamma \in \widehat{\mathbb{R}^k} : L_k(\gamma,\gamma) = -r, \gamma_1 < 0\} \qquad (r > 0),$$
$$\mathcal{O}_r^- = \{\gamma \in \widehat{\mathbb{R}^k} : L_k(\gamma,\gamma) = -r, \gamma_1 > 0\} \qquad (r > 0),$$
$$\mathcal{U}_r = \{\gamma \in \widehat{\mathbb{R}^k} : L_k(\gamma,\gamma) = r, \gamma_1 > 0\} \qquad (r > 0).$$

   Clearly the first three orbits may be neglected, leaving us with three families of orbits, each parameterized by a ray. Hence, under the action of $\mathbb{R}^+$ on $\widehat{\mathbb{R}^k}$ used in 5.14, we have essentially two A-orbits in $\widehat{\mathbb{R}^k}/H$, i.e., the $\mathcal{O}_r^\pm$-families yield one orbit, and the remaining A-orbit is obtained from the $\mathcal{U}_r$-family. Hence, $\overline{\lambda}$ is more or less completely determined by the action of A, and the $\mu_{\mathcal{O}(\gamma)}$ have to be computed for at most two H-orbits, which are represented by $\gamma_1 := (1,0,\ldots,0)$ and $\gamma_2 := (0,\ldots,0,1)$.
2. Now let us determine Ω_{rc}. It is easily seen that the fixed group H_{γ_1} is canonically isomorphic to $SO(k-1)$, whereas H_{γ_2} is isomorphic to $SO(k-2,1)$. In particular the former is compact and the latter is not. Clearly Ω_{rc} is not a null set, hence we may continue.
3. The orbit $\mathcal{O}(\gamma)_1$ can be parametrized by

$$\Psi : \mathbb{R} \times S^{k-2} \ni (u, \mathbf{y}) \mapsto (\sinh u, y_1 \cosh u, \ldots, y_{k-1} \cosh u).$$

The measure $d\mu_{\mathcal{O}(\gamma)_1} = du d\mathbf{y}$, where du is the usual measure on $\mathbb{R}$ and $d\mathbf{y}$ is the rotation-invariant surface measure on the sphere, is easily verified to be invariant under the action of H. Hence it is the image of μ_H under the quotient map, since H is unimodular. (Note that here we have in fact fixed a normalization of μ_H.) The parametrization of $\mathcal{O}(a\gamma_1)$ is $a \cdot \Psi$, and $\mu_{\mathcal{O}(a\gamma_1)} = a^*(\mu_{\mathcal{O}(\gamma_1)}$.

4. As we have seen, Ω_{rc}/H may be identified with $\mathbb{R} \setminus \{0\}$, and the measure $\overline{\lambda}$ has to fulfill the relation $\overline{\lambda}_a = |a|^k \overline{\lambda}$, whence we immediately obtain $d\overline{\lambda}(r) = |r|^{k-1} dr$.
5. G is unimodular, and we have chosen the $\mu_{\mathcal{O}(\gamma)}$ so as to ensure $\kappa \equiv 1$.
6. We obtain the following admissibility condition: An invariant subspace $\mathcal{H}_U \subset \mathrm{L}^2(\mathbb{R}^k)$ has an admissible vector iff the corresponding H-invariant subset $U \subset \widehat{\mathbb{R}^k}$ is contained in Ω_{rc} and its projection $\overline{U} := \{r \in \mathbb{R} : (r, 0, \ldots, 0) \in U\}$ satisfies

$$\int_{\overline{U}} |r|^{k-1} dr < \infty. \tag{5.12}$$

A vector $f \in \mathcal{H}_U$ is admissible for $\mathcal{H}_U$ iff

$$\int_{\mathbb{R}} \int_{S^{k-2}} |\widehat{f}(r \sinh u, r y_1 \cosh u, \ldots, r y_{k-1} \cosh u)|^2 d\mathbf{y} du = 1 \text{ for a.e. } r \in \overline{U}.$$

This normalization refers to the fixed choice of Haar measure μ_H made in step 3.

7. It is now simple to produce admissible vectors g for arbitrary H-invariant sets $U \subset \widehat{\mathbb{R}^k}$ satisfying (5.12): Fix a function g_0 on $\mathbb{R} \times S^{k-2}$ with

$$\int_{\mathbb{R}} \int_{S^{k-2}} |g_0(u, \mathbf{y})|^2 d\mathbf{y} du = 1$$

and let

$$\widehat{g}(r \sinh u, r y_1 \cosh u, \ldots, r y_{k-1} \cosh u) = g_0(u, \mathbf{y}) \quad .$$

5.3 Concrete and Abstract Admissibility Conditions

We will now work out the connection between the admissibility criteria of the previous section and the Plancherel transform of G. We consider a more general setting, i.e., $G = N \rtimes H$, where N is a unimodular group. Note that since G/N carries an invariant measure, Δ_N is the restriction of Δ_G to N. Hence $N \subset \mathrm{Ker}(\Delta_G)$, so that Δ_G can be lifted to a function on H.

We denote by $h \mapsto \alpha_h$ the associated homomorphism $H \to \mathrm{Aut}(N)$. If we identify H with a subgroup of G, then $\alpha_h(n) = hnh^{-1}$. Elements of G are

pairs $(x,h) \in N \times H$, with the group law $(x,h)(x',h') = (x\alpha_h(x'), hh')$. The **quasiregular representation** $\pi = \mathrm{Ind}_H^G 1$ acts on $\mathrm{L}^2(N)$ by

$$(\pi(x,h)f)(y) = \Delta_G(h)^{-1/2} f(\alpha_h^{-1}(x^{-1}y)) \quad .$$

We want to establish when $\pi \leq \lambda_G$, by decomposing π into irreducibles and checking for absolute continuity with respect to Plancherel measure.

Next we compute the decomposition of π. Since the restriction of π to N is λ_N, it is no surprise that the Plancherel transform of N is a useful tool for the decomposition of π into irreducible representations. Thus far the discussion runs completely parallel to the one in the previous two sections.

Proposition 5.18. *Let*

$$\mathcal{P}_N : \mathrm{L}^2(N) \to \int_{\widehat{N}}^{\oplus} \mathcal{B}_2(\mathcal{H}_\sigma) d\nu_N(\sigma)$$

be the Plancherel transform of N. Define a representation $\widehat{\pi}$ of G acting on the right hand side by $\widehat{\pi}(x,h) = \mathcal{P}_N \circ \pi(x,h) \circ \mathcal{P}_N^{-1}$. Then

$$(\widehat{\pi}(x,h)F)(\sigma) = \Delta_G(h)^{1/2} \sigma(x) \circ F(h^{-1}.\sigma) \tag{5.13}$$

Proof. For $F = \widehat{f}$ with $f \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G)$, and $\varphi, \eta \in \mathcal{H}_\sigma$, we compute

$$\begin{aligned}
\langle (\widehat{\pi}(x,t)F)(\sigma)\varphi, \eta \rangle &= \int_N \Delta_G^{-1/2}(h) f(\alpha_h^{-1}(x^{-1}y)) \langle \sigma(y)\varphi, \eta \rangle dy \\
&= \int_N \Delta_G^{-1/2}(h) f(\alpha_h^{-1}(y)) \langle \sigma(x)\sigma(y)\varphi, \eta \rangle dy \\
&= \Delta_G^{1/2}(h) \int_N f(y) \langle \sigma(x)\sigma(\alpha_h(y))\varphi, \eta \rangle dy \\
&= \Delta_G^{1/2}(h) \sigma(x) \widehat{f}(h^{-1}.\sigma) \quad .
\end{aligned}$$

Proposition 5.19. *Assume that for a ν_N-conull subset the Mackey obstructions are particularly trivial in the sense of 3.38. For almost every orbit $\mathcal{O}(\sigma) \subset \widehat{N}$, let $\mu_{\mathcal{O}(\sigma)}$ denote the measure arising from the measure decomposition*

$$d\nu_N = d\mu_{\mathcal{O}(\sigma)} d\overline{\nu_N} \quad . \tag{5.14}$$

By standardness of ν_N/H, these measures exist and are unique ν_N-almost everywhere. We define the representations ϱ_σ acting on $\mathrm{L}^2(\mathcal{O}(\sigma), \mu_{\mathcal{O}(\sigma)}; \mathcal{B}_2(\mathcal{H}_\sigma))$ via

$$(\varrho_\sigma(x,h)F)(\omega) = \Delta_G(h)^{1/2} \omega(x) \circ F(h^{-1}.\omega) \quad . \tag{5.15}$$

Then the following statements hold:

(a) $\varrho_\sigma \simeq \dim(\mathcal{H}_\sigma) \cdot \mathrm{Ind}_{G_\sigma}^G \sigma \times 1$.

(b) The Plancherel transform of N effects a decomposition

$$\pi \simeq \int_{\Sigma/H}^{\oplus} \dim(\mathcal{H}_\sigma) \cdot \left(\mathrm{Ind}_{G_\sigma}^G \sigma \times 1\right) d\overline{\nu_N}(\sigma) \quad . \tag{5.16}$$

This is a decomposition into irreducibles.

Proof. We first show that ϱ_σ is induced from the left action via σ on $\mathcal{B}_2(\mathcal{H}_\sigma)$, by comparing (5.15) with (3.29). First observe that by relation (5.14) and

$$\frac{d\nu_{N(x,h)^{-1}}}{d\nu_N}(\omega) = \Delta_G(h) \quad ,$$

we obtain that

$$\frac{d\mu_{\mathcal{O}(\sigma)(x,h)^{-1}}}{d\mu_{\mathcal{O}(\sigma)}}(\omega) = \Delta_G(h) \quad .$$

In other words, the Radon-Nikodym derivative used in the definition of ϱ_σ coincides with the one employed in (3.29).

Next pick a cross-section $\alpha_0 : \mathcal{O}(\sigma) \to H$, and let $\alpha : G.\sigma \to G$ be defined by $\alpha(\omega) = (e_N, \alpha_0(\omega))$, where e_N denotes the neutral element in N. Then we find for $(x,h) \in G$ that

$$\begin{aligned} \alpha(\omega)^{-1}(x,h)\alpha((n,h)^{-1}\omega)) &= (e_N, \alpha_0(\omega)^{-1})(x,h)(e_N, \alpha_0((x,h)^{-1}\omega)) \\ &= (\alpha_0(\omega)^{-1}(x), \alpha_0(\omega)^{-1}h\alpha_0(h^{-1}.\omega)) \quad , \end{aligned}$$

with $\alpha_0(\omega)^{-1}h\alpha_0(h^{-1}.\omega) \in H_\sigma$. By assumption, σ extends trivially to a representation of the big fixed group $G_\sigma = N \rtimes H_\sigma$. Hence

$$(\sigma \times 1)\left(\alpha(\omega)^{-1}(x,h)\alpha((x,h)^{-1}\omega)\right) = (\alpha_0(\omega).\sigma)(x) = \omega(x)$$

Hence the second term, $\omega(x)$, in the definition of ϱ_σ also coincides with the second term of the right hand side of (3.29). Since the same is obviously true for the third terms, we are finished.

The left action via σ on $\mathcal{B}_2(\mathcal{H}_\sigma)$ is clearly a $\dim(\mathcal{H}_\sigma)$-fold multiple of σ, and induction commutes with taking direct sums [45, 6.9], hence (a) follows. In view of this and Proposition 3.29, part (b) follows from a comparison of the definition of the ϱ_σ with (5.13). It is a decomposition into irreducibles by Mackey's theorem.

Next we show the desired containment result. As in the above discussion the set $\Omega_c = \{\sigma \in \widehat{N} : H_\sigma \text{ is compact } \}$ plays a crucial role. We have again, with the same arguments as before:

Lemma 5.20. *Ω_c is a Borel set.*

Lemma 5.21. *The mapping $\Phi : \mathcal{O}(\gamma) \mapsto \mathrm{Ind}_{G_\gamma}^G \gamma \times 1$ is a Borel isomorphism onto a Borel subset Σ of $\widehat{G}$.*

Proof. This follows from the decomposition in 5.19 and the uniqueness theorem 3.25.

Theorem 5.22. *Let $G = N \rtimes H$, and suppose that G and N are as in Theorem 3.40. Letting*

$$\pi_1 \simeq \int_{\Omega_c}^{\oplus} \dim(\mathcal{H}_\sigma) \cdot (\mathrm{Ind}_{G_\sigma}^G \sigma \times 1) d\overline{\nu}(\mathcal{O}(\gamma)) \tag{5.17}$$

and π_2 the orthogonal complement in π, then $\pi_1 < \lambda_G$, and π_2 is disjoint from λ_G.

