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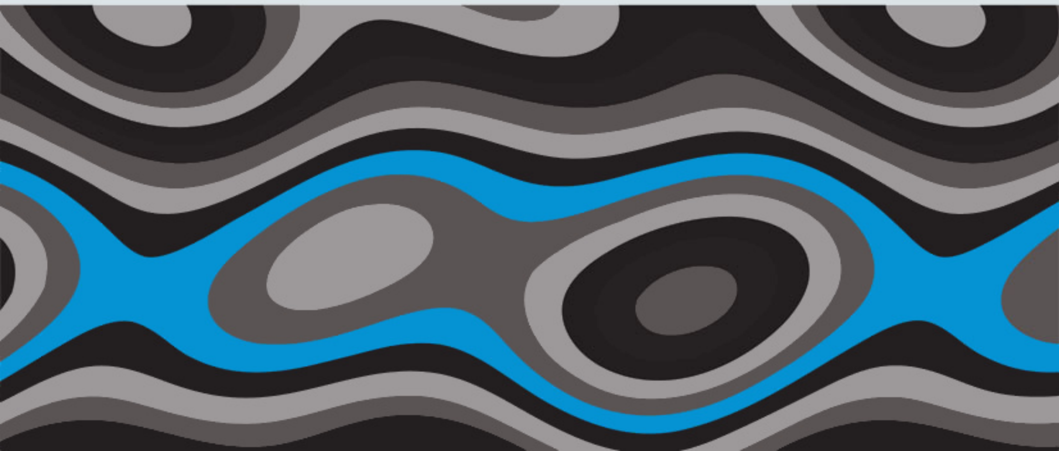
Fluid Mechanics

Eduard Feireisl
Antonín Novotný

Singular Limits in Thermodynamics of Viscous Fluids



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Advances in Mathematical Fluid Mechanics

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Eduard Feireisl
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Singular Limits in Thermodynamics of Viscous Fluids

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Preface

Another advantage of a mathematical statement is
that it is so definite that it might be definitely wrong...
Some verbal statements have not this merit.

L.F. Richardson (1881–1953)

Many interesting problems in mathematical fluid mechanics involve the behavior of solutions to systems of nonlinear partial differential equations as certain parameters vanish or become infinite. Frequently the solutions converge, provided the limit exists, to a solution of a limit problem represented by a qualitatively different system of differential equations. The simplest physically relevant example of this phenomenon is the behavior of a compressible fluid flow in the situation when the Mach number tends to zero, where the limit solution formally satisfies a system describing the motion of an incompressible fluid. Other interesting phenomena occur in the equations of magnetohydrodynamics, when either the Mach or the Alfven number, or both, tend to zero. As a matter of fact, most, if not all mathematical models used in fluid mechanics rely on formal asymptotic analysis of more complex systems. The concept of incompressible fluid itself should be viewed as a convenient idealization of a medium in which the speed of sound dominates the characteristic velocity.

Singular limits are closely related to scale analysis of differential equations. Scale analysis is an efficient tool used both theoretically and in numerical experiments to reduce the undesirable and mostly unnecessary complexity of investigated physical systems. The simplified asymptotic limit equations may provide a deeper insight into the dynamics of the original, mathematically more complicated, system. They reduce considerably the costs of computations, or offer a suitable alternative in the case when these fail completely or become unacceptably expensive when applied to the original problem. However, we should always keep in mind that these simplified equations are associated with singular asymptotic limits of the full governing equations, this fact having an important impact on the behavior of their solutions, for which degeneracies as well as other significant changes of the character of the governing equations become imminent.

Despite the vast amount of existing literature, most of the available studies devoted to scale analysis are based on *formal* asymptotic expansion of (hypothetical) solutions with respect to one or several singular parameters. Although this might seem wasted or at least misguided effort from the purely theoretical point of view, such an approach proved to be exceptionally efficient in real world applications. On the other hand, progress at the purely theoretical level has been hampered for many years by almost complete absence of a rigorous existence theory that would be applicable to the complex nonlinear systems arising in mathematical fluid dynamics. Although these problems are essentially well posed on short time intervals or for small, meaning close to equilibrium states, initial data, a universal existence theory is still out of reach of modern mathematical methods. Still, understanding the theoretical aspects of singular limits in systems of partial differential equations in general, and in problems of mathematical fluid mechanics in particular, is of great interest because of its immediate impact on the development of the theory. Last but not least, a rigorous identification of the asymptotic problem provides a justification of the mathematical model employed.

The concept of *weak solution* based on direct integral formulation of the underlying physical principles provides the only available framework for studying the behavior of solutions to problems in fluid mechanics in the large. The class of weak solutions is reasonably wide in order to accommodate all possible singularities that may develop in a finite time because of the highly nonlinear structures involved. Although optimality of this class of solutions may be questionable and still not completely accepted by the whole community, we firmly believe that the mathematical theory elaborated in this monograph will help to promote this approach and to contribute to its further development.

The book is designed as an introduction to problems of singular limits and scale analysis of systems of differential equations describing the motion of compressible, viscous, and heat conducting fluids. Accordingly, the primitive problem is always represented by the NAVIER-STOKES-FOURIER SYSTEM of equations governing the time evolution of three basic state variables: the density, the velocity, and the absolute temperature associated to the fluid. In addition we assume the fluid is linearly viscous, meaning the viscous stress is determined through Newton's rheological law, while the internal energy flux obeys Fourier's law of heat conduction. The state equation is close to that of a perfect gas, at least for moderate values of density and temperature. General ideas as well as the variational formulation of the problem based on a system of integral identities rather than partial differential equations are introduced and properly motivated in Chapter 1.

Chapters 2, 3 contain a complete existence theory for the full Navier-Stokes-Fourier system without any essential restriction imposed on the size of the data as well as the length of the existence interval. The ideas developed in this part are of fundamental importance for the forthcoming analysis of singular limits.

Chapter 4 resumes the basic concepts and methods to be used in the study of singular limits. The underlying principle used amply in all future considerations is a decomposition of each quantity as a sum of its *essential* part, relevant in the limit system, and a *residual* part, where the latter admits uniform bounds induced by the available *a priori* estimates and vanishes in the asymptotic limit. This chapter also reveals an intimate relation between certain results obtained in this book and the so-called Lighthill's Acoustic Analogy used in numerous engineering applications.

Chapter 5 gives a comprehensive treatment of the low Mach number limit for the Navier-Stokes-Fourier system in the regime of low stratification, that means, the Froude number is strongly dominated by the Mach number. As a limit system, we recover the well-known Oberbeck-Boussinesq Approximation widely used in many applications. Remarkably, we establish uniform estimates of the set of weak solutions of the primitive system derived by help of the so-called dissipation inequality. This can be viewed as a direct consequence of the *Second Law of Thermodynamics* expressed in terms of the entropy balance equation, and the hypothesis of *thermodynamic stability* imposed on the constitutive relations. Convergence toward the limit system in the field equations is then obtained by means of the nowadays well-established technique based on compensated compactness. Another non-standard aspect of the analysis is a detailed description of propagation of the acoustic waves that arise as an inevitable consequence of *ill-prepared* initial data. In contrast with all previous studies, the underlying acoustic equation is driven by an external force whose distribution is described by a non-negative Borel measure. This is one of the intrinsic features encountered in the framework of weak solutions, where a piece of information concerning energy transfer through possible singularities is lost.

Chapter 6 is primarily concerned with the strongly stratified fluids arising in astrophysics and meteorology. The central issue discussed here is *anisotropy* of the propagation of sound waves resulting from the strong stratification imposed by the gravitational field. Accordingly, the asymptotic analysis of the acoustic waves must be considerably modified in order to take into account the dispersion effects. As a model example, we identify the asymptotic system proposed by several authors as a suitable Model of Stellar Radiative Zones.

Most wave motions, in particular the sound waves propagation examined in this book, are strongly influenced by the effect of the boundary of the underlying physical space. If viscosity is present, a strong attenuation of sound waves is expected, at least in the case of the so-called no-slip boundary conditions imposed on the velocity field. These phenomena are studied in detail in Chapter 7. In particular, it is shown that under certain geometrical conditions imposed on the physical boundary, the convergence of the velocity field in the low Mach number regime is strong, meaning free of time oscillations. Although our approach parallels other recent studies based on boundary layer analysis, we tried to minimize the number of necessary steps in the asymptotic expansion to make it relatively simple, concise, and applicable without any extra effort to a larger class of problems.

Another interesting aspect of the problem arises when singular limits are considered on large or possibly even unbounded spatial domains, where "large" is to be quantified with regard to the size of other singular dimensionless parameters. Such a situation is examined in Chapter 8. It is shown that the acoustic waves redistribute rapidly the energy and, leaving any fixed bounded subset of the physical space during a short time as the speed of sound becomes infinite, render the velocity field strongly (pointwise) convergent. Although the result is formally similar to those achieved in Chapter 7, the methods are rather different based on dispersive estimates of Strichartz type and finite speed of propagation for the acoustic equation.

Chapter 9 interprets the results on singular limits in terms of the acoustic analogies used frequently in numerical analysis. We identify the situations where these methods are likely to provide reliable results and point out their limitations. Our arguments here rely on the uniform estimates obtained in Chapter 5.

The book is appended by two supplementary parts. In order to follow the subsequent discussion, the reader is recommended first to turn to the preliminary chapter, where the basic notation, function spaces, and other useful concepts, together with the fundamental mathematical theorems used in the book, are reviewed. The material is presented in a concise form and provided with relevant references when necessary. Appendix (Chapter 10) provides for reader's convenience some background material, with selected proofs, of more advanced but mostly standard results widely applicable in the mathematical theory of viscous compressible fluids in general, and, in the argumentation throughout this monograph, in particular. Besides providing a list of the relevant literature, Appendix offers a comprehensive and self-contained introduction to various specific recent mathematical tools designed to handle the problems arising in the mathematical theory of compressible fluids. As far as these results are concerned, the proofs are performed in full detail.

Since the beginning of this project we have greatly profited from a number of seminal works and research studies. Although the most important references are included directly in the text of Chapters 1–10, Chapter 11 is designed to take the reader through the available literature on the topics addressed elsewhere in the book. In particular, a comprehensive list of reference material is given, with a clear indication of the corresponding part discussed in the book. The reader is encouraged to consult these resources, together with the references cited therein, for a more complex picture of the problem as well as a more comprehensive exposition of some special topics.

The authors sincerely appreciate all who have offered comments and criticisms concerning the content of this book, in particular, many thanks go to Giovanni P. Galdi for many fruitful discussions, and to Jan Březina, Petr Kaplický, Josef Málek, William Layton, Šárka Nečasová, Hana Petzeltová, Milan Pokorný, Dalibor Pražák, Jan Stebel, and Ivan Straškraba for careful reading of several chapters of the manuscript.

Eduard Feireisl and Antonín Novotný
Praha and Toulon, September 2008

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Notation, Definitions, and Function Spaces

0.1 Notation

Unless otherwise indicated, the symbols in the book are used as follows:

(i) The symbols const, c, c_i denote generic constants, usually found in inequalities. They do not have the same value when used in different parts of the text.

(ii) $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{C}$ are the sets of integers, positive integers, and complex numbers, respectively. The symbol $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^N$ is the N-dimensional Euclidean space.

(iii) The symbol $\Omega \subset \mathbb{R}^N$ stands for a *domain* – an open connected subset of $\mathbb{R}^N$. The closure of a set $Q \subset \mathbb{R}^N$ is denoted by $\overline{Q}$, its boundary is ∂Q. By the symbol 1_Q we denote the characteristic function of the set Q. The outer normal vector to ∂Q, if it exists, is denoted by $\mathbf{n}$.

The symbol $\mathcal{T}^N$ denotes the *flat torus*,

$$\mathcal{T}^N = \left([-\pi, \pi]|_{\{-\pi;\pi\}} \right)^N = (\mathbb{R}|2\pi\mathbb{Z})^N$$

considered as a factor space of the Euclidean space $\mathbb{R}^N$, where $x \approx y$ whenever all coordinates of x differ from those of y by an integer multiple of 2π. Functions defined on $\mathcal{T}^N$ can be viewed as 2π-periodic in $\mathbb{R}^N$.

The symbol $B(a;r)$ denotes an (open) ball in $\mathbb{R}^N$ of center $a \in R^N$ and radius $r > 0$.

(iv) Vectors and functions ranging in a Euclidean space are represented by symbols beginning by a boldface minuscule, for example $\mathbf{u}$, $\mathbf{v}$. Matrices (tensors) and matrix-valued functions are represented by special Roman characters such as $\mathbb{S}$, $\mathbb{T}$, in particular, the identity matrix is denoted by $\mathbb{I} = \{\delta_{i,j}\}_{i,j=1}^N$. The symbol $\mathbb{I}$ is also used to denote the identity operator in a general setting.

The transpose of a square matrix $\mathbb{A} = \{a_{i,j}\}_{i,j=1}^N$ is $\mathbb{A}^T = \{a_{j,i}\}_{i,j=1}^N$. The trace of a square matrix $\mathbb{A} = \{a_{i,j}\}_{i,j=1}^N$ is $\mathrm{trace}[\mathbb{A}] = \sum_{i=1}^N a_{i,i}$.

(v) The scalar product of vectors $\mathbf{a} = [a_1, \dots, a_N]$, $\mathbf{b} = [b_1, \dots, b_N]$ is denoted by

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^N a_i b_i,$$

the scalar product of tensors $\mathbb{A} = \{A_{i,j}\}_{i,j=1}^N$, $\mathbb{B} = \{B_{i,j}\}_{i,j=1}^N$ reads

$$\mathbb{A} : \mathbb{B} = \sum_{i,j=1}^N A_{i,j} B_{j,i}.$$

The symbol $\mathbf{a} \otimes \mathbf{b}$ denotes the tensor product of vectors $\mathbf{a}$, $\mathbf{b}$, specifically,

$$\mathbf{a} \otimes \mathbf{b} = \{\mathbf{a} \otimes \mathbf{b}\}_{i,j} = a_i b_j.$$

The vector product $\mathbf{a} \times \mathbf{b}$ is the antisymmetric part of $\mathbf{a} \otimes \mathbf{b}$. If $N = 3$, the vector product of vectors $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$ is identified with a vector

$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1).$$

The product of a matrix $\mathbb{A}$ with a vector $\mathbf{b}$ is a vector $\mathbb{A}\mathbf{b}$ whose components are

$$[\mathbb{A}\mathbf{b}]_i = \sum_{j=1}^N A_{i,j} b_j \text{ for } i = 1, \dots, N,$$

while the product of a matrix $\mathbb{A} = \{A_{i,j}\}_{i,j=1}^{N,M}$ and a matrix $\mathbb{B} = \{B_{i,j}\}_{i,j=1}^{M,S}$ is a matrix $\mathbb{A}\mathbb{B}$ with components

$$[\mathbb{A}\mathbb{B}]_{i,j} = \sum_{k=1}^M A_{i,k} B_{k,j}.$$

(vi) The Euclidean norm of a vector $\mathbf{a} \in \mathbb{R}^N$ is denoted by

$$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{\sum_{i=1}^N a_i^2}.$$

The distance of a vector $\mathbf{a}$ to a set $K \subset \mathbb{R}^N$ is denoted as

$$\mathrm{dist}[\mathbf{a}, K] = \inf\{|\mathbf{a} - \mathbf{k}| \mid \mathbf{k} \in K\},$$

and the diameter of K is

$$\mathrm{diam}[K] = \sup_{(x,y) \in K^2} |x - y|.$$

The closure of K is denoted by $\mathrm{closure}[K]$, the Lebesgue measure of a set Q is $|Q|$.

0.2 Differential operators

The symbol

$$\partial_{y_i} g(y) \equiv \frac{\partial g}{\partial y_i}(y), \ i = 1, \dots, N,$$

denotes the partial derivative of a function $g = g(y)$, $y = [y_1, \dots, y_N]$, with respect to the (real) variable y_i calculated at a point $y \in \mathbb{R}^N$. The same notation is used for distributional derivatives introduced below. Typically, we consider functions $g = g(t, x)$ of the *time variable* $t \in (0, T)$ and the *spatial coordinate* $x = [x_1, x_2, x_3] \in \Omega \subset \mathbb{R}^3$. We use italics rather than boldface minuscules to denote the independent variables, although they may be vectors in many cases.

(i) The *gradient* of a scalar function $g = g(y)$ is a vector

$$\nabla g = \nabla_y g = [\partial_{y_1} g(y), \dots, \partial_{y_N} g(y)];$$

$\nabla^T g$ denotes the transposed vector to ∇g.

The *gradient* of a scalar function $g = g(t, x)$ with respect to the spatial variable x is a vector

$$\nabla_x g(t, x) = [\partial_{x_1} g(t, x), \partial_{x_2} g(t, x), \partial_{x_3} g(t, x)].$$

The *gradient* of a vector function $\mathbf{v} = [v_1(y), \dots, v_N(y)]$ is the matrix

$$\nabla \mathbf{v} = \nabla_y \mathbf{v} = \{\partial_{y_j} v_i\}_{i,j=1}^N;$$

$\nabla^T \mathbf{v}$ denotes the transposed matrix to $\nabla \mathbf{v}$. Similarly, the *gradient* of a vector function $\mathbf{v} = [v_1(t, x), v_2(t, x), v_3(t, x)]$ with respect to the space variables x is the matrix

$$\nabla_x \mathbf{v}(t, x) = \{\partial_{x_j} v_i(t, x)\}_{i,j=1}^3.$$

(ii) The *divergence* of a vector function $\mathbf{v} = [v_1(y), \dots, v_N(y)]$ is a scalar

$$\mathrm{div}\mathbf{v} = \mathrm{div}_y \mathbf{v} = \sum_{i=1}^N \partial_{y_i} v_i.$$

The *divergence* of a vector function depending on spatial and temporal variables $\mathbf{v} = [v_1(t, x), v_2(t, x), v_3(t, x)]$ with respect to the space variable x is a scalar

$$\mathrm{div}_x \mathbf{v}(t, x) = \sum_{i=1}^3 \partial_{x_i} v_i(t, x).$$

The *divergence* of a tensor (matrix-valued) function $\mathbb{B} = \{B_{i,j}(t, x)\}_{i,j=1}^3$ with respect to the space variable x is a vector

$$[\mathrm{div}\mathbb{B}]_i = [\mathrm{div}_x \mathbb{B}(t, x)]_i = \sum_{j=1}^3 \partial_{x_j} B_{i,j}(t, x), \ i = 1, \dots, 3.$$

(iii) The symbol $\Delta = \Delta_x$ denotes the *Laplace operator*,

$$\Delta_x = \mathrm{div}_x \nabla_x.$$

(iv) The *vorticity* (rotation) **curl** of a vectorial function $\mathbf{v} = [v_1(y), \dots, v_N(y)]$ is an antisymmetric matrix

$$\mathbf{curl}\ \mathbf{v} = \mathbf{curl}_y \mathbf{v} = \nabla \mathbf{v} - \nabla^T \mathbf{v} = \left\{ \partial_{y_j} v_i - \partial_{y_i} v_j \right\}_{i,j=1}^{N}.$$

The vorticity of a vectorial function $\mathbf{v} = [v_1(t,x), \dots, v_3(t,x)]$ is an antisymmetric matrix

$$\mathbf{curl}_x \mathbf{v} = \nabla_x \mathbf{v} - \nabla_x^T \mathbf{v} = \left\{ \partial_{x_j} v_i - \partial_{x_i} v_j \right\}_{i,j=1}^{3}.$$

The vorticity operator in $\mathbb{R}^3$ is sometimes interpreted as a vector $\mathbf{curl}\ \mathbf{v} = \nabla_x \times \mathbf{v}$.

(v) For a surface $\mathrm{S} \subset \mathbb{R}^3$, with an outer normal $\mathbf{n}$, we introduce the *normal gradient* of a scalar function $g : G \to \mathbb{R}^3$ defined on an open set $G \subset \mathbb{R}^3$ containing S as

$$\partial_{\mathbf{n}} g = \nabla_x g \cdot \mathbf{n},$$

and the tangential gradient as

$$[\partial_S]_i g = \partial_{x_i} g - (\nabla_x g \cdot \mathbf{n}) n_i, \ \ i = 1, 2, 3.$$

The *Laplace-Beltrami operator* on S is defined as

$$\Delta_s g = \sum_{i=1}^{3} [\partial_S]_i [\partial_S]_i g$$

(see Gilbarg and Trudinger [96, Chapter 16]).

0.3 Function spaces

If not otherwise stated, all function spaces considered in this book are real. For a normed linear space X, we denote by $\| \cdot \|_X$ the *norm* on X. The duality pairing between an abstract vector space X and its dual X^* is denoted as $\langle \cdot ; \cdot \rangle_{X^*;X}$, or simply $\langle \cdot ; \cdot \rangle$ in case the underlying spaces are clearly identified in the context. In particular, if X is a Hilbert space, the symbol $\langle \cdot ; \cdot \rangle$ denotes the scalar product in X.

The symbol $\mathrm{span}\{M\}$, where M is a subset of a vector space X, denotes the space of all finite linear combinations of vectors contained in M.

(i) For $Q \subset \mathbb{R}^N$, the symbol $C(Q)$ denotes the set of continuous functions on Q. For a bounded set Q, the symbol $C(\overline{Q})$ denotes the Banach space of functions continuous on the closure $\overline{Q}$ endowed with norm

$$\|g\|_{C(\overline{Q})} = \sup_{y \in \overline{Q}} |g(y)|.$$

Similarly, $C(\overline{Q}; X)$ is the Banach space of vectorial functions continuous in $\overline{Q}$ and ranging in a Banach space X with norm

$$\|g\|_{C(\overline{Q})} = \sup_{y \in \overline{Q}} \|g(y)\|_X.$$

(ii) The symbol $C_{\text{weak}}(\overline{Q}; X)$ denotes the space of all vector-valued functions on $\overline{Q}$ ranging in a Banach space X continuous with respect to the weak topology. More specifically, $g \in C_{\text{weak}}(\overline{Q}; X)$ if the mapping $y \mapsto \|g(y)\|_X$ is bounded and

$$y \mapsto \langle f; g(y) \rangle_{X^*;X}$$

is continuous on $\overline{Q}$ for any linear form f belonging to the dual space X^*.

We say that $g_n \to g$ in $C_{\text{weak}}(\overline{Q}; X)$ if

$$\langle f; g_n \rangle_{X^*;X} \to \langle f; g \rangle_{X^*;X} \text{ in } C(\overline{Q}) \text{ for all } g \in X^*.$$

(iii) The symbol $C^k(\overline{Q})$, $Q \subset \mathbb{R}^N$, where k is a non-negative integer, denotes the space of functions on $\overline{Q}$ that are restrictions of k-times continuously differentiable functions on $\mathbb{R}^N$. $C^{k,\nu}(\overline{Q})$, $\nu \in (0,1)$ is the subspace of $C^k(\overline{Q})$ of functions having their k-th derivatives ν-Hölder continuous in $\overline{Q}$. $C^{k,1}(\overline{Q})$ is a subspace of $C^k(\overline{Q})$ of functions whose k-th derivatives are Lipschitz on $\overline{Q}$. For a bounded domain Q, the spaces $C^k(\overline{Q})$ and $C^{k,\nu}(\overline{Q})$, $\nu \in (0,1]$ are Banach spaces with norms

$$\|u\|_{C^k(\overline{Q})} = \max_{|\alpha| \le k} \sup_{x \in Q} |\partial^\alpha u(x)|$$

and

$$\|u\|_{C^{k,\nu}(\overline{Q})} = \|u\|_{C^k(\overline{Q})} + \max_{|\alpha| = k} \sup_{(x,y) \in Q^2,\, x \neq y} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^\nu},$$

where $\partial^\alpha u$ stands for the partial derivative $\partial_{x_1}^{\alpha_1} \ldots \partial_{x_N}^{\alpha_N} u$ of order $|\alpha| = \sum_{i=1}^N \alpha_i$. The spaces $C^{k,\nu}(\overline{Q}; \mathbb{R}^M)$ are defined in a similar way. Finally, we set $C^\infty = \cap_{k=0}^\infty C^k$.

(iv)

■ ARZELÀ-ASCOLI THEOREM:

Theorem 0.1. *Let $Q \subset \mathbb{R}^M$ be compact and X a compact topological metric space endowed with a metric d_X. Let $\{v_n\}_{n=1}^\infty$ be a sequence of functions in $C(Q; X)$*

that is equi-continuous, meaning, for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$d_X\left[v_n(y), v_n(z)\right] \leq \varepsilon \text{ provided } |y - z| < \delta \text{ independently of } n = 1, 2, \dots .$$

Then $\{v_n\}_{n=1}^{\infty}$ is precompact in $C(Q;X)$, that is, there exists a subsequence (not relabeled) and a function $v \in C(Q;X)$ such that

$$\sup_{y \in Q} d_X\left[v_n(y), v(y)\right] \to 0 \text{ as } n \to \infty.$$

See Kelley [119, Chapter 7, Theorem 17]. □

(v) For $Q \subset \mathbb{R}^N$ an open set and a function $g : Q \to \mathbb{R}$, the symbol $\text{supp}[g]$ denotes the *support* of g in Q, specifically,

$$\text{supp}[g] = \text{closure}\left[\{y \in Q \mid g(y) \neq 0\}\right].$$

(vi) The symbol $C_c^k(Q;\mathbb{R}^M)$, $k \in \{0, 1, \dots, \infty\}$ denotes the vector space of functions belonging to $C^k(Q;\mathbb{R}^M)$ and having compact support in Q. If $Q \subset \mathbb{R}^N$ is an open set, the symbol $\mathcal{D}(Q;\mathbb{R}^M)$ will be used alternatively for the space $C_c^\infty(Q;\mathbb{R}^M)$ endowed with the topology induced by the convergence:

$$\begin{aligned} \varphi_n \to \varphi \in \mathcal{D}(Q) \quad \text{if} \quad & \text{supp}[\varphi_n] \subset K, \ K \subset Q \text{ a compact set}, \\ & \varphi_n \to \varphi \text{ in } C^k(K) \text{ for any } k = 0, 1, \dots . \end{aligned} \tag{1}$$

We write $\mathcal{D}(Q)$ instead of $\mathcal{D}(Q;\mathbb{R})$.

The dual space $\mathcal{D}'(Q;\mathbb{R}^M)$ is the space of *distributions* on Ω with values in $\mathbb{R}^M$. Continuity of a linear form belonging to $\mathcal{D}'(Q)$ is understood with respect to the convergence introduced in (1).

(vii) A differential operator ∂^α of order $|\alpha|$ can be identified with a distribution

$$\langle \partial^\alpha v; \varphi \rangle_{\mathcal{D}'(Q);\mathcal{D}(Q)} = (-1)^{|\alpha|} \int_Q v \partial^\alpha \varphi \, \mathrm{d}y$$

for any locally integrable function v.

(viii) The *Lebesgue spaces* $L^p(Q;X)$ are spaces of (Bochner) measurable functions v ranging in a Banach space X such that the norm

$$\|v\|_{L^p(Q;X)}^p = \int_Q \|v\|_X^p \, \mathrm{d}y \text{ is finite, } 1 \leq p < \infty.$$

Similarly, $v \in L^\infty(Q;X)$ if v is (Bochner) measurable and

$$\|v\|_{L^\infty(Q;X)} = \operatorname*{ess\,sup}_{y \in Q} \|v(y)\|_X < \infty.$$

The symbol $L^p_{\text{loc}}(Q;X)$ denotes the vector space of locally L^p-integrable functions, meaning

$$v \in L^p_{\text{loc}}(Q;X) \text{ if } v \in L^p(K;X) \text{ for any compact set } K \text{ in } Q.$$

We write $L^p(Q)$ for $L^p(Q;\mathbb{R})$.

Let $f \in L^1_{\text{loc}}(Q)$ where Q is an open set. A *Lebesgue point* $a \in Q$ *of* f in Q is characterized by the property

$$\lim_{r\to 0+} \frac{1}{|B(a,r)|} \int_{B(a;r)} f(x)\mathrm{d}x = f(a). \tag{2}$$

For $f \in L^1(Q)$ the set of all Lebesgue points is of full measure, meaning its complement in Q is of zero Lebesgue measure. A similar statement holds for vector-valued functions $f \in L^1(Q;X)$, where X is a Banach space (see Brezis [34]).

If $f \in C(Q)$, then identity (2) holds for all points a in Q.

(ix)

■ LINEAR FUNCTIONALS ON $L^p(Q;X)$:

Theorem 0.2. *Let $Q \subset \mathbb{R}^N$ be a measurable set, X a Banach space that is reflexive and separable, $1 \le p < \infty$.*

Then any continuous linear form $\xi \in [L^p(Q;X)]^*$ admits a unique representation $w_\xi \in L^{p'}(Q;X^*)$,

$$\langle \xi; v\rangle_{L^{p'}(Q,X^*);L^p(Q;X)} = \int_Q \langle w_\xi(y); v(y)\rangle_{X^*;X}\, \mathrm{d}y \text{ for all } v \in L^p(Q;X),$$

where

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Moreover the norm on the dual space is given as

$$\|\xi\|_{[L^p(Q;X)]^*} = \|w_\xi\|_{L^{p'}(Q;X^*)}.$$

Accordingly, the spaces $L^p(Q;X)$ are reflexive for $1 < p < \infty$ as soon as X is reflexive and separable.

See Gajewski, Gröger, Zacharias [91, Chapter IV, Theorem 1.14, Remark 1.9]. □

Identifying ξ with w_ξ, we write

$$[L^p(Q;\mathbb{R}^N)]^* = L^{p'}(Q;\mathbb{R}^N),\ \|\xi\|_{[L^p(Q;\mathbb{R}^N)]^*} = \|\xi\|_{L^{p'}(Q;\mathbb{R}^N)},\ 1 \le p < \infty.$$

This formula is known as the *Riesz representation theorem*.

(x) If the Banach space X in Theorem 0.2 is merely separable, we have

$$[L^p(Q;X)]^* = L^{p'}_{\text{weak}-(*)}(Q;X^*) \text{ for } 1 \le p < \infty,$$

where

$$L^{p'}_{\text{weak}-(*)}(Q;X^*)$$
$$:= \Big\{ \xi : Q \to X^* \;\Big|\; y \in Q \mapsto \langle \xi(y); v\rangle_{X^*;X} \text{ measurable for any fixed } v \in X,$$
$$y \mapsto \|\xi(y)\|_{X^*} \in L^{p'}(Q) \Big\}$$

(see Edwards [69, Theorem 8.20.3], Pedregal [172, Chapter 6, Theorem 6.14]).

(xi) *Hölder's inequality* reads

$$\|uv\|_{L^r(Q)} \le \|u\|_{L^p(Q)} \|v\|_{L^q(Q)}, \; \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$$

for any $u \in L^p(Q)$, $v \in L^q(Q)$, $Q \subset \mathbb{R}^N$ (see Adams [1, Chapter 2]).

