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EMMANUEL HEBEY

LECTURE NOTES

Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities

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Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities

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Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities

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Preface

These notes deal with the theory of Sobolev spaces on Riemannian manifolds. Though Riemannian manifolds are natural extensions of Euclidean space, the naive idea that what is valid for Euclidean space must be valid for manifolds is false. Several surprising phenomena appear when studying Sobolev spaces on manifolds. Questions that are elementary for Euclidean space become challenging and give rise to sophisticated mathematics, where the geometry of the manifold plays a central role. The reader will find many examples of this in the text.

These notes have their origin in a series of lectures given at the Courant Institute of Mathematical Sciences in 1998. For the sake of clarity, I decided to deal only with manifolds without boundary. An appendix concerning manifolds with boundary can be found at the end of these notes. To illustrate some of the results or concepts developed here, I have included some discussions of a special family of PDEs where these results and concepts are used. These PDEs are generalizations of the scalar curvature equation. As is well known, geometric problems often lead to limiting cases of known problems in analysis.

The study of Sobolev spaces on Riemannian manifolds is a field currently undergoing great development. Nevertheless, several important questions still puzzle mathematicians today. While the theory of Sobolev spaces for noncompact manifolds has its origin in the 1970s with the work of Aubin, Cantor, Hoffman, and Spruck, many of the results presented in these lecture notes have been obtained in the 1980s and 1990s. This is also the case for the applications already mentioned to scalar curvature and generalized scalar curvature equations. A substantial part of these notes is devoted to the concept of best constants. This concept appeared very early on to be crucial for solving limiting cases of some partial differential equations. A striking example of this was the major role that best constants played in the Yamabe problem.

These lecture notes are intended to be as self-contained as possible. In particular, it is not assumed that the reader is familiar with differentiable manifolds and Riemannian geometry. The present notes should be accessible to a large audience, including graduate students and specialists of other fields.

The present notes are organized into nine chapters. Chapter 1 is a quick introduction to differential and Riemannian geometry. Chapter 2 deals with the general theory of Sobolev spaces for compact manifolds, while Chapter 3 deals with the general theory of Sobolev spaces for complete, noncompact manifolds. Best constants problems for compact manifolds are discussed in Chapters 4 and 5, while Chapter 6 deals with some special type of Sobolev inequalities under constraints. Best constants problems for complete noncompact manifolds are discussed in Chapter 7. Chapter 8 deals with Euclidean-type Sobolev inequalities. The influence of symmetries on Sobolev embeddings is discussed in Chapter 9. An appendix at the end of these notes briefly discusses the case of manifolds with boundaries.

It is my pleasure to thank my friend Jalal Shatah for encouraging me to write these notes. It is also my pleasure to express my deep thanks to my friends and colleagues Tobias Colding, Zindine Djadli, Olivier Druet, Antoinette Jourdain, Michel Ledoux, Frédéric Robert, and Michel Vaugon for stimulating discussions and valuable comments about the manuscript. Finally, I wish to thank Reeva Goldsmith, Paul Monsour, and Joe Shearer for the wonderful job they did in the preparation of the manuscript.

Emmanuel Hebey
Paris, September 1998

CHAPTER 1

Elements of Riemannian Geometry

The purpose of this chapter is to recall some basic facts concerning Riemannian geometry. Needless to say, for dimension reasons, we are obliged to be succinct and partial. For those who have only a slight acquaintance with Riemannian geometry, we recommend the following books: Chavel [**45**], Do-Carmo [**70**], Gallot-Hulin-Lafontaine [**88**], Hebey [**109**], Jost [**127**], Kobayashi-Nomizu [**136**], Sakai [**171**], and Spivak [**181**]. Of course, many other excellent books on the subject do exist. We mention that Einstein's summation convention is adopted: an index occurring twice in a product is to be summed. This also holds for the rest of the book.

1.1. Smooth Manifolds

Paraphrasing a sentence of Elie Cartan, a manifold is really made of small pieces of Euclidean space. More precisely, let M be a Hausdorff topological space. We say that M is a topological manifold of dimension n if each point of M possesses an open neighborhood that is homeomorphic to some open subset of the Euclidean space $\mathbb{R}^n$. A chart of M is then a couple (Ω, φ) where Ω is an open subset of M, and φ is a homeomorphism of Ω onto some open subset of $\mathbb{R}^n$. For $y \in \Omega$, the coordinates of $\varphi(y)$ in $\mathbb{R}^n$ are said to be the coordinates of y in (Ω, φ). An atlas of M is a collection of charts (Ω_i, φ_i), $i \in I$, such that $M = \bigcup_{i \in I} \Omega_i$. Given $(\Omega_i, \varphi_i)_{i \in I}$ an atlas, the transition functions are

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(\Omega_i \cap \Omega_j) \to \varphi_j(\Omega_i \cap \Omega_j)$$

with the obvious convention that we consider $\varphi_j \circ \varphi_i^{-1}$ if and only if $\Omega_i \cap \Omega_j \neq \emptyset$. The atlas is then said to be of class C^k if the transition functions are of class C^k, and it is said to be C^k-complete if it is not contained in a (strictly) larger atlas of class C^k. As one can easily check, every atlas of class C^k is contained in a unique C^k-complete atlas.

For our purpose, we will always assume in what follows that $k = +\infty$ and that M is connected. One then gets the following definition of a smooth manifold: A smooth manifold M of dimension n is a connected topological manifold M of dimension n together with a C^∞-complete atlas.

Classical examples of smooth manifolds are the Euclidean space $\mathbb{R}^n$ itself, the torus T^n, the unit sphere S^n of $\mathbb{R}^{n+1}$, and the real projective space $\mathbb{P}^n(\mathbb{R})$.

Given M and N two smooth manifolds, and $f : M \to N$ some map from M to N, we say that f is differentiable (or of class C^k) if for any charts (Ω, φ) and $(\tilde{\Omega}, \tilde{\varphi})$ of M and N such that $f(\Omega) \subset \tilde{\Omega}$, the map

$$\tilde{\varphi} \circ f \circ \varphi^{-1} : \varphi(\Omega) \to \tilde{\varphi}(\tilde{\Omega})$$

is differentiable (or of class C^k). In particular, this allows us to define the notion of diffeomorphism and the notion of diffeomorphic manifolds. Independently, one can define the rank $R(f)_x$ of f at some point x of M as the rank of $\tilde{\varphi} \circ f \circ \varphi^{-1}$ at $\varphi(x)$, where (Ω, φ) and $(\tilde{\Omega}, \tilde{\varphi})$ are as above, with the additional property that $x \in \Omega$. This is an intrinsic definition in the sense that it does not depend on the choice of the charts. The map f is then said to be an immersion if, for any $x \in M$, $R(f)_x = m$, where m is the dimension of M, and a submersion if for any $x \in M$, $R(f)_x = n$, where n is the dimension of N. It is said to be an embedding if it is an immersion that realizes a homeomorphism onto its image.

We refer to the above definition of a manifold as the abstract definition of a smooth manifold. Looking carefully to what it says, and to the questions it raises, things appear to be less clear than they may seem at first glance. Given M a connected topological manifold, one can ask if there always exists a structure of smooth manifold on M, and if this structure is unique. Here, uniqueness has to be understood in the following sense: given M a connected topological manifold, and $\mathcal{A}$ a C^∞-complete atlas of M, the smooth structure of M is said to be unique if, for any other C^∞-complete atlas $\tilde{\mathcal{A}}$ of M, the smooth manifolds $(M, \mathcal{A})$ and $(M, \tilde{\mathcal{A}})$ are diffeomorphic. With this definition of uniqueness, the only reasonable definition for that notion, one gets surprising answers to the questions we asked above. From the works of Moïse, developed in the 1950s, one has that up to dimension 3, any topological manifold possesses one, and only one, smooth structure. But starting from dimension 4, one gets that there exist topological manifolds which do not possess smooth structures (this was shown by Freedman in the 1980s), and that there exist topological manifolds which possess many smooth structures. Coming back to the works of Milnor in the 1950s, and to the works of Kervaire and Milnor, one has that S^7 possesses 28 smooth structures, while S^{11} possesses 992 smooth structures! Perhaps more surprising are the consequences of the works of Donaldson and Taubes: While $\mathbb{R}^n$ possesses a unique smooth structure for $n \neq 4$, there exist infinitely many smooth structures on $\mathbb{R}^4$!

Up to now, we have adopted the abstract definition of a manifold. As a surface gives the idea of a two-dimensional manifold, a more concrete approach would have been to define manifolds as submanifolds of Euclidean space. Given M and N two manifolds, one will say that N is a submanifold of M if there exists a smooth embedding $f : N \to M$. According to a well-known result of Whitney, the two approaches (concrete and abstract) are equivalent, at least when dealing with paracompact manifolds, since for any paracompact manifold M of dimension n, there exists a smooth embedding $f : M \to \mathbb{R}^{2n+1}$. In other words, any paracompact (abstract) manifold of dimension n can be seen as a submanifold of some Euclidean space.

Let us now say some words about the tangent space of a manifold. Given M a smooth manifold and $x \in M$, let $\mathcal{F}_x$ be the vector space of functions $f : M \to \mathbb{R}$ which are differentiable at x. For $f \in \mathcal{F}_x$, we say that f is flat at x if for some chart (Ω, φ) of M at x, $D(f \circ \varphi^{-1})_{\varphi(x)} = 0$. Let $\mathcal{N}_x$ be the vector space of such functions. A linear form X on $\mathcal{F}_x$ is then said to be a tangent vector of M at x if $\mathcal{N}_x \subset \operatorname{Ker} X$. We let $T_x(M)$ be the vector space of such tangent vectors. Given

(Ω, φ) some chart at x, of associated coordinates x^i, we define $\left(\frac{\partial}{\partial x_i}\right)_x \in T_x(M)$ by: for any $f \in \mathcal{F}_x$,

$$\left(\frac{\partial}{\partial x_i}\right)_x \cdot (f) = D_i\left(f \circ \varphi^{-1}\right)_{\varphi(x)}$$

As a simple remark, one gets that the $\left(\frac{\partial}{\partial x_i}\right)_x$'s form a basis of $T_x(M)$. Now, one defines the tangent bundle of M as the disjoint union of the $T_x(M)$'s, $x \in M$. If M is n-dimensional, one can show that $T(M)$ possesses a natural structure of a $2n$-dimensional smooth manifold. Given (Ω, φ) a chart of M,

$$\left(\bigcup_{x \in \Omega} T_x(M), \Phi\right)$$

is a chart of $T(M)$, where for $X \in T_x(M)$, $x \in \Omega$,

$$\Phi(X) = \left(\varphi^1(x), \ldots, \varphi^n(x), X(\varphi^1), \ldots, X(\varphi^n)\right)$$

(the coordinates of x in (Ω, φ) and the components of X in (Ω, φ), that is, the coordinates of X in the basis of $T_x(M)$ associated to (Ω, φ) by the process described above). By definition, a vector field on M is a map $X : M \to T(M)$ such that for any $x \in M$, $X(x) \in T_x(M)$. Since M and $T(M)$ are smooth manifolds, the notion of a vector field of class C^k makes sense.

Given M, N two smooth manifolds, x a point of M, and $f : M \to N$ differentiable at x, the tangent linear map of f at x (or the differential map of f at x), denoted by $f_\star(x)$, is the linear map from $T_x(M)$ to $T_{f(x)}(N)$ defined by: For $X \in T_x(M)$ and $g : N \to \mathbb{R}$ differentiable at $f(x)$,

$$\left(f_\star(x) \cdot (X)\right) \cdot (g) = X(g \circ f)$$

By extension, if f is differentiable on M, one gets the tangent linear map of f, denoted by $f_\star$. That is the map $f_\star : T(M) \to T(N)$ defined by: For $X \in T_x(M)$, $f_\star(X) = f_\star(x).(X)$. As one can easily check, $f_\star$ is C^{k-1} if f is C^k. For $f : M_1 \to M_2$, $g : M_2 \to M_3$, and $x \in M_1$, $(g \circ f)_\star(x) = g_\star(f(x)) \circ f_\star(x)$.

Similar to the construction of the tangent bundle, one can define the cotangent bundle of a smooth manifold M. For $x \in M$, let $T_x(M)^\star$ be the dual space of $T_x(M)$. If (Ω, φ) is a chart of M at x of associated coordinates x^i, one gets a basis of $T_x(M)^\star$ by considering the dx^i_x's defined by $dx^i_x \cdot \left(\frac{\partial}{\partial x_j}\right)_x = \delta^i_j$. As for the tangent bundle, the cotangent bundle of M, denoted by $T^\star(M)$, is the disjoint union of the $T_x(M)^\star$'s, $x \in M$. Here again, if M is n-dimensional, $T^\star(M)$ possesses a natural structure of $2n$-dimensional smooth manifold. Given (Ω, φ) a chart of M,

$$\left(\bigcup_{x \in \Omega} T_x(M)^\star, \Phi\right)$$

is a chart of $T(M)$, where for $\eta \in T_x(M)^\star$, $x \in \Omega$,

$$\Phi(\eta) = \left(\varphi^1(x), \ldots, \varphi^n(x), \eta\left(\frac{\partial}{\partial x_1}\right)_x, \ldots, \eta\left(\frac{\partial}{\partial x_n}\right)_x\right)$$

(the coordinates of x in (Ω, φ) and the components of η in (Ω, φ), that is, the coordinates of η in the basis of $T_x(M)^\star$ associated to (Ω, φ) by the process described above). By definition, a 1-form on M is a map $\eta : M \to T^*(M)$ such that for any $x \in M$, $\eta(x) \in T_x(M)^*$. Here again, since M and $T^*(M)$ are smooth manifolds, the notion of a 1-form of class C^k makes sense. For f a function of class C^k on M, let df be defined by: For $x \in M$ and $X \in T_x(M)$, $df(x) \cdot X = X(f)$. Then df is a 1-form of class C^{k-1}.

Given M a smooth n-manifold, and $1 \le q \le n$ an integer, let $\bigwedge^q T_x(M)^*$ be the space of skew-symmetric q-linear forms on $T_x(M)$. If (Ω, φ) is a chart of M at x, of associated coordinates x^i, $\{dx_x^{i_1} \wedge \cdots \wedge dx_x^{i_q}\}_{i_1 < \cdots < i_q}$ is a basis of $\bigwedge^q T_x(M)^*$. With similar constructions to the ones made above, one gets that $\bigwedge^q(M)$, the disjoint union of the $\bigwedge^q T_x(M)^*$'s, possesses a natural structure of a smooth manifold. Its dimension is $n + C_n^q$, where $C_n^q = n!/(q!(n-q)!)$. Some map $\eta : M \to \bigwedge^q(M)$ is then said to be an exterior form of degree q, or just an exterior q-form, if for any $x \in M$, $\eta(x) \in \bigwedge^q T_x(M)^*$. Here again, the notion of an exterior q-form of class C^k makes sense. Given (Ω, φ) some chart of M, and η a q-form of class C^k whose expression in (Ω, φ) is

$$\eta = \sum_{i_1 < \cdots < i_q} \eta_{i_1 \ldots i_q} dx^{i_1} \wedge \cdots \wedge dx^{i_q}$$

the exterior derivative of η, denoted by $d\eta$, is the exterior $(q+1)$-form of class C^{k-1} whose expression in (Ω, φ) is

$$d\eta = \sum_{i_1 < \cdots < i_q} d\eta_{i_1 \ldots i_q} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_q}$$

One then gets that for any exterior q-form η, $d(d\eta) = 0$. Conversely, by the Poincaré lemma, if η is an exterior q-form such that $d\eta = 0$, that is, a closed exterior q-form, around any point in M, there exists an exterior $(q-1)$-form $\tilde{\eta}$ such that $d\tilde{\eta} = \eta$. One says that a closed exterior form is locally exact.

As another generalization, given M a smooth n-manifold, x some point of M, and p, q two integers, one can define $T_p^q(T_x(M))$ as the space of (p,q)-tensors on $T_x(M)$, that is, the space of $(p+q)$-linear forms

$$\eta : \underbrace{T_x(M) \times \cdots \times T_x(M)}_{p} \times \underbrace{T_x(M)^* \times \cdots \times T_x(M)^*}_{q} \longrightarrow \mathbb{R}$$

An element of $T_p^q\big(T_x(M)\big)$ is said to be p-times covariant and q-times contravariant. If (Ω, φ) is a chart of M at x, of associated coordinates x^i, the family

$$\left\{dx_x^{i_1} \otimes \cdots \otimes dx_x^{i_p} \otimes \left(\frac{\partial}{\partial x_{j_1}}\right)_x \otimes \cdots \otimes \left(\frac{\partial}{\partial x_{j_q}}\right)_x\right\}_{i_1, \ldots, i_p, j_1, \ldots, j_q}$$

is a basis of $T_p^q\big(T_x(M)\big)$. Here again, one gets that the disjoint union $T_p^q(M)$ of the $T_p^q\big(T_x(M)\big)$'s possesses a natural structure of a smooth manifold. Its dimension is $n\big(1 + n^{p+q-1}\big)$. A map $T : M \to T_p^q(M)$ is then said to be a (p,q)-tensor field on M if for any $x \in M$, $T(x) \in T_p^q\big(T_x(M)\big)$. It is said to be of class C^k if it is of class C^k from the manifold M to the manifold $T_p^q(M)$. Given (Ω, φ) and (Ω, ψ) two charts of M of associated coordinates x^i and y^i, and T a (p,q)-tensor field, let us denote by $T^{j_1\dots j_q}_{i_1\dots i_p}$ and $\tilde{T}^{j_1\dots j_q}_{i_1\dots i_p}$ its components in (Ω, φ) and (Ω, ψ). Then, for any $i_1, \dots, i_p, j_1, \dots, j_q$, and any $x \in \Omega$,

$$\tilde{T}^{j_1\dots j_q}_{i_1\dots i_p}(x) = T^{\beta_1\dots\beta_q}_{\alpha_1\dots\alpha_p}(x)\left(\frac{\partial x^{\alpha_1}}{\partial y_{i_1}}\right)_x \cdots \left(\frac{\partial x^{\alpha_p}}{\partial y_{i_p}}\right)_x \left(\frac{\partial y^{j_1}}{\partial x_{\beta_1}}\right)_x \cdots \left(\frac{\partial y^{j_q}}{\partial x_{\beta_q}}\right)_x \tag{1.1}$$

As a remark, given M and N two manifolds, $f : M \to N$ a map of class C^{k+1}, and T a $(p, 0)$-tensor field of class C^k on N, one can define the pullback f^*T of T by f, that is, the $(p, 0)$-tensor field of class C^k on M defined by: For $x \in M$ and $X_1, \dots, X_p \in T_x(M)$,

$$(f^*T)(x) \cdot (X_1, \dots, X_p) = T\big(f(x)\big) \cdot \big(f_\star(x) \cdot X_1, \dots, f_\star(x) \cdot X_p\big)$$

As one can easily check, for $f : M_1 \to M_2$ and $g : M_2 \to M_3$, $(g \circ f)^* = f^* \circ g^*$.

Let us now define the notion of a linear connection. Denote by $\Gamma(M)$ the space of differentiable vector fields on M. A linear connection D on M is a map

$$D : T(M) \times \Gamma(M) \to T(M)$$

such that

1. $\forall x \in M, \forall X \in T_x(M), \forall Y \in \Gamma(M), D(X, Y) \in T_x(M)$,
2. $\forall x \in M, D : T_x(M) \times \Gamma(M) \to T_x(M)$ is bilinear,
3. $\forall x \in M$, $\forall X \in T_x(M)$, $\forall f : M \to \mathbb{R}$ differentiable, $\forall Y \in \Gamma(M)$, $D(X, fY) = X(f)Y(x) + f(x)D(X, Y)$, and
4. $\forall X, Y \in \Gamma(M)$, and $\forall k$ integer, if X is of class C^k and Y is of class C^{k+1}, then $D(X, Y)$ is of class C^k, where $D(X, Y)$ is the vector field $x \to D\big(X(x), Y\big)$.

Given D a linear connection, the usual notation for $D(X, Y)$ is $D_X(Y)$. One says that $D_X(Y)$ is the covariant derivative of Y with respect to X. Let (Ω, φ) be a chart of M of associated coordinates x^i. Set

$$\nabla_i = D_{(\frac{\partial}{\partial x_i})}$$

As one can easily check, there exist n^3 smooth functions $\Gamma^k_{ij} : \Omega \to \mathbb{R}$ such that for any i, j, and any $x \in \Omega$,

$$\nabla_i\left(\frac{\partial}{\partial x_j}\right)(x) = \Gamma^k_{ij}(x)\left(\frac{\partial}{\partial x_k}\right)_x$$

Such functions, the Christoffel symbols of D in (Ω, φ), characterize the connection in the sense that for $X \in T_x(M)$, $x \in \Omega$, and $Y \in \Gamma(M)$,

$$D_X(Y) = X^i(\nabla_i Y)(x) = X^i\left(\left(\frac{\partial Y^j}{\partial x_i}\right)_x + \Gamma^j_{i\alpha}(x)Y^\alpha(x)\right)\left(\frac{\partial}{\partial x_j}\right)_x$$

where the X^i's and Y^i's denote the components of X and Y in the chart (Ω, φ), and for $f : M \to \mathbb{R}$ differentiable at x,

$$\left(\frac{\partial f}{\partial x_i}\right)_x = D_i\big(f \circ \varphi^{-1}\big)_{\varphi(x)}$$

As one can easily check, since (1.1) is not satisfied by the Γ^k_{ij}'s, the Γ^k_{ij}'s are not the components of a (2, 1)-tensor field. An important remark is that linear connections have natural extensions to differentiable tensor fields. Given T a differentiable (p, q)-tensor field, x a point of M, $X \in T_x(M)$, and (Ω, φ) a chart of M at x, $D_X(T)$ is the (p, q)-tensor on $T_x(M)$ defined by $D_X(T) = X^i(\nabla_i T)(x)$, where

$$(\nabla_i T)(x)^{j_1\dots j_q}_{i_1\dots i_p} = \left(\frac{\partial T^{j_1\dots j_q}_{i_1\dots i_p}}{\partial x_i}\right)_x - \sum_{k=1}^{p} \Gamma^{\alpha}_{i i_k}(x) T(x)^{j_1\dots j_q}_{i_1\dots i_{k-1}\alpha i_{k+1}\dots i_p}$$
$$+ \sum_{k=1}^{q} \Gamma^{j_k}_{i\alpha}(x) T(x)^{j_1\dots j_{k-1}\alpha j_{k+1}\dots j_q}_{i_1\dots i_p}$$

The covariant derivative commutes with the contraction in the sense that

$$D_X\left(C^{k_2}_{k_1} T\right) = C^{k_2}_{k_1} D_X(T)$$

where $C^{k_2}_{k_1} T$ stands for the contraction of T of order (k_1, k_2). More, for $X \in T_x(M)$, and T and $\tilde{T}$ two differentiable tensor fields, one has that

$$D_X\left(T \otimes \tilde{T}\right) = \left(D_X(T)\right) \otimes \tilde{T}(x) + T(x) \otimes \left(D_X(\tilde{T})\right)$$

Given T a (p, q)-tensor field of class C^{k+1}, we let ∇T be the $(p + 1, q)$-tensor field of class C^k whose components in a chart are given by

$$\left(\nabla T\right)^{j_1\dots j_q}_{i_1\dots i_{p+1}} = \left(\nabla_{i_1} T\right)^{j_1\dots j_q}_{i_2\dots i_{p+1}}$$

By extension, one can then define $\nabla^2 T$, $\nabla^3 T$, and so on. For $f : M \to \mathbb{R}$ a smooth function, one has that $\nabla f = df$ and, in any chart (Ω, φ) of M,

$$(\nabla^2 f)(x)_{ij} = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_x - \Gamma^k_{ij}(x)\left(\frac{\partial f}{\partial x_k}\right)_x$$

where

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_x = D^2_{ij}\left(f \circ \varphi^{-1}\right)_{\varphi(x)}$$

In the Riemannian context, $\nabla^2 f$ is called the Hessian of f and is sometimes denoted by $\mathrm{Hess}(f)$.

Finally, let us define the torsion and the curvature of a linear connection D. The torsion T of D can be seen as the smooth $(2, 1)$-tensor field on M whose components in any chart are given by the relation $T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}$. One says that the connection is torsion-free if $T \equiv 0$. The curvature R of D can be seen as the smooth $(3, 1)$-tensor field on M whose components in any chart are given by the relation

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ki}}{\partial x_j} - \frac{\partial \Gamma^l_{ji}}{\partial x_k} + \Gamma^l_{j\alpha}\Gamma^{\alpha}_{ki} - \Gamma^l_{k\alpha}\Gamma^{\alpha}_{ji}$$

As one can easily check, $R^l_{ijk} = -R^l_{ikj}$. Moreover, when the connection is torsion-free, one has that

$$R^l_{ijk} + R^l_{kij} + R^l_{jki} = 0$$
$$(\nabla_i R)^l_{mjk} + (\nabla_k R)^l_{mij} + (\nabla_j R)^l_{mki} = 0$$

Such relations are referred to as the first Bianchi's identity, and the second Bianchi's identity.

1.2. Riemannian Manifolds

Let M be a smooth manifold. A Riemannian metric g on M is a smooth $(2,0)$-tensor field on M such that for any $x \in M$, $g(x)$ is a scalar product on $T_x(M)$. A smooth Riemannian manifold is a pair (M, g) where M is a smooth manifold and g a Riemannian metric on M. According to Whitney, for any paracompact smooth n-manifold there exists a smooth embedding $f : M \to \mathbb{R}^{2n+1}$. One then gets that any smooth paracompact manifold possesses a Riemannian metric. Just think to $g = f^\star e$, e the Euclidean metric. Two Riemannian manifolds (M_1, g_1) and (M_2, g_2) are said to be isometric if there exists a diffeomorphism $f : M_1 \to M_2$ such that $f^\star g_2 = g_1$.

Given (M, g) a smooth Riemannian manifold, and $\gamma : [a, b] \to M$ a curve of class C^1, the length of γ is

$$L(\gamma) = \int_a^b \sqrt{g(\gamma(t)) \cdot \left(\left(\frac{d\gamma}{dt} \right)_t, \left(\frac{d\gamma}{dt} \right)_t \right)} dt$$

where $(\frac{d\gamma}{dt})_t \in T_{\gamma(t)}(M)$ is such that $(\frac{d\gamma}{dt})_t \cdot f = \left(f \circ \gamma\right)'(t)$ for any $f : M \to \mathbb{R}$ differentiable at $\gamma(t)$. If γ is piecewise C^1, the length of γ may be defined as the sum of the lengths of its C^1 pieces. For x and y in M, let $\mathcal{C}_{xy}$ be the space of piecewise C^1 curves $\gamma : [a, b] \to M$ such that $\gamma(a) = x$ and $\gamma(b) = y$. Then

$$d_g(x, y) = \inf_{\gamma \in \mathcal{C}_{xy}} L(\gamma)$$

defines a distance on M whose topology coincides with the original one of M. In particular, by Stone's theorem, a smooth Riemannian manifold is paracompact. By definition, d_g is the distance associated to g.

Let (M, g) be a smooth Riemannian manifold. There exists a unique torsion-free connection on M having the property that $\nabla g = 0$. Such a connection is the Levi-Civita connection of g. In any chart (Ω, φ) of M, of associated coordinates x^i, and for any $x \in \Omega$, its Christoffel symbols are given by the relations

$$\Gamma_{ij}^k(x) = \frac{1}{2} \left(\left(\frac{\partial g_{mj}}{\partial x_i} \right)_x + \left(\frac{\partial g_{mi}}{\partial x_j} \right)_x - \left(\frac{\partial g_{ij}}{\partial x_m} \right)_x \right) g(x)^{mk}$$

where the g^{ij}'s are such that $g_{im} g^{mj} = \delta_i^j$. Let R be the curvature of the Levi-Civita connection as introduced above. One defines:

1. the *Riemann curvature* $\mathrm{Rm}_{(M,g)}$ of g as the smooth $(4,0)$-tensor field on M whose components in a chart are $R_{ijkl} = g_{i\alpha} R^\alpha_{jkl}$,
2. the *Ricci curvature* $\mathrm{Rc}_{(M,g)}$ of g as the smooth $(2,0)$-tensor field on M whose components in a chart are $R_{ij} = R_{\alpha i \beta j} g^{\alpha\beta}$, and
3. the *scalar curvature* $\mathrm{Scal}_{(M,g)}$ of g as the smooth real-valued function on M whose expression in a chart is $\mathrm{Scal}_{(M,g)} = R_{ij} g^{ij}$.

As one can check, in any chart,

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij}$$

and the two Bianchi identities are

$$\begin{aligned}&R_{ijkl} + R_{iljk} + R_{iklj} = 0\,,\\ &\left(\nabla_i \operatorname{Rm}_{(M,g)}\right)_{jklm} + \left(\nabla_m \operatorname{Rm}_{(M,g)}\right)_{jkil} + \left(\nabla_l \operatorname{Rm}_{(M,g)}\right)_{jkmi} = 0\,.\end{aligned}$$

In particular, the Ricci curvature $\operatorname{Rc}_{(M,g)}$ of g is symmetric, so that in any chart $R_{ij} = R_{ji}$. For $x \in M$, let $G_x^2(M)$ be the 2-Grassmannian of $T_x(M)$. The sectional curvature $K_{(M,g)}$ of g is the real-valued function defined on $\bigcup_{x\in M} G_x^2(M)$ by: For $P \in G_x^2(M)$,

$$K_{(M,g)}(P) = \frac{\operatorname{Rm}_{(M,g)}(x)(X,Y,X,Y)}{g(x)(X,X)g(x)(Y,Y) - g(x)(X,Y)^2}$$

where (X,Y) is a basis of P. As one can easily check, such a definition does not depend on the choice of the basis. Moreover, one can prove that the sectional curvature determines the Riemann curvature.

Given (M,g) a smooth Riemannian manifold, and D its Levi-Civita connection, a smooth curve $\gamma : [a,b] \to M$ is said to be a geodesic if for all t,

$$D_{\left(\frac{d\gamma}{dt}\right)_t}\left(\frac{d\gamma}{dt}\right) = 0$$

This means again that in any chart, and for all k,

$$\left(\gamma^k\right)''(t) + \Gamma_{ij}^k\left(\gamma(t)\right)\left(\gamma^i\right)'(t)\left(\gamma^j\right)'(t) = 0$$

For any $x \in M$, and any $X \in T_x(M)$, there exists a unique geodesic $\gamma : [0,\varepsilon] \to M$ such that $\gamma(0) = x$ and $(\frac{d\gamma}{dt})_0 = X$. Let $\gamma_{x,X}$ be this geodesic. For $\lambda > 0$ real, $\gamma_{x,\lambda X}(t) = \gamma_{x,X}(\lambda t)$. Hence, for $\|X\|$ sufficiently small, where $\|\cdot\|$ stands for the norm in $T_x(M)$ associated to $g(x)$, one has that $\gamma_{x,X}$ is defined on $[0,1]$. The exponential map at x is the map from a neighborhood of 0 in $T_x(M)$, with values in M, defined by $\exp_x(X) = \gamma_{x,X}(1)$. If M is n-dimensional and up to the assimilation of $T_x(M)$ to $\mathbb{R}^n$ via the choice of an orthonormal basis, one gets a chart $\left(\Omega, \exp_x^{-1}\right)$ of M at x. This chart is normal at x in the sense that the components g_{ij} of g in this chart are such that $g_{ij}(x) = \delta_{ij}$, with the additional property that the Christoffel symbols Γ_{ij}^k of the Levi-Civita connection in this chart are such that $\Gamma_{ij}^k(x) = 0$. The coordinates associated to this chart are referred to as geodesic normal coordinates.

Let (M,g) be a smooth Riemannian manifold. The Hopf-Rinow theorem states that the following assertions are equivalent:

1. the metric space (M, d_g) is complete,
2. any closed-bounded subset of M is compact,
3. there exists $x \in M$ for which $\exp_x$ is defined on the whole of $T_x(M)$, and
4. for any $x \in M$, $\exp_x$ is defined on the whole of $T_x(M)$.

Moreover, one gets that any of the above assertions implies that any two points in M can be joined by a minimizing geodesic. Here, a curve γ from x to y is said to be minimizing if $L(\gamma) = d_g(x, y)$.

Given (M, g) a smooth Riemannian n-manifold, one can define a natural positive Radon measure on M. In particular, the theory of the Lebesgue integral can be applied. For $(\Omega_i, \varphi_i)_{i\in I}$ some atlas of M, we shall say that a family $(\Omega_j, \varphi_j, \alpha_j)_{j\in J}$ is a partition of unity subordinate to $(\Omega_i, \varphi_i)_{i\in I}$ if the following holds:

1. $(\alpha_j)_j$ is a smooth partition of unity subordinate to the covering $(\Omega_i)_i$,
2. $(\Omega_j, \varphi_j)_j$ is an atlas of M, and
3. for any j, $\operatorname{supp}\alpha_j \subset \Omega_j$.

As one can easily check, for any atlas $(\Omega_i, \varphi_i)_{i\in I}$ of M, there exists a partition of unity $(\Omega_j, \varphi_j, \alpha_j)_{j\in J}$ subordinate to $(\Omega_i, \varphi_i)_{i\in I}$. One can then define the Riemannian measure as follows: Given $f : M \to \mathbb{R}$ continuous with compact support, and given $(\Omega_i, \varphi_i)_{i\in I}$ an atlas of M,

$$\int_M f\,dv(g) = \sum_{j\in J}\int_{\varphi_j(\Omega_j)}\left(\alpha_j\sqrt{|g|}f\right)\circ\varphi_j^{-1}\,dx$$

where $(\Omega_j, \varphi_j, \alpha_j)_{j\in J}$ is a partition of unity subordinate to $(\Omega_i, \varphi_i)_{i\in I}$, $|g|$ stands for the determinant of the matrix whose elements are the components of g in (Ω_j, φ_j), and dx stands for the Lebesgue volume element of $\mathbb{R}^n$. One can prove that such a construction does not depend on the choice of the atlas $(\Omega_i, \varphi_i)_{i\in I}$ and the partition of unity $(\Omega_j, \varphi_j, \alpha_j)_{j\in J}$.

The Laplacian acting on functions of a smooth Riemannian manifold (M, g) is the operator Δ_g whose expression in a local chart of associated coordinates x^i is

$$\Delta_g u = -g^{ij}\left(\frac{\partial^2 u}{\partial x_i\partial x_j} - \Gamma_{ij}^k\frac{\partial u}{\partial x_k}\right)$$

For u and v of class C^2 on M, on then has the following integration by parts formula

$$\int_M (\Delta_g u)v\,dv(g) = \int_M \langle\nabla u, \nabla v\rangle dv(g) = \int_M u(\Delta_g v)dv(g)$$

where $\langle\cdot, \cdot\rangle$ is the scalar product associated with g for 1-forms.

Coming back to geodesics, one can define the injectivity radius of (M, g) at some point x, denoted by $\operatorname{inj}_{(M,g)}(x)$, as the largest positive real number r for which any geodesic starting from x and of length less than r is minimizing. One can then define the (global) injectivity radius by

$$\operatorname{inj}_{(M,g)} = \inf_{x\in M}\operatorname{inj}_{(M,g)}(x)$$

One has that $\operatorname{inj}_{(M,g)} > 0$ for a compact manifold, but it may be zero for a complete noncompact manifold. More generally, one can define the cut locus $\operatorname{Cut}(x)$ of x as a subset of M and prove that $\operatorname{Cut}(x)$ has measure zero, that $\operatorname{inj}_{(M,g)}(x) = d_g(x, \operatorname{Cut}(x))$, and that $\exp_x$ is a diffeomorphism from some star-shaped domain of $T_x(M)$ at 0 onto $M\setminus\operatorname{Cut}(x)$. In particular, one gets that the distance function r

to a given point is differentiable almost everywhere, with the additional property that $|\nabla r| = 1$ almost everywhere.

1.3. Curvature and Topology

As is well-known, curvature assumptions may give topological and diffeomorphic information on the manifold. A striking example of the relationship that exists between curvature and topology is given by the Gauss-Bonnet theorem, whose present form is actually due to the works of Allendoerfer [2], Allendoerfer-Weil [3], Chern [49], and Fenchel [81]. One has here that the Euler-Poincaré characteristic $\chi(M)$ of a compact manifold can be expressed as the integral of a universal polynomial in the curvature. For instance, when the dimension of M is 2,

$$\chi(M) = \frac{1}{4\pi}\int_M \mathrm{Scal}_{(M,g)}\, dv(g)$$

and when the dimension of M is 4, as shown by Avez [15],

$$\chi(M) = \frac{1}{16\pi^2}\int_M \left(\frac{1}{2}|\mathrm{Weyl}_{(M,g)}|^2 + \frac{1}{12}\mathrm{Scal}^2_{(M,g)} - |E_{(M,g)}|^2\right) dv(g)$$

where $|\cdot|$ stands for the norm associated to g for tensors, and where $\mathrm{Weyl}_{(M,g)}$ and $E_{(M,g)}$ are, respectively, the Weyl tensor of g and the traceless Ricci tensor of g. In a local chart, the components of $\mathrm{Weyl}_{(M,g)}$ are

$$\begin{aligned} W_{ijkl} = R_{ijkl} &- \frac{1}{n-2}\left(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il}\right) \\ &+ \frac{\mathrm{Scal}_{(M,g)}}{(n-1)(n-2)}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right) \end{aligned}$$

where n stands for the dimension of the manifold. As another striking example of the relationship that exists between curvature and topology, one can refer to Myer's theorem (see, for instance, [88]). This theorem states that a smooth, complete Riemannian n-manifold (M, g) whose Ricci curvature satisfies

$$\mathrm{Rc}_{(M,g)} \geq (n-1)k^2 g$$

as bilinear forms, and for some $k > 0$ real, must be compact, with the additional property that its diameter $\mathrm{diam}_{(M,g)}$ is less than or equal to $\frac{\pi}{k}$. Moreover, by Hamilton's work [99], any 3-dimensional, compact, simply connected Riemannian manifold of positive Ricci curvature must be diffeomorphic to the unit sphere S^3 of $\mathbb{R}^4$. Conversely, by recent results of Lohkamp [153], negative sign assumptions on the Ricci curvature have no effect on the topology, since any compact manifold possesses a Riemannian metric of negative Ricci curvature. This does not hold anymore when dealing with sectional curvature. By the Cartan-Hadamard theorem (see, for instance, [88]), one has that any complete, simply connected, n-dimensional Riemannian manifold of nonpositive sectional curvature is diffeomorphic to $\mathbb{R}^n$.

As other examples of the relationship that exists between curvature and topology, let us mention the well-known sphere theorem of Berger [26], Klingenberg

[**133**, **134**], Rauch [**168**], and Tsukamoto [**187**]. Given (M, g) some smooth, compact, simply connected Riemannian n-manifold (M, g), and δ, Δ two positive real numbers, if the sectional curvature of g is such that $\delta \le K_{(M,g)} \le \Delta$, and if $\frac{\delta}{\Delta} > \frac{1}{4}$, then M is homeomorphic to the unit sphere S^n of $\mathbb{R}^{n+1}$. Moreover, as shown, for instance, by Im-Hoff and Ruh [**125**], one gets the existence of diffeomorphisms provided that $\frac{\delta}{\Delta} > \alpha_n$ for some $\alpha_n \in (\frac{1}{4}, 1)$ sufficiently close to 1.

1.4. From Local to Global Analysis

We prove here a packing lemma that will be used many times in the sequel. Such a lemma was first proved by Calabi (unpublished) under the assumptions that the sectional curvature of the manifold is bounded and that the injectivity radius of the manifold is positive (see Aubin [**8**] and Cantor [**37**]). By Croke's result [**59**] it was then possible to replace the assumption on the sectional curvature by a lower bound on the Ricci curvature. Finally, by an ingenious use of Gromov's theorem, Theorem 1.1 below, one obtains the result under the more general form of Lemma 1.1. When we discuss Sobolev inequalities on complete manifolds, this lemma will be an important tool in the process of passing from local to global inequalities.

As a starting point, we mention the following result, generally referred to as Gromov's volume comparison theorem. Under the present form, it is actually due to Bishop and Gromov. We refer the reader to the excellent references Chavel [**45**] and Gallot-Hulin-Lafontaine [**88**] for details on the proof of this theorem.

THEOREM 1.1 *Let (M, g) be a smooth, complete Riemannian n-manifold whose Ricci curvature satisfies $\mathrm{Rc}_{(M,g)} \ge (n-1)kg$ as bilinear forms, for some k real. Then, for any $0 < r < R$ and any $x \in M$,*

$$\mathrm{Vol}_g\big(B_x(R)\big) \le \frac{V_k(R)}{V_k(r)} \mathrm{Vol}_g\big(B_x(r)\big)$$

where $\mathrm{Vol}_g\big(B_x(t)\big)$ denotes the volume of the geodesic ball of center x and radius t, and where $V_k(t)$ denotes the volume of a ball of radius t in the complete simply connected Riemannian n-manifold of constant curvature k. In particular, for any $r > 0$ and any $x \in M$, $\mathrm{Vol}_g\big(B_x(r)\big) \le V_k(r)$.

As a remark, let b_n be the volume of the Euclidean ball of radius one. It is well-known (see, for instance, [**88**]), that for any $t > 0$,

$$V_{-1}(t) = nb_n \int_0^t (\sinh s)^{n-1}\, ds$$

where, according to the notation of Theorem 1.1, $V_{-1}(t)$ denotes the volume of a ball of radius t in the simply connected hyperbolic space of dimension n. It is then easy to prove that for any $k \ge 0$ and any $t > 0$,

$$b_n t^n \le V_{-k}(t) \le b_n t^n e^{(n-1)\sqrt{k}t}$$

One just has to note here that for $s \ge 0$, $s \le \sinh s \le se^s$, and that if $g' = \alpha^2 g$ are Riemannian metrics on a n-manifold M, where α is some positive real number, then for any $x \in M$ and any $t > 0$,

$$\mathrm{Vol}_{g'}\big(B'_x(t)\big) = \alpha^n \,\mathrm{Vol}_g\big(B_x(t/\alpha)\big)$$

As a consequence, by Theorem 1.1 and what we just said, we get that if (M, g) is a complete Riemannian n-manifold whose Ricci curvature satisfies $\mathrm{Rc}_{(M,g)} \geq kg$ for some k real, then for any $x \in M$ and any $0 < r < R$,

$$\mathrm{Vol}_g\big(B_x(R)\big) \leq e^{\sqrt{(n-1)|k|}R}\left(\frac{R}{r}\right)^n \mathrm{Vol}_g\big(B_x(r)\big)$$

Such an explicit inequality will be used occasionally in the sequel.

Given (M, g) a Riemannian manifold, we say that a family (Ω_k) of open subsets of M is a uniformly locally finite covering of M if the following holds: (Ω_k) is a covering of M, and there exists an integer N such that each point $x \in M$ has a neighborhood which intersects at most N of the Ω_k's. One then has the following result:

LEMMA 1.1 *Let (M, g) be a smooth, complete Riemannian n-manifold with Ricci curvature bounded from below by some k real, and let $\rho > 0$ be given. There exists a sequence (x_i) of points of M such that for any $r \geq \rho$:*

(i) *the family $\big(B_{x_i}(r)\big)$ is a uniformly locally finite covering of M, and there is an upper bound for N in terms of n, ρ, r, and k*

(ii) *for any $i \neq j$, $B_{x_i}(\frac{\rho}{2}) \cap B_{x_j}(\frac{\rho}{2}) = \emptyset$*

where, for $x \in M$ and $r > 0$, $B_x(r)$ stands for the geodesic ball of center x and radius r.

PROOF: By Theorem 1.1 and the remark following this theorem, for any $x \in M$ and any $0 < r < R$,

$$\mathrm{Vol}_g\big(B_x(r)\big) \geq e^{-\sqrt{(n-1)|k|}R}\left(\frac{r}{R}\right)^n \mathrm{Vol}_g\big(B_x(R)\big) \tag{1.2}$$

Independently, we claim there exists a sequence (x_i) of points of M such that

$$M = \bigcup_i B_{x_i}(\rho) \tag{1.3}$$

$$\forall i \neq j,\ B_{x_i}\left(\frac{\rho}{2}\right) \cap B_{x_j}\left(\frac{\rho}{2}\right) = \emptyset \tag{1.4}$$

Let

$$X_\rho = \big\{(x_i)_I,\ x_i \in M,\ \text{s.t. } I \text{ is countable and } \forall i \neq j,\ d_g(x_i, x_j) \geq \rho\big\}$$

where d_g is the Riemannian distance associated to g. As one can easily check, X_ρ is partially ordered by inclusion and every chain in X_ρ has an upper bound. Hence, by Zorn's lemma, X_ρ contains a maximal element (x_i), and (x_i) satisfies (1.3) and (1.4). This proves the above claim. From now on, let (x_i) be such that (1.3) and (1.4) are satisfied. For $r > 0$ and $x \in M$ we define

$$I_r(x) = \big\{i \text{ s.t. } x \in B_{x_i}(r)\big\}$$

By (1.2) we get that for $r \geq \rho$

$$\begin{aligned}\mathrm{Vol}_g\big(B_x(r)\big) &\geq \frac{1}{2^n}e^{-2\sqrt{(n-1)|k|}r}\,\mathrm{Vol}_g\big(B_x(2r)\big)\\ &\geq \frac{1}{2^n}e^{-2\sqrt{(n-1)|k|}r}\sum_{i\in I_r(x)}\mathrm{Vol}_g\left(B_{x_i}\left(\frac{\rho}{2}\right)\right)\end{aligned}$$

since

$$\bigcup_{i\in I_r(x)} B_{x_i}\left(\frac{\rho}{2}\right)\subset B_x(2r)$$

$$B_{x_i}\left(\frac{\rho}{2}\right)\cap B_{x_j}\left(\frac{\rho}{2}\right)=\emptyset \quad \text{if } i\neq j$$

But, again by (1.2),

$$\mathrm{Vol}_g\big(B_{x_i}(\rho/2)\big)\geq e^{-2\sqrt{(n-1)|k|}r}\left(\frac{\rho}{4r}\right)^n\mathrm{Vol}_g\big(B_{x_i}(2r)\big)$$

and since for any $i\in I_r(x)$, $B_x(r)\subset B_{x_i}(2r)$, we get that

$$\mathrm{Vol}_g\big(B_x(r)\big)\geq\left(\frac{\rho}{8r}\right)^n e^{-4\sqrt{(n-1)|k|}r}\,\mathrm{Card}\,I_r(x)\,\mathrm{Vol}_g\big(B_x(r)\big)$$

where Card stands for the cardinality. As a consequence, for any $r\geq\rho$ there exists $C=C(n,\rho,r,k)$ such that for any $x\in M$, $\mathrm{Card}\,I_r(x)\leq C$. Now, let $B_{x_i}(r)$ be given, $r\geq\rho$, and suppose that N balls $B_{x_j}(r)$ have a nonempty intersection with $B_{x_i}(r)$, $j\neq i$. Then, obviously, $\mathrm{Card}\,I_{2r}(x_i)\geq N+1$. Hence,

$$N\leq C(n,\rho,2r,k)-1$$

and this proves the lemma. □

1.5. Special Coordinates

Given (M,g) a smooth Riemannian manifold, some chart (Ω,φ) of M of associated coordinates x^i is said to be harmonic if for any i, $\Delta_g x^i=0$, where Δ_g is the Laplacian of g. As one can easily check from the expression of Δ_g, this means again that for any k, $g^{ij}\Gamma_{ij}^k=0$, where the Γ_{ij}^k's stand for the Christoffel symbols of the Levi-Civita connection in the chart. A simple assertion to prove is that for any x in M, there exists a harmonic chart (Ω,φ) at x. This comes from the classical fact that there always exists a smooth solution of $\Delta_g u=0$ with $u(x)$ and $\partial_i u(x)$ prescribed. The solutions y^j of

$$\begin{cases}\Delta_g y^j=0\\ y^j(x)=0\\ \partial_i y^j(x)=\delta_i^j\end{cases}$$

are then the desired harmonic coordinates. Furthermore, since composing with linear transformations do not affect the fact that coordinates are harmonic, one easily sees that we can choose the harmonic coordinate system such that $g_{ij}(x)=\delta_{ij}$ for any i,j.

A key idea when dealing with harmonic coordinates, as first noticed by Lanczos **[139]**, is that they simplify the formula for the Ricci tensor. In harmonic coordinates, one has that

$$R_{ij} = -\frac{1}{2} g^{\alpha\beta} \frac{\partial^2 g_{ij}}{\partial x_\alpha \partial x_\beta} + \cdots$$

where the dots indicate lower-order terms involving at most one derivative of the metric. Very nice results based on such a formula can be found in DeTurck-Kazdan **[63]**.

For our purpose, let us now define the concept of harmonic radius.

DEFINITION 1.1 Let (M, g) be a smooth Riemannian n-manifold and let $x \in M$. Given $Q > 1$, $k \in \mathbb{N}$, and $\alpha \in (0, 1)$, we define the $C^{k,\alpha}$ harmonic radius at x as the largest number $r_H = r_H(Q, k, \alpha)(x)$ such that on the geodesic ball $B_x(r_H)$ of center x and radius r_H, there is a harmonic coordinate chart such that the metric tensor is $C^{k,\alpha}$ controlled in these coordinates. Namely, if g_{ij}, $i, j = 1, \dots, n$, are the components of g in these coordinates, then

1. $Q^{-1}\delta_{ij} \le g_{ij} \le Q\delta_{ij}$ as bilinear forms
2. $\displaystyle\sum_{1 \le |\beta| \le k} r_H^{|\beta|} \sup_y |\partial_\beta g_{ij}(y)| + \sum_{|\beta| = k} r_H^{k+\alpha} \sup_{y \ne z} \frac{|\partial_\beta g_{ij}(z) - \partial_\beta g_{ij}(y)|}{d_g(y, z)^\alpha} \le Q - 1$

where d_g is the distance associated to g. We now define the (global) harmonic radius $r_H(Q, k, \alpha)(M)$ of (M, g) by

$$r_H(Q, k, \alpha)(M) = \inf_{x \in M} r_H(Q, k, \alpha)(x)$$

where $r_H(Q, k, \alpha)(x)$ is as above.

As one can easily check, the function

$$x \to r_H(Q, k, \alpha)(x)$$

is 1-Lipschitz on M, since by definition, for any $x, y \in M$,

$$r_H(Q, k, \alpha)(y) \ge r_H(Q, k, \alpha)(x) - d_g(x, y)$$

One then gets that the harmonic radius is positive for any fixed, smooth, compact Riemannian manifold. The purpose of Theorem 1.2 below is to show that one obtains lower bounds on the harmonic radius in terms of bounds on the Ricci curvature and the injectivity radius. Roughly speaking, when changing from geodesic normal coordinates to harmonic coordinates, one controls the components of the metric in terms of the Ricci curvature instead of the whole Riemann curvature. As it is stated below, Theorem 1.2 can be found in the survey article of Hebey-Herzlich **[111]**. For original references, we refer to Anderson **[5]**, Anderson-Cheeger **[6]**, and also to Jost-Karcher **[128]**. Concerning its proof, let us just say that the general idea is to construct a sequence of Riemannian n-manifolds with harmonic radius less than or equal to 1 to prove that such a sequence converges to the Euclidean space $\mathbb{R}^n$, and to get the contradiction by noting that this would imply that the harmonic radius of $\mathbb{R}^n$ is less than or equal to 1. (Obviously, $\mathbb{R}^n$ has an infinite harmonic radius). Key steps in such a proof are the above formula for the Ricci tensor in harmonic coordinates, and properties of the harmonic radius when passing to the limit in a converging sequence of metrics.

THEOREM 1.2 *Let $\alpha \in (0,1)$, $Q > 1$, $\delta > 0$. Let (M, g) be a smooth Riemannian n-manifold, and Ω an open subset of M. Set*

$$\Omega(\delta) = \left\{x \in M \text{ s.t. } d_g(x, \Omega) < \delta\right\}$$

where d_g is the distance associated to g. Suppose that for some λ real and some $i > 0$ real, we have that for all $x \in \Omega(\delta)$,

$$\mathrm{Rc}_{(M,g)}(x) \geq \lambda g(x) \quad \textit{and} \quad \mathrm{inj}_{(M,g)}(x) \geq i$$

Then there exists a positive constant $C = C(n, Q, \alpha, \delta, i, \lambda)$, depending only on n, Q, α, δ, i, and λ, such that for any $x \in \Omega$, $r_H(Q, 0, \alpha)(x) \geq C$. In addition, if instead of the bound $\mathrm{Rc}_{(M,g)}(x) \geq \lambda g(x)$ we assume that for some k integer, and some positive constants $C(j)$,

$$|\nabla^j \mathrm{Rc}_{(M,g)}(x)| \leq C(j) \quad \textit{for all } j = 0, \ldots, k \textit{ and all } x \in \Omega(\delta)$$

then, there exists a positive constant $C = C(n, Q, k, \alpha, \delta, i, C(j)_{0 \leq j \leq k})$, depending only on n, Q, k, α, δ, i, and the $C(j)$'s, $0 \leq j \leq k$, such that for any $x \in \Omega$, $r_H(Q, k+1, \alpha)(x) \geq C$.

Let (M, g) be a smooth, complete Riemannian n-manifold, $\alpha \in (0, 1)$ real, and $Q > 1$ real. Suppose that for λ real and some $i > 0$ real,

$$\mathrm{Rc}_{(M,g)} \geq \lambda g \quad \text{and} \quad \mathrm{inj}_{(M,g)} \geq i$$

on M. As an immediate consequence of Theorem 1.2, one gets that there exists a positive constant $C = C(n, Q, \alpha, i, \lambda)$, depending only on n, Q, α, i, and λ, such that the (global) harmonic radius of (M, g) satisfies $r_H(Q, 0, \alpha)(M) \geq C$. Similarly, if instead of the bound $\mathrm{Rc}_{(M,g)} \geq \lambda g$ we assume that for some k integer and some positive constants $C(j)$,

$$|\nabla^j \mathrm{Rc}_{(M,g)}| \leq C(j) \quad \text{for all } j = 0, \ldots, k$$

then there exists a positive constant $C = C(n, Q, k, \alpha, i, C(j)_{0 \leq j \leq k})$, depending only on n, Q, k, α, i, and the $C(j)$'s, $0 \leq j \leq k$, such that the (global) harmonic radius of (M, g) satisfies $r_H(Q, k+1, \alpha)(M) \geq C$.

Coming back to geodesic normal coordinates, analogous estimates to those of Theorem 1.2 are available. Such estimates are rougher. On the one hand, they involve the Riemann curvature instead of the Ricci curvature. On the other hand, one recovers the type of phenomena that was illustrated by DeTurck-Kazdan [**63**]: Changing from harmonic coordinates to geodesic normal coordinates involves loss of derivatives. Nevertheless, such results are sometimes useful, because of special properties that geodesic normal coordinates have with respect to harmonic coordinates. For the sake of clarity, when dealing with geodesic normal coordinates, we will restrict ourselves to the following result, as it appeared in Hebey-Vaugon [**117**].

THEOREM 1.3 *Let (M, g) be a smooth Riemannian n-manifold. Suppose that for some point $x \in M$ there exist positive constants Λ_1 and Λ_2 such that*

$$|\mathrm{Rm}_{(M,g)}| \leq \Lambda_1 \quad \textit{and} \quad |\nabla \mathrm{Rm}_{(M,g)}| \leq \Lambda_2$$

on the geodesic ball $B_x\big(\mathrm{inj}_{(M,g)}(x)\big)$ of center x and radius $\mathrm{inj}_{(M,g)}(x)$. Then there exist positive constants $K = K(n, \Lambda_1, \Lambda_2)$ and $\delta = \delta(n, \Lambda_1, \Lambda_2)$, depending only on n, Λ_1, and Λ_2, such that the components g_{ij} of g in geodesic normal coordinates at x satisfy: For any $i, j, k = 1, \dots, n$ and any $y \in B_0(\min(\delta, \mathrm{inj}_{(M,g)}(x)))$,

(i) $\frac{1}{4}\delta_{ij} \leq g_{ij}\big(\exp_x(y)\big) \leq 4\delta_{ij}$ *(as bilinear forms) and*
(ii) $|g_{ij}\big(\exp_x(y)\big) - \delta_{ij}| \leq K|y|^2$ *and* $|\partial_k g_{ij}\big(\exp_x(y)\big)| \leq K|y|$

where for $t > 0$, $B_0(t)$ denotes the Euclidean ball of $\mathbb{R}^n$ with center 0 and radius t, and $|y|$ is the Euclidean distance from 0 to y. In addition, one has that

$$\lim_{\Lambda \to 0} \delta(n, \Lambda_1, \Lambda_2) = +\infty \quad \textit{and} \quad \lim_{\Lambda \to 0} K(n, \Lambda_1, \Lambda_2) = 0$$

where $\Lambda = (\Lambda_1, \Lambda_2)$.

PROOF: Let B be the Euclidean ball of $\mathbb{R}^n$ of radius $\mathrm{inj}_{(M,g)}(x)$ and centered at 0. We still denote by g the metric when transported on B by $\exp_x$. Let S be a segment in B joining 0 to some point P on ∂B. Then, S is a geodesic for g. Let $(\rho, \theta_1, \dots, \theta_{n-1})$ be a polar coordinate system defined in a neighborhood of S, and let $Q \in S^{n-1}$, the unit sphere of $\mathbb{R}^n$, be such that $\overrightarrow{OQ} = \lambda \overrightarrow{OP}$ for some $\lambda > 0$. We choose $(\theta_1, \dots, \theta_{n-1})$ such that it is a normal coordinate system at Q for the standard metric of S^{n-1}. By the Gauss lemma (see, for instance, [**88**]),

$$g = d\rho^2 + \rho^2 h_{ij}(\rho, \theta) d\theta^i d\theta^j$$

We let $g_{ij} = \rho^2 h_{ij}$. It is then easy to see that for any i, j,

$$R_{i\rho j\rho} = -\frac{1}{2}\partial_\rho \partial_\rho g_{ij} + \frac{1}{4} g^{\alpha\beta} \partial_\rho g_{\alpha i} \partial_\rho g_{\beta j} \tag{1.5}$$

where obvious notation is used in this relation. Independently (see, for instance, [**12**]), there exist positive constants $\delta_1(n, \Lambda_1)$ and $C_1(n, \Lambda_1)$, satisfying

$$\begin{cases} \lim_{\Lambda_1 \to 0} \delta_1(n, \Lambda_1) = +\infty \\ \lim_{\Lambda_1 \to 0} C_1(n, \Lambda_1) = 0 \end{cases}$$

and such that for any $\rho < \delta_1(n, \Lambda_1)$, and any i, j,

$$\left|\partial_\rho h_{ij}\right| \leq C_1(n, \Lambda_1)\rho \tag{1.6}$$

Since, when passing to the limit along S, $h_{ij}(0) = \delta_{ij}$, we get that for any $\rho < \delta_1(n, \Lambda_1)$, and any i, j,

$$\left|h_{ij} - \delta_{ij}\right| \leq C_1(n, \Lambda_1)\rho^2 \tag{1.7}$$

on S. There exists then a positive constant

$$\delta_2(n, \Lambda_1) = \min\Big(\delta_1(n, \Lambda_1), \big(2C_1(n, \Lambda_1)\big)^{-\frac{1}{2}}\Big)$$

satisfying

$$\begin{cases} \delta_2(n, \Lambda_1) \le \delta_1(n, \Lambda_1) \\ \lim_{\Lambda_1 \to 0} \delta_2(n, \Lambda_1) = +\infty \end{cases}$$

and such that for any $\rho < \delta_2(n, \Lambda_1)$,

$$\frac{1}{2}\delta_{ij} \le \left(1 - \frac{n}{2}C_1(n, \Lambda_1)\rho^2\right)\delta_{ij} \le h_{ij} \le \left(1 + \frac{n}{2}C_1(n, \Lambda_1)\rho^2\right)\delta_{ij} \le \frac{3}{2}\delta_{ij}$$

as bilinear forms, and on S. Independently, it is easy to see that there exists a positive constant A such that for any i, j, k, $|\partial_\rho \partial_k g_{ij}| \le A\rho^3$ on S. Hence, for any i, j, k, $|\partial_k g_{ij}| \le (A/4)\rho^4$ on S. In the following, we show that A can be chosen such that it depends only on n, Λ_1, and Λ_2. First, by the derivation of (1.5), we get that for any i, j, k,

$$\partial_k R_{i\rho j\rho} = -\frac{1}{2}\partial_\rho \partial_\rho \partial_k g_{ij} + \frac{1}{4}\partial_k\left(g^{\alpha\beta}\partial_\rho g_{\alpha i}\partial_\rho g_{\beta j}\right) \tag{1.8}$$

Independently, since $|\mathrm{Rm}_{(M,g)}| \le \Lambda_1$ and $|\nabla \mathrm{Rm}_{(M,g)}| \le \Lambda_2$, we get that there exist positive constants $\delta_3(n, \Lambda_1, \Lambda_2) \le \delta_2(n, \Lambda_1)$, $C_2(n, \Lambda_1)$, and $C_3(n, \Lambda_1)$, such that

$$\begin{cases} \lim_{\Lambda \to 0} \delta_3(n, \Lambda_1, \Lambda_2) = +\infty \\ \lim_{\Lambda_1 \to 0} C_2(n, \Lambda_1) = 0 \\ \lim_{\Lambda_1 \to 0} C_3(n, \Lambda_1) = 0 \end{cases}$$

and such that for any $\rho < \delta_3(n, \Lambda_1, \Lambda_2)$, and any i, j, k,

$$\left|\partial_k R_{i\rho j\rho}\right| \le C_2(n, \Lambda_1)\rho^2 + C_3(n, \Lambda_1)A\rho^4$$

on S, where $\Lambda = (\Lambda_1, \Lambda_2)$. On the other hand, it is possible to prove that there exist positive constants $\delta_4(n, \Lambda_1, \Lambda_2) \le \delta_3(n, \Lambda_1, \Lambda_2)$ and $C_4(n, \Lambda_1)$ such that

$$\begin{cases} \lim_{\Lambda \to 0} \delta_4(n, \Lambda_1, \Lambda_2) = +\infty \\ \lim_{\Lambda_1 \to 0} C_4(n, \Lambda_1) = 0 \end{cases}$$

and such that for any $\rho < \delta_4(n, \Lambda_1, \Lambda_2)$, and any i, j, k,

$$\left|\partial_k\left(g^{\alpha\beta}\partial_\rho g_{\alpha i}\partial_\rho g_{\beta j}\right)\right| \le 5A\rho^2 + C_4(n, \Lambda_1)A\rho^4$$

on S. Now, combining these estimates with (1.8), we get that there exist positive constants $C_5(n, \Lambda_1)$ and $C_6(n, \Lambda_1)$, such that

$$\begin{cases} \lim_{\Lambda_1 \to 0} C_5(n, \Lambda_1) = 0 \\ \lim_{\Lambda_1 \to 0} C_6(n, \Lambda_1) = 0 \end{cases}$$

and such that for any $\rho < \delta_4(n, \Lambda_1, \Lambda_2)$, and any i, j, k,

$$\left|\partial_\rho \partial_\rho \partial_k g_{ij}\right| \le \frac{5}{2}A\rho^2 + C_5(n, \Lambda_1)\rho^2 + C_6(n, \Lambda_1)A\rho^4$$

on S. Hence,

$$\left|\partial_\rho \partial_k g_{ij}\right| \le \frac{5}{6}A\rho^3 + \frac{1}{3}C_5(n, \Lambda_1)\rho^3 + \frac{1}{5}C_6(n, \Lambda_1)A\rho^5 \quad \text{on } S$$

and there exist positive constants $\delta_5(n, \Lambda_1, \Lambda_2) \le \delta_4(n, \Lambda_1, \Lambda_2)$, and $C_7(n, \Lambda_1)$, such that

$$\begin{cases} \lim_{\Lambda \to 0} \delta_5(n, \Lambda_1, \Lambda_2) = +\infty \\ \lim_{\Lambda_1 \to 0} C_7(n, \Lambda_1) = 0 \end{cases}$$

and such that for any $\rho < \delta_5(n, \Lambda_1, \Lambda_2)$, and any i, j, k,

$$\left|\partial_\rho \partial_k g_{ij}\right| \le \frac{6}{7} A\rho^3 + C_7(n, \Lambda_1)\rho^3$$

on S. Therefore, by induction, we get that for any $\rho < \delta_5(n, \Lambda_1, \Lambda_2)$, and any i, j, k,

$$\left|\partial_\rho \partial_k g_{ij}\right| \le 7C_7(n, \Lambda_1)\rho^3$$

on S. As a consequence, for any $\rho < \delta_5(n, \Lambda_1, \Lambda_2)$, and any i, j, k,

$$\left|\partial_k h_{ij}\right| \le \frac{7}{4} C_7(n, \Lambda_1)\rho^2 \tag{1.9}$$

on S. Rewriting the inequalities (1.6), (1.7), and (1.9) in the Euclidean coordinate system of $\mathbb{R}^n$ ends the proof of the result. □

CHAPTER 2

Sobolev Spaces: The Compact Setting

We start in this chapter with the theory of Sobolev spaces on Riemannian manifolds. Section 2.1 recalls some elementary facts about Sobolev spaces for open subsets of the Euclidean space. Section 2.2 introduces Sobolev spaces on Riemannian manifolds. Here, in these sections, the compactness of the manifold is not assumed to hold. In Section 2.3, we start dealing with Sobolev embeddings and Sobolev inequalities. General results are proved there. Here again, the compactness of the manifold is not assumed to hold. In Section 2.4, we present the proof of Gagliardo [**85**] and Nirenberg [**162**] on what concerns the validity of Sobolev embeddings for Euclidean space. Sections 2.5 and 2.6 deal with the validity of such embeddings and such inequalities for compact manifolds, while Section 2.7 deals with the compactness of these embeddings, still for compact manifolds. We discuss in Section 2.8 the so-called Poincaré and Sobolev-Poincaré inequalities. A finiteness theorem is proved in Section 2.9.

2.1. Background Material

Let Ω be an open subset of $\mathbb{R}^n$, α a multi-index of length $|\alpha|$, and $u \in L^1_{loc}(\Omega)$ a locally integrable, real-valued function on Ω. A function $v_\alpha \in L^1_{loc}(\Omega)$ is said to be the α^{th} weak (or distributional) derivative of u; we write $v_\alpha = D_\alpha u$, if, for any $\varphi \in \mathcal{D}(\Omega)$,

$$\int_\Omega u(D_\alpha \varphi)dx = \int_\Omega v_\alpha \varphi\, dx$$

where $\mathcal{D}(\Omega)$ denotes the space of smooth functions with compact support in Ω, and dx is the Lebesgue's volume element. If such a v_α exists, it is unique up to sets of measure zero. When all the first weak derivatives of u exist, namely, when $D_\alpha u$ exists for any α such that $|\alpha| = 1$, u is said to be weakly differentiable on Ω. It is said to be k times weakly differentiable if all its weak derivatives $D_\alpha u$ exist for $|\alpha| \leq k$.

Let us now recall what we mean when speaking of an absolutely continuous function. Given $u : \mathbb{R} \to \mathbb{R}$, and $a < b$ real, we shall say that u is absolutely continuous on $[a, b]$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for any finite sequence

$$a \leq x_1 < y_1 \leq x_2 < y_2 \leq \cdots \leq x_m < y_m \leq b\,,$$

one has that

$$\sum_{j=1}^{m}(y_j - x_j) \leq \delta \Rightarrow \sum_{j=1}^{m}|u(y_j) - u(x_j)| \leq \varepsilon$$

As one can easily check, u is absolutely continuous on $[a, b]$ if and only if there exists v integrable on $[a, b]$ such that for any $a \leq x \leq b$,

$$u(x) - u(a) = \int_a^x v(t)dt$$

In particular, u is differentiable almost everywhere and $u' = v$. By extension, given Ω some open subset of $\mathbb{R}^n$, and $u : \Omega \to \mathbb{R}$ a real-valued function, we shall say that u is absolutely continuous on all (resp. almost all) line segments in Ω parallel to the coordinate axes, if for all (resp. almost all) $x = (x_1, \dots, x_n)$ in Ω, all $i = 1, \dots, n$, and all $a < x_i < b$ such that

$$\{(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n),\ x \in [a, b]\} \subset \Omega$$

the function

$$x \to u(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n)$$

is absolutely continuous on $[a, b]$. According to what has been said above, if u is absolutely continuous on almost all line segments in Ω parallel to the coordinate axes, then u possesses partial derivatives of first order almost everywhere. We recall here the well-known following result. For its proof, one can look at the celebrated book of Schwartz [**178**].

THEOREM 2.1 *Let Ω be an open subset of $\mathbb{R}^n$ and $u \in L^1_{loc}(\Omega)$. Then u is weakly differentiable on Ω if and only if (up to modifications on a set of measure zero):*

(i) *u is absolutely continuous on almost all line segments in Ω parallel to the coordinate axes, and*
(ii) *the first partial derivatives of u (which exist almost everywhere) belong to $L^1_{loc}(\Omega)$.*

Let us now recall some material on what concerns the theory of Sobolev spaces in the Euclidean context. The origin of such a theory goes back to the work of Sobolev [**180**] developed in the 1940s. Let Ω be some open subset of $\mathbb{R}^n$, k an integer, $p \geq 1$ real, and $u : \Omega \to \mathbb{R}$ a smooth, real-valued function. We let

$$\|u\|_{k,p} = \sum_{0 \leq |\alpha| \leq k} \left(\int_\Omega |D_\alpha u|^p dx \right)^{1/p}$$

and we define then the Sobolev spaces

$$H_k^p(\Omega) = \text{the completion of } \{u \in C^\infty(\Omega) \,/\, \|u\|_{k,p} < +\infty\} \quad \text{for } \|\cdot\|_{k,p}$$
$$W_k^p(\Omega) = \{u \in L^p(\Omega) \,/\, \forall |\alpha| \leq k,\ D_\alpha u \text{ exists and belongs to } L^p(\Omega)\}$$

where $D_\alpha u$ denotes the α^{th} weak partial derivative of u as defined above. For many years, there has been considerable confusion in the mathematical literature about the relationship between these spaces. The following result of Meyers-Serrin [**156**] dispelled such confusion. For its proof, we refer the reader to Adams [**1**].

THEOREM 2.2 *For any Ω, any k, and any $p \geq 1$, $H_k^p(\Omega) = W_k^p(\Omega)$.*

In order to end this section, we recall basic properties of Sobolev spaces with respect to Lipschitz functions. This is the purpose of the following result. For its proof we refer once more the reader to Adams [**1**]. See also Ziemer [**203**].

THEOREM 2.3 (i) *If Ω is a bounded, open subset of $\mathbb{R}^n$, and if $u : \Omega \to \mathbb{R}$ is Lipschitz, then $u \in H_1^p(\Omega)$ for all $p \geq 1$.*

(ii) *Let Ω be an open subset of $\mathbb{R}^n$, $h : \mathbb{R} \to \mathbb{R}$ a Lipschitz function, and $u \in H_1^p(\Omega)$ for some $p \geq 1$. If $h \circ u \in L^p(\Omega)$, then $h \circ u \in H_1^p(\Omega)$ and*

$$D_i\big(h \circ u\big)(x) = h'\big(u(x)\big) D_i u(x)$$

for all $i = 1, \dots, n$, and almost all $x \in \Omega$.

2.2. Sobolev Spaces on Riemannian Manifolds

Let (M, g) be a smooth Riemannian manifold. For k integer, and $u : M \to \mathbb{R}$ smooth, we denote by $\nabla^k u$ the k^{th} covariant derivative of u, and $|\nabla^k u|$ the norm of $\nabla^k u$ defined in a local chart by

$$|\nabla^k u| = g^{i_1 j_1} \dots g^{i_k j_k} (\nabla^k u)_{i_1 \dots i_k} (\nabla^k u)_{j_1 \dots j_k}$$

Recall that $(\nabla u)_i = \partial_i u$, while

$$(\nabla^2 u)_{ij} = \partial_{ij} u - \Gamma_{ij}^k \partial_k u$$

Given k an integer, and $p \geq 1$ real, set

$$\mathcal{C}_k^p(M) = \left\{ u \in C^\infty(M) \,/ \forall j = 0, \dots, k, \int_M |\nabla^j u|^p \, dv(g) < +\infty \right\}$$

When M is compact, one clearly has that $\mathcal{C}_k^p(M) = C^\infty(M)$ for any k and any $p \geq 1$. For $u \in \mathcal{C}_k^p(M)$, set also

$$\|u\|_{H_k^p} = \sum_{j=0}^k \left(\int_M |\nabla^j u|^p \, dv(g) \right)^{1/p}$$

We define the Sobolev space $H_k^p(M)$ as follows:

DEFINITION 2.1 Given (M, g) a smooth Riemannian manifold, k an integer, and $p \geq 1$ real, the Sobolev space $H_k^p(M)$ is the completion of $\mathcal{C}_k^p(M)$ with respect to $\|\cdot\|_{H_k^p}$.

Note here that one can look at these spaces as subspaces of $L^p(M)$. Let $\|\cdot\|_p$ be the norm of $L^p(M)$ defined by

$$\|u\|_p = \left(\int_M |u|^p \, dv(g) \right)^{1/p}$$

As one can easily check:

1. any Cauchy sequence in $\big(\mathcal{C}_k^p(M), \|\cdot\|_{H_k^p}\big)$ is a Cauchy sequence in the Lebesgue space $\big(L^p(M), \|\cdot\|_p\big)$ and
2. any Cauchy sequence in $\big(\mathcal{C}_k^p(M), \|\cdot\|_{H_k^p}\big)$ that converges to 0 in the Lebesgue space $\big(L^p(M), \|\cdot\|_p\big)$, also converges to 0 in $\big(\mathcal{C}_k^p(M), \|\cdot\|_{H_k^p}\big)$.

As a consequence, one can look at $H_k^p(M)$ as the subspace of $L^p(M)$ made of functions $u \in L^p(M)$ which are limits in $\left(L^p(M), \|\cdot\|_p\right)$ of a Cauchy sequence (u_m) in $\left(\mathcal{C}_k^p(M), \|\cdot\|_{H_k^p}\right)$, and define $\|u\|_{H_k^p}$ as above, where $|\nabla^j u|$, $0 \le j \le k$, is now the limit in $\left(L^p(M), \|\cdot\|_p\right)$ of the Cauchy sequence $(|\nabla^j u_m|)$.

Coming back to Definition 2.1, one can, of course, replace $\|\cdot\|_{H_k^p}$ by any other equivalent norm. In particular, the following holds.

PROPOSITION 2.1 *For any k integer, $H_k^2(M)$ is a Hilbert space when equipped with the equivalent norm*

$$\|u\| = \sqrt{\sum_{j=0}^{k} \int_M |\nabla^j u|^2 \, dv(g)}$$

The scalar product $\langle \cdot, \cdot \rangle$ associated to $\|\cdot\|$ is defined by

$$\langle u, v \rangle = \sum_{j=0}^{k} \int_M \langle \nabla^j u, \nabla^j v \rangle dv(g)$$

where, in such an expression, $\langle \cdot, \cdot \rangle$ is the scalar product on covariant tensor fields associated to g.

In the same order of ideas, let M be a compact manifold endowed with two Riemannian metrics g and $\tilde{g}$. As one can easily check, there exists $C > 1$ such that

$$\frac{1}{C} g \le \tilde{g} \le C g$$

on M, where such inequalities have to be understood in the sense of bilinear forms. This leads to the following:

PROPOSITION 2.2 *If M is compact, $H_k^p(M)$ does not depend on the metric.*

Such a proposition is of course not anymore true if the manifold is not assumed to be compact. Let, for instance, g and $\tilde{g}$ be two Riemannian metrics on $\mathbb{R}^n$, $(\mathbb{R}^n, g)$ being of finite volume, $(\mathbb{R}^n, \tilde{g})$ being of infinite volume. As an example, one can take

$$g = \frac{4}{(1 + |x|^2)^2} e$$

(the standard metric of S^n after stereographic projection), and $\tilde{g} = e$, where e is the Euclidean metric of $\mathbb{R}^n$. Then the constant function $u = 1$ belongs to the Sobolev spaces associated to g, while it does not belong to the Sobolev spaces associated to $\tilde{g}$. This proves the claim. Independently, noting that $\left(L^p(M), \|\cdot\|_p\right)$ is reflexive if $p > 1$, one gets the following:

PROPOSITION 2.3 *If $p > 1$, $H_k^p(M)$ is reflexive.*

Still when dealing with general results, let us now prove the following one. Given (M, g) a Riemannian manifold, $u : M \to \mathbb{R}$ is said to be Lipschitz on M if there exists $\Lambda > 0$ such that for any $x, y \in M$,

$$\left|u(y) - u(x)\right| \le \Lambda d_g(x, y)$$

where d_g is the distance associated to g.

PROPOSITION 2.4 *Let (M, g) be a smooth Riemannian manifold, and $u : M \to \mathbb{R}$ a Lipschitz function on M with compact support. Then $u \in H_1^p(M)$ for any $p \geq 1$. In particular, if M is compact, any Lipschitz function on M belongs to the Sobolev spaces $H_1^p(M)$, $p \geq 1$.*

PROOF: Let $u : M \to \mathbb{R}$ be a Lipschitz function on M such that $u = 0$ outside a compact subset K of M. Let also $(\Omega_k, \varphi_k)_{k=1,\dots,N}$ be a family of charts such that $K \subset \bigcup_{k=1}^N \Omega_k$ and such that for any $k = 1, \dots, N$,

$$\varphi_k(\Omega_k) = B_0(1) \quad \text{and} \quad \frac{1}{C}\delta_{ij} \leq g_{ij}^k \leq C\delta_{ij}$$

as bilinear forms, where $C > 1$ is given, $B_0(1)$ denotes the Euclidean ball of $\mathbb{R}^n$ of center 0 and radius 1, and where the g_{ij}^k's stand for the components of g in (Ω_k, φ_k). Consider $(\eta_k)_{k=1,\dots,N+1}$ a smooth partition of unity subordinate to the covering

$$(\Omega_k)_{k=1,\dots,N} \cup (M \backslash K)$$

For $k \in \{1, \dots, N\}$, it is clear that the function

$$u_k = (\eta_k u) \circ \varphi_k^{-1}$$

is Lipschitz on $B_0(1)$ for the Euclidean metric. According to Theorem 2.3 one then gets that $u_k \in H_1^p\big(B_0(1)\big)$ for any $p \geq 1$. Clearly, this implies that $\eta_k u \in H_1^p(M)$. Since

$$u = \sum_{k=1}^N \eta_k u$$

this ends the proof of the proposition. □

On what concerns Proposition 2.4, note that given (M, g) a smooth Riemannian manifold, a differentiable function $u : M \to \mathbb{R}$ for which $|\nabla u|$ is bounded, is Lipschitz on M. In order to fix ideas, suppose that (M, g) is complete. Let x and y be two points on M, and let $\gamma : [0, 1]Ê \to M$ be the minimizing geodesic from x to y. One has that there exists $t \in (0, 1)$ such that

$$\begin{aligned} \big|u(y) - u(x)\big| &= \big|u\big(\gamma(1)\big) - u\big(\gamma(0)\big)\big| \\ &= \big|(u \circ \gamma)'(t)\big| \end{aligned}$$

Hence, if d_g denotes the distance associated to g,

$$\begin{aligned} \big|u(y) - u(x)\big| &= \big|(u \circ \gamma)'(t)\big| \\ &= \left|du\big(\gamma(t)\big) \cdot \left(\frac{d\gamma}{dt}\right)_t\right| \\ &\leq \big|\nabla u\big(\gamma(t)\big)\big| \times \left|\left(\frac{d\gamma}{dt}\right)_t\right| \\ &\leq \Big(\sup_M |\nabla u|\Big) d_g(x, y) \end{aligned}$$

This proves the claim. Independently, one has the following result:

PROPOSITION 2.5 *Let (M, g) be a smooth complete Riemannian manifold, $h : \mathbb{R} \to \mathbb{R}$ a Lipschitz function, and $u \in H_1^p(M)$, $p \geq 1$. If $h \circ u \in L^p(M)$, then $h \circ u \in H_1^p(M)$ and*

$$\left|\left(\nabla(h \circ u)\right)(x)\right| = \left|h'\left(u(x)\right)\right| \cdot \left|(\nabla u)(x)\right|$$

for almost all x in M. In particular, for any $u \in H_1^p(M)$, $|u| \in H_1^p(M)$, and $\left|\nabla|u|\right| = \left|\nabla u\right|$ almost everywhere.

PROOF: Let $x \in M$ be given. Let also $v = h \circ u$. With similar arguments to those used in the proof of Proposition 2.4, one can easily get that $v \in H_1^p\left(B_x(r)\right)$ for all $r > 0$. Moreover, coming back to Theorem 2.3, one sees that

$$\left|\nabla v(y)\right| = \left|h'\left(u(y)\right)\right| \cdot \left|\nabla u(y)\right|$$

for almost all y in M. In particular, and by assumption, v and $|\nabla v|$ both belong to $L^p(M)$. One must still check that $v \in H_1^p(M)$. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(t) = 1 \quad \text{if } t \leq 0, \qquad f(t) = 1 - t \quad \text{if } 0 \leq t \leq 1, \qquad f(t) = 0 \quad \text{if } t \geq 1$$

For j an integer, and if d_g denotes the distance associated to g, we let f_j be the function defined by

$$f_j(y) = f\left(d_g(x, y) - j\right)$$

Clearly, f_j is Lipschitz with compact support the closure of $B_x(j+1)$. Moreover, since the cut-locus of x is negligible, one has that f_j is differentiable almost everywhere, with the additional property that $|\nabla f_j| \leq 1$. We claim here that $v_j = f_j v$ belongs to $H_1^p(M)$. Indeed, given $r > j + 1$, let (v_m) be a sequence in $C^\infty\left(B_x(r)\right)$ that converges to v in $H_1^p\left(B_x(r)\right)$. Clearly, $f_j v_m$ is Lipschitz with compact support in M, so that by Proposition 2.4, $f_j v_m \in H_1^p(M)$. Moreover, since f_j and $|\nabla f_j|$ are bounded, and since

$$\nabla v_j = \left(\nabla f_j\right)v + f_j\left(\nabla v\right)$$

almost everywhere, one easily gets that $\left(f_j v_m\right)$ converges, in $H_1^p(M)$ and as m goes to $+\infty$, to v_j. Hence, $v_j \in H_1^p(M)$ and this proves the claim. Here, one easily checks that for any j,

$$\left(\int_M \left|v_j - v\right|^p dv(g)\right)^{1/p} \leq \left(\int_{M\backslash B_x(j)} |v|^p \, dv(g)\right)^{1/p}$$

and

$$\left(\int_M \left|\nabla\left(v_j - v\right)\right|^p dv(g)\right)^{1/p}$$
$$\leq \left(\int_{M\backslash B_x(j)} |\nabla v|^p \, dv(g)\right)^{1/p} + \left(\int_{M\backslash B_x(j)} |v|^p \, dv(g)\right)^{1/p}$$

As an easy consequence of such inequalities, one gets that $v \in H_1^p(M)$, and that (v_j) converges to v in $H_1^p(M)$ as j goes to $+\infty$. This proves the proposition. □

In order to end this section, we now prove the following, whose first appearance seems to be in Aubin [**8**]. Similar density questions for higher-order Sobolev spaces will be treated in Chapter 3, Section 3.1.

THEOREM 2.4 *Given (M, g) a smooth, complete Riemannian manifold, the set $\mathcal{D}(M)$ of smooth functions with compact support in M is dense in $H_1^p(M)$ for any $p \geq 1$.*

PROOF: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(t) = 1 \quad \text{if } t \leq 0, \qquad f(t) = 1 - t \quad \text{if } 0 \leq t \leq 1, \qquad f(t) = 0 \quad \text{if } t \geq 1$$

and let $u \in \mathcal{C}_1^p(M)$ where $p \geq 1$ is some given real number. Let x be some point of M and set

$$u_j(y) = u(y) f\big(d_g(x, y) - j\big)$$

where d_g is the distance associated to g, j is an integer, and $y \in M$. By Proposition 2.4, $u_j \in H_1^p(M)$ for any j, and since $u_j = 0$ outside a compact subset of M, one easily gets that for any j, u_j is the limit in $H_1^p(M)$ of some sequence of functions in $\mathcal{D}(M)$. One just has to note here that if $(u_m) \in \mathcal{C}_1^p(M)$ converges to u_j in $H_1^p(M)$, and if $\alpha \in \mathcal{D}(M)$, then (αu_m) converges to αu_j in $H_1^p(M)$. Then, one can choose $\alpha \in \mathcal{D}(M)$ such that $\alpha = 1$ where $u_j \neq 0$. Independently, one clearly has that for any j,

$$\left(\int_M |u_j - u|^p \, dv(g)\right)^{1/p} \leq \left(\int_{M \setminus B_x(j)} |u|^p \, dv(g)\right)^{1/p}$$

and

$$\left(\int_M |\nabla(u_j - u)|^p \, dv(g)\right)^{1/p}$$
$$\leq \left(\int_{M \setminus B_x(j)} |\nabla u|^p \, dv(g)\right)^{1/p} + \left(\int_{M \setminus B_x(j)} |u|^p \, dv(g)\right)^{1/p}$$

Hence, (u_j) converges to u in $H_1^p(M)$ as j goes to $+\infty$. According to what has been said above, one then gets that u is the limit in $H_1^p(M)$ of some sequence in $\mathcal{D}(M)$. This ends the proof of the theorem. □

2.3. Sobolev Embeddings: General Results

As a starting point, let us fix a convention that will be used in the sequel. Given $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ two normed vector spaces with the property that E is a subspace of F, we write that $E \subset F$ and say that the inclusion is continuous if there exists $C > 0$ such that for any $x \in E$,

$$\|x\|_F \leq C \|x\|_E$$

Now, let (M, g) be a smooth Riemannian n-manifold. By Sobolev embeddings, at least in their first part, we refer to the following:

Sobolev embeddings: given p, q two real numbers with $1 \leq q < p$, and given k, m two integers with $0 \leq m < k$, if $1/p = 1/q - (k - m)/n$, then $H_k^q(M) \subset H_m^p(M)$.

As mentioned above, the notation $H_k^q(M) \subset H_m^p(M)$ includes the continuity of the embedding, namely, the existence of a positive constant C such that for any $u \in H_k^q(M)$, $\|u\|_{H_m^p} \le C\|u\|_{H_k^q}$. Such embeddings were first proved by Sobolev [**180**] for the Euclidean space $(\mathbb{R}^n, e)$. The validity of such embeddings, which may or may not hold in the general case of a Riemannian manifold, is often referred to as the Sobolev embedding theorem. We will see later on in this chapter that the Sobolev embedding theorem does hold for compact manifolds, while we will see in the next chapter that the situation is more intricate on what concerns complete noncompact manifolds. Note here that for $k = 1$, and hence $m = 0$, the Sobolev embeddings reduce to the assertion that for any $q \in [1, n)$, $H_1^q(M) \subset L^p(M)$ with $p = nq/(n-q)$. Note also that the exponents in the Sobolev embeddings are optimal. Think, for instance, of the Euclidean space $(\mathbb{R}^n, e)$; let $k = 1$, and let $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $\varphi \not\equiv 0$, be smooth with compact support in $\mathbb{R}^n$. For $\lambda \ge 1$, set $\varphi_\lambda(x) = \varphi(\lambda x)$. Then, as one can easily check,

$$\|\varphi_\lambda\|_p = \lambda^{-n/p}\|\varphi\|_p$$
$$\|\varphi_\lambda\|_{H_1^q} \le \lambda^{1-\frac{n}{q}}\|\varphi\|_{H_1^q}$$

By passing to the limit $\lambda \to +\infty$, one sees that the existence of $C > 0$ such that for any $u \in \mathcal{D}(\mathbb{R}^n)$,

$$\|u\|_p \le C\|u\|_{H_1^q}$$

leads to the inequality $1/p \ge 1/q - 1/n$. This proves the claim. For convenience, all the manifolds in what follows will be assumed to be at least complete. We start here by proving the following result:

LEMMA 2.1 *Let (M, g) be a smooth, complete Riemannian n-manifold. Suppose that $H_1^1(M) \subset L^{n/(n-1)}(M)$. Then for any real numbers $1 \le q < p$ and any integers $0 \le m < k$ such that $1/p = 1/q - (k-m)/n$, $H_k^q(M) \subset H_m^p(M)$.*

PROOF: We prove that if $H_1^1(M) \subset L^{n/(n-1)}(M)$, then for any $q \in [1, n)$, $H_1^q(M) \subset L^p(M)$ where $1/p = 1/q - 1/n$. We refer to Aubin [**12**], Proposition 2.11, for the proof that the other embeddings are also valid. Let $C > 0$ be such that for any $u \in H_1^1(M)$,

$$\left(\int_M |u|^{n/(n-1)}\,dv(g)\right)^{(n-1)/n} \le C\int_M \Big(|\nabla u| + |u|\Big)dv(g)$$

Let also $q \in (1, n)$, $p = nq/(n-q)$, and $u \in \mathcal{D}(M)$. Set $\varphi = |u|^{p(n-1)/n}$. By Hölder's inequalities we get that

$$\begin{aligned}
&\left(\int_M |u|^p\,dv(g)\right)^{(n-1)/n}\\
&= \left(\int_M |\varphi|^{n/(n-1)}\,dv(g)\right)^{(n-1)/n}\\
&\le C\int_M (|\nabla\varphi| + |\varphi|)dv(g)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{Cp(n-1)}{n} \int_M |u|^{p'} |\nabla u| dv(g) + C \int_M |u|^{p(n-1)/n} dv(g) \\
&\leq \frac{Cp(n-1)}{n} \left(\int_M |u|^{p'q'} dv(g) \right)^{1/q'} \left(\int_M |\nabla u|^q dv(g) \right)^{1/q} \\
&\quad + C \left(\int_M |u|^{p'q'} dv(g) \right)^{1/q'} \left(\int_M |u|^q dv(g) \right)^{1/q}
\end{aligned}
$$

where $1/q + 1/q' = 1$ and $p' = p(n-1)/n - 1$. But $p'q' = p$ since $1/p = 1/q - 1/n$. As a consequence, for any $u \in \mathcal{D}(M)$,

$$
\begin{aligned}
&\left(\int_M |u|^p dv(g) \right)^{1/p} \\
&\quad \leq \frac{Cp(n-1)}{n} \left(\left(\int_M |\nabla u|^q dv(g) \right)^{1/q} + \left(\int_M |u|^q dv(g) \right)^{1/q} \right)
\end{aligned}
$$

By Theorem 2.4, this ends the proof of the lemma. □

Note here that with the same arguments as those developed in the proof of Lemma 2.1, one gets a hierarchy for Sobolev embeddings. More precisely, one can prove that if for some $q \in [1, n)$, $H_1^q(M) \subset L^{nq/(n-q)}(M)$, then $H_1^{q'}(M) \subset L^{nq'/(n-q')}(M)$ for any $q' \in [q, n)$. Indeed, let $C > 0$ be such that for any $u \in H_1^q(M)$,

$$
\left(\int_M |u|^p dv(g) \right)^{1/p} \leq C \left(\left(\int_M |\nabla u|^q dv(g) \right)^{1/q} + \left(\int_M |u|^q dv(g) \right)^{1/q} \right)
$$

where $1/p = 1/q - 1/n$. Given $q' \in (q, n)$, and $u \in \mathcal{D}(M)$, let also $\varphi = |u|^{p'(n-q)/nq}$ where p' is such that $1/p' = 1/q' - 1/n$. Then, as in the proof of Lemma 2.1, one gets with Hölder's inequalities that

$$
\begin{aligned}
&\left(\int_M |u|^{p'} dv(g) \right)^{1/p} \\
&= \left(\int_M |\varphi|^p dv(g) \right)^{1/p} \\
&\leq C \left(\left(\int_M |\nabla \varphi|^q dv(g) \right)^{1/q} + \left(\int_M |\varphi|^q dv(g) \right)^{1/q} \right) \\
&= C(s+1) \left(\int_M |u|^{sq} |\nabla u|^q dv(g) \right)^{1/q} + C \left(\int_M |u|^{p'(n-q)/n} dv(g) \right)^{1/q} \\
&\leq C(s+1) \left(\int_M |u|^{qsq'/(q'-q)} dv(g) \right)^{(q'-q)/qq'} \left(\int_M |\nabla u|^{q'} dv(g) \right)^{1/q'} \\
&\quad + C \left(\int_M |u|^{qsq'/(q'-q)} dv(g) \right)^{(q'-q)/qq'} \left(\int_M |u|^{q'} dv(g) \right)^{1/q'}
\end{aligned}
$$

where $s = \frac{p'(n-q)}{nq} - 1$. But

$$\frac{1}{p} - \frac{q'-q}{qq'} = \frac{1}{p'} \quad \text{and} \quad \frac{qsq'}{q'-q} = p'$$

Hence, for any $u \in \mathcal{D}(M)$,

$$\left(\int_M |u|^{p'} dv(g)\right)^{1/p'}$$
$$\leq \frac{p'(n-q)C}{nq}\left(\left(\int_M |\nabla u|^{q'} dv(g)\right)^{1/q'} + \left(\int_M |u|^{q'} dv(g)\right)^{1/q'}\right)$$

By Theorem 2.4, this proves the above claim.

Let us now discuss an important consequence of the validity of Sobolev embeddings. As a starting point, suppose that

$$H_1^1(M) \subset L^{n/(n-1)}(M)$$

Let $C > 0$ be such that for any $u \in H_1^1(M)$,

$$\left(\int_M |u|^{n/(n-1)} dv(g)\right)^{(n-1)/n} \leq C \int_M (|\nabla u| + |u|) dv(g)$$

From such an inequality (see, for instance, chapter 6 of Chavel [45]), one gets that for any $x \in M$, and almost all $r > 0$,

$$\mathrm{Vol}_g\left(B_x(r)\right)^{(n-1)/n} \leq C\frac{d}{dr}\mathrm{Vol}_g\left(B_x(r)\right) + C\,\mathrm{Vol}_g\left(B_x(r)\right)$$

where $B_x(r)$ is the ball of center x and radius r in M, and $\mathrm{Vol}_g\left(B_x(r)\right)$ stands for its volume with respect to g. From now on, let $R > 0$ be given. Either $\mathrm{Vol}_g\left(B_x(R)\right) \geq (1/2C)^n$, or $\mathrm{Vol}_g\left(B_x(R)\right) \leq (1/2C)^n$. In the last case, one gets that for almost all $r \in (0, R]$,

$$\frac{1}{2C}\mathrm{Vol}_g\left(B_x(r)\right)^{1-1/n} \leq \frac{d}{dr}\mathrm{Vol}_g\left(B_x(r)\right)$$

Integrating this last inequality one then gets that for any $x \in M$ and any $R > 0$,

$$\mathrm{Vol}_g\left(B_x(R)\right) \geq \min\left(\left(\frac{1}{2C}\right)^n, \left(\frac{R}{2nC}\right)^n\right)$$

In other words, the fact that $H_1^1(M) \subset L^{n/(n-1)}(M)$ implies that there is a lower bound for the volume of balls with respect to their center. The following important lemma, due to Carron [39], extends this result to the other embeddings $H_1^q(M) \subset L^p(M)$, $q \in [1, n)$, $1/p = 1/q - 1/n$.

LEMMA 2.2 *Let (M, g) be a smooth, complete Riemannian n-manifold. Suppose that $H_1^q(M) \subset L^p(M)$ for some $q \in [1, n)$, where $1/p = 1/q - 1/n$. Then for any $r > 0$ there exists a positive constant $v = v(M, q, r)$ such that for any $x \in M$, $\mathrm{Vol}_g\left(B_x(r)\right) \geq v$.*

PROOF: Let $q \in [1, n)$, and suppose that $H_1^q(M) \subset L^p(M)$ where $1/p = 1/q - 1/n$. One then gets the existence of $A > 0$ such that for any $u \in H_1^q(M)$,

$$\left(\int_M |u|^p \, dv(g)\right)^{1/p} \le A\left(\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} + \left(\int_M |u|^q \, dv(g)\right)^{1/q}\right)$$

Let $r > 0$, let x be some point of M, and let $v \in H_1^q(M)$ be such that $v = 0$ on $M \backslash B_x(r)$. By Hölder's inequality,

$$\left(\int_M |v|^q \, dv(g)\right)^{1/q} \le \mathrm{Vol}_g\big(B_x(r)\big)^{1/n}\left(\int_M |v|^p \, dv(g)\right)^{1/p}$$

Hence,

$$\frac{1}{\mathrm{Vol}_g\big(B_x(r)\big)^{1/n}} - A \le A \frac{\left(\int_M |\nabla v|^q \, dv(g)\right)^{1/q}}{\left(\int_M |v|^q \, dv(g)\right)^{1/q}}$$

Fix $x \in M$ and let $R > 0$ be given. Then, either $\mathrm{Vol}_g\big(B_x(R)\big) > (1/2A)^n$ or $\mathrm{Vol}_g\big(B_x(R)\big) \le (1/2A)^n$, in which case we get that for any $r \in (0, R]$,

$$\frac{1}{\mathrm{Vol}_g\big(B_x(r)\big)^{1/n}} - A \ge \frac{1}{2\,\mathrm{Vol}_g\big(B_x(r)\big)^{1/n}}$$

Suppose that $\mathrm{Vol}_g\big(B_x(R)\big) \le (1/2A)^n$. We then have that for any $r \in (0, R]$ and any $v \in H_1^q(M)$ such that $v = 0$ on $M \backslash B_x(r)$,

$$\frac{1}{(2A)^q}\mathrm{Vol}_g\big(B_x(r)\big)^{-q/n} \le \frac{\int_M |\nabla v|^q \, dv(g)}{\int_M |v|^q \, dv(g)}$$

From now on, let

$$\begin{aligned} v(y) &= r - d_g(x, y) && \text{if } d_g(x, y) \le r \\ v(y) &= 0 && \text{if } d_g(x, y) \ge r \end{aligned}$$

where d_g is the distance associated to g. Clearly,v is Lipschitz and $v = 0$ on $M \backslash B_x(r)$. Hence (see Proposition 2.4), v belongs to $H_1^q(M)$. As a consequence,

$$\frac{1}{(2A)^q}\mathrm{Vol}_g\big(B_x(r)\big)^{-q/n} \le \frac{\mathrm{Vol}_g\big(B_x(r)\big)}{\int_{B_x(r/2)} v^q \, dv(g)} \le \frac{2^q \,\mathrm{Vol}_g\big(B_x(r)\big)}{r^q \,\mathrm{Vol}_g\big(B_x(r/2)\big)}$$

and we get that for any $r \le R$,

$$\mathrm{Vol}_g\big(B_x(r)\big) \ge \left(\frac{r}{4A}\right)^{nq/(n+q)} \mathrm{Vol}_g\big(B_x(r/2)\big)^{n/(n+q)}$$

By induction we then get that for any $m \in \mathbb{N}\backslash\{0\}$,

$$\mathrm{Vol}_g\big(B_x(R)\big) \ge \left(\frac{R}{2A}\right)^{q\alpha(m)} \left(\frac{1}{2}\right)^{q\beta(m)} \mathrm{Vol}_g\big(B_x(R/2^m)\big)^{\gamma(m)} \tag{2.1}$$

where

$$\alpha(m) = \sum_{i=1}^{m}\left(\frac{n}{n+q}\right)^i, \quad \beta(m) = \sum_{i=1}^{m} i\left(\frac{n}{n+q}\right)^i, \quad \text{and} \quad \gamma(m) = \left(\frac{n}{n+q}\right)^m$$

But (see, for instance, [**88**]),

$$\mathrm{Vol}_g\big(B_x(r)\big) = b_n r^n\big(1 + o(r)\big)$$

where b_n is the volume of the Euclidean ball of radius one. Hence,

$$\lim_{m\to\infty} \mathrm{Vol}_g\big(B_x(R/2^m)\big)^{\gamma(m)} = 1$$

In addition, we have that

$$\sum_{i=1}^{\infty}\left(\frac{n}{n+q}\right)^i = \frac{n}{q} \quad \text{and} \quad \sum_{i=1}^{\infty} i\left(\frac{n}{n+q}\right)^i = \frac{n(n+q)}{q^2}$$

As a consequence, letting $m \to \infty$ in (2.1) we get that

$$\mathrm{Vol}_g\big(B_x(R)\big) \geq \left(\frac{1}{2^{(n+2q)/q}A}\right)^n R^n$$

Finally, for any $x \in M$ and any $R > 0$,

$$\mathrm{Vol}_g\big(B_x(R)\big) \geq \min\big(1/2A, R/2^{(n+2q)/q}A\big)^n$$

and this ends the proof of the lemma. □

Note here that one gets from the above proof the exact dependence of v. Namely, v depends on n, q, r, and the constant C of the embedding of $H_1^q(M)$ in $L^p(M)$. Independently, we used in the above proof the fact that $\mathrm{Vol}_g\big(B_x(r)\big) = b_n r^n\big(1 + o(r)\big)$ where b_n is the volume of the Euclidean ball of radius one. More precisely (see, for instance, Gallot-Hulin-Lafontaine [**88**]), one has that

$$\mathrm{Vol}_g\big(B_x(r)\big) = b_n r^n\left(1 - \frac{1}{6(n+2)}\mathrm{Scal}_{(M,g)}(x)r^2 + o(r^2)\right)$$

where $\mathrm{Scal}_{(M,g)}$ stands for the scalar curvature of (M, g).

2.4. The Case of the Euclidean Space

The purpose of this section is to recall how one can prove the well-known fact that Sobolev embeddings in their first part are valid for the Euclidean space $(\mathbb{R}^n, e)$. The original proof, given by Sobolev [**180**], was based on quite a difficult lemma. We present here the proof of Gagliardo [**85**] and Nirenberg [**162**]. We start with the following lemma:

LEMMA 2.3 *For any $u \in \mathcal{D}(\mathbb{R}^n)$,*

$$\left(\int_{R^n} |u|^{n/(n-1)}\,dx\right)^{(n-1)/n} \leq \frac{1}{2}\prod_{i=1}^{n}\left(\int_{R^n}\left|\frac{\partial u}{\partial x_i}\right| dx\right)^{1/n}$$

where dx is the Lebesgue's volume element of $\mathbb{R}^n$.

PROOF: We present the proof for $n = 3$. The proof for $n \neq 3$ is similar. Let P be a point of $\mathbb{R}^3$, (x, y, z) the coordinates in $\mathbb{R}^3$, (x_0, y_0, z_0) the coordinates of P, and D_x (respectively, D_y, D_z) the straight line through P parallel to the x-axis (respectively, y-, z-axis). With such notation, the Lebesgue volume element dx of the lemma is $dx\,dy\,dz$. Let $u \in \mathcal{D}(\mathbb{R}^n)$. We then have that

$$u(P) = \int_{-\infty}^{x_0} (\partial_x u)(x, y_0, z_0)dx = -\int_{x_0}^{+\infty} (\partial_x u)(x, y_0, z_0)dx$$

As a consequence, $|u(P)| \le \frac{1}{2}\int_{D_x} |(\partial_x u)(x, y_0, z_0)|dx$. With similar arguments for $\partial_y u$ and $\partial_z u$ we get that

$$\begin{aligned}|u(P)|^{3/2} \le \left(\frac{1}{2}\right)^{3/2} &\left(\int_{D_x} |(\partial_x u)(x, y_0, z_0)|dx\right)^{1/2} \\ &\times \left(\int_{D_y} |(\partial_y u)(x_0, y, z_0)|dy\right)^{1/2} \left(\int_{D_z} |(\partial_z u)(x_0, y_0, z)|dz\right)^{1/2}\end{aligned}$$

Now, integrating x_0 over $\mathbb{R}$ yields, by Hölder's inequality,

$$\begin{aligned}\int_{D_x} |u(x, y_0, z_0)|^{3/2} dx \le \left(\frac{1}{2}\right)^{3/2} &\left(\int_{D_x} |(\partial_x u)(x, y_0, z_0)|dx\right)^{1/2} \\ &\times \left(\int_{D_{xy}} |(\partial_y u)(x, y, z_0)|dx\,dy\right)^{1/2} \\ &\times \left(\int_{D_{xz}} |(\partial_z u)(x, y_0, z)|dx\,dz\right)^{1/2}\end{aligned}$$

where D_{xy} (resp. D_{xz}) is the plane through P parallel to the x- and y-axes (resp. x- and z-axes). Integrating y_0 over $\mathbb{R}$ then yields, by Hölder's inequality,

$$\begin{aligned}\int_{D_{xy}} |u(x, y, z_0)|^{3/2} dx\,dy \le \left(\frac{1}{2}\right)^{3/2} &\left(\int_{D_{xy}} |(\partial_x u)(x, y, z_0)|dx\,dy\right)^{1/2} \\ &\times \left(\int_{D_{xy}} |(\partial_y u)(x, y, z_0)|dx\,dy\right)^{1/2} \\ &\times \left(\int_{R^3} |(\partial_z u)(x, y, z)|dx\,dy\,dz\right)^{1/2}\end{aligned}$$

Finally, integrating z_0 over $\mathbb{R}$, leads to the inequality of the lemma. □

With such a result we are now in position to prove that the Sobolev embeddings are valid for $(\mathbb{R}^n, e)$.

THEOREM 2.5 *Let $q \in [1, n)$ and let p be such that $1/p = 1/q - 1/n$. Then for any $u \in H_1^q(\mathbb{R}^n)$,*

$$\left(\int_{R^n} |u|^p\,dx\right)^{1/p} \le \frac{p(n-1)}{2n}\left(\int_{R^n} |\nabla u|^q\,dx\right)^{1/q}$$

In particular, for any real numbers $1 \le q < p$ and any integers $0 \le m < k$ satisfying $1/p = 1/q - (k-m)/n$, $H_k^q(\mathbb{R}^n) \subset H_m^p(\mathbb{R}^n)$.

PROOF: As a direct consequence of Lemma 2.3, $H_1^1(\mathbb{R}^n) \subset L^{n/(n-1)}(\mathbb{R}^n)$ with the additional property that for any $u \in H_1^1(\mathbb{R}^n)$,

$$\left(\int_{R^n} |u|^{n/(n-1)}\,dx\right)^{(n-1)/n} \leq \frac{1}{2}\int_{R^n} |\nabla u|dx$$

By Lemma 2.1 this proves that for any real numbers $1 \leq q < p$ and any integers $0 \leq m < k$ satisfying $1/p = 1/q - (k-m)/n$, $H_k^q(\mathbb{R}^n) \subset H_m^p(\mathbb{R}^n)$. Moreover, a similar computation to the one made in the proof of Lemma 2.1 shows that for any $1 \leq q < n$ and any $u \in H_1^q(\mathbb{R}^n)$,

$$\left(\int_{R^n} |u|^p\,dx\right)^{1/p} \leq \frac{p(n-1)}{2n}\left(\int_{R^n} |\nabla u|^q\,dx\right)^{1/q}$$

where $1/p = 1/q - 1/n$. This proves the theorem. □

As a remark, note here that the value $\frac{p(n-1)}{2n}$ given by Theorem 2.5 of the constant K in the inequality

$$\left(\int_{R^n} |u|^p\,dx\right)^{1/p} \leq K\left(\int_{R^n} |\nabla u|^q\,dx\right)^{1/q}$$

is not optimal. We refer to Chapter 4, Theorem 4.4, for the best value of K in such an inequality.

2.5. Sobolev Embeddings I

We prove in this section that Sobolev embeddings in their first part do hold for compact manifolds. This is the subject of the following theorem:

THEOREM 2.6 *Let (M, g) be a smooth, compact Riemannian n-manifold. The Sobolev embeddings in their first part do hold on (M, g) in the sense that for any real numbers $1 \leq q < p$ and any integers $0 \leq m < k$, if $1/p = 1/q - (k-m)/n$, then $H_k^q(M) \subset H_m^p(M)$. In particular, for any $q \in [1, n)$ real, $H_1^q(M) \subset L^p(M)$ where $1/p = 1/q - 1/n$.*

PROOF: By Lemma 2.1 we have only to prove that the embedding $H_1^1(M) \subset L^{n/(n-1)}(M)$ is valid. Since M is compact, M can be covered by a finite number of charts

$$(\Omega_m, \varphi_m)_{m=1,\dots,N}$$

such that for any m the components g_{ij}^m of g in (Ω_m, φ_m) satisfy

$$\frac{1}{2}\delta_{ij} \leq g_{ij}^m \leq 2\delta_{ij}$$

as bilinear forms. Let (η_m) be a smooth partition of unity subordinate to the covering (Ω_m). For any $u \in C^\infty(M)$ and any m, one has that

$$\int_M |\eta_m u|^{n/(n-1)}\,dv(g) \leq 2^{n/2}\int_{R^n} \left|(\eta_m u)\circ\varphi_m^{-1}(x)\right|^{n/(n-1)}dx$$

and

$$\int_M |\nabla(\eta_m u)|dv(g) \geq 2^{-(n+1)/2}\int_{R^n} \left|\nabla\left((\eta_m u)\circ\varphi_m^{-1}\right)(x)\right|dx$$

Independently, by Theorem 2.5,

$$\left(\int_{R^n} |(\eta_m u)\circ\varphi_m^{-1}(x)|^{n/(n-1)}\,dx\right)^{(n-1)/n} \le \frac{1}{2}\int_{R^n} \left|\nabla\big((\eta_m u)\circ\varphi_m^{-1}\big)(x)\right|dx$$

for any m. As a consequence, for any $u \in C^\infty(M)$,

$$\begin{aligned}\left(\int_M |u|^{n/(n-1)}\,dv(g)\right)^{(n-1)/n} &\le \sum_{m=1}^{N}\left(\int_M |\eta_m u|^{n/(n-1)}\,dv(g)\right)^{(n-1)/n}\\ &\le 2^{n-1}\sum_{m=1}^{N}\int_M |\nabla(\eta_m u)|\,dv(g)\\ &\le 2^{n-1}\int_M |\nabla u|\,dv(g)\\ &\quad + 2^{n-1}\left(\max_M \sum_{m=1}^{N}|\nabla\eta_m|\right)\int_M |u|\,dv(g)\end{aligned}$$

Hence, for any $u \in C^\infty(M)$,

$$\left(\int_M |u|^{n/(n-1)}\,dv(g)\right)^{(n-1)/n} \le A\left(\int_M |\nabla u|\,dv(g) + \int_M |u|\,dv(g)\right)$$

where

$$A = 2^{n-1}\left(1 + \max_M \sum_{m=1}^{N}|\nabla\eta_m|\right)$$

This ends the proof of the theorem. □

As an immediate corollary to Theorem 2.6, one has the following: Just note here that since M is assumed to be compact, (M, g) has finite volume. Hence, for $1 \le q \le q'$, $L^{q'}(M) \subset L^q(M)$.