Proof. Denote the part corresponding to Ω_c by Σ_c. The image measure of $\overline{\nu_N}$ under this map is a standard measure $\widetilde{\nu}$ on $\widehat{G}$. The key observation is now that, with respect to the measure decomposition (3.32), Σ meets each fibre in exactly one point, namely in $\sigma \times 1$. In computing $\nu_N(B)$, for subsets B of Σ, the inner integral is simply $\nu_{H_\gamma}(\{1\})$, which is positive iff $\mathcal{H}_\gamma$ is compact. In short, $\widetilde{\nu}$ is ν_G-absolutely continuous on Σ_c, and disjoint with ν_G on $\Sigma \setminus \Sigma_c$. The containment statement $\pi_1 < \lambda_G$ is obtained by checking the conditions in 3.26: The absolute continuity requirement has already been verified. For the comparison of multiplicities note that the representation space of $\mathrm{Ind}_{G_\gamma}^G \sigma \times 1$ is a nontrivial $\mathcal{H}_\sigma$-valued L^2-space, thus its dimension is necessarily $\geq \dim(\mathcal{H}_\sigma)$. The former is the multiplicity of $\mathrm{Ind}_{G_\gamma}^G \sigma \times 1$ in λ_G, the latter (smaller) is the multiplicity of the same representation in π. Thus follows the containment statement.

The disjointness part is due to Corollary 3.18.

The disjointness statement means that π_1 is the maximal subrepresentation of π which is contained in λ_G.

Let us now take a second look at the case $N = \mathbb{R}^k$. The following theorem explains how the different admissibility conditions are related:

Theorem 5.23. *Let $G = \mathbb{R}^k \rtimes H$, and assume that G and $N = \mathbb{R}^k$ fulfill all requirements of Theorem 3.40. For $\mathcal{O}(\gamma) \subset \Omega_c$ let $\mathcal{K}_{\mathcal{O}(\gamma)}$ denote the operator on $\mathrm{L}^2(\mathcal{O}(\gamma), d\beta_{\mathcal{O}(\gamma)})$ given by pointwise multiplication with $\kappa|_{\mathcal{O}(\gamma)}$. The map Φ from Lemma 5.21 gives rise to the following correspondences between the objects in Section 5.2 and those appearing in the Plancherel decomposition:*

$$\begin{aligned}
\Omega_c/H &\longleftrightarrow \Sigma \quad , \\
\mathcal{O}(\gamma) &\longleftrightarrow \sigma \quad , \\
\mathrm{L}^2(\mathcal{O}(\gamma), d\beta_{\mathcal{O}(\gamma)}) &\longleftrightarrow \mathcal{H}_\sigma \quad , \\
\widehat{f}|_{\mathcal{O}(\gamma)} &\longleftrightarrow \eta_\sigma \quad , \\
S_H(\widehat{f})(\mathcal{O}(\gamma)) &\longleftrightarrow \|\eta_\sigma\| \quad , \\
\overline{\lambda} &\longleftrightarrow \nu_G \quad , \\
\mathcal{K}_{\mathcal{O}(\gamma)} &\longleftrightarrow C_\sigma^{-2} \quad , \\
T_H(\widehat{f})(\mathcal{O}(\gamma)) &\longleftrightarrow \|C_\sigma \eta_\sigma\| \quad .
\end{aligned}$$

In particular, the admissibility criterion from Theorem 5.8 is a special case of (4.15).

Proof. It remains to check that the C_σ^{-2} corresponds to $\mathcal{K}_{\mathcal{O}(\gamma)}$, and that the Plancherel measure ν_G belonging to this particular choice of Duflo-Moore operators corresponds to $\overline{\lambda}$. Straightforward calculation, using relation (5.6) from Lemma 5.9, shows that $\mathcal{K}_{\mathcal{O}(\gamma)}$ satisfies the semi-invariance relation

$$\left(\mathrm{Ind}_{G_\gamma}^G(\gamma \times 1)(x,h)\right) \; \mathcal{K}_{\mathcal{O}(\gamma)} \; \left(\mathrm{Ind}_{G_\gamma}^G(\gamma \times 1)(x,h)\right)^* = \Delta_G(x,h)^{-1}\mathcal{K}_{\mathcal{O}(\gamma)} \quad .$$

It follows that $K_{\mathcal{O}(\gamma)}^{-1/2}$ obeys the semi-invariance relation (3.51), hence Theorem 3.48(e) entails the desired correspondence.

It remains to prove that, given this particular choice of Duflo-Moore operators, the measure $\overline{\lambda}$ is the corresponding Plancherel measure. But in view of the identifications we already established, (4.15) yields for every H-invariant measurable $U \subset \widehat{\mathbb{R}^k}$ that

$$g \in \mathcal{H}_U \text{ is admissible } \iff \left(\frac{d\overline{\lambda}}{d\nu_G}(\mathcal{O}(\gamma))\right)^{1/2} \|K_{\mathcal{O}(\gamma)}^{-1/2}(\widehat{g}|_{\mathcal{O}(\gamma)})\|_2 = 1 \;,\; \text{(a.e.)} \quad .$$

On the other hand, (5.5) provides

$$g \in \mathcal{H}_U \text{ is admissible } \iff \|K_{\mathcal{O}(\gamma)}^{-1/2}(\widehat{g}|_{\mathcal{O}(\gamma)})\|_2 = 1 \;,\; \text{(a.e.)} \quad .$$

Hence the Radon-Nikodym derivative is trivial.

As mentioned before, the majority of authors dealing with wavelets from semidirect products of the form $\mathbb{R}^k \rtimes H$ concentrated on discrete series representations occurring as subrepresentations of the quasiregular representation. In this context, a well-known result relates the existence of such representations to open dual orbits with associated compact fixed groups, see e.g. [22, 48]. This condition can now be retrieved, by restricting the discrete series criterion in Corollary 3.41 to representations of the form $\mathrm{Ind}_{G_\sigma}^G \sigma \times 1$, as they appear in the decomposition of the quasiregular representation. Note that a group is compact iff the Haar measure of the group is finite, iff the trivial representation is square-integrable.

Corollary 5.24. *The quasiregular representation π contains a discrete series representation iff there exists a dual orbit $\mathcal{O}(\gamma) \subset \widehat{N}$ of positive Plancherel measure, such that in addition H_γ is compact.*

Note that if N is a vector group and H a closed matrix group, then Sard's theorem implies that all orbits of positive measure are indeed open. See [49] for a proof.

5.4 Wavelets on Homogeneous Groups**

In this section we use Theorem 4.22 to prove that there exists a continuous wavelet transform on homogeneous Lie groups. Starting from a homogeneous Lie group N with a one-parameter group H of dilations, we show that the quasiregular representation of $G = N \rtimes H$ on $\mathrm{L}^2(N)$ has admissible vectors. Since G is nonunimodular, containment in λ_G will be sufficient for that.

The resolution of the identity provided by the wavelet transform can also be read as a continuous Calderon reproducing formula. Discrete versions of the Calderon reproducing formula have been employed for the analysis of pseudodifferential operators on these groups [47], and it is conceivable that the wavelet transform we present below could be useful for these purposes also. Note however that we only provide the **existence** of admissible vectors. Unless N is a vector group, the arising representation has infinite multiplicity, and an explicit characterization of admissible vectors will be a tough problem.

Now for the definition of homogeneous Lie groups.

Definition 5.25. *A connected simply connected Lie group N with Lie algebra $\mathfrak{n}$ is called* **homogeneous**, *if there exists a one-parameter group $H = \{\delta_r : r \geq 0\}$ of Lie algebra automorphisms of the form $\delta_r = \mathrm{e}^{A \log r}$, such that in addition A is diagonalizable with strictly positive eigenvalues. The elements of H are called* **dilations**.

We define $G = N \rtimes H$, and π as the quasiregular representation $\pi = \mathrm{Ind}_H^G 1$ of G. The homogenous structure of the group allows a rather nice geometric interpretation of G and π: It can be shown that N carries a "homogeneous norm" $|\cdot| : N \to \mathbb{R}_0^+$, on which the dilations act in the expected way, i.e., $|\delta_r(x)| = r|x|$. See [47, Chapter 1] for details. Hence the group G consists of shifts on N and "zooms" with respect to the norm, and the interpretation of the continuous wavelet transform as a "mathematical microscope" carries over from $\mathbb{R}$ to N. However, the "mathematical microscope" view requires some sort of decay behaviour of the admissible vectors (ideally, compact support), and whether admissible vectors exist with these properties is unclear.

Lemma 5.26. *Let N be homogeneous with dilation group H and the associated infinitesimal generator A. For $a \in \mathbb{R}$, let $E_a = \mathrm{Ker}(A - a\mathrm{Id})$. Then we have:*

(i) If $X \in E_a$ and $Y \in E_b$, then $[X, Y] \in E_{a+b}$.
(ii) N is nilpotent. H acts as a group of automorphisms of N.
(iii) The Haar measure on N is Lebesgue measure. We have $\mu_N(\delta_r(E)) = r^Q \mu_N(E)$, where $Q := \mathrm{trace}(A) > 0$. Hence G is nonunimodular, with $N = \mathrm{Ker}(\Delta_G)$.
(iv) Let $0 < a_1 \leq a_2 \leq \ldots \leq a_n$ be the eigenvalues of A, each occurring with its geometric multiplicity. Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{n}$ with $AX_i = a_i X_i$. Given $Y \in \mathfrak{n}$ arbitrary, $\mathrm{Ad}(Y)$ is properly upper triangular with respect to this basis.

(v) For almost all $\sigma \in \widehat{N}$ the little fixed group H_σ is trivial.

Proof. See [47] for $(i), (ii)$ and the formula $\mu_N(\delta_r(E)) = r^Q \mu_N(E)$. But this formula entails that $\Delta_G(x, r) = r^Q$, and since $Q > 0$, we find $N = \mathrm{Ker}(\Delta_G)$. (iv) is immediate from (i). (v) follows from (iii) and Proposition 3.50(ii).

We need one more auxiliary result to show that the quasiregular representation on $\mathrm{L}^2(N)$ is contained in λ_G, namely the fact that G is type I.

Proposition 5.27. *G is an exponential Lie group. In particular, G is type I.*

Proof. Let $\mathfrak{g}$ denote the Lie algebra of G. In order to prove that G is exponential, we need to show that there does not exist a $Z \in \mathfrak{g}$ for which $\mathrm{ad}(Z)$ has a purely imaginary eigenvalue [34]. Let $a_1, \ldots, a_n$ and the associated eigenbasis $X_1, \ldots, X_n$ of $\mathfrak{n}$ be as in Lemma 5.26. Then a basis of $\mathfrak{g}$ is given by $X_1, \ldots, X_n, Y$, with $[Y, X_i] = a_i X_i$. It is therefore straightforward to compute that, for an arbitrary $Z = tY + \sum_{i=1}^n s_i X_i = tY + X$, the matrix of $\mathrm{ad}(Z)$ with respect to our basis is

$$\begin{pmatrix} tA + M & v \\ 0 & 0 \end{pmatrix} \quad , \tag{5.18}$$

where M is the matrix of $\mathrm{ad}(X)$ (acting on $\mathfrak{n}$), in particular (properly) upper triangular by Lemma 5.26 (iv), and v is some column vector. But this matrix clearly has the eigenvalues $ta_1, \ldots, ta_n$ and 0. Hence G is exponential, and [23, Chap. VI, 2.11] implies that G is type I.

Now Remark 5.26 (v) and Theorem 5.22 allow to conclude that π is contained in the regular representation. Hence 4.22 provides the desired existence result:

Corollary 5.28. *The quasi-regular representation $\pi = \mathrm{Ind}_H^G 1$ is contained in λ_G. Hence there exists a continuous wavelet transform on N arising from the action of N by left translations and the action of the dilations.*

The bad news is that, unless N is abelian (in which case we are back to the first sections of this chapter) the problem of computing concrete admissible vectors is essentially equivalent to that of computing admissible vectors for λ_G:

Proposition 5.29. *We have $\pi \approx \lambda_G$. Moreover, $\pi \simeq \lambda_G$ iff N is nonabelian.*

Proof. For the first statement we observe that ν_G is precisely the quotient measure on $\widehat{N}/H$, which also underlies the decomposition of π, by (5.16). Hence $\nu_G \approx \pi$.

If N is abelian, then π is multiplicity-free; already the restriction $\pi|_N$ is. But G is nonabelian, and therefore λ_G is not multiplicity-free.