(xii) *Interpolation inequality* for L^p-spaces reads

$$\|v\|_{L^r(Q)} \le \|v\|^{\lambda}_{L^p(Q)} \|v\|^{(1-\lambda)}_{L^q(Q)}, \; \frac{1}{r} = \frac{\lambda}{p} + \frac{1-\lambda}{q}, \; p < r < q, \; \lambda \in (0,1)$$

for any $v \in L^p \cap L^q(Q)$, $Q \subset \mathbb{R}^N$ (see Adams [1, Chapter 2]).

(xiii)

■ GRONWALL'S LEMMA:

Lemma 0.1. *Let $a \in L^1(0,T)$, $a \ge 0$, $\beta \in L^1(0,T)$, $b_0 \in \mathbb{R}$, and*

$$b(\tau) = b_0 + \int_0^\tau \beta(t) \; \mathrm{d}t$$

be given. Let $r \in L^\infty(0,T)$ satisfy

$$r(\tau) \le b(\tau) + \int_0^\tau a(t) r(t) \; \mathrm{d}t \text{ for a.a. } \tau \in [0,T].$$

Then

$$r(\tau) \le b_0 \exp\left(\int_0^\tau a(t) \; \mathrm{d}t \right) + \int_0^\tau \beta(t) \exp\left(\int_t^\tau a(s) \; \mathrm{d}s \right) \; \mathrm{d}t$$

for a.a. $\tau \in [0,T]$.

See Carroll [41]. □

0.4 Sobolev spaces

(i) A domain $\Omega \subset \mathbb{R}^N$ is of class $\mathcal{C}$ if for each point $x \in \partial\Omega$, there exist $r > 0$ and a mapping $\gamma : \mathbb{R}^{N-1} \to \mathbb{R}$ belonging to a function class $\mathcal{C}$ such that – upon rotating and relabeling the coordinate axes if necessary – we have

$$\left.\begin{aligned} \Omega \cap B(x;r) &= \{y \mid \gamma(y') < y_N\} \cap B(x,r) \\ \partial\Omega \cap B(x;r) &= \{y \mid \gamma(y') = y_N\} \cap B(x,r) \end{aligned}\right\}, \quad \text{where } y' = (y_1, \dots, y_{N-1}).$$

In particular, Ω is called a *Lipschitz domain* if γ is Lipschitz.

If $A \subset \Gamma := \partial\Omega \cap B(x;r)$, γ is Lipschitz and $f : A \to R$, then one can define the surface integral

$$\int_A f \, \mathrm{d}S_x := \int_{\Phi_\gamma(A)} f(y', \gamma(y')) \sqrt{1 + \sum_{i=1}^{N-1} \left(\frac{\partial \gamma}{\partial y_i} \right)^2} \, \mathrm{d}y',$$

where $\Phi_\gamma : \mathbb{R}^N \mapsto \mathbb{R}^N$, $\Phi_\gamma(y', y_N) = (y', y_N - \gamma(y'))$, whenever the (Lebesgue) integral at the right-hand side exists. If $f = 1_A$ then $S_{N-1}(A) = \int_A \mathrm{d}S_x$ is the surface measure on $\partial\Omega$ of A that can be identified with the $(N-1)$-Hausdorff measure on $\partial\Omega$ of A (cf. Evans and Gariepy [75, Chapter 4.2]). In the general case of $A \subset \partial\Omega$, one can define $\int_A f \, \mathrm{d}S_x$ using a covering $\mathcal{B} = \{B(x_i; r)\}_{i=1}^M$, $x_i \in \partial\Omega$, $M \in \mathbb{N}$ of $\partial\Omega$ by balls of radii r and subordinated partition of unity $\mathcal{F} = \{\varphi_i\}_{i=1}^M$, and set

$$\int_A f \, \mathrm{d}S_x = \sum_{i=1}^M \int_{\Gamma_i} \varphi_i f \, \mathrm{d}S_x, \quad \Gamma_i = \partial\Omega \cap B(x_i; r),$$

see Nečas [162, Section I.2] or Kufner, Fučik, John [125, Section 6.3].

A Lipschitz domain Ω admits the outer normal vector $\mathbf{n}(x)$ for a.a. $x \in \partial\Omega$. Here *a.a.* refers to the surface measure on $\partial\Omega$.

The distance function$d(x) = \mathrm{dist}[x, \partial\Omega]$ is Lipschitz continuous. Moreover, d is differentiable a.a. in $\mathbb{R}^3$, and

$$\nabla_x d(x) = \frac{x - \xi(x)}{d(x)}$$

whenever d is differentiable at $x \in \mathbb{R}^3 \setminus \Omega$, where ξ denotes the nearest point to x on $\partial\Omega$ (see Ziemer [207, Chapter 1]). Moreover, if the boundary $\partial\Omega$ is of class C^k, then d is k-times continuously differentiable in a neighborhood of $\partial\Omega$ (see Foote [88]).

(ii) The *Sobolev spaces* $W^{k,p}(Q; \mathbb{R}^M)$, $1 \le p \le \infty$, k a positive integer, are the spaces of functions having all distributional derivatives up to order k in $L^p(Q; \mathbb{R}^M)$.

The norm in $W^{k,p}(Q;\mathbb{R}^M)$ is defined as

$$\|\mathbf{v}\|_{W^{k,p}(Q;\mathbb{R}^M)} = \left\{ \begin{array}{ll} \Big(\sum_{i=1}^{M} \sum_{|\alpha|\le k} \|\partial^\alpha v_i\|^p_{L^p(Q)} \Big)^{1/p} & \text{if } 1 \le p < \infty \\ \max_{1\le i\le M,\, |\alpha|\le k} \{\|\partial^\alpha v_i\|_{L^\infty(Q)}\} & \text{if } p = \infty \end{array} \right\},$$

where the symbol ∂^α stands for any partial derivative of order $|\alpha|$.

If Q is a bounded domain with boundary of class $C^{k-1,1}$, then there exists a continuous linear operator which maps $W^{k,p}(Q)$ to $W^{k,p}(\mathbb{R}^N)$; it is called an *extension operator*. If, in addition, $1 \le p < \infty$, then $W^{k,p}(Q)$ is separable and the space $C^k(\overline{Q})$ is its dense subspace.

The space $W^{1,\infty}(Q)$, where Q is a bounded domain, is isometrically isomorphic to the space $C^{0,1}(\overline{Q})$ of Lipschitz functions on $\overline{Q}$.

For basic properties of Sobolev functions, see Adams [1] or Ziemer [207].

(iii) The symbol $W_0^{k,p}(Q;\mathbb{R}^M)$ denotes the completion of $C_c^\infty(Q;\mathbb{R}^M)$ with respect to the norm $\|\cdot\|_{W^{k,p}(Q;\mathbb{R}^M)}$. In what follows, we identify $W^{0,p}(\Omega;R^N) = W_0^{0,p}(\Omega;\mathbb{R}^N)$ with $L^p(\Omega;\mathbb{R}^N)$.

We denote $\dot{L}^p(Q) = \{u \in L^p(Q) \mid \int_Q u \,\mathrm{d}y = 0\}$ and $\dot{W}^{1,p}(Q) = W^{1,p}(Q) \cap \dot{L}^p(Q)$. If $Q \subset \mathbb{R}^N$ is a bounded domain, then $\dot{L}^p(Q)$ and $\dot{W}^{1,p}(Q)$ can be viewed as closed subspaces of $L^p(Q)$ and $W^{1,p}(Q)$, respectively.

(iv) Let $Q \subset \mathbb{R}^N$ be an open set, $1 \le p \le \infty$ and $v \in W^{1,p}(Q)$. Then we have:

(a) $|v|^+$, $|v|^- \in W^{1,p}(Q)$ and

$$\partial_{x_j}|v|^+ = \left\{ \begin{array}{c} \partial_{x_k} v \text{ a.a. in } \{v > 0\} \\ 0 \text{ a.a. in } \{v \le 0\} \end{array} \right\},$$

$$\partial_{x_j}|v|^- = \left\{ \begin{array}{c} \partial_{x_k} v \text{ a.a. in } \{v < 0\} \\ 0 \text{ a.a. in } \{v \ge 0\} \end{array} \right\},$$

$j = 1, \ldots, N$, where $|v|^+ = \max\{u, 0\}$ denotes a positive part and $|v|^- = \min\{u, 0\}$ a negative part of v.

(b) If $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function and $f \circ v \in L^p(Q)$, then $f \circ v \in W^{1,p}(Q)$ and

$$\partial_{x_j}[f \circ v](x) = f'(v(x))\partial_{x_j} v(x) \text{ for a.a. } x \in Q.$$

For more details see Ziemer [207, Section 2.1].

(v) *Dual spaces to Sobolev spaces.*

■ DUAL SOBOLEV SPACES:

Theorem 0.3. *Let $\Omega \subset \mathbb{R}^N$ be a domain, and let $1 \leq p < \infty$. Then the dual space $[W_0^{k,p}(\Omega)]^*$ is a proper subspace of the space of distributions $\mathcal{D}'(\Omega)$. Moreover, any linear form $f \in [W_0^{k,p}(\Omega)]^*$ admits a representation*

$$\langle f; v \rangle_{[W_0^{k,p}(\Omega)]^*; W_0^{k,p}(\Omega)} = \sum_{|\alpha| \leq k} \int_\Omega (-1)^{|\alpha|} w_\alpha \, \partial^\alpha v \, \mathrm{d}x, \tag{3}$$

where $w_\alpha \in L^{p'}(\Omega)$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

The norm of f in the dual space is given as

$$\|f\|_{[W_0^{k,p}(\Omega)]^*} = \begin{cases} \inf \left\{ \left(\sum_{|\alpha| \leq k} \|w_\alpha\|^{p'}_{L^{p'}(\Omega)} \right)^{1/p'} \middle| \, w_\alpha \text{ satisfy } (3) \right\} \\ \text{for } 1 < p < \infty; \\ \inf \left\{ \max_{|\alpha| \leq k} \{ \|w_\alpha\|_{L^\infty(Q)} \} \middle| \, w_\alpha \text{ satisfy } (3) \right\} \\ \text{if } p = 1. \end{cases}$$

The infimum is attained in both cases.

See Adams [1, Theorem 3.8], Mazya [154, Section 1.1.14]. □

The dual space to the Sobolev space $W_0^{k,p}(\Omega)$ is denoted as $W^{-k,p'}(\Omega)$.

The dual to the Sobolev space $W^{k,p}(\Omega)$ admits formally the same representation formula as (3). However it cannot be identified as a space of distributions on Ω. A typical example is the linear form

$$\langle f; v \rangle = \int_\Omega \mathbf{w}_f \cdot \nabla_x v \, \mathrm{d}x, \text{ with } \mathrm{div}_x \mathbf{w}_f = 0$$

that vanishes on $\mathcal{D}(\Omega)$ but generates a non-zero linear form when applied to $v \in W^{1,p}(\Omega)$.

(vi)

■ RELLICH-KONDRACHOV EMBEDDING THEOREM:

Theorem 0.4. *Let $\Omega \subset R^N$ be a bounded Lipschitz domain.*

(i) *Then, if $kp < N$ and $p \geq 1$, the space $W^{k,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any*

$$1 \leq q \leq p^* = \frac{Np}{N - kp}.$$

Moreover, the embedding is compact if $k > 0$ and $q < p^*$.

(ii) *If* $kp = N$*, the space* $W^{k,p}(\Omega)$ *is compactly embedded in* $L^q(\Omega)$ *for any* $q \in [1, \infty)$.

(iii) *If* $kp > N$*, then* $W^{k,p}(\Omega)$ *is continuously embedded in* $C^{k-[N/p]-1,\nu}(\overline{\Omega})$*, where* $[\,]$ *denotes the integer part and*

$$\nu = \begin{cases} [\frac{N}{p}] + 1 - \frac{N}{p} \text{ if } \frac{N}{p} \notin \mathbb{Z}, \\ \text{arbitrary positive number in } (0,1) \text{ if } \frac{N}{p} \in \mathbb{Z}. \end{cases}$$

Moreover, the embedding is compact if $0 < \nu < [\frac{N}{p}] + 1 - \frac{N}{p}$.

See Ziemer [207, Theorem 2.5.1, Remark 2.5.2]. □

The symbol $\hookrightarrow$ will denote continuous embedding, $\hookrightarrow\hookrightarrow$ indicates compact embedding.

(vii) The following result may be regarded as a direct consequence of Theorem 0.4.

■ EMBEDDING THEOREM FOR DUAL SOBOLEV SPACES:

Theorem 0.5. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain. Let* $k > 0$ *and* $q < \infty$ *satisfy*

$$q > \frac{p^*}{p^* - 1}, \quad \text{where} \quad p^* = \frac{Np}{N - kp} \quad \text{if} \quad kp < N,$$

$$q > 1 \quad \text{for} \quad kp = N, \quad \text{or} \quad q \geq 1 \quad \text{if} \quad kp > N.$$

Then the space $L^q(\Omega)$ *is compactly embedded into the space* $W^{-k,p'}(\Omega)$, $1/p + 1/p' = 1$.

(viii) The *Sobolev-Slobodeckii spaces* $W^{k+\beta,p}(Q)$, $1 \leq p < \infty$, $0 < \beta < 1$, $k = 0, 1, \ldots$, where Q is a domain in $\mathbb{R}^L$, $L \in \mathbb{N}$, are Banach spaces of functions with finite norm

$$W^{k+\beta,p}(Q) = \left(\|v\|^p_{W^{k,p}(Q)} + \sum_{|\alpha|=k} \int_Q \int_Q \frac{|\partial^\alpha v(y) - \partial^\alpha v(z)|^p}{|y - z|^{L+\beta p}} \,\mathrm{d}y \,\mathrm{d}z \right)^{\frac{1}{p}},$$

see, e.g., Nečas[162, Section 2.3.8].

Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Referring to the notation introduced in **(i)**, we say that $f \in W^{k+\beta,p}(\partial\Omega)$ if $(\varphi f) \circ (\mathbb{I}', \gamma) \in W^{k+\beta,p}(\mathbb{R}^{N-1})$ for any $\Gamma = \partial\Omega \cap B$ with B belonging to the covering $\mathcal{B}$ of $\partial\Omega$ and φ the corresponding term in the partition of unity $\mathcal{F}$. The space $W^{k+\beta,p}(\partial\Omega)$ is a Banach space endowed with an equivalent norm $\|\cdot\|_{W^{k+\beta,p}(\partial\Omega)}$, where

$$\|v\|^p_{W^{k+\beta,p}(\partial\Omega)} = \sum_{i=1}^{M} \|(v\varphi_i) \circ (\mathbb{I}', \gamma)\|^p_{W^{k+\beta,p}(\mathbb{R}^{N-1})}.$$

In the above formulas $(\mathbb{I}', \gamma) : \mathbb{R}^{N-1} \to \mathbb{R}^N$ maps y' to $(y', \gamma(y'))$. For more details see, e.g., Nečas [162, Section 3.8].

In the situation when $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, the Sobolev-Slobodeckii spaces admit similar embeddings as classical Sobolev spaces. Namely, the embeddings

$$W^{k+\beta,p}(\Omega) \hookrightarrow L^q(\Omega) \text{ and } W^{k+\beta,p}(\Omega) \hookrightarrow C^s(\overline{\Omega})$$

are compact provided $(k+\beta)p < N$, $1 \le q < \frac{Np}{N-(k+\beta)p}$, and $s = 0, 1, \dots, k$, $(k - s + \beta)p > N$, respectively. The former embedding remains continuous (but not compact) at the border case $q = \frac{Np}{N-(k+\beta)p}$.

(ix)

■ TRACE THEOREM FOR SOBOLEV SPACES AND GREEN'S FORMULA:

Theorem 0.6. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain.*

Then there exists a linear operator γ_0 with the following properties:

$$[\gamma_0(v)](x) = v(x) \textit{ for } x \in \partial\Omega \textit{ provided } v \in C^\infty(\overline{\Omega}),$$

$$\|\gamma_0(v)\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \le c\|v\|_{W^{1,p}(\Omega)} \textit{ for all } v \in W^{1,p}(\Omega),$$

$$\ker[\gamma_0] = W_0^{1,p}(\Omega)$$

provided $1 < p < \infty$.

Conversely, there exists a continuous linear operator

$$\ell : W^{1-\frac{1}{p},p}(\partial\Omega) \to W^{1,p}(\Omega)$$

such that

$$\gamma_0(\ell(v)) = v \textit{ for all } v \in W^{1-\frac{1}{p},p}(\partial\Omega)$$

provided $1 < p < \infty$.

In addition, the following formula holds:

$$\int_\Omega \partial_{x_i} uv \, \mathrm{d}x + \int_\Omega u\partial_{x_i} v \, \mathrm{d}x = \int_{\partial\Omega} \gamma_0(u)\gamma_0(v)n_i \, \mathrm{dS}_x, \ i = 1, \dots, N,$$

for any $u \in W^{1,p}(\Omega)$, $v \in W^{1,p'}(\Omega)$, where $\mathbf{n}$ is the outer normal vector to the boundary $\partial\Omega$.

See Nečas [162, Theorems 5.5, 5.7]. □

The dual $[W^{1-\frac{1}{p},p}(\partial\Omega)]^*$ to the Sobolev-Slobodeckii space $W^{1-\frac{1}{p},p}(\partial\Omega)) = W^{\frac{1}{p'},p}(\partial\Omega)$ is denoted simply by $W^{-\frac{1}{p'},p'}(\partial\Omega)$.

(ix) If $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, then we have the interpolation-inequality

$$\|v\|_{W^{\alpha,r}(\Omega)} \leq c\|v\|^{\lambda}_{W^{\beta,p}(\Omega)}\|v\|^{1-\lambda}_{W^{\gamma,q}(\Omega)},\ 0 \leq \lambda \leq 1, \tag{4}$$

for

$$0 \leq \alpha, \beta, \gamma \leq 1,\ 1 < p, q, r < \infty,\ \alpha = \lambda\beta + (1-\lambda)\gamma,\ \frac{1}{r} = \frac{\lambda}{p} + \frac{1-\lambda}{q}$$

(see Sections 2.3.1, 2.4.1, 4.3.2 in Triebel [190]).

0.5 Fourier transform

Let $v = v(x)$ be a complex-valued function integrable on $\mathbb{R}^N$. The *Fourier transform* of v is a complex-valued function $\mathcal{F}_{x\to\xi}[v]$ of the variable $\xi \in \mathbb{R}^N$ defined as

$$\mathcal{F}_{x\to\xi}[v](\xi) = \left(\frac{1}{2\pi}\right)^{N/2} \int_{\mathbb{R}^N} v(x) \exp(-\mathrm{i}\xi \cdot x)\ \mathrm{d}x. \tag{5}$$

Therefore, the Fourier transform $\mathcal{F}_{x\to\xi}$ can be viewed as a continuous linear mapping defined on $L^1(\mathbb{R}^N)$ with values in $L^\infty(\mathbb{R}^N)$.

(i) For u, v complex-valued square integrable functions on $\mathbb{R}^N$, we have *Parseval's identity*:

$$\int_{\mathbb{R}^N} u(x)\overline{v}(x)\ \mathrm{d}x = \int_{\mathbb{R}^N} \mathcal{F}_{x\to\xi}[u](\xi)\overline{\mathcal{F}_{x\to\xi}[v]}(\xi)\ \mathrm{d}\xi,$$

where bar denotes the complex conjugate. Parseval's identity implies that $\mathcal{F}_{x\to\xi}$ can be extended as a continuous linear mapping defined on $L^2(\mathbb{R}^N)$ with values in $L^2(\mathbb{R}^N)$.

(ii) The symbol $\mathcal{S}(\mathbb{R}^N)$ denotes the space of smooth rapidly decreasing (complex-valued) functions, specifically, $\mathcal{S}(\mathbb{R}^N)$ consists of functions u such that

$$\sup_{|\alpha|\leq m} \sup_{x\in\mathbb{R}^N} (1+|x|^2)^s|\partial^\alpha u| < \infty$$

for all $s, m = 0, 1, \ldots$. We say that $u_n \to u$ in $\mathcal{S}(\mathbb{R}^N)$ if

$$\sup_{|\alpha|\leq m} \sup_{x\in\mathbb{R}^N} (1+|x|^2)^s|\partial^\alpha(u_n - u)| \to 0,\ s, m = 0, 1, \ldots. \tag{6}$$

The space of *tempered distributions* is identified as the dual $\mathcal{S}'(\mathbb{R}^N)$. Continuity of a linear form belonging to $\mathcal{S}'(\mathbb{R}^N)$ is understood with respect to convergence introduced in (6).

The Fourier transform introduced in (5) can be extended as a bounded linear operator defined on $\mathcal{S}(\mathbb{R}^N)$ with values in $\mathcal{S}(\mathbb{R}^N)$. Its inverse reads

$$\mathcal{F}^{-1}_{\xi\to x}[f] = \left(\frac{1}{2\pi}\right)^{N/2} \int_{\mathbb{R}^N} f(\xi)\exp(ix\cdot\xi)\mathrm{d}\xi. \tag{7}$$

(iii) The Fourier transform of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^N)$ is defined as

$$\langle \mathcal{F}_{x\to\xi}[f]; g\rangle = \langle f; \mathcal{F}_{x\to\xi}[g]\rangle \text{ for any } g \in \mathcal{S}(\mathbb{R}^N). \tag{8}$$

It is a continuous linear operator defined on $\mathcal{S}'(\mathbb{R}^N)$ onto $\mathcal{S}'(\mathbb{R}^N)$ with the inverse $\mathcal{F}^{-1}_{\xi\to x}$,

$$\left\langle \mathcal{F}^{-1}_{\xi\to x}[f]; g\right\rangle = \left\langle f; \mathcal{F}^{-1}_{\xi\to x}[g]\right\rangle, \quad f \in \mathcal{S}'(R^N), g \in \mathcal{S}(\mathbb{R}^N). \tag{9}$$

(iv) We recall formulas

$$\partial_{\xi_k}\mathcal{F}_{x\to\xi}[f] = \mathcal{F}_{x\to\xi}[-\mathrm{i}x_k f], \quad \mathcal{F}_{x\to\xi}[\partial_{x_k} f] = \mathrm{i}\xi_k \mathcal{F}_{x\to\xi}[f], \tag{10}$$

where $f \in \mathcal{S}'(\mathbb{R}^N)$, and

$$\mathcal{F}_{x\to\xi}[f * g] = \Big(\mathcal{F}_{x\to\xi}[f]\Big) \times \Big(\mathcal{F}_{x\to\xi}[g]\Big), \tag{11}$$

where $f \in \mathcal{S}(\mathbb{R}^N)$, $g \in \mathcal{S}'(\mathbb{R}^N)$ and $*$ denotes *convolution.*

(v) A partial differential operator D of order m,

$$D = \sum_{|\alpha|\le m} a_\alpha \partial^\alpha,$$

can be associated to a *Fourier multiplier*in the form

$$\tilde{D} = \sum_{|\alpha|\le m} a_\alpha (\mathrm{i}\xi)^\alpha, \ \xi^\alpha = \xi_1^{\alpha_1}\dots\xi_N^{\alpha_N}$$

in the sense that

$$D[v](x) = \mathcal{F}^{-1}_{\xi\to x}\left[\sum_{|\alpha|\le m} a_\alpha (\mathrm{i}\xi)^\alpha \mathcal{F}_{x\to\xi}[v](\xi)\right], \ v \in \mathcal{S}(\mathbb{R}^N).$$

The operators defined through the right-hand side of the above expression are called *pseudodifferential operators.*

(vi) Various *pseudodifferential operators* used in the book are identified through their Fourier symbols:

- *Riesz transform*:

$$\mathcal{R}_j \approx \frac{-i\xi_j}{|\xi|},\ j = 1, \ldots, N.$$

- *Inverse Laplacian*:

$$(-\Delta)^{-1} \approx \frac{1}{|\xi|^2}.$$

- *The "double" Riesz transform*:

$$\{\mathcal{R}\}_{i,j=1}^{N},\ \mathcal{R} = \Delta^{-1}\nabla_x \otimes \nabla_x,\ \mathcal{R}_{i,j} \approx \frac{\xi_i \xi_j}{|\xi|^2},\ i, j = 1, \ldots, N.$$

- *Inverse divergence*:

$$\mathcal{A}_j = \partial_{x_j}\Delta^{-1} \approx \frac{\mathrm{i}\xi_j}{|\xi|^2},\ j = 1, \ldots, N.$$

We write

$$\mathbb{A} : \mathcal{R} \equiv \sum_{i,j=1}^{3} A_{i,j}\mathcal{R}_{i,j},\ \mathcal{R}[\mathbf{v}]_i \equiv \sum_{j=1}^{3} \mathcal{R}_{i,j}[v_j],\ i = 1, 2, 3.$$

(vii)

■ HÖRMANDER-MIKHLIN THEOREM:

Theorem 0.7. *Consider an operator $\mathcal{L}$ defined by means of a Fourier multiplier $m = m(\xi)$,*

$$\mathcal{L}[v](x) = \mathcal{F}^{-1}_{\xi \to x}\left[m(\xi)\mathcal{F}_{x\to\xi}[v](\xi)\right],$$

where $m \in L^\infty(\mathbb{R}^N)$ has classical derivatives up to order $[N/2] + 1$ in $\mathbb{R}^N \setminus \{0\}$ and satisfies

$$|\partial_\xi^\alpha m(\xi)| \leq c_\alpha |\xi|^{-|\alpha|},\ \xi \neq 0,$$

for any multiindex α such that $|\alpha| \leq [N/2] + 1$, where $[\,]$ denotes the integer part.

Then $\mathcal{L}$ is a bounded linear operator on $L^p(\mathbb{R}^N)$ for any $1 < p < \infty$.

See Stein [186, Chapter 4, Theorem 3]. □

0.6 Weak convergence of integrable functions

Let X be a Banach space, B_X the (closed) unit ball in X, and B_{X^*} the (closed) unit ball in the dual space X^*.

(i) Here are some known facts concerning *weak compactness*:

(1) B_X is weakly compact only if X is reflexive. This is stated in Kakutani's theorem, see Theorem III.6 in Brezis [35].
(2) B_{X^*} is weakly-(*) compact. This is the Banach-Alaoglu-Bourbaki theorem, see Theorem III.15 in Brezis [35].
(3) If X is separable, then B_{X^*} is sequentially weakly-(*) compact, see Theorem III.25 in Brezis [35].
(4) A non-empty subset of a Banach space X is weakly relatively compact only if it is sequentially weakly relatively compact. This is stated in the Eberlein-Shmuliyan-Grothendieck theorem, see Kothe [124], Paragraph 24, 3.(8); 7.

(ii) In view of these facts:

(1) Any bounded sequence in $L^p(Q)$, where $1 < p < \infty$ and $Q \subset \mathbb{R}^N$ is a domain, is weakly (relatively) compact.
(2) Any bounded sequence in $L^\infty(Q)$, where $Q \subset \mathbb{R}^N$ is a domain, is weakly-(*) (relatively) compact.

(iii) Since L^1 is neither reflexive nor dual of a Banach space, the uniformly bounded sequences in L^1 are in general not weakly relatively compact in L^1. On the other hand, the property of weak compactness is equivalent to the property of sequential weak compactness.

■ WEAK COMPACTNESS IN THE SPACE L^1:

Theorem 0.8. *Let $\mathcal{V} \subset L^1(Q)$, where $Q \subset \mathbb{R}^M$ is a bounded measurable set. Then the following statements are equivalent:*

(i) *any sequence $\{v_n\}_{n=1}^\infty \subset \mathcal{V}$ contains a subsequence weakly converging in $L^1(Q)$;*

(ii) *for any $\varepsilon > 0$ there exists $k > 0$ such that*

$$\int_{\{|v|\geq k\}} |v(y)| \,\mathrm{d}y \leq \varepsilon \text{ for all } v \in \mathcal{V};$$

(iii) *for any $\varepsilon > 0$ there exists $\delta > 0$ such for all $v \in \mathcal{V}$*

$$\int_M |v(y)| \mathrm{d}y < \varepsilon$$

for any measurable set $M \subset Q$ such that

$$|M| < \delta;$$

(iv) *there exists a non-negative function* $\Phi \in C([0,\infty))$,

$$\lim_{z\to\infty} \frac{\Phi(z)}{z} = \infty,$$

such that

$$\sup_{v\in\mathcal{V}} \int_Q \Phi(|v(y)|)\ \mathrm{d}y \le c.$$

See Ekeland and Temam [70, Ch. 8, Thm. 1.3], Pedregal [172, Lem. 6.4]. □

Condition (iii) is termed *equi-integrability* of a given set of integrable functions and the equivalence of (i) is the *Dunford-Pettis theorem* (cf., e.g., Diestel [62, p.93]. Condition (iv) is called the De la Vallé-Poussin criterion, see Pedregal [172, Lemma 6.4]. The statement "there exists a non-negative function $\Phi \in C([0,\infty))$" in condition (iv) can be replaced by "there exists a non-negative convex function on $[0,\infty)$".

0.7 Non-negative Borel measures

(i) The symbol $C_c(Q)$ denotes the space of continuous functions with compact support in a locally compact Hausdorff metric space Q.

(ii) The symbol $\mathcal{M}(Q)$ stands for the space of signed Borel measures on Q. The symbol $\mathcal{M}^+(Q)$ denotes the cone of non-negative Borel measures on Q. A measure $\nu \in \mathcal{M}^+(Q)$ such that $\nu[Q] = 1$ is called *probability measure.*

(iii)

■ The Riesz Representation Theorem:

Theorem 0.9. *Let Q be a locally compact Hausdorff metric space. Let f be a non-negative linear functional defined on the space $C_c(Q)$.*

Then there exist a σ-algebra of measurable sets containing all Borel sets and a unique non-negative measure on $\mu_f \in \mathcal{M}^+(Q)$ such that

$$\langle f; g\rangle = \int_Q g\ \mathrm{d}\mu_f \text{ for any } g \in C_c(Q). \tag{12}$$

Moreover, the measure μ_f enjoys the following properties:

- $\mu_f[K] < \infty$ *for any compact* $K \subset Q$.
- $\mu_f[E] = \sup\{\mu_f[K] \mid K \subset E\}$ *for any open set* $E \subset Q$.
- $\mu_f[V] = \inf\{\mu(E) \mid V \subset E,\ E \text{ open}\}$ *for any Borel set* V.
- *If E is μ_f measurable, $\mu_f(E) = 0$, and $A \subset E$, then A is μ_f measurable.*

See Rudin [175, Chapter 2, Theorem 2.14]. □

(iv)

Corollary 0.1. *Assume that $f : C_c^\infty(Q) \to \mathbb{R}$ is a linear and non-negative functional, where Q is a domain in $\mathbb{R}^N$.*

Then there exists a measure μ_f enjoying the same properties as in Theorem 0.9 *such that f is represented through* (12)*.*

See Evans and Gariepy [75, Chapter 1.8, Corollary 1].