COROLLARY 2.1 *Let (M, g) be a smooth, compact Riemannian n-manifold, and let q and p_0 real be such that $q \in [1, n)$ and $1/p_0 = 1/q - 1/n$. Then $H_1^q(M) \subset L^p(M)$ for any $p \in [1, p_0]$.*

2.6. Sobolev Embeddings II

The Sobolev embeddings in their first part have been discussed in the preceding sections. The purpose of this section is to discuss Sobolev embeddings in their second part. For dimension reasons and the sake of clarity, we will be brief on the subject. Let $q \ge 1$ be real and let $m < k$ be two integers. If $1/q - (k-m)/n > 0$, one has by Theorem 2.5 that $H_k^q(\mathbb{R}^n) \subset H_m^p(\mathbb{R}^n)$ where $1/p = 1/q - (k-m)/n$. Suppose now that $1/q - (k-m)/n < 0$. Sobolev **[180]** proved that in such a situation, $H_k^q(\mathbb{R}^n) \subset C_B^m(\mathbb{R}^n)$, where $C_B^m(\mathbb{R}^n)$ denotes the space of functions $u : \mathbb{R}^n \to \mathbb{R}$ of class C^m for which the norm

$$\|u\|_{C^m} = \sum_{|\alpha|=0}^{m} \sup_{x\in R^n} \left|D_\alpha u(x)\right|$$

is finite. Refinements were then obtained by Morrey [159] with embeddings in Hölder spaces. A very good reference on the subject is Adams [1]. Embeddings such as $H_k^q \subset C^m$ are referred to as Sobolev embeddings in their second part. Given (M, g) a smooth, compact Riemannian manifold, we define the norm $\|\cdot\|_{C^m}$ on $C^m(M)$ by

$$\|u\|_{C^m} = \sum_{j=0}^{m} \max_{x \in M} \left|\left(\nabla^j u\right)(x)\right|$$

One then has the following:

THEOREM 2.7 *Let (M, g) be a smooth, compact Riemannian n-manifold, $q \geq 1$ real, and $m < k$ two integers. If $1/q < (k-m)/n$, then $H_k^q(M) \subset C^m(M)$.*

PROOF: First we prove that for $q > n$, $H_1^q(M) \subset C^0(M)$. Since M is compact, M can be covered by a finite number of charts

$$(\Omega_s, \varphi_s)_{s=1,\dots,N}$$

such that for any s the components g_{ij}^s of g in (Ω_s, φ_s) satisfy

$$\frac{1}{2}\delta_{ij} \leq g_{ij}^s \leq 2\delta_{ij}$$

as bilinear forms. Let (η_s) be a smooth partition of unity subordinate to the covering (Ω_s). Given $u \in C^\infty(M)$,

$$\|\eta_s u\|_{C^0} = \|\left(\eta_s u\right) \circ \varphi_s^{-1}\|_{C^0}$$

for all s. Independently, starting from the inequalities satisfied by the g_{ij}^s's, one easily gets that there exists $C > 0$ such that for any s and any $u \in C^\infty(M)$,

$$\|\left(\eta_s u\right) \circ \varphi_s^{-1}\|_{H_1^q} \leq C\|\eta_s u\|_{H_1^q}$$

where the H_1^q-norm in the left-hand side of this inequality is with respect to the Euclidean space. Since $H_1^q(\mathbb{R}^n) \subset C_B^0(\mathbb{R}^n)$, this leads to the existence of some $A > 0$ such that for any s and any $u \in C^\infty(M)$,

$$\|\eta_s u\|_{C^0} \leq A\|\eta_s u\|_{H_1^q}$$

Clearly, there exists $B > 0$ such that for any $u \in C^\infty(M)$,

$$\sum_{s=1}^{N} \|\eta_s u\|_{H_1^q} \leq B\|u\|_{H_1^q}$$

For instance, one can take

$$B = \sum_{s=1}^{N} \left(\max_{x \in M} \eta_s + \max_{x \in M} |\nabla \eta_s|\right)$$

Hence,

$$\|u\|_{C^0} \leq \sum_{s=1}^{N} \|\eta_s u\|_{C^0} \leq A \sum_{s=1}^{N} \|\eta_s u\|_{H_1^q} \leq AB\|u\|_{H_1^q}$$

This proves the above claim, i.e., that for (M, g) compact and $q > n$, $H_1^q(M) \subset C^0(M)$. Let us now prove that for q, k, and m as in the theorem, $H_k^q(M) \subset C^m(M)$. Given $u \in C^\infty(M)$, one has by Kato's inequality that for any integer s,

$$\left|\nabla|\nabla^s u|\right| \leq \left|\nabla^{s+1} u\right|$$

Let $s \in \{0, \dots, m\}$. According to the first part of the Sobolev embedding theorem, Theorem 2.6, one has that $H_{k-s}^q(M) \subset H_1^{p_s}(M)$ where

$$\frac{1}{p_s} = \frac{1}{q} - \frac{k-s-1}{n}$$

In particular, $p_s > n$, so that, according to what has been said above, $H_1^{p_s}(M) \subset C^0(M)$. Hence, for any $s \in \{0, \dots, m\}$, and any $u \in C^\infty(M)$,

$$\|\nabla^s u\|_{C^0} \leq C_1(s)\|\nabla^s u\|_{H_1^{p_s}} \leq C_2(s)\|\nabla^s u\|_{H_{k-s}^q} \leq C_2(s)\|u\|_{H_k^q}$$

by Kato's inequality, and where $C_1(s)$ and $C_2(s)$ do not depend on u. As an immediate consequence of such inequalities, one gets that $H_k^q(M) \subset C^m(M)$ for k, q, and m as above. This ends the proof of the theorem. □

As already mentioned, improvements of the above result involving Hölder spaces can be obtained. Such improvements will be discussed now. For the sake of clarity, we restrict ourselves to the case $k = 1$ and $m = 0$. Given (M, g) a smooth, compact Riemannian manifold, and $\lambda \in (0, 1)$, let $C^\lambda(M)$ be the set of continuous functions $u : M \to \mathbb{R}$ for which the norm

$$\|u\|_{C^\lambda} = \max_{x \in M} \left|u(x)\right| + \max_{x \neq y \in M} \frac{\left|u(y) - u(x)\right|}{d_g(x, y)^\lambda}$$

is finite, where d_g denotes the distance associated to g. One then has the following:

THEOREM 2.8 *Let (M, g) be a smooth, compact Riemannian n-manifold, $q \geq 1$ real, and $\lambda \in (0, 1)$ real. If $1/q \leq (1 - \lambda)/n$, then $H_1^q(M) \subset C^\lambda(M)$.*

PROOF: Let $C_B^\lambda(\mathbb{R}^n)$ be the space of smooth functions $u : \mathbb{R}^n \to \mathbb{R}$ for which the norm

$$\|u\|_{C^\lambda} = \max_{x \in R^n} \left|u(x)\right| + \max_{x \neq y \in R^n} \frac{\left|u(y) - u(x)\right|}{\left|y - x\right|^\lambda}$$

is finite. By classical results of Morrey, see Adams [1], one has that for $q \geq 1$ real such that $1/q \leq (1 - \lambda)/n$, $H_1^q(\mathbb{R}^n) \subset C_B^\lambda(\mathbb{R}^n)$. Consider now a smooth, compact Riemannian n-manifold (M, g), and $q \geq 1$ as above. Since M is compact, one can once more assume that M is covered by a finite number of charts

$$(\Omega_s, \varphi_s)_{s=1,\dots,N}$$

such that for any s the components g_{ij}^s of g in (Ω_s, φ_s) satisfy

$$\frac{1}{2}\delta_{ij} \leq g_{ij}^s \leq 2\delta_{ij}$$

as bilinear forms. Without loss of generality, one can also assume that the Ω_s's are convex with respect to g. Let (η_s) be a smooth partition of unity subordinate to the covering (Ω_s). Starting from the inequalities satisfied by the g_{ij}^s's, one clearly gets that there exist $C_1 > 0$ and $C_2 > 0$ such that for any s and any $u \in C^\infty(M)$,

$$\|\eta_s u\|_{C^\lambda} \leq C_1 \|(\eta_s u) \circ \varphi_s^{-1}\|_{C^\lambda}$$
$$\|(\eta_s u) \circ \varphi_s^{-1}\|_{H_1^q} \leq C_2 \|\eta_s u\|_{H_1^q}$$

where the norms in the right-hand side of the first inequality, and in the left-hand side of the second inequality, are with respect to the Euclidean space. Since $H_1^q(\mathbb{R}^n) \subset C_B^\lambda(\mathbb{R}^n)$, one gets from the above inequalities that there exists $C_3 > 0$ such that for any s and any $u \in C^\infty(M)$,

$$\|\eta_s u\|_{C^\lambda} \leq C_3 \|\eta_s u\|_{H_1^q}$$

Independently, one clearly has that there exists $B > 0$ such that for any $u \in C^\infty(M)$,

$$\sum_{s=1}^{N} \|\eta_s u\|_{H_1^q} \leq B \|u\|_{H_1^q}$$

For instance, one can take

$$B = \sum_{s=1}^{N} \left(\max_{x \in M} \eta_s + \max_{x \in M} |\nabla \eta_s| \right)$$

Hence, for any $u \in C^\infty(M)$,

$$\|u\|_{C^\lambda} \leq \sum_{s=1}^{N} \|\eta_s u\|_{C^\lambda} \leq B C_3 \|u\|_{H_1^q}$$

This ends the proof of the theorem. □

In order to end this section, let us now say some words about the exceptional case of Sobolev embeddings. For that purpose, let (M, g) be a smooth, compact Riemannian n-manifold. We restrict our attention to the Sobolev space $H_1^n(M)$. Here, Sobolev embeddings in their second part give no information about the possible embeddings of $H_1^n(M)$. On the contrary, noting that for any $q \in [1, n)$, $H_1^n(M) \subset H_1^q(M)$, one gets from the Sobolev embeddings in their first part that $H_1^n(M) \subset L^p(M)$ for any $p \geq 1$. One can then hope that $H_1^n(M)$ is continuously embedded in $L^\infty(M)$. The answer to such a question is negative, as shown by the following example: Consider the function $u : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\begin{cases} u(x) = \log \big| \log |x| \big| & \text{if } 0 < |x| < \frac{1}{e} \\ u(x) = 0 & \text{otherwise} \end{cases}$$

As one can easily check, $u \in H_1^2(\mathbb{R}^2)$, but $u \notin L^\infty(\mathbb{R}^2)$. This proves the claim. On the other hand, given (M, g) a smooth, compact Riemannian n-manifold, one can prove (see Aubin [12] for more details) that if $u \in H_1^n(M)$, then $e^u \in L^1(M)$. Moreover, there exists C and μ such that

$$\int_M e^u \, dv(g) \leq C e^{\mu \int_M \left(|\nabla u|^n + |u|^n\right) dv(g)}$$

for any $u \in H_1^n(M)$. Such results were first proved by Trudinger [**185**] when dealing with bounded domains of the Euclidean space.

2.7. Compact Embeddings

We discuss in this section compactness properties of Sobolev embeddings. Given $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ two normed vector spaces, E being a subspace of F, recall that the embedding of E in F is said to be compact if bounded subsets of $(E, \|\cdot\|_E)$ are relatively compact in $(F, \|\cdot\|_F)$. This means, again, that bounded sequences in $(E, \|\cdot\|_E)$ possess convergent subsequences in $(F, \|\cdot\|_F)$. Clearly, if the embedding of E in F is compact, it is also continuous. We prove here the following result. On what concerns its second part, we restrict once more our attention to the case $k = 1$ and $m = 0$.

THEOREM 2.9 *Let (M, g) be a smooth, compact Riemannian n-manifold.*

(i) *For any integers $j \geq 0$ and $m \geq 1$, any real number $q \geq 1$, and any real number p such that $1 \leq p < nq/(n - mq)$, the embedding of $H_{j+m}^q(M)$ in $H_j^p(M)$ is compact. In particular, for any $q \in [1, n)$ real and any $p \geq 1$ such that $1/p > 1/q - 1/n$, the embedding of $H_1^q(M)$ in $L^p(M)$ is compact.*

(ii) *For $q > n$, the embedding of $H_1^q(M)$ in $C^\lambda(M)$ is compact for any $\lambda \in (0, 1)$ such that $(1 - \lambda)q > n$. In particular, the embedding of $H_1^q(M)$ in $C^0(M)$ is compact.*

Such a theorem is often referred to as the Rellich-Kondrakov theorem, in memory of the works developed by Rellich [**169**] and Kondrakov [**137**]. In order to prove Theorem 2.9, we need first the following lemma. Such a lemma can be seen as the analogue of the Ascoli theorem. Given A and B two subsets of $\mathbb{R}^n$, $\operatorname{dist}(A, B)$ denotes the distance from A to B.

LEMMA 2.4 *Let Ω be an open subset of $\mathbb{R}^n$, $p \geq 1$ real, and $\mathcal{H}$ a bounded subset of $L^p(\Omega)$. Then $\mathcal{H}$ is relatively compact in $L^p(\Omega)$ if and only if for any $\varepsilon > 0$, there exists a compact subset $K \subset \Omega$, and there exists $0 < \delta < \operatorname{dist}(K, \partial\Omega)$ such that*

$$\int_{\Omega\setminus K} |u(x)|^p\, dx < \varepsilon \quad \textit{and} \quad \int_K |u(x+y) - u(x)|^p\, dx < \varepsilon$$

for any $u \in \mathcal{H}$ and any y such that $|y| < \delta$.

We refer the reader to Adams [**1**], Theorem 2.21, for the proof of this result. Given Ω an open subset of $\mathbb{R}^n$, and $q \geq 1$ real, we denote by $H_{0,1}^q(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $H_1^q(\Omega)$. Then we prove the following:

LEMMA 2.5 *Let Ω be a bounded, open subset of $\mathbb{R}^n$, $q \in [1, n)$ real, and $p \geq 1$ real such that $1/p > 1/q - 1/n$. Then the embedding of $H_{0,1}^q(\Omega)$ in $L^p(\Omega)$ is compact.*

PROOF: Let $q \in [1, n)$ and $p \geq 1$ such that $1/p > 1/q - 1/n$ be given. Let also p_0 be defined by $p_0 = nq/(n - q)$. By Theorem 2.5, one has that $H_{0,1}^q(\Omega) \subset$

$L^{p_0}(\Omega)$. Moreover, there exists $A > 0$ such that for any $u \in H^q_{0,1}(\Omega)$,

$$\left(\int_\Omega |u|^{p_0}\,dx\right)^{1/p_0} \le A\left(\int_\Omega |\nabla u|^q\,dx\right)^{1/q}$$

Noting that Ω has finite volume, one clearly gets that $H^q_{0,1}(\Omega) \subset L^p(\Omega)$. There is still to prove that this embedding is compact. Let $\mathcal{H}$ be a bounded subset of $H^q_{0,1}(\Omega)$. There exists $C > 0$ such that for any $u \in \mathcal{H}$,

$$\int_\Omega |\nabla u|^q\,dx + \int_\Omega |u|^q\,dx \le C$$

For j integer, set

$$K_j = \left\{x \in \Omega \text{ s.t. } \operatorname{dist}(x, \partial\Omega) \ge \frac{2}{j}\right\}$$

Given $u \in \mathcal{H}$, and according to Hölder's inequality,

$$\begin{aligned}\int_{\Omega\backslash K_j} |u|dx &\le \left(\int_{\Omega\backslash K_j} |u|^{p_0}\,dx\right)^{1/p_0}\left(\int_{\Omega\backslash K_j} dx\right)^{1-\frac{1}{p_0}}\\ &\le AC^{1/q}\left(\int_{\Omega\backslash K_j} dx\right)^{1-\frac{1}{p_0}}\end{aligned}$$

Let $\varepsilon > 0$ be given. One then gets that for j big enough, and any $u \in \mathcal{H}$,

$$\int_{\Omega\backslash K_j} |u|dx < \varepsilon$$

From now on, let y be such that $|y| < 1/j$. If $x \in K_j$, then $x + y \in K_{2j}$. For $u \in \mathcal{D}(\Omega)$ one can then write that

$$\begin{aligned}\int_{K_j} |u(x+y) - u(x)|dx &\le \int_{K_j} dx \int_0^1 \left|\frac{d}{dt}u(x+ty)\right|dt\\ &\le |y| \int_{K_{2j}} |\nabla u|dx\\ &\le |y| \int_\Omega |\nabla u|dx\end{aligned}$$

Since $\mathcal{D}(\Omega)$ is dense in $H^q_{0,1}(\Omega)$, one gets that for any $u \in H^q_{0,1}(\Omega)$, and any y such that $|y| < 1/j$,

$$\int_{K_j} |u(x+y) - u(x)|dx \le |y| \int_\Omega |\nabla u|dx$$

By Hölder's inequality

$$\int_\Omega |\nabla u|dx \le \left(\int_\Omega |\nabla u|^q\,dx\right)^{1/q}\left(\int_\Omega dx\right)^{1-\frac{1}{q}}$$

One then gets that there exists $B > 0$ such that for any $u \in \mathcal{H}$,

$$\int_\Omega |\nabla u|dx \le B$$

Hence, for any $u \in \mathcal{H}$, and any y such that $|y| < \min\left(\frac{\varepsilon}{B}, \frac{1}{j}\right)$,

$$\int_{K_j} \left|u(x+y) - u(x)\right| dx < \varepsilon$$

By Lemma 2.4 this implies that $\mathcal{H}$ is relatively compact in $L^1(\Omega)$. One gets that $\mathcal{H}$ is relatively compact in $L^p(\Omega)$ as follows: If (u_m) is a sequence in $\mathcal{H}$, then, by Hölder's inequality,

$$\begin{aligned}\int_\Omega |u_{m_1} - u_{m_2}|^p\,dx &\le \left(\int_\Omega |u_{m_1} - u_{m_2}|dx\right)^k \left(\int_\Omega |u_{m_1} - u_{m_2}|^{p_0}dx\right)^{1-k} \\ &\le \left(2AC^{1/q}\right)^{(1-k)p_0}\left(\int_\Omega |u_{m_1} - u_{m_2}|dx\right)^k\end{aligned}$$

where $k = \frac{p_0-p}{p_0-1}$. From such an inequality, and from the fact that $\mathcal{H}$ is relatively compact in $L^1(\Omega)$, one easily gets that $\mathcal{H}$ is also relatively compact in $L^p(\Omega)$. This ends the proof of the lemma. □

Now that such results have been stated, we prove Theorem 2.9. For the sake of clarity, concerning point (i), we restrict ourselves to the case $j = 0$ and $m = 1$. In other words, we prove that for any smooth, compact Riemannian n-manifold, any $q \in [1, n)$, and any $p \ge 1$ such that $p < nq/(n-q)$, the embedding of $H_1^q(M)$ in $L^p(M)$ is compact. We refer the reader to Aubin **[12]** for the proof that the other embeddings are also compact.

PROOF OF THEOREM 2.9: (i) Since M is compact, M can be covered by a finite number of charts

$$(\Omega_s, \varphi_s)_{s=1,\dots,N}$$

such that for any s the components g_{ij}^s of g in (Ω_s, φ_s) satisfy

$$\frac{1}{2}\delta_{ij} \le g_{ij}^s \le 2\delta_{ij}$$

as bilinear forms. Let (η_s) be a smooth partition of unity subordinate to the covering (Ω_s). Given (u_m) a bounded sequence in $H_1^q(M)$, and for any s, we let

$$u_m^s = \left(\eta_s u_m\right) \circ \varphi_s^{-1}$$

Clearly, (u_m^s) is a bounded sequence in $H_{0,1}^q(\varphi_s(\Omega_s))$ for any s. By Lemma 2.5 one then gets that a subsequence (u_m^s) of (u_m^s) is a Cauchy sequence in $L^p\big(\varphi_s(\Omega_s)\big)$. Let (u_m) be a subsequence of (u_m) chosen such that for any s, (u_m^s) is a Cauchy sequence in $L^p\big(\varphi_s(\Omega_s)\big)$. Coming back to the inequalities satisfied by the g_{ij}^s's, one easily gets that for any s, $(\eta_s u_m)$ is a Cauchy sequence in $L^p(M)$. But for any m_1 and m_2,

$$\|u_{m_2} - u_{m_1}\|_p \le \sum_{s=1}^N \|\eta_s u_{m_2} - \eta_s u_{m_1}\|_p$$

where $\|\cdot\|_p$ stands for the L^p-norm. Hence, (u_m) is a Cauchy sequence in $L^p(M)$. This proves the result.

(ii) Let $\lambda \in (0,1)$ be such that $(1-\lambda)q > n$, and let $\alpha \in (0,1)$ be such that $\lambda < \alpha$ and $(1-\alpha)q > n$. By Theorem 2.8, one has that $H_1^q(M) \subset C^\alpha(M)$. Given $\mathcal{H}$ a bounded subset in $H_1^q(M)$, one then gets that there exists $C > 0$ such that for any $u \in \mathcal{H}$, $\|u\|_{C^\alpha} \le C$. By Ascoli's theorem, $\mathcal{H}$ is relatively compact in $C^0(M)$. From now on, let (u_m) be a sequence in $\mathcal{H}$. Up to the extraction of a subsequence, (u_m) converges to some u in $C^0(M)$. Clearly, $u \in C^\alpha(M)$ and $\|u\|_{C^\alpha} \le C$. Setting $v_m = u_m - u$, one then gets that $\|v_m\|_{C^\alpha} \le 2C$, and that for all x and y in M, $x \neq y$,

$$\begin{aligned}\frac{|v_m(y) - v_m(x)|}{d_g(x,y)^\lambda} &= \left(\frac{|v_m(y) - v_m(x)|}{d_g(x,y)^\alpha}\right)^{\frac{\lambda}{\alpha}} |v_m(y) - v_m(x)|^{1-\frac{\lambda}{\alpha}} \\ &\le (2C)^{\frac{\lambda}{\alpha}} |v_m(y) - v_m(x)|^{1-\frac{\lambda}{\alpha}} \\ &\le (2C)^{\frac{\lambda}{\alpha}} \left(2\|v_m\|_{C^0}\right)^{1-\frac{\lambda}{\alpha}}\end{aligned}$$

Since (v_m) converges to 0 in $C^0(M)$, one gets from such inequalities that (v_m) converges to 0 in $C^\lambda(M)$. This proves the theorem. □

2.8. Poincaré and Sobolev-Poincaré Inequalities

We establish in this section the so-called Poincaré and Sobolev-Poincaré inequalities. First we prove that the Poincaré inequality does hold for compact Riemannian manifolds. This is the subject of the following result.

THEOREM 2.10 *Let (M, g) be a smooth, compact Riemannian n-manifold, and let $q \in [1, n)$ be real. There exists a positive constant C such that for any $u \in H_1^q(M)$,*

$$\left(\int_M |u - \overline{u}|^q \, dv(g)\right)^{1/q} \le C \left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q}$$

where $\overline{u} = \frac{1}{\mathrm{Vol}_{(M,g)}} \int_M u \, dv(g)$.

PROOF: Suppose first that $q > 1$. To prove Theorem 2.10 we just have to prove that

$$\inf_{u \in \mathcal{H}} \int_M |\nabla u|^q \, dv(g) > 0$$

where

$$\mathcal{H} = \left\{ u \in H_1^q(M) \text{ s.t. } \int_M |u|^q \, dv(g) = 1 \text{ and } \int_M u \, dv(g) = 0 \right\}$$

Let $(u_k) \in \mathcal{H}$ be such that

$$\lim_{k \to \infty} \int_M |\nabla u_k|^q \, dv(g) = \inf_{u \in \mathcal{H}} \int_M |\nabla u|^q \, dv(g)$$

By combining the fact that $H_1^q(M)$ is reflexive for $q > 1$ with the Rellich-Kondrakov theorem, there exists a subsequence (u_k) of (u_k) which converges weakly in $H_1^q(M)$ and strongly in $L^q(M)$. Let v be its limit. The strong convergence in $L^q(M)$ implies that $v \in \mathcal{H}$, while we get with the weak convergence that

$$\int_M |\nabla v|^q \, dv(g) \leq \lim_{k\to\infty} \int_M |\nabla u_k|^q \, dv(g)$$

As a consequence, $\inf_{u\in\mathcal{H}} \int_M |\nabla u|^q \, dv(g)$ is attained by v, and since v cannot be constant,

$$\inf_{u\in\mathcal{H}} \int_M |\nabla u|^q \, dv(g) > 0$$

This proves the Poincaré inequalities for $q > 1$. When $q = 1$ we can use the well-known fact that on a compact manifold there always exists a Green function for the Laplacian. More precisely (see, for instance, **[12]**), if (M, g) is a compact Riemannian n-manifold there exists $G : M \times M \to \mathbb{R}$ such that:

(i) for any $u \in C^\infty(M)$ and any $x \in M$,

$$u(x) = \frac{1}{\mathrm{Vol}_{(M,g)}} \int_M u \, dv(g) + \int_M G(x, y)\Delta_g u(y) dv_g(y)$$

(ii) $G(x, y) = G(y, x)$ and $G(x, y)$ is C^∞ on $M \times M\backslash\Delta$ where Δ is the diagonal

$$\Delta = \left\{(x, y) \in M \times M \text{ s.t. } x = y\right\}$$

(iii) there exists a constant $K > 0$ such that for any $(x, y) \in M \times M\backslash\Delta$,

$$|G(x, y)| \leq \frac{K}{r^{n-2}} \quad \text{and} \quad |\nabla_y G(x, y)| \leq \frac{K}{r^{n-1}}$$

where $r = d_g(x, y)$ is the Riemannian distance from x to y.

From now on, let $u \in C^\infty(M)$ be such that $\int_M u \, dv(g) = 0$. We then have that for any x,

$$u(x) = \int_M G(x, y)\Delta_g u(y) dv_g(y)$$

Hence,

$$|u(x)| \leq \int_M |\nabla_y G(x, y)||\nabla u(y)| dv_g(y)$$

and

$$\begin{aligned} \int_M |u(x)| dv_g(x) &\leq \int_M \int_M |\nabla_y G(x, y)||\nabla u(y)| dv_g(x) dv_g(y) \\ &\leq C \int_M |\nabla u(y)| dv_g(y) \end{aligned}$$

where $C > 0$ is such that for any $y \in M$, $\int_M |\nabla_y G(x, y)| dv_g(x) \leq C$. Recall here that G satisfies $|\nabla_y G(x, y)| \leq K/r^{n-1}$. As a consequence, for any $u \in C^\infty(M)$ such that $\int_M u \, dv(g) = 0$,

$$\int_M |u| dv(g) \leq C \int_M |\nabla u| dv(g)$$

and the Poincaré inequality for $q = 1$ is proved. This ends the proof of the theorem. □

Inequalities such as the ones in question in Theorem 2.10 are referred to as Poincaré inequalities. Now that such inequalities have been proved, one easily gets the so-called Sobolev-Poincaré inequalities. This is the subject of the following:

THEOREM 2.11 *Let (M, g) be a smooth, compact Riemannian n-manifold, $q \in [1, n)$ real, and p real such that $1/p = 1/q - 1/n$. There exists a positive constant C such that for any $u \in H_1^q(M)$,*

$$\left(\int_M |u - \overline{u}|^p \, dv(g)\right)^{1/p} \le C\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q}$$

where $\overline{u} = \frac{1}{\mathrm{Vol}_{(M,g)}} \int_M u \, dv(g)$.

PROOF: By Theorem 2.6 there exists a positive constant B such that for any $u \in H_1^q(M)$,

$$\left(\int_M |u - \overline{u}|^p \, dv(g)\right)^{1/p} \le B\left(\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} + \left(\int_M |u - \overline{u}|^q \, dv(g)\right)^{1/q}\right)$$

Independently, by Theorem 2.10, there exists $C > 0$ such that for any $u \in H_1^q(M)$,

$$\left(\int_M |u - \overline{u}|^q \, dv(g)\right)^{1/q} \le C\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q}$$

Hence, for any $u \in H_1^q(M)$,

$$\left(\int_M |u - \overline{u}|^p \, dv(g)\right)^{1/p} \le B(1 + C)\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q}$$

and this proves the theorem. □

Of course, Sobolev embeddings, compactness properties of these embeddings, and such Poincaré and Sobolev-Poincaré inequalities are very useful when studying PDEs. As an application, we mention the following result of Druet (oral communication), Proposition 2.6, dealing with generalized Laplace equations. Given (M, g) a smooth, compact Riemannian n-manifold, and $q \in (1, n)$, we denote by $\Delta_{q,g}$ the q-Laplacian of g defined by

$$\Delta_{q,g} u = -\operatorname{div}_g \left(|\nabla u|^{q-2} \nabla u\right)$$

With such a definition, $\Delta_{2,g} = \Delta_g$, the usual Laplacian of g. Given f some smooth function on M, we study the existence of solutions $u \in H_1^q(M)$ to the equations

$$\Delta_{q,g} u = f$$

Such equations will be referred to as generalized Laplace equations. By regularity results (see, for instance, Druet [73]), one has that any solution $u \in H_1^q(M)$ of such an equation is $C^{1,\alpha}$ for some $\alpha \in (0, 1)$. Furthermore, such a regularity is in general optimal, as shown by the following situation. For $(\mathbb{R}^n, e)$ the Euclidean space,

$$u = \frac{q-1}{q}|x|^{1+\frac{1}{q-1}}$$

is a solution of $\Delta_{q,e}u = -n$ in $\mathbb{R}^n$. In the particular case $q = 2$, one gets the full regularity for the solutions of $\Delta_{q,g}u = f$, that is, the C^∞ regularity. The result of Druet (private communication) we present here is the following:

PROPOSITION 2.6 *For any smooth, compact Riemannian n-dimensional manifold (M, g), any $q \in (1, n)$ real, and any $f \in C^\infty(M)$, the generalized Laplace equation $\Delta_{q,g}u = f$ possesses a solution $u \in H_1^q(M)$ if and only if $\int_M f\,dv(g) = 0$. Moreover, the solution is unique up to the addition of a constant, and it is of class $C^{1,\alpha}$ for some $\alpha \in (0, 1)$.*

PROOF: It is clear that the condition $\int_M f\,dv(g) = 0$ is a necessary condition. Conversely, let

$$\mathcal{H} = \left\{u \in H_1^q(M) \text{ s.t. } \int_M fu\,dv(g) = 1 \text{ and } \int_M u\,dv(g) = 0\right\}$$

Set

$$\lambda = \inf_{u\in\mathcal{H}} \int_M |\nabla u|^q\,dv(g)$$

Clearly, $\mathcal{H} \neq \emptyset$ since, up to a constant scale factor, $f \in \mathcal{H}$. Let $(u_i) \in \mathcal{H}$ be a minimizing sequence for λ. By Poincaré's inequalities,

$$\int_M |u_i|^q\,dv(g) \leq C \int_M |\nabla u_i|^q\,dv(g)$$

for some $C > 0$ independent of i. As a consequence (u_i) is bounded in $H_1^q(M)$. By classical arguments, based on the Rellich-Kondrakov theorem and similar to those used in the proof of Theorem 2.10, one then gets that there exists $u \in \mathcal{H}$ such that

$$\int_M |\nabla u|^q\,dv(g) = \lambda$$

In particular, $\lambda > 0$. Moreover, one gets that there exist $\alpha, \beta \in \mathbb{R}$ such that for any $\varphi \in H_1^q(M)$,

$$\int_M |\nabla u|^{q-2}\langle\nabla u, \nabla\varphi\rangle dv(g) = \alpha\int_M \varphi\,dv(g) + \beta\int_M f\varphi\,dv(g)$$

where $\langle\cdot,\cdot\rangle$ is the scalar product associated to g. By taking $\varphi = 1$, we get that $\alpha = 0$. Taking $\varphi = u$, we get that $\beta = \lambda$. Since $u \not\equiv 0$, since $\lambda > 0$, and up to rescaling, u is the solution we were looking for. Moreover, one gets the uniqueness by noting that if $\Delta_{q,g}u = \Delta_{q,g}v$, then

$$\int_M \Big\langle\big(|\nabla u|^{q-2}\nabla u - |\nabla v|^{q-2}\nabla v\big)(\nabla u - \nabla v)\Big\rangle dv(g) = 0$$

and by noting that for any X, Y

$$\Big\langle\big(|X|^{q-2}X - |Y|^{q-2}Y\big), (X - Y)\Big\rangle \geq 0$$

with equality if and only if $X = Y$. This ends the proof of the proposition. □

2.9. A Finiteness Theorem

We prove a kind of Cheeger's finiteness theorem for the class of compact manifolds with bounded sectional curvature, volume bounded from above, and that satisfy a given Sobolev inequality. Given $n \geq 2$ an integer, $q \in [1, n)$ real, and Λ, V, and A positive real numbers, let $\mathcal{M} = \mathcal{M}(n, q, \Lambda, V, A)$ be the class of smooth, compact Riemannian n-manifolds (M, g) such that $K_{(M,g)} \leq \Lambda$ and $\mathrm{Vol}_{(M,g)} \leq V$, where $K_{(M,g)}$ stands for the sectional curvature of (M, g), and such that for any $u \in C^\infty(M)$,

$$\left(\int_M |u|^p \, dv(g)\right)^{1/p} \leq A\left(\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} + \left(\int_M |u|^q \, dv(g)\right)^{1/q}\right)$$

We prove here the following:

THEOREM 2.12 *For any n, q, Λ, V, and A as above, there are only finitely many diffeomorphism types of manifolds in $\mathcal{M}$. In other words, there exists a finite number m of smooth, compact manifolds $M_1, \ldots, M_m$ such that if $(M, g) \in \mathcal{M}$, then M is diffeomorphic to one of the M_i's.*

PROOF: Let $\alpha \in (0, 1)$ real. In order to prove Theorem 2.12, we just need to prove that $\mathcal{M}$ is precompact in the $C^{1,\alpha}$-topology. By Lemma 2.2, there exists $v : \mathbb{R}^{+*} \to \mathbb{R}^{+*}$ such that for any $(M, g) \in \mathcal{M}$, any $r > 0$, and any $x \in M$,

$$\mathrm{Vol}_g\left(B_x(r)\right) \geq v(r)$$

Hence, for any $(M, g) \in \mathcal{M}$, and any $\varepsilon > 0$, the maximal number of disjoint balls of radius ε that M can contain is bounded above by

$$N = \left[\frac{V}{v(\varepsilon)}\right] + 1$$

where $[\frac{V}{v(\varepsilon)}]$ stands for the greatest integer not exceeding $\frac{V}{v(\varepsilon)}$. In particular, this shows that there exists $d > 0$ such that for any $(M, g) \in \mathcal{M}$, $\mathrm{diam}_{(M,g)} \leq d$, where $\mathrm{diam}_{(M,g)}$ stands for the diameter of (M, g). Hence, there exist $v' > 0$ and $d > 0$ such that $\mathcal{M} \subset \tilde{\mathcal{M}}$ where

$$\begin{aligned}\tilde{\mathcal{M}} = \big\{&(M, g) \text{ compact } n\text{-manifolds s.t. } |K_g| \leq \Lambda, \\ &\mathrm{Vol}_{(M,g)} \geq v',\ \mathrm{diam}_{(M,g)} \leq d\big\}\end{aligned}$$

Furthermore, one has by Cheeger-Gromov-Taylor [46] that under the bound $|K_g| \leq \Lambda$, the bounds $\mathrm{Vol}_{(M,g)} \geq v'$ and $\mathrm{diam}_{(M,g)} \leq d$ are equivalent to the bounds $\mathrm{inj}_{(M,g)} \geq i$ and $\mathrm{Vol}_{(M,g)} \leq v$ where $\mathrm{inj}_{(M,g)}$ stands for the injectivity radius of (M, g). One then gets from Anderson's results [5] that $\tilde{\mathcal{M}}$ is precompact in the $C^{1,\alpha}$-topology. Since $\mathcal{M} \subset \tilde{\mathcal{M}}$, $\mathcal{M}$ is also precompact in the $C^{1,\alpha}$-topology. As already mentioned, this ends the proof of the theorem. □

As a consequence of this result, one easily gets that for any n, $q \in [1, n)$, $\Lambda > 0$, and $A > 0$, there are only finitely many diffeomorphism types of smooth, compact Riemannian n-manifolds (M, g) such that

$$\left|K_{(M,g)}\right| \mathrm{Vol}_{(M,g)}^{2/n} \leq \Lambda$$

and such that for any $u \in C^\infty(M)$,

$$\left(\int_M |u|^p \, dv(g)\right)^{1/p} \le A\left(\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} + \mathrm{Vol}_{(M,g)}^{-1/n}\left(\int_M |u|^q \, dv(g)\right)^{1/q}\right)$$

Concerning such an assertion, just note that the condition

$$\left|K_{(M,g)}\right| \mathrm{Vol}_{(M,g)}^{2/n} \le \Lambda\,,$$

and that the above Sobolev inequality are scale invariant.

CHAPTER 3

Sobolev Spaces: The Noncompact Setting

We mainly discuss in this chapter the validity of Sobolev embeddings and Sobolev inequalities for complete manifolds. As one will see, surprising phenomena appear when dealing with such a question. Density problems are first discussed in Section 3.1. Sobolev embeddings and Sobolev inequalities are then studied in Sections 3.2 and 3.3, while disturbed inequalities are presented in Section 3.4. For Euclidean-type Sobolev inequalities, we refer the reader to Chapter 8.

Given (M, g) a smooth Riemannian manifold, k an integer, and $p \geq 1$ real, recall that we defined the Sobolev space $H_k^p(M)$ as the completion of $\mathcal{C}_k^p(M)$ with respect to the norm

$$\|u\|_{H_k^p} = \sum_{j=0}^{k} \left(\int_M |\nabla^j u|^p \, dv(g) \right)^{1/p}$$

Here,

$$\mathcal{C}_k^p(M) = \left\{ u \in C^\infty(M) \, / \forall j = 0, \ldots, k, \int_M |\nabla^j u|^p \, dv(g) < +\infty \right\}$$

that is, the set of smooth functions on M for which $\|u\|_{H_k^p}$ is finite. Given (M, g) a smooth, complete Riemannian n-manifold, recall also that:

1. if $H_1^1(M) \subset L^{n/(n-1)}(M)$, then $H_k^q(M) \subset H_m^p(M)$ for any real numbers $1 \leq q < p$ and any integers $0 \leq m < k$ such that $1/p = 1/q - (k-m)/n$, and
2. if for some $q \in [1, n)$ real, $H_1^q(M) \subset L^p(M)$ where $1/p = 1/q - 1/n$, then for any $r > 0$ there exists $v > 0$ such that for any $x \in M$, $\mathrm{Vol}_g(B_x(r)) \geq v$ where $\mathrm{Vol}_g(B_x(r))$ stands for the volume of $B_x(r)$ with respect to g.

These two statements are, respectively, the ones of Lemma 2.1 and Lemma 2.2.

Given (M, g) a smooth, complete Riemannian manifold, k an integer, and $p \geq 1$ real, we define

$$H_{0,k}^q(M) = \text{closure of } \mathcal{D}(M) \text{ in } H_k^q(M)$$

where $\mathcal{D}(M)$ is the space of smooth functions with compact support in M. As already mentioned, we start in this chapter with density problems for Sobolev spaces.

3.1. Density Problems

Let (M, g) be a smooth, complete Riemannian manifold. For $H_{0,k}^q(M)$ defined as above, we discuss in this section the case of equality $H_{0,k}^q(M) = H_k^q(M)$. In other words, we try to find for which complete manifolds (M, g) one has that

$\mathcal{D}(M)$ is dense in $H_k^q(M)$. The completion for such a study is necessary, in the sense that one can construct many noncomplete manifolds for which $H_{0,k}^q \neq H_k^q$. Think, for instance, of Ω a bounded, open subset of $\mathbb{R}^n$ endowed with the Euclidean metric e. One easily checks that in such a situation, $H_{0,1}^2(\Omega) \neq H_1^2(\Omega)$. Consider for this purpose the scalar product $\langle \cdot, \cdot \rangle$ of Proposition 2.1 (with $g = e$ and $k = 1$), and let $u \in C^\infty(\Omega) \cap H_1^2(\Omega)$ be such that $\Delta_e u + u = 0$, $u \not\equiv 0$, where Δ_e is the Laplacian of e (with the minus sign convention). For instance, one can take $u = \sinh x_1$, x_1 the first coordinate in $\mathbb{R}^n$. Then for any $v \in \mathcal{D}(\Omega)$,

$$\langle u, v \rangle = \int_\Omega (\Delta_e u + u) v \, dx = 0$$

so that $u \notin H_{0,1}^2(\Omega)$. This proves the above claim. On the contrary, one has the following result:

PROPOSITION 3.1 *For any k an integer and any $q \geq 1$ real, $H_{0,k}^q(\mathbb{R}^n) = H_k^q(\mathbb{R}^n)$.*

PROOF: Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth decreasing function such that

$$f(t) = 1 \text{ if } t \leq 0 \quad \text{and} \quad f(t) = 0 \text{ if } t \geq 1$$

As one can easily check, it is sufficient to prove that any $u \in C^\infty(\mathbb{R}^n) \cap H_k^q(\mathbb{R}^n)$ can be approximated in $H_k^q(\mathbb{R}^n)$ by functions of $\mathcal{D}(\mathbb{R}^n)$. For m an integer and u some smooth function in $H_k^q(\mathbb{R}^n)$, set

$$u_m(x) = u(x) f(r - m)$$

where r denotes the distance from 0 to x. Clearly, $u_m \in \mathcal{D}(\mathbb{R}^n)$ for any m. On the one hand, one has by Leibnitz's formula that for any s integer and any m,

$$\left|\nabla^s (u_m - u)\right| \leq C_1 \sum_{j=0}^{s} |\nabla^j u| \cdot \left|\nabla^{s-j}(f_m - 1)\right|$$

where $C_1 > 0$ is independent of m, and $f_m(x) = f(r - m)$. In particular, noting that $|\nabla^s r|$ is bounded for $s \geq 1$ and $r \geq 1$, one gets that for any s integer, and any $m \geq 1$,

$$\left|\nabla^s (u_m - u)\right| \leq C_2 \sum_{j=0}^{s} |\nabla^j u|$$

where $C_2 > 0$ is independent of m. On the other hand, one clearly has that for any s integer, and with respect to the pointwise convergence,

$$\lim_{m \to +\infty} \nabla^s u_m = \nabla^s u$$

Since for any $s \leq k$, $|\nabla^s u| \in L^p(\mathbb{R}^n)$, the proposition easily follows from the Lebesgue dominated convergence theorem. □

When dealing with arbitrary, complete Riemannian manifolds (M, g), one can hope that the equality $H_{0,k}^q(M) = H_k^q(M)$ still holds. As surprising as it may seem, such a question is open for $k \geq 2$. For $k = 1$, this is the content of Theorem 2.4 of Chapter 2; things work for the best and one has the following result of Aubin [8].

THEOREM 3.1 *Given (M, g) a smooth, complete Riemannian manifold, $H_{0,1}^q(M) = H_1^q(M)$ for any $q \geq 1$ real.*

The situation for $k \geq 2$ seems to be more complicated, and assumptions on the manifolds are now needed (at least at the present state of the field). Aubin [**8**] proved that for any $q \geq 1$ and $k \geq 2$, $\mathcal{D}(M)$ is dense in $H_k^q(M)$ provided that (M, g) has a positive injectivity radius and that the Riemann curvature of (M, g) is bounded up to the order $k-2$. Hebey [**108**] proved that the above result still holds if the assumptions on the Riemann curvature are replaced by similar assumptions on the Ricci curvature. Moreover, thanks to the Bochner-Lichnerowicz-Weitzenböck formula, something special happens in the case $k = p = 2$, where only a lower bound on the Ricci curvature is needed instead of a global bound. This is what we are going to discuss now. Let us start with the general case.

PROPOSITION 3.2 *Let (M, g) be a smooth, complete Riemannian manifold with positive injectivity radius, and let $k \geq 2$ be an integer. We assume that for $j = 0, \dots, k-2$, $|\nabla^j \mathrm{Rc}_{(M,g)}|$ is bounded. Then for any $q \geq 1$ real, $H_{0,k}^q(M) = H_k^q(M)$.*

PROOF: Suppose that the injectivity radius $\mathrm{inj}_{(M,g)}$ of (M, g) is positive, and that there exists $C > 0$ such that for any $j = 0, \dots, k-2$, $|\nabla^j \mathrm{Rc}_{(M,g)}| \leq C$. By Theorem 1.2 one has that for any real numbers $Q > 1$ and $\alpha \in (0, 1)$, the $C^{k-1,\alpha}$-harmonic radius $r_H = r_H(Q, k-1, \alpha)$ is positive. Fix, for instance, $Q = 4$ and $\alpha = 1/2$. (As one will see, α plays no role in the following of the proof). For any $x \in M$ one then has that there exists some harmonic chart $\varphi : B_x(r_H) \to \mathbb{R}^n$ such that the points 1 and 2 of Definition 1.1 are satisfied with $Q = 4$ and $\alpha = 1/2$. (Without loss of generality, we can also assume that $\varphi(x) = 0$). In particular, we get that for any $r \leq r_H$

$$B_0(r/2) \subset \varphi\big(B_x(r)\big) \subset B_0(2r)$$

where for $t \geq 0$ real $B_0(t)$ denotes the Euclidean ball of center 0 and radius t. Let $\beta \in \mathcal{D}(\mathbb{R}^n)$ be such that

$$0 \leq \beta \leq 1\,, \qquad \beta = 1 \quad \text{on } B_0\left(\frac{r_H}{8}\right), \qquad \beta = 0 \quad \text{on } \mathbb{R}^n \backslash B_0\left(\frac{r_H}{4}\right)$$

As a consequence of the above inclusions, we get that $\beta \circ \varphi \in \mathcal{D}(M)$ satisfies

$$0 \leq \beta \circ \varphi \leq 1\,, \qquad \beta \circ \varphi = 1 \quad \text{on } B_x\left(\frac{r_H}{16}\right),$$
$$\beta \circ \varphi = 0 \quad \text{on } M \backslash B_x\left(\frac{r_H}{2}\right)$$

From now on, let (x_i) be a sequence of points of M such that

$$M = \bigcup_i B_{x_i}\left(\frac{r_H}{16}\right), \qquad \left(B_{x_i}\left(\frac{r_H}{2}\right)\right)_i \text{ is uniformly locally finite}$$

The existence of such a sequence is given by Lemma 1.1. Let $\varphi_i : B_{x_i}(r_H) \to \mathbb{R}^n$ be as above and set $\beta_i = \beta \circ \varphi_i$. Since the components of the metric tensor are

C^{k-1}-controlled in the charts $(B_{x_i}(r_H), \varphi_i)$, one easily gets that there exists $C > 0$ such that for any i and any $m = 0, \dots, k$, $|\nabla^m \beta_i| \le C$. Let us now set

$$\eta_i = \frac{\beta_i}{\sum_j \beta_j}$$

As a consequence of what we have said above, (η_i) is a smooth partition of unity subordinate to the covering $(B_{x_i}(\frac{r_H}{2}))$, and since this covering is uniformly locally finite, one easily obtains that there exists some constant $\tilde{C} \ge 1$ such that for any $m = 0, \dots, k$, $\sum_i |\nabla^m \eta_i| \le \tilde{C}$. Now fix $u \in \mathcal{C}_k^p(M)$ where $p \ge 1$ is some given real number. The proposition will obviously be proved if we show that for any $\varepsilon > 0$ there exists $u_0 \in \mathcal{D}(M)$ such that $\|u - u_0\|_{H_k^p} < \varepsilon$. Fix $\varepsilon > 0$ and let $\Omega \subset M$ be some bounded subset of M such that

$$\sum_{m=0}^{k} C_{k+1}^{m+1} \left(\int_{M \backslash \Omega} |\nabla^m u|^p \, dv(g) \right)^{1/p} < \varepsilon / \tilde{C}$$

where $\tilde{C}$ is as above and

$$C_{k+1}^{m+1} = \frac{(k+1)!}{(m+1)!(k-m)!}$$

Since the covering $(B_{x_i}(\frac{r_H}{2}))$ is uniformly locally finite, one easily obtains that there exists some integer N such that for any $i \ge N+1$, $B_{x_i}(\frac{r_H}{2}) \cap \Omega = \emptyset$. Set $u_0 = (1-\eta)u$ where $1 - \eta = \sum_{i=1}^{N} \eta_i$. Then $u_0 \in \mathcal{D}(M)$ and

$$\|u - u_0\|_{H_k^p} \le \sum_{m=0}^{k} \|\nabla^m (\eta u)\|_p$$

where $\|\cdot\|_p$ stands for the norm of $L^p(M)$. But

$$|\nabla^m (\eta u)| \le \sum_{j=0}^{m} C_m^j |\nabla^j \eta| |\nabla^{m-j} u|$$

and since $\operatorname{supp} \eta \subset M \backslash \Omega$ and $\sum_i |\nabla^j \eta_i| \le \tilde{C}$ for any $j = 0, \dots, k$, we get that

$$\|\nabla^m (\eta u)\|_p \le \tilde{C} \sum_{j=0}^{m} C_m^j \left(\int_{M \backslash \Omega} |\nabla^j u|^p \, dv(g) \right)^{1/p}$$

As a consequence, noting that for any $0 \le m \le k$, $\sum_{j=m}^{k} C_j^m = C_{k+1}^{m+1}$, we get that

$$\begin{aligned}
\|u - u_0\|_{H_k^p} &\le \tilde{C} \sum_{m=0}^{k} \sum_{j=0}^{m} C_m^j \left(\int_{M \backslash \Omega} |\nabla^j u|^p \, dv(g) \right)^{1/p} \\
&= \tilde{C} \sum_{m=0}^{k} \left(\sum_{j=m}^{k} C_j^m \right) \left(\int_{M \backslash \Omega} |\nabla^m u|^p \, dv(g) \right)^{1/p} \\
&= \tilde{C} \sum_{m=0}^{k} C_{k+1}^{m+1} \left(\int_{M \backslash \Omega} |\nabla^m u|^p \, dv(g) \right)^{1/p}
\end{aligned}$$

Since

$$\sum_{m=0}^{k} C_{k+1}^{m+1}\left(\int_{M\setminus\Omega} |\nabla^m u|^p \, dv(g)\right)^{1/p} < \varepsilon/\tilde{C}$$

we have shown that for any $\varepsilon > 0$ and any $u \in \mathcal{C}_k^p(M)$ there exists $u_0 \in \mathcal{D}(M)$ such that $\|u - u_0\|_{H_k^p} < \varepsilon$. As already mentioned, this ends the proof of the proposition. □

As a straightforward corollary to Proposition 3.2 one gets the following:

COROLLARY 3.1 *For any Riemannian covering $(\tilde{M}, \tilde{g})$ of a compact Riemannian manifold (M, g), for any k integer, and any $q \geq 1$ real, $H_{0,k}^q(M) = H_k^q(M)$.*

As already mentioned, thanks to the so-called Bochner-Lichnerowicz-Weitzenböck formula, something special happens in the case $k = p = 2$. Here, one can replace the global bound on the Ricci curvature by a lower bound on the Ricci curvature.

PROPOSITION 3.3 *For any smooth, complete Riemannian manifold (M, g) with positive injectivity radius and Ricci curvature bounded from below, $H_{0,2}^2(M) = H_2^2(M)$.*

PROOF: Let $K_2^2(M)$ be the completion of

$$\tilde{\mathcal{C}}_2^2(M) = \left\{u \in C^\infty(M) \,/ u,\ |\nabla u|,\ \Delta_g u \in L^2(M)\right\}$$

with respect to

$$\|u\|_{K_2^2} = \left(\int_M u^2 \, dv(g)\right)^{1/2} + \left(\int_M |\nabla u|^2 \, dv(g)\right)^{1/2} + \left(\int_M |\Delta_g u|^2 \, dv(g)\right)^{1/2}$$

Let also $K_{0,2}^2(M)$ be the closure of $\mathcal{D}(M)$ in $K_2^2(M)$. We assume that the Ricci curvature of (M, g) is bounded from below by some λ, and that the injectivity radius of (M, g) is positive. By Theorem 1.2 one then gets that for any $Q > 1$ real, and any $\alpha \in (0, 1)$ real, the $C^{0,\alpha}$-harmonic radius $r_H = r_H(Q, 0, \alpha)$ is positive. Noting that in a harmonic coordinate chart,

$$\Delta_g u = -g^{ij}\partial_{ij} u$$

for any $u \in C^\infty(M)$, similar arguments to those used in the proof of Proposition 3.2 prove that

$$K_{0,2}^2(M) = K_2^2(M)$$

Independently, one clearly has that for any $u \in C^\infty(M)$, $|\Delta_g u|^2 \leq n|\nabla^2 u|^2$. Hence,

$$H_2^2(M) \subset K_2^2(M)$$

with the property that this embedding is continuous. Recall now that by the Bochner-Lichnerowicz-Weitzenböck formula, for any smooth function u on M,

$$\langle \Delta_g(du), du \rangle = \frac{1}{2}\Delta_g(|du|^2) + |\nabla(du)|^2 + \mathrm{Rc}_{(M,g)}(\nabla u, \nabla u)$$

(see, for instance, [**109**]). Integrating this formula, one then gets that for any $u \in \mathcal{D}(M)$,

$$\begin{aligned}\int_M |\nabla^2 u|^2 dv(g) &= \int_M |\Delta_g u|^2 dv(g) - \int_M \mathrm{Rc}_{(M,g)}(\nabla u, \nabla u) dv(g) \\ &\leq \int_M |\Delta_g u|^2 dv(g) + |\lambda| \int_M |\nabla u|^2 dv(g)\end{aligned}$$

Hence,

$$\|u\|_{H_2^2} \leq \left(1 + \sqrt{|\lambda|}\right) \|u\|_{K_2^2}$$

for any $u \in \mathcal{D}(M)$, and according to what we have just said, we get that

$$H_{0,2}^2(M) = K_{0,2}^2(M)$$

As a consequence,

$$H_{0,2}^2(M) \subset H_2^2(M) \subset K_2^2(M) = K_{0,2}^2(M) = H_{0,2}^2(M)$$

and this ends the proof of the proposition. □

3.2. Sobolev Embeddings I

We discuss in this section the validity of Sobolev embeddings in their first part for complete manifolds. As one will see, surprising phenomena appear there. While such embeddings do hold for the Euclidean space (see Theorem 2.5 of Chapter 2), there exist complete manifolds for which they do not hold. For the sake of clarity, recall that by Sobolev embeddings in their first part we refer to the following: Given p, q two real numbers with $1 \leq q < p$, and given k, m two integers with $0 \leq m < k$, if $1/p = 1/q - (k-m)/n$, then $H_k^q(M) \subset H_m^p(M)$ where n is the dimension of M. As already mentioned, such embeddings are necessarily valid if the first one is valid, that is, if $H_1^1(M) \subset L^{n/(n-1)}(M)$. For clarity, and when discussing counterexamples, we restrict ourselves in this section to the case $k = 1$, that is, to the scale of embeddings $H_1^q \subset L^p$. We start with the following result:

PROPOSITION 3.4 *For any integer $n \geq 2$, there exist smooth, complete Riemannian n-manifolds (M, g) for which for any $q \in [1, n)$, $H_1^q(M) \not\subset L^p(M)$ where $1/p = 1/q - 1/n$.*

PROOF: Consider the warped product

$$M = \mathbb{R} \times S^{n-1}, \qquad g(x, \theta) = \xi_x + u(x) h_\theta$$

where ξ is the Euclidean metric of $\mathbb{R}$, h is the standard metric of the unit sphere S^{n-1} of $\mathbb{R}^n$, and $u : \mathbb{R} \to (0, 1]$ is smooth and such that $u(x) = 1$ when $x \leq 0$, $u(x) = e^{-2x}$ when $x \geq 1$. Clearly, if $y = (x_1, \theta_1)$ and $z = (x_2, \theta_2)$ are two points of M, then $d_g(y, z) \geq |x_2 - x_1|$. This implies that (M, g) is complete. In addition, if $y = (x, \theta)$ is a point of $M = \mathbb{R} \times S^{n-1}$, then $B_y(1) \subset (x-1, x+1) \times S^{n-1}$. As a consequence, when $x \geq 2$,

$$\begin{aligned}\mathrm{Vol}_g\left(B_{(x,\theta)}(1)\right) &\leq \mathrm{Vol}_g\left((x-1,x+1)\times S^{n-1}\right)\\ &\leq \omega_{n-1}\int_{x-1}^{x+1} e^{-(n-1)t}\,dt\\ &\leq C(n)e^{-(n-1)x}\end{aligned}$$

where ω_{n-1} denotes the volume of (S^{n-1},h) and

$$C(n) = \frac{\omega_{n-1}}{(n-1)}\left(e^{n-1} - e^{1-n}\right)$$

Therefore, for any $\theta \in S^{n-1}$,

$$\lim_{x\to\infty} \mathrm{Vol}_g\left(B_{(x,\theta)}(1)\right) = 0$$

and by Lemma 2.2 we get that $H_1^q(M) \not\subset L^p(M)$ for any $1 \leq q < n$ real and $1/p = 1/q - 1/n$. This ends the proof of the proposition. □

As a remark, note that the Ricci curvature of the manifold (M,g) constructed in the proof of Proposition 3.4 is bounded from below. Indeed, since

$$\mathrm{Rc}_{(S^{n-1},h)} = (n-2)h$$

one easily sees that $\mathrm{Rc}_{(M,g)}$ will be bounded from below if there exists A real such that for any $\theta \in S^{n-1}$ and any $x > 1$, $\mathrm{Rc}_{(M,g)}(x,\theta) \geq Ag_{(x,\theta)}$. Let $g'_{(x,\theta)} = e^{2x}\xi_x + h_\theta$, R'_{ij} be the components of $\mathrm{Rc}_{(M,g')}$ in some chart $(\mathbb{R}\times\Omega, Id\times\varphi)$, and R_{ij} be the components of $\mathrm{Rc}_{(M,g)}$ in the same chart. We have $R'_{ij} = 0$ if $i = 1$ or $j = 1$, while $R'_{ij} = (n-2)h_{ij}$ if $i \geq 2$ and $j \geq 2$. Independently, if $g' = e^f g$ are conformal metrics on a n-dimensional manifold, then

$$\begin{aligned}R'_{ij} = R_{ij} &- \frac{n-2}{2}(\nabla^2 f)_{ij} + \frac{n-2}{4}(\nabla f)_i(\nabla f)_j\\ &- \frac{1}{2}\left(-\Delta_g f + \frac{n-2}{2}|\nabla f|^2\right)g_{ij}\end{aligned}$$

Hence, since $g' = e^{2x}g$ if $x > 1$, we get that for $x > 1$,

$$\begin{aligned}&R_{11} = -(n-1) \quad \text{and} \quad R_{1j} = 0 \text{ when } j \geq 2\\ &R_{ij} = \left((n-2)e^{2x} - 1\right)g_{ij} \text{ when } i \geq 2 \text{ and } j \geq 2\end{aligned}$$

As a consequence, $\mathrm{Rc}_{(M,g)} \geq -(n-1)g$ for $x > 1$, and the Ricci curvature of (M,g) is bounded from below. Independently, note that with the same ideas as the ones developed in the proof of Proposition 3.4, one gets the following:

PROPOSITION 3.5 *For any integer $n \geq 2$, any smooth, complete noncompact Riemannian n-manifold (M,g) of finite volume, and any $q \in [1,n)$, one has that $H_1^q(M) \not\subset L^p(M)$ where $1/p = 1/q - 1/n$.*

PROOF: Here again, the proof of such a result is based on Lemma 2.2. Let (M,g) be a smooth, complete Riemannian n-manifold. Suppose that for some $q \in [1,n)$, one has that $H_1^q(M) \subset L^p(M)$ where $1/p = 1/q - 1/n$. Then by Lemma 2.2, there exists $v > 0$ such that for any $x \in M$, $\mathrm{Vol}_g(B_x(1)) \geq v$ where

$\mathrm{Vol}_g(B_x(1))$ stands for the volume of $B_x(1)$ with respect to g. Let (x_i), given by Zorn's lemma, be a sequence of points in M such that

$$M = \bigcup_i B_{x_i}(2) \quad \text{and} \quad B_{x_i}(1) \cap B_{x_j}(1) = \emptyset \quad \text{if } i \neq j \quad .$$

Since (M, g) is complete, the $B_{x_i}(2)$'s are relatively compact, while clearly

$$\mathrm{Vol}_{(M,g)} \geq \sum_i \mathrm{Vol}_g\left(B_{x_i}(1)\right)$$

where $\mathrm{Vol}_{(M,g)}$ stands for the volume of (M, g). Suppose now that (M, g) has finite volume. According to what has been said above, (x_i) must then be finite and hence, M must be compact. This proves the proposition. □

Let us now discuss results where Sobolev embeddings do hold. As one will see, the situation is well understood when dealing with manifolds having the property that their Ricci curvature is bounded from below. In the 1970s, Aubin [**7**] and Cantor [**37**] proved that Sobolev embeddings were valid for complete manifolds with bounded sectional curvature and positive injectivity radius. About ten years later, Varopoulos [**192**] proved that Sobolev embeddings do hold if the Ricci curvature of the manifold is bounded from below and if one has a lower bound for the volume of small balls which is uniform with respect to their center. By Croke's result [**59**] a lower bound on the injectivity radius implies a lower bound on the volume of small balls which is uniform with respect to their center. One then has the following generalization of the result of Aubin and Cantor. The assumption that there is a bound on the sectional curvature is here replaced by the weaker assumption that there is a lower bound for the Ricci curvature.

PROPOSITION 3.6 *The Sobolev embeddings in their first part are valid for any smooth, complete Riemannian manifold with Ricci curvature bounded from below and positive injectivity radius. In particular, given (M, g) a smooth, complete Riemannian n-manifold with Ricci curvature bounded from below and positive injectivity radius, and for any $q \in [1, n)$ real, $H_1^q(M) \subset L^p(M)$ where $1/p = 1/q - 1/n$.*

Let us now state and prove the more general result of Varopoulos [**192**] mentioned above. The original proof of this result was based on rather intricate semigroup techniques. The proof we present here is somehow more natural. It has its origins in Coulhon and Saloff-Coste [**58**]. For the exact statement of Varopoulos result, where no lower bounds on the volume of small balls are assumed and where disturbed Sobolev inequalities are obtained, we refer to Section 3.4.

THEOREM 3.2 *Let (M, g) be a smooth, complete Riemannian n-manifold with Ricci curvature bounded from below. Assume that*

$$\inf_{x \in M} \mathrm{Vol}_g\left(B_x(1)\right) > 0$$

where $\mathrm{Vol}_g(B_x(1))$ stands for the volume of $B_x(1)$ with respect to g. Then the Sobolev embeddings in their first part are valid for (M, g). In particular, for any $q \in [1, n)$ real, $H_1^q(M) \subset L^p(M)$ where $1/p = 1/q - 1/n$.

As a remark on the statement of Theorem 3.2, note that the assumption

$$\inf_{x\in M} \mathrm{Vol}_g\left(B_x(1)\right) > 0$$

implies that for any $r > 0$, there exists $v_r > 0$ such that for any $x \in M$, $\mathrm{Vol}_g(B_x(r)) \geq v_r$. Such a claim is a straightforward consequence of Gromov's result, Theorem 1.1. Now, the proof of Theorem 3.2 proceeds in several steps. As a starting point, we prove the following:

LEMMA 3.1 *Let (M, g) be a smooth, complete Riemannian n-manifold such that its Ricci curvature satisfies $\mathrm{Rc}_{(M,g)} \geq kg$ for some $k \in \mathbb{R}$. Let also $R > 0$ be some positive real number. There exists a positive constant $C = C(n, k, R)$, depending only on n, k, and R, such that for any $r \in (0, R)$, and any $u \in \mathcal{D}(M)$,*

$$\int_M |u - \overline{u}_r| dv(g) \leq Cr \int_M |\nabla u| dv(g)$$

where $\overline{u}_r(x) = \frac{1}{\mathrm{Vol}_g\left(B_x(r)\right)} \int_{B_x(r)} u\, dv(g)$, $x \in M$.

PROOF: Let (M, g) be a smooth, complete Riemannian n-manifold such that $\mathrm{Rc}_{(M,g)} \geq kg$ for some $k \in \mathbb{R}$, and let $R > 0$. By the work of Buser [35], there exists a positive constant $C = C(n, k, R)$, depending only on n, k, and R, such that for any $x \in M$, any $r \in (0, 2R)$, and any $u \in C^\infty(B_x(r))$,

$$\int_{B_x(r)} |u - \overline{u}_r(x)| dv(g) \leq Cr \int_{B_x(r)} |\nabla u| dv(g) \tag{3.1}$$

Let $r \in (0, R)$ be given and let $(x_i)_{i\in I}$ be a sequence of points of M such that simultaneously

$$M = \bigcup_i B_{x_i}(r) \quad \text{and} \quad B_{x_i}\left(\frac{r}{2}\right) \cap B_{x_j}\left(\frac{r}{2}\right) = \emptyset \quad \text{if } i \neq j$$

With the same arguments as used in the proof of Lemma 1.1, one gets that

$$\mathrm{Card}\left\{i \in I \,/\, x \in B_{x_i}(2r)\right\} \leq N = N(n, k, R) = (16)^n e^{8\sqrt{(n-1)|k|}R}$$

where Card stands for the cardinality. Let $u \in \mathcal{D}(M)$. We have

$$\begin{aligned}
\int_M |u - \overline{u}_r| dv(g) &\leq \sum_i \int_{B_{x_i}(r)} |u - \overline{u}_r| dv(g) \\
&\leq \sum_i \int_{B_{x_i}(r)} |u - \overline{u}_r(x_i)| dv(g) \\
&\quad + \sum_i \int_{B_{x_i}(r)} |\overline{u}_r(x_i) - \overline{u}_{2r}(x_i)| dv(g) \\
&\quad + \sum_i \int_{B_{x_i}(r)} |\overline{u}_r - \overline{u}_{2r}(x_i)| dv(g)
\end{aligned}$$

By (3.1), we get that

$$\sum_i \int_{B_{x_i}(r)} |u - \overline{u}_r(x_i)| dv(g) \leq Cr \sum_i \int_{B_{x_i}(r)} |\nabla u| dv(g) \leq NCr \int_M |\nabla u| dv(g)$$

while

$$\begin{aligned} \sum_i \int_{B_{x_i}(r)} |\overline{u}_r(x_i) - \overline{u}_{2r}(x_i)| dv(g) &= \sum_i \mathrm{Vol}_g\left(B_{x_i}(r)\right) |\overline{u}_r(x_i) - \overline{u}_{2r}(x_i)| \\ &\leq \sum_i \int_{B_{x_i}(r)} |u - \overline{u}_{2r}(x_i)| dv(g) \\ &\leq \sum_i \int_{B_{x_i}(2r)} |u - \overline{u}_{2r}(x_i)| dv(g) \\ &\leq 2NCr \int_M |\nabla u| dv(g) \end{aligned}$$

Independently, we have

$$\begin{aligned} &\sum_i \int_{B_{x_i}(r)} |\overline{u}_r - \overline{u}_{2r}(x_i)| dv(g) \\ &\leq \sum_i \int_{x \in B_{x_i}(r)} \left\{ \frac{1}{\mathrm{Vol}_g(B_x(r))} \int_{y \in B_x(r)} |u(y) - \overline{u}_{2r}(x_i)| dv_g(y) \right\} dv_g(x) \\ &\leq \sum_i \int_{x \in B_{x_i}(r)} \left\{ \frac{1}{\mathrm{Vol}_g(B_x(r))} \int_{y \in B_{x_i}(2r)} |u(y) - \overline{u}_{2r}(x_i)| dv_g(y) \right\} dv_g(x) \\ &\leq \sum_i \int_{B_{x_i}(2r)} |u(y) - \overline{u}_{2r}(x_i)| dv_g(y) \int_{B_{x_i}(r)} \frac{1}{\mathrm{Vol}_g(B_x(r))} dv_g(x) \end{aligned}$$

But, by (3.1),

$$\int_{B_{x_i}(2r)} |u(y) - \overline{u}_{2r}(x_i)| dv_g(y) \leq 2Cr \int_{B_{x_i}(2r)} |\nabla u| dv(g)$$

while by Gromov's result,

$$\frac{1}{\mathrm{Vol}_g\left(B_x(r)\right)} \leq \frac{K}{\mathrm{Vol}_g\left(B_x(2r)\right)}$$

where $K = K(n, k, R) = 2^n e^{2\sqrt{(n-1)|k|}R}$. Since $x \in B_{x_i}(r)$ implies that $B_{x_i}(r)$ is a subset of $B_x(2r)$, we get that

$$\int_{B_{x_i}(r)} \frac{1}{\mathrm{Vol}_g\left(B_x(r)\right)} dv_g(x) \leq K$$

Hence,

$$\sum_i \int_{B_{x_i}(r)} |\overline{u}_r - \overline{u}_{2r}(x_i)| dv(g) \leq 2KCNr \int_M |\nabla u| dv(g)$$

and for any $u \in \mathcal{D}(M)$,

$$\int_M |u - \overline{u}_r| dv(g) \leq 3(1+K)NCr \int_M |\nabla u| dv(g)$$

This ends the proof of the lemma. □

We now prove the following lemma:

LEMMA 3.2 *Let (M, g) be a smooth, complete Riemannian n-manifold. Suppose that its Ricci curvature satisfies $\mathrm{Rc}_{(M,g)} \geq kg$ for some $k \in \mathbb{R}$, and suppose that there exists $v > 0$ such that $\mathrm{Vol}_g(B_x(1)) \geq v$ for any $x \in M$. There exist two positive constants $C = C(n, k, v)$ and $\eta = \eta(n, k, v)$, depending only on n, k, and v, such that for any open subset Ω of M with smooth boundary and compact closure, if $\mathrm{Vol}_g(\Omega) \leq \eta$, then $\mathrm{Vol}_g(\Omega)^{(n-1)/n} \leq C\,\mathrm{Area}_g(\partial\Omega)$.*

PROOF: By Theorem 1.1 and the remark following this theorem, we have that for any $x \in M$ and any $0 < r < R$,

$$\mathrm{Vol}_g\left(B_x(r)\right) \geq \left(\frac{1}{R^n} e^{-\sqrt{(n-1)|k|}R}\,\mathrm{Vol}_g\left(B_x(R)\right)\right) r^n$$

Fix $R = 1$. Then we get that for any $x \in M$ and any $r \in (0, 1)$,

$$\mathrm{Vol}_g\left(B_x(r)\right) \geq \left(e^{-\sqrt{(n-1)|k|}} v\right) r^n$$

Set

$$\eta = \frac{1}{16} e^{-\sqrt{(n-1)|k|}} v \quad \text{and} \quad C_1 = e^{-\sqrt{(n-1)|k|}} v$$

Let Ω be some open subset of M with smooth boundary, compact closure, and such that $\mathrm{Vol}_g(\Omega) \leq \eta$. For sufficiently small $\varepsilon > 0$, consider the function

$$u_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in \Omega \\ 1 - \frac{1}{\varepsilon} d_g(x, \partial\Omega) & \text{if } x \in M\backslash\Omega \text{ and } d_g(x, \partial\Omega) \leq \varepsilon \\ 0 & \text{if } x \in M\backslash\Omega \text{ and } d_g(x, \partial\Omega) \geq \varepsilon \end{cases}$$

Then u_ε is Lipschitz for every ε and one easily sees that

$$\lim_{\varepsilon \to 0} \int_M u_\varepsilon \, dv(g) = \mathrm{Vol}_g(\Omega)$$

while

$$|\nabla u_\varepsilon| = \begin{cases} \frac{1}{\varepsilon} & \text{if } x \in M\backslash\overline{\Omega} \text{ and } d_g(x, \partial\Omega) < \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

which implies that

$$\lim_{\varepsilon \to 0} \int_M |\nabla u_\varepsilon| dv(g) = \mathrm{Area}_g(\partial\Omega)$$

Furthermore, for every $\varepsilon > 0$,

$$\mathrm{Vol}_g(\Omega) = \mathrm{Vol}_g\left(\left\{x \in M / u_\varepsilon(x) \geq 1\right\}\right)$$

and for any $\varepsilon > 0$ and any $r > 0$,

$$\begin{aligned}\mathrm{Vol}_g\left(\{x \in M / u_\varepsilon(x) \geq 1\}\right) \leq \mathrm{Vol}_g\left(\left\{x \in M \,/\, |u_\varepsilon(x) - \overline{u}_{\varepsilon,r}(x)| \geq \frac{1}{2}\right\}\right) \\ + \mathrm{Vol}_g\left(\left\{x \in M \,/\, \overline{u}_{\varepsilon,r}(x) \geq \frac{1}{2}\right\}\right)\end{aligned}$$

where

$$\overline{u}_{\varepsilon,r}(x) = \frac{1}{\mathrm{Vol}_g(B_x(r))}\int_{B_x(r)} u_\varepsilon dv(g)$$

Now note that for $r > 0$ and $\varepsilon \ll 1$,

$$\overline{u}_{\varepsilon,r}(x) \leq \frac{2\,\mathrm{Vol}_g(\Omega)}{\mathrm{Vol}_g(B_x(r))}$$

Fix $r = (\frac{8\,\mathrm{Vol}_g(\Omega)}{C_1})^{1/n}$. Since $\mathrm{Vol}_g(\Omega) \leq \eta = \frac{C_1}{16}$, we get that $r \in (0, 1)$ and that

$$\frac{2\,\mathrm{Vol}_g(\Omega)}{\mathrm{Vol}_g\left(B_x(r)\right)} \leq \frac{1}{4}$$

(according to what we have said above). Hence,

$$\left\{x \in M \,/\, \overline{u}_{\varepsilon,r}(x) \geq \frac{1}{2}\right\} = \emptyset$$

and for every $0 < \varepsilon \ll 1$,

$$\mathrm{Vol}_g(\Omega) \leq \mathrm{Vol}_g\left(\left\{x \in M \,/\, |u_\varepsilon(x) - \overline{u}_{\varepsilon,r}(x)| \geq \frac{1}{2}\right\}\right)$$

But

$$\mathrm{Vol}_g\left(\left\{x \in M \,/\, |u_\varepsilon(x) - \overline{u}_{\varepsilon,r}(x)| \geq \frac{1}{2}\right\}\right) \leq 2\int_M |u_\varepsilon - \overline{u}_{\varepsilon,r}| dv(g)$$

and by Lemma 3.1 there exists a positive constant $C_2 = C_2(n, k)$ such that

$$\int_M |u_\varepsilon - \overline{u}_{\varepsilon,r}| dv(g) \leq C_2 r \int_M |\nabla u_\varepsilon| dv(g)$$

Hence,

$$\begin{aligned}\mathrm{Vol}_g(\Omega) &\leq 2C_2\left(\frac{8\,\mathrm{Vol}_g(\Omega)}{C_1}\right)^{1/n} \lim_{\varepsilon\to 0}\int_M |\nabla u_\varepsilon| dv(g) \\ &\leq C_3\,\mathrm{Vol}_g(\Omega)^{1/n}\,\mathrm{Area}_g(\partial\Omega)\end{aligned}$$

where C_3 depends only on n, k, and v. Clearly, this ends the proof of the lemma. □

Lemma 3.2 has the following consequence: The ideas used in the proof of Lemma 3.3 are by now standard. One will find them in the celebrated works of Federer [79] and Federer-Fleming [80]. For an exposition in book form of such ideas, we refer the reader to Chavel [45].

LEMMA 3.3 *Let (M, g) be a smooth, complete Riemannian n-manifold. Suppose that its Ricci curvature satisfies $\mathrm{Rc}_{(M,g)} \geq kg$ for some $k \in \mathbb{R}$ and suppose that there exists $v > 0$ such that $\mathrm{Vol}_g(B_x(1)) \geq v$ for any $x \in M$. There exist two positive constants $\delta = \delta(n, k, v)$ and $A = A(n, k, v)$, depending only on n, k, and v, such that*

$$\left(\int_M |u|^{n/(n-1)} dv(g)\right)^{(n-1)/n} \leq A \int_M |\nabla u| dv(g)$$

for any $x \in M$ and any $u \in \mathcal{D}(B_x(\delta))$.

PROOF: Let $\eta = \eta(n, k, v)$ be as in Lemma 3.2. By Theorem 1.1 there exists $\delta = \delta(n, k, v)$ such that for any $x \in M$, $\mathrm{Vol}_g(B_x(\delta)) \leq \eta$. Let $x \in M$ and let $u \in \mathcal{D}(B_x(\delta))$. For $t \geq 0$, let

$$\Omega(t) = \{x \in M \,/\, |u(x)| > t\} \quad \text{and} \quad V(t) = \mathrm{Vol}_g(\Omega(t))$$

Clearly, $V(t) \leq \eta$ for any $t \geq 0$. Then the co-area formula and Lemma 3.2 imply that

$$\int_M |\nabla u| dv(g) \geq \frac{1}{C} \int_0^\infty V(t)^{1-1/n} dt$$

where C is the constant given by Lemma 3.2. Independently,

$$\int_M |u|^{n/(n-1)} dv(g) = \frac{n}{n-1} \int_0^\infty t^{1/(n-1)} V(t) dt$$

Noting that

$$\int_0^\infty V(t)^{1-1/n} dt \geq \left(\frac{n}{n-1} \int_0^\infty t^{1/(n-1)} V(t) dt\right)^{1-1/n}$$

we end the proof of the lemma. □

With Lemma 3.3 we are now in position to prove Theorem 3.2.

PROOF OF THEOREM 3.2: Let (M, g) be a smooth, complete Riemannian n-manifold such that $\mathrm{Rc}_{(M,g)} \geq kg$ for some $k \in \mathbb{R}$ and such that there exists $v > 0$ with the property that $\mathrm{Vol}_g(B_x(1)) \geq v$ for any $x \in M$. We want to prove that the Sobolev embeddings are valid on M. By Lemma 2.1 we just have to prove that $H_1^1(M) \subset L^{n/(n-1)}(M)$. Let $\delta = \delta(n, k, v)$ be as in Lemma 3.3 and let (x_i) be a sequence of points of M such that

1. $M = \bigcup_i B_{x_i}(\frac{\delta}{2})$
2. $B_{x_i}(\frac{\delta}{4}) \cap B_{x_j}(\frac{\delta}{4}) = \emptyset$ if $i \neq j$, and
3. there exists $N = N(n, k, v)$ depending only on n, k, and v, such that each point of M has a neighborhood that intersects at most N of the $B_{x_i}(\delta)$'s.

The existence of such a sequence is given by Lemma 1.1. Let also

$$\rho : [0, \infty) \to [0, 1]$$

be defined by

$$\rho(t) = \begin{cases} 1 & \text{if } 0 \le t \le \frac{\delta}{2} \\ 3 - \frac{4}{\delta}t & \text{if } \frac{\delta}{2} \le t \le \frac{3\delta}{4} \\ 0 & \text{if } t \ge \frac{3\delta}{4} \end{cases}$$

and let

$$\alpha_i(x) = \rho\big(d_g(x_i, x)\big)$$

where d_g denotes the distance associated to g and $x \in M$. Clearly, α_i is Lipschitz with compact support. Hence, by Proposition 2.4, α_i belongs to $H_1^1(M)$. Furthermore, since $\operatorname{supp}\alpha_i \subset B_{x_i}(\frac{3\delta}{4})$, we get without any difficulty that $\alpha_i \in H_{0,1}^1(B_{x_i}(\delta))$. Let

$$\eta_i = \frac{\alpha_i}{\sum_m \alpha_m}$$

Then, since $|\nabla\alpha_i| \le 4/\delta$ a.e., we get by (3) that for any i, $\eta_i \in H_{0,1}^1(B_{x_i}(\delta))$, (η_i) is a partition of unity subordinate to the covering $(B_{x_i}(\delta))$, $\nabla\eta_i$ exists almost everywhere, and there exists a positive constant $H = H(n, k, v)$ such that $|\nabla\eta_i| \le H$ a.e. Let $u \in \mathcal{D}(M)$. We have

$$\begin{aligned}\left(\int_M |u|^{n/(n-1)}\,dv(g)\right)^{(n-1)/n} &\le \sum_i \left(\int_M |\eta_i u|^{n/(n-1)}\,dv(g)\right)^{(n-1)/n} \\ &\le A\sum_i \int_M |\nabla(\eta_i u)|\,dv(g)\end{aligned}$$

where A is the constant of Lemma 3.3. Hence,

$$\begin{aligned}&\left(\int_M |u|^{n/(n-1)}\,dv(g)\right)^{(n-1)/n} \\ &\le A\sum_i \int_M \eta_i|\nabla u|\,dv(g) + A\sum_i \int_M |u||\nabla\eta_i|\,dv(g) \\ &\le A\int_M |\nabla u|\,dv(g) + ANH\int_M |u|\,dv(g) \\ &\le A(1+NH)\left(\int_M |\nabla u|\,dv(g) + \int_M |u|\,dv(g)\right)\end{aligned}$$

and there exists $\tilde{A} > 0$ such that for any $u \in \mathcal{D}(M)$,

$$\left(\int_M |u|^{n/(n-1)}\,dv(g)\right)^{(n-1)/n} \le \tilde{A}\left(\int_M |\nabla u|\,dv(g) + \int_M |u|\,dv(g)\right)$$

By Theorem 3.1 we then get that $H_1^1(M) \subset L^{n/(n-1)}(M)$. As already mentioned, this ends the proof of the theorem. □

Given (M, g) a smooth, complete Riemannian n-manifold, we refer to the scale of Sobolev embeddings when considering the embeddings $H_1^q(M) \subset L^p(M)$, $q \in [1, n)$, $1/p = 1/q - 1/n$. As already mentioned in Section 2.3 of Chapter 2, the validity of one of these embeddings implies the validity of the ones after: if

$H_1^{q_0}(M) \subset L^{p_0}(M)$ for some $q_0 \in [1, n)$ and $1/p_0 = 1/q_0 - 1/n$, then $H_1^q(M) \subset L^p(M)$ for any $q \in [q_0, n)$ and $1/p = 1/q - 1/n$. A natural question is to know if such a scale is coherent, that is, if the validity of one of these embeddings implies the validity of all the other ones. In other words, if the validity of one of the embeddings $H_1^q(M) \subset L^p(M)$, $q \in [1, n)$, implies the validity of the embedding $H_1^1(M) \subset L^{n/(n-1)}(M)$. Combining Theorem 3.2 and Lemma 2.2, one gets that the scale of Sobolev embeddings is coherent for complete manifolds with Ricci curvature bounded from below. More precisely, one has the following consequence of Theorem 3.2 and Lemma 2.2:

THEOREM 3.3 *Let (M, g) be a smooth, complete Riemannian n-manifold with Ricci curvature bounded from below.*

(i) *Suppose that for some $q_0 \in [1, n)$, $H_1^{q_0}(M) \subset L^{p_0}(M)$ where $1/p_0 = 1/q_0 - 1/n$. Then for any $q \in [1, n)$, $H_1^q(M) \subset L^p(M)$ where $1/p = 1/q - 1/n$. In particular, one has that $H_1^1(M) \subset L^{n/(n-1)}(M)$.*

(ii) *Given $q \in [1, n)$, one has that $H_1^q(M) \subset L^p(M)$, where $1/p = 1/q - 1/n$ if and only if there exists a lower bound for the volume of small balls which is uniform with respect to their center.*

Point (ii) in such a theorem means that for any $r > 0$ there exists $v_r > 0$ such that for any $x \in M$, $\mathrm{Vol}_g(B_x(r)) \geq v_r$. By Gromov's result, Theorem 1.1, since the manifolds considered have their Ricci curvature bounded from below, it is sufficient to have such a lower bound for one $r_0 > 0$. Independently, let (M, g) be a smooth, complete Riemannian n-manifold satisfying the assumptions of Theorem 3.2. Namely, its Ricci curvature satisfies that $\mathrm{Rc}_{(M,g)} \geq kg$ for some $k \in \mathbb{R}$, and there exists $v > 0$ such that for any $x \in M$, $\mathrm{Vol}_g(B_x(1)) \geq v$. By Theorem 3.2, for any $q \in [1, n)$, there exists $A > 0$ such that for any $u \in H_1^q(M)$,

$$\left(\int_M |u|^p \, dv(g)\right)^{1/p} \leq A\left(\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} + \left(\int_M |u|^q \, dv(g)\right)^{1/q}\right)$$

Note here that the proof of Theorem 3.2 gives the exact dependence of A: it depends only on n, q, k, and v. Finally, we have seen in Chapter 2 that for compact n-manifolds, $H_1^q \subset L^p$ for any $q \in [1, n)$ and any $p \geq 1$ such that $p \leq nq/(n-q)$, that is, for any p such that $1/p \geq 1/q - 1/n$. One can ask here if such a result still holds for complete manifolds. As a first remark, one can note that for complete, noncompact manifolds, one must have that $p \geq q$. Indeed, given $(\mathbb{R}^n, e)$ the Euclidean space, let $u_\alpha \in C^\infty(\mathbb{R}^n)$ be some smooth function such that $u_\alpha(x) = 1/|x|^\alpha$ if $|x| \geq 1$. As one can easily check, for $p \in [1, q)$, $u_{n/p} \in H_1^q(\mathbb{R}^n)$ while $u_{n/p} \notin L^p(\mathbb{R}^n)$. This proves the above claim. On the contrary, one can prove that the embeddings $H_1^q \subset L^p$ do hold for complete n-manifolds as soon as $p \geq q$. This is the subject of the following result:

PROPOSITION 3.7 *Let (M, g) be a smooth, complete n-dimensional Riemannian manifold such that its Ricci curvature is bounded from below and such that there exists $v > 0$ with the property that for any $x \in M$, $\mathrm{Vol}_g(B_x(1)) \geq v$. For any $q \in [1, n)$ real and any $p \in [q, nq/(n-q)]$, $H_1^q(M) \subset L^p(M)$.*

PROOF: Set $q^* = nq/(n-q)$, and let $p \in [q, q^*]$. As a simple application of Hölder's inequality, one gets that for any $u \in \mathcal{D}(M)$,

$$\left(\int_M |u|^p \, dv(g)\right)^{1/p} \leq \left(\int_M |u|^q \, dv(g)\right)^{\alpha/q} \left(\int_M |u|^{q^*} \, dv(g)\right)^{(1-\alpha)/q^*}$$

where $\alpha \in [0, 1]$ is given by

$$\alpha = \frac{1/p - 1/q^*}{1/q - 1/q^*}$$

By Theorem 3.2, there exists $A = A(n, q, k, v)$ such that for any $u \in \mathcal{D}(M)$,

$$\left(\int_M |u|^{q^*} \, dv(g)\right)^{1/q^*} \leq A\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} + A\left(\int_M |u|^q \, dv(g)\right)^{1/q}$$

Since for any x and y nonnegative, and any $\alpha \in [0, 1]$, $x^\alpha y^{1-\alpha} \leq x + y$, one gets that for any $p \in [q, q^*]$, and any $u \in \mathcal{D}(M)$,

$$\begin{aligned}
&\left(\int_M |u|^p \, dv(g)\right)^{1/p} \\
&\quad \leq \left(\int_M |u|^q \, dv(g)\right)^{1/q} + \left(\int_M |u|^{q^*} \, dv(g)\right)^{1/q^*} \\
&\quad \leq A\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} + (A+1)\left(\int_M |u|^q \, dv(g)\right)^{1/q}
\end{aligned}$$

Clearly, this proves the proposition. □

To end this section, let us now make some remarks. As a first remark, note that the assumption we made till now on the Ricci curvature is satisfactory but certainly not necessary. Indeed, there exist complete manifolds for which the whole scale of Sobolev embeddings $H_1^q \subset L^p$ is valid, but for which the Ricci curvature is not bounded from below. Just consider the space $\mathbb{R}^n$ with a conformal metric $g = e^u e$ to the Euclidean metric e, the conformal factor u being bounded and chosen such that the Ricci curvature of g is not bounded from below. With such a choice, one gets examples of the kind mentioned above. As a second remark, recall that we have seen in Chapter 2 that for compact manifolds, the embeddings $H_1^q \subset L^p$ with $p < nq/(n-q)$ are compact. One can ask here if such a property still holds for complete manifolds. The answer is negative. Just think to $\mathbb{R}^n$ with its Euclidean metric e, and let $u \in \mathcal{D}(\mathbb{R}^n)$ be such that $0 \leq u \leq 1$, $u = 1$ in $B_0(1)$, and $u = 0$ in $\mathbb{R}^n \backslash B_0(2)$. For m an integer, set $u_m(x) = u(x - x_m)$ where $x_m \in \mathbb{R}^n$ is such that $|x_m| = m$. Clearly, (u_m) is bounded in $H_1^q(\mathbb{R}^n)$ by $\|u\|_{H_1^q}$, while for any m, $\|u_m\|_p = \|u\|_p > 0$. Since (u_m) converges to 0 for the pointwise convergence, one gets as a consequence of what has been said that (u_m) does not converge in $L^p(M)$. This proves the above claim. On the contrary, we will see in Chapter 9 that symmetries may help to reverse the situation.

3.3. Sobolev Embeddings II

We briefly discuss in this section the validity of Sobolev embeddings in their second part for complete manifolds. Recall that by Sobolev embeddings in their second part, we refer to embeddings such as $H_k^q \subset C^m$. In the 1970s, Aubin [7] and Cantor [37] proved that such embeddings were valid for complete manifolds with bounded sectional curvature and positive injectivity radius. We prove here that the result still holds under the weaker assumption that the Ricci curvature is bounded from below and that the injectivity radius is positive. Extensions will be discussed at the end of the section. Given (M, g) a smooth, complete manifold and m an integer, we denote by $C_B^m(M)$ the space of functions $u : M \to \mathbb{R}$ of class C^m for which the norm

$$\|u\|_{C^m} = \sum_{j=0}^{m} \sup_{x \in M} \left|(\nabla^j u)(x)\right|$$

is finite. In the same order of ideas, given $\lambda \in (0, 1)$, we denote by $C_B^\lambda(M)$ the space of continuous functions $u : M \to \mathbb{R}$ for which the norm

$$\|u\|_{C^\lambda} = \sup_{x \in M} \left|u(x)\right| + \sup_{x \neq y \in M} \frac{|u(y) - u(x)|}{d_g(x, y)^\lambda}$$

is finite, where d_g denotes the distance associated to g. The first result we prove is the following:

THEOREM 3.4 *Let (M, g) be a smooth, complete Riemannian n-manifold with Ricci curvature bounded from below and positive injectivity radius. For $q \geq 1$ real and $m < k$ two integers, if $1/q < (k - m)/n$, then $H_k^q(M) \subset C_B^m(M)$.*

PROOF: First we prove that for $q > n$, $H_1^q(M) \subset C_B^0(M)$. By Theorem 1.2, one has that for any $Q > 1$ and $\alpha \in (0, 1)$, the $C^{0,\alpha}$-harmonic radius $r_H = r_H(Q, 0, \alpha)$ is positive. Fix, for instance, $Q = 2$ and $\alpha = 1/2$. For any $x \in M$ one then has that there exists some harmonic chart $\varphi_x : B_x(r_H) \to \mathbb{R}^n$ such that the components g_{ij} of g in this chart satisfy

$$\frac{1}{2}\delta_{ij} \leq g_{ij} \leq 2\delta_{ij}$$

as bilinear forms. Let (x_i) be a sequence of points of M such that

1. $M = \bigcup_i B_{x_i}(\frac{r_H}{2})$ and
2. there exists N such that each point of M has a neighborhood which intersects at most N of the $B_{x_i}(r_H)$'s.

The existence of such a sequence is given by Lemma 1.1. Let also

$$\rho : [0, \infty) \to [0, 1]$$

be defined by

$$\rho(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq \frac{r_H}{2} \\ 3 - \frac{4}{r_H}t & \text{if } \frac{r_H}{2} \leq t \leq \frac{3r_H}{4} \\ 0 & \text{if } t \geq \frac{3r_H}{4} \end{cases}$$

and let

$$\alpha_i(x) = \rho\big(d_g(x_i, x)\big)$$

where d_g denotes the distance associated to g and $x \in M$. Clearly, α_i is Lipschitz and bounded, with compact support in $B_{x_i}(r_h)$. Set

$$\eta_i = \frac{\alpha_i^{[q]+1}}{\sum_m \alpha_m^{[q]+1}}$$

where $[q]$ is the greatest integer not exceeding q. As one can easily check, η_i and $\eta_i^{1/q}$ are also Lipschitz with compact support in $B_{x_i}(r_H)$. In particular, one gets by Proposition 2.4 that $\eta_i^{1/q} \in H_{0,1}^q(B_{x_i}(r_H))$. Moreover, one has that (η_i) is a partition of unity subordinate to the covering $(B_{x_i}(r_H))$, that $\nabla\eta_i^{1/q}$ exists almost everywhere, and that there exists a positive constant H such that for all i, $|\nabla\eta_i^{1/q}| \le H$ a.e. Given $u \in \mathcal{D}(M)$, one clearly has that

$$\|\eta_i^{1/q}u\|_{C^0} = \|\big(\eta_i^{1/q}u\big) \circ \varphi_{x_i}^{-1}\|_{C^0}$$

for all i. Independently, starting from the inequalities satisfied by the g_{ij}'s, one easily gets that there exists $C > 0$ such that for any i and any $u \in \mathcal{D}(M)$,

$$\|\big(\eta_i^{1/q}u\big) \circ \varphi_{x_i}^{-1}\|_{H_1^q} \le C\|\eta_i^{1/q}u\|_{H_1^q}$$

where the norm in the left-hand side of this inequality is with respect to the Euclidean metric. Since $H_1^q(\mathbb{R}^n) \subset C_B^0(\mathbb{R}^n)$, this leads to the existence of some $A > 0$ such that for any i and any $u \in \mathcal{D}(M)$,

$$\|\eta_i^{1/q}u\|_{C^0} \le A\|\eta_i^{1/q}u\|_{H_1^q}$$

Given $u \in \mathcal{D}(M)$ one can write that

$$\begin{aligned}\|u\|_{C^0}^q = \Big\|\sum_i \eta_i|u|^q\Big\|_{C^0} &\le \sum_i \|\eta_i|u|^q\|_{C^0} = \sum_i \|\eta_i^{1/q}u\|_{C^0}^q \\ &\le A^q\sum_i \|\eta_i^{1/q}u\|_{H_1^q}^q\end{aligned}$$

Let $\mu = \mu(q)$ be such that for $x \ge 0$ and $y \ge 0$, $(x+y)^q \le \mu(x^q + y^q)$. Then, for $u \in \mathcal{D}(M)$,

$$\|u\|_{C^0}^q \le A^q\mu\sum_i\left(\int_M |\nabla(\eta_i^{1/q}u)|^q\,dv(g) + \int_M \eta_i|u|^q\,dv(g)\right)$$

Here, one has that

$$\begin{aligned}&\sum_i\int_M |\nabla(\eta_i^{1/q}u)|^q\,dv(g)\\ &\quad\le \mu\sum_i\int_M |\nabla\eta_i^{1/q}|^q|u|^q\,dv(g) + \mu\sum_i\int_M \eta_i|\nabla u|^q\,dv(g)\end{aligned}$$

$$\leq \mu N H^q \int_M |u|^q \, dv(g) + \mu \int_M |\nabla u|^q \, dv(g)$$

$$\leq \mu(NH^q + 1)\left(\int_M |\nabla u|^q \, dv(g) + \int_M |u|^q \, dv(g) \right)$$

Hence, there exists $B > 0$ such that for any $u \in \mathcal{D}(M)$,

$$\|u\|^q_{C^0} \leq B \|u\|^q_{H_1^q}$$

Clearly, by Theorem 3.1, this proves that $H_1^q(M) \subset C_B^0(M)$. Let us now prove that for q, k, and m as in the theorem, $H_k^q(M) \subset C_B^m(M)$. Given $u \in \mathcal{C}_k^q(M)$, one has by Kato's inequality that for any integer s,

$$\big|\nabla |\nabla^s u|\big| \leq |\nabla^{s+1} u|$$

Let $s \in \{0, \dots, m\}$. By Proposition 3.6 one has that $H_{k-s}^q(M) \subset H_1^{p_s}(M)$ where

$$\frac{1}{p_s} = \frac{1}{q} - \frac{k - s - 1}{n}$$

In particular, $p_s > n$. Hence, according to what has been said above, $H_1^{p_s}(M) \subset C^0(M)$. Given $s \in \{0, \dots, m\}$ and $u \in \mathcal{C}_k^q(M)$ one then gets that

$$\|\nabla^s u\|_{C^0} \leq C_1(s) \|\nabla^s u\|_{H_1^{p_s}} \leq C_2(s) \|\nabla^s u\|_{H_{k-s}^q} \leq C_2(s) \|u\|_{H_k^q}$$

by Kato's inequality, and where $C_1(s)$ and $C_2(s)$ do not depend on u. As an immediate consequence of such inequalities, one gets that $H_k^q(M) \subset C_B^m(M)$ for q, k, and m as above. This ends the proof of the theorem. □

Let us now prove the following result:

THEOREM 3.5 *Let (M, g) be a smooth, complete Riemannian n-manifold with Ricci curvature bounded from below and positive injectivity radius. For $q \geq 1$ real and $\lambda \in (0, 1)$ real, if $1/q \leq (1 - \lambda)/n$, then $H_1^q(M) \subset C_B^\lambda(M)$.*

PROOF: Here again, given $Q > 1$ and $\alpha \in (0, 1)$, one has by Theorem 1.2 that the $C^{0,\alpha}$-harmonic radius $r_H = r_H(Q, 0, \alpha)$ is positive. Fix, for instance, $Q = 2$ and $\alpha = 1/2$. For any $x \in M$ one then has that there exists some harmonic chart $\varphi_x : B_x(r_H) \to \mathbb{R}^n$ such that the components g_{ij} of g in this chart satisfy

$$\frac{1}{2}\delta_{ij} \leq g_{ij} \leq 2\delta_{ij} \tag{3.2}$$

as bilinear forms. Let also $r \in (0, r_H)$ sufficiently small, for instance, $r < r_H/3$, such that for any $x \in M$, the minimizing geodesic joining two points in $B_x(r)$ lies in $B_x(r_H)$. We use in what follows that for Ω, a regular, bounded, open subset of $\mathbb{R}^n$, and q, λ as in the theorem, $H_1^q(\Omega) \subset C_B^\lambda(\Omega)$. For such an assertion, we refer the reader to Adams [1]. Given q and λ as in the theorem, let x and y be two points of M such that $x \neq y$.

Suppose first that $d_g(x, y) \geq r$. Then for any $u \in \mathcal{D}(M)$,

$$\frac{|u(y) - u(x)|}{d_g(x, y)^\lambda} \leq \frac{2}{r^\lambda} \|u\|_{C^0}$$

By Theorem 3.4, this leads to the existence of $C_1 > 0$ such that for any $u \in \mathcal{D}(M)$,

$$\frac{|u(y) - u(x)|}{d_g(x, y)^\lambda} \le C_1 \|u\|_{H_1^q}$$

Suppose now that $d_g(x, y) < r$. By (3.2) one easily gets that

$$\left|\varphi_x(y) - \varphi_x(x)\right| \le \sqrt{2} d_g(x, y)$$

Hence,

$$\frac{|u(y) - u(x)|}{d_g(x, y)^\lambda} \le 2^{\frac{\lambda}{2}} \frac{|(u \circ \varphi_x^{-1})(\varphi_x(y)) - (u \circ \varphi_x^{-1})(\varphi_x(x))|}{|\varphi_x(y) - \varphi_x(x)|^\lambda}$$

Similarly, one easily gets from (3.2) that there exists $C_2 > 0$ such that for any $u \in \mathcal{D}(M)$,

$$\int_\Omega \left|\nabla\left(u \circ \varphi_x^{-1}\right)\right|^q dx \le C_2 \int_{B_x(r)} |\nabla u|^q \, dv(g)$$

$$\int_\Omega \left|u \circ \varphi_x^{-1}\right|^q dx \le C_2 \int_{B_x(r)} |u|^q \, dv(g)$$

where $\Omega = \varphi_x(B_x(r))$, and dx stands for the Euclidean volume element. Since $H_1^q(\Omega) \subset C_B^\lambda(\Omega)$, such inequalities lead to the existence of $C_3 > 0$ such that for any $u \in \mathcal{D}(M)$,

$$\frac{|u(y) - u(x)|}{d_g(x, y)^\lambda} \le C_3 \|u\|_{H_1^q}$$

Take $C_4 = \max(C_1, C_3)$. Then, for any x and y in M, with the property that $x \ne y$, and for any $u \in \mathcal{D}(M)$,

$$\frac{|u(y) - u(x)|}{d_g(x, y)^\lambda} \le C_4 \|u\|_{H_1^q}$$

Such an inequality, combined with Theorem 3.4, leads to the existence of $C_5 > 0$ such that for any $u \in \mathcal{D}(M)$,

$$\|u\|_{C^\lambda} \le C_5 \|u\|_{H_1^q}$$

By Theorem 3.1, one then gets that $H_1^q(M) \subset C_B^\lambda(M)$. This ends the proof of the theorem. □

Theorem 3.5 has been generalized by Coulhon **[56]** in the spirit of what has been said in the preceding section. More precisely, it is proved in **[56]** that for (M, g) a smooth, complete Riemannian n-manifold, for $q \ge 1$ real and $\lambda \in (0, 1)$ real, if $1/q \le (1 - \lambda)/n$, then the embedding of $H_1^q(M)$ in $C_B^\lambda(M)$ does hold as soon as the Ricci curvature of (M, g) is bounded from below and that for any $r_0 > 0$, there exists $C(r_0) > 1$, such that for any $x \in M$ and any $r \in (0, r_0)$,

$$C(r_0)^{-1} r^n \le \mathrm{Vol}_g\left(B_x(r)\right) \le C(r_0) r^n$$

Under the assumption that

$$\inf_{x \in M} \mathrm{Vol}_g\left(B_x(1)\right) > 0$$

this last property is an easy consequence of Gromov's theorem, Theorem 1.1. One then gets the following generalization of Theorem 3.5. We refer the interested reader to Coulhon [56] for its proof.

THEOREM 3.6 *Let (M, g) be a smooth, complete Riemannian n-manifold with Ricci curvature bounded from below. Assume that*

$$\inf_{x\in M} \mathrm{Vol}_g\left(B_x(1)\right) > 0$$

where $\mathrm{Vol}_g(B_x(1))$ stands for the volume of $B_x(1)$ with respect to g. For $q \geq 1$ real and $\lambda \in (0, 1)$ real, if $1/q \leq (1-\lambda)/n$, then $H_1^q(M) \subset C_B^\lambda(M)$.

As a remark, note that with the same arguments as the ones used in the second part of the proof of Theorem 3.4, one gets from Theorem 3.6 that for (M, g) as in the statement of Theorem 3.6, for $q \geq 1$ real and for $m < k$ two integers, if $1/q < (k-m)/n$, then $H_k^q(M) \subset C_B^m(M)$.

3.4. Disturbed Sobolev Inequalities

As already mentioned, Theorem 3.2 is less general than the result obtained by Varopoulos in [192]. The exact setting of this result is that for any complete Riemannian n-manifold (M, g) satisfying $\mathrm{Rc}_{(M,g)} \geq kg$ for some $k \in \mathbb{R}$, there exists a positive constant $A = A(n, k)$, depending only on n and k, such that for any $u \in \mathcal{D}(M)$,

$$\left(\int_M |u|^{n/(n-1)} v\, dv(g)\right)^{(n-1)/n} \leq A \int_M (|\nabla u| + |u|) v\, dv(g)$$

where

$$v(x) = \frac{1}{\mathrm{Vol}_g(B_x(1))}$$

for $x \in M$. It is easy to see that one recovers Theorem 3.2 from such a result. Indeed, the assumptions of Theorem 3.2 imply that v is bounded above while by Theorem 1.1 v is bounded from below. This leads to the statement of Theorem 3.2. The proof presented by Varopoulos of such a result was based on rather intricate semi-group techniques. A more natural proof, together with generalizations of this result, were obtained in Hebey [108]. We present them here. As a starting point, one can use the following result of Maheux and Saloff-Coste [154]. In a certain sense, this result generalizes the one of Buser [35] to Sobolev inequalities.

THEOREM 3.7 *Let (M, g) be a smooth, complete Riemannian n-manifold. Suppose that its Ricci curvature satisfies $\mathrm{Rc}_{(M,g)} \geq kg$ for some $k \in \mathbb{R}$, and let p, q be two real numbers such that $1 \leq q < n$ and $p \in [q, nq/(n-q)]$. There exists a positive constant $A = A(n, p, q, k)$, depending only on n, p, q, and k, such that for any $x \in M$, any $r \in (0, 1]$, and any $u \in C^\infty(B_x(r))$,*

$$\left(\int_{B_x(r)} |u - \overline{u}_r(x)|^p\, dv(g)\right)^{1/p}$$
$$\leq Ar\, \mathrm{Vol}_g\left(B_x(r)\right)^{1/p-1/q} \left(\int_{B_x(r)} |\nabla u|^q\, dv(g)\right)^{1/q}$$

where $\overline{u}_r(x) = \frac{1}{\mathrm{Vol}_g(B_x(r))}\int_{B_x(r)} u\,dv(g)$.

PROOF: We only sketch the proof and refer to Maheux and Saloff-Coste **[154]** for more details. Fix $R \in (0, 1]$ real. By Gromov's theorem, Theorem 1.1, there exists some positive constant $C_0 = C_0(n, k)$ such that for any $x \in M$ and any $r \in (0, R)$,

$$V(x, 2r) \le C_0 V(x, r)$$

where $V(x, r)$ stands for the volume of $B_x(r)$ with respect to g. Independently, by the work of Buser **[35]**, for any $q \ge 1$ real, there exists some positive constant $C_1 = C_1(n, k, q)$ such that for any $x \in M$, any $r \in (0, R)$, and any $u \in C^\infty(M)$,

$$\int_{B_x(r)} |u - \overline{u}_r(x)|^q\,dv(g) \le C_1 r^q \int_{B_x(r)} |\nabla u|^q\,dv(g)$$

From such inequalities (this is the main point in the argument), one gets that for any $q \ge 1$ real, there exists a positive constant $C_2 = C_2(n, k, q)$ such that the following holds: For any $x \in M$ satisfying the property that

$$\forall y \in B_x(R)\,,\ \forall t \in (0, R]\,,\ V(y, t) \ge K(x, R)t^n$$

one has that for any $u \in C^\infty(B_x(r))$,

$$\begin{aligned}\sup_{\lambda>0}&\left\{\lambda\,\mathrm{Vol}_g\big(B_x(R) \cap \{|u| \ge \lambda\}\big)^{\frac{1}{q(1-1/n)}}\right\}\\ &\le C_2\left[K(x, R)^{-q/n}\int_{B_x(R)}\big(|\nabla u|^q + R^{-q}|u|^q\big)dv(g)\right]^{\frac{1}{q(1+1/n)}}\\ &\quad\times\left[\int_{B_x(r)}|u|dv(g)\right]^{\frac{1}{n(1+1/n)}}\end{aligned}$$

As in Maz'ja **[155]** (see also Bakry, Coulhon, Ledoux, and Saloff-Coste **[18]**), this leads to the fact that for any $q \in [1, n)$ real, there exists a positive constant $C_3 = C_3(n, k, q)$ such that the following holds: For any $x \in M$ satisfying the property that

$$\forall y \in B_x(R)\,,\ \forall t \in (0, R]\,,\quad V(y, t) \ge K(x, R)t^n$$

one has that for any $u \in C^\infty(B_x(R))$,

$$\left(\int_{B_x(R)} |u - \overline{u}_R(x)|^p\,dv(g)\right)^{1/p} \le C_3 K(x, R)^{-1/n}\left(\int_{B_x(R)} |\nabla u|^q\,dv(g)\right)^{1/q}$$

where $p = nq/(n - q)$. By Gromov's theorem, Theorem 1.1, one can take

$$K(x, R) = C(n, k)R^{-n}V(x, R)$$

This ends the proof of the theorem. □

Coming back to Theorem 3.7, for $q = 1$ and $p = n/(n - 1)$, one gets, in particular, that for any $r \in (0, 1]$ there exists a positive constant $A = A(n, k)$, depending only on n and k, such that for any $x \in M$ and any $u \in C^\infty(B_x(r))$,

$$(\mathrm{I}_r) \quad \begin{aligned} &\left(\int_{B_x(r)} |u|^p \, dv(g)\right)^{1/p} \\ &\le A \operatorname{Vol}_g\left(B_x(r)\right)^{-1/n}\left(\int_{B_x(r)} |\nabla u| dv(g) + \int_{B_x(r)} |u| dv(g)\right) \end{aligned}$$

Starting from such inequalities, we then get the following extension of Varopoulos's result. In particular, as already mentioned, this provides us with a simple and more natural proof of this result.

THEOREM 3.8 *Let (M, g) be a smooth, complete Riemannian n-manifold. Suppose that its Ricci curvature satisfies* $\mathrm{Rc}_{(M,g)} \ge kg$ *for some $k \in \mathbb{R}$, and let p, q be two real numbers such that $1 \le q < n$ and $1/p = 1/q - 1/n$. Then for any $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$ satisfying $\beta/q - \alpha/p \ge 1/n$, there exists a positive constant $A = A(n, q, k, \alpha, \beta)$, depending only on n, q, k, α, and β, such that for any $u \in \mathcal{D}(M)$,*

$$\begin{aligned} &\left(\int_M |u|^p v^\alpha \, dv(g)\right)^{1/p} \\ &\le A\left(\left(\int_M |\nabla u|^q v^\beta \, dv(g)\right)^{1/q} + \left(\int_M |u|^q v^\beta \, dv(g)\right)^{1/q}\right) \end{aligned}$$

where

$$v(x) = \frac{1}{\operatorname{Vol}_g\left(B_x(1)\right)}$$

for $x \in M$. In particular, there exists a positive constant $A = A(n, k)$, depending only on n and k, such that the Varopoulos inequality

$$\left(\int_M |u|^{n/(n-1)} v \, dv(g)\right)^{(n-1)/n} \le A \int_M (|\nabla u| + |u|) v \, dv(g)$$

holds for any $u \in \mathcal{D}(M)$.

First we prove the following result, that is, the case $q = 1$ in Theorem 3.8:

LEMMA 3.4 *Let (M, g) be a smooth, complete Riemannian n-manifold. Suppose that its Ricci curvature satisfies* $\mathrm{Rc}_{(M,g)} \ge kg$ *for some $k \in \mathbb{R}$. Then for any $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$ satisfying $\beta - \frac{n-1}{n}\alpha \ge 1/n$, there exists a positive constant $A = A(n, k, \alpha, \beta)$, depending only on n, k, α, and β, such that for any $u \in \mathcal{D}(M)$,*

$$\left(\int_M |u|^{n/(n-1)} v^\alpha \, dv(g)\right)^{(n-1)/n} \le A \int_M (|\nabla u| + |u|) v^\beta \, dv(g)$$

where $v(x) = \frac{1}{\operatorname{Vol}_g(B_x(1))}$, $x \in M$.

PROOF: Let (x_i) be a sequence of points of M such that

1. $M = \bigcup_i B_{x_i}(\frac{1}{2})$,
2. $B_{x_i}(\frac{1}{4}) \cap B_{x_j}(\frac{1}{4}) = \emptyset$ if $i \ne j$, and
3. there exists $N = N(n, k)$, depending only on n and k, such that each point of M has a neighborhood which intersects at most N of the $B_{x_i}(1)$'s.

The existence of such a sequence is given by Lemma 1.1. Let also

$$\rho : [0, \infty) \to [0, 1]$$

be defined by

$$\rho(t) = \begin{cases} 1 & \text{if } 0 \le t \le \frac{1}{2} \\ 3 - 4t & \text{if } \frac{1}{2} \le t \le \frac{3}{4} \\ 0 & \text{if } t \ge \frac{3}{4} \end{cases}$$

and let

$$\alpha_i(x) = \rho\big(d_g(x_i, x)\big)$$

where d_g denotes the distance associated to g and $x \in M$. Clearly, α_i is Lipschitz with compact support. Hence, by Proposition 2.4, $\alpha_i \in H_1^1(M)$. Furthermore, since $\operatorname{supp} \alpha_i \subset B_{x_i}(\frac{3}{4})$, we get without any difficulty that $\alpha_i \in H_{0,1}^1(B_{x_i}(1))$. Let

$$\eta_i = \frac{\alpha_i}{\sum_m \alpha_m}$$

Then, since $|\nabla \alpha_i| \le 4$ a.e., we get by (3) that for any i, $\eta_i \in H_{0,1}^1(B_{x_i}(1))$, (η_i) is a partition of unity subordinate to the covering $(B_{x_i}(1))$, $\nabla \eta_i$ exists almost everywhere, and there exists a positive constant $H = H(N)$ such that $|\nabla \eta_i| \le H$ a.e. Independently, by Theorem 1.1, we get that there exists a positive constant $C = C(n, k)$ such that for any $x \in M$,

$$\operatorname{Vol}_g\big(B_x(1)\big) \ge C \operatorname{Vol}_g\big(B_x(2)\big)$$

As a consequence, for any $x \in M$ and any $y \in B_x(1)$,

$$\operatorname{Vol}_g\big(B_x(1)\big) \ge C \operatorname{Vol}_g\big(B_y(1)\big)$$

(since $y \in B_x(1)$ implies that $B_y(1) \subset B_x(2)$). Similarly, for any $x \in M$ and any $y \in B_x(1)$, we clearly have (by symmetry) that

$$\operatorname{Vol}_g\big(B_y(1)\big) \ge C \operatorname{Vol}_g\big(B_x(1)\big)$$

Furthermore, once more by Theorem 1.1, there exists a positive constant $V = V(n, k)$ such that for any $x \in M$,

$$\operatorname{Vol}_g\big(B_x(1)\big) \le V$$

Let $\alpha, \beta \in \mathbb{R}$ be such that

$$\beta - \frac{n-1}{n}\alpha \ge \frac{1}{n}$$

Multiplying (I_1) by $\operatorname{Vol}_g(B_x(1))^{-(n-1)\alpha/n}$, and according to what we have just said, we get that there exists a positive constant $\tilde{A} = \tilde{A}(n, k)$ such that for any $x \in M$

and any $u \in C^\infty(B_x(1))$,

$$\begin{aligned}
&\left(\int_{B_x(1)} |u|^{n/(n-1)} v^\alpha \, dv(g)\right)^{(n-1)/n} \\
&\le \tilde{A}\,\mathrm{Vol}_g\left(B_x(1)\right)^{-((n-1)\alpha+1)/n} \int_{B_x(1)} (|\nabla u| + |u|) dv(g) \\
&\le \tilde{A}\,\mathrm{Vol}_g\left(B_x(1)\right)^{\beta-((n-1)\alpha+1)/n}\,\mathrm{Vol}_g\left(B_x(1)\right)^{-\beta} \int_{B_x(1)} (|\nabla u| + |u|) dv(g) \\
&\le A' \int_{B_x(1)} (|\nabla u| + |u|) v^\beta \, dv(g)
\end{aligned}$$

where

$$A' = \tilde{A} V^{\beta-((n-1)\alpha+1)/n} C^{-|\beta|}$$

depends only on n, k, α, and β. As a consequence, for any i and any $u \in C^\infty(B_{x_i}(1))$,

$$\left(\int_{B_{x_i}(1)} |u|^{n/(n-1)} v^\alpha \, dv(g)\right)^{(n-1)/n} \le A' \int_{B_{x_i}(1)} (|\nabla u| + |u|) v^\beta \, dv(g)$$

Let $u \in \mathcal{D}(M)$. We then have

$$\begin{aligned}
&\left(\int_M |u|^{n/(n-1)} v^\alpha \, dv(g)\right)^{(n-1)/n} \\
&\le \sum_i \left(\int_{B_{x_i}(1)} |\eta_i u|^{n/(n-1)} v^\alpha \, dv(g)\right)^{(n-1)/n} \\
&\le A' \sum_i \int_M (|\nabla(\eta_i u)| + \eta_i |u|) v^\beta \, dv(g) \\
&\le A' \sum_i \int_M |\nabla \eta_i||u| v^\beta \, dv(g) + A' \sum_i \int_M \eta_i |\nabla u| v^\beta \, dv(g) + A' \int_M |u| v^\beta \, dv(g) \\
&\le A'NH \int_M |u| v^\beta \, dv(g) + A' \int_M |\nabla u| v^\beta \, dv(g) + A' \int_M |u| v^\beta \, dv(g) \\
&\le A \int_M (|\nabla u| + |u|) v^\beta \, dv(g)
\end{aligned}$$

where $A = A'(1 + NH)$ depends only on n, k, α, and β. Clearly, this ends the proof of the lemma. □

Let us now prove Theorem 3.8.

PROOF OF THEOREM 3.8: We proceed as in the proof of Lemma 2.1, but starting from Lemma 3.4. Let α, β be as in Lemma 3.4, and let (p, q) be two real numbers such that $1 < q < n$ and $1/p = 1/q - 1/n$. By Lemma 3.4 there exists $A = A(n, k, \alpha, \beta) > 0$ such that for any $u \in \mathcal{D}(M)$,

$$\left(\int_M |u|^{n/(n-1)} v^\alpha \, dv(g)\right)^{(n-1)/n} \le A \int_M \Big(|\nabla u| + |u|\Big) v^\beta \, dv(g)$$

Let $u \in \mathcal{D}(M)$ and set $\varphi = |u|^{p(n-1)/n}$. Applying Hölder's inequality, we get that

$$\begin{aligned}
&\left(\int_M |u|^p v^\alpha \, dv(g)\right)^{(n-1)/n} \\
&= \left(\int_M |\varphi|^{n/(n-1)} v^\alpha \, dv(g)\right)^{(n-1)/n} \\
&\leq A \int_M (|\nabla\varphi| + |\varphi|) v^\beta \, dv(g) \\
&\leq \frac{Ap(n-1)}{n} \int_M |u|^{p'} |\nabla u| v^\beta \, dv(g) + A \int_M |u|^{p(n-1)/n} v^\beta \, dv(g) \\
&\leq \frac{Ap(n-1)}{n} \left(\int_M |u|^{p'q'} v^\alpha \, dv(g)\right)^{1/q'} \left(\int_M |\nabla u|^q v^{q(\beta-\alpha/q')} dv(g)\right)^{1/q} \\
&\quad + A\left(\int_M |u|^{p'q'} v^\alpha \, dv(g)\right)^{1/q'} \left(\int_M |u|^q v^{q(\beta-\alpha/q')} \, dv(g)\right)^{1/q}
\end{aligned}$$

where $1/q + 1/q' = 1$ and $p' = p(n-1)/n - 1$. But $p'q' = p$ since $1/p = 1/q - 1/n$. As a consequence, for any $u \in \mathcal{D}(M)$,

$$\begin{aligned}
&\left(\int_M |u|^p v^\alpha \, dv(g)\right)^{1/p} \\
&\leq \frac{Ap(n-1)}{n}\left(\left(\int_M |\nabla u|^q v^\gamma dv(g)\right)^{1/q} + \left(\int_M |u|^q v^\gamma dv(g)\right)^{1/q}\right)
\end{aligned}$$

where $\gamma = q(\beta - \alpha/q')$. Noting that $\gamma/q - \alpha/p = \beta - \frac{n-1}{n}\alpha$, we end the proof of the theorem. □

To end this section, we now say some words about a result proved by Schoen and Yau [177]. Here again, the norms are disturbed by some function of the geometry of the manifold. Recall that given $(\tilde{M}, \tilde{g})$ and (M, g) two Riemannian manifolds, an immersion $\varphi : \tilde{M} \to M$ is said to be conformal if $\varphi^* g$ is a conformal metric to $\tilde{g}$, that is, of the form $\varphi^* g = e^f \tilde{g}$ for some $f \in C^\infty(\tilde{M})$. In what follows, (S^n, h) stands for the standard unit sphere of $\mathbb{R}^{n+1}$.