Conversely, assume N to be nonabelian. It is enough to prove that $\pi \simeq \infty \cdot \pi$, which in view of (5.16) amounts to proving $\dim(\mathcal{H}_\sigma) = \infty$, for ν_N-almost every σ. Since N is not abelian, there exist coadjoint orbits of positive dimensions, and the coadjoint orbits of maximal dimension are a conull subset of $\mathfrak{n}^*$. Hence ν_N-almost every σ corresponds to a coadjoint orbit of positive – and necessarily even – dimension, say of dimension $2k$. By construction of the Kirillov map, σ is therefore induced from a character of a subgroup M of codimension k, hence $\mathcal{H}_\sigma = \mathrm{L}^2(N/M)$ is infinite-dimensional. Since the action of H on $\widehat{N}$ is free ν_N-almost everywhere, each dual orbit in $\widehat{N}$ contributes precisely one representation, which also occurs in (5.16).

5.5 Zak Transform Conditions for Weyl-Heisenberg Frames

This section deals with another class of examples. The results presented here are taken from [54]. We consider a characterization of tight Weyl-Heisenberg frames via the Zak transform. It turns out that it can be seen as a special instance of the Plancherel transform criterion, where the underlying group is discrete and type I. This makes the example somewhat remarkable. As in the semidirect product case, the admissibility conditions which arise are already known and obtainable by less involved machinery also.

Admissibility Conditions and Weyl-Heisenberg Frames

Weyl-Heisenberg systems are discretizations of the windowed Fourier transform introduced in Example 2.27. To make things precise, define the translation operators T_x and modulation operators M_ω on $\mathrm{L}^2(\mathbb{R})$ by

$$(T_x f)(y) = f(y-x) \quad , \quad (M_\omega f)(y) = \mathrm{e}^{2\pi \mathrm{i} \omega y} f(y) \quad .$$

Now a **Weyl-Heisenberg system** $\mathcal{G}(\alpha, \beta, g)$ of $\mathrm{L}^2(\mathbb{R})$ is a family

$$g_{k,m} = M_{\alpha k} T_{\beta m} g \quad (m, k \in \mathbb{Z}),$$

arising from a fixed vector $g \in \mathrm{L}^2(\mathbb{R})$ and $\alpha, \beta \neq 0$. A **(normalized, tight) Weyl-Heisenberg frame** is a Weyl-Heisenberg system that is a (normalized, tight) frame of $\mathrm{L}^2(\mathbb{R})$. There exist several alternative definitions, with varying indexing and ordering of operators. However, up to phase factors which clearly do not affect any of the frame properties, the resulting systems are identical.

Here we focus on normalized tight Weyl-Heisenberg frames **with integer oversampling L** ($L \in \mathbb{N}$), which corresponds to choosing $\alpha = 1$ and $\beta = 1/L$. Given L, the problem is to decide for a given g whether it induces a normalized tight Weyl-Heisenberg frame or not. As we will see in the next subsection, the Zak transform allows a precise answer to this question. Our next aim is to show

that the condition is in fact an admissibility condition for ψ. Note that for $L = 1$, this is obvious: The set $\{T_n M_m : n, m \in \mathbb{Z}\}$ is an abelian subgroup of the unitary group of $\mathrm{L}^2(\mathbb{R})$, and the normalized tight frame condition precisely means admissibility in this case. For $L > 1$ however, $\{T_n M_{m/L} : n, m \in \mathbb{Z}\}$ is not a subgroup, and we have to deal with the nonabelian group G generated by this set. Hence, for the duration of this section, we fix an integer oversampling rate $L \geq 1$, and define the underlying group G as

$$G = \mathbb{Z} \times \mathbb{Z} \times (\mathbb{Z}/L\mathbb{Z}) \quad ,$$

with the group law

$$(n, k, \overline{\ell})(n', k', \overline{\ell'}) = (n + n', k + k', \overline{\ell} + \overline{\ell'} + \overline{k'n}) \tag{5.19}$$

and inverse given by $(n, k, \overline{\ell})^{-1} = (-n, -k, \overline{-\ell + kn})$. Here we used the notation $\overline{n} = n + L\mathbb{Z}$. The representation π of G acts on $\mathrm{L}^2(\mathbb{R})$ by

$$\pi(n, k, \overline{\ell}) = \mathrm{e}^{2\pi \mathrm{i}(\ell - nk)/L} M_{n/L} T_k = \mathrm{e}^{2\pi \mathrm{i}\ell/L} T_k M_{n/L} \quad .$$

It is straightforward to check how normalized tight frames with oversampling L relate to admissibility for π:

Lemma 5.30. *Let $\psi \in \mathrm{L}^2(\mathbb{R})$. Then $(M_{n/L} T_k \psi)_{n,k \in \mathbb{Z}}$ is a normalized tight frame iff $\frac{1}{\sqrt{L}} \psi$ is admissible for π.*

Proof. The relation

$$M_{n/L} T_k \psi = \mathrm{e}^{-2\pi \mathrm{i}(\ell - nk)/L} \pi(k, n, \overline{\ell}) \psi$$

implies for all $g \in \mathrm{L}^2(\mathbb{R})$ that

$$\sum_{n,k,\overline{\ell}} \left| \langle g, \pi(k, n, \overline{\ell}) f \rangle \right|^2 = L \sum_{n,k} \left| \langle g, M_{n/L} T_k f \rangle \right|^2 \quad ,$$

which shows the claim.

The following lemma establishes that G is a finite extension of an abelian normal subgroup N. It is central for our purposes: It ensures that G is type I, and it allows to compute ν_G via Theorem 3.40.

Lemma 5.31. *Let*

$$N = \{(nL, k, \overline{\ell}) : k, n, \ell \in \mathbb{Z}\} \quad .$$

Then N is an abelian normal subgroup of G with $G/N \cong \mathbb{Z}/L\mathbb{Z}$. In particular, G is type I.

Proof. The statements concerning N are obvious from (5.19); for the description of G/N use the representatives $(0, 0, 0), (1, 0, 0), \ldots, (L - 1, 0, 0)$ of the N-cosets. The type I property of G follows by Theorem 3.39 (c), observing that the orbit space of a standard Borel space by a measurable finite group action is standard.

Zak Transform Criteria for Tight Weyl-Heisenberg Frames

In this subsection we introduce the Zak transform and formulate the criterion for normalized tight Weyl-Heisenberg frames. Our main reference for the following will be [58].

Definition 5.32. *For $f \in C_c(\mathbb{R})$, define the Zak transform of f as the function $\mathcal{Z}f : \mathbb{R}^2 \to \mathbb{C}$ given by*

$$\mathcal{Z}f(x,\omega) = \sum_{m\in\mathbb{Z}} f(x-m)\mathrm{e}^{2\pi \mathrm{i} m\omega} \quad .$$

The definition of the Zak transform immediately implies a quasi-periodicity condition for $F = \mathcal{Z}f$:

$$\forall m,n \in \mathbb{Z} \quad : \quad F(x+m,\omega+n) = \mathrm{e}^{2\pi \mathrm{i} m\omega} F(x,\omega) \quad . \tag{5.20}$$

In particular, the Zak transform of a function f is uniquely determined by its restriction to the unit square $[0,1]^2$. We next extend the Zak transform to a unitary operator $\mathcal{Z} : \mathrm{L}^2(\mathbb{R}) \to \mathcal{H}$, where $\mathcal{H}$ is a suitably defined Hilbert space. For the proof of the following see [58, Theorem 8.2.3]. In the proposition $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)$ denotes the space of all measurable functions which are square-integrable on compact sets.

Proposition 5.33. *Let the Hilbert space $\mathcal{H}$ be defined by*

$$\mathcal{H} = \{F \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2) : F \text{ satisfies (5.20) almost everywhere on } \mathbb{R}^2\} \quad ,$$

with norm

$$\|F\|_{\mathcal{H}} = \|F\|_{\mathrm{L}^2([0,1]^2)} \quad .$$

The Zak transform extends uniquely to a unitary operator $\mathcal{Z} : \mathrm{L}^2(\mathbb{R}) \to \mathcal{H}$.

The next lemma describes how the representation π operates on the Zak transform side. It is easily verified on $\mathcal{Z}(C_c(\mathbb{R}))$, and extends to $\mathcal{H}$ by density.

Proposition 5.34. *Let $\widehat{\pi}$ be the representation acting on $\mathcal{H}$, obtained by conjugating π with $\mathcal{Z}$, i.e., $\widehat{\pi}(n,k,\overline{\ell}) = \mathcal{Z} \circ \pi(n,k,\overline{\ell}) \circ \mathcal{Z}^*$. Then*

$$\widehat{\pi}(n,k,\overline{\ell})F(x,\omega) = \mathrm{e}^{2\pi \mathrm{i}(\ell - nk)/L} \mathrm{e}^{2\pi \mathrm{i} n x/L} F(x-k,\omega - n/L) \quad . \tag{5.21}$$

Now we can cite the Zak transform criterion for normalized tight Weyl-Heisenberg frames with integer oversampling. For a sketch of the proof confer [58], more details are contained in [33]. Our discussion provides an alternative proof, see Corollary 5.41.

Theorem 5.35. *Let $f \in \mathrm{L}^2(\mathbb{R})$. Then $(M_{n/L}T_k f)_{n,k\in\mathbb{Z}}$ is a normalized tight frame of $\mathrm{L}^2(\mathbb{R})$ iff*

$$\sum_{i=0}^{L-1} |\mathcal{Z}f(x,\omega + i/L)|^2 = 1 \quad \textit{almost everywhere.} \tag{5.22}$$

There exist more general versions of this criterion, which allow more complicated sets of time-frequency translations for the construction of the Gabor frames. While we have restricted our attention to the simple time-frequency lattice $\mathbb{Z}\times(1/L)\mathbb{Z}$ mostly for reasons of notational simplicity, the more general statements can be obtained by use of suitable symplectic automorphisms of the time-frequency plane.

Computing the Plancherel Measure

G is a unimodular group extension, thus ν_G is obtainable from Theorem 3.40 by computing $\widehat{G}$ with the aid of the Mackey machine, and keeping track of the various measures on duals and quotient spaces.

Note that since G/N is finite, N is regularly embedded in G. N is the direct product of three cyclic groups, hence the character group $\widehat{N}$ is conveniently parametrized by $[0,1[\times[0,1[\times\{0,1,\ldots,L-1\}$, by letting

$$\chi_{\omega_1,\omega_2,j}(nL,k,\overline{\ell}) = \mathrm{e}^{2\pi\mathrm{i}(\omega_1 n+\omega_2 k+j\ell/L)} \quad .$$

Since

$$(n,k,\overline{\ell})(n'L,k',\overline{\ell'})(n,k,\overline{\ell})^{-1} = (n'L,k',\overline{\ell'+k'n}) \quad ,$$

we compute the dual action as

$$(\omega_1,\omega_2,j).(n,k,\overline{\ell}) = (\omega_1,\omega_2+jn/L-\lfloor\omega_2+jn/L\rfloor,j) \quad .$$

Here $\lfloor x\rfloor$ denotes the largest integer $\leq x$. Hence, defining

$$\Omega_j = [0,1[\times[0,\gcd(j,L)/L[\times\{j\} \quad ,$$

a measurable transversal of the orbits under the dual action is given by $\Omega = \bigcup_{j=0}^{L-1}\Omega_j$. Here $\gcd(j,L)$ is the greatest common divisor of j and L. The fact that the subgroup of $\mathbb{Z}/L\mathbb{Z}$ generated by $\overline{j}$ coincides with the subgroup generated by $\overline{\gcd(j,L)}$ accounts for this choice of transversal. With the respect to the dual action, $(\omega_1,\omega_2,j)\in\Omega_j$ has $N_j = \{(nL/\gcd(j,L),k,\overline{\ell}) : k,n,\ell\in\mathbb{Z}\}$ as fixed group. The associated little fixed group is $N_j/N \cong \mathbb{Z}/\gcd(j,L)\mathbb{Z}$. For a convenient parametrization of $\widehat{G}$ in terms of Ω and the duals of the N_j we need to establish that every $(\omega_1,\omega_2,j)\in\widehat{N}$ has trivial Mackey obstruction.

Lemma 5.36. *Let $(\omega_1,\omega_2,j)\in\Omega_j$ and $m\in\{0,1,\ldots,\gcd(j,L)-1\}$. Then*

$$\varrho_{m,\omega_1,\omega_2,j}(nL/\gcd(j,L),k,\overline{\ell}) = \mathrm{e}^{2\pi\mathrm{i}((\omega_1+m)n/\gcd(j,L)+\omega_2 k+j\ell/L)}$$

defines a character of N_j with $\varrho_{m,\omega_1,\omega_2,j}|_N = \chi_{\omega_1,\omega_2,j}$. Moreover, every irreducible representation of N_j whose restriction to N is a multiple of $\chi_{\omega_1,\omega_2,j}$ is equivalent to some $\varrho_{m,\omega_1,\omega_2,j}$.

Proof. The character property is verified by straightforward computation. The last statement is [45, Proposition 6.40].

Note that the additional parameter m indexes the characters of the little fixed group N_j/N. Now Theorem 3.39 implies that $\widehat{G}$ is obtained by inducing the $\varrho_{m,\boldsymbol{\omega},j}$.