(v) If $Q \subset \mathbb{R}^M$ is a bounded set, the space $\mathcal{M}(\overline{Q})$ can be identified with the dual to the Banach space $C(\overline{Q})$ via (12). The space $\mathcal{M}(\overline{Q})$ is compactly embedded into the dual Sobolev space $W^{-k,p'}(Q)$ as soon as $Q \subset \mathbb{R}^M$ is a bounded Lipschitz domain and $kp > M$, $1/p + 1/p' = 1$ (see Evans [73, Chapter 1, Theorem 6]).

(vi) If μ is a probability measure on Ω and g a μ-measurable real-valued function, then we have *Jensen's inequality*

$$\Phi\left(\int_\Omega g \, \mathrm{d}\mu\right) \leq \int_\Omega \Phi(g) \, \mathrm{d}\mu \tag{13}$$

for any convex Φ defined on $\mathbb{R}$.

0.8 Parametrized (Young) measures

(i) Let $Q \subset \mathbb{R}^N$ be a domain. We say that $\psi : Q \times \mathbb{R}^M$ is a *Carathéodory function* on $Q \times \mathbb{R}^M$ if

$$\left\{ \begin{array}{c} \text{for a. a. } x \in Q, \text{ the function } \lambda \mapsto \psi(x,\lambda) \text{ is continuous on } \mathbb{R}^M; \\ \text{for all } \lambda \in \mathbb{R}^M, \text{ the function } x \mapsto \psi(x,\lambda) \text{ is measurable on } Q. \end{array} \right\} \tag{14}$$

We say that $\{\nu_x\}_{x \in Q}$ is a *family of parametrized measures* if ν_x is a probability measure for a.a. $x \in Q$, and if

$$\left\{ \begin{array}{c} \text{the function } x \to \int_{\mathbb{R}^M} \phi(\lambda) \, \mathrm{d}\nu_x(\lambda) := \langle \nu_x; \phi \rangle \text{ is measurable on } Q \\ \text{for all } \phi : \mathbb{R}^M \to \mathbb{R}, \, \phi \in \big(C(\mathbb{R}^M) \cap L^\infty(\mathbb{R}^M) \big). \end{array} \right\} \tag{15}$$

(ii)

■ FUNDAMENTAL THEOREM OF THE THEORY OF PARAMETERIZED (YOUNG) MEASURES:

Theorem 0.10. *Let $\{\mathbf{v}_n\}_{n=1}^\infty$, $\mathbf{v}_n : Q \subset \mathbb{R}^N \to \mathbb{R}^M$ be a sequence of functions bounded in $L^1(Q; \mathbb{R}^M)$, where Q is a domain in $\mathbb{R}^N$.*

Then there exist a subsequence (not relabeled) and a parameterized family $\{\nu_y\}_{y \in Q}$ *of probability measures on* $\mathbb{R}^M$ *depending measurably on* $y \in Q$ *with the following property:*

For any Carathéodory function $\Phi = \Phi(y, z)$, $y \in Q$, $z \in \mathbb{R}^M$ *such that*

$$\Phi(\cdot, \mathbf{v}_n) \to \overline{\Phi} \text{ weakly in } L^1(Q),$$

we have

$$\overline{\Phi}(y) = \int_{\mathbb{R}^M} \psi(y, z) \, \mathrm{d}\nu_y(z) \text{ for a.a. } y \in Q.$$

See Pedregal [172, Chapter 6, Theorem 6.2]. □

(iii) The family of measures $\{\nu_y\}_{y \in Q}$ associated to a sequence $\{\mathbf{v}_n\}_{n=1}^{\infty}$,

$$\mathbf{v}_n \to \mathbf{v} \text{ weakly in } L^1(Q; \mathbb{R}^M),$$

is termed *Young measure*. We shall systematically denote by the symbol $\overline{\Phi(\cdot, \mathbf{v})}$ the weak limit associated to $\{\Phi(\cdot, \mathbf{v}_n)\}_{n=1}^{\infty}$ via the corresponding Young measure constructed in Theorem 0.10. Note that a Young measure need not be unique for a given sequence.

Chapter 1

Fluid Flow Modeling

Physics distinguishes four basic forms of matter: solids, liquids, gases, and plasmas. The last three forms fall in the category of *fluids*. Fluid is a material that can flow, meaning fluids cannot sustain stress in the equilibrium state. Any time a force is applied to a fluid, the latter starts and keeps moving even when the force is no longer active. Fluid mechanics studies flows of fluids under the principal laws of mechanics. Examples of real fluid flows are numerous ranging from oceans and atmosphere to gaseous stars. The relevant applications include meteorology, engineering, and astrophysics to name only a few.

There are several qualitative levels of models studied in *mathematical fluid mechanics*. The main conceptual idea is the fundamental hypothesis that matter is made of atoms and molecules, viewed as solid objects with several degrees of freedom, that obey the basic principles of *classical mechanics*.

- MOLECULAR DYNAMICS (MD) studies typically a very large number of ordinary differential equations that govern the time evolution of each single particle of the fluid coupled through the interaction forces of different kinds. Numerical simulations based on (MD) are of fundamental importance when determining the physical properties of "macroscopic" fluids, for instance their interaction with a solid wall. Models based on (MD) are fully *reversible* in time.
- KINETIC MODELS are based on *averaging* with respect to particles having the same velocity. The basic state variable is the *density* of the fluid particles at a given time and spatial position with the same velocity. Accordingly, the evolution is governed by a transport equation of Boltzmann's type including the so-called collision operator. The presence of collisions results in *irreversibility* of the process in time.
- CONTINUUM FLUID MECHANICS is a phenomenological theory based on macroscopic (observable) state variables such as density, fluid velocity, and temperature. The time evolution of these quantities is described through a

system of *partial differential equations*. The objective *existence* of the macroscopic quantities (fields) is termed the *continuum hypothesis*. The theory is widely used in numerical analysis and real world applications. The processes, in general, are irreversible in time.

- MODELS OF TURBULENCE are based on further averaging of the macroscopic models studied in continuum fluid mechanics. According to the present state of knowledge, there is no universally accepted theory of turbulence. The evolution of state variables is described by a system of partial differential equations and is irreversible in time.

The mathematical theory of *continuum fluid mechanics* developed in this book is based on fundamental physical principles that can be expressed in terms of *balance laws*. These may be written by means of either a Lagrangian or a Eulerian reference system. In Lagrangian coordinates, the description is associated to particles moving in space and time. The Eulerian reference system is based on a fixed frame attached to the underlying physical space. We will use systematically the *Eulerian description* more suitable for fluids which undergo unlimited displacements. Accordingly, the independent variables are associated to the physical space represented by a spatial domain $\Omega \subset \mathbb{R}^3$, and a time interval $I \subset \mathbb{R}$, typically, $I = (0, T)$, $T > 0$.

1.1 Fluids in continuum mechanics

We adopt the standard mathematical description of a *fluid* as found in many classical textbooks on continuum fluid mechanics. Unlike certain recently proposed alternative theories based on a largely *extended number* of state variables, we assume the *state* of a fluid at a given instant can be characterized by its density and temperature distribution whereas the *motion* is completely determined by a velocity field. Simplifying further we focus on chemically inert homogeneous fluids that may be characterized through the following quantities.

■ FLUIDS IN CONTINUUM MECHANICS:

(a) a domain $\Omega \subset \mathbb{R}^3$ occupied by a fluid in an ambient space;

(b) a non-negative measurable function $\varrho = \varrho(t, x)$ defined for $t \in (0, T)$, $x \in \Omega$, yielding the *mass density*;

(c) a vector field $\mathbf{u} = \mathbf{u}(t, x)$, $t \in (0, T)$, $x \in \Omega$, defining the *velocity* of the fluid;

(d) a positive measurable function $\vartheta = \vartheta(t, x)$, $t \in (0, T)$, $x \in \Omega$, describing the distribution of *temperature* measured in the absolute Kelvin scale;

(e) the *thermodynamic functions*: the *pressure* $p = p(\varrho, \vartheta)$, the specific *internal energy* $e = e(\varrho, \vartheta)$, and the specific *entropy* $s = s(\varrho, \vartheta)$;

(f) a *stress tensor* $\mathbb{T} = \{T_{i,j}\}_{i,j=1}^3$ yielding the force per unit surface that the part of a fluid in contact with an ideal surface element imposes on the part of the fluid on the other side of the same surface element;

(g) a vector field $\mathbf{q}$ giving the *flux of the internal energy*;

(h) a vector field $\mathbf{f} = \mathbf{f}(t,x)$, $t \in (0,T)$, $x \in \Omega$, defining the distribution of a *volume force* acting on a fluid;

(i) a function $\mathcal{Q} = \mathcal{Q}(t,x)$, $t \in (0,T)$, $x \in \Omega$, yielding the rate of *production of internal energy.*

The trio $\{\varrho, \mathbf{u}, \vartheta\}$ introduced in (b)–(d) represents the basic *state variables* characterizing completely the state of a fluid at a given instant t. The remaining quantities are determined in terms of the state variables by means of a set of constitutive relations.

Fluids are characterized among other materials through *Stokes' law*

$$\mathbb{T} = \mathbb{S} - p\mathbb{I}, \tag{1.1}$$

where $\mathbb{S}$ denotes the *viscous stress tensor*. Viscosity is a property associated to the relative motion of different parts of the fluid; whence $\mathbb{S}$ is always interrelated with the velocity gradient $\nabla_x \mathbf{u}$ or rather its symmetric part $\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u}$. In particular, the viscous stress vanishes whenever $\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} = 0$, that means, when the fluid exhibits a rigid motion with respect to a fixed coordinate system. In accordance with the *Second law of thermodynamics*, viscosity is responsible for the irreversible transfer of the mechanical energy associated to the motion into heat. Although omitted in certain mathematical idealizations (Euler system), viscosity is always present and must be taken into account when modeling the motion of fluids in the long run.

The *pressure* p, similarly to the *specific energy* e and the *specific entropy* s, are typical thermostatic variables attributed to a system in thermodynamic equilibrium and as such can be evaluated as numerical functions of the density and the absolute temperature. Moreover, in accordance with the *Second law of thermodynamics*, $p = p(\varrho, \vartheta)$, $e = e(\varrho, \vartheta)$, and $s = s(\varrho, \vartheta)$ are interrelated through

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■ GIBBS' EQUATION:

$$\vartheta D s(\varrho, \vartheta) = De(\varrho, \vartheta) + p(\varrho, \vartheta) D\Big(\frac{1}{\varrho}\Big). \tag{1.2}$$

The symbol D in (1.2) stands for the differential with respect to the variables ϱ, ϑ. A common hypothesis tacitly assumed in many mathematical models asserts that the time scale related to the macroscopic motion of a fluid is so large that the latter can be considered at thermodynamic equilibrium at any instant t of the "real" time, in particular, the temperature ϑ is well determined and can be taken as a state variable even if the system may be quite far from the equilibrium state (see Öttinger [168]).

Gibbs' equation (1.2) can be equivalently written in the form of *Maxwell's relation*

$$\frac{\partial e(\varrho,\vartheta)}{\partial \varrho} = \frac{1}{\varrho^2}\Big(p(\varrho,\vartheta) - \vartheta \frac{\partial p(\varrho,\vartheta)}{\partial \vartheta}\Big). \tag{1.3}$$

The precise meaning of (1.3) is that the expression $1/\vartheta(De + pD(1/\varrho))$ is a perfect gradient of a scalar function termed *entropy*.

1.2 Balance laws

Classical continuum mechanics describes a fluid by means of a family of *state variables* – observable and measurable macroscopic quantities – a representative sample of which has been introduced in the preceding part. The basic physical principles are then expressed through a system of *balance laws*. A general *balance law* takes the form of an integral identity

$$\begin{aligned} \int_B d(t_2,x)\ \mathrm{d}x - \int_B d(t_1,x)\ \mathrm{d}x + \int_{t_1}^{t_2}\int_{\partial B} \mathbf{F}(t,x)\cdot \mathbf{n}(x)\ \mathrm{dS}_x\ \mathrm{d}t \\ = \int_{t_1}^{t_2}\int_B \sigma(t,x)\ \mathrm{d}x\ \mathrm{d}t \end{aligned} \tag{1.4}$$

to be satisfied for any $t_1 \leq t_2$ and any subset $B \subset \Omega$, where the symbol d stands for the volumetric (meaning per unit volume) density of an observable property, $\mathbf{F}$ denotes its flux, $\mathbf{n}$ is the outer normal vector to ∂B, and σ denotes the rate of production of d per unit volume. The principal idea, pursued and promoted in this book, asserts that (1.4) is the most natural and correct mathematical formulation of any balance law in continuum mechanics.

The expression on the left-hand side of (1.4) can be interpreted as the integral mean of the *normal trace* of the four-component vector field $[d, \mathbf{F}]$ on the boundary of the time-space cylinder $(t_1, t_2) \times B$. On the other hand, by means of the Gauss-Green theorem, we can write

$$\begin{aligned} &\int_B d(t_2,x)\varphi(t_2,x)\ \mathrm{d}x - \int_B d(t_1,x)\varphi(t_1,x)\ \mathrm{d}x + \int_{t_1}^{t_2}\int_{\partial B} \mathbf{F}(t,x)\cdot \mathbf{n}(x)\varphi(t,x)\mathrm{dS}_x\ \mathrm{d}t \\ &= \int_{t_1}^{t_2}\int_B \Big(\partial_t d(t,x) + \mathrm{div}_x \mathbf{F}(t,x)\Big)\varphi(t,x)\ \mathrm{d}x\ \mathrm{d}t \\ &\quad + \int_{t_1}^{t_2}\int_B \Big(d(t,x)\partial_t\varphi(t,x) + \mathbf{F}(t,x)\cdot\nabla_x\varphi(t,x)\Big)\ \mathrm{d}x\ \mathrm{d}t \end{aligned} \tag{1.5}$$

for any smooth test function φ defined on $\mathbb{R}\times\mathbb{R}^3$. If all quantities are continuously differentiable, it is easy to check that relations (1.4), (1.5) are compatible as soon as

$$\partial_t d(t,x) + \mathrm{div}_x \mathbf{F}(t,x) = \sigma(t,x) \tag{1.6}$$

yielding

$$\int_B d(t_2,x)\varphi(t_2,x)\ \mathrm{d}x - \int_B d(t_1,x)\varphi(t_1,x)\ \mathrm{d}x + \int_{t_1}^{t_2}\int_{\partial B} \mathbf{F}(t,x)\cdot\mathbf{n}(x)\varphi(t,x)\mathrm{dS}_x\ \mathrm{d}t$$
$$= \int_{t_1}^{t_2}\int_B \sigma(t,x)\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2}\int_B \Big(d(t,x)\partial_t\varphi(t,x) + \mathbf{F}(t,x)\cdot\nabla_x\varphi(t,x)\Big)\,\mathrm{d}x\,\mathrm{d}t. \tag{1.7}$$

The integral identity (1.7) can be used as a proper *definition* of the *normal trace* of the field $[d, \mathbf{F}]$ as long as

$$\int_0^T \int_\Omega \Big(d(t,x)\partial_t\varphi(t,x) + \mathbf{F}(t,x)\cdot\nabla_x\varphi(t,x)\Big)\ \mathrm{d}x\ \mathrm{d}t$$
$$+ \int_0^T \int_\Omega \sigma(t,x)\varphi(t,x)\ \mathrm{d}x\ \mathrm{d}t = 0 \tag{1.8}$$

for any $\varphi \in C_c^\infty((0,T)\times\Omega)$. In the terminology of the modern theory of partial differential equations, relation (1.8) represents a *weak formulation* of the differential equation (1.6). If (1.8) holds for any test function $\varphi \in C_c^\infty((0,T)\times\Omega)$, we say that equation (1.6) is satisfied in $\mathcal{D}'((0,T)\times\Omega)$, or, in the sense of distributions.

The satisfaction of the initial condition $d(0,\cdot) = d_0$, together with the prescribed normal component of the flux $F_b = \mathbf{F}\cdot\mathbf{n}|_{\partial\Omega}$ on the boundary, can be incorporated into the weak formulation by means of the integral identity

$$-\int_\Omega d_0(x)\varphi(0,x)\ \mathrm{d}x + \int_0^T\int_{\partial\Omega} F_b(t,x)\varphi(t,x)\mathrm{dS}_x\ \mathrm{d}t \tag{1.9}$$
$$= \int_0^T\int_\Omega \sigma(t,x)\varphi(t,x)\ \mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \Big(d(t,x)\partial_t\varphi(t,x) + \mathbf{F}(t,x)\cdot\nabla_x\varphi(t,x)\Big)\ \mathrm{d}x\,\mathrm{d}t$$

to be satisfied for any $\varphi \in C_c^\infty([0,T)\times\overline{\Omega})$.

As a matter of fact, the source term σ need not be an integrable function. The normal trace of $[d,\mathbf{F}]$ is still well defined through (1.7) even if σ is merely a signed measure, more specifically, $\sigma = \sigma^+ - \sigma^-$, where $\sigma^+, \sigma^- \in \mathcal{M}^+([0,T]\times\overline{\Omega})$ are non-negative regular Borel measures defined on the compact set $[0,T]\times\overline{\Omega}$. Accordingly, relation (1.9) takes the form of a general

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■ BALANCE LAW:

$$\langle\sigma;\varphi\rangle_{[\mathcal{M};C]([0,T]\times\overline{\Omega})} + \int_0^T\int_\Omega \Big(d(t,x)\partial_t\varphi(t,x) + \mathbf{F}(t,x)\cdot\nabla_x\varphi(t,x)\Big)\ \mathrm{d}x\ \mathrm{d}t$$
$$= \int_0^T\int_{\partial\Omega} F_b(t,x)\varphi(t,x)\ \mathrm{dS}_x\ \mathrm{d}t - \int_\Omega d_0(x)\varphi(0,x)\ \mathrm{d}x \tag{1.10}$$

for any test function $\varphi \in C_c^\infty([0,T)\times\overline{\Omega})$.

If (1.10) holds, the (outer) normal trace of the field $[d, \mathbf{F}]$ can be identified through (1.7), in particular, the *instantaneous values* of d at a time t can be defined. However, these may exhibit jumps if the rate of production σ is not absolutely continuous with respect to the Lebesgue measure. Specifically, using (1.7), (1.10), we can define the *left instantaneous value* of d at a time $\tau \in (0, T]$ as

$$\begin{aligned} &\langle d(\tau-, \cdot); \varphi \rangle_{[\mathcal{M}; C](\overline{\Omega})} \qquad (1.11)\\ &= \int_\Omega d_0(x)\varphi(x)\ \mathrm{d}x + \int_0^\tau \int_\Omega \mathbf{F}(t,x) \cdot \nabla_x \varphi(x)\ \mathrm{d}x\,\mathrm{d}t + \lim_{\delta \to 0+} \langle \sigma; \psi_\delta \varphi \rangle_{[\mathcal{M}, C]([0,T]\times\overline{\Omega})}, \end{aligned}$$

for any $\varphi \in C_c^\infty(\Omega)$, where $\psi_\delta = \psi_\delta(t)$ is non-increasing,

$$\psi_\delta \in C^1(\mathbb{R}),\ \ \psi_\delta(t) = \begin{cases} 1 \text{ for } t \in (-\infty, \tau - \delta], \\ 0 \text{ for } t \in [\tau, \infty). \end{cases}$$

Similarly, we define the *right instantaneous value* of d at a time $\tau \in [0, T)$ as

$$\begin{aligned} &\langle d(\tau+, \cdot); \varphi \rangle_{[\mathcal{M}; C](\overline{\Omega})} \qquad (1.12)\\ &= \int_\Omega d_0(x)\varphi(x)\ \mathrm{d}x + \int_0^\tau \int_\Omega \mathbf{F}(t,x) \cdot \nabla_x \varphi(x)\ \mathrm{d}x\,\mathrm{d}t + \lim_{\delta \to 0+} \langle \sigma; \psi_\delta \varphi \rangle_{[\mathcal{M}, C]([0,T]\times\overline{\Omega})}, \end{aligned}$$

where $\psi_\delta = \psi_\delta(t)$ is non-increasing,

$$\psi_\delta \in C^1(\mathbb{R}),\ \ \psi_\delta(t) = \begin{cases} 1 \text{ for } t \in (-\infty, \tau], \\ 0 \text{ for } t \in [\tau + \delta, \infty). \end{cases}$$

Note that, at least for $d \in L^\infty(0, T; L^1(\Omega))$, the left and right instantaneous values are represented by signed measures on Ω that coincide with $d(\tau, \cdot) \in L^1(\Omega)$ at any Lebesgue point of the mapping $\tau \mapsto d(\tau, \cdot)$. Moreover, $d(\tau-, \cdot) = d(\tau+, \cdot)$ for any $\tau \in [0, T]$ and the mapping $\tau \mapsto d(\tau, \cdot)$ is weakly-(*) continuous as soon as σ is absolutely continuous with respect to the standard Lebesgue measure on $(0, T) \times \Omega$.

Under certain circumstances, notably when identifying the *entropy production rate*, the piece of information that is provided by the available mathematical theory enables us only to show that

$$\int_0^T \int_\Omega d(t,x)\partial_t \varphi(t,x) + \mathbf{F}(t,x) \cdot \nabla_x \varphi(t,x)\ \mathrm{d}x \le 0 \qquad (1.13)$$

for any *non-negative* test function $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$. Intuitively, this means

$$\partial_t d + \mathrm{div}_x(\mathbf{F}) \ge 0$$

though a rigorous verification requires differentiability of d and $\mathbf{F}$.

Let us show that (1.13) is in fact *equivalent* to the integral identity

$$\int_0^T \int_\Omega d(t,x)\partial_t\varphi(t,x)+\mathbf{F}(t,x)\cdot\nabla_x\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t+\langle\sigma;\varphi\rangle_{[\mathcal{M}^+;C]([0,T]\times\Omega)}=0 \quad (1.14)$$

for any $\varphi \in C_c^\infty([0,T)\times\overline{\Omega})$, where $\sigma \in \mathcal{M}^+([0,T]\times\overline{\Omega})$ is a non-negative regular Borel measure on the set $[0,T]\times\overline{\Omega}$. This fact may be viewed as a variant of the well-known statement that any non-negative distribution is representable by a measure.

In order to see (1.14), assume that

$$d \in L^\infty(0,T;L^1(\Omega)) \text{ and } \mathbf{F} \in L^p((0,T)\times\Omega;R^3) \text{ for a certain } p>1.$$

Consider a linear form

$$\langle\sigma;\varphi\rangle = -\int_0^T \int_\Omega \Big(d(t,x)\partial_t\varphi(t,x)+\mathbf{F}(t,x)\cdot\nabla_x\varphi(t,x)\Big)\,\mathrm{d}x$$

which is well defined for any $\varphi \in C_c^1([0,T)\times\overline{\Omega})$. Moreover, it follows from (1.13), that

$$\langle\sigma;\varphi\rangle \ \geq 0 \text{ for any } \varphi \in C_c^\infty([0,T)\times\overline{\Omega}),\ \varphi\geq 0. \quad (1.15)$$

Next, for any compact set $K \subset [0,T)\times\overline{\Omega}$ we can find a function χ_K such that

$$\chi_K=\chi_K(t)\in C_c^\infty[0,T),\ 0\leq\chi_K\leq 1,\ \partial_t\chi_K\leq 0,\ \chi_K=1 \text{ on } K. \quad (1.16)$$

In particular, as a direct consequence of (1.15), we get

$$\langle\sigma;\chi_K\rangle \leq \operatorname{ess}\sup_{t\in(0,T)} \|d(t,\cdot)\|_{L^1(\Omega)} \text{ for any } K. \quad (1.17)$$

We claim that σ can be extended in a unique way as a bounded non-negative linear form on the vector space $C_c([0,T)\times\overline{\Omega})$. Indeed for any sequence $\{\varphi_n\}_{n=1}^\infty$ of (smooth) functions supported by a fixed compact set $K\subset[0,T)\times\overline{\Omega}$, we have

$$|\,\langle\sigma;\varphi_n\rangle-\langle\sigma;\varphi_m\rangle\,| \leq \ \langle\sigma;\chi_K\rangle\,\|\varphi_n-\varphi_m\|_{C(K)},$$

with χ_K constructed in (1.16).

By virtue of Riesz's representation theorem (Theorem 0.9), the linear form σ can be identified with a non-negative Borel measure on the set $[0,T)\times\overline{\Omega}$. Finally, because of the uniform estimate (1.17) on the value of $\sigma[K]$ for any compact set $K\subset[0,T)\times\overline{\Omega}$, the measure $\sigma[[0,T)\times\overline{\Omega}]$ of the full domain is finite, in particular σ can be trivially extended (by zero) to the set $[0,T]\times\overline{\Omega}$. Let us point out, however, that such an extension represents only a suitable convention (the measure σ is defined on a *compact* set $[0,T]\times\overline{\Omega}$) without any real impact on formula (1.14).

To conclude, we recall that the *weak formulation* of a balance law introduced in (1.10) is deliberately expressed in the space-fixed, Eulerian form rather than a "body-fixed" material description. This convention avoids the ambiguous notion of trajectory in the situation where $\mathbf{F}$, typically proportional to the velocity of the fluid, is not regular enough to give rise to a unique system of streamlines.

1.3 Field equations

In accordance with the general approach delineated in Section 1.2, the basic physical principles formulated in terms of balance laws will be understood in the sense of integral identities similar to (1.10) rather than systems of partial differential equations set forth in classical textbooks on fluid mechanics. Nonetheless, in the course of formal discussion, we stick to the standard terminology "equation" or "field equation" even if these mathematical objects are represented by an infinite system of integral identities to be satisfied for a suitable class of test functions rather than a single equation. Accordingly, the macroscopic quantities characterizing the state of a material in continuum mechanics are called *fields*, the balance laws they obey are termed *field equations*.

1.3.1 Conservation of mass

The fluid *density* $\varrho = \varrho(t,x)$ is a fundamental state variable describing the distribution of mass. The integral

$$M(B) = \int_B \varrho(t,x)\ \mathrm{d}x$$

represents the total amount of mass of the fluid contained in a set $B \subset \Omega$ at an instant t. In a broader sense, the density could be a non-negative measure defined on a suitable system of subsets of the ambient space Ω. However, for the purposes of this study, we content ourselves with $\varrho(t,\cdot)$ that is absolutely continuous with respect to the standard Lebesgue measure on $\mathbb{R}^3$, therefore representable by a non-negative measurable function.

Motivated by the general approach described in the previous part, we write the physical principle of *mass conservation* in the form

$$\int_B \varrho(t_2,x)\ \mathrm{d}x - \int_B \varrho(t_1,x)\ \mathrm{d}x + \int_{t_1}^{t_2}\int_{\partial B} \varrho(t,x)\mathbf{u}(t,x)\cdot\mathbf{n}\ \mathrm{dS}_x\ \mathrm{d}t = 0$$

for any (smooth) subset $B \subset \Omega$, where $\mathbf{u} = \mathbf{u}(t,x)$ is the velocity field determining the motion of the fluid. Thus assuming, for a moment, that all quantities are smooth, we deduce the *equation of continuity* in the differential form

$$\partial_t \varrho(t,x) + \mathrm{div}_x(\varrho(t,x)\mathbf{u}(t,x)) = 0 \text{ in } (0,T)\times\Omega. \tag{1.18}$$

In addition, we impose *impermeability* of the boundary $\partial\Omega$, meaning,

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0. \tag{1.19}$$

Multiplying (1.18) on $B(\varrho) + \varrho B'(\varrho)$, where B is a continuously differentiable function, we easily deduce that

$$\partial_t(\varrho B(\varrho)) + \mathrm{div}_x(\varrho B(\varrho)\mathbf{u}) + b(\varrho)\mathrm{div}_x\mathbf{u} = 0 \tag{1.20}$$

for any $b \in BC[0,\infty)$ (bounded and continuous functions), where

$$B(\varrho) = B(1) + \int_1^{\varrho} \frac{b(z)}{z^2}\ \mathrm{d}z. \tag{1.21}$$

Equation (1.20) can be viewed as a *renormalized variant* of (1.18).

Summing up the previous discussion and returning to the weak formulation, we introduce

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■ RENORMALIZED EQUATION OF CONTINUITY:

$$\int_0^T \int_\Omega \Big(\varrho B(\varrho)\partial_t \varphi + \varrho B(\varrho)\mathbf{u}\cdot\nabla_x\varphi - b(\varrho)\mathrm{div}_x\mathbf{u}\ \varphi \Big)\ \mathrm{d}x\,\mathrm{d}t = -\int_\Omega \varrho_0 B(\varrho_0)\varphi(0,\cdot)\ \mathrm{d}x \tag{1.22}$$

to be satisfied for any test function $\varphi \in C_c^\infty([0,T)\times\overline{\Omega})$, and any B, b interrelated through (1.21), where b is continuous and uniformly bounded function on $\mathbb{R}$.

The family of integral identities (1.22) represents a mathematical formulation of the physical principle of *mass conservation*. Formally, relation (1.22) reduces to (1.20) provided all quantities are smooth, and, furthermore, to (1.18) if we take $b \equiv 0$, $B(1) = 1$. The initial distribution of the density is determined by a given function $\varrho_0 = \varrho(0,\cdot)$, while the boundary conditions (1.19) are satisfied implicitly through the choice of test functions in (1.22) in the spirit of (1.10).

In a certain sense, the renormalized equation (1.22) can be viewed as a very weak formulation of (1.18) since, at least for $B(1) = 0$, the density ϱ need not be integrable. On the other hand, relation (1.22) requires integrability of the velocity field $\mathbf{u}$ at the level of first derivatives, specifically, $\mathrm{div}_x\mathbf{u}$ must be integrable on the set $[0,T)\times\overline{\Omega}$.

In contrast to (1.18), relation (1.22) provides a useful piece of information on the mass transport and possible density oscillations in terms of the initial data. It is important to note that (1.22) can be deduced from (1.18) even at the level of the weak formulation as soon as the density is a bounded measurable function (see Section 10.18 in Appendix).