PROPOSITION 3.8 *Let (M, g) be a smooth Riemannian n-manifold, $n \geq 3$, not necessarily complete. Assume that there exists a conformal immersion φ from (M, g) to (S^n, h). Then for any $u \in \mathcal{D}(M)$,*

$$\begin{aligned}
&\left(\int_M |u|^{2n/(n-2)} \, dv(g)\right)^{(n-2)/n} \\
&\leq \frac{4}{n(n-2)\omega_n^{2/n}}\left(\int_M |\nabla u|^2 \, dv(g) + \frac{n-2}{4(n-1)} \int_M \mathrm{Scal}_{(M,g)} u^2 \, dv(g)\right)
\end{aligned}$$

where ω_n is the volume of (S^n, h) and $\mathrm{Scal}_{(M,g)}$ is the scalar curvature of (M, g).

PROOF: We proceed as in Schoen and Yau [177]. Define

$$Q(M) = \inf_{\{u \in \mathcal{D}(M),\, \int_M |u|^{2n/(n-2)} dv(g) = 1\}} \int_M u(L_g u) dv(g)$$

where

$$L_g u = \Delta_g u + \frac{n-2}{4(n-1)} \operatorname{Scal}_{(M,g)} u$$

is the conformal Laplacian of g. As one can easily check, for any $v \in C^\infty(M)$, $v > 0$, and any $u \in C^\infty(M)$,

$$L_g(uv) = v^{(n+2)/(n-2)} L_{g'}(u)$$

where $g' = v^{4/(n-2)} g$. By Obata [**163**],

$$Q(S^n) = \frac{n(n-2)\omega_n^{2/n}}{4}$$

The inequality of the proposition is then equivalent to $Q(M) \geq Q(S^n)$. Let $(\Omega_i)_{i \in \mathbb{N}}$ be an exhaustion of M by compact domains with smooth boundary. We then have

$$Q(M) = \lim_{i \to \infty} Q(\Omega_i)$$

Thus, in order to show that $Q(M) \geq Q(S^n)$, it is enough to show that $Q(\Omega) \geq Q(S^n)$ for any domain $\Omega \subset M$ with $\overline{\Omega}$ compact and $\partial\Omega$ smooth. Now the proof is by contradiction. Hence, we suppose that for some Ω as above, $Q(\Omega) < Q(S^n)$. By standard variational techniques, one then gets that there exists a smooth function $u > 0$ in Ω satisfying $\int_\Omega u^{2n/(n-2)} dv(g) = 1$ as well as

$$L_g u = Q(\Omega) u^{(n+2)/(n-2)} \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

If we extend u by defining $u \equiv 0$ in $M \backslash \Omega$, we then have

$$L_g u \leq Q(\Omega) u^{(n+2)/(n-2)} \quad \text{in } M, \qquad \int_M u^{2n/(n-2)} dv(g) = 1$$

where the inequality is understood in the distributional sense. Let φ be the conformal immersion of the proposition, $\varphi : (M, g) \to (S^n, h)$. We define a function $\tilde{u}$ on S^n by $\tilde{u} \equiv 0$ in $S^n \backslash \varphi(\overline{\Omega})$, and for $y \in \varphi(\overline{\Omega})$,

$$\tilde{u}(y) = \max_{x \in \varphi^{-1}(y) \cap \overline{\Omega}} \alpha(x)^{-(n-2)/4} u(x)$$

where $\varphi^* h = \alpha g$. Since φ is an immersion, the set $\varphi^{-1}(y) \cap \overline{\Omega}$ is finite, and for each $x \in \varphi^{-1}(y) \cap \overline{\Omega}$ there is a neighborhood U_x of x such that φ is a diffeomorphism of U_x onto $\varphi(U_x)$, a neighborhood of y. Let φ_x^{-1} denote the inverse of this local diffeomorphism. By the conformal invariance property of the conformal Laplacian, if

$$\tilde{u}_x(y) = \beta_x(y)^{(n-2)/4} u\big(\varphi_x^{-1}(y)\big), \quad y \in \varphi(U_x)$$

where $(\varphi_x^{-1})^* g = \beta_x h$, then $L_h \tilde{u}_x \leq Q(\Omega) \tilde{u}_x^{(n+2)/(n-2)}$ in $\varphi(U_x)$. Hence, we see that $\tilde{u}$ is a nonnegative Lipschitz function on S^n satisfying

$$L_h \tilde{u} \leq Q(\Omega) \tilde{u}^{(n+2)/(n-2)}$$

on S^n. Again by conformal invariance, we have

$$\int_{\varphi(U_x)} \tilde{u}_x^{2n/(n-2)} dv(h) = \int_{U_x} u^{2n/(n-2)} dv(g)$$

and hence we see that $\int_{S^n} \tilde{u}^{2n/(n-2)} dv(h) \leq 1$. By integrating the differential inequality satisfied by $\tilde{u}$, one then gets that

$$\int_{S^n} \tilde{u}(L_h \tilde{u}) dv(h) \leq Q(\Omega) \int_{S^n} \tilde{u}^{2n/(n-2)} dv(h)$$

Since $\int_{S^n} \tilde{u}^{2n/(n-2)} dv(h) \leq 1$, this inequality implies that $Q(S^n) \leq Q(\Omega)$, a contradiction. This ends the proof of the proposition. □

Concerning the assumptions of Proposition 3.8, one has (see, for instance, Kulkarni in [**138**]) that for any simply connected, conformally flat Riemannian n-manifold (M, g), there exists a conformal immersion from (M, g) to the standard sphere (S^n, h). Recall here that a Riemannian n-manifold (M, g) is said to be conformally flat if for any $x \in M$, there exist $f \in C^\infty(M)$ and Ω an open neighborhood of x such that $e^f g$ is flat on Ω. Spaces of constant curvature are conformally flat. More generally, when $n \geq 4$, (M, g) is conformally flat if and only if its Weyl curvature is zero. Combining the above fact with Proposition 3.8, one gets the following result:

COROLLARY 3.2 *Let (M, g) be a smooth, complete, conformally flat n-manifold, $n \geq 3$. For any simply connected domain Ω of M, and any $u \in \mathcal{D}(M)$,*

$$\left(\int_M |u|^{2n/(n-2)} dv(g) \right)^{(n-2)/n}$$
$$\leq \frac{4}{n(n-2)\omega_n^{2/n}} \left(\int_M |\nabla u|^2 dv(g) + \frac{n-2}{4(n-1)} \int_M \mathrm{Scal}_{(M,g)} u^2 dv(g) \right)$$

where ω_n is the volume of (S^n, h) and $\mathrm{Scal}_{(M,g)}$ is the scalar curvature of (M, g).

As a final remark, consider (M, g) a smooth, compact Riemannian n-manifold. Suppose g is Einstein. By Obata [**163**] one has that if (M, g) is not conformally diffeomorphic to the standard sphere (S^n, h), then, up to a constant scale factor, g is the unique metric of constant scalar curvature in its conformal class. As a consequence of the resolution of the Yamabe problem by Aubin [**9**] and Schoen [**175**], one easily gets that for any compact Einstein n-manifold (M, g), any g' in the conformal class of g, and any $u \in C^\infty(M)$,

$$\frac{n-2}{4(n-1)} \mathrm{Scal}_{(M,g)} \mathrm{Vol}_{(M,g)}^{2/n} \left(\int_M |u|^{2n/(n-2)} dv(g') \right)^{(n-2)/n}$$
$$\leq \int_M |\nabla u|^2 dv(g') + \frac{n-2}{4(n-1)} \int_M \mathrm{Scal}_{(M,g')} u^2 dv(g')$$

When $\mathrm{Scal}_{(M,g)} > 0$, this provides us with disturbed Sobolev inequalities. Such inequalities can be useful. We refer the reader to Hebey-Vaugon [**118**] for an example of an application.

CHAPTER 4

Best Constants in the Compact Setting I

Let (M, g) be a smooth, compact Riemannian n-manifold. By the Sobolev embedding theorem one has that for any $q \in [1, n)$ real, $H_1^q(M) \subset L^p(M)$ where $1/p = 1/q - 1/n$. We write here that for any $q \in [1, n)$, there exist two real numbers A and B, that may depend, of course, on the metric, such that for any $u \in H_1^q(M)$,

$(\mathrm{I}^1_{q.\mathrm{gen}})$
$$\left(\int_M |u|^p \, dv(g)\right)^{1/p} \leq A\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} + B\left(\int_M |u|^q \, dv(g)\right)^{1/q}$$

In such a notation, gen stands for generic, and $(\mathrm{I}^1_{q.\mathrm{gen}})$ will be referred to as the generic Sobolev inequality of order q. Let us now start with some definitions. First we define

$$\mathcal{A}_q(M) = \left\{A \in \mathbb{R} \text{ s.t. } \exists B \in \mathbb{R} \text{ for which } (\mathrm{I}^1_{q.\mathrm{gen}}) \text{ is valid}\right\}$$

and in a parallel manner, we define

$$\mathcal{B}_q(M) = \left\{B \in \mathbb{R} \text{ s.t. } \exists A \in \mathbb{R} \text{ for which } (\mathrm{I}^1_{q.\mathrm{gen}}) \text{ is valid}\right\}$$

Clearly, if $A \in \mathcal{A}_q(M)$, and if $A' \geq A$, then $A' \in \mathcal{A}_q(M)$. In the same way, if $B \in \mathcal{B}_q(M)$, and if $B' \geq B$, then $B' \in \mathcal{B}_q(M)$. As a consequence, $\mathcal{A}_q(M)$ and $\mathcal{B}_q(M)$ are intervals of right extremity $+\infty$. The relevant real numbers for $\mathcal{A}_q(M)$ and $\mathcal{B}_q(M)$ are then

$$\begin{cases} \alpha_q(M) = \inf \mathcal{A}_q(M) \\ \beta_q(M) = \inf \mathcal{B}_q(M) \end{cases}$$

By definition, $\alpha_q(M)$ and $\beta_q(M)$ are the best constants associated to the generic inequality $(\mathrm{I}^1_{q.\mathrm{gen}})$ of order q.

Two parallel research programs are associated to this notion of best constants. In the first one, priority is given to the best (first) constant $\alpha_q(M)$. In the second one, priority is given to the best (second) constant $\beta_q(M)$. For the sake of clarity, we start with the first two questions of these programs. Two more questions will be asked in Chapter 5.

Before stating the two questions we note that to say that $\mathcal{A}_q(M)$ is a closed set, in other words, to say that $\alpha_q(M) \in \mathcal{A}_q(M)$, means that there exists $B \in \mathbb{R}$ such that for any $u \in H_1^q(M)$,

$(\mathrm{I}^1_{q,\mathrm{opt}})$

$$\left(\int_M |u|^p \, dv(g)\right)^{1/p} \le \alpha_q(M)\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} + B\left(\int_M |u|^q \, dv(g)\right)^{1/q}$$

In a parallel manner, to say that $\mathcal{B}_q(M)$ is a closed set, in other words, to say that $\beta_q(M) \in \mathcal{B}_q(M)$, means that there exists $A \in \mathbb{R}$ such that for any $u \in H_1^q(M)$,

$(\mathrm{J}^1_{q,\mathrm{opt}})$

$$\left(\int_M |u|^p \, dv(g)\right)^{1/p} \le A\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} + \beta_q(M)\left(\int_M |u|^q \, dv(g)\right)^{1/q}$$

In what follows, we use the letter I when dealing with inequalities where priority is given to the best (first) constant $\alpha_q(M)$, and the letter J when dealing with inequalities where priority is given to the best (second) constant $\beta_q(M)$. We refer to these two inequalities as the optimal Sobolev inequalities of order q.

Program $\mathcal{A}$, Part I	Program $\mathcal{B}$, Part I
Question 1$\mathcal{A}$: Is it possible to compute explicitly $\alpha_q(M)$?	Question 1$\mathcal{B}$: Is it possible to compute explicitly $\beta_q(M)$?
Question 2$\mathcal{A}$: Is $\mathcal{A}_q(M)$ a closed set? In other words, is $(\mathrm{I}^1_{q,\mathrm{opt}})$ valid?	Question 2$\mathcal{B}$: Is $\mathcal{B}_q(M)$ a closed set? In other words, is $(\mathrm{J}^1_{q,\mathrm{opt}})$ valid?

Now we start with the discussion of these two programs. As one will see, questions 1$\mathcal{B}$ and 2$\mathcal{B}$ are very simple. This will not be the case for questions 1$\mathcal{A}$ and 2$\mathcal{A}$, which are much more difficult.

4.1. Program $\mathcal{B}$, Part I

As said above, the mathematics involved in questions 1$\mathcal{B}$ and 2$\mathcal{B}$ are very simple. The result that answers these questions is the following: Given (M, g) a smooth, compact Riemannian manifold, $\mathrm{Vol}_{(M,g)}$ denotes the volume of (M, g).

THEOREM 4.1 *For any smooth, compact Riemannian n-manifold (M, g), and for any $q \in [1, n)$ real, $\beta_q(M) = \mathrm{Vol}_{(M,g)}^{-1/n}$. Moreover, $\mathcal{B}_q(M)$ is a closed set, so that for any smooth, compact Riemannian n-manifold (M, g), and any $q \in [1, n)$ real, there exists $A \in \mathbb{R}$ such that for any $u \in H_1^q(M)$,*

$$(\mathrm{J}^1_{q,\mathrm{opt}}) \qquad \begin{aligned} &\left(\int_M |u|^p \, dv(g)\right)^{1/p} \le \\ &A\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} + \mathrm{Vol}_{(M,g)}^{-1/n}\left(\int_M |u|^q \, dv(g)\right)^{1/q} \end{aligned}$$

where $1/p = 1/q - 1/n$. In other words, $\beta_q(M) = \mathrm{Vol}_{(M,g)}^{-1/n}$ and the optimal inequality $(\mathrm{J}^1_{q,\mathrm{opt}})$ is valid.

PROOF: Let (M, g) and $q \in [1, n)$ be given. On the one hand, by taking $u = 1$ in $(I^1_{q,\text{gen}})$, one gets that $B \geq \mathrm{Vol}_{(M,g)}^{-1/n}$. In particular, $\beta_q(M) \geq \mathrm{Vol}_{(M,g)}^{-1/n}$. On the other hand, one has by the Sobolev-Poincaré inequality (see Section 2.8 of Chapter 2) that there exists some positive real number $A = A(M, g, q)$ such that for any $u \in H_1^q(M)$,

$$\left(\int_M |u - \overline{u}|^p dv(g)\right)^{1/p} \leq A\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q}$$

where $1/p = 1/q - 1/n$ and $\overline{u} = \mathrm{Vol}_{(M,g)}^{-1} \int_M u \, dv(g)$. As a consequence, for any $u \in H_1^q(M)$,

$$\left(\int_M |u|^p \, dv(g)\right)^{1/p} \leq A\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} + \mathrm{Vol}_{(M,g)}^{(1/p)-1}\left|\int_M u \, dv(g)\right|$$

But, by Hölder's inequality,

$$\left|\int_M u \, dv(g)\right| \leq \mathrm{Vol}_{(M,g)}^{1-(1/q)}\left(\int_M |u|^q \, dv(g)\right)^{1/q}$$

Since $1/p = 1/q - 1/n$, these two inequalities imply that for any $u \in H_1^q(M)$,

$$\left(\int_M |u|^p \, dv(g)\right)^{1/p} \leq A\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} + \mathrm{Vol}_{(M,g)}^{-1/n}\left(\int_M |u|^q \, dv(g)\right)^{1/q}$$

Combining this inequality with the fact that $\beta_q(M) \geq \mathrm{Vol}_{(M,g)}^{-1/n}$, one sees that $\beta_q(M) = \mathrm{Vol}_{(M,g)}^{-1/n}$ and that $\mathcal{B}_q(M)$ is a closed set. This ends the proof of the theorem. □

Concerning such a result, one knows on which geometric quantities the remaining constant A depends. As shown by Ilias **[123]**, A depends only on n, q, a lower bound for the Ricci curvature, a lower bound for the volume, and an upper bound for the diameter. More precisely, given (M, g) a smooth, compact Riemannian n-manifold, suppose that its Ricci curvature, volume, and diameter satisfy

$$\mathrm{Rc}_{(M,g)} \geq kg\,, \ \mathrm{Vol}_{(M,g)} \geq v\,, \ \mathrm{diam}_{(M,g)} \leq d$$

where k, $v > 0$ and $d > 0$ are real numbers. Then for any $q \in [1, n)$, there exists a positive constant $A = A(n, q, k, v, d)$, depending only on n, q, k, v, and d, such that for any $u \in H_1^q(M)$,

$$(J^1_{q,\text{opt}}) \quad \left(\int_M |u|^p \, dv(g)\right)^{1/p} \leq A\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} + \mathrm{Vol}_{(M,g)}^{-1/n}\left(\int_M |u|^q \, dv(g)\right)^{1/q}$$

where $1/p = 1/q - 1/n$. In other words, if two compact Riemannian manifolds of same dimension satisfy the same lower bounds on the Ricci curvature and volume, and the same upper bound on the diameter, then they satisfy the same optimal inequality ($J^1_{q,\mathrm{opt}}$).

Independently, we considered inequality ($I^1_{q,\mathrm{gen}}$) as a starting inequality. One can consider instead inequality ($I^q_{q,\mathrm{gen}}$) below. Clearly, there exist $A, B \in \mathbb{R}$ such that for any $u \in H_1^q(M)$,

$$(I^q_{q,\mathrm{gen}}) \qquad \left(\int_M |u|^p\,dv(g)\right)^{q/p} \le A\int_M |\nabla u|^q\,dv(g) + B\int_M |u|^q\,dv(g)$$

where $1/p = 1/q - 1/n$. Roughly speaking, $(I^q_{q,\mathrm{gen}}) = (I^1_{q,\mathrm{gen}})^q$, in the sense that we elevate each term in ($I^1_{q,\mathrm{gen}}$) to the power q. As one can easily check, by Theorem 4.1, the best constant B in such an inequality is $\beta_q(M)^q = \mathrm{Vol}_{(M,g)}^{-q/n}$. Instead of considering inequality ($J^1_{q,\mathrm{opt}}$), one can now consider the stronger inequality $(J^q_{q,\mathrm{opt}}) = (J^1_{q,\mathrm{opt}})^q$. That is, there exists $A \in \mathbb{R}$ such that for any $u \in H_1^q(M)$,

$$(J^q_{q,\mathrm{opt}}) \qquad \left(\int_M |u|^p\,dv(g)\right)^{q/p} \le A\int_M |\nabla u|^q\,dv(g) + \beta_q(M)^q\int_M |u|^q\,dv(g)$$

In the spirit of what we did above, one can ask if such an inequality is valid. The case $q = 2$ received an affirmative answer by Bakry **[17]**. Its proof extends to the case $\hat{2}^\star \le q \le 2$, with $q \ne 2$ if $n = 2$, where

$$\hat{2}^\star = 2n/(n+2)$$

is the conjugate exponent of $2^\star = 2n/(n-2)$. The remaining case where $1 \le q \le \hat{2}^\star$ has been treated by Druet, so that, as stated below, Theorem 4.2 is due to Bakry **[17]** and Druet (oral communication). As one will see, the arguments involved in the proof of such a result are still simple, though more delicate than the ones we used to solve questions $1\mathcal{B}$ and $2\mathcal{B}$.

THEOREM 4.2 *Let (M, g) be a smooth, compact Riemannian n-manifold. For any $q \in [1, 2)$ if $n = 2$, and any $q \in [1, 2]$ if $n \ge 3$, there exists $A \in \mathbb{R}$ such that for any $u \in H_1^q(M)$,*

$$(J^q_{q,\mathrm{opt}}) \qquad \left(\int_M |u|^p\,dv(g)\right)^{q/p} \le A\int_M |\nabla u|^q\,dv(g) + \mathrm{Vol}_{(M,g)}^{-q/n}\int_M |u|^q\,dv(g)$$

where $1/p = 1/q - 1/n$. In particular, the optimal inequality ($J^q_{q,\mathrm{opt}}$) is valid for any q on any 2-dimensional compact Riemannian manifold, and for any $q \le 2$ on any compact Riemannian n-dimensional manifold if $n \ge 3$.

PROOF: Let (M, g) be a smooth, compact Riemannian n-manifold, $n \ge 2$. Following Bakry **[17]** and as a starting point, we prove that for any $p \ge 2$, and any $u \in L^p(M)$,

$$\left(\int_M |u|^p\,dv(g)\right)^{2/p} \le$$
$$\mathrm{Vol}_{(M,g)}^{-2(p-1)/p}\left(\int_M u\,dv(g)\right)^2 + (p-1)\left(\int_M |u-\overline{u}|^p\,dv(g)\right)^{2/p}$$

where

$$\overline{u} = \frac{1}{\mathrm{Vol}_{(M,g)}} \int_M u \, dv(g)$$

As one can easily check, it is enough to prove such an inequality for any $u \in C^0(M)$. For homogeneity reasons, and since the inequality is obviously satisfied if $\int_M u \, dv(g) = 0$, we can then restrict ourselves to functions $u \in C^0(M)$ for which $\int_M u \, dv(g) = \mathrm{Vol}_{(M,g)}$. For such functions, one can write that

$$u = 1 + tv \quad \text{where } t \geq 0 \text{ is real and } v \in C^0(M)$$

is such that $\int_M v \, dv(g) = 0$ and $\int_M v^2 \, dv(g) = 1$. The above inequality then becomes

$$\left(\int_M |1 + tv|^p \, dv(g) \right)^{2/p} \leq \mathrm{Vol}_{(M,g)}^{2/p} + t^2(p-1) \left(\int_M |v|^p \, dv(g) \right)^{2/p}$$

Let

$$\varphi(t) = \left(\int_M |1 + tv|^p \, dv(g) \right)^{2/p}$$

Then $\varphi(0) = \mathrm{Vol}_{(M,g)}^{2/p}$ and $\varphi'(0) = 0$. As a consequence, one just has to prove that

$$\varphi''(t) \leq 2(p-1) \left(\int_M |v|^p \, dv(g) \right)^{2/p}$$

to get the inequality. Here, a simple computation shows that

$$\begin{aligned} \varphi''(t) = 2p\left(\frac{2}{p} - 1\right) &\left(\int_M |1 + tv|^{p-1} v \, dv(g) \right)^2 \left(\int_M |1 + tv|^p \, dv(g) \right)^{(2/p)-2} \\ &+ 2(p-1) \left(\int_M |1 + tv|^p \, dv(g) \right)^{(2/p)-1} \int_M |1 + tv|^{p-2} v^2 \, dv(g) \end{aligned}$$

But $p \geq 2$, so that the first member in the right-hand side of this equality is nonpositive, while by Hölder's inequality

$$\int_M |1 + tv|^{p-2} v^2 \, dv(g) \leq \left(\int_M |1 + tv|^p \, dv(g) \right)^{1-2/p} \left(\int_M |v|^p \, dv(g) \right)^{2/p}$$

Hence,

$$\varphi''(t) \leq 2(p-1) \left(\int_M |v|^p \, dv(g) \right)^{2/p}$$

and we get the inequality we were looking for. From now on, let q be such that $\hat{2}^\star \leq q \leq 2$, with $q \neq 2$ if $n = 2$, and let p be given by $1/p = 1/q - 1/n$. Since $q \geq \hat{2}^\star$, $p \geq 2$. Let $A > 0$ be such that for any $u \in H_1^q(M)$,

$$\left(\int_M |u - \overline{u}|^p \, dv(g) \right)^{q/p} \leq A \int_M |\nabla u|^q \, dv(g)$$

The existence of A comes from the Sobolev-Poincaré inequality we discussed in Section 2.8 of Chapter 2. Let $u \in H_1^q(M)$. Since $q \leq 2$, $q/2 \leq 1$, and for a

and b two nonnegative real numbers, $(a+b)^{q/2} \le a^{q/2} + b^{q/2}$. According to the inequality of the beginning of the proof, we can then write that

$$
\begin{aligned}
&\left(\int_M |u|^p \, dv(g)\right)^{q/p} \\
&\le \mathrm{Vol}_{(M,g)}^{-q(p-1)/p} \left|\int_M u \, dv(g)\right|^q + (p-1)^{q/2}\left(\int_M |u-\overline{u}|^p \, dv(g)\right)^{q/p} \\
&\le \mathrm{Vol}_{(M,g)}^{-q(p-1)/p} \left|\int_M u \, dv(g)\right|^q + (p-1)^{q/2} A \int_M |\nabla u|^q \, dv(g) \\
&\le \mathrm{Vol}_{(M,g)}^{q-1-(q(p-1)/p)} \int_M |u|^q \, dv(g) + (p-1)^{q/2} A \int_M |\nabla u|^q \, dv(g)
\end{aligned}
$$

and since

$$q - 1 - \frac{q(p-1)}{p} = -\frac{q}{n}$$

this proves that $(\mathrm{J}^q_{q,\mathrm{opt}})$ is valid for $q \in [\hat{2}^\star, 2]$, with $q \ne 2$ if $n = 2$. Suppose now that $q \in [1, \hat{2}^\star]$ and let us follow the argument of Druet. Since $q \le \hat{2}^\star$, for p such that $1/p = 1/q - 1/n$, one has that $p \le 2$. Following Druet (oral communication), we start with the proof that for any $u \in L^p(M)$,

$$
\begin{aligned}
\left(\int_M |u|^p \, dv(g)\right)^{q/p} \le\ & \mathrm{Vol}_{(M,g)}^{(q/p)-q} \left|\int_M u \, dv(g)\right|^q \\
& + \left(1 + p(p-1)^{(p-1)}\right)^{q/p}\left(\int_M |u-\overline{u}|^p \, dv(g)\right)^{q/p}
\end{aligned}
$$

Here, as one can easily check, we can assume that $u \in C^0(M)$ and that $\int_M u \, dv(g) \ne 0$. Let us write that

$$u = \overline{u}(1+v)$$

where v is such that $\int_M v \, dv(g) = 0$. With obvious notation,

$$
\begin{aligned}
\int_M |1+v|^p \, dv(g) = & \int_{\{v \ge 0\}} |1+v|^p \, dv(g) + \int_{\{-1 \le v < 0\}} |1+v|^p \, dv(g) \\
& + \int_{\{v < -1\}} |1+v|^p \, dv(g)
\end{aligned}
$$

Since $p \le 2$, the following holds:

$$
\begin{cases}
\text{for } x \ge 0 \text{ real}, \ (1+x)^p \le 1 + px + x^p \,, \\
\text{for } 0 \le x \le 1, \ (1-x)^p \le 1 - px + x^p \,, \\
\text{for } x \ge 1 \text{ real}, \ (x-1)^p \le x^p
\end{cases}
$$

Hence,

$$\begin{aligned}\int_M |1+v|^p\,dv(g) &\le \int_{\{v\ge 0\}} dv(g) + \int_{\{v\ge 0\}} |v|^p\,dv(g) + p\int_{\{v\ge 0\}} v\,dv(g) + \int_{\{-1\le v<0\}} dv(g)\\ &\quad + \int_{\{-1\le v<0\}} |v|^p\,dv(g) + p\int_{\{-1\le v<0\}} v\,dv(g) + \int_{\{v<-1\}} |v|^p\,dv(g)\end{aligned}$$

so that

$$\begin{aligned}\int_M |1+v|^p\,dv(g) &= \mathrm{Vol}_g(\{v\ge -1\}) + \int_M |v|^p\,dv(g) + p\int_{\{v\ge -1\}} v\,dv(g)\\ &= \mathrm{Vol}_g(\{v\ge -1\}) + \int_M |v|^p\,dv(g) + p\int_M v\,dv(g) - p\int_{\{v<-1\}} v\,dv(g)\\ &= \mathrm{Vol}_g(\{v\ge -1\}) + \int_M |v|^p\,dv(g) + p\int_{\{v<-1\}} |v|dv(g)\end{aligned}$$

Then, by Hölder's inequality,

$$\begin{aligned}\int_M |1+v|^p\,dv(g) \le \mathrm{Vol}_g(\{v\ge -1\}) &+ \int_M |v|^p\,dv(g)\\ &+ p\,\mathrm{Vol}_g(\{v<-1\})^{(p-1)/p}\left(\int_M |v|^p\,dv(g)\right)^{1/p}\end{aligned}$$

Set

$$X_0 = \mathrm{Vol}_g(\{v\ge -1\})$$

and for $X \in [0, V)$ real, let

$$f(X) = X + \|v\|_p^p + p\|v\|_p(V-X)^{(p-1)/p}$$

where V stands for $\mathrm{Vol}_{(M,g)}$ and $\|\cdot\|_p$ for the L^p-norm. Then,

$$f'(X) = 1 - (p-1)\|v\|_p(V-X)^{-1/p}$$

so that

$$\lim_{X\to V} f'(X) = -\infty$$

Suppose first that $f'(0) < 0$, in other words, that

$$1 < (p-1)\|v\|_p V^{-1/p}$$

Since f' is nonincreasing, one gets that $f(X_0) \le f(0)$. Hence,

$$\int_M |1+v|^p\,dv(g) \le \|v\|_p^p + p\|v\|_p V^{(p-1)/p} \le \left(1 + p(p-1)^{p-1}\right)\|v\|_p^p$$

Suppose now that $f'(0) \ge 0$, in other words, that

$$1 \ge (p-1)\|v\|_p V^{-1/p}$$

Let

$$X_c = V - (p-1)^p \|v\|_p^p$$

Then X_c is such that $f'(X_c) = 0$ and one has that

$$f(X_0) \le f(X_c)$$

As one can easily check, this leads to

$$\int_M |1+v|^p \, dv(g) \le V + \|v\|_p^p + p(p-1)^{p-1}\|v\|_p^p$$

Summarizing,

$$\int_M |1+v|^p \, dv(g) \le \mathrm{Vol}_{(M,g)} + \left(1 + p(p-1)^{p-1}\right) \int_M |v|^p \, dv(g)$$

and since $q/p \le 1$, one can write that

$$\left(\int_M |1+v|^p \, dv(g)\right)^{q/p} \le$$

$$\mathrm{Vol}_{(M,g)}^{q/p} + \left(1 + p(p-1)^{p-1}\right)^{q/p} \left(\int_M |v|^p \, dv(g)\right)^{q/p}$$

Multiplying such an inequality by $|\overline{u}|^q$ one then gets the inequality we were looking for: $\forall u \in L^p(M)$,

$$\left(\int_M |u|^p \, dv(g)\right)^{q/p} \le \mathrm{Vol}_{(M,g)}^{(q/p)-q} \left|\int_M u \, dv(g)\right|^q$$
$$+ \left(1 + p(p-1)^{(p-1)}\right)^{q/p} \left(\int_M |u - \overline{u}|^p \, dv(g)\right)^{q/p}$$

Independently, by Sobolev-Poincaré, there exists $A > 0$ such that for any $u \in H_1^q(M)$,

$$\left(\int_M |u - \overline{u}|^p \, dv(g)\right)^{q/p} \le A \int_M |\nabla u|^q \, dv(g)$$

while, by Hölder's inequality,

$$\left|\int_M u \, dv(g)\right|^q \le \mathrm{Vol}_{(M,g)}^{q-1} \int_M |u|^q \, dv(g)$$

Hence, for any $u \in H_1^q(M)$,

$$\left(\int_M |u|^p \, dv(g)\right)^{q/p} \le \mathrm{Vol}_{(M,g)}^{(q/p)-1} \int_M |u|^q \, dv(g)$$
$$+ \left(1 + p(p-1)^{p-1}\right)^{q/p} A \int_M |\nabla u|^q \, dv(g)$$

and since

$$\frac{q}{p} - 1 = -\frac{q}{n}$$

one gets that $(\mathrm{J}_{q,\mathrm{opt}}^q)$ is also valid if $q \in [1, \hat{2}^\star]$. This ends the proof of the theorem. □

Let us now deal with the validity of $(J^q_{q.\mathrm{opt}})$ for $q > 2$. A very simple argument, somehow inspired by the argument of Bakry, shows that $(J^q_{q.\mathrm{opt}})$ is never valid in such a case. As far as we know, and as surprising as it may seem, this result is stated there for the first time.

PROPOSITION 4.1 *For any smooth, compact Riemannian n-manifold, $n \geq 3$, and any $q \in (2, n)$, $(J^q_{q.\mathrm{opt}})$ is not valid.*

PROOF: Let (M, g) be a smooth, compact Riemannian n-manifold, $n \geq 3$, and let $q \in (2, n)$ be given. Let also $u \in C^\infty(M)$ be some nonconstant function. For $t > 2$ real, and $\varepsilon > 0$, we define

$$\varphi_t(\varepsilon) = \int_M |1 + \varepsilon u|^t \, dv(g)$$

Clearly, one has that

$$\varphi_t(\varepsilon) = \mathrm{Vol}_{(M,g)} + t\left(\int_M u \, dv(g)\right)\varepsilon + \frac{t(t-1)}{2}\left(\int_M u^2 \, dv(g)\right)\varepsilon^2 + o(\varepsilon^2)$$

Hence,

$$\begin{aligned}\int_M |1 + \varepsilon u|^q \, dv(g) &= \mathrm{Vol}_{(M,g)} + q\left(\int_M u \, dv(g)\right)\varepsilon \\ &\quad + \frac{q(q-1)}{2}\left(\int_M u^2 \, dv(g)\right)\varepsilon^2 + o(\varepsilon^2)\end{aligned}$$

and

$$\begin{aligned}\left(\int_M |1 + \varepsilon u|^p \, dv(g)\right)^{q/p} &= \mathrm{Vol}^{q/p}_{(M,g)} + q\,\mathrm{Vol}^{\frac{q}{p}-1}_{(M,g)}\left(\int_M u \, dv(g)\right)\varepsilon \\ &\quad + \frac{q(p-1)}{2}\mathrm{Vol}^{\frac{q}{p}-1}_{(M,g)}\left(\int_M u^2 \, dv(g)\right)\varepsilon^2 \\ &\quad + \frac{q(q-p)}{2}\mathrm{Vol}^{\frac{q}{p}-2}_{(M,g)}\left(\int_M u \, dv(g)\right)^2\varepsilon^2 + o(\varepsilon^2)\end{aligned}$$

Suppose now that $(J^q_{q.\mathrm{opt}})$ is valid. Noting that for $q > 2$,

$$\int_M |\nabla(1 + \varepsilon u)|^q \, dv(g) = o(\varepsilon^2)$$

one would get that for any $\varepsilon > 0$,

$$\begin{aligned}
&\mathrm{Vol}_{(M,g)}^{q/p} + q\,\mathrm{Vol}_{(M,g)}^{\frac{q}{p}-1}\left(\int_M u\,dv(g)\right)\varepsilon \\
&\quad + \frac{q(p-1)}{2}\mathrm{Vol}_{(M,g)}^{\frac{q}{p}-1}\left(\int_M u^2\,dv(g)\right)\varepsilon^2 \\
&\quad + \frac{q(q-p)}{2}\mathrm{Vol}_{(M,g)}^{\frac{q}{p}-2}\left(\int_M u\,dv(g)\right)^2\varepsilon^2 \\
&\quad \leq \mathrm{Vol}_{(M,g)}^{1-\frac{q}{n}} + q\,\mathrm{Vol}_{(M,g)}^{-\frac{q}{n}}\left(\int_M u\,dv(g)\right)\varepsilon \\
&\qquad + \frac{q(q-1)}{2}\mathrm{Vol}_{(M,g)}^{-\frac{q}{n}}\left(\int_M u^2\,dv(g)\right)\varepsilon^2 + o(\varepsilon^2)
\end{aligned}$$

But

$$\frac{q}{p} = 1 - \frac{q}{n}$$

so that, as one can easily check, such an inequality implies that

$$(p-1)\int_M u^2\,dv(g) \leq (p-q)\frac{1}{\mathrm{Vol}_{(M,g)}}\left(\int_M u\,dv(g)\right)^2 + (q-1)\int_M u^2\,dv(g)$$

This means again that

$$\mathrm{Vol}_{(M,g)}\int_M u^2\,dv(g) \leq \left(\int_M u\,dv(g)\right)^2$$

which is impossible as soon as u is nonconstant. This ends the proof of the proposition. □

Before studying questions $1\mathcal{A}$ and $2\mathcal{A}$, let us now say some words about the role of $\alpha_q(M)$ when studying partial differential equations on Riemannian manifolds. This is the subject of the next section. This section can be omitted from a first reading.

4.2. The Role of $\alpha_q(M)$

Let (M, g) be a smooth, compact Riemannian n-manifold, $n \geq 3$. The conformal class $[g]$ of g is

$$[g] = \left\{\tilde{g} = u^{4/(n-2)}g,\ u \in C^\infty(M),\ u > 0\right\}$$

If $\tilde{g} = u^{4/(n-2)}g$ is a conformal metric to g, one has that

$$\text{(E}_1\text{)}\qquad \frac{4(n-1)}{n-2}\Delta_g u + \mathrm{Scal}_g\, u = \mathrm{Scal}_{\tilde{g}}\, u^{(n+2)/(n-2)}$$

where Scal_g and $\mathrm{Scal}_{\tilde{g}}$ are the scalar curvatures of g and $\tilde{g}$. In 1960, Yamabe [199] attempted to prove that given a compact Riemannian n-manifold (M, g), $n \geq 3$, there always exists a metric conformal to g of constant scalar curvature. Coming back to the transformation law (E_1), this means that for any compact Riemannian n-manifold (M, g), $n \geq 3$, there exists λ real, and $u \in C^\infty(M)$, $u > 0$, solution of

$$\frac{4(n-1)}{n-2}\Delta_g u + \mathrm{Scal}_g\, u = \lambda u^{(n+2)/(n-2)}$$

Yamabe's claim was that such λ and u always exist. Unfortunately, his proof contained an error, discovered in 1968 by Trudinger [**186**]. Trudinger was able to repair the proof, but only with a rather restrictive assumption on the manifold. Finally, the problem was solved in two steps, by Aubin [**9**] in 1976 and Schoen [**175**] in 1984. In particular, Schoen discovered the unexpected relevance of the positive mass theorem of general relativity. This marked a milestone in the development of the theory of nonlinear partial differential equations. While semilinear equations of Yamabe type arise in many contexts and have long been studied by analysts, this was the first time that such an equation was completely solved. A rather complete discussion of the Yamabe problem in book form can be found in Hebey [**109**].

Now let (M, g) be a smooth, compact Riemannian n-manifold, $n \geq 2$. Let also $q \in (1, n)$, and a, f two smooth functions on M. Following Druet [**73**], we discuss here the existence of positive solutions $u \in H_1^q(M)$ to the equation

$$(E_2) \qquad \Delta_{q,g} u + a(x)u^{q-1} = f(x)u^{p-1}$$

where $p = \frac{nq}{n-q}$ and

$$\Delta_{q,g} u = -\mathrm{div}_g\left(|\nabla u|^{q-2}\nabla u\right)$$

is the q-Laplacian associated to g. (Note that $\Delta_{2,g} = \Delta_g$). Such equations will be referred to as "generalized scalar curvature type equations." By regularity results (see, for instance, Druet [**73**]), any solution u of (E_2) is $C^{1,\alpha}$ for some $\alpha \in (0, 1)$. By Trudinger [**186**] it is C^∞ if $q = 2$. As a remark, one can note that the $C^{1,\alpha}$-regularity is, in general, optimal. Think of $(\mathbb{R}^n, e)$, the Euclidean space, and note that

$$u = \frac{q-1}{q}|x|^{1+\frac{1}{q-1}}$$

is a solution of $\Delta_{q,e} u = -n$ in $\mathbb{R}^n$.

We present here the following result of Druet [**73**], the formal generalization to all q of Aubin's result [**9**] proved in the case $q = 2$. It clearly illustrates the role $\alpha_q(M)$ plays when studying such type of equations. Let $L_{q,g}$ be defined by

$$L_{q,g} u = \Delta_{q,g} u + a(x)|u|^{q-2}u$$

We say that $L_{q,g}$ is coercive if there exists $\lambda > 0$ such that for any $u \in H_1^q(M)$,

$$I(u) \geq \lambda \int_M |u|^q \, dv(g)$$

where $I(u) = \int_M (L_{q,g} u)u \, dv(g)$, so that

$$I(u) = \int_M \left(|\nabla u|^q + a(x)|u|^q\right) dv(g)$$

As one can easily check, $L_{q,g}$ is certainly coercive if $a > 0$ on M. We also let

$$\Lambda = \left\{ u \in H_1^q(M) \,/\, \int_M f|u|^p \, dv(g) = 1 \right\}$$

where $p = \frac{nq}{n-q}$. One then has the following result of Druet [73]:

THEOREM 4.3 *Let (M, g) be a smooth, compact Riemannian n-manifold, $n \geq 2$, $q \in (1, n)$ some real number, and let a, f be smooth, real-valued functions on M. We assume that $L_{q,g}$ is coercive and that f is positive somewhere on M. If*

$$\left(\max_{x \in M} f(x) \right)^{\frac{q}{p}} \inf_{u \in \Lambda} I(u) < \frac{1}{\alpha_q(M)^q}$$

then equation (E_2) *possesses a positive solution $u \in C^{1,\alpha}(M)$, $\alpha \in (0, 1)$.*

The proof of Theorem 4.3 proceeds in several steps. In the first step, one gets solutions for "subcritical" equations. For $s \in (q, p)$ real, let

$$\Lambda_s = \left\{ u \in H_1^q(M) \,/\, \int_M f|u|^s \, dv(g) = 1 \right\}$$

and

$$\mu_s = \inf_{u \in \Lambda_s} I(u)$$

Then the following holds:

LEMMA 4.1 *Let (M, g) be a smooth, compact Riemannian n-manifold, $n \geq 2$, $q \in (1, n)$ some real number, and let a, f be smooth, real-valued functions on M. We assume that $L_{q,g}$ is coercive and that f is positive somewhere on M. Set $p = nq/(n - q)$. For any $s \in (q, p)$ real, the equation*

$$\Delta_{q,g} u + a(x) u^{q-1} = \mu_s f(x) u^{s-1}$$

possesses a positive solution $u_s \in \Lambda_s \cap C^{1,\alpha}(M)$, $\alpha \in (0, 1)$.

PROOF: Let (u_i) be a minimizing sequence in Λ_s for μ_s. Namely, $u_i \in \Lambda_s$ for any i, and

$$\lim_{i \to +\infty} I(u_i) = \mu_s$$

Without loss of generality, up to replacing u_i by $|u_i|$, one can assume that the u_i's are nonnegative. Since $L_{q,g}$ is coercive, (u_i) is a bounded sequence in $H_1^q(M)$. Up to the extraction of a subsequence, since $H_1^q(M)$ is reflexive, and by the Rellich-Kondrakov theorem, this leads to the existence of some $u_s \in H_1^q(M)$ such that

$$u_i \rightharpoonup u_s \quad \text{in } H_1^q(M)\,, \qquad u_i \to u_s \quad \text{in } L^s(M)\,, \qquad u_i \to u_s \quad \text{a.e.}$$

One then gets that $u_s \geq 0$ a.e., and that $u_s \in \Lambda_s$. Moreover, the weak convergence in $H_1^q(M)$ implies that

$$I(u_s) \leq \liminf_{i \to +\infty} I(u_i)$$

Hence, $I(u_s) = \mu_s$. By Euler's equation, the fact that u_s is a minimizer for I on Λ_s leads to the fact that u_s is a solution of

$$\Delta_{q,g} u_s + a(x) u_s^{q-1} = \mu_s f(x) u_s^{s-1}$$

The result then easily follows from maximum principles and regularity results. □

From now on, the general idea is to get the solution u of Theorem 4.3 as the limit of (a subsequence of) (u_s), $s \to p$. As a first remark, one can prove here that

$$\limsup_{s \to p} \mu_s \leq \inf_{u \in \Lambda} I(u)$$

For such an assertion, let $\varepsilon > 0$ be given, and let $v \in \Lambda$, v nonnegative and such that

$$I(v) \leq \inf_{u \in \Lambda} I(u) + \varepsilon$$

For s close to p,

$$v_s = \left(\int_M f(x) v^s \, dv(g) \right)^{-\frac{1}{s}} v$$

makes sense and belongs to Λ_s. Hence, $I(v_s) \geq \mu_s$. Noting that $I(v_s) \to I(v)$ as $s \to p$, one gets that

$$\limsup_{s \to p} \mu_s \leq I(v) \leq \inf_{u \in \Lambda} I(u) + \varepsilon$$

The fact that such an inequality holds for any $\varepsilon > 0$ proves the above claim.

In what follows, up to the extraction of a subsequence, we assume that there exists $\lim_{s \to p} \mu_s$. We let

$$\mu = \lim_{s \to p} \mu_s$$

One then has the following:

LEMMA 4.2 *Let (M, g) be a smooth, compact Riemannian n-manifold, $n \geq 2$, $q \in (1, n)$ some real number, and let a, f be smooth, real-valued functions on M. We assume that $L_{q,g}$ is coercive and that f is positive somewhere on M. Set $p = nq/(n-q)$. For $s \in (q, p)$ real, let (u_s) be as in Lemma 4.1 with the additional property that (μ_s) has a limit μ as $s \to p$. Suppose that a subsequence of (u_s) converges in some $L^k(M)$, $k > 1$, to a function $u \not\equiv 0$. Then $u \in C^{1,\alpha}(M)$, $\alpha \in (0, 1)$, u is positive, and*

$$\Delta_{q,g} u + a(x) u^{q-1} = \mu f(x) u^{p-1}$$

In particular, $\mu > 0$ and, up to rescaling, u is a solution of (E₂).

PROOF: Clearly, (u_s) is bounded in $H_1^q(M)$. Up to the extraction of a subsequence, and as $s \to p$, we can assume that

$$u_s \rightharpoonup u \quad \text{in } H_1^q(M), \qquad u_s \to u \quad \text{in } L^q(M), \qquad u_s \to u \quad \text{a.e.}$$

In particular, u is nonnegative. Moreover, since $|\nabla u_s|$ is bounded in $L^q(M)$, we can assume that for $s \to p$,

$$|\nabla u_s|^{q-2} \nabla u_s \rightharpoonup \Sigma$$

in $L^{p/(p-1)}(M)$. Similarly, we can assume that

$$u_s^{s-1} \rightharpoonup u^{p-1} \quad \text{in } L^{p/(p-1)}(M)$$

since (u_s^{s-1}) is bounded in $L^{p/(s-1)}(M) \subset L^{p/(p-1)}(M)$. By passing to the limit as s tends to p in the equation satisfied by the u_s's, one then gets that

$$-\operatorname{div}(\Sigma) + a(x)u^{q-1} = \mu f(x)u^{p-1}$$

One can prove here that $\Sigma = |\nabla u|^{q-2}\nabla u$, as in Demengel-Hebey [**61**] (see also Druet [**73**]). Hence, u is a solution of

$$\Delta_{q,g}u + a(x)u^{q-1} = \mu f(x)u^{p-1}$$

By maximum principles and regularity results, one then gets that u is positive and that $u \in C^{1,\alpha}(M)$ for some $\alpha \in (0,1)$. Moreover, multiplying the above equation by u and integrating over M shows that $\mu > 0$. This proves the lemma. □

As a general remark on this result, one can note that

$$\mu = \inf_{u\in\Lambda} I(u)$$

and that u of Lemma 4.2 belongs to Λ, so that u realizes the infimum of I on Λ. Indeed, multiplying the equation of Lemma 4.2 by u and integrating the result over M, one gets that

$$\begin{aligned}\mu \int_M f(x)u^p\,dv(g) &= \int_M \left(|\nabla u|^q + a(x)u^q\right)dv(g)\\ &\le \liminf_{s\to p} \int_M \left(|\nabla u_s|^q + a(x)u_s^q\right)dv(g)\\ &= \liminf_{s\to p} \mu_s\end{aligned}$$

Hence, $\int_M f(x)u^p\,dv(g) \le 1$. Let

$$v = \frac{u}{\left(\int_M f(x)u^p\,dv(g)\right)^{\frac{1}{p}}}$$

Then $v \in \Lambda$ and, according to what has been said above,

$$\mu \le I(v) = \mu\left(\int_M f(x)u^p\,dv(g)\right)^{1-\frac{q}{p}}$$

As a consequence, $\int_M f(x)u^p\,dv(g) \ge 1$, so that $\int_M f(x)u^p\,dv(g) = 1$ and μ is the infimum of I on Λ. This proves the above claim. Now the proof of Theorem 4.3 proceeds as follows:

PROOF OF THEOREM 4.3: Let (u_s) be as in Lemma 4.1. Up to the extraction of a subsequence, one can assume that

$$\lim_{s\to p} \mu_s = \mu$$

exists. Moreover, by the Rellich-Kondrakov theorem, and still up to the extraction of a subsequence, one can also assume that (u_s) converges to some u in $L^q(M)$ as s tends to p. By Lemma 4.2, Theorem 4.3 reduces to the proof that $u \not\equiv 0$. By definition of $\alpha_q(M)$, one has that for any $\varepsilon > 0$, there exists $B_\varepsilon > 0$ such that for any $u \in H_1^q(M)$,

$$\left(\int_M |u|^p dv(g)\right)^{q/p} \leq \left(\alpha_q(M)^q + \varepsilon\right) \int_M |\nabla u|^q dv(g) + B_\varepsilon \int_M |u|^q dv(g)$$

By assumption, there exists $\varepsilon > 0$ such that

$$\left(\alpha_q(M)^q + \varepsilon\right) \inf_{u\in\Lambda} I(u) < \frac{1}{\left(\max_{x\in M} f(x)\right)^{\frac{q}{p}}}$$

Fix such an ε. Then, for any s,

$$\left(\int_M u_s^p dv(g)\right)^{q/p} \leq \left(\alpha_q(M)^q + \varepsilon\right)\mu_s + \tilde{B}_\varepsilon \int_M u_s^q dv(g)$$

for some $\tilde{B}_\varepsilon$ independent of s. Moreover, one has that

$$\begin{aligned}\frac{1}{\max_{x\in M} f(x)} = \frac{1}{\max_{x\in M} f(x)} \int_M f(x) u_s^s dv(g) &\leq \int_M u_s^s dv(g) \\ &\leq \left(\int_M u_s^p dv(g)\right)^{\frac{s}{p}} \mathrm{Vol}_{(M,g)}^{1-\frac{s}{p}}\end{aligned}$$

Hence,

$$\left(\int_M u_s^p dv(g)\right)^{q/p} \geq \frac{1}{\mathrm{Vol}_{(M,g)}^{\frac{q}{s}-\frac{q}{p}}} \frac{1}{\left(\max_{x\in M} f(x)\right)^{\frac{q}{s}}}$$

and one gets that

$$\frac{1}{\mathrm{Vol}_{(M,g)}^{\frac{q}{s}-\frac{q}{p}}} \frac{1}{\left(\max_{x\in M} f(x)\right)^{\frac{q}{s}}} \leq \left(\alpha_q(M)^q + \varepsilon\right)\mu_s + \tilde{B}_\varepsilon \int_M u_s^q dv(g)$$

Since

$$\limsup_{s\to p} \mu_s \leq \inf_{u\in\Lambda} I(u)$$

one gets by passing to the limit as s tends to p that

$$\frac{1}{\left(\max_{x\in M} f(x)\right)^{\frac{q}{p}}} \leq \left(\alpha_q(M)^q + \varepsilon\right) \inf_{u\in\Lambda} I(u) + \tilde{B}_\varepsilon \int_M u^q dv(g)$$

According to the choice of ε, one then gets that $\int_M u^q dv(g) > 0$. In particular, $u \not\equiv 0$. As already mentioned, this proves Theorem 4.3. □

As a straightforward application of Theorem 4.3, Druet [73] obtains the following result. Here, for Λ as above, one just has to remark that the constant function

$$u_0 \equiv \left(\int_M f(x) dv(g)\right)^{-1/p}$$

belongs to Λ. Writing that

$$\inf_{u\in\Lambda} I(u) \leq I(u_0)$$

one then gets the following corollary:

COROLLARY 4.1 *Let (M, g) be a smooth, compact Riemannian n-manifold, $n \geq 2$, $q \in (1, n)$ be some real number, and let a, f be smooth, real-valued functions on M. We assume that $L_{q,g}$ is coercive and that f is such that $\int_M f(x)dv(g) > 0$. If*

$$\left(\frac{\max_{x\in M} f(x)}{\int_M f(x)dv(g)}\right)^{\frac{q}{p}} \int_M a(x)dv(g) < \frac{1}{\alpha_q(M)^q}$$

then equation (E$_2$) *possesses a positive solution $u \in C^{1,\alpha}(M)$, $\alpha \in (0, 1)$.*

As another example of the kind of results that can be derived from Theorem 4.3, we mention the following result of Druet [73]. Consider the $u'_\varepsilon s$ defined by

$$u_\varepsilon = \left(\varepsilon + r^{q/(q-1)}\right)^{1-(n/q)} \varphi(r)$$

where r is the distance to some point x_0 where f attains its maximum, and where φ is a smooth cutoff function. By studying the expansion of

$$J(u_\varepsilon) = \frac{I(u_\varepsilon)}{\left(\int_M f(x)u_\varepsilon^p\, dv(g)\right)^{q/p}}$$

for $\varepsilon \ll 1$, one gets the following corollary: In such a result, one has to use the explicit value of $\alpha_q(M)$ given by Theorems 4.4 and 4.5. We refer to Druet [73] for more details on the proof of this result. However, note that similar computations to the ones involved in such a proof will be developed in Sections 4.3 and 4.4 below.

COROLLARY 4.2 *Let (M, g) be a smooth, compact Riemannian n-manifold, $q \in (1, n)$ real such that $q^2 < n$, and let a, f be smooth, real-valued functions on M. We assume that a is such that $L_{q,g}$ is coercive, that f is positive somewhere, and that there exists $x_0 \in M$ such that $f(x_0) = \max_{x\in M} f(x)$ and for which we are in one of the following cases:*

1. $q < 2$, $n > 3q - 2$, *and* $a(x_0) < 0$.
2. $q = 2$ *and* $\frac{8(n-1)}{(n-2)(n-4)}a(x_0) < \frac{-\Delta_g f(x_0)}{f(x_0)} + \frac{2\,\mathrm{Scal}_g(x_0)}{n-4}$.
3. $q > 2$ *and* $(\frac{n+2-3q}{q})\frac{\Delta_g f(x_0)}{f(x_0)} < \mathrm{Scal}_g(x_0)$.

Then equation (E$_2$) *possesses a positive solution $u \in C^{1,\alpha}(M)$, $\alpha \in (0, 1)$.*

Such a corollary has two very interesting consequences. One will be discussed later on in Section 4.3. The other one concerns the Nirenberg problem. Let (S^n, h) be the standard n-dimensional unit sphere of $\mathbb{R}^{n+1}$, $n \geq 3$. The Nirenberg problem consists in characterizing the scalar curvatures of conformal metrics to h. Given $f \in C^\infty(S^n)$ and coming back to (E$_1$), this means that one will have to find conditions on f for (E_1) with h in place of g and f in place of $\mathrm{Scal}_{\tilde{g}}$ to have a solution. In other words, since $\mathrm{Scal}_h = n(n-1)$, and up to some harmless constant, one will have to find conditions on f that ensure the existence of $u \in C^\infty(S^n)$, $u > 0$, solution of

$$(E_3) \qquad \Delta_h u + \frac{n(n-2)}{4} u = f u^{(n+2)/(n-2)}$$

When looking to such a problem, as discovered by Kazdan and Warner [**130**] (see, also, Section 6.3 of Chapter 6), obstructions do exist. More precisely, if u is a solution of the above equation, then necessarily

$$\int_{S^n} \langle \nabla f, \nabla \xi \rangle u^{2n/(n-2)} \, dv(h) = 0$$

for all first spherical harmonics ξ on S^n, where $\langle \cdot, \cdot \rangle$ is the scalar product associated to h. In particular, for any $\varepsilon > 0$ and any first spherical harmonic ξ, functions of the form $f = 1 + \varepsilon\xi$, though as closed as we want to the constant function 1 for which (E_3) has a solution, are not the scalar curvature of some conformal metric to h. In particular, for any $\varepsilon > 0$, there exist smooth functions f on S^n such that $\|f - 1\|_{C^2} < \varepsilon$ and such that (E_3) does not possess positive solutions. Conversely, such an equation can be seen as the limiting case for $q \to 2$, $q > 2$, of the generalized scalar curvature equations

$$(E_4) \qquad \Delta_{q,g} u + \frac{n(n-2)}{4} u^{q-1} = f u^{p-1}$$

where $p = \frac{nq}{n-q}$. By Corollary 4.2, one easily gets that there exists some $\varepsilon_0 > 0$ such that for any $q \in (2, \sqrt{n})$, and any $f \in C^\infty(S^n)$, if $\|f - 1\|_{C^2} < \varepsilon_0$, then (E_4) possesses a positive solution. As unexpected as it may seem, Corollary 4.2 shows that the well-known Kazdan-Warner obstructions are specific properties of the limiting equation (E_3).

Many references that illustrate the role of $\alpha_2(M)$ when studying PDEs do exist. Among others, let us mention Aubin [**9**], Brezis-Nirenberg [**34**], and Schoen [**175**], but also Djadli [**68**], Escobar-Schoen [**78**], Jourdain [**129**], and works of the author. A survey reference on the subject could be Hebey [**103**].

4.3. Program $\mathcal{A}$, Part I

Contrary to Program $\mathcal{B}$, Part I, Program $\mathcal{A}$, Part I seems to involve serious difficulties. As surprising as it may seem, one has to face serious problems even when dealing with the first question of this program. Many authors have worked on this question. We mention Aubin [**10**], Federer-Fleming [**80**], Fleming-Rishel [**83**], Rosen [**170**], and Talenti [**183**]. The first definitive and important result on the subject was obtained independently by Aubin [**10**] and Talenti [**183**]. It is stated as follows:

THEOREM 4.4 *Let* $1 \leq q < n$ *and* $1/p = 1/q - 1/n$.

1. *For any* $u \in \mathcal{D}(\mathbb{R}^n)$,

$$\left(\int_{R^n} |u|^p \, dx\right)^{1/p} \leq K(n,q)\left(\int_{R^n} |\nabla u|^q \, dx\right)^{1/q} \tag{4.1}$$

where

$$K(n,1) = \frac{1}{n}\left(\frac{n}{\omega_{n-1}}\right)^{1/n},$$
$$K(n,q) = \frac{1}{n}\left(\frac{n(q-1)}{n-q}\right)^{1-1/q}\left(\frac{\Gamma(n+1)}{\Gamma(n/q)\Gamma(n+1-n/q)\omega_{n-1}}\right)^{1/n}$$

when $q > 1$, *and* ω_{n-1} *is the volume of the standard unit sphere of* $\mathbb{R}^n$.

2. $K(n,q)$ *is the best constant in* (4.1) *and if* $q > 1$, *the equality in* (4.1) *is attained by the functions*

$$u_\lambda(x) = \left(\frac{1}{\lambda + |x|^{q/(q-1)}}\right)^{n/q-1}$$

where λ *is any positive real number.*

PROOF: We only sketch the proof of the result. For more details, we refer to [**10**], [**12**], or [**183**]. Let us consider the case $q > 1$. The second part of point 2 is easy to check. So we are left with the proof of (4.1), together with the fact that $K(n,q)$ is the best constant. By standard Morse theory (see, for instance, Aubin [**12**] for the following claim), it suffices to prove (4.1) for continuous nonnegative functions u with compact support K, K being itself smooth, u being smooth in K and such that it has only nondegenerate critical points in K. For such a u, let $u^\star : \mathbb{R}^n \to \mathbb{R}$, radially symmetric, nonnegative, and decreasing with respect to $|x|$, be defined by:

$$\mathrm{Vol}_e\big(\{x \in \mathbb{R}^n \,/\, u^\star(x) \geq t\}\big) = \mathrm{Vol}_e\big(\{x \in \mathbb{R}^n \,/\, u(x) \geq t\}\big)$$

where e stands for the Euclidean metric. One can check that $u^\star$ has compact support and is Lipschitz. Moreover, one easily gets from the co-area formula that for any $m \geq 1$,

$$\int_{R^n} |\nabla u|^m \, dx \geq \int_{R^n} |\nabla u^\star|^m \, dx \quad \text{and} \quad \int_{R^n} u^m \, dx = \int_{R^n} (u^\star)^m \, dx$$

As a consequence, it suffices to prove (4.1) for decreasing absolutely continuous radially symmetric functions which equal zero at infinity. Denote by g such functions. The problem then reduces to compute the maximum of

$$I(g) = \int_0^\infty |g(r)|^p r^{n-1} \, dr \quad \text{when} \quad J(g) = \int_0^\infty |g'(r)|^q r^{n-1} \, dr$$

is a given positive constant. Set

$$y = \left(\frac{1}{\lambda + r^{q/(q-1)}}\right)^{n/q-1}$$

By Bliss [30] one has that the corresponding value $I(y)$ is an absolute maximum. One then gets that the best value of K in the Sobolev inequality

$$\left(\int_{R^n} |u|^p\, dx\right)^{1/p} \leq K\left(\int_{R^n} |\nabla u|^q\, dx\right)^{1/q}$$

is

$$K = \frac{1}{\omega_{n-1}^{1/n}} I(y)^{1/p} J(y)^{-1/q}$$

Simple computations give that $K = K(n, q)$. This proves the theorem. □

Regarding Theorem 4.4, note that when $q = 2$,

$$K(n, 2) = \sqrt{\frac{4}{n(n-2)\omega_n^{2/n}}}$$

where ω_n denotes the volume of the standard unit sphere of $\mathbb{R}^{n+1}$. Such a result is an easy consequence of the properties satisfied by the Γ function, that is, $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$, and $\Gamma(x+1) = x\Gamma(x)$, and of the fact that

$$\omega_{2n} = \frac{(4\pi)^n (n-1)!}{(2n-1)!}, \qquad \omega_{2n+1} = \frac{2\pi^{n+1}}{n!}$$

Note also that when $q = 1$, (4.1) is the usual isoperimetric inequality [80], [83], and [79]. A very nice proof of such an inequality is presented in Gromov [96] (see, also, Chavel [45]). The extremum functions are here the characteristic functions of the balls of $\mathbb{R}^n$. When $q = 1$, (4.1) is sharp, for an easy computation shows that

$$\left(\int_{R^n} |u_k|^{n/(n-1)}\, dx\right)^{(n-1)/n} \left(\int_{R^n} |\nabla u_k| dx\right)^{-1} = K(n, 1)(1 + 0(1/k))$$

where the u_k's are defined by: $u_k(x) = 1$ when $0 \leq |x| \leq 1$, $u_k(x) = 1 + k(1 - |x|)$ when $1 \leq |x| \leq 1 + 1/k$, and $u_k(x) = 0$ when $|x| \geq 1 + 1/k$. Note here that $K(n, 1)$ is the limiting value of $K(n, q)$ as $q \to 1$.

REMARK 4.1. Okikiolu [164], Glaser-Martin-Grosse-Thirring [92], and Lieb [148] generalized Theorem 4.4 when $q = 2$. One has that for any real number $0 \leq b < 1$ and any $u \in \mathcal{D}(\mathbb{R}^n)$,

$$\left(\int_{R^n} |x|^{-bp} |u|^p\, dx\right)^{1/p} \leq K_{n,p}\left(\int_{R^n} |\nabla u|^2\, dx\right)^{1/2} \tag{4.2}$$

Moreover, the equality in (4.2) is attained by the function

$$u(x) = \left(1 + |x|^{2t/r}\right)^{-r}$$

where $p = 2n/(2b + n - 2)$, $r = 2/(p-2)$, $t = (n-2)/2$,

$$K_{n,p} = \omega_{n-1}^{-(p-2)/2p} t^{-(p+2)/2p} M_p^{1/2},$$

and

$$M_p = \left((2r+1)\Gamma(2r)/r\Gamma(r)^2\right)^{1-2/p} (r/4)^{2/p} (r+1)^{-1}$$

We refer the reader to [148] for more details on this result.

Starting from Theorem 4.4, one gets the answer to question $1\mathcal{A}$. The first result one has to prove here is the following:

PROPOSITION 4.2 *Let (M, g) be a Riemannian n-manifold (not necessarily compact), and let $q \in [1, n)$ be some real number. Suppose that there exist $A, B \in \mathbb{R}$ such that for any $u \in \mathcal{D}(M)$,*

$$\left(\int_M |u|^p \, dv(g)\right)^{1/p} \le A\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} + B\left(\int_M |u|^q \, dv(g)\right)^{1/q}$$

where $1/p = 1/q - 1/n$. Then $A \ge K(n, q)$, where $K(n, q)$ is as in Theorem 4.4.

PROOF: In order to prove the proposition, one can use truncated Bliss functions [30] brought to zero at the edge of a ball. Such an argument is carried out explicitly in [117] for $q = 2$. We present here the following proof by contradiction. Suppose that there exist a Riemannian n-manifold (M, g) and real numbers $q \in [1, n)$, $A < K(n, q)$, and B, such that for any $u \in \mathcal{D}(M)$,

$$\left(\int_M |u|^p \, dv(g)\right)^{1/p} \le A\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} + B\left(\int_M |u|^q \, dv(g)\right)^{1/q} \tag{4.3}$$

where $1/p = 1/q - 1/n$. Let $x \in M$. It is easy to see that for any $\varepsilon > 0$ there exists a chart (Ω, φ) of M at x, and there exists $\delta > 0$ such that $\varphi(\Omega) = B_0(\delta)$ the Euclidean ball of center 0 and radius δ in $\mathbb{R}^n$, and such that the components g_{ij} of g in this chart satisfy

$$(1 - \varepsilon)\delta_{ij} \le g_{ij} \le (1 + \varepsilon)\delta_{ij}$$

as bilinear forms. Choosing ε small enough we then get by (4.3) that there exist $\delta_0 > 0$, $A' < K(n, q)$, and $B' \in \mathbb{R}$ such that for any $\delta \in (0, \delta_0)$ and any $u \in \mathcal{D}(B_0(\delta))$,

$$\left(\int_{R^n} |u|^p \, dx\right)^{1/p} \le A'\left(\int_{R^n} |\nabla u|^q \, dx\right)^{1/q} + B'\left(\int_{R^n} |u|^q \, dx\right)^{1/q}$$

But by Hölder,

$$\left(\int_{B_0(\delta)} |u|^q \, dx\right)^{1/q} \le \mathrm{Vol}_e\left(B_0(\delta)\right)^{1/n}\left(\int_{B_0(\delta)} |u|^p \, dx\right)^{1/p}$$

where e denotes the Euclidean metric. Hence, choosing δ small enough, we get that there exist $\delta > 0$ and $A'' < K(n, q)$ such that for any $u \in \mathcal{D}(B_0(\delta))$,

$$\left(\int_{R^n} |u|^p \, dx\right)^{1/p} \le A''\left(\int_{R^n} |\nabla u|^q \, dx\right)^{1/q}$$

Let $u \in \mathcal{D}(\mathbb{R}^n)$. Set $u_\lambda(x) = u(\lambda x)$, $\lambda > 0$. For λ large enough, $u_\lambda \in \mathcal{D}(B_0(\delta))$. Hence,

$$\left(\int_{R^n} |u_\lambda|^p \, dx\right)^{1/p} \le A''\left(\int_{R^n} |\nabla u_\lambda|^q \, dx\right)^{1/q}$$

But

$$\left(\int_{R^n} |u_\lambda|^p \, dx\right)^{1/p} = \lambda^{-n/p}\left(\int_{R^n} |u|^p \, dx\right)^{1/p}$$

while

$$\left(\int_{R^n} |\nabla u_\lambda|^q\,dx\right)^{1/q} = \lambda^{1-n/q}\left(\int_{R^n} |\nabla u|^q\,dx\right)^{1/q}$$

Since $1/p = 1/q - 1/n$, we get that for any $u \in \mathcal{D}(\mathbb{R}^n)$,

$$\left(\int_{R^n} |u|^p\,dx\right)^{1/p} \le A''\left(\int_{R^n} |\nabla u|^q\,dx\right)^{1/q}$$

Since $A'' < K(n,q)$, such an inequality is in contradiction with Theorem 4.4. This ends the proof of the proposition. □

Let us now give the answer to question $1\mathcal{A}$ as first obtained by Aubin in [**10**].

THEOREM 4.5 *Let (M,g) be a smooth, compact Riemannian n-manifold. For any $\varepsilon > 0$ and any $q \in [1,n)$ real, there exists $B \in \mathbb{R}$ such that for any $u \in H_1^q(M)$,*

$$\left(\int_M |u|^p\,dv(g)\right)^{q/p} \le \left(K(n,q)^q + \varepsilon\right)\int_M |\nabla u|^q\,dv(g) + B\int_M |u|^q\,dv(g)$$

where $1/p = 1/q - 1/n$ and $K(n,q)$ is as in Theorem 4.4. In particular, for any smooth, compact Riemannian n-manifold (M,g), and any $q \in [1,n)$ real, $\alpha_q(M) = K(n,q)$.

PROOF: Let $\varepsilon > 0$ be given, and let $q \in [1,n)$ be given. For any x in M, and any $\eta > 0$, there exists some chart (Ω,φ) at x such that the components g_{ij} of g in this chart satisfy

$$\frac{1}{1+\eta}\delta_{ij} \le g_{ij} \le (1+\eta)\delta_{ij}$$

as bilinear forms. Coming back to the inequality of Theorem 4.4, and by choosing $\eta > 0$ small enough, one can then assume that for any smooth function u with compact support in Ω,

$$\left(\int_M |u|^p\,dv(g)\right)^{q/p} \le \left(K(n,q)^q + \frac{\varepsilon}{2}\right)\int_M |\nabla u|^q\,dv(g) \tag{4.4}$$

M can be covered by a finite number of charts $(\Omega_i,\varphi)_{i=1,\dots,N}$ because it is compact. Denote by $(\alpha_i)_{i=1,\dots,N}$ a smooth partition of unity subordinate to the covering $(\Omega_i)_{i=1,\dots,N}$. Set

$$\eta_i = \frac{\alpha_i^{[q]+1}}{\sum_{m=1}^N \alpha_m^{[q]+1}}$$

where $[q]$ is the greatest integer not exceeding q. Clearly, $\eta_i^{1/q} \in C^1(M)$ and η_i has compact support in Ω_i for any i. For $u \in C^\infty(M)$, we write that

$$\|u\|_p^q = \|u^q\|_{p/q} = \left\|\sum_{i=1}^N \eta_i u^q\right\|_{p/q} \le \sum_{i=1}^N \|\eta_i u^q\|_{p/q} = \sum_{i=1}^N \|\eta_i^{1/q} u\|_p^q$$

where $\|\cdot\|_p$ stands for the norm of $L^p(M)$. Coming back to (4.4), one then has that for any $u \in C^\infty(M)$,

$$
\begin{aligned}
&\left(\int_M |u|^p \, dv(g)\right)^{q/p} \\
&\leq \left(K(n,q)^q + \frac{\varepsilon}{2}\right) \sum_{i=1}^N \int_M \left(\eta_i^{1/q}|\nabla u| + |u||\nabla \eta_i^{1/q}|\right)^q dv(g) \\
&\leq \left(K(n,q)^q + \frac{\varepsilon}{2}\right) \int_M \sum_{i=1}^N \Big(|\nabla u|^q \eta_i + \mu |\nabla u|^{q-1} |\nabla \eta_i^{1/q}| \eta_i^{(q-1)/q} |u| \\
&\qquad\qquad + \nu |u|^q |\nabla \eta_i^{1/q}|^q\Big) dv(g) \\
&\leq \left(K(n,q)^q + \frac{\varepsilon}{2}\right)\left(\|\nabla u\|_q^q + \mu N H \|\nabla u\|_q^{q-1} \|u\|_q + \nu N H^q \|u\|_q^q\right)
\end{aligned}
$$

by Hölder's inequality, where μ and ν are such that

$$(1+t)^q \leq 1 + \mu t + \nu t^q$$

for any $t \geq 0$, for instance, $\mu = q\max(1, 2^{q-2})$ and $\nu = \max(1, 2^{q-2})$, and where H is such that for any i, $|\nabla \eta_i^{1/q}| \leq H$. From now on, let $\varepsilon_0 > 0$ be such that

$$\left(K(n,q)^q + \frac{\varepsilon}{2}\right)(1+\varepsilon_0) \leq K(n,q)^q + \varepsilon$$

For any positive real numbers x, y, and λ,

$$q x^{q-1} y \leq \lambda (q-1) x^q + \lambda^{1-q} y^q$$

By taking $x = \|\nabla u\|_q$, $y = \|u\|_q$, and

$$\lambda = \frac{q\varepsilon_0}{\mu(q-1)NH}$$

one then gets that for any $u \in C^\infty(M)$,

$$\mu N H \|\nabla u\|_q^{q-1} \|u\|_q \leq \varepsilon_0 \|\nabla u\|_q^q + C\|u\|_q^q$$

where

$$C = \frac{\mu N H}{q}\left(\frac{q\varepsilon_0}{\mu(q-1)NH}\right)^{1-q}$$

Hence, for any $u \in C^\infty(M)$,

$$
\begin{aligned}
&\left(\int_M |u|^p \, dv(g)\right)^{q/p} \\
&\leq \left(K(n,q)^q + \frac{\varepsilon}{2}\right)(1+\varepsilon_0) \int_M |\nabla u|^q \, dv(g) + B \int_M |u|^q \, dv(g) \\
&\leq \left(K(n,q)^q + \varepsilon\right) \int_M |\nabla u|^q \, dv(g) + B \int_M |u|^q \, dv(g)
\end{aligned}
$$

where

$$B = \left(K(n,q)^q + \frac{\varepsilon}{2}\right)(C + \nu N H^q)$$

In particular, the inequality of Theorem 4.5 is valid. Now, noting that

$$(x+y)^{1/q} \le x^{1/q} + y^{1/q}$$

for x and y nonnegative, one gets that for any $\varepsilon > 0$, there exists $B > 0$ such that for any $u \in H_1^q(M)$,

$$\begin{aligned}
&\left(\int_M |u|^p \, dv(g)\right)^{1/p} \\
&\le \left(K(n,q)^q + \varepsilon\right)^{1/q}\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} + B^{1/q}\int_M |u|^q \, dv(g) \\
&\le \left(K(n,q) + \varepsilon^{1/q}\right)\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} + B^{1/q}\int_M |u|^q \, dv(g)
\end{aligned}$$

Clearly, such an inequality combined with Proposition 4.2 shows that $\alpha_q(M) = K(n,q)$. This ends the proof of the theorem. □

Concerning Theorem 4.5, we will see in Section 7.1 of Chapter 7, when dealing with complete manifolds, that B depends only on n, q, ε, a lower bound for the Ricci curvature of (M,g), and a lower bound for the injectivity radius of (M,g). Anyway, now that the answer to question 1$\mathcal{A}$ has been given, let us deal with question 2$\mathcal{A}$. We start with the following result of Hebey-Vaugon **[119]**, which fully answers the question when $q = 2$:

THEOREM 4.6 *Let (M,g) be a smooth, compact Riemannian n-manifold, $n \ge 3$. There exists $B \in \mathbb{R}$ such that for any $u \in H_1^2(M)$,*

$$(\mathrm{I}^2_{2,\mathrm{opt}}) \qquad \left(\int_M |u|^p \, dv(g)\right)^{2/p} \le K(n,2)^2 \int_M |\nabla u|^2 \, dv(g) + B\int_M u^2 \, dv(g)$$

where $1/p = 1/2 - 1/n$. In particular, for any smooth, compact Riemannian n-manifold (M,g), $n \ge 3$, inequality $(I^1_{2,\mathrm{opt}})$ is valid and $\mathcal{A}_2(M)$ is a closed set.

PROOF: We give only a very general idea of how the proof works and will come back to the complete proof when discussing the case of complete manifolds in Section 7.3 of Chapter 7. Let $\alpha > 0$ be some positive real number, and for $u \in H_1^2(M)$, let

$$I_\alpha(u) = \frac{\int_M |\nabla u|^2 \, dv(g) + \alpha \int_M u^2 \, dv(g)}{\left(\int_M |u|^p \, dv(g)\right)^{2/p}}$$

Clearly, the result is equivalent to the existence of some $\alpha_0 > 0$ such that

$$\inf_{u \in H_1^2(M)\setminus\{0\}} I_{\alpha_0}(u) \ge \frac{1}{K(n,2)^2}$$

The proof here proceeds by contradiction: We assume that for any $\alpha > 0$,

$$\inf_{u\in H_1^2(M)\setminus\{0\}} I_\alpha(u) < \frac{1}{K(n,2)^2}$$

By standard variational techniques, as used in the proof of Theorem 4.3 of Section 4.2, such an inequality leads to the following: For any $\alpha > 0$, there exists $u_\alpha \in C^\infty(M)$, $u_\alpha > 0$, and there exists $\lambda_\alpha \in (0, K(n,2)^{-2})$, such that

$$\Delta_g u_\alpha + \alpha u_\alpha = \lambda_\alpha u_\alpha^{p-1}$$

on M, and $\int_M u_\alpha^p \, dv(g) = 1$. The idea then is to prove that for α large enough, the u_α's do not exist. This looks very much to what is given for free in the Euclidean context by the Euclidean Pohozaev identity (see the remark below). In order to prove that the u_α's do not exist for α large enough, we let $\alpha \to +\infty$. Here, the u_α's develop one concentration point. By a careful study of what happens at this point, involving a rather sophisticated blowup argument, and surprisingly by coming back to the Euclidean Pohozaev identity, one gets the contradiction. A major point in the final argument is to estimate the difference that exists between the Euclidean metric and the Riemannian metric after rescaling. Details on such an approach will be given in Section 7.3 of Chapter 7 when proving the more general Theorem 7.2. □

REMARK 4.2. Suppose that M is some smooth, bounded, star-shaped domain Ω of $\mathbb{R}^n$ with respect to 0, that g is the Euclidean metric, and that $u_\alpha = 0$ on $\partial\Omega$. The Euclidean Pohozaev identity states that for any smooth function u such that $u = 0$ on $\partial\Omega$,

$$\int_{\partial\Omega} (x,\nu)(\partial_\nu u)^2 \, d\sigma = -2\int_\Omega (\nabla u, x)\Delta_e u \, dx - (n-2)\int_\Omega u\Delta_e u \, dx$$

where ν is the unit outer normal to $\partial\Omega$, $(\cdot,\cdot)$ is the Euclidean scalar product, dx stands for the Euclidean volume element, and Δ_e stands for the Euclidean Laplacian. For u_α a positive solution of

$$\Delta_e u_\alpha + \alpha u_\alpha = \lambda_\alpha u_\alpha^{p-1}$$

in Ω with respect to the Euclidean metric, such that $u_\alpha = 0$ on $\partial\Omega$, one then gets that

$$\int_{\partial\Omega} (x,\nu)(\partial_\nu u_\alpha)^2 \, d\sigma = -2\alpha \int_\Omega u_\alpha^2 \, dx$$

The fact that Ω is star-shaped with respect to 0 implies that the left-hand side member of this equality is nonnegative. Clearly, this proves that the u_α's do not exist as soon as $\alpha > 0$. In the more general context of an arbitrary, compact manifold, coming back to the sketch of the proof of Theorem 4.6 we discussed above, one gets the existence of some $\alpha_0 > 0$ such that the u_α's do not exist as soon as $\alpha \geq \alpha_0$.

Concerning Theorem 4.6, we will see in Section 7.2, when dealing with complete manifolds, that B depends only on n, a lower bound for the injectivity radius of (M,g), and an upper bound for the norm of the Riemann curvature Rm_g of (M,g) and for the norm of the first covariant derivative $\nabla \mathrm{Rm}_g$ of Rm_g. Anyway, now that we have answered question 2$\mathcal{A}$ for $q = 2$, one can ask what happens for $q \neq 2$. The first result we mention is the following one of Aubin [10]:

THEOREM 4.7 *Let (M, g) be a smooth, compact Riemannian n-manifold, and let $q \in [1, n)$ real. Suppose either that $n = 2$ or that (M, g) has constant sectional curvature. Then inequality $(I^1_{q.\mathrm{opt}})$ is valid and $\mathcal{A}_q(M)$ is a closed set.*

PROOF: Let us just sketch the proof. As shown by Aubin [10] when $n = 2$, or by Schmidt and Dinghas [66] when g has constant sectional curvature, for K an upper bound of the sectional curvature of g in a ball (in spirit small) of M, and for Ω a smooth, bounded domain in such a ball, the area of $\partial\Omega$ is greater than or equal to the area in a space of constant curvature K of the boundary of a ball having the same volume as Ω. From such a result, by standard arguments of Morse theory, and with a symmetrization process via the co-area formula, somehow similar to the one we described in the proof of Theorem 4.4, one gets that any point in M possesses some open neighborhood Ω where $(\mathrm{I}^1_{q.\mathrm{opt}})$ is valid for all $u \in \mathcal{D}(\Omega)$. (We refer to Aubin [10] for details on such an assertion. The point is that $(\mathrm{I}^1_{q.\mathrm{opt}})$ is actually valid on the standard sphere.) A localization process, similar to the one used in the proof of Theorem 4.5, then shows that $(\mathrm{I}^1_{q.\mathrm{opt}})$ must be valid for any $u \in H^q_1(M)$. This proves the theorem. □

As in Section 4.1, one can start now with inequality $(\mathrm{I}^q_{q.\mathrm{gen}})$ instead of inequality $(\mathrm{I}^1_{q.\mathrm{gen}})$. Given (M, g) a smooth, compact Riemannian n-manifold, and $q \in [1, n)$ real, one can then consider the (possibly valid) following inequality: There exists $B \in \mathbb{R}$ such that for any $u \in H^q_1(M)$,

$$(\mathrm{I}^q_{q.\mathrm{opt}}) \qquad \left(\int_M |u|^p \, dv(g)\right)^{q/p} \leq K(n, q)^q \int_M |\nabla u|^q \, dv(g) + B \int_M |u|^q \, dv(g)$$

where $1/p = 1/q - 1/n$. Clearly, the validity of $(\mathrm{I}^q_{q.\mathrm{opt}})$ implies that of $(\mathrm{I}^1_{q.\mathrm{opt}})$ since $q \geq 1$ and, roughly speaking, $(\mathrm{I}^q_{q.\mathrm{opt}}) = (\mathrm{I}^1_{q\,\mathrm{opt}})^q$ in the sense that we elevate each term in $(\mathrm{I}^1_{q.\mathrm{opt}})$ to the power q. One can now ask if such an inequality is valid. According to Theorem 4.6, $(\mathrm{I}^2_{2.\mathrm{opt}})$ is valid on any smooth, compact Riemannian manifold. Independently, and when dealing with the standard unit sphere (S^n, h) of $\mathbb{R}^{n+1}$, one has the following result of Aubin [10]:

PROPOSITION 4.3 *Let (S^n, h) be the standard unit sphere of $\mathbb{R}^{n+1}$. Inequality $(\mathrm{I}^q_{q.\mathrm{opt}})$ is valid on (S^n, h) for any $q \in [1, n)$ if $n = 2$ and any $q \in [1, 2]$ if $n \geq 3$.*

PROOF: Here again, let us just sketch the proof. Let P be some point in S^n, and let r the distance to P. Once more by arguments from standard Morse theory, and by a symmetrization process, it suffices to prove that $(\mathrm{I}^q_{q.\mathrm{opt}})$ is valid for functions of the form $g(x) = g(r)$, g a nonnegative, absolutely continuous, decreasing function on $[0, \pi]$. In geodesic normal coordinates at P,

$$r^{n-1}\sqrt{|h|} = \sin^{n-1} r$$

By Theorem 4.4, as one can easily be convinced, one has that for g as above,

$$\left[\omega_{n-1}\int_0^\pi \left|g\left(\frac{\sin r}{r}\right)^{(n-1)/q}\right|^p r^{n-1}\,dr\right]^{q/p} \leq$$
$$\omega_{n-1}K(n,q)^q\int_0^\pi \left|\nabla\left[g\left(\frac{\sin r}{r}\right)^{(n-1)/q}\right]\right|^q r^{n-1}\,dr$$

As shown by Aubin [10], one can then prove that there exists some constant C_1 such that

$$\omega_{n-1}\int_0^\pi \left|\nabla\left[g\left(\frac{\sin r}{r}\right)^{(n-1)/q}\right]\right|^q r^{n-1}\,dr \leq$$
$$\int_{S^n}|\nabla g|^q\,dv(h) + C_1\int_{S^n}|u|^q\,dv(h)$$

and that there exists some constant C_2 such that

$$\left[\omega_{n-1}\int_0^\pi \left|g\left(\frac{\sin r}{r}\right)^{(n-1)/q}\right|^p r^{n-1}\,dr\right]^{q/p} \geq$$
$$\left(\int_{S^n} g^p\,dv(h)\right)^{q/p} - C_2\int_{S^n} g^q\,dv(h)$$

Clearly, this proves the proposition. □

Let us now deal with $(\mathrm{I}^q_{q,\mathrm{opt}})$ for $q > 2$. The first result on the subject is the following very nice, though simple, result of Druet [74].

THEOREM 4.8 *Let (M,g) be a smooth, compact Riemannian n-manifold, and let $q \in [1,n)$ real. Assume that $q > 2$, that $q^2 < n$, and that the scalar curvature of (M,g) is positive somewhere. Then inequality $(\mathrm{I}^q_{q,\mathrm{opt}})$ is false on (M,g).*

PROOF: Let $x_0 \in M$ be such that $\mathrm{Scal}_{(M,g)}(x_0) > 0$, where $\mathrm{Scal}_{(M,g)}$ stands for the scalar curvature of g. For $\varepsilon > 0$, we set

$$u_\varepsilon = \left(\varepsilon + r^{\frac{q}{q-1}}\right)^{1-\frac{n}{q}}\varphi(r)$$

where r denotes the distance to x_0, φ is smooth such that $0 \leq \varphi \leq 1$, $\varphi = 1$ on $(-\frac{\delta}{2},\frac{\delta}{2})$, and $\varphi = 0$ if $r \geq \delta$, and $\delta > 0$, δ small, is real. In order to prove Theorem 4.8, one just has to prove that for any $\alpha > 0$, and for ε small enough, $J(u_\varepsilon) < K(n,q)^{-q}$, where

$$J(u_\varepsilon) = \frac{\int_M |\nabla u_\varepsilon|^q\,dv(g) + \alpha\int_M u_\varepsilon^q\,dv(g)}{\left(\int_M u_\varepsilon^p\,dv(g)\right)^{\frac{q}{p}}}$$

Here, rather standard computations lead to

$$\alpha \int_M u_\varepsilon^q \, dv(g) = \left(\alpha\omega_{n-1}\int_0^\infty (1+s^{\frac{q}{q-1}})^{q-n}s^{n-1}\,ds\right)\varepsilon^{\frac{q^2-n}{q}} + o\left(\varepsilon^{\frac{q^2-n}{q}}\right)$$

$$\int_M u_\varepsilon^p \, dv(g) = \left(\omega_{n-1}\int_0^\infty \left(1+s^{\frac{q}{q-1}}\right)^{-n}s^{n-1}\,ds\right)\varepsilon^{-\frac{n}{q}}$$
$$- \left(\frac{\omega_{n-1}}{6n}\mathrm{Scal}_{(M,g)}(x_0)\int_0^\infty \left(1+s^{\frac{q}{q-1}}\right)^{-n}s^{n+1}\,ds\right)$$
$$\times\, \varepsilon^{\frac{-n+2(q-1)}{q}} + o\left(\varepsilon^{\frac{-n+2(q-1)}{q}}\right)$$

$$\int_M |\nabla u_\varepsilon|^q \, dv(g) \le C + K(n,q)^{-q}\left(\omega_{n-1}\int_0^\infty \left(1+s^{\frac{q}{q-1}}\right)^{-n}s^{n-1}\,ds\right)^{\frac{q}{p}}\varepsilon^{1-\frac{n}{q}}$$
$$- \left(\left(\frac{n-q}{q-1}\right)^q \omega_{n-1}\frac{\mathrm{Scal}_{(M,g)}(x_0)}{6n}\int_0^\infty \left(1+s^{\frac{q}{q-1}}\right)^{-n}\right.$$
$$\left.\times\, s^{\frac{q}{q-1}+n+1}\,ds\right)\varepsilon^{\frac{3q-2-n}{q}} + o\left(\varepsilon^{\frac{3q-2-n}{q}}\right)$$

where ω_{n-1} stands for the volume of the standard unit sphere (S^{n-1},h) of $\mathbb{R}^n$. As a remark, note that the above integrals do exist as soon as $q>2$ and $q^2<n$. Now, as a consequence of such inequalities, one gets that

$$K(n,q)^q J(u_\varepsilon) \le 1 + \varepsilon^{\frac{n}{q}-1}\times\left(A_1 + A_2\varepsilon^{\frac{q^2-n}{q}} + A_3\varepsilon^{2\frac{q-1}{q}+1-\frac{n}{q}}\right.$$
$$\left. + o\left(\varepsilon^{\frac{q^2-n}{q}}\right) + o\left(\varepsilon^{2\frac{q-1}{q}+1-\frac{n}{q}}\right)\right)$$

where $A_1>0$, $A_2>0$ real are independent of ε, and

$$A_3 = \frac{\mathrm{Scal}_{(M,g)}(x_0)}{6n}\left(\frac{q}{p}\frac{\int_0^\infty(1+s^{\frac{q}{q-1}})^{-n}s^{n+1}\,ds}{\int_0^\infty(1+s^{\frac{q}{q-1}})^{-n}s^{n-1}\,ds} - \frac{\int_0^\infty(1+s^{\frac{q}{q-1}})^{-n}s^{\frac{q}{q-1}+n+1}\,ds}{\int_0^\infty(1+s^{\frac{q}{q-1}})^{-n}s^{\frac{q}{q-1}+n-1}\,ds}\right)$$

Since $q>2$ and $q^2<n$,

$$1-\frac{n}{q}+2\frac{q-1}{q} < \frac{q^2-n}{q} < 0$$

Hence, one will find $\varepsilon>0$ small enough such that

$$J(u_\varepsilon) < \frac{1}{K(n,q)^q}$$

if $A_3<0$. But, as one can easily check,

$$A_3 = -\frac{q}{2n^2}\frac{\Gamma(n-\frac{n}{q}-\frac{2}{q}+2)\Gamma(\frac{n}{q}+\frac{2}{q}-3)}{\Gamma(n-\frac{n}{q})\Gamma(\frac{n}{q}-1)}\mathrm{Scal}_{(M,g)}(x_0)$$

As a consequence, $A_3 < 0$ as soon as $\mathrm{Scal}_{(M,g)}(x_0) > 0$. Clearly, this ends the proof of the theorem. □

As a consequence of this result of Druet [74] and of Theorem 4.7, one gets that inequality ($\mathrm{I}^1_{q,\mathrm{opt}}$) is valid on the standard unit sphere (S^n, h), while inequality ($\mathrm{I}^q_{q,\mathrm{opt}}$) is not valid on (S^n, h) if $q > 2$ and $q^2 < n$. This leads to the following corollary:

COROLLARY 4.3 *For $q > 2$, there exist smooth, compact Riemannian manifolds for which ($\mathrm{I}^1_{q,\mathrm{opt}}$) is true while ($\mathrm{I}^q_{q,\mathrm{opt}}$) is false.*

Coming back to Theorem 4.8, one can ask if its assumptions are sharp or not. Concerning the assumption that $q > 2$, we already know by Theorem 4.6 and Proposition 4.3 that it is sharp. A very surprising fact (think, for instance, of Proposition 4.1) is that the assumption on the scalar curvature is also sharp, as proved by the following result of Druet [74]:

PROPOSITION 4.4 *Let (T^n, g) be a smooth, flat, compact n-dimensional torus. For any $q \in [1, n)$ real, inequality ($\mathrm{I}^q_{q,\mathrm{opt}}$) is true on (T^n, g).*

PROOF: The proof of such a result is more subtle than that of Theorem 4.8. It proceeds by contradiction. We closely follow the lines of Druet [74]. By Theorem 4.7, we may assume that $q > 1$. Let $q \in (1, n)$ real be given. For $\alpha > 0$, set

$$\lambda_\alpha = \inf_{u \in \Lambda} \left(\int_{T^n} |\nabla u|^q \, dv(g) + \alpha \int_{T^n} |u|^q \, dv(g) \right)$$

where

$$\Lambda = \left\{ u \in H_1^q(T^n) \,/\, \int_{T^n} |u|^p \, dv(g) = 1 \right\}$$

Assume that ($\mathrm{I}^q_{q,\mathrm{opt}}$) is false on (T^n, g). Then for any $\alpha > 0$,

$$\lambda_\alpha < \frac{1}{K(n, q)^q}$$

By standard variational techniques, as developed in the proof of Theorem 4.3, one gets from such an inequality that for any $\alpha > 0$, there exists some function $u_\alpha \in \Lambda$, $u_\alpha \geq 0$, solution of

$$\Delta_{q,g} u_\alpha + \alpha u_\alpha^{q-1} = \lambda_\alpha u_\alpha^{p-1} \tag{4.5}$$

where $\Delta_{q,g}$ is the q-Laplacian of g. By maximum principles $u_\alpha > 0$, while by regularity results $u_\alpha \in C^{1,\lambda}(T^n)$ for some $\lambda \in (0, 1)$. In particular, $u_\alpha \in C^1(T^n)$. Let us now say that $x \in T^n$ is a point of concentration of (u_α) if for all $\delta > 0$,

$$\limsup_{\alpha \to \infty} \int_{B_x(\delta)} u_\alpha^p \, dv(g) > 0$$

First, we claim that, up to a subsequence, (u_α) has a unique point of concentration. The existence of such a point is evident since T^n is compact. Conversely, let $x \in T^n$ be a point of concentration of (u_α). Let us prove that x is unique. For that purpose, let $\delta > 0$, δ small, and let $\eta \in \mathcal{D}\big(B_x(\delta)\big)$ such that $0 \le \eta \le 1$ and $\eta = 1$ on $B_x(\delta/2)$. Multiplying (4.5) by $\eta^q u_\alpha^k$, $k > 1$ real, and integrating over T^n lead to

$$\int_{T^n} \eta^q u_\alpha^k \Delta_{q,g} u_\alpha \, dv(g) + \alpha \int_{T^n} \eta^q u_\alpha^{k+q-1} \, dv(g) = \lambda_\alpha \int_{T^n} \eta^q u_\alpha^{k+p-1} \, dv(g)$$

Standard computations, using Theorem 4.5, then lead to the following situation: for any $\varepsilon > 0$, there exist positive constants B_ε and C_ε such that for any $\alpha > 0$,

$$\begin{aligned} &\left[1 - \left(\frac{k+q-1}{q}\right)^q \frac{1+\varepsilon}{k} \lambda_\alpha \left(K(n,q)^q + \varepsilon\right)\left(\int_{B_x(\delta)} u_\alpha^p \, dv(g)\right)^{\frac{p-q}{p}}\right] \\ &\quad \times \left(\int_{T^n} \left(\eta u_\alpha^{\frac{k+q-1}{q}}\right)^p dv(g)\right)^{\frac{q}{p}} \\ (4.6) \qquad &\quad \le \int_{T^n} \left[C_\varepsilon\big(K(n,q)^q + \varepsilon\big)|\nabla\eta|^q + B_\varepsilon \eta^q\right] u_\alpha^{k+q-1} \, dv(g) \\ &\quad + \left(\frac{k+q-1}{q}\right)^q (1+\varepsilon)\big(K(n,q)^q + \varepsilon\big) \\ &\quad \times \left(\int_{T^n} |\nabla(\eta^q)|^q u_\alpha^{kq} \, dv(g)\right)^{\frac{1}{q}} \left(\int_{T^n} |\nabla u_\alpha|^q \, dv(g)\right)^{\frac{q-1}{q}} \end{aligned}$$

Since x is assumed to be a point of concentration of (u_α), one has that for $\delta > 0$

$$\limsup_{\alpha \to \infty} \left(\int_{B_x(\delta)} u_\alpha^p \, dv(g)\right)^{\frac{p-q}{p}} = a > 0$$

where $a \le 1$. Assume that $a < 1$ for some $\delta > 0$. Then we may take $\varepsilon > 0$ small enough, and $k > 1$ sufficiently close to 1, such that

$$1 - \left(\frac{k+q-1}{q}\right)^q \frac{1+\varepsilon}{k} \lambda_\alpha \big(K(n,q)^q + \varepsilon\big) a > 0$$

Since the right-hand side of (4.6) is bounded for $k > 1$ close to 1, this leads to the existence of some $M > 0$ such that for all $\alpha \gg 1$

$$\left(\int_{T^n} \left(\eta u_\alpha^{\frac{k+q-1}{q}}\right)^p dv(g)\right)^{\frac{q}{p}} \le M$$

By Hölder's inequalities,

$$\begin{aligned} \int_{B_x(\frac{\delta}{2})} u_\alpha^p \, dv(g) &= \int_{B_x(\frac{\delta}{2})} u_\alpha^{p-q-k+1} u_\alpha^{q+k-1} \, dv(g) \\ &\le \left(\int_{T^n} \left(\eta u_\alpha^{\frac{q+k-1}{q}}\right)^p dv(g)\right)^{\frac{q}{p}} \left(\int_{T^n} u_\alpha^{p - \frac{(k-1)p}{p-q}} \, dv(g)\right)^{\frac{p-q}{p}} \end{aligned}$$

Hence,

$$\int_{B_x(\frac{\delta}{2})} u_\alpha^p \, dv(g) \le M \left(\int_{T^n} u_\alpha^{p - \frac{(k-1)p}{p-q}} \, dv(g)\right)^{\frac{p-q}{p}}$$

and since for $k > 1$ sufficiently close to 1,

$$0 < p - \frac{(k-1)p}{p-q} < p,$$

one gets that

$$\limsup_{\alpha\to\infty} \int_{B_x(\frac{\delta}{2})} u_\alpha^p \, dv(g) = 0 \tag{4.7}$$

Indeed, multiplying (4.5) by u_α, integrating over T^n, and letting α go to $+\infty$ leads to

$$\lim_{\alpha\to\infty} \int_{T^n} u_\alpha^q \, dv(g) = 0$$

By compactness, one then gets that

$$\limsup_{\alpha\to\infty} \int_{T^n} u_\alpha^s \, dv(g) = 0$$

for any $0 < s < p$. But (4.7) is absurd. As a consequence, $a = 1$ and

$$\limsup_{\alpha\to\infty} \int_{B_x(\delta)} u_\alpha^p \, dv(g) = 1$$

for all $\delta > 0$. As one can easily check, up to the extraction of a subsequence, this shows that the concentration point x of (u_α) is unique. The above claim is proved. Now, thanks to inequality (4.6), we easily get that there exist $\varepsilon > 0$ and $\tilde{M} > 0$ such that for any $\overline{\Omega} \Subset T^n \backslash \{x\}$, and any $\alpha \gg 1$,

$$\int_\Omega u_\alpha^{p(1+\varepsilon)} \, dv(g) \le \tilde{M}$$

By Moser's iterative scheme, as developed, for instance, in Serrin [**179**] (see, also, Trudinger [**184**] and Vcron [**197**]), one then gets that

$$u_\alpha \to 0 \quad \text{in } C^0_{\text{loc}}\left(T^n \backslash \{x\}\right)$$

as $\alpha \to +\infty$. All we have said till now holds on arbitrary, compact Riemannian manifolds. Starting from now, we use the specificity of (T^n, g). Since (T^n, g) is flat, there exists some small ball B, centered at x, such that (B, g) is isometric to the Euclidean ball of same radius. By Theorem 4.4, one then gets that for any $u \in H^q_{0,1}(B)$,

$$\left(\int_B |u|^p \, dv(g)\right)^{\frac{q}{p}} \le K(n,q)^q \int_B |\nabla u|^q \, dv(g)$$

The goal now is to prove that such an inequality, combined with the facts that

$$\Delta_{q,g} u_\alpha + \alpha u_\alpha^{q-1} = \lambda_\alpha u_\alpha^{p-1},$$

that $u_\alpha \in \Lambda$, and that $\lambda_\alpha < K(n,q)^{-q}$, leads to a contradiction. Clearly, this will end the proof of the proposition. In what follows, let η be a smooth function on T^n such that $\eta = 1$ on $B' \subset B$, $\eta = 0$ on $T^n \backslash B$, B' another ball centered at x. Then

$$\left(\int_B (\eta u_\alpha)^p \, dv(g)\right)^{\frac{q}{p}} \le K(n,q)^q \int_B |\nabla(\eta u_\alpha)|^q \, dv(g)$$

so that, setting $\eta' = 1 - \eta$,

$$\left(\int_{B'} u_\alpha^p \, dv(g)\right)^{\frac{q}{p}} \le \left(\int_B (\eta u_\alpha)^p \, dv(g)\right)^{\frac{q}{p}}$$
$$\le K(n,q)^q \int_{T^n} \left|\nabla\big((1-\eta')u_\alpha\big)\right|^q dv(g)$$

In what follows, C will always denote some constant independent of α. We have that

$$\begin{aligned}\left|\nabla\big((1-\eta')u_\alpha\big)\right|^q &\le \big(|\nabla u_\alpha| + |\nabla(\eta' u_\alpha)|\big)^q \\ &\le |\nabla u_\alpha|^q + C|\nabla u_\alpha|^{q-1}|\nabla(\eta' u_\alpha)| + C|\nabla(\eta' u_\alpha)|^q \\ &\le \big(1 + C\eta'\big)|\nabla u_\alpha|^q + C u_\alpha |\nabla u_\alpha|^{q-1}|\nabla \eta'| + C|\nabla(\eta' u_\alpha)|^q\end{aligned}$$

But

$$|\nabla(\eta' u_\alpha)|^q \le C(\eta')^q|\nabla u_\alpha|^q + C u_\alpha^q |\nabla \eta'|^q$$

so that

$$\begin{aligned}&\int_{T^n} \left|\nabla\big((1-\eta')u_\alpha\big)\right|^q dv(g) \\ &\quad\le \int_{T^n} |\nabla u_\alpha|^q \, dv(g) + C\int_{T^n\setminus B'} |\nabla u_\alpha|^q \, dv(g) \\ &\quad\quad + C\int_{T^n\setminus B'} u_\alpha |\nabla u_\alpha|^{q-1} \, dv(g) + C\int_{T^n\setminus B'} u_\alpha^q \, dv(g)\end{aligned}$$

Hence,

$$\begin{aligned}\left(\int_{B'} u_\alpha^p \, dv(g)\right)^{\frac{q}{p}} &\le K(n,q)^q \int_{T^n} |\nabla u_\alpha|^q \, dv(g) + C\int_{T^n\setminus B'} |\nabla u_\alpha|^q \, dv(g) \\ &\quad + C\int_{T^n\setminus B'} u_\alpha^q \, dv(g) + C\int_{T^n\setminus B'} u_\alpha |\nabla u_\alpha|^{q-1} \, dv(g)\end{aligned}$$

Multiplying (4.5) by u_α and integrating over T^n leads to

$$\int_{T^n} |\nabla u_\alpha|^q \, dv(g) + \alpha \int_{T^n} u_\alpha^q \, dv(g) = \lambda_\alpha$$

As a consequence, we get that

$$\begin{aligned}\left(\int_{B'} u_\alpha^p \, dv(g)\right)^{\frac{q}{p}} &\le \lambda_\alpha K(n,q)^q - \alpha K(n,q)^q \int_M u_\alpha^q \, dv(g) \\ &\quad + C\int_{M\setminus B'} |\nabla u_\alpha|^q \, dv(g) + C\int_{M\setminus B'} u_\alpha^q \, dv(g) \\ &\quad + C\int_{M\setminus B'} u_\alpha |\nabla u_\alpha|^{q-1} \, dv(g)\end{aligned}$$

Since $\lambda_\alpha K(n,q)^q < 1$, this leads to

$$\begin{aligned}
&\left(\alpha K(n,q)^q - C\right)\int_{T^n} u_\alpha^q\, dv(g)\\
&\le 1 - \left(\int_{B'} u_\alpha^p\, dv(g)\right)^{\frac{q}{p}} + C\int_{T^n\backslash B'} |\nabla u_\alpha|^q\, dv(g)\\
&\quad + C\int_{T^n\backslash B'} u_\alpha |\nabla u_\alpha|^{q-1}\, dv(g)\\
&\le \int_{T^n\backslash B'} u_\alpha^p\, dv(g) + C\int_{T^n\backslash B'} |\nabla u_\alpha|^q\, dv(g)\\
&\quad + C\int_{T^n\backslash B'} u_\alpha |\nabla u_\alpha|^{q-1}\, dv(g)
\end{aligned}$$

that is to say,

$$\begin{aligned}
\alpha K(n,q)^q - C \le \frac{\int_{T^n\backslash B'} u_\alpha^p\, dv(g)}{\int_{T^n} u_\alpha^q\, dv(g)} &+ C\frac{\int_{T^n\backslash B'} |\nabla u_\alpha|^q dv(g)}{\int_{T^n} u_\alpha^q\, dv(g)}\\
&+ C\frac{\int_{T^n\backslash B'} u_\alpha |\nabla u_\alpha|^{q-1}\, dv(g)}{\int_{T^n} u_\alpha^q\, dv(g)}
\end{aligned}$$

By Hölder's inequalities, one then gets that

$$\begin{aligned}
\alpha K(n,q)^q - C \le \frac{\int_{T^n\backslash B'} u_\alpha^p\, dv(g)}{\int_{T^n} u_\alpha^q\, dv(g)} &+ C\frac{\int_{T^n\backslash B'} |\nabla u_\alpha|^q\, dv(g)}{\int_{T^n} u_\alpha^q\, dv(g)}\\
&+ C\left(\frac{\int_{T^n\backslash B'} |\nabla u_\alpha|^q\, dv(g)}{\int_{T^n} u_\alpha^q\, dv(g)}\right)^{\frac{q-1}{q}}
\end{aligned} \tag{4.8}$$

Here,

$$\frac{\int_{T^n\backslash B'} u_\alpha^p\, dv(g)}{\int_{T^n} u_\alpha^q\, dv(g)} \le \sup_{T^n\backslash B'} u_\alpha^{p-q}$$

which tends to 0 as $\alpha \to +\infty$, since

$$u_\alpha \to 0 \quad \text{in } C^0_{\mathrm{loc}}\left(T^n\backslash\{x\}\right)$$

as $\alpha \to +\infty$.

Let us now get estimates on the expression

$$\frac{\int_{T^n\backslash B'} |\nabla u_\alpha|^q\, dv(g)}{\int_{T^n} u_\alpha^q\, dv(g)}$$

Let $B'' \subset B'$ a ball centered at x, and let $\eta'' \ge 0$ be a smooth function on T^n such that $\eta'' = 0$ on B'' and $\eta'' = 1$ on $T^n\backslash B'$. Multiplying (4.5) by $(\eta'')^q u_\alpha$ and integrating over T^n, we obtain

$$\begin{aligned}
&\int_{T^n} (\eta'')^q |\nabla u_\alpha|^q\, dv(g) + q\int_{T^n} (\eta'')^{q-1} u_\alpha |\nabla u_\alpha|^{q-2}\langle\nabla\eta'', \nabla u_\alpha\rangle dv(g) \le\\
&\quad K(n,q)^{-q}\int_{T^n} (\eta'')^q u_\alpha^p\, dv(g)
\end{aligned}$$

Hence,

$$\begin{aligned}\int_{T''}(\eta'')^q|\nabla u_\alpha|^q\,dv(g) &\le C\int_{T''}|\eta''\nabla u_\alpha|^{q-1}u_\alpha\,dv(g)+C\int_{T''}(\eta'')^q u_\alpha^p\,dv(g)\\ &\le C\left(\int_{T''}u_\alpha^q\,dv(g)\right)^{\frac{1}{q}}\left(\int_{T''}|\eta''\nabla u_\alpha|^q\,dv(g)\right)^{\frac{q-1}{q}}\\ &\quad+C\int_{T''}(\eta'')^q u_\alpha^p\,dv(g)\end{aligned}$$

and as a consequence

$$\begin{aligned}\frac{\int_{T''}(\eta'')^q|\nabla u_\alpha|^q\,dv(g)}{\int_{T''}u_\alpha^q\,dv(g)} &\le C\left(\frac{\int_{T''}(\eta'')^q|\nabla u_\alpha|^q\,dv(g)}{\int_{T''}u_\alpha^q\,dv(g)}\right)^{\frac{q-1}{q}}\\ &\quad+C\,\frac{\int_{T''\setminus B''}u_\alpha^p\,dv(g)}{\int_{T''}u_\alpha^q\,dv(g)}\end{aligned}$$

Here again, one has that

$$\frac{\int_{T''\setminus B''}u_\alpha^p\,dv(g)}{\int_{T''}u_\alpha^q\,dv(g)}\le\sup_{T''\setminus B''}u_\alpha^{p-q}$$

which tends to 0 as $\alpha\to+\infty$, since

$$u_\alpha\to 0\quad\text{in } C^0_{\mathrm{loc}}\big(T''\setminus\{x\}\big)$$

as $\alpha\to+\infty$. Hence,

$$\frac{\int_{T''\setminus B'}|\nabla u_\alpha|^q\,dv(g)}{\int_{T''}u_\alpha^q\,dv(g)}\le\frac{\int_{T''}(\eta'')^q|\nabla u_\alpha|^q\,dv(g)}{\int_{T''}u_\alpha^q\,dv(g)}\le C$$

Coming back to (4.8), and letting $\alpha\to+\infty$, one gets the desired contradiction. As already mentioned, this ends the proof of the proposition. □

Looking carefully at the proof of Proposition 4.4, one sees that the arguments involved in such a proof are very general. Such arguments provide us with a localization process for the $(\mathrm{I}^q_{q,\mathrm{opt}})$ optimal inequality. This is expressed in the following result:

THEOREM 4.9 (Druet's localization) *Let (M,g) be a smooth compact n-dimensional Riemannian manifold. Suppose that for some $q\in[1,n)$ the $(\mathrm{I}^q_{q,\mathrm{opt}})$ inequality is locally valid in the sense that any x in M possesses an open neighborhood Ω with the property that for any $u\in\mathcal{D}(\Omega)$,*

$$(\mathrm{I}^q_{q,\mathrm{opt}})\qquad\left(\int_M|u|^p\,dv(g)\right)^{q/p}\le K(n,q)^q\int_M|\nabla u|^q\,dv(g)+B_x\int_M|u|^q\,dv(g)$$

for some $B_x\in\mathbb{R}$ independent of u. Then inequality $(\mathrm{I}^q_{q,\mathrm{opt}})$ is globally valid on (M,g): There exists $B\in\mathbb{R}$ such that

$$(\mathrm{I}^q_{q,\mathrm{opt}})\qquad\left(\int_M|u|^p\,dv(g)\right)^{q/p}\le K(n,q)^q\int_M|\nabla u|^q\,dv(g)+B\int_M|u|^q\,dv(g)$$

for any $u\in H^q_1(M)$.