Proposition 5.37. *Define, for* $(\omega_1, \omega_2, j) \in \Omega$ *and* $m \in \{0, \ldots, \gcd(j, L) - 1\}$ *the representation*

$$\sigma_{m,\omega_1,\omega_2,j} = \mathrm{Ind}_{N_j}^{G} \varrho_{m,\omega_1,\omega_2,j} \quad .$$

If we let

$$\Sigma_j = \{\sigma_{m,\omega_1,\omega_2,j} : (\omega_1, \omega_2, j) \in \Omega_j, m \in \{0, 1, \ldots, \gcd(j, L) - 1\} \quad ,$$

then the dual of G *is the disjoint union*

$$\widehat{G} = \bigcup_{j=0}^{L-1} \Sigma_j$$

We normalize all Haar measures on discrete groups occurring in the following by $|\{e\}| = 1$. This choice fixes the Plancherel measures uniquely, and implies in particular for all abelian groups H arising in the following that $\nu_H(\widehat{H}) = 1$. Moreover, Weil's integral formulae are automatically ensured by these choices, whence we will obtain the correct normalizations.

Recall that we have the identification

$$\widehat{G} = \bigcup_{j=0}^{L-1} \Sigma_j = \bigcup_{j=0}^{L-1} (N_j/N)^\wedge \times \Omega_j \quad .$$

On each of the Σ_j, Plancherel measure is a product measure: The $(N_j/N)^\wedge$ carry the Plancherel measure of the finite quotient group, which is simply counting measure weighted with $1/|N_j/N| = 1/\gcd(j, L)$. For the missing parts, we decompose Plancherel measure of N on $\widehat{N}$ along orbits of the dual action. This results in a measure on $\Omega \simeq \widehat{N}/G$, and the restrictions to the Ω_j provide the second factors. In order to explicitly compute these we note that the Plancherel measure on $\widehat{N} \cong [0, 1[\times[0, 1[\times\{0, 1, \ldots, L-1\}$ is $1/L$ times the product measure of Lebesgue measure on the first two factors and counting measure on the third. Since each orbit carries counting measure, the measure on the quotient is simply Lebesgue measure on the transversal $[0, 1[\times[0, \gcd(j, L)/L[$, for each j. Thus we arrive at:

Proposition 5.38. *The Plancherel measure of* G *is given by*

$$d\nu_G(\sigma_{m,\omega_1,\omega_2,j}) = \frac{1}{L\gcd(j, L)} dm d\omega_1 d\omega_2 dj \quad . \tag{5.23}$$

Here $d\omega_1$ *and* $d\omega_2$ *are Lebesgue measure on the intervals* $[0, 1[$ *and* $[0, \gcd(j, L)/L[$, *and* dm, dj *are counting measure on* $\{0, \ldots, \gcd(j, L) - 1\}$ *and* $\{0, \ldots, L - 1\}$, *respectively.*

As we will see in the next subsection, only the set Σ_1 will be of interest for the Weyl-Heisenberg frame setting. Here the indexing somewhat simplifies: $N_1 = N$, and m can only take the value 0. So we can identify Σ_1 with $\{0\} \times [0,1[\times[0,1/L[\times\{1\} \cong [0,1[\times[0,1/L[$.

Zak Transform and Plancherel Transform

The aim in this subsection is to exhibit the representation $\widehat{\pi}$ obtained by conjugating π with the Zak transform as a direct integral of irreducibles. This is done by taking a second look at (5.21), which is a twisted action by translations along $\mathbb{Z} \times (1/L)\mathbb{Z}$. Hence a decomposition of Lebesgue measure along cosets of $\mathbb{Z} \times (1/L)\mathbb{Z}$ gives rise to a decomposition into representations acting on the cosets, and the twisted action of the latter representations reveals them as induced representations.

To make this more precise, we let for $\boldsymbol{\omega} \in [0,1[\times[0,1/L[$ denote $\mathcal{O}_{\boldsymbol{\omega}} = \boldsymbol{\omega} + \mathbb{Z} \times (1/L)\mathbb{Z}$. The following lemma exhibits the direct integral structure of $\widehat{\pi}$; it is a direct consequence of Proposition 3.29.

Lemma 5.39. *Define for $\boldsymbol{\omega} \in [0,1[\times[0,1/L[$ the Hilbert space*

$$\mathcal{H}_{\boldsymbol{\omega}} = \{F : \mathcal{O}_{\boldsymbol{\omega}} \to \mathbb{C} : F \text{ fulfills } (5.20)\} \quad ,$$

with the norm defined by

$$\|F\|^2_{\mathcal{H}_{\boldsymbol{\omega}}} = \sum_{i=0}^{L-1} |F(\boldsymbol{\omega} + (0, i/L))|^2. \tag{5.24}$$

Let $\widehat{\pi}_{\boldsymbol{\omega}}$ be the representation acting on $\mathcal{H}_{\boldsymbol{\omega}}$ by

$$\widehat{\pi}_{\boldsymbol{\omega}}(k,n,\overline{\ell})F(\boldsymbol{\gamma}) = \mathrm{e}^{2\pi\mathrm{i}(\ell+nk)/L}\mathrm{e}^{2\pi\mathrm{i}nx/L}F(\boldsymbol{\gamma} - (k, n/L)) \quad .$$

Then

$$\widehat{\pi} \simeq \int_{[0,1[\times[0,1/L[}^{\oplus} \widehat{\pi}_{\boldsymbol{\omega}} \, d\boldsymbol{\omega} \quad , \tag{5.25}$$

via the map

$$F \mapsto (F|_{\mathcal{O}_{\boldsymbol{\omega}}})_{\boldsymbol{\omega} \in [0,1[\times[0,1/L[} \tag{5.26}$$

As a first glimpse of the connection between conditions (5.22) and (4.15) note that the right-hand side of (5.22) can now be reformulated as

$$\|(\mathcal{Z}f)\,|_{\mathcal{O}_{\boldsymbol{\omega}}}\|_{\mathcal{H}_{\boldsymbol{\omega}}} = 1, \quad \text{for almost every } \boldsymbol{\omega} \in [0,1[\times[0,1/L[\quad .$$

Hence the final step is to note that (5.25) is in fact a decomposition into irreducibles:

Lemma 5.40. *If $\boldsymbol{\omega} \in [0,1[\times[0,1/L[$, then $\widehat{\pi}_{\boldsymbol{\omega}} \simeq \sigma_{0,\boldsymbol{\omega},1} \in \Sigma_1$.*

Proof. We will use the imprimitivity theorem to show that $\widehat{\pi}_{\boldsymbol{\omega}}$ is induced from a character of N. For this purpose consider the set $S = \{\boldsymbol{\omega} + (0, i/L) : i = 0, \ldots, L-1\}$, with an action of G on S given by

$$\boldsymbol{\gamma}.(n, k, \overline{\ell}) = (\gamma_1, \gamma_2 - n/L - \lfloor \gamma_2 - n/L \rfloor) \quad .$$

The action is transitive with N as associated stabilizer. To any subset $A \subset S$ we associate a projection operator P_A on $\mathcal{H}_{\boldsymbol{\omega}}$ defined by pointwise multiplication with the characteristic function of $A + \mathbb{Z} \times \mathbb{Z}$. It is then straightforward to check that $A \mapsto P_A$ is a projection-valued measure on S satisfying

$$\widehat{\pi}_{\boldsymbol{\omega}}(n, k, \overline{\ell}) P_A \widehat{\pi}_{\boldsymbol{\omega}}(n, k, \overline{\ell})^* = P_{A.(n,k,\ell)} \quad .$$

In other words, $A \mapsto P_A$ defines a transitive system of imprimitivity. Hence the imprimitivity theorem [45, Theorem 6.31] applies to show that $\widehat{\pi}_{\boldsymbol{\omega}} \simeq \mathrm{Ind}_N^G \varrho$ for a suitable representation ϱ of N. Since the system of imprimitivity is based on a discrete set, we can follow the procedure outlined in [45] immediately after Theorem 6.31, which identifies ϱ as the representation of N acting on $P_{\{\boldsymbol{\omega}\}}(\mathcal{H}_\omega)$. For this purpose consider the function $F \in \mathcal{H}_{\boldsymbol{\omega}}$ defined by

$$F(\boldsymbol{\omega} + (0, m/L)) = \delta_{m,0} \quad \text{for } m = 0, \ldots, L-1 \quad .$$

Now the fact that

$$\widehat{\pi}_{\boldsymbol{\omega}}(nL, k, \overline{\ell}) F = \mathrm{e}^{2\pi \mathrm{i} \ell/L} \mathrm{e}^{2\pi \mathrm{i} \omega_1 n} \mathrm{e}^{2\pi \mathrm{i} \omega_2 k} F = \chi_{\omega_1, \omega_2, 1}(nL, k, \overline{\ell}) F$$

shows that

$$\widehat{\pi}_{\boldsymbol{\omega}} \simeq \mathrm{Ind}_N^G \chi_{\omega_1, \omega_2, 1} = \sigma_{0, \omega_1, \omega_2, 1,} \quad .$$

By the last lemma and Mackey's theory, no two representations appearing in (5.25) are equivalent. Since G is type I, (5.25) is central, and 3.20 implies that π is multiplicity-free.

Summarizing our findings in the language of Remark 4.30, we have verified the following:

Corollary 5.41. *(a) $\mathcal{Z}$ implements a unitary equivalence $\pi \simeq \int_{\Sigma_1}^{\oplus} \sigma d\nu_G(\sigma)$. The multiplicity function is computed as $m_\pi(\sigma) = \mathbf{1}_{\Sigma_1}(\sigma)$.*

(b) The necessary conditions for the existence of admissible vectors, in terms of absolute continuity of the underlying measure, the multiplicity function and the finite Plancherel measure condition are trivially fulfilled.

(c) The Zak transform criterion (5.22) is a special instance of (4.17).

In view of Theorem 4.32, it is in fact enough to establish the direct integral decomposition of π. However, the calculations in this section also serve as an illustration of the use of Theorem 3.40 for the explicit computation of ν_G.

6

Sampling Theorems for the Heisenberg Group

In this chapter we characterize the sampling spaces of the Heisenberg group by use of the Plancherel transform. We are aware of two previous authors, Dooley and Pesenson, dealing with sampling theorems on nonabelian groups, for two rather different settings [37, 98, 99, 100]. What these two approaches have in common with each other (and with the one developed here) is the fact that they deal with the problem of reconstructing elements of a fixed leftinvariant subspace of $\mathrm{L}^2(G)$ from sampled values.

The difference between the approaches of Dooley and Pesenson initiate from different notions of bandlimitation. Dooley [37] considers groups of the form $G = \mathbb{R}^k \rtimes H$, with H compact. His concept of bandlimitedness is representation-theoretic, that is, defined as a condition on the support of the subspace on the Plancherel transform side. The condition was inspired by the Mackey picture. Recall that by Mackey's theory the dual has the form

$$\widehat{G} = \bigcup_{\mathcal{O}(\gamma)\in\mathbb{R}^k/H} \{\mathcal{O}(\gamma)\} \times \widehat{H_\gamma} \quad .$$

Now, given a leftinvariant subspace $\mathcal{H} \subset \mathrm{L}^2(G)$ with associated projection field $(P_{\mathcal{O}(\gamma),\sigma})_{\mathcal{O}(\gamma),\sigma}$, the space $\mathcal{H}$ is declared bandlimited if there exists a bounded set $B \subset \widehat{\mathbb{R}^k}$ such that $P_{\mathcal{O}(\gamma),\sigma} = 0$ for almost all $\gamma \notin B$.

By contrast, Pesenson [98, 99, 100] studies stratified Lie groups G, and his notion of bandlimitedness is of a geometric rather than a representation-theoretic nature. It involves a (leftinvariant) sub-Laplacian L on G, which is a particular selfadjoint differential operator on G. This time, a leftinvariant space $\mathcal{H} \subset \mathrm{L}^2(G)$ is called bandlimited if the projection onto $\mathcal{H}$ was given by a spectral projection of L corresponding to a bounded interval in $\mathbb{R}$.

In both cases, the sampling theorems state that the elements of the spaces under consideration are uniquely determined by their sampled values. Put differently, the restriction maps were shown to be injective, but no other functional-analytic property of the restriction map was studied. Accordingly,

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the reconstruction procedures, if they were at all given, were not shown to be stable with respect to the L^2-norm.

The approach presented here is rather different in this respect: We start from our notion of sampling space, which is tied to the existence of a sampling expansion with convergence in the norm, and our goal is to characterize the subspaces that admit this type of expansions; i.e., to solve **T5**. We only consider one particular group, the Heisenberg group, but we will be able to give a complete characterization of sampling spaces in terms of conditions on the associated projection fields. These conditions, in turn, can be read as bandlimitation requirements.