1.3.2 Balance of linear momentum

In accordance with *Newton's second law*, the flux associated to the momentum $\varrho\mathbf{u}$ in the Eulerian coordinate system can be written in the form $(\varrho\mathbf{u}\otimes\mathbf{u} - \mathbb{T})$, where the symbol $\mathbb{T}$ stands for the stress tensor introduced in Section 1.1. In accordance with Stokes' law (1.1), the *balance of linear momentum* reads

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x p = \mathrm{div}_x\mathbb{S} + \varrho\mathbf{f}\ \text{ in } \mathcal{D}'((0,T)\times\Omega;\mathbb{R}^3), \tag{1.23}$$

or,

$$\int_0^T \int_\Omega \Big((\varrho \mathbf{u}) \cdot \partial_t \varphi + \varrho (\mathbf{u} \otimes \mathbf{u}) : \nabla_x \varphi + p \mathrm{div}_x \varphi \Big) \, \mathrm{d}x \, \mathrm{d}t$$
$$= \int_0^T \int_\Omega \Big(\mathbb{S} : \nabla_x \varphi - \varrho \mathbf{f} \cdot \varphi \Big) \, \mathrm{d}x - \int_\Omega (\varrho \mathbf{u})_0 \cdot \varphi(0, \cdot) \, \mathrm{d}x, \tag{1.24}$$

to be satisfied by any test function $\varphi \in C_c^\infty([0,T) \times \Omega; \mathbb{R}^3)$. Note that relation (1.24) already includes the initial condition

$$\varrho \mathbf{u}(0, \cdot) = (\varrho \mathbf{u})_0 \text{ in } \Omega. \tag{1.25}$$

Analogously, as in the previous sections, the variational formulation (1.24) may include implicit satisfaction of boundary conditions provided the class of admissible test functions is extended "up to the boundary". Roughly speaking, the test functions should belong to the same regularity class as the velocity field $\mathbf{u}$. Accordingly, in order to enforce the impermeability condition (1.19), we take

$$\varphi \in C_c^\infty([0,T) \times \overline{\Omega}; \mathbb{R}^3), \ \ \varphi \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{1.26}$$

Postulating relation (1.24) for any test function satisfying (1.26), we deduce formally that

$$(\mathbb{S}\mathbf{n}) \times \mathbf{n}|_{\partial\Omega} = 0, \tag{1.27}$$

which means, the tangential component of the normal stress forces vanishes on the boundary. This behavior of the stress characterizes *complete slip* of the fluid against the boundary.

In the theory of *viscous fluids*, however, it is more customary to impose the *no-slip boundary condition*

$$\mathbf{u}|_{\partial\Omega} = 0, \tag{1.28}$$

together with the associated class of test functions

$$\varphi \in C_c^\infty([0,T) \times \Omega; \mathbb{R}^3). \tag{1.29}$$

The no-slip boundary condition (1.28) and even the impermeability condition (1.19) require a concept of *trace* of the field $\mathbf{u}$ on the boundary $\partial\Omega$. Therefore the velocity field $\mathbf{u}$ must belong to a "better" space than just $L^p(\Omega; \mathbb{R}^3)$. As for the impermeability hypothesis (1.19), we recall the Gauss-Green theorem yielding

$$\int_{\partial\Omega} \varphi \mathbf{u} \cdot \mathbf{n} \, \mathrm{dS}_x = \int_\Omega \nabla_x \varphi \cdot \mathbf{u} \, \mathrm{d}x + \int_\Omega \varphi \, \mathrm{div}_x \mathbf{u} \, \mathrm{d}x. \tag{1.30}$$

Consequently, we need both $\mathbf{u}$ and $\mathrm{div}_x \mathbf{u}$ to be at least integrable on Ω for (1.19) to make sense. The no-slip boundary condition (1.28) requires the partial derivatives of $\mathbf{u}$ to be at least (locally) integrable in Ω (cf. Theorem 0.6).

Before leaving this section, we give a concise formulation of *Newton's second law* in terms of

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■ BALANCE OF MOMENTUM:

$$\int_0^T \int_\Omega \Big((\varrho \mathbf{u}) \cdot \partial_t \varphi + \varrho (\mathbf{u} \otimes \mathbf{u}) : \nabla_x \varphi + p \mathrm{div}_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$= \int_0^T \int_\Omega \Big(\mathbb{S} : \nabla_x \varphi - \varrho \mathbf{f} \cdot \varphi \Big) \ \mathrm{d}x - \int_\Omega (\varrho \mathbf{u})_0 \cdot \varphi(0, \cdot) \ \mathrm{d}x \tag{1.31}$$

must be satisfied by any test function φ belonging to the class $C_c^\infty([0,T) \times \Omega; \mathbb{R}^3)$ if the no-slip boundary conditions (1.28) are imposed, or

$$\varphi \in C_c^\infty([0,T) \times \overline{\Omega}; \mathbb{R}^3), \ \ \varphi \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

in the case of complete slip boundary conditions (1.19), (1.27).

1.3.3 Total energy

The *energy density* $\mathcal{E}$ can be written in the form

$$\mathcal{E} = \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta), \tag{1.32}$$

where the symbol e denotes the specific internal energy introduced in Section 1.1.

Multiplying equation (1.23) on $\mathbf{u}$ we deduce the kinetic energy balance

$$\partial_t \Big(\frac{1}{2} \varrho |\mathbf{u}|^2 \Big) + \mathrm{div}_x \Big(\frac{1}{2} \varrho |\mathbf{u}|^2 \mathbf{u} \Big) = \mathrm{div}_x (\mathbb{T} \mathbf{u}) - \mathbb{T} : \nabla_x \mathbf{u} + \varrho \mathbf{f} \cdot \mathbf{u}, \tag{1.33}$$

where the stress tensor $\mathbb{T}$ is related to $\mathbb{S}$ and p by means of Stokes' law (1.1). On the other hand, by virtue of the *First law of thermodynamics*, the changes of the energy of the system are caused only by external sources, in particular, the internal energy balance reads

$$\partial_t (\varrho e) + \mathrm{div}_x (\varrho e \mathbf{u}) + \mathrm{div}_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u} - p \mathrm{div}_x \mathbf{u} + \varrho \mathcal{Q}, \tag{1.34}$$

where the term $\varrho \mathcal{Q}$ represent the volumetric rate of the internal energy production, and $\mathbf{q}$ is the internal energy flux.

Consequently, the *energy balance* equation may be written in the form

$$\partial_t \mathcal{E} + \mathrm{div}_x (\mathcal{E} \mathbf{u}) + \mathrm{div}_x \Big(\mathbf{q} - \mathbb{S} \mathbf{u} + p \mathbf{u} \Big) = \varrho \mathbf{f} \cdot \mathbf{u} + \varrho \mathcal{Q}. \tag{1.35}$$

Relation (1.35) can be integrated over the whole domain Ω in order to obtain the balance of total energy. Performing by parts integration of the resulting expression we finally arrive at

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■ TOTAL ENERGY BALANCE:

$$\int_\Omega \mathcal{E}(t_2,\cdot)\ \mathrm{d}x - \int_\Omega \mathcal{E}(t_1,\cdot)\ \mathrm{d}x = \int_{t_1}^{t_2}\int_\Omega \Big(\varrho\mathbf{f}\cdot\mathbf{u} + \varrho\mathcal{Q}\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{1.36}$$

for any $0 \le t_1 \le t_2 \le T$ provided

$$\mathbf{q}\cdot\mathbf{n}|_{\partial\Omega} = 0, \tag{1.37}$$

and either the no-slip boundary condition (1.28) or the complete slip boundary conditions (1.19), (1.27) hold.

In the previous considerations, the internal energy e has been introduced to balance the dissipative terms in (1.33). Its specific form required by Gibbs' equation (1.2) is a consequence of the *Second law of thermodynamics* discussed in the next section.

1.3.4 Entropy

The *Second law of thermodynamics* is the central principle around which we intend to build up the mathematical theory used in this study. As a matter of fact, Gibbs' equation (1.2) should be viewed as a constraint imposed on p and e by the principles of statistical physics, namely $\frac{1}{\vartheta}(De + pD\frac{1}{\varrho})$ must be a perfect gradient. Accordingly, the internal energy balance equation (1.34) can be rewritten in the form of *entropy balance*

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma + \frac{\varrho}{\vartheta}\mathcal{Q}, \tag{1.38}$$

with the *entropy production rate*

$$\sigma = \frac{1}{\vartheta}\Big(\mathbb{S}:\nabla_x\mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\Big). \tag{1.39}$$

The *Second law of thermodynamics* postulates that the entropy production rate σ must be nonnegative for any admissible thermodynamic process. As we will see below, this can be viewed as a restriction imposed on the constitutive relations for $\mathbb{S}$ and $\mathbf{q}$.

A weak formulation of equation (1.38) reads

■ ENTROPY BALANCE EQUATION:

$$\int_0^T \int_\Omega \left(\varrho s \partial_t \varphi + \varrho s \mathbf{u} \cdot \nabla_x \varphi + \left(\frac{\mathbf{q}}{\vartheta} \right) \cdot \nabla_x \varphi \ \mathrm{d}x \right) \mathrm{d}t \tag{1.40}$$
$$= - \int_\Omega (\varrho s)_0 \varphi \ \mathrm{d}x - \int_0^T \int_\Omega \sigma \varphi \ \mathrm{d}x \ \mathrm{d}t - \int_0^T \int_\Omega \frac{\varrho}{\vartheta} \mathcal{Q} \varphi \ \mathrm{d}x \ \mathrm{d}t$$

must be satisfied for any test function $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$. Note that (1.40) already includes the no-flux boundary condition (1.37) as well as the initial condition $\varrho s(0, \cdot) = (\varrho s)_0$.

In the framework of the weak solutions considered in this book, the entropy production rate σ will be a non-negative measure satisfying

$$\sigma \geq \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

in place of (1.39). Such a stipulation reflects one of the expected features of the weak solutions, namely they produce maximal dissipation rate of the kinetic energy enhanced by the presence of singularities that are not captured by the "classical" formula (1.39). As we will see in Chapter 2, this approach still leads to a (formally) well-posed problem.

1.4 Constitutive relations

The field equations derived in Section 1.3 must be supplemented with a set of *constitutive relations* characterizing the material properties of a concrete fluid. In particular, the viscous stress tensor $\mathbb{S}$, the internal energy flux $\mathbf{q}$ as well as the thermodynamic functions p, e, and s must be determined in terms of the independent state variables $\{\varrho, \mathbf{u}, \vartheta\}$.

1.4.1 Molecular energy and transport terms

The *Second law of thermodynamics*, together with its implications on the sign of the entropy production rate discussed in Section 1.3.4, gives rise to further restrictions that must be imposed on the transport terms $\mathbb{S}$, $\mathbf{q}$. In particular, as the entropy production is non-negative for any admissible physical process, we deduce from (1.39) that

$$\mathbb{S} : \nabla_x \mathbf{u} \geq 0, \ -\mathbf{q} \cdot \nabla_x \vartheta \geq 0. \tag{1.41}$$

A fundamental hypothesis of the mathematical theory developed in this book asserts that the constitutive equations relating $\mathbb{S}$, $\mathbf{q}$ to the affinities $\nabla_x \mathbf{u}$, $\nabla_x \vartheta$ are *linear*. Such a stipulation gives rise to

■ NEWTON'S RHEOLOGICAL LAW:

$$\mathbb{S} = \mu \Big(\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \ \mathbb{I} \Big) + \eta \mathrm{div}_x \mathbf{u} \ \mathbb{I}; \tag{1.42}$$

and

■ FOURIER'S LAW:

$$\mathbf{q} = -\kappa \nabla_x \vartheta. \tag{1.43}$$

The specific form of $\mathbb{S}$ can be deduced from the physical principle of the material frame indifference, see Chorin and Marsden [47] for details.

Writing

$$\mathbb{S} : \nabla_x \mathbf{u} = \frac{\mu}{2} \Big| \nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \Big|^2 + \eta |\mathrm{div}_x \mathbf{u}|^2,$$

we conclude, by virtue of (1.41), that the *shear viscosity coefficient* μ, the *bulk viscosity coefficient* η, as well as the *heat conductivity coefficient* κ must be non-negative. As our theory is primarily concerned with viscous and heat conducting fluids, we shall always assume that the shear viscosity coefficient μ as well as the heat conductivity coefficient κ are strictly positive. On the other hand, it is customary, at least for certain gases, to neglect the second term in (1.42) setting the bulk viscosity coefficient $\eta = 0$.

1.4.2 State equations

Gibbs' equation (1.2) relates the *thermal equation of state*

$$p = p(\varrho, \vartheta)$$

to the *caloric equation of state*

$$e = e(\varrho, \vartheta),$$

in particular, p and e must obey Maxwell's relation (1.3).

The mathematical theory of singular limits developed in this book leans essentially on

■ HYPOTHESIS OF THERMODYNAMIC STABILITY:

$$\frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0, \ \frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0. \tag{1.44}$$

The meaning of (1.44) is that both the *specific heat at constant volume* $c_v = \partial e/\partial\vartheta$ and the *compressibility* of the fluid $\partial p/\partial\varrho$ are positive although the latter condition is apparently violated by the standard Van der Waals equation of state.

In order to fix ideas, we focus on the simplest possible situation supposing the fluid is a monoatomic gas. In this case, it can be deduced by the methods of statistical physics that the molecular pressure $p = p_M$ and the associated internal energy $e = e_M$ are interrelated through

$$p_M(\varrho, \vartheta) = \frac{2}{3}\varrho e_M(\varrho, \vartheta) \tag{1.45}$$

(see Eliezer et al. [71]). It is a routine matter to check that (1.45) is compatible with (1.3) only if there is a function P such that

$$p_M(\varrho, \vartheta) = \vartheta^{5/2} P\Big(\frac{\varrho}{\vartheta^{3/2}}\Big). \tag{1.46}$$

Indeed inserting (1.45) into (1.3) gives rise to a first-order partial differential equation that can be solved by means of the change of variables $q(Z, \vartheta) = p(Z\vartheta^{3/2}, \vartheta)$.

If P is linear, we recover the standard *Boyle-Marriot state equation* of perfect gas,

$$p_M(\varrho, \vartheta) = \mathrm{R}\varrho\vartheta \quad \text{with a positive gas constant } R. \tag{1.47}$$

As a matter of fact, formula (1.46) applies to any real gas, monoatomic or not, at least in the following two domains of the (ϱ, ϑ)-plane:

- Non-Degenerate Region, where the density is low and/or the temperature is sufficiently large, specifically,

$$\frac{\varrho}{\vartheta^{\frac{3}{2}}} < \underline{Z} \tag{1.48}$$

  for a certain positive constant $\underline{Z}$. Here the fluid can be considered as a mixture of classical gases that obeys Dalton's law, hence the pressure p is given by the state equation (1.47) (see Galavotti [93]);

- Degenerate Area

$$\frac{\varrho}{\vartheta^{\frac{3}{2}}} > \overline{Z}, \text{ with } \overline{Z} \gg \underline{Z}, \tag{1.49}$$

  where the gas is completely ionized, and the nuclei as well as the free electrons behave like a monoatomic gas satisfying (1.46). If, in addition, we assume that in the degenerate area at least one of the gas constituents, for instance the cloud of free electrons, behaves as a Fermi gas, we obtain

$$\lim_{\vartheta \to 0} e_M(\varrho, \vartheta) > 0 \text{ for any fixed } \varrho > 0 \tag{1.50}$$

  (see Müller and Ruggeri [160]).

Finally, we suppose that the specific heat at constant volume is uniformly bounded, meaning

$$c_v = \frac{\partial e_M(\varrho,\vartheta)}{\partial\vartheta} \le c \text{ for all } \varrho,\ \vartheta > 0, \tag{1.51}$$

with obvious implications on the specific form of the function P in (1.46) discussed in detail in Chapter 2.

It is worth noting that, unlike (1.47), the previous assumptions are in perfect agreement with the *Third law of thermodynamics* requiring the entropy to vanish when the absolute temperature approaches zero (see Callen [40]).

1.4.3 Effect of thermal radiation

Before starting our discussion, let us point out that the interaction of matter and *radiation* (photon gas) occurring in the high temperature regime is a complex problem, a complete discussion of which goes beyond the scope of the present study. Here we restrict ourselves to the very special but still physically relevant situation, where the emitted photons are in thermal equilibrium with the other constituents of the fluid, in particular, the whole system admits a single temperature ϑ (see the monograph by Oxenius [169]).

Under these circumstances, it is well known that the heat conductivity is substantially enhanced by the radiation effect, in particular, the *heat conductivity coefficient* κ takes the form

$$\kappa = \kappa_M + \kappa_R,\ \kappa_R = k\vartheta^3,\ k > 0, \tag{1.52}$$

where κ_M denotes the standard "molecular" transport coefficient and κ_R represents the contribution due to radiation. The influence of the radiative transport is particularly relevant in some astrophysical models studied in the asymptotic limit of small Péclet (Prandtl) number in Chapter 6.

Similarly, the standard molecular pressure p_M is augmented by its radiation counterpart p_R so that, finally,

$$p(\varrho,\vartheta) = p_M(\varrho,\vartheta) + p_R(\vartheta),\ \text{where } p_R(\vartheta) = \frac{a}{3}\vartheta^4,\ a > 0; \tag{1.53}$$

whence, in accordance with Gibbs' equation (1.2),

$$e(\varrho,\vartheta) = e_M(\varrho,\vartheta) + e_R(\varrho,\vartheta),\ \text{where } \varrho e_R(\varrho,\vartheta) = a\vartheta^4, \tag{1.54}$$

and

$$s(\varrho,\vartheta) = s_M(\varrho,\vartheta) + s_R(\varrho,\vartheta),\ \text{with } \varrho s_R(\varrho,\vartheta) = \frac{4}{3}a\vartheta^3. \tag{1.55}$$

1.4.4 Typical values of some physical coefficients

In order to get better insight concerning the magnitude and proportionality of the different material forces acting on a fluid, we conclude this introductory part by reviewing the typical values of several physical constants introduced in the preceding text.

The quantity R appearing in formula (1.47) is the specific gas constant, the value of which for a gas (or a mixture of gases) equals $\overline{R}/M$, where $\overline{R}$ is the universal gas constant ($\overline{R} = 8.314 JK^{-1}\text{mol}^{-1}$), and M is the molar mass (or a weighted average of molar masses of the mixture components). For dry air, we get $R = 2.87 J\text{kg}^{-1}K^{-1}$.

In formulas (1.53–1.55), the symbol a stands for the Stefan-Boltzmann constant ($a = 5.67 \cdot 10^{-8} JK^{-4}m^{-2}s^{-1}$), while the coefficient k in formula (1.52) is related to a by $k = \frac{4}{3}alc$, where l denotes the mean free path of photons (typically $l \approx 10^{-7} - 10^{-8}m$), and c is the speed of light ($c = 3 \cdot 10^8 ms^{-1}$).

The specific heat at constant volume c_v takes the value $c_v = 2.87 J\text{kg}^{-1}K^{-1}$ for the dry air, in particular, $e_R \approx 1 J\text{kg}^{-1}$, $e_M \approx 10^2 - 10^3 J\text{kg}^{-1}$ at the atmospheric temperature, while at the temperature of order $10^3 K$ attained, for instance, in the solar radiative zone, $e_R \approx 10^3 - 10^4 J\text{kg}^{-1}$ and $e_M \approx 10^3 - 10^4 J\text{kg}^{-1}$. Accordingly, the effect of radiation is often negligible under the "normal" laboratory conditions on the Earth ($e_M/e_R \approx 10^2 - 10^3$) but becomes highly significant in the models of hot stars studied in astrophysics ($e_M/e_R \approx 10^{-1} - 10$). However, radiation plays an important role in certain meteorological models under specific circumstances.

The kinetic theory predicts the viscosity of gases to be proportional to $\sqrt{\vartheta}$ or a certain power of ϑ varying with the specific model and characteristic temperatures. This prediction is confirmed by experimental observations; a generally accepted formula is the so-called Shutherlang correlation yielding

$$\mu = \frac{A\sqrt{\vartheta}}{1 + B/\vartheta} \text{ for } \vartheta \text{ ``large''},$$

where A and B are experimentally determined constants. For the air in the range of pressures between $1 - 10$atm, we have $A = 1.46\text{kgm}^{-1}\text{s}^{-1}\text{K}^{-1/2}$, $B = 100.4$K. The dependence of the transport coefficients on the temperature plays a significant role in the mathematical theory developed in this book.

The specific values of physical constants presented in this part are taken over from Bolz and Tuve [26].

Chapter 2

Weak Solutions, A Priori Estimates

The fundamental laws of continuum mechanics interpreted as infinite families of integral identities introduced in Chapter 1, rather than systems of partial differential equations, give rise to the concept of weak (or variational) solutions that can be vastly extended to extremely divers physical systems of various sorts. The main stumbling block of this approach when applied to the field equations of fluid mechanics is the fact that the available *a priori estimates* are not strong enough in order to control the flux of the total energy and/or the dissipation rate of the kinetic energy. This difficulty has been known since the seminal work of Leray [132] on the incompressible Navier-Stokes system, where the validity of the so-called energy equality remains an open problem, even in the class of suitable weak solutions introduced by Caffarelli et al. [37]. The question is whether or not the rate of decay of the kinetic energy equals the dissipation rate due to viscosity as predicted by formula (1.39). It seems worth noting that certain weak solutions to *hyperbolic conservation laws* indeed dissipate the kinetic energy whereas classical solutions of the same problem, provided they exist, do not. On the other hand, however, we are still very far from complete understanding of possible singularities, if any, that may be developed by solutions to dissipative systems studied in fluid mechanics. The problem seems even more complex in the framework of compressible fluids, where Hoff [113] showed that singularities survive in the course of evolution provided they were present in the initial data. However, it is still not known if the density may develop "blow up" (gravitational collapse) or vanish (vacuum state) in a finite time. Quite recently, Brenner [28] proposed a daring new approach to fluid mechanics, where at least some of the above mentioned difficulties are likely to be eliminated.

Given the recent state of the art, we anticipate the hypothetical possibility that the weak solutions may indeed dissipate more kinetic energy than indicated by (1.33), thereby replacing the classical expression of the entropy production rate (1.39) by an *inequality*

$$\sigma \geq \frac{1}{\vartheta}\Big(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}}{\vartheta} \cdot \nabla_x \vartheta\Big). \tag{2.1}$$

Similarly to the theory of hyperbolic systems, the entropy production rate σ is now to be understood as a non-negative measure on the set $[0,T] \times \overline{\Omega}$, whereas the term

$$\int_0^T \int_\Omega \sigma\varphi \, \mathrm{d}x \text{ is replaced by } \langle \sigma; \varphi \rangle_{[\mathcal{M}^+; C]([0,T]\times\overline{\Omega})} \text{ in (1.40).}$$

Although it may seem that changing *equation* to mere *inequality* may considerably extend the class of possible solutions, it is easy to verify that inequality (2.1) reduces to the classical formula (1.39) as soon as the weak solution is regular and satisfies the global energy balance (1.36). By a regular solution we mean that all state variables ϱ, $\mathbf{u}$, ϑ are continuously differentiable up to the boundary of the space-time cylinder $[0,T] \times \overline{\Omega}$, possess all the necessary derivatives in $(0,T) \times \Omega$, and ϱ, ϑ are strictly positive. Indeed if ϑ is smooth we are allowed to use the quantity $\vartheta\varphi$ as a test function in (1.40) to obtain

$$\begin{aligned}
&\int_0^T \int_\Omega \varrho s \Big(\partial_t \vartheta + \mathbf{u} \cdot \nabla_x \vartheta\Big)\varphi \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_\Omega \varrho s \vartheta \Big(\partial_t \varphi + \mathbf{u} \cdot \nabla_x \varphi\Big) \, \mathrm{d}x \, \mathrm{d}t \\
&\qquad + \int_0^T \int_\Omega \mathbf{q} \cdot \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t + \langle \sigma; \vartheta\varphi \rangle + \int_0^T \int_\Omega \frac{\mathbf{q}}{\vartheta} \cdot \nabla_x \vartheta \varphi \, \mathrm{d}x \, \mathrm{d}t \\
&= - \int_0^T \int_\Omega \varrho \mathcal{Q} \varphi \, \mathrm{d}x \, \mathrm{d}t
\end{aligned}$$

for any $\varphi \in C_c^\infty((0,T) \times \overline{\Omega})$. Moreover, as ϱ, $\mathbf{u}$ satisfy the equation of continuity (1.22), we get

$$\begin{aligned}
&\int_0^T \int_\Omega \varrho s \Big(\partial_t \vartheta + \mathbf{u} \cdot \nabla_x \vartheta\Big)\varphi \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_\Omega \varrho s \vartheta \Big(\partial_t \varphi + \mathbf{u} \cdot \nabla_x \varphi\Big) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad = - \int_0^T \int_\Omega \varrho \vartheta \Big(\partial_t s + \mathbf{u} \cdot \nabla_x s\Big)\varphi \, \mathrm{d}x \, \mathrm{d}t \\
&\quad = - \int_0^T \int_\Omega \varrho \Big(\partial_t e + \mathbf{u} \cdot \nabla_x e\Big)\varphi \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \int_\Omega p \mathrm{div}_x \mathbf{u} \varphi \, \mathrm{d}x,
\end{aligned}$$

where we have used Gibbs' relation (1.2). Consequently, we deduce

$$\begin{aligned}
&\int_\Omega \varrho e(\varrho, \vartheta)(t_2) \, \mathrm{d}x - \int_\Omega \varrho e(\varrho, \vartheta)(t_1) \, \mathrm{d}x \\
&\quad = \int_{t_1}^{t_2} \int_\Omega \Big(\varrho \mathcal{Q} - p \mathrm{div}_x \mathbf{u}\Big) \, \mathrm{d}x \, \mathrm{d}t + \int_{t_1}^{t_2} \int_\Omega \Big(\vartheta\sigma + \frac{\mathbf{q}}{\vartheta} \cdot \nabla_x \vartheta\Big) \, \mathrm{d}x \, \mathrm{d}t
\end{aligned}$$

for $0 < t_1 \leq t_2 < T$.

Conversely, since regular solutions necessarily satisfy the kinetic energy equation (1.33), we can use the total energy balance (1.36) in order to conclude that

$$\int_\Omega \varrho e(\varrho,\vartheta)(t_2)\ \mathrm{d}x - \int_\Omega \varrho e(\varrho,\vartheta)(t_1)\ \mathrm{d}x = \int_{t_1}^{t_2} \int_\Omega \Big(\varrho \mathcal{Q} + \mathbb{S} : \nabla_x \mathbf{u} - p \mathrm{div}_x \mathbf{u} \Big)\ \mathrm{d}x\ \mathrm{d}t;$$

whence, by means of (2.1),

$$\sigma = \frac{1}{\vartheta} \Big(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}}{\vartheta} \cdot \nabla_x \vartheta \Big) \text{ in } [t_1, t_2] \times \overline{\Omega}.$$

Note that our approach based on postulating *inequality* (2.1), together with *equality* (1.36) is reminiscent of the concept of *weak solutions with defect measure* elaborated by DiPerna and Lions [64] and Alexandre and Villani [5] in the context of Boltzmann's equation. Although uniqueness in terms of the data is probably out of reach of such a theory, the piece of information provided is sufficient in order to study the qualitative properties of solutions, in particular, the long-time behavior and singular limits for several scaling parameters tending to zero. Starting from these ideas, we develop a thermodynamically consistent mathematical model based on the *state variables* $\{\varrho, \mathbf{u}, \vartheta\}$ and enjoying the following properties:

- The problem admits global-in-time solutions for any initial data of finite energy.
- The changes of the total energy of the system are only due to the action of the external source terms represented by $\mathbf{f}$ and $\mathcal{Q}$. In the absence of external sources, the total energy is a constant of motion.
- The total entropy is increasing in time as soon as $\mathcal{Q} \geq 0$, the system evolves to a state maximizing the entropy.
- Weak solutions coincide with classical ones provided they are smooth, notably the entropy production rate σ is equal to the expression on the right-hand side of (2.1).

2.1 Weak formulation

For reader's convenience and future use, let us summarize in a concise form the *weak formulation* of the problem identified in Chapter 1. The problem consists of finding a trio $\{\varrho, \mathbf{u}, \vartheta\}$ satisfying a family of integral identities referred to in the future as a Navier-Stokes-Fourier system. We also specify the minimal regularity of solutions required, and interpret formally the integral identities in terms of standard partial differential equations provided all quantities involved in the weak formulation are smooth enough.