PROOF: If $q = 1$, the proof goes in a very simple way via a partition-of-unity argument. One just has to glue together the local inequalities to get the global one. Suppose now that $q > 1$. The proof then goes as in the proof of Proposition 4.4. Assuming that $(\mathrm{I}^q_{q,\mathrm{opt}})$ is not globally valid, one gets some sequence $(u_\alpha)_{\alpha>0}$ of positive $C^{1,\lambda}$ functions on M, $\lambda \in (0, 1)$, such that for any α,

$$\Delta_{q,g}u_\alpha + \alpha u_\alpha^{q-1} = \lambda_\alpha u_\alpha^{p-1}$$

for some real number $\lambda_\alpha \in (0, K(n,q)^{-q})$, and such that for any α,

$$\int_M u_\alpha^p \, dv(g) = 1$$

As in the proof of Proposition 4.4, one then gets that, up to a subsequence, (u_α) has a unique point of concentration $x \in M$, with the property that

$$u_\alpha \to 0 \quad \text{in } C^0_{\mathrm{loc}}\big(M\backslash\{x\}\big)$$

as $\alpha \to +\infty$. Suppose now that there is some open neighborhood Ω of x, and that there exists $B_x \in \mathbb{R}$ such that for any $u \in \mathcal{D}(\Omega)$,

$$\left(\int_M |u|^p \, dv(g)\right)^{q/p} \le K(n,q)^q \int_M |\nabla u|^q \, dv(g) + B_x \int_M |u|^q \, dv(g)$$

Without loss of generality, we can assume that $\Omega = B$ is a ball centered at x. Following what has been done in the proof of Proposition 4.4, the goal now is to prove that such an inequality, combined with the fact that

$$\Delta_{q,g}u_\alpha + \alpha u_\alpha^{q-1} = \lambda_\alpha u_\alpha^{p-1}$$

that $\int_M u_\alpha^p \, dv(g) = 1$, and that $\lambda_\alpha < K(n,q)^{-q}$, lead to a contradiction. Clearly, this will end the proof of the theorem. In what follows, let η be a smooth function on M such that $\eta = 1$ on $B' \subset B$, $\eta = 0$ on $M\backslash B$, B' another ball centered at x. Then

$$\left(\int_B (\eta u_\alpha)^p \, dv(g)\right)^{\frac{q}{p}} \le K(n,q)^q \int_B |\nabla(\eta u_\alpha)|^q \, dv(g) + B_x \int_B (\eta u_\alpha)^q \, dv(g)$$

so that, setting $\eta' = 1 - \eta$,

$$\begin{aligned}
&\left(\int_{B'} u_\alpha^p \, dv(g)\right)^{\frac{q}{p}} \\
&\le \left(\int_B (\eta u_\alpha)^p \, dv(g)\right)^{\frac{q}{p}} \\
&\le K(n,q)^q \int_M \big|\nabla\big((1-\eta')u_\alpha\big)\big|^q \, dv(g) + B_x \int_M (\eta u_\alpha)^q \, dv(g) \\
&\le K(n,q)^q \int_M \big|\nabla\big((1-\eta')u_\alpha\big)\big|^q \, dv(g) + B_x \int_M u_\alpha^q \, dv(g)
\end{aligned}$$

As in the proof of Proposition 4.4,

$$\int_M \left|\nabla\big((1-\eta')u_\alpha\big)\right|^q dv(g) \le \int_M |\nabla u_\alpha|^q dv(g) + C\int_{M\backslash B'} |\nabla u_\alpha|^q dv(g) + C\int_{M\backslash B'} u_\alpha |\nabla u_\alpha|^{q-1} dv(g) + C\int_{M\backslash B'} u_\alpha^q dv(g)$$

for some constant C independent of α, while

$$\int_M |\nabla u_\alpha|^q dv(g) + \alpha\int_M u_\alpha^q dv(g) = \lambda_\alpha$$

As a consequence, we get that

$$\left(\int_{B'} u_\alpha^p dv(g)\right)^{\frac{q}{p}} \le \lambda_\alpha K(n,q)^q - \alpha K(n,q)^q \int_M u_\alpha^q dv(g) + C\int_{M\backslash B'} |\nabla u_\alpha|^q dv(g) + C\int_{M\backslash B'} u_\alpha^q dv(g) + C\int_{M\backslash B'} u_\alpha |\nabla u_\alpha|^{q-1} dv(g) + B_x \int_M u_\alpha^q dv(g)$$

for some other constant C independent of α. Since $\lambda_\alpha K(n,q)^q < 1$, and noting that

$$1 - \left(\int_{B'} u_\alpha^p dv(g)\right)^{\frac{q}{p}} \le \int_{M\backslash B'} u_\alpha^p dv(g)$$

this leads to

(4.9)

$$\begin{aligned} &\alpha K(n,q)^q - (B_x + C) \\ &\quad\le \frac{\int_{M\backslash B'} u_\alpha^p dv(g)}{\int_M u_\alpha^q dv(g)} + C\frac{\int_{M\backslash B'} |\nabla u_\alpha|^q dv(g)}{\int_M u_\alpha^q dv(g)} + C\frac{\int_{M\backslash B'} u_\alpha |\nabla u_\alpha|^{q-1} dv(g)}{\int_M u_\alpha^q dv(g)} \\ &\quad\le \frac{\int_{M\backslash B'} u_\alpha^p dv(g)}{\int_M u_\alpha^q dv(g)} + C\frac{\int_{M\backslash B'} |\nabla u_\alpha|^q dv(g)}{\int_M u_\alpha^q dv(g)} \\ &\qquad + C\left(\frac{\int_{M\backslash B'} |\nabla u_\alpha|^q dv(g)}{\int_M u_\alpha^q dv(g)}\right)^{\frac{q-1}{q}} \end{aligned}$$

With the same arguments used in the proof of Proposition 4.4, and since

$$u_\alpha \to 0 \quad \text{in } C^0_{\text{loc}}\big(M\backslash\{x\}\big)$$

as $\alpha \to +\infty$, one has that

$$\lim_{\alpha\to+\infty} \frac{\int_{M\backslash B'} u_\alpha^p dv(g)}{\int_M u_\alpha^q dv(g)} = 0$$

and that

$$\frac{\int_{M\backslash B'} |\nabla u_\alpha|^q \, dv(g)}{\int_M u_\alpha^q \, dv(g)} \leq C'$$

for some constant C' independent of α. By taking the limit as $\alpha \to +\infty$ in (4.9), one then gets the desired contradiction. This ends the proof of the theorem. □

Theorem 4.9 leads to several important results. One gets, for instance, the following proposition of Druet [**74**]:

PROPOSITION 4.5 *Let (H^n, h_0) be a smooth, compact, n-dimensional hyperbolic space. For any $q \in [1, n)$ real, inequality $(\mathrm{I}^q_{q,\mathrm{opt}})$ is true on (H^n, h_0).*

PROOF: Let $(\tilde{H}^n, \tilde{h}_0)$ be the simply connected hyperbolic space of n-dimensions. Any point in H^n possesses some neighborhood that is isometric to an open subset of $\tilde{H}^n$. By a result of Aubin [**10**], for any $u \in \mathcal{D}(\tilde{H}^n)$, and any $q \in [1, n)$,

$$\left(\int_{\tilde{H}^n} |u|^p \, dv(\tilde{h}_0)\right)^{q/p} \leq K(n, q)^q \int_{\tilde{H}^n} |\nabla u|^q \, dv(\tilde{h}_0)$$

Hence, any point in H^n possesses some open neighborhood Ω such that for any $q \in [1, n)$, and any $u \in \mathcal{D}(\Omega)$,

$$\left(\int_{H^n} |u|^p \, dv(h_0)\right)^{q/p} \leq K(n, q)^q \int_{H^n} |\nabla u|^q \, dv(h_0)$$

By Druet's localization, Theorem 4.9, this proves the result. □

As another striking example of application of Druet's localization, the following result holds (Druet, oral communication):

THEOREM 4.10 *For any smooth, compact Riemannian 2-manifold, and any $q \in [1, 2)$, $(\mathrm{I}^q_{q,\mathrm{opt}})$ is valid.*

PROOF: Let (M, g) be a smooth, compact Riemannian manifold of dimension 2. Without loss of generality, up to rescaling, we can assume that the sectional curvature of (M, g) is less than or equal to 1. As shown by Aubin [**10**], for any x in M, there exists some $\delta_x > 0$ such that for any smooth, bounded domain $\Omega \subset B_x(\delta_x)$, the area of $\partial\Omega$ is greater than or equal to the area in the standard sphere (S^2, h) of the boundary of a ball having the same volume than Ω. Let $q \in [1, 2)$ be given. From such a result and with a symmetrization process via the co-area formula similar to the one we described in the proof of Theorem 4.4, one gets that any point x in M possesses some open neighborhood Ω_x with the following property: For any smooth, nonnegative continuous function u on M, with compact support $K \subset \Omega_x$, K being itself smooth, u being smooth in K and such that it has only nondegenerate critical points in K, and for any $P \in S^2$, there exists some Lipschitz function $u^\star : S^2 \to \mathbb{R}$, depending only on the distance r to P, and decreasing with respect to r such that $\|\nabla u^\star\|_q \leq \|\nabla u\|_q$, $\|u^\star\|_q = \|u\|_q$, and $\|u^\star\|_p = \|u\|_p$. By Proposition 4.3, one has that there exists $B \in \mathbb{R}$ such that for any $f \in H_1^q(S^2)$,

$$\left(\int_{S^2} |f|^p \, dv(h)\right)^{q/p} \le K(n,q)^q \int_{S^2} |\nabla f|^q \, dv(h) + B \int_{S^2} |f|^q \, dv(h)$$

Hence,

$$\left(\int_{S^2} |u^\star|^p \, dv(h)\right)^{q/p} \le K(n,q)^q \int_{S^2} |\nabla u^\star|^q \, dv(h) + B \int_{S^2} |u^\star|^q \, dv(h)$$

and for u as above,

$$\left(\int_M |u|^p \, dv(g)\right)^{q/p} \le K(n,q)^q \int_M |\nabla u|^q \, dv(g) + B \int_M |u|^q \, dv(g)$$

By standard arguments from Morse theory, one then easily gets that such an inequality is actually valid for any $u \in \mathcal{D}(\Omega_x)$. In other words, and for any $q \in [1,2)$, $(\mathrm{I}^q_{q,\mathrm{opt}})$ is locally true. By Theorem 4.9, this proves the result. □

Coming back to Propositions 4.4 and 4.5, these propositions suggest that the optimal inequality $(\mathrm{I}^q_{q,\mathrm{opt}})$ should be valid when dealing with manifolds of nonpositive curvature. As recently noticed by Aubin, Druet, and the author **[13]**, this is basically the case. This is the subject of the following theorem, another very nice application of Druet's localization. More precisely, one gets that the validity of the Cartan-Hadamard conjecture in dimension n (as discussed in Chapter 8) implies the validity of $(\mathrm{I}^q_{q,\mathrm{opt}})$ on compact Riemannian n-dimensional manifolds of nonpositive sectional curvature. Apart from the 2-dimensional case which has already been discussed in Theorem 4.10, and since the Cartan-Hadamard conjecture is true in dimensions 3 and 4, $(\mathrm{I}^q_{q,\mathrm{opt}})$ is valid in such dimensions.

THEOREM 4.11 *Let (M,g) be a smooth, compact Riemannian n-manifold of nonpositive sectional curvature, and let $q \in [1,n)$. Suppose $n = 3$ or 4. Then inequality $(\mathrm{I}^q_{q,\mathrm{opt}})$ is valid on (M,g).*

PROOF: Let $(\tilde M, \tilde g)$ be the universal Riemannian covering of (M,g). Since the sectional curvature of g is nonpositive, the sectional curvature of $\tilde g$ is also nonpositive. In particular, $(\tilde M, \tilde g)$ is a Cartan-Hadamard manifold. As shown in Section 8.2 of Chapter 8, the validity of the Cartan-Hadamard conjecture implies that for any $u \in \mathcal{D}(\tilde M)$, and any $q \in [1,n)$,

$$\left(\int_{\tilde M} |u|^p \, dv(\tilde g)\right)^{q/p} \le K(n,q)^q \int_{\tilde M} |\nabla u|^q \, dv(\tilde g)$$

Hence, under the assumption that the Cartan-Hadamard conjecture is true, any point in M possesses some open neighborhood Ω such that for any $q \in [1,n)$ and any $u \in \mathcal{D}(\Omega)$,

$$\left(\int_M |u|^p \, dv(g)\right)^{q/p} \le K(n,q)^q \int_M |\nabla u|^q \, dv(g)$$

By Druet's localization, Theorem 4.9, one then gets that ($I^q_{q,\mathrm{opt}}$) is globally valid. Since the Cartan-Hadamard conjecture is true in dimensions 3 and 4 (see Section 8.2 of Chapter 8), this proves the theorem. □

Let us now come back to Proposition 4.4. In this result, the torus may be seen as a limit case of the manifolds in question in Theorem 4.8. Looking more precisely to the developments involved in the proof of Theorem 4.8, as shown by Druet [74], the torus also appears to be a limit case of compact Ricci flat manifolds which are not flat. As shown by Berger [25], $4n$-dimensional Riemannian manifolds whose holonomy group is contained in $\mathrm{Sp}(n)$ are Ricci flat. Such manifolds are also called hyperkählerian manifolds. Explicit examples of compact hyperkählerian manifolds that are not flat have been given by Beauville [20] in all dimensions. We refer the reader to the excellent reference by Besse [28] for more details on the subject. The following result is once more due to Druet [74]:

THEOREM 4.12 *Let (M, g) be a smooth, compact Riemannian n-manifold, and let $q \in [1, n)$ real. Assume that (M, g) is Ricci flat, but not flat. If $q > 4$ and $q^2 < n$, then inequality ($I^q_{q,\mathrm{opt}}$) is false on (M, g).*

PROOF: The proof is similar to that of Theorem 4.8. Let $\mathrm{Weyl}_{(M,g)}$ be the Weyl curvature of g, and let $x_0 \in M$ be such that $|\mathrm{Weyl}_{(M,g)}(x_0)| > 0$. For $\varepsilon > 0$, we set

$$u_\varepsilon = \left(\varepsilon + r^{\frac{q}{q-1}}\right)^{1-\frac{n}{q}} \varphi(r)$$

where r denotes the distance to x_0, φ is smooth such that $0 \le \varphi \le 1$, $\varphi = 1$ on $(-\frac{\delta}{2}, \frac{\delta}{2})$, and $\varphi = 0$ if $r \ge \delta$, and $\delta > 0$, δ small, is real. In order to prove the result, one just has to prove that for any $\alpha > 0$, and for ε small enough, $J(u_\varepsilon) < K(n, q)^{-q}$, where

$$J(u_\varepsilon) = \frac{\int_M |\nabla u_\varepsilon|^q \, dv(g) + \alpha \int_M u_\varepsilon^q \, dv(g)}{\left(\int_M u_\varepsilon^p \, dv(g)\right)^{\frac{q}{p}}}$$

Similar computations to those involved in the proof of Theorem 4.8 then lead to the following:

$$K(n, q)^q J(u_\varepsilon) \le 1 + \varepsilon^{\frac{n}{q}-1}\Big(B_1 + B_2 \varepsilon^{\frac{q^2-n}{q}} + B_3 \varepsilon^{1-\frac{n}{q}+4\frac{q-1}{q}} + o\Big(\varepsilon^{1-\frac{n}{q}+4\frac{q-1}{q}}\Big) + o\Big(\varepsilon^{\frac{q^2-n}{q}}\Big)\Big)$$

where $B_1 > 0$, $B_2 > 0$, are independent of ε, and

$$B_3 = \frac{|\mathrm{Weyl}_{(M,g)}(x_0)|^2}{120n(n+2)} \left(\frac{n-q}{n} \frac{\int_0^\infty \left(1 + s^{\frac{q}{q-1}}\right)^{-n} s^{n+3} \, ds}{\int_0^\infty \left(1 + s^{\frac{q}{q-1}}\right)^{-n} s^{n-1} \, ds} - \frac{\int_0^\infty \left(1 + s^{\frac{q}{q-1}}\right)^{-n} s^{\frac{q}{q-1}+n+3} \, ds}{\int_0^\infty \left(1 + s^{\frac{q}{q-1}}\right)^{-n} s^{\frac{q}{q-1}+n-1} \, ds} \right)$$

Since $q > 4$ and $q^2 < n$, one has that

$$1 - \frac{n}{q} + 4\,\frac{q-1}{q} < \frac{q^2 - n}{q} < 0$$

Hence, one will find $\varepsilon > 0$ small enough such that

$$J(u_\varepsilon) < \frac{1}{K(n,q)^q}$$

if $B_3 < 0$. As one can easily check, $B_3 < 0$ if $|\mathrm{Weyl}_{(M,g)}(x_0)|^2 > 0$. This ends the proof of the theorem. □

As a direct consequence of such a result, and of Theorem 4.8, one gets the following corollary:

COROLLARY 4.4 *Let (M, g) be a smooth, compact, n-dimensional Riemannian manifold of nonnegative Ricci curvature. Assume that for some q real with $q > 4$ and $q^2 < n$, inequality $(\mathrm{I}^q_{q,\mathrm{opt}})$ is true on (M, g). Then g is flat, and M is covered by a torus.*

Corollary 4.4 can be seen as the compact version of the result of Ledoux **[140]** that we will discuss in Chapter 8: For (M, g) a smooth, complete Riemannian n-manifold of nonnegative Ricci curvature, and $q \in [1, n)$ real, if for any $u \in \mathcal{D}(M)$,

$$\left(\int_M |u|^p \, dv(g)\right)^{q/p} \le K(n,q)^q \int_M |\nabla u|^q \, dv(g),$$

then (M, g) must be isometric to the Euclidean space $(\mathbb{R}^n, e)$. As a remark, note that it would be nice in Corollary 4.4 to improve the assumptions $q > 4$ and $q^2 < n$. However, by Proposition 4.3, it is necessary to assume that $q > 2$. Finally, note that by Theorem 4.4 and Theorem 4.9, if (M, g) is compact and flat, then $(\mathrm{I}^q_{q,\mathrm{opt}})$ is true on (M, g) for any $q \in [1, n)$ real. The argument goes here as in the proof of Proposition 4.5, noting that any compact flat manifold is covered by the Euclidean space.

By summarizing what has been said in this section, one gets a complete answer to question 1$\mathcal{A}$. According to the works of Aubin and Talenti, the value of $\alpha_q(M)$ is known, explicit, and depends only on n and q. Then, concerning question 2$\mathcal{A}$ and its extension to the $(\mathrm{I}^q_{q,\mathrm{gen}})$ inequality, one has that the validity of $(\mathrm{I}^q_{q,\mathrm{opt}})$ implies the validity of $(\mathrm{I}^1_{q,\mathrm{opt}})$, and that the following results hold:

1. $(\mathrm{I}^1_{q,\mathrm{opt}})$ is valid for all q on any smooth, compact Riemannian manifold of constant sectional curvature (Aubin).
2. $(\mathrm{I}^2_{2,\mathrm{opt}})$ is valid on any smooth, compact Riemannian n-manifold, $n \ge 3$ (Hebey-Vaugon).
3. Given (M, g) a smooth, compact Riemannian n-manifold, $(\mathrm{I}^q_{q,\mathrm{opt}})$ is not valid as soon as $q > 2$, $q^2 < n$, and the scalar curvature of g is positive somewhere (Druet).
4. Given (M, g) a smooth, compact Riemannian n-manifold of nonnegative Ricci curvature, $(\mathrm{I}^q_{q,\mathrm{opt}})$ is not valid as soon as $q > 4$, $q^2 < n$, and g is not flat (Druet).

5. $(\mathrm{I}_{q,\mathrm{opt}}^q)$ is valid for all q on any 2-dimensional smooth, compact Riemannian manifold (Druet).
6. $(\mathrm{I}_{q,\mathrm{opt}}^q)$ is valid for all q on compact flat spaces, compact hyperbolic spaces, and smooth, compact n-manifolds of nonpositive sectional curvature as long as the Cartan-Hadamard n-dimensional conjecture is true, so, in particular, if $n = 3$ or 4 (Druet, Aubin-Druet-Hebey).

Contrary to what we said about the optimal inequality $(\mathrm{J}_{q,\mathrm{opt}}^q)$ of program $\mathcal{B}$ (see Proposition 4.1), the $(\mathrm{I}_{q,\mathrm{opt}}^q)$ optimal inequality may be valid for $q > 2$. On the one hand, as when dealing with Program $\mathcal{B}$, there is really a difference between the optimal inequalities $(\mathrm{I}_{q,\mathrm{opt}}^1)$ and $(\mathrm{I}_{q,\mathrm{opt}}^q)$. As an example, $(\mathrm{I}_{q,\mathrm{opt}}^1)$ is valid on the standard unit sphere, while $(\mathrm{I}_{q,\mathrm{opt}}^q)$ is not for $q > 2$. On the other hand, the geometry interferes with the validity of $(\mathrm{I}_{q,\mathrm{opt}}^q)$.

4.4. On the Scale of Optimal Inequalities

According to the preceding section, there exist smooth, compact manifolds for which the optimal inequality $(\mathrm{I}_{q,\mathrm{opt}}^1)$ is valid while the stronger optimal inequality $(\mathrm{I}_{q,\mathrm{opt}}^q)$ is not valid. Given (M, g) a smooth, compact Riemannian n-manifold, $q \in [1, n)$ real, and $\theta \in [1, q]$ real, let us now consider the possible validity of the following inequality: There exists $B \in \mathbb{R}$ such that for any $u \in H_1^q(M)$,

$$(\mathrm{I}_{q,\mathrm{opt}}^\theta) \qquad \left(\int_M |u|^p \, dv(g)\right)^{\theta/p} \le K(n,q)^\theta \left(\int_M |\nabla u|^q \, dv(g)\right)^{\theta/q} + B\left(\int_M |u|^q \, dv(g)\right)^{\theta/q}$$

where $p = nq/(n-q)$. A conjecture of Aubin [10] on the subject is that $(\mathrm{I}_{q,\mathrm{opt}}^\theta)$ is valid on any smooth, compact Riemannian manifold with $\theta = q/(q-1)$ when $q > 2$. Note here that the validity of $(\mathrm{I}_{q,\mathrm{opt}}^{\theta_1})$ implies that of $(\mathrm{I}_{q,\mathrm{opt}}^{\theta_2})$ as soon as $\theta_1 \ge \theta_2$. The first result we have on the subject is the following result of **[10]**.

PROPOSITION 4.6 *For $q > 2$, the optimal inequality $(I_{q,\mathrm{opt}}^\theta)$ is valid with $\theta = q/(q-1)$ on the standard unit sphere (S^n, h) of $\mathbb{R}^{n+1}$.*

PROOF: Basically, the proof of such a result proceeds as in the proof of Proposition 4.3. For $q > 2$, let $\theta = q/(q-1)$. Let also P be some point in S^n, and let r denote the distance to P. Here again, by arguments from standard Morse theory, and by a symmetrization process, it suffices to prove that $(\mathrm{I}_{q,\mathrm{opt}}^\theta)$ is valid for functions of the form $g(x) = g(r)$, g a nonnegative, absolutely continuous, decreasing function on $[0, \pi]$. By Theorem 4.4, one has that

$$\left[\omega_{n-1} \int_0^\pi \left| g\left(\frac{\sin r}{r}\right)^{(n-1)/q} \right|^p r^{n-1} \, dr\right]^{q/p} \le \omega_{n-1} K(n,q)^q \int_0^\pi \left|\nabla\left[g\left(\frac{\sin r}{r}\right)^{(n-1)/q}\right]\right|^q r^{n-1} \, dr$$

As shown by Aubin [10], one can now prove that there exists some constant C_1 such that

$$\left(\omega_{n-1}\int_0^\pi \left|\nabla\left[g\left(\frac{\sin r}{r}\right)^{(n-1)/q}\right]'\right|^q r^{n-1}\,dr\right)^{1/(q-1)} \le$$
$$\left(\int_{S^n}|\nabla g|^q\,dv(h)\right)^{1/(q-1)} + C_1\left(\int_{S^n}|u|^q\,dv(h)\right)^{1/(q-1)}$$

and that there exists some constant C_2 such that

$$\left[\omega_{n-1}\int_0^\pi \left|g\left(\frac{\sin r}{r}\right)^{(n-1)/q}\right|^p r^{n-1}\,dr\right]^{q/(q-1)p} \ge$$
$$\left(\int_{S^n} g^p\,dv(h)\right)^{q/(q-1)p} - C_2\left(\int_{S^n} g^q\,dv(h)\right)^{1/(q-1)}$$

Clearly, as one can easily be convinced, the optimal inequality $(\mathrm{I}^\theta_{q,\mathrm{opt}})$ follows from such inequalities. This proves the proposition. □

For more details on the proof of Proposition 4.6, we refer the reader to [10]. Independently, as another result on the subject, one has the following theorem of Druet [74]:

THEOREM 4.13 *Let (M, g) be a smooth, compact Riemannian n-manifold, and let $q \in [1, n)$ real. Assume that $q > 2$, that $q^2 < n$, and that the scalar curvature of (M, g) is positive somewhere. Then inequality $(\mathrm{I}^\theta_{q,\mathrm{opt}})$ is false on (M, g) for any $\theta > 2$.*

PROOF: Here again, the proof is similar to that of Theorem 4.8. Using the same notation used there, we now have that

$$K(n,q)^\theta J_\theta(u_\varepsilon) \le 1 + \varepsilon^{\frac{n}{q}-1}\Big(C_1 + C_2\varepsilon^{\theta\frac{q-1}{q}+1-\frac{n}{q}} + C_3\varepsilon^{2\frac{q-1}{q}+1-\frac{n}{q}}$$
$$+ o\Big(\varepsilon^{\theta\frac{q-1}{q}+1-\frac{n}{q}}\Big) + o\Big(\varepsilon^{2\frac{q-1}{q}+1-\frac{n}{q}}\Big)\Big)$$

Here, $C_1 > 0$, $C_2 > 0$ real are independent of ε,

$$J_\theta(u) = \frac{\left(\int_M |\nabla u|^q\,dv(g)\right)^{\frac{\theta}{q}} + \alpha\left(\int_M |u|^q\,dv(g)\right)^{\frac{\theta}{q}}}{\left(\int_M |u|^p\,dv(g)\right)^{\frac{\theta}{p}}}$$

and

$$C_3 = \frac{\theta\,\mathrm{Scal}_{(M,g)}(x_0)}{6nq}\left(\frac{q}{p}\frac{\int_0^\infty\left(1+s^{\frac{q}{q-1}}\right)^{-n}s^{n+1}\,ds}{\int_0^\infty\left(1+s^{\frac{q}{q-1}}\right)^{-n}s^{n-1}\,ds}\right.$$
$$\left. - \frac{\int_0^\infty\left(1+s^{\frac{q}{q-1}}\right)^{-n}s^{\frac{q}{q-1}+n+1}\,ds}{\int_0^\infty\left(1+s^{\frac{q}{q-1}}\right)^{-n}s^{\frac{q}{q-1}+n-1}\,ds}\right)$$

Hence, $C_3 < 0$ if $\mathrm{Scal}_{(M,g)}(x_0) > 0$. As one can easily check, for $\theta > 2$, such an expansion leads to the existence of $\varepsilon > 0$ small such that $J_\theta(u_\varepsilon) < K(n,q)^{-\theta}$. This proves the theorem. □

When priority is given to the second constant, one can also deal with the scale of optimal inequalities. For (M, g) a smooth, compact Riemannian n-manifold, $q \in [1, n)$ real, and $\theta \in [1, q]$ real, let us now consider the following inequality: There exists $A \in \mathbb{R}$ such that for any $u \in H_1^q(M)$,

$$(J_{q,\mathrm{opt}}^\theta) \qquad \begin{aligned} &\left(\int_M |u|^p \, dv(g)\right)^{\theta/p} \leq \\ &A\left(\int_M |\nabla u|^q \, dv(g)\right)^{\theta/q} + \mathrm{Vol}_{(M,g)}^{-\theta/n}\left(\int_M |u|^q \, dv(g)\right)^{\theta/q} \end{aligned}$$

where $p = nq/(n-q)$. As for the $(I_{q,\mathrm{opt}}^\theta)$ inequality, if $(J_{q,\mathrm{opt}}^{\theta_1})$ is valid, then $(J_{q,\mathrm{opt}}^{\theta_2})$ is also valid for $\theta_2 \leq \theta_1$. Regarding the scale $(J_{q,\mathrm{opt}}^\theta)$ of optimal inequalities, using the same kind of arguments as used in Section 4.1, one gets the following result, which was observed by Druet:

THEOREM 4.14 *Let (M, g) be a smooth, compact Riemannian n-manifold, and let $q \in [1, n)$ real. If $q \leq 2$, then for any $\theta \in [1, q]$, $(J_{q,\mathrm{opt}}^\theta)$ is valid. Conversely, if $q > 2$, $(J_{q,\mathrm{opt}}^\theta)$ is valid if and only if $\theta \leq 2$.*

PROOF: By Theorem 4.2, $(J_{q,\mathrm{opt}}^q)$ is valid if $q \leq 2$. Hence, for any $\theta \in [1, q]$, $(J_{q,\mathrm{opt}}^\theta)$ is also valid. Suppose now that $q > 2$. By Bakry's inequality, discussed in the beginning of the proof of Theorem 4.2, one has that for any $u \in L^p(M)$,

$$\begin{aligned} \left(\int_M |u|^p \, dv(g)\right)^{2/p} &\leq \mathrm{Vol}_{(M,g)}^{-2(p-1)/p}\left(\int_M u \, dv(g)\right)^2 \\ &\quad + (p-1)\left(\int_M |u - \overline{u}|^p \, dv(g)\right)^{2/p} \end{aligned}$$

where

$$\overline{u} = \frac{1}{\mathrm{Vol}_{(M,g)}} \int_M u \, dv(g)$$

Independently, by the Sobolev-Poincaré inequality we discussed in Section 2.8 of Chapter 2, there exists $A > 0$ such that for any $u \in H_1^q(M)$,

$$\left(\int_M |u - \overline{u}|^p \, dv(g)\right)^{2/p} \leq A\left(\int_M |\nabla u|^q \, dv(g)\right)^{2/q}$$

Hence, for any $u \in H_1^q(M)$,

$$\begin{aligned} \left(\int_M |u|^p \, dv(g)\right)^{2/p} &\leq \mathrm{Vol}_{(M,g)}^{-2(p-1)/p}\left(\int_M u \, dv(g)\right)^2 \\ &\quad + (p-1)A\left(\int_M |\nabla u|^q \, dv(g)\right)^{2/q} \end{aligned}$$

and since by Hölder's inequality,

$$\left(\int_M u \, dv(g)\right)^2 \leq \mathrm{Vol}_{(M,g)}^{2(q-1)/q}\left(\int_M |u|^q \, dv(g)\right)^{2/q}$$

one gets that there exists $\tilde{A} > 0$ such that for any $u \in H_1^q(M)$,

$$\left(\int_M |u|^p \, dv(g)\right)^{2/p} \leq \tilde{A}\left(\left(\int_M |\nabla u|^q \, dv(g)\right)^{2/q} + \mathrm{Vol}_{(M,g)}^{-2/n}\left(\int_M |u|^q \, dv(g)\right)^{2/q}\right)$$

In other words, $(\mathrm{J}^2_{q,\mathrm{opt}})$ is valid. Hence, $(\mathrm{J}^\theta_{q,\mathrm{opt}})$ is valid for $\theta \leq 2$. Summarizing, we are left with the proof that for $q > 2$ and $\theta > 2$, $(\mathrm{J}^\theta_{q,\mathrm{opt}})$ is false. The argument here goes as in the proof of Proposition 4.1. Let $u \in C^\infty(M)$ be some nonzero function such that $\int_M u \, dv(g) = 0$. Then, for $\varepsilon > 0$,

$$\left(\int_M |1 + \varepsilon u|^p \, dv(g)\right)^{\theta/p} =$$

$$\mathrm{Vol}_{(M,g)}^{\theta/p} + \frac{\theta(p-1)}{2} \mathrm{Vol}_{(M,g)}^{\frac{\theta}{p}-1}\left(\int_M u^2 \, dv(g)\right)\varepsilon^2 + o(\varepsilon^2)$$

$$\left(\int_M |1 + \varepsilon u|^q \, dv(g)\right)^{\theta/q} =$$

$$\mathrm{Vol}_{(M,g)}^{\theta/q} + \frac{\theta(q-1)}{2} \mathrm{Vol}_{(M,g)}^{\frac{\theta}{q}-1}\left(\int_M u^2 \, dv(g)\right)\varepsilon^2 + o(\varepsilon^2)$$

while, since $\theta > 2$,

$$\left(\int_M |\nabla(1 + \varepsilon u)|^q \, dv(g)\right)^{\theta/q} = o(\varepsilon^2)$$

Assume now that $(\mathrm{J}^\theta_{q,\mathrm{opt}})$ is valid. One would get that for any $\varepsilon > 0$,

$$\mathrm{Vol}_{(M,g)}^{\theta/p} + \frac{\theta(p-1)}{2} \mathrm{Vol}_{(M,g)}^{\frac{\theta}{p}-1}\left(\int_M u^2 \, dv(g)\right)\varepsilon^2 \leq$$

$$\mathrm{Vol}_{(M,g)}^{\frac{\theta}{q}-\frac{\theta}{n}} + \frac{\theta(q-1)}{2} \mathrm{Vol}_{(M,g)}^{\frac{\theta}{q}-1-\frac{\theta}{n}}\left(\int_M u^2 \, dv(g)\right)\varepsilon^2 + o(\varepsilon^2)$$

But

$$\frac{\theta}{p} = \frac{\theta}{q} - \frac{\theta}{n}$$

so that, as one can easily check, such an inequality is impossible. This ends the proof of the theorem. □

Note added in proof: After the manuscript was completed, O. Druet ("The best constants problem in Sobolev inequalities," preprint of the University of Cergy-Pontoise, October 1998, and to appear in *Mathematische Annalen*) got the complete answer to the conjecture of Aubin **[10]** we mentioned at the beginning of Section 4.4. He even proved more since for $q > 2$, $\theta = q/(q-1)$ can be replaced by $\theta = 2$. With the notation of Section 4.4, the result of Druet is the following: *Given any smooth, compact Riemannian n-dimensional manifold* (M, g), $n \geq 2$, *and* $q \in (1, n)$ *real, the optimal inequality* $(\mathrm{I}^{\theta}_{q,\mathrm{opt}})$ *is valid on* (M, g) *with* $\theta = q$ *if* $q \le 2$, *and* $\theta = 2$ *if* $q \ge 2$. Together with Theorem 4.13, one sees that the relevant exponent in the $(\mathrm{I}^{\theta}_{q,\mathrm{opt}})$ scale of optimal inequalities is really $\theta = 2$ for $q > 2$, and not $\theta = q/(q-1)$ as suggested by the original conjecture. The idea that $\theta = 2$ would be the relevant exponent appeared to Druet when he got Theorem 4.13, but also Theorem 4.14, stating that the $(\mathrm{J}^{\theta}_{q,\mathrm{opt}})$ optimal inequality is valid with $\theta = q$ if $q \le 2$ or with $\theta = 2$ if $q \ge 2$, and is not valid with $\theta > 2$ if $q > 2$. One has to note here that the above-mentioned result of Druet implies that:

1. For any smooth, compact n-dimensional Riemannian manifold, $n \ge 2$, and any $q \in (1, n)$, the optimal inequality $(\mathrm{I}^{1}_{q,\mathrm{opt}})$ is valid on (M, g), and
2. for any smooth, compact n-dimensional Riemannian manifold, $n \ge 2$, and any $q \in (1, 2]$, the optimal inequality $(\mathrm{I}^{q}_{q,\mathrm{opt}})$ is valid on (M, g).

As we recently learned, a similar result was also announced by Aubin and Li (to appear in *Comptes Rendus de l'Académie des Sciences de Paris*, 1999).

CHAPTER 5

Best Constants in the Compact Setting II

We are concerned in this chapter with the continuation of Programs $\mathcal{A}$ and $\mathcal{B}$ of Chapter 4. Two more questions will be asked. For the sake of clarity, and according to what has been said in Sections 4.1 and 4.3, we restrict ourselves to the case $q = 2$.

In what follows, let (M, g) be a smooth, compact Riemannian n-manifold, with $n \geq 3$. According to Theorem 4.6, there exists some $B \in \mathbb{R}$ such that for any $u \in H_1^2(M)$,

$$(\mathrm{I}_{2,\mathrm{opt}}^2) \qquad \left(\int_M |u|^p \, dv(g)\right)^{2/p} \leq \alpha_2(M)^2 \int_M |\nabla u|^2 \, dv(g) + B \int_M u^2 \, dv(g)$$

where $p = 2n/(n-2)$, and $\alpha_2(M) = K(n, 2)$ has its precise value given by Theorem 4.4 (see, also, 4.5). Namely,

$$\alpha_2(M) = \sqrt{\frac{4}{n(n-2)\omega_n^{2/n}}}$$

where ω_n is the volume of the standard unit sphere of $\mathbb{R}^{n+1}$. One can now define $B_0(g)$ as the smallest possible B in $(\mathrm{I}_{2,\mathrm{opt}}^2)$. Namely, one can define

$$B_0(g) = \inf\left\{B \in \mathbb{R} \text{ s.t. } (\mathrm{I}_{2,\mathrm{opt}}^2) \text{ is valid}\right\}$$

Clearly, $(\mathrm{I}_{2,\mathrm{opt}}^2)$ holds with $B_0(g)$ in place of B. One then has that for any $u \in H_1^2(M)$,

$$(\mathrm{I}_{2,\mathrm{OPT}}^2) \quad \left(\int_M |u|^p \, dv(g)\right)^{2/p} \leq K(n, 2)^2 \int_M |\nabla u|^2 \, dv(g) + B_0(g) \int_M u^2 \, dv(g)$$

Here, OPT refers to the fact that $(\mathrm{I}_{2,,\mathrm{OPT}}^2)$ is totally optimal, in the sense that the two constants $K(n, 2)^2$ and $B_0(g)$ cannot be lowered. As a remark, one can note that by taking $u \equiv 1$ in $(\mathrm{I}_{2,\mathrm{OPT}}^2)$, one gets that necessarily, $B_0(g) \geq \mathrm{Vol}_{(M,g)}^{-2/n}$, where $\mathrm{Vol}_{(M,g)}$ denotes the volume of (M, g).

Similarly, when giving the priority to the second constant, one has by Theorem 4.2 that there exists $A \in \mathbb{R}$ such that for any $u \in H_1^2(M)$,

$$(\mathrm{J}_{2,\mathrm{opt}}^2) \qquad \left(\int_M |u|^p \, dv(g)\right)^{2/p} \leq A \int_M |\nabla u|^2 \, dv(g) + \beta_2(M)^2 \int_M u^2 \, dv(g)$$

where $p = 2n/(n-2)$, and $\beta_2(M) = \mathrm{Vol}_{(M,g)}^{-1/n}$. One can now define $A_0(g)$ as the smallest possible A in $(\mathrm{J}^2_{2,\mathrm{opt}})$. Namely, one can define

$$A_0(g) = \inf\left\{A \in \mathbb{R} \text{ s.t. } (\mathrm{J}^2_{2,\mathrm{opt}}) \text{ is valid}\right\}$$

Clearly, $(\mathrm{J}^2_{2,\mathrm{opt}})$ holds with $A_0(g)$ in place of A. One then has that for any $u \in H_1^2(M)$,

$$(\mathrm{J}^2_{2,\mathrm{OPT}}) \quad \left(\int_M |u|^p \, dv(g)\right)^{2/p} \le A_0(g) \int_M |\nabla u|^2 \, dv(g) + \mathrm{Vol}_{(M,g)}^{-2/n} \int_M u^2 \, dv(g)$$

As above, OPT refers to the fact that $(\mathrm{J}^2_{2,\mathrm{OPT}})$ is "totally optimal" in the sense that the two constants $A_0(g)$ and $\mathrm{Vol}_{(M,g)}^{-2/n}$ cannot be lowered. Note that by Proposition 4.2, $A_0(g) \ge K(n,2)^2$.

In what follows, we say that some nonzero function $u \in H_1^2(M)$ is an extremum function for $(\mathrm{I}^2_{2,\mathrm{OPT}})$ if

$$\left(\int_M |u|^p \, dv(g)\right)^{2/p} = K(n,2)^2 \int_M |\nabla u|^2 \, dv(g) + B_0(g) \int_M u^2 \, dv(g)$$

Similarly, we say that some function $u \in H_1^2(M)$, u nonconstant, is an extremum function for $(\mathrm{J}^2_{2,\mathrm{OPT}})$ if

$$\left(\int_M |u|^p \, dv(g)\right)^{2/p} = A_0(g) \int_M |\nabla u|^2 \, dv(g) + \mathrm{Vol}_{(M,g)}^{-2/n} \int_M u^2 \, dv(g)$$

One can then ask two more questions in each program. They are stated as follows:

Program $\mathcal{A}$, Part II	Program $\mathcal{B}$, Part II
Question 3$\mathcal{A}$: Can one compute or have estimates on $B_0(g)$?	Question 3$\mathcal{B}$: Can one compute or have estimates on $A_0(g)$?
Question 4$\mathcal{A}$: Under which conditions does one have that $(\mathrm{I}^2_{2,\mathrm{OPT}})$ possesses extremum functions?	Question 4$\mathcal{B}$: Under which conditions does one have that $(\mathrm{J}^2_{2,\mathrm{OPT}})$ possesses extremum functions?

Once more, when looking to these questions, one can see that more complete answers have been obtained concerning Program $\mathcal{B}$, Part II, than concerning Program $\mathcal{A}$, Part II. First, we start with the case of the standard unit sphere. For such a manifold, $A_0(g) = \alpha_2(M)^2$ and $B_0(g) = \beta_2(M)^2$, so that $(\mathrm{I}^2_{2,\mathrm{OPT}}) = (\mathrm{J}^2_{2,\mathrm{OPT}})$. Moreover, all the extremum functions here are explicitly known.

5.1. The Case of the Standard Unit Sphere

Let (S^n, h), $n \ge 3$, be the standard unit sphere of $\mathbb{R}^{n+1}$. For such a specific manifold, as one will see in Theorem 5.1 below, special phenomena occur when dealing with Program $\mathcal{A}$, Part II, and Program $\mathcal{B}$, Part II. Except for its more recent very last part, Theorem 5.1 is due to Aubin [9].

THEOREM 5.1 *Let (S^n, h) be the standard unit sphere of $\mathbb{R}^{n+1}$, $n \geq 3$. For any $u \in H_1^2(S^n)$,*

$$\text{(I)} \quad \left(\int_{S^n} |u|^p \, dv(h)\right)^{2/p} \leq K(n,2)^2 \int_{S^n} |\nabla u|^2 \, dv(h) + \omega_n^{-2/n} \int_{S^n} u^2 \, dv(h)$$

where ω_n is the volume of (S^n, h) and $p = 2n/(n-2)$. In particular, $A_0(h) = K(n,2)^2$, $B_0(h) = \omega_n^{-2/n}$, and $(\mathrm{I}^2_{2,\mathrm{OPT}}) = (\mathrm{J}^2_{2,\mathrm{OPT}}) = (\mathrm{I})$. Moreover, given $x_0 \in S^n$ and $\beta > 1$ real,

$$u_{x_0,\beta}(x) = (\beta - \cos r)^{1-\frac{n}{2}}$$

where r denotes the distance from x_0 to x is an extremum function for (I). *Conversely, if u is a nonconstant extremum function for* (I), *then up to a constant scale factor, u is one of the $u_{x_0,\beta}$'s.*

PROOF: Let us start with the fact that (I) is true. As an easy consequence of Proposition 4.2,

$$\inf_{u \in H_1^2(S^n)^*} \frac{\int_{S^n} \left(|\nabla u|^2 + \frac{n(n-2)}{4} u^2\right) dv(h)}{\left(\int_{S^n} |u|^{2n/(n-2)} \, dv(h)\right)^{(n-2)/n}} \leq \frac{1}{K(n,2)^2}$$

where $H_1^2(S^n)^*$ stands for the set of nonzero functions of $H_1^2(S^n)$. Suppose here that

$$\inf_{u \in H_1^2(S^n)^*} \frac{\int_{S^n} \left(|\nabla u|^2 + \frac{n(n-2)}{4} u^2\right) dv(h)}{\left(\int_{S^n} |u|^{2n/(n-2)} \, dv(h)\right)^{(n-2)/n}} < \frac{1}{K(n,2)^2}$$

One clearly gets from such an inequality that for $f \in C^\infty(S^n)$, f close to 1 in the C^0-norm,

$$\inf_{u \in H_1^2(S^n)^*} \frac{\int_{S^n} \left(|\nabla u|^2 + \frac{n(n-2)}{4} u^2\right) dv(h)}{\left(\int_{S^n} f |u|^{2n/(n-2)} \, dv(h)\right)^{(n-2)/n}} < \frac{1}{(\max f)^{(n-2)/n} K(n,2)^2}$$

As shown in Section 4.2 of Chapter 4, this leads to the existence of $u \in C^\infty(S^n)$, $u > 0$, a solution of

$$\Delta_h u + \frac{n(n-2)}{4} u = f u^{(n+2)/(n-2)}$$

In other words, one gets that there exists $\varepsilon > 0$ such that for any $f \in C^\infty(S^n)$, if f is such that $\|f - 1\|_{C^0} < \varepsilon$, then f is the scalar curvature of some conformal metric to h. The point, then (see, for instance, Section 6.3 of Chapter 6 or the discussion at the end of Section 4.2), is that such a result is in contradiction with the obstructions of Kazdan and Warner [**130**]. As a conclusion, one has that

$$\inf_{u \in H_1^2(S^n)^*} \frac{\int_{S^n} \left(|\nabla u|^2 + \frac{n(n-2)}{4} u^2\right) dv(h)}{\left(\int_{S^n} |u|^{2n/(n-2)} \, dv(h)\right)^{(n-2)/n}} = \frac{1}{K(n,2)^2}$$

and inequality (I) is true.

Let us now prove that the $u_{x_0,\beta}$'s are extremum functions for (I). In geodesic normal coordinates,

$$\Delta_h f = -\frac{1}{r^{n-1}\sqrt{|h|}} \partial_r \left(r^{n-1} \sqrt{|h|} \partial_r f\right)$$

and one has here that

$$r^{n-1}\sqrt{|h|} = \sin^{n-1} r$$

Simple computations then lead to the fact that

$$\frac{4}{n(n-2)}\Delta_h u_{x_0,\beta} + u_{x_0,\beta} = (\beta^2 - 1)u_{x_0,\beta}^{(n+2)/(n-2)}$$

Moreover,

$$\int_{S^n} u_{x_0,\beta}^{2n/(n-2)}\,dv(h) = \omega_{n-1}\int_0^\pi \frac{\sin^{n-1} t}{(\beta - \cos t)^n}\,dt$$

and one then easily gets that

$$(\beta^2 - 1)^{n/2}\int_{S^n} u_{x_0,\beta}^{2n/(n-2)}\,dv(h) = \omega_n$$

Recall now that

$$K(n,2)^2 = \frac{4}{n(n-2)\omega_n^{2/n}}$$

Hence,

$$\begin{aligned}
&K(n,2)^2\int_{S^n}|\nabla u_{x_0,\beta}|^2\,dv(h) + \omega_n^{-2/n}\int_{S^n} u_{x_0,\beta}^2\,dv(h)\\
&= \frac{1}{\omega_n^{2/n}}\int_{S^n}\left(\frac{4}{n(n-2)}\Delta_h u_{x_0,\beta} + u_{x_0,\beta}\right)u_{x_0,\beta}\,dv(h)\\
&= \frac{\beta^2 - 1}{\omega_n^{2/n}}\int_{S^n} u_{x_0,\beta}^{2n/(n-2)}\,dv(h)\\
&= \left(\int_{S^n} u_{x_0,\beta}^{2n/(n-2)}\,dv(h)\right)^{1-\frac{2}{n}}
\end{aligned}$$

and this proves that the $u_{x_0,\beta}$'s are extremum functions for (I). In order to end the proof of the theorem, let us now consider some nonconstant extremum function u of (I). Then $v = |u|$ is also a nonconstant extremum function for (I). Up to multiplication by a positive constant scale factor, we can assume that

$$\int_{S^n} v^{2n/(n-2)}\,dv(h) = \omega_n$$

Since v realizes the infimum of

$$J(u) = \frac{\int_{S^n}\left(|\nabla u|^2 + \frac{n(n-2)}{4}u^2\right)dv(h)}{\left(\int_{S^n}|u|^{2n/(n-2)}\,dv(h)\right)^{(n-2)/n}}$$

it is a solution of

$$\frac{4(n-1)}{n-2}\Delta_h v + n(n-1)v = n(n-1)v^{(n+2)/(n-2)}$$

By regularity results and the maximum principle, one then gets that v is smooth and positive. On the one hand, this shows that $|u| > 0$, so that u is either everywhere positive or everywhere negative. On the other hand, the fact that v is a positive, smooth solution of the above equation implies that $g = v^{4/(n-2)}h$ has constant scalar curvature $n(n-1)$. By Obata **[163]**, one then gets that g and h are isometric.

The proof of the last part of the theorem then reduces to the fact that for $u_\varphi > 0$ given by $\varphi^\star h = u_\varphi^{4/(n-2)} h$, where $\varphi \in \mathrm{Conf}(S^n)$ is a conformal diffeomorphism of (S^n, h), there exist $\lambda > 0$, $\beta > 1$, and $x_0 \in S^n$ such that $u_\varphi = \lambda u_{x_0,\beta}$. Since the group of isometries of (S^n, h) acts transitively on S^n, it suffices to restrict our attention to conformal diffeomorphisms φ such that $\varphi(P) = P$ for some fixed P in S^n. Indeed, given $\varphi \in \mathrm{Conf}(S^n)$, choose $A \in O(n+1)$ an isometry of (S^n, h) such that $A(\varphi(P)) = P$, and note that $(A \circ \varphi)^\star h = \varphi^\star h$. From now on, let $\mathrm{Conf}_P(S^n)$ be the group of conformal diffeomorphisms φ of (S^n, h) such that $\varphi(P) = P$. Let Pr_P be the stereographic projection of pole P, and let $\varphi \in \mathrm{Conf}_P(S^n)$. Then $\psi = Pr_P \circ \varphi \circ Pr_P^{-1}$ is a conformal diffeomorphism of the Euclidean space $(\mathbb{R}^n, e)$. As is well-known,

$$\psi = A \circ B \circ C$$

where $A \in O(n)$, B is a translation, and C is a dilatation. As usually done, we assimilate $\mathbb{R}^n$ with $P^\perp$, that is, the hyperplane of $\mathbb{R}^{n+1}$ that is orthogonal to the line passing through P and $-P$. Here, one has that

$$\left(Pr_P^{-1}\right)^\star h(x) = \frac{4}{(1+|x|^2)^2} e$$

Take B under the form $B(x) = x + a$ and take C under the form $C(x) = \lambda x$, $\lambda \neq 0$. Then,

$$\psi^\star\left(Pr_P^{-1}\right)^\star h(x) = \frac{4\lambda^2}{(1+|\lambda x + a|^2)^2} e$$

As one can easily check, this leads to

$$\varphi^\star h(x) = \lambda^2 \left(\frac{1 + |Pr_P(x)|^2}{1 + |\lambda Pr_P(x) + a|^2} \right)^2 h(x)$$

Set

$$\alpha(x) = \frac{1 + |Pr_P(x)|^2}{1 + |\lambda Pr_P(x) + a|^2}$$

The goal then is to compute α. Without loss of generality, we can assume that if $a = 0$, then $\lambda \neq \pm 1$. Since

$$Pr_P(x) = P + \frac{1}{1 - \langle P, x \rangle}(x - P)$$

and since P and $Pr_P(x)$ are orthogonal for any $x \in S^n$, one has that

$$1 + |Pr_P(x)|^2 = |Pr_P(x) - P|^2 = \frac{2}{1 - \langle P, x \rangle}$$

In the same order of ideas, and since $a \in P^\perp$, one has that

$$\begin{aligned} 1 + |\lambda Pr_P(x) + a|^2 &= 1 - \lambda^2 + |\lambda(Pr_P(x) - P) + a|^2 \\ &= 1 + |a|^2 - \lambda^2 + \lambda^2 |Pr_P(x) - P|^2 + 2\lambda \langle Pr_P(x), a \rangle \\ &= 1 + |a|^2 - \lambda^2 + \frac{2\lambda}{1 - \langle P, x \rangle}\langle a, x \rangle + \frac{2\lambda^2}{1 - \langle P, x \rangle} \end{aligned}$$

Hence,

$$\alpha(x) = \frac{2}{R - \langle Q, x \rangle}$$

where $R = 1 + |a|^2 + \lambda^2$ and

$$Q = \left(1 + |a|^2 - \lambda^2\right)P - 2\lambda a$$

Set $x_0 = \frac{1}{|Q|}Q$. As one can easily check (see below), $Q \neq 0$ since either $a \neq 0$ or $a = 0$ and then $\lambda \neq \pm 1$. Clearly, $x_0 \in S^n$ and

$$\alpha(x) = \frac{2}{|Q|}\left(\beta - \langle x_0, x \rangle\right)^{-1}$$

where $\beta = R/|Q|$. Noting that

$$|Q|^2 = 4\lambda^2|a|^2 + \left(1 + |a|^2 - \lambda^2\right)^2$$

one easily gets that $\beta > 1$. Moreover, for r the distance between x_0 and x on S^n, one has that $\cos r = \langle x_0, x \rangle$. Hence,

$$u_\varphi(x) = \left(\frac{4\lambda^2}{|Q|^2}\right)^{(n-2)/4} (\beta - \cos r)^{1-\frac{n}{2}}$$

and this proves the theorem. □

Before ending this section, we mention that the above inequality (I) has been extended by Beckner [**21**] to powers $2 \leq k \leq p$, where $p = 2n/(n-2)$. This is the subject of the result below. We refer to [**21**] for its proof, but also to Bakry-Ledoux [**19**] and Fontenas [**84**], where the result is proved in the more general context of an abstract Markov generator.

THEOREM 5.2 *Let (S^n, h) be the standard unit sphere of $\mathbb{R}^{n+1}$, $n \geq 3$. For any $k \in [2, p]$, and any $u \in H_1^2(S^n)$,*

$$\left(\int_{S^n} |u|^k \, dv(h)\right)^{2/k} \leq \frac{k-2}{n\omega_n^{1-\frac{2}{k}}} \int_{S^n} |\nabla u|^2 \, dv(h) + \frac{1}{\omega_n^{1-\frac{2}{k}}} \int_{S^n} u^2 \, dv(h)$$

where ω_n is the volume of (S^n, h) and $p = 2n/(n-2)$.

5.2. Program $\mathcal{B}$, Part II

Three main results are available on the subject. Let us start with question 3$\mathcal{B}$. We prove first the following general result. As in the case of the remaining constant in $(J^1_{2,\mathrm{opt}})$, $A_0(g)$ has an upper bound depending only on the dimension n of the manifold, and on a lower bound for the Ricci curvature, a lower bound for the volume, and an upper bound for the diameter.

THEOREM 5.3 *Let (M, g) be a smooth, compact Riemannian n-manifold, $n \geq 3$. Suppose that its Ricci curvature, volume, and diameter satisfy*

$$\mathrm{Rc}_{(M,g)} \geq kg\,, \qquad \mathrm{Vol}_{(M,g)} \geq v\,, \qquad \mathrm{diam}_{(M,g)} \leq d$$

for some k, $v > 0$, and $d > 0$ real. There exists $A = A(n, k, v, d)$, depending only on n, k, v, and d, such that $A_0(g) \leq A$. In other words, for any smooth, compact

Riemannian n-manifold (M, g), $n \geq 3$, *such that* $\mathrm{Rc}_{(M,g)} \geq kg$, $\mathrm{Vol}_{(M,g)} \geq v$, *and* $\mathrm{diam}_{(M,g)} \leq d$, *one has that for any* $u \in H_1^2(M)$,

$$\left(\int_M |u|^p \, dv(g)\right)^{2/p} \leq A \int_M |\nabla u|^2 \, dv(g) + \mathrm{Vol}_{(M,g)}^{-2/n} \int_M u^2 \, dv(g)$$

where $p = 2n/(n-2)$.

PROOF: Let λ_1 be the first nonzero eigenvalue of the Laplacian Δ_g associated to g. As shown by Yau [**200**], there exists some positive constant $\lambda = \lambda(n, k, v, d)$, depending only on n, k, v, and d, such that $\lambda_1 \geq \lambda$. In particular, this implies that for any $u \in H_1^2(M)$,

$$\int_M |u - \overline{u}|^2 \, dv(g) \leq \frac{1}{\lambda} \int_M |\nabla u|^2 \, dv(g)$$

where

$$\overline{u} = \frac{1}{\mathrm{Vol}_{(M,g)}} \int_M u \, dv(g)$$

Independently, one clearly has by Gromov's theorem, Theorem 1.1, that there exists some positive constant $a = a(n, k, v, d)$, depending only on n, k, v, and d, such that for any $x \in M$, $\mathrm{Vol}_g(B_x(1)) \geq a$. According to what has been said in Chapter 3, Section 3.2, one then gets that there exists some positive constant $A = A(n, k, v, d)$, depending only on n, k, v, and d, such that for any $u \in H_1^2(M)$,

$$\left(\int_M |u|^p \, dv(g)\right)^{2/p} \leq A \int_M |\nabla u|^2 \, dv(g) + A \int_M u^2 \, dv(g)$$

where $p = 2n/(n-2)$. As a consequence, combining these two inequalities, there exists a positive constant $\tilde{A} = \tilde{A}(n, k, v, d)$, depending only on n, k, v , and d, such that for any $u \in H_1^2(M)$,

$$\left(\int_M |u - \overline{u}|^2 \, dv(g)\right)^{2/p} \leq \tilde{A} \int_M |\nabla u|^2 \, dv(g)$$

Independently, as shown in the proof of Theorem 4.2, for any $u \in L^p(M)$,

$$\left(\int_M |u|^p \, dv(g)\right)^{2/p} \leq \mathrm{Vol}_{(M,g)}^{-2(p-1)/p} \left(\int_M u \, dv(g)\right)^2 + (p-1)\left(\int_M |u - \overline{u}|^p \, dv(g)\right)^{2/p}$$

Hence, for any $u \in H_1^2(M)$,

$$\begin{aligned}
&\left(\int_M |u|^p\,dv(g)\right)^{2/p}\\
&\le \mathrm{Vol}_{(M,g)}^{-2(p-1)/p}\left(\int_M u\,dv(g)\right)^2 + (p-1)\left(\int_M |u-\overline{u}|^p\,dv(g)\right)^{2/p}\\
&\le \mathrm{Vol}_{(M,g)}^{-2(p-1)/p}\left(\int_M u\,dv(g)\right)^2 + (p-1)\tilde{A}\int_M |\nabla u|^2\,dv(g)\\
&\le \mathrm{Vol}_{(M,g)}^{1-(2(p-1)/p)}\int_M u^2\,dv(g) + (p-1)\tilde{A}\int_M |\nabla u|^2\,dv(g)
\end{aligned}$$

and this proves the theorem. □

In the case of positive Ricci curvature, one can give an explicit expression for the upper bound A of Theorem 5.3. This is the subject of the following result, due to Ilias **[123]**. We refer also the reader to Bakry-Ledoux **[19]** and Fontenas **[84]**.

THEOREM 5.4 *Let (M,g) be a smooth, compact Riemannian n-manifold, $n \ge 3$. Suppose that its Ricci curvature satisfies that* $\mathrm{Rc}_{(M,g)} \ge (n-1)kg$ *for some $k > 0$. Then*

$$A_0(g) \le \frac{4}{n(n-2)k\,\mathrm{Vol}_{(M,g)}^{2/n}}$$

so that for any $u \in H_1^2(M)$,

$$\left(\int_M |u|^p\,dv(g)\right)^{2/p} \le \frac{4}{n(n-2)k\,\mathrm{Vol}_{(M,g)}^{2/n}}\int_M |\nabla u|^2\,dv(g) + \mathrm{Vol}_{(M,g)}^{-2/n}\int_M |u|^2\,dv(g)$$

where $p = 2n/(n-2)$.

PROOF: Without loss of generality, we can assume that $k = 1$. In other words, we can assume that $\mathrm{Rc}_{(M,g)} \ge (n-1)g$. Let $\mathcal{M}(M)$ be the space of smooth functions having only nondegenerate critical points. By standard arguments of Morse theory, it suffices to prove the inequality of the theorem for $u \in \mathcal{M}(M)$, u nonnegative. Let $\beta = \mathrm{Vol}_{(M,g)}/\omega_n$. As shown by Gromov **[95]** (see also Berard, Besson, and Gallot **[22]**), for Ω a smooth domain in M, and B a geodesic ball in the standard sphere (S^n, h), if $\mathrm{Vol}_g(\Omega) = \beta\,\mathrm{Vol}_h(B)$, then

$$\mathrm{Area}_g(\partial\Omega) \ge \beta\,\mathrm{Area}_h(\partial B)$$

By rather standard arguments of symmetrization, one can then associate to each u in $\mathcal{M}(M)$, u nonnegative, some radially symmetric function u^* on S^n such that for any $m \ge 1$ real,

$$\int_M |u|^m dv(g) = \beta \int_{S^n} |u^*|^m dv(h)$$

$$\int_M |\nabla u|^m dv(g) \ge \beta \int_{S^n} |\nabla u^*|^m dv(h)$$

Just define u^* by

$$\mathrm{Vol}_g(\{x \in M \,/\, u(x) > t\}) = \beta\,\mathrm{Vol}_h\big(\{x \in S^n \,/\, u^*(x) > t\}\big)$$

By Theorem 5.1, one then has that

$$\begin{aligned}
&\left(\int_M |u|^p\,dv(g)\right)^{2/p}\\
&= \left(\beta\int_{S^n} |u^*|^p\,dv(h)\right)^{2/p}\\
&\le K(n,2)^2\beta^{2/p}\int_{S^n}|\nabla u^*|^2\,dv(h) + \beta^{2/p}\omega_n^{-2/n}\int_{S^n}|u^*|^2\,dv(h)\\
&\le K(n,2)^2\beta^{\frac{2}{p}-1}\int_M|\nabla u|^2\,dv(g) + \beta^{\frac{2}{p}-1}\omega_n^{-2/n}\int_M u^2\,dv(g)\\
&= \frac{K(n,2)^2\omega_n^{2/n}}{\mathrm{Vol}_{(M,g)}^{2/n}}\int_M|\nabla u|^2\,dv(g) + \mathrm{Vol}_{(M,g)}^{-2/n}\int_M u^2\,dv(g)\\
&= \frac{4}{n(n-2)\,\mathrm{Vol}_{(M,g)}^{2/n}}\int_M|\nabla u|^2\,dv(g) + \mathrm{Vol}_{(M,g)}^{-2/n}\int_M u^2\,dv(g)
\end{aligned}$$

This proves the theorem. □

Let us now deal with question 4$\mathcal{B}$. We prove here the following result, due to Bakry and Ledoux [19].

THEOREM 5.5 *Let (M, g) be a smooth, compact Riemannian n-manifold, $n \ge 3$. We assume that for all $u \in H_1^2(M)$,*

$$\left(\int_M |u|^p\,dv(g)\right)^{2/p} \le \frac{4}{n(n-2)\,\mathrm{Vol}_{(M,g)}^{2/n}}\int_M|\nabla u|^2\,dv(g) + \mathrm{Vol}_{(M,g)}^{-2/n}\int_M|u|^2\,dv(g)$$

and that there exists some real-valued Lipschitz function f on M such that

$$\max_{x\in M}|\nabla f(x)| \le 1 \quad \text{and} \quad \max_{x,y\in M}|f(y)-f(x)| = \pi$$

Then

$$A_0(g) = \frac{4}{n(n-2)\,\mathrm{Vol}_{(M,g)}^{2/n}}$$

and there exist nonconstant extremum functions for $(\mathrm{J}^2_{2,\mathrm{OPT}})$. More precisely, if we translate f such that $\int_M \sin(f)dv(g) = 0$, for every $\lambda \in (-1,+1)$,

$$\left(\int_M |f_\lambda|^p\,dv(g)\right)^{2/p} = \frac{4}{n(n-2)\,\mathrm{Vol}_{(M,g)}^{2/n}}\int_M|\nabla f_\lambda|^2\,dv(g) + \mathrm{Vol}_{(M,g)}^{-2/n}\int_M|f_\lambda|^2\,dv(g)$$

where $f_\lambda = (1+\lambda\sin(f))^{1-n/2}$ and $p = 2n/(n-2)$.

PROOF: Let f be a real-valued Lipschitz function on M such that

$$\max_{x\in M}|\nabla f(x)|\le 1 \quad \text{and} \quad \max_{x,y\in M}|f(y)-f(x)|=\pi$$

As one can easily check, there exists $\theta\in\mathbb{R}$ such that

$$\int_M \sin(f+\theta)dv(g)=0$$

In what follows, let f stand for $f+\theta$. Following Bakry and Ledoux [19], for $\lambda\in(-1,1)$, we set

$$F(\lambda)=\int_M(1+\lambda\sin(f))^{2-n}\,dv(g)$$

while for $k>0$, we let D_k be the differential operator on $(-1,1)$ defined by

$$D_k=\frac{1}{k}\lambda\frac{d}{d\lambda}+I$$

We set also $\alpha=(n-2)/n=2/p$, and let

$$G=D_{n-1}F$$

The proof then proceeds in several steps.

STEP 1: For $\lambda\in(-1,1)$,

$$\left(D_{n-2}G\right)^{\alpha}+\alpha(1-\lambda^2)V^{-2/n}D_{n-2}G\le(1+\alpha)V^{-2/n}G$$

where $V=\mathrm{Vol}_{(M,g)}$.

In order to prove this claim, let

$$f_\lambda=(1+\lambda\sin(f))^{1-\frac{n}{2}}$$

Then, since

$$\max_{x\in M}|\nabla f(x)|\le 1\,,$$

one gets that

$$\int_M|\nabla f_\lambda|^2\,dv(g)\le\left(\frac{n}{2}-1\right)^2\lambda^2\int_M(1+\sin(f))^{-n}(1-\cos^2(f))dv(g)$$

Applying the Sobolev inequality

$$\left(\int_M|u|^p\,dv(g)\right)^{2/p}\le\frac{4}{n(n-2)V^{2/n}}\int_M|\nabla u|^2\,dv(g)+V^{-2/n}\int_M|u|^2\,dv(g)$$

with $u=f_\lambda$, this leads to

$$\begin{aligned}&\left(\int_M(1+\lambda\sin(f))^{-n}\,dv(g)\right)^{\alpha}\\&\le V^{-2/n}\int_M(1+\lambda\sin(f))^{2-n}\,dv(g)\\&\quad+\alpha\lambda^2V^{-2/n}\int_M(1+\sin(f))^{-n}(1-\cos^2(f))dv(g)\end{aligned}$$

where, as above, V stands for $\mathrm{Vol}_{(M,g)}$. Now, note that

$$F(\lambda) = \int_M (1 + \lambda \sin(f))^{-n} A(\lambda) dv(g)$$

$$F'(\lambda) = (n-2) \int_M (1 + \lambda \sin(f))^{-n} B(\lambda) dv(g)$$

$$F''(\lambda) = (n-1)(n-2) \int_M (1 + \lambda \sin(f))^{-n} C(\lambda) dv(g)$$

where

$$A(\lambda) = 1 + 2\lambda \sin(f) + \lambda^2 \sin^2(f)$$
$$B(\lambda) = -\sin(f) - \lambda \sin^2(f)$$
$$C(\lambda) = \sin^2(f)$$

Clearly,

$$A(\lambda) + 2\lambda B(\lambda) = 1 - \lambda^2 C$$

where $C = C(\lambda)$. Hence, coming back to the above inequality,

$$\begin{aligned}\left(\int_M (1 + \lambda \sin(f))^{-n} dv(g)\right)^\alpha &+ \alpha(1-\lambda^2) V^{-2/n} \int_M (1 + \lambda \sin(f))^{-n} dv(g) \\ &\le (1+\alpha) V^{-2/n} F(\lambda) + \frac{2\lambda\alpha}{n-2} V^{-2/n} F'(\lambda) \\ &= (1+\alpha) V^{-2/n} G(\lambda)\end{aligned}$$

Noting that

$$\begin{aligned}\int_M (1 + \lambda \sin(f))^{-n} dv(g) &= F(\lambda) + \frac{2}{n-2} \lambda F'(\lambda) + \frac{1}{(n-1)(n-2)} \lambda^2 F''(\lambda) \\ &= D_{n-2} G(\lambda)\end{aligned}$$

this proves Step 1.

Let us now consider the differential equation

$$\left(D_{n-2} H\right)^\alpha + \alpha(1-\lambda^2) V^{-2/n} D_{n-2} H = (1+\alpha) V^{-2/n} H$$

where $\lambda \in (-1, 1)$ and $V = \mathrm{Vol}_{(M,g)}$. For $c \in \mathbb{R}$, set

$$H_c(\lambda) = \frac{V}{1+\alpha} U_c(\lambda)^{\frac{2\alpha}{1-\alpha}} + \frac{\alpha V}{1+\alpha} (1-\lambda^2) U_c(\lambda)^{\frac{2}{1-\alpha}}$$

where

$$U_c(\lambda) = \frac{c\lambda + \sqrt{c^2\lambda^2 + (1-\lambda^2)}}{1-\lambda^2}$$

As one can easily check, the H_c's are solutions of the above differential equation. The second step in the proof of Theorem 5.5 is as follows:

STEP 2: Assume that $G(\lambda_0) < H_c(\lambda_0)$ for some $\lambda_0 \in [0, 1)$. Then $G(\lambda) \le H_c(\lambda)$ for every $\lambda \in [\lambda_0, 1)$.

In order to prove this claim, let $v = v(t, \lambda)$ be the unique nonnegative solution of

$$v^\alpha + \alpha(1-\lambda^2) V^{-2/n} v = (1+\alpha) V^{-2/n} t$$

One has that v is increasing in t. In addition,

$$D_{n-2}(G - H_c) \leq v(G, \lambda) - v(H_c, \lambda)$$

Hence, $D_{n-2}(G - H_c) \leq 0$ on the set $\{G \leq H_c\}$. Suppose that for some $\lambda_1 \in (\lambda_0, 1)$, $G(\lambda_1) > H_c(\lambda_1)$. Set

$$\lambda_* = \inf\left\{\lambda > \lambda_0 \,/\, G(\lambda) = H_c(\lambda)\right\}$$

Then $\lambda_* \in (\lambda_0, 1)$ and $G \leq H_c$ on $[\lambda_0, \lambda_*]$. Hence, $D_{n-2}(G - H_c) \leq 0$ on this interval. Integrating between λ_0 and λ_*, one gets that

$$\lambda_0^{n-2}\big(G - H_c\big)(\lambda_0) \geq \lambda_*^{n-2}\big(G - H_c\big)(\lambda_*)$$

Since $G(\lambda_0) < H_c(\lambda_0)$, this contradicts the fact that $G(\lambda_*) = H_c(\lambda_*)$. This proves Step 2.

As a third and last step, one has the following:

STEP 3: Assume that

$$\int_M (1 + \sin(f))^{1-n}\, dv(g) < +\infty$$

Then $\|(1 + \sin(f))^{-1}\|_\infty < +\infty$

In order to prove this claim, we apply the Sobolev inequality

$$\left(\int_M |u|^p\, dv(g)\right)^{2/p} \leq \frac{4}{n(n-2)\,\mathrm{Vol}_{(M,g)}^{2/n}} \int_M |\nabla u|^2\, dv(g) + \mathrm{Vol}_{(M,g)}^{-2/n} \int_M |u|^2\, dv(g)$$

to the family of functions

$$u = (1 + \sin(f))^{-s/2}$$

with $s \geq n - 2$. Set

$$\hat{F}(s) = \int_M (1 + \sin(f))^{-s}\, dv(g)$$

Since

$$\max_{x \in M} |\nabla f(x)| \leq 1\,,$$

one gets from the above inequality that

$$\hat{F}(\beta s)^{1/\beta} \leq V^{-2/n}\hat{F}(s) + C V^{-2/n} s^2 \hat{F}(s+1)$$

where $\beta = 1/\alpha$ and $C = 2/n(n-2)$. Noting that $\beta s > s + 1$ when $s \geq n - 2$, it already follows by iteration that $\hat{F}(s) < +\infty$ for every $s \geq n - 1$. We aim to prove that

$$\sup_{s \geq n-1} \hat{F}(s)^{1/s} < +\infty$$

from which the conclusion follows. Here, it may be assumed that $\hat{F}(s) \geq 1$ for some s large enough. Otherwise, there is nothing to prove. But then, by Jensen's inequality,

$$\hat{F}(\beta s)^{1/\beta} \leq \left(1 + Cs^2\right) V^{-2/n} \hat{F}(s+1)$$

By a simple iteration procedure, this yields Step 3.

Let us now prove Theorem 5.5. We claim here that $G = H_0$. As a starting point, since

$$\int_M \sin(f)dv(g) = 0\,,$$

one has that $G'(0) = 0$. By Step 2 one then easily gets that for every $c > 0$, $G \le H_c$ on $[0, 1)$. By continuity, this leads to $G \le H_0$ on $[0, 1)$. Suppose now that there is some $\lambda_0 > 0$ such that $G(\lambda_0) < H_0(\lambda_0)$. Then there exists $c < 0$ such that $G(\lambda_0) < H_c(\lambda_0)$. Again by Step 2, this implies that $G(\lambda) \le H_c(\lambda)$ for every $\lambda \in [\lambda_0, 1)$. Letting $\lambda \to 1$, one then gets that $G(1) < +\infty$. But

$$G(1) = \int_M (1+\sin(f))^{1-n}\left(1 + \frac{1}{n-1}\sin(f)\right)dv(g)$$

so that

$$\int_M (1+\sin(f))^{1-n}\,dv(g) \le \frac{n-1}{n-2}G(1)$$

Hence, by Step 3, we get that

$$\|(1+\sin(f))^{-1}\|_\infty < +\infty$$

Since

$$\max_{x,y\in M} |f(y) - f(x)| = \pi$$

there is some $x_0 \in M$ with the property that $[f(x_0), f(x_0)+\pi] \subset \operatorname{Im} f$. Clearly, this contradicts the above inequality. Hence, $G = H_0$ on $[0, 1)$. Replacing f by $-f$, one then gets that $G = H_0$ on $(-1, 1)$. As a consequence,

$$\left(D_{n-2}G\right)^\alpha + \alpha(1-\lambda^2)V^{-2/n}D_{n-2}G = (1+\alpha)V^{-2/n}G$$

on $(-1, 1)$. Coming back to what we said in the proof of Step 1, this means again that

$$\begin{aligned}\left(\int_M |f_\lambda|^p\,dv(g)\right)^{2/p} &= \frac{4}{n(n-2)\operatorname{Vol}_{(M,g)}^{2/n}}\int_M |\nabla f_\lambda|^2\,dv(g)\\ &\quad + \operatorname{Vol}_{(M,g)}^{-2/n}\int_M |f_\lambda|^2\,dv(g)\end{aligned}$$

for every $\lambda \in (-1, 1)$. Clearly, this proves the theorem. □

Concerning Theorem 5.5, it has been established by Bakry and Ledoux [**19**] for an abstract Markov generator L. As stated above, namely, in the Riemannian context with $L = \Delta_g$, take care that combined with Theorem 5.4, it basically gives Theorem 5.1. Under the assumption that $\mathrm{Rc}_{(M,g)} \ge (n-1)g$, which ensures by Theorem 5.4 the validity of the Sobolev inequality in question in Theorem 5.5, the existence of f as in Theorem 5.5 implies that (M, g) is isometric to the standard unit sphere (S^n, h). Indeed, the existence of f as in Theorem 5.5 implies that $\mathrm{diam}_{(M,g)} \ge \pi$. On the other hand, one has by Myer's theorem that $\mathrm{diam}_{(M,g)} \le \pi$. Hence, $\mathrm{diam}_{(M,g)} = \pi$, and by the Toponogov-Cheng maximal diameter theorem (see [**45**]), one gets that (M, g) is isometric to (S^n, h).

5.3. Program $\mathcal{A}$, Part II

Let us start with question 3$\mathcal{A}$. First, as already mentioned in Chapter 4, one has the following result of Hebey and Vaugon [117]. Such a result will be proved in Chapter 7, Section 7.2, when dealing with complete manifolds.

THEOREM 5.6 *Let (M, g) be a smooth, compact Riemannian n-manifold, $n \geq 3$. Suppose that its Riemann curvature $\mathrm{Rm}_{(M,g)}$ and its injectivity radius $\mathrm{inj}_{(M,g)}$ satisfy*

$$|\mathrm{Rm}_{(M,g)}| \leq \Lambda_1\,,\ |\nabla \mathrm{Rm}_{(M,g)}| \leq \Lambda_2\,,\ \mathrm{inj}_{(M,g)} \geq i$$

for some $\Lambda_1 > 0$, $\Lambda_2 > 0$, and $i > 0$ real. There exists $B = B(n, \Lambda_1, \Lambda_2, i)$, depending only on n, Λ_1, Λ_2, and i, such that $B_0(g) \leq B$. In other words, for any smooth, compact Riemannian n-manifold (M, g), $n \geq 3$, such that $|\mathrm{Rm}_{(M,g)}| \leq \Lambda_1$, $|\nabla \mathrm{Rm}_{(M,g)}| \leq \Lambda_2$, and $\mathrm{inj}_{(M,g)} \geq i$, one has that for any $u \in H_1^2(M)$,

$$\left(\int_M |u|^p\, dv(g)\right)^{2/p} \leq K(n,2)^2 \int_M |\nabla u|^2\, dv(g) + B \int_M u^2\, dv(g)$$

where $p = 2n/(n-2)$.

As will be discussed in Chapter 7, the role played by $\mathrm{Rm}_{(M,g)}$ could certainly be replaced by an analogous one but with $\mathrm{Rc}_{(M,g)}$ in place of $\mathrm{Rm}_{(M,g)}$. The best result here, as one will see in Chapter 7, would be that B depends only on n, a bound for $|\mathrm{Rc}_{(M,g)}|$, and a lower bound for $\mathrm{inj}_{(M,g)}$. The reason for such a fact comes from the following result:

PROPOSITION 5.1 *Let (M, g) be a smooth Riemannian n-manifold (not necessarily compact) of dimension $n \geq 4$. Suppose that there exists $B \in \mathbb{R}$ such that for any $u \in \mathcal{D}(M)$,*

$$\left(\int_M |u|^{2n/(n-2)}\, dv(g)\right)^{(n-2)/n} \leq K(n,2)^2\left(\int_M |\nabla u|^2\, dv(g) + B \int_M u^2\, dv(g)\right) \tag{5.1}$$

where $K(n, 2)$ is as in Theorem 4.4. Then, for any $x \in M$,

$$B \geq \frac{n-2}{4(n-1)} \mathrm{Scal}_{(M,g)}(x)$$

where $\mathrm{Scal}_{(M,g)}$ is the scalar curvature of (M, g). In particular,

$$B_0(g) \geq \frac{n-2}{4(n-1)} K(n,2)^2 \left(\max_M \mathrm{Scal}_{(M,g)}\right)$$

when M is compact.

PROOF: We proceed as in Aubin [9]. Let $x \in M$ and let $r > 0$ be such that $r < \mathrm{inj}_{(M,g)}(x)$ where $\mathrm{inj}_{(M,g)}(x)$ is the injectivity radius at x. Then in geodesic normal coordinates

$$\frac{1}{\omega_{n-1}} \int_{S(r)} \sqrt{\det(g_{ij})}\, ds = 1 - \frac{1}{6n} \mathrm{Scal}_{(M,g)}(x) r^2 + O(r^4)$$

where $S(r) = \{y \in M \,/\, d_g(x, y) = r\}$. For $\varepsilon > 0$, we define

$$\begin{cases} u_\varepsilon = \left(\varepsilon + r^2\right)^{1-n/2} - \left(\varepsilon + \delta^2\right)^{1-n/2} & \text{if } r \leq \delta \\ u_\varepsilon = 0 & \text{otherwise} \end{cases}$$

where $\delta \in (0, \mathrm{inj}_{(M,g)}(x))$ is given and $r = d_g(x, \cdot)$. Easy computations lead to

(1) $\int_M |\nabla u_\varepsilon|^2 \, dv(g)$

$$= \frac{(n-2)^2 \omega_{n-1}}{2} I_n^{n/2} \, \varepsilon^{1-n/2} \left(1 - \frac{(n+2)}{6n(n-4)} \mathrm{Scal}_{(M,g)}(x) \, \varepsilon + o(\varepsilon)\right) \quad \text{if } n > 4$$

$$= \frac{(n-2)^2 \omega_{n-1}}{2} \, \varepsilon^{1-n/2} \left(I_n^{n/2} + \frac{1}{6n} \mathrm{Scal}_{(M,g)}(x) \varepsilon \log \varepsilon + o(\varepsilon \log \varepsilon)\right) \quad \text{if } n = 4$$

(2) $\int_M u_\varepsilon^2 dv(g)$

$$= \frac{2(n-2)(n-1)\omega_{n-1}}{n(n-4)} I_n^{n/2} \varepsilon^{2-n/2} + o(\varepsilon^{2-n/2}) \quad \text{if } n > 4$$

$$= -\frac{\omega_{n-1}}{2} \log \varepsilon + o(\log \varepsilon) \quad \text{if } n = 4$$

(3) $\int_M u_\varepsilon^{2n/(n-2)} dv(g)$

$$\geq \frac{(n-2)\omega_{n-1}}{2n} I_n^{n/2} \, \varepsilon^{-n/2} \left(1 - \frac{1}{6(n-2)} \mathrm{Scal}_{(M,g)}(x) \varepsilon + o(\varepsilon)\right) \quad \text{if } n > 4$$

$$\geq \frac{(n-2)\omega_{n-1}}{2n} I_n^{n/2} \, \varepsilon^{-n/2} (1 + o(\varepsilon \log \varepsilon)) \quad \text{if } n = 4$$

where $I_p^q = \int_0^{+\infty} (1+t)^{-p} t^q \, dt$. Independently, one easily checks that

$$\frac{\omega_n}{2^{n-1}\omega_{n-1}} = I_n^{n/2-1} = \frac{(n-2)}{n} I_n^{n/2}$$

Hence,

$$\frac{(n-2)^2 \omega_{n-1}}{2} I_n^{n/2} = \frac{1}{K(n,2)^2} \left(\frac{(n-2)\omega_{n-1}}{2n} I_n^{n/2}\right)^{(n-2)/n}$$

and as a consequence of the developments made above, we get that

$$\frac{\int_M |\nabla u_\varepsilon|^2 \, dv(g) + B \int_M u_\varepsilon^2 dv(g)}{\left(\int_M |u_\varepsilon|^{2n/(n-2)} \, dv(g)\right)^{(n-2)/n}}$$

$$\leq \frac{1}{K(n,2)^2} \left(1 + \frac{\varepsilon}{n(n-4)} \left(\frac{4(n-1)}{n-2} B - \mathrm{Scal}_{(M,g)}(x)\right) + o(\varepsilon)\right) \quad \text{if } n > 4$$

$$\leq \frac{1}{K(4,2)^2} \left(1 + \frac{1}{8} \left(\mathrm{Scal}_{(M,g)}(x) - 6B\right) \varepsilon \log \varepsilon + o(\varepsilon \log \varepsilon)\right) \quad \text{if } n = 4$$

Since (5.1) implies that

$$\frac{\int_M |\nabla u_\varepsilon|^2 dv(g) + B \int_M u_\varepsilon^2 dv(g)}{\left(\int_M |u_\varepsilon|^{2n/(n-2)} dv(g)\right)^{(n-2)/n}} \geq \frac{1}{K(n,2)^2},$$

we must have

$$\frac{4(n-1)}{n-2} B \geq \mathrm{Scal}_{(M,g)}(x)$$

This ends the proof of the proposition. □

Going further on and when looking for analogues of Theorem 5.4, namely, when looking for explicit upper bounds of $B_0(g)$ for a large class of manifolds, one has to face a kind of no man's land. No such results are available. Until now, the best we were able to do was to get explicit expressions of upper bounds for some specific manifolds. The following result is due to Hebey and Vaugon **[113]**:

PROPOSITION 5.2 *Let $(\mathbb{P}^n(\mathbb{R}), g)$ be the standard real projective space of dimension n, $n \geq 3$. Then*

$$B_0(g) \leq \frac{n+2}{(n-2)\omega_n^{2/n}}$$

where ω_n is the volume of the standard unit sphere (S^n, h).

PROOF: First we claim that there exist $n+1$ open subsets Ω_i of $\mathbb{P}^n(\mathbb{R})$ and $n+1$ functions $\eta_i : \Omega_i \to \mathbb{R}$ such that

1. $(\Omega_i)_{i=1,\dots,n+1}$ is an open covering of $\mathbb{P}^n(\mathbb{R})$,
2. for all i, (Ω_i, g) is conformally diffeomorphic to some connected, open subset of $\mathbb{R}^n$ endowed with the Euclidean metric,
3. for all i, η_i and $\sqrt{\eta_i}$ belong to $H_{0,1}^2(\Omega_i) \cap C^0(\overline{\Omega_i})$,
4. for all i, $0 \leq \eta_i \leq 1$ and $|\nabla\sqrt{\eta_i}| \in C^0(\overline{\Omega_i})$, and
5. $\sum_{i=1}^{n+1} \eta_i = 1$ and $\sum_{i=1}^{n+1} |\nabla\sqrt{\eta_i}|^2 = n$.

In order to prove the claim, let us denote by $P_1, \dots, P_{n+1}$ the $n+1$ points of S^n whose coordinates in $\mathbb{R}^{n+1}$ are

$$(1,0,\dots,0,0), (0,1,\dots,0,0), \dots, (0,0,\dots,1,0), (0,0,\dots,0,1)$$

Let also G be the subgroup of $O(n+1)$ whose elements are Id and $-Id$, the antipodal map. We denote by $\tilde{\eta}_i$, $i = 1, \dots, n+1$, the functions on S^n defined by

$$\tilde{\eta}_i(x) = \cos^2\left(d(P_i, x)\right)$$

where d is the distance on S^n. Let also $\tilde{\Omega}_i$, $i = 1, \dots, n+1$, the half-spheres centered at P_i defined by

$$\tilde{\Omega}_i = \left\{ x \in S^n \,/\, d(P_i, x) < \frac{\pi}{2} \right\}$$

If $\Pi : S^n \to \mathbb{P}^n(\mathbb{R})$ is the canonical projection, then, as one can easily check, the following holds:

6. for any i, the restriction of Π to $\tilde{\Omega}_i$ is an isometry from $\tilde{\Omega}_i$ onto $\Pi(\tilde{\Omega}_i)$,
7. for any i, $\tilde{\eta}_i$ is G-invariant so that it defines some $\eta_i : \mathbb{P}^n(\mathbb{R}) \to \mathbb{R}$,
8. if $\Omega_i = \Pi(\tilde{\Omega}_i)$, then $(\Omega_i)_{i=1,\dots,n+1}$ is an open covering of $\mathbb{P}^n(\mathbb{R})$,

9. for any i, (Ω_i, g) is conformally diffeomorphic to some connected, open subset of $\mathbb{R}^n$ endowed with the Euclidean metric,
10. for any i, $\eta_i = 0$ on $\mathbb{P}^n(\mathbb{R})\backslash\Omega_i$,
11. $\sum_{i=1}^{n+1} \tilde{\eta}_i = 1$ and $\sum_{i=1}^{n+1} \eta_i = 1$, and
12. for any i, $|\nabla\sqrt{\eta_i}|^2 \circ \Pi = |\nabla\sqrt{\tilde{\eta}_i}|^2$ on $\tilde{\Omega}_i$.

Noting that

$$|\nabla\tilde{\eta}_i|^2(x) = 1 - \cos^2\big(d(P_i, x)\big) = 1 - \tilde{\eta}_i$$

one gets the second part of point (5). This proves the above claim.

Starting now from the existence of $(\Omega_i, \eta_i)_{i=1,\dots,n+1}$, we prove the proposition. Given $u \in C^\infty(\mathbb{P}^n(\mathbb{R}))$, one has that

$$\begin{aligned}\|u\|^2_{2n/(n-2)} = \|u^2\|_{n/(n-2)} &= \|\sum_{i=1}^{n+1} \eta_i u^2\|_{n/(n-2)} \\ &\leq \sum_{i=1}^{n+1} \|\eta_i u^2\|_{n/(n-2)} = \sum_{i=1}^{n+1} \|\sqrt{\eta_i} u\|^2_{2n/(n-2)}\end{aligned}$$

where $\|\cdot\|_p$ stands for the norm of $L^p(\mathbb{P}^n(\mathbb{R}))$. Independently, by point (6) above, and Theorem 5.1, one has that

$$\begin{aligned}&\left(\int_{P^n(R)} |\sqrt{\eta_i} u|^{2n/(n-2)}\, dv(g)\right)^{(n-2)/n} \leq \\ &\quad K(n,2)^2 \int_{P^n(R)} \left|\nabla(\sqrt{\eta_i} u)\right|^2 dv(g) + \frac{1}{\omega_n^{2/n}} \int_{P^n(R)} \eta_i u^2\, dv(g)\end{aligned}$$

Using now point (5) above, one gets that for any $u \in C^\infty(\mathbb{P}^n(\mathbb{R}))$,

$$\begin{aligned}&\left(\int_{P^n(R)} |u|^{2n/(n-2)}\, dv(g)\right)^{(n-2)/n} \\ &\leq \sum_{i=1}^{n+1} \left(\int_{P^n(R)} |\sqrt{\eta_i} u|^{2n/(n-2)}\, dv(g)\right)^{(n-2)/n} \\ &\leq K(n,2)^2 \sum_{i=1}^{n+1} \int_{P^n(R)} |\nabla(\sqrt{\eta_i} u)|^2\, dv(g) + \omega_n^{-2/n} \sum_{i=1}^{n+1} \int_{P^n(R)} \eta_i u^2\, dv(g) \\ &= K(n,2)^2 \sum_{i=1}^{n+1} \int_{P^n(R)} \left(\eta_i |\nabla u|^2 + |\nabla\sqrt{\eta_i}|^2 u^2 + u \nabla^\nu \eta_i \nabla_\nu u\right) dv(g) \\ &\quad + \omega_n^{-2/n} \int_{P^n(R)} u^2\, dv(g)\end{aligned}$$

$$= K(n,2)^2 \int_{P^n(R)} |\nabla u|^2 \, dv(g) + nK(n,2)^2 \int_{P^n(R)} u^2 \, dv(g)$$

$$+ \omega_n^{-2/n} \int_{P^n(R)} u^2 \, dv(g)$$

$$= K(n,2)^2 \int_{P^n(R)} |\nabla u|^2 \, dv(g) + \frac{(n+2)}{(n-2)\omega_n^{2/n}} \int_{P^n(R)} u^2 \, dv(g)$$

This ends the proof of the proposition. □

Concerning Proposition 5.2, recall that $B_0(g)$ has to be greater than or equal to the volume of $(\mathbb{P}^n(\mathbb{R}), g)$ to the power $-2/n$. Hence, for the real projective space $(\mathbb{P}^n(\mathbb{R}), g)$,

$$\left(\frac{2}{\omega_n}\right)^{2/n} \leq B_0(g) \leq \frac{n+2}{(n-2)\omega_n^{2/n}}$$

Since

$$\lim_{n\to+\infty} \frac{n+2}{(n-2)2^{2/n}} = 1$$

the above estimation is asymptotically sharp. On the other hand, we have no idea what the exact value of $B_0(g)$ for $(\mathbb{P}^n(\mathbb{R}), g)$ is. More generally, one has the following extension of Proposition 5.2. Such a result is also due to Hebey and Vaugon **[113]**.

PROPOSITION 5.3 *Let $G \subset O(n+1)$ be a cyclic group of order k acting freely on the unit sphere S^n of $\mathbb{R}^{n+1}$, $n \geq 3$. Set $M = S^n/G$ and let g be the metric on M obtained as the quotient metric of the standard metric h of S^n. Then*

$$B_0(g) \leq \frac{4}{n(n-2)\omega_n^{2/n}} \left[\left(1 + \frac{k^2}{4}\right)\left(\frac{n+1}{2}\right) - 1 + \frac{n(n-2)}{4}\right]$$

where ω_n denotes the volume of the standard unit sphere (S^n, h).

PROOF: Here again, one can prove the existence of $n+1$ open subsets Ω_i of M, and $n+1$ functions $\eta_i : \Omega_i \to \mathbb{R}$ such that

1. $(\Omega_i)_{i=1,\dots,n+1}$ is an open covering of M,
2. for all i, (Ω_i, g) is conformally diffeomorphic to some connected open subset of $\mathbb{R}^n$ endowed with the Euclidean metric,
3. for all i, η_i and $\sqrt{\eta_i}$ belong to $H_{0,1}^2(\Omega_i) \cap C^0(\overline{\Omega_i})$, and
4. for all i, $0 \leq \eta_i \leq 1$ and $|\nabla\sqrt{\eta_i}| \in C^0(\overline{\Omega_i})$

with the additional properties that $\sum_{i=1}^{n+1} \eta_i = 1$ and

$$\sum_{i=1}^{n+1} |\nabla\sqrt{\eta_i}|^2 = \left(1 + \frac{k^2}{4}\right)\left(\frac{n+1}{2}\right) - 1$$

For such a claim, we refer the reader to [113]. Then the proof proceeds as in the proof of Proposition 5.2. Given $u \in C^\infty(M)$, one has that

$$\|u\|^2_{2n/(n-2)} \leq \sum_{i=1}^{n+1} \|\sqrt{\eta_i}u\|^2_{2n/(n-2)}$$

while for any i,

$$\left(\int_M |\sqrt{\eta_i}u|^{2n/(n-2)}\,dv(g)\right)^{(n-2)/n} \leq$$
$$K(n,2)^2 \int_M \left|\nabla(\sqrt{\eta_i}u)\right|^2 dv(g) + \frac{1}{\omega_n^{2/n}} \int_M \eta_i u^2\,dv(g)$$

By the properties of (η_i), one then gets with similar computations to those used in the proof of Proposition 5.2 that for any $u \in C^\infty(M)$,

$$\begin{aligned}\left(\int_M |u|^{2n/(n-2)}\,dv(g)\right)^{(n-2)/n}\\ &\leq K(n,2)^2 \int_M |\nabla u|^2\,dv(g)\\ &\quad + K(n,2)^2\left[\left(1+\frac{k^2}{4}\right)\left(\frac{n+1}{2}\right) - 1 + \frac{n(n-2)}{4}\right]\int_M u^2\,dv(g)\end{aligned}$$

This ends the proof of the proposition. □

Regarding Proposition 5.3, recall that if n is even, the only nontrivial subgroup of $O(n+1)$ that acts freely is the antipodal group $\{Id, -Id\}$. For n even, Proposition 5.3 is just Proposition 5.2. On the other hand, for n odd, one has that for any integer $k \geq 1$, there exists some cyclic subgroup of $O(n+1)$ of order k that acts freely on S^n. Moreover, according to Zassenhaus [201] (see, also, Kobayashi and Nomizu [136]), any subgroup of $O(n+1)$ acting freely on S^n and of order ab where a and b are prime integers (not necessarily distinct) must be cyclic.

Let us now come back to specific results. By Hebey-Vaugon [113], one also has the following:

PROPOSITION 5.4 *Let $S^1(T)$, $T > 0$, be the circle of radius T centered at 0 in $\mathbb{R}^2$. We consider $S^1(T) \times S^{n-1}$ endowed with its standard product metric g_T. Then*

$$B_0(g_T) \leq \frac{1+(n-2)^2T^2}{n(n-2)T^2\omega_n^{2/n}}$$

where ω_n denotes the volume of the standard unit sphere (S^n, h).

PROOF: Let P be some point on $S^1(T)$. Let also $Q = -P$. Then $S^1(T)\backslash\{P\}$ and $S^1(T)\backslash\{Q\}$ are isometric to $(0, 2\pi T)$ so that

$$\left(S^1(T)\backslash\{P\}\right) \times S^{n-1} \quad \text{and} \quad \left(S^1(T)\backslash\{Q\}\right) \times S^{n-1}$$

are isometric to $(0, 2\pi T) \times S^{n-1}$. Note now that $(0, 2\pi T) \times S^{n-1}$ is conformally diffeomorphic to the annulus

$$\mathcal{C} = \left\{x \in \mathbb{R}^n \,/\, 1 < |x| < e^{2\pi T}\right\}$$

the inverse diffeomorphism φ from $\mathbb{R}^n\backslash\{0\}$ to $\mathbb{R} \times S^{n-1}$ being given by $\varphi(x) = (\log|x|, \frac{x}{|x|})$. Set $\Omega_1 = (S^1(T)\backslash\{P\}) \times S^{n-1}$ and $\Omega_2 = (S^1(T)\backslash\{Q\}) \times S^{n-1}$. Define also η_1 and η_2 by

$$\eta_1(Te^{i\theta}, x) = \cos^2\left(\frac{\theta}{2}\right), \qquad \eta_2(Te^{i\theta}, x) = \sin^2\left(\frac{\theta}{2}\right)$$

where θ at Q equals 0. Then

1. $(\Omega_i)_{i=1,2}$ is an open covering of $S^1(T) \times S^{n-1}$,
2. for all $i = 1, 2$, (Ω_i, g_T) is conformally diffeomorphic to some connected open subset of $\mathbb{R}^n$ endowed with the Euclidean metric,
3. for all $i = 1, 2$, η_i and $\sqrt{\eta_i}$ belong to $H^2_{0,1}(\Omega_i) \cap C^0(\overline{\Omega_i})$,
4. for all $i = 1, 2$, $0 \le \eta_i \le 1$ and $|\nabla\sqrt{\eta_i}| \in C^0(\overline{\Omega_i})$, and
5. $\sum_{i=1}^2 \eta_i = 1$ and $\sum_{i=1}^2 |\nabla\sqrt{\eta_i}|^2 = 1/4T^2$.

As in the proof of Proposition 5.2, one then has that for u smooth on $S^1(T) \times S^{n-1}$

$$\|u\|^2_{2n/(n-2)} \le \sum_{i=1}^2 \|\sqrt{\eta_i}u\|^2_{2n/(n-2)}$$

while for any $i = 1, 2$,

$$\begin{aligned}\left(\int_{S^1(T)\times S^{n-1}} |\sqrt{\eta_i}u|^{2n/(n-2)}\, dv(g_T)\right)^{(n-2)/n} &\le K(n,2)^2 \int_{S^1(T)\times S^{n-1}} \left|\nabla(\sqrt{\eta_i}u)\right|^2 dv(g_T)\\ &+ \frac{(n-2)^2}{4}K(n,2)^2 \int_{S^1(T)\times S^{n-1}} \eta_i u^2\, dv(g_T)\end{aligned}$$

Note here that the scalar curvature of g_T equals $(n-1)(n-2)$. By (5) above and similar computations to those made in the proof of Proposition 5.2, one then gets that for any u smooth on $S^1(T) \times S^{n-1}$,

$$\begin{aligned}\left(\int_{S^1(T)\times S^{n-1}} |u|^{2n/(n-2)}\, dv(g_T)\right)^{(n-2)/n} &\le K(n,2)^2 \int_{S^1(T)\times S^{n-1}} |\nabla u|^2\, dv(g_T)\\ &+ \frac{(n-2)^2}{4}K(n,2)^2 \int_{S^1(T)\times S^{n-1}} u^2\, dv(g_T)\\ &+ \frac{1}{4T^2}K(n,2)^2 \int_{S^1(T)\times S^{n-1}} u^2\, dv(g_T)\end{aligned}$$

This ends the proof of the proposition. $\square$

Finally, we prove the following result of Hebey-Vaugon [113]:

Proposition 5.5 *Let (H^q, h_0) be a compact, q-dimensional hyperbolic space, let (S^p, h) be the standard unit sphere of $\mathbb{R}^{p+1}$, and let $M = H^q \times S^p$ be endowed* *with its product metric $g_{q,p}$. There exists some constant $C(H^q)$, independent of p, such that for any p,*

$$B_0(g_{q,p}) \le K(n_{q,p}, 2)^2 \left[\frac{n_{q,p} - 2}{4(n_{q,p} - 1)} \operatorname{Scal}_{(M, g_{q,p})} + C(H^q) \right]$$

where $n_{q,p} = p + q$, $\operatorname{Scal}_{(M, g_{q,p})} = p(p-1) - q(q-1)$ is the scalar curvature of $(M, g_{q,p})$, and $K(n_{q,p}, 2)$ is as in Theorem 4.4.

PROOF: Let $(U_i)_{i=1,\dots,m}$ be a covering of H^q by simply connected open subsets, and let $(\alpha_i)_{i=1,\dots,m}$ be a smooth partition of unity subordinate to this covering. On $U_i \times S^p$ we define

$$\eta_i(x, y) = \frac{\alpha_i^2(x)}{\sum_{j=1}^{m} \alpha_j^2(x)}$$

As in the proof of Proposition 5.2, given $u \in C^\infty(H^q \times S^p)$, one has that

$$\|u\|_{2n/(n-2)}^2 \le \sum_{i=1}^{m} \|\sqrt{\eta_i} u\|_{2n/(n-2)}^2$$

Set $n = n_{q,p}$. By corollary 3.2, one has that for any i

$$\begin{aligned}
&\left(\int_{H^q \times S^p} |\sqrt{\eta_i} u|^{2n/(n-2)} \, dv(g_{q,p}) \right)^{(n-2)/n} \\
&\quad \le K(n,2)^2 \int_{H^q \times S^p} \left|\nabla(\sqrt{\eta_i} u)\right|^2 dv(g_{q,p}) \\
&\qquad + \frac{(n-2)}{4(n-1)} l(p(p-1) - q(q-1)) K(n,2)^2 \int_{H^q \times S^p} \eta_i u^2 \, dv(g_{q,p})
\end{aligned}$$

Similar computations to those made in the proof of Proposition 5.2 then lead to the following: For any $u \in C^\infty(H^q \times S^p)$,

$$\begin{aligned}
&\left(\int_{H^q \times S^p} |u|^{2n/(n-2)} \, dv(g_{q,p}) \right)^{(n-2)/n} \\
&\quad \le K(n,2)^2 \int_{H^q \times S^p} |\nabla u|^2 dv(g_{q,p}) + C(H^q) K(n,2)^2 \int_{H^q \times S^p} u^2 \, dv(g_{q,p}) \\
&\qquad + \frac{(n-2)}{4(n-1)} (p(p-1) - q(q-1)) K(n,2)^2 \int_{H^q \times S^p} u^2 \, dv(g_{q,p})
\end{aligned}$$

where

$$C(H^q) = \max_{H^q} \sum_{i=1}^{m} |\nabla \sqrt{\varphi_i}|^2$$

the function $\varphi_i : H^q \to \mathbb{R}$ being defined by $\varphi_i(x) = \alpha_i^2(x) / \sum_{j=1}^{m} \alpha_j^2(x)$. Clearly, this ends the proof of the proposition. □

Let us now deal with question 4.$\mathcal{A}$. By Theorem 5.1, $B_0(h) = \omega_n^{-2/n}$ for the standard unit sphere (S^n, h), and one knows explicitly all the extremum functions of ($I^2_{2,\mathrm{OPT}}$). Since the scalar curvature $\mathrm{Scal}_{(S^n,h)}$ is equal to $n(n-1)$, one can also write that

$$B_0(h) = \frac{n-2}{4(n-1)} K(n,2)^2 \Big(\max_{S^n} \mathrm{Scal}_{(S^n,h)} \Big)$$

We will be concerned in what follows with the conformal class of h. As one can see, though simple, such a context leads to interesting and surprising phenomena. Let $\big[[h]\big]$ be the set of conformal metrics to h having the same volume as h. Namely,

$$\big[[h]\big] = \big\{ g \in [h] \,/\, \mathrm{Vol}_{(S^n,g)} = \omega_n \big\}$$

where $[h]$ denotes the conformal class of h, and, as usual, $\omega_n = \mathrm{Vol}_{(S^n,h)}$. The question of the explicit value of $B_0(g)$ and of the existence of extremum functions for ($I^2_{2,\mathrm{OPT}}$) is not affected by rescaling. In other words, answering such questions for $g \in \big[[h]\big]$ is equivalent to answering such questions for $g \in [h]$. The point here is that for $\lambda > 0$ real, $B_0(\lambda g) = \lambda^{-1} B_0(g)$, and that if u is an extremum function for ($I^2_{2,\mathrm{OPT}}$) with respect to g, then $\lambda^{-(n-2)/4} u$ is an extremum function for ($I^2_{2,\mathrm{OPT}}$) with respect to λg. Hence, without loss of generality, we can restrict ourselves to $\big[[h]\big]$. The first result we prove here, due to Hebey **[110]**, completely answers the question for $n \geq 4$.

THEOREM 5.7 *Let (S^n, h) be the standard unit sphere of $\mathbb{R}^{n+1}$, $n \geq 4$, and let $g \in \big[[h]\big]$. Then*

$$B_0(g) = \frac{n-2}{4(n-1)} K(n,2)^2 \Big(\max_{S^n} \mathrm{Scal}_{(S^n,g)} \Big)$$

and one has that there exist nonzero extremum functions for ($I^2_{2,\mathrm{OPT}}$) if and only if the scalar curvature $\mathrm{Scal}_{(S^n,g)}$ of g is constant. In such a case, g and h are isometric, and if φ is an isometry from (S^n,h) onto (S^n,g), then u is an extremum function for ($I^2_{2,\mathrm{OPT}}$) with respect to g if and only if $u \circ \varphi$ is an extremum function for ($I^2_{2,\mathrm{OPT}}$) with respect to h.

PROOF: Let $g \in \big[[h]\big]$ and I_g be the functional defined on $H_1^2(S^n)\backslash\{0\}$ by

$$I_g(u) = \frac{\int_{S^n} |\nabla u|^2 \, dv(g) + \frac{n-2}{4(n-1)} \int_{S^n} \mathrm{Scal}_{(S^n,g)} u^2 \, dv(g)}{\Big(\int_{S^n} |u|^{2n/(n-2)} \, dv(g) \Big)^{(n-2)/n}}$$

where $\mathrm{Scal}_{(S^n,g)}$ stands for the scalar curvature of g. As is well-known (see, for instance, **[109]**), $\inf_u I_g(u)$ is a conformal invariant. Hence, by Theorem 5.1,

$$\inf_u I_g(u) = \inf_u I_h(u) = \frac{1}{K(n,2)^2} \tag{5.2}$$

Independently, and by Proposition 5.1,

$$B_0(g) \geq \frac{n-2}{4(n-1)} K(n,2)^2 \Big(\max_{S^n} \mathrm{Scal}_{(S^n,g)} \Big) \tag{5.3}$$

In order to prove the first part of the theorem, we proceed by contradiction. Suppose that

$$B_0(g) > \frac{n-2}{4(n-1)} K(n,2)^2 \Big(\max_{S^n} \mathrm{Scal}_{(S^n,g)} \Big) \tag{5.4}$$

Coming back to the definition of $B_0(g)$, one has that for $B < B_0(g)$ there exists u in $H_1^2(S^n)\backslash\{0\}$ such that

$$\int_{S^n} |\nabla u|^2 \, dv(g) + \frac{B}{K(n,2)^2} \int_{S^n} u^2 \, dv(g) < \frac{1}{K(n,2)^2} \Big(\int_{S^n} |u|^{2n/(n-2)} \, dv(g) \Big)^{(n-2)/n}$$

Hence, (5.4) implies that

$$\inf_u I_g(u) < \frac{1}{K(n,2)^2}$$

This is in contradiction with (5.2). As a consequence, and coming back to (5.3),

$$B_0(g) = \frac{n-2}{4(n-1)} K(n,2)^2 \Big(\max_{S^n} \mathrm{Scal}_{(S^n,g)} \Big)$$

This proves the first part of the theorem. Let us now prove its second part. Suppose that $(\mathrm{I}^2_{2,\mathrm{OPT}})$ with respect to g possesses some extremum function u_0. By definition,

$$\frac{\int_{S^n} |\nabla u_0|^2 \, dv(g) + \frac{n-2}{4(n-1)} \Big(\max_{S^n} \mathrm{Scal}_{(S^n,g)} \Big) \int_{S^n} u_0^2 \, dv(g)}{\Big(\int_{S^n} |u_0|^{2n/(n-2)} \, dv(g) \Big)^{(n-2)/n}} = \frac{1}{K(n,2)^2}$$

and u_0 realizes the minimum of the functional

$$J(u) = \frac{\int_{S^n} |\nabla u|^2 \, dv(g) + \frac{n-2}{4(n-1)} \Big(\max_{S^n} \mathrm{Scal}_{(S^n,g)} \Big) \int_{S^n} u^2 \, dv(g)}{\Big(\int_{S^n} |u|^{2n/(n-2)} \, dv(g) \Big)^{(n-2)/n}}$$

Without loss of generality, one can assume that $u_0 \geq 0$ a.e, and that

$$\int_{S^n} u_0^{2n/(n-2)} \, dv(g) = 1$$

By classical variational techniques, one then gets that u_0 is a weak solution of

$$\Delta_g u_0 + \frac{n-2}{4(n-1)} \Big(\max_{S^n} \mathrm{Scal}_{(S^n,g)} \Big) u_0 = \frac{1}{K(n,2)^2} u_0^{(n+2)/(n-2)}$$

where Δ_g is the Laplacian of g. By maximum principles and regularity results, u_0 is everywhere positive and smooth. Hence,

$$\int_{S^n} \mathrm{Scal}_{(S^n,g)} u_0^2 \, dv(g) < \Big(\max_{S^n} \mathrm{Scal}_{(S^n,g)} \Big) \int_{S^n} u_0^2 \, dv(g)$$

as soon as $\mathrm{Scal}_{(S^n,g)}$ is nonconstant. In such a case, one gets that

$$\frac{1}{K(n,2)^2} = \inf_u I_g(u) \leq I_g(u_0) < J(u_0) = \frac{1}{K(n,2)^2}$$

which is absurd. As a consequence, $(I^2_{2,OPT})$ with respect to g does not possess extremum functions if $\mathrm{Scal}_{(S^n,g)}$ is nonconstant. Conversely, if $\mathrm{Scal}_{(S^n,g)}$ is constant, then, by a well-known result of Obata [**163**], g and h are isometric. In such a situation, it is clear that $(I^2_{2,OPT})$ with respect to g possesses extremum functions, and that if φ is an isometry from (S^n, h) onto (S^n, g), then u is an extremum function for $(I^2_{2,OPT})$ with respect to g if and only if $u \circ \varphi$ is an extremum function for $(I^2_{2,OPT})$ with respect to h. This proves the theorem. □

Now that Theorem 5.7 is proved, it is natural to ask what happens when $n = 3$. Here, it seems that there does not exist a complete answer in the spirit of the answer given above for $n \geq 4$. More precisely, when dealing with the case $n = 3$, one has the following result of Hebey [**110**]. Note that $\frac{n-2}{4(n-1)} = \frac{1}{8}$ for $n = 3$.

THEOREM 5.8 *Let (S^3, h) be the standard unit sphere of $\mathbb{R}^4$. For any $g \in [[h]]$,*

$$B_0(g) \leq \frac{1}{8} K(3,2)^2 \left(\max_{S^3} \mathrm{Scal}_{(S^3,g)} \right)$$

but there now exists some $g \in [[h]]$ such that

$$B_0(g) < \frac{1}{8} K(3,2)^2 \left(\max_{S^3} \mathrm{Scal}_{(S^3,g)} \right)$$

Independently, in the case of equality, there exist nonzero extremum functions for $(I^2_{2,OPT})$ if and only if the scalar curvature $\mathrm{Scal}_{(S^3,g)}$ of g is constant. In such a case, g and h are isometric, and if φ is an isometry from (S^3, h) onto (S^3, g), then u is an extremum function for $(I^2_{2,OPT})$ with respect to g if and only if $u \circ \varphi$ is an extremum function for $(I^2_{2,OPT})$ with respect to h.

The proof of Theorem 5.8 proceeds in several steps. Its main ingredient is the following result of Brezis and Nirenberg [**34**]: For any bounded domain Ω of $\mathbb{R}^3$, and any $u \in \mathcal{D}(\Omega)$,

$$\left(\int_{R^3} |u|^6 \, dx \right)^{1/3} \leq K(3,2)^2 \int_{R^3} |\nabla u|^2 \, dx - \lambda(\Omega) \int_{R^3} u^2 \, dx$$

where

$$\lambda(\Omega) = \frac{\pi^2}{4} K(3,2)^2 \left(\frac{3 \,\mathrm{Vol}_e(\Omega)}{4\pi} \right)^{-2/3}$$

and $\mathrm{Vol}_e(\Omega)$ stands for the Euclidean volume of Ω. In what follows, let $\mathcal{B}$ be the unit ball of $\mathbb{R}^3$ centered at 0. The first result we prove is the following: It is an easy consequence of the above result of Brezis and Nirenberg.

LEMMA 5.1 *There exists $\theta_1 \in C^\infty(\mathbb{R}^3)$ such that for any $u \in \mathcal{D}(\mathbb{R}^3)$*

$$\left(\int_{R^3} |u|^6 \, dx \right)^{1/3} \leq K(3,2)^2 \int_{R^3} |\nabla u|^2 \, dx + \int_{R^3} \theta_1 u^2 \, dx$$

with the property that $\theta_1(0) = -\lambda(\mathcal{B})$ and $\theta_1(x) = 0$ as soon as $|x| \geq 2$.

PROOF: Let $\eta \in C^\infty(\mathbb{R}^3)$, $0 \le \eta \le 1$, be such that $\eta(x) = 1$ if $|x| \le 1/2$, and $\eta(x) = 0$ if $|x| \ge 3/4$. Set

$$\eta_1 = \frac{\eta^2}{\eta^2 + (1-\eta)^2}, \qquad \eta_2 = \frac{(1-\eta)^2}{\eta^2 + (1-\eta)^2}$$

For any $u \in \mathcal{D}(\mathbb{R}^3)$,

$$\begin{aligned}\left(\int_{R^3} |u|^6\, dx\right)^{1/3} &= \|u\|^2_{L^6} = \|u^2\|_{L^3} = \|(\sqrt{\eta_1}u)^2 + (\sqrt{\eta_2}u)^2\|_{L^3} \\ &\le \|(\sqrt{\eta_1}u)^2\|_{L^3} + \|(\sqrt{\eta_2}u)^2\|_{L^3} \le \|\sqrt{\eta_1}u\|^2_{L^6} + \|\sqrt{\eta_2}u\|^2_{L^6}\end{aligned}$$

On the one hand, one has by the result of Brezis and Nirenberg that

$$\|\sqrt{\eta_1}u\|^2_{L^6} \le K(3,2)^2 \int_{R^3} |\nabla(\sqrt{\eta_1}u)|^2\, dx - \lambda(\mathcal{B}) \int_{R^3} \eta_1 u^2\, dx$$

On the other hand, one has by Theorem 4.4 that

$$\|\sqrt{\eta_2}u\|^2_{L^6} \le K(3,2)^2 \int_{R^3} |\nabla(\sqrt{\eta_2}u)|^2\, dx$$

As a consequence,

$$\begin{aligned}&\left(\int_{R^3} |u|^6\, dx\right)^{1/3} \\ &\le K(3,2)^2 \int_{R^3} |\nabla(\sqrt{\eta_1}u)|^2\, dx + K(3,2)^2 \int_{R^3} |\nabla(\sqrt{\eta_2}u)|^2\, dx \\ &\quad - \lambda(B) \int_{R^3} \eta_1 u^2\, dx \\ &= K(3,2)^2 \int_{R^3} \eta_1 |\nabla u|^2\, dx + K(3,2)^2 \int_{R^3} |\nabla\sqrt{\eta_1}|^2 u^2\, dx \\ &\quad + \int_{R^3} u\big(\nabla^\nu \eta_1 \nabla_\nu u\big) dx + K(3,2)^2 \int_{R^3} \eta_2 |\nabla u|^2\, dx \\ &\quad + K(3,2)^2 \int_{R^3} |\nabla\sqrt{\eta_2}|^2 u^2\, dx + \int_{R^3} u\big(\nabla^\nu \eta_2 \nabla_\nu u\big) dx \\ &\quad - \lambda(\mathcal{B}) \int_{R^3} \eta_1 u^2\, dx \\ &= K(3,2)^2 \int_{R^3} |\nabla u|^2\, dx \\ &\quad + \int_{R^3} \big(K(3,2)^2\big(|\nabla\sqrt{\eta_1}|^2 + |\nabla\sqrt{\eta_2}|^2\big) - \lambda(\mathcal{B})\eta_1\big) u^2\, dx\end{aligned}$$

since $\eta_1 + \eta_2 = 1$. Setting

$$\theta_1 = K(3,2)^2\big(|\nabla\sqrt{\eta_1}|^2 + |\nabla\sqrt{\eta_2}|^2\big) - \lambda(B)\eta_1$$

this proves the lemma. □

As an easy consequence of Lemma 5.1, one gets the following result:

LEMMA 5.2 *Let $x_0 \in S^3$. There exists $\theta_2 \in C^\infty(S^3)$ such that for any $u \in C^\infty(S^3)$,*

$$\left(\int_{S^3} |u|^6 \, dv(h)\right)^{1/3} \leq K(3,2)^2 \int_{S^3} |\nabla u|^2 \, dv(h) + B_0(h) \int_{S^3} u^2 \, dv(h) + \int_{S^3} \theta_2 u^2 \, dv(h)$$

with the property that $\theta_2(x_0) < 0$.