It is instructive to compare Dooley's notion of bandlimited subspaces with our notion of sampling spaces. It is easy to check by Theorem 3.39(c) that the semidirect product group $G = \mathbb{R}^k \rtimes H$ is type I if H is compact. Picking any bounded measurable set $B \subset \widehat{\mathbb{R}^k}$, we can consider the twosided invariant subspace $\mathcal{H} \subset L^2(G)$, supported by

$$\widetilde{B} = \{\mathrm{Ind}_{G_\gamma}^G(\gamma \times \sigma) : \gamma \in B, \sigma \in \widehat{H_\gamma}\} \subset \widehat{G} \quad ,$$

which is bandlimited in the sense of [37]. We will show that if $\mathcal{H}$ is a sampling space in the sense of 2.51, then H is finite.

By Theorem 2.56, $\mathcal{H}$ is generated by an L^2-convolution idempotent. Theorem 4.22 then implies that

$$\int_{\widetilde{B}} \dim(\mathcal{H}_\pi) d\nu_G(\pi) < \infty \quad .$$

Before we continue the computation, observe that the representation space of $\mathrm{Ind}_{G_\gamma}^G(\gamma \times \sigma)$ has Hilbert space dimension $\min(\infty, [G : G_\gamma]) \cdot \dim(\mathcal{H}_\sigma)$. Theorem 3.40 by Kleppner and Lipsman allows to compute

$$\begin{aligned} \infty &> \int_{\widetilde{B}} \dim(\mathcal{H}_\pi) d\nu_G(\pi) \\ &= \int_B \int_{\widehat{H_\gamma}} \min(\infty, [G : G_\gamma]) \cdot \dim(\mathcal{H}_\sigma) d\nu_{H_\gamma}(\sigma) d\overline{\lambda}(\gamma) \qquad (6.1) \\ &\geq \int_B \int_{\widehat{H_\gamma}} \dim(\mathcal{H}_\sigma) d\nu_{H_\gamma}(\sigma) d\overline{\lambda}(\gamma) \qquad (6.2) \end{aligned}$$

The finiteness of (6.1) entails that almost all G_γ have finite index, which means that almost all orbits are finite. In addition, the finiteness of (6.2) requires that for almost all γ the inner integral is finite. Then Corollary 4.25 yields for these γ that H_γ is discrete (and compact of course), hence finite. But now the orbits and the fixed groups are finite for almost all γ, i.e., H is finite. Hence we see that Dooley's concept and ours are identical only for finite extensions of $\mathbb{R}^k$.

6.1 The Heisenberg Group and Its Lattices

Let us quickly recall the main properties of the Heisenberg group, outlined in Section 4.5. As a set $\mathbb{H} = \mathbb{R}^3$, with group law

$$(p, q, t) * (p', q', t') = (p + p', q + q', t + t' + (pq' - qp')/2) \quad .$$

It is a unimodular group, with the usual Lebesgue measure on as Haar measure.

The center of $\mathbb{H}$ is given by $Z(\mathbb{H}) = \{(0, 0, t) : t \in \mathbb{R}\}$. We denote the group of topological automorphisms of $\mathbb{H}$ by $\mathrm{Aut}(\mathbb{H})$. Recall the definition of the Schrödinger representations, acting via

$$[\varrho_h(p, q, t)f](x) = \mathrm{e}^{2\pi \mathrm{i} ht} \mathrm{e}^{2\pi \mathrm{i} qx} \mathrm{e}^{\pi \mathrm{i} hpq} f(x + hp) \quad .$$

The Schrödinger representations do not exhaust the dual of $\mathbb{H}$, which in addition contains the characters of the abelian factor group $\mathbb{H}/Z(\mathbb{H})$. However, for the decomposition of the regular representation of $\mathbb{H}$, we may concentrate on the Schrödinger representations. More precisely, the set $(\varrho_h)_{h \in \mathbb{R}'}$ is a $\nu_{\mathbb{H}}$-conull subset of $\widehat{\mathbb{H}}$, and Plancherel measure is given by $|h|dh$, where dh denotes Lebesgue measure [45, Section 7.6].

To close our survey of the Heisenberg group, we cite a result classifying the lattices of $\mathbb{H}$. We associate to such a lattice Γ two numbers $d(\Gamma) \in \mathbb{N}'$, $r(\Gamma) \in \mathbb{R}^+$ which contain sufficient information for our purposes. The two parameters provide a measure of the density of Γ in $\mathbb{H}$. We first single out a particular family of lattices, which turns out to be exhaustive (up to automorphisms of $\mathbb{H}$).

Definition 6.1. *For any positive integer d let Γ_d be the subgroup generated by $(1, 0, 0), (0, d, 0), (0, 0, 1)$. Γ_d is a lattice, with*

$$\Gamma_d = \left\{ (m, dk, \ell + \frac{1}{2} dmk) : m, k, \ell \in \mathbb{Z} \right\} \quad .$$

It is convenient to introduce the **reduced lattice** Γ_d^r *which is the subset*

$$\Gamma_d^r = \{(m, dk, dmk/2) : m, k \in \mathbb{Z}\} \quad .$$

Note that Γ_d^r is not a lattice, not even a subgroup.

Let us next give a classification of lattices. It has been attributed (in more generality) to Maltsev. Since we were not able to locate the original source, we sketch a short proof for the sake of completeness.

Theorem 6.2. *Let Γ be a lattice of $\mathbb{H}$. Then there exists a strictly positive integer d and $\alpha \in \mathrm{Aut}(\mathbb{H})$ with $\alpha(\Gamma_d) = \Gamma$. The integer d is uniquely determined by these properties.*

Proof. By [30, Theorem 5.1.6], there exist a basis $\widetilde{P}, \widetilde{Q}, \widetilde{Z}$ of $\mathfrak{h}$ with $\widetilde{Z} \in Z(\mathbb{H})$, and $\Gamma = \mathbb{Z}\widetilde{Z} * \mathbb{Z}\widetilde{P} * \mathbb{Z}\widetilde{Q}$. Now $[\widetilde{P}, \widetilde{Q}] = \widetilde{P}\widetilde{Q}\widetilde{P}^{-1}\widetilde{Q}^{-1} \in \Gamma \cap Z(\mathbb{H}) = \mathbb{Z}\widetilde{Z}$ implies $[\widetilde{P}, \widetilde{Q}] = d\widetilde{Z}$ for some $d \in \mathbb{Z}$, w.l.o.g. $d \geq 0$ (otherwise exchange $\widetilde{Q}, \widetilde{P}$). In fact, $d > 0$ since $\mathfrak{h}$ is not abelian. It is immediately checked that the linear isomorphism defined by

$$(1,0,0) \mapsto \widetilde{P} \quad , \quad (0,d,0) \mapsto \widetilde{Q} \quad , \quad (0,0,1) \mapsto \widetilde{Z}$$

is in $\mathrm{Aut}(\mathbb{H})$. That d is unique is due to the fact that each automorphism α mapping Γ_d to $\Gamma_{d'}$ maps Z onto $\pm Z$: Indeed, from Proposition 6.17 (a) below follows that α leaves the Haar measure of $\mathbb{H}$ invariant, and this implies that $\mathrm{covol}(\Gamma_d) = \mathrm{covol}(\Gamma_{d'})$. On the other hand, $\mathrm{covol}(\Gamma_d) = d$, hence $d = d'$.

We denote by $d(\Gamma)$ the unique integer d from the theorem. For the definition of $r(\Gamma)$ we take the unique positive real satisfying $\Gamma \cap Z(\mathbb{H}) = r(\Gamma)\mathbb{Z}Z$.

6.2 Main Results

Now we can state the main results of this chapter. In this section, $\mathcal{H}$ always denotes a closed, leftinvariant subspace of $\mathrm{L}^2(\mathbb{H})$, and $\Gamma < \mathbb{H}$ a lattice. Recall from Theorem 2.56 that we may assume $\mathcal{H} = \mathrm{L}^2(G) * S$, where S is a selfadjoint convolution idempotent, and that $\mathcal{H}$ is a sampling space iff $\lambda_{\mathbb{H}}(\Gamma)S$ is a tight frame of $\mathcal{H}$.

Definition 6.3. *We associate to a leftinvariant subspace $\mathcal{H} \subset \mathrm{L}^2(\mathbb{H})$ with associated projection field $(\widehat{P}_h)_{h \in \mathbb{R}'}$ the multiplicity function $m_{\mathcal{H}} : \mathbb{R}' \to \mathbb{N}_0 \cup \{\infty\}$ of the associated subrepresentation. This function is given by $m_{\mathcal{H}}(h) = \mathrm{rank}(\widehat{P}_h)$. The support of $m_{\mathcal{H}}$ is denoted by $\Sigma(\mathcal{H})$. $\mathcal{H}$ is called* **bandlimited** *is $\Sigma(\mathcal{H})$ is bounded in $\mathbb{R}$.*

Similarly, if a representation π is equivalent to a subrepresentation of $\lambda_{\mathbb{H}}$, say to the restriction of $\lambda_{\mathbb{H}}$ to $\mathcal{H}$, we let $m_\pi = m_{\mathcal{H}}$ and $\Sigma(\pi) = \Sigma(\mathcal{H})$. This is obviously well-defined.

The main theorem characterizes the subspaces admitting tight frames.

Theorem 6.4. *(i) There exists a tight frame of the form $\lambda_{\mathbb{H}}(\Gamma)\Phi$ with suitable $\Phi \in \mathcal{H}$ iff the multiplicity function m associated to $\mathcal{H}$ satisfies almost everywhere*

$$m(h) \cdot |h| + m\left(h - \frac{1}{r(\Gamma)}\right) \cdot \left|h - \frac{1}{r(\Gamma)}\right| \leq \frac{1}{d(\Gamma) r(\Gamma)} \quad . \tag{6.3}$$

In particular, $\Sigma(\mathcal{H}) \subset \left[-\frac{1}{d(\Gamma)r(\Gamma)}, \frac{1}{d(\Gamma)r(\Gamma)}\right]$ (up to a set of measure zero).
(ii) There does not exist an orthonormal basis of the form $\lambda_{\mathbb{H}}(\Gamma)\Phi$ for $\mathcal{H}$.

Remark 6.5. Note that if $d(\Gamma) > 1$ and $m(h) > 0$ and $m\left(h - \frac{1}{r(\Gamma)}\right) > 0$, inequality (6.3) entails the inequality

$$|h| + \left|h - \frac{1}{r(\Gamma)}\right| \leq \frac{1}{2r(\Gamma)}$$

which is impossible to satisfy. Hence in the case $d(\Gamma) > 1$ relation (6.3) simplifies to

$$m(h) \cdot |h| \leq \frac{1}{d(\Gamma)r(\Gamma)} \quad . \tag{6.4}$$

Corollary 6.6. *Assume that $m_{\mathcal{H}}$ is essentially bounded. There exists a tight frame of the form $\lambda_{\mathbb{H}}(\Gamma)\Phi$, (with a suitable lattice Γ and suitable $\Phi \in \mathcal{H}$) iff $\mathcal{H}$ is bandlimited.*

The following is a rephrasing for discretization of continuous wavelet transforms.

Corollary 6.7. *Let $(\pi, \mathcal{H}_\pi)$ be a representation of $\mathbb{H}$ with admissible vector. There exists a tight frame of the form $\pi(\Gamma)\eta$, (with suitable lattice Γ and suitable $\eta \in \mathcal{H}_\pi$) if $\Sigma(\pi)$ and m_π are bounded.*

That bounded multiplicity cannot be dispensed with in the previous corollary is shown by the next result:

Corollary 6.8. *There exists a bandlimited leftinvariant subspace $\mathcal{H} = \mathrm{L}^2(G) * S$, with a selfadjoint convolution idempotent $S \in \mathrm{L}^2(\mathbb{H})$, admitting no tight frame of the form $\lambda_{\mathbb{H}}(\Gamma)\Phi$.*

With regard to the existence of sampling subspaces, we have:

Corollary 6.9. *Not every space admitting a tight frame of the form $\lambda_{\mathbb{H}}(\Gamma)S$ is a sampling subspace for Γ. However, for such a space $\mathcal{H}$ there exists $\Phi \in \mathcal{H}$ such that $f \mapsto f * \Phi^*$ is an isometry on $\mathcal{H}$, mapping $\mathcal{H}$ onto a sampling space. There does not exist a sampling space $\mathcal{H}$ with the interpolation property.*

The proofs for these results will be given in Section 6.5 below. The following remarks discuss similarities and differences to the case of the reals.