2.1.1 Equation of continuity

(i) Weak (renormalized) formulation:

$$\int_0^T \int_\Omega \varrho B(\varrho)\Big(\partial_t \varphi + \mathbf{u}\cdot\nabla_x \varphi\Big)\ \mathrm{d}x\ \mathrm{d}t$$
$$= \int_0^T \int_\Omega b(\varrho)\mathrm{div}_x \mathbf{u}\varphi\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega \varrho_0 B(\varrho_0)\varphi(0,\cdot)\ \mathrm{d}x. \tag{2.2}$$

(ii) Admissible test functions:

$$b \in L^\infty \cap C[0,\infty),\ B(\varrho) = B(1) + \int_1^\varrho \frac{b(z)}{z^2}\ \mathrm{d}z, \tag{2.3}$$
$$\varphi \in C_c^1([0,T)\times\overline{\Omega}). \tag{2.4}$$

(iii) Minimal regularity of solutions required:

$$\varrho \ge 0,\ \varrho \in L^1((0,T)\times\Omega), \tag{2.5}$$
$$\varrho\mathbf{u} \in L^1((0,T)\times\Omega;\mathbb{R}^3),\ \mathrm{div}_x\mathbf{u} \in L^1((0,T)\times\Omega). \tag{2.6}$$

(iv) Formal interpretation:

$$\partial_t(\varrho B(\varrho)) + \mathrm{div}_x(\varrho B(\varrho)\mathbf{u}) + b(\varrho)\mathrm{div}_x\mathbf{u} = 0 \text{ in } (0,T)\times\Omega, \tag{2.7}$$
$$\varrho(0,\cdot) = \varrho_0,\ \mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0. \tag{2.8}$$

2.1.2 Balance of linear momentum

(i) Weak formulation:

$$\int_0^T \int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho[\mathbf{u}\otimes\mathbf{u}] : \nabla_x\varphi + p\mathrm{div}_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t$$
$$= \int_0^T \int_\Omega \Big(\mathbb{S} : \nabla_x\varphi - \varrho\mathbf{f}\cdot\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega (\varrho\mathbf{u})_0\cdot\varphi(0,\cdot)\ \mathrm{d}x. \tag{2.9}$$

(ii) Admissible test functions:

$$\varphi \in C_c^1([0,T)\times\overline{\Omega};\mathbb{R}^3), \tag{2.10}$$

and either

$$\varphi\cdot\mathbf{n}|_{\partial\Omega} = 0 \text{ in the case of the complete slip boundary conditions,} \tag{2.11}$$

or

$$\varphi|_{\partial\Omega} = 0 \text{ in the case of the no-slip boundary conditions.} \tag{2.12}$$

(iii) Minimal regularity of solutions required:

$$\varrho\mathbf{u} \in L^1((0,T)\times\Omega;\mathbb{R}^3),\ \ \varrho|\mathbf{u}|^2 \in L^1((0,T)\times\Omega), \tag{2.13}$$

$$p \in L^1((0,T)\times\Omega),\ \mathbb{S} \in L^1((0,T)\times\Omega;\mathbb{R}^{3\times 3}),\ \varrho\mathbf{f} \in L^1((0,T)\times\Omega;\mathbb{R}^3), \tag{2.14}$$

$$\nabla_x\mathbf{u} \in L^1(0,T;L^q(\Omega;\mathbb{R}^{3\times 3})),\ \text{for a certain } q>1; \tag{2.15}$$

and, either

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0 \text{ in the case of the complete slip boundary conditions,} \tag{2.16}$$

or

$$\mathbf{u}|_{\partial\Omega} = 0 \text{ in the case of the no-slip boundary conditions.} \tag{2.17}$$

(iv) Formal interpretation:

$$\partial_t(\varrho\mathbf{u}) + \operatorname{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x p = \operatorname{div}_x\mathbb{S} + \varrho\mathbf{f} \text{ in } (0,T)\times\Omega, \tag{2.18}$$

$$(\varrho\mathbf{u})(0,\cdot) = (\varrho\mathbf{u})_0, \tag{2.19}$$

together with the complete slip boundary conditions

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0,\ \ (\mathbb{S}\mathbf{n})\times\mathbf{n}|_{\partial\Omega} = 0, \tag{2.20}$$

or, alternatively, the no-slip boundary condition

$$\mathbf{u}|_{\partial\Omega} = 0. \tag{2.21}$$

2.1.3 Balance of total energy

(i) Weak formulation:

$$\int_0^T\int_\Omega \mathcal{E}(t)\ \mathrm{d}x\ \partial_t\psi(t)\ \mathrm{d}t = -\int_0^T\int_\Omega \Big(\varrho\mathbf{u}\cdot\mathbf{f}(t) + \varrho\mathcal{Q}(t)\Big)\psi(t)\ \mathrm{d}x\ \mathrm{d}t - \psi(0)E_0 \tag{2.22}$$

$$\mathcal{E}(t) = \frac{1}{2}\varrho|\mathbf{u}|^2(t) + \varrho e(t) \text{ for a.a. } t\in(0,T). \tag{2.23}$$

(ii) Admissible test functions:

$$\psi \in C^1_c[0,T). \tag{2.24}$$

(iii) Minimal regularity of solutions required:

$$\mathcal{E},\ \ \varrho\mathbf{u}\cdot\mathbf{f},\ \ \varrho\mathcal{Q} \in L^1((0,T)\times\Omega). \tag{2.25}$$

(iv) Formal interpretation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathcal{E}\ \mathrm{d}x = \int_\Omega \Big(\varrho\mathbf{u}\cdot\mathbf{f} + \varrho\mathcal{Q}\Big)\ \mathrm{d}x \text{ in } (0,T),\ \int_\Omega \mathcal{E}(0)\ \mathrm{d}x = E_0. \tag{2.26}$$

2.1.4 Entropy production

(i) Weak formulation:

$$\int_0^T \int_\Omega \varrho s \Big(\partial_t \varphi + \mathbf{u} \cdot \nabla_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t + \int_0^T \int_\Omega \frac{\mathbf{q}}{\vartheta} \cdot \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t + \langle \sigma ; \varphi \rangle_{[\mathcal{M}^+; C]([0,T]\times\overline{\Omega})}$$
$$= - \int_\Omega (\varrho s)_0 \varphi(0, \cdot) \ \mathrm{d}x - \int_0^T \int_\Omega \frac{\varrho}{\vartheta} \mathcal{Q} \varphi \ \mathrm{d}x \ \mathrm{d}t, \tag{2.27}$$

where $\sigma \in \mathcal{M}^+([0,T] \times \overline{\Omega})$,

$$\sigma \geq \frac{1}{\vartheta} \Big(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}}{\vartheta} \cdot \nabla_x \vartheta \Big). \tag{2.28}$$

(ii) Admissible test functions

$$\varphi \in C^1_c([0,T) \times \overline{\Omega}). \tag{2.29}$$

(iii) Minimal regularity of solutions required:

$$\begin{gathered} \vartheta > 0 \text{ a.a. on } (0,T) \times \Omega, \vartheta \in L^q((0,T) \times \Omega), \\ \nabla_x \vartheta \in L^q((0,T) \times \Omega; \mathbb{R}^3), \ q > 1, \end{gathered} \tag{2.30}$$

$$\begin{gathered} \varrho s \in L^1((0,T) \times \Omega), \ \varrho s \mathbf{u}, \ \frac{\mathbf{q}}{\vartheta} \in L^1((0,T) \times \Omega; \mathbb{R}^3), \\ \frac{\varrho}{\vartheta} \mathcal{Q} \in L^1((0,T) \times \Omega), \end{gathered} \tag{2.31}$$

$$\frac{1}{\vartheta} \mathbb{S} : \nabla_x \mathbf{u}, \ \frac{1}{\vartheta^2} \mathbf{q} \cdot \nabla_x \vartheta \in L^1((0,T) \times \Omega). \tag{2.32}$$

(iv) Formal interpretation:

$$\partial_t (\varrho s) + \mathrm{div}_x (\varrho s \mathbf{u}) + \mathrm{div}_x \Big(\frac{\mathbf{q}}{\vartheta} \Big)$$
$$\geq \frac{1}{\vartheta} \Big(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}}{\vartheta} \cdot \nabla_x \vartheta \Big) + \frac{\varrho}{\vartheta} \mathcal{Q} \text{ in } (0,T) \times \Omega, \tag{2.33}$$

$$\varrho s(0+, \cdot) \geq (\varrho s)_0, \ \mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} \leq 0. \tag{2.34}$$

2.1.5 Constitutive relations

(i) Gibbs' equation:

$$p = p(\varrho, \vartheta), \ e = e(\varrho, \vartheta), \ s = s(\varrho, \vartheta) \text{ a.a. in } (0,T) \times \Omega,$$

where

$$\vartheta D s = D e + p D \Big(\frac{1}{\varrho} \Big). \tag{2.35}$$

(ii) Newton's law:

$$\mathbb{S} = \mu\Big(\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\ \mathbb{I}\Big) + \eta \mathrm{div}_x \mathbf{u}\ \mathbb{I} \text{ a.a. in } (0,T)\times\Omega, \tag{2.36}$$

(iii) Fourier's law:

$$\mathbf{q} = -\kappa \nabla_x \vartheta \text{ a.a. in } (0,T)\times\Omega. \tag{2.37}$$

2.2 A priori estimates

A priori estimates represent a corner stone of any mathematical theory related to a system of nonlinear partial differential equations. The remarkable informal rule asserts that "if we can establish *sufficiently strong* estimates for solutions of a nonlinear partial differential equation under the assumption that such a solution exists, then the solution does exist". *A priori* estimates are natural bounds imposed on the family of all admissible solutions through the system of equations they obey, the boundary conditions, and the given data. The modern theory of partial differential equations is based on function spaces, notably the Sobolev spaces, that have been identified by means of the corresponding *a priori* bounds for certain classes of elliptic equations.

Strictly speaking, *a priori* estimates are *formal*, being derived under the hypothesis that all quantities in question are smooth. However, as we shall see below, all bounds obtained for the NAVIER-STOKES-FOURIER SYSTEM hold even within the class of the weak solutions introduced in Section 2.1. This is due to the fact that all nowadays available *a priori* estimates follow from the physical principle of conservation of the total amount of certain quantities as mass and total energy, or they result from the dissipative mechanism enforced by means of the *Second law of thermodynamics*.

2.2.1 Total mass conservation

Taking $b \equiv 0$, $B = B(1) = 1$ in the renormalized equation of continuity (2.2) we deduce that

$$\int_\Omega \varrho(t,\cdot)\ \mathrm{d}x = \int_\Omega \varrho_0\ \mathrm{d}x = M_0 \text{ for a.a. } t \in (0,T), \tag{2.38}$$

more specifically, for any $t \in (0,T)$ which is a Lebesgue point of the vector-valued mapping $t \mapsto \varrho(t,\cdot) \in L^1(\Omega)$. As a matter of fact, in accordance with the property of weak continuity in time of solutions to abstract balance laws discussed in Section 1.2, relation (2.38) holds for any $t \in [0,T]$ provided ϱ was redefined on a set of times of zero measure . Formula (2.38) rigorously confirms the intuitively obvious fact that the total mass M_0 of the fluid contained in a physical domain Ω is a constant of motion provided the normal component of the velocity field $\mathbf{u}$ vanishes on the boundary $\partial\Omega$.

2.2.2 Energy estimates

The balance of total energy expressed through (2.22) provides another sample of *a priori* estimates. Indeed assuming, for simplicity, that both $\mathbf{f}$ and $\mathcal{Q}$ are uniformly bounded we get

$$\Big| \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u} + \varrho \mathcal{Q} \ \mathrm{d}x \Big|$$
$$\leq \|\mathbf{f}\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^3)} \sqrt{M_0} \|\sqrt{\varrho}\mathbf{u}\|_{L^2(\Omega;\mathbb{R}^3)} + M_0 \|\mathcal{Q}\|_{L^\infty((0,T)\times\Omega)};$$

whence a straightforward application of Gronwall's lemma to (2.22) gives rise to

$$\operatorname{ess} \sup_{t\in(0,T)} \int_\Omega \Big(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) \Big)(t) \ \mathrm{d}x$$
$$\leq c\Big(T, E_0, M_0, \|\mathbf{f}\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^3)}, \|\mathcal{Q}\|_{L^\infty((0,T)\times\Omega)}\Big). \tag{2.39}$$

In particular,

$$\operatorname{ess} \sup_{t\in(0,T)} \int_\Omega \varrho |\mathbf{u}|^2(t) \ \mathrm{d}x \leq c(\text{data}), \tag{2.40}$$

where the symbol $c(\text{data})$ denotes a generic positive constant depending solely on the *data*

$$T, E_0, M_0, \|\mathbf{f}\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^3)}, \|\mathcal{Q}\|_{L^\infty((0,T)\times\Omega)}, \text{ and } S_0 = \int_\Omega (\varrho s)_0 \ \mathrm{d}x. \tag{2.41}$$

In order to get more information, we have to exploit the specific structure of the internal energy function e. In accordance with hypotheses (1.44), (1.50), (1.54), we have

$$\varrho e(\varrho,\vartheta) \geq a\vartheta^4 + \varrho \lim_{\vartheta\to 0} e_M(\varrho,\vartheta). \tag{2.42}$$

On the other hand, the molecular component e_M is given through (1.45), (1.46) in the degenerate area $\varrho > \overline{Z}\vartheta^{3/2}$, therefore

$$\lim_{\vartheta\to 0} e_M(\varrho,\vartheta) = \frac{3\varrho^{\frac{2}{3}}}{2} \lim_{\vartheta\to 0} \frac{\vartheta^{\frac{5}{2}}}{\varrho^{\frac{5}{3}}} P\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big) = \frac{3\varrho^{\frac{2}{3}}}{2} \lim_{Z\to\infty} \frac{P(Z)}{Z^{\frac{5}{3}}}, \tag{2.43}$$

where, in accordance with (1.50),

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{\frac{5}{3}}} = p_\infty > 0. \tag{2.44}$$

Consequently, going back to (2.42) we conclude

$$\varrho e(\varrho,\vartheta) \geq a\vartheta^4 + \frac{3p_\infty}{2} \varrho^{\frac{5}{3}}, \tag{2.45}$$

in particular, it follows from (2.39) that

$$\text{ess}\sup_{t\in(0,T)}\int_\Omega \left(\vartheta^4+\varrho^{\frac{5}{3}}\right)(t)\ \mathrm{d}x \le c(\text{data}). \tag{2.46}$$

It is important to note that estimate (2.46) yields a uniform bound on the pressure $p = p_M + p_R$. Indeed the pressure is obviously bounded in the degenerate area (1.49), where p_M satisfies (1.45) and the appropriate bound is provided by (2.39). Otherwise, using the hypothesis of thermodynamic stability (1.44), we obtain

$$0 \le p_M(\varrho,\vartheta) \le p_M(\overline{Z}\vartheta^{\frac{3}{2}},\vartheta) = \vartheta^{\frac{5}{2}}P(\overline{Z});$$

whence the desired bound follows from (2.46) as soon as Ω is bounded. Consequently, we have shown that the energy estimate (2.39) gives rise to

$$\text{ess}\sup_{t\in(0,T)}\int_\Omega p(\varrho,\vartheta)(t)\ \mathrm{d}x \le c(\text{data}) \tag{2.47}$$

at least for a bounded domain Ω.

2.2.3 Estimates based on the Second law of thermodynamics

The *Second law of thermodynamics* asserts the irreversible transfer of the mechanical energy into heat valid for all physical systems. This can be expressed mathematically by means of the entropy production equation (2.27). In order to utilize this relation for obtaining *a priori* bounds, we introduce a remarkable quantity which will play a crucial role not only in the existence theory but also in the study of singular limits.

■ HELMHOLTZ FUNCTION:

$$H_{\overline{\vartheta}}(\varrho,\vartheta) = \varrho\left(e(\varrho,\vartheta) - \overline{\vartheta}s(\varrho,\vartheta)\right), \tag{2.48}$$

where $\overline{\vartheta}$ is a positive constant.

Obviously, the quantity $H_{\overline{\vartheta}}$ is reminiscent of the *Helmholtz free energy* albeit in the latter $\overline{\vartheta}$ must be replaced by ϑ.

It follows from Gibbs' relation (2.35) that

$$\frac{\partial^2 H_{\overline{\vartheta}}(\varrho,\overline{\vartheta})}{\partial\varrho^2} = \frac{1}{\varrho}\frac{\partial p(\varrho,\overline{\vartheta})}{\partial\varrho} = \frac{1}{\varrho}\frac{\partial p_M(\varrho,\overline{\vartheta})}{\partial\varrho}, \tag{2.49}$$

while

$$\frac{\partial H_{\overline{\vartheta}}(\varrho,\vartheta)}{\partial\vartheta} = \frac{\varrho}{\vartheta}(\vartheta-\overline{\vartheta})\frac{\partial e(\varrho,\vartheta)}{\partial\vartheta} = 4a\vartheta^2(\vartheta-\overline{\vartheta}) + \frac{\varrho}{\vartheta}(\vartheta-\overline{\vartheta})\frac{\partial e_M(\varrho,\vartheta)}{\partial\vartheta}. \tag{2.50}$$

Thus, as a direct consequence of the hypothesis of thermodynamic stability (1.44), we thereby infer that

- $\varrho \mapsto H_{\overline{\vartheta}}(\varrho, \overline{\vartheta})$ is a strictly convex function, which, being augmented by a suitable affine function of ϱ, attains its global minimum at some positive $\overline{\varrho}$,
- the function $\vartheta \mapsto H_{\overline{\vartheta}}(\varrho, \vartheta)$ is decreasing for $\vartheta < \overline{\vartheta}$ and increasing for $\vartheta > \overline{\vartheta}$, in particular, it attains its (global) minimum at $\vartheta = \overline{\vartheta}$ for any fixed ϱ.

The total energy balance (2.22), together with the entropy production equation (2.27), gives rise to

$$\int_\Omega \Big(\frac{1}{2} \varrho |\mathbf{u}|^2 + H_{\overline{\vartheta}}(\varrho, \vartheta) \Big)(\tau) \ \mathrm{d}x + \overline{\vartheta} \sigma \Big[[0, \tau] \times \overline{\Omega} \Big]$$
$$= E_0 - \overline{\vartheta} S_0 + \int_0^\tau \int_\Omega \Big[\varrho \Big(\mathcal{Q} - \frac{\overline{\vartheta}}{\vartheta} \mathcal{Q} \Big) + \varrho \mathbf{f} \cdot \mathbf{u} \Big] \ \mathrm{d}x \ \mathrm{d}t \tag{2.51}$$

for a.a. $\tau \in (0, T)$, where we have introduced the symbol $\sigma[Q]$ to denote the value of the measure σ applied to a Borel set Q.

Now suppose there exists a positive number $\overline{\varrho} > 0$ such that

$$\int_\Omega (\varrho - \overline{\varrho})(t) \ \mathrm{d}x = 0 \text{ for any } t \in [0, T].$$

Clearly, if Ω is a bounded domain, we have $\overline{\varrho} = M_0 / |\Omega|$, where M_0 is the total mass of the fluid. Accordingly, relation (2.51) can be rewritten as

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■ Total Dissipation Balance:

$$\int_\Omega \Big(\frac{1}{2} \varrho |\mathbf{u}|^2 + H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho}) \frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \Big)(\tau) \ \mathrm{d}x + \overline{\vartheta} \sigma \Big[[0, \tau] \times \overline{\Omega} \Big]$$
$$= E_0 - \overline{\vartheta} S_0 - \int_\Omega \Big((\varrho_0 - \overline{\varrho}) \frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} + H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \Big) \ \mathrm{d}x$$
$$+ \int_0^\tau \int_\Omega \Big(\varrho \Big(\mathcal{Q} - \frac{\overline{\vartheta}}{\vartheta} \mathcal{Q} \Big) + \varrho \mathbf{f} \cdot \mathbf{u} \Big) \ \mathrm{d}x \ \mathrm{d}t \tag{2.52}$$

for a.a. $\tau \in (0, T)$.

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at least if Ω is a bounded domain. In contrast with (2.51), the quantity $H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho}) \frac{\partial H_{\overline{\vartheta}}}{\partial \varrho}(\overline{\varrho}, \overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})$ at the left-hand side is obviously non-negative as a direct consequence of the hypothesis of thermodynamic stability.

Consequently, assuming $\mathcal{Q} \geq 0$, we can use (2.28), together with (2.52), in order to obtain

$$\int_0^T \int_\Omega \frac{1}{\vartheta} \Big(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \Big) \ \mathrm{d}x \leq c(\text{data}). \tag{2.53}$$

As the transport terms $\mathbb{S}$, $\mathbf{q}$ are given by (1.42), (1.43), notably they are linear functions of the affinities $\nabla_x \mathbf{u}$, $\nabla_x \vartheta$, respectively, we get

$$\int_0^T \int_\Omega \frac{\mu}{\vartheta} \Big| \nabla_x \mathbf{u} + \nabla_x^\perp \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \, \mathbb{I} \Big|^2 \, \mathrm{d}x \, \mathrm{d}t \leq c(\mathrm{data}), \tag{2.54}$$

and

$$\int_0^T \int_\Omega \frac{\kappa}{\vartheta^2} |\nabla_x \vartheta|^2 \, \mathrm{d}x \, \mathrm{d}t \leq c(\mathrm{data}). \tag{2.55}$$

In order to continue, we have to specify the structural properties to be imposed on the transport coefficients μ and κ. In view of (1.52), it seems reasonable to assume that the heat conductivity coefficient $\kappa = \kappa_M + \kappa_R$ satisfies

$$\begin{gathered} 0 < \underline{\kappa}_M (1 + \vartheta^\alpha) \leq \kappa_M(\vartheta) \leq \overline{\kappa}_M (1 + \vartheta^\alpha), \\ 0 < \underline{\kappa}_R \vartheta^3 \leq \kappa_R(\vartheta) \leq \overline{\kappa}_R (1 + \vartheta^3), \end{gathered} \tag{2.56}$$

where $\underline{\kappa}_M$, $\overline{\kappa}_M$, $\underline{\kappa}_R$, $\overline{\kappa}_R$ are positive constants.

Similarly, the shear viscosity coefficient μ obeys

$$0 < \underline{\mu}(1 + \vartheta^\alpha) \leq \mu(\vartheta) \leq \overline{\mu}(1 + \vartheta^\alpha) \tag{2.57}$$

for any $\vartheta \geq 0$, positive constants $\underline{\mu}$, $\overline{\mu}$, and a positive exponent α specified below. Note that κ_M, μ are not allowed to depend explicitly on ϱ – a hypothesis that is crucial in existence theory but entirely irrelevant in the study of singular limits. We remark that such a stipulation is physically relevant at least for gases (see Becker [20]) and certain liquids.

Keeping (2.56) in mind we deduce from (2.55) that

$$\int_0^T \int_\Omega \Big(|\nabla_x \log(\vartheta)|^2 + |\nabla_x \vartheta^{\frac{3}{2}}|^2 \Big) \, \mathrm{d}x \, \mathrm{d}t \leq c(\mathrm{data}). \tag{2.58}$$

Combining (2.58) with (2.46) we conclude that the temperature $\vartheta(t, \cdot)$ belongs to $W^{1,2}(\Omega)$ for a.a. $t \in (0, T)$, where the symbol $W^{1,2}(\Omega)$ stands for the *Sobolev space* of functions belonging with their gradients to the Lebesgue space $L^2(\Omega)$ (cf. the relevant part in Section 0.3). More specifically, we have, by the standard Poincaré's inequality (Theorem 10.14),

$$\| \vartheta^\beta \|_{L^2(0,T;W^{1,2}(\Omega))} \leq c(\mathrm{data}) \text{ for any } 1 \leq \beta \leq \frac{3}{2}. \tag{2.59}$$

A similar estimate for $\log(\vartheta)$ is more delicate and is postponed to the next section.

From estimate (2.54) and Hölder's inequality we get

$$\left\| \nabla_x \mathbf{u} + \nabla_x^\perp \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \, \mathbb{I} \right\|_{L^p(\Omega;\mathbb{R}^{3\times 3})}$$

$$\leq \left\| \sqrt{\frac{\vartheta}{\mu(\vartheta)}} \right\|_{L^q(\Omega)} \left\| \sqrt{\frac{\mu(\vartheta)}{\vartheta}} \Big(\nabla_x \mathbf{u} + \nabla_x^\perp \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \, \mathbb{I} \Big) \right\|_{L^2(\Omega;\mathbb{R}^{3\times 3})}$$

$$\leq c \| (1 + \vartheta^{\frac{1-\alpha}{2}}) \|_{L^q(\Omega)} \left\| \sqrt{\frac{\mu(\vartheta)}{\vartheta}} \Big(\nabla_x \mathbf{u} + \nabla_x^\perp \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \, \mathbb{I} \Big) \right\|_{L^2(\Omega;\mathbb{R}^{3\times 3})}$$

provided

$$\frac{1}{p} = \frac{1}{q} + \frac{1}{2}.$$

Thus we deduce from estimates (2.46), (2.54) that

$$\left\| \nabla_x \mathbf{u} + \nabla_x^\perp \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \, \mathbb{I} \right\|_{L^2(0,T;L^p(\Omega;\mathbb{R}^{3\times 3}))} \leq c(\text{data}) \tag{2.60}$$

for

$$p = \frac{8}{5-\alpha}, \ 0 \leq \alpha \leq 1. \tag{2.61}$$

Similarly, in accordance with (2.59) and the standard embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ (see Theorem 0.4), we have

$$\| \ \vartheta \ \|_{L^3(0,T;L^9(\Omega))} \leq c(\text{data}); \tag{2.62}$$

whence, following the arguments leading to (2.60),

$$\left\| \nabla_x \mathbf{u} + \nabla_x^\perp \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \, \mathbb{I} \right\|_{L^q(0,T;L^p(\Omega;\mathbb{R}^{3\times 3}))} \leq c(\text{data}) \tag{2.63}$$

for

$$q = \frac{6}{4-\alpha}, \ p = \frac{18}{10-\alpha}, \ 0 \leq \alpha \leq 1. \tag{2.64}$$

As we will see below, the range of suitable values of the parameter α in (2.61), (2.62) is subjected to further restrictions.

The previous estimates concern only certain components of the velocity gradient. In order to get uniform bounds on $\nabla_x \mathbf{u}$, we need the following version of *Korn's inequality* proved in Theorem 10.17 in the Appendix.

■ GENERALIZED KORN-POINCARÉ INEQUALITY:

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Assume that r is a non-negative function such that*

$$0 < M_0 \leq \int_\Omega r \ \mathrm{d}x, \ \int_\Omega r^\gamma \ \mathrm{d}x \leq K \ \ \textit{for a certain} \ \ \gamma > 1.$$

Then

$$\|\mathbf{v}\|_{W^{1,p}(\Omega;\mathbb{R}^3)} \leq c(p, M_0, K)\Big(\Big\|\nabla_x \mathbf{v} + \nabla_x^\perp \mathbf{v} - \frac{2}{3}\mathrm{div}_x \mathbf{v}\ \mathbb{I}\Big\|_{L^p(\Omega;\mathbb{R}^3)} + \int_\Omega r|\mathbf{v}|\ \mathrm{d}x\Big)$$

for any $\mathbf{v} \in W^{1,p}(\Omega;\mathbb{R}^3)$, $1 < p < \infty$.

Applying Proposition 2.1 with $r = \varrho$, $\gamma = \frac{5}{3}$, $\mathbf{v} = \mathbf{u}$, we can use estimates (2.40), (2.46), (2.60), and (2.63) to conclude that

$$\| \mathbf{u} \|_{L^2(0,T;W^{1,p}(\Omega;\mathbb{R}^3))} \leq c(\text{data}) \text{ for } p = \frac{8}{5-\alpha}, \tag{2.65}$$

and

$$\| \mathbf{u} \|_{L^q(0,T;W^{1,p}(\Omega;\mathbb{R}^3))} \leq c(\text{data}) \text{ for } q = \frac{6}{4-\alpha},\ p = \frac{18}{10-\alpha}. \tag{2.66}$$

Estimates (2.65), (2.66) imply uniform bounds on the viscous stress tensor $\mathbb{S}$. To see this, write

$$\mu(\vartheta)\Big(\nabla_x \mathbf{u} + \nabla_x^\perp \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\ \mathbb{I}\Big) = \sqrt{\vartheta\mu(\vartheta)}\sqrt{\frac{\mu(\vartheta)}{\vartheta}}\Big(\nabla_x \mathbf{u} + \nabla_x^\perp \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\ \mathbb{I}\Big),$$

where $\sqrt{\frac{\mu(\vartheta)}{\vartheta}}\Big(\nabla_x \mathbf{u} + \nabla_x^\perp \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\ \mathbb{I}\Big)$ admits the bound established in (2.54). On the other hand, in view of estimates (2.46), (2.62), ϑ is bounded in $L^{\frac{17}{3}}((0,T)\times\Omega)$. This fact combined with hypothesis (2.57) yields boundedness of $\sqrt{\vartheta\mu(\vartheta)}$ in $L^p((0,T)\times\Omega)$ for a certain $p > 2$. Assuming the bulk viscosity η satisfies

$$0 \leq \eta(\vartheta) \leq c(1 + \vartheta^\alpha), \tag{2.67}$$

with the same exponent α as in (2.57), we obtain

$$\| \mathbb{S} \|_{L^q(0,T;L^q(\Omega;\mathbb{R}^{3\times 3}))} \leq c(\text{data}) \text{ for a certain } q > 1. \tag{2.68}$$

In a similar way, we can deduce estimates on the linear momentum and the kinetic energy. By virtue of the standard embedding relation $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $q \leq 3p/(3-p)$ (Theorem 0.4), we get

$$\| \mathbf{u} \|_{L^2(0,T;L^{\frac{24}{7-3\alpha}}(\Omega;\mathbb{R}^3))} + \| \mathbf{u} \|_{L^{\frac{6}{4-\alpha}}(0,T;L^{\frac{18}{4-\alpha}}(\Omega;\mathbb{R}^3))} \leq c(\text{data}), \tag{2.69}$$

see (2.65), (2.66). On the other hand, by virtue of (2.40), (2.46),

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\varrho\mathbf{u}\|_{L^{\frac{5}{4}}(\Omega;\mathbb{R}^3)} \leq c(\text{data}). \tag{2.70}$$

Combining the last two estimates, we get

$$\| \varrho\mathbf{u}\otimes\mathbf{u} \|_{L^q((0,T)\times\Omega;\mathbb{R}^{3\times 3}))} \leq c(\text{data}) \text{ for a certain } q > 1, \tag{2.71}$$

provided

$$\alpha > \frac{2}{5}. \tag{2.72}$$

It is worth noting that (2.72) allows for the physically relevant exponent $\alpha = 1/2$ (cf. Section 1.4.4).

2.2.4 Positivity of the absolute temperature

Our goal is to exploit estimate (2.58) in order to show

$$\int_0^T \int_\Omega \Big(|\log \vartheta|^2 + |\nabla_x \log \vartheta|^2 \Big) \, \mathrm{d}x \, \mathrm{d}t \leq c(\mathrm{data}). \tag{2.73}$$

Formula (2.73) not only facilitates future analysis but is also physically relevant as it implies positivity of the absolute temperature with a possible exception of a set of Lebesgue measure zero.

In order to establish (2.73), we introduce the following version of *Poincaré's inequality* proved in Theorem 10.14 in the Appendix.

■ POINCARÉ'S INEQUALITY:

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Let $V \subset \Omega$ be a measurable set such that*

$$|V| \geq V_0 > 0.$$

Then there exists a positive constant $c = c(V_0)$ such that

$$\| \, v \, \|_{W^{1,2}(\Omega)} \leq c(V_0) \Big(\|\nabla_x v\|_{L^2(\Omega;\mathbb{R}^3)} + \int_V |v| \mathrm{d}x \Big)$$

for any $v \in W^{1,2}(\Omega)$.

In view of Proposition 2.2 the desired relation (2.73) will follow from (2.58) as soon as we show that the temperature ϑ cannot vanish identically in the physical domain Ω. As the hypothetical state of a system with zero temperature minimizes the entropy, it is natural to evoke the *Second law of thermodynamics* expressed in terms of the entropy balance (2.27).

The total entropy of the system $\int_\Omega \varrho s(\varrho, \vartheta) \, \mathrm{d}x$ is a non-decreasing function of time provided the heat source $\mathcal{Q}$ is non-negative. In particular,

$$\int_\Omega \varrho s(\varrho, \vartheta)(t, \cdot) \, \mathrm{d}x \geq \int_\Omega (\varrho s)_0 \, \mathrm{d}x \text{ for a.a. } t \in (0, T), \tag{2.74}$$

where we assume that the initial distribution of the entropy is compatible with that for the density, that means, $(\varrho s)_0 = \varrho_0 s(\varrho_0, \vartheta_0)$ for a suitable initial temperature distribution ϑ_0.

If $\varrho \geq \overline{Z}\vartheta^{\frac{3}{2}}$, meaning if (ϱ, ϑ) belong to the degenerate region introduced in (1.49), the pressure p and the internal energy e are interrelated through (1.45), (1.46). Then it is easy to check, by means of Gibbs' equation (2.35), that the specific entropy s can be written in the form $s = s_M + s_R$, where

$$s_M(\varrho, \vartheta) = S(Z),\ Z = \frac{\varrho}{\vartheta^{\frac{3}{2}}},\ S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2},\ Z \geq \overline{Z}. \tag{2.75}$$

The quantity

$$\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z}$$

plays a role of the specific heat at constant volume and is strictly positive in accordance with the hypothesis of thermodynamic stability (1.44). In particular, we can set

$$s_\infty = \lim_{Z \to \infty} S(Z) = \lim_{\vartheta \to 0} s_M(\varrho, \vartheta) \geq -\infty \text{ for any fixed } \varrho > 0. \tag{2.76}$$

Moreover, modifying S by a suitable additive constant, we can assume $s_\infty = 0$ in the case when the limit is finite.