PROOF: Let $P : S^3 \backslash \{-x_0\} \to \mathbb{R}^3$ be the stereographic projection of pole $-x_0$. Then

$$(P^{-1})^\star h(x) = \frac{4}{(1+|x|^2)^2} e$$

where e stands for the Euclidean metric of $\mathbb{R}^3$. Set $g = (P^{-1})^\star h$ and let φ be the function defined by

$$\varphi(x) = \left(\frac{4}{(1+|x|^2)^2}\right)^{1/4}$$

One has that $g = \varphi^4 e$. By conformal invariance of the conformal Laplacian

$$u \to \Delta_g u + \frac{1}{8} \mathrm{Scal}_{(S^3,g)} u$$

(see, for instance, [**109**]), one gets that for any $u \in \mathcal{D}(\mathbb{R}^3)$,

$$\begin{aligned}
&\int_{R^3} |\nabla u|^2 \, dv(g) + \frac{1}{8} \int_{R^3} \mathrm{Scal}_{(S^3,g)} u^2 \, dv(g) \\
&= \int_{R^3} u\left(\Delta_g u + \frac{1}{8} \mathrm{Scal}_{(S^3,g)} u\right) dv(g) \\
&= \int_{R^3} u\varphi^{-5}\left(\Delta_e(u\varphi)\right)\varphi^6 \, dx \\
&= \int_{R^3} |\nabla(u\varphi)|^2 \, dx
\end{aligned}$$

where Δ_g is the Laplacian of g and Δ_e is the Euclidean Laplacian. Independently, it is clear that

$$\int_{R^3} |u|^6 \, dv(g) = \int_{R^3} |u\varphi|^6 \, dx$$

By Lemma 5.1 one then gets that for any $u \in \mathcal{D}(\mathbb{R}^3)$,

$$\begin{aligned}
&\left(\int_{R^3} |u|^6 \, dx\right)^{1/3} \\
&= \left(\int_{R^3} |u\varphi|^6 \, dx\right)^{1/3} \\
&\leq K(3,2)^2 \int_{R^3} |\nabla(u\varphi)|^2 \, dx + \int_{R^3} \theta_1 (u\varphi)^2 \, dx
\end{aligned}$$

$$= K(3,2)^2 \int_{R^3} |\nabla u|^2 dv(g) + \frac{K(3,2)^2}{8} \int_{R^3} \mathrm{Scal}_{(S^3,g)} u^2 dv(g)$$
$$+ \int_{R^3} (\theta_1 \varphi^{-4}) u^2 dv(g)$$
$$= K(3,2)^2 \int_{R^3} |\nabla u|^2 dv(g) + B_0(h) \int_{R^3} u^2 dv(g)$$
$$+ \int_{R^3} (\theta_1 \varphi^{-4}) u^2 dv(g)$$

Note here that since g and h are isometric, $\mathrm{Scal}_{(S^3,g)} = 6$. Set

$$\theta = \left(\theta_1 \varphi^{-4}\right) \circ P$$

Then θ is defined on $S^3 \backslash \{-x_0\}$, and as an easy remark, one has that

$$\theta(x_0) = -\lambda(\mathcal{B})\varphi(0)^{-4} < 0$$

Independently, and since g are h isometric, one gets from the above developments that for any $u \in \mathcal{D}(S^3 \backslash \{-x_0\})$,

$$\left(\int_{S^3} |u|^6 dv(h)\right)^{1/3} \leq K(3,2)^2 \int_{S^3} |\nabla u|^2 dv(h)$$
$$+ B_0(h) \int_{S^3} u^2 dv(h) + \int_{S^3} \theta u^2 dv(h)$$

But $\theta_1(x) = 0$ for $|x| \gg 1$. Hence, $\theta = 0$ near $-x_0$. We extend θ by 0 at $-x_0$. Let $r > 0$ real be such that $\theta = 0$ on $B_{-x_0}(r)$, where $B_{-x_0}(r)$ stands for the ball of center $-x_0$ and radius r in S^3. Let also $\eta \in C^\infty(S^3)$, $0 \leq \eta \leq 1$, be such that $\eta(x) = 1$ if $d_h(-x_0, x) \leq r/2$, and $\eta(x) = 0$ if $d_h(-x_0, x) \geq 3r/4$, where d_h is the distance on S^3 associated to h. As in the proof of Lemma 5.1, we set

$$\eta_1 = \frac{\eta^2}{\eta^2 + (1-\eta)^2} \quad \text{and} \quad \eta_2 = \frac{(1-\eta)^2}{\eta^2 + (1-\eta)^2}$$

Given $u \in C^\infty(S^3)$, one can write that

$$\|\sqrt{\eta_1} u\|_{L^6}^2 \leq K(3,2)^2 \int_{S^3} |\nabla(\sqrt{\eta_1} u)|^2 dv(h) + B_0(h) \int_{S^3} \eta_1 u^2 dv(h)$$

Independently, and according to what has been said above,

$$\|\sqrt{\eta_2} u\|_{L^6}^2 \leq K(3,2)^2 \int_{S^3} |\nabla(\sqrt{\eta_2} u)|^2 dv(h)$$
$$+ B_0(h) \int_{S^3} \eta_2 u^2 dv(h) + \int_{S^3} \theta \eta_2 u^2 dv(h)$$

Similar computations to the ones involved in the proof of Lemma 5.1 then lead to the following: For any $u \in C^\infty(S^3)$,

$$\left(\int_{S^3} |u|^6\, dv(h)\right)^{1/3} \le K(3,2)^2 \int_{S^3} |\nabla u|^2\, dv(h) + B_0(h) \int_{S_3} u^2\, dv(h) + \int_{S^3} \theta_2 u^2\, dv(h)$$

where $\theta_2 = K(3,2)^2(|\nabla\sqrt{\eta_1}|^2 + |\nabla\sqrt{\eta_2}|^2) + \theta\eta_2$. As one can easily be convinced, this proves the lemma. □

In order to prove Theorem 5.8, we also need the following result:

LEMMA 5.3 *Let $A \in C^\infty(S^3)$ be some smooth function on S^3. Given $x_0 \in S^3$ and $\delta > 0$ real, there exists $u \in C^\infty(S^3)$, u positive, and there exists $\lambda > 0$ real such that*

$$\begin{cases} \Delta_h u + Au = -\lambda u^5 & \text{on } S^3 \backslash B_{x_0}(\delta) \\ \int_{S^3} u^6\, dv(h) = \omega_3 \end{cases}$$

where Δ_h is the Laplacian of h, and $B_{x_0}(\delta)$ is the ball in S^3 of center x_0 and radius δ.

PROOF: Let $\tilde{A} \in C^\infty(S^3)$ be such that

$$\tilde{A} = A \text{ on } S^3 \backslash B_{x_0}\left(\frac{\delta}{2}\right) \quad \text{and} \quad \int_{S^3} \tilde{A}\, dv(h) < 0$$

For $q \in (2,6)$ real, let H_q be the functional defined on $H_1^2(S^3)\backslash\{0\}$ by

$$H_q(u) = \frac{\int_{S^3} |\nabla u|^2\, dv(h) + \int_{S^3} \tilde{A} u^2\, dv(h)}{\left(\int_{S^3} |u|^q\, dv(h)\right)^{2/q}}$$

and let

$$\mathcal{H}_q = \left\{ u \in H_1^2(S^3) \,/ \int_{S^3} |u|^q\, dv(h) = 1 \right\}$$

Let also

$$\lambda_q = \inf_{u \in \mathcal{H}_q} H_q(u)$$

Then clearly $\lambda_q < 0$ since $H_q(1) < 0$. Independently, as one can easily check, λ_q is finite, and $(\lambda_q)_q$ is bounded. By standard variational techniques, and since the embedding of H_1^2 in L^q is compact (Theorem 2.9), one easily gets that there exists $u_q \in C^\infty(S^3)$, $u_q > 0$, such that

$$\begin{cases} \Delta_h u_q + \tilde{A} u_q = \lambda_q u_q^{q-1} \\ \int_{S^3} u_q^q\, dv(h) = 1 \end{cases}$$

Note now that the u_q's are bounded in $H_1^2(S^3)$. Hence, up to the extraction of a subsequence, there exists $u \in H_1^2(S^3)$ such that as $q \to 6$,

1. $u_q \rightharpoonup u$ in $H_1^2(S^3)$,
2. $u_q \to u$ in $L^2(S^3)$,
3. $u_q \to u$ a.e., and

4. $u_q^{q-1} \rightharpoonup u^5$ in $L^{6/5}(S^3)$.

Similarly, and without loss of generality, one can assume that

$$\lim_{q\to 6} \lambda_q = \lambda$$

where $\lambda < 0$ is real. As an easy consequence of such properties, one then gets that u is a weak solution of

$$\Delta_h u + \tilde{A}u = \lambda u^5$$

By maximum principles, either $u \equiv 0$ or $u > 0$, while by standard regularity results, $u \in C^\infty(S^3)$. In order to prove the lemma, as one can easily be convinced, we are left with the proof that $u \not\equiv 0$. For that purpose, let us write that

$$\begin{aligned}
1 &= \left(\int_{S^3} u_q^q \, dv(h)\right)^{2/q} \\
&\le \left(\int_{S^3} u_q^6 \, dv(h)\right)^{1/3} \omega_3^{2(1/q-1/6)} \\
&\le \omega_3^{2(1/q-1/6)} K(3,2)^2 \int_{S^3} |\nabla u_q|^2 \, dv(h) + B_0(h)\omega_3^{2(1/q-1/6)} \int_{S^3} u_q^2 \, dv(h) \\
&\le \omega_3^{2(1/q-1/6)} K(3,2)^2 \lambda_q - \omega_3^{2(1/q-1/6)} K(3,2)^2 \int_{S^3} \tilde{A} u_q^2 \, dv(h) \\
&\quad + B_0(h)\omega_3^{2(1/q-1/6)} \int_{S^3} u_q^2 \, dv(h) \\
&\le \omega_3^{2(1/q-1/6)} K(3,2)^2 \lambda_q + C_q \int_{S^3} u_q^2 \, dv(h)
\end{aligned}$$

where $C_q > 0$ is given by

$$C_q = \omega_3^{2(1/q-1/6)} K(3,2)^2 \max_{S^3} |\tilde{A}| + B_0(h)\omega_3^{2(1/q-1/6)}$$

From what has been said above, and by passing to the limit as $q \to 6$, one gets from the above inequality that

$$1 \le \lambda K(3,2)^2 + C \int_{S^3} u^2 \, dv(h)$$

where $C = \lim_{q\to 6} C_q$ is given by

$$C = K(3,2)^2 \max_{S^3} |\tilde{A}| + B_0(h)$$

As a consequence, since $\lambda < 0$, $\int_{S^3} u^2 \, dv(h) \neq 0$ and $u \not\equiv 0$. As already mentioned, this proves the lemma. □

With such lemmas, Lemmas 5.1 to 5.3, we are in position to prove Theorem 5.8.

PROOF OF THEOREM 5.8: Let $g \in [[h]]$. With the same arguments as those used in the proof of Theorem 5.7, one gets that

$$B_0(g) \le \frac{1}{8} K(3,2)^2 \left(\max_{S^3} \mathrm{Scal}_{(S^3,g)}\right)$$

From now on, let $\varphi \in C^\infty(S^3)$, $\varphi > 0$, such that $g = \varphi^4 h$, and let B be real. Given $u \in C^\infty(S^3)$, we write that

$$\int_{S^3} |\nabla u|^2 dv(g) + B \int_{S^3} u^2 dv(g) =$$
$$\int_{S^3} u\left(\Delta_g u + \frac{1}{8}\mathrm{Scal}_{(S^3,g)}u\right)dv(g) + \int_{S^3}\left(B - \frac{1}{8}\mathrm{Scal}_{(S^3,g)}\right)u^2 dv(g)$$

where Δ_g is the Laplacian of g. By conformal invariance of the conformal Laplacian

$$u \to \Delta_g u + \frac{1}{8}\mathrm{Scal}_{(S^3,g)}u$$

(see, for instance, [**109**]), one has that

$$\begin{aligned}\int_{S^3} |\nabla u|^2 dv(g) &+ \alpha \int_{S^3} u^2 dv(g)\\ &= \int_{S^3} |\nabla (u\varphi)|^2 dv(h) + \frac{3}{4}\int_{S^3} (u\varphi)^2 dv(h)\\ &\quad + \int_{S^3}\left(\alpha - \frac{1}{8}\mathrm{Scal}_{(S^3,g)}\right)\varphi^4 (u\varphi)^2 dv(h)\end{aligned}$$

while, as one can easily check,

$$\int_{S^3} |u|^6 dv(g) = \int_{S^3} |u\varphi|^6 dv(h)$$

From such relations, one gets that

$$\inf_u \frac{\int_{S^3} |\nabla u|^2 dv(g) + B\int_{S^3} u^2 dv(g)}{\left(\int_{S^3}|u|^6 dv(g)\right)^{1/3}} =$$
$$\inf_u \frac{\int_{S^3} |\nabla u|^2 dv(h) + \frac{3}{4}\int_{S^3} u^2 dv(h) + \int_{S^3}\left(B - \frac{1}{8}\mathrm{Scal}_{(S^3,g)}\right)\varphi^4 u^2 dv(h)}{\left(\int_{S^3}|u|^6 dv(h)\right)^{1/3}}$$

and by Lemma 5.2,

$$\inf_u \frac{\int_{S^3} |\nabla u|^2 dv(g) + B\int_{S^3} u^2 dv(g)}{\left(\int_{S^3}|u|^6 dv(g)\right)^{1/3}} \geq \frac{1}{K(3,2)^2}$$

as long as

$$\left(B - \frac{1}{8}\mathrm{Scal}_{(S^3,g)}\right)\varphi^4 \geq \frac{\theta_2}{K(3,2)^2}$$

Set

$$B = \max_{S^3}\left(\frac{\theta_2}{K(3,2)^2\varphi^4} + \frac{1}{8}\mathrm{Scal}_{(S^3,g)}\right) \tag{5.5}$$

As a consequence of what has been said up to now, one will get that

$$B_0(g) < \frac{1}{8}K(3,2)^2\left(\max_{S^3}\mathrm{Scal}_{(S^3,g)}\right)$$

if g is such that

$$B < \frac{1}{8} \max_{S^3} \mathrm{Scal}_{(S^3,g)} \tag{5.6}$$

where B is as in (5.5). In order to prove the existence of such a g, let $x_0 \in S^3$ and let $\delta > 0$ be real such that $\theta_2 < 0$ on $B_{x_0}(2\delta)$, where $B_{x_0}(2\delta)$ is the ball in S^3 of center x_0 and radius 2δ. Set

$$A = \frac{3}{4} + \frac{\theta_2}{K(3,2)^2}$$

and let $\varphi \in C^\infty(S^3)$, $\varphi > 0$, the function given by Lemma 5.3, solution of

$$\begin{cases} \Delta_h \varphi + A\varphi = -\lambda\varphi^5 & \text{in } S^3\backslash B_{x_0}(\delta) \\ \int_{S^3} \varphi^6 \, dv(h) = \omega_3 \end{cases}$$

where λ real is positive. Setting $g = \varphi^4 h$, one gets that

$$\begin{aligned} \frac{\theta_2(x)}{K(3,2)^2\varphi^4(x)} + \frac{1}{8}\mathrm{Scal}_{(S^3,g)}(x) &= \frac{\theta_2(x)}{K(3,2)^2\varphi^4(x)} + \left(\frac{8\Delta_h\varphi + 6\varphi}{8\varphi^5}\right)(x) \\ &= \frac{\left(\Delta_h\varphi + A\varphi\right)(x)}{\varphi^5(x)} \end{aligned}$$

so that

$$\frac{\theta_2(x)}{K(3,2)^2\varphi^4(x)} + \frac{1}{8}\mathrm{Scal}_{(S^3,g)}(x) = -\lambda < \frac{1}{8}\max_{S^3}\mathrm{Scal}_{(S^3,g)} \tag{5.7}$$

for all $x \in S^3\backslash B_{x_0}(\delta)$. Independently, let $\varepsilon > 0$ be such that $\theta_2 \le -\varepsilon$ and $\varphi^4(x) \le 1/\varepsilon$ in $B_{x_0}(\delta)$. Then, for all $x \in B_{x_0}(\delta)$,

$$\begin{aligned} \frac{\theta_2(x)}{K(3,2)^2\varphi^4(x)} + \frac{1}{8}\mathrm{Scal}_{(S^3,g)}(x) &\le \frac{1}{8}\mathrm{Scal}_{(S^3,g)}(x) - \frac{1}{K(3,2)^2}\varepsilon^2 \\ &< \frac{1}{8}\max_{S^3}\mathrm{Scal}_{(S^3,g)} \end{aligned} \tag{5.8}$$

From (5.7) and (5.8), one gets that (5.6) is satisfied by g. Hence, there exists $g \in \big[[h]\big]$ such that

$$B_0(g) < \frac{1}{8}K(3,2)^2\Big(\max_{S^3}\mathrm{Scal}_{(S^3,g)}\Big)$$

Let us now prove the last part of Theorem 5.8. Let $g \in \big[[h]\big]$ be such that

$$B_0(g) = \frac{1}{8}K(3,2)^2\Big(\max_{S^3}\mathrm{Scal}_{(S^3,g)}\Big)$$

One gets, as in the proof of Theorem 5.7, that $(\mathrm{I}^2_{2,\mathrm{OPT}})$ possesses extremum functions if and only if $\mathrm{Scal}_{(S^3,g)}$ is constant. Here again, by Obata **[163]**, g and h are isometric when $\mathrm{Scal}_{(S^3,g)}$ is constant. In such a situation, it is clear that $(\mathrm{I}^2_{2,\mathrm{OPT}})$ with respect to g possesses extremum functions, and that if φ is an isometry from (S^3,h) onto (S^3,g), then u is an extremum function for $(\mathrm{I}^2_{2,\mathrm{OPT}})$ with respect to g if and only if $u \circ \varphi$ is an extremum function for $(I^2_{2,\mathrm{OPT}})$ with respect to h. This ends the proof of the theorem. □

Finally, note that one can prove that for any $g \in [[h]]$, h the standard metric of S^3,

$$B_0(g) \geq \frac{1}{8} K(3,2)^2 \left(\min_{S^3} \mathrm{Scal}_{(S^3,g)} \right)$$

with equality if and only if $\mathrm{Scal}_{(S^3,g)}$ is constant, hence if and only if g and h are isometric (by Obata [163]). To see this, one can use the conformal invariance of the conformal Laplacian

$$u \longrightarrow \Delta_g u + \frac{1}{8} \mathrm{Scal}_{(S^3,g)}$$

If $g = v^4 h$, one gets, as in the proof of Theorem 5.8, that for any $u \in C^\infty(S^3)$,

$$\begin{aligned} &\frac{\int_{S^3} |\nabla u|^2 \, dv(g) + \frac{1}{K(3,2)^2} B_0(g) \int_{S^3} u^2 \, dv(g)}{\left(\int_{S^3} |u|^6 \, dv(g) \right)^{1/3}} \\ &= \frac{\int_{S^3} |\nabla (uv)|^2 \, dv(h) + \frac{3}{4} \int_{S^3} (uv)^2 \, dv(h)}{\left(\int_{S^3} |uv|^6 \, dv(h) \right)^{1/3}} \\ &\quad + \frac{\int_{S^3} \left(\frac{1}{K(3,2)^2} \alpha_0(g) - \frac{1}{8} \mathrm{Scal}_{(S^3,g)} \right) v^4 (uv)^2 \, dv(h)}{\left(\int_{S^3} |uv|^6 \, dv(h) \right)^{1/3}} \end{aligned}$$

Hence, taking $u = \frac{1}{v}$ in this equality, one gets that

$$\frac{1}{K(3,2)^2} \leq \frac{1}{K(3,2)^2} + \frac{1}{\omega_3^{1/3}} \int_{S^3} \left(\frac{1}{K(3,2)^2} B_0(g) - \frac{1}{8} \mathrm{Scal}_{(S^3,g)} \right) v^4 \, dv(h)$$

As a consequence,

$$\max_{S^3} \left(\frac{1}{K(3,2)^2} B_0(g) - \frac{1}{8} \mathrm{Scal}_{(S^3,g)} \right) \geq 0$$

with the property that in the case of equality to 0, $\mathrm{Scal}_{(S^3,g)}$ has to be constant. This proves the above claim.

5.4. The Role of $B_0(g)$

We have already seen in Chapter 4, Section 4.2, the role played by $\alpha_2(M)$. Namely, $\alpha_2(M)$ is connected with the existence of solutions to scalar curvature type equations

$$\Delta_g u + au = f u^{(n+2)/(n-2)}$$

where a and f are smooth functions on M. We will see here, as initiated by Hebey-Vaugon [113], that $B_0(g)$ is connected with the existence of multiple solutions for such equations. For the sake of clarity, we deal here with the multiplicity attached to the Yamabe problem. Similar results have been obtained in [113] when dealing with the Nirenberg problem.

Given (M, g) a smooth, compact Riemannian n-manifold, $n \geq 3$, let $[g]$ be the conformal class of g. Namely,

$$[g] = \left\{\tilde{g} = u^{4/(n-2)}g, u \in C^\infty(M), u > 0\right\}$$

According to Aubin [**9**] and Schoen [**175**], $[g]$ possesses at least one metric of constant scalar curvature. Up to some harmless constant, this means that for any smooth, compact Riemannian n-manifold, $n \geq 3$, there exists $\lambda \in \mathbb{R}$, and there exists $u \in C^\infty(M)$, $u > 0$, such that

$$\text{(E)} \qquad \Delta_g u + \frac{n-2}{4(n-1)} \mathrm{Scal}_{(M,g)}\, u = \lambda u^{(n+2)/(n-2)}$$

where $\mathrm{Scal}_{(M,g)}$ is the scalar curvature of (M, g). The problem here is to find conditions on the manifold for $[g]$ to possess several metrics having the same constant scalar curvature. This reduces again to finding conditions on the manifold for (E) to possess several solutions. On such a problem, the main results available are negative ones. Namely, $[g]$ possesses, up to constant scale factors, a unique metric of constant scalar curvature in each of the following cases:

1. $[g]$ possesses some metric $\tilde{g}$ with the property that $\int_M \mathrm{Scal}_{(M,\tilde{g})}\, dv(\tilde{g}) \leq 0$ (Aubin [**9**]), or
2. $[g]$ possesses an Einstein metric, and (M, g) is not conformally diffeomorphic to the standard sphere (S^n, h) of the same dimension (Obata [**163**]).

Given (M, g) a smooth, compact Riemannian n-manifold, $n \geq 3$, its Yamabe functional J is defined on $H_1^2(M)\backslash\{0\}$ by

$$J(u) = \frac{\int_M |\nabla u|^2\, dv(g) + \frac{n-2}{4(n-1)} \int_M \mathrm{Scal}_{(M,g)}\, u^2\, dv(g)}{\left(\int_M |u|^{2n/(n-2)}\, dv(g)\right)^{(n-2)/n}}$$

According to the resolution of the Yamabe problem by Aubin [**9**] and Schoen [**175**], one has that

$$\inf_u J(u) \leq \frac{1}{K(n,2)^2}$$

this inequality being strict if (M, g) is not conformally diffeomorphic to the standard n-dimensional sphere. We also define $C_0(g)$ as the smallest constant C such that for any $u \in H_1^2(M)$,

$$\frac{1}{K(n,2)^2}\left(\int_M |u|^{2n/(n-2)}\, dv(g)\right)^{(n-2)/n} \leq$$
$$\int_M |\nabla u|^2\, dv(g) + \frac{n-2}{4(n-1)} \int_M \mathrm{Scal}_{(M,g)}\, u^2\, dv(g) + C \int_M u^2\, dv(g)$$

where $K(n, 2)$ is as in Theorem 4.4. Clearly,

$$K(n,2)^2 C_0(g) \leq B_0(g) - \frac{n-2}{4(n-1)} K(n,2)^2 \left(\min_M \mathrm{Scal}_{(M,g)}\right)$$

with equality if $\mathrm{Scal}_{(M,g)}$ is constant. The first result we prove is the following one of Hebey-Vaugon [**113**]:

THEOREM 5.9 *Let $\Pi : (M, g) \to (M_1, g_1)$ be a Riemannian covering with b sheets, $b > 1$, where (M, g) and (M_1, g_1) are smooth, compact Riemannian n-manifolds, $n \geq 3$. Suppose that (M, g) is not conformally diffeomorphic to the standard n-dimensional sphere and that*

$$C_0(g_1)\,\mathrm{Vol}_{(M,g)}^{2/n} \leq \frac{1}{K(n,2)^2}\left(b^{2/n} - 1\right)$$

Then $[g]$ possesses two distinct metrics having the same constant scalar curvature.

PROOF: Let J be the functional defined on $H_1^2(M)\backslash\{0\}$ by

$$J(u) = \frac{\int_M |\nabla u|^2\, dv(g) + \frac{n-2}{4(n-1)} \int_M \mathrm{Scal}_{(M,g)}\, u^2\, dv(g)}{\left(\int_M |u|^{2n/(n-2)}\, dv(g)\right)^{(n-2)/n}}$$

and let J_1 be the functional defined on $H_1^2(M_1)\backslash\{0\}$ by

$$J_1(u) = \frac{\int_{M_1} |\nabla u|^2\, dv(g_1) + \frac{n-2}{4(n-1)} \int_{M_1} \mathrm{Scal}_{(M_1,g_1)}\, u^2\, dv(g_1)}{\left(\int_{M_1} |u|^{2n/(n-2)}\, dv(g_1)\right)^{(n-2)/n}}$$

From the resolution of the Yamabe problem by Aubin **[9]** and Schoen **[175]**, one gets that there exists $u \in C^\infty(M)$, $u > 0$, and $u_1 \in C^\infty(M_1)$, $u_1 > 0$, such that

$$J(u) = \inf_v J(v)\,, \qquad J_1(u_1) = \inf_v J_1(v)$$

Without loss of generality, since J and J_1 are homogeneous, one can assume that u and u_1 are both of norm 1 in $L^p(M)$ and $L^p(M_1)$, where $p = 2n/(n-2)$. Then u and u_1 are solutions of

$$\Delta_g u + \frac{n-2}{4(n-1)} \mathrm{Scal}_{(M,g)}\, u = \lambda u^{p-1}$$
$$\Delta_{g_1} u_1 + \frac{n-2}{4(n-1)} \mathrm{Scal}_{(M_1,g_1)}\, u_1 = \lambda_1 u_1^{p-1}$$

where $\lambda = J(u)$ and $\lambda_1 = J(u_1)$. Set $\tilde{u} = u_1 \circ \Pi$. Then $\tilde{u}$ is a solution of

$$\Delta_g \tilde{u} + \frac{n-2}{4(n-1)} \mathrm{Scal}_{(M,g)}\, \tilde{u} = \lambda_1 \tilde{u}^{p-1}$$

on M. On the one hand, still from the resolution of the Yamabe problem by Aubin **[9]** and Schoen **[175]**, one has that

$$J(u) < \frac{1}{K(n,2)^2}$$

On the other hand, by the definition of $C_0(g_1)$,

$$J(\tilde{u}) = b^{2/n} J_1(u_1) \geq b^{2/n}\left(\frac{1}{K(n,2)^2} - C_0(g_1)\int_{M_1} u_1^2\, dv(g_1)\right)$$

Since u_1 is of norm 1 in $L^p(M_1)$,

$$\int_{M_1} u_1^2\, dv(g_1) \leq \mathrm{Vol}_{(M_1,g_1)}^{2/n} = b^{-2/n}\,\mathrm{Vol}_{(M,g)}^{2/n}$$

Hence, one will have that $J(u) < J(\tilde{u})$ if

$$b^{2/n}\left(\frac{1}{K(n,2)^2} - C_0(g_1)b^{-2/n}\operatorname{Vol}_{(M,g)}^{2/n}\right) \geq \frac{1}{K(n,2)^2}$$

This is the inequality of the theorem. Under such an inequality, one then gets that, up to a constant scale factor, $u^{4/(n-2)}g$ and $\tilde{u}^{4/(n-2)}g$ are distinct but have the same constant scalar curvature. Clearly, this proves the theorem. □

As a remark, note that the value of the constant scalar curvature has no interest when dealing with such a problem. The point here is that if $\lambda > 0$, then

$$\operatorname{Scal}_{(M,\lambda g)} = \frac{1}{\lambda}\operatorname{Scal}_{(M,g)}$$

so that any value can be prescribed. More generally, with the same arguments than those used in the proof of Theorem 5.9, one easily gets the following result, also due to Hebey-Vaugon **[113]**.

THEOREM 5.10 *Let (M,g) be a smooth, compact Riemannian n-manifold, $n \geq 3$, and let $\Pi_i : (M,g) \to (M_i,g_i)$ be m Riemannian coverings with b_i sheets, $1 < b_1 < \cdots < b_m$. Set $b_0 = 1$ and assume that for any $i = 1,\ldots,m$,*

$$C_0(g_i)\operatorname{Vol}_{(M,g)}^{2/n} \leq \frac{1}{K(n,2)^2}\left(b_i^{2/n} - b_{i-1}^{2/n}\right)$$

with the additional property that (M,g) is not conformally diffeomorphic to the standard n-dimensional sphere. Then $[g]$ possesses $m+1$ distinct metrics having the same constant scalar curvature.

A simple computation shows that for J the Yamabe functional on (M,g), and u a smooth positive function on M,

$$\frac{4(n-1)}{n-2}J(u) = \operatorname{Vol}_{(M,\tilde{g})}^{-(n-2)/n}\int_M \operatorname{Scal}_{(M,\tilde{g})}\,dv(\tilde{g})$$

where $\tilde{g} = u^{4/(n-2)}g$. For u such that it realizes the minimum of J, one then gets that $\tilde{g} = u^{4/(n-2)}g$ has constant scalar curvature λ, and that

$$J(u) = \frac{(n-2)\lambda}{4(n-1)}\operatorname{Vol}_{(M,\tilde{g})}^{2/n}$$

Coming back to the proof of Theorem 5.9, and by extension of Theorem 5.10, we set $M_0 = M$ and $\Pi_0 = Id$. As one can easily check, the $m+1$ metrics of Theorem 5.10 are of the form $g_i = (u_i \circ \Pi_i)^{4/(n-2)}g$, $i = 0,\ldots,m$, where the u_i's realize the minimum of the Yamabe functional J_i on M_i, and if $J_0 = J$ is the Yamabe functional on M,

$$J(v_0) < J(v_1) < \cdots < J(v_m)$$

where $v_i = u_i \circ \Pi_i$. Noting that $J(v_i) = b_i^{2/n}J_i(u_i)$, and according to what has been said above, one then gets that

$$\operatorname{Vol}_{(M,g_0)} < \operatorname{Vol}_{(M,g_1)} < \cdots < \operatorname{Vol}_{(M,g_m)}$$

In other words, the metrics of Theorems 5.9 and 5.10 are distinguished by their volumes. In particular, one can get more distinct metrics having the same constant scalar curvature in the presence of symmetries.

When dealing with multiplicity for the Yamabe problem, only very few explicit examples are known. First, one has the case of the standard unit sphere (S^n, h) where the structure of the set of conformal metrics to h having the same constant scalar curvature is explicitly known. Namely, $g \in [h]$ is such that $\mathrm{Scal}_{(S^n,g)} = \mathrm{Scal}_{(S^n,h)}$ if and only if there exists a conformal diffeomorphism φ of (S^n, h) such that $g = \varphi^\star h$. In particular, g and h have the same volume. Then, one has the case of the product manifold $S^1(T) \times S^{n-1}$, as first studied by Schoen **[176]**. In Schoen's study of $S^1(T) \times S^{n-1}$, the main point is that the scalar curvature equation on $S^1(T) \times S^{n-1}$ reduces to some equation on $\mathbb{R}$. Here again, one has a rather explicit description, depending on the parameter T, of the set of conformal metrics having the same constant scalar curvature. As done in Hebey-Vaugon **[113]**, one recovers the multiplicity part of Schoen's result by using Theorem 5.10. This is the subject of the following:

COROLLARY 5.1 *Let $S^1(T) \times S^{n-1}$, $n \geq 3$, be endowed with its standard product metric g_T. For any integer k, there exists $T(k) > 0$ such that for any $T \geq T(k)$, $[g_T]$ possesses k distinct metrics having the same constant scalar curvature. Moreover, these metrics have distinct volumes.*

PROOF: Let G_i, $i = 1, \dots, k$, be k finite groups of rotations on S^1 of order $b_i = i$. One then gets k Riemannian coverings

$$\Pi_i : \left(S^1(T) \times S^{n-1}, g_T\right) \to \left(S^1\left(\frac{T}{i}\right) \times S^{n-1}, g_{\frac{T}{i}}\right)$$

By Proposition 5.4, as one can easily check,

$$C_0\left(g_{\frac{T}{i}}\right) \leq \frac{i^2}{4T^2}$$

Hence, applying Theorem 5.10, one will get the result if for all $i = 2, \dots, k$,

$$\frac{i^2}{4T^2}\left(2\pi T \omega_{n-1}\right)^{2/n} \leq \frac{1}{K(n,2)^2}\left(i^{2/n} - (i-1)^{2/n}\right)$$

Clearly, such inequalities are satisfied for T large enough. This proves the lemma.

□

Other specific examples can be deduced from the approach presented here. Think, for instance, of (H^q, h_0) some compact, hyperbolic Riemannian q-manifold, $q \geq 2$, having the property that it is a nontrivial Riemannian covering of some other compact, hyperbolic Riemannian q-manifold. Let $g_{q,p} = h_0 + h$ be the product metric on $H^q \times S^p$, where (S^p, h) stands for the standard p-dimensional sphere. By Theorem 5.9 and Proposition 5.5, as one can easily check, $[g_{q,p}]$ possesses two distinct metrics having the same constant scalar curvature provided that p is large enough. Moreover, one will get a third metric by noting that for p large enough, $J(1) > \inf_u J(u)$, where J stands for the Yamabe functional on $H^q \times S^p$. For p large enough, $[g_{q,p}]$ then possesses three distinct metrics having the same constant scalar curvature. For more details on the role that $B_0(g)$ can play when dealing with the existence of several solutions to scalar curvature equations, including the case of the Nirenberg problem, we refer the reader to Hebey-Vaugon [**113**].

5.5. One More Question

By Theorem 5.1, the totally optimal inequality

$$\left(\int_M |u|^p \, dv(g)\right)^{2/p} \le \alpha_2(M)^2 \int_M |\nabla u|^2 \, dv(g) + \beta_2(M)^2 \int_M u^2 \, dv(g)$$

is valid on the standard unit sphere (S^n, h), $n \ge 3$. This leads to the following question: For which compact Riemannian n-manifold (M, g), $n \ge 3$, do we have that for any $u \in H_1^2(M)$,

$$\text{(I)} \quad \left(\int_M |u|^p \, dv(g)\right)^{2/p} \le K(n,2)^2 \int_M |\nabla u|^2 \, dv(g) + \mathrm{Vol}_{(M,g)}^{-2/n} \int_M u^2 \, dv(g)$$

where $p = 2n/(n-2)$. A rather natural guess here (which may or may not be true) would be that if (M, g) is such that (I) is valid, then, up to a constant scale factor, (M, g) and (S^n, h) are isometric. On such a question, whose first appearance can be found in Hebey [**108**], only very partial answers have been obtained. Let us start with the following one of Hebey-Vaugon [**113**]. Regarding terminology, we say that U is a conical neighborhood of some subset A of $H_1^2(M)$ if it is a neighborhood of A in $H_1^2(M)$ which satisfies that for any $u \in U$ and any $\lambda > 0$, $\lambda u \in U$.

PROPOSITION 5.6 *Let (M, g) be a smooth, compact Riemannian manifold of dimension n, $n \ge 3$, and of constant scalar curvature. Let λ_1 be the first nonzero eigenvalue of the Laplacian Δ_g associated to g, let $v = \mathrm{Vol}_{(M,g)}$ be the volume of (M, g), and let ω_n be the volume of the standard unit sphere (S^n, h).*

(i) *If* (I) *is valid, or more generally, if there exists some neighborhood U of $A = \{1\}$ in $H_1^2(M)$ such that* (I) *is valid for any $u \in U$, then $\lambda_1 \ge n \frac{\omega_n}{v}^{2/n}$.*

(ii) *Conversely, if $\lambda_1 > n(\frac{\omega_n}{v})^{2/n}$, there exists a conical neighborhood U of $A = \{-1, 1\}$ in $H_1^2(M)$ such that* (I) *is valid for any $u \in U$.*

PROOF: Let H be the functional defined on $H_1^2(M) \backslash \{0\}$ by

$$H(u) = \left(\frac{1}{K(n,2)^2} - J(u)\right) \frac{\|u\|_{2n/(n-2)}^2}{\|u\|_2^2}$$

where $\|\cdot\|_p$ stands for the L^p-norm of (M, g), and J is the Yamabe functional

$$J(u) = \frac{\int_M |\nabla u|^2 \, dv(g) + \frac{n-2}{4(n-1)} \int_M \mathrm{Scal}_{(M,g)} u^2 \, dv(g)}{\left(\int_M |u|^{2n/(n-2)} dv(g)\right)^{(n-2)/n}}$$

For $C_0(g)$ as in the preceding section, $C_0(g) = \sup_u H(u)$, so that

$$B_0(g) = \frac{n-2}{4(n-1)} K(n,2)^2 \left(\mathrm{Scal}_{(M,g)}\right) + K(n,2)^2 \left(\sup_u H(u)\right)$$

when $\mathrm{Scal}_{(M,g)}$ is constant.

First, we prove point (ii) of the proposition. For $u \in H_1^2(M)$, as one can easily check,

$$H'(1) \cdot u = \frac{n-2}{2(n-1)v}\left(\frac{\int_M \mathrm{Scal}_{(M,g)}\, dv(g)}{v} \int_M u\, dv(g) - \int_M \mathrm{Scal}_{(M,g)}\, u dv(g)\right)$$

Hence, 1 is a critical point of H if and only if $\mathrm{Scal}_{(M,g)}$ is constant. In such a case, one has that

$$H''(1) \cdot (u,u) = -\frac{1}{v}\int_M |\nabla u|^2\, dv(g) + \frac{n}{v}\left(\frac{\omega_n}{v}\right)^{2/n}\left(\int_M u^2\, dv(g) - \frac{\left(\int_M u\, dv(g)\right)^2}{v}\right)$$

Set

$$W = \left\{1 + u,\ u \in H_1^2(M),\ \int_M u\, dv(g) = 0\right\}$$

When restricted to W, H is such that

$$\begin{aligned} H''(1) \cdot (u,u) &= \frac{1}{v}\left(n\left(\frac{\omega_n}{v}\right)^{2/n} \int_M u^2\, dv(g) - \int_M |\nabla u|^2\, dv(g)\right) \\ &\le \frac{1}{v}\left(\lambda_1^{-1} n\left(\frac{\omega_n}{v}\right)^{2/n} - 1\right)\int_M |\nabla u|^2\, dv(g) \end{aligned}$$

Assuming that $\lambda_1 > n(\frac{\omega_n}{v})^{2/n}$, one then gets that $-H''(1)$ is coercive. Hence, the constant function 1 realizes a local maximum of H on W, and since H is homogeneous, this leads to point (ii).

Let us now prove point (i). Suppose that 1 is a local maximum of H. Then $H''(1) \cdot (u,u) \le 0$ for all $u \in H_1^2(M)$. In particular, for any $u \in H_1^2(M)$ such that $\int_M u\, dv(g) = 0$,

$$n\left(\frac{\omega_n}{v}\right)^{2/n} \int_M u^2\, dv(g) \le \int_M |\nabla u|^2\, dv(g)$$

Hence,

$$\lambda_1 \ge n\left(\frac{\omega_n}{v}\right)^{\frac{2}{n}}$$

and this ends the proof of the proposition. □

As a consequence of Proposition 5.6, (I) is satisfied by an infinite number of nonhomothetic functions on the projective space $(\mathbb{P}^n(\mathbb{R}), g)$. Conversely, one has the following result. In its statement, $n = 3$ can be replaced by n odd, and the idea extends to products $S^1(t) \times S^{n-1}$, $t \ll 1$.

Proposition 5.7 *There exist standard quotients of the standard 3-sphere where* (I) *is not valid.*

PROOF: Consider $S^3 \subset \mathbb{R}^4 = \mathbb{C}^2$, and for k integer, let $G_k = \{\sigma_k^j\}$ be the finite group generated by σ_k, where

$$\sigma_k(z, z') = e^{\frac{2i\pi}{k}}(z, z')$$

It is clear that G_k acts freely on S^3. Set $P_k = S^3/G_k$, and let g_k be its standard metric induced from the standard metric h of S^3. Let also u be the function $u(z, z') = |z|^2$. Since u is G_k-invariant for any k, it defines some function u_k on P_k. Suppose now that for any k, (I) holds on (P_k, g_k). One would have that for any k,

$$\left(\int_{P_k} |u_k|^6 dv(g_k)\right)^{1/3} \leq K(3,2)^2 \int_{P_k} |\nabla u_k|^2 dv(g_k) + \mathrm{Vol}_{(P_k, g_k)}^{-2/3} \int_{P_k} u_k^2 dv(g_k)$$

and hence that for any k,

$$\left(\int_{S^3} u^6 dv(h)\right)^{1/3} \leq K(n,2)^2 \frac{1}{k^{2/3}} \int_{S^3} |\nabla u|^2 dv(h) + \omega_3^{-2/3} \int_{S^3} u^2 dv(h)$$

Letting k goes to $+\infty$, this would mean that

$$\left(\int_{S^3} u^6 dv(h)\right)^{1/3} \leq \omega_3^{-2/3} \int_{S^3} u^2 dv(h)$$

Since u is nonconstant, such an inequality is false. This proves the proposition. $\square$

Given (M, g) a smooth, compact Riemannian n-manifold, $n \geq 3$, denote by $\mathrm{Yam}(M, g)$ its Yamabe energy. By definition,

$$\mathrm{Yam}(M, g) = \frac{1}{\mathrm{Vol}_{(M,g)}^{(n-2)/n}} \int_M \mathrm{Scal}_{(M,g)}\, dv(g)$$

where $\mathrm{Scal}_{(M,g)}$ is the scalar curvature of (M, g). As an easy consequence of Proposition 5.1, one has the following result:

PROPOSITION 5.8 *Let (M, g) be a smooth, compact Riemannian manifold of dimension n, $n \geq 4$. If* (I) *is valid, then* $\mathrm{Yam}(M, g) \leq \mathrm{Yam}(S^n, h)$.

PROOF: If (I) is valid, one gets by Proposition 5.1 that for any $x \in M$,

$$\mathrm{Vol}_{(M,g)}^{-2/n} \geq \frac{n-2}{4(n-1)} K(n,2)^2\, \mathrm{Scal}_{(M,g)}(x)$$

Integrating such an inequality over M leads to

$$\mathrm{Yam}(M, g) \leq \frac{4(n-1)}{(n-2)K(n,2)^2}$$

that is, $\mathrm{Yam}(M, g) \leq \mathrm{Yam}(S^n, h)$. This proves the proposition. $\square$

Combining Proposition 5.8 and the well-known fact that for any g in the conformal class of h one has that $\mathrm{Yam}(S^n, g) \geq \mathrm{Yam}(S^n, h)$ with equality if and only if g has constant sectional curvature (see, for instance, [**142**]), we then get that the guess mentioned at the beginning of this Section is true in the conformal class of the standard metric h of S^n, $n \geq 4$.

PROPOSITION 5.9 *Let g be a Riemannian metric on S^n, $n \geq 4$, conformal to the standard metric h. If* (I) *is valid for (S^n, g), then, up to a constant scale factor, g and h are isometric.*

Independently, combining Propositions 5.6 and 5.8, we get the following:

PROPOSITION 5.10 *Let (M, g) be a smooth, compact Riemannian manifold of dimension n, $n \geq 4$, and let λ_1 be the first nonzero eigenvalue of the Laplacian Δ_g associated to g. Assume that the scalar curvature of (M, g) is constant. If* (I) *is valid, then*

$$\lambda_1 \geq \frac{1}{n-1} \mathrm{Scal}_{(M,g)}$$

where $\mathrm{Scal}_{(M,g)}$ *is the scalar curvature of (M, g).*

With regards to such a proposition, note that, as proved by Aubin [**9**], the inequality

$$\lambda_1 \geq \frac{1}{n-1} \mathrm{Scal}_{(M,g)}$$

is also satisfied by Yamabe metrics. By definition, a Yamabe metric is a metric which realizes the infimum of the Yamabe energy in its conformal class. On the one hand, Yamabe metrics have constant scalar curvature. On the other hand, thanks to the resolution of the Yamabe problem by Aubin and Schoen, every conformal class possesses at least one Yamabe metric. Coming back to Proposition 5.8, one can also prove that any Yamabe metric g on a compact n-manifold M is such that

$$\mathrm{Yam}(M, g) \leq \mathrm{Yam}(S^n, h)$$

With respect to Propositions 5.8 and 5.10, metrics for which (I) is valid look very much like Yamabe metrics. The following result is an easy consequence of Theorem 5.5 of Bakry and Ledoux.

PROPOSITION 5.11 *Let (M, g) be a smooth, compact Riemannian manifold of dimension n, $n \geq 3$. If* (I) *is valid, then*

$$\mathrm{diam}_{(M,g)} \mathrm{Vol}_{(M,g)}^{-1/n} \leq \mathrm{diam}_{(S^n,h)} \mathrm{Vol}_{(S^n,h)}^{-1/n}$$

where diam *stands for the diameter,* Vol *for the volume, and (S^n, h) is the standard unit sphere of $\mathbb{R}^{n+1}$, for which* $\mathrm{diam}_{(S^n,h)} = \pi$ *and* $\mathrm{Vol}_{(S^n,h)} = \omega_n$.

PROOF: The proof is by contradiction. Suppose that (I) is valid and that

$$\mathrm{diam}_{(M,g)} \mathrm{Vol}_{(M,g)}^{-1/n} > \pi \omega_n^{-1/n}$$

Clearly, (I) and $\mathrm{diam}_{(M,g)} \mathrm{Vol}_{(M,g)}^{-1/n}$ are scale invariant. Up to rescaling, one can then assume that $\mathrm{diam}_{(M,g)} = \pi$. As a consequence, $\mathrm{Vol}_{(M,g)}^{-1/n} > \omega_n^{-1/n}$ and we get that for any $u \in H_1^2(M)$,

$$\left(\int_M |u|^{2n/(n-2)}\,dv(g)\right)^{(n-2)/n} \leq \frac{4}{n(n-2)\,\mathrm{Vol}_{(M,g)}^{2/n}}\int_M |\nabla u|^2\,dv(g) + \mathrm{Vol}_{(M,g)}^{-2/n}\int_M u^2\,dv(g)$$

By Theorem 5.5, letting $f = r$, the distance function to some suitable point in M, leads to the existence of some nonconstant $u_0 \in H_1^2(M)$ such that

$$\left(\int_M |u_0|^{2n/(n-2)}\,dv(g)\right)^{(n-2)/n} = \frac{4}{n(n-2)\,\mathrm{Vol}_{(M,g)}^{2/n}}\int_M |\nabla u_0|^2\,dv(g) + \mathrm{Vol}_{(M,g)}^{-2/n}\int_M u_0^2\,dv(g)$$

Independently, and since (I) is valid, one also has that

$$\left(\int_M |u_0|^{2n/(n-2)}\,dv(g)\right)^{(n-2)/n} \leq \frac{4}{n(n-2)\omega_n^{2/n}}\int_M |\nabla u_0|^2\,dv(g) + \mathrm{Vol}_{(M,g)}^{-2/n}\int_M u_0^2\,dv(g)$$

The fact that $\int_M |\nabla u_0|^2\,dv(g) \neq 0$ then implies that $\mathrm{Vol}_{(M,g)}^{-2/n} \leq \omega_n^{-2/n}$, which is the contradiction we were looking for. This ends the proof of the proposition. □

Finally, as a straightforward application of what has been said in Section 2.9, one gets the following:

PROPOSITION 5.12 *For any* $\Lambda > 0$ *there are only finitely many diffeomorphism types of compact Riemannian n-manifolds* (M, g) *for which* $|K_{(M,g)}|\mathrm{Vol}_{(M,g)}^{2/n} \leq \Lambda$ *and* (I) *is valid simultaneously, where* $K_{(M,g)}$ *denotes the sectional curvature of* (M, g).

CHAPTER 6

Optimal Inequalities with Constraints

We discuss in this short chapter conditions under which one can lower the value $K(n, q)$ of the best possible A in the generic Sobolev inequality $(\mathrm{I}^1_{q,\mathrm{gen}})$ of Chapter 4. More precisely, we show that orthogonality conditions allow one to lower the value $K(n, q)$ of the best possible A in $(\mathrm{I}^1_{q,\mathrm{gen}})$. The discussion includes the general case of a compact manifold in the first section, and the special case of the sphere in the second section. In the third section we discuss simple applications of these results to the Nirenberg problem.

6.1. The Case of an Arbitrary Compact Manifold

The results of this section have their origin in the work of Aubin [**11**]. Extensions to complete manifolds can be found in Hebey [**108**]. We start with the following result (Aubin [**11**]):

THEOREM 6.1 *Let (M, g) be a smooth, compact Riemannian n-manifold, let $q \in [1, n)$ be real, and let p be such that $1/p = 1/q - 1/n$. Let also f_i, $i = 1, \dots, N$, be N changing sign functions of class C^1 satisfying that $\sum_{i=1}^N |f_i|^q = 1$. For any $\varepsilon > 0$ there exists $B \in \mathbb{R}$ such that*

$$\left(\int_M |u|^p \, dv(g)\right)^{q/p} \le \left(\frac{K(n, q)^q}{2^{q/n}} + \varepsilon\right) \int_M |\nabla u|^q \, dv(g) + B \int_M |u|^q \, dv(g)$$

for any $u \in H_1^q(M)$ satisfying

$$\int_M f_i |f_i|^{p-1} |u|^p \, dv(g) = 0$$

for all $i = 1, \dots, N$.

PROOF: We proceed as in Aubin [**11**]. For $f : M \to \mathbb{R}$ set $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$ so that $f = f_+ - f_-$. If $u \in H_1^q(M)$ satisfies the orthogonality conditions of Theorem 6.1, then, for any i,

$$\int_M (f_{i+})^p |u|^p \, dv(g) = \int_M (f_{i-})^p |u|^p \, dv(g)$$

Independently, $f_{i+}u$, as well as $f_{i-}u$, belong to $H_1^q(M)$. By Theorem 4.5 we then get that for any $\varepsilon > 0$ there exists $B' \in \mathbb{R}$ such that for any i and any $u \in H_1^q(M)$,

$$\left(\int_M |f_{i\pm} u|^p \, dv(g)\right)^{q/p} \le \left(K(n, q)^q + \varepsilon\right) \int_M |\nabla (f_{i\pm} u)|^q \, dv(g) + B' \int_M |f_{i\pm} u|^q \, dv(g)$$

Suppose now that u satisfies the orthogonality conditions of Theorem 6.1 and that

$$\int_M |\nabla(f_{i+}u)|^q \, dv(g) \geq \int_M |\nabla(f_{i-}u)|^q \, dv(g)$$

Then,

$$\begin{aligned}
&\left(\int_M |f_i u|^p \, dv(g)\right)^{q/p} \\
&= 2^{q/p}\left(\int_M |f_{i-}u|^p \, dv(g)\right)^{q/p} \\
&\leq 2^{q/p}\left(K(n,q)^q + \varepsilon\right)\int_M |\nabla(f_{i-}u)|^q \, dv(g) + 2^{q/p}B'\int_M |f_{i-}u|^q \, dv(g) \\
&\leq 2^{q/p-1}\left(K(n,q)^q + \varepsilon\right)\left(\int_M |\nabla(f_{i-}u)|^q \, dv(g) + \int_M |\nabla(f_{i+}u)|^q \, dv(g)\right) \\
&\quad + 2^{q/p}B'\int_M |f_i u|^q \, dv(g) \\
&\leq 2^{-q/n}\left(K(n,q)^q + \varepsilon\right)\int_M |\nabla(f_i u)|^q \, dv(g) + 2^{q/p}B'\int_M |f_i u|^q \, dv(g)
\end{aligned}$$

since $1/p = 1/q - 1/n$ and

$$|\nabla(f_{i-}u)|^q + |\nabla(f_{i+}u)|^q = |\nabla(f_i u)|^q$$

almost everywhere. Noting that the result would have been the same under the assumption

$$\int_M |\nabla(f_{i-}u)|^q \, dv(g) \geq \int_M |\nabla(f_{i+}u)|^q \, dv(g)$$

we get that for any $\varepsilon > 0$ there exists $B'' \in \mathbb{R}$ such that for any i and any $u \in H_1^q(M)$ satisfying the orthogonality conditions of Theorem 6.1,

$$\begin{aligned}
\left(\int_M |f_i u|^p \, dv(g)\right)^{q/p} &\leq \left(\frac{K(n,q)^q}{2^{q/n}} + \varepsilon\right)\int_M |\nabla(f_i u)|^q \, dv(g) \\
&\quad + B''\int_M |f_i u|^q \, dv(g)
\end{aligned} \tag{6.1}$$

One can then proceed as in the proof of Theorem 4.5, with $|f_i|^q$ in place of η_i. Let $\varepsilon > 0$ be given, and let $u \in H_1^q(M)$ satisfying the orthogonality conditions of Theorem 6.1. Then

$$\|u\|_p^q = \|u^q\|_{p/q} = \left\|\sum_{i=1}^N |f_i|^q u^q\right\|_{p/q} \leq \sum_{i=1}^N \||f_i|^q u^q\|_{p/q} = \sum_{i=1}^N \|f_i u\|_p^q$$

where $\|\cdot\|_s$ stands for the norm of $L^s(M)$. Coming back to (6.1) with $\frac{\varepsilon}{2}$ in place of ε, one then gets that for any $u \in H_1^q(M)$ satisfying the orthogonality conditions of

Theorem 6.1,

$$
\begin{aligned}
\left(\int_M |u|^p \, dv(g)\right)^{q/p}
&\le \left(\frac{K(n,q)^q}{2^{q/n}} + \frac{\varepsilon}{2}\right) \sum_{i=1}^N \int_M \left(|f_i||\nabla u| + |u||\nabla f_i|\right)^q dv(g) \\
&\quad + B'' \sum_{i=1}^N \int_M |f_i u|^q \, dv(g) \\
&\le \left(\frac{K(n,q)^q}{2^{q/n}} + \frac{\varepsilon}{2}\right) \int_M \sum_{i=1}^N \left(|\nabla u|^q |f_i|^q + \mu |\nabla u|^{q-1} |\nabla f_i| |f_i|^{(q-1)} |u|\right. \\
&\qquad\qquad \left. + \nu |u|^q |\nabla f_i|^q\right) dv(g) + B'' \int_M |u|^q \, dv(g) \\
&\le \left(\frac{K(n,q)^q}{2^{q/n}} + \frac{\varepsilon}{2}\right)\left(\|\nabla u\|_q^q + \mu N H \|\nabla u\|_q^{q-1} \|u\|_q + \nu N H^q \|u\|_q^q\right) \\
&\quad + B'' \int_M |u|^q \, dv(g)
\end{aligned}
$$

by Hölder's inequality, where μ and ν are such that

$$(1+t)^q \le 1 + \mu t + \nu t^q$$

for any $t \ge 0$ (for instance, $\mu = q \max(1, 2^{q-2})$ and $\nu = \max(1, 2^{q-2})$) and where H is such that for any i, $|\nabla f_i| \le H$. From now on, let $\varepsilon_0 > 0$ be such that

$$\left(\frac{K(n,q)^q}{2^{q/n}} + \frac{\varepsilon}{2}\right)(1 + \varepsilon_0) \le \frac{K(n,q)^q}{2^{q/n}} + \varepsilon$$

For any positive real numbers x, y, and λ,

$$q x^{q-1} y \le \lambda (q-1) x^q + \lambda^{1-q} y^q$$

By taking $x = \|\nabla u\|_q$, $y = \|u\|_q$, and

$$\lambda = \frac{q \varepsilon_0}{\mu (q-1) N H}$$

one then gets that for any $u \in H_1^q(M)$,

$$\mu N H \|\nabla u\|_q^{q-1} \|u\|_q \le \varepsilon_0 \|\nabla u\|_q^q + C \|u\|_q^q$$

where

$$C = \frac{\mu N H}{q}\left(\frac{q \varepsilon_0}{\mu (q-1) N H}\right)^{1-q}$$

Hence, for any $u \in H_1^q(M)$ satisfying the orthogonality conditions of Theorem 6.1,

$$\begin{aligned}
&\left(\int_M |u|^p \, dv(g)\right)^{q/p} \\
&\quad \le \left(\frac{K(n,q)^q}{2^{q/n}} + \frac{\varepsilon}{2}\right)(1+\varepsilon_0)\int_M |\nabla u|^q \, dv(g) + \tilde{B}\int_M |u|^q \, dv(g) \\
&\quad \le \left(\frac{K(n,q)^q}{2^{q/n}} + \varepsilon\right)\int_M |\nabla u|^q \, dv(g) + \tilde{B}\int_M |u|^q \, dv(g)
\end{aligned}$$

where

$$\tilde{B} = \left(\frac{K(n,q)^q}{2^{q/n}} + \frac{\varepsilon}{2}\right)(C + \nu N H^q) + B''$$

Clearly, this ends the proof of the theorem. □

In the case $q = 2$, one gets more than Theorem 6.1. More precisely, one has the following:

THEOREM 6.2 *Let (M, g) be a smooth, compact Riemannian n-manifold, $n \ge 3$, let $p = 2n/(n-2)$, and let f_i, $i = 1, \dots, N$, be N changing sign functions of class C^1 satisfying that $\sum_{i=1}^N f_i^2 = 1$. There exists $B \in \mathbb{R}$ such that*

$$\left(\int_M |u|^p \, dv(g)\right)^{2/p} \le \frac{K(n,2)^2}{2^{2/n}}\int_M |\nabla u|^2 \, dv(g) + B\int_M u^2 \, dv(g)$$

for any $u \in H_1^2(M)$ satisfying

$$\int_M f_i |f_i|^{p-1} |u|^p \, dv(g) = 0$$

for all $i = 1, \dots, N$.

PROOF: We proceed as in the proof of Theorem 6.1, but using Theorem 4.6 instead of Theorem 4.5. For $f : M \to \mathbb{R}$, set $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$ so that $f = f_+ - f_-$. If $u \in H_1^2(M)$ satisfies the orthogonality conditions of Theorem 6.2, then for any i,

$$\int_M (f_{i+})^p |u|^p \, dv(g) = \int_M (f_{i-})^p |u|^p \, dv(g)$$

Independently, $f_{i+}u$ as well as $f_{i-}u$ belong to $H_1^2(M)$. By Theorem 4.6 we then get that there exists $B' \in \mathbb{R}$ such that for any i and any $u \in H_1^2(M)$,

$$\begin{aligned}
\left(\int_M |f_{i\pm} u|^p \, dv(g)\right)^{2/p} &\le K(n,2)^2 \int_M |\nabla (f_{i\pm} u)|^2 \, dv(g) \\
&\quad + B' \int_M (f_{i\pm} u)^2 \, dv(g)
\end{aligned}$$

Suppose now that u satisfies the orthogonality conditions of Theorem 6.2 and that

$$\int_M |\nabla (f_{i+} u)|^2 \, dv(g) \ge \int_M |\nabla (f_{i-} u)|^2 \, dv(g)$$

Then,

$$
\begin{aligned}
&\left(\int_M |f_i u|^p\, dv(g)\right)^{2/p}\\
&= 2^{2/p}\left(\int_M |f_{i-}u|^p\, dv(g)\right)^{2/p}\\
&\le 2^{2/p} K(n,2)^2 \int_M |\nabla(f_{i-}u)|^2\, dv(g) + 2^{2/p} B' \int_M (f_{i-}u)^2\, dv(g)\\
&\le 2^{2/p-1} K(n,2)^2 \left(\int_M |\nabla(f_{i-}u)|^2\, dv(g) + \int_M |\nabla(f_{i+}u)|^2\, dv(g)\right)\\
&\quad + 2^{2/p} B' \int_M (f_i u)^2\, dv(g)\\
&\le 2^{-2/n} K(n,2)^2 \int_M |\nabla(f_i u)|^2\, dv(g) + 2^{2/p} B' \int_M (f_i u)^2\, dv(g)
\end{aligned}
$$

since $1/p = 1/2 - 1/n$ and

$$|\nabla(f_{i-}u)|^2 + |\nabla(f_{i+}u)|^2 = |\nabla(f_i u)|^2$$

almost everywhere. Noting that the result would have been the same under the assumption

$$\int_M |\nabla(f_{i-}u)|^2\, dv(g) \ge \int_M |\nabla(f_{i+}u)|^2\, dv(g)$$

we get that there exists $B'' \in \mathbb{R}$ such that for any i and any $u \in H_1^2(M)$ satisfying the orthogonality conditions of Theorem 6.2,

$$\left(\int_M |f_i u|^p\, dv(g)\right)^{2/p} \le \frac{K(n,2)^2}{2^{2/n}} \int_M |\nabla(f_i u)|^2\, dv(g) + B'' \int_M (f_i u)^2\, dv(g) \tag{6.2}$$

One can then proceed as in the proof of Theorem 4.5 or Theorem 6.1, though the argument here is slightly simpler. Given $u \in H_1^2(M)$ satisfying the orthogonality conditions of Theorem 6.2, we write that

$$\|u\|_p^2 = \|u^2\|_{p/2} = \left\|\sum_{i=1}^N f_i^2 u^2\right\|_{p/2} \le \sum_{i=1}^N \|f_i^2 u^2\|_{p/2} = \sum_{i=1}^N \|f_i u\|_p^2$$

where $\|\cdot\|_s$ stands for the norm of $L^s(M)$. Coming back to (6.2), one then gets that for any $u \in H_1^2(M)$ satisfying the orthogonality conditions of Theorem 6.2,

$$\begin{aligned}\left(\int_M |u|^p\,dv(g)\right)^{2/p} &\le \frac{K(n,2)^2}{2^{2/n}} \sum_{i=1}^{N} \int_M \left(f_i^2|\nabla u|^2 + u^2|\nabla f_i|^2 + u\nabla^\nu u \nabla_\nu f_i\right) dv(g)\\ &\quad + B'' \sum_{i=1}^{N} \int_M (f_i u)^2\,dv(g)\\ &\le \frac{K(n,2)^2}{2^{2/n}} \int_M |\nabla u|^2\,dv(g) + \left(B'' + NH^2\right) \int_M u^2\,dv(g)\end{aligned}$$

where H is such that for any i, $|\nabla f_i| \le H$. Clearly, this ends the proof of the theorem. □

6.2. The Case of the Sphere

We present here a result of Aubin [**11**]. As one will see in the next section, such a result has nice applications when dealing with the Nirenberg problem. In what follows, (S^n, h) denotes the standard unit sphere of $\mathbb{R}^{n+1}$.

THEOREM 6.3 *Let $(\xi_i)_{i=1,\dots,n+1}$ be the first spherical harmonics obtained by restricting the coordinates x_i of $\mathbb{R}^{n+1}$ to S^n, and let (S^n, h) be the standard unit sphere of $\mathbb{R}^{n+1}$. Let $\varepsilon > 0$, let $q \in [1, n)$ be real, and let p be such that $1/p = 1/q - 1/n$. There exists $B \in \mathbb{R}$ such that*

$$\left(\int_{S^n} |u|^p\,dv(h)\right)^{q/p} \le \left(\frac{K(n,q)^q}{2^{q/n}} + \varepsilon\right) \int_{S^n} |\nabla u|^q\,dv(h) + B \int_{S^n} |u|^q\,dv(h)$$

for any $u \in H_1^q(S^n)$ satisfying $\forall i = 1, \dots, n+1$, $\int_{S^n} \xi_i |u|^p\,dv(h) = 0$.

PROOF: Let $\eta \in (0, \frac{1}{2})$ real to be chosen later on. Let also Λ be the vector space of first spherical harmonics. Following Aubin [**11**], we claim first that there exists a family $(\xi_i)_{i=1,\dots,k} \in \Lambda$ such that

$$1 + \eta < \sum_{i=1}^{k} |\xi_i|^{q/p} < 1 + 2\eta$$

with the additional property that $|\xi_i| < 2^{-p}$ for any i. Indeed, let P be in S^n and let r_P denote the distance to P. The function $\xi_P = \cos(r_P)$ belongs to Λ and, as one can easily check,

$$\int_{S^n} |\xi_P|^{q/p}\,dv(h) = \text{const}$$

in the sense that the integral does not depend on P. From such a property of the family $(\xi_P)_{P\in S^n}$, one easily gets the existence of $(\xi_i)_{i=1,\dots,k}$. From now on, let h_i

of class C^1 be such that $h_i\xi_i \geq 0$ everywhere and such that

$$\left||h_i|^q - |\xi_i|^{q/p}\right| < \left(\frac{\eta}{k}\right)^p$$

Then,

$$1 < \sum_{i=1}^{k} |h_i|^q < 1 + 3\eta$$

while by the mean-value theorem,

$$\left||h_i|^p - |\xi_i|\right| \leq \frac{p}{q}\left(|\xi_i|^{q/p} + \left(\frac{\eta}{k}\right)^p\right)^{\frac{p}{q}-1} \left||h_i|^q - |\xi_i|^{q/p}\right| \leq \frac{p}{q}\left(\frac{\eta}{k}\right)^p$$

Let B_η real be such that for any $u \in H_1^q(S^n)$,

$$\left(\int_{S^n} |u|^p \, dv(h)\right)^{q/p} \leq K(n,q)^q(1+\eta)\int_{S^n} |\nabla u|^q \, dv(h) + B_\eta \int_{S^n} |u|^q \, dv(h)$$

For u nonnegative in $H_1^q(S^n)$, one can write that

$$\begin{aligned}
\left(\int_{S^n} u^p \, dv(h)\right)^{q/p} &= \left(\int_{S^n} (u^q)^{p/q} \, dv(h)\right)^{q/p} \\
&\leq \left(\int_{S^n} \left(\sum_{i=1}^{k} |h_i|^q u^q\right)^{p/q} dv(h)\right)^{q/p} \\
&\leq \sum_{i=1}^{k} \left(\int_{S^n} (|h_i|^q u^q)^{p/q} \, dv(h)\right)^{q/p} \\
&= \sum_{i=1}^{k} \left(\int_{S^n} |h_i|^p u^p \, dv(h)\right)^{q/p}
\end{aligned}$$

Given $f : S^n \to \mathbb{R}$, let $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$. For u nonnegative in $H_1^q(S^n)$ such that for all $\xi \in \Lambda$

$$\int_{S^n} \xi u^p \, dv(h) = 0$$

one has that

$$\int_{S^n} \xi_{i+} u^p \, dv(h) = \int_{S^n} \xi_{i-} u^p \, dv(h)$$

Hence, for u as above,

$$\begin{aligned}
&\left(\int_{S^n} |h_i|^p u^p \, dv(h)\right)^{q/p} \\
&= \left(\int_{S^n} (h_{i+})^p u^p \, dv(h) + \int_{S^n} (h_{i-})^p u^p \, dv(h)\right)^{q/p} \\
&\leq \left(\int_{S^n} \xi_{i+} u^p \, dv(h) + \varepsilon_0^p \int_{S^n} u^p \, dv(h) + \int_{S^n} (h_{i-})^p u^p \, dv(h)\right)^{q/p}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_{S^n} \xi_{i-} u^p \, dv(h) + \varepsilon_0^p \int_{S^n} u^p \, dv(h) + \int_{S^n} (h_{i-})^p u^p \, dv(h) \right)^{q/p} \\
&\le 2^{q/p} \left(\int_{S^n} \left((h_{i-})^p + \varepsilon_0^p \right) u^p \, dv(h) \right)^{q/p} \\
&\le 2^{q/p} \left(\int_{S^n} \left(h_{i-} + \varepsilon_0 \right)^p u^p \, dv(h) \right)^{q/p} \\
&\le 2^{q/p} K(n,q)^q (1+\eta) \int_{S^n} \left| \nabla \left((h_{i-} + \varepsilon_0) u \right) \right|^q dv(h) \\
&\quad + B_\eta \int_{S^n} (h_{i-} + \varepsilon_0)^q u^q \, dv(h)
\end{aligned}
$$

where $\varepsilon_0 = (\frac{p}{q})^{1/p} \frac{\eta}{k}$. Similarly,

$$
\begin{aligned}
\left(\int_{S^n} |h_i|^p u^p \, dv(h) \right)^{q/p} &\le 2^{q/p} K(n,q)^q (1+\eta) \int_{S^n} \left| \nabla \left((h_{i+} + \varepsilon_0) u \right) \right|^q dv(h) \\
&\quad + B_\eta \int_{S^n} (h_{i+} + \varepsilon_0)^q u^q \, dv(h)
\end{aligned}
$$

Set

$$
H = \max_{i=1,\dots,k} \max_{S^n} |\nabla h_i|
$$

and let μ and ν independent of i and u such that

$$
\begin{aligned}
&\int_{S^n} \left| \nabla \left((h_{i\pm} + \varepsilon_0) u \right) \right|^q dv(h) \\
&\quad \le \int_{S^n} (h_{i\pm} + \varepsilon_0)^q |\nabla u|^q \, dv(h) \\
&\qquad + \mu H \left(\int_{S^n} |\nabla u|^q \, dv(h) \right)^{(q-1)/q} \left(\int_{S^n} u^q \, dv(h) \right)^{1/q} \\
&\qquad + \nu H^q \int_{S^n} u^q \, dv(h)
\end{aligned}
$$

Let also $M_\eta > 0$ be such that for x and y nonnegative,

$$
x^{q-1} y \le \frac{\eta}{\mu H k} x^q + M_\eta y^q
$$

Then,

$$
\begin{aligned}
\int_{S^n} \left| \nabla \left((h_{i\pm} + \varepsilon_0) u \right) \right|^q dv(h) &\le \int_{S^n} (h_{i\pm} + \varepsilon_0)^q |\nabla u|^q \, dv(h) \\
&\quad + \frac{\eta}{k} \int_{S^n} |\nabla u|^q \, dv(h) + M_\eta \int_{S^n} u^q \, dv(h) \\
&\quad + \nu H^q \int_{S^n} u^q \, dv(h)
\end{aligned}
$$

Noting that for η small, $h_{i\pm} + \varepsilon_0 \le 1$, one gets that

$$
\left(h_{i\pm} + \varepsilon_0 \right)^q \le h_{i\pm}^q + q \left(h_{i\pm} + \varepsilon_0 \right)^{q-1} \varepsilon_0 \le h_{i\pm}^q + q \varepsilon_0
$$

As a consequence, for u nonnegative in $H_1^q(S^n)$ such that for all $\xi \in \Lambda$

$$\int_{S^n} \xi u^p \, dv(h) = 0$$

one gets that

$$\begin{aligned}
&\left(\int_{S^n} u^p \, dv(h)\right)^{q/p} \\
&\le \sum_{i=1}^{k} \left(\int_{S^n} |h_i|^p u^p \, dv(h)\right)^{q/p} \\
&\le \sum_{i=1}^{k} \Bigg[2^{\frac{q}{p}-1} K(n,q)^q (1+\eta) \int_{S^n} \left|\nabla\big((h_{i-} + \varepsilon_0)u\big)\right|^q dv(h) \\
&\qquad + \frac{B_\eta}{2} \int_{S^n} (h_{i-} + \varepsilon_0)^q u^q \, dv(h) \\
&\qquad + 2^{\frac{q}{p}-1} K(n,q)^q (1+\eta) \int_{S^n} \left|\nabla\big((h_{i+} + \varepsilon_0)u\big)\right|^q dv(h) \\
&\qquad + \frac{B_\eta}{2} \int_{S^n} (h_{i+} + \varepsilon_0)^q u^q \, dv(h) \Bigg] \\
&\le 2^{\frac{q}{p}-1} K(n,q)^q (1+\eta) \sum_{i=1}^{k} \Bigg[\int_{S^n} (h_{i-} + \varepsilon_0)^q |\nabla u|^q \, dv(h) \\
&\qquad + \int_{S^n} (h_{i+} + \varepsilon_0)^q |\nabla u|^q \, dv(h) \\
&\qquad + 2\frac{\eta}{k} \int_{S^n} |\nabla u|^q \, dv(h) + C_\eta \int_{S^n} u^q \, dv(h) \Bigg] \\
&\le 2^{\frac{q}{p}-1} K(n,q)^q (1+\eta) \Bigg[\int_{S^n} \left(\sum_{i=1}^{k} |h_i|^q\right) |\nabla u|^q \, dv(h) \\
&\qquad + 2qk\varepsilon_0 \int_{S^n} |\nabla u|^q \, dv(h) + 2\eta \int_{S^n} |\nabla u|^q \, dv(h) \Bigg] \\
&\quad + kC'_\eta \int_{S^n} u^q \, dv(h)
\end{aligned}$$

where C_η and C'_η do not depend on u. Hence, for u nonnegative in $H_1^q(S^n)$ such that for all $\xi \in \Lambda$

$$\int_{S^n} \xi u^p \, dv(h) = 0$$

one gets that

$$\begin{aligned}
&\left(\int_{S^n} u^p\,dv(h)\right)^{q/p}\\
&\le 2^{\frac{q}{p}-1}K(n,q)^q(1+\eta)\big[1+3\eta+2qk\varepsilon_0+2\eta\big]\int_{S^n}|\nabla u|^q\,dv(h)\\
&\quad+\tilde{C}_\eta\int_{S^n}u^q\,dv(h)\\
&\le 2^{\frac{q}{p}-1}K(n,q)^q(1+\eta)\left[1+3\eta+2q\left(\frac{p}{q}\right)^{1/p}\eta+2\eta\right]\int_{S^n}|\nabla u|^q\,dv(h)\\
&\quad+\tilde{C}_\eta\int_{S^n}u^q\,dv(h)
\end{aligned}$$

where $\tilde{C}_\eta$ does not depend on u. Given $\varepsilon > 0$, one can now choose η small enough such that

$$2^{\frac{q}{p}-1}K(n,q)^q(1+\eta)\left[1+3\eta+2q\left(\frac{p}{q}\right)^{1/p}\eta+2\eta\right]\le\frac{K(n,q)^q}{2^{q/n}}+\varepsilon$$

Clearly, this proves the theorem. □

6.3. Applications to the Nirenberg Problem

Let (S^n, h) be the standard unit sphere of $\mathbb{R}^{n+1}$. For the sake of clarity, we assume in what follows that $n \ge 3$. The conformal class of h, denoted by $[h]$, is

$$[h] = \left\{g = u^{4/(n-2)}h\,,\ u \in C^\infty(S^n),\ u > 0\right\}$$

As already mentioned in Chapter 4, if $g = u^{4/(n-2)}h$ is a conformal metric to h, then

$$\Delta_h u + \frac{n-2}{4(n-1)}\operatorname{Scal}_h u = \frac{n-2}{4(n-1)}\operatorname{Scal}_g u^{(n+2)/(n-2)}$$

where Scal_h and Scal_g denote the scalar curvatures of h and g. Set

$$\mathcal{S}([h]) = \big\{f \in C^\infty(S^n) \text{ s.t. } f \text{ is the scalar curvature of some metric conformal to } h\big\}$$

The Nirenberg problem, also called the Kazdan and Warner problem, consists of describing the set $\mathcal{S}([h])$ of scalar curvature functions of conformal metrics to h. In other words, one will have to find conditions on some smooth function f on S^n for f to belong to $\mathcal{S}([h])$. Up to some harmless constant, and since $\operatorname{Scal}_h = n(n-1)$, this is equivalent to finding conditions on f for the existence of $u \in C^\infty(S^n)$, $u > 0$, solution of the equation

$$\text{(E)}\qquad \Delta_h u + \frac{n(n-2)}{4}u = f u^{(n+2)/(n-2)}$$

Multiplying (E) by u and integrating over S^n, one sees that $\max_{S^n} f > 0$ is a necessary condition for f to belong to $\mathcal{S}([h])$. Contrary to similar problems on compact manifolds distinct from the sphere, such a condition is not the only necessary one, as discovered by Kazdan and Warner [**130**]. As one can easily check, given (M, g) some Riemannian manifold, and u, v smooth functions on M,

$$2\langle\nabla u, \nabla v\rangle\Delta_g u \equiv \left(2\nabla^2 v + (\Delta_g v)g\right)\cdot(\nabla u, \nabla u)$$

where $\langle\cdot,\cdot\rangle$ is the scalar product associated to g, and where $\equiv$ means that the equality holds up to divergence terms. Suppose now that we are on the sphere, and let ξ be some first spherical harmonic on S^n. Namely, ξ is an eigenfunction associated to the first nonzero eigenvalue $\lambda_1 = n$ of Δ_h. One has that

$$2\nabla^2\xi + (\Delta_h\xi)h = (n-2)\xi h$$

As a consequence of what has been said, one gets that for any smooth function u on S^n, and any first spherical harmonic ξ on S^n,

$$2\langle\nabla u, \nabla\xi\rangle\Delta_h u \equiv (n-2)\xi|\nabla u|^2$$

But

$$\xi|\nabla u|^2 \equiv -\frac{1}{2}\xi(\Delta_h u)^2 + \xi u\Delta_h u \equiv -\frac{1}{2}nu^2\xi + \xi u\Delta_h u$$

so that for any smooth function u on S^n, and any first spherical harmonic ξ on S^n,

$$2\langle\nabla u, \nabla\xi\rangle\Delta_h u \equiv -(n-2)\left(\frac{1}{2}nu^2\xi - \xi u\Delta_h u\right)$$

Suppose now that u is a solution of equation (E) for some $f \in C^\infty(S^n)$. Noting that

$$2fu^{(n+2)/(n-2)}\langle\nabla u, \nabla\xi\rangle \equiv -\frac{n-2}{n}u^{2n/(n-2)}\langle\nabla f, \nabla\xi\rangle + (n-2)fu^{2n/(n-2)}\xi$$

and that

$$u\langle\nabla u, \nabla\xi\rangle \equiv \frac{n}{2}u^2\xi$$

one gets that

$$u^{2n/(n-2)}\langle\nabla f, \nabla\xi\rangle \equiv 0$$

In other words, if $f \in C^\infty(S^n)$ and $u \in C^\infty(S^n)$, $u > 0$, satisfy (E), then for any first spherical harmonic ξ on S^n,

$$\int_{S^n}\langle\nabla f, \nabla\xi\rangle u^{2n/(n-2)}\,dv(h) = 0$$

Such a condition is known as the Kazdan-Warner condition. In particular, one sees that for any $\varepsilon > 0$ and any first spherical harmonic ξ, functions of the form $f = 1 + \varepsilon\xi$, though as close as we want to the constant function 1 for which (E) has a solution, are not the scalar curvature of some metric conformal to h. Moreover, by conformal invariance of the problem, one gets that for any conformal diffeomorphism φ of (S^n, h), and any first spherical harmonic ξ on S^n, $f = 1+\xi\circ\varphi$ does not belong to $\mathcal{S}([h])$. Conversely, as a nice and simple application of Theorem 6.3, one can prove the following result (Hebey [**104**]):

THEOREM 6.4 *Let $f \in C^\infty(S^n)$ be such that* $\max_{S^n} f > 0$. *There exists a first spherical harmonic ξ on S^n and a conformal diffeomorphism φ of (S^n, h) such that $f - (\xi\circ\varphi)$ is the scalar curvature of some conformal metric to h.*

The history of this result goes back to the work of Aubin [**11**]. It was proved there that for $f \in C^\infty(S^n)$, f everywhere positive and such that

$$\max_M f < 4^{1/(n-2)} \min_M f$$

there exists a first spherical harmonic ξ on S^n with the property that $f - \xi$ is the scalar curvature of some conformal metric to h. In what follows, let Λ be the space of first spherical harmonics, and for $f \in C^\infty(S^n)$ and $q \in (1, p]$, let

$$A_{f,q} = \left\{ u \in H_1^2(S^n),\ u \geq 0,\ \int_{S^n} f u^q \, dv(h) = 1,\ \int_{S^n} \xi u^q \, dv(h) = 0,\ \forall \xi \in \Lambda \right\}$$

where $p = 2n/(n-2)$. Let also

$$\lambda_{f,q} = \inf_{u \in A_{f,q}} I(u)$$

where I is the functional defined on $H_1^2(S^n)$ by

$$I(u) = \int_{S^n} |\nabla u|^2 \, dv(h) + \frac{n(n-2)}{4} \int_{S^n} u^2 \, dv(h)$$

In order to prove Theorem 6.4, we first prove the following:

LEMMA 6.1 *Let f be a smooth function on S^n and $q \in (1, p)$ real. Assume either that $\int_{S^n} f \, dv(h) > 0$ or that f is positive at two antipodal points of S^n. Then $\lambda_{f,q}$ as defined above is attained. In particular, there exists $u_q \in A_{f,q}$, u_q smooth and positive such that $I(u_q) = \lambda_{f,q}$ and such that the Euler-Lagrange equation*

$$\Delta_h u_q + \frac{n(n-2)}{4} u_q = \lambda_{f,q} (f - \xi_{f,q}) u_q^{q-1}$$

is satisfied for some $\xi_{f,q} \in \Lambda$.

PROOF: As a first remark, note that $A_{f,q}$ is not empty. Indeed, the condition that f is positive at two antipodal points of S^n implies that there are globally symmetrical functions in $A_{f,q}$, i.e., functions such that $u(x) = u(-x)$ for all $x \in S^n$, while the condition $\int_{S^n} f \, dv(h) > 0$ implies that there is at least one positive constant in $A_{f,q}$. Let us now consider $(u_i) \in A_{f,q}$ a minimizing sequence for $\lambda_{f,q}$. Since the embedding of $H_1^2(S^n) \subset L^q(S^n)$ is compact (Theorem 2.9), we may suppose, up to the extraction of a subsequence, that there exists $u_q \in H_1^2(S^n)$ such that

1. $u_i \rightharpoonup u_q$ in $H_1^2(S^n)$,
2. $u_i \to u_q$ in $L^2(S^n)$,
3. $u_i \to u_q$ in $L^q(S^n)$, and
4. $u_i \to u_q$ a.e.

The strong convergence in $L^q(S^n)$ together with the convergence almost everywhere implies that $u_q \in A_{f,q}$, while the weak convergence in $H_1^2(S^n)$ together with the strong convergence in $L^2(S^n)$ implies that $I(u_q) \leq \lambda_{f,q}$. As a consequence, u_q realizes $\lambda_{f,q}$. Maximum principles and regularity results then end the proof of the lemma. □

Together with such a result, one gets easily that

$$\limsup_{q\to p} \lambda_{f,q} \le \lambda_{f,p} \tag{6.3}$$

Indeed, let $u \in A_{f,p}$, $u > 0$ and bounded, be such that $I(u) \le \lambda_{f,p} + \varepsilon$, $\varepsilon > 0$. Set $v_q = u^{p/q}$. Then, as one can easily check, $v_q \in A_{f,q}$. Hence, $\lambda_{f,q} \le I(v_q)$. But

$$\lim_{q\to p} I(v_q) = I(u)$$

As a consequence,

$$\limsup_{q\to p} \lambda_{f,q} \le \lambda_{f,p} + \varepsilon$$

for all $\varepsilon > 0$. This proves (6.3). Independently, and as an easy consequence of Theorem 6.3, one has the following:

LEMMA 6.2 *For any $\varepsilon > 0$, there exists $B_\varepsilon \in \mathbb{R}$, and $q_\varepsilon \in (1, p)$ real, such that*

$$\begin{aligned}\left(\int_{S^n} |u|^q\, dv(h)\right)^{n/(n+q)} \le &\left(\frac{K(n,2)^2}{2^{2/n}} + \varepsilon\right)\left(\int_{S^n} |\nabla u|^2\, dv(h)\right)^{nq/2(n+q)} \\ &+ B_\varepsilon\left(\int_{S^n} u^2\, dv(h)\right)^{nq/2(n+q)}\end{aligned} \tag{6.4}$$

for any $q \in (q_\varepsilon, p]$ and any $u \in H_1^2(S^n)$ such that $\int_{S^n} \xi |u|^q\, dv(h) = 0$ for all $\xi \in \Lambda$.

PROOF: For $s \in [1, n)$, let $\chi(s) = \frac{ns}{n-s}$. Then χ is strictly increasing and goes from $\frac{n}{n-1}$ to $+\infty$. For $q < p$ close to p, let $s_q < 2$ be such that $\chi(s_q) = q$, that is, $s_q = \frac{nq}{n+q}$. Given $q_0 < p$ close to p, and $\varepsilon > 0$, one gets from Theorem 6.3 that there exists $B_0 > 0$ such that for any $u \in H_1^2(S^n)$ satisfying $\int_{S^n} \xi |u|^{q_0}\, dv(h) = 0$ for all $\xi \in \Lambda$,

$$\begin{aligned}\left(\int_{S^n} |u|^{q_0}\, dv(h)\right)^{1/q_0} \le &\left(\frac{K(n,s_{q_0})^{s_{q_0}}}{2^{s_{q_0}/n}} + \frac{\varepsilon}{2}\right)^{1/s_{q_0}}\left(\int_{S^n} |\nabla u|^{s_{q_0}}\, dv(h)\right)^{1/s_{q_0}} \\ &+ B_0\left(\int_{S^n} u^{s_{q_0}}\, dv(h)\right)^{1/s_{q_0}}\end{aligned} \tag{6.5}$$

Let $q \in (q_0, p]$, and for $u \in H_1^2(S^n)$ satisfying $\int_{S^n} \xi |u|^q\, dv(h) = 0$ for all $\xi \in \Lambda$, set $\varphi = |u|^{q/q_0}$. Then $\int_{S^n} \xi |\varphi|^{q_0}\, dv(h) = 0$ for all $\xi \in \Lambda$. Moreover, one has by

Hölder's inequalities and (6.5) that

$$
\begin{aligned}
&\left(\int_{S^n} |u|^q \, dv(h)\right)^{1/q_0} \\
&= \left(\int_{S^n} |\varphi|^{q_0} \, dv(h)\right)^{1/q_0} \\
&\le \left(\frac{K(n, s_{q_0})^{s_{q_0}}}{2^{s_{q_0}/n}} + \frac{\varepsilon}{2}\right)^{1/s_{q_0}} \left(\int_{S^n} |\nabla\varphi|^{s_{q_0}} \, dv(h)\right)^{1/s_{q_0}} \\
&\quad + B_0\left(\int_{S^n} \varphi^{s_{q_0}} \, dv(h)\right)^{1/s_{q_0}} \\
&= \left(\frac{K(n, s_{q_0})^{s_{q_0}}}{2^{s_{q_0}/n}} + \frac{\varepsilon}{2}\right)^{1/s_{q_0}} \frac{q}{q_0}\left(\int_{S^n} |u|^{(\frac{q}{q_0}-1)s_{q_0}} |\nabla u|^{s_{q_0}} \, dv(h)\right)^{1/s_{q_0}} \\
&\quad + B_0\left(\int_{S^n} |u|^{(\frac{q}{q_0}-1)s_{q_0}} |u|^{s_{q_0}} \, dv(h)\right)^{1/s_{q_0}} \\
&\le \left(\frac{K(n, s_{q_0})^{s_{q_0}}}{2^{s_{q_0}/n}} + \frac{\varepsilon}{2}\right)^{1/s_{q_0}} \frac{q}{q_0}\left(\int_{S^n} |u|^{(\frac{q}{q_0}-1)s_{q_0}s_q/(s_q-s_{q_0})}\right)^{(s_q-s_{q_0})/s_q s_{q_0}} \\
&\quad \times\left(\int_{S^n} |\nabla u|^{s_q} \, dv(h)\right)^{1/s_q} \\
&\quad + B_0\left(\int_{S^n} |u|^{(\frac{q}{q_0}-1)s_{q_0}s_q/(s_q-s_{q_0})}\right)^{(s_q-s_{q_0})/s_q s_{q_0}} \left(\int_{S^n} |u|^{s_q} \, dv(h)\right)^{1/s_q}
\end{aligned}
$$

Hence, noting that

$$
\frac{(\frac{q}{q_0} - 1)s_{q_0}s_q}{(s_q - s_{q_0})} = q \quad \text{and} \quad \frac{s_q - s_{q_0}}{s_q s_{q_0}} = \frac{1}{q_0} - \frac{1}{q}
$$

one gets that for any $u \in H_1^2(S^n)$ satisfying that $\int_{S^n} \xi |u|^q \, dv(h) = 0$ for all $\xi \in \Lambda$, and for any $q \in (q_0, p]$,

$$
\begin{aligned}
\left(\int_{S^n} |u|^q \, dv(h)\right)^{1/q} \le{}& \left(\frac{K(n, s_{q_0})^{s_{q_0}}}{2^{s_{q_0}/n}} + \frac{\varepsilon}{2}\right)^{1/s_{q_0}} \frac{q}{q_0}\left(\int_{S^n} |\nabla u|^{s_q} \, dv(h)\right)^{1/s_q} \\
&+ B_0\left(\int_{S^n} |u|^{s_q} \, dv(h)\right)^{1/s_q}
\end{aligned}
$$

Independently, and by Hölder's inequalities,

$$
\begin{aligned}
\int_{S^n} |\nabla u|^{s_q} \, dv(h) &\le \left(\int_{S^n} |\nabla u|^2 \, dv(h)\right)^{s_q/2} \omega_n^{(2-s_q)/2} \\
\int_{S^n} |u|^{s_q} \, dv(h) &\le \left(\int_{S^n} u^2 \, dv(h)\right)^{s_q/2} \omega_n^{(2-s_q)/2}
\end{aligned}
$$

Hence, for any $q \in (q_0, p]$, and any $u \in H_1^2(S^n)$ such that $\int_{S^n} \xi |u|^q \, dv(h) = 0$ for all $\xi \in \Lambda$,

$$\begin{aligned}\left(\int_{S^n} |u|^q \, dv(h)\right)^{1/q} &\\ \leq \omega_n^{(2-s_q)/2s_q} &\left(\frac{K(n, s_{q_0})^{s_{q_0}}}{2^{s_{q_0}/n}} + \frac{\varepsilon}{2}\right)^{1/s_{q_0}} \left(\int_{S^n} |\nabla u|^2 \, dv(h)\right)^{1/2} \\ &+ B_0 \omega_n^{(2-s_q)/2s_q} \left(\int_{S^n} u^2 \, dv(h)\right)^{1/2}\end{aligned}$$

We now choose $q_0 < p$ sufficiently close to p such that for any $q \in (q_0, p]$,

$$\omega_n^{(2-s_q)/2s_q} \left(\frac{K(n, s_{q_0})^{s_{q_0}}}{2^{s_{q_0}/n}} + \frac{\varepsilon}{2}\right)^{1/s_{q_0}} \leq \left(\frac{K(n, 2)^2}{2^{2/n}} + \frac{3\varepsilon}{4}\right)^{1/s_q}$$
$$B_0 \omega_n^{(2-s_q)/2s_q} \leq 2B_0$$

One then gets that for any $u \in H_1^2(S^n)$ such that $\int_{S^n} \xi |u|^q \, dv(h) = 0$ for all $\xi \in \Lambda$ and for any $q \in (q_0, p]$,

$$\left(\int_{S^n} |u|^q \, dv(h)\right)^{1/q} \leq \left(\frac{K(n, 2)^2}{2^{2/n}} + \frac{3\varepsilon}{4}\right)^{1/s_q} \left(\int_{S^n} |\nabla u|^2 \, dv(h)\right)^{1/2} + 2B_0 \left(\int_{S^n} u^2 \, dv(h)\right)^{1/2} \tag{6.6}$$

Choose now $\eta > 0$ real such that

$$\left(\frac{K(n, 2)^2}{2^{2/n}} + \frac{3\varepsilon}{4}\right)(1 + \eta) < \frac{K(n, 2)^2}{2^{2/n}} + \varepsilon$$

Clearly, there exists $C_\eta > 0$ such that for any $q \in [q_0, p]$, and any x, y nonnegative,

$$(x + y)^{s_q} \leq (1 + \eta)x^{s_q} + C_\eta y^{s_q}$$

Combined with (6.6), one gets that for any $q \in (q_0, p]$, and any $u \in H_1^2(S^n)$ such that $\int_{S^n} \xi |u|^q \, dv(h) = 0$ for all $\xi \in \Lambda$,

$$\begin{aligned}\left(\int_{S^n} |u|^q \, dv(h)\right)^{s_q/q} \leq \left(\frac{K(n, 2)^2}{2^{2/n}} + \varepsilon\right) &\left(\int_{S^n} |\nabla u|^2 \, dv(h)\right)^{s_q/2} \\ &+ (2B_0)^{s_q} C_\eta \left(\int_{S^n} u^2 \, dv(h)\right)^{s_q/2}\end{aligned}$$

Noting that for $q \in (q_0, p]$,

$$(2B_0)^{s_q} C_\eta \leq (1 + 2B_0)^2 C_\eta$$

one gets (6.4) with, as a possible value for B_ε,

$$B_\varepsilon = (1 + 2B_0)^2 C_\eta$$

Setting $q_\varepsilon = q_0$, this ends the proof of the lemma. □

Now, we prove the following result:

LEMMA 6.3 *Let f be a smooth function on S^n. Assume either that $\int_{S^n} f\,dv(h) > 0$ or that f is positive at two antipodal points of S^n. If*

$$\inf_{u \in A_{f\,p}} I(u) < \frac{2^{2/n}}{K(n,2)^2 \left(\max_{S^n} f\right)^{(n-2)/n}}$$

then there exists $\xi_f \in \Lambda$ such that $f - \xi_f \in \mathcal{S}([h])$.

PROOF: Let u_q be given by Lemma 6.1. As one can easily check, (u_q) is bounded in $H_1^2(S^n)$. Without loss of generality, up to the extraction of a subsequence, one can then choose the u_q's such that

1. $u_q \rightharpoonup u$ in $H_1^2(S^n)$,
2. $u_q \to u$ in $L^2(S^n)$,
3. $u_q^{q-1} \rightharpoonup u^{p-1}$ in $L^{p/(p-1)}(M)$, and
4. $u_q \to u$ a.e.

as $q \to p$ and for some $u \in H_1^2(S^n)$, $u \geq 0$. Moreover, one can choose the u_q's such that

$$\lim_{q \to p} \lambda_{f,q} = \lambda$$

does exist. Let $\varepsilon > 0$ be given. By Lemma 6.2, one easily gets the existence of $B_\varepsilon > 0$, independent of q, such that for q close to p,

$$\begin{aligned} 1 &= \left(\int_{S^n} f u_q^q \,dv(h)\right)^{n/(n+q)} \\ &\leq \left(\max_{S^n} f\right)^{n/(n+q)} \left(\int_{S^n} u_q^q \,dv(h)\right)^{n/(n+q)} \\ &\leq \left(\max_{S^n} f\right)^{n/(n+q)} \left(\frac{K(n,2)^2}{2^{2/n}} + \varepsilon\right) \lambda_{f,q}^{nq/2(n+q)} \\ &\quad + B_\varepsilon \left(\int_{S^n} u_q^2 \,dv(h)\right)^{nq/2(n+q)} \end{aligned} \tag{6.7}$$

Moreover, one has by (6.3) that

$$\begin{aligned} &\lim_{q \to p} \left(\left(\max_{S^n} f\right)^{n/(n+q)} \left(\frac{K(n,2)^2}{2^{2/n}} + \varepsilon\right) \lambda_{f,q}^{nq/2(n+q)} \right) \\ &\leq \left(\max_{S^n} f\right)^{1-\frac{2}{n}} \left(\frac{K(n,2)^2}{2^{2/n}} + \varepsilon\right) \lambda_{f,p} \end{aligned}$$

Let us now choose $\varepsilon > 0$ such that

$$\left(\max_{S^n} f\right)^{1-\frac{2}{n}} \left(\frac{K(n,2)^2}{2^{2/n}} + \varepsilon\right) \lambda_{f,p} < 1$$

The existence of such an ε is given by the assumption of the lemma. Coming back to (6.7), one gets that there exists $C > 0$ such that for q close to p,

$$\int_{S^n} u_q^2 \,dv(h) \geq C$$

Hence, u is not identically zero. In particular, $\lambda > 0$, since $\lambda = 0$ would imply that (u_q) converges to 0 in $H_1^2(S^n)$ as $q \to p$. Let us now write that $\xi_{f,q} = \mu_q \xi_q$ for $\mu_q \geq 0$ real and $\xi_q \in \Lambda$ such that $\|\xi_q\|_{C^0} = 1$. Since Λ is of finite dimension, we may assume that (ξ_q) converges C^0 to some $\xi \in \Lambda$ as $q \to p$. Thus, $\|\xi\|_{C^0} = 1$. Multiplying the equation of Lemma 6.1 by $\xi_q u_q$ and integrating over S^n leads to

$$\int_{S^n} \xi_q u_q (\Delta_h u_q) dv(h) + \frac{n(n-2)}{4} \int_{S^n} \xi_q u_q^2 \, dv(h) =$$
$$\lambda_{f,q} \int_{S^n} f \xi_q u_q^q \, dv(h) - \mu_q \lambda_{f,q} \int_{S^n} \xi_q^2 u_q^2$$

Noting that

$$\int_{S^n} \xi_q u_q (\Delta_h u_q) dv(h) = \int_{S^n} \xi_q |\nabla u_q|^2 \, dv(h) + \frac{n}{2} \int_{S^n} \xi_q u_q^2 \, dv(h)$$

one gets that

$$\mu_q \lambda_{f,q} \int_{S^n} \xi_q^2 u_q^2 = \lambda_{f,q} \int_{S^n} f \xi_q u_q^q \, dv(h) - \int_{S^n} \xi_q |\nabla u_q|^2 \, dv(h)$$
$$- \frac{n^2}{4} \int_{S^n} \xi_q u_q^2 \, dv(h)$$

Clearly, the right-hand side member of this equality is bounded, while by Hölder's inequalities

$$\mu_q \lambda_{f,q} \int_{S^n} \xi_q^2 u_q^2 \geq \mu_q \lambda_{f,q} \left(\omega_n^{-(q-2)/q} \int_{S^n} |\xi_q|^{4/q} u_q^2 \, dv(h) \right)^{q/2}$$

Hence, there exists $M > 0$, independent of q, such that

$$\mu_q \lambda_{f,q} \left(\omega_n^{-(q-2)/q} \int_{S^n} |\xi_q|^{4/q} u_q^2 \, dv(h) \right)^{q/2} \leq M \tag{6.8}$$

As a consequence,

$$\limsup_{q \to p} \mu_q < +\infty \tag{6.9}$$

Indeed, suppose by contradiction that it is not the case. By passing to the limit as $q \to p$ in (6.8), one would get that

$$\int_{S^n} |\xi|^{2(n-2)/n} u^2 \, dv(h) = 0 \tag{6.10}$$

But $\{x \in S^n \,/\, \xi(x) = 0\}$ has measure zero, and $u \not\equiv 0$. Hence, (6.10) is impossible, so that (6.9) holds. As a consequence of (6.9), and up to the extraction of a subsequence, one can now assume that (μ_q) converges to some μ as $q \to p$. By (1) to (4) above, and by passing to the limit as $q \to p$ in the equation satisfied by the u_q's, one then gets that u is a solution of

$$\Delta_h u + \frac{n(n-2)}{4} u = \lambda (f - \mu \xi) u^{p-1}$$

Just note here that for any $\varphi \in C^\infty(S^n)$,

$$\lim_{q\to p}\int_{S^n}(\xi_q - \xi)u_q^{q-1}\varphi\, dv(h) = 0$$

so that $\xi_q u_q^{q-1} \rightharpoonup \xi u^{p-1}$ in $L^{p/(p-1)}(M)$. By maximum principles and regularity results, $u > 0$ and $u \in C^\infty(S^n)$. This ends the proof of the lemma. □

With such a lemma, one easily gets the above-mentioned result of Aubin [**11**]: For f smooth and positive on S^n, if

$$\max_M f < 4^{1/(n-2)} \min_M f$$

then there exists $\xi \in \Lambda$ such that $f - \xi \in \mathcal{S}([h])$. Just note here that the constant function $u_0 = (\int_{S^n} f\, dv(h))^{-1/p}$ belongs to $A_{f,p}$, and that

$$I(u_0) \le \frac{1}{K(n,2)^2\left(\min_{S^n} f\right)^{(n-2)/n}}$$

Let us now prove Theorem 6.4.

PROOF OF THEOREM 6.4: Let $x \in S^n$. By choosing $\varphi_t(y) = e^t y$ in the stereographical model, one sees that there exists a one-parameter subgroup $\{\varphi_t\}$ of conformal diffeomorphisms of (S^n, h) such that for all $y \neq \pm x$,

$$\lim_{t\to+\infty}\varphi_t(y) = x \quad \text{and} \quad \lim_{t\to-\infty}\varphi_t(y) = -x$$

Moreover, given $w \in C^0(S^n)$, set $w_t = w \circ \varphi_t$. Then, $w_t \to w(x)$ as $t \to +\infty$ uniformly on compact subset of $S^n\backslash\{-x\}$ and in $L^s(S^n)$ for all $s \in [1, +\infty)$. Let f be as in the theorem. We choose x such that $f(x) = \max_{S^n} f$. For $\{\varphi_t\}$ as above, and $f_t = f \circ \varphi_t$,

$$\lim_{t\to+\infty}\frac{1}{\omega_n}\int_{S^n} f_t\, dv(h) = \max_{S^n} f$$

In particular, one has that for t large enough, $\int_{S^n} f_t\, dv(h) > 0$ and

$$\frac{\omega_n}{2^{2/(n-2)}}\left(\max_{S^n} f\right) < \int_{S^n} f_t\, dv(h) \tag{6.11}$$

Fix such a t so that (6.11) is true. Note in addition that $\max_{S^n} f = \max_{S^n} f_t$ and that the constant function $u_0 = (\int_{S^n} f_t\, dv(h))^{-1/p}$ belongs to $A_{f_t,p}$. Then,

$$\lambda_{f_t,p} \le \frac{n(n-2)\omega_n}{4\left(\int_{S^n} f_t\, dv(h)\right)^{(n-2)/n}}$$

and one gets by (6.11) that the assumptions of Lemma 6.3 are satisfied for f_t. As a consequence, there exists $\xi \in \Lambda$ such that $f_t - \xi \in \mathcal{S}([h])$. By conformal invariance, this means again that $f - (\xi \circ \varphi_t^{-1}) \in \mathcal{S}([h])$. The theorem is proved. □

Coming back to Lemma 6.3, and with a somehow subtler argument, one can prove the following (Hebey [**104**]):

THEOREM 6.5 *Let f be a smooth function on S^3. Suppose that there exists $x \in S^3$ such that $f(x) = f(-x) = \max_{S^3} f > 0$. Then there exists a first spherical harmonic ξ on S^3 such that $f - \xi$ is the scalar curvature of some metric conformal to h.*

PROOF: Let f and x be as in the theorem. For r the distance to x, and $\beta > 1$ real, we set

$$w_\beta^+ = (\beta - \cos r)^{-1/2} \quad \text{and} \quad w_\beta^- = (\beta + \cos r)^{-1/2}$$

Let $v_\beta = w_\beta^+ + w_\beta^-$ and

$$u_\beta = \left(\int_{S^3} f v_\beta^6 \, dv(h)\right)^{-1/6} v_\beta$$

For $\beta > 1$ sufficiently close 1, $u_\beta \in A_{f,6}$. Moreover, rather simple computations lead to the following expansion:

$$\begin{aligned} I(u_\beta) = \frac{3}{4}(2\omega_3)^{2/3}\left(\max_{S^3} f\right)^{-1/3} & \\ \times\Bigg[1 + \frac{1}{\omega_3}(\beta^2 - 1)^{1/2}\Bigg(&(\beta^2 - 1)\int_{S^3} (w_\beta^+)^5 w_\beta^- \, dv(h) \\ &- \frac{1}{\left(\max_{S^3} f\right)}(\beta^2 - 1)\int_{S^3} f (w_\beta^+)^5 w_\beta^- \, dv(h) \\ &- \frac{1}{\left(\max_{S^3} f\right)}(\beta^2 - 1)\int_{S^3} f (w_\beta^-)^5 w_\beta^+ \, dv(h)\Bigg) \\ + (\beta^2 - 1)^{1/2}\varepsilon(\beta - 1)\Bigg] & \end{aligned}$$

where $\varepsilon(\beta - 1)$ tends to 0 as $\beta \to 1$. Thus, one will find $\beta > 1$ close to 1 such that

$$I(u_\beta) < \frac{3}{4}(2\omega_3)^{2/3}\left(\max_{S^3} f\right)^{1/3} \tag{6.12}$$

if

$$\begin{aligned} \lim_{\beta \to 1^+} \Bigg[&(\beta^2 - 1)\int_{S^3} (w_\beta^+)^5 w_\beta^- \, dv(h) \\ &- \frac{1}{\left(\max_{S^3} f\right)}(\beta^2 - 1)\int_{S^3} f (w_\beta^+)^5 w_\beta^- \, dv(h) \\ &- \frac{1}{\left(\max_{S^3} f\right)}(\beta^2 - 1)\int_{S^3} f (w_\beta^-)^5 w_\beta^+ \, dv(h)\Bigg] < 0 \end{aligned}$$

As one can check, such a limit equals $-16\pi/3$ since

$$\lim_{\beta \to 1^+} (\beta^2 - 1)\int_{S^3} f (w_\beta^+)^5 w_\beta^- \, dv(h) = \frac{16\pi}{3} f(x)$$

and

$$\lim_{\beta\to 1^+} (\beta^2 - 1) \int_{S^3} f(w_\beta^-)^5 w_\beta^+ \, dv(h) = \frac{16\pi}{3} f(-x)$$

for any $f \in C^\infty(S^3)$. As a consequence, there exists $\beta > 1$ close to 1 such that (6.12) is true. In particular, the inequality of Lemma 6.3 holds. This ends the proof of the theorem. □

Regarding Theorem 6.5, note that by Escobar-Schoen **[78]** (see also Hebey **[101, 103]** for an extension to groups acting without fixed points), if f is such that $f(x) = f(-x)$ for all $x \in S^3$, then $f \in \mathcal{S}([h])$. In other words, under the assumption that f is (globally) invariant for the action of the antipodal map, one can take $\xi = 0$ in Theorem 6.5. Independently, and for a much more sophisticated application to the Nirenberg problem of Sobolev inequalities with constraints like those in Theorem 6.3, we refer the reader to Chang-Yang **[44]** and Chang-Gursky-Yang **[43]**.

REMARK 6.1. The results stated above are of course not the only ones available on the Nirenberg problem. Such results have been chosen in order to illustrate some simple possible applications of Sobolev inequalities with constraints as developed in the two first sections of this chapter. For more details on the Nirenberg problem, we refer the reader to the books or survey type articles **[42, 103, 109, 176]** and the references they contain.

CHAPTER 7

Best Constants in the Noncompact Setting

In this chapter, we deal with complete manifolds, not necessarily compact, and ask again some of the questions of Program $\mathcal{A}$ we considered in Chapters 4 and 5. For an analogue to the questions involved in Program $\mathcal{B}$, we refer the reader to Chapter 8. Given (M, g) a smooth, complete Riemannian n-manifold, and $q \in [1, n)$ real, we say that the generic Sobolev inequality $(\mathrm{I}^1_{q,\mathrm{gen}})$ of order q is valid if there exist A and B real such that for any $u \in H^q_1(M)$,

$(\mathrm{I}^1_{q,\mathrm{gen}})$

$$\left(\int_M |u|^p \, dv(g)\right)^{1/p} \le A\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} + B\left(\int_M |u|^q \, dv(g)\right)^{1/q}$$

where $1/p = 1/q - 1/n$. As already mentioned in Chapter 3, there are no reasons when dealing with complete manifolds for the generic Sobolev inequalities to be valid. Indeed, as seen in Chapter 3, there exist complete manifolds for which all the $(\mathrm{I}^1_{q,\mathrm{gen}})$'s are false. Anyway, since the manifolds we will consider all have their Ricci curvature bounded from below, one has by Theorem 3.3 that for such manifolds

1. the scale $(\mathrm{I}^1_{q,\mathrm{gen}})$, $1 \le q < n$, of generic Sobolev inequalities is coherent and,
2. one, and hence all, of the $(\mathrm{I}^1_{q,\mathrm{gen}})$'s is valid if and only if the volume of any ball of radius 1 is bounded from below by some positive real number independent of its center.

Clearly, the validity of $(\mathrm{I}^1_{q,\mathrm{gen}})$ is equivalent to the validity of $(\mathrm{I}^q_{q,\mathrm{gen}})$, where we say that $(\mathrm{I}^q_{q,\mathrm{gen}})$ is valid if there exist A and B real such that for any $u \in H^q_1(M)$,

$$(\mathrm{I}^q_{q,\mathrm{gen}}) \qquad \left(\int_M |u|^p \, dv(g)\right)^{q/p} \le A\int_M |\nabla u|^q \, dv(g) + B\int_M |u|^q \, dv(g)$$

where $1/p = 1/q - 1/n$. As in Chapter 4, given $q \in [1, n)$ real, we define

$$\mathcal{A}_q(M) = \left\{A \in \mathbb{R} \text{ s.t. } \exists B \in \mathbb{R} \text{ for which } (\mathrm{I}^1_{q,\mathrm{gen}}) \text{ is valid}\right\}$$

Here again, one clearly has that $\mathcal{A}_q(M)$ is an interval of right extremity $+\infty$. Then, we define

$$\alpha_q(M) = \inf \mathcal{A}_q(M)$$

which is, by definition, the best first constant associated to $(\mathrm{I}^1_{q,\mathrm{gen}})$. We deal here with the following questions:

Question 1. Is it possible to compute explicitly $\alpha_q(M)$?

Question 2. For $A \in \mathcal{A}_q(M)$ close to $\alpha_q(M)$, can one compute or have estimates on the remaining constant B of $(\mathrm{I}^1_{q.\mathrm{gen}})$?

Question 3. Is $\mathcal{A}_q(M)$ a closed set? Namely, one has that $\alpha_q(M) \in \mathcal{A}_q(M)$?

Question 4. When $\mathcal{A}_q(M)$ is a closed set, and for $A = \alpha_q(M)$, can one compute or have estimates on the remaining constant B of $(\mathrm{I}^1_{q.\mathrm{gen}})$?

Note that to say that $\mathcal{A}_q(M)$ is a closed set means that there exists $B \in \mathbb{R}$ such that for any $u \in H_1^q(M)$,

$(\mathrm{I}^1_{q.\mathrm{opt}})$

$$\left(\int_M |u|^p \, dv(g)\right)^{1/p} \le \alpha_q(M)\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} + B\left(\int_M |u|^q \, dv(g)\right)^{1/q}$$

Such an inequality will be referred to as the optimal Sobolev inequality of order q. Getting estimates for the remaining constant B of $(\mathrm{I}^1_{q,\mathrm{gen}})$ when $A = \alpha_q(M)$, as asked in question 4, means that one gets estimates for the remaining constant B of $(\mathrm{I}^1_{q.\mathrm{opt}})$.

7.1. Questions 1 and 2

Proposition 4.2 of Chapter 4 still holds in this context. For the sake of clarity, we recall it here.

PROPOSITION 7.1 *Let (M, g) be a Riemannian n-manifold (not necessarily complete), and let $q \in [1, n)$ be some real number. Suppose that there exist $A, B \in \mathbb{R}$ such that for any $u \in \mathcal{D}(M)$,*

$$\left(\int_M |u|^p \, dv(g)\right)^{1/p} \le A\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} + B\left(\int_M |u|^q \, dv(g)\right)^{1/q}$$

where $1/p = 1/q - 1/n$. Then $A \ge K(n, q)$, where $K(n, q)$ is as in Theorem 4.4.

As a consequence of such a result, one gets that necessarily, $\alpha_q(M) \ge K(n, q)$. Conversely, Aubin **[10]** was able to prove that $\alpha_q(M) \le K(n, q)$, and hence that $\alpha_q(M) = K(n, q)$ when the manifold considered has bounded sectional curvature and positive injectivity radius. In Hebey **[107]**, we were able to prove that we still have that $\alpha_q(M) = K(n, q)$ if the bound on the sectional curvature is replaced by a lower bound on the Ricci curvature. This is, of course, a much weaker assumption. In particular, the result becomes very sharp if one compares it with what has been said in Chapter 3 (see Theorem 3.3 and Proposition 3.6). Given (M, g) a smooth, complete Riemannian n-manifold, and $q \in [1, n)$ real, let us say that the quasi-optimal Sobolev inequality of order q is valid if for any $\varepsilon > 0$, there exists $B_\varepsilon \in \mathbb{R}$ such that for any $u \in H_1^q(M)$,

$$(\mathrm{I}^1_{q,\varepsilon-\mathrm{opt}}) \qquad \begin{aligned} \left(\int_M |u|^p \, dv(g)\right)^{1/p} &\le \left(K(n, q) + \varepsilon\right)\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} \\ &\quad + B_\varepsilon\left(\int_M |u|^q \, dv(g)\right)^{1/q} \end{aligned}$$

where $1/p = 1/q - 1/n$. Roughly speaking, Hebey's result mentioned above states that for complete manifolds with Ricci curvature bounded from below, the validity of the generic Sobolev inequalities is equivalent to the validity of the quasi-optimal Sobolev inequalities. This holds also for $(\mathrm{I}^q_{q,\mathrm{gen}})$ and $(\mathrm{I}^q_{q,\varepsilon-\mathrm{opt}})$, where we say that the quasi-optimal Sobolev inequality $(\mathrm{I}^q_{q,\varepsilon-\mathrm{opt}})$ is valid if for any $\varepsilon > 0$ there exists $B_\varepsilon \in \mathbb{R}$ such that for any $u \in H_1^q(M)$,

$$(\mathrm{I}^q_{q,\varepsilon-\mathrm{opt}})$$

$$\left(\int_M |u|^p \, dv(g)\right)^{q/p} \leq \left(K(n,q)^q + \varepsilon\right) \int_M |\nabla u|^q \, dv(g) + B_\varepsilon \int_M |u|^q \, dv(g)$$

We can thus write that for any $q \in [1, n)$,

$$(\mathrm{I}^q_{q,\mathrm{gen}}) \Leftrightarrow (\mathrm{I}^q_{q,\varepsilon-\mathrm{opt}})$$

As one will see, the result also gives an answer to question 2, by showing that for $A = K(n,q) + \varepsilon$, the remaining constant B of $(\mathrm{I}^1_{q,\mathrm{gen}})$ depends only on n, ε, q, a lower bound for the Ricci curvature, and a lower bound for the injectivity radius. More precisely, Hebey's result **[107]** can be stated as follows:

THEOREM 7.1 *Let (M, g) be a smooth, complete Riemannian n-manifold. Suppose that its Ricci curvature $\mathrm{Rc}_{(M,g)}$ is such that $\mathrm{Rc}_{(M,g)} \geq kg$ for some $k \in \mathbb{R}$, and that its injectivity radius $\mathrm{inj}_{(M,g)}$ is such that $\mathrm{inj}_{(M,g)} \geq i$ for some $i > 0$. For any $\varepsilon > 0$, and any $q \in [1, n)$, there exists $B = B(\varepsilon, n, q, k, i)$, depending only on ε, n, q, k, and i, such that for any $u \in H_1^q(M)$,*

$$(\mathrm{I}^q_{q,\varepsilon-\mathrm{opt}})$$

$$\left(\int_M |u|^p \, dv(g)\right)^{q/p} \leq \left(K(n,q)^q + \varepsilon\right) \int_M |\nabla u|^q \, dv(g) + B \int_M |u|^q \, dv(g)$$

In particular, for any $\varepsilon > 0$, and any $q \in [1, n)$, there exists $B = B(\varepsilon, n, q, k, i)$, depending only on ε, n, q, k, and i, such that for any $u \in H_1^q(M)$,

$$(\mathrm{I}^1_{q,\varepsilon-\mathrm{opt}}) \quad \begin{aligned} \left(\int_M |u|^p \, dv(g)\right)^{1/p} &\leq \left(K(n,q) + \varepsilon\right)\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} \\ &\quad + B\left(\int_M |u|^q \, dv(g)\right)^{1/q} \end{aligned}$$

Here, $1/p = 1/q - 1/n$, and $K(n,q)$ is as in Theorem 4.4.