Remark 6.10.
1. The main similarity lies in the notion of bandwidth, and the fact that it can be interpreted as inversely proportional to the density of the lattice. Note that over $\mathbb{H}$ the bandwidth restriction is much more rigid: The set $\Sigma(\mathcal{H})$ is contained in a fixed interval, whereas the analog of that set in the real case can be shifted arbitrarily and still give a sampling subspace; compare Theorem 2.71.
2. Corollaries 6.8 and 6.9 provide an interesting contrast between the sampling theories of $\mathbb{H}$ and $\mathbb{R}$. None of the counterexamples given in the corollaries has an analog in the real setting. In particular, in the Heisenberg group case the question whether a given space is a sampling space

is much more subtle than deciding whether it has a frame. For the first problem, a close inspection of the projection operator field $(\widehat{P}_h)_{h\in\mathbb{R}'}$ is necessary (using the criteria in Proposition 6.11 below), for the second, only the ranks of these operators are needed. By contrast, 2.71 (a) $\Leftrightarrow$ (a') shows that for the reals the two properties are equivalent. This is not so surprising after all: Projection fields on $\int_{\widehat{\mathbb{R}}}^{\oplus} \mathbb{C} d\omega$ are obviously uniquely determined by their supports.

3. While Theorem 6.4 shows that the Plancherel transform can be used to characterize sampling spaces and frames, it is not clear how it can be generalized to a larger class of locally compact groups. Indeed, as far as we are aware, among the entities entering the central relation (6.3), only the multiplicity function m has an abstract interpretation. Also, the techniques proving Theorem 6.4 are rather specific to the Heisenberg group, combining known results concerning Weyl-Heisenberg with the Plancherel transform of $\mathbb{H}$ (see the arguments in the next section), which is a further illustration how the
4. We use lattices as sampling sets simply because they are easily accessible. In particular, we do not exploit the representation theory of the lattices at all. Recall that the tight frame condition is nothing but an admissibility condition for the restriction of $\lambda_{\mathbb{H}}$. However, the lattices in $\mathbb{H}$ are not type I, hence a better understanding of the non-type I setting will be needed.

6.3 Reduction to Weyl-Heisenberg Systems*

In this section we start the discussion of normalized tight frames for left-invariant subspaces. On the Plancherel transform side, the space $\mathcal{H}$ under consideration decomposes into a direct integral. In this section, we reduce the complexity of the problem in two ways: We get rid of the direct integral on the one hand, and the central variable of the lattice on the other, and are faced with the problem of constructing certain normalized tight frames in the fibres, arising from the action of the reduced lattice. The latter problem is equivalent to the construction of Weyl-Heisenberg (super-)frames.

Proposition 6.11. *Let $\Gamma = \Gamma_d$. Let $\Phi \in \mathcal{H}$ be such that $\lambda_G(\Gamma)\Phi$ is a normalized tight frame of $\mathcal{H}$. Then, for almost every $h \in \Sigma(\mathcal{H})$, the reduced lattice satisfies the following condition:*

$$\left(|h|^{1/2}\varrho_h(\gamma)\widehat{\Phi}(h)\right)_{\gamma\in\Gamma_d^r} \text{ is a normalized tight frame of } \mathcal{B}_2(\mathrm{L}^2(\mathbb{R}))\circ\widehat{P}_h\ . \quad (6.5)$$

Conversely, if both (6.5) (for almost every h) and the support condition

$$\forall m \in \mathbb{Z}\setminus\{0\} \quad : \quad \Sigma(\mathcal{H}) \cap m + \Sigma(\mathcal{H}) \text{ has measure zero} \qquad (6.6)$$

hold, then $\lambda_G(\Gamma)\Phi$ is a normalized tight frame of $\mathcal{H}$.

Proof. Assume first that $|\Sigma(f) \cap m + \Sigma(f)| = 0$. We calculate

$$\begin{aligned}\sum_{\gamma\in\Gamma} |\langle f, \lambda_{\mathbb{H}}(\gamma)\Phi\rangle|^2 &= \sum_{\gamma\in\Gamma} \left| \int_{\Sigma(f)} \langle \widehat{f}(h), \varrho_h(\gamma)\widehat{\Phi}(h)\rangle |h| dh \right|^2 \\ &= \sum_{\gamma\in\Gamma_d^r} \sum_{\ell\in\mathbb{Z}} \left| \int_{\Sigma(f)} \mathrm{e}^{-2\pi \mathrm{i} h\ell} \langle \widehat{f}(h), \varrho_h(\gamma)\widehat{\Phi}(h)\rangle |h| dh \right|^2 \\ &= \sum_{\gamma\in\Gamma_d^r} \int_{\Sigma(f)} \left| \langle \widehat{f}(h), \varrho_h(\gamma)\widehat{\Phi}(h)\rangle \right|^2 |h|^2 dh \\ &= \int_{\Sigma(f)} \sum_{\gamma\in\Gamma_d^r} \left| \langle \widehat{f}(h), \varrho_h(\gamma)|h|^{1/2}\widehat{\Phi}(h)\rangle \right|^2 |h| dh \quad .\end{aligned}$$

Here we used the assumption on $\Sigma(f)$ to apply the Plancherel Theorem for Fourier series on $\Sigma(f)$ and thereby discard the summation over ℓ. On the other hand, the tight frame condition together with the Plancherel formula for $\mathbb{H}$ implies that

$$\sum_{\gamma\in\Gamma} |\langle f, \lambda(\gamma)s\rangle|^2 = \int_{\Sigma(f)} \|\widehat{f}(h)\|^2 |h| dh \quad ,$$

and thus

$$\int_{\Sigma(f)} \sum_{\gamma\in\Gamma_d^r} \left| \langle \widehat{f}(h), \varrho_h(\gamma)|h|^{1/2}\widehat{\Phi}(h)\rangle \right|^2 |h| dh = \int_{\Sigma(f)} \|\widehat{f}(h)\|^2 |h| dh \quad . \tag{6.7}$$

Replacing f by g with $\widehat{g}(h) = \mathbf{1}_B(h)\widehat{f}(h)$, we see that we may replace $\Sigma(f)$ in (6.7) by any Borel subset B. Hence the integrands must be equal almost everywhere:

$$\sum_{\gamma\in\Gamma_d^r} \left| \langle \widehat{f}(h), \varrho_h(\gamma)|h|^{1/2}\widehat{\Phi}(h)\rangle \right|^2 = \|\widehat{f}(h)\|^2 \quad . \tag{6.8}$$

By covering $\Sigma(\mathcal{H})$ with sets of the form $\Sigma(f)$ fulfilling the initial support condition we see that (6.8) holds for every $f \in \mathcal{H}$ and almost every $h \in \mathbb{R}'$. However, it remains to show that the relation holds for all h in a common conull subset, independent of f. For this purpose we pick a countable dense $\mathbb{Q}$-subspace $\mathcal{A} \subset \mathrm{L}^2(\mathbb{H})$. Then there exists a conull subset $C \subset \mathbb{R}'$ such that, for all $h \in C$, $\{\widehat{f}(h) : f \in \mathcal{A}\}$ is dense in $\mathcal{B}_2(\mathrm{L}^2(\mathbb{R})) \circ \widehat{P}_h$, and in addition (6.8) holds for all $f \in \mathcal{A}$ and $h \in C$. Now, for every $h \in C$, the coefficient map

$$\widehat{g}(h) \mapsto \left(\langle \widehat{g}(h), \varrho_h(\gamma)|h|^{1/2}\widehat{\Phi}(h)\rangle \right)_{\gamma\in\Gamma_d^r}$$

is a closed linear operator, by Proposition 2.53 (d), coinciding with an isometry on a dense subset, hence it is an isometry.

Finally, we note that the argument can be reversed to prove the sufficiency of condition (6.5) under the additional assumption (6.6).

6.4 Weyl-Heisenberg Frames*

For any $g \in \mathrm{L}^2(\mathbb{R})$, the operation of the reduced lattice Γ_d^r on g via ϱ_h gives the system

$$(\varrho_h(m, dk, dmk/2)g)(x) = \mathrm{e}^{\pi \mathrm{i} hmdk} \mathrm{e}^{2\pi \mathrm{i} kx} g(x + hm) \quad ,$$

hence $\varrho_h(\Gamma_d^r)g$ and the Weyl-Heisenberg system $\mathcal{G}(d, h, g)$, as defined in Section 5.5, only differ up to phase factors. Clearly these phase factors do not influence any normalized tight frame or ONB properties of the system, hence we may and will switch freely between the Weyl-Heisenberg system and the orbit of the reduced lattice.

We cite the following existence result:

Theorem 6.12. *A normalized tight Weyl-Heisenberg frame $\mathcal{G}(d, h, g)$ of $\mathrm{L}^2(\mathbb{R})$ exists iff $|h|d \leq 1$. For any such frame we have $\|g\|_2^2 = |h|d$.*

Proof. The "only-if"-part is [58, Corollary 7.5.1]. The "if"-part follows from [58, Theorem 6.4.1], applied to a suitably chosen characteristic function. The norm equality is due to [58, Corollary 7.3.2].

In dealing with subspaces of Hilbert-Schmidt spaces, we have to consider a more general setting: We will be interested in normalized tight frames of $\left(\mathrm{L}^2(\mathbb{R})\right)^r$ consisting of vectors of the type

$$g_{k,m} = (\mathrm{e}^{2\pi \mathrm{i} dkx} g^j(x + h_j m))_{j=1,\ldots,r} = (g^j_{k,m})_{j=1,\ldots,r} \tag{6.9}$$

where $g = (g^j)_{j=1,\ldots,r} \in \left(\mathrm{L}^2(\mathbb{R})\right)^r$ is suitably chosen, and $\mathbf{h} = (h_j)_j \in \mathbb{R}^r$ is a vector of nonzero real numbers. This problem has already been considered by other authors, see [21] and the references therein. The following two lemmata extend the results on $\mathrm{L}^2(\mathbb{R})$ to the more general situation. The first one is quite obvious and does not reflect the special structure of Weyl-Heisenberg frames. We already used a similar argument for the proof of 2.23. A similar result for arbitrary frames is given in [21].

Lemma 6.13. *Let $\mathbf{h} = (h_1, \ldots, h_r) \in \mathbb{R}^r$ and $g = (g^j)_{j=1,\ldots,r} \in \left(\mathrm{L}^2(\mathbb{R})\right)^r$. Then $(g_{k,m})_{k,m\in\mathbb{Z}}$, defined as in equation (6.9), is a normalized tight frame of $\left(\mathrm{L}^2(\mathbb{R})\right)^r$ iff*

(i) for $j = 1, \ldots, r$, $\mathcal{G}(h_j, d, g^j)$ is a normalized tight frame of $\mathrm{L}^2(\mathbb{R})$; and
(ii) for $i \neq j$, and for all $f_1, f_2 \in \mathrm{L}^2(\mathbb{R})$,

$$\left\langle \left(\langle f_1, g^j_{m,n}\rangle\right)_{m,n}, \left(\langle f_2, g^i_{m,n}\rangle\right)_{m,n} \right\rangle_{\ell^2(\mathbb{Z}\times\mathbb{Z})} = 0 \quad . \tag{6.10}$$

i.e., the coefficient operators belonging to $\mathcal{G}(h_j, d, g^j)$ and $\mathcal{G}(h_i, d, g^i)$ have orthogonal ranges in $\ell^2(\mathbb{Z} \times \mathbb{Z})$.

Proof. Consider the subspace $\mathcal{H}_j \subset \left(\mathrm{L}^2(\mathbb{R})\right)^r$ whose elements are nonzero at most on the jth component. The necessity of property (i) follows immediately from Proposition 2.53(a), applied to the $\mathcal{H}_j$. Property (ii) is necessary because the (pairwise orthogonal) $\mathcal{H}_j$ need to have orthogonal images in $\ell^2(\mathbb{Z} \times \mathbb{Z})$. The converse is clear.

Necessary and sufficient conditions for the existence of such frames are given in the next proposition.

Proposition 6.14. *Let $(h_j)_{j=1,\dots,r}$, $d \in \mathbb{N}'$ be given.*

(a) There exists a normalized tight frame of $\left(\mathrm{L}^2(\mathbb{R})\right)^r$ of the form (6.9) iff $d\sum_{j=1}^r |h_j| \le 1$.

(b) Assume that $h_j = h$, for all $j = 1, \dots, r$, and $g = (g^j)_{j=1,\dots,r}$ is such that (6.9) is a normalized tight frame. Then $g^i \perp g^j$, for $i \neq j$.

Proof. For the necessity in part (a), observe that Lemma 6.13 together with Theorem 6.12 yields that $\|g^j\|^2 = |h_j|d$, and thus $\|g\|^2 = d\sum_{j=1}^r |h_j|$. Now Proposition 2.53 (c) entails the desired inequality.