In order to proceed we need the following assertion that may be of independent interest. The claim is that the absolute temperature ϑ must remain strictly positive at least on a set of positive measure.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Assume that non-negative functions ϱ, ϑ satisfy*

$$0 < M_0 = \int_\Omega \varrho \ \mathrm{d}x,\ \int_\Omega \left(\vartheta^4 + \varrho^{\frac{5}{3}}\right) \ \mathrm{d}x \leq K,$$

and

$$\int_\Omega \varrho s(\varrho, \vartheta) \ \mathrm{d}x \geq S_0 > M_0 s_\infty \text{ for a certain } S_0, \tag{2.77}$$

where $s_\infty \in \{0, -\infty\}$ is determined by (2.76).

Then there are $\underline{\vartheta} > 0$ and $V_0 > 0$, depending only on M_0, K, and S_0 such that

$$\left|\left\{x \in \Omega \ \middle| \ \vartheta(x) > \underline{\vartheta}\right\}\right| \geq V_0.$$

Proof. Arguing by contradiction we construct a sequence ϱ_n, ϑ_n satisfying (2.77) and such that

$$\begin{gathered} \varrho_n \to \varrho \text{ weakly in } L^{\frac{5}{3}}(\Omega),\ \textstyle\int_\Omega \varrho \ \mathrm{d}x = M_0, \\ \left|\left\{x \in \Omega \,|\, \vartheta_n > \frac{1}{n}\right\}\right| < \frac{1}{n}. \end{gathered} \tag{2.78}$$

In particular,

$$\begin{gathered} \vartheta_n \to 0 \text{ (strongly) in } L^p(\Omega) \text{ for any } 1 \leq p < 4, \\ \varrho_n s_R(\varrho_n, \vartheta_n) = \frac{4}{3} a \vartheta_n^3 \to 0 \text{ in } L^1(\Omega). \end{gathered} \tag{2.79}$$

Next we claim that

$$\limsup_{n\to\infty} \int_{\{\varrho_n \leq \overline{Z}\vartheta_n^{\frac{3}{2}}\}} \varrho_n s_M(\varrho_n, \vartheta_n) \ \mathrm{d}x \leq 0. \tag{2.80}$$

In order to see (2.80), we first observe that the specific (molecular) entropy s_M is increasing in ϑ; whence

$$s_M(\varrho,\vartheta) \leq \begin{cases} s_M(\varrho,1) \text{ if } \vartheta < 1, \\ s_M(\varrho,1) + \int_1^\vartheta \frac{\partial s_M(\varrho,z)}{\partial z} \ \mathrm{d}z \leq s_M(\varrho,1) + c\log\vartheta \text{ for } \vartheta \geq 1, \end{cases}$$

where we have used hypothesis (1.51). On the other hand, it follows from Gibbs' equation (2.35) that

$$\frac{\partial s_M(\varrho,\vartheta)}{\partial \varrho} = -\frac{1}{\varrho^2}\frac{\partial p_M(\varrho,\vartheta)}{\partial \vartheta};$$

whence

$$|s_M(\varrho,1)| \leq c(\overline{Z})(1 + |\log(\varrho)|) \text{ for all } \varrho \leq \overline{Z}.$$

Resuming the above inequalities yields

$$|s_M(\varrho,\vartheta)| \leq c(1 + |\log(\varrho)| + |\log(\vartheta)|). \tag{2.81}$$

Returning to (2.80) we get

$$\begin{aligned} \int_{\{\varrho_n \leq \overline{Z}\vartheta_n^{\frac{3}{2}}\}} \varrho_n s_M(\varrho_n,\vartheta_n) \ \mathrm{d}x &\leq c \int_{\{\varrho_n \leq \overline{Z}\vartheta_n^{\frac{3}{2}}\}} \varrho_n (1 + |\log(\varrho_n)| + |\log(\vartheta_n)|) \ \mathrm{d}x \\ &\leq c(\overline{Z}) \int_\Omega (\vartheta_n^{\frac{3}{2}} + \vartheta_n^{\frac{3}{4}}\sqrt{\varrho_n}|\log(\sqrt{\varrho_n})| + \vartheta_n\sqrt{\vartheta_n}|\log(\sqrt{\vartheta_n})|) \ \mathrm{d}x \to 0, \end{aligned}$$

where we have used (2.78), (2.79).

Finally, we have

$$\varrho s_M(\varrho,\vartheta) = \varrho S\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big)$$

in the degenerate area $\varrho > \overline{Z}\vartheta^{\frac{3}{2}}$, and, consequently,

$$\begin{aligned} &\int_{\{\varrho_n > \overline{Z}\vartheta_n^{\frac{3}{2}}\}} \varrho_n s_M(\varrho_n,\vartheta_n) \ \mathrm{d}x \\ &\quad = \int_{\{Z\vartheta_n^{\frac{3}{2}} > \varrho_n > \overline{Z}\vartheta_n^{\frac{3}{2}}\}} \varrho_n S\Big(\frac{\varrho_n}{\vartheta_n^{\frac{3}{2}}}\Big) \ \mathrm{d}x + \int_{\{\varrho_n \geq Z\vartheta_n^{\frac{3}{2}}\}} \varrho_n S\Big(\frac{\varrho_n}{\vartheta_n^{\frac{3}{2}}}\Big) \ \mathrm{d}x, \end{aligned}$$

where

$$\int_{\{Z\vartheta_n^{\frac{3}{2}} > \varrho_n \geq \overline{Z}\vartheta_n^{\frac{3}{2}}\}} \varrho_n S\Big(\frac{\varrho_n}{\vartheta_n^{\frac{3}{2}}}\Big) \ \mathrm{d}x \leq S(\overline{Z})Z \int_\Omega \vartheta_n^{\frac{3}{2}} \ \mathrm{d}x \to 0. \tag{2.82}$$

Combining (2.79–2.82), together with hypothesis (2.77), we conclude that

$$\liminf_{n\to\infty} \int_{\{\varrho_n > Z\vartheta_n^{\frac{3}{2}}\}} \varrho_n S\Big(\frac{\varrho_n}{\vartheta_n^{\frac{3}{2}}}\Big)\ \mathrm{d}x > M_0 s_\infty \text{ for any } Z > \overline{Z}. \tag{2.83}$$

However, relation (2.83) leads immediately to contradiction as

$$\int_{\{\varrho_n > Z\vartheta_n^{\frac{3}{2}}\}} \varrho_n S\Big(\frac{\varrho_n}{\vartheta_n^{\frac{3}{2}}}\Big)\ \mathrm{d}x \le S(Z) \int_{\{\varrho_n > Z\vartheta_n^{\frac{3}{2}}\}} \varrho_n\ \mathrm{d}x \to S(Z)M_0.$$

Indeed write $\int_\Omega \varrho_n\ \mathrm{d}x$ as $\int_{\{\varrho_n \le Z\vartheta_n^{\frac{3}{2}}\}} \varrho_n\ \mathrm{d}x + \int_{\{\varrho_n > Z\vartheta_n^{\frac{3}{2}}\}} \varrho_n\ \mathrm{d}x$, and observe that

$$0 \le \int_{\{\varrho_n \le Z\vartheta_n^{\frac{3}{2}}\}} \varrho_n\ \mathrm{d}x = \int_{\{\varrho_n \le Z(\frac{1}{n})^{\frac{3}{2}}\}} \varrho_n\ \mathrm{d}x + \int_{\{\vartheta_n > \frac{1}{n}\}} \varrho_n\ \mathrm{d}x,$$

where the right-hand side tends to 0 by virtue of (2.77). □

By means of Proposition 2.2 and Lemma 2.1, it is easy to check that estimates (2.46), (2.58) give rise to (2.73).

2.2.5 Pressure estimates

The central problem of the mathematical theory of the NAVIER-STOKES-FOURIER SYSTEM is to control the pressure. Under the constitutive relations considered in this book, the pressure p is proportional to the volumetric density of the internal energy ϱe that is *a priori* bounded in $L^1(\Omega)$ uniformly with respect to time, see (2.45–2.47). This section aims to find *a priori* estimates for p in the *weakly closed* reflexive space $L^q((0,T)\times\Omega)$ for a certain $q > 1$. To this end, the basic idea is to "compute" p by means of the momentum equation (2.9) and use the available estimates in order to control the remaining terms. Such an approach, however, faces serious technical difficulties, in particular because of the presence of the time derivative $\partial_t(\varrho\mathbf{u})$ in the momentum equation. Instead we use the quantities

$$\varphi(t,x) = \psi(t)\phi(t,x), \text{ with } \phi = \mathcal{B}\Big[h(\varrho) - \frac{1}{|\Omega|}\int_\Omega h(\varrho)\ \mathrm{d}x\Big],\ \psi \in C_c^\infty(0,T) \tag{2.84}$$

as test functions in the momentum equation (2.9), where $\mathcal{B}$ is a suitable branch of the inverse div_x^{-1}.

There are several ways to construct the operator $\mathcal{B}$, here we adopt the formula proposed by Bogovskii (see Section 10.5 in the Appendix). In particular, the operator $\mathcal{B}$ enjoys the following properties.

■ Bogovskii Operator $\mathcal{B} \approx \mathrm{div}_x^{-1}$:

(b1) Given

$$g \in C_c^\infty(\Omega), \ \int_\Omega g \ \mathrm{d}x = 0,$$

the vector field $\mathcal{B}[g]$ satisfies

$$\mathcal{B}[g] \in C_c^\infty(\Omega; \mathbb{R}^3), \ \mathrm{div}_x \mathcal{B}[g] = g. \tag{2.85}$$

(b2) For any non-negative integer m and any $1 < q < \infty$,

$$\| \ \mathcal{B}[g] \ \|_{W^{m+1,q}(\Omega;\mathbb{R}^3)} \leq c \|g\|_{W^{m,q}(\Omega)} \tag{2.86}$$

provided $\Omega \subset \mathbb{R}^3$ is a Lipschitz domain, in particular, the operator $\mathcal{B}$ can be extended to functions $g \in L^q(\Omega)$ with zero mean satisfying

$$\mathcal{B}[g]|_{\partial\Omega} = 0 \text{ in the sense of traces.} \tag{2.87}$$

(b3) If $g \in L^q(\Omega)$, $1 < q < \infty$, and, in addition,

$$g = \mathrm{div}_x \mathbf{G}, \ \mathbf{G} \in L^p(\Omega; \mathbb{R}^3), \ \mathbf{G} \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

then

$$\| \ \mathcal{B}[g] \ \|_{L^p(\Omega;\mathbb{R}^3)} \leq c \|\mathbf{G}\|_{L^p(\Omega;\mathbb{R}^3)}. \tag{2.88}$$

In order to render the test functions (2.84) admissible, we take

$$\varphi_\alpha(t,x) = \psi(t)[\phi]^\alpha(t,x), \ \text{with } [\phi]^\alpha = \mathcal{B}\Big[h(\varrho) - \frac{1}{|\Omega|}\int_\Omega h(\varrho) \ \mathrm{d}x\Big]^\alpha, \ \psi \in C_c^\infty(0,T), \tag{2.89}$$

where h is a smooth bounded function, and the symbol $[v]^\alpha$ denotes convolution in the *time* variable t with a suitable family of regularizing kernels (see Section 10.1 in Appendix). Here, we have extended $h(\varrho)$ to be zero outside the interval $[0,T]$.

Since ϱ, $\mathbf{u}$ satisfy the renormalized equation (2.2), we easily deduce that

$$\begin{gathered} \partial_t \Big[h(\varrho)\Big]^\alpha + \mathrm{div}_x \Big[h(\varrho)\mathbf{u}\Big]^\alpha + \Big[(\varrho h'(\varrho) - h(\varrho))\mathrm{div}_x \mathbf{u}\Big]^\alpha = 0 \\ \text{for any } t \in (\alpha, T-\alpha) \text{ and a.a. } x \in \Omega, \end{gathered} \tag{2.90}$$

in particular, from the properties (b2), (b3) we may infer that

$$\begin{aligned} \partial_t[\phi]^\alpha = & - \mathcal{B}\Big[\mathrm{div}_x(h(\varrho)\mathbf{u})\Big]^\alpha \\ & - \mathcal{B}\Big[\Big(\varrho h'(\varrho) - h(\varrho)\Big)\mathrm{div}_x\mathbf{u} - \frac{1}{|\Omega|}\int_\Omega \Big(\varrho h'(\varrho) - h(\varrho)\Big)\mathrm{div}_x \mathbf{u} \ \ \mathrm{d}x\Big]^\alpha \end{aligned} \tag{2.91}$$

(cf. Section 10.5 in Appendix).

By virtue of (2.86–2.88), we obtain

$$\| [\phi]^\alpha(t,\cdot) \|_{W^{1,p}(\Omega;\mathbb{R}^3)} \le c(p,\Omega)\| [h(\varrho)]^\alpha(t,\cdot) \|_{L^p(\Omega)},\ 1 < p < \infty, \tag{2.92}$$

and

$$\| [\partial_t \phi]^\alpha(t,\cdot) \|_{L^p(\Omega;\mathbb{R}^3)} \le c(p,s,\Omega) \| [h(\varrho)\mathbf{u}]^\alpha(t,\cdot) \|_{L^p(\Omega)} \tag{2.93}$$

$$+ \begin{cases} \| [(\varrho h'(\varrho) - h(\varrho))\mathrm{div}\mathbf{u}]^\alpha(t,\cdot) \|_{L^{\frac{3p}{3+p}}(\Omega)} \text{ if } \frac{3}{2} < p < \infty, \\ \|[(\varrho h'(\varrho) - h(\varrho))\mathrm{div}\mathbf{u}]^\alpha(t,\cdot) \|_{L^s(\Omega)} \text{ for any } 1 < s < \infty \text{ if } 1 \le p \le \frac{3}{2}, \end{cases}$$

for any $t \in [\alpha, T-\alpha]$.

Having completed the preliminary considerations we take the quantities φ_α specified in (2.89) as test functions in the momentum equation (2.9) to obtain

$$\int_0^T \left(\psi \int_\Omega p(\varrho,\vartheta)[h(\varrho)]^\alpha \ \mathrm{d}x \right) \mathrm{d}t = \sum_{j=1}^5 I_j, \tag{2.94}$$

where

$$I_1 = \frac{1}{|\Omega|} \int_0^T \left(\psi \int_\Omega [h(\varrho)]^\alpha \int_\Omega p(\varrho,\vartheta)\ \mathrm{d}x \right) \mathrm{d}t,$$

$$I_2 = -\int_0^T \left(\psi \int_\Omega \varrho\mathbf{u}\cdot\partial_t[\phi]^\alpha \ \mathrm{d}x \right) \mathrm{d}t,$$

$$I_3 = -\int_0^T \left(\psi \int_\Omega \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x[\phi]^\alpha \ \mathrm{d}x \right) \mathrm{d}t,$$

$$I_4 = \int_0^T \left(\psi \int_\Omega \mathbb{S} : \nabla_x[\phi]^\alpha \ \mathrm{d}x \right) \mathrm{d}t,$$

$$I_5 = -\int_0^T \left(\psi \int_\Omega \varrho\mathbf{f}\cdot[\phi]^\alpha \ \mathrm{d}x \right) \mathrm{d}t,$$

and

$$I_6 = -\int_0^T \left(\psi' \int_\Omega \varrho\mathbf{u}\cdot[\phi]^\alpha \ \mathrm{d}x \right) \mathrm{d}t.$$

Now, our intention is to use the uniform bounds established in Section 2.2.3, together with the integral identity (2.94), in order to show that

$$\int_0^T \int_\Omega p(\varrho,\vartheta)\varrho^\nu \ \mathrm{d}x\ \mathrm{d}t \le c(\text{data}) \text{ for a certain } \nu > 0. \tag{2.95}$$

To this end, the integrals $I_1,\dots,I_6$ are estimated by means of Hölder's inequality as follows:

$$|I_1| \le \|\psi\|_{L^\infty(0,T)} \| [h(\varrho)]^\alpha \|_{L^1((0,T)\times\Omega)} \|p(\varrho,\vartheta)\|_{L^\infty(0,T;L^1(\Omega))},$$
$$|I_2| \le \|\psi\|_{L^\infty(0,T)} \|\varrho\mathbf{u}\|_{L^\infty(0,T;L^{\frac{5}{4}}(\Omega;\mathbb{R}^3))} \|\partial_t[\phi]^\alpha\|_{L^1(0,T;L^5(\Omega;\mathbb{R}^3))},$$

$$|I_3| \le \|\psi\|_{L^\infty(0,T)} \, \|\varrho\mathbf{u}\otimes\mathbf{u}\|_{L^p((0,T)\times\Omega;\mathbb{R}^{3\times 3})} \, \|\nabla_x[\phi]^\alpha\|_{L^{p'}((0,T)\times\Omega;\mathbb{R}^3)},$$
where p is the same as in (2.71),
$$|I_4| \le \|\psi\|_{L^\infty(0,T)} \, \| \, \mathbb{S} \, \|_{L^q((0,T)\times\Omega;\mathbb{R}^{3\times 3})} \, \|\nabla_x[\phi]^\alpha\|_{L^{q'}((0,T)\times\Omega;\mathbb{R}^{3\times 3})},$$
$$\frac{1}{q}+\frac{1}{q'}=1, \text{ with the same } q \text{ as in (2.68)},$$
$$|I_5| \le \|\psi\|_{L^\infty(0,T)} \, \|\mathbf{f}\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^3)}\|\varrho\|_{L^\infty(0,T;L^{\frac{5}{3}}(\Omega))} \, \|[\phi]^\alpha\|_{L^1(0,T;L^{\frac{5}{2}}(\Omega;\mathbb{R}^3))},$$
$$|I_6| \le \|\psi'\|_{L^1(0,T)} \, \|\varrho\mathbf{u}\|_{L^\infty(0,T;L^{\frac{5}{4}}(\Omega;\mathbb{R}^3))} \, \| \, [\phi]^\alpha \, \|_{L^\infty(0,T;L^5(\Omega;\mathbb{R}^3))}.$$

Furthermore, by virtue of the uniform bounds established in (2.92), (2.93), the above estimates are independent of the value of the parameter α, specifically,

$$|I_1| \le \|\psi\|_{L^\infty(0,T)} \, \|h(\varrho)\|_{L^1((0,T)\times\Omega)} \, \|p(\varrho,\vartheta)\|_{L^\infty(0,T;L^1(\Omega))},$$
$$|I_2| \le \|\psi\|_{L^\infty(0,T)} \, \|\varrho\mathbf{u}\|_{L^\infty(0,T;L^{\frac{5}{4}}(\Omega;\mathbb{R}^3))}$$
$$\times \Big(\|h(\varrho)\mathbf{u}\|_{L^1(0,T;L^5(\Omega;\mathbb{R}^3))} + \|(\varrho h'(\varrho)-h(\varrho))\mathrm{div}_x\mathbf{u}\|_{L^1(0,T;L^{\frac{15}{8}}(\Omega))}\Big),$$
$$|I_3| \le \|\psi\|_{L^\infty(0,T)} \, \|\varrho\mathbf{u}\otimes\mathbf{u}\|_{L^p((0,T)\times\Omega;\mathbb{R}^{3\times 3})} \, \|h(\varrho)\|_{L^{p'}((0,T)\times\Omega)},$$
with p as in (2.71),
$$|I_4| \le \|\psi\|_{L^\infty(0,T)} \, \|\mathbb{S}\|_{L^q((0,T)\times\Omega;\mathbb{R}^{3\times 3})} \, \|h(\varrho)\|_{L^{q'}((0,T)\times\Omega)},$$
with q as in (2.68),
$$|I_5| \le \|\psi\|_{L^\infty(0,T)} \, \|\mathbf{f}\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^3)}\|\varrho\|_{L^\infty(0,T;L^{\frac{5}{3}}(\Omega))} \, \|h(\varrho)\|_{L^1(0,T;L^{\frac{15}{11}}(\Omega))},$$
$$|I_6| \le \|\psi'\|_{L^1(0,T)} \, \|\varrho\mathbf{u}\|_{L^\infty(0,T;L^{\frac{5}{4}}(\Omega;\mathbb{R}^3))} \, \|h(\varrho)\|_{L^\infty(0,T;L^{\frac{15}{8}}(\Omega))}.$$

Consequently, taking $h(\varrho) \approx \varrho^\nu$ in (2.94) for a sufficiently small $\nu > 0$ and sufficiently large values of ϱ, we can use estimates (2.46), (2.47), (2.68–2.71), together with the bounds on the integrals $I_1, \dots, I_6$ established above, in order to obtain the desired estimate (2.95).

Furthermore, as

$$\underline{c}\varrho^{\frac{5}{3}} \le p_M(\varrho,\vartheta) \le \overline{c}\begin{cases} \varrho\vartheta \text{ for } \varrho \le \overline{Z}\vartheta^{\frac{3}{2}}, \\ \varrho^{\frac{5}{3}} \text{ for } \varrho \ge \overline{Z}\vartheta^{\frac{3}{2}}, \end{cases} \tag{2.96}$$

estimate (2.95) implies

$$\|\varrho\|_{L^{\frac{5}{3}+\nu}((0,T)\times\Omega)} \le c(\text{data}). \tag{2.97}$$

Finally (2.97) together with (2.46) and (2.96) yields

$$\|p_M(\varrho,\vartheta)\|_{L^p((0,T)\times\Omega)} \le c(\text{data}) \quad \text{for some } p > 1. \tag{2.98}$$

2.2.6 Pressure estimates, an alternative approach

The approach to pressure estimates based on the operator $\mathcal{B} \approx \mathrm{div}_x^{-1}$ requires a certain minimal regularity of the boundary $\partial\Omega$. In the remaining part of this chapter, we briefly discuss an alternative method yielding uniform estimates in the interior of the physical domain together with equi-integrability of the pressure up to the boundary. In particular, the interior estimates may be of independent interest since they are sufficient for resolving the problem of global existence for the NAVIER-STOKES-FOURIER SYSTEM provided the equality sign in the total energy balance (2.22) is relaxed to inequality "$\leq$".

Local pressure estimates. Similarly to the preceding part, the basic idea is to "compute" the pressure by means of the momentum equation (2.9). In order to do it locally, we introduce a family of test functions

$$\varphi(t,x) = \psi(t)\eta(x)(\nabla_x\Delta_x^{-1})[1_\Omega h(\varrho)], \tag{2.99}$$

where $\psi \in C_c^\infty(0,T)$, $\eta \in C_c^\infty(\Omega)$, $h \in C_c^\infty(0,\infty)$,

$$0 \leq \psi, \eta \leq 1, \text{ and } h(r) = r^\nu \text{ for } r \geq 1$$

for a suitable exponent $\nu > 0$. Here the symbol Δ_x^{-1} stands for the inverse of the Laplace operator on the whole space $\mathbb{R}^3$, specifically, in terms of the Fourier transform $\mathcal{F}_{x\to\xi}$,

$$\Delta_x^{-1}[v](x) = -\mathcal{F}_{\xi\to x}^{-1}\Big[\frac{\mathcal{F}_{x\to\xi}[v]}{|\xi|^2}\Big], \tag{2.100}$$

see Sections 0.5 and 10.16.

Note that

$$\nabla_x\varphi = \psi\nabla_x\eta \otimes \nabla_x\Delta_x^{-1}[1_\Omega h(\varrho)] + \psi\eta\mathcal{R}[1_\Omega h(\varrho)],$$

where

$$\mathcal{R} = [\nabla_x \otimes \nabla_x]\Delta_x^{-1},\ \mathcal{R}_{i,j}[v](x) = \mathcal{F}^{-1}\Big[\frac{\xi_i\xi_j\mathcal{F}_{x\to\xi}[v]}{|\xi|^2}\Big] \tag{2.101}$$

is a superposition of two Riesz maps. By virtue of the classical Calderón-Zygmund theory, the operator $\mathcal{R}_{i,j}$ is bounded on $L^p(\mathbb{R}^3)$ for any $1 < p < \infty$. In particular, $\varphi \in L^q(0,T;W_0^{1,p}(\Omega;\mathbb{R}^3))$ whenever $h(\varrho) \in L^q(0,T;L^p(\Omega))$ for certain $1 \leq q \leq \infty$, $1 < p < \infty$, see Section 10.16 in Appendix.

Similarly, using the renormalized equation (2.2) with $b(\varrho) = h'(\varrho)\varrho - h(\varrho)$ we "compute"

$$\partial_t\varphi = \partial_t\psi\eta\nabla_x\Delta_x^{-1}[1_\Omega h(\varrho)] \\ + \psi\eta\Big(\nabla_x\Delta_x^{-1}\Big[1_\Omega(h(\varrho) - h'(\varrho)\varrho)\mathrm{div}_x\mathbf{u}\Big] - \nabla_x\Delta_x^{-1}[\mathrm{div}_x(1_\Omega h(\varrho)\mathbf{u})]\Big).$$

Let us point out that equation (2.2) holds on the whole space $\mathbb{R}^3$ provided $\mathbf{u}$ has been extended outside Ω and h replaced by $1_\Omega h(\varrho)$. Note that functions belonging to $W^{1,p}(\Omega)$ can be extended outside Ω to be in the space $W^{1,p}(\mathbb{R}^3)$ as soon as Ω is a bounded Lipschitz domain.

It follows from the above discussion that the quantity φ specified in (2.99) can be taken as a test function in the momentum equation (2.9), more precisely, the function φ, together with its first derivatives, can be approximated in the L^p-norm by a suitable family of regular test functions satisfying (2.10), (2.12). Thus we get

$$\int_0^T \int_\Omega \psi\eta\Big(ph(\varrho) - \mathbb{S} : \mathcal{R}[1_\Omega h(\varrho)]\Big) \, \mathrm{d}x \, \mathrm{d}t = \sum_{j=1}^7 I_j, \tag{2.102}$$

where

$$I_1 = \int_0^T \int_\Omega \psi\eta\Big(\varrho\mathbf{u}\cdot\mathcal{R}[1_\Omega h(\varrho)\mathbf{u}] - (\varrho\mathbf{u}\otimes\mathbf{u}) : \mathcal{R}[1_\Omega h(\varrho)]\Big) \, \mathrm{d}x \, \mathrm{d}t,$$
$$I_2 = -\int_0^T \int_\Omega \psi\eta \; \varrho\mathbf{u}\cdot\nabla_x\Delta_x^{-1}\Big[1_\Omega(h(\varrho) - h'(\varrho)\varrho)\mathrm{div}_x\mathbf{u}\Big] \, \mathrm{d}x \, \mathrm{d}t,$$
$$I_3 = -\int_0^T \int_\Omega \psi\eta\varrho\mathbf{f}\cdot\nabla_x\Delta_x^{-1}[1_\Omega h(\varrho)] \, \mathrm{d}x \, \mathrm{d}t,$$
$$I_4 = -\int_0^T \int_\Omega \psi p\nabla_x\eta\cdot\nabla_x\Delta_x^{-1}[1_\Omega h(\varrho)] \, \mathrm{d}x \, \mathrm{d}t,$$
$$I_5 = \int_0^T \int_\Omega \psi\mathbb{S} : \nabla_x\eta\otimes\nabla_x\Delta_x^{-1}[1_\Omega h(\varrho)] \, \mathrm{d}x \, \mathrm{d}t,$$
$$I_6 = -\int_0^T \int_\Omega \psi(\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla_x\eta\otimes\nabla_x\Delta^{-1}[1_\Omega h(\varrho)] \, \mathrm{d}x \, \mathrm{d}t,$$

and

$$I_7 = -\int_0^T \int_\Omega \partial_t\psi \; \eta\varrho\mathbf{u}\cdot\nabla_x\Delta_x^{-1}[1_\Omega h(\varrho)] \, \mathrm{d}x \, \mathrm{d}t.$$

Here, we have used the notation

$$\mathbb{A} : \mathcal{R} \equiv \sum_{i,j=1}^3 A_{i,j}\mathcal{R}_{i,j}, \;\; \mathcal{R}[\mathbf{v}]_i \equiv \sum_{j=1}^3 \mathcal{R}_{i,j}[v_j], \;\; i = 1,2,3.$$

Exactly as in Section 2.2.5, the integral identity (2.102) can be used to establish a bound

$$\int_0^T \int_K p(\varrho,\vartheta)\varrho^\nu \, \mathrm{d}x \, \mathrm{d}t \le c(\mathrm{data}, K) \text{ for a certain } \nu > 0, \tag{2.103}$$

and, consequently,

$$\int_0^T \int_K \varrho^{\frac{5}{3}+\nu} \, \mathrm{d}x \, \mathrm{d}t \le c(\mathrm{data}, K), \tag{2.104}$$

$$\int_0^T \int_K |p(\varrho,\vartheta)|^r \, \mathrm{d}x \, \mathrm{d}t \le c(\mathrm{data}, K) \text{ for a certain } r > 1 \tag{2.105}$$

for any compact $K \subset \Omega$.

Pressure estimates near the boundary. Our ultimate goal is to extend, in a certain sense, the local estimates established in Section 2.2.6 up to the boundary $\partial\Omega$. In particular, our aim is to show that the pressure is equi-integrable in Ω, where the bound can be determined in terms of the data. To this end, it is enough to solve the following auxiliary problem:

Given $q > 1$ arbitrary, find a function $\mathbf{G} = \mathbf{G}(x)$ such that

$$\mathbf{G} \in W_0^{1,q}(\Omega; \mathbb{R}^3), \ \mathrm{div}_x \mathbf{G}(x) \to \infty \ \ \textit{uniformly for} \ \ \mathrm{dist}(x, \partial\Omega) \to 0. \tag{2.106}$$

If Ω is a bounded Lipschitz domain, the function $\mathbf{G}$ can be taken as a solution of the problem

$$\mathrm{div}_x \mathbf{G} = g \text{ in } \Omega, \ \mathbf{G}|_{\partial\Omega} = 0, \tag{2.107}$$

where

$$g = \mathrm{dist}^{-\beta}(x, \partial\Omega) - \frac{1}{|\Omega|} \int_\Omega \mathrm{dist}^{-\beta}(x, \partial\Omega) \, \mathrm{d}x, \text{ with } 0 < \beta < \frac{1}{q},$$

so that (2.106) is satisfied. Problem (2.107) can be solved by means of the operator $\mathcal{B}$ introduced in Section 2.2.5 as soon as Ω is a Lipschitz domain. For less regular domains, an explicit solution may be found by an alternative method (see Kukučka [126]).

Pursuing step by step the procedure developed in the preceding section we take the quantity

$$\varphi(t,x) = \psi(t)\mathbf{G}(x), \ \psi \in C_c^\infty(0,T),$$

as a test function in the momentum equation (2.9). Assuming $\mathbf{G}$ belongs to $W_0^{1,q}(\Omega; \mathbb{R}^3)$, with $q > 1$ large enough, we can deduce, exactly as in Section 2.2.6, that

$$\int_0^T \int_\Omega p(\varrho,\vartheta) \mathrm{div}_x \mathbf{G} \, \mathrm{d}x \, \mathrm{d}t \le c(\mathrm{data}). \tag{2.108}$$

Note that this step can be fully justified via a suitable approximation of $\mathbf{G}$ by a family of smooth, compactly supported functions. As $\mathrm{div}_x \mathbf{G}(x) \to \infty$ whenever $x \to \partial\Omega$, relation (2.108) yields equi-integrability of the pressure in a neighborhood of the boundary (cf. Theorem 0.8).