As a straightforward consequence of this result, one has that $\alpha_q(M) = K(n,q)$ for complete manifolds with Ricci curvature bounded from below and positive injectivity radius.

COROLLARY 7.1 *For any smooth, complete Riemannian n-manifold (M, g) with Ricci curvature bounded from below and positive injectivity radius, and for any $q \in [1, n)$ real, $\alpha_q(M) = K(n,q)$, where $K(n,q)$ is as in Theorem* 4.4.

In order to prove Theorem 7.1, we first establish the following lemma:

LEMMA 7.1 *Let (M, g) be a smooth, complete Riemannian n-manifold. Suppose that its Ricci curvature $\mathrm{Rc}_{(M,g)}$ is such that $\mathrm{Rc}_{(M,g)} \geq kg$ for some $k \in \mathbb{R}$, and that its injectivity radius $\mathrm{inj}_{(M,g)}$ is such that $\mathrm{inj}_{(M,g)} \geq i$ for some $i > 0$. For any $\varepsilon > 0$ there exists a positive constant $\delta = \delta(\varepsilon, n, k, i)$, depending only on ε, n, k, and i, such that for any $x \in M$, any $q \in [1, n)$, and any $u \in \mathcal{D}(B_x(\delta))$,*

$$\left(\int_M |u|^p \, dv(g)\right)^{q/p} \leq K(n, q)^q (1 + \varepsilon) \int_M |\nabla u|^q \, dv(g)$$

where $1/p = 1/q - 1/n$, and $K(n, q)$ is as in Theorem 4.4.

PROOF: According to Theorem 1.2, there exists $\delta = \delta(\varepsilon, n, k, i) > 0$ for any $\varepsilon > 0$ with the following property: For any $x \in M$ there exists a harmonic coordinate chart $\varphi : B_x(\delta) \to \mathbb{R}^n$ such that the components g_{ij} of g in this chart satisfy

$$(1 + \varepsilon)^{-1} \delta_{ij} \leq g_{ij} \leq (1 + \varepsilon) \delta_{ij}$$

as bilinear forms. One then has that for any $x \in M$, any $1 \leq q < n$, and any $u \in \mathcal{D}(B_x(\delta))$,

$$\int_M |\nabla u|^q \, dv(g) \geq (1 + \varepsilon)^{-(n+q)/2} \int_{R^n} |\nabla (u \circ \varphi^{-1})(x)|^q \, dx$$

and

$$\int_M |u|^p \, dv(g) \leq (1 + \varepsilon)^{n/2} \int_{R^n} |(u \circ \varphi^{-1})(x)|^p \, dx$$

where $1/p = 1/q - 1/n$. Independently, by Theorem 4.4,

$$\left(\int_{R^n} |(u \circ \varphi^{-1})(x)|^p \, dx\right)^{q/p} \leq K(n, q)^q \int_{R^n} |\nabla (u \circ \varphi^{-1})(x)|^q \, dx$$

As a consequence, we get that for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n, k, i) > 0$ such that for any $x \in M$, any $1 \leq q < n$, and any $u \in \mathcal{D}(B_x(\delta))$,

$$\left(\int_M |u|^p \, dv(g)\right)^{q/p} \leq K(n, q)^q (1 + \varepsilon)^n \int_M |\nabla u|^q \, dv(g)$$

where $1/p = 1/q - 1/n$. This ends the proof of the lemma. □

With such a result, we are now in position to prove Theorem 7.1.

PROOF OF THEOREM 7.1: Let $1 \leq q < n$ be given and let $p = nq/(n - q)$. By Lemma 7.1 there exists $\delta = \delta(\varepsilon, n, q, k, i) > 0$ such that for any $u \in \mathcal{D}(B_x(\delta))$,

$$\left(\int_M |u|^p \, dv(g)\right)^{q/p} \leq \left(K(n, q)^q + \frac{\varepsilon}{2}\right) \int_M |\nabla u|^q \, dv(g)$$

Without loss of generality, we can assume that $\delta < i$ for any $\varepsilon > 0$. Independently, by Lemma 1.1 we get that for any $\varepsilon > 0$ there exists a sequence (x_j) of points of M such that

1. $M = \bigcup_j B_{x_j}(\delta/2)$ and

2. there exists $N = N(\varepsilon, n, q, k, i)$ such that each point of M has a neighborhood which intersects at most N of the $B_{x_j}(\delta)$'s

where $\delta = \delta(\varepsilon, n, q, k, i)$ is as above. Let

$$\eta_j = \frac{\alpha_j^{[q]+1}}{\sum_m \alpha_m^{[q]+1}}$$

where $[q]$ is the greatest integer not exceeding q, and the function $\alpha_j \in \mathcal{D}(B_{x_j}(\delta))$ is chosen such that

$$0 \le \alpha_j \le 1,\ \alpha_j = 1 \text{ in } B_{x_j}(\delta/2) \quad \text{and} \quad |\nabla \alpha_j| \le 4/\delta$$

As one can easily check, (η_j) is a smooth partition of unity subordinate to the covering $(B_{x_j}(\delta))$, $\eta_j^{1/q} \in C^1(M)$ for any j, and there exists $H = H(\varepsilon, n, q, k, i) > 0$ such that for any j, $|\nabla \eta_j^{1/q}| \le H$. Fix $\varepsilon > 0$ and let $u \in \mathcal{D}(M)$. On the one hand,

$$\|u\|_p^q = \|u^q\|_{p/q} = \Big\| \sum_j \eta_j u^q \Big\|_{p/q} \le \sum_j \|\eta_j u^q\|_{p/q} = \sum_j \big\|\eta_j^{1/q} u\big\|_p^q$$

where $\|\cdot\|_s$ stands for the norm of $L^s(M)$. On the other hand, for any j,

$$\|\eta_j^{1/q} u\|_p^q \le \left(K(n,q)^q + \frac{\varepsilon}{2}\right) \|\nabla(\eta_j^{1/q} u)\|_p^q$$

As a consequence we get that

$$\begin{aligned}
&\left(\int_M |u|^p \, dv(g)\right)^{q/p} \\
&\le \left(K(n,q)^q + \frac{\varepsilon}{2}\right) \sum_j \int_M \left(\eta_j |\nabla u| + |u||\nabla \eta_j^{1/q}|\right)^q dv(g) \\
&\le \left(K(n,q)^q + \frac{\varepsilon}{2}\right) \int_M \sum_j \Big(|\nabla u|^q \eta_j + \mu |\nabla u|^{q-1} |\nabla \eta_j^{1/q}| \eta_j^{(q-1)/q} |u| \\
&\qquad\qquad + \nu |u|^q |\nabla \eta_j^{1/q}|^q \Big) dv(g) \\
&\le \left(K(n,q)^q + \frac{\varepsilon}{2}\right) \left(\|\nabla u\|_q^q + \mu N H \|\nabla u\|_q^{q-1} \|u\|_q + \nu N H^q \|u\|_q^q \right)
\end{aligned}$$

by Hölder's inequality and where μ and ν are such that for any $t \ge 0$,

$$(1+t)^q \le 1 + \mu t + \nu t^q$$

For instance, one can choose $\mu = q \max(1, 2^{q-2})$ and $\nu = \max(1, 2^{q-2})$. Now, let $\varepsilon_0 > 0$ be such that

$$\left(K(n,q)^q + \frac{\varepsilon}{2}\right)(1 + \varepsilon_0) \le K(n,q)^q + \varepsilon$$

Since for any positive real numbers x, y, and λ,

$$q x^{q-1} y \le \lambda (q-1) x^q + \lambda^{1-q} y^q$$

if we take $x = \|\nabla u\|_q$, $y = \|u\|_q$, and $\lambda = q\varepsilon_0/\mu(q-1)NH$, we get that

$$\mu NH\|\nabla u\|_q^{q-1}\|u\|_q \le \varepsilon_0\|\nabla u\|_q^q + C\|u\|_q^q$$

where

$$C = \left(\frac{\mu NH}{q}\right)\left(\frac{q\varepsilon_0}{\mu(q-1)NH}\right)^{1-q}$$

Hence, for any $u \in \mathcal{D}(M)$,

$$\begin{aligned}\left(\int_M |u|^p\,dv(g)\right)^{q/p} &\le \left(K(n,q)^q + \frac{\varepsilon}{2}\right)(1+\varepsilon_0)\int_M |\nabla u|^q\,dv(g) + B\int_M |u|^q\,dv(g)\\ &\le \left(K(n,q)^q + \varepsilon\right)\int_M |\nabla u|^q\,dv(g) + B\int_M |u|^q\,dv(g)\end{aligned}$$

where

$$B = \left(K(n,q)^q + \frac{\varepsilon}{2}\right)\left(C + \nu NH^q\right)$$

This ends the proof of the theorem. □

7.2. Questions 3 and 4

When dealing with Question 3, Aubin [10] was able to prove that $\mathcal{A}_2(M)$ is a closed set for complete manifolds with constant sectional curvature and positive injectivity radius. On the one hand, this result has been extended by Hebey-Vaugon [113] to complete, conformally flat manifolds with bounded sectional curvature and positive injectivity radius. A more sophisticated statement, where one only needs the conformal flatness at infinity, was then found in [117]. This is the subject of Theorem 7.4 below. On the other hand, the above-mentioned result of Aubin has been extended to complete manifolds with Riemann curvature bounded up to the order 1 and positive injectivity radius. This is the subject of the following theorem of Hebey-Vaugon [117], from which one easily gets Theorem 4.6 of Chapter 4. For the sake of clarity, the proof of this theorem is postponed to the following section.

THEOREM 7.2 *Let (M,g) be a smooth, complete Riemannian n-manifold, $n \ge 3$. Suppose that its Riemann curvature $\mathrm{Rm}_{(M,g)}$ is such that*

$$|\mathrm{Rm}_{(M,g)}| \le \Lambda_1 \quad \text{and} \quad |\nabla \mathrm{Rm}_{(M,g)}| \le \Lambda_2$$

for some nonnegative constants Λ_1 and Λ_2, and that its injectivity radius is such that $\mathrm{inj}_{(M,g)} \ge i$ for some $i > 0$. There exists $B = B(n, \Lambda_1, \Lambda_2, i)$, depending only on n, Λ_1, Λ_2, and i, such that for any $u \in H_1^2(M)$,

$$(\mathrm{I}_{2,\mathrm{opt}}^2) \qquad \left(\int_M |u|^p\,dv(g)\right)^{2/p} \le K(n,2)^2\int_M |\nabla u|^2\,dv(g) + B\int_M u^2\,dv(g)$$

In particular, there exists $B = B(n, \Lambda_1, \Lambda_2, i)$, depending only on n, Λ_1, Λ_2, and i, such that for any $u \in H_1^2(M)$,

$(\mathrm{I}^1_{2,\mathrm{opt}})$

$$\left(\int_M |u|^p\,dv(g)\right)^{1/p} \le K(n,2)\left(\int_M |\nabla u|^2\,dv(g)\right)^{1/2} + B\left(\int_M u^2\,dv(g)\right)^{1/2}$$

Here $1/p = 1/2 - 1/n$, and $K(n,2)$ is as in Theorem 4.4.

As a straightforward consequence of such a result, one gets the following:

COROLLARY 7.2 *For any smooth, complete Riemannian manifold with Riemann curvature bounded up to the order 1, and with positive injectivity radius, $\mathcal{A}_2(M)$ is a closed set.*

As another straightforward consequence of Theorem 7.2, note that the following holds:

COROLLARY 7.3 *Let (M,g) be a smooth, compact Riemannian n-manifold, $n \ge 3$. For any Riemannian covering $(\tilde{M}, \tilde{g})$ of (M,g), there exists $B \in \mathbb{R}$ such that for any $u \in H_1^2(\tilde{M})$,*

$$\left(\int_{\tilde{M}} |u|^p\,dv(\tilde{g})\right)^{2/p} \le K(n,2)^2 \int_{\tilde{M}} |\nabla u|^2\,dv(\tilde{g}) + B\int_{\tilde{M}} u^2\,dv(\tilde{g})$$

where $1/p = 1/2 - 1/n$ and $K(n,2)$ is as in Theorem 4.4. In particular, $\mathcal{A}_2(\tilde{M})$ is a closed set for any Riemannian covering $(\tilde{M}, \tilde{g})$ of a compact Riemannian manifold.

Let us now come back to the statements of Theorems 7.1 and 7.2. When comparing Theorem 7.2 and Theorem 7.1, one can ask if the conclusion of Theorem 7.2 still holds under the assumptions that the Ricci curvature of the manifold is bounded from below, and that the injectivity radius of the manifold is positive. In other words, one can ask if it is possible to take $\varepsilon = 0$ in Theorem 7.1. A surprising fact here is that the answer to such a question is negative. This is the subject of the following result:

THEOREM 7.3 *For any integer $n \ge 4$, there exist smooth, complete Riemannian n-manifolds with Ricci curvature bounded from below and positive injectivity radius for which $(\mathrm{I}^2_{2,\mathrm{opt}})$ is not valid.*

PROOF: Let (M,g) be a smooth, complete Riemannian n-manifold, $n \ge 4$, with Ricci curvature bounded from below and positive injectivity radius. Suppose that $(\mathrm{I}^2_{2,\mathrm{opt}})$ is valid. By Proposition 5.1 one gets that for any $x \in M$,

$$\mathrm{Scal}_{(M,g)}(x) \le \frac{4(n-1)}{n-2}\frac{B}{K(n,2)^2} \tag{7.1}$$

where B is the second constant of $(\mathrm{I}^2_{2,\mathrm{opt}})$, and $\mathrm{Scal}_{(M,g)}$ is the scalar curvature of g. Noting that the existence of a lower bound for the Ricci curvature and of an upper bound for the scalar curvature leads to the existence of a global bound for the Ricci curvature, one gets from (7.1) that the Ricci curvature of g is bounded. In other words, for $n \geq 4$, and for (M, g) a smooth, complete Riemannian n-manifold with Ricci curvature bounded from below and positive injectivity radius, if $(\mathrm{I}^2_{2,\mathrm{opt}})$ is valid, then its Ricci curvature is bounded. Independently, one can show that in any dimension there exist complete manifolds with Ricci curvature bounded from below and positive injectivity radius having the property that their Ricci curvature is not bounded. (References on the construction of such manifolds will be found in Anderson-Cheeger [**6**]). According to what has been said above, $(\mathrm{I}^2_{2,\mathrm{opt}})$ must be false for such manifolds. Clearly, this ends the proof of the theorem. □

Roughly speaking, coming back to what has been said in the preceding section, one has that

$$(\mathrm{I}^2_{2,\mathrm{gen}}) \Leftrightarrow (\mathrm{I}^2_{2,\varepsilon-\mathrm{opt}}) \not\Leftrightarrow (\mathrm{I}^2_{2,\mathrm{opt}})$$

A natural guess here would be that the conclusion of Theorem 7.2 holds under the assumptions that the Ricci curvature of the manifold is bounded and that the injectivity radius of the manifold is positive. As a first step, it would certainly be simpler to prove that the conclusion of Theorem 7.2 is valid under the assumptions that the Ricci curvature of the manifold is bounded up to the order 1 and that the injectivity radius of the manifold is positive. The following result is due to Hebey-Vaugon [**117**].

THEOREM 7.4 *Let (M, g) be a smooth, complete Riemannian n-manifold, $n \geq 3$. Suppose that its Ricci curvature is bounded, that its injectivity radius is positive, and that g is conformally flat outside some compact subset of M. There exists $B \in \mathbb{R}$ such that for any $u \in H_1^2(M)$,*

$$(\mathrm{I}^2_{2,\mathrm{opt}}) \qquad \left(\int_M |u|^p \, dv(g)\right)^{2/p} \leq K(n,2)^2 \int_M |\nabla u|^2 \, dv(g) + B \int_M u^2 \, dv(g)$$

In particular, there exists $B \in \mathbb{R}$ such that for any $u \in H_1^2(M)$,

$(\mathrm{I}^1_{2,\mathrm{opt}})$

$$\left(\int_M |u|^p \, dv(g)\right)^{1/p} \leq K(n,2)\left(\int_M |\nabla u|^2 \, dv(g)\right)^{1/2} + B\left(\int_M u^2 \, dv(g)\right)^{1/2}$$

Here $1/p = 1/2 - 1/n$, and $K(n,2)$ is as in Theorem 4.4.

PROOF: Let $\rho > 0$ be such that $\rho < \mathrm{inj}_{(M,g)}$, the injectivity radius of (M, g). Given $x \in M$, $B_x(\rho)$ is simply connected. For $x \in M$ such that g is conformally flat on $B_x(\rho)$, one gets by Proposition 3.8 that for any $u \in \mathcal{D}(B_x(\rho))$,

$$\left(\int_M |u|^p \, dv(g)\right)^{2/p} \leq K(n,2)^2\left(\int_M |\nabla u|^2 \, dv(g) + \frac{n-2}{4(n-1)} \int_M \mathrm{Scal}_{(M,g)} u^2 \, dv(g)\right) \tag{7.2}$$

where $\mathrm{Scal}_{(M,g)}$ stands for the scalar curvature of g. Independently, and as an easy consequence of what will be said in the next section, for any $\rho \in (0, \mathrm{inj}_{(M,g)})$, and any $x \in M$, there exists $B \in \mathbb{R}$ such that for any $u \in \mathcal{D}(B_x(\rho))$,

$$\left(\int_M |u|^p \, dv(g)\right)^{2/p} \le K(n,2)^2 \int_M |\nabla u|^2 \, dv(g) + B \int_M u^2 \, dv(g) \tag{7.3}$$

Fix $\rho < \mathrm{inj}_{(M,g)}$, $\rho > 0$. By Lemma 1.1 one gets the existence of a sequence (x_j) of points of M such that

1. $M = \bigcup_j B_{x_j}(\frac{\rho}{2})$ and
2. there exists N integer such that each point of M has a neighborhood which intersects at most N of the $B_{x_j}(\rho)$'s.

Let

$$\eta_j = \frac{\alpha_j^2}{\sum_m \alpha_m^2}$$

where $\alpha_j \in \mathcal{D}(B_{x_j}(\rho))$ is chosen such that

$$0 \le \alpha_j \le 1, \ \alpha_j = 1 \text{ in } B_{x_j}\left(\frac{\rho}{2}\right) \quad \text{and} \quad |\nabla \alpha_j| \le 4/\rho$$

As one can easily check, (η_j) is a smooth partition of unity subordinate to the covering $(B_{x_j}(\rho))$, $\sqrt{\eta_j} \in C^1(M)$ for any j, and there exists $H > 0$ such that for any j, $|\nabla\sqrt{\eta_j}| \le H$. From now on, let K be a compact subset of M such that g is conformally flat outside of K. Let j_0 be such that

$$B_{x_j}(\rho) \cap K = \emptyset$$

for $j > j_0$. By (7.3) one has that for any $j \le j_0$ there exists $B_j \in \mathbb{R}$ such that for any $u \in \mathcal{D}(M)$,

$$\begin{aligned}\left(\int_M |\sqrt{\eta_j} u|^p \, dv(g)\right)^{2/p} \le{}& K(n,2)^2 \int_M |\nabla(\sqrt{\eta_j} u)|^2 \, dv(g) \\ &+ B_j \int_M \eta_j u^2 \, dv(g)\end{aligned}$$

Independently, and by (7.2), one has that for any $j > j_0$, and any $u \in \mathcal{D}(M)$,

$$\left(\int_M |\sqrt{\eta_j} u|^p \, dv(g)\right)^{2/p} \le K(n,2)^2 \int_M |\nabla(\sqrt{\eta_j} u)|^2 \, dv(g) + S \int_M \eta_j u^2 \, dv(g)$$

where

$$S = \frac{n-2}{4(n-1)} K(n,2)^2 \left(\sup_M |\mathrm{Scal}_{(M,g)}|\right)$$

Clearly $S < +\infty$ since the Ricci curvature of g is bounded. Given $u \in \mathcal{D}(M)$, one has that

$$\|u\|_p^2 = \|u^2\|_{p/2} = \left\| \sum_j \eta_j u^2 \right\|_{p/2} \le \sum_j \|\eta_j u^2\|_{p/2} = \sum_j \|\sqrt{\eta_j} u\|_p^2$$

where $\|\cdot\|_s$ stands for the norm of $L^s(M)$. Set

$$B = \max\left(B_1, \ldots, B_{j_0}, S\right)$$

Then, for any $u \in \mathcal{D}(M)$,

$$\begin{aligned}\left(\int_M |u|^p \, dv(g)\right)^{2/p} & \\ &\le K(n,2)^2 \sum_j \int_M \left|\nabla\left(\eta_j^{1/2} u\right)\right|^2 dv(g) + B \sum_j \int_M \eta_j u^2 \, dv(g) \\ &= K(n,2)^2 \Bigg(\sum_j \int_M \eta_j |\nabla u|^2 \, dv(g) + \sum_j \int_M u(\nabla^\nu u \nabla_\nu \eta_j) dv(g) \\ &\qquad + \sum_j \int_M u^2 \left|\nabla \eta_j^{1/2}\right|^2 dv(g) \Bigg) + B \sum_j \int_M \eta_j u^2 \, dv(g) \\ &\le K(n,2)^2 \int_M |\nabla u|^2 \, dv(g) + \left(B + K(n,2)^2 N H^2\right) \int_M u^2 \, dv(g)\end{aligned}$$

This ends the proof of the theorem. □

As a straightforward consequence of Theorem 7.4, coming back to Question 3, one has the following:

COROLLARY 7.4 *For any smooth, complete Riemannian manifold (M, g) with bounded Ricci curvature and positive injectivity radius that has the property that it is conformally flat outside some compact subset of M, $\mathcal{A}_2(M)$ is a closed set.*

Now that such results have been stated, one can ask what happens for $q \neq 2$. As in Chapter 4, we say that the optimal Sobolev inequality $(\mathrm{I}^q_{q,\mathrm{opt}})$ is valid if there exists $B \in \mathbb{R}$ such that for any $u \in H_1^q(M)$,

$$(\mathrm{I}^q_{q,\mathrm{opt}}) \qquad \left(\int_M |u|^p \, dv(g)\right)^{q/p} \le K(n,q)^q \int_M |\nabla u|^q \, dv(g) + B \int_M |u|^q \, dv(g)$$

where $1/p = 1/q - 1/n$, and $K(n, q)$ is as in Theorem 4.4. Here again, one has the following result of Druet [**74**]. The proof of such a result is the same than that of Theorem 4.8.

THEOREM 7.5 *Let (M, g) be a smooth, complete Riemannian n-manifold, and let $q \in [1, n)$ real. Assume that $q > 2$, that $q^2 < n$, and that the scalar curvature of (M, g) is positive somewhere. Then inequality $(\mathrm{I}^q_{q,\mathrm{opt}})$ is false on (M, g).*

Conversely, one has the following result of Aubin [**10**]:

THEOREM 7.6 *Let (M, g) be a smooth, complete Riemannian n-manifold with positive injectivity radius. Let also $q \in [1, n)$ real. Suppose either that $n = 2$ and that (M, g) has bounded sectional curvature, or that $n \ge 3$ and that (M, g) has constant sectional curvature. Then inequality $(\mathrm{I}^1_{q,\mathrm{opt}})$ is valid and $\mathcal{A}_q(M)$ is a closed set.*

The proof of such a result proceeds as in the proof of Theorem 4.7. We refer the reader to [**10**] for more details.

7.3. Proof of Theorem 7.2

The proof of Theorem 7.2 proceeds in several steps. As one can see, it mixes PDE and geometric arguments. Let g be a smooth, Riemannian metric on $\mathbb{R}^n$. Suppose that for some $\Lambda_1 > 0$ and $\Lambda_2 > 0$, $|\mathrm{Rm}_{(R^n,g)}| \le \Lambda_1$ and $|\nabla \mathrm{Rm}_{(R^n,g)}| \le \Lambda_2$ on $B_0(4)$, the Euclidean ball of center 0 and radius 4. We say that g satisfies $(\star)$ if the following holds:

$$(\star)\qquad \begin{cases} \text{(i) the canonical coordinate system of } \mathbb{R}^n \text{ when restricted to } B_0(2), \\ \text{the Euclidean ball of center 0 and radius 2,} \\ \text{is a normal geodesic coordinate system at 0 for } g \\ \text{(ii) for any } x \in B_0(1), \text{ the Euclidean ball of center 0 and} \\ \text{radius } 1,\ 2 < \min\big(\delta, \mathrm{inj}_g(x)\big) \end{cases}$$

where δ is as in Theorem 1.3 of Chapter 1, and $\mathrm{inj}_g(x)$ stands for the injectivity radius of $(\mathbb{R}^n, g)$ at x. Let us denote by $\mathcal{B} = B_0(1)$ the Euclidean ball of center 0 and radius 1. If (ii) holds, then for any $x \in \mathcal{B}$, $\overline{\mathcal{B}}$ is contained in the geodesic ball for g of center x and radius $\min(\delta, \mathrm{inj}_g(x))$. The first result we prove is the following:

LEMMA 7.2 *Let $n \ge 3$ be given. Suppose that for any positive constants Λ_1 and Λ_2, and any smooth, Riemannian metric g on $\mathbb{R}^n$ such that*

$$\begin{cases} (a)\ |\mathrm{Rm}_{(R^n,g)}| \le \Lambda_1 \text{ and } |\nabla \mathrm{Rm}_{(R^n,g)}| \le \Lambda_2 \text{ in } B_0(4), \\ \text{the Euclidean ball of center 0 and radius 4} \\ (b)\ g \text{ satisfies } (\star) \end{cases}$$

there exists some $B = B(n, \Lambda_1, \Lambda_2)$ real, depending only on n, Λ_1, and Λ_2, with the property that for any $u \in \mathcal{D}(\mathcal{B})$,

$$\left(\int_{\mathcal{B}} |u|^p\, dv(g)\right)^{2/p} \le K(n,2)^2 \int_{\mathcal{B}} |\nabla u|^2\, dv(g) + B \int_{\mathcal{B}} u^2\, dv(g)$$

where $1/p = 1/2 - 1/n$. Then for any positive constants Λ_1, Λ_2 and i, and for any smooth, complete Riemannian n-manifold (M,g) satisfying that $|\mathrm{Rm}_{(M,g)}| \le \Lambda_1$, $|\nabla \mathrm{Rm}_{(M,g)}| \le \Lambda_2$, and $\mathrm{inj}_{(M,g)} \ge i$, there exists some $\tilde{B} = \tilde{B}(n, \Lambda_1, \Lambda_2, i)$ real, depending only on n, Λ_1, Λ_2, and i, such that for any $u \in H_1^2(M)$,

$$\left(\int_M |u|^p\, dv(g)\right)^{2/p} \le K(n,2)^2 \int_M |\nabla u|^2\, dv(g) + \tilde{B} \int_M u^2\, dv(g)$$

where $1/p = 1/2 - 1/n$.

PROOF OF LEMMA 7.2: Let (M,g) be a smooth, complete Riemannian n-manifold such that $|\mathrm{Rm}_{(M,g)}| \le \Lambda_1$, $|\nabla \mathrm{Rm}_{(M,g)}| \le \Lambda_2$, and $\mathrm{inj}_{(M,g)} \ge i$ for some positive constants Λ_1, Λ_2, and i. Let δ be given by Theorem 1.3 of Chapter 1. For $\lambda > 0$ real, as one can easily check,

$$|\mathrm{Rm}_{(M,\lambda g)}| = \lambda^{-1} |\mathrm{Rm}_{(M,g)}|$$
$$|\nabla \mathrm{Rm}_{(M,\lambda g)}| = \lambda^{-3/2} |\nabla \mathrm{Rm}_{(M,g)}|$$
$$\mathrm{inj}_{(M,\lambda g)} = \sqrt{\lambda}\, \mathrm{inj}_{(M,g)}$$

Since

$$\lim_{(\Lambda_1,\Lambda_2)\to(0,0)} \delta(n, \Lambda_1, \Lambda_2) = +\infty$$

there exists $\lambda = \lambda(n, \Lambda_1, \Lambda_2, i)$, $\lambda \gg 1$ depending only on n, Λ_1, Λ_2, and i, such that

$$\min\left(\delta, \mathrm{inj}_{(M,\lambda g)}\right) \geq 5$$

Set $\tilde{g} = \lambda g$. Clearly, the Ricci curvature of $\tilde{g}$ is bounded from below by some real number depending only on n, Λ_1, Λ_2, and i. By Lemma 1.1 of Chapter 1, one then gets that there exists a sequence (x_m) of points of M, and some integer $N = N(n, \Lambda_1, \Lambda_2, i)$, depending only on n, Λ_1, Λ_2, and i, such that

1. the family $(B_{x_m}(\frac{1}{2}))$ is a covering of M and
2. every point in M has a neighborhood that intersects at most N of the $B_{x_m}(1)$'s

where $B_{x_m}(\frac{1}{2})$ and $B_{x_m}(1)$ refer to $\tilde{g}$. Let $\alpha_m \in \mathcal{D}(B_{x_m}(1))$ be such that $0 \leq \alpha_m \leq 1$, $\alpha_m = 1$ in $B_{x_m}(\frac{1}{2})$, and $|\nabla\alpha_m| \leq 4$ (for the norm with respect to $\tilde{g}$). Set

$$\eta_m = \frac{\alpha_m^2}{\sum_j \alpha_j^2}$$

One then gets that (η_m) is a partition of unity subordinate to the covering $(B_{x_m}(1))$ such that for any m, $\eta_m^{1/2}$ is smooth and $|\nabla\eta_m^{1/2}| \leq H$ (for the norm with respect to $\tilde{g}$), where $H = H(n, \Lambda_1, \lambda_2, i)$ is some positive real number depending only on n, Λ_1, Λ_2, and i. By considering the pullback of $\tilde{g}$ by the exponential map of $\tilde{g}$ at x_m, and by gluing this metric with the Euclidean metric in $B_0(5)\backslash B_0(4)$, we get some metric defined on $\mathbb{R}^n$. Clearly, this metric satisfies the assumptions of the lemma. Hence, for any m, and any $u \in C^\infty(M)$,

$$\begin{aligned}\left(\int_M |(\eta_m^{1/2}u)|^p \, dv(\tilde{g})\right)^{2/p} &\leq K(n,2)^2 \int_M |\nabla(\eta_m^{1/2}u)|^2 \, dv(\tilde{g}) \\ &\quad + B\int_M (\eta_m^{1/2}u)^2 \, dv(\tilde{g})\end{aligned}$$

where $B = B(n, \Lambda_1, \Lambda_2, i)$ depends only on n, Λ_1, Λ_2, and i. As in the proof of Theorem 7.1, one has that for any $u \in C^\infty(M)$,

$$\left(\int_M |u|^p \, dv(\tilde{g})\right)^{2/p} \leq \sum_m \left(\int_M |(\eta_m^{1/2}u)|^p \, dv(\tilde{g})\right)^{2/p}$$

As a consequence, for any $u \in C^\infty(M)$,

$$\begin{aligned}
&\left(\int_M |u|^p \, dv(\tilde{g})\right)^{2/p} \\
&\quad \le K(n,2)^2 \sum_m \int_M |\nabla(\eta_m^{1/2} u)|^2 \, dv(\tilde{g}) + B \sum_m \int_M (\eta_m^{1/2} u)^2 \, dv(\tilde{g}) \\
&\quad = K(n,2)^2 \sum_m \int_M \eta_m |\nabla u|^2 \, dv(\tilde{g}) + K(n,2)^2 \sum_m \int_M u(\nabla^\nu u \nabla_\nu \eta_m) dv(\tilde{g}) \\
&\qquad + \sum_m \int_M |\nabla \eta_m^{1/2}|^2 u^2 \, dv(\tilde{g}) + B \sum_m \int_M (\eta_m^{1/2} u)^2 \, dv(\tilde{g})
\end{aligned}$$

Since $\sum_m \eta_m = 1$, this leads to

$$\left(\int_M |u|^p \, dv(\tilde{g})\right)^{2/p} \le K(n,2)^2 \int_M |\nabla u|^2 \, dv(\tilde{g}) + B' \int_M u^2 \, dv(\tilde{g})$$

where $B' = NH^2 K(n,2)^2 + B$. Clearly, B' depends only on n, Λ_1, Λ_2, and i, since this is the case for N, H, and B. Coming back to g, one then gets that for any $u \in C^\infty(M)$,

$$\left(\int_M |u|^p \, dv(g)\right)^{2/p} \le K(n,2)^2 \int_M |\nabla u|^2 \, dv(g) + B'' \int_M u^2 \, dv(g)$$

where $B'' = \lambda B'$. Here again, B'' depends only on n, Λ_1, Λ_2, and i. This ends the proof of the lemma. □

By Lemma 7.2, the proof of Theorem 7.2 reduces to proving that for any $n \ge 3$, $\Lambda_1 > 0$, and $\Lambda_2 > 0$, there exists some constant $B = B(n, \Lambda_1, \Lambda_2)$, depending only on n, Λ_1, and Λ_2, such that for any smooth, Riemannian metric g on $\mathbb{R}^n$ satisfying the points (a) and (b) of Lemma 7.2, and for any $u \in \mathcal{D}(\mathcal{B})$,

$$\left(\int_{R^n} |u|^p \, dv(g)\right)^{2/p} \le K(n,2)^2 \int_{R^n} |\nabla u|^2 \, dv(g) + B \int_{R^n} u^2 \, dv(g)$$

where $1/p = 1/2 - 1/n$. This is what we are going to do now.

For g some smooth metric on $\mathbb{R}^n$, let $H_{0,1}^2(\mathcal{B})$ be the completion of $\mathcal{D}(\mathcal{B})$ with respect to the standard norm

$$\|u\| = \sqrt{\int_{\mathcal{B}} |\nabla u|^2 \, dv(g) + \int_{\mathcal{B}} u^2 \, dv(g)}$$

Since $\mathcal{B}$ is relatively compact, $H_{0,1}^2(\mathcal{B})$ does not depend on g. For $\alpha > 0$ real, and $u \in H_{0,1}^2(\mathcal{B})$, $u \not\equiv 0$, we define

$$I_{g,\alpha}(u) = \frac{\int_{\mathcal{B}} |\nabla u|^2 \, dv(g) + \alpha \int_{\mathcal{B}} u^2 \, dv(g)}{\left(\int_{\mathcal{B}} |u|^p \, dv(g)\right)^{2/p}}$$

Lemma 7.2 can be stated as follows: For any $n \ge 3$, $\Lambda_1 > 0$, and $\Lambda_2 > 0$, there exists $\alpha = \alpha(n, \Lambda_1, \Lambda_2)$, depending only on n, Λ_1, and Λ_2, such that for any smooth, Riemannian metric g on $\mathbb{R}^n$ satisfying the points (a) and (b) of Lemma

7.2, and for any $u \in H_{0,1}^2(\mathcal{B})$, $u \neq 0$, $I_{g,\alpha}(u) \geq K(n,2)^{-2}$. To see this, just set $B = \alpha K(n,2)^2$.

From now on, we proceed by contradiction. Hence we suppose that there exist $n \geq 3$, $\Lambda_1 > 0$, and $\Lambda_2 > 0$ such that the following holds: For any $\alpha > 0$, there exists a smooth, Riemannian metric g_α on $\mathbb{R}^n$ such that

$$(\star\star) \qquad \begin{cases} \text{(i) } |\mathrm{Rm}_{g_\alpha}| \leq \Lambda_1 \text{ and } |\nabla \mathrm{Rm}_{g_\alpha}| \leq \Lambda_2 \text{ in } B_0(4) \\ \text{(ii) } g_\alpha \text{ satisfies } (\star) \\ \text{(iii) } \inf_u I_{g_\alpha,\alpha}(u) < \frac{1}{K(n,2)^2} \end{cases}$$

where Rm_{g_α} stands for the Riemann curvature of g_α, and the infimum $\inf_u I_{g_\alpha,\alpha}(u)$ is taken over the $u \in H_{0,1}^2(\mathcal{B})$, $u \neq 0$. For convenience, we set $I_\alpha = I_{g_\alpha,\alpha}$. The first result we prove is the following:

LEMMA 7.3 *Let $\alpha > 0$ and g_α be as in $(\star\star)$. There exists $\varphi_\alpha \in C^2(\overline{\mathcal{B}}) \cap H_{0,1}^2(\mathcal{B})$, $\varphi_\alpha > 0$ in $\mathcal{B}$, and there exists $\lambda_\alpha \in (0, K(n,2)^{-2})$, such that*

$$\Delta_{g_\alpha}\varphi_\alpha + \alpha\varphi_\alpha = \lambda_\alpha \varphi_\alpha^{p-1} \quad \text{in } \mathcal{B} \tag{7.4}$$

$$\int_{\mathcal{B}} \varphi_\alpha^p \, dv(g_\alpha) = 1 \tag{7.5}$$

where Δ_{g_α} is the Laplacian of g_α.

PROOF: For $q \in (1, p]$, let

$$\mathcal{H}_q = \left\{ u \in H_{0,1}^2(\mathcal{B}) \,/ \int_{\mathcal{B}} |u|^q \, dv(g_\alpha) = 1 \right\}$$

Let also μ_q be defined by

$$\mu_q = \inf_{u \in \mathcal{H}_q} E_\alpha(u)$$

where

$$E_\alpha(u) = \int_{\mathcal{B}} |\nabla u|^2 \, dv(g_\alpha) + \alpha \int_{\mathcal{B}} u^2 \, dv(g_\alpha)$$

For $q < p$, the embedding of $H_{0,1}^2(\mathcal{B})$ in $L^q(\mathcal{B})$ is compact. Fix such a q, and let $(\varphi_i) \in \mathcal{H}_q$ be a minimizing sequence for μ_q. Without loss of generality, up to replacing φ_i by $|\varphi_i|$, we can assume that the φ_i's are nonnegative. Since $\alpha > 0$, (φ_i) is a bounded sequence in $H_{0,1}^2(\mathcal{B})$. Up to the extraction of a subsequence, since $H_{0,1}^2(\mathcal{B})$ is reflexive, and since the embedding of $H_{0,1}^2(\mathcal{B})$ in $L^q(\mathcal{B})$ is compact, this leads to the existence of $\varphi_q \in H_{0,1}^q(\mathcal{B})$ such that

$$\varphi_i \rightharpoonup \varphi_q \text{ in } H_{0,1}^q(\mathcal{B}), \quad \varphi_i \to \varphi_q \text{ in } L^q(\mathcal{B}), \quad \varphi_i \to \varphi_q \text{ a.e.}$$

One then gets that $\varphi_q \geq 0$ a.e., and that $\varphi_q \in \mathcal{H}_q$. Moreover, the weak convergence in $H_{0,1}^2(\mathcal{B})$ implies that

$$E_\alpha(\varphi_q) \leq \liminf_{i \to +\infty} E_\alpha(\varphi_i)$$

Hence, $E_\alpha(\varphi_q) = \mu_q$, and φ_q is a solution of

$$\Delta_{g_\alpha}\varphi_q + \alpha\varphi_q = \mu_q \varphi_q^{q-1}$$

in $\mathcal{B}$. By maximum principles and regularity results, φ_q is positive in $\mathcal{B}$ and $\varphi_q \in C^2(\overline{\mathcal{B}})$.

Let us now get the solution φ_α we are looking for as the limit of the φ_q's, $q \to p$. As a first remark, let $\varepsilon > 0$ be given, and let $\psi \in \mathcal{H}_p$, ψ nonnegative, be such that

$$E_\alpha(\psi) \leq \inf_{u \in \mathcal{H}_p} E_\alpha(u) + \varepsilon$$

For q as above, $\psi_q = \|\psi\|_q^{-1}\psi$ belongs to $\mathcal{H}_q$. Hence, $E_\alpha(\psi_q) \geq \mu_q$. Noting that $E_\alpha(\psi_q) \to E_\alpha(\psi)$ as $q \to p$, one gets that

$$\limsup_{q \to p} \mu_q \leq E_\alpha(\psi) \leq \inf_{u \in \mathcal{H}_p} E_\alpha(u) + \varepsilon$$

Since this inequality holds for any $\varepsilon > 0$,

$$\limsup_{q \to p} \mu_q \leq \inf_{u \in \mathcal{H}_p} E_\alpha(u)$$

From now on, and up to the extraction of a subsequence, we assume that

$$\mu = \lim_{q \to p} \mu_q$$

exists. As one can easily check, (φ_q) is a bounded sequence in $H_{0,1}^2(\mathcal{B})$. Since $H_{0,1}^2(\mathcal{B})$ is reflexive, and since for $s < p$ the embedding of $H_{0,1}^2(\mathcal{B})$ in $L^s(\mathcal{B})$ is compact, we get the existence of some $\varphi_\alpha \in H_{0,1}^2(\mathcal{B})$ such that, up to a subsequence,

$$\varphi_q \rightharpoonup \varphi_\alpha \text{ in } H_{0,1}^2(\mathcal{B}), \quad \varphi_q \to \varphi_\alpha \text{ in } L^2(\mathcal{B}), \quad \varphi_q \to \varphi_\alpha \text{ a.e.}$$

In particular, φ_α is nonnegative and, since (φ_q^{q-1}) is bounded in $L^{p/(q-1)}(\mathcal{B}) \subset L^{p/(p-1)}(\mathcal{B})$, we can assume that

$$\varphi_q^{q-1} \rightharpoonup \varphi_\alpha^{p-1} \quad \text{in } L^{p/(p-1)}(\mathcal{B})$$

By passing to the limit as q tends to p in the equation satisfied by φ_q, one then gets that

$$\Delta_{g_\alpha}\varphi_\alpha + \alpha\varphi_\alpha = \mu\varphi_\alpha^{p-1}$$

By maximum principles and regularity results, $\varphi_\alpha \in C^2(\overline{\mathcal{B}})$ and either $\varphi_\alpha \equiv 0$ or $\varphi_\alpha > 0$ in $\mathcal{B}$.

Let us now prove that $\varphi_\alpha \not\equiv 0$. As for compact manifolds without boundary, the best first constant for the embedding of $H_{0,1}^2(\mathcal{B})$ in $L^p(\mathcal{B})$ is $K(n, 2)$. Hence, for any $\varepsilon > 0$, there exists some $B_\varepsilon \in \mathbb{R}$ such that for any $u \in H_{0,1}^2(\mathcal{B})$,

$$\left(\int_{\mathcal{B}} |u|^p \, dv(g_\alpha)\right)^{2/p} \leq \left(K(n,2)^2 + \varepsilon\right) \int_{\mathcal{B}} |\nabla u|^2 \, dv(g_\alpha) + B_\varepsilon \int_{\mathcal{B}} u^2 \, dv(g_\alpha)$$

The proof of such a claim goes in a very standard way (see, for instance, Aubin [12] for details). By our contradiction assumption,

$$\inf_{u \in \mathcal{H}_p} E_\alpha(u) < \frac{1}{K(n,2)^2}$$

Let $\varepsilon > 0$ be such that

$$\left(K(n,2)^2 + \varepsilon\right) \inf_{u \in \mathcal{H}_p} E_\alpha(u) < 1$$

and take $u = \varphi_q$ in the above inequality. Then

$$\left(\int_{\mathcal{B}} \varphi_q^p \, dv(g_\alpha)\right)^{2/p} \leq \left(K(n,2)^2 + \varepsilon\right)\mu_q + B_\varepsilon \int_{\mathcal{B}} \varphi_q^2 \, dv(g_\alpha)$$

Moreover, one has that

$$\int_{\mathcal{B}} \varphi_q^q \, dv(g_\alpha) \leq \left(\int_{\mathcal{B}} \varphi_q^p \, dv(g_\alpha)\right)^{q/p} \mathrm{Vol}_{g_\alpha}(\mathcal{B})^{1-\frac{q}{p}}$$

where $\mathrm{Vol}_{g_\alpha}(\mathcal{B})$ stands for the volume of $\mathcal{B}$ with respect to g_α. Since $\varphi_q \in \mathcal{H}_q$,

$$\mathrm{Vol}_{g_\alpha}(\mathcal{B})^{\frac{2}{p}-\frac{2}{q}} \leq \left(\int_{\mathcal{B}} \varphi_q^p \, dv(g_\alpha)\right)^{2/p}$$

and one has that

$$\mathrm{Vol}_{g_\alpha}(\mathcal{B})^{\frac{2}{p}-\frac{2}{q}} \leq \left(K(n,2)^2 + \varepsilon\right)\mu_q + B_\varepsilon \int_{\mathcal{B}} \varphi_q^2 \, dv(g_\alpha)$$

By passing to the limit as $q \to p$ in this inequality, and since

$$\mu \leq \inf_{u \in \mathcal{H}_p} E_\alpha(u)$$

we get that

$$1 \leq \left(K(n,2)^2 + \varepsilon\right) \inf_{u \in \mathcal{H}_p} E_\alpha(u) + B_\varepsilon \int_{\mathcal{B}} \varphi_\alpha^2 \, dv(g_\alpha)$$

By the above choice of ε, this implies that

$$\int_{\mathcal{B}} \varphi_\alpha^2 \, dv(g_\alpha) > 0$$

Hence, $\varphi_\alpha \not\equiv 0$, so that, as already mentioned, φ_α is positive in $\mathcal{B}$. In particular, multiplying by φ_α the equation satisfied by φ_α and integrating over $\mathcal{B}$ shows that $\mu > 0$.

In order to end the proof of the lemma, let us now prove that

$$\mu = \inf_{u \in \mathcal{H}_p} E_\alpha(u)$$

and that $\varphi_\alpha \in \mathcal{H}_p$, so that φ_α realizes the infimum of E_α on $\mathcal{H}_p$. Multiplying by φ_α the equation satisfied by φ_α, and integrating the result over $\mathcal{B}$, one gets that

$$\begin{aligned}\mu \int_{\mathcal{B}} \varphi_\alpha^p \, dv(g_\alpha) &= \int_{\mathcal{B}} \left(|\nabla \varphi_\alpha|^2 + \alpha \varphi_\alpha^2\right) dv(g_\alpha) \\ &\leq \liminf_{q \to p} \int_{\mathcal{B}} \left(|\nabla \varphi_q|^2 + \alpha \varphi_q^2\right) dv(g_\alpha) = \liminf_{q \to p} \mu_q\end{aligned}$$

Hence,

$$\int_{\mathcal{B}} \varphi_\alpha^p \, dv(g_\alpha) \leq 1$$

Let $\psi = \|\varphi_\alpha\|_p^{-1}\varphi_\alpha$. Then $\psi \in \mathcal{H}_p$, and

$$\mu \le E_\alpha(\psi) = \mu\left(\int_{\mathcal{B}} \varphi_\alpha^p \, dv(g_\alpha)\right)^{1-\frac{2}{p}}$$

As a consequence,

$$\int_{\mathcal{B}} \varphi_\alpha^p \, dv(g_\alpha) \ge 1$$

so that $\varphi_\alpha \in \mathcal{H}_p$ and μ is the infimum of E_α over $\mathcal{H}_p$. Setting

$$\lambda_\alpha = \inf_{u\in\mathcal{H}_p} E_\alpha(u)$$

this ends the proof of the lemma. □

Since the proof of Theorem 7.2 is by contradiction, we assume the existence of g_α and φ_α, as in Lemma 7.3. We start with the study of some of the basic properties satisfied by the φ_α's. In what follows, we consider a sequence of real numbers α which tends to $+\infty$, and we successively pass to subsequences. As a first result, we prove the following:

LEMMA 7.4 *Up to a subsequence,*

(i) $\lim_{\alpha\to+\infty} \varphi_\alpha = 0$ *a.e.*,
(ii) $\lim_{\alpha\to+\infty} \int_{\mathcal{B}} \varphi_\alpha^2 \, dv(g_\alpha) = 0$, *and*
(iii) $\lim_{\alpha\to+\infty} \lambda_\alpha = \frac{1}{K(n,2)^2}$.

where λ_α and φ_α are as in Lemma 7.3.

PROOF: As a starting point, note that

$$\alpha\int_{\mathcal{B}} \varphi_\alpha^2 \, dv(g_\alpha) \le \int_{\mathcal{B}} |\nabla\varphi_\alpha|^2 \, dv(g_\alpha) + \alpha\int_{\mathcal{B}} \varphi_\alpha^2 \, dv(g_\alpha) = \lambda_\alpha \le \frac{1}{K(n,2)^2}$$

Hence,

$$\lim_{\alpha\to+\infty} \int_{\mathcal{B}} \varphi_\alpha^2 \, dv(g_\alpha) = 0$$

Moreover, since g_α satisfies ($\star\star$), one gets from Theorem 1.3 that for any $x \in \mathcal{B}$

$$\frac{1}{4}e \le g_\alpha(x) \le 4e$$

where e stands for the Euclidean metric of $\mathbb{R}^n$, and the inequality has to be understood in the sense of bilinear forms. As a consequence,

$$\int_{\mathcal{B}} \varphi_\alpha^2 \, dv(g_\alpha) \ge \frac{1}{2^n}\int_{\mathcal{B}} \varphi_\alpha^2 \, dx \quad \text{and} \quad \lim_{\alpha\to+\infty}\int_{\mathcal{B}} \varphi_\alpha^2 \, dx = 0$$

After passing to a subsequence, one then gets that

$$\lim_{\alpha\to+\infty} \varphi_\alpha = 0 \text{ a.e.}$$

Suppose now that there exists some subsequence (λ_α) of (λ_α) such that

$$\lim_{\alpha\to+\infty} \lambda_\alpha = \lambda < \frac{1}{K(n,2)^2}$$

Let $\varepsilon > 0$ be such that

$$(1+\varepsilon)\lambda < \frac{1}{K(n,2)^2}$$

In the spirit of Theorem 7.1 (see Hebey [**107**] for more details), there exists a positive constant B_ε, independent of α, depending only on ε, n, and Λ_1, such that for any $u \in H_{0,1}^2(\mathcal{B})$,

$$\left(\int_{\mathcal{B}} |u|^p \, dv(g_\alpha)\right)^{2/p} \le (1+\varepsilon)K(n,2)^2\left(\int_{\mathcal{B}} |\nabla u|^2 \, dv(g_\alpha) + B_\varepsilon \int_{\mathcal{B}} u^2 \, dv(g_\alpha)\right)$$

From such an inequality, one gets that

$$\begin{aligned}\lambda_\alpha &= \int_{\mathcal{B}} |\nabla\varphi_\alpha|^2 \, dv(g_\alpha) + \alpha \int_{\mathcal{B}} \varphi_\alpha^2 \, dv(g_\alpha)\\ &\ge \frac{1}{(1+\varepsilon)K(n,2)^2} + (\alpha - B_\varepsilon)\int_{\mathcal{B}} \varphi_\alpha^2 \, dv(g_\alpha)\end{aligned}$$

since

$$\int_{\mathcal{B}} \varphi_\alpha^p \, dv(g_\alpha) = 1$$

As a consequence,

$$\left(\frac{1}{K(n,2)^2} - (1+\varepsilon)\lambda_\alpha\right) + (1+\varepsilon)(\alpha - B_\varepsilon)\int_{\mathcal{B}} \varphi_\alpha^2 \, dv(g_\alpha) \le 0$$

Noting that such an inequality is obviously false for $\alpha \gg 1$, one gets that

$$\lim_{\alpha\to+\infty} \lambda_\alpha = \frac{1}{K(n,2)^2}$$

This ends the proof of the lemma. □

A slight improvement of point (ii) of Lemma 7.4 is the following:

LEMMA 7.5 $\displaystyle\lim_{\alpha\to+\infty} \alpha \int_{\mathcal{B}} \varphi_\alpha^2 \, dv(g_\alpha) = 0$

PROOF: Let $\varepsilon > 0$. In the spirit of Theorem 7.1 (see Hebey [**107**] for more details), there exists $B_\varepsilon > 0$, depending only on ε, n, and Λ_1, such that for any $u \in H_{0,1}^2(\mathcal{B})$,

$$\left(\int_{\mathcal{B}} |u|^p \, dv(g_\alpha)\right)^{2/p} \le \left(K(n,2)^2 + \varepsilon\right)\int_{\mathcal{B}} |\nabla u|^2 \, dv(g_\alpha) + B_\varepsilon \int_{\mathcal{B}} u^2 \, dv(g_\alpha)$$

By (7.4) and (7.5),

$$\int_{\mathcal{B}} |\nabla\varphi_\alpha|^2 \, dv(g_\alpha) + \alpha \int_{\mathcal{B}} \varphi_\alpha^2 \, dv(g_\alpha) = \lambda_\alpha\left(\int_{\mathcal{B}} \varphi_\alpha^p \, dv(g_\alpha)\right)^{2/p}$$

Hence,

$$\begin{aligned}&\int_{\mathcal{B}} |\nabla\varphi_\alpha|^2 \, dv(g_\alpha) + \alpha \int_{\mathcal{B}} \varphi_\alpha^2 \, dv(g_\alpha) \le\\ &\qquad \lambda_\alpha\left(K(n,2)^2 + \varepsilon\right)\int_{\mathcal{B}} |\nabla\varphi_\alpha|^2 \, dv(g_\alpha) + \lambda_\alpha B_\varepsilon \int_{\mathcal{B}} \varphi_\alpha^2 \, dv(g_\alpha)\end{aligned}$$

Noting that

$$\lambda_\alpha < \frac{1}{K(n,2)^2} \quad \text{and} \quad \int_{\mathcal{B}} |\nabla \varphi_\alpha|^2 \, dv(g_\alpha) \leq \lambda_\alpha$$

one gets that

$$\alpha \int_{\mathcal{B}} \varphi_\alpha^2 \, dv(g_\alpha) \leq \frac{\varepsilon}{K(n,2)^4} + \frac{B_\varepsilon}{K(n,2)^2} \int_{\mathcal{B}} \varphi_\alpha^2 \, dv(g_\alpha)$$

By Lemma 7.4, point (ii), this leads to

$$\limsup_{\alpha \to +\infty} \alpha \int_{\mathcal{B}} \varphi_\alpha^2 \, dv(g_\alpha) \leq \frac{\varepsilon}{K(n,2)^4}$$

Since $\varepsilon > 0$ is arbitrary, one gets the lemma. □

As a definition, let us say that $x \in \overline{\mathcal{B}}$ is a *concentration point* for (φ_α) if the following holds: For any $\delta > 0$,

$$\limsup_{\alpha \to +\infty} \int_{V_x(\delta)} \varphi_\alpha^p \, dv(g_\alpha) > 0$$

where $V_x(\delta) = B_x(\delta) \cap \mathcal{B}$, $B_x(\delta)$ the Euclidean ball of center x and radius δ.

LEMMA 7.6 *If x is a concentration point for (φ_α), then, for any $\delta > 0$,*

$$\limsup_{\alpha \to +\infty} \int_{V_x(\delta)} \varphi_\alpha^p \, dv(g_\alpha) = 1$$

where φ_α is as in Lemma 7.3, and $V_x(\delta)$ is as above.

PROOF: Let $x \in \overline{\mathcal{B}}$ and let $\eta \in \mathcal{D}(\mathbb{R}^n)$ be such that $0 \leq \eta \leq 1$ and $\eta = 1$ in $B_x(\delta/2)$, where $B_x(\delta/2)$ stands for the Euclidean ball of center x and radius $\delta/2$, $\delta > 0$ small. Let also $k \geq 1$ real. As one can easily check, multiplying (7.4) by $\eta^2 \varphi_\alpha^k$ and integrating by parts lead to

$$\begin{aligned}
&\frac{4k}{(k+1)^2} \int_{\mathcal{B}} \left|\nabla\left(\eta \varphi_\alpha^{(k+1)/2}\right)\right|^2 dv(g_\alpha) - \frac{2(k-1)}{(k+1)^2} \int_{\mathcal{B}} \eta (\Delta_{g_\alpha} \eta) \varphi_\alpha^{k+1} \, dv(g_\alpha) \\
&\quad - \frac{2}{k+1} \int_{\mathcal{B}} |\nabla \eta|^2 \varphi_\alpha^{k+1} \, dv(g_\alpha) + \alpha \int_{\mathcal{B}} \eta^2 \varphi_\alpha^{k+1} \, dv(g_\alpha) \\
&\quad = \lambda_\alpha \int_{\mathcal{B}} \eta^2 \varphi_\alpha^{k+p-1} \, dv(g_\alpha)
\end{aligned} \tag{7.6}$$

Since g_α satisfies $(\star\star)$, one gets by Theorem 1.3 that there exists $C > 0$, independent of α, such that $|\nabla \eta| \leq C$ and $|\Delta_{g_\alpha} \eta| \leq C$ for all α. For $\varepsilon > 0$, let $B_\varepsilon > 0$, independent of α, be such that for any $u \in H_{0,1}^2(\mathcal{B})$,

$$\left(\int_{\mathcal{B}} |u|^p \, dv(g_\alpha)\right)^{2/p} \leq \left(K(n,2)^2 + \varepsilon\right) \int_{\mathcal{B}} |\nabla u|^2 \, dv(g_\alpha) + B_\varepsilon \int_{\mathcal{B}} u^2 \, dv(g_\alpha)$$

As above, we refer to Hebey [107] for details on the proof of the existence of such a B_ε. By Hölder's inequalities,

$$\begin{aligned}&\int_{\mathcal{B}} \eta^2 \varphi_\alpha^{k+p-1}\, dv(g_\alpha)\\ &\quad = \int_{\mathcal{B}} (\eta^2 \varphi_\alpha^{k+1}) \varphi_\alpha^{p-2}\, dv(g_\alpha)\\ &\quad \le \left(\int_{\mathcal{B}} (\eta \varphi_\alpha^{(k+1)/2})^p\, dv(g_\alpha) \right)^{2/p} \left(\int_{\mathcal{B}} \varphi_\alpha^p\, dv(g_\alpha) \right)^{(p-2)/p}\end{aligned}$$

Combining (7.6), the fact that $|\nabla \eta|$ and $|\Delta_{g_\alpha} \eta|$ are uniformly bounded, and the above Sobolev inequality, we get the following: $\forall \varepsilon > 0$, $\exists C_1 > 0$, independent of α, such that $\forall \alpha$, $\forall \delta \ll 1$, $\forall k \in [1, p-1]$,

$$\begin{aligned}&\int_{\mathcal{B}} |\nabla(\eta \varphi_\alpha^{(k+1)/2})|^2\, dv(g_\alpha)\\ &\quad \le C_1 + \frac{(k+1)^2}{4k} \lambda_\alpha \left(K(n,2)^2 + \varepsilon \right) \left(\int_{V_\mathfrak{r}(\delta)} \varphi_\alpha^p\, dv(g_\alpha) \right)^{(p-2)/p}\\ &\qquad \times \int_{\mathcal{B}} |\nabla(\eta \varphi_\alpha^{(k+1)/2})|^2\, dv(g_\alpha)\end{aligned}$$

Since by Lemma 7.4,

$$\lim_{\alpha \to +\infty} \lambda_\alpha = \frac{1}{K(n,2)^2}$$

we get that: $\forall \varepsilon > 0$, $\exists C_1 > 0$, independent of α, such that $\forall \alpha \gg 1$, $\forall \delta \ll 1$, $\forall k \in [1, p-1]$,

$$\begin{aligned}&\int_{\mathcal{B}} |\nabla(\eta \varphi_\alpha^{(k+1)/2})|^2\, dv(g_\alpha)\\ &\quad \le C_1 + \frac{(k+1)^2}{4k} (1+\varepsilon) \left(\int_{V_\mathfrak{r}(\delta)} \varphi_\alpha^p\, dv(g_\alpha) \right)^{(p-2)/p}\\ &\qquad \times \int_{\mathcal{B}} |\nabla(\eta \varphi_\alpha^{(k+1)/2})|^2\, dv(g_\alpha)\end{aligned} \tag{7.7}$$

Let us now suppose that for some $\delta_0 > 0$,

$$\limsup_{\alpha \to +\infty} \int_{V_\mathfrak{r}(\delta_0)} \varphi_\alpha^p\, dv(g_\alpha) < 1$$

By (7.7), and up to the extraction of a subsequence, we get that for $k > 1$ sufficiently close to 1, and for $\delta \ll 1$,

$$\int_{\mathcal{B}} |\nabla(\eta \varphi_\alpha^{(k+1)/2})|^2\, dv(g_\alpha) \le C_2 \int_{\mathcal{B}} |\nabla(\eta \varphi_\alpha^{(k+1)/2})|^2\, dv(g_\alpha) + C_3$$

where $C_2 \in (0,1)$ and $C_3 > 0$ are independent of α. Hence,

$$\int_{\mathcal{B}} |\nabla(\eta \varphi_\alpha^{(k+1)/2})|^2\, dv(g_\alpha) \le C_4$$

where $C_4 > 0$ is independent of α. For $\delta \ll 1$, set

$$A = \limsup_{\alpha \to +\infty} \int_{V_x(\delta/2)} \varphi_\alpha^p \, dv(g_\alpha)$$

By definition, if x is a concentration point for (φ_α), then $A > 0$. Independently, by Hölder's inequalities,

$$\begin{aligned}\int_{V_x(\delta/2)} \varphi_\alpha^p \, dv(g_\alpha) &\leq \left(\int_{V_x(\delta/2)} \varphi_\alpha^{n(k+1)/(n-2)} \, dv(g_\alpha) \right)^{(n+2)/n(k+1)} \\ &\quad \times \left(\int_{V_x(\delta/2)} \varphi_\alpha^{n(k+1)/(nk-2)} \, dv(g_\alpha) \right)^{(nk-2)/n(k+1)} \\ &\leq C_5 \left(\int_{V_x(\delta/2)} \varphi_\alpha^{n(k+1)/(nk-2)} \, dv(g_\alpha) \right)^{(nk-2)/n(k+1)}\end{aligned}$$

since

$$\int_{\mathcal{B}} |\nabla(\eta \varphi_\alpha^{(k+1)/2})|^2 \, dv(g_\alpha) \leq C_4$$

and where $C_5 > 0$ is independent of α. Hence, if we set

$$k_1 = \frac{n(k+1)}{nk-2}$$

we get that $k_1 \in (1, p)$ and that, up to a subsequence,

$$\int_{\mathcal{B}} \varphi_\alpha^{k_1} \, dv(g_\alpha) \geq C_6$$

where $C_6 > 0$ is independent of α. We claim now that this is in contradiction with Lemma 7.4. Indeed, for $k > 1$ sufficiently close to 1, $k_1 > 2$, and by Hölder's inequality,

$$\int_{\mathcal{B}} \varphi_\alpha^{k_1} \, dv(g_\alpha) \leq \left(\int_{\mathcal{B}} \varphi_\alpha^2 \, dv(g_\alpha) \right)^{k_1\theta/2} \left(\int_{\mathcal{B}} \varphi_\alpha^p \, dv(g_\alpha) \right)^{k_1(1-\theta)/p}$$

where

$$\theta = \frac{\frac{1}{k_1} - \frac{1}{p}}{\frac{1}{2} - \frac{1}{p}}$$

Since

$$\int_{\mathcal{B}} \varphi_\alpha^p \, dv(g_\alpha) = 1$$

and since by Lemma 7.4

$$\lim_{\alpha \to +\infty} \int_{\mathcal{B}} \varphi_\alpha^2 \, dv(g_\alpha) = 0$$

this proves the claim. As a consequence, if x is a concentration point for (φ_α), then for any $\delta > 0$,

$$\limsup_{\alpha \to +\infty} \int_{V_x(\delta)} \varphi_\alpha^p \, dv(g_\alpha) = 1$$

This ends the proof of the lemma. □

Lemma 7.6 leads to the following:

LEMMA 7.7 *Up to a subsequence, (φ_α) has one and only one concentration point. Moreover, $\|\varphi_\alpha\|_{L^\infty(\mathcal{B})} \to +\infty$ as $\alpha \to +\infty$.*

PROOF: Since

$$\int_{\mathcal{B}} \varphi_\alpha^p \, dv(g_\alpha) = 1$$

it is clear that (φ_α) has at least one concentration point. Let x be such a point. By Lemma 7.6, for any $q \in \mathbb{N}^*$ there exists α such that

$$1 - \frac{1}{q} \le \int_{V_x(1/q)} \varphi_\alpha^p \, dv(g_\alpha) \le 1$$

The subsequence $(\varphi_\alpha) = (\varphi_{\alpha_q})$ we just defined then satisfies that for any $\delta > 0$,

$$\lim_{\alpha \to +\infty} \int_{V_x(\delta)} \varphi_\alpha^p \, dv(g_\alpha) = 1$$

As one can easily check, this implies that (φ_α) has one and only one concentration point. Independently, and since by Theorem 1.3,

$$\frac{1}{4} e \le g_\alpha \le 4e$$

where e is the Euclidean metric, one has that

$$\int_{V_x(\delta)} \varphi_\alpha^p \, dv(g_\alpha) \le \frac{2^n \omega_{n-1}}{n} \delta^n \|\varphi_\alpha\|_{L^\infty(\mathcal{B})}^p$$

Since for any $\delta > 0$,

$$\lim_{\alpha \to +\infty} \int_{V_x(\delta)} \varphi_\alpha^p \, dv(g_\alpha) = 1$$

one clearly gets by passing to the limit, and by taking δ smaller and smaller, that

$$\lim_{\alpha \to +\infty} \|\varphi_\alpha\|_{L^\infty(\mathcal{B})} = +\infty$$

This ends the proof of the lemma. □

Going on with the study of the behavior of the φ_α's with respect to the notion of concentration point, one gets the following:

LEMMA 7.8 *Let x be the concentration point of (φ_α) given by Lemma 7.7. As $\alpha \to +\infty$, $\varphi_\alpha \to 0$ in $C^1_{\mathrm{loc}}(\overline{\mathcal{B}} \backslash \{x\})$.*

PROOF: Let $y \in \overline{\mathcal{B}}$, $y \neq x$. Since y is not a concentration point for (φ_α), there exists some $0 < \delta \ll 1$ such that

$$\limsup_{\alpha \to +\infty} \int_{V_y(\delta)} \varphi_\alpha^p \, dv(g_\alpha) = 0$$

Independently (see (7.6) and (7.7)), for $\alpha \gg 1$

$$\int_{\mathcal{B}} |\nabla(\eta \varphi_\alpha^{(k+1)/2})|^2 \, dv(g_\alpha) \le C_1 \int_{\mathcal{B}} |\nabla(\eta \varphi_\alpha^{(k+1)/2})|^2 \, dv(g_\alpha) + C_2 \tag{7.8}$$

where

$$C_1 = \frac{(k+1)^2}{4k}(1+\varepsilon)\left(\int_{V_y(\delta)} \varphi_\alpha^p \, dv(g_\alpha)\right)^{(p-2)/p}$$

and where

$$C_2 \le \frac{k-1}{2k}\int_{\mathcal{B}} \eta |\Delta_{g_\alpha}\eta| \varphi_\alpha^{k+1} \, dv(g_\alpha) + \frac{k+1}{2k}\int_{\mathcal{B}} |\nabla\eta|^2 \varphi_\alpha^{k+1} \, dv(g_\alpha)$$

Let e be the Euclidean metric. Since g_α satisfies $(\star\star)$,

$$\frac{1}{4}e \le g_\alpha \le 4e$$

for all α. Hence, there exists a constant $C > 0$, independent of α, such that $|\nabla\eta| \le C$ and $|\Delta_{g_\alpha}\eta| \le C$ for all α. As a consequence, we get that for all $k \in [1, p-1]$,

(i) $\lim_{\alpha\to+\infty} \int_{\mathcal{B}} |\nabla(\eta\varphi_\alpha^{(k+1)/2})|^2 \, dv(g_\alpha) = 0$ and
(ii) $\lim_{\alpha\to+\infty} \int_{V_y(\delta/2)} \varphi_\alpha^{p(k+1)/2} \, dv(g_\alpha) = 0$.

According to (ii), we can now use (7.8) with

$$k = \frac{p^2}{2} - 1$$

Therefore,

$$\lim_{\alpha\to+\infty} \int_{V_y(\delta')} \varphi_\alpha^{p^3/4} \, dv(g_\alpha) = 0$$

for some $0 < \delta' \ll 1$. More generally, and by induction, we get that for any $y \neq x$, and any q, there exists $0 < \delta \ll 1$ such that

$$\lim_{\alpha\to+\infty} \int_{V_y(\delta)} \varphi_\alpha^q \, dv(g_\alpha) = 0$$

Hence, see Gilbarg-Trudinger [**91**], Theorem 8.25, since

$$\Delta_{g_\alpha}\varphi_\alpha \le \lambda_\alpha \varphi_\alpha^{p-1}$$

and since g_α satisfies $(\star\star)$, we obtain that for any relatively compact subset ω of $\overline{\mathcal{B}}\backslash\{x\}$,

$$\lim_{\alpha\to+\infty} \varphi_\alpha = 0 \quad \text{in } L^\infty(\omega)$$

The result then easily follows from Lemma 7.9 below and Gilbarg-Trudinger [**91**], theorem 8.32 and corollary 8.36. □

Let x be the concentration point of (φ_α). By $\omega \Subset \overline{\mathcal{B}}\backslash\{x\}$ we mean that ω is a relatively compact subset of $\overline{\mathcal{B}}\backslash\{x\}$. One then has the following:

LEMMA 7.9 *Let x be the concentration point of (φ_α). For any q and any $\omega \Subset \overline{\mathcal{B}}\backslash\{x\}$, $\alpha^q\|\varphi_\alpha\|_{L^\infty(\mathcal{B})} \to 0$ as $\alpha \to +\infty$. In particular, for any $\omega \Subset \overline{\mathcal{B}}\backslash\{x\}$, $\alpha\|\varphi_\alpha\|_{L^\infty(\mathcal{B})} \to 0$ as $\alpha \to +\infty$.*

PROOF: Let $y \in \overline{\mathcal{B}}$, $y \neq x$, and let $\delta > 0$ be such that $x \notin \overline{V_y(\delta)}$. By (7.6),

$$\alpha^{q+1} \int_{\mathcal{B}} \eta^2 \varphi_\alpha^{k+1}\, dv(g_\alpha) \leq C_1 \alpha^q \int_{V_y(\delta)} \varphi_\alpha^{k+1}\, dv(g_\alpha) + C_2 \alpha^q \int_{V_y(\delta)} \varphi_\alpha^{k+p}\, dv(g_\alpha)$$

where C_1 and C_2 can be chosen independent of α. Moreover, according to what we proved above, for any $\omega \Subset \overline{\mathcal{B}}\backslash\{x\}$ and any m,

$$\lim_{\alpha \to +\infty} \int_\omega \varphi_\alpha^m dv(g_\alpha) = 0$$

Hence, by induction on q, we get that for any $y \neq x$, any m, and any q, there exists $0 < \delta \ll 1$, such that

$$\lim_{\alpha \to +\infty} \alpha^q \int_{V_y(\delta)} \varphi_\alpha^m\, dv(g_\alpha) = 0$$

Finally (see Gilbarg-Trudinger [91], theorem 8.25), since

$$\Delta_{g_\alpha} \varphi_\alpha \leq \lambda_\alpha \varphi_\alpha^{p-1}$$

and since g_α satisfies ($\star\star$), we obtain that for any q and any $\omega \Subset \overline{\mathcal{B}}\backslash\{x\}$,

$$\lim_{\alpha \to +\infty} \alpha^q \|\varphi_\alpha\|_{L^\infty(\omega)} = 0$$

This ends the proof of the lemma. □

From now on, we set

$$u_\alpha = \left(\frac{\lambda_\alpha}{n(n-2)}\right)^{(n-2)/4} \varphi_\alpha$$

where λ_α and φ_α are as in Lemma 7.3. Here again, (u_α) concentrates at x, and, as one can easily check,

$$\Delta_{g_\alpha} u_\alpha + \alpha u_\alpha = n(n-2) u_\alpha^{p-1} \tag{7.9}$$

in $\mathcal{B}$. Moreover, one has that

$$\lim_{\alpha \to +\infty} \int_{\mathcal{B}} u_\alpha^p\, dv(g_\alpha) = \frac{\omega_n}{2} \tag{7.10}$$

$$\lim_{\alpha \to +\infty} \int_{\mathcal{B}} |\nabla u_\alpha|^2\, dv(g_\alpha) = \frac{n(n-2)\omega_n}{2^n} \tag{7.11}$$

Let x_α be some point of $\mathcal{B}$ such that

$$u_\alpha(x_\alpha) = \|u_\alpha\|_{L^\infty(\mathcal{B})}$$

and let $\mu_\alpha \in (0, +\infty)$ be such that

$$\|u_\alpha\|_{L^\infty(\mathcal{B})} = \mu_\alpha^{-(n-2)/2}$$

According to Lemma 7.7 and Lemma 7.8,

$$\lim_{\alpha \to +\infty} x_\alpha = x \quad \text{and} \quad \lim_{\alpha \to +\infty} \mu_\alpha = 0$$

As a first remark, one has the following:

LEMMA 7.10 *There exists $C > 0$, independent of α, such that for any α, $\alpha \mu_\alpha^2 \leq C$.*

PROOF: Since $\Delta_{g_\alpha} u_\alpha(x_\alpha) \geq 0$,

$$\alpha \mu_\alpha^{-(n-2)/2} \leq n(n-2)\mu_\alpha^{-(n+2)/2}$$

so that $\alpha\mu_\alpha^2 \leq n(n-2)$. This proves the lemma. □

An important step in the proof of Theorem 7.2 is now given by the following lemma:

LEMMA 7.11 *Up to a subsequence,*

$$\lim_{\alpha\to+\infty} \frac{d(x_\alpha, \partial\mathcal{B})}{\mu_\alpha} = +\infty$$

where d stands for the Euclidean distance, and x_α and μ_α are as above.

PROOF: Since $x_\alpha \to x$ as $\alpha \to +\infty$, the result is immediate if $x \notin \partial\mathcal{B}$. Without loss of generality, we can then assume that $x = (0, \ldots, 0, 1) \in \partial\mathcal{B}$. Since g_α satisfies $(\star\star)$, one gets by Theorem 1.3 that there exists a constant K, independent of α, such that for any α and any $i, j, k = 1, \ldots, n$,

$$\frac{1}{4}\delta_{ij} \leq g_{ij}^\alpha \leq 4\delta_{ij} \quad \text{in } B_0(2) \tag{7.12}$$

$$|g_{ij}^\alpha| \leq K\,, \ |\partial_k g_{ij}^\alpha| \leq K \quad \text{in } B_0(2) \tag{7.13}$$

where $B_0(2)$ stands for the Euclidean ball of center 0 and radius 2, where (7.12) has to be understood in the sense of bilinear forms, and where the g_{ij}^α's stand for the components of g_α in the canonical chart of $\mathbb{R}^n$. Hence, by Ascoli, there exists a $C^{0,1/2}$ Riemannian metric g in $B_0(3/2)$ such that, after passing to a subsequence,

$$\lim_{\alpha\to+\infty} g_\alpha = g \quad \text{in } C^{0,1/2}\big(B_0(3/2)\big)\,,$$

for any α, g_α satisfies (7.12) and (7.13),

$$\frac{1}{4}\delta_{ij} \leq g_{ij} \leq 4\delta_{ij} \quad \text{in } B_0(3/2)\,.$$

From now on, let $\sigma_\alpha \in O(n)$ be such that $\sigma_\alpha(x_\alpha) = x_\alpha^R \in [0, x]$, where $[0, x]$ stands for the segment $[0, x] = \{tx, 0 \leq t \leq 1\}$. We still denote by g_α the metric $(\sigma_\alpha^{-1})^\star g_\alpha$, and by u_α the function $u_\alpha \circ \sigma_\alpha^{-1}$. Since $x_\alpha \to x$ as $\alpha \to +\infty$, we get that $\sigma_\alpha \to Id$ as $\alpha \to +\infty$. Hence, we still have

(7.14) for any α, g_α satisfies (7.12) and (7.13)

$$\lim_{\alpha\to+\infty} g_\alpha = g \quad \text{in } C^{0,1/2}\big(B_0(3/2)\big) \tag{7.15}$$

In addition, according to (7.9), (7.10), and (7.11), the following holds:

(7.16) for any α, $u_\alpha \in C^2(\overline{\mathcal{B}}) \cap H_{0,1}^2(\mathcal{B})$ and $u_\alpha > 0$ in $\mathcal{B}$

(7.17) for any α, $\Delta_{g_\alpha} u_\alpha + \alpha u_\alpha = n(n-2)u_\alpha^{p-1}$ in $\mathcal{B}$

$$\lim_{\alpha\to+\infty} \int_{\mathcal{B}} u_\alpha^p \, dv(g_\alpha) = \frac{\omega_n}{2^n} \tag{7.18}$$

$$\lim_{\alpha\to+\infty} \int_{\mathcal{B}} |\nabla u_\alpha|^2 \, dv(g_\alpha) = \frac{n(n-2)\omega_n}{2^n} \tag{7.19}$$

$$\text{for any } \alpha,\ u_\alpha(x_\alpha^R) = \mu_\alpha^{-(n-2)/2} = \|u_\alpha\|_{L^\infty(\mathcal{B})} \tag{7.20}$$

Since $d(x_\alpha, \partial\mathcal{B}) = d(x_\alpha^R, x)$, we have to prove that

$$\lim_{\alpha\to+\infty} \frac{d(x_\alpha^R, x)}{\mu_\alpha} = +\infty$$

Let v_α be defined in $\mathcal{B}_\alpha = B_{-x/\mu_\alpha}(1/\mu_\alpha)$ by

$$v_\alpha(y) = \mu_\alpha^{(n-2)/2} u_\alpha(\mu_\alpha y + x)$$

We have $0 \le v_\alpha \le 1$ and $\bigcup \mathcal{B}_\alpha = E$ where

$$E = \left\{(x_1, \dots, x_n) \in \mathbb{R}^n \,/\, x_n < 0\right\}$$

In addition, $0 \in \overline{\mathcal{B}_\alpha}$ for all α, and if we assume that (μ_α) is decreasing, we get that $\mathcal{B}_\alpha \subset \mathcal{B}_{\alpha'}$ as soon as $\alpha < \alpha'$. Let us also define the metric h_α by

$$h_\alpha(y) = g_\alpha(\mu_\alpha y + x)$$

for $y \in \mathcal{B}_\alpha$. By (7.14), we get that for any α

$$\frac{1}{4}\delta_{ij} \le h_{ij}^\alpha \le 4\delta_{ij} \quad \text{in } \mathcal{B}_\alpha \tag{7.21}$$

in the sense of bilinear forms, and we get that

(7.22) there exists a constant K independent of α, such that for any α and any $i, j, k = 1, \dots, n$, $|h_{ij}^\alpha| \le K$ and $|\partial_k h_{ij}^\alpha| \le K\mu_\alpha$ in $\mathcal{B}_\alpha$

Combining (7.15) with (7.22) leads to the fact that for any $\omega \Subset E$,

$$\lim_{\alpha\to+\infty} h_\alpha = g(x) \quad \text{in } C^1(\omega)$$

Moreover, since

$$\Delta_{g_\alpha} u_\alpha + \alpha u_\alpha = n(n-2)u_\alpha^{p-1}$$

in $\mathcal{B}$, we get that for any α,

$$\Delta_{h_\alpha} v_\alpha + (\alpha\mu_\alpha^2)v_\alpha = n(n-2)v_\alpha^{p-1} \tag{7.23}$$

in $\mathcal{B}_\alpha$. Note here that (7.23) may also be written in the following form:

$$-D_i\left(h_\alpha^{ij}\sqrt{|h_\alpha|}D_j v_\alpha\right) + \left(\alpha\mu_\alpha^2\right)v_\alpha = n(n-2)v_\alpha^{p-1} \tag{7.24}$$

where $|h_\alpha|$ stands for the determinant of the matrix (h_{ij}^α), and where $(h_\alpha^{ij}) = (h_{ij}^\alpha)^{-1}$. Let us now set

$$y_\alpha = \frac{x_\alpha^R - x}{\mu_\alpha}$$

For any α, $y_\alpha \in \mathcal{B}_\alpha$ and $v_\alpha(y_\alpha) = 1$. As a starting point, we claim that

$$\liminf_{\alpha\to+\infty} \frac{d(x_\alpha^R, x)}{\mu_\alpha} > 0$$

The proof here is by contradiction. Suppose that, after passing to a subsequence,

$$\liminf_{\alpha\to+\infty} \frac{d(x_\alpha^R, x)}{\mu_\alpha} = 0$$

Then $y_\alpha \to 0$ as $\alpha \to +\infty$. Moreover, by a slight modification of corollary 8.36 of Gilbarg-Trudinger [**91**] that can be found in Hebey-Vaugon [**119**], by (7.21), (7.22), and (7.23), and since $0 \le v_\alpha \le 1$, we get that the v_α's are C^1 bounded in a neighborhood of 0. Hence, by the mean value theorem,

$$1 = |v_\alpha(y_\alpha) - v_\alpha(0)| \le Cd(0, y_\alpha)$$

which is impossible. This proves the above claim. Let us now prove that

$$\lim_{\alpha\to+\infty} \frac{d(x_\alpha^R, x)}{\mu_\alpha} = +\infty$$

Here again the proof is by contradiction. We assume that, after passing to a subsequence,

$$\lim_{\alpha\to+\infty} \frac{d(x_\alpha^R, x)}{\mu_\alpha} = A$$

where $A > 0$ is real. Up to the extraction of another subsequence, we may also assume that

$$\lim_{\alpha\to+\infty} y_\alpha = y_0$$

for some $y_0 \in E$. By Gilbarg-Trudinger [**91**], theorem 8.32, by (7.21), (7.22), and (7.24), and since $0 \le v_\alpha \le 1$, the v_α's are equicontinuous in any compact subset of E. Hence, by Ascoli, there exists $v \in C^0(E)$ such that for any $\omega \Subset E$, some subsequence of (v_α) converges to v in $L^\infty(\omega)$. In particular, $0 \le v \le 1$, $v \not\equiv 0$, and $v(y) = 1$. Now, note that for any $\omega \Subset E$, and any $\alpha \gg 1$,

$$\alpha \int_{\mathcal{B}} u_\alpha^2\, dv(g_\alpha) = \alpha\mu_\alpha^2 \int_{\mathcal{B}_\alpha} v_\alpha^2\, dv(h_\alpha) \ge \alpha\mu_\alpha^2 \int_\omega v_\alpha^2\, dv(h_\alpha)$$

Hence, since h_α satisfies (7.21), since $v \not\equiv 0$, and since

$$\int_{\mathcal{B}} u_\alpha^2\, dv(g_\alpha) = \left(\frac{\lambda_\alpha}{n(n-2)}\right)^{(n-2)/2} \int_{\mathcal{B}} \varphi_\alpha^2\, dv(g_\alpha)$$

we get from Lemma 7.5 that

$$\lim_{\alpha\to+\infty} \alpha\mu_\alpha^2 = 0$$

As a consequence, by (7.24), and since

$$\lim_{\alpha\to+\infty} h_\alpha = g(x) \quad \text{in } C^1_{\mathrm{loc}}(E)$$

we get that

$$\Delta_{g(x)} v = n(n-2) v^{p-1} \tag{7.25}$$

in E. Now, note that there exists $\sigma \in Gl(n)$ such that $\sigma(E) = E$ and $\sigma^\star g(x) = e$, where e is the Euclidean metric of $\mathbb{R}^n$. For convenience, we still denote by v the function $v \circ \sigma$. Then, we get from (7.25) that

$$\Delta_e v = n(n-2) v^{p-1} \tag{7.26}$$

in E. Let

$$w(x) = \left(\frac{1}{1 + |x|^2}\right)^{(n-2)/2}$$

where $|x| = d(0, x)$. We set

$$\begin{cases} h = w^{4/(n-2)}e \\ \tilde{v} = \frac{v}{w} \\ \tilde{v}_\alpha = \frac{v_\alpha}{w} \end{cases}$$

Since $\mathrm{Scal}_h = 4n(n-1)$, where Scal_h is the scalar curvature of h, we get from (7.26) that

$$\Delta_h \tilde{v} + n(n-2)\tilde{v} = n(n-2)\tilde{v}^{p-1} \tag{7.27}$$

in E. Moreover, by (7.18), (7.21), and (7.23),

$$\begin{aligned} \int_{\mathcal{B}_\alpha} |\nabla \tilde{v}_\alpha|^2\, dv(h) + n(n-2) \int_{\mathcal{B}_\alpha} \tilde{v}_\alpha^2\, dv(h) &= \int_{\mathcal{B}_\alpha} |\nabla v_\alpha|^2\, dx \\ &\le C_1 \int_{\mathcal{B}_\alpha} |\nabla v_\alpha|^2\, dv(h_\alpha) \\ &\le C_2 \int_{\mathcal{B}_\alpha} v_\alpha^p\, dv(h_\alpha) \\ &= C_2 \int_{\mathcal{B}} u_\alpha^p\, dv(g_\alpha) \le C_3 \end{aligned}$$

where C_1, C_2, and C_3 are positive constants independent of α. As a consequence, the sequence $(\tilde{v}_\alpha)$ is bounded in $H_{0,1}^2(E)$, the Sobolev space $H_{0,1}^2(E)$ being considered with respect to h. On the other hand, for any $\omega \Subset E$, there exists a subsequence $(\tilde{v}_\alpha)$ of $(\tilde{v}_\alpha)$ such that

$$\lim_{\alpha \to +\infty} \tilde{v}_\alpha = \tilde{v} \quad \text{in } L^\infty(\omega)$$

Hence, since any bounded sequence in a Hilbert space possesses a subsequence which converges weakly, we get that $\tilde{v} \in H_{0,1}^2(E)$. In what follows, let S^n be the unit sphere of $\mathbb{R}^{n+1}$, and, with a slight modification of the notation we have adopted until now, let c be the canonical metric of S^n induced from the Euclidean metric e of $\mathbb{R}^{n+1}$. By stereographic projection, E becomes a half-sphere S^+ of S^n. By (7.27) we then get a positive solution $v_1 \in H_{0,1}^2(S^+)$ of

$$\Delta_c v_1 + \frac{n(n-2)}{4} v_1 = \frac{n(n-2)}{4} v_1^{p-1}$$

in S^+. Let P be the pole of S^+ and let Φ be the stereographic projection of pole $-P$. As one can easily check, $v_2 = v_1 \circ \Phi^{-1}$ satisfies

$$\begin{cases} v_2 \in H_{0,1}^2(\mathcal{B}) \quad \text{and} \quad v_2 > 0 \quad \text{in } \mathcal{B} \\ \Delta_e v_2 = \frac{n(n-2)}{4} v_2^{p-1} \quad \text{in } \mathcal{B} \end{cases}$$

By Pohozaev [**167**] (see also Kazdan-Warner [**131**]), such a v_2 does not exist. As a consequence,

$$\lim_{\alpha \to +\infty} \frac{d(x_\alpha^R, x)}{\mu_\alpha} = +\infty$$

This ends the proof of the lemma. □

From now on, let $\exp_{x_\alpha}$ be the exponential map of g_α at x_α. We set $\Psi_\alpha = \exp_{x_\alpha}$ and

$$\begin{cases} \omega_\alpha = \Psi_\alpha^{-1}(\mathcal{B}) \\ \tilde{g}_\alpha = \Psi_\alpha^\star g_\alpha \\ \tilde{u}_\alpha = u_\alpha \circ \Psi_\alpha \end{cases}$$

Since g_α satisfies ($\star\star$), $\tilde{g}_\alpha$ is defined in an open neighborhood of $\overline{B_0(2)}$ where $B_0(2)$ stands for the Euclidean ball of center 0 and radius 2. Moreover, one has that for any α, $\overline{\omega_\alpha} \subset B_0(2)$. By Theorem 1.3 of Chapter 1, there exists a constant K, independent of α, such that for any α and any $i, j, k = 1, \dots, n$,

$$\frac{1}{4}\delta_{ij} \le \tilde{g}_{ij}^\alpha \le 4\delta_{ij} \quad \text{in } B_0(2) \tag{7.28}$$

$$|\tilde{g}_{ij}^\alpha| \le K \quad \text{and} \quad |\partial_k \tilde{g}_{ij}^\alpha| \le K \quad \text{in } \overline{B_0(2)} \tag{7.29}$$

where $B_0(2)$ stands for the Euclidean ball of center 0 and radius 2, where (7.28) has to be understood in the sense of bilinear forms, and where the $\tilde{g}_{ij}^\alpha$'s stand for the components of $\tilde{g}_\alpha$ in the canonical chart of $\mathbb{R}^n$. Along the same line of thinking, one easily gets that for any α

(7.30) ω_α is star shaped at 0

(7.31) for any $i, j, k = 1, \dots, n$, $\tilde{g}_{ij}^\alpha(0) = \delta_{ij}$, $\partial_k \tilde{g}_{ij}^\alpha(0) = 0$

(7.32) $\tilde{u}_\alpha \in C^2(\overline{\omega_\alpha}) \cap H_{0,1}^2(\omega_\alpha)$ and $\tilde{u}_\alpha > 0$ in ω_α

(7.33) $\Delta_{\tilde{g}_\alpha}\tilde{u}_\alpha + \alpha\tilde{u}_\alpha = n(n-2)\tilde{u}_\alpha^{p-1}$ in ω_α

(7.34) $\tilde{u}_\alpha(0) = \|\tilde{u}_\alpha\|_{L^\infty(\omega_\alpha)} = \mu_\alpha^{-(n-2)/2}$

and that

$$\lim_{\alpha \to +\infty} \int_{\omega_\alpha} \tilde{u}_\alpha^p \, dv(\tilde{g}_\alpha) = \frac{\omega_n}{2^n} \tag{7.35}$$

$$\lim_{\alpha \to +\infty} \int_{\omega_\alpha} |\nabla \tilde{u}_\alpha|^2 \, dv(\tilde{g}_\alpha) = \frac{n(n-2)\omega_n}{2^n} \tag{7.36}$$

Since $\tilde{g}_\alpha$ satisfies (7.29), one gets by Ascoli that there exists a $C^{0,1/2}$ Riemannian metric $\tilde{g}$ in $B_0(2)$ such that, up to the extraction of a subsequence,

$$\lim_{\alpha \to +\infty} \tilde{g}_\alpha = \tilde{g} \quad \text{in } C^{0,1/2}\big(B_0(2)\big) \tag{7.37}$$

In what follows, we assume that $\tilde{g}_\alpha$ satisfies (7.37). By (7.31), one then gets that $\tilde{g}_{ij}(0) = \delta_{ij}$. Let us now set

$$\begin{cases} \tilde{v}_\alpha(y) = \mu_\alpha^{(n-2)/2}\tilde{u}_\alpha(\mu_\alpha y) \\ h_\alpha(y) = \tilde{g}_\alpha(\mu_\alpha y) \end{cases}$$

for $y \in \Omega_\alpha$ where $\Omega_\alpha = \frac{1}{\mu_\alpha}\omega_\alpha$ is given by

$$\Omega_\alpha = \left\{ \frac{1}{\mu_\alpha}x, \; x \in \omega_\alpha \right\}$$

As one can easily check,

$$0 \le \tilde{v}_\alpha \le 1 \quad \text{and} \quad \tilde{v}_\alpha(0) = 1 \tag{7.38}$$

$$\Delta_{h_\alpha} \tilde{v}_\alpha + (\alpha \mu_\alpha^2)\tilde{v}_\alpha = n(n-2)\tilde{v}_\alpha^{p-1} \quad \text{in } \Omega_\alpha \tag{7.39}$$

Independently, since g_α and $\tilde{g}_\alpha$ satisfy (7.12) and (7.28), one gets by Lemma 7.11 that

$$\lim_{\alpha \to +\infty} d(0, \partial\Omega_\alpha) = +\infty$$

where d stands for the Euclidean distance. Hence, $\bigcup \Omega_\alpha = \mathbb{R}^n$. More precisely, for any $\omega \Subset \mathbb{R}^n$, there exists α_0 such that for any $\alpha > \alpha_0$, $\omega \subset \Omega_\alpha$. On the other hand, by (7.29), one gets that for any α and any $i, j, k = 1, \dots, n$,

$$|h_{ij}^\alpha| \le K \quad \text{and} \quad |\partial_k h_{ij}^\alpha| \le K\mu_\alpha \quad \text{in } \Omega_\alpha \tag{7.40}$$

As above, let e be the Euclidean metric of $\mathbb{R}^n$. Since $\tilde{g}(0) = e$, one gets by combining (7.37) and (7.40) that for any $\omega \Subset \mathbb{R}^n$,

$$\lim_{\alpha \to +\infty} h_\alpha = e \quad \text{in } C^1(\omega) \tag{7.41}$$

We start here by the study of some of the basic properties that the $\tilde{v}_\alpha$'s satisfy. On such a subject, first note that by (7.28), (7.38), (7.39), (7.41), and Gilbarg-Trudinger [**91**], theorem 8.32, the $\tilde{v}_\alpha$'s are equicontinuous on any compact subset of $\mathbb{R}^n$. Hence, by Ascoli, there exists $\tilde{v} \in C^0(\mathbb{R}^n)$ such that for any $\omega \Subset \mathbb{R}^n$, a subsequence of $(\tilde{v}_\alpha)$ converges to $\tilde{v}$ in $L^\infty(\omega)$. As a consequence, $0 \le \tilde{v} \le 1$, and $\tilde{v}(0) = 1$. In particular, $\tilde{v} \not\equiv 0$.

LEMMA 7.12 *Up to a subsequence, $\alpha\mu_\alpha^2 \to 0$ as $\alpha \to +\infty$.*

PROOF: For any $\omega \Subset \mathbb{R}^n$,

$$\alpha \int_{\mathcal{B}} u_\alpha^2 \, dv(g_\alpha) = \alpha \int_{\omega_\alpha} \tilde{u}_\alpha^2 \, dv(\tilde{g}_\alpha) = \alpha\mu_\alpha^2 \int_{\Omega_\alpha} \tilde{v}_\alpha^2 \, dv(h_\alpha) \ge \alpha\mu_\alpha^2 \int_\omega \tilde{v}_\alpha^2 \, dv(h_\alpha)$$

On the other hand,

$$u_\alpha = \left(\frac{\lambda_\alpha}{n(n-2)}\right)^{(n-2)/4} \varphi_\alpha$$

Since $\tilde{v} \not\equiv 0$, one easily gets by Lemma 7.5 that, up to the extraction of a subsequence,

$$\lim_{\alpha \to +\infty} \alpha\mu_\alpha^2 = 0$$

On such an assertion, recall that h_α satisfies (7.41). This ends the proof of the lemma. □

By (7.39), (7.41), and Lemma 7.12, we get that $\tilde{v} \in C^\infty(\mathbb{R}^n)$ and that

$$\Delta_e \tilde{v} = n(n-2)\tilde{v}^{p-1}$$

in $\mathbb{R}^n$, where Δ_e stands for the Euclidean Laplacian. Hence, according to Caffarelli-Gidas-Spruck [**36**] (see also Obata [**163**]), one has that

$$\tilde{v}(y) = \left(\frac{1}{1+|y|^2}\right)^{(n-2)/2}$$

where $|y|$ stands for the Euclidean distance from 0 to y. Let us now prove the following:

LEMMA 7.13 *For $\tilde{u}_\alpha$ as above,*

$$\lim_{\alpha\to+\infty}\int_{\omega_\alpha}|\nabla\tilde{u}_\alpha|^2\,dx=\frac{n(n-2)\omega_n}{2^n}$$

where dx stands for the Euclidean volume element, and the norm in the integral is with respect to the Euclidean metric.

PROOF: By Lemma 7.8,

$$\lim_{\alpha\to+\infty}u_\alpha=0\quad\text{in } C^1_{loc}(\overline{\mathcal{B}}\backslash\{0\})$$

Hence, for any $\delta>0$, $\delta\ll 1$,

$$\lim_{\alpha\to+\infty}\int_{\omega_\alpha}|\nabla\tilde{u}_\alpha|^2\,dx=\lim_{\alpha\to+\infty}\int_{B_0(\delta)}|\nabla\tilde{u}_\alpha|^2\,dx$$

where $B_0(\delta)$ stands for the Euclidean ball of center 0 and radius δ. Since $\tilde{g}(0)=e$, the Euclidean metric, we get by (7.37) that for any $\varepsilon>0$, there exist $\delta>0$ and $\alpha_0\gg 1$, such that for any $\alpha\geq\alpha_0$,

$$\|\tilde{g}_\alpha-e\|_{C^0(B_0(\delta))}<\varepsilon$$

Hence, for any $\varepsilon>0$, there exists $\delta>0$, $\delta\ll 1$, such that

$$(1-\varepsilon)\int_{B_0(\delta)}|\nabla\tilde{u}_\alpha|^2\,dv(\tilde{g}_\alpha)\leq\int_{B_0(\delta)}|\nabla\tilde{u}_\alpha|^2\,dx$$

and

$$\int_{B_0(\delta)}|\nabla\tilde{u}_\alpha|^2\,dx\leq(1+\varepsilon)\int_{B_0(\delta)}|\nabla\tilde{u}_\alpha|^2\,dv(\tilde{g}_\alpha)$$

Independently, by (7.36),

$$\lim_{\alpha\to+\infty}\int_{\omega_\alpha}|\nabla\tilde{u}_\alpha|^2\,dv(\tilde{g}_\alpha)=\frac{n(n-2)\omega_n^{2/n}}{2^n}$$

while by Lemma 7.8 and, for instance, (7.28), one gets that for any $\delta>0$ small,

$$\lim_{\alpha\to+\infty}\int_{\omega_\alpha}|\nabla\tilde{u}_\alpha|^2\,dv(\tilde{g}_\alpha)=\lim_{\alpha\to+\infty}\int_{B_0(\delta)}|\nabla\tilde{u}_\alpha|^2\,dv(\tilde{g}_\alpha)$$

This ends the proof of the lemma. □

Lemma 7.13 leads to the following:

LEMMA 7.14 *For $\tilde{v}_\alpha$ as above, with the convention that $\tilde{v}_\alpha=0$ outside Ω_α, and as $\alpha\to+\infty$, $\tilde{v}_\alpha\to\tilde{v}$ in $L^p(\mathbb{R}^n)$.*

PROOF: We prove that

$$\lim_{\alpha\to+\infty}\int_{R^n}|\nabla(\tilde{v}_\alpha-\tilde{v})|^2\,dx=0$$

Clearly, the lemma follows from such a result. As one can easily check,

$$\int_{R^n} |\nabla(\tilde{v}_\alpha - \tilde{v})|^2 dx = \int_{\Omega_\alpha} |\nabla \tilde{v}_\alpha|^2 dx + \int_{R^n} |\nabla \tilde{v}|^2 dx - 2\int_{\Omega_\alpha} (\nabla \tilde{v}_\alpha, \nabla \tilde{v}) dx$$

where $(\cdot, \cdot)$ stands for the Euclidean scalar product of $\mathbb{R}^n$. But

$$\int_{\Omega_\alpha} (\nabla \tilde{v}_\alpha, \nabla \tilde{v}) dx = \int_{\Omega_\alpha} \tilde{v}_\alpha \Delta_e \tilde{v}\, dx = n(n-2)\int_{\Omega_\alpha} \tilde{v}_\alpha \tilde{v}^{p-1} dx$$

and, since $0 \le \tilde{v}_\alpha \le 1$, we get that

$$\lim_{\alpha \to +\infty} \int_{\Omega_\alpha} \tilde{v}_\alpha \tilde{v}^{p-1} dx = \int_{R^n} \tilde{v}^p dx$$

Independently, by Lemma 7.13,

$$\lim_{\alpha \to +\infty} \int_{\Omega_\alpha} |\nabla \tilde{v}_\alpha|^2 dx = \lim_{\alpha \to +\infty} \int_{\omega_\alpha} |\nabla \tilde{u}_\alpha|^2 dx = \frac{n(n-2)\omega_n}{2^n}$$

Finally,

$$\int_{R^n} |\nabla \tilde{v}|^2 dx = n(n-2)\int_{R^n} \tilde{v}^p dx = \frac{n(n-2)\omega_n}{2^n}$$

Therefore,

$$\lim_{\alpha \to +\infty} \int_{R^n} |\nabla(\tilde{v}_\alpha - \tilde{v})|^2 dx = \frac{n(n-2)\omega_n}{2^n} + \frac{n(n-2)\omega_n}{2^n} - 2\frac{n(n-2)\omega_n}{2^n} = 0$$

This ends the proof of the lemma. □

From now on, let

$$\begin{cases} \tilde{h}_\alpha = \tilde{v}^{4/(n-2)} h_\alpha \\ \tilde{h} = \tilde{v}^{4/(n-2)} e \end{cases}$$

where e stands for the Euclidean metric of $\mathbb{R}^n$. Let also $\tilde{w}_\alpha = \tilde{v}_\alpha / \tilde{v}$. By (7.39), one has that

$$\frac{4(n-1)}{n-2}\Delta_{h_\alpha} \tilde{v}_\alpha + \frac{4(n-1)}{n-2}\alpha\mu_\alpha^2 \tilde{v}_\alpha = 4n(n-1)\tilde{v}_\alpha^{p-1}$$

and if S_{h_α} stands for the scalar curvature of h_α, we can write that

$$\frac{4(n-1)}{n-2}\Delta_{h_\alpha} \tilde{v}_\alpha + S_{h_\alpha}\tilde{v}_\alpha + \left(\frac{4(n-1)}{n-2}\alpha\mu_\alpha^2 - S_{h_\alpha}\right)\tilde{v}_\alpha = 4n(n-1)\tilde{v}_\alpha^{p-1}$$

On the other hand, we have

$$\frac{4(n-1)}{n-2}\Delta_{h_\alpha} \tilde{v}_\alpha + S_{h_\alpha}\tilde{v}_\alpha = \left(\frac{4(n-1)}{n-2}\Delta_{\tilde{h}_\alpha} \tilde{w}_\alpha + S_{\tilde{h}_\alpha}\tilde{w}_\alpha\right)\tilde{v}^{p-1}$$

and

$$\frac{4(n-1)}{n-2}\Delta_{h_\alpha} \tilde{v} + S_{h_\alpha}\tilde{v} = S_{\tilde{h}_\alpha}\tilde{v}^{p-1}$$

Hence, we get that

$$\Delta_{\tilde{h}_\alpha} \tilde{w}_\alpha + \left(\frac{\Delta_{h_\alpha}\tilde{v}}{\tilde{v}^{p-1}} + \frac{\alpha\mu_\alpha^2}{\tilde{v}^{p-2}}\right)\tilde{w}_\alpha = n(n-2)\tilde{w}_\alpha^{p-1} \tag{7.42}$$

in Ω_α. Let us now prove the following:

LEMMA 7.15 *For $\alpha \gg 1$,*

$$\frac{\Delta_{h_\alpha}\tilde{v}}{\tilde{v}^{p-1}} + \frac{\alpha\mu_\alpha^2}{\tilde{v}^{p-2}} \geq 0$$

in Ω_α.

PROOF: Set $\phi_\alpha(y) = \mu_\alpha y$. As one can easily check,

$$\phi_\alpha : \left(\Omega_\alpha, \mu_\alpha^2 h_\alpha\right) \to \left(\omega_\alpha, \tilde{g}_\alpha\right)$$

is an isometry. Hence,

$$\Delta_{h_\alpha}\tilde{v} = \mu_\alpha^2\left(\Delta_{\tilde{g}_\alpha}(\tilde{v} \circ \phi_\alpha^{-1})\right) \circ \phi_\alpha$$

Independently, since the canonical coordinate system of $\mathbb{R}^n$ is a normal geodesic coordinate system at 0 for $\tilde{g}_\alpha$, we get that for any radial function $u = u(r)$,

$$\Delta_{\tilde{g}_\alpha}u = \Delta_e u - u'\partial_r \log\sqrt{|\tilde{g}_\alpha|}$$

where $r = |x|$ is the Euclidean distance from 0 to x, Δ_e stands for the Euclidean Laplacian, and $|\tilde{g}_\alpha|$ is as in Aubin **[12]**, theorem 1.53. Noting that

$$\exp_{x_\alpha} : \left(\omega_\alpha, \tilde{g}_\alpha\right) \to (\mathcal{B}, g)$$

is an isometry, one gets by Aubin **[12]**, theorem 1.53, that there exists a constant $A > 0$, independent of α, such that

$$\left|\partial_r \log\sqrt{|\tilde{g}_\alpha|}\right| \leq Ar$$

As a consequence,

$$\Delta_{h_\alpha}\tilde{v} \geq \Delta_e\tilde{v} - A\mu_\alpha^2 r|\tilde{v}'|$$

and since

$$\Delta_e\tilde{v} = n(n-2)\tilde{v}^{p-1} \quad \text{and} \quad \tilde{v}'(y) = -(n-2)|y|\left(1+|y|^2\right)^{-n/2}$$

we get that

$$\begin{aligned}\frac{\Delta_{h_\alpha}\tilde{v}}{\tilde{v}^{p-1}} + \frac{\alpha\mu_\alpha^2}{\tilde{v}^{p-2}} &\geq n(n-2) + \alpha\mu_\alpha^2\left(1+|y|^2\right)^2 - (n-2)A\mu_\alpha^2|y|^2\left(1+|y|^2\right) \\ &= n(n-2) + \alpha\mu_\alpha^2\left(1+|y|^2\right) + (\alpha-(n-2)A)\mu_\alpha^2|y|^2\left(1+|y|^2\right)\end{aligned}$$

Hence,

$$\frac{\Delta_{h_\alpha}\tilde{v}}{\tilde{v}^{p-1}} + \frac{\alpha\mu_\alpha^2}{\tilde{v}^{p-2}} \geq 0$$

if $\alpha \geq (n-2)A$. This ends the proof of the lemma. □

From (7.37) and Lemma 7.15, we get that

$$\Delta_{\tilde{h}_\alpha}\tilde{w}_\alpha \leq n(n-2)\tilde{w}_\alpha^{p-1} \tag{7.43}$$

in Ω_α. We set

$$\begin{cases} \varphi(y) = \frac{1}{|y|^2}y \\ \Theta_\alpha = \varphi(\Omega_\alpha) \\ W_\alpha = \tilde{w}_\alpha \circ \varphi \\ H_\alpha = \varphi^\star\tilde{h}_\alpha \end{cases}$$

Clearly, $\Theta_\alpha = \mathbb{R}^n \backslash \theta_\alpha$ where θ_α is an open neighborhood of 0 such that for any $\delta > 0$ and any $\alpha \gg 1$, $\theta_\alpha \subset B_0(\delta)$, where $B_0(\delta)$ stands for the Euclidean ball of center 0 and radius δ. In addition, since

$$\varphi : \left(\Omega_\alpha, h_\alpha\right) \to \left(\Theta_\alpha, H_\alpha\right)$$

is an isometry, we get by (7.43) that

$$\Delta_{H_\alpha} W_\alpha \le n(n-2) W_\alpha^{p-1} \tag{7.44}$$

in Θ_α. Finally, for any $y \in \Theta_\alpha$, and any $i, j = 1, \dots, n$,

$$H_{ij}^\alpha(y) = \left(\frac{|y|^2}{1+|y|^2}\right)^2 \sum_{k,m} \tilde{g}_{km}^\alpha\left(\mu_\alpha \varphi(y)\right) \frac{(\delta_{ik}|y|^2 - 2y_i y_k)}{|y|^4} \frac{(\delta_{jm}|y|^2 - 2y_j y_m)}{|y|^4}$$

where the H_{ij}^α stands for the components of H_α in the canonical chart of $\mathbb{R}^n$, and where the $\tilde{g}_{ij}^\alpha$ stand for the components of $\tilde{g}_\alpha$ in the canonical chart of $\mathbb{R}^n$. Hence, by (7.28) and (7.29) we get that

(7.45) $\exists \lambda > 1$ such that for any α and any $y \in \Theta_\alpha \cap \mathcal{B}$, $\lambda^{-1}\delta_{ij} \le H_{ij}^\alpha(y) \le \lambda \delta_{ij}$ as bilinear forms

(7.46) $\exists K > 0$ such that for any $\alpha > 0$ and any $i, j = 1, \dots, n$, $|H_{ij}^\alpha| \le K$ in $\Theta_\alpha \cap \mathcal{B}$

(7.47) for any $\delta > 0$, $\exists K' > 0$ such that for any $\alpha > 0$ and any $i, j, k = 1, \dots, n$, $|\partial_k H_{ij}^\alpha| \le K'$ in $\Theta_\alpha \cap \left(\mathcal{B} \backslash B_0(\delta)\right)$

where $B_0(\delta)$ stands for the Euclidean ball of center 0 and radius δ.

LEMMA 7.16 *For W_α as above, with the convention that $W_\alpha = 0$ outside Θ_α, and as $\alpha \to +\infty$, $W_\alpha \to 1$ in $L^p(\mathcal{B})$.*

PROOF: We have

$$\begin{aligned}\int_{\mathcal{B}} |W_\alpha - 1|^p \, dx \le C_1 \int_{\mathcal{B}} |W_\alpha - 1|^p \, dv(H_\alpha) &= C_1 \int_{\varphi(\mathcal{B})} |\tilde{w}_\alpha - 1|^p \, dv(\tilde{h}_\alpha) \\ &= C_1 \int_{\varphi(\mathcal{B})} |\tilde{v}_\alpha - \tilde{v}|^p \, dv(h_\alpha) \\ &\le C_2 \int_{\Omega_\alpha} |\tilde{v}_\alpha - \tilde{v}|^p \, dx\end{aligned}$$

since, by (7.28) and (7.45), h_α and H_α are uniformly equivalent to the Euclidean metric. Hence, according to Lemma 7.14, we get that

$$\lim_{\alpha \to +\infty} \int_{\mathcal{B}} |W_\alpha - 1|^p \, dx = 0$$

This ends the proof of the lemma. □

Now we prove the following main estimate. Note that as a consequence of such an estimate, and since $\tilde{v}(y) \to 0$ as $|y| \to +\infty$, one gets that $\tilde{v}_\alpha \to \tilde{v}$ in $L^\infty(\mathbb{R}^n)$.