The proof for sufficiency is a slight modification of a construction given by Balan [21, Example 13]. Define $c_i = \sum_{j=1}^i |h_j|$, and let $g^i = \sqrt{d}\mathbf{1}_{[c_{i-1},c_i]}$. Given $f = (f^i) \in \left(\mathrm{L}^2(\mathbb{R})\right)^r$, we compute

$$\begin{aligned}\langle f, g_{k,m}\rangle &= \sum_{i=1}^r \langle f^i, g^i_{k,m}\rangle \\ &= \sum_{i=1}^r \sqrt{d}\int_{c_{i-1}}^{c_i} \mathrm{e}^{-2\pi \mathrm{i} m d x} f^i(x + h_i k)dx \\ &= \sqrt{d}\int_0^{1/d} \mathrm{e}^{-2\pi \mathrm{i} m d x} H_k(x)dx \quad ,\end{aligned}$$

where

$$H_k(x) = \begin{cases} f^i(x - h_i k) & x \in [c_{i-1}, c_i] \\ 0 & \text{elsewhere} \end{cases} \quad .$$

Fixing k, we compute

$$\begin{aligned}\sum_{m\in\mathbb{Z}} |\langle f, g_{k,m}\rangle|^2 &= \sum_{m\in\mathbb{Z}} d\left|\int_0^{1/d} \mathrm{e}^{-2\pi \mathrm{i} m d x} H_k(x)dx\right|^2 \\ &= \int_0^{1/d} |H_k(x)|^2 dx \\ &= \sum_{i=1,\dots,r} \int_{c_{i-1}}^{c_i} |f^i(x - h_i k)|^2 dx \quad .\end{aligned}$$

Since the $h_i\mathbb{Z}$-translates of $[c_{i-1}, c_i]$ tile $\mathbb{R}$, summing over k yields the desired normequality. This closes the proof of (a).

For the proof of (b), pick $f_1, f_2 \in L^\infty(\mathbb{R})$ with supports in $[0, |h|]$. Then we calculate

$$\begin{aligned}
&\sum_{m,k\in\mathbb{Z}} \langle f_1, g^j_{m,k}\rangle\overline{\langle f_2, g^i_{m,k\in\mathbb{Z}}\rangle} = \\
&= \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} \left(\sum_{k\in\mathbb{Z}} \langle f_1 g^j_{m,0}, \mathrm{e}^{2\pi i dk\cdot}\rangle \mathrm{e}^{2\pi i dkx} \right) \overline{f_2(x)} g^i(x+hm)dx \\
&= \sum_{m\in\mathbb{Z}} d^{-1} \int_0^{|h|} f_1(x)\overline{f_2(x) g^j(x+hm)} g^i(x+hm)dx \\
&= d^{-1} \int_0^{|h|} \left(\sum_{m\in\mathbb{Z}} \overline{g^j(x+hm)} g^i(x+hm) \right) f_1(x)\overline{f_2(x)}dx
\end{aligned}$$

Here the Fourier series

$$\sum_{k\in\mathbb{Z}} \langle f_1 g^j_{m,0}, \mathrm{e}^{2\pi i dk\cdot}\rangle \mathrm{e}^{2\pi dkx} = d^{-1} f_1(x) g^j(x+hm)$$

is valid on $[0, |h|]$, at least in the L^2-sense, because of $|h| \le d^{-1}$, the latter being a consequence of Theorem 6.12. Now, for arbitrary f_1, f_2, the scalar product we started with has to be zero, whence we obtain for almost every $x \in [0, |h|]$,

$$\sum_{m\in\mathbb{Z}} g^i(x+hm)\overline{g^j(x+hm)} = 0 \quad .$$

Integrating over $[0, |h|]$ and applying Fubini's theorem yields $\langle g^i, g^j\rangle = 0$.

Remark 6.15. Note that the vectors $(g^i)_{i=1,\dots,r}$ constructed in the proof of part (a) depend measurably on $\mathbf{h}$, i.e., if we let $(g^i_{\mathbf{h}})$ be the vector of functions constructed from $\mathbf{h}$, then $(x, \mathbf{h}) \mapsto (g^i_{\mathbf{h}}(x))_{i=1,\dots,r}$ is a measurable mapping.

6.5 Proofs of the Main Results*

The general proof strategy consists in explicit calculation for the Γ_d and then transferring the results to arbitrary lattices by the action of $\mathrm{Aut}(\mathbb{H})$. For this purpose we need a more detailed description of $\mathrm{Aut}(\mathbb{H})$ and its action on the Plancherel transform side. Most of the results are standard, and we only sketch the proofs.

Proposition 6.16. *(a) For $r > 0$ let $\alpha_r(p, q, t) := (\sqrt{r}p, \sqrt{r}q, rt)$. Then $\alpha_r \in \mathrm{Aut}(\mathbb{H})$. In addition, $\alpha_{inv} : (p, q, t) \mapsto (q, p, -t)$ defines an involutory automorphism of $\mathbb{H}$.*

(b) Each $\alpha \in \mathrm{Aut}(\mathbb{H})$ can be written uniquely as $\alpha = \alpha_r \alpha^i_{inv} \alpha'$, where $r \in \mathbb{R}'$, $i \in \{0, 1\}$ and α' leaves the center of $\mathbb{H}$ pointwise fixed.

(c) Suppose that $\alpha(\Gamma_d) = \Gamma$ *for some* d, α, *and let* $\alpha = \alpha_s \alpha_{inv}^i \alpha'$ *be the decomposition from part (b). Then* $r(\Gamma) = s$.

Proof. For parts (a), (b) see [46, Theorem 1.22]. Part (c) follows directly from the definition of $r(\Gamma)$ and the fact that α' and α_{inv} map every discrete subgroup of $Z(\mathbb{H})$ onto itself.

Next let us consider the action on the Fourier transform side.

Proposition 6.17. *(a) Define* $\Delta : \mathrm{Aut}(\mathbb{H}) \to \mathbb{R}^+$ *by*

$$\Delta(\alpha) = \frac{\mu_{\mathbb{H}}(\alpha(B))}{\mu_{\mathbb{H}}(B)} \quad ,$$

where B *is a measurable set of positive Haar measure.* Δ *does not depend on the choice of* B, *and it is a continuous group homomorphism. For* $\alpha = \alpha_r \alpha_{inv}^i \alpha'$ *as in 6.16(b),* $\Delta(\alpha) = r^2$.

(b) For $\alpha \in \mathrm{Aut}(\mathbb{H})$, *let* $\mathcal{D}_\alpha : \mathrm{L}^2(\mathbb{H}) \to \mathrm{L}^2(\mathbb{H})$ *be defined as* $(\mathcal{D}_\alpha f)(x) := \Delta(\alpha)^{1/2} f(\alpha(x))$. *This defines a unitary operator.*

(c) Let $\mathcal{H} \subset \mathrm{L}^2(G)$ *be a closed, leftinvariant subspace with multiplicity function* m. *Then* $\widetilde{\mathcal{H}} = \mathcal{D}_\alpha(\mathcal{H})$ *is closed and leftinvariant as well. Let* $\widetilde{m}$ *denote the multiplicity function related to* $\widetilde{\mathcal{H}}$. *If* $\alpha = \alpha_r \alpha_{inv}^i \alpha'$ *then* $\widetilde{m}$ *satisfies*

$$\widetilde{m}(h) = m((-1)^i r^{-1} h) \text{ (almost everywhere)} \quad . \tag{6.11}$$

(d) Let Γ *be a lattice,* $\alpha \in \mathrm{Aut}(\mathbb{H})$ *such that* $\alpha(\Gamma_d) = \Gamma$. *Let* $\mathcal{H} \subset \mathrm{L}^2(\mathbb{H})$ *be a closed, leftinvariant subspace. Then* $\lambda_{\mathbb{H}}(\Gamma)\Phi$ *is a normalized tight frame (an ONB) for* $\mathcal{H}$ *iff* $\lambda_{\mathbb{H}}(\Gamma_d)(\mathcal{D}_\alpha \Phi)$ *is a normalized tight frame (an ONB) for* $\mathcal{D}_\alpha(\mathcal{H})$.

Proof. Parts (a) and (b) are standard results concerning the action of automorphisms on locally compact groups, see [64]. The explicit formula for $\Delta(\alpha)$ follows from the fact that every automorphism leaving the center invariant factors into an inner and a symplectic automorphism [46, Theorem 1.22]; both do not affect the Haar measure.

For part (c), we first note that by the Stone-von Neumann theorem [46, Theorem 1.50], any automorphism α' keeping the center pointwise fixed acts trivially on the dual of $\mathbb{H}$. Hence,

$$(\mathcal{D}_{\alpha'} f)^\wedge(h) = U_{\alpha',h} \circ \widehat{f}(h) \circ U_{\alpha',h}^* \quad ,$$

where $U_{\alpha',h}$ is a unitary operator on $\mathrm{L}^2(\mathbb{R})$. Hence the action of α' does not affect the multiplicity function, and from now on, we only consider $\alpha = \alpha_r \alpha_{inv}^i$. In this case, letting

$$(D_r f)(x) = r^{1/2} f(rx) \quad ,$$

we obtain by straightforward computation that

$$(\mathcal{D}_\alpha f)^\wedge(h) = r^{-1} \cdot D_r \circ \widehat{f}((-1)^i r^{-1} h) \circ D_r^* \quad . \tag{6.12}$$

This immediately implies (6.11).

To prove (d), observe that the unitarity of $\mathcal{D}_\alpha$ implies that $\mathcal{D}_\alpha(\lambda_{\mathbb{H}}(\Gamma))$ is a normalized tight frame of $\mathcal{D}_\alpha(\mathcal{H})$, and check the equality

$$\mathcal{D}_\alpha(\lambda_{\mathbb{H}}(x)S) = \lambda_{\mathbb{H}}(\alpha^{-1}(x))(\mathcal{D}_\alpha S) \quad .$$

Proof of Theorem 6.4. We first prove the theorem for the case $\Gamma = \Gamma_d$. Writing

$$\widehat{\Phi}(h) = \sum_{i \in I_h} \varphi_i^h \otimes \eta_i^h \quad ,$$

we find by Proposition 6.11, that for almost every h, $(\varrho_h(\gamma) \circ |h|^{1/2} \widehat{\Phi}(h))_{\gamma \in \Gamma}$ has to be a normalized tight frame of $\mathrm{L}^2(\mathbb{R}) \circ \widehat{P}_h$, or equivalently, that the vector $(\varphi_i^h)_{i=1,\dots,m(h)}$ generates a frame of $\left(\mathrm{L}^2(\mathbb{R})\right)^{m(h)}$, for $\mathbf{h} = (h, \dots, h)$. Then 6.13 (a) implies that $\mathcal{G}(h, d, |h|^{1/2}\varphi_i^h)$ is a normalized tight frame of $\mathrm{L}^2(\mathbb{R})$. In particular, Theorem 6.12 entails

$$\|\varphi_i^h\|^2 = d \quad , \tag{6.13}$$

as well as $\Sigma(\mathcal{H}) \subset \left[-\frac{1}{d}, \frac{1}{d}\right]$. If $d > 1$, the support condition (6.6) in Proposition 6.11 is fulfilled. Hence Proposition 6.14 (a), applied to $\mathbf{h} = (h, \dots, h)$, shows that (6.4) is necessary and sufficient for the existence of a normalized tight frame for $\mathcal{H}$. (Note that by Remark 6.15, 6.14 (a) provides a *measurable* vector field.)