Chapter 3

Existence Theory

The informal notion of a *well-posed problem* captures many of the desired features of what we mean by solving a system of partial differential equations. Usually a given problem is *well posed* if

- the problem has a solution;
- the solution is unique in a given class;
- the solution depends continuously on the data.

The first condition is particularly important for us as we want to perform the singular limits on *existing objects*. It is a peculiar feature of non-linear problems that *existence* of solutions can be rigorously established only in the class determined by *a priori* estimates. Without any extra assumption concerning the magnitude of the initial data and/or the length of the existence interval $(0,T)$, all available and known *a priori* bounds on solutions to the NAVIER-STOKES-FOURIER SYSTEM have been collected in Chapter 2. Accordingly, the existence theory to be developed in the forthcoming chapter necessarily uses the framework of the *weak solutions* introduced in Chapter 1 and identified in Chapter 2. To begin, let us point out that the existence theory is not the main objective of this book, and, strictly speaking, all results concerning the singular limits can be stated without referring to any specific solution. On the other hand, however, it seems important to know that the class of objects we deal with is not void.

The complete proof of existence for the initial-boundary value problem associated to the NAVIER-STOKES-FOURIER SYSTEM is rather technical and considerably long. The following text aims to provide a concise and self-contained treatment starting directly with the approximate problem and avoiding completely the nowadays popular "approach" based on reducing the task of *existence* to showing the *weak sequential stability* of the set of hypothetical solutions.

The principal tools to be employed in the existence proof can be summarized as follows:

- Nowadays "classical" arguments based on compactness of embeddings of Sobolev spaces (the Rellich-Kondrashov theorem);
- a generalized Arzelà-Ascoli compactness result for weakly continuous functions and its variants including the Lions-Aubin lemma;
- the Div-Curl lemma developed in the theory of compensated compactness;
- the "weak continuity" property of the so-called effective viscous flux established by P.-L. Lions and its generalization to the case of non-constant viscosity coefficients via a commutator lemma;
- the theory of parametrized (Young) measures, in particular, its application to compositions of weakly converging sequences with a Carathéodory function;
- the analysis of density oscillations via oscillations defect measures in weighted Lebesgue spaces.

3.1 Hypotheses

Before formulating our main existence result, we present a concise list of hypotheses imposed on the data. To see their interpretation, the reader may consult Chapter 1 for the physical background and the relevant discussion.

(i) Initial data: The initial state of the system is determined through the choice of the quantities ϱ_0, $(\varrho\mathbf{u})_0$, E_0, and $(\varrho s)_0$.

The initial density ϱ_0 is a *non-negative* measurable function such that

$$\varrho_0 \in L^{\frac{5}{3}}(\Omega), \ \int_\Omega \varrho_0 \, \mathrm{d}x = M_0 > 0. \tag{3.1}$$

The initial distribution of the momentum satisfies a compatibility condition

$$(\varrho\mathbf{u})_0 = 0 \text{ a.a. on the set } \{x \in \Omega \mid \varrho_0(x) = 0\}, \tag{3.2}$$

notably the total amount of the kinetic energy is finite, meaning,

$$\int_\Omega \frac{|(\varrho\mathbf{u})_0|^2}{\varrho_0} \, \mathrm{d}x < \infty. \tag{3.3}$$

The initial temperature is determined by a measurable function ϑ_0 satisfying

$$\vartheta_0 > 0 \text{ a.a. in } \Omega, \ (\varrho s)_0 = \varrho_0 s(\varrho_0, \vartheta_0), \ \varrho_0 s(\varrho_0, \vartheta_0) \in L^1(\Omega). \tag{3.4}$$

Finally, we assume that the initial energy of the system is finite, specifically,

$$E_0 = \int_\Omega \Big(\frac{1}{2\varrho_0} |(\varrho\mathbf{u})_0|^2 + \varrho_0 e(\varrho_0, \vartheta_0) \Big) \, \mathrm{d}x < \infty. \tag{3.5}$$

(ii) Source terms: For the sake of simplicity, we suppose that

$$\mathbf{f} \in L^\infty((0,T)\times\Omega;\mathbb{R}^3),\ \ \mathcal{Q} \geq 0,\ \ \mathcal{Q} \in L^\infty((0,T)\times\Omega). \tag{3.6}$$

(iii) Constitutive relations: The quantities p, e, and s are continuously differentiable functions for positive values of ϱ, ϑ satisfying *Gibbs' equation*

$$\vartheta Ds(\varrho,\vartheta) = De(\varrho,\vartheta) + p(\varrho,\vartheta)D\Big(\frac{1}{\varrho}\Big) \text{ for all } \varrho,\vartheta > 0. \tag{3.7}$$

In addition,

$$p(\varrho,\vartheta) = p_M(\varrho,\vartheta) + p_R(\vartheta),\ \ p_R(\vartheta) = \frac{a}{3}\vartheta^4,\ \ a > 0, \tag{3.8}$$

and

$$e(\varrho,\vartheta) = e_M(\varrho,\vartheta) + e_R(\varrho,\vartheta),\ \ \varrho e_R(\varrho,\vartheta) = a\vartheta^4, \tag{3.9}$$

where, in accordance with the hypothesis of thermodynamic stability (1.44), the molecular components satisfy

$$\frac{\partial p_M(\varrho,\vartheta)}{\partial\varrho} > 0 \text{ for all } \varrho,\vartheta > 0, \tag{3.10}$$

and

$$0 < \frac{\partial e_M(\varrho,\vartheta)}{\partial\vartheta} \leq c \text{ for all } \varrho,\vartheta > 0. \tag{3.11}$$

Furthermore,

$$\lim_{\vartheta\to 0+} e_M(\varrho,\vartheta) = \underline{e}_M(\varrho) > 0 \text{ for any fixed } \varrho > 0, \tag{3.12}$$

and,

$$\Big|\varrho\frac{\partial e_M(\varrho,\vartheta)}{\partial\varrho}\Big| \leq c\ e_M(\varrho,\vartheta) \text{ for all } \varrho,\vartheta > 0. \tag{3.13}$$

Finally, we suppose that there is a function P satisfying

$$P \in C^1[0,\infty),\ P(0) = 0,\ P'(0) > 0, \tag{3.14}$$

and two positive constants $0 < \underline{Z} < \overline{Z}$ such that

$$p_M(\varrho,\vartheta) = \vartheta^{\frac{5}{2}}P\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big) \text{ whenever } 0 < \varrho \leq \underline{Z}\vartheta^{\frac{3}{2}},\ \text{or},\ \varrho > \overline{Z}\vartheta^{\frac{3}{2}}, \tag{3.15}$$

where, in addition,

$$p_M(\varrho,\vartheta) = \frac{2}{3}\varrho e_M(\varrho,\vartheta) \text{ for } \varrho > \overline{Z}\vartheta^{\frac{3}{2}}. \tag{3.16}$$

(iv) Transport coefficients: The viscosity coefficients μ, η are continuously differentiable functions of the absolute temperature ϑ, more precisely $\mu,\ \eta \in C^1[0,\infty)$, satisfying

$$0 < \underline{\mu}(1+\vartheta^\alpha) \le \mu(\vartheta) \le \overline{\mu}(1+\vartheta^\alpha), \tag{3.17}$$

$$\sup_{\vartheta\in[0,\infty)} |\mu'(\vartheta)| \le \overline{m}, \tag{3.18}$$

$$0 \le \eta(\vartheta) \le \overline{\eta}(1+\vartheta^\alpha). \tag{3.19}$$

The heat conductivity coefficient κ can be decomposed as

$$\kappa(\vartheta) = \kappa_M(\vartheta) + \kappa_R(\vartheta), \tag{3.20}$$

where $\kappa_M,\ \kappa_R \in C^1[0,\infty)$, and

$$0 < \underline{\kappa}_R \vartheta^3 \le \kappa_R(\vartheta) \le \overline{\kappa}_R(1+\vartheta^3), \tag{3.21}$$

$$0 < \underline{\kappa}_M(1+\vartheta^\alpha) \le \kappa_M(\vartheta) \le \overline{\kappa}_M(1+\vartheta^\alpha). \tag{3.22}$$

In formulas (3.17–3.22), $\underline{\mu}$, $\overline{\mu}$, $\overline{m}$, $\overline{\eta}$, $\underline{\kappa}_R$, $\overline{\kappa}_R$, $\underline{\kappa}_M$, $\overline{\kappa}_M$ are positive constants and

$$\frac{2}{5} < \alpha \le 1. \tag{3.23}$$

Remark: *Some of the above hypotheses, in particular those imposed on the thermodynamic functions, are rather technical and may seem awkward at first glance. The reader should always keep in mind the prototype example*

$$p(\varrho,\vartheta) = \vartheta^{\frac{5}{2}} P\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big) + \frac{a}{3}\vartheta^4,\ \ P(0)=0,\ \ P'(0)>0,\quad P(Z)\approx Z^{\frac{5}{3}} \ \textit{for}\ Z >> 1$$

which meets all the hypotheses stated above. Note that if $a > 0$ *is small and* $P(Z)$ *is close to a linear function for moderate values of* Z*, the above formula approaches the standard* Boyle-Marriot law *of a perfect gas.*

The present hypotheses cover, in particular, the physically reasonable case when the constitutive law for the molecular pressure is that of the monoatomic gas, meaning

$$p_M = \frac{2}{3}\varrho e_M;$$

for more details see Section 1.4.2.

Very roughly indeed, we can say that the pressure is regularized in the area where either ϱ or ϑ are close to zero. The radiation component p_R prevents the temperature field from oscillating in the *vacuum zone* where ϱ vanishes, while the superlinear growth of P for large arguments guarantees strong enough *a priori* estimates on the density ϱ in the "cold" regime $\vartheta \approx 0$.

3.2 Structural properties of constitutive functions

The hypotheses on constitutive relationsfor the pressure, the internal energy and the entropy entail further restrictions imposed on the structural properties of the functions p, e, and s. Some of them have already been identified and used in Chapter 2. For reader's convenience, they are recorded and studied in a systematic way in the text below.

(i) The first observation is that for (3.15), (3.16) to be compatible with the hypothesis of thermodynamic stability expressed through (3.10), (3.11), the function P must obey certain structural restrictions. In particular, relation (3.10) yields

$$P'(Z) > 0 \ \text{ whenever } \ 0 < Z < \underline{Z}, \ \text{or}, \ Z > \overline{Z},$$

which, together with (3.14), yields

$$P'(Z) > 0 \text{ for all } Z \geq 0, \tag{3.24}$$

where P has been extended to be strictly increasing on the interval $[\underline{Z}, \overline{Z}]$.

Similarly, a direct inspection of (3.11), (3.15), (3.16) gives rise to

$$0 < \frac{3}{2} \frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z} := c_{v,M} < c, \quad \text{whenever } Z = \frac{\varrho}{\vartheta^{3/2}} \geq \overline{Z}. \tag{3.25}$$

In particular $P(Z)/Z^{5/3}$ possesses a limit for $Z \to \infty$, specifically, in accordance with (3.15), (3.16),

$$\lim_{\vartheta \to 0+} e_M(\varrho, \vartheta) = \frac{3}{2} \lim_{\vartheta \to 0+} \frac{\vartheta^{5/2}}{\varrho} P\Big(\frac{\varrho}{\vartheta^{3/2}} \Big) = \frac{3}{2} \varrho^{\frac{2}{3}} \lim_{Z \to \infty} \frac{P(Z)}{Z^{5/3}} \text{ for any fixed } \varrho > 0.$$

Moreover, in agreement with (3.12),

$$\lim_{Z \to \infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0, \tag{3.26}$$

and

$$\lim_{\vartheta \to 0+} e_M(\varrho, \vartheta) = \underline{e}_M(\varrho) = \frac{3}{2} \varrho^{2/3} p_\infty. \tag{3.27}$$

(ii) By virtue of (3.11), the function $\vartheta \mapsto e_M(\varrho, \vartheta)$ is strictly increasing on the whole interval $(0, \infty)$ for any fixed $\varrho > 0$. This fact, together with (3.9), (3.27), gives rise to the lower bound

$$\varrho e(\varrho, \vartheta) \geq \frac{3p_\infty}{2} \varrho^{\frac{5}{3}} + a\vartheta^4. \tag{3.28}$$

On the other hand,

$$e_M(\varrho, \vartheta) = \underline{e}_M(\varrho) + \int_0^\vartheta \frac{\partial e_M}{\partial \vartheta}(\varrho, \tau) \, \mathrm{d}\tau, \tag{3.29}$$

which, together with (3.11) and (3.27), yields

$$0 \le e_M(\varrho,\vartheta) \le c(\varrho^{\frac{2}{3}} + \vartheta). \tag{3.30}$$

Similarly, relation (3.24), together with (3.14–3.16), and (3.26), yield the following bounds on the molecular pressure p_M:

$$\underline{c}\varrho\vartheta \le p_M(\varrho,\vartheta) \le \overline{c}\varrho\vartheta \quad \text{ if } \varrho < \overline{Z}\vartheta^{\frac{3}{2}}, \tag{3.31}$$

and

$$\underline{c}\varrho^{\frac{5}{3}} \le p_M(\varrho,\vartheta) \le \overline{c}\left\{ \begin{array}{c} \vartheta^{\frac{5}{2}} \text{ if } \varrho < \overline{Z}\vartheta^{\frac{3}{2}} \\ \varrho^{\frac{5}{3}} \text{ if } \varrho > \overline{Z}\vartheta^{\frac{3}{2}}. \end{array} \right\} \tag{3.32}$$

Here, we have used the monotonicity of p_M in ϱ in order to control the behavior of the pressure in the region

$$\underline{Z}\vartheta^{\frac{3}{2}} \le \varrho \le \overline{Z}\vartheta^{\frac{3}{2}}.$$

Moreover, in accordance with (3.30), (3.32), it is easy to observe that

$$e_M, p_M \quad \text{are bounded on bounded sets of } [0,\infty)^2. \tag{3.33}$$

(iii) In agreement with Gibbs' relation (3.7), the specific entropy s can be written as

$$s = s_M + s_R, \quad \frac{\partial s_M}{\partial \vartheta} = \frac{1}{\vartheta}\frac{\partial e_M}{\partial \vartheta}, \quad \varrho s_R(\varrho,\vartheta) = \frac{4}{3}a\vartheta^3, \tag{3.34}$$

where the molecular component s_M satisfies

$$s_M(\varrho,\vartheta) = S(Z),\ Z = \frac{\varrho}{\vartheta^{3/2}},\ S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z^2} < 0 \tag{3.35}$$

in the degenerate area $\varrho > \overline{Z}\vartheta^{\frac{3}{2}}$. Note that the function S is determined up to an additive constant.

On the other hand, due to (3.11), the function $\vartheta \mapsto s_M(\varrho,\vartheta)$ is increasing on $(0,\infty)$ for any fixed ϑ. Accordingly,

$$s_M(\varrho,\vartheta) \le \left\{ \begin{array}{c} s_M(\varrho,1) \quad \text{if } \vartheta \le 1 \\ s_M(\varrho,1) + \int_1^\vartheta \frac{\partial s_M}{\partial \vartheta}(\varrho,\tau)\,\mathrm{d}\tau \le s_M(\varrho,1) + c\log\vartheta \quad \text{if } \vartheta > 1 \end{array} \right\}, \tag{3.36}$$

where we have exploited (3.11) combined with (3.34) in order to control

$$\left| \int_1^\vartheta \frac{\partial s_M}{\partial \vartheta}(\varrho,\tau)\ \mathrm{d}\tau \right| \le c|\log\vartheta| \text{ for all } \vartheta > 0. \tag{3.37}$$

Another application of Gibbs' relation (3.7) yields

$$\frac{\partial s_M}{\partial \varrho} = -\frac{1}{\varrho^2}\frac{\partial p_M}{\partial \vartheta},$$

see also (1.3); therefore

$$s_M(\varrho, 1) = s_M(1,1) + \int_1^{\varrho} \frac{1}{\tau^2}\frac{\partial p_M}{\partial \vartheta}(\tau, 1)\,\mathrm{d}\tau.$$

By virtue of (3.15) and (3.25),

$$\frac{\partial p_M}{\partial \vartheta}(\rho, 1) = \frac{5}{2}P(\varrho) - \frac{3}{2}\varrho P'(\varrho) \leq c\varrho \text{ for all } \varrho \in (0, \underline{Z}] \cup [\overline{Z}, \infty),$$

whereas

$$|\frac{\partial p_M}{\partial \vartheta}(\rho, 1)| \quad \text{is bounded in } [\underline{Z}, \overline{Z}].$$

Consequently,

$$|s_M(\varrho, 1)| \leq c(1 + |\log \varrho|) \text{ for all } \varrho \in (0, \infty). \tag{3.38}$$

Writing

$$s_M(\varrho, \vartheta) = s_M(\varrho, 1) + \int_1^{\varrho} \frac{\partial s_M}{\partial \vartheta}(\varrho, \tau)\ \mathrm{d}\tau$$

and resuming the previous estimates, we conclude that

$$|s_M(\varrho, \vartheta)| \leq c(1 + |\log \varrho| + |\log \vartheta|) \text{ for all } \varrho, \vartheta > 0. \tag{3.39}$$

(iv) It follows from (3.35) that

$$\begin{gathered} \lim\nolimits_{Z \to \infty} S(Z) = s_\infty = \left\{ \begin{array}{c} -\infty \\ 0 \end{array} \right\}; \\ \text{whence} \\ \lim\nolimits_{\vartheta \to 0+} s_M(\varrho, \vartheta) = s_\infty \text{ for any fixed } \varrho > 0. \end{gathered} \tag{3.40}$$

where, in the latter case, we have fixed the free additive constant in the definition of S in (3.35) to obtain $s_\infty = 0$.

(v) Finally, as a direct consequence of (3.15),

$$\frac{\partial p_M}{\partial \varrho}(\varrho, \vartheta) = \vartheta P'\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right) \text{if } \varrho < \underline{Z}\vartheta^{\frac{3}{2}},\ \text{or},\ \varrho > \overline{Z}\vartheta^{\frac{3}{2}},$$

where, by virtue of (3.24), (3.25), and (3.26),

$$P'(Z) \geq c(1 + Z^{\frac{2}{3}}),\ c > 0,\ \text{for all } Z \geq 0. \tag{3.41}$$

Thus we can write

$$p_M(\varrho,\vartheta) = \vartheta^{\frac{5}{2}} P\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right) + p_b(\varrho,\vartheta),$$

with

$$p_b(\varrho,\vartheta) = p_M(\varrho,\vartheta) - \vartheta^{\frac{5}{2}} P\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right).$$

In accordance with (3.15), (3.32), we have

$$|p_b(\varrho,\vartheta)| \le c(1+\vartheta^{\frac{5}{2}}). \tag{3.42}$$

Finally, we conclude with help of (3.41) that there exists $d > 0$ such that

$$p_M(\varrho,\vartheta) = d\varrho^{\frac{5}{3}} + p_m(\varrho,\vartheta) + p_b(\varrho,\vartheta), \tag{3.43}$$

where

$$\frac{\partial p_m}{\partial \varrho}(\varrho,\vartheta) > 0 \text{ for all } \varrho,\vartheta > 0. \tag{3.44}$$

3.3 Main existence result

Having collected all the preliminary material, we are in a position to formulate our main existence result concerning the weak solutions of the Navier-Stokes-Fourier system.

■ Global Existence for the Navier-Stokes-Fourier System:

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$, $\nu \in (0,1)$. Assume that*

- *the data ϱ_0, $(\varrho\mathbf{u})_0$, E_0, $(\varrho s)_0$ satisfy* (3.1–3.5)*;*
- *the source terms $\mathbf{f}$, $\mathcal{Q}$ are given by* (3.6)*;*
- *the thermodynamic functions p, e, s, and the transport coefficients μ, η, κ obey the structural hypotheses* (3.7–3.23)*.*

Then for any $T > 0$ the Navier-Stokes-Fourier system admits a weak solution $\{\varrho, \mathbf{u}, \vartheta\}$ on $(0,T)\times\Omega$ in the sense specified in Section 2.1*. More precisely, $\{\varrho, \mathbf{u}, \vartheta\}$ satisfy relations* (2.2–2.6), (2.9–2.17), (2.22–2.25), (2.27–2.32), *with* (2.35–2.37).

The complete proof of Theorem 3.1 presented in the remaining part of this chapter is tedious, rather technical, consisting in four steps:

- The momentum equation (2.9) is replaced by a Faedo-Galerkin approximation, the equation of continuity (2.2) is supplemented with an artificial viscosity term, and the entropy production equation (2.27) is replaced by the balance of internal energy. The approximate solutions are obtained by help of the Schauder fixed point theorem, first locally in time, and then extended on the full interval $(0,T)$ by means of suitable uniform estimates.

- Performing the limit in the Faedo-Galerkin approximation scheme we recover the momentum equation supplemented with an artificial pressure term. Simultaneously, the balance of internal energy is converted to the entropy production equation (2.27), together with the total energy balance (2.22) containing some extra terms depending on small parameters.
- We pass to the limit in the regularized equation of continuity sending the artificial viscosity terms to zero.
- Finally, the proof of Theorem 3.1 is completed letting the artificial pressure term go to zero.

3.3.1 Approximation scheme

(i) The equation of continuity (2.2) is regularized by means of an artificial viscosity term:

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = \varepsilon \Delta \varrho \text{ in } (0,T)\times\Omega, \tag{3.45}$$

and supplemented with the homogeneous Neumann boundary condition

$$\nabla_x \varrho \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{3.46}$$

and the initial condition

$$\varrho(0,\cdot) = \varrho_{0,\delta}, \tag{3.47}$$

where

$$\varrho_{0,\delta} \in C^{2,\nu}(\overline{\Omega}),\ \inf_{x\in\Omega} \varrho_{0,\delta}(x) > 0,\ \nabla_x \varrho_{0,\delta}\cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{3.48}$$

(ii) The momentum balance expressed through the integral identity (2.9) is replaced by a *Faedo-Galerkin approximation*:

$$\int_0^T \int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho[\mathbf{u}\otimes\mathbf{u}] : \nabla_x\varphi + \Big(p(\varrho,\vartheta) + \delta(\varrho^\Gamma + \varrho^2)\Big)\operatorname{div}_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t$$
$$= \int_0^T \int_\Omega \Big(\varepsilon(\nabla_x\varrho\nabla_x\mathbf{u})\cdot\varphi + \mathbb{S}_\delta : \nabla_x\varphi - \varrho\mathbf{f}_\delta\cdot\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega (\varrho\mathbf{u})_0\cdot\varphi\ \mathrm{d}x, \tag{3.49}$$

to be satisfied for any test function $\varphi \in C^1_c([0,T);X_n)$, where

$$X_n \subset C^{2,\nu}(\overline{\Omega};\mathbb{R}^3) \subset L^2(\Omega;\mathbb{R}^3) \tag{3.50}$$

is a finite-dimensional vector space of functions satisfying either

$$\varphi\cdot\mathbf{n}|_{\partial\Omega} = 0 \text{ in the case of the complete slip boundary conditions}, \tag{3.51}$$

or

$$\varphi|_{\partial\Omega} = 0 \text{ in the case of the no-slip boundary conditions}. \tag{3.52}$$

The space X_n is endowed with the Hilbert structure induced by the scalar product of the Lebesgue space $L^2(\Omega;\mathbb{R}^3)$.

Furthermore, we set

$$\begin{aligned}\mathbb{S}_\delta &= \mathbb{S}_\delta(\vartheta, \nabla_x \mathbf{u}) \\ &= (\mu(\vartheta) + \delta\vartheta)\Big(\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\ \mathbb{I}\Big) + \eta(\vartheta)\mathrm{div}_x \mathbf{u}\ \mathbb{I},\end{aligned} \tag{3.53}$$

while the function

$$\mathbf{f}_\delta \in C^1([0,T] \times \overline{\Omega}; \mathbb{R}^3) \tag{3.54}$$

is a suitable approximation of the driving force $\mathbf{f}$.

(iii) Instead of the entropy balance (2.27), we consider a modified internal energy equation in the form:

$$\begin{aligned}&\partial_t(\varrho e_\delta(\varrho,\vartheta)) + \mathrm{div}_x(\varrho e_\delta(\varrho,\vartheta)\mathbf{u}) - \mathrm{div}_x \nabla_x \mathcal{K}_\delta(\vartheta) \\ &= \mathbb{S}_\delta(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - p(\varrho,\vartheta)\mathrm{div}_x \mathbf{u} + \varrho \mathcal{Q}_\delta + \varepsilon\delta(\Gamma \varrho^{\Gamma-2} + 2)|\nabla_x \varrho|^2 + \delta\frac{1}{\vartheta^2} - \varepsilon\vartheta^5,\end{aligned} \tag{3.55}$$

supplemented with the Neumann boundary condition

$$\nabla_x \vartheta \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{3.56}$$

and the initial condition

$$\vartheta(0,\cdot) = \vartheta_{0,\delta}, \tag{3.57}$$

$$\vartheta_{0,\delta} \in W^{1,2}(\Omega) \cap L^\infty(\Omega),\ \operatorname*{ess\,inf}_{x\in\Omega} \vartheta_{0,\delta}(x) > 0. \tag{3.58}$$

Here

$$\begin{aligned}&e_\delta(\varrho,\vartheta) = e_{M,\delta}(\varrho,\vartheta) + a\vartheta^4, \quad e_{M,\delta}(\varrho,\vartheta) = e_M(\varrho,\vartheta) + \delta\vartheta, \\ &\mathcal{K}_\delta(\vartheta) = \int_1^\vartheta \kappa_\delta(z)\ \mathrm{d}z,\ \kappa_\delta(\vartheta) = \kappa_M(\vartheta) + \kappa_R(\vartheta) + \delta\Big(\vartheta^\Gamma + \frac{1}{\vartheta}\Big),\end{aligned} \tag{3.59}$$

and

$$\mathcal{Q}_\delta \geq 0,\ \mathcal{Q}_\delta \in C^1([0,T] \times \overline{\Omega}). \tag{3.60}$$

In problem (3.45–3.60), the quantities ε, δ are small positive parameters, while $\Gamma > 0$ is a sufficiently large fixed number. The meaning of the extra terms will become clear in the course of the proof. Loosely speaking, the ε-dependent quantities provide more regularity of the approximate solutions modifying the type of the field equations, while the δ-dependent quantities prevent concentrations yielding better estimates on the amplitude of the approximate solutions. For technical reasons, the limit passage must be split up in two steps letting first $\varepsilon \to 0$ and then $\delta \to 0$.

3.4 Solvability of the approximate system

We claim the following result concerning solvability of the approximate problem (3.45–3.60).

■ Global Existence for the Approximate System:

Proposition 3.1. *Let ε, δ be given positive parameters.*

Under the hypotheses of Theorem 3.1*, there exists* $\Gamma_0 > 0$ *such that for any* $\Gamma > \Gamma_0$ *the approximate problem* (3.45–3.60) *admits a strong solution* $\{\varrho, \mathbf{u}, \vartheta\}$ *belonging to the following regularity class:*

$$\varrho \in C([0,T]; C^{2,\nu}(\overline{\Omega})),\ \partial_t \varrho \in C([0,T]; C^{0,\nu}(\overline{\Omega})),\ \inf_{[0,T]\times\overline{\Omega}} \varrho > 0,$$
$$\mathbf{u} \in C^1([0,T]; X_n),$$
$$\vartheta \in C([0,T]; W^{1,2}(\Omega)) \cap L^\infty((0,T)\times\Omega),\ \partial_t\vartheta,\ \Delta\mathcal{K}_\delta(\vartheta) \in L^2((0,T)\times\Omega),$$
$$\operatorname{ess}\inf_{(0,T)\times\Omega} \vartheta > 0. \tag{3.61}$$

Remark: *As a matter of fact, since the velocity field* $\mathbf{u}$ *is continuously differentiable, a bootstrap argument could be used in order to show that* ϑ *is smooth, hence a classical solution of* (3.55) *for* $t > 0$*, as soon as the thermodynamic functions* p*,* e *as well as the transport coefficients* μ*,* λ*, and* κ *are smooth functions of* ϱ*,* ϑ *on the set* $(0,\infty)^2$*.*

In spite of a considerable number of technicalities, the proof of Proposition 3.1 is based on standard arguments. We adopt the following strategy:

- The solution $\mathbf{u}$ of the approximate momentum equation (3.49) is looked for as a fixed point of a suitable integral operator in the Banach space $C([0,T]; X_n)$. Consequently, the functions ϱ, ϑ have to be determined in terms of $\mathbf{u}$. This is accomplished in the following manner:
- Given $\mathbf{u}$, the approximate continuity equation (3.45) is solved directly by means of the standard theory of linear parabolic equations.
- Having solved (3.45–3.47) we determine the temperature ϑ as a solution of the quasilinear parabolic problem (3.55–3.57), where ϱ, $\mathbf{u}$ play a role of given data.

3.4.1 Approximate continuity equation

The rest of this section is devoted to the proof of Proposition 3.1. We start with a series of preparatory steps. Following the strategy delineated in the previous paragraph, we fix a vector field $\mathbf{u}$ and discuss solvability of the Neumann-initial value problem (3.45–3.47).

■ APPROXIMATE CONTINUITY EQUATION:

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$, $\nu \in (0,1)$ and let $\mathbf{u} \in C([0,T]; X_n)$ be a given vector field. Suppose that $\varrho_{0,\delta}$ belongs to the class of regularity specified in* (3.48).

Then problem (3.45–3.47) *possesses a unique classical solution $\varrho = \varrho_{\mathbf{u}}$, more specifically,*

$$\varrho_{\mathbf{u}} \in V \equiv \left\{ \begin{array}{c} \varrho \in C([0,T]; C^{2,\nu}(\overline{\Omega})), \\ \partial_t \varrho \in C([0,T]; C^{0,\nu}(\overline{\Omega})). \end{array} \right\} \tag{3.62}$$

Moreover, the mapping $\mathbf{u} \in C([0,T]; X_n) \mapsto \varrho_{\mathbf{u}}$ maps bounded sets in $C([0,T]; X_n)$ into bounded sets in V and is continuous with values in $C^1([0,T] \times \overline{\Omega})$.

Finally,

$$\underline{\varrho}_0 \exp\Big(- \int_0^\tau \|\mathrm{div}_x \mathbf{u}\|_{L^\infty(\Omega)} \ \mathrm{d}t\Big) \leq \varrho_{\mathbf{u}}(\tau, x) \tag{3.63}$$
$$\leq \overline{\varrho}_0 \exp\Big(\int_0^\tau \|\mathrm{div}_x \mathbf{u}\|_{L^\infty(\Omega)} \ \mathrm{d}t\Big) \ \textit{for all } \tau \in [0,T], \ x \in \Omega,$$

where $\underline{\varrho}_0 = \inf_\Omega \varrho_{0,\delta}$, $\overline{\varrho}_0 = \sup_\Omega \varrho_{0,\delta}$.

Proof. Step 1. The unique strong solution of problem (3.45–3.48)

$$\varrho \in L^2(0,T; W^{2,2}(\Omega)) \cap C([0,T]; W^{1,2}(\Omega)), \ \partial_t \varrho \in L^2((0,T) \times \Omega)$$

that satisfies the estimate

$$\|\varrho\|_{C([0,T];W^{1,2}(\Omega))} + \|\varrho\|_{L^2(0,T;W^{2,2}(\Omega))} + \|\partial_t \varrho\|_{L^2((0,T)\times\Omega)} \leq c \|\varrho_{0,\delta}\|_{W^{1,2}(\Omega)},$$

with $c = c(\varepsilon, T, \|\mathbf{u}\|_{C([0,T];C^\nu(\overline{\Omega}))}) > 0$, may be constructed by means of the standard Galerkin approximation within the standard L^2 theory.

The maximal $L^p - L^q$ regularity resumed in Theorem 10.22 in Appendix applied to the problem

$$\partial_t \varrho - \varepsilon \Delta_x \varrho = f := -\mathrm{div}_x(\varrho \mathbf{u}), \ \nabla_x \varrho \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \varrho(0) = \varrho_{0,\delta} \tag{3.64}$$

combined with a bootstrap argument gives the bound

$$\|\varrho\|_{C([0,T];W^{2-\frac{2}{p},p}(\Omega))} + \|\varrho\|_{L^p(0,T;W^{2,p}(\Omega))} + \|\partial_t \varrho\|_{L^p((0,T)\times\Omega)} \leq c \|\varrho_{0,\delta}\|_{W^{2-\frac{2}{p},p}(\Omega)}$$

for any $p > 3$.

Since $W^{2-\frac{2}{p},p}(\Omega) \hookrightarrow C^{1,\nu}(\overline{\Omega})$ for any sufficiently large p, we have $\mathrm{div}_x(\varrho \mathbf{u}) \in C([0,T]; C^{1,\nu}(\overline{\Omega}))$ and may employ Theorem 10.23 from Appendix to show relation (3.62) as well as boundedness of the map $\mathbf{u} \mapsto \varrho_{\mathbf{u}}$: $C([0,T]; X_n) \to V$.

Step 2. The difference $\omega = \varrho_{\mathbf{u}_1} - \varrho_{\mathbf{u}_2}$ satisfies

$$\partial_t \omega - \varepsilon \Delta \omega + \mathrm{div}_x(\omega \mathbf{u}_1) = f := \mathrm{div}_x(\varrho_{\mathbf{u}_2}(\mathbf{u}_1 - \mathbf{u}_2)),\ \nabla_x \omega \cdot \mathbf{n}|_{\partial\Omega} = 0,\ \omega(0) = 0.$$

Similar reasoning as in the first step applied to this equation yields the continuity of the map $\mathbf{u} \mapsto \varrho_{\mathbf{u}}$ from $C([0,T];X_n)$ to $C^1([0,T] \times \overline{\Omega})$.

Step 3. The difference

$$\omega(t,x) = \varrho_{\mathbf{u}}(\tau,x) - \overline{\varrho}_0 \exp\left(\int_0^\tau \|\mathrm{div}_x \mathbf{u}\|_{L^\infty(\Omega)}\,\mathrm{d}t\right)$$

obeys a differential inequality

$$\partial_t \omega + \mathrm{div}_x(\omega \mathbf{u}) - \varepsilon \Delta_x \omega \le 0,\ \nabla_x \omega \cdot \mathbf{n}|_{\partial\Omega} = 0,\ \omega(0) = \varrho_0 - \overline{\varrho}_0 \le 0.$$

When multiplied on the positive part $|\omega|^+$ of ω and integrated over Ω, the first relation gives $\|\,|\omega|^+(t)\|_{L^2(\Omega)} \le 0$ which shows the right inequality in (3.63). The left inequality can be obtained in a similar way. Lemma 3.1 is thus proved. The reader may consult [79, Chapter 7.3] or [166, Section 7.2] for more details. □

3.4.2 Approximate internal energy equation

Having fixed $\mathbf{u}$, together with $\varrho = \varrho_{\mathbf{u}}$ – the unique solution of problem (3.45–3.47) – we focus on the approximate internal energy equation (3.55) that can be viewed as a quasilinear parabolic problem for the unknown ϑ.

Comparison principle. To begin, we establish a comparison principle in the class of strong (super, sub) solutions of problem (3.55–3.57). We recall that a function ϑ is termed a super (sub) solution if it satisfies (3.55) with "=" sign replaced by "$\ge$" ("$\le$").

Lemma 3.2. *Given the quantities*

$$\begin{gathered} \mathbf{u} \in C([0,T];X_n),\ \ \varrho \in C([0,T];C^2(\overline{\Omega})), \\ \partial_t \varrho \in C([0,T] \times \overline{\Omega}),\ \ \inf_{(0,T)\times\Omega} \varrho > 0, \end{gathered} \tag{3.65}$$

assume that $\underline{\vartheta}$ *and* $\overline{\vartheta}$ *are respectively a sub and super-solution to problem* (3.55–3.57) *belonging to the class*

$$\left\{ \begin{array}{c} \underline{\vartheta},\ \overline{\vartheta} \in L^2(0,T;W^{1,2}(\Omega)),\ \partial_t \underline{\vartheta},\ \partial_t \overline{\vartheta} \in L^2((0,T)\times\Omega), \\ \Delta \mathcal{K}_\delta(\underline{\vartheta}),\ \Delta \mathcal{K}_\delta(\overline{\vartheta}) \in L^2((0,T)\times\Omega), \end{array} \right\}, \tag{3.66}$$

$$\left\{ \begin{array}{c} 0 < \mathrm{ess\ inf}_{(0,T)\times\Omega} \underline{\vartheta} \le \mathrm{ess\ sup}_{(0,T)\times\Omega} \underline{\vartheta} < \infty, \\ 0 < \mathrm{ess\ inf}_{(0,T)\times\Omega} \overline{\vartheta} \le \mathrm{ess\ sup}_{(0,T)\times\Omega} \overline{\vartheta} < \infty, \end{array} \right\} \tag{3.67}$$

and satisfying

$$\underline{\vartheta}(0,\cdot) \le \overline{\vartheta}(0,\cdot) \ \textit{a.a. in } \Omega. \tag{3.68}$$

Then

$$\underline{\vartheta}(t,x) \le \overline{\vartheta}(t,x) \ \textit{a.a. in } (0,T)\times\Omega.$$

Proof. As $\underline{\vartheta}$, $\overline{\vartheta}$ belong to the regularity class specified in (3.66), we can compute

$$\begin{aligned}
&\mathrm{sgn}^+\Big(\varrho e_\delta(\varrho,\underline{\vartheta}) - \varrho e_\delta(\varrho,\overline{\vartheta})\Big)\Big[\Big(\partial_t\Big(\varrho e_\delta(\varrho,\underline{\vartheta}) - \varrho e_\delta(\varrho,\overline{\vartheta})\Big) \\
&\qquad + \nabla_x\Big(\varrho e_\delta(\varrho,\underline{\vartheta}) - \varrho e_\delta(\varrho,\overline{\vartheta})\Big)\cdot\mathbf{u}\Big] \\
&\qquad + \Delta_x\Big(\mathcal{K}_\delta(\overline{\vartheta}) - \mathcal{K}_\delta(\underline{\vartheta})\Big)\mathrm{sgn}^+\Big(\varrho e(\varrho,\underline{\vartheta}) - \varrho e(\varrho,\overline{\vartheta})\Big) \\
&\le |F(t,x,\underline{\vartheta}) - F(t,x,\overline{\vartheta})| \ \mathrm{sgn}^+\Big(\varrho e_\delta(\varrho,\underline{\vartheta}) - \varrho e_\delta(\varrho,\overline{\vartheta})\Big),
\end{aligned} \tag{3.69}$$

where we have introduced

$$\mathrm{sgn}^+(z) = \begin{cases} 0 \text{ if } z \le 0, \\ 1 \text{ if } z > 0, \end{cases}$$

and where we have set

$$\begin{aligned}
F(t,x,\vartheta) &= \mathbb{S}_\delta(\vartheta,\nabla_x\mathbf{u}(t,x)) : \nabla_x\mathbf{u}(t,x) + \big(\varepsilon\delta(\Gamma\varrho^{\Gamma-2}+2)|\nabla_x\varrho|^2\big)(t,x) \\
&\quad - \varrho(t,x)e_\delta(\varrho(t,x),\vartheta)\mathrm{div}_x\mathbf{u}(t,x) \\
&\quad - p(\varrho(t,x),\vartheta)\mathrm{div}_x\mathbf{u}(t,x) + \delta\frac{1}{\vartheta^2} - \varepsilon\vartheta^5 + \varrho\mathcal{Q}_\delta.
\end{aligned}$$

In accordance with our hypotheses, we may assume that $F = F(t,x,\vartheta)$ is globally Lipschitz with respect to ϑ.

Denoting by $|z|^+ = \max\{z,0\}$ the positive part, we have

$$\partial_t|w|^+ = \mathrm{sgn}^+(w)\partial_t w, \ \nabla_x|w|^+ = \mathrm{sgn}^+(w)\nabla_x w \text{ a.a. in } (0,T)\times\Omega$$

for any $w \in W^{1,2}((0,T)\times\Omega)$, in particular,

$$\begin{aligned}
&\mathrm{sgn}^+\Big(\varrho e_\delta(\varrho,\underline{\vartheta}) - \varrho e_\delta(\varrho,\overline{\vartheta})\Big) \\
&\qquad \times\Big[\Big(\partial_t\Big(\varrho e_\delta(\varrho,\underline{\vartheta}) - \varrho e_\delta(\varrho,\overline{\vartheta})\Big) + \nabla_x\Big(\varrho e_\delta(\varrho,\underline{\vartheta}) - \varrho e_\delta(\varrho,\overline{\vartheta})\Big)\cdot\mathbf{u}\Big] \\
&= \partial_t\Big|\varrho e_\delta(\varrho,\underline{\vartheta}) - \varrho e_\delta(\varrho,\overline{\vartheta})\Big|^+ + \nabla_x\Big|\varrho e_\delta(\varrho,\underline{\vartheta}) - \varrho e_\delta(\varrho,\overline{\vartheta})\Big|^+\cdot\mathbf{u}.
\end{aligned}$$

Moreover, as both e_δ and $\mathcal{K}_\delta$ are increasing functions of ϑ, we have

$$\mathrm{sgn}^+\Big(\varrho e_\delta(\varrho,\underline{\vartheta}) - \varrho e_\delta(\varrho,\overline{\vartheta})\Big) = \mathrm{sgn}^+\Big(\mathcal{K}_\delta(\underline{\vartheta}) - \mathcal{K}_\delta(\overline{\vartheta})\Big).$$

Seeing that

$$\int_\Omega \Delta_x w \ \mathrm{sgn}^+(w)\ \mathrm{d}x \le 0 \text{ whenever } w \in W^{2,2}(\Omega),\ \nabla_x w \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

we can integrate (3.69) in order to deduce

$$\int_\Omega \Big| \varrho e_\delta(\varrho, \underline{\vartheta}) - \varrho e_\delta(\varrho, \overline{\vartheta}) \Big|^+ (\tau)\ \mathrm{d}x$$
$$\le c \int_0^\tau \int_\Omega (1 + |\mathrm{div}_x \mathbf{u}|) \Big| \varrho e_\delta(\varrho, \underline{\vartheta}) - \varrho e_\delta(\varrho, \overline{\vartheta}) \Big|^+ \ \mathrm{d}x\ \mathrm{d}t$$

for any $\tau > 0$. Here we have used Lipschitz continuity of $F(t, x, \cdot)$ and the fact that $|\underline{\vartheta} - \overline{\vartheta}|\ \mathrm{sgn}^+[\varrho e_\delta(\varrho, \underline{\vartheta}) - \varrho e_\delta(\varrho, \overline{\vartheta})] \le c|\varrho e_\delta(\varrho, \underline{\vartheta}) - \varrho e_\delta(\varrho, \overline{\vartheta})|^+$ which follows from (3.9), (3.11), (3.65), (3.67). Thus a direct application of Gronwall's lemma, together with the monotonicity of e_δ with respect to ϑ, completes the proof. □

Corollary 3.1. *For given data ϱ, $\mathbf{u}$ satisfying* (3.65)*, and a measurable function $\vartheta_{0,\delta}$ such that*

$$0 < \underline{\vartheta}_0 = \operatorname*{ess\,inf}_\Omega \vartheta_{0,\delta} \le \operatorname*{ess\,sup}_\Omega \vartheta_{0,\delta} = \overline{\vartheta}_0 < \infty, \tag{3.70}$$

problem (3.55–3.57) *admits at most one* (*strong*) *solution ϑ in the class specified in* (3.66–3.67)*.*

Another application of Lemma 3.2 gives rise to uniform bounds on the function ϑ in terms of the data.

Corollary 3.2. *Let ϱ, $\mathbf{u}$ belong to the regularity class* (3.65)*, and let $\vartheta_{0,\delta}$ satisfy* (3.70)*. Suppose that ϑ is a* (*strong*) *solution of problem* (3.55–3.57) *belonging to the regularity class* (3.66)*.*

Then there exist two constants $\underline{\vartheta}$, $\overline{\vartheta}$ depending only on the quantities

$$\|\mathbf{u}\|_{C([0,T];X_n)},\ \|\varrho\|_{C^1([0,T]\times\overline{\Omega})}, \tag{3.71}$$

satisfying

$$0 < \underline{\vartheta} \le \underline{\vartheta}_0 \le \overline{\vartheta}_0 \le \overline{\vartheta}, \tag{3.72}$$

and

$$\underline{\vartheta} \le \vartheta(t, x) \le \overline{\vartheta} \text{ for a.a. } (t, x) \in (0, T) \times \Omega. \tag{3.73}$$

Proof. It is a routine matter to check that a *constant* function $\underline{\vartheta}$ is a subsolution of (3.55–3.57) as soon as

$$\frac{\delta}{\underline{\vartheta}^2} \ge \Big[\varepsilon \underline{\vartheta}^5 + p_M(\varrho, \underline{\vartheta}) \mathrm{div}_x \mathbf{u} + a \underline{\vartheta}^4 \mathrm{div}_x \mathbf{u} \tag{3.74}$$
$$+ \varrho \frac{\partial e_M(\varrho, \underline{\vartheta})}{\partial \varrho} \Big(\partial_t \varrho + \mathbf{u} \cdot \nabla_x \varrho \Big) + \Big(e_M(\varrho, \underline{\vartheta}) + a \underline{\vartheta}^4 + \delta \underline{\vartheta} \Big) \Big(\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) \Big)$$
$$- \mathbb{S}_\delta(\underline{\vartheta}, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \varepsilon\delta(\Gamma \varrho^{\Gamma-2} + 2)|\nabla_x \varrho|^2 - \varrho Q_\delta \Big].$$

Revoking (3.30) we can use hypotheses (3.65), (3.13), together with estimate (3.32), in order to see that all quantities on the right-hand side of (3.74) are bounded in terms of $\|\varrho\|_{C^1([0,T]\times\overline{\Omega})}$ and $\|\mathbf{u}\|_{C([0,T];X_n)}$ provided, say, $0 < \underline{\vartheta} < 1$. Note that all norms are equivalent when restricted to the finite-dimensional space X_n.

Consequently, a direct application of the comparison principle established in Lemma 3.2 yields the left inequality in (3.73).

Following step by step with obvious modifications the above procedure, the upper bound claimed in (3.73) can be established by help of the dominating term $-\varepsilon\vartheta^5$ in (3.55). □

Remark: *Corollary* 3.2 *reveals the role of the extra term* δ/ϑ^2 *in equation* (3.55), *namely to keep the absolute temperature* ϑ *bounded below away from zero at this stage of the approximation procedure. Positivity of* ϑ *is necessary for the passage from* (3.55) *to the entropy balance equation used in the weak formulation of the Navier-Stokes-Fourier system.*

A priori estimates. We shall derive *a priori* estimates satisfied by any strong solution of problem (3.55–3.57).

Lemma 3.3. *Let the data* ϱ, $\mathbf{u}$ *belong to the regularity class* (3.65), *and let* $\vartheta_{0,\delta} \in W^{1,2}(\Omega)$ *satisfy* (3.70).

Then any strong solution ϑ *of problem* (3.55–3.57) *belonging to the class* (3.66–3.67) *satisfies the estimate*

$$\begin{aligned} \operatorname*{ess\,sup}_{t\in(0,T)} \|\vartheta\|^2_{W^{1,2}(\Omega)} + \int_0^T \Big(\|\partial_t\vartheta\|^2_{L^2(\Omega)} + \|\Delta_x\mathcal{K}_\delta(\vartheta)\|^2_{L^2(\Omega)} \Big) \mathrm{d}t \\ \leq h\Big(\|\varrho\|_{C^1([0,T]\times\overline{\Omega})}, \|\mathbf{u}\|_{C([0,T;X_n)}, (\inf_{(0,T)\times\Omega} \varrho)^{-1}, \|\vartheta_{0,\delta}\|_{W^{1,2}(\Omega)} \Big), \end{aligned} \tag{3.75}$$

where h *is bounded on bounded sets.*

Proof. Note that relation (3.75) represents the standard *energy* estimates for problem (3.55–3.57). These are easily deduced via multiplying equation (3.55) by ϑ and integrating the resulting expression by parts in order to obtain

$$\begin{aligned} \frac{1}{2}\int_\Omega \varrho\frac{\partial e_\delta}{\partial\vartheta}(\varrho,\vartheta)\partial_t\vartheta^2 \, \mathrm{d}x - \int_\Omega \varrho e_\delta(\varrho,\vartheta)\nabla_x\vartheta\cdot\mathbf{u} \, \mathrm{d}x + \int_\Omega \kappa_\delta(\vartheta)|\nabla_x\vartheta|^2 \, \mathrm{d}x \\ = \int_\Omega F_1(t,x)\vartheta \, dx, \end{aligned} \tag{3.76}$$

where

$$\begin{aligned} F_1 = & -\frac{\partial(\varrho e_\delta)}{\partial\varrho}(\varrho,\vartheta)\partial_t\varrho + \mathbb{S}_\delta(\vartheta,\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} \\ & + \varepsilon\delta(\Gamma\varrho^{\Gamma-2}+2)|\nabla_x\varrho|^2 - p(\varrho,\vartheta)\mathrm{div}_x\mathbf{u} + \delta\frac{1}{\vartheta^2} - \varepsilon\vartheta^5 + \varrho\mathcal{Q}_\delta. \end{aligned}$$

In view of the uniform bounds already proved in (3.73), the function F_1 is bounded in $L^\infty((0,T)\times\Omega)$ in terms of the data.

Similarly, multiplying (3.55) on $\partial_t \mathcal{K}_\delta(\vartheta)$ gives rise to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}|\nabla_x \mathcal{K}_\delta(\vartheta)|^2 \ \mathrm{d}x + \int_\Omega \varrho\kappa_\delta(\vartheta)\frac{\partial e_\delta}{\partial\vartheta}(\varrho,\vartheta)|\partial_t\vartheta|^2 \ \mathrm{d}x \\ + \int_\Omega \varrho\frac{\partial e_\delta}{\partial\vartheta}(\varrho,\vartheta) \ \partial_t\vartheta\nabla_x\mathcal{K}_\delta(\vartheta)\cdot\mathbf{u} \ \mathrm{d}x = \int_\Omega F_2(t,x)\partial_t\vartheta \ \mathrm{d}x \tag{3.77}$$

where

$$F_2 = -\kappa_\delta(\vartheta)\Big(\partial_\varrho[\varrho e_\delta](\varrho,\vartheta)\partial_t\varrho - \partial_\varrho[\varrho e_\delta](\varrho,\vartheta)\nabla_x\varrho\cdot\mathbf{u} \\ - \varrho e_\delta(\varrho,\vartheta)\mathrm{div}_x\mathbf{u}\Big) + \mathbb{S}_\delta(\vartheta,\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} + \varepsilon\delta(\Gamma\varrho^{\Gamma-2}+2)|\nabla_x\varrho|^2 \\ - p(\varrho,\vartheta)\mathrm{div}_x\mathbf{u} + \delta\frac{1}{\vartheta^2} - \varepsilon\vartheta^5 + \varrho\mathcal{Q}_\delta$$

is bounded in $L^\infty((0,T)\times\Omega)$ in terms of the data.

Taking the sum of (3.76), (3.77), and using Young's inequality and Gronwall's lemma, we conclude that

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\nabla_x\mathcal{K}_\delta(\vartheta)\|^2_{L^2(\Omega;R^3)} + \int_0^T \|\partial_t\vartheta\|^2_{L^2(\Omega)} \ \mathrm{d}t \\ \leq h\Big(\|\varrho\|_{C^1([0,T]\times\overline{\Omega})}, \|\mathbf{u}\|_{C([0,T;X_n)}, (\inf_{(0,T)\times\Omega}\varrho)^{-1}, \|\vartheta_0\|_{W^{1,2}(\Omega)}\Big).$$

Finally, evaluating $\Delta_x\mathcal{K}_\delta(\vartheta)$ by means of equation (3.55), we get (3.75). □

Existence for the approximate internal energy equation. Having prepared the necessary material, we are ready to show existence of strong solutions to problem (3.55–3.57). In fact, the *a priori* bounds (3.73), (3.75) imply compactness of solutions in the space $C([0,T];W^{1,2}(\Omega))$, in particular, any accumulation point of a family of strong solutions is another solution of the same problem. Under these circumstances, showing *existence* is a routine matter. Regularizing the data ϱ, $\mathbf{u}$ with respect to the time variable, and approximating the quantities μ, η, κ_δ, e, p by smooth ones as the case may be, we can construct a family of approximate solutions to problem (3.55–3.57) via the classical results for quasilinear parabolic equations. Then we pass to the limit in a suitable sequence of approximate solutions to recover the (unique) solution of problem (3.55–3.57). The relevant theory of quasilinear parabolic equations taken over from the book Ladyzhenskaya et al. [129, Chapter V]) is summarized in Section 10.15 in Appendix.

Hereafter we describe a possible way to construct the approximations to problem (3.55–3.57).

(i) Let $\nu\in(0,1)$ be the same parameter as in Lemma 3.1. To begin, we extend $\varrho\in C([0,T];C^{2,\nu}(\overline{\Omega}))\cap C^1([0,T];C^{0,\nu}(\overline{\Omega}))$, $\mathbf{u}\in C([0,T];X_n)$, continuously to

$\varrho \in C(\mathbb{R}; C^{2,\nu}(\overline{\Omega})) \cap C^1(\mathbb{R}; C^{0,\nu}(\overline{\Omega}))$, $\operatorname{supp}\varrho \subset (-2T, 2T) \times \overline{\Omega}$, $\mathbf{u} \in C(\mathbb{R}, X_n)$, $\operatorname{supp}\mathbf{u} \subset (-2T, 2T) \times \overline{\Omega}$. We approximate $\mathcal{Q}_\delta$ by smooth functions $\mathcal{Q}_\omega$ on $[0,T] \times \overline{\Omega}$ and we take a sequence of initial conditions

$$C^{2,\nu}(\overline{\Omega}) \ni \vartheta_{0,\omega} \to \vartheta_{0,\delta} \text{ in } W^{1,2}(\Omega) \cap L^\infty(\Omega)$$

such that $\inf_{x\in\Omega} \vartheta_{0,\omega}(x) > \underline{\vartheta}_0$ uniformly with respect to $\omega \to 0+$, where $\underline{\vartheta}_0$ is a positive constant.

(ii) We denote

$$E_M(\varrho, \vartheta) = \varrho e_M(\varrho, \vartheta)$$

and set

$$\begin{aligned}
E_{\delta,\omega}(\varrho, \vartheta) &= [\langle E_M\rangle]^\omega(\varrho, \theta_\omega) + a\theta_\omega^4 + \delta\varrho\vartheta, \\
\{\partial_\vartheta E\}_{\delta,\omega}(\varrho, \vartheta) &= [\langle \partial_\vartheta E_M\rangle]^\omega(\varrho, \vartheta) + 4a\,\frac{\vartheta^4}{\sqrt{\vartheta^2+\omega^2}} + \delta\varrho, \\
\kappa_{\delta,\omega}(\vartheta) &= [\langle \kappa_M\rangle]^\omega(\theta_\omega) + [\langle \kappa_R\rangle]^\omega(\theta_\omega) + \delta(\theta_\omega^\Gamma + \frac{1}{\sqrt{\vartheta^2+\omega^2}}), \\
\mathcal{K}_{\delta,\omega}(\vartheta) &= \int_1^\vartheta \kappa_{\delta,\omega}(\tau)\,d\tau, \\
p_\omega(\varrho, \vartheta) &= [\langle p_M\rangle]^\omega(\varrho, \theta_\omega) + \frac{a}{3}\theta_\omega^4, \\
G(t,x) &= \Big((\Gamma\varrho^{\Gamma-2} + 2)|\nabla_x\varrho|^2\Big)(t,x),\ G_\omega(t,x) = G^\omega(t,x), \\
\mathbb{S}_{\delta,\omega}(\vartheta, \nabla_x\mathbf{u}^\omega) &= \langle\mu\rangle^\omega(\theta_\omega)\Big(\nabla\mathbf{u}^\omega + \nabla^T\mathbf{u}^\omega - \frac{2}{3}\operatorname{div}\mathbf{u}^\omega\mathbb{I}\Big) + \langle\eta\rangle^\omega(\theta_\omega)\operatorname{div}\mathbf{u}^\omega\mathbb{I},
\end{aligned} \tag{3.78}$$

where

$$\begin{aligned}
\theta_\omega = \theta_\omega(\vartheta) &= \frac{\sqrt{\vartheta^2+\omega^2}}{1+\omega\sqrt{\vartheta^2+\omega^2}}, \\
\langle a\rangle(z) &= \left\{ \begin{array}{c} a(z) \text{ if } z \in (0,\infty)^N \\ \max\{\inf_{z\in(0,\infty)^N} a(z)\,,\,0\} \end{array} \right\},\ N = 1,2.
\end{aligned} \tag{3.79}$$

The operator $b \mapsto b^\omega$, $\omega > 0$ is the standard regularizing operator, see (10.2) in Appendix 10.1, that applies to all independent variables in the case of functions $\langle E_M\rangle$, $\langle\partial_\vartheta E_M\rangle$, $\langle p\rangle$, $\langle\mu\rangle$, $\langle\eta\rangle$, $\langle\kappa_M\rangle$, and to the variable t in the case of functions $\varrho(t,x)$, $\mathbf{u}(t,x)$, $G(t,x)$. Notice that by virtue of hypotheses (3.21–3.23) and (3.11),

$$\kappa_{\delta,\omega}(\vartheta) \ge \underline{\kappa}_M > 0, \quad \{\partial_\vartheta E\}_{\delta,\omega}(\varrho,\vartheta) > \delta\underline{\varrho} > 0 \tag{3.80}$$

for all $(\varrho, \vartheta) \in \mathbb{R}^2$, where $\underline{\varrho} = \inf_{(0,T)\times\Omega}\varrho$.

(iii) We will find a solution of problem (3.55–3.57), as a limit of the sequence $\{\vartheta_\omega\}_{\omega>0}$ of solutions to the equation

$$
\begin{aligned}
&\{\partial_\vartheta E\}_{\delta,\omega}(\varrho^\omega,\vartheta)\partial_t\vartheta + \operatorname{div}\Big(E_{\delta,\omega}(\varrho^\omega,\vartheta)\mathbf{u}\Big) - \Delta_x\mathcal{K}_{\delta,\omega}(\vartheta)\\
&= -\partial_\varrho E_{\delta,\omega}(\varrho^\omega,\vartheta)\partial_t\varrho^\omega + \mathbb{S}_{\delta,\omega}(\nabla_x\mathbf{u}^\omega,\vartheta) : \nabla\mathbf{u}^\omega\\
&\qquad + \varepsilon\delta G_\omega - p_\omega(\varrho^\omega,\vartheta) - \frac{\delta}{\vartheta^2+\omega^2} + \varepsilon\theta_\omega^5 + \varrho^\omega\mathcal{Q}_\omega,\\
&\nabla_x\vartheta\cdot\mathbf{n}|_{\partial\Omega} = 0,\ \vartheta(0,x) = \vartheta_{0,\omega}(x).
\end{aligned}
\tag{3.81}
$$

Problem (3.81) for the unknown ϑ has the following quasilinear parabolic equation form:

$$
\begin{aligned}
\partial_t\vartheta - \sum_{i,j=1}^3 a_{ij}(t,x,\vartheta)\partial_{x_i}\partial_{x_j}\vartheta + b(t,x,\vartheta,\nabla_x\vartheta) = 0 \quad \text{in } (0,T)\times\Omega,\\
\Big(\sum_{i,j=1}^3 a_{ij}\partial_{x_j}\vartheta\, n_i + \psi\Big)\Big|_{(0,T)\times\partial\Omega} = 0,\\
\vartheta|_{\{0\}\times\Omega} = 0,
\end{aligned}
\tag{3.82}
$$

where

$$
a_{ij}(t,x,\vartheta) = \frac{\kappa_{\delta,\omega}(\vartheta)}{[\partial_\vartheta E]_{\delta,\omega}(\varrho^\omega(t,x),\vartheta)}\delta_{ij}, \quad i,j = 1,2,3, \quad \psi = 0 \tag{3.83}
$$

and

$$
\begin{aligned}
b(t,x,\vartheta,\mathbf{z}) = &\frac{1}{\{\partial_\vartheta E\}_{\delta,\omega}(\varrho^\omega(t,x),\vartheta)}\\
&\times\Big[-\kappa'_{\delta,\omega}(\vartheta)|\mathbf{z}|^2 + \partial_\varrho E_{\delta,\omega}(\varrho^\omega(t,x),\vartheta)\partial_t\varrho^\omega(t,x)\\
&\quad + \partial_\varrho E_{\delta,\omega}(\varrho^\omega(t,x),\vartheta)\big(\nabla\varrho^\omega\cdot\mathbf{u}^\omega\big)(t,x)\\
&\quad - \mathbb{S}_{\delta,\omega}(\nabla_x\mathbf{u}^\omega(t,x),\vartheta) : \nabla\mathbf{u}^\omega(t,x)\\
&\quad + \partial_\vartheta E_{\delta,\omega}(\varrho^\omega(t,x),\vartheta)\big(\mathbf{z}\cdot\mathbf{u}^\omega\big)(t,x)\\
&\quad + E_{\delta,\omega}(\varrho^