LEMMA 7.17 *There exists a positive constant C, independent of α, such that for $\tilde{v}_\alpha$ and $\tilde{v}$ as above, and for $\alpha \gg 1$, $\tilde{v}_\alpha \le C\tilde{v}$ on Ω_α.*

PROOF: As one can easily check, the inequality of the lemma is equivalent to the existence of a constant $C > 0$ such that for any $\alpha \gg 1$ and any $y \in \Omega_\alpha$, $\tilde{w}_\alpha(y) \leq C$. Since for any $\Theta \Subset \mathbb{R}^n$,

$$\lim_{\alpha \to +\infty} \tilde{w}_\alpha = 1$$

in $L^\infty(\Theta)$, we just have to prove that there exist a constant $C > 0$ and $R > 0$ such that for any $\alpha \gg 1$ and any $y \in \Omega_\alpha$, $\tilde{w}_\alpha(y) \leq C$ as soon as $|y| \geq R$. Obviously, this is equivalent to the existence of a constant $C > 0$ and to the existence of $\delta_0 > 0$ such that for any $\alpha \gg 1$ and any $y \in \Theta_\alpha$, $W_\alpha(y) \leq C$ as soon as $|y| \leq \delta_0$. From now on, let $\eta \in C^\infty(\mathbb{R}^n)$ be such that $0 \leq \eta \leq 1$, $\eta = 1$ in $B_0(\delta/2)$, $\eta = 0$ in $\mathbb{R}^n \backslash B_0(\delta)$, $0 < \delta < 1$. As one can easily check, multiplying (7.44) by $\eta^2 W_\alpha^k$, $k \geq 1$, and integrating by parts lead to

(7.48)

$$\begin{aligned} &\frac{4k}{(k+1)^2} \int_{\Theta_\alpha} |\nabla(\eta W_\alpha^{(k+1)/2})|^2 \, dv(H_\alpha) \\ &\quad - \frac{2(k-1)}{(k+1)^2} \int_{\Theta_\alpha} \eta(\Delta_{H_\alpha}\eta) W_\alpha^{k+1} \, dv(H_\alpha) - \frac{2}{k+1} \int_{\Theta_\alpha} |\nabla \eta|^2 W_\alpha^{k+1} \, dv(H_\alpha) \\ &\quad \leq n(n-2) \int_{\Theta_\alpha} \eta^2 W_\alpha^{k+p-1} \, dv(H_\alpha) \end{aligned}$$

On the other hand, Hölder's inequalities for the right-hand side of (7.48) give

$$\begin{aligned} &\int_{\Theta_\alpha} \eta^2 W_\alpha^{k+p-1} \, dv(H_\alpha) \leq \\ &\quad \left(\int_{\Theta_\alpha} (\eta W_\alpha^{(k+1)/2})^p \, dv(H_\alpha) \right)^{2/p} \left(\int_{B_0(\delta)} W_\alpha^p \, dv(H_\alpha) \right)^{(p-2)/p} \end{aligned}$$

where $B_0(\delta)$ stands for the Euclidean ball of center 0 and radius δ. Finally, since for any $u \in \mathcal{D}(\mathbb{R}^n)$,

$$\left(\int_{R^n} |u|^p \, dx \right)^{2/p} \leq K(n,2)^2 \int_{R^n} |\nabla u|^2 \, dx$$

we get by (7.45) that there exists a constant $C_1 > 0$, independent of α, such that

$$\left(\int_{\Theta_\alpha} (\eta W_\alpha^{(k+1)/2})^p \, dv(H_\alpha) \right)^{2/p} \leq C_1 \int_{\Theta_\alpha} |\nabla(\eta W_\alpha^{(k+1)/2})|^2 \, dv(H_\alpha)$$

As a consequence, we get that

$$\begin{aligned} &\int_{\Theta_\alpha} |\nabla(\eta W_\alpha^{(k+1)/2})|^2 \, dv(H_\alpha) \leq \\ &\quad C_2 + C_3 \left(\int_{B_0(\delta)} W_\alpha^p \, dv(H_\alpha) \right)^{(p-2)/p} \int_{\Theta_\alpha} |\nabla(\eta W_\alpha^{(k+1)/2})|^2 \, dv(H_\alpha) \end{aligned}$$

where $C_3 > 0$ is independent of α, and

$$C_2 \leq \frac{k-1}{4k} \int_{\Theta_\alpha} \eta |\Delta_{H_\alpha} \eta| W_\alpha^{k+1} \, dv(H_\alpha) + \frac{k+1}{2k} \int_{\Theta_\alpha} |\nabla \eta|^2 W_\alpha^{k+1} \, dv(H_\alpha)$$

By Lemma 7.16, for any k we can choose $\delta > 0$, $\delta \ll 1$, such that

$$C_3\left(\int_{B_0(\delta)} W_\alpha^p \, dv(H_\alpha)\right)^{(p-2)/p} \leq C_4$$

where $C_4 < 1$. Hence, if we proceed as in the proof of Lemma 7.8, we get that

$$\text{for any } q \in \mathbb{N}, \exists \delta > 0, \delta \ll 1, \text{ such that for any } \alpha \gg 1,$$

$$\int_{B_0(\delta)} W_\alpha^q \, dx \leq C_5 \tag{7.49}$$

On such an assertion, recall that H_α satisfies (7.45) and (7.47). Let us now write (7.44) in the following form:

$$D_i\left(H_\alpha^{ij}\sqrt{|H_\alpha|}D_j W_\alpha\right) \geq -n(n-2)\sqrt{|W_\alpha|}W_\alpha^{p-1}$$

where (H_α^{ij}) stands for the inverse matrix of (H_{ij}^α), and $|H_\alpha|$ stands for the determinant of (H_{ij}^α). By (7.45), (7.46), (7.49), and Gilbarg-Trudinger **[91]**, theorem 8.25, we get the existence of a constant $C_6 > 0$, and the existence of $\delta_0 > 0$, $\delta_0 \ll 1$, such that for any $y \in \Theta_\alpha$, $W_\alpha(y) \leq C_6$ if $|y| \leq \delta_0$. As already mentioned, this ends the proof of the lemma. □

With such an estimate we are now able to get the contradiction we were looking for and hence prove Theorem 7.2. The argument starts here with the Pohozaev identity **[167]**. For the sake of completeness, let us say some words about this identity. Given Ω a smooth, bounded domain of $\mathbb{R}^n$, let ν be the unit outer normal to $\partial\Omega$. As one can easily check, for any smooth functions u and v,

$$\begin{aligned}-2(\Delta_e u)\langle\nabla u, \nabla v\rangle &= \operatorname{div}\left(2\langle\nabla u, \nabla v\rangle\nabla u - |\nabla u|^2\nabla v\right)\\ &\quad - |\nabla u|^2(\Delta_e v) - 2\langle\nabla u, \operatorname{Hess}(v).\nabla u\rangle\end{aligned}$$

where Δ_e stands for the Euclidean Laplacian, and the scalar product $\langle\cdot,\cdot\rangle$ is with respect to the Euclidean metric. Let $v(x) = \frac{1}{2}|x|^2$, and assume that $u = 0$ on $\partial\Omega$. Integration by parts then leads to

$$\int_{\partial\Omega}\langle x, \nu\rangle(\partial_\nu u)^2 \, d\sigma = -2\int_\Omega\langle\nabla u, x\rangle(\Delta_e u)dx - (n-2)\int_\Omega u(\Delta_e u)dx$$

Such a relation is referred to as the Pohozaev identity. Starting from this identity, the proof of Theorem 7.2 goes in the following way:

PROOF OF THEOREM 7.2 (FINAL ARGUMENT): In what follows, the C_i's are positive constants independent of α. We let e be the Euclidean metric, and ν_α be the unit outer normal to $\partial\Omega_\alpha$ for e. Since g_α satisfies ($\star\star$), by Theorem 1.3 of Chapter 1 we get that there exists a constant K, independent of α, such that for any α and any $i, j, k = 1, \ldots, n$,

$$\frac{1}{4}\delta_{ij} \leq \tilde{g}_{ij}^\alpha \leq 4\delta_{ij} \quad \text{in } \omega_\alpha \tag{7.50}$$

$$|\tilde{g}_{ij}^\alpha| \leq K \quad \text{and} \quad |\partial_k\tilde{g}_{ij}^\alpha| \leq K \quad \text{in } \omega_\alpha \tag{7.51}$$

where (7.50) has to be understood in the sense of bilinear forms. By the Pohozaev identity,

$$\int_{\Omega_\alpha} (y^k \partial_k \tilde{v}_\alpha) \Delta_e \tilde{v}_\alpha \, dx + \frac{n-2}{2} \int_{\Omega_\alpha} \tilde{v}_\alpha (\Delta_e \tilde{v}_\alpha) dx = -\frac{1}{2} \int_{\partial\Omega_\alpha} \langle y, \nu_\alpha \rangle (\partial_{\nu_\alpha} \tilde{v}_\alpha)^2 \, d\sigma$$

But $\langle y, \nu_\alpha \rangle \geq 0$ since Ω_α is star shaped at 0. Hence,

$$\int_{\Omega_\alpha} (y^k \partial_k \tilde{v}_\alpha) \Delta_e \tilde{v}_\alpha \, dx + \frac{n-2}{2} \int_{\Omega_\alpha} \tilde{v}_\alpha (\Delta_e \tilde{v}_\alpha) dx \leq 0 \tag{7.52}$$

Independently, since $\tilde{g}_\alpha$ satisfies (7.50), for any $X \in \mathbb{R}^n$,

$$h_\alpha^{ij} X_i X_j \geq \lambda |X|^2$$

where $\lambda = 1/4$ and (h_α^{ij}) stands for the inverse matrix of (h_{ij}^α). Let us now write that

$$\lambda \Delta_e \tilde{v}_\alpha = \Delta_{h_\alpha} \tilde{v}_\alpha + \left(h_\alpha^{ij} - \lambda \delta^{ij}\right) \partial_{ij} \tilde{v}_\alpha - h_\alpha^{ij} \Gamma(h_\alpha)_{ij}^k \partial_k \tilde{v}_\alpha$$

where the $\Gamma(h_\alpha)_{ij}^k$'s stand for the Christoffel symbols of h_α. Multiplying (7.52) by λ, we get that

(7.53)

$$\begin{aligned}
&\int_{\Omega_\alpha} (y^k \partial_k \tilde{v}_\alpha) \Delta_{h_\alpha} \tilde{v}_\alpha \, dx + \frac{n-2}{2} \int_{\Omega_\alpha} \tilde{v}_\alpha \Delta_{h_\alpha} \tilde{v}_\alpha \, dx \\
&\quad + \int_{\Omega_\alpha} (y^k \partial_k \tilde{v}_\alpha)(h_\alpha^{ij} - \lambda \delta^{ij}) \partial_{ij} \tilde{v}_\alpha \, dx - \int_{\Omega_\alpha} (y^k \partial_k \tilde{v}_\alpha)(h_\alpha^{ij} \Gamma(h_\alpha)_{ij}^m) \partial_m \tilde{v}_\alpha \, dx \\
&\quad + \frac{n-2}{2} \int_{\Omega_\alpha} \tilde{v}_\alpha (h_\alpha^{ij} - \lambda \delta^{ij}) \partial_{ij} \tilde{v}_\alpha \, dx - \frac{n-2}{2} \int_{\Omega_\alpha} \tilde{v}_\alpha (h_\alpha^{ij} \Gamma(h_\alpha)_{ij}^m) \partial_m \tilde{v}_\alpha \, dx \\
&\leq 0
\end{aligned}$$

Since $\tilde{v}_\alpha$ satisfies (7.39),

$$\begin{aligned}
&\int_{\Omega_\alpha} (y^k \partial_k \tilde{v}_\alpha) \Delta_{h_\alpha} \tilde{v}_\alpha \, dx + \frac{n-2}{2} \int_{\Omega_\alpha} \tilde{v}_\alpha \Delta_{h_\alpha} \tilde{v}_\alpha \, dx \\
&\quad = \int_{\Omega_\alpha} (y^k \partial_k \tilde{v}_\alpha)\left(n(n-2) \tilde{v}_\alpha^{p-1} - (\alpha \mu_\alpha^2) \tilde{v}_\alpha\right) dx \\
&\qquad + \frac{n-2}{2} \int_{\Omega_\alpha} \tilde{v}_\alpha \left(n(n-2) \tilde{v}_\alpha^{p-1} - (\alpha \mu_\alpha^2) \tilde{v}_\alpha\right) dx
\end{aligned}$$

In addition, since $\tilde{v}_\alpha = 0$ on $\partial\Omega_\alpha$, integration by parts gives

$$\int_{\Omega_\alpha} (y^k \partial_k \tilde{v}_\alpha) \tilde{v}_\alpha^{p-1} \, dx = -\frac{n-2}{2} \int_{\Omega_\alpha} \tilde{v}_\alpha^p \, dx$$

$$\int_{\Omega_\alpha} (y^k \partial_k \tilde{v}_\alpha) \tilde{v}_\alpha \, dx = -\frac{n}{2} \int_{\Omega_\alpha} \tilde{v}_\alpha^2 \, dx$$

Hence,

$$\int_{\Omega_\alpha}(y^k\partial_k\tilde{v}_\alpha)\Delta_{h_\alpha}\tilde{v}_\alpha\,dx+\frac{n-2}{2}\int_{\Omega_\alpha}\tilde{v}_\alpha\Delta_{h_\alpha}\tilde{v}_\alpha\,dx=\alpha\mu_\alpha^2\int_{\Omega_\alpha}\tilde{v}_\alpha^2\,dx$$

and (7.53) can be written in the following form

(7.54)
$$\begin{aligned}
&\alpha\mu_\alpha^2\int_{\Omega_\alpha}\tilde{v}_\alpha^2\,dx+\int_{\Omega_\alpha}(y^k\partial_k\tilde{v}_\alpha)(h_\alpha^{ij}-\lambda\delta^{ij})\partial_{ij}\tilde{v}_\alpha\,dx\\
&-\int_{\Omega_\alpha}(y^k\partial_k\tilde{v}_\alpha)(h_\alpha^{ij}\Gamma(h_\alpha)_{ij}^m)\partial_m\tilde{v}_\alpha\,dx+\frac{n-2}{2}\int_{\Omega_\alpha}\tilde{v}_\alpha(h_\alpha^{ij}-\lambda\delta^{ij})\partial_{ij}\tilde{v}_\alpha\,dx\\
&-\frac{n-2}{2}\int_{\Omega_\alpha}\tilde{v}_\alpha(h_\alpha^{ij}\Gamma(h_\alpha)_{ij}^m)\partial_m\tilde{v}_\alpha\,dx\\
&\le 0
\end{aligned}$$

Let us now concentrate on the different terms of (7.54). As a starting point, we get by integration by parts that

$$\begin{aligned}
&\int_{\Omega_\alpha}(y^k\partial_k\tilde{v}_\alpha)(h_\alpha^{ij}-\lambda\delta^{ij})\partial_{ij}\tilde{v}_\alpha\,dx\\
&=\int_{\partial\Omega_\alpha}\langle y,\nu_\alpha\rangle(\partial_{\nu_\alpha}\tilde{v}_\alpha)^2\big((h_\alpha^{ij}-\lambda\delta^{ij})\nu_i^\alpha\nu_j^\alpha\big)d\sigma-\int_{\Omega_\alpha}(\partial_i h_\alpha^{ij})y^k\partial_j\tilde{v}_\alpha\partial_k\tilde{v}_\alpha\,dx\\
&\quad-\int_{\Omega_\alpha}(h_\alpha^{kj}-\lambda\delta^{kj})\partial_k\tilde{v}_\alpha\partial_j\tilde{v}_\alpha\,dx-\int_{\Omega_\alpha}(h_\alpha^{ij}-\lambda\delta^{ij})x^k\partial_{ik}\tilde{v}_\alpha\partial_j\tilde{v}_\alpha\,dx
\end{aligned}$$

where the ν_i^α's stand for the coordinates of ν. Similarly,

$$\begin{aligned}
&\int_{\Omega_\alpha}(h_\alpha^{ij}-\lambda\delta^{ij})x^k\partial_{ik}\tilde{v}_\alpha\partial_j\tilde{v}_\alpha\,dx\\
&=\int_{\partial\Omega_\alpha}\langle y,\nu_\alpha\rangle(\partial_{\nu_\alpha}\tilde{v}_\alpha)^2\big((h_\alpha^{ij}-\lambda\delta^{ij})\nu_i^\alpha\nu_j^\alpha\big)d\sigma-\int_{\Omega_\alpha}(y^k\partial_k h_\alpha^{ij})\partial_i\tilde{v}_\alpha\partial_j\tilde{v}_\alpha\,dx\\
&\quad-n\int_{\Omega_\alpha}(h_\alpha^{ij}-\lambda\delta^{ij})\partial_i\tilde{v}_\alpha\partial_j\tilde{v}_\alpha\,dx-\int_{\Omega_\alpha}(h_\alpha^{ij}-\lambda\delta^{ij})y^k\partial_i\tilde{v}_\alpha\partial_{jk}\tilde{v}_\alpha\,dx
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_{\Omega_\alpha}(h_\alpha^{ij}-\lambda\delta^{ij})y^k\partial_{ik}\tilde{v}_\alpha\partial_j\tilde{v}_\alpha\,dx\\
&=\frac{1}{2}\int_{\partial\Omega_\alpha}\langle y,\nu_\alpha\rangle(\partial_{\nu_\alpha}\tilde{v}_\alpha)^2\big((h_\alpha^{ij}-\lambda\delta^{ij})\nu_i^\alpha\nu_j^\alpha\big)d\sigma\\
&\quad-\frac{1}{2}\int_{\Omega_\alpha}(y^k\partial_k h_\alpha^{ij})\partial_i\tilde{v}_\alpha\partial_j\tilde{v}_\alpha\,dx-\frac{n}{2}\int_{\Omega_\alpha}(h_\alpha^{ij}-\lambda\delta^{ij})\partial_i\tilde{v}_\alpha\partial_j\tilde{v}_\alpha\,dx
\end{aligned}$$

and we get that

$$
\begin{aligned}
&\int_{\Omega_\alpha} (y^k \partial_k \tilde{v}_\alpha)(h_\alpha^{ij} - \lambda\delta^{ij})\partial_{ij}\tilde{v}_\alpha \, dx \\
&= \frac{1}{2}\int_{\partial\Omega_\alpha} \langle y, \nu_\alpha\rangle (\partial_{\nu_\alpha}\tilde{v}_\alpha)^2\big((h_\alpha^{ij} - \lambda\delta^{ij})\nu_i^\alpha \nu_j^\alpha\big) d\sigma \\
&\quad - \int_{\Omega_\alpha} (\partial_i h_\alpha^{ij}) y^k \partial_j \tilde{v}_\alpha \partial_k \tilde{v}_\alpha \, dx \\
&\quad + \frac{n-2}{2}\int_{\Omega_\alpha} (h_\alpha^{ij} - \lambda\delta^{ij})\partial_i\tilde{v}_\alpha \partial_j \tilde{v}_\alpha \, dx + \frac{1}{2}\int_{\Omega_\alpha} (y^k \partial_k h_\alpha^{ij})\partial_i \tilde{v}_\alpha \partial_j \tilde{v}_\alpha \, dx
\end{aligned} \tag{7.55}
$$

Similarly,

$$
\begin{aligned}
\int_{\Omega_\alpha} \tilde{v}_\alpha (h_\alpha^{ij} - \lambda\delta^{ij})\partial_{ij}\tilde{v}_\alpha \, dx = &- \int_{\Omega_\alpha} (h_\alpha^{ij} - \lambda\delta^{ij})\partial_i \tilde{v}_\alpha \partial_j \tilde{v}_\alpha \, dx \\
&- \int_{\Omega_\alpha} (\partial_i h_\alpha^{ij})\tilde{v}_\alpha \partial_j \tilde{v}_\alpha \, dx
\end{aligned}
$$

and

$$
\int_{\Omega_\alpha} (\partial_i h_\alpha^{ij})\tilde{v}_\alpha \partial_j \tilde{v}_\alpha \, dx = -\frac{1}{2}\int_{\Omega_\alpha} (\partial_{ij} h_\alpha^{ij})\tilde{v}_\alpha^2 \, dx
$$

Therefore,

$$
\begin{aligned}
\int_{\Omega_\alpha} \tilde{v}_\alpha (h_\alpha^{ij} - \lambda\delta^{ij})\partial_{ij}\tilde{v}_\alpha \, dx = &- \int_{\Omega_\alpha} (h_\alpha^{ij} - \lambda\delta^{ij})\partial_i \tilde{v}_\alpha \partial_j \tilde{v}_\alpha \, dx \\
&+ \frac{1}{2}\int_{\Omega_\alpha} (\partial_{ij} h_\alpha^{ij})\tilde{v}_\alpha^2 \, dx
\end{aligned} \tag{7.56}
$$

Finally,

$$
\int_{\Omega_\alpha} \tilde{v}_\alpha (h_\alpha^{ij}\Gamma(h_\alpha)_{ij}^m)\partial_m \tilde{v}_\alpha \, dx = -\frac{1}{2}\int_{\Omega_\alpha} \big(\partial_m (h_\alpha^{ij}\Gamma(h_\alpha)_{ij}^m)\big)\tilde{v}_\alpha^2 \, dx \tag{7.57}
$$

Hence, by (7.54), (7.55), (7.56), and (7.57), we get that

$$
\begin{aligned}
&\alpha\mu_\alpha^2 \int_{\Omega_\alpha} \tilde{v}_\alpha^2 \, dx + \frac{1}{2}\int_{\partial\Omega_\alpha} \langle y, \nu_\alpha\rangle (\partial_{\nu_\alpha}\tilde{v}_\alpha)^2\big((h_\alpha^{ij} - \lambda\delta^{ij})\nu_i^\alpha \nu_j^\alpha\big) d\sigma \\
&+ \frac{1}{2}\int_{\Omega_\alpha} (y^k \partial_k h_\alpha^{ij})\partial_i \tilde{v}_\alpha \partial_j \tilde{v}_\alpha \, dx - \int_{\Omega_\alpha} (\partial_i h_\alpha^{ij}) y^k \partial_j \tilde{v}_\alpha \partial_k \tilde{v}_\alpha \, dx \\
&+ \frac{n-2}{4}\int_{\Omega_\alpha} (\partial_{ij} h_\alpha^{ij})\tilde{v}_\alpha^2 \, dx + \frac{n-2}{4}\int_{\Omega_\alpha} \big(\partial_m (h_\alpha^{ij}\Gamma(h_\alpha)_{ij}^m)\big)\tilde{v}_\alpha \, dx \\
&- \int_{\Omega_\alpha} (y^k h_\alpha^{ij}\Gamma(h_\alpha)_{ij}^m)\partial_k \tilde{v}_\alpha \partial_m \tilde{v}_\alpha \, dx \\
&\leq 0
\end{aligned}
$$

For any $y \in \partial\Omega_\alpha$,

$$
\langle y, \nu_\alpha\rangle \geq 0 \quad \text{and} \quad (h_\alpha^{ij} - \lambda\delta^{ij})\nu_i^\alpha \nu_j^\alpha \geq 0
$$

As a consequence,

$$\int_{\partial\Omega_\alpha} \langle y, \nu_\alpha\rangle (\partial_{\nu_\alpha}\tilde{v}_\alpha)^2\left((h_\alpha^{ij} - \lambda\delta^{ij})\nu_i^\alpha \nu_j^\alpha\right) d\sigma \geq 0$$

and we get that

$$\begin{aligned} &\alpha\mu_\alpha^2 \int_{\Omega_\alpha} \tilde{v}_\alpha^2\,dx + \frac{1}{2}\int_{\Omega_\alpha} (y^k\partial_k h_\alpha^{ij})\partial_i\tilde{v}_\alpha\partial_j\tilde{v}_\alpha\,dx \\ &- \int_{\Omega_\alpha} (\partial_i h_\alpha^{ij}) y^k \partial_j\tilde{v}_\alpha\partial_k\tilde{v}_\alpha\,dx + \frac{n-2}{4}\int_{\Omega_\alpha} (\partial_{ij}h_\alpha^{ij})\tilde{v}_\alpha^2\,dx \\ &+ \frac{n-2}{4}\int_{\Omega_\alpha} \left(\partial_m (h_\alpha^{ij}\Gamma(h_\alpha)_{ij}^m)\right)\tilde{v}_\alpha^2\,dx - \int_{\Omega_\alpha} (y^k h_\alpha^{ij}\Gamma(h_\alpha)_{ij}^m)\partial_k\tilde{v}_\alpha\partial_m\tilde{v}_\alpha\,dx \\ &\leq 0 \end{aligned} \tag{7.58}$$

By (7.50) and (7.51),

$$\left|\partial_k h_\alpha^{ij}(y)\right| = \mu_\alpha\left|(\partial_k\tilde{g}_\alpha^{ij})(\mu_\alpha y)\right| \leq C_1\mu_\alpha^2|y|$$

Hence, by (7.39), (7.50), and (7.51), integration by parts leads to

$$\begin{aligned} &\left|\int_{\Omega_\alpha} y^k(\partial_k h_\alpha^{ij})\partial_i\tilde{v}_\alpha\partial_j\tilde{v}_\alpha\,dx\right| \\ &\leq C_2\mu_\alpha^2\int_{\Omega_\alpha} |y|^2|\nabla\tilde{v}_\alpha|^2\,dx \\ &\leq C_3\mu_\alpha^2\int_{\Omega_\alpha} |y|^2|\nabla\tilde{v}_\alpha|^2\,dv(h_\alpha) \\ &= -C_3\mu_\alpha^2\int_{\Omega_\alpha} \left(\nabla(|y|^2\nabla\tilde{v}_\alpha)\right)\tilde{v}_\alpha\,dv(h_\alpha) \\ &= C_3\mu_\alpha^2\int_{\Omega_\alpha} |y|^2(\Delta_{h_\alpha}\tilde{v}_\alpha)\tilde{v}_\alpha\,dv(h_\alpha) - C_3\mu_\alpha^2\int_{\Omega_\alpha} \left(\nabla|y|^2\nabla\tilde{v}_\alpha\right)\tilde{v}_\alpha\,dv(h_\alpha) \\ &= C_3\mu_\alpha^2\int_{\Omega_\alpha} |y|^2(\Delta_{h_\alpha}\tilde{v}_\alpha)\tilde{v}_\alpha\,dv(h_\alpha) - C_4\mu_\alpha^2\int_{\Omega_\alpha} (\Delta_{h_\alpha}|x|^2)\tilde{v}_\alpha^2\,dv(h_\alpha) \\ &\leq C_3\mu_\alpha^2\int_{\Omega_\alpha} |y|^2(\Delta_{h_\alpha}\tilde{v}_\alpha)\tilde{v}_\alpha\,dv(h_\alpha) + C_5\mu_\alpha^2\int_{\Omega_\alpha} \tilde{v}_\alpha^2\,dv(h_\alpha) \\ &= C_3\mu_\alpha^2\int_{\Omega_\alpha} |y|^2\tilde{v}_\alpha\left(n(n-2)\tilde{v}_\alpha^{p-1} - \alpha\mu_\alpha^2\tilde{v}_\alpha\right)dv(h_\alpha) + C_5\mu_\alpha^2\int_{\Omega_\alpha} \tilde{v}_\alpha^2\,dv(h_\alpha) \\ &\leq C_5\mu_\alpha^2\int_{\Omega_\alpha} \tilde{v}_\alpha^2\,dx + C_6\mu_\alpha^2\int_{\Omega_\alpha} |y|^2\tilde{v}_\alpha^p\,dx \end{aligned}$$

Therefore,

$$\left|\int_{\Omega_\alpha} y^k(\partial_k h_\alpha^{ij})\partial_i\tilde{v}_\alpha\partial_j\tilde{v}_\alpha\,dx\right| \leq C_5\mu_\alpha^2\int_{\Omega_\alpha} \tilde{v}_\alpha^2\,dx + C_6\mu_\alpha^2\int_{\Omega_\alpha} |y|^2\tilde{v}_\alpha^p\,dx \tag{7.59}$$

and with the same arguments we get that

$$\left|\int_{\Omega_\alpha} (\partial_i h_\alpha^{ij}) y^k\partial_j\tilde{v}_\alpha\partial_k\tilde{v}_\alpha\,dx\right| \leq C_7\mu_\alpha^2\int_{\Omega_\alpha} \tilde{v}_\alpha^2\,dx + C_8\mu_\alpha^2\int_{\Omega_\alpha} |y|^2\tilde{v}_\alpha^p\,dx \tag{7.60}$$

Independently, by (7.50) and (7.51), for any $X \in \mathbb{R}^n$,

$$\left|(y^k h_\alpha^{ij}\Gamma(h_\alpha)_{ij}^m)X_k X_m\right| \le C_9\mu_\alpha^2|y|^2|X|^2$$

Hence,

$$\left|\int_{\Omega_\alpha}(y^k h_\alpha^{ij}\Gamma(h_\alpha)_{ij}^m)\partial_k\tilde{v}_\alpha\partial_m\tilde{v}_\alpha\,dx\right| \le C_{10}\mu_\alpha^2\int_{\Omega_\alpha}|y|^2|\nabla\tilde{v}_\alpha|^2\,dx$$

and, here again,

(7.61)
$$\left|\int_{\Omega_\alpha}(y^k h_\alpha^{ij}\Gamma(h_\alpha)_{ij}^m)\partial_k\tilde{v}_\alpha\partial_m\tilde{v}_\alpha\,dx\right| \le C_{11}\mu_\alpha^2\int_{\Omega_\alpha}\tilde{v}_\alpha^2\,dx + C_{12}\mu_\alpha^2\int_{\Omega_\alpha}|y|^2\tilde{v}_\alpha^p\,dx$$

Now, we are left with two terms in the study of (7.58). First, by (7.51),

$$\begin{aligned}\left|\int_{\Omega_\alpha}(\partial_{ij}h_\alpha^{ij})\tilde{v}_\alpha^2\,dx\right| &= \left|\int_{\Omega_\alpha}(\partial_j h_\alpha^{ij})(\partial_i\tilde{v}_\alpha^2)dx\right| \\ &\le C_{13}\mu_\alpha^2\int_{\Omega_\alpha}\left(|y|\sum_i|\partial_i\tilde{v}_\alpha|\right)\tilde{v}_\alpha\,dx \\ &\le C_{14}\mu_\alpha^2\sqrt{\int_{\Omega_\alpha}|y|^2|\nabla\tilde{v}_\alpha|^2\,dx}\sqrt{\int_{\Omega_\alpha}\tilde{v}_\alpha^2\,dx}\end{aligned}$$

Therefore, since

$$\int_{\Omega_\alpha}|y|^2|\nabla\tilde{v}_\alpha|^2\,dx \le C_{15}\int_{\Omega_\alpha}\tilde{v}_\alpha^2\,dx + C_{16}\int_{\Omega_\alpha}|y|^2\tilde{v}_\alpha^p\,dx$$

we get that

$$\left|\int_{\Omega_\alpha}(\partial_{ij}h_\alpha^{ij})\tilde{v}_\alpha^2\,dx\right| \le C_{17}\mu_\alpha^2\int_{\Omega_\alpha}\tilde{v}_\alpha^2\,dx + C_{18}\mu_\alpha^2\int_{\Omega_\alpha}|y|^2\tilde{v}_\alpha^p\,dx \tag{7.62}$$

Similarly, since

$$\left|\int_{\Omega_\alpha}\partial_m(h_\alpha^{ij}\Gamma(h_\alpha)_{ij}^m)\tilde{v}_\alpha^2\,dx\right| = \left|\int_{\Omega_\alpha}(h_\alpha^{ij}\Gamma(h_\alpha)_{ij}^m)(\partial_m\tilde{v}_\alpha^2)dx\right|$$

we get with the same arguments as those used to establish (7.62) that

$$\left|\int_{\Omega_\alpha}\partial_m(h_\alpha^{ij}\Gamma(h_\alpha)_{ij}^m)\tilde{v}_\alpha^2\,dx\right| \le C_{19}\mu_\alpha^2\int_{\Omega_\alpha}\tilde{v}_\alpha^2\,dx + C_{20}\mu_\alpha^2\int_{\Omega_\alpha}|y|^2\tilde{v}_\alpha^p\,dx \tag{7.63}$$

Now, combining (7.58) with the estimates (7.59) to (7.63), we get that

$$\alpha\int_{\Omega_\alpha}\tilde{v}_\alpha^2\,dx \le C_{21}\int_{\Omega_\alpha}\tilde{v}_\alpha^2\,dx + C_{22}\int_{\Omega_\alpha}|y|^2\tilde{v}_\alpha^p\,dx \tag{7.64}$$

Since

$$\lim_{\alpha\to+\infty}\tilde{v}_\alpha = \tilde{v}$$

uniformly in any compact subset of $\mathbb{R}^n$, for $\alpha \gg 1$,

$$\int_{\Omega_\alpha}\tilde{v}_\alpha^2\,dx \ge \int_{\mathcal{B}}\tilde{v}^2\,dx > 0 \tag{7.65}$$

Moreover, by the fundamental estimate of Lemma 7.17, and since $n \geq 3$,

$$\int_{\Omega_\alpha} |x|^2 \tilde{v}_\alpha^p \, dx \leq C_{23} \int_{R^n} |x|^2 \tilde{v}^p \, dx \leq C_{24} \tag{7.66}$$

Combining (7.64), (7.65), and (7.66), we then get that $\alpha \leq C_{25}$, the contradiction we were looking for. This ends the proof of the theorem. □

7.4. Explicit Inequalities

Let $(\mathbb{R}^n, e)$ be the n-Euclidean space, (S^n, h) be the standard unit sphere of $\mathbb{R}^{n+1}$, (H^n, h_0) be the n-dimensional, simply connected hyperbolic space, and $(\mathbb{P}^n(\mathbb{R}), p)$ be the n-dimensional projective space with its canonical metric induced from h. We consider here the following inequality: For (M, g) as in Theorem 7.2, and for any $u \in H_1^2(M)$,

$$(\mathrm{I}^2_{2,\mathrm{opt}}) \qquad \left(\int_M |u|^p \, dv(g)\right)^{2/p} \leq K(n,2)^2 \int_M |\nabla u|^2 \, dv(g) + B \int_M u^2 \, dv(g)$$

where $1/p = 1/2 - 1/n$, and $K(n, 2)$ is as in Theorem 4.4.

The purpose of this section is to prove the following result, where, in the spirit of Question 4, explicit optimal inequalities are obtained for some specific manifolds. As already mentioned, case (i) below is due to Aubin [10] and Talenti [183].

THEOREM 7.7 *The optimal inequality* $(\mathrm{I}^2_{2,\mathrm{opt}})$ *is valid with*

(i) $B = 0$ *for* $(\mathbb{R}^n, e)$

(ii) $B = -\frac{1}{\omega_n^{2/n}}$ *for* (H^n, h_0)

(iii) $B = \frac{m-n}{(m+n)\omega_{m+n}^{2/(m+n)}}$ *for* $(S^m \times H^n, h + h_0)$, $m \geq 2$, $n \geq 2$

(iv) $B = \frac{m-n+2}{(m+n-2)\omega_{m+n}^{2/(m+n)}}$ *for* $(\mathbb{P}^m(\mathbb{R}) \times H^n, p + h_0)$, $m \geq 2$, $n \geq 2$

(v) $B = \frac{n-1}{(n+1)\omega_{n+1}^{2/(n+1)}}$ *for* $(S^n \times \mathbb{R}, h + e)$, $n \geq 2$

(vi) $B = \frac{n+1}{(n-1)\omega_{n+1}^{2/(n+1)}}$ *for* $(\mathbb{P}^n(\mathbb{R}) \times \mathbb{R}, p + e)$, $n \geq 2$

(vii) $B = -\frac{n-1}{(n+1)\omega_{n+1}^{2/(n+1)}}$ *for* $(H^n \times \mathbb{R}, h_0 + e)$, $n \geq 2$

where ω_n *denotes the volume of* (S^n, h). *Furthermore, at least when the dimension of the manifold is greater than or equal to* 4, *these values are the best possible for* $(\mathbb{R}^n, e)$, (H^n, h_0), $(S^m \times H^n, h + h_0)$, $(S^n \times \mathbb{R}, h + e)$, *and* $(H^n \times \mathbb{R}, h_0 + e)$.

PROOF: Regarding points (i), (ii), (iii), (v), and (vii), the result is an easy consequence of Proposition 5.1 of Chapter 5 and Proposition 3.8 of Chapter 3, since the manifolds in question in these points are conformally flat, simply connected, and of constant scalar curvature. Hence, we are left with the proof of points (iv) and (vi). We present the proof of (iv). The proof of (vi) proceeds with the same arguments.

According to what has been said in the proof of Proposition 5.2, there exist $m+1$ simply connected open subsets Ω_i of $\mathbb{P}^m(\mathbb{R})$, and $m+1$ functions $\eta_i : \Omega_i \to \mathbb{R}$ such that

1. $(\Omega_i)_{i=1,\ldots,n+1}$ is an open covering of $\mathbb{P}^m(\mathbb{R})$,

2. for all i, η_i and $\sqrt{\eta_i}$ belong to $H_{0,1}^2(\Omega_i) \cap C^0(\overline{\Omega_i})$,
3. for all i, $0 \leq \eta_i \leq 1$ and $|\nabla\sqrt{\eta_i}| \in C^0(\overline{\Omega_i})$, and
4. $\sum_{i=1}^{m+1} \eta_i = 1$ and $\sum_{i=1}^{m+1} |\nabla\sqrt{\eta_i}|^2 = m$.

Let $\tilde{\eta}_i : \mathbb{P}^m(\mathbb{R}) \times H^n \to \mathbb{R}$ be defined by $\tilde{\eta}_i(x, y) = \eta_i(x)$. As in the proof of Theorem 7.4, one has that for $u \in \mathcal{D}(\mathbb{P}^m(\mathbb{R}) \times H^n)$,

$$\|u\|_{2N/(N-2)}^2 \leq \sum_{i=1}^{m+1} \|\sqrt{\tilde{\eta}_i} u\|_{2N/(N-2)}^2$$

where $N = m + n$, and $\|\cdot\|_s$ stands for the norm of $L^s(M)$. Independently, since $\Omega_i \times H^n$ is simply connected, since $p + h_0$ is conformally flat, and since the scalar curvature of $p + h_0$ equals $m(m-1) - n(n-1)$, one gets from Proposition 3.8 of Chapter 3 that for any i and any $u \in \mathcal{D}(\mathbb{P}^m(\mathbb{R}) \times H^n)$,

$$\|\sqrt{\tilde{\eta}_i}\|_p^2 \leq K(N,2)^2 \|\nabla(\sqrt{\tilde{\eta}_i} u)\|_2^2 + B(m,n) \|\sqrt{\tilde{\eta}_i} u\|_2^2$$

where

$$B(m,n) = \frac{m(m-1) - n(n-1)}{(m+n)(m+n-1)\omega_{m+n}^{2/(m+n)}}$$

Similar computations to those made in the proof of Theorem 7.4 then lead to the following: For any $u \in \mathcal{D}(M)$,

$$\|u\|_p^2 \leq K(N,2)^2 \|\nabla u\|_2^2 + B\|u\|_2^2$$

where

$$B = K(N,2)^2 \left(\sup_{P^m(R) \times H^n} \sum_{i=1}^{m+1} |\nabla\sqrt{\tilde{\eta}_i}|^2 \right) + B(m,n)$$

By point (4) above,

$$\sum_{i=1}^{m+1} |\nabla\sqrt{\tilde{\eta}_i}|^2 = m$$

Hence,

$$B = \frac{(m-n)(m+n-2) + 4m}{(m+n)(m+n-2)\omega_{m+n}^{2/(m+n)}} = \frac{m-n+2}{(m+n-2)\omega_{m+n}^{2/(m+n)}}$$

This ends the proof of the theorem. □

CHAPTER 8

Euclidean-Type Sobolev Inequalities

Let (M, g) be a smooth, complete Riemannian n-manifold of infinite volume, and let $q \in [1, n)$ real. We say that the Euclidean-type Sobolev inequality of order q is valid if there exists $C_q > 0$ real such that for any $u \in \mathcal{D}(M)$,

$$(\mathrm{I}^{\mathrm{eucl.}}_{q.\mathrm{gen}}) \qquad \left(\int_M |u|^p \, dv(g)\right)^{q/p} \leq C_q \int_M |\nabla u|^q \, dv(g)$$

where $1/p = 1/q - 1/n$. As shown by Theorem 2.5, such an inequality is satisfied by the Euclidean space. In the first section of this chapter, we try to find some nice conditions on (M, g) for such inequalities to be valid. Note here that the study of $(\mathrm{I}^{\mathrm{eucl.}}_{q.\mathrm{gen}})$ can be seen as the analogue of Program $\mathcal{B}$ we studied in Chapters 4 and 5, since for infinite-volume manifolds,

$$\mathrm{Vol}_{(M,g)}^{-1/n} = 0$$

Regarding such an assertion, given (M, g) some smooth, complete Riemannian n-manifold of finite volume, recall that by Proposition 3.5 there do not exist real numbers A and B such that for any $u \in \mathcal{D}(M)$,

$$\left(\int_M |u|^p \, dv(g)\right)^{1/p} \leq A\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} + B\left(\int_M |u|^q \, dv(g)\right)^{1/q}$$

unless M is compact. In particular, one cannot expect to get inequalities such as

$$\left(\int_M |u|^p \, dv(g)\right)^{1/p} \leq A\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} + \mathrm{Vol}_{(M,g)}^{-1/n}\left(\int_M |u|^q \, dv(g)\right)^{1/q}$$

In other words, one cannot expect to extend Program $\mathcal{B}$ to complete manifolds of finite volume, and a natural extension of Program $\mathcal{B}$ to complete manifolds is the one mentioned above, dealing with infinite-volume complete manifolds and Euclidean-type Sobolev inequalities. As a starting point (this is an arbitrary choice), one fixes here the value of $\beta_q(M)$ to be zero. The study of the validity of $(\mathrm{I}^{\mathrm{eucl.}}_{q.\mathrm{gen}})$ then refers to Question $2\mathcal{B}$, while getting estimates on C_2 refers to Question $3\mathcal{B}$. Still when dealing with complete, noncompact manifolds of finite volume, as one can easily check, the Euclidean-type Sobolev inequalities must be false. One can use here the preceding argument, or note that there is a serious problem when taking $u = 1$ in

$(I_{q,\mathrm{gen}}^{\mathrm{eucl.}})$. One may ask instead if the Sobolev-Poincaré inequalities

$$\left(\int_M |u-\bar{u}|^p \, dv(g)\right)^{q/p} \leq A \int_M |\nabla u|^q \, dv(g)$$

are valid for complete manifolds of finite volume. This could be motivated by the idea that the Euclidean-type Sobolev inequalities are the infinite-volume versions of the Sobolev-Poincaré inequalities. As one can easily check, the answer to such a question is negative, unless once more the manifold is compact. The point here is that the validity of such a Sobolev-Poincaré inequality implies the validity of the generic Sobolev inequality of same order, so we are back to Proposition 3.5. Independently, and in the second section of this chapter, we will discuss the question of the best value of C_q in $(I_{q,\mathrm{gen}}^{\mathrm{eucl.}})$. By Theorem 4.4, one has that the best value of C_q is $K(n,q)^q$ for the Euclidean space.

8.1. Euclidean-Type Generic Sobolev Inequalities

Let (M,g) be a smooth, complete Riemannian n-manifold of infinite volume. We discuss in this section conditions on the manifold for the Euclidean-type generic Sobolev inequality $(I_{q,\mathrm{gen}}^{\mathrm{eucl.}})$ to be valid. Clearly, since the validity of $(I_{q,\mathrm{gen}}^{\mathrm{eucl.}})$ implies that of $(I_{q,\mathrm{gen}}^{1})$, there exist complete manifolds for which all the $(I_{q,\mathrm{gen}}^{\mathrm{eucl.}})$'s are false. As a first result, one has the following analogue of Lemma 2.1.

LEMMA 8.1 *Let (M,g) be a smooth, complete Riemannian n-manifold of infinite volume. Suppose that for some $q \in [1,n)$, $(I_{q,\mathrm{gen}}^{\mathrm{eucl.}})$ is valid. Then, for all $s \in [q,n)$, $(I_{s,\mathrm{gen}}^{\mathrm{eucl.}})$ is valid. In particular, if $(I_{1,\mathrm{gen}}^{\mathrm{eucl.}})$ is valid, then for all $s \in [1,n)$, $(I_{s,\mathrm{gen}}^{\mathrm{eucl.}})$ is valid.*

PROOF: Let $q \in [1,n)$ and let $C > 0$ be such that for any $u \in \mathcal{D}(M)$,

$$\left(\int_M |u|^p \, dv(g)\right)^{1/p} \leq C\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q}$$

where $1/p = 1/q - 1/n$. Given $s \in (q,n)$, and $u \in \mathcal{D}(M)$, let also $\varphi = |u|^{t(n-q)/nq}$ where t is such that $1/t = 1/s - 1/n$. Then, as in the proof of Lemma 2.1, one gets with Hölder's inequalities that

$$\begin{aligned}
&\left(\int_M |u|^t \, dv(g)\right)^{1/p} \\
&= \left(\int_M |\varphi|^p \, dv(g)\right)^{1/p} \\
&\leq C\left(\int_M |\nabla\varphi|^q \, dv(g)\right)^{1/q} \\
&= C(\alpha+1)\left(\int_M |u|^{\alpha q}|\nabla u|^q \, dv(g)\right)^{1/q} \\
&\leq C(\alpha+1)\left(\int_M |u|^{q\alpha s/(s-q)} \, dv(g)\right)^{(s-q)/qs} \left(\int_M |\nabla u|^s \, dv(g)\right)^{1/s}
\end{aligned}$$

where $\alpha = \frac{t(n-q)}{nq} - 1$. But

$$\frac{1}{p} - \frac{s-q}{qs} = \frac{1}{t} \quad \text{and} \quad \frac{q\alpha s}{s-q} = t$$

Hence, for any $u \in \mathcal{D}(M)$,

$$\left(\int_M |u|^t\, dv(g)\right)^{1/t} \leq \frac{t(n-q)C}{nq}\left(\int_M |\nabla u|^s\, dv(g)\right)^{1/s}$$

This proves the lemma. □

Following Carron [39], we start here with the discussion of the existence of C_2. Namely, we discuss the validity of $(I_{2,\mathrm{gen}}^{\mathrm{eucl.}})$: There exists C_2 such that for any $u \in \mathcal{D}(M)$,

$$(I_{2,\mathrm{gen}}^{\mathrm{eucl.}}) \qquad \left(\int_M |u|^p\, dv(g)\right)^{2/p} \leq C_2 \int_M |\nabla u|^2\, dv(g)$$

where $1/p = 1/2 - 1/n$. Let Ω be some subset of M. By $\Omega \Subset M$ we mean that Ω is a regular, bounded, open subset of M. Then, $\lambda_1^D(\Omega)$ denotes the first eigenvalue of the Laplacian Δ_g for the Dirichlet problem on Ω, while $\mathrm{Vol}_g(\Omega)$ denotes the volume of Ω with respect to g. Recall here that

$$\lambda_1^D(\Omega) = \inf_{u \in H_{0,1}^2(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\, dv(g)}{\int_M u^2\, dv(g)}$$

We say in what follows that the Faber-Krahn inequality is valid if there exists $\Lambda > 0$ such that for any $\Omega \Subset M$,

$$(\mathrm{FK}) \qquad \lambda_1^D(\Omega) \geq \Lambda\, \mathrm{Vol}_g(\Omega)^{-2/n}$$

The following proposition is due to Carron [39]:

PROPOSITION 8.1 *Let (M, g) be a smooth, complete Riemannian manifold of dimension n, $n \geq 3$, and of infinite volume. The following two propositions are equivalent:*

(i) *The Euclidean-type generic Sobolev inequality* $(I_{2,\mathrm{gen}}^{\mathrm{eucl.}})$ *is valid.*
(ii) *The Faber-Krahn inequality* (FK) *is valid.*

Furthermore, taking C_2 to be the best constant in $(I_{2,\mathrm{gen}}^{\mathrm{eucl.}})$, and Λ to be the best constant in (FK), *one has that $C_2^{-1} \leq \Lambda \leq C(n)C_2^{-1}$ where $C(n) > 1$ is explicit and depends only on n.*

PROOF: The proof follows the lines of [39]. Suppose first that there exists $C_2 > 0$ such that for any $u \in \mathcal{D}(M)$,

$$\left(\int_M |u|^p\, dv(g)\right)^{2/p} \leq C_2 \int_M |\nabla u|^2\, dv(g)$$

By Hölder's inequality one easily gets that if Ω is some regular, bounded, open subset of M and if $u \not\equiv 0$ satisfies

$$\begin{cases} \Delta_g u = \lambda_1^D(\Omega) u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

then

$$\frac{\int_\Omega |\nabla u|^2 dv(g)}{\left(\int_\Omega |u|^p dv(g)\right)^{2/p}} \le \lambda_1^D(\Omega)\,\mathrm{Vol}_g(\Omega)^{2/n}$$

Hence, for any regular, bounded, open subset Ω of M,

$$\lambda_1^D(\Omega) \ge C_2^{-1}\,\mathrm{Vol}_g(\Omega)^{-2/n} \tag{8.1}$$

This proves that (i) implies (ii). Suppose now that there exists $\Lambda > 0$ such that for any $\Omega \Subset M$, $\lambda_1^D(\Omega) \ge \Lambda\,\mathrm{Vol}_g(\Omega)^{-2/n}$. Let $\Omega \Subset M$ be given. For $s > n$ we set

$$\Lambda_s(\Omega) = \inf_{U \Subset \Omega} \lambda_1^D(U)\,\mathrm{Vol}_g(U)^{2/s}$$

and we set

$$\mu_s(\Omega) = \inf_{u \in \mathcal{D}(\Omega)} \frac{\int_M |\nabla u|^2 dv(g)}{\left(\int_M |u|^{2s/(s-2)} dv(g)\right)^{(s-2)/s}}$$

Since $2s/(s-2) < p$ when $s > n$, one easily gets by standard variational techniques that for any $s > n$, there exists $u_s \in C^\infty(\Omega) \cap H_{0,1}^2(\Omega)$ such that

$$\begin{cases} \Delta_g u_s = \mu_s(\Omega) u_s^{(s+2)/(s-2)} \text{ in } \Omega \\ u_s > 0 \text{ in } \Omega, \ \int_\Omega u_s^{2s/(s-2)} dv(g) = 1 \end{cases}$$

One can then prove (see [39]) that for any $0 \le t \le \|u_s\|_\infty$,

$$\mathrm{Vol}_g(\{x \in \Omega \text{ s.t. } u_s(x) > \|u_s\|_\infty - t\}) \ge \left(\frac{\Lambda_s(\Omega)}{2^{(s+4)/4}}\right)^{s/2} \left(\frac{t}{\mu_s(\Omega)\|u_s\|_\infty^{(s+2)/(s-2)}}\right)^{s/2}$$

Hence, if we set $L = \|u_s\|_\infty$, we get that

$$\begin{aligned} 1 &= \int_\Omega u_s^{2s/(s-2)} dv(g) \\ &= \frac{2s}{(s-2)} \int_0^L \mathrm{Vol}_g(\{x \in \Omega \quad \text{s.t. } u_s(x) > t\}) t^{(s+2)/(s-2)} dt \\ &= \frac{2s}{(s-2)} \int_0^L \mathrm{Vol}_g(\{x \in \Omega \quad \text{s.t. } u_s(x) > L - t\}) (L-t)^{(s+2)/(s-2)} dt \\ &\ge \frac{2s}{(s-2)} \left(\frac{\Lambda_s(\Omega)}{2^{(s+4)/4}\mu_s(\Omega) L^{(s+2)/(s-2)}}\right)^{s/2} \int_0^L t^{s/2}(L-t)^{(s+2)/(s-2)} dt \end{aligned}$$

But

$$\int_0^1 \theta^{x-1}(1-\theta)^{y-1} d\theta = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ is the Euler function, and

$$\int_0^L t^{s/2}(L-t)^{(s+2)/(s-2)} dt = L^{s(s+2)/2(s-2)} \int_0^1 \theta^{s/2}(1-\theta)^{(s+2)/(s-2)} d\theta$$

As a consequence, we get that for any $s > n$,

$$\Lambda_s(\Omega) \le C(s)\mu_s(\Omega)$$

where

$$C(s) = 2^{(s+4)/4}\left(\frac{2s\Gamma(1+s/2)\Gamma(2s/(s-2))}{(s-2)\Gamma\big((s^2+4s-4)/2(s-2)\big)}\right)^{-2/s}$$

Now, by assumption,

$$\Lambda_s(\Omega) \geq \Lambda\, \mathrm{Vol}_g(\Omega)^{2/s-2/n}$$

Hence, if $u \in \mathcal{D}(\Omega)$,

$$\begin{aligned}\left(\int_M |u|^p\, dv(g)\right)^{2/p} &= \lim_{s\to n^+}\left(\int_M |u|^{2s/(s-2)}\, dv(g)\right)^{(s-2)/s}\\ &\leq \lim_{s\to n^+}\left(C(s)\Lambda_s(\Omega)^{-1}\right)\int_M |\nabla u|^2\, dv(g)\\ &\leq \frac{1}{\Lambda}\left(\lim_{s\to n^+} C(s)\, \mathrm{Vol}_g(\Omega)^{2/n-2/s}\right)\int_M |\nabla u|^2\, dv(g)\\ &\leq \frac{C(n)}{\Lambda}\int_M |\nabla u|^2\, dv(g)\end{aligned}$$

and we get that for any $\Omega \Subset M$ and any $u \in \mathcal{D}(\Omega)$,

$$\left(\int_M |u|^p\, dv(g)\right)^{2/p} \leq \frac{C(n)}{\Lambda}\int_M |\nabla u|^2\, dv(g) \tag{8.2}$$

This ends the proof of the first part of the proposition. The second part easily follows from (8.1) and (8.2). □

Before stating the next result, let us say some words about the existence of positive Green functions on complete, noncompact Riemannian manifolds. Let (M, g) be a complete, noncompact Riemannian manifold and let x be some point of M. One can then prove that, uniformly with respect to x, either there exist positive Green functions of pole x, and in particular there exists a positive minimal Green's function of pole x, or there does not exist any positive Green function of pole x. More precisely, let $\Omega \Subset M$ be such that $x \in \Omega$ and let G be the solution of

$$\begin{cases}\Delta_g G = \delta_x & \text{in } \Omega\\ G = 0 & \text{on } \partial\Omega\end{cases}$$

Set $G_x^\Omega(y) = G(y)$ when $y \in \Omega$, $G_x^\Omega(y) = 0$ otherwise. Obviously, $G_x^\Omega \leq G_x^{\Omega'}$ if $\Omega \subset \Omega'$.

One then has the following:

THEOREM 8.1 *Set* $G_x(y) = \sup_{\{\Omega \text{ s.t. } x\in\Omega\}} G_x^\Omega(y)$, $y \in M$. *Then,*

(i) *either* $G_x(y) = +\infty$, $\forall y \in M$, *or*
(ii) $G_x(y) < +\infty$, $\forall y \in M\backslash\{x\}$.

This alternative does not depend on x *and in case* (ii), G_x *is the positive minimal Green function of pole* x.

In case (i) the manifold is said to be parabolic; in case (ii) the manifold is said to be nonparabolic. By Cheng-Yau [48], one has that if for some $x \in M$,

$$\liminf_{r \to +\infty} \frac{\mathrm{Vol}_g\left(B_x(r)\right)}{r^2} < +\infty$$

then (M, g) is parabolic. This explains, for instance, why $\mathbb{R}^2$ is parabolic while $\mathbb{R}^3$ is not. More generally, it is proved in Grigor'yan [94] and Varopoulos [189] that if for some $x \in M$,

$$\int_1^{+\infty} \frac{r\,dr}{\mathrm{Vol}_g\left(B_x(r)\right)} = +\infty$$

then again, (M, g) is parabolic. Conversely, Varopoulos proved in [189] that if the Ricci curvature of (M, g) is nonnonnegative and if

$$\int_1^{+\infty} \frac{r\,dr}{\mathrm{Vol}_g\left(B_x(r)\right)} < +\infty$$

then (M, g) is nonparabolic. Independently (see Grigor'yan [94]), one has that if

$$\int_1^{+\infty} \frac{dr}{h(r)^2} < +\infty$$

where

$$h(r) = \inf_{\{\Omega \Subset M \text{ s.t. } \mathrm{Vol}_g(\Omega) \le r\}} \mathrm{Area}_g(\partial\Omega)$$

then (M, g) is nonparabolic. For more details on these questions we refer the reader to Cheng-Yau [48], Grigor'yan [94], Varopoulos [189], and the references contained in these papers.

Let us now prove the following result. Extracted from Carron [39], it gives a very nice answer to the question we asked at the beginning of this section.

THEOREM 8.2 *Let (M, g) be a smooth, complete Riemannian n-manifold of infinite volume, $n \ge 3$. The following two propositions are equivalent*:

(i) *The Euclidean-type generic Sobolev inequality* $(\mathrm{I}_{2,\mathrm{gen}}^{\mathrm{eucl.}})$ *is valid.*

(ii) (M, g) *is nonparabolic and there exists $K > 0$ such that for any $x \in M$ and any $t > 0$,*

$$\mathrm{Vol}_g(\{y \in M \text{ s.t. } G_x(y) > t\}) \le K t^{-n/(n-2)}$$

where G_x is the positive minimal Green function of pole x.

PROOF: The proof follows the lines of [39]. Suppose first that there exists $C_2 > 0$ such that for any $u \in \mathcal{D}(M)$,

$$\left(\int_M |u|^p\,dv(g)\right)^{2/p} \le C_2 \int_M |\nabla u|^2\,dv(g)$$

Let $x \in M$, let Ω be some regular, bounded, open subset of M such that $x \in \Omega$, and set

$$\Phi_t(y) = \min\left(G_x^\Omega(y), t\right)$$

where G_x^Ω is as in Theorem 8.1 and where $t > 0$ is given. Applying the above inequality to Φ_t we get that

$$\begin{aligned}\int_M |\nabla \Phi_t|^2 \, dv(g) &\geq C_2^{-1}\left(\int_M \Phi_t^{2n/(n-2)} \, dv(g)\right)^{(n-2)/n}\\ &\geq C_2^{-1} \operatorname{Vol}_g\left(\left\{y \in M \text{ s.t. } G_x^\Omega(y) > t\right\}\right)^{(n-2)/n} t^2\end{aligned}$$

while if $\Theta = \{y \in M \text{ s.t. } G_x^\Omega(y) < t\}$ and $\theta = \{y \in M \text{ s.t. } G_x^\Omega(y) = t\}$,

$$\begin{aligned}\int_M |\nabla \Phi_t|^2 \, dv(g) &= \int_\Theta |\nabla G_x^\Omega|^2 \, dv(g)\\ &= \int_\Theta G_x^\Omega (\Delta_g G_x^\Omega) dv(g) - t \int_\theta (\partial_n G_x^\Omega) ds = t\end{aligned}$$

since $\Delta_g G_x^\Omega = 0$ in $\Omega \backslash \{x\}$. As a consequence, for any $x \in M$, any $t > 0$, and any bounded, open subset Ω of M such that $x \in \Omega$,

$$\operatorname{Vol}_g\left(\left\{y \in M \text{ s.t. } G_x^\Omega(y) > t\right\}\right) \leq C_2^{n/(n-2)} t^{-n/(n-2)}$$

By Theorem 8.1 one then gets that (M, g) is nonparabolic and that for any $x \in M$ and any $t > 0$,

$$\operatorname{Vol}_g\left(\left\{y \in M \text{ s.t. } G_x(y) > t\right\}\right) \leq C_2^{n/(n-2)} t^{-n/(n-2)}$$

where G_x is the positive minimal Green function of pole x. This proves that (i) implies (ii).

Suppose now that (M, g) is nonparabolic and that there exists $K > 0$ such that for any $x \in M$ and any $t > 0$,

$$\operatorname{Vol}_g\left(\left\{y \in M \text{ s.t. } G_x(y) > t\right\}\right) \leq K t^{-n/(n-2)}$$

Let Ω be some regular, bounded, open subset of M and let $u \not\equiv 0$ be such that

$$\begin{cases}\Delta_g u = \lambda_1^D(\Omega) u & \text{in } \Omega\\ u = 0 & \text{on } \partial\Omega\end{cases}$$

For any $x \in \Omega$,

$$u(x) = \lambda_1^D(\Omega) \int_\Omega G_x^\Omega u \, dv(g) \leq \lambda_1^D(\Omega)\left(\int_\Omega G_x^\Omega \, dv(g)\right)\left(\sup_{y\in\Omega} u(y)\right)$$

where G_x^Ω is as in Theorem 8.1. We now choose $x \in \Omega$ such that

$$u(x) = \sup_{y\in\Omega} u(y)$$

Since $G_x^\Omega \leq G_x$, we get that

$$1 \leq \lambda_1^D(\Omega) \int_\Omega G_x \, dv(g) = \lambda_1^D(\Omega) \int_0^{+\infty} \operatorname{Vol}_g\left(\left\{y \in \Omega \text{ s.t. } G_x(y) > t\right\}\right) dt$$

while, by assumption,

$$\operatorname{Vol}_g\left(\left\{y \in \Omega \text{ s.t. } G_x(y) > t\right\}\right) \leq \min\left(\operatorname{Vol}_g(\Omega), K t^{-n/(n-2)}\right)$$

As a consequence,

$$\int_0^{+\infty} \mathrm{Vol}_g(\{y \in \Omega \text{ s.t. } G_x(y) > t\})dt \leq T\,\mathrm{Vol}_g(\Omega) + K\int_T^{+\infty} t^{-n/(n-2)}\,dt$$

where T is such that $KT^{-n/(n-2)} = \mathrm{Vol}_g(\Omega)$. Hence,

$$\int_0^{+\infty} \mathrm{Vol}_g\left(\{y \in \Omega \text{ s.t. } G_x(y) > t\}\right)dt \leq \frac{n}{2}K^{(n-2)/n}\,\mathrm{Vol}_g(\Omega)^{2/n}$$

and we get that for any $\Omega \Subset M$,

$$\lambda_1^D(\Omega) \geq \frac{2}{n}K^{-(n-2)/n}\,\mathrm{Vol}_g(\Omega)^{-2/n}$$

By Proposition 8.1, this ends the proof of the theorem. □

Now that we have answered our question for $q = 2$, and hence for $q \geq 2$, let us discuss the case $q = 1$, namely, the validity of $(\mathrm{I}_{1,\mathrm{gen}}^{\mathrm{eucl.}})$: There exists C_1 such that for any $u \in \mathcal{D}(M)$,

$$(\mathrm{I}_{1,\mathrm{gen}}^{\mathrm{eucl.}}) \qquad \left(\int_M |u|^p\,dv(g)\right)^{1/p} \leq C_1\int_M |\nabla u|\,dv(g)$$

where $1/p = 1 - 1/n$. We start here by a result of Hoffman and Spruck **[121]**. As a consequence of their work (see also Michael and Simon **[157]**), one has the following:

THEOREM 8.3 *The Euclidean-type generic Sobolev inequality* $(\mathrm{I}_{1,\mathrm{gen}}^{\mathrm{eucl.}})$ *is valid on any smooth, complete, simply connected Riemannian manifold of nonpositive sectional curvature.*

PROOF: In **[121]**, Hoffman and Spruck studied the case of submanifolds M of a Riemannian manifold $\overline{M}$. For $M \to \overline{M}$ an isometric immersion of Riemannian manifolds of dimension n and m, respectively, let

$\overline{K}$ = sectional curvature in $\overline{M}$
H = mean curvature vector field of the immersion
$\overline{R}$ = injectivity radius of $\overline{M}$ restricted to M

We assume that $\overline{K} \leq \alpha^2$, where α is a positive real number or a pure imaginary one. Let g be the Riemannian metric on M. As a main point, that we assume here, Hoffman and Spruck **[121]** got that for any C^1 function u vanishing on ∂M

$$\left(\int_M |u|^{n/(n-1)}\,dv(g)\right)^{(n-1)/n} \leq C(n,\theta)\int_M \left(|\nabla u| + |H| \times |u|\right)dv(g)$$

provided that

$$\alpha^2(1-\theta)^{-2/n}\left(\omega_n^{-1}\,\mathrm{Vol}_g(\mathrm{supp}\,u)\right)^{2/n} \leq 1$$

and $2\rho_0 \leq \overline{R}$, where

$$\rho_0 = \begin{cases} \alpha^{-1}\sin^{-1}\alpha(1-\theta)^{-1/n}\left(\omega_n^{-1}\,\mathrm{Vol}_g(\mathrm{supp}\,u)\right)^{1/n} & \text{for } \alpha \text{ real} \\ (1-\theta)^{-1/n}\left(\omega_n^{-1}\,\mathrm{Vol}_g(\mathrm{supp}\,u)\right)^{1/n} & \text{for } \alpha \text{ imaginary} \end{cases}$$

In such a result, $\theta \in (0, 1)$ is a free parameter, $C(n, \theta) > 0$ depends only on n and θ, and $\mathrm{Vol}_g(\mathrm{supp}\, u)$ stands for the volume with respect to g of the support of u. Starting from the above inequality, and noting that for $M = \overline{M}$ as in the theorem, $\overline{R} = +\infty$, one easily gets that the $(\mathrm{I}^{\mathrm{eucl.}}_{1,\mathrm{gen}})$ inequality is valid. □

REMARK 8.1. Let (M, g) be a smooth, complete, simply connected Riemannian n-manifold with sectional curvature less than or equal to $K < 0$. Define the Cheeger constant $I_\infty(M)$ of M by

$$I_\infty(M) = \inf_\Omega \frac{\mathrm{Area}_g(\Omega)}{\mathrm{Vol}_g(\Omega)}$$

where Ω ranges over smooth bounded domains of M. As a simple application of the divergence theorem, Yau [200] got that $I_\infty(M) \geq (n-1)\sqrt{-K}$. By standard arguments, as used in Federer [79] and Federer-Fleming [80] (see, for instance, Chavel [45]), one then gets that for any $u \in \mathcal{D}(M)$,

$$\int_M |u| dv(g) \leq \frac{1}{(n-1)\sqrt{-K}} \int_M |\nabla u| dv(g)$$

As a remark, by taking $\varphi = |u|^q$ in such an inequality, this in turn implies that for any $q \geq 1$ and any $u \in \mathcal{D}(M)$,

$$\int_M |u|^q \, dv(g) \leq \left(\frac{q}{(n-1)\sqrt{-K}}\right)^q \int_M |\nabla u|^q \, dv(g)$$

In particular, for such manifolds,

$$\|u\| = \left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q}$$

is a norm on $H_1^q(M)$ which is equivalent to the standard one.

Since the validity of $(\mathrm{I}^{\mathrm{eucl.}}_{1,\mathrm{gen}})$ implies that of $(\mathrm{I}^{\mathrm{eucl.}}_{2,\mathrm{gen}})$, one has that the validity of $(\mathrm{I}^{\mathrm{eucl.}}_{1,\mathrm{gen}})$ implies that (M, g) is nonparabolic and that there exists $K > 0$ such that for any $x \in M$ and any $t > 0$,

$$\mathrm{Vol}_g(\{y \in M \text{ s.t. } G_x(y) > t\}) \leq K t^{-n/(n-2)}$$

where G_x is the positive minimal Green function of pole x. As a remark, this result was already contained in Grigor'yan [94]. One can now ask if such a necessary condition is also sufficient. The answer is positive if the Ricci curvature is nonnegative, but negative in general. Such results are due to Carron [40], Coulhon-Ledoux [57], and Varopoulos [191]. More precisely, one has the following:

THEOREM 8.4 (i) *Let (M, g) be a smooth, complete Riemannian n-manifold of infinite volume, $n \geq 3$. If (M, g) has a nonnegative Ricci curvature, then $(\mathrm{I}^{\mathrm{eucl.}}_{1,\mathrm{gen}})$ is valid if and only if (M, g) is nonparabolic and there exists $K > 0$ such that for any $x \in M$ and any $t > 0$,*

$$\mathrm{Vol}_g\left(\left\{y \in M \text{ s.t. } G_x(y) > t\right\}\right) \leq K t^{-n/(n-2)}$$

where G_x is the positive minimal Green function of pole x. In particular, the validity of $(\mathrm{I}^{\mathrm{eucl.}}_{1,\mathrm{gen}})$ is equivalent to the validity of $(\mathrm{I}^{\mathrm{eucl.}}_{2,\mathrm{gen}})$.

(ii) *For any $n \geq 3$, there exist smooth, complete Riemannian n-manifolds of infinite volume for which $(I_{2,\mathrm{gen}}^{\mathrm{eucl.}})$ is valid but $(I_{1,\mathrm{gen}}^{\mathrm{eucl.}})$ is not. Furthermore, one can choose these manifolds such that the sectional curvature is bounded and the injectivity radius is positive.*

PROOF: We restrict ourselves to the proof of point (i). For point (ii), we refer to Coulhon-Ledoux [57]. By Lemma 8.1 and Theorem 8.2, if $(I_{1,\mathrm{gen}}^{\mathrm{eucl.}})$ is valid, then (M, g) is nonparabolic and there exists $K > 0$ such that for any $x \in M$ and any $t > 0$,

$$\mathrm{Vol}_g\left(\left\{y \in M \text{ s.t. } G_x(y) > t\right\}\right) \leq K t^{-n/(n-2)}$$

where G_x is the positive minimal Green function of pole x. Conversely, let us assume that (M, g) is nonparabolic and that there exists $K > 0$ such that for any $x \in M$ and any $t > 0$,

$$\mathrm{Vol}_g\left(\left\{y \in M \text{ s.t. } G_x(y) > t\right\}\right) \leq K t^{-n/(n-2)}$$

where G_x is the positive minimal Green function of pole x. Under such assumptions, and according to Theorem 8.2, $(I_{2,\mathrm{gen}}^{\mathrm{eucl.}})$ is valid. What we have to prove is that $(I_{1,\mathrm{gen}}^{\mathrm{eucl.}})$ is also valid. Following Carron [40], this will be a consequence of the work of Buser [35]. Let Ω be some domain in M, and let $r > 0$ real. We define

$$\tilde{\Omega} = \left\{x \in M \text{ s.t. } \mathrm{Vol}_g\left(B_x(r) \cap \Omega\right) \geq \frac{1}{2}\mathrm{Vol}_g\left(B_x(r)\right)\right\}$$

By the work of Buser [35], and since (M, g) has nonnegative Ricci curvature,

$$\mathrm{Vol}_g(\tilde{\Omega}) \geq \mathrm{Vol}_g(\Omega) - c_1(n) r \,\mathrm{Area}_g(\partial\Omega)$$

Set

$$r = \frac{\mathrm{Vol}_g(\Omega)}{2c_1(n)\,\mathrm{Area}_g(\partial\Omega)}$$

Then,

$$\mathrm{Vol}_g(\tilde{\Omega}) \geq \frac{1}{2}\mathrm{Vol}_g(\Omega)$$

and $\tilde{\Omega} \neq \emptyset$ if $\Omega \neq \emptyset$. Let $x_0 \in \tilde{\Omega}$ be such that

$$\frac{1}{2}\mathrm{Vol}_g\left(B_{x_0}(r)\right) \leq \mathrm{Vol}_g\left(B_{x_0}(r) \cap \Omega\right) \leq \mathrm{Vol}_g(\Omega) \tag{8.3}$$

Since $(I_{2,\mathrm{gen}}^{\mathrm{eucl.}})$ is valid, and according to Proposition 8.2, the Faber-Krahn inequality (FK) is also valid. We claim here that this implies that

$$\mathrm{Vol}_g\left(B_{x_0}(r)\right) \geq \left(\frac{\Lambda}{2^{n+2}}\right)^{n/2} r^n \tag{8.4}$$

where Λ is given by (FK). Indeed, one has by (FK) that

$$\Lambda\,\mathrm{Vol}_g\left(B_x(r)\right)^{-2/n} \leq \lambda_1^D\left(B_x(r)\right)$$

Let $u(y) = r - d_g(x_0, y)$, d_g the distance associated to g, and take u in the Rayleigh quotient which defines $\lambda_1^D(B_x(r))$. One gets that

$$\lambda_1^D\left(B_x(r)\right) \le \frac{4\,\mathrm{Vol}_g\left(B_{x_0}(r)\right)}{r^2\,\mathrm{Vol}_g\left(B_{x_0}(r/2)\right)}$$

Hence,

$$\mathrm{Vol}_g\left(B_{x_0}(r)\right) \ge \left(\frac{\Lambda r^2}{4}\right)^{\frac{n}{n+2}} \mathrm{Vol}_g\left(B_{x_0}\left(\frac{r}{2}\right)\right)^{\frac{n}{n+2}}$$

By induction as in the proof of Lemma 2.2, this proves the above claim. Going on with the proof of Theorem 8.4, one then gets by (8.3) and (8.4) that

$$c_2(n)\Lambda^{n/2}r^n \le 2\,\mathrm{Vol}_g(\Omega)$$

According to the choice of r, this leads to the existence of $c_3(n) > 0$ such that

$$\mathrm{Vol}_g(\Omega)^{(n-1)/n} \le c_3(n)\frac{1}{\sqrt{\Lambda}}\,\mathrm{Area}_g(\partial\Omega)$$

By classical arguments based on the co-area formula, as developed in the proof of Lemma 3.3 of Chapter 3, one easily gets from such an inequality that $(\mathrm{I}_{1,\mathrm{gen}}^{\mathrm{eucl.}})$ is valid. □

Finally (see, for instance, [40] for more details), we mention that if (M, g) is a nonparabolic, complete Riemannian n-manifold whose Ricci curvature is bounded from below, and if there exists $K > 0$ such that for any $x \in M$ and any $t > 0$ the positive minimal Green's function G_x of pole x satisfies

$$\mathrm{Vol}_g\left(\left\{y \in M \text{ s.t. } G_x(y) > t\right\}\right) \le K t^{-n/(n-1)}$$

then the Euclidean-type generic Sobolev inequality $(\mathrm{I}_{1,\mathrm{gen}}^{\mathrm{eucl.}})$ is valid. However, such a result is not sharp. Indeed, $(\mathrm{I}_{1,\mathrm{gen}}^{\mathrm{eucl.}})$ is valid for the Euclidean space $\mathbb{R}^n$ while the condition above is obviously not satisfied by the positive minimal Green function G_x of $\mathbb{R}^n$. Recall here that

$$G_x(y) = \frac{1}{(n-2)\omega_{n-1}|y-x|^{n-2}}$$

where ω_n denotes the volume of the standard unit sphere (S^n, h) of $\mathbb{R}^{n+1}$.

8.2. Euclidean-Type Optimal Sobolev Inequalities

We discuss in this section the value of the best constant C_q in $(\mathrm{I}_{q,\mathrm{gen}}^{\mathrm{eucl.}})$. By Theorem 4.4, one has that $C_q = K(n,q)^q$ for the Euclidean n-dimensional space. Using the same arguments as the ones used in the proof of Proposition 4.2 of Chapter 4, one can easily get that for any smooth, complete Riemannian n-manifold of infinite volume, if $(\mathrm{I}_{q,\mathrm{gen}}^{\mathrm{eucl.}})$ is valid, then $C_q \ge K(n,q)^q$. The point we would like to discuss here is the so-called Cartan-Hadamard conjecture. By definition, a Cartan-Hadamard manifold is a smooth, complete, simply connected Riemannian manifold of nonpositive sectional curvature. For such manifolds, as already seen in theorem 8.3 of Hoffman and Spruck, $(\mathrm{I}_{1,\mathrm{gen}}^{\mathrm{eucl.}})$ is valid. The Cartan-Hadamard conjecture, a longstanding conjecture in the mathematical literature, states that for Cartan-Hadamard manifolds, ($I^{\text{eucl.}}_{1,\text{gen}}$) holds with the best possible value $C_1 = K(n, 1)$. In other words, the Cartan-Hadamard conjecture states that for Cartan-Hadamard n-dimensional manifolds, the Euclidean-type optimal Sobolev inequality ($I^{\text{eucl.}}_{1,\text{opt}}$) is valid. Concerning the terminology, we say that ($I^{\text{eucl.}}_{1,\text{opt}}$) is valid if for any $u \in \mathcal{D}(M)$,

$$(I^{\text{eucl.}}_{1,\text{opt}}) \qquad \left(\int_M |u|^p \, dv(g)\right)^{1/p} \le K(n, 1) \int_M |\nabla u| dv(g)$$

where $1/p = 1 - 1/n$, and $K(n, 1)$ is as in Theorem 4.4. As a remark, this is equivalent to saying that for any smooth, bounded domain Ω on a Cartan-Hadamard n-dimensional manifold (M, g),

$$\text{Area}_g(\partial\Omega) \ge \frac{1}{K(n, 1)} \text{Vol}_g(\Omega)^{1-\frac{1}{n}}$$

The proof of such an assertion is quite standard, and goes back to Federer and Fleming [**80**]. In order to see this, let us prove that

$$\inf_u \frac{\int_M |\nabla u| dv(g)}{\left(\int_M |u|^{n/(n-1)} dv(g)\right)^{(n-1)/n}} = \inf_\Omega \frac{\text{Area}_g(\partial\Omega)}{\text{Vol}_g(\Omega)^{1-\frac{1}{n}}} \tag{8.5}$$

As a starting point, consider Ω a smooth bounded domain in (M, g). For sufficiently small $\varepsilon > 0$, let u_ε be the function

$$u_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in \Omega \\ 1 - \frac{1}{\varepsilon} d_g(x, \partial\Omega) & \text{if } x \in M\backslash\Omega, d_g(x, \partial\Omega) < \varepsilon \\ 0 & \text{if } x \in M\backslash\Omega, d_g(x, \partial\Omega) \ge \varepsilon \end{cases}$$

where d_g stands for the distance associated to g. Clearly, u_ε is Lipschitz for all $\varepsilon > 0$. As one easily sees,

$$\lim_{\varepsilon \to 0} \int_M u_\varepsilon^{n/(n-1)} dv(g) = \text{Vol}_g(\Omega)$$

Furthermore,

$$|\nabla u_\varepsilon| = \begin{cases} \frac{1}{\varepsilon} & \text{if } x \in M\backslash\overline{\Omega}, d_g(x, \partial\Omega) < \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

Hence,

$$\lim_{\varepsilon \to 0} \int_M |\nabla u_\varepsilon| dv(g) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \text{Vol}_g\left(\left\{x \notin \Omega / d_g(x, \partial\Omega) < \varepsilon\right\}\right) = \text{Area}_g(\partial\Omega)$$

and one gets that

$$\inf_u \frac{\int_M |\nabla u| dv(g)}{\left(\int_M |u|^{n/(n-1)} dv(g)\right)^{(n-1)/n}} \le \inf_\Omega \frac{\text{Area}_g(\partial\Omega)}{\text{Vol}_g(\Omega)^{1-\frac{1}{n}}}$$

Let us now prove the opposite inequality, that is

$$\inf_u \frac{\int_M |\nabla u| dv(g)}{\left(\int_M |u|^{n/(n-1)} dv(g)\right)^{(n-1)/n}} \ge \inf_\Omega \frac{\text{Area}_g(\partial\Omega)}{\text{Vol}_g(\Omega)^{1-\frac{1}{n}}} \tag{8.6}$$

Given $u \in \mathcal{D}(M)$, let

$$\Omega(t) = \{x \,/\, |u(x)| > t\}$$

and $V(t) = \mathrm{Vol}_g(\Omega(t))$, for $t \in R_u$, the regular values of u. The proof of (8.6) is based on the co-area formula (see Chavel [45]):

$$\int_M f|\nabla u| dv(g) = \int_0^\infty \left(\int_{\Sigma_t} f\, d\sigma\right) dt$$

where $\Sigma_t = |u|^{-1}(t)$. Indeed, one gets by the co-area formula that

$$\int_M |\nabla u| dv(g) \geq \left(\inf_\Omega \frac{\mathrm{Area}_g(\partial\Omega)}{\mathrm{Vol}_g(\Omega)^{1-\frac{1}{n}}}\right) \int_0^\infty V(t)^{1-\frac{1}{n}}\, dt$$

and

$$\begin{aligned}\int_M |u|^{n/(n-1)}\, dv(g) &= \int_M \left(\int_0^{|u|} \frac{n}{n-1} t^{1/(n-1)} dt\right) dv(g)\\ &= \frac{n}{n-1} \int_0^\infty \left(\int_{\Omega(t)} dv(g)\right) t^{1/(n-1)} dt\\ &= \frac{n}{n-1} \int_0^\infty t^{1/(n-1)} V(t) dt\end{aligned}$$

In order to prove (8.6), it suffices then to prove that

$$\int_0^\infty V(t)^{1-\frac{1}{n}}\, dt \geq \left(\frac{n}{n-1}\int_0^\infty t^{1/(n-1)} V(t) dt\right)^{1-\frac{1}{n}} \tag{8.7}$$

To establish (8.7), set

$$F(s) = \int_0^s V(t)^{1-\frac{1}{n}}\, dt\,, \qquad G(s) = \left(\frac{n}{n-1}\int_0^s t^{1/(n-1)} V(t) dt\right)^{1-\frac{1}{n}}$$

One has that $F(0) = G(0)$, and since $V(s)$ is a decreasing function of s,

$$\begin{aligned}G'(s) &= \frac{n-1}{n}\left(\frac{n}{n-1}\right)^{1-\frac{1}{n}} \left(\int_0^s t^{1/(n-1)} V(t) dt\right)^{-1/n} s^{1/(n-1)} V(s)\\ &\leq \left(\frac{n}{n-1}\right)^{-1/n} \left(\int_0^s t^{1/(n-1)} dt\right)^{-1/n} s^{1/(n-1)} V(s)^{1-\frac{1}{n}}\\ &= V(s)^{1-\frac{1}{n}} = F'(s)\end{aligned}$$

Clearly, (8.7) easily follows. Hence, (8.6) is true, and then (8.5) is also true. This proves the claim.

Given (M, g) some Cartan-Hadamard n-dimensional manifold, and $q \in [1, n)$, let us denote by $(\mathrm{I}_{q,\mathrm{opt}}^{\mathrm{eucl.}})$ the following optimal inequality: For any $u \in \mathcal{D}(M)$,

$$\left(\int_M |u|^p\, dv(g)\right)^{1/p} \leq K(n,q)\left(\int_M |\nabla u|^q\, dv(g)\right)^{1/q} \tag{$\mathrm{I}_{q,\mathrm{opt}}^{\mathrm{eucl.}}$}$$

where $1/p = 1/q - 1/n$ and $K(n, q)$ is as in Theorem 4.4. Coming back to ideas developed in Aubin [10], one gets the following result:

PROPOSITION 8.2 *Let (M, g) be a n-dimensional Cartan-Hadamard manifold. Suppose that $(I_{1,\mathrm{opt}}^{\mathrm{eucl.}})$ is valid on (M, g). Then for any $q \in [1, n)$, $(I_{q,\mathrm{opt}}^{\mathrm{eucl.}})$ is valid on (M, g).*

PROOF: Let (M, g) be an n-dimensional Cartan-Hadamard manifold. By assumption, according to what has been said above, one has that for any smooth bounded domain Ω in M,

$$\frac{\mathrm{Area}_g(\partial\Omega)}{\mathrm{Vol}_g(\Omega)^{1-\frac{1}{n}}} \geq \frac{1}{K(n,1)}$$

Let e be the Euclidean metric in $\mathbb{R}^n$. Since for any ball B in $\mathbb{R}^n$,

$$\frac{\mathrm{Area}_e(\partial B)}{\mathrm{Vol}_e(B)^{1-\frac{1}{n}}} = \frac{1}{K(n,1)}$$

one gets that for any smooth bounded domain Ω in M, and any ball B in $\mathbb{R}^n$,

$$\frac{\mathrm{Area}_g(\partial\Omega)}{\mathrm{Vol}_g(\Omega)^{1-\frac{1}{n}}} \geq \frac{\mathrm{Area}_e(\partial B)}{\mathrm{Vol}_e(B)^{1-\frac{1}{n}}} \tag{8.8}$$

By classical Morse theory (see, for instance, Aubin [12] for the following claim), it suffices to prove $(I_{q,\mathrm{opt}}^{\mathrm{eucl.}})$ for continuous nonnegative functions u with compact support K, K being itself smooth, u being smooth in K and such that it has only nondegenerate critical points in K. For such a u, let $u^\star : \mathbb{R}^n \to \mathbb{R}$, radially symmetric, nonnegative, and decreasing with respect to $|x|$, be defined by:

$$\mathrm{Vol}_e\left(\left\{x \in \mathbb{R}^n \,/\, u^\star(x) \geq t\right\}\right) = \mathrm{Vol}_g(\{x \in M \,/\, u(x) \geq t\}) \tag{8.9}$$

One can check that $u^\star$ has compact support and is Lipschitz. Set

$$V(t) = \mathrm{Vol}_g(\{x \in M \,/\, u(x) \geq t\})$$

and let $\Sigma_t = u^{-1}(t)$. All that follows is based on the co-area formula (see Chavel [45]):

$$\int_M f\, dv(g) = \int_0^\infty \left(\int_{\Sigma_t} \frac{f}{|\nabla u|}\, d\sigma\right) dt$$

One has here that

$$V'(t) = -\int_{\Sigma_t} |\nabla u|^{-1}\, d\sigma$$

Independently, by Hölder's inequality, and for $m \geq 1$ real,

$$\int_{\Sigma_t} d\sigma \leq \left(\int_{\Sigma_t} |\nabla u|^{-1}\, d\sigma\right)^{(m-1)/m} \left(\int_{\Sigma_t} |\nabla u|^{m-1}\, d\sigma\right)^{1/m}$$

According to (8.8),

$$\mathrm{Area}_g(\Sigma_t) \geq \mathrm{Area}_e(\Sigma_t^\star)$$

where $\Sigma_t^\star = (u^\star)^{-1}(t)$. Since $|\nabla u^\star|$ is constant on $\Sigma_t^\star$, one gets with (8.9) that

$$\begin{aligned}\int_{\Sigma_t} |\nabla u|^{m-1}\, d\sigma &\geq \mathrm{Area}_e(\Sigma_t^\star)^m |\nabla u^\star|^{m-1}\, \mathrm{Area}_e(\Sigma_t^\star)^{1-m} \\ &= \int_{\Sigma_t^\star} |\nabla u^\star|^{m-1}\, d\sigma\end{aligned}$$

Integrating with respect to t then gives by the co-area formula that

$$\int_M |\nabla u|^m \, dv(g) \geq \int_{R^n} |\nabla u^\star|^m \, dx$$

Similarly, for $m \geq 1$ real, one gets with the co-area formula that

$$\int_M u^m \, dv(g) = -\int_0^\infty t^m V'(t) dt$$

so that, as above, one gets with (8.9) that

$$\int_M u^m \, dv(g) = \int_{R^n} (u^\star)^m \, dx$$

By Theorem 4.4, one can then write that for $q \in [1, n)$ real, and u and $u^\star$ as above,

$$\begin{aligned}\left(\int_M |u|^p \, dv(g)\right)^{1/p} = \left(\int_{R^n} |u^\star|^p \, dx\right)^{1/p} &\leq K(n,q)\left(\int_{R^n} |\nabla u^\star|^q \, dx\right)^{1/q} \\ &\leq K(n,q)\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q}\end{aligned}$$

As already mentioned, this proves the proposition. □

As one can see, the Cartan-Hadamard conjecture has been proved to be true for 2-, 3-, and 4-dimensional Cartan-Hadamard manifolds. Such results are given here without any proof, apart for the 4-dimensional case due to Croke **[60]** that we discuss. The 2-dimensional case is due to Weil **[198]**. One then has the following:

THEOREM 8.5 *The Euclidean-type optimal Sobolev inequality* $(\mathrm{I}^{\mathrm{eucl.}}_{1,\mathrm{opt}})$ *is valid on any 2-dimensional Cartan-Hadamard manifold.*

The 3-dimensional case of the Cartan-Hadamard conjecture is due to Kleiner **[132]**. One then has the following:

THEOREM 8.6 *The Euclidean-type optimal Sobolev inequality* $(\mathrm{I}^{\mathrm{eucl.}}_{1,\mathrm{opt}})$ *is valid on any 3-dimensional Cartan-Hadamard manifold.*

The 4-dimensional case of the Cartan-Hadamard conjecture is due to Croke **[60]**. Here, Croke gets some explicit Euclidean-type generic Sobolev inequality $(\mathrm{I}^{\mathrm{eucl.}}_{1,\mathrm{gen}})$ for all $n \geq 3$, with the property that one recovers $(\mathrm{I}^{\mathrm{eucl.}}_{1,\mathrm{opt}})$ for $n = 4$. For $n \geq 3$, let

$$C(n) = \frac{\omega_{n-2}^{n-2}}{\omega_{n-1}^{n-1}} \left(\int_0^{\pi/2} \cos^{n/(n-2)}(t) \sin^{n-2}(t) dt\right)^{n-2}$$

where ω_n denotes the volume of the standard unit sphere (S^n, h) of $\mathbb{R}^{n+1}$. As one can easily check, $C(4)^{1/4} = K(4, 1)$. Croke's result **[60]** can then be stated as follows:

THEOREM 8.7 *Let* (M, g) *be a* n*-dimensional Cartan-Hadamard manifold,* $n \geq 3$. *For any* $u \in \mathcal{D}(M)$,

$$\left(\int_M |u|^p \, dv(g)\right)^{1/p} \leq C(n)^{\frac{1}{n}} \int_M |\nabla u| dv(g)$$

where $1/p = 1 - 1/n$, and $C(n)$ is as above. In particular, the Euclidean-type optimal Sobolev inequality ($\mathrm{I}^{\text{eucl.}}_{1,\text{opt}}$) is valid on any 4-dimensional Cartan-Hadamard manifold.

PROOF: According to what has been said above, it suffices to prove that for any smooth, bounded domain Ω in M,

$$\frac{\mathrm{Area}_g(\partial\Omega)}{\mathrm{Vol}_g(\Omega)^{1-\frac{1}{n}}} \geq \frac{1}{C(n)^{\frac{1}{n}}}$$

We follow the lines of Croke [60]. Let Ω be a smooth, bounded domain in M. Every geodesic ray in Ω minimizes length up to the point it hits the boundary. Let $\Pi : U\Omega \to \Omega$ represent the unit sphere bundle with the canonical (local product) measure. For $v \in U\Omega$, let γ_v be the geodesic with $\gamma_v'(0) = v$ and let $\xi^t(v)$ represent the geodesic flow, that is, $\xi^t(v) = \gamma_v'(t)$. For $v \in U\Omega$, we let

$$l(v) = \max\left\{t \,/\, \gamma_v(t) \in \Omega\right\}$$

For $x \in \partial\Omega$, we define N_x as the inwardly pointing unit normal vector to $\partial\Omega$ at x. In addition, let $\Pi : U^+\partial\Omega \to \partial\Omega$ be the bundle of inwardly pointing vectors, that is,

$$U^+\partial\Omega = \left\{u \in U\Omega \,/\, \Pi(u) \in \partial\Omega,\ \langle u, N_{\Pi(u)}\rangle > 0\right\}$$

The main tool in the proof of the Theorem is a formula due to Santalo [174]. In such a context, one has that for all integrable functions f,

$$\int_{U\Omega} f(u)du = \int_{U^+\partial\Omega}\left(\int_0^{l(u)} f\left(\xi^t(u)\right)\cos(u)dt\right)du$$

where $\cos(u)$ represents $\langle u, N_{\Pi(u)}\rangle$, and the measure on $U^+\partial\Omega$ is the local product measure du where the measure of the fiber is that of the unit upper hemisphere. From this formula, one gets that

$$\mathrm{Vol}_g(\Omega) = \frac{1}{\omega_{n-1}}\int_{U^+\partial\Omega} l(u)\cos(u)du$$

Moreover, one can prove that

$$\int_{U^+\partial\Omega}\frac{l(u)^{n-1}}{\cos(\mathrm{ant}(u))}du \leq \mathrm{Area}_g(\partial\Omega)^2$$

and that

$$\int_{U^+\partial\Omega}\cos^{\frac{1}{n-2}}(\mathrm{ant}(u))\cos^{\frac{n-1}{n-2}}(u)du \leq A(n)\,\mathrm{Area}_g(\partial\Omega)$$

where

$$A(n) = \omega_{n-2}\int_0^{\frac{\pi}{2}}\cos^{n/(n-2)}(t)\sin^{n-2}(t)dt$$

and $\mathrm{ant}(u) = -\gamma_u'(l(u))$. We refer to Croke [60] for such assertions. By Hölder's inequality, one has that

$$\begin{aligned}\mathrm{Vol}_g(\Omega) &= \frac{1}{\omega_{n-1}}\int_{U^+\partial\Omega} l(u)\cos(u)du \\ &= \frac{1}{\omega_{n-1}}\int_{U^+\partial\Omega} \frac{l(u)}{\cos^{\frac{1}{n-1}}(\mathrm{ant}(u))}\cos^{\frac{1}{n-1}}(\mathrm{ant}(u))\cos(u)du \\ &\le \frac{1}{\omega_{n-1}}\left(\int_{U^+\partial\Omega}\frac{l(u)^{n-1}}{\cos(\mathrm{ant}(u))}du\right)^{\frac{1}{n-1}} \\ &\quad\times\left(\int_{U^+\partial\Omega}\cos^{\frac{1}{n-2}}(\mathrm{ant}(u))\cos^{\frac{n-1}{n-2}}(u)du\right)^{\frac{n-2}{n-1}}\end{aligned}$$

Hence,

$$\mathrm{Vol}_g(\Omega) \le \frac{1}{\omega_{n-1}}\mathrm{Area}_g(\partial\Omega)^{\frac{2}{n-1}}A(n)^{\frac{n-2}{n-1}}\mathrm{Area}_g(\partial\Omega)^{\frac{n-2}{n-1}}$$

and one gets that

$$\frac{\mathrm{Area}_g(\partial\Omega)}{\mathrm{Vol}_g(\Omega)^{1-\frac{1}{n}}} \ge \frac{1}{C(n)^{\frac{1}{n}}}$$

As already mentioned, this proves the theorem. □

Now that these results have been stated, one can ask what happens to the Euclidean-type optimal Sobolev inequality $(\mathrm{I}_{1,\mathrm{opt}}^{\mathrm{eucl.}})$, and more generally to $(\mathrm{I}_{q,\mathrm{opt}}^{\mathrm{eucl.}})$, for the opposite sign of the curvature, namely, for manifolds of nonnegative curvature. By Theorem 7.5, one already knows that for $q > 2$ such that $q^2 < n$, $(\mathrm{I}_{q,\mathrm{opt}}^{\mathrm{eucl.}})$ must be false if the scalar curvature of the manifold is positive somewhere. Similarly, one has by Proposition 5.1 that if $n \ge 4$ and $(\mathrm{I}_{2,\mathrm{opt}}^{\mathrm{eucl.}})$ is valid, then the scalar curvature of the manifold must be nonpositive. The following rigidity result of Ledoux [140] answers the question we just asked:

THEOREM 8.8 *Let (M, g) be a smooth, complete n-dimensional Riemannian manifold with nonnegative Ricci curvature. Suppose that for some $q \in [1, n)$, $(\mathrm{I}_{q,\mathrm{opt}}^{\mathrm{eucl.}})$ is valid. Then (M, g) is isometric to the Euclidean space $(\mathbb{R}^n, e)$ of the same dimension.*

PROOF: Following Ledoux [140], suppose that $(\mathrm{I}_{q,\mathrm{opt}}^{\mathrm{eucl.}})$ is valid for some $q \in [1, n)$. We assume here that $q > 1$, and refer to what is said below for the case $q = 1$. Let x_0 be some given point in M and let r be the distance to x_0. For $\lambda > 0$ real, and $\theta > 1$ real, we set

$$u_{\lambda,\theta}(x) = \left(\lambda + \left(\frac{r}{\theta}\right)^{q/(q-1)}\right)^{1-\frac{n}{q}}$$

Set also

$$F_\theta(\lambda) = \frac{1}{n-1}\int_M \frac{1}{\left(\lambda + (\frac{r}{\theta})^{q/(q-1)}\right)^{n-1}}dv(g)$$

Clearly, F_θ is well defined and of class C^1. Indeed, as one can easily check, an integration by parts leads to

$$F_\theta(\lambda) = \frac{1}{n-1}\int_0^{+\infty} V'(r)\frac{1}{\left(\lambda + (\frac{r}{\theta})^{q/(q-1)}\right)^{n-1}}\,dr = \frac{q}{q-1}\int_0^{+\infty} V(\theta s)\frac{s^{1/(q-1)}}{(\lambda + s^{q/(q-1)})^n}\,ds$$

where $V(s)$ stands for the volume of $B_{x_0}(s)$ with respect to g. From Gromov's comparison theorem (see Theorem 1.1), one has that $V(s) \le V_0(s)$ for any s, where $V_0(s) = \frac{1}{n}\omega_{n-1}s^n$ is the volume of the Euclidean ball $B_0(s)$ in $\mathbb{R}^n$. This proves the above claim. Take now $u = u_{\lambda,\theta}$ in ($\mathrm{I}^{\mathrm{eucl.}}_{q,\mathrm{opt}}$). Since $\theta > 1$ we get that

$$\left(\int_M \frac{1}{\left(\lambda + (\frac{r}{\theta})^{q/(q-1)}\right)^n}\,dv(g)\right)^{1/p} \le K(n,q)\left(\frac{n-q}{q-1}\right)\left(\int_M \frac{(\frac{r}{\theta})^{q/(q-1)}}{\left(\lambda + (\frac{r}{\theta})\right)^n}\,dv(g)\right)^{1/q}$$

Set

$$\alpha = \frac{1}{K(n,q)^q}\left(\frac{q-1}{n-q}\right)^q$$

Then, as one can easily check,

$$\alpha\left(-F'_\theta(\lambda)\right)^{q/p} - \lambda F'_\theta(\lambda) \le (n-1)F_\theta(\lambda)$$

for every $\lambda > 0$. Set now

$$H(\lambda) = \frac{1}{n-1}\int_{R^n} \frac{1}{\left(\lambda + |x|^{q/(q-1)}\right)^{n-1}}\,dx$$

By Theorem 4.4,

$$\alpha\left(-H'(\lambda)\right)^{q/p} - \lambda H'(\lambda) = (n-1)H(\lambda)$$

and one can write that $H(\lambda) = A\lambda^{1-\frac{n}{q}}$, where

$$A = \frac{1}{n-1}\int_{R^n} \frac{1}{(1 + |x|^{q/(q-1)})^{n-1}}\,dx$$

that is, $A = H(1)$. The claim here is that if $F_\theta(\lambda_0) < H(\lambda_0)$ for some $\lambda_0 > 0$, then $F_\theta(\lambda) < H(\lambda)$ for all $\lambda \in (0,\lambda_0)$. In order to see this, suppose that there exists some $\lambda_1 \in (0,\lambda_0)$ such that $F_\theta(\lambda_1) \ge H(\lambda_1)$. Let

$$\lambda_* = \sup\left\{\lambda < \lambda_0 \,/\, F_\theta(\lambda) = H(\lambda)\right\}$$

For $\lambda > 0$, $\varphi_\lambda(X) = \alpha X^{q/p} + \lambda X$ is strictly increasing in $X \ge 0$ so that the differential inequality satisfied by F_θ reads as

$$-F'_\theta(\lambda) \le \varphi_\lambda^{-1}\left((n-1)F_\theta(\lambda)\right)$$

while the differential equality satisfied by H reads as

$$-H'(\lambda) = \varphi_\lambda^{-1}((n-1)H(\lambda))$$

As a consequence,

$$\big(F_\theta - H\big)'(\lambda) \geq \varphi_\lambda^{-1}((n-1)H(\lambda)) - \varphi_\lambda^{-1}\big((n-1)F_\theta(\lambda)\big) \geq 0$$

for λ such that $F_\theta(\lambda) \leq H(\lambda)$. In particular, $(F_\theta - H)' \geq 0$ on $[\lambda_*, \lambda_0]$, and one gets that

$$0 = \big(F_\theta - H\big)(\lambda_*) \leq \big(F_\theta - H\big)(\lambda_0) < 0$$

which is a contradiction. This proves the above claim.

Let us now come back to the expression of F_θ. As one can easily check,

$$\begin{aligned} F_\theta(\lambda) &= \frac{q}{q-1}\int_0^{+\infty} V(\theta s)\frac{s^{1/(q-1)}}{\big(\lambda + s^{q/(q-1)}\big)^n}\,ds \\ &= \frac{q}{q-1}\theta^{\frac{(n-1)q}{q-1}}\int_0^{+\infty} V(s)\frac{s^{1/(q-1)}}{\big(\theta^{q/(q-1)}\lambda + s^{q/(q-1)}\big)^n}\,ds \end{aligned}$$

For $V(s)$ and $V_0(s)$ as above, $V(s) \sim V_0(s)$ as $s \to 0$. One then gets that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\lambda > 0$,

$$\begin{aligned} &\int_0^{+\infty} V(s)\frac{s^{1/(q-1)}}{\big(\theta^{q/(q-1)}\lambda + s^{q/(q-1)}\big)^n}\,ds \\ &\geq (1-\varepsilon)\int_0^{\delta} V_0(s)\frac{s^{1/(q-1)}}{\big(\theta^{q/(q-1)}\lambda + s^{q/(q-1)}\big)^n}\,ds \\ &= \frac{1-\varepsilon}{\big(\theta^{q/(q-1)}\lambda\big)^{\frac{n}{q}-1}}\int_0^{\delta/\theta\lambda^{(q-1)/q}} V_0(s)\frac{s^{1/(q-1)}}{\big(1 + s^{q/(q-1)}\big)^n}\,ds \end{aligned}$$

Recall now that $H(\lambda) = A\lambda^{1-\frac{n}{q}}$, where

$$\begin{aligned} A &= \frac{1}{n-1}\int_{R^n}\frac{1}{\big(1 + |x|^{q/(q-1)}\big)^{n-1}}\,dx \\ &= \frac{q}{q-1}\int_0^{+\infty} V_0(s)\frac{s^{1/(q-1)}}{\big(1 + s^{q/(q-1)}\big)^n}\,ds \end{aligned}$$

Hence,

$$\liminf_{\lambda\to 0}\frac{F_\theta(\lambda)}{H(\lambda)} \geq \theta^n > 1$$

From the above claim, it follows that $F_\theta(\lambda) \geq H(\lambda)$ for all $\lambda > 0$, so that

$$\int_0^{+\infty}\big(V(\theta s) - V_0(s)\big)\frac{s^{1/(q-1)}}{\big(\lambda + s^{q/(q-1)}\big)^n}\,ds \geq 0$$

for all $\lambda > 0$. Letting $\theta \to 1$, this leads to

$$\int_0^{+\infty}\big(V(s) - V_0(s)\big)\frac{s^{1/(q-1)}}{\big(\lambda + s^{q/(q-1)}\big)^n}\,ds \geq 0$$

for all $\lambda > 0$. By Gromov's comparison theorem, Theorem 1.1, $V(s) \leq V_0(s)$ for all s. As a consequence, $V(s) = V_0(s)$ for all s, and by the case of equality in Bishop's comparison theorem (see the Bishop-Gromov's comparison theorem in

Chavel [45]), one gets that (M, g) is isometric to the Euclidean space $(\mathbb{R}^n, e)$ of same dimension. This proves the theorem. □

Regarding Theorem 8.8, note that the particular case $q = 1$ in such a result was well known. In this case, as already mentioned, the Euclidean-type optimal Sobolev inequality is equivalent to the isoperimetric inequality

$$\mathrm{Area}_g(\partial\Omega) \geq \frac{1}{K(n, 1)} \mathrm{Vol}_g(\Omega)^{1-\frac{1}{n}}$$

If we let $V(s)$ be the volume of $B_{x_0}(s)$ with respect to g, we have that

$$\frac{dV(s)}{ds} = \mathrm{Area}_g\left(\partial B_{x_0}(s)\right)$$

Hence, setting $\Omega = B_{x_0}(s)$ in the isoperimetric inequality, we get that

$$\frac{1}{K(n, 1)} V(s)^{(n-1)/n} \leq \frac{dV(s)}{ds}$$

for all s. Integrating yields

$$V(s) \geq \frac{1}{n^n K(n, 1)^n} s^n$$

and since $K(n, 1) = \frac{1}{n} b_n^{-1/n}$, one gets that for every x_0 and for every s,

$$\mathrm{Vol}_g\left(B_{x_0}(s)\right) \geq \mathrm{Vol}_e\left(B_0(s)\right)$$

where $B_0(s)$ is the ball of center 0 and radius s in the Euclidean space $(\mathbb{R}^n, e)$. Under the assumption that (M, g) has nonnegative Ricci curvature, one gets from Gromov's comparison theorem, Theorem 1.1, that for every x_0 and every s,

$$\mathrm{Vol}_g\left(B_{x_0}(s)\right) \leq \mathrm{Vol}_e\left(B_0(s)\right)$$

Hence, for every x_0 and every s,

$$\mathrm{Vol}_g\left(B_{x_0}(s)\right) = \mathrm{Vol}_e\left(B_0(s)\right)$$

and one gets from the case of equality in Bishop's comparison theorem (see the Bishop-Gromov's comparison theorem in Chavel [45]) that (M, g) is isometric to the Euclidean space $(\mathbb{R}^n, e)$.

8.3. Nash's Inequality

Many inequalities may be derived from the Euclidean-type generic Sobolev inequalities $(\mathrm{I}^{\mathrm{eucl.}}_{q,\mathrm{gen}})$. As a very small part of a much more general situation, we restrict our attention here to the case $q = 2$, and discuss the equivalence that exists between the Euclidean-type generic Sobolev inequality $(\mathrm{I}^{\mathrm{eucl.}}_{2,\mathrm{gen}})$ of Section 8.1, and the Nash inequality (N) of Nash [161] (as stated below). We refer the reader to the exhaustive [18] by Bakry, Coulhon, Ledoux, and Saloff-Coste for discussions on Gagliardo-Nirenberg type inequalities

$$\|u\|_r \leq C\|\nabla u\|_q^\alpha \|u\|_s^{1-\alpha}$$

and more information on the subject.

Let (M, g) be a smooth, complete Riemannian n-manifold of infinite volume, $n \geq 3$. We say that the Nash inequality (N) is valid if there exists a constant $C > 0$ such that for any $u \in \mathcal{D}(M)$,

$$\text{(N)} \qquad \left(\int_M u^2\, dv(g)\right)^{1+\frac{2}{n}} \leq C\left(\int_M |\nabla u|^2\, dv(g)\right)\left(\int_M |u|\, dv(g)\right)^{\frac{4}{n}}$$

Such an inequality first appeared in the celebrated Nash **[161]** when discussing the Hölder regularity of solutions of divergence from uniformly elliptic equations. Following what is done in the above-mentioned **[18]** by Bakry, Coulhon, Ledoux, and Saloff-Coste, we prove here the following:

THEOREM 8.9 *Let (M, g) be a smooth, complete Riemannian n-manifold, $n \geq 3$. The Euclidean-type generic Sobolev inequality* $(\mathrm{I}_{2,\mathrm{gen}}^{\mathrm{eucl.}})$ *and the Nash inequality* (N) *are equivalent in the sense that if one of them is valid, the other one is also valid.*

PROOF: The implication $(\mathrm{I}_{2,\mathrm{gen}}^{\mathrm{eucl.}}) \Rightarrow$ (N) easily follows from Hölder's inequality, since for any $u \in \mathcal{D}(M)$,

$$\int_M u^2\, dv(g) \leq \left(\int_M |u|^p\, dv(g)\right)^{\frac{1}{p-1}}\left(\int_M |u|\, dv(g)\right)^{\frac{p-2}{p-1}}$$

where $p = 2n/(n-2)$. The converse implication, (N) $\Rightarrow (\mathrm{I}_{2,\mathrm{gen}}^{\mathrm{eucl.}})$, is a little bit more subtle. Given $u \in \mathcal{D}(M)$, and $k \in \mathbb{Z}$, we let u_k be defined by

$$\begin{cases} u_k(x) = 0 & \text{if } |u(x)| < 2^k \\ u_k(x) = |u(x)| - 2^k & \text{if } 2^k \leq |u(x)| < 2^{k+1} \\ u_k(x) = 2^k & \text{if } |u(x)| \geq 2^{k+1} \end{cases}$$

and we let

$$B_k = \left\{x \in M \,/\, 2^k \leq |u(x)| < 2^{k+1}\right\}$$

Applying Nash's inequality to u_k, one gets that

$$\left(\int_M u_k^2\, dv(g)\right)^{1+\frac{2}{n}} \leq C\left(\int_{B_k} |\nabla u_k|^2\, dv(g)\right)\left(\int_M u_k\, dv(g)\right)^{\frac{4}{n}}$$

Hence,

$$\begin{gathered}\left(2^{2k}\operatorname{Vol}_g\left(\left\{|u| \geq 2^{k+1}\right\}\right)\right)^{1+2/n} \leq \\ C\left(\int_{B_k} |\nabla u|^2\, dv(g)\right)\left(2^k \operatorname{Vol}_g\left(\left\{|u| \geq 2^k\right\}\right)\right)^{4/n}\end{gathered}$$

Set $\alpha = n/(n+2)$, and for every $k \in \mathbb{Z}$,

$$a_k = 2^{pk}\operatorname{Vol}_g\left(\left\{|u| \geq 2^k\right\}\right), \qquad b_k = \int_{B_k} |\nabla u|^2\, dv(g)$$

where p is as above. With this notation, the preceding inequality raised to the power α, and multiplied by 2^{p-2}, yields

$$a_{k+1} \leq 2^p C^\alpha b_k^\alpha a_k^{2(1-\alpha)}$$

By Hölder's inequality,

$$\sum_{k\in\mathbb{Z}} a_k = \sum_{k\in\mathbb{Z}} a_{k+1} \le 2^p C^\alpha \Big(\sum_{k\in\mathbb{Z}} b_k\Big)^\alpha \Big(\sum_{k\in\mathbb{Z}} a_k^2\Big)^{1-\alpha}$$
$$\le 2^p C^\alpha \Big(\sum_{k\in\mathbb{Z}} b_k\Big)^\alpha \Big(\sum_{k\in\mathbb{Z}} a_k\Big)^{2(1-\alpha)}$$

so that

$$\sum_{k\in\mathbb{Z}} a_k \le \left[2^p C^\alpha \Big(\sum_{k\in\mathbb{Z}} b_k\Big)^\alpha\right]^{1/(2\alpha-1)}$$

Clearly,

$$\sum_{k\in\mathbb{Z}} b_k \le \int_M |\nabla u|^2\, dv(g)$$

while

$$\int_M |u|^p\, dv(g) \le (2^p-1)\sum_{k\in\mathbb{Z}} a_k$$

Hence, for any $u \in \mathcal{D}(M)$,

$$\left(\int_M |u|^p\, dv(g)\right)^{2/p} \le (2^p-1)^{2/p} 2^{2(p-1)} C \int_M |\nabla u|^2\, dv(g)$$

so that (N) $\Rightarrow$ ($\mathrm{I}^{\mathrm{eucl.}}_{2,\mathrm{gen}}$). The theorem is proved. □

Concerning the optimal version of Nash's inequality, the sharp constant in (N) has been computed by Carlen and Loss [38] for the Euclidean space. The argument they used is very elegant. Here is their result.

THEOREM 8.10 *For any $u \in \mathcal{D}(\mathbb{R}^n)$,*

$$\left(\int_{R^n} u^2\, dv(g)\right)^{1+\frac{2}{n}} \le C_n \left(\int_{R^n} |\nabla u|^2\, dv(g)\right)\left(\int_{R^n} |u|\, dv(g)\right)^{\frac{4}{n}}$$

where

$$C_n = \frac{(n+2)^{(n+2)/n}}{2^{2/n} n \lambda_1^N(\mathcal{B}) |\mathcal{B}|^{2/n}}$$

is sharp. In such an expression, $|\mathcal{B}|$ denotes the volume of the unit ball $\mathcal{B}$ in $\mathbb{R}^n$, and $\lambda_1^N(\mathcal{B})$ denotes the first nonzero Neumann eigenvalue of the Laplacian on radial functions on $\mathcal{B}$.

PROOF: Let us first prove that the inequality of the theorem does hold. As in the proof of Theorem 4.4, it suffices to establish this inequality for nonnegative, radially symmetric, decreasing functions. Following Carlen and Loss [38], let u be such a function. For $r > 0$ arbitrary, let

$$v(x) = \begin{cases} u(x) & \text{if } |x| \le r \\ 0 & \text{if } |x| > r \end{cases} \quad \text{and} \quad w(x) = \begin{cases} 0 & \text{if } |x| \le r \\ u(x) & \text{if } |x| > r \end{cases}$$

Clearly,

$$\|u\|_2^2 = \|v\|_2^2 + \|w\|_2^2$$

and since u is radially decreasing,

$$w(x) \le u(r) \le \frac{1}{|\mathcal{B}|r^n}\|v\|_1$$

In particular,

$$\|w\|_2^2 \le \frac{1}{|\mathcal{B}|r^n}\|v\|_1\|w\|_1$$

Let

$$\bar{v} = \frac{1}{|\mathcal{B}|r^n}\|v\|_1$$

be the average of v. One gets from the variational characterization of λ_1^N that

$$\begin{aligned}
\|v\|_2^2 &= \int_{\mathcal{B}_0(r)} (v-\bar{v})^2\,dx + \int_{\mathcal{B}_0(r)} \bar{v}^2\,dx \\
&\le \frac{1}{\lambda_1^N(\mathcal{B}_0(r))}\int_{\mathcal{B}_0(r)} |\nabla v|^2\,dx + \frac{1}{|\mathcal{B}|r^n}\|v\|_1^2 \\
&= \frac{r^2}{\lambda_1^N(\mathcal{B})}\int_{\mathcal{B}_0(r)} |\nabla v|^2\,dx + \frac{1}{|\mathcal{B}|r^n}\|v\|_1^2 \\
&\le \frac{r^2}{\lambda_1^N(\mathcal{B})}\|\nabla u\|_2^2 + \frac{1}{|\mathcal{B}|r^n}\|v\|_1^2
\end{aligned}$$

where $\mathcal{B}_0(r)$ stands for the Euclidean ball of center 0 and radius r. According to what we said above, and noting that

$$\|u\|_1^2 \ge \|v\|_1\left(\|v\|_1 + \|h\|_1\right)$$

this leads to

$$\|u\|_2^2 \le \frac{r^2}{\lambda_1^N(\mathcal{B})}\|\nabla u\|_2^2 + \frac{1}{|\mathcal{B}|r^n}\|u\|_1^2 \tag{8.10}$$

The right-hand side in this inequality is minimized at

$$r_{\min} = \left(\frac{n\lambda_1^N(\mathcal{B})}{2|\mathcal{B}|}\right)^{1/(n+2)}\left(\frac{\|u\|_1}{\|\nabla u\|_2}\right)^{2/(n+2)} \tag{8.11}$$

As one can easily check, taking $r = r_{\min}$ in (8.10) gives the inequality of the theorem. To see that this inequality is sharp, let u_0 be some eigenfunction associated to $\lambda_1^N(\mathcal{B})$. Set

$$u(x) = \begin{cases} u_0(|x|) - u_0(1) & \text{if } |x| \le 1 \\ 0 & \text{if } |x| \ge 1 \end{cases}$$

Clearly, u saturates (8.10) with $r = 1$. For such a function, $r_{\min} = 1$. One then easily gets from (8.11) that u also saturates the Nash inequality we just got. This ends the proof of the theorem. □

In addition to Theorem 8.10, Carlen and Loss [38] also determined the cases of equality in the optimal Nash inequality. As in the above proof, let u_0 be some eigenfunction associated to $\lambda_1^N(\mathcal{B})$, and set

$$u(x) = \begin{cases} u_0(|x|) - u_0(1) & \text{if } |x| \le 1 \\ 0 & \text{if } |x| \ge 1 \end{cases}$$

Then $\tilde{u}$ is an extremum function for the optimal Nash inequality if and only if after a possible translation, scaling, and normalization, $\tilde{u} = u$. As one can easily check, a particularly striking feature of this result is that all of the extremals have compact support. Another reference on the subject is Beckner ("Geometric proof of Nash's inequality," *IMRN*, 2, 1998, 67–71).

Concerning complete, noncompact Riemannian manifolds, note here that by the arguments used in the proof of Proposition 8.2, the optimal version of Nash's inequality does hold on n-dimensional Cartan-Hadamard manifolds as soon as the Cartan-Hadamard conjecture in dimension n is true. In particular, by Theorems 8.5, 8.6, and 8.7, the optimal version of Nash's inequality does hold on 2-dimensional, 3-dimensional, and 4-dimensional Cartan-Hadamard manifolds. On the contrary (see the recent Druet-Hebey-Vaugon [75]), the optimal version of Nash's inequality is false as soon as the scalar curvature is positive somewhere. In a way similar to Theorem 8.8 (see once more Druet-Hebey-Vaugon [75]), one can also prove that if the Ricci curvature of the manifold is nonnegative and the optimal version of Nash's inequality does hold, then the manifold must be flat.

CHAPTER 9

The Influence of Symmetries

The idea in this chapter is to show that Sobolev embeddings can be improved in the presence of symmetries. This includes embeddings in higher L^p spaces and compactness properties of these embeddings. Such phenomena have been observed in specific context by several authors. We especially point out the work of Lions **[150]** dealing with the Euclidean space, where the first systematic study of the subject was carried out. The goal here is to study the question in the more general context of arbitrary Riemannian manifolds. For the sake of clarity, we decided to separate the compact setting from the noncompact one. As one can see, when dealing with compact manifolds, one just has to consider the minimum orbit dimension of the group considered. On the other hand, when dealing with noncompact manifolds, one has also to consider the geometry of the action of the group at infinity.