The case $d = 1$ requires a somewhat more involved argument. Assume that $\lambda_{\mathbb{H}}(\Gamma)\Phi$ is a normalized tight frame, and let $f \in \mathcal{H}$. Condition (6.5) from Proposition 6.11 yields

$$\begin{aligned} \|f\|^2 = \int_0^1 \sum_{\gamma \in \Gamma_d^r} & \left| \langle \widehat{f}(h), \varrho_h(\gamma) \widehat{\Phi}(h) \rangle \right|^2 |h|^2 \\ & + \left| \langle \widehat{f}(h-1), \varrho_{h-1}(\gamma) \widehat{\Phi}(h-1) \rangle \right|^2 |h-1|^2 dh \quad . \end{aligned} \tag{6.14}$$

On the other hand,

$$\begin{aligned} \|f\|^2 &= \sum_{\gamma \in \Gamma_d^r} \sum_{\ell \in \mathbb{Z}} \left| \int_0^1 \mathrm{e}^{-2\pi \mathrm{i} h \ell} \left(\langle \widehat{f}(h), \varrho_h(\gamma) \widehat{\Phi}(h) \rangle |h| + \right. \right. \\ &\qquad \left. \left. + \langle \widehat{f}(h-1), \varrho_{h-1}(\gamma) \widehat{\Phi}(h-1) \rangle |h-1| \right) dh \right|^2 \\ &= \int_0^1 \left(\sum_{\gamma \in \Gamma_d^r} |\langle \widehat{f}(h), \varrho_h(\gamma) \widehat{\Phi}(h) \rangle |h| \right. \\ &\qquad \left. + \langle \widehat{f}(h-1), \varrho_{h-1}(\gamma) \widehat{\Phi}(h-1) \rangle |h-1| \, |^2 \right) dh \quad . \end{aligned} \tag{6.15}$$

As in the proof of Proposition 6.11, the fact that the two equations hold for all $f \in \mathcal{H}$ allows to equate the integrands of (6.14) and (6.15). But this implies the orthogonality of the coefficient families:

$$\left\langle \left(\langle \widehat{f}(h), \varrho_h(\gamma)\widehat{\Phi}(h)\rangle \right)_{\gamma\in\Gamma_d^r}, \left(\langle \widehat{f}(h-1), \varrho_{h-1}(\gamma)\widehat{\Phi}(h-1)\rangle \right)_{\gamma\in\Gamma_d^r} \right\rangle_{\ell^2(\Gamma_d^r)} = 0 \quad .$$

Plugging this fact, together with condition (6.5) from Proposition 6.11, into Proposition 6.13, we finally obtain that the system

$$\Big(\varrho_{h-1}(\gamma)|h-1|^{1/2}\varphi_1^{h-1} \ , \dots, \ \varrho_{h-1}(\gamma)|h-1|^{1/2}\varphi_{m(h-1)}^{h-1}, \varrho_h(\gamma)|h|^{1/2}\varphi_1^h, \\ \dots, \ \varrho_h(\gamma)|h|^{1/2}\varphi_{m(h)}^h \Big)_{\gamma\in\Gamma_d^r}$$

has to be a normalized tight frame of $\left(\mathrm{L}^2(\mathbb{R})\right)^{m(h)+m(h-1)}$. An application of Proposition 6.14 (a) with $\mathbf{h} = (h-1, \dots, h-1, h, \dots, h)$ yields that such a frame exists iff $m(h)|h|+m(h-1)|h-1| \leq 1$. This shows the necessity of (6.3). The sufficiency is obtained by running the proof backward; the measurability of the constructed operator field is again ensured by Remark 6.15.

For the proof of (ii) we need to show, by 2.53(c), that $\|\Phi\| < 1$, for every Φ for which $\lambda_{\mathbb{H}}(\Gamma)\Phi$ is a normalized tight frame. Recalling that

$$|h|\|\widehat{\Phi}(h)\|_{\mathcal{B}_2}^2 = |h|m(h)d \quad ,$$

and using the fact that the inequality $m(h)|h|d+m(h-1)|h-1|d \leq 1$ is strict almost everywhere (say, for h irrational) we can estimate

$$\begin{aligned} \|\Phi\|^2 &= \int_{-1}^{1} \|\widehat{\Phi}(h)\|_{\mathcal{B}_2}^2 |h| dh \\ &= \int_0^1 m(h)|h|d + m(h-1)|h-1|d \ dh \\ &< 1 \quad . \end{aligned}$$

This closes the proof for $\Gamma = \Gamma_d$. For $\Gamma = \alpha(\Gamma_d)$, write $\alpha = \alpha_{r(\Gamma)}\alpha_{inv}^i\alpha'$ as in Proposition 6.16 (c). By 6.17 (d), we may consider Γ_d and $\widetilde{\mathcal{H}} = \mathcal{D}_\alpha(\mathcal{H})$ instead of Γ and $\mathcal{H}$. Part (ii) immediately follows from this observation. For part (i), we find by Proposition 6.17(c) that the associated multiplicity function $\widetilde{m}$ fulfills $\widetilde{m}(h) = m((-1)^i r(\Gamma)^{-1}h)$. Hence, (6.3) for $\Gamma_d, \widetilde{\mathcal{H}}$ becomes

$$m((-1)^i r(\Gamma)^{-1}h)|h|+m((-1)^i r(\Gamma)^{-1}(h-1))|h-1| \leq \frac{1}{d} \ \text{(almost everywhere)}$$

which after dividing both sides by $r(\Gamma)$ and passing to the variable $\widetilde{h} = (-1)^i r(\Gamma)^{-1}h$ is the desired inequality (6.3). □

Proof of Corollary 6.6. The assumptions imply that $m(h)|h| \leq c$, for all $h \in \mathbb{R}'$, and c a constant. Hence picking $s \geq \frac{2c}{d}$ and defining $\Gamma = \alpha_s(\Gamma_d)$ ensures that (6.3) is fulfilled. □

Proof of Corollary 6.7. Pick an admissible vector η and transfer the corresponding results from Theorem 6.4 and Corollary 6.6 to $\mathcal{H}_\pi$ via V_η^{-1}. □

Proof of Corollary 6.8. Pick any measurable function $m : [-1, 1] \to \mathbb{N}'$ such that $h \mapsto m(h)|h|$ is integrable but unbounded. Pick a closed, leftinvariant space $\mathcal{H}$ with multiplicity function m. The space can be constructed by realizing that each representation in the Plancherel decomposition of $\mathbb{H}$ enters with infinite multiplicity, hence the projection field

$$P_h = \sum_{n=1}^{m(h)} e_n(h) \otimes e_n(h)$$

constructed from a measurable field $(e_n)_{n\in\mathbb{N}}$ of ONB's is well-defined and measurable. $\mathcal{H}$ is of the desired form, but violates (6.3), for all lattices Γ. □

Proof of Corollary 6.9. To give an example proving the first statement, let $\Gamma = \Gamma_d$; using the appropriate $\alpha \in \mathrm{Aut}(\mathbb{H})$ the argument can be adapted to suit any other lattice. For $h \in \left[0, \frac{1}{d}\right]$, define

$$\eta^h = \frac{1}{\sqrt{h/2}} \mathbf{1}_{[0,h/2]} \quad .$$

and $S \in \mathrm{L}^2(\mathbb{H})$ with $\widehat{S}(h) = \eta_h \otimes \eta_h$. Then S is a selfadjoint convolution idempotent, and $\mathcal{H} = \mathrm{L}^2(\mathbb{H}) * S$ has a tight frame of the form $\lambda_{\mathbb{H}}(\Gamma)\Phi$. However, for $\mathcal{H}$ to be a sampling space, $\lambda_{\mathbb{H}}(\Gamma)S$ must be a tight frame, and condition (6.5) implies that $\mathcal{G}(h, d, \eta_h)$ is a tight frame of $\mathrm{L}^2(\mathbb{R})$, for almost every h. But $\mathbf{1}_{[h/2,h]}$ has disjoint support with all elements of that system, hence $\mathcal{G}(h, d, \eta_h)$ is not even total.

The second statement is obvious from Proposition 2.54. The last statement follows from Theorem 6.4 (ii). □

6.6 A Concrete Example

In this section we explicitly compute a sinc-type function for $\Gamma = \Gamma_1$. The construction proceeds backwards, starting on the Plancherel transform side by giving a field of rank-one projection operators fulfilling the additional requirements for the sampling space property. Fourier inversion yields the sinc-type function S. As a consequence, the sampling space is given as $\mathrm{L}^2(\mathbb{H}) * S$. In order to minimize tedium, we have shortened some of the more straightforward calculations. The three steps carry out the abstract program developed above.

1. Construction on the Plancherel transform side.

For $h \in [-0.5, 0.5]$ let $\eta_h = |h|^{-1/2} \mathbf{1}_{[-|h|/2,|h|/2]}$, and

$$\widehat{S}(h) = \eta_h \otimes \eta_h \quad ,$$

and let $\widehat{S}$ be zero outside of $[-0.5, 0.5]$. $\widehat{S}$ is a measurable field of rank-one projection operators, with integrable trace, hence has an inverse image $S \in \mathrm{L}^2(\mathbb{H})$ which is a selfadjoint convolution idempotent. Moreover, it is straightforward to check that $\varrho_h(\Gamma_1^r)|h|^{1/2}\eta_h = \varrho_h(\Gamma_1^r)\mathbf{1}_{[-0.5,0.5]}$ is a normalized tight frame of $\mathrm{L}^2(\mathbb{R})$, (compare the proof of Proposition 6.14 (a)). Hence, by Proposition 6.11, $\lambda_{\mathbb{H}}(\Gamma)S$ is a normalized tight frame of $\mathcal{H} = \mathrm{L}^2(\mathbb{H}) * S$, and $\mathcal{H}$ is a sampling space.

2. Plancherel inversion.
An application of the Plancherel inversion formula (4.5) yields

$$\begin{aligned} &S(p,q,t) \\ &= \int_{-0.5}^{0.5} \langle \eta_h, \varrho_h(p,q,t)\eta_h \rangle |h| dh \\ &= \int_{-0.5}^{0.5} \mathrm{e}^{-2\pi i h(t+pq/2)} \int_{-\frac{|h|}{2}}^{\frac{|h|}{2}} \mathrm{e}^{-2\pi i q x} \mathbf{1}_{[-|h|/2,|h|/2]}(x+hp) dx \; dh \quad . \end{aligned} \tag{6.16}$$

3. Computing integrals.
Let $\widetilde{S}(p,q,h)$ denote the inner integral. In the following, we assume that $q \neq 0$ and $p \geq 0$. The missing values will be obtained by taking limits (for $q = 0$) and reflection (for $p < 0$). Observe further that $S(p,q,t) = 0$ for $|p| > 1$, hence we will use $|p| \leq 1$ wherever we may need it. Integration yields

$$\widetilde{S}(p,q,h) = \begin{cases} \dfrac{\mathrm{e}^{2\pi i q|h|/2} - \mathrm{e}^{-2\pi i q(|h|/2 - hp)}}{2\pi i q} & h \geq 0 \\ \dfrac{\mathrm{e}^{2\pi i q(|h|/2 + hp)} - \mathrm{e}^{-2\pi i q|h|/2}}{2\pi i q} & h < 0 \end{cases} \quad ,$$

After plugging this into (6.16) and integrating, straightforward simplifications lead to

$$S(p,q,t) = \frac{1}{2\pi q} \left(\frac{\cos\left(\pi(t + (p-1)q/2)\right) - 1}{\pi(t + (p-1)q/2)} - \frac{\cos\left(\pi(t - (p-1)q/2)\right) - 1}{\pi(t - (p-1)q/2)} \right).$$

In order to further simplify this expression, we use the relation

$$\frac{\cos(\pi\alpha) - 1}{\pi\alpha} = -\frac{\pi\alpha}{2} \mathrm{sinc}^2\left(\frac{\alpha}{2}\right)$$

by which means we finally arrive at

$$\begin{aligned} S(p,q,t) = \frac{1}{4} \Bigg[&\left(\frac{t}{q} + \frac{1-p}{2} \right) \mathrm{sinc}^2 \left(\frac{t}{2} + \frac{1-p}{4} q \right) \\ &- \left(\frac{t}{q} - \frac{1-p}{2} \right) \mathrm{sinc}^2 \left(\frac{t}{2} - \frac{1-p}{4} q \right) \Bigg] \quad . \end{aligned} \tag{6.17}$$

For $p < 0$ we use that $S(p,q,t) = S^*(p,q,t) = S(-p,-q,-t)$. It turns out that replacing p by $|p|$ in (6.17) is the only necessary adjustment for the formula to hold in the general case. Finally, sending q to 0 allows to compute the values $S(p,0,t)$, since S is continuous. The following theorem summarizes our calculations:

Theorem 6.18. *Define* $S \in \mathrm{L}^2(\mathbb{H})$ *by*

$$S(p,q,t) = \begin{cases} 0 & \text{for } |p| > 1 \\ \frac{1}{4}\left[\left(\frac{t}{q} + \frac{1-|p|}{2}\right)\operatorname{sinc}^2\left(\frac{t}{2} + \frac{1-|p|}{4}q\right)\right. & \\ \left. - \left(\frac{t}{q} - \frac{1-|p|}{2}\right)\operatorname{sinc}^2\left(\frac{t}{2} - \frac{1-|p|}{4}q\right)\right] & \text{for } |p| \leq 1, q \neq 0 \\ \frac{1-|p|}{4}(2\operatorname{sinc}(t) - \operatorname{sinc}^2(t/2)) & \text{for } |p| \leq 1, q = 0 \end{cases}$$

Let $\mathcal{H} \subset \mathrm{L}^2(\mathbb{H})$ *be the leftinvariant closed subspace generated by* S, $\mathcal{H} = \mathrm{L}^2(\mathbb{H}) * S$. *Then* $\mathcal{H}$ *is a sampling space for the lattice* Γ_1, *with* $c_{\mathcal{H}} = 1$, *and* S *the associated sinc-type function.* $\lambda_{\mathbb{H}}(\Gamma_1)S$ *is a normalized tight frame, but not an orthonormal basis of* $\mathcal{H}$, *because of* $\|S\|_2 = \frac{1}{2}$.

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