9.1. Geometric Preliminaries

For the sake of clarity, we introduce here the notation and the background material we will use in the sequel. Though many of the results of the next sections do hold for arbitrary Riemannian manifolds, we assume in what follows, as done in the rest of the book, that the manifolds considered are at least complete. We refer to Hebey-Vaugon **[120]** for analogous results when dealing with noncomplete Riemannian manifolds.

Given (M, g) a smooth, complete Riemannian n-manifold, we denote its group of isometries by $\mathrm{Isom}_g(M)$. It is well-known that $\mathrm{Isom}_g(M)$ is a Lie group with respect to the compact open topology, and that $\mathrm{Isom}_g(M)$ acts differentiably on M (see, for instance, **[135]**). Since (this is actually due to E. Cartan) any closed subgroup of a compact Lie group is a Lie group, we get that any compact subgroup of $\mathrm{Isom}_g(M)$ is a sub-Lie group of $\mathrm{Isom}_g(M)$. Moreover, one has that $\mathrm{Isom}_g(M)$ is compact if M is compact. For G some subgroup of $\mathrm{Isom}_g(M)$, let $C_G^\infty(M)$ be the set of functions $u \in C^\infty(M)$ for which $u \circ \sigma = u$ for all $\sigma \in G$, and let $\mathcal{D}_G(M)$ be the set of functions $u \in \mathcal{D}(M)$ for which $u \circ \sigma = u$ for all $\sigma \in G$. Similarly, for $p \geq 1$, let

$$H_{1,G}^p(M) = \left\{u \in H_1^p(M) \,/\, \forall \sigma \in G,\ u \circ \sigma = u\right\}$$

One has that $\mathcal{D}_G(M) \subset H_{1,G}^p(M)$, and if G is compact, one gets from the existence of the Haar measure that $\mathcal{D}_G(M)$ is dense in $H_{1,G}^p(M)$. From now on, let (M, g) and (N, h) be smooth Riemannian manifolds, and let $\Pi : M \to N$ be a submersion. We recall that Π is said to be a Riemannian submersion if for any x

in M,

$$\Pi_*(x) : \big(H_x, g(x)\big) \longrightarrow \big(T_y(N), h(y)\big)$$

is an isometry, where $y = \Pi(x)$ and H_x denotes the orthogonal complement of $T_x(\Pi^{-1}(y))$ in $T_x(M)$. Assume now that $\dim M > \dim N$, that $\Pi : M \to N$ is a Riemannian submersion, and that for any $y \in N$, $\Pi^{-1}(y)$ is compact. Let $v : N \to \mathbb{R}$ be the function defined by

$$v(y) \stackrel{\text{def}}{=} \text{volume of } \Pi^{-1}(y) \text{ for the metric induced by } g$$

Then (see, for instance, [27]), for any $\phi : N \to \mathbb{R}$ such that $\phi v \in L^1(N)$, one has that

$$\int_M (\phi \circ \Pi) dv(g) = \int_N (\phi v) dv(h) \tag{9.1}$$

Independently, by O'Neill's formula (see, for instance, [28] or [88]), if $\Pi : M \to N$ is a Riemannian submersion, for any orthonormal vector fields X and Y on N with horizontal lifts $\tilde{X}$ and $\tilde{Y}$,

$$K_{(N,h)}(X, Y) = K_{(M,g)}(\tilde{X}, \tilde{Y}) + \frac{3}{4}|[\tilde{X}, \tilde{Y}]^v|^2$$

where $K_{(M,g)}$ and $K_{(N,h)}$ stand for the sectional curvatures of (M, g) and (N, h), and where the superscript v means that we are concerned with the vertical component of $[\tilde{X}, \tilde{Y}]$. As an immediate consequence of this formula, one gets that the sectional curvature of (N, h) is bounded from below if that of (M, g) is bounded from below. This in turn implies that the Ricci curvature of (N, h) is bounded from below if the sectional curvature of (M, g) is bounded from below.

We now recall some facts about the action of compact subgroups of $\mathrm{Isom}_g(M)$. For G a compact subgroup of $\mathrm{Isom}_g(M)$, and x a point of M, we denote by

$$O_G^x = \{\sigma(x),\ \sigma \in G\}$$

the orbit of x under the action of G, and we denote by

$$S_G^x = \{\sigma \in G \,/\, \sigma(x) = x\}$$

the isotropy group of x. It is by now classical (see, [32] and [64]), that for any x in M, O_G^x is a smooth, compact submanifold of M, the quotient manifold G/S_G^x exists, and the canonical map $\Phi_x : G/S_G^x \to O_G^x$ is a diffeomorphism. (The isotropy group of any other point in O_G^x is actually conjugate to S_G^x). An orbit O_G^x is said to be principal if for any $y \in M$, S_G^y possesses a subgroup that is conjugate to S_G^x. Principal orbits are then of maximal dimension (but there may exist orbits of maximal dimension that are not principal). We refer to [32] for more details on the subject. Anyway, we will use the following basic facts in the sequel:

1. $\Omega = \bigcup_{\{x \text{ s.t. } O_G^x \text{ is principal}\}} O_G^x$ is a dense, open subset of M,

and if $\Pi : M \to M/G$ denotes the canonical surjection,

2. the quotient space M/G is Hausdorff and Π is a proper map, and
3. $\Pi(\Omega) = \Omega/G$ possesses the structure of a smooth, connected manifold for which Π, when restricted to Ω, becomes a submersion.

Here again, these points can be found in [32]. Furthermore, one clearly has that the metric g on M induces a quotient metric h on Ω/G for which Π, when restricted to Ω, becomes a Riemannian submersion from (Ω, g) to $(\Omega/G, h)$. The distance d_h associated to h then extends to M/G by

$$d_h(\Pi(x), \Pi(y)) = d_g\left(O_G^x, O_G^y\right)$$

for any $x, y \in M$, where d_g denotes the distance associated to g. We refer to [88] for the constructions involved in these statements.

9.2. Compact Manifolds

For the sake of clarity, we start discussing improvement of Sobolev embeddings in the presence of symmetries by considering the case of compact manifolds. The more general case of complete manifolds will be treated in the sequel. Let (M, g) be a smooth, compact Riemannian manifold, and let G be a subgroup of $\mathrm{Isom}_g(M)$. As already mentioned, up to replacing G by its closure in $\mathrm{Isom}_g(M)$, one can assume that G is compact. The first result we prove is the following:

LEMMA 9.1 *Let (M, g) be a smooth Riemannian n-manifold (not necessarily compact or complete), and let G be a compact subgroup of* $\mathrm{Isom}_g(M)$. *Let $x \in M$ and set $k = \dim O_G^x$. Assume $k \geq 1$. There exists a coordinate chart (Ω, φ) of M at x such that:*

(i) *$\varphi(\Omega) = U \times V$, where U is some open subset of $\mathbb{R}^k$ and V is some open subset of $\mathbb{R}^{n-k}$ and*
(ii) *$\forall y \in \Omega$, $U \times \Pi_2(\varphi(y)) \subset \varphi(O_G^y \cap \Omega)$ where $\Pi_2 : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ is the second projection.*

PROOF: Let $\Phi : G \to M$ be defined by $\Phi(\sigma) = \sigma(x)$. It is by now classical that Φ has constant rank (see, for instance, [64]). Since $S_G^x = \Phi^{-1}(x)$, we get that

$$\dim S_G^x = \dim G - \mathrm{Rank}\, \Phi$$

On the other hand (see Section 9.1 and [64]),

$$\dim\left(G/S_G^x\right) = \dim O_G^x = \dim G - \dim S_G^x$$

Hence, $\mathrm{Rank}\,\Phi = k$. As a consequence, there exists a k-dimensional submanifold H of G such that $Id \in H$ and $\Phi_{|H}$ is an embedding. Let N be an $(n-k)$-dimensional submanifold of M such that

$$T_x\Phi(H) \oplus T_xN = T_xM\,,$$

and let $\Psi : H \times N \to M$ be defined by $\Psi(\sigma, y) = \sigma(y)$. Clearly, Ψ is smooth and $D\Psi_{(Id,x)}$ is an isomorphism. Let (U', φ_1) be a chart of H at Id and (V', φ_2) be a chart of N at x, U' and V' being such that $\Psi_{|U'\times V'}$ is a diffeomorphism. To get the lemma one just has to set $\Omega = \Psi(U' \times V')$ and $\varphi = (\varphi_1 \circ \Psi_1^{-1}, \varphi_2 \circ \Psi_2^{-1})$, where $\Psi^{-1} = (\Psi_1^{-1}, \Psi_2^{-1})$. □

With such a lemma we are now in position to prove our first result (Hebey-Vaugon [108, 120]). As one can see, for functions that possess enough symmetries, Sobolev embeddings are valid in higher L^p spaces. Similar results have been obtained in specific contexts by Cotsiolis-Iliopoulos **[57]**, Lions **[151]**, and Ding **[65]**.

THEOREM 9.1 *Let (M, g) be a smooth, compact Riemannian n-manifold, and $q \geq 1$ real. Let G be a compact subgroup of* $\mathrm{Isom}_g(M)$. *Assume that for any $x \in M$,* $\mathrm{Card}\, O_G^x = +\infty$ *where* Card *stands for the cardinality, and set $k = \min_{x \in M} \dim O_G^x$. Then, $k \geq 1$ and*

(i) *if $n - k \leq q$, for any real number $p \geq 1$, $H_{1,G}^q(M) \subset L^p(M)$ and the embedding is continuous and compact and*

(ii) *if $n - k > q$, for any real number $1 \leq p \leq (n-k)q/(n-k-q)$, $H_{1,G}^q(M) \subset L^p(M)$, the embedding is continuous, and compact provided that $p < (n-k)q/(n-k-q)$.*

In particular, there exists $p_0 > nq/(n-q)$ such that for any $1 \leq p \leq p_0$, $H_{1,G}^q(M) \subset L^p(M)$, the embedding being continuous and compact.

PROOF: By Lemma 9.1, and since M is compact, it is covered by a finite number of charts $(\Omega_m, \varphi_m)_{m=1,\dots,N}$ with the properties that for any m:

(i) $\varphi_m(\Omega_m) = U_m \times V_m$, where U_m is some open subset of $\mathbb{R}^{k_m}$, V_m is some open subset of $\mathbb{R}^{n-k_m}$, and k_m integer is such that $k_m \geq k$,

(ii) U_m, V_m are bounded, and V_m has smooth boundary,

(iii) $\forall y \in \Omega_m$, $U_m \times \Pi_2(\varphi_m(y)) \subset \varphi_m(O_G^y \cap \Omega_m)$, and

(iv) $\exists \alpha_m > 0$ with $\alpha_m^{-1}\delta_{ij} \leq g_{ij}^m \leq \alpha_m \delta_{ij}$ as bilinear forms.

In (iii), $\Pi_2 : \mathbb{R}^{k_m} \times \mathbb{R}^{n-k_m} \to \mathbb{R}^{n-k_m}$ denotes the second projection, and in (iv), the g_{ij}^m's denote the components of g in (Ω_m, φ_m). Let $u \in C_G^\infty(M)$. Since $u \circ \sigma = u$ for any $\sigma \in G$, we get that for any m, any $x, x' \in U_m$, and any $y \in V_m$,

$$u \circ \varphi_m^{-1}(x, y) = u \circ \varphi_m^{-1}(x', y)$$

As a consequence, for any m there exists $\tilde{u}_m \in C^\infty(\mathbb{R}^{n-k_m})$ such that for any $x \in U_m$ and any $y \in V_m$, one has that

$$u \circ \varphi_m^{-1}(x, y) = \tilde{u}_m(y)$$

(Without loss of generality, one can assume that φ_m is actually defined on some open set $\tilde{\Omega}_m$ containing $\overline{\Omega}_m$ such that $\varphi_m(\tilde{\Omega}_m) = \tilde{U}_m \times \tilde{V}_m$ with $\overline{V}_m \subset \tilde{V}_m$.) We then get that for any m and any real number $p \geq 1$,

$$\begin{aligned}\int_{\Omega_m} |u|^p \, dv(g) &= \int_{U_m \times V_m} \left(|u|^p \sqrt{\det g_{ij}^m}\right) \circ \varphi_m^{-1}(x, y) dx\, dy \\ &\leq A_m \int_{U_m \times V_m} |u \circ \varphi_m^{-1}(x, y)|^p \, dx\, dy \\ &= \tilde{A}_m \int_{V_m} |\tilde{u}_m(y)|^p \, dy\end{aligned}$$

where A_m and $\tilde{A}_m$ are positive constants that do not depend on u. Similarly, one has that for any m and any $p \geq 1$,

$$\int_{\Omega_m} |u|^p \, dv(g) \geq B_m \int_{V_m} |\tilde{u}_m(y)|^p \, dy$$

and

$$\int_{\Omega_m} |\nabla u|^p \, dv(g) \geq \tilde{B}_m \int_{V_m} |\nabla \tilde{u}_m(y)|^p \, dy$$

where $B_m > 0$ and $\tilde{B}_m > 0$ do not depend on u. Combining these inequalities and the Sobolev embedding theorem for bounded domains of Euclidean spaces, we get that for any m and any real number $q \geq 1$,

(v) if $n - k_m \leq q$, then for any real number $p \geq 1$ there exists $C_m > 0$ such that for any $u \in C_G^\infty(M)$,

$$\left(\int_{\Omega_m} |u|^p \, dv(g)\right)^{1/p} \leq C_m\left(\left(\int_{\Omega_m} |\nabla u|^q \, dv(g)\right)^{1/q} + \left(\int_{\Omega_m} |u|^q \, dv(g)\right)^{1/q}\right)$$

(vi) if $n - k_m > q$, then for any real number $1 \leq p \leq \frac{(n-k_m)q}{n-k_m-q}$ there exists $C_m > 0$ such that for any $u \in C_G^\infty(M)$,

$$\left(\int_{\Omega_m} |u|^p \, dv(g)\right)^{1/p} \leq C_m\left(\left(\int_{\Omega_m} |\nabla u|^q \, dv(g)\right)^{1/q} + \left(\int_{\Omega_m} |u|^q \, dv(g)\right)^{1/q}\right)$$

But:

(vii) $n - k_m \leq n - k$ so that for $q < n - k_m$, $\frac{(n-k_m)q}{n-k_m-q} \geq \frac{(n-k)q}{n-k-q}$,

(viii) $\left(\int_M |u|^p \, dv(g)\right)^{1/p} \leq \sum_{m=1}^N \left(\int_{\Omega_m} |u|^p \, dv(g)\right)^{1/p}$,

(ix) $\sum_{m=1}^N \left(\left(\int_{\Omega_m} |\nabla u|^q \, dv(g)\right)^{1/q} + \left(\int_{\Omega_m} |u|^q \, dv(g)\right)^{1/q}\right)$
$\leq N\left(\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} + \left(\int_M |u|^q \, dv(g)\right)^{1/q}\right)$

As a consequence, for any real number $q \geq 1$:

(x) if $n - k \leq q$, then for any real number $p \geq 1$, $H_{1,G}^q(M) \subset L^p(M)$ and

(xi) if $n - k > q$, then for any real number $1 \leq p \leq (n-k)q/(n-k-q)$, $H_{1,G}^q(M) \subset L^p(M)$.

This proves the validity and the continuity of the embeddings in question in the theorem. By standard arguments, as developed in the proof of Theorem 2.9, one then easily gets that these embeddings are compact for any $p \geq 1$ in case (x), and any $p < (n-k)q/(n-k-q)$ in case (xi). This ends the proof of the theorem. □

9.3. Optimal Inequalities for Compact Manifolds

When G has finite orbits, as one may easily check, there is no hope to get embeddings in higher L^p spaces. In such a situation, one has to deal with optimal inequalities. Given (M, g) a smooth, compact Riemannian n-manifold, and $q \in [1, n)$, let $A, B \in \mathbb{R}$ be such that for any $u \in H_1^q(M)$,

$(\mathrm{I}_{q,\mathrm{gen}}^1)$

$$\left(\int_M |u|^p \, dv(g)\right)^{1/p} \leq A\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} + B\left(\int_M |u|^q \, dv(g)\right)^{1/q}$$

where $1/p = 1/q - 1/n$. Given G a compact subgroup of $\mathrm{Isom}_g(M)$ that possesses finite orbits, let us say that $(\mathrm{I}^1_{q,\mathrm{gen}})$ is G-valid if it holds for all $u \in H^q_{1,G}(M)$. Mimicking what was done in Chapter 4, we define the best constants

$$\alpha_{q,G}(M) = \inf\left\{A \in \mathbb{R} \text{ s.t. } \exists B \in \mathbb{R} \text{ for which } (\mathrm{I}^1_{q,\mathrm{gen}}) \text{ is } G-\text{valid}\right\}$$
$$\beta_{q,G}(M) = \inf\left\{B \in \mathbb{R} \text{ s.t. } \exists A \in \mathbb{R} \text{ for which } (\mathrm{I}^1_{q,\mathrm{gen}}) \text{ is } G-\text{valid}\right\}$$

Following what was done in Section 4.1 of Chapter 4, one clearly gets that $\beta_{q,G}(M) = \mathrm{Vol}_{(M,g)}^{-1/n}$ and that there exists $A \in \mathbb{R}$ such that $(\mathrm{I}^1_{q,\mathrm{gen}})$ with $B = \mathrm{Vol}_{(M,g)}^{-1/n}$ is G-valid. As one may easily check, the arguments used in the proofs of Theorems 4.1, 4.2, and Proposition 4.1, are G-invariant arguments. The challenging question in such a context is to know what is the exact value of the best constant $\alpha_{q,G}(M)$, and if there exists $B \in \mathbb{R}$ such that $(\mathrm{I}^1_{q,\mathrm{gen}})$ with $A = \alpha_{q,G}(M)$ is G-valid. The first result we prove here is the following (Hebey-Vaugon [**108**]):

LEMMA 9.2 *Let (M, g) be a smooth, compact Riemannian n-manifold, G a compact subgroup of* $\mathrm{Isom}_g(M)$, *G_0 the connected component of the identity in G, and p, q two real numbers such that $1 \le q < n$ and $p = nq/(n-q)$. Let O be a compact subset of M such that O is stable under the action of G_0 (i.e., $\sigma O = O$, for any $\sigma \in G_0$), and such that for any $x \in O$,* Card $O_{G_0}(x) = +\infty$ *where* Card *stands for the cardinality. Then there exists $\delta > 0$ such that for any $\varepsilon > 0$ there exists $B \in \mathbb{R}$ with the following property: For any $u \in C^\infty_G(M)$,*

$$\left(\int_M |u|^p \, dv(g)\right)^{q/p} \le \varepsilon \int_M |\nabla u|^q \, dv(g) + B \int_M |u|^q \, dv(g)$$

as soon as $\mathrm{supp}\, u \subset O_\delta = \{y \in M \text{ s.t. } d_g(y, O) \le \delta\}$, *where d_g is the distance associated to g.*

PROOF: Because O is compact, it is covered by a finite number of charts (Ω_m, φ_m) satisfying assumptions (i) to (iv) of the proof of Theorem 9.1, with $k \ge 1$ given by

$$k = \min_{x \in O} \dim O^x_{G_0}$$

We choose $\delta > 0$ such that $O_\delta \subset \cup \Omega_m$. Let $1 \le q < n$ and $p = nq/(n-q)$. Set

$$H_q(M) = \left\{u \in H^q_{1,G}(M) \text{ s.t. } \mathrm{supp}\, u \subset O_\delta\right\}$$

With similar arguments to those developed in the proof of Theorem 9.1, one can get that the embedding of $H_q(M)$ in $L^p(M)$ is compact. Independently, by a classical result of Lions [**149**], if $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ are three Banach spaces, $u : \mathcal{B}_1 \to \mathcal{B}_2$ is a compact linear operator, and $v : \mathcal{B}_2 \to \mathcal{B}_3$ is a continuous one to one linear operator, then, for any $\varepsilon > 0$ there exists $B > 0$ such that for any $x \in \mathcal{B}_1$,

$$\|u(x)\|_{\mathcal{B}_2} \le \varepsilon \|x\|_{\mathcal{B}_1} + B \|v \circ u(x)\|_{\mathcal{B}_3}$$

Applying this result with $\mathcal{B}_1 = H_q(M)$, $\mathcal{B}_2 = L^p(M)$, and $\mathcal{B}_3 = L^q(M)$, one gets the lemma. □

With such a lemma, we are in position to prove the following result of Hebey-Vaugon [**108**]. It gives the answer to the first part of the question we asked.

THEOREM 9.2 *Let (M, g) be a smooth, compact Riemannian n-manifold and let G be a compact subgroup of* $\mathrm{Isom}_g(M)$. *Let* $k = \inf_{x\in M} \mathrm{Card}\, O_G^x$, *where* Card *stands for the cardinality. For any $q \in [1, n)$ real, and any $\varepsilon > 0$, there exists $B \in \mathbb{R}$ such that for any $u \in H_{1,G}^q(M)$,*

$$\left(\int_M |u|^p \, dv(g)\right)^{q/p} \le \left(\frac{K(n,q)^q}{k^{q/n}} + \varepsilon\right) \int_M |\nabla u|^q \, dv(g) + B \int_M |u|^q \, dv(g)$$

where $1/p = 1/q - 1/n$, $K(n, q)$ is as in Theorem 4.4, and, as a convention, $K(n,q)^q/k^{q/n} = 0$ if $k = +\infty$. Moreover, $K(n,q)^q/k^{q/n}$ is the best constant in such an inequality, so that $\alpha_{q,G}(M) = K(n,q)/k^{1/n}$.

PROOF: If $k = +\infty$, Theorem 9.2 is an easy consequence of Theorem 9.1 and the result of Lions mentioned above. One can then suppose that $k < +\infty$. Let $1 \le q < n$ be given, and G_0 be the connected component of the identity in G. Let $x \in M$.

If $\mathrm{Card}\, O_G^x < +\infty$, let $O_G^x = \{x_1, \ldots, x_m\}$ with the convention that $x_1 = x$. We then choose $\delta = \delta(x) \in (0, \mathrm{inj}_{(M,g)})$ small enough such that for any $i \ne j$, $B_{x_i}(\delta) \cap B_{x_j}(\delta) = \emptyset$, and we define $U_x = \cup_{j=1}^m B_{x_j}(\delta)$.

Suppose now that $\mathrm{Card}\, O_G^x = +\infty$. One has that O_G^x is a smooth, compact submanifold of M of dimension greater than or equal to 1. Let $O_G^x = O_1 \cup \cdots \cup O_m$, with the convention that $x \in O_1$, the O_i's being the connected components of O_G^x. The O_i's are compact since O_G^x is compact. Furthermore, O_1 is clearly stable under the action of G_0, and for any $y \in O_1$, one has that $\mathrm{Card}\, O_{G_0}^y = +\infty$. We now choose $\delta = \delta(x)$ small enough such that

(i) δ is less than the one given by Lemma 9.2 (with $O = O_1$),

(ii) for any $i \ne j$, $B_i^\delta \cap B_j^\delta = \emptyset$ where $B_i^\delta = \{y \in M \text{ s.t. } d_g(y, O_i) \le \delta\}$, and

(iii) for any i, $\varphi_i : \overset{\circ}{B}{}_i^{2\delta} \to \mathbb{R}$ defined by $\varphi_i(y) = d_g(y, O_i)^2$ is smooth.

Here again, we define $U_x = \bigcup_{j=1}^m B_j^\delta$. One then has that for any $x \in M$, U_x is stable under the action of G. Now since M is compact, let $x_1, \ldots, x_N \in M$ be such that

$$M = \bigcup_{i=1}^N \overset{\circ}{U}_{x_i}$$

For any $\varepsilon > 0$, let $f_\varepsilon \in C^\infty(\mathbb{R})$ be such that $f_\varepsilon(t) > 0$ if $t < \varepsilon$ and $f_\varepsilon(t) = 0$ if $t \ge 0$. For any $i = 1, \ldots, N$ we set

$$\alpha_{ij}(x) = f_{\delta_i}(d_g(x, x_{ij})^2) \quad \text{if } U_{x_i} = \cup_{j=1}^{m_i} B_{x_{ij}}(\delta_i),\ \delta_i = \delta(x_i)$$

$$\alpha_{ij}(x) = f_{\delta_i}(d_g(x, O_{ij})^2) \quad \text{if } U_{x_i} = \cup_{j=1}^{m_i} B_j^{\delta_i},\ \delta_i = \delta(x_i)$$

The α_{ij}'s, $i = 1, \ldots, N$, $j = 1, \ldots, m_i$, are smooth. We set

$$\eta_{ij} = \frac{\alpha_{ij}^{[q]+1}}{\sum_{\mu,\nu} \alpha_{\mu\nu}^{[q]+1}}$$

where $[q]$ is the greatest integer not exceeding q, $i = 1, \ldots, N$, $j = 1, \ldots, m_i$. Clearly, η_{ij} is a smooth partition of unity of M such that:

(iv) for any i, j, $\eta_{ij}^{1/q} \in C^1(M)$,

(v) there exists $H \in \mathbb{R}$ such that for any i, j, $|\nabla(\eta_{ij}^{1/q})| \le H$, and

(vi) for any $i = 1, \dots, N$ and any $j \neq j' = 1, \dots, m_i$, $\eta_{ij}\eta_{ij'} = 0$.

Furthermore, as one easily can easily check, for any $i = 1, \dots, N$ and any $j, j' = 1, \dots, m_i$, there exists $\sigma \in G$ such that $\eta_{ij'} = \eta_{ij} \circ \sigma$. According to what we have just said, one then has that for any $u \in C_G^\infty(M)$,

$$\begin{aligned}\left(\int_M |u|^p \, dv(g)\right)^{q/p} &= \left(\int_M \left|\sum_{i,j} \eta_{ij}|u|^q\right|^{p/q} dv(g)\right)^{q/p} \\ &\le \sum_{i=1}^N \left(\int_M \left|\sum_{j=1}^{m_i} \eta_{ij}|u|^q\right|^{p/q} dv(g)\right)^{q/p} \\ &= \sum_{i=1}^N m_i^{q/p}\left(\int_M |\eta_{i1}^{1/q} u|^p \, dv(g)\right)^{q/p}\end{aligned}$$

Let $i \in \{1, \dots, N\}$ be given and suppose that Card $O_G^{x_i} < +\infty$. By Theorem 4.5, for any $\varepsilon_i > 0$, there exists $B_i \in \mathbb{R}$ such that for any $u \in C_G^\infty(M)$,

$$\begin{aligned}&\left(\int_M |\eta_{i1}^{1/q} u|^p \, dv(g)\right)^{q/p} \le \\ &\qquad (K(n,q)^q + \varepsilon_i) \int_M \left|\nabla\left(\eta_{i1}^{1/q} u\right)\right|^q dv(g) + B_i \int_M \eta_{i1}|u|^q \, dv(g)\end{aligned}$$

Independently, suppose that Card $O_G^{x_i} = +\infty$. Since η_{i1} is G_0-invariant (as one easily checks), we get by Lemma 9.2 that for any $\varepsilon_i > 0$ there exists $B_i \in \mathbb{R}$ such that for any $u \in C_G^\infty(M)$,

$$\left(\int_M |\eta_{i1}^{1/q} u|^p \, dv(g)\right)^{q/p} \le \varepsilon_i \int_M \left|\nabla\left(\eta_{i1}^{1/q} u\right)\right|^q dv(g) + B_i \int_M \eta_{i1}|u|^q \, dv(g)$$

Let

$$I_1 = \left\{i \text{ s.t. Card } O_G^{x_i} < +\infty\right\} \quad \text{and} \quad I_2 = \left\{i \text{ s.t. Card } O_G^{x_i} = +\infty\right\}$$

Similar computations to the ones involved in the proof of Theorem 4.5 lead to the following: For any $u \in C_G^\infty(M)$,

$$\begin{aligned}\|u\|_p^q &\le \sum_{i \in I_1} m_i^{q/p}\left(K(n,q)^q + \varepsilon_i\right) \\ &\quad \times \left(\int_M |\nabla u|^q \eta_{i1} \, dv(g) + \mu H \|\nabla u\|_q^{q-1} \|u\|_q + \nu H^q \|u\|_q^q\right) \\ &\quad + \sum_{i \in I_2} m_i^{q/p} \varepsilon_i \left(\int_M |\nabla u|^q \eta_{i1} \, dv(g) + \mu H \|\nabla u\|_q^{q-1} \|u\|_q + \nu H^q \|u\|_q^q\right) \\ &\quad + N\left(\max_{i=1,\dots,N} B_i m_i^{q/p}\right) \|u\|_q^q\end{aligned}$$

where $\mu > 0$ and $\nu > 0$ depend only on q, and where $\|\cdot\|_p$ stands for the norm of $L^p(M)$. Independently, for any $i = 1, \dots, N$,

$$\int_M |\nabla u|^q \eta_{i1}\, dv(g) = \frac{1}{m_i} \sum_{j=1}^{m_i} \int_M |\nabla u|^q \eta_{ij}\, dv(g)$$

while for any $i \in I_1$,

$$m_i \geq k = \min_{x \in M} \operatorname{Card} O_G(x)$$

Since $1 - q/p = q/n$, choosing $\varepsilon_i = \varepsilon$ when $i \in I_1$ and $\varepsilon_i \leq K(n,q)^q (m_i/k)^{q/n}$ when $i \in I_2$ gives that for any $\varepsilon > 0$, there exists $B \in \mathbb{R}$ such that for any $u \in C_G^\infty(M)$,

$$\|u\|_p^q \leq \frac{K(n,q)^q + \varepsilon}{k^{q/n}} \left(\|\nabla u\|_q^q + \mu H\left(\sum_{i=1}^N m_i\right) \|\nabla u\|_q^{q-1} \|u\|_q \right.$$
$$\left. + \nu H^q \left(\sum_{i=1}^N m_i\right) \|u\|_q^q \right) + B\|u\|_q^q$$

Noting that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for any positive real numbers x and y, $x^{q-1}y \leq \varepsilon x^q + C_\varepsilon y^q$, one easily obtains the inequality of Theorem 9.2 from this last inequality. This ends the proof of the first part of the theorem. Concerning the second part, namely, that $K(n,q)^q/k^{q/n}$ is the best constant, one can use test functions centered at some minimal orbit. Suppose that $k < +\infty$, and let $\{x_1, \dots, x_k\}$ some minimal orbit of G. For $x \in M$ and $\varepsilon > 0$, define

$$u_{\varepsilon,x} = \left(\varepsilon + r^{\frac{q}{q-1}}\right)^{1-\frac{n}{q}} \varphi(r)$$

where r denotes the distance to x, φ is smooth such that $0 \leq \varphi \leq 1$, $\varphi = 1$ on $(-\frac{\delta}{2}, \frac{\delta}{2})$, and $\varphi = 0$ if $r \geq \delta$, and $\delta > 0$, δ small, is real. With similar computations to the ones involved in the proof of Theorem 4.8, one gets that for any B real

$$\lim_{\varepsilon \to 0} \frac{\int_M |\nabla u_{\varepsilon,x}|^q\, dv(g) + B \int_M u_{\varepsilon,x}^q\, dv(g)}{\left(\int_M u_{\varepsilon,x}^p\, dv(g)\right)^{q/p}} = \frac{1}{K(n,q)^q}$$

For $\varepsilon > 0$, set

$$u_\varepsilon = \sum_{i=1}^k u_{\varepsilon,x_i}$$

For $\delta > 0$ small enough, u_ε is G-invariant. Moreover,

$$\int_M |\nabla u_\varepsilon|^q\, dv(g) = k \int_M |\nabla u_{\varepsilon,x_1}|^q\, dv(g)$$
$$\int_M u_\varepsilon^q\, dv(g) = k \int_M u_{\varepsilon,x_1}^q\, dv(g)$$
$$\int_M u_\varepsilon^p\, dv(g) = k \int_M u_{\varepsilon,x_1}^p\, dv(g)$$

Hence, for any B real,

$$\lim_{\varepsilon\to 0}\frac{\int_M |\nabla u_\varepsilon|^q\,dv(g) + B\int_M u_\varepsilon^q\,dv(g)}{\left(\int_M u_\varepsilon^p\,dv(g)\right)^{q/p}} = \frac{k^{1-\frac{q}{p}}}{K(n,q)^q}$$

Clearly, this proves the second part of the theorem. □

Regarding Theorem 9.2, one can take $\varepsilon = 0$ in the case $q = 2$. This is the subject of the following result of Hebey-Vaugon [**108**]:

THEOREM 9.3 *Let (M, g) be a smooth, compact Riemannian n-manifold, $n \geq 3$, and let G be a compact subgroup of* $\mathrm{Isom}_g(M)$ *that possesses at least one finite orbit. Let $k = \min_{x\in M}$* Card *$O_G(x)$ where* Card *stands for the cardinality. There exists $B \in \mathbb{R}$ such that for any $u \in H_{1,G}^2(M)$,*

$$\left(\int_M |u|^p\,dv(g)\right)^{2/p} \leq \frac{K(n,2)^2}{k^{2/n}}\int_M |\nabla u|^2\,dv(g) + B\int_M u^2\,dv(g)$$

where $1/p = 1/2 - 1/n$ and $K(n,2)$ is as in Theorem 4.4. In particular, there exists $B \in \mathbb{R}$ such that $(\mathrm{I}_{2,\mathrm{gen}}^1)$ with $A = \alpha_{2,G}(M)$ is G-valid.

PROOF: We proceed as in the proof of Theorem 9.2, using Theorem 4.6 instead of Theorem 4.5. We then have that for any $i \in I_1$, there exists $B_i \in \mathbb{R}$ such that for any $u \in C_G^\infty(M)$,

$$\left(\int_M |\eta_{i1}^{1/2}u|^p\,dv(g)\right)^{2/p} \leq K(n,2)^2\int_M |\nabla(\eta_{i1}^{1/2}u)|^2\,dv(g) + B_i\int_M \eta_{i1}u^2\,dv(g)$$

As a consequence, for any $u \in C_G^\infty(M)$,

$$\begin{aligned}
&\left(\int_M |u|^p\,dv(g)\right)^{2/p}\\
&\leq \sum_{i\in I_1} m_i^{(n-2)/n}K(n,2)^2\Big(\int_M |\nabla u|^2\eta_{i1}\,dv(g) + \int_M u^2|\nabla(\eta_{i1}^{1/2}u)|^2\,dv(g)\\
&\qquad + \int_M u\nabla^\nu u\nabla_\nu\eta_{i1}\,dv(g)\Big)\\
&\quad + \sum_{i\in I_2} m_i^{(n-2)/n}\varepsilon_i\Big(\int_M |\nabla u|^2\eta_{i1}\,dv(g) + \int_M u^2|\nabla(\eta_{i1}^{1/2}u)|^2\,dv(g)\\
&\qquad + \int_M u\nabla^\nu u\nabla_\nu\eta_{i1}\,dv(g)\Big)\\
&\quad + N\Big(\max_{i=1,\dots,N} B_i m_i^{(n-2)/n}\Big)\int_M u^2\,dv(g)
\end{aligned}$$

Independently, for any $i = 1, \dots, N$,

$$\int_M |\nabla u|^2 \eta_{i1}\, dv(g) = \frac{1}{m_i} \sum_{j=1}^{m_i} \int_M |\nabla u|^2 \eta_{ij}\, dv(g)$$

and

$$\int_M u \nabla^\nu u \nabla_\nu \eta_{i1}\, dv(g) = \frac{1}{m_i} \sum_{j=1}^{m_i} \int_M u \nabla^\nu u \nabla_\nu \eta_{ij}\, dv(g)$$

while for any $i \in I_1$,

$$m_i \geq k = \min_{x \in M} \operatorname{Card} O_G^x$$

Choosing

$$\varepsilon_i \leq K(n,2)^2 \left(\frac{m_i}{k}\right)^{2/n}$$

we then get that for any $u \in C_G^\infty(M)$,

$$\begin{aligned}
\left(\int_M |u|^p\, dv(g)\right)^{2/p} &\leq \frac{K(n,2)^2}{k^{2/n}} \sum_{i,j} \int_M |\nabla u|^2 \eta_{ij}\, dv(g) \\
&\quad + \frac{K(n,2)^2}{k^{2/n}} \sum_{i,j} \int_M u \nabla^\nu u \nabla_\nu \eta_{ij}\, dv(g) \\
&\quad + \frac{K(n,2)^2}{k^{2/n}} \left(\sum_{i=1}^N m_i\right) H^2 \int_M u^2\, dv(g) \\
&\quad + N\left(\max_{i=1,\dots,N} B_i m_i^{(n-2)/n}\right) \int_M u^2\, dv(g) \\
&= \frac{K(n,2)^2}{k^{2/n}} \sum_{i,j} \int_M |\nabla u|^2 \eta_{ij}\, dv(g) \\
&\quad + \frac{K(n,2)^2}{k^{2/n}} \left(\sum_{i=1}^N m_i\right) H^2 \int_M u^2\, dv(g) \\
&\quad + N\left(\max_{i=1,\dots,N} B_i m_i^{(n-2)/n}\right) \int_M u^2\, dv(g)
\end{aligned}$$

since $\sum_{i,j} \eta_{ij} = 1$. Let

$$B = \frac{K(n,2)^2}{k^{2/n}} \left(\sum_{i=1}^N m_i\right) H^2 + N\left(\max_{i=1,\dots,N} B_i m_i^{(n-2)/n}\right)$$

Then for any $u \in C_G^\infty(M)$,

$$\left(\int_M |u|^p\, dv(g)\right)^{2/p} \leq \frac{K(n,2)^2}{k^{2/n}} \int_M |\nabla u|^2\, dv(g) + B \int_M u^2\, dv(g)$$

This ends the proof of the theorem. □

Let (M, g) be a smooth, compact Riemannian n-manifold, let $q \in [1, n)$ real, and let G be a compact subgroup of $\mathrm{Isom}_g(M)$. Suppose that G does not possess finite orbits and let k be the minimum dimension of O_G^x, $x \in M$. Suppose that $n - k > q$. By Theorem 9.1, $H_{1,G}^q(M) \subset L^p(M)$ for $p = (n-k)q/(n-k-q)$. In other words, there exist A and B real such that for any $u \in C_G^\infty(M)$,

$$\left(\int_M |u|^p \, dv(g)\right)^{1/p} \le A\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} + B\left(\int_M |u|^q \, dv(g)\right)^{1/q}$$

where p is as above. Mimicking what has been done in the proof of Theorem 4.1, the best constant B in such an inequality, denoted by $\beta_{q,G}(M)$, is

$$\beta_{q.G}(M) = \mathrm{Vol}_{(M,g)}^{-1/(n-k)}$$

Moreover, there exists $A \in \mathbb{R}$ such that the above inequality with $B = \beta_{q,G}(M)$ is G-valid. Following what we did in Theorems 9.2 and 9.3, the challenging question there is to know what is the value of the best constant A in the above inequality. An answer to this question has recently been announced by Iliopoulos [**124**] in the following particular case: $(M, g) = (S^n, h)$, the standard unit sphere of $\mathbb{R}^{n+1}$, and $G = O(m_1) \times O(m_2)$, with m_1 and m_2 two integers such that $m_1 + m_2 = n + 1$, $m_1 \ge m_2 \ge 2$. In such a context, $k = m_2 - 1$ so that $n - k = m_1$. For $q < m_1$, one then gets that there exist A and B real such that for any $u \in C_G^\infty(S^n)$,

$$\left(\int_{S^n} |u|^p \, dv(h)\right)^{1/p} \le A\left(\int_{S^n} |\nabla u|^q \, dv(h)\right)^{1/q} + B\left(\int_{S^n} |u|^q \, dv(h)\right)^{1/q} \tag{9.2}$$

where $p = m_1 q/(m_1 - q)$. The result announced by Iliopoulos [**124**] is that the best constant A in (9.2), denoted by $\alpha = \alpha_{q,G}(S^n)$, is such that

$$\begin{cases} \alpha \ge \omega_{m_2-1}^{-\frac{1}{m_1}} K(m_1, q) \\ \alpha \le \max\left(\omega_{m_2-1}^{-\frac{1}{m_1}} K(m_1, q), \omega_{m_1-1}^{-\frac{1}{m_2}} K(m_2, q_n)\omega_n^{\frac{1}{m_2}-\frac{1}{m_1}}\right) \end{cases}$$

where $1/q_n = 1/q + 1/m_2 - 1/m_1$, ω_s stands for the volume of (S^s, h), and $K(\cdot, \cdot)$ is as in Theorem 4.4. In particular, one gets from this result that

$$\alpha_{q.G}(S^n) = \omega_{m-1}^{-1/m} K(m, q)$$

when $m_1 = m_2 = m$. We refer the reader to Iliopoulos [**124**] for more details on this result.

9.4. Compactness for Radially Symmetric Functions

In this section, we start dealing with complete, not necessarily compact, manifolds. As an example of the compactness results one can get in the presence of symmetries, we present a result first obtained by Berestycki-Lions [**24**], Coleman-Glazer-Martin [**53**], and Strauss [**182**]. Their proof was based on the special structure that radially symmetric functions on $\mathbb{R}^n$ have. The proof we present here is slightly different. For $q \ge 1$ real, set

$$H_{1,r}^q(\mathbb{R}^n) = \left\{u \in H_1^q(\mathbb{R}^n) \text{ s.t. } u \text{ is radially symmetric}\right\}$$

By u is radially symmetric, we mean that u is invariant under the action of $O(n)$. Recall that by Proposition 3.7, for any $q \in [1, n)$ and any $p \in [q, nq/(n-q)]$, $H_1^q(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$. In particular, for q and p as above, $H_{1,r}^q(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$. The result of Berestycki-Lions **[24]**, Coleman-Glazer-Martin **[53]**, and Strauss **[182]** can then be stated as follows:

THEOREM 9.4 *For any $q \in [1, n)$ real, and any $p \in (q, \frac{nq}{n-q})$ real, the embedding of $H_{1,r}^q(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$ is compact.*

The proof of Theorem 9.4 is based on the following compactness lemma, an easy consequence of Lemma 2.4 of Chapter 2. We refer the reader to Adams **[1]** for its proof.

LEMMA 9.3 *Let Ω be an open subset of $\mathbb{R}^n$, $p \geq 1$ real, and $\mathcal{H} \subset L^p(\Omega)$. Suppose there exists a sequence (Ω_j) of subdomains of Ω having the following properties:*

(i) *For each j, $\Omega_j \subset \Omega_{j+1}$,*
(ii) *for each j, $\mathcal{H}$ is precompact in $L^p(\Omega_j)$, and*
(iii) *for every $\varepsilon > 0$, there exists j such that $\int_{\Omega\setminus\Omega_j} |u|^p\,dx < \varepsilon$ for every $u \in \mathcal{H}$.*

Then $\mathcal{H}$ is precompact in $L^p(\Omega)$.

With such a result, we are now in position to prove Theorem 9.4.

PROOF OF THEOREM 9.4: Let $1 \leq q < n$ and $q < p < nq/(n-q)$ be given. By the mean value theorem for integrals one easily gets that there exists a positive constant C such that for any $f \in C^\infty([0, 1])$,

$$\int_0^1 |f(t)|^p\,dt \leq C\left(\int_0^1 \left(|f'(t)|^q + |f(t)|^q\right)dt\right)^{p/q}$$

It is then easy to see that for any integer k and any $f \in C^\infty([k, k+1])$,

$$\int_k^{k+1} |f(t)|^p\,dt \leq C\left(\int_k^{k+1} \left(|f'(t)|^q + |f(t)|^q\right)dt\right)^{p/q}$$

Let k be an integer and set

$$C_k = \{x \in \mathbb{R}^n \text{ s.t. } k \leq |x| \leq k+1\}$$

If $\mathcal{D}_r(\mathbb{R}^n)$ stands for $\mathcal{D}_{O(n)}(\mathbb{R}^n)$, $\mathcal{D}_r(\mathbb{R}^n)$ is dense in $H_{1,r}^q(\mathbb{R}^n)$. Noting that $p/q > 1$, one then has that for any $u \in \mathcal{D}_r(\mathbb{R}^n)$ and any real number $R \geq 1$,

$$\begin{aligned}
&\int_{R^n\setminus B_0^e(R)} |u(x)|^p\,dx \\
&\leq \sum_{k\geq[R]} \int_{C_k} |u(x)|^p\,dx \\
&\leq \omega_{n-1} \sum_{k\geq[R]} (k+1)^{n-1} \int_k^{k+1} |u(t)|^p\,dt \\
&\leq C\omega_{n-1} \sum_{k\geq[R]} (k+1)^{n-1} \left(\int_k^{k+1} \left(|u'(t)|^q + |u(t)|^q\right)dt\right)^{p/q}
\end{aligned}$$

$$
\begin{aligned}
&= C\omega_{n-1} \sum_{k\geq[R]} (k+1)^{(n-1)(1-p/q)} \\
&\quad \times \left(\left(\frac{k+1}{k}\right)^{n-1} k^{n-1} \int_k^{k+1} (|u'(t)|^q + |u(t)|^q)dt \right)^{p/q} \\
&\leq 2^{(n-1)p/q} C\omega_{n-1}^{1-p/q} \sum_{k\geq[R]} (k+1)^{(n-1)(1-p/q)} \\
&\quad \times \left(\int_{C_k} (|\nabla u(x)|^q + |u(x)|^q)dx \right)^{p/q} \\
&\leq \frac{2^{(n-1)p/q} C\omega_{n-1}^{1-p/q}}{([R]+1)^{(n-1)(p/q-1)}} \left(\int_{R^n\setminus B_0^e(R)} (|\nabla u(x)|^q + |u(x)|^q)dx \right)^{p/q}
\end{aligned}
$$

where $[R]$ is the greatest integer not exceeding R, and ω_{n-1} stands for the volume of the standard unit sphere (S^{n-1}, h) of $\mathbb{R}^n$. As a consequence, we get that there exists a positive constant A such that for any $R \geq 1$ and any $u \in \mathcal{D}_r(\mathbb{R}^n)$,

$$
\left(\int_{R^n\setminus B_0^e(R)} |u(x)|^p \, dx \right)^{1/p} \leq A([R]+1)^{-(n-1)(1/q-1/p)} \|u\|_{H_1^q}
$$

By density such an inequality is then valid for any $u \in H_{1,r}^q(\mathbb{R}^n)$. Independently, since $1/q - 1/p > 0$, one has that

$$
\lim_{R\to+\infty} ([R]+1)^{-(n-1)(1/q-1/p)} = 0
$$

By Lemma 9.3, this ends the proof of the theorem. □

As an important remark, note that the compactness of $H_{1,r}^q(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ is not true anymore for $p = q$ or $p = nq/(n-q)$. Suppose first that $p = nq/(n-q)$. Let $u \in \mathcal{D}(\mathbb{R}^n)$, $u \not\equiv 0$, be radially symmetric. For $\lambda \geq 1$ real, set

$$
u_\lambda(x) = \lambda^{\frac{n}{q}-1} u(\lambda x)
$$

As one can easily check, u_λ is also radially symmetric, and

(i) $\forall\lambda,\ \|u_\lambda\|_{H_1^q} \leq \|u\|_{H_1^q}$,
(ii) $\forall\lambda,\ \|u_\lambda\|_p = \|u\|_p$, and
(iii) $\lim_{\lambda\to+\infty} u_\lambda = 0$ a.e.

where $\|\cdot\|_p$ stands for the L^p-norm in $\mathbb{R}^n$. By (i), (u_λ) is bounded in $H_{1,r}^q(\mathbb{R}^n)$, while by (ii) and (iii), there does not exist a subsequence of (u_λ) that converges in $L^p(\mathbb{R}^n)$ as $\lambda \to +\infty$. Suppose now that $p = q$. For u as above, and for $\lambda \in (0, 1]$ real, set

$$
u_\lambda(x) = \lambda^{\frac{n}{q}} u(\lambda x)
$$

Here again, u_λ is radially symmetric, and

(iv) $\forall\lambda,\ \|u_\lambda\|_{H_1^q} \leq \|u\|_{H_1^q}$,
(v) $\forall\lambda,\ \|u_\lambda\|_p = \|u\|_p$, and
(vi) $\lim_{\lambda\to 0^+} u_\lambda = 0$ a.e.

By (iv), (u_λ) is bounded in $H^q_{1,r}(\mathbb{R}^n)$, while by (v) and (vi), there does not exist a subsequence of (u_λ) that converges in $L^p(\mathbb{R}^n)$ as $\lambda \to 0^+$. As mentioned above, this proves that the compactness of the embedding of $H^q_{1,r}(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$ is not true anymore for $p = q$ or $p = nq/(n-q)$.

9.5. A Main Lemma for Complete Manifolds

Let (M, g) be a smooth, complete Riemannian n-manifold. For $q \geq 1$ real, and n, k two integers, we define $p^\star = p^\star(n, k, q)$ by

$$\begin{cases} p^\star = \frac{(n-k)q}{n-k-q} & \text{if } n-k > q \\ p^\star = +\infty & \text{if } n-k \leq q \end{cases}$$

When $k \geq 1$ and $q < n$, one then has that $p^\star > nq/(n-q)$. The purpose of this section is to prove the following result of Hebey-Vaugon [120]:

LEMMA 9.4 (Main Lemma) *Let (M, g) be a smooth, complete Riemannian n-manifold, and G be a compact subgroup of* $\mathrm{Isom}_g(M)$. *Let $q \geq 1$ be given,*

$$k = \min_{x \in M} \dim O^x_G \,,$$

and $p^\star = p^\star(n, k, q)$ be as above. For $p \geq 1$ real, consider the two following conditions:

A_p. *There exists $C > 0$ and there exists a compact subset K of M such that for any $u \in \mathcal{D}_G(M)$,*

$$\left(\int_{M\setminus K} |u|^p \, dv(g)\right)^{\frac{1}{p}} \leq C \left(\left(\int_M |\nabla u|^q \, dv(g)\right)^{\frac{1}{q}} + \left(\int_M |u|^q \, dv(g)\right)^{\frac{1}{q}}\right)$$

B_p. *For any $\varepsilon > 0$ there exists a compact subset K_ε of M such that for any $u \in \mathcal{D}_G(M)$,*

$$\left(\int_{M\setminus K_\varepsilon} |u|^p \, dv(g)\right)^{\frac{1}{p}} \leq \varepsilon \left(\left(\int_M |\nabla u|^q \, dv(g)\right)^{\frac{1}{q}} + \left(\int_M |u|^q \, dv(g)\right)^{\frac{1}{q}}\right)$$

If $1 \leq p \leq p^\star$ and A_p holds, then $H^q_{1,G}(M) \subset L^p(M)$ and the embedding is continuous. If $1 \leq p < p^\star$ and B_p holds, then the embedding is compact.

In order to prove the main lemma, we need first the following result:

LEMMA 9.5 *Let (M, g) be a smooth, complete Riemannian n-manifold, K be a compact subset of M, and G be a compact subgroup of* $\mathrm{Isom}_g(M)$. *Let $q \geq 1$ be given,*

$$k = \min_{x \in K} \dim O^x_G \,,$$

and $p^\star = p^\star(n, k, q)$ be as above. Noting that functions on M can be seen as functions on K, for any $1 \leq p \leq p^\star$, $H^q_{1,G}(M) \subset L^p(K)$, and the embedding is continuous. Furthermore, the embedding becomes compact if $p < p^\star$.

PROOF: If $k = 0$, the result is a straightforward consequence of the standard Sobolev embedding theorem. We assume in the sequel that $k \geq 1$. By Lemma 9.1 one has that K is covered by a finite number of charts $(\Omega_m, \varphi_m)_{m=1,\dots,N}$ such that for any m:

(i) $\varphi_m(\Omega_m) = U_m \times V_m$, where U_m is some open subset of $\mathbb{R}^{k_m}$, V_m is some open subset of $\mathbb{R}^{n-k_m}$, and $k_m \in \mathbb{N}$ satisfies $k_m \geq k$;

(ii) U_m and V_m are bounded, and V_m has a smooth boundary;

(iii) $\forall y \in \Omega_m$, $U_m \times \Pi_2(\varphi_m(y)) \subset \varphi_m(O_G^y \cap \Omega_m)$ where $\Pi_2 : \mathbb{R}^{k_m} \times \mathbb{R}^{n-k_m}$ is the second projection; and

(iv) $\exists \alpha_m > 0$ with $\alpha_m^{-1}\delta_{ij} \leq g_{ij}^m \leq \alpha_m \delta_{ij}$ as bilinear forms, where the g_{ij}^m's are the components of g in (Ω_m, φ_m).

From now on, let $u \in \mathcal{D}_G(M)$. According to (iii), and since u is G-invariant, one has that for any m, any $x, x' \in U_m$, and any $y \in V_m$,

$$u \circ \varphi_m^{-1}(x, y) = u \circ \varphi_m^{-1}(x', y)$$

As a consequence, for any m there exists $\tilde{u}_m \in C^\infty(\mathbb{R}^{n-k_m})$ such that for any $x \in U_m$ and any $y \in V_m$,

$$u \circ \varphi_m^{-1}(x, y) = \tilde{u}_m(y)$$

(Without loss of generality, one can assume that φ_m is actually defined on some open set $\tilde{\Omega}_m$ containing $\overline{\Omega}_m$ such that $\varphi_m(\tilde{\Omega}_m) = \tilde{U}_m \times \tilde{V}_m$ with $\overline{V}_m \subset \tilde{V}_m$). We then get that for any m and any real number $p \geq 1$,

$$\begin{aligned}\int_{\Omega_m} |u|^p \, dv(g) &= \int_{U_m \times V_m} \left(|u|^p \sqrt{\det g_{ij}^m}\right) \circ \varphi_m^{-1}(x, y) dx\, dy \\ &\leq A_m \int_{U_m \times V_m} |u \circ \varphi_m^{-1}(x, y)|^p \, dx\, dy \\ &= \tilde{A}_m \int_{V_m} |\tilde{u}_m(y)|^p \, dy\end{aligned}$$

where A_m and $\tilde{A}_m$ are positive constants that do not depend on u. Similarly, one has that for any m and any $p \geq 1$,

$$\int_{\Omega_m} |u|^p \, dv(g) \geq B_m \int_{V_m} |\tilde{u}_m(y)|^p \, dy$$

and

$$\int_{\Omega_m} |\nabla u|^p \, dv(g) \geq \tilde{B}_m \int_{V_m} |\nabla \tilde{u}_m(y)|^p \, dy$$

where $B_m > 0$ and $\tilde{B}_m > 0$ do not depend on u. Combining these inequalities and the Sobolev embedding theorem for bounded domains of Euclidean spaces, we get that for any m and any real number $q \geq 1$,

(v) if $n - k_m \leq q$, then for any real number $p \geq 1$ there exists $C_m > 0$ such that for any $u \in \mathcal{D}_G(M)$,

$$\left(\int_{\Omega_m} |u|^p \, dv(g)\right)^{\frac{1}{p}} \leq C_m \left(\left(\int_{\Omega_m} |\nabla u|^q \, dv(g)\right)^{\frac{1}{q}} + \left(\int_{\Omega_m} |u|^q \, dv(g)\right)^{\frac{1}{q}}\right)$$

(vi) if $n - k_m > q$, then for any real number $1 \le p \le \frac{(n-k_m)q}{n-k_m-q}$, there exists $C_m > 0$ such that for any $u \in \mathcal{D}_G(M)$,

$$\left(\int_{\Omega_m} |u|^p \, dv(g)\right)^{\frac{1}{p}} \le C_m \left(\left(\int_{\Omega_m} |\nabla u|^q \, dv(g)\right)^{\frac{1}{q}} + \left(\int_{\Omega_m} |u|^q \, dv(g)\right)^{\frac{1}{q}}\right)$$

But:

(vii) $n - k_m \le n - k$ so that $p^\star(n, k_m, q) \ge p^\star(n, k, q)$,

(viii) $\left(\int_K |u|^p \, dv(g)\right)^{1/p} \le \sum_{m=1}^N \left(\int_{\Omega_m} |u|^p \, dv(g)\right)^{1/p}$, and

(ix) $\sum_{m=1}^N \left(\left(\int_{\Omega_m} |\nabla u|^q \, dv(g)\right)^{1/q} + \left(\int_{\Omega_m} |u|^q \, dv(g)\right)^{1/q}\right)$
$\le N\left(\left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} + \left(\int_M |u|^q \, dv(g)\right)^{1/q}\right)$

As a consequence, for any $q \ge 1$ and any real number $1 \le p \le p^\star$, $H_{1,G}^q(M) \subset L^p(K)$, and the embedding is continuous. By standard arguments, as developed in the proof of Theorem 2.9, one then easily gets that these embeddings are compact provided that $p < p^\star$. This ends the proof of the lemma. □

We are now in position to prove the main lemma.

PROOF OF LEMMA 9.4: Suppose that A_p holds for $1 \le p \le p^\star$, p real. Then there exists a positive constant C_1 and a compact subset K of M such that for any $u \in H_{1,G}^q(M)$,

$$\int_{M\setminus K} |u|^p \, dv(g) \le C_1 \left(\left(\int_M |\nabla u|^q \, dv(g)\right)^{\frac{1}{q}} + \left(\int_M |u|^q \, dv(g)\right)^{\frac{1}{q}}\right)^p$$

while by Lemma 9.5, there exists some positive constant C_2 such that for any $u \in H_{1,G}^q(M)$,

$$\int_K |u|^p \, dv(g) \le C_2 \left(\left(\int_M |\nabla u|^q \, dv(g)\right)^{\frac{1}{q}} + \left(\int_M |u|^q \, dv(g)\right)^{\frac{1}{q}}\right)^p$$

Hence, for any $u \in H_{1,G}^q(M)$,

$$\left(\int_M |u|^p \, dv(g)\right)^{\frac{1}{p}} \le (C_1 + C_2)^{\frac{1}{p}} \|u\|_{H_1^q}$$

so that $H_{1,G}^q(M) \subset L^p(M)$, and the embedding is continuous. Suppose now that B_p holds for some $1 \le p < p^\star$. Let (K_i) be a sequence of compact subsets of M such that $K_i \subset K_{i+1}$, $\bigcup_i K_i = M$, and such that for any $u \in H_{1,G}^q(M)$,

$$\left(\int_{M\setminus K_i} |u|^p \, dv(g)\right)^{\frac{1}{p}} \le \frac{1}{i} \left(\left(\int_M |\nabla u|^q \, dv(g)\right)^{\frac{1}{q}} + \left(\int_M |u|^q \, dv(g)\right)^{\frac{1}{q}}\right)$$

Let (u_k) be some sequence of functions in $H_{1,G}^q(M)$ such that for any k, $\|u_k\|_{H_1^q} \le C_0$. By induction, and with Lemma 9.5, one easily gets that for any i there exists a subsequence (u_k^i) of (u_k) such that

1. if $i \le j$, (u_k^j) is a subsequence of (u_k^i), and
2. (u_k^i) converges in $L^p(K_i)$.

Let u^i be the limit of (u^i_k) in $L^p(K_i)$. For any i, denote by u_i one of the u^i_k's such that

$$\left(\int_{K_i} |u_i - u^i|^p \, dv(g)\right)^{\frac{1}{p}} \leq \frac{1}{i}$$

Then, (u_i) is a subsequence of (u_k). We now assert that (u_i) converges in $L^p(M)$. (Obviously, this will end the proof of Lemma 9.4). In order to prove the claim, we first remark that for $j \geq i$, $u^j = u^i$ in K_i. We then note that the u^i's, when extended by 0 in $M \backslash K_i$, form a Cauchy sequence in $L^p(M)$. This comes from the fact that for $j \geq i$,

$$\begin{aligned}
&\left(\int_M |u^j - u^i|^p \, dv(g)\right)^{\frac{1}{p}} \\
&= \left(\int_{K_j \backslash K_i} |u^j|^p \, dv(g)\right)^{\frac{1}{p}} \\
&\leq \left(\int_{K_j} |u_j - u^j|^p \, dv(g)\right)^{\frac{1}{p}} + \left(\int_{M \backslash K_i} |u_j|^p \, dv(g)\right)^{\frac{1}{p}} \leq \frac{1}{j} + \frac{C_0}{i}
\end{aligned}$$

Let u be the limit of the u^i's (extended by 0 in $M \backslash K_i$) in $L^p(M)$. According to what we have just said,

$$\left(\int_M |u^i - u|^p \, dv(g)\right)^{\frac{1}{p}} \leq \frac{C_0}{i}$$

for any i. One then gets the claim by noting that for any i,

$$\begin{aligned}
&\left(\int_M |u_i - u|^p \, dv(g)\right)^{\frac{1}{p}} \\
&\leq \left(\int_M |u_i - u^i|^p \, dv(g)\right)^{\frac{1}{p}} + \left(\int_M |u^i - u|^p \, dv(g)\right)^{\frac{1}{p}} \\
&\leq \frac{C_0}{i} + \left(\int_{K_i} |u_i - u^i|^p \, dv(g)\right)^{\frac{1}{p}} + \left(\int_{M \backslash K_i} |u_i|^p \, dv(g)\right)^{\frac{1}{p}} \leq \frac{2C_0 + 1}{i}
\end{aligned}$$

This ends the proof of the lemma. □

9.6. The Codimension 1 Case

Let (M, g) be a smooth, complete Riemannian n-manifold, and G be a compact subgroup of $\mathrm{Isom}_g(M)$. In what follows, the action of G is said to be of codimension 1 if

$$\max_{x \in M} \dim O^x_G = n - 1$$

One can then prove (see [32]) that the quotient M/G is homeomorphic to an interval of $\mathbb{R}$. For $x \in M$, let $v(O^x_G)$ be the volume of O^x_G for the metric induced by g.

As in Section 9.5, let also $p^\star = p^\star(n,k,q)$ be defined

$$p^\star = \begin{cases} \frac{(n-k)q}{n-k-q} & \text{if } n-k > q \\ +\infty & \text{if } n-k \le q \end{cases}$$

The purpose of this section is to prove the following result of Hebey-Vaugon [**120**]:

THEOREM 9.5 *Let (M,g) be a smooth, complete Riemannian n-manifold, let G be a compact subgroup of* $\mathrm{Isom}_g(M)$ *whose action is of codimension 1, and set*

$$k = \min_{x \in M} \dim O_G^x$$

Consider the two following assumptions:

H$_1$. *There exist $C > 0$ and a compact subset K of M such that for any $x \in M\backslash K$, $v(O_G^x) \ge C$.*

H$_2$. *For any $\varepsilon > 0$ there exists a compact subset K_ε of M such that for any $x \in M\backslash K_\varepsilon$, $v(O_G^x) \ge \frac{1}{\varepsilon}$.*

For $q \ge 1$, let $p^\star = p^\star(n,k,q)$ be as above. If H$_1$ *holds, then for any $q \ge 1$ and any real number $p \in [q, p^\star]$, $H_{1,G}^q(M) \subset L^p(M)$ and the embedding is continuous. If* H$_2$ *holds, then for any $q \ge 1$ and any $p \in (q, p^\star)$, the embedding of $H_{1,G}^q(M)$ in $L^p(M)$ is compact.*

PROOF: If M is compact, the result is already contained in Theorem 9.1 (or in the standard Sobolev embedding theorem for compact manifolds if G has a fixed point). We assume in what follows that M is not compact. Let $\Pi : M \to M/G$ be the canonical projection from M to M/G. As already mentioned, M/G is homeomorphic to some interval I of $\mathbb{R}$. Since Π is a proper map (see Section 9.1), M/G is noncompact and I is homeomorphic either to $\mathbb{R}$ itself, or to $[0,+\infty)$. In what follows, we assume that I is homeomorphic to $[0,+\infty)$. (The difficulties involved in the case where I is homeomorphic to $\mathbb{R}$ are all contained in the case where I is homeomorphic to $[0,+\infty)$). Let us identify I with $[0,+\infty)$. By [**32**], one has that for any $t \in (0,+\infty)$, $\Pi^{-1}(t)$ is a principal orbit (of dimension $n-1$), and that $O = \Pi^{-1}(0)$ has dimension $k \le n-1$. Furthermore (see Section 9.1) one has that

$$\Pi : M\backslash O \to (0,+\infty)$$

is a Riemannian submersion with respect to g and the quotient metric h (induced from g) on $(0,+\infty)$. In what follows, v denotes the function on $(0,+\infty)$ defined by $v(\Pi(x)) = v(O_G^x)$, and we set $\tilde h = v^2 h$. Suppose now that H$_1$ holds. In order to prove the first part of the theorem, by the main lemma of Section 9.5, one has to prove that for $p \ge q$, there exist $\tilde C > 0$ and a compact subset $\tilde K$ of M such that for any $u \in \mathcal{D}_G(M)$,

$$\left(\int_{M\backslash\tilde K} |u|^p \, dv(g)\right)^{\frac{1}{p}} \le \tilde C \left(\left(\int_M |\nabla u|^q \, dv(g)\right)^{\frac{1}{q}} + \left(\int_M |u|^q \, dv(g)\right)^{\frac{1}{q}}\right)$$

Let K be the compact subset of M given by H$_1$. Then, $\Pi(K)$ is contained in some interval $[0,R]$, $\tilde K = \Pi^{-1}([0,R])$ is a compact subset of M such that $K \subset \tilde K$, and $\Pi(M\backslash\tilde K) = (R,+\infty)$. If $u \in \mathcal{D}_G(M)$ we denote by $\tilde u$ the function on $[0,+\infty)$

defined by $\tilde{u} \circ \Pi = u$. Let $1 \le q \le p$ be given. By (9.1) we get that for any $u \in \mathcal{D}_G(M)$,

$$\int_{M \backslash \tilde{K}} |u|^p \, dv(g) = \int_R^{+\infty} |\tilde{u}|^p \, dv(\tilde{h})$$

On the other hand, since we are in dimension 1, and since again by (9.1),

$$\int_0^{+\infty} dv(\tilde{h}) = \mathrm{Vol}_g(M) \,,$$

one easily gets that $((0, +\infty), \tilde{h})$ and $((0, \mathrm{Vol}_g(M)), e)$ are isometric, where e denotes the Euclidean metric of $\mathbb{R}$ and $\mathrm{Vol}_g(M)$ denotes the volume of (M, g). Hence, by the standard Sobolev inequality for intervals of $\mathbb{R}$ (see Lemma 9.6 for a slight improvement of such an inequality), we get that there exists $C > 0$ such that for any $u \in \mathcal{D}_G(M)$,

$$\left(\int_R^{+\infty} |\tilde{u}|^p \, dv(\tilde{h}) \right)^{\frac{q}{p}} \le C \left(\int_R^{+\infty} \left(|\nabla \tilde{u}|_{\tilde{h}}^q + |\tilde{u}|^q \right) dv(\tilde{h}) \right)$$

(Since there might be some possible confusion in what follows, the subscript $\tilde{h}$ in $|\nabla \tilde{u}|_{\tilde{h}}$ means that we take the norm of $\nabla \tilde{u}$ with respect to the metric $\tilde{h}$). Since by H_1 one has that v is bounded from below on $[R, +\infty)$, we get that for any $u \in \mathcal{D}_G(M)$,

$$\begin{aligned} \left(\int_{M \backslash \tilde{K}} |u|^p \, dv(g) \right)^{\frac{q}{p}} &\le C \left(\int_R^{+\infty} \left(v^{-2} |\nabla \tilde{u}|_h^q + |\tilde{u}|^q \right) dv(\tilde{h}) \right) \\ &\le \tilde{C} \left(\int_R^{+\infty} \left(|\nabla \tilde{u}|_h^q + |\tilde{u}|^q \right) dv(\tilde{h}) \right) \end{aligned}$$

But $\Pi : (M \backslash O, g) \to ((0, +\infty), h)$ is a Riemannian submersion. Hence, for any $x \in M \backslash O$ and any $u \in \mathcal{D}_G(M)$, $|\nabla \tilde{u}|_h(\Pi(x)) = |\nabla u|_g(x)$. As a consequence, and again by (9.1), we get that for any $u \in \mathcal{D}_G(M)$,

$$\int_{M \backslash \tilde{K}} |\nabla u|^q \, dv(g) = \int_R^{+\infty} |\nabla \tilde{u}|_h^q \, dv(\tilde{h})$$

Clearly, one also has that for any $u \in \mathcal{D}_G(M)$,

$$\int_{M \backslash \tilde{K}} |u|^q \, dv(g) = \int_R^{+\infty} |\tilde{u}|^q \, dv(\tilde{h})$$

so that we get the existence of some $\tilde{C} > 0$ and some compact subset $\tilde{K}$ of M such that for any $u \in \mathcal{D}_G(M)$,

$$\left(\int_{M \backslash \tilde{K}} |u|^p \, dv(g) \right)^{\frac{1}{p}} \le \tilde{C} \left(\left(\int_M |\nabla u|^q \, dv(g) \right)^{\frac{1}{q}} + \left(\int_M |u|^q \, dv(g) \right)^{\frac{1}{q}} \right)$$

As already mentioned, this proves the first part of Theorem 9.5. □

In order to prove the second part of Theorem 9.5, we need the following lemma:

LEMMA 9.6 *Let $\mathbb{R}$ be endowed with its standard metric e, and let I be some noncompact interval of $\mathbb{R}$.*

(i) *If I is bounded and of length δ, then for any $p, q \geq 1$,*

$$\left(\int_I |u|^p \, dv(e)\right)^{\frac{1}{p}} \leq \delta^{1+\frac{1}{p}-\frac{1}{q}}\left(\int_I |u'|^q \, dv(e)\right)^{\frac{1}{q}} \quad \textit{for any } u \in \mathcal{D}(I)$$

(ii) *If I is not bounded, then for any $1 \leq q < p$, and any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ (depending only on ε, p, and q) such that,*

$$\left(\int_I |u|^p \, dv(e)\right)^{\frac{q}{p}} \leq C_\varepsilon \int_I |u'|^q \, dv(e) + \varepsilon \int_I |u|^q \, dv(e) \quad \textit{for any } u \in \mathcal{D}(I)$$

PROOF: Suppose that I is bounded. Without loss of generality, one can assume that $I = (0, \delta]$ or $I = (0, \delta)$. Let $u \in \mathcal{D}(I)$. Then, for any $x \in (0, \delta)$,

$$\begin{aligned}|u(x)| = \left|\int_0^x u'(t)dt\right| &\leq \left(\int_0^x |u'(t)|^q \, dt\right)^{\frac{1}{q}}\left(\int_0^x dt\right)^{1-\frac{1}{q}} \\ &\leq \delta^{1-\frac{1}{q}}\left(\int_I |u'|^q \, dv(e)\right)^{\frac{1}{q}}\end{aligned}$$

As a consequence, we get that for any $u \in \mathcal{D}(I)$,

$$\left(\int_I |u|^p \, dv(e)\right)^{\frac{1}{p}} \leq \delta^{1+\frac{1}{p}-\frac{1}{q}}\left(\int_I |u'|^q \, dv(e)\right)^{\frac{1}{q}}$$

that proves (i). Suppose now that I is not bounded, and let $1 \leq q < p$ and $\varepsilon > 0$ be given. Without loss of generality, we can assume that $I = [0, +\infty)$ or $I = (0, +\infty)$. For $\delta > 0$ real, consider the covering

$$\mathbb{R} = \bigcup_{m \in \mathbb{Z}} (m\delta, (m+2)\delta)$$

and let (η_m) be a smooth partition of unity subordinate to this covering such that for any m, $\eta_m^{1/q} \in C^\infty(\mathbb{R})$ and $|(\eta_m^{1/q})'| \leq C_0/\delta$ for some $C_0 > 0$ that does not depend on m and δ. For any $u \in \mathcal{D}(I)$ one then has that

$$\begin{aligned}\left(\int_I |u|^p \, dv(e)\right)^{\frac{q}{p}} &= \left(\int_I \left(\sum_{m \in \mathbb{Z}} \eta_m |u|^q\right)^{\frac{p}{q}} dv(e)\right)^{\frac{q}{p}} \\ &\leq \sum_{m \in \mathbb{Z}} \left(\int_I |\eta_m^{1/q} u|^p \, dv(e)\right)^{\frac{q}{p}}\end{aligned}$$

while by (i), one easily gets that

$$\left(\int_I |\eta_m^{1/q} u|^p \, dv(e)\right)^{\frac{1}{p}} \leq (2\delta)^{1+\frac{1}{p}-\frac{1}{q}} \left(\int_{I_m} \left|(\eta_m^{1/q})u' + (\eta_m^{1/q})'u\right|^q dv(e)\right)^{\frac{1}{q}}$$

where $I_m = I \cap (m\delta, (m+2)\delta)$. From now on, let $\mu > 0$, depending only on q, be such that for $x, y \geq 0$, $(x+y)^q \leq \mu(x^q + y^q)$. One then has that for any

$u \in \mathcal{D}(I)$,

$$\left(\int_I |\eta_m^{1/q} u|^p \, dv(e)\right)^{\frac{1}{p}} \leq$$

$$(2\delta)^{1+\frac{1}{p}-\frac{1}{q}} \mu^{\frac{1}{q}} \left(\int_{I_m} \left(\eta_m |u'|^q + |(\eta_m^{1/q})'|^q |u|^q\right) dv(e)\right)^{\frac{1}{q}}$$

and since any t in I meets at most two of the I_m's, we get that for any $u \in \mathcal{D}(I)$,

$$\left(\int_I |u|^p \, dv(e)\right)^{\frac{q}{p}} \leq (2\delta)^{q+\frac{q}{p}-1} \mu \left\{ \int_I |u'|^q \, dv(e) + \frac{2C_0^q}{\delta^q} \int_I |u|^q \, dv(e) \right\}$$

One then obtains the result by choosing δ such that $(2^{q+\frac{q}{p}} C_0^q \mu) \delta^{\frac{q}{p}-1} = \varepsilon$. This ends the proof of the lemma. □

We now return to the proof of Theorem 9.5.

PROOF OF THEOREM 9.5 (CONTINUED): Suppose that H_2 holds, and let $\varepsilon > 0$ and $1 \leq q < p$ be given. Assume first that $\mathrm{Vol}_g(M) = +\infty$. By Lemma 9.6, there exists $C_\varepsilon > 0$ such that for any unbounded interval I of $\mathbb{R}$ and any $\tilde{u} \in \mathcal{D}(I)$,

$$\left(\int_I |\tilde{u}|^p \, dv(e)\right)^{\frac{q}{p}} \leq C_\varepsilon \int_I |\tilde{u}'|^q \, dv(e) + \varepsilon \int_I |\tilde{u}|^q \, dv(e)$$

Let K_ε (given by H_2) be some compact subset of M such that for any $x \in M \backslash K_\varepsilon$,

$$v(O_G^x) \geq \sqrt{\frac{C_\varepsilon}{\varepsilon}}$$

With the notation of the first part of the proof of Theorem 9.5, $\Pi(K_\varepsilon)$ is contained in some interval $[0, R_\varepsilon]$, $\tilde{K}_\varepsilon = \Pi^{-1}([0, R_\varepsilon])$ is a compact subset of M such that $K_\varepsilon \subset \tilde{K}_\varepsilon$, and $\Pi(M \backslash \tilde{K}_\varepsilon) = (R_\varepsilon, +\infty)$. Noting that $((0, +\infty), \tilde{h})$ and $((0, \mathrm{Vol}_g(M)), e)$ are isometric, and that by (9.1),

$$\mathrm{Vol}_{\tilde{h}}\left((R_\varepsilon, +\infty)\right) = \mathrm{Vol}_g(M \backslash \tilde{K}_\varepsilon) = +\infty$$

one then gets, with the same kind of arguments as those used in the first part of the proof of Theorem 9.5, that for any $u \in \mathcal{D}_G(M)$,

$$\begin{aligned}
\left(\int_{M \backslash \tilde{K}_\varepsilon} |u|^p \, dv(g)\right)^{\frac{q}{p}} &\leq C_\varepsilon \int_{R_\varepsilon}^{+\infty} |\nabla \tilde{u}|_{\tilde{h}}^q \, dv(\tilde{h}) + \varepsilon \int_{R_\varepsilon}^{+\infty} |\tilde{u}|^q \, dv(\tilde{h}) \\
&= C_\varepsilon \int_{R_\varepsilon}^{+\infty} |\nabla \tilde{u}|_h^q v^{-2} \, dv(\tilde{h}) + \varepsilon \int_{R_\varepsilon}^{+\infty} |\tilde{u}|^q \, dv(\tilde{h}) \\
&\leq C_\varepsilon \frac{\varepsilon}{C_\varepsilon} \int_{R_\varepsilon}^{+\infty} |\nabla \tilde{u}|_h^q \, dv(\tilde{h}) + \varepsilon \int_{R_\varepsilon}^{+\infty} |\tilde{u}|^q \, dv(\tilde{h}) \\
&= \varepsilon \left(\int_{M \backslash \tilde{K}_\varepsilon} |\nabla u|^q \, dv(g) + \int_{M \backslash \tilde{K}_\varepsilon} |u|^q \, dv(g)\right)
\end{aligned}$$

As a consequence, condition B_p of the main lemma is satisfied, and we get that the embedding of $H_{1,G}^q(M)$ in $L^p(M)$ is compact provided that $p < p^*$. Assume now that $\mathrm{Vol}_g(M) < +\infty$. Let K_ε (given by H_2) be some compact subset of M such that for any $x \in M\backslash K_\varepsilon$,

$$v(O_G^x) \geq \sqrt{\frac{\lambda}{\varepsilon}}$$

where $\lambda = (\mathrm{Vol}_g(M))^{1+1/p-1/q}$. Let R_ε and $\tilde{K}_\varepsilon$ be as above. Here,

$$\mathrm{Vol}_{\tilde{h}}\left((R_\varepsilon, +\infty)\right) = \mathrm{Vol}_g(M\backslash\tilde{K}_\varepsilon) \leq \mathrm{Vol}_g(M) < +\infty$$

so that by part (i) of Lemma 9.6, we get that for any $u \in \mathcal{D}_G(M)$,

$$\begin{aligned}\left(\int_{M\backslash\tilde{K}_\varepsilon} |u|^p\, dv(g)\right)^{\frac{q}{p}} \leq \lambda \int_{R_\varepsilon}^{+\infty} |\nabla\tilde{u}|_{\tilde{h}}^q\, dv(\tilde{h}) &= \lambda \int_{R_\varepsilon}^{+\infty} |\nabla\tilde{u}|_h^q v^{-2}\, dv(\tilde{h})\\ &\leq \lambda\left(\frac{\varepsilon}{\lambda}\right) \int_{R_\varepsilon}^{+\infty} |\nabla\tilde{u}|_h^q\, dv(\tilde{h})\\ &= \varepsilon \int_{M\backslash\tilde{K}_\varepsilon} |\nabla u|^q\, dv(g)\end{aligned}$$

Hence, condition B_p of the main lemma is satisfied, and the embedding of $H_{1,G}^q(M)$ in $L^p(M)$ is again compact provided that $p < p^\star$. This ends the proof of the theorem. □

As a concrete and easy example of applying Theorem 9.5, note that one recovers Theorem 9.4 dealing with functions on $\mathbb{R}^n$ that are radially symmetric.

9.7. The General Case

Let (M, g) be a smooth, complete Riemannian n-manifold, and G be a compact subgroup of $\mathrm{Isom}_g(M)$. We treat here the case where the action of G is not necessarily of codimension 1. Following Hebey-Vaugon [120], for $x \in M$ and $r > 0$ we set

$$T_r(O_G^x) = \left\{y \in M \,/\, d_g(y, O_G^x) < r\right\}$$

where d_g is the distance associated to g. If O_G^x is principal, we define the principal radius $R_{\mathrm{pr}}(O_G^x)$ by

$$\begin{aligned}R_{\mathrm{pr}}(O_G^x) = \sup\big\{r > 0 \,/\, \forall y \in T_r(O_G^x),\ &O_G^y \text{ is principal,}\\ &\text{and } \forall r' < r,\ \overline{T_{r'}(O_G^x)} \text{ is compact}\big\}\end{aligned}$$

and the principal tube $T_{\mathrm{pr}}(O_G^x)$ by

$$T_{\mathrm{pr}}(O_G^x) = T_\kappa(O_G^x) \quad \text{where} \quad \kappa = \min\left(1, \frac{R_{\mathrm{pr}}(O_G^x)}{2}\right)$$

The action of G on M is then said to be uniform at infinity if there exist $\alpha \geq 1$ and a compact subset K of M such that the following holds: For any $x \in M\backslash K$ such that O_G^x is principal, and for any $y, y' \in T_{\mathrm{pr}}(O_G^x)$,

$$v(O_G^y) \leq \alpha v(O_G^{y'})$$

where, as in Theorem 9.5, $v(O_g^y)$ and $v(O_g^{y'})$ denote the volume of O_G^y and $O_G^{y'}$ for the metric induced by g. Independently, we will say that the action of G on M is of bounded geometry type if the Ricci curvature of $(\Omega/G, h)$ is bounded from below, where

$$\Omega = \bigcup_{\{x \text{ s.t } O_G^x \text{ is principal}\}} O_G^x$$

and h is the quotient metric (induced from g) on Ω/G (see Section 9.1). By O'Neill's formula (Section 9.1), the action of G on M is of bounded geometry type if the sectional curvature of (M, g) is bounded from below.

We prove here the following result of Hebey-Vaugon [120]. As in Section 9.5, we define $p^\star = p^\star(n, k, q)$ by

$$p^\star = \begin{cases} \frac{(n-k)q}{n-k-q} & \text{if } n-k > q \\ +\infty & \text{if } n-k \le q \end{cases}$$

Since several groups are involved in the statement of Theorem 9.6, G_i-principal means principal for G_i (and for $x \in M$, $T_{\text{pr}}(O_{G_i}^x)$ is the principal tube with respect to G_i). In what follows, $\text{Vol}_g(T_{\text{pr}}(O_{G_i}^x))$ stands for the volume of $T_{\text{pr}}(O_{G_i}^x)$ with respect to g.

THEOREM 9.6 *Let (M, g) be a smooth, complete Riemannian n-manifold, G be a compact subgroup of* $\text{Isom}_g(M)$, *and $G_1, \ldots, G_s$ be s compact subgroups of G such that the actions of the G_i's on M are of bounded geometry type and uniform at infinity, $i = 1, \ldots, s$. Let $k_{\min} = \min_{x \in M} \dim O_G^x$, $k_i = \max_{x \in M} \dim O_{G_i}^x$ be the dimension of the principal orbits of G_i, and $k = \min\{k_{\min}, k_1, \ldots, k_s\}$. Consider the two following assumptions:*

H$_1$. *There exist $C > 0$ and a compact subset K of M such that for any point x in $M \backslash K$ there is some i for which $O_{G_i}^x$ is G_i-principal and for which* $\text{Vol}_g(T_{\text{pr}}(O_{G_i}^x)) \ge C$.

H$_2$. *For any $\varepsilon > 0$ there exists a compact subset K_ε of M such that for any point x in $M \backslash K_\varepsilon$ there is some i for which $O_{G_i}^x$ is G_i-principal and for which* $\text{Vol}_g(T_{\text{pr}}(O_{G_i}^x)) \ge \frac{1}{\varepsilon}$.

For $q \ge 1$ let $p^\star = p^\star(n, k, q)$ be as above. If H$_1$ *holds, then for any $q \ge 1$ and any real number $p \in [q, p^\star]$, $H_{1,G}^q(M) \subset L^p(M)$, and the embedding is continuous. If* H$_2$ *holds, then for any $q \ge 1$ and any $p \in (q, p^\star)$, the embedding of $H_{1,G}^q(M)$ in $L^p(M)$ is compact.*

In order to prove Theorem 9.6, we first need the following result:

LEMMA 9.7 *Let (M, g) be a smooth Riemannian n-manifold, not necessarily complete, such that* $\text{Rc}_g \ge \lambda g$ *for some $\lambda \in \mathbb{R}$. For $x \in M$ set*

$$\delta_x = \sup\left\{\delta > 0 \,/\, \overline{B_x(\delta)} \textit{ is compact}\right\}$$

and let $\varepsilon_x = \min(1, \frac{\delta_x}{10})$. For any subset $\mathcal{V}$ of M, there exists an integer $N = N(n, \lambda)$, depending only on n and λ, and there exists $I \subset \mathcal{V}$, such that $\mathcal{V} \subset \bigcup_{x \in I} B_x(\varepsilon_x)$, and such that for any $y \in \mathcal{V}$,

$$\text{Card}\left\{x \in I \,/\, y \in B_x(\varepsilon_x)\right\} \le N$$

where Card *stands for the cardinality.*

PROOF: First, we claim that for any $x, x' \in M$,

$$|\varepsilon_x - \varepsilon_{x'}| \leq \frac{1}{10} d_g(x, x') \tag{9.3}$$

In order to prove the claim, one can assume that $\delta_x < +\infty$ for any x. (If not, M is complete by Hopf-Rinow's theorem, $\varepsilon_x = 1$ for any x, and the claim is trivial). One can then note that the claim will be proved if we show that for any $x, x' \in M$,

$$|\delta_x - \delta_{x'}| \leq d_g(x, x')$$

Assume for that purpose that $\delta_x \geq \delta_{x'}$. Then, either $d_g(x, x') \geq \delta_x$, and the inequality above is trivial, or $d_g(x, x') < \delta_x - \eta$ for some $\eta > 0$, and one gets that

$$B_{x'}\big(\delta_x - d_g(x, x')\big) \subset B_x\big(\delta_x - \eta\big)$$

so that $\delta_{x'} \geq \delta_x - d_g(x, x')$ by definition of δ_x and $\delta_{x'}$. In any case, this proves the claim. Now, consider

$$X = \left\{I \subset \mathcal{V} \,/\, \forall x \neq x' \in I,\ d_g(x, x') \geq \tfrac{10}{21}(\varepsilon_x + \varepsilon_{x'})\right\}$$

Then X is partially ordered by inclusion, and obviously every chain in X has an upper bound. Hence, by Zorn's lemma, X contains a maximal element I. We now prove that $(B_x(\varepsilon_x))$, $x \in I$, is the covering we are looking for. First, we claim that

$$\mathcal{V} \subset \bigcup_{x \in I} B_x(\varepsilon_x)$$

In order to prove the claim, let us consider y some point in $\mathcal{V}$. If for any $x \in I$, $d_g(x, y) \geq \frac{10}{21}(\varepsilon_x + \varepsilon_y)$, then $I \cup \{y\} \in X$, so that by the maximality of I, $y \in I$. If not, there exists some $x \in I$ such that $d_g(x, y) < \frac{10}{21}(\varepsilon_x + \varepsilon_y)$. But by (9.3), $\varepsilon_y \leq \frac{1}{10} d_g(x, y) + \varepsilon_x$, so that

$$d_g(x, y) < \frac{10}{21}\varepsilon_x + \frac{1}{21} d_g(x, y) + \frac{10}{21}\varepsilon_x$$

$d_g(x, y) < \varepsilon_x$, and $y \in B_x(\varepsilon_x)$. This proves the claim. Now, let $y \in \mathcal{V}$ and suppose that y belongs to N balls $B_{x_i}(\varepsilon_{x_i})$ of the covering. Set $\varepsilon_i = \varepsilon_{x_i}$ and assume that the ε_i's are ordered so that $\varepsilon_1 \geq \varepsilon_2 \geq \cdots \geq \varepsilon_N$. Clearly, one has that

$$\bigcup_{i=1}^{N} B_{x_i}(\varepsilon_i) \subset B_y(2\varepsilon_1)$$

and since for $i \neq j$, $d_g(x_i, x_j) \geq \frac{10}{21}(\varepsilon_i + \varepsilon_j)$, one gets that for $i \neq j$,

$$B_{x_i}\left(\frac{10}{21}\varepsilon_i\right) \cap B_{x_j}\left(\frac{10}{21}\varepsilon_j\right) = \emptyset$$

Independently, note that by (9.3),

$$\varepsilon_1 - \varepsilon_N \leq \frac{1}{10} d_g(x_1, x_N) \leq \frac{1}{10}\big(d_g(x_1, y) + d_g(y, x_N)\big) \leq \frac{1}{10}\big(\varepsilon_1 + \varepsilon_N\big)$$

so that $\varepsilon_N \geq \frac{9}{11}\varepsilon_1$. Note also that for any i, the balls $B_{x_i}(3\varepsilon_1)$ are relatively compact, with the additional property that $B_y(2\varepsilon_1) \subset B_{x_i}(3\varepsilon_1)$. According to all these remarks and by Gromov's result, Theorem 1.1 of Chapter 1, one gets that there exists $C(n,\lambda) > 0$, depending only on n and λ, such that

$$\begin{aligned}\mathrm{Vol}_g\left(B_y(2\varepsilon_1)\right) &\geq \sum_{i=1}^{N} \mathrm{Vol}_g\left(B_{x_i}\left(\tfrac{10}{21}\varepsilon_i\right)\right) \geq \sum_{i=1}^{N} \mathrm{Vol}_g\left(B_{x_i}\left(\tfrac{90}{231}\varepsilon_1\right)\right)\\ &\geq C(n,\lambda)\sum_{i=1}^{N} \mathrm{Vol}_g\left(B_{x_i}(3\varepsilon_1)\right)\\ &\geq NC(n,\lambda)\,\mathrm{Vol}_g\left(B_y(2\varepsilon_1)\right)\end{aligned}$$

(We use here a different version of Theorem 1.1, namely, that the completeness of the manifold can be dropped in such a result provided that $B_x(R)$ is relatively compact. In such a situation, as one can easily check, the proof of Theorem 1.1 is unchanged). Hence, $N \leq \frac{1}{C(n,\lambda)}$, and this ends the proof of the lemma. □

We are now in position to prove Theorem 9.6.

Proof of Theorem 9.6: For any $i = 1,\dots,s$, let

$$\Omega_i = \bigcup_{\{x \text{ s.t. } O^x_{G_i} \text{ is } G_i\text{-principal}\}} O^x_{G_i}$$

and denote by h_i the quotient metric (induced from g) on Ω_i/G_i. Set $n_i = \dim(\Omega_i/G_i)$, and if $\Pi_i : \Omega_i \to \Omega_i/G_i$ is the canonical submersion, let v_i be the function on Ω_i/G_i defined by $v_i(\Pi_i(x)) = v(O^x_{G_i})$. Suppose now that H_1 holds. Then there exist $C > 0$, $\alpha \geq 1$, and a compact subset K of M, such that for any $x \in M\backslash K$, $O^x_{G_i}$ is G_i-principal for some $i \in \{1,\dots,s\}$, with the additional properties that:

(i) $\forall y, y' \in T_{\mathrm{pr}}(O^x_{G_i})$, $v(O^y_{G_i}) \leq \alpha v(O^{y'}_{G_i})$, and
(ii) $\mathrm{Vol}_g(T_{\mathrm{pr}}(O^x_{G_i})) \geq C$.

Let $\mathcal{U}_i \subset M\backslash K$ be such that for any $x \in \mathcal{U}_i$, $O^x_{G_i}$ is G_i-principal and (i) and (ii) hold. By assumption one has that

$$M\backslash K = \bigcup_{i=1}^{s} \mathcal{U}_i \tag{9.4}$$

Independently, and since for any $z, z' \in \Omega_i$,

$$d_{h_i}\left(\Pi_i(z), \Pi_i(z')\right) = d_g\left(O^z_{G_i}, O^{z'}_{G_i}\right)$$

one has that for any $x \in \Omega_i$ and any $\eta > 0$,

$$T_\eta(O^x_{G_i}) = \Pi_i^{-1}\left(B_{\Pi_i(x)}(\eta)\right) \tag{9.5}$$

Noting that Π_i is a proper map, and that Π_i is surjective, one then gets that for $x \in \Omega_i$, $y = \Pi_i(x)$, and δ_y as in Lemma 9.7,

$$R_{\mathrm{pr}}(O^x_{G_i}) = \delta_y \tag{9.6}$$

From now on, let $\mathcal{V}_i = \Pi_i(\mathcal{U}_i)$, and $(B_y(\varepsilon_y))$, $y \in I_i$, be the covering of $\mathcal{V}_i \subset \Omega_i/G_i$ given by Lemma 9.7. Let also $1 \leq q \leq p$ and $i \in \{1,\dots,s\}$ be given. For the sake of clarity, we assume in what follows that $k_i \geq 1$. (If $k_i = 0$, G_i is finite and $\Pi_i : \Omega_i \to \Omega_i/G_i$ is a finite covering. We then proceed as below, noting that (9.1) is still valid with the convention that $v(O^x_{G_i}) = \text{Card}\, O^x_{G_i}$). For $u \in \mathcal{D}_G(M)$, let $\tilde{u}_i$ be the function on M/G_i defined by $\tilde{u}_i \circ \Pi_i = u$. By (9.1) one has that

$$\int_{\mathcal{U}_i} |u|^p \, dv(g) = \int_{\mathcal{V}_i} |\tilde{u}_i|^p v \, dv(h_i)$$

while

$$\int_{\mathcal{V}_i} |\tilde{u}_i|^p v \, dv(h_i) \leq \sum_{y \in I_i} \int_{B_y(\varepsilon_y)} |\tilde{u}_i|^p v \, dv(h_i)$$

Hence,

$$\begin{aligned} \int_{\mathcal{U}_i} |u|^p \, dv(g) &\leq \sum_{y \in I_i} \int_{B_y(\varepsilon_y)} |\tilde{u}_i|^p v \, dv(h_i) \\ &\leq \sum_{y \in I_i} \left(\sup_{B_y(\varepsilon_y)} v \right) \int_{B_y(\varepsilon_y)} |\tilde{u}_i|^p \, dv(h_i) \end{aligned}$$

By Maheux and Saloff-Coste's result, Theorem 3.7 of Chapter 3 (see also **[154]**), and since the action of G_i on M is of bounded geometry type, one then obtains that for $p \leq p^\star(n, k_i, q)$, there exists $C_i > 0$ such that

$$\int_{\mathcal{U}_i} |u|^p \, dv(g) \leq$$
$$C_i \sum_{y \in I_i} \left(\sup_{B_y(\varepsilon_y)} v \right) \left(\text{Vol}_{h_i} B_y(\varepsilon_y) \right)^{1-\frac{p}{q}} \left(\int_{B_y(\varepsilon_y)} \left(|\nabla \tilde{u}_i|^q + |\tilde{u}_i|^q \right) dv(h_i) \right)^{\frac{p}{q}}$$

(Theorem 3.7, and what is done in **[154]**, hold for noncomplete manifolds, provided that the ball considered is relatively compact). One can then write that

$$\begin{aligned} \int_{\mathcal{U}_i} |u|^p \, dv(g) \leq C_i \sum_{y \in I_i} & \left(\frac{\sup_{B_y(\varepsilon_y)} v}{\left(\inf_{B_y(\varepsilon_y)} v \right)^{\frac{p}{q}}} \right) \left(\text{Vol}_{h_i} B_y(\varepsilon_y) \right)^{1-\frac{p}{q}} \\ & \times \left(\int_{B_y(\varepsilon_y)} \left(|\nabla \tilde{u}_i|^q + |\tilde{u}_i|^q \right) v \, dv(h_i) \right)^{\frac{p}{q}} \end{aligned}$$

Now note that by Gromov's result, as used in the proof of Lemma 9.7, there exists $\beta_i > 0$ (depending only on $(n - k_i)$ and a lower bound for Rc_{h_i}) such that

$$\text{Vol}_{h_i} \left(B_y(\varepsilon_y) \right) \geq \beta_i \, \text{Vol}_{h_i} \left(B_y(\kappa_y) \right)$$

where $\kappa_y = \min(1, \frac{\delta_y}{2})$. Since $1 \leq \frac{p}{q}$, one then gets by (i), (9.1), (9.5), and (9.6) that there exists $\hat{C}_i > 0$ such that for any $y \in I_i$,

$$\left(\frac{\sup_{B_y(\varepsilon_y)} v}{\left(\inf_{B_y(\varepsilon_y)} v \right)^{\frac{p}{q}}} \right) \left(\text{Vol}_{h_i} B_y(\varepsilon_y) \right)^{1-\frac{p}{q}} \leq \hat{C}_i \, \text{Vol}_g \left(T_{\text{pr}}(O^x_{G_i}) \right)^{1-\frac{p}{q}}$$

where $x \in \mathcal{U}_i$ is such that $\Pi_i(x) = y$. By (ii), Lemma 9.7, and (9.1), one then has that there exists $\tilde{C}_i > 0$ and an integer N_i such that

$$\begin{aligned}\int_{\mathcal{U}_i} |u|^p \, dv(g) &\le \tilde{C}_i \sum_{y \in I_i} \left(\int_{B_y(\varepsilon_y)} \left(|\nabla \tilde{u}_i|^q + |\tilde{u}_i|^q \right) v \, dv(h_i) \right)^{\frac{p}{q}} \\ &\le \tilde{C}_i \left(\sum_{y \in I_i} \int_{B_y(\varepsilon_y)} \left(|\nabla \tilde{u}_i|^q + |\tilde{u}_i|^q \right) v \, dv(h_i) \right)^{\frac{p}{q}} \\ &\le N_i^{\frac{p}{q}} \tilde{C}_i \left(\int_{\Omega_i / G_i} \left(|\nabla \tilde{u}_i|^q + |\tilde{u}_i|^q \right) v \, dv(h_i) \right)^{\frac{p}{q}} \\ &= N_i^{\frac{p}{q}} \tilde{C}_i \left(\int_{\Omega_i} \left(|\nabla u|^q + |u|^q \right) dv(g) \right)^{\frac{p}{q}}\end{aligned}$$

(Since $\Pi_i : (\Omega_i, g) \to (\Omega_i / G_i, h_i)$ is a Riemannian submersion, for any $x \in \Omega_i$ and any $u \in \mathcal{D}_G(M)$, $|\nabla \tilde{u}_i|(\Pi_i(x)) = |\nabla u|(x)$). As a consequence, we have proved that for any $i \in \{1, \dots, s\}$, any $q \ge 1$, and any p such that $q \le p \le p^\star(n, k_i, q)$, there exists a positive constant μ_i such that for any $u \in \mathcal{D}_G(M)$,

$$\int_{\mathcal{U}_i} |u|^p \, dv(g) \le \mu_i \int_{\Omega_i} \left(|\nabla u|^q + |u|^q \right) dv(g) \Big)^{\frac{p}{q}}$$

By (9.4), and since

$$p^\star(n, k, q) \le \min \left(p^\star(n, k_{\min}, q), p^\star(n, k_i, q) \right)$$

for any i, this implies that for any $q \ge 1$ and any p such that $q \le p \le p^\star(n, k, q)$, there exists $\mu > 0$ such that for any $u \in \mathcal{D}_G(M)$,

$$\left(\int_{M \setminus K} |u|^p \, dv(g) \right)^{\frac{1}{p}} \le \mu \left(\left(\int_M |\nabla u|^q \, dv(g) \right)^{\frac{1}{q}} + \left(\int_M |u|^q \, dv(g) \right)^{\frac{1}{q}} \right)$$

By the main lemma of Section 9.5, this proves the first part of Theorem 9.6.

Let us now prove the second part of Theorem 9.6. We assume here that H_2 holds. Let $\varepsilon > 0$ be given. Then, there exists $\alpha \ge 1$ and a compact subset K_ε of M such that for any $x \in M \setminus K_\varepsilon$, $O^x_{G_i}$ is G_i-principal for some $i \in \{1, \dots, s\}$, with the additional properties that

(iii) $\forall y, y' \in T_{\mathrm{pr}}(O^x_{G_i})$, $v(O^y_{G_i}) \le \alpha v(O^{y'}_{G_i})$ and
(iv) $\mathrm{Vol}_g(T_{\mathrm{pr}}(O^x_{G_i})) \ge \frac{1}{\varepsilon}$.

Let $\mathcal{U}_i \subset M \setminus K_\varepsilon$ be such that for any $x \in \mathcal{U}_i$, $O^x_{G_i}$ is G_i-principal and (iii) and (iv) hold. Here again, (9.4) is valid. By (iv) and the computations developed above, one then easily obtains that for any $q \ge 1$ and any p such that $q < p < p^\star(n, k, q)$, there exists $\mu > 0$, independent of ε, such that for any $u \in \mathcal{D}_G(M)$,

$$\left(\int_{M \setminus K_\varepsilon} |u|^p \, dv(g) \right)^{\frac{1}{p}} \le \mu \varepsilon^{\frac{p}{q} - 1} \left(\left(\int_M |\nabla u|^q \, dv(g) \right)^{\frac{1}{q}} + \left(\int_M |u|^q \, dv(g) \right)^{\frac{1}{q}} \right)$$

Since $\varepsilon > 0$ is arbitrary, such an inequality implies that condition B_p of the main lemma is satisfied. This ends the proof of the theorem. □

As a first remark on Theorem 9.6, note that when G is reduced to the identity, all the orbits are principal and the principal tubes $T_{\mathrm{pr}}(O_G^x)$ are just the balls $B_x(1)$. Condition H_1 of Theorem 9.6 is then optimal by Theorem 3.3. Here, and by convention, $v(O_G^x) = \operatorname{Card} O_G^x$ if G is a finite group, while $\Pi : \Omega \to \Omega/G$ is a finite covering, so that the action of G on M is of bounded geometry type if and only if the Ricci curvature of (M, g) is bounded from below. As another remark, note that an interesting property of Theorem 9.6 is that it allows the study of product manifolds. This can be seen as a kind of atomic decomposition. We illustrate this fact in the following proposition. For the sake of simplicity, the assumptions are not as general as they could be.

PROPOSITION 9.1 *Let (M_i, g_i), $i = 1, \dots, m$, be m complete Riemannian manifolds of dimensions n_i, and for any $i \in \{1, \dots, m\}$, let G_i be a compact subgroup of $\operatorname{Isom}_{g_i}(M_i)$. Suppose that for any $i \in \{1, \dots, m\}$:*

(i) *the Ricci curvature of (M_i, g_i) is bounded from below,*
(ii) *there exists $c_i > 0$ such that for any $x \in M_i$, $\operatorname{Vol}_{g_i}(B_x(1)) \geq c_i$,*
(iii) *the action of G_i on M_i is of bounded geometry type,*
(iv) *there exists $\alpha_i \geq 1$ such that for any principal orbit $O_{G_i}^x$, $x \in M_i$, and any $y, y' \in T_{\mathrm{pr}}(O_{G_i}^x)$, $v(O_{G_i}^y) \leq \alpha_i v(O_{G_i}^{y'})$, and*
(v) *there exist $r_i > 0$ and a compact subset K_i of M_i such that for any principal orbit $O_{G_i}^x$, $x \in M_i \backslash K_i$, $R_{\mathrm{pr}}(O_{G_i}^x) \geq r_i$.*

Consider the two following assumptions:

H_1. *For any $i \in \{1, \dots, m\}$, there exist $C_i > 0$ and a compact subset K_i of M_i such that for any $x \in M_i \backslash K_i$, $O_{G_i}^x$ is principal and $\operatorname{Vol}_{g_i}(T_{\mathrm{pr}}(O_{G_i}^x)) \geq C_i$.*

H_2. *For any $i \in \{1, \dots, m\}$ and any $\varepsilon > 0$, there exists a compact subset K_ε^i of M_i such that for any $x \in M_i \backslash K_\varepsilon^i$, $O_{G_i}^x$ is principal and $\operatorname{Vol}_{g_i}(T_{\mathrm{pr}}(O_{G_i}^x)) \geq \frac{1}{\varepsilon}$.*

Let $M = M_1 \times \cdots \times M_m$, $g = g_1 + \cdots + g_m$, and G be the compact subgroup of $\operatorname{Isom}_g(M)$ defined by $G = G_1 \times \cdots \times G_m$. For any $i \in \{1, \dots, m\}$, let also $k_{\min}^i$ be the minimum orbit dimension of G_i, and $k_{\max}^i$ be the maximum orbit dimension of G_i. Set

$$k = \min\left\{\sum_{i=1}^m k_{\min}^i, k_{\max}^1, \dots, k_{\max}^m\right\}$$

and for $q \geq 1$, let $p^* = p^*(n, k, q)$ be as above, where $n = \sum_{i=1}^m n_i$. If H_1 holds, then for any $q \geq 1$ and any real number $p \in [q, p^*]$, $H_{1,G}^q(M) \subset L^p(M)$, and the embedding is continuous. If H_2 holds, then for any $q \geq 1$ and any $p \in (q, p^*)$, the embedding of $H_{1,G}^q(M)$ in $L^p(M)$ is compact.

PROOF: Let $\tilde{G}_i$, $i = 1, \dots, m$, be the compact subgroups of G defined by

$$\tilde{G}_1 = G_1 \times \{Id_2\} \times \cdots \times \{Id_m\}$$

$$\tilde{G}_i = \{Id_1\} \times \cdots \times \{Id_{i-1}\} \times G_i \times \{Id_{i+1}\} \times \cdots \times \{Id_m\}, \quad 2 \leq i \leq m-1$$

$$\tilde{G}_m = \{Id_1\} \times \cdots \times \{Id_{m-1}\} \times G_m$$

where Id_i denotes the identity of M_i. One easily checks that by (i) and (iii), the action of the $\tilde{G}_i$'s on M is of bounded geometry type, and that by (iv), the action of the $\tilde{G}_i$'s on M is uniform (at infinity). The result then comes from the fact that by (ii) and (v), H_1 (respectively, H_2) of Theorem 9.6 holds for M and the $\tilde{G}_i$'s, if H_1 (respectively, H_2) of Proposition 9.1 holds. This ends the proof of the proposition. □

As a concrete and easy example of application of Proposition 9.1, hence of Theorem 9.6, one recovers a result of Lions **[150]** dealing with functions on $\mathbb{R}^n$ that are cylindrically symmetric. More precisely, one has the following:

COROLLARY 9.1 *Let $m \geq 2$ and $n_1, \dots, n_m$ be integers such that $n_i \geq 2$ for all i. Let also G be the subgroup of* $\mathrm{Isom}_\delta(\mathbb{R}^n)$ *defined by*

$$G = O(n_1) \times \cdots \times O(n_m)$$

where $n = \sum_{i=1}^m n_i$, δ is the Euclidean metric, and $\mathbb{R}^n = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$. For $q \geq 1$, set $p^\star = nq/(n-q)$ if $q < n$, and $p^\star = +\infty$ if $q \geq n$. Then for any $q \geq 1$ and any $p \in (q, p^\star)$, the embedding of $H^q_{1,G}(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$ is compact.

CHAPTER 10

Manifolds with Boundary

For dimension reasons, we decided in these notes to deal only with closed manifolds, that is, with manifolds without boundaries. For completeness, we briefly comment here on some results in the case of manifolds with boundaries. We refer the reader to the references appearing below for more details on the subject.

Let (M, g) be a smooth, compact n-dimensional Riemannian manifold with boundary. For $q \in [1, n)$, and $u \in C^\infty(M)$, we let

$$\|u\|_{H_1^q} = \left(\int_M |\nabla u|^q \, dv(g)\right)^{1/q} + \left(\int_M |u|^q \, dv(g)\right)^{1/q}$$

and we set

$$H_1^q(M) = \text{completion of } C^\infty(M) \text{ with respect to } \|\cdot\|_{H_1^q}$$
$$H_{0,1}^q(M) = \text{completion of } \mathcal{D}(M) \text{ with respect to } \|\cdot\|_{H_1^q}$$

As one can easily check, a simple adaptation of what we said in Chapter 2 leads to the fact that the Sobolev and Rellich-Kondrakov theorems do hold for these spaces. For more detail on such an assertion, we refer the reader to Aubin [**12**]. In particular, one has the following:

THEOREM 10.1 *Let (M, g) be a smooth, compact, n-dimensional Riemannian manifold with boundary. For $q \in [1, n)$ real, set $p = nq/(n-q)$. Then, for any $q \in [1, n)$, and any $k \in [1, p]$, $H_1^q(M) \subset L^k(M)$ and $H_{0,1}^q(M) \subset L^k(M)$, with the additional property that these embeddings are compact if $k \in [1, p)$.*

Regarding the notion of best constants for such spaces, here again a simple adaptation of what we said in Chapter 4 shows that the best first constant for the embedding of $H_{0,1}^q$ in $L^p(M)$ is $K(n, q)$, where $K(n, q)$ is as in Theorem 4.4. With more subtle arguments, as shown by Cherrier [**51**], one gets that the best first constant for the embedding of $H_1^q(M)$ in $L^p(M)$ is $2^{1/n}K(n, q)$.

THEOREM 10.2 *Let (M, g) be a smooth, compact, n-dimensional Riemannian manifold with boundary. For $q \in [1, n)$ real, set $p = nq/(n-q)$. For any $q \in [1, n)$, and any $\varepsilon > 0$, there exist $B_1, B_2 \in \mathbb{R}$ such that for any $u \in H_{0,1}^q(M)$,*

$$\left(\int_M |u|^p \, dv(g)\right)^{q/p} \le \left(K(n, q)^q + \varepsilon\right) \int_M |\nabla u|^q \, dv(g) + B_1 \int_M |u|^q \, dv(g)$$

with the additional property that $K(n,q)$ is the best constant in such an inequality, and such that for any $u \in H_1^q(M)$,

$$\left(\int_M |u|^p \, dv(g)\right)^{q/p} \le \left(2^{\frac{q}{n}} K(n,q)^q + \varepsilon\right) \int_M |\nabla u|^q \, dv(g) + B_2 \int_M |u|^q \, dv(g)$$

with the additional property that $2^{q/n} K(n,q)^q$ is the best constant in such an inequality.

Following what was done in Chapter 4, one can now ask if the above inequalities do hold with $\varepsilon = 0$. By the work of Druet **[74]**, the answer to such a question is no for the embedding of $H_{0,1}^q(M)$ in $L^p(M)$ if $q > 2$, $q^2 < n$, and the scalar curvature of g is positive somewhere. Conversely, the following was proved in Hebey-Vaugon **[119]**:

THEOREM 10.3 *Let (M,g) be a smooth, compact, n-dimensional Riemannian manifold with boundary, $n \ge 3$, and let $p = 2n/(n-2)$. There exists $B \in \mathbb{R}$ such that*

$$\left(\int_M |u|^p \, dv(g)\right)^{2/p} \le K(n,2)^2 \int_M |\nabla u|^2 \, dv(g) + B \int_M u^2 \, dv(g)$$

for any $u \in H_{0,1}^2(M)$.

Still when dealing with best-constants problems, one can ask for sharp trace inequalities. Set

$$\tilde{K}(n) = \frac{2}{(n-2)\omega_{n-1}^{1/(n-1)}}$$

As shown by Beckner **[21]** and Escobar **[76]**,

$$\frac{1}{\tilde{K}(n)} = \inf \frac{\int_{R_+^n} |\nabla u|^2 \, dx}{\left(\int_{\partial R_+^n} |u|^q \, d\sigma\right)^{2/q}}$$

where $q = 2(n-1)/(n-2)$, and the infimum is taken over functions u such that $\nabla u \in L^2(\mathbb{R}_+^n)$ and $u \in L^q(\partial \mathbb{R}_+^n) \backslash \{0\}$. Moreover, this infimum is achieved (Lions **[151]**). The extremum functions there were found independently in **[21]** and **[76]**. The following theorem is due to Li and Zhu **[145]**:

THEOREM 10.4 *Let (M,g) be a smooth, compact, n-dimensional Riemannian manifold with boundary, $n \ge 3$, and let $q = 2(n-1)/(n-2)$. There exists $B \in \mathbb{R}$ such that for any $u \in H_1^2(M)$,*

$$\left(\int_{\partial M} |u|^q \, dv(g)\right)^{2/q} \le \tilde{K}(n) \int_M |\nabla u|^2 \, dv(g) + B \int_{\partial M} u^2 \, dv(g)$$

with the additional property that $\tilde{K}(n)$ is the best constant in such an inequality.

Somewhat more closely related to the inequalities we discussed in these notes, Li and Zhu **[146]** also got the following:

THEOREM 10.5 *Let (M, g) be a smooth, compact, n-dimensional Riemannian manifold with boundary, $n \geq 3$, and let $p = 2n/(n-2)$. There exists $B_1, B_2 \in \mathbb{R}$ such that for any $u \in H_1^2(M)$,*

$$\left(\int_M |u|^p \, dv(g)\right)^{2/p} \leq 2^{\frac{2}{n}} K(n,2)^2 \int_M |\nabla u|^2 \, dv(g) + B_1 \int_M u^2 \, dv(g) + B_2 \int_{\partial M} u^2 \, d\sigma$$

with the additional property that $2^{2/n} K(n,2)^2$ is the best constant in such an inequality. Moreover, one can take $B_1 = 0$ in the above inequality if and only if

$$\left(\int_M |u|^p \, dv(g)\right)^{2/p} < 2^{\frac{2}{n}} K(n,2)^2 \int_M |\nabla u|^2 \, dv(g)$$

for any $u \in H_{0,1}^2(M) \backslash \{0\}$.

As an easy consequence of this result of Li and Zhu, one gets that for Ω a smooth, bounded domain in $\mathbb{R}^n$, there exists $B \in \mathbb{R}$ such that for any $u \in H_1^2(\Omega)$,

$$\left(\int_\Omega |u|^p \, dx\right)^{2/p} \leq 2^{\frac{2}{n}} K(n,2)^2 \int_\Omega |\nabla u|^2 \, dx + B \int_{\partial\Omega} u^2 \, d\sigma$$

Extensions of such an inequality are studied in Zhu **[202]**. Again when dealing with open subsets of Euclidean space, sharp Sobolev inequalities with remainder terms are studied in Brezis and Lieb **[33]**, while sharp Sobolev inequalities for functions vanishing on some part of the boundary are studied in Lions, Pacella, and Tricarico **[152]**.

When dealing with best-constants problems, one may also discuss the value of the best second constant. The arguments when studying such a question are similar to the ones presented in Section 4.1 of Chapter 4. The value of the best second constant depends here on whether we are concerned with the embedding $H_{0,1}^q(M) \subset L^p(M)$ or the embedding $H_1^q(M) \subset L^p(M)$, $p = nq/(n-q)$. Given (M, g) a smooth, compact, n-dimensional Riemannian manifold with boundary, and $q \in [1, n)$ real, one gets with similar arguments to the ones used in Section 2.8 that there exists $A \in \mathbb{R}$ such that for any $u \in H_{0,1}^q(M)$,

$$\left(\int_M |u|^p \, dv(g)\right)^{q/p} \leq A \int_M |\nabla u|^q \, dv(g)$$

The best second constant for the embedding of $H_{0,1}^q(M)$ in $L^p(M)$ is then nonpositive. It may be zero or negative. Just consider smooth, bounded, open subsets of either the Euclidean space, or the hyperbolic space, and look at what we said in Section 7.4. On the contrary, as one can easily check by considering some nonzero constant function, the value of the best second constant for the embedding of $H_1^q(M)$ in $L^p(M)$ has to be greater than or equal to $\mathrm{Vol}_{(M,g)}^{-1/n}$, where $\mathrm{Vol}_{(M,g)}$ stands for the volume of (M, g).

For results on the influence of symmetries on manifolds with boundary in the spirit of the work in Chapter 9, we refer the reader to Hebey-Vaugon **[120]**.

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Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities

EMMANUEL HEBEY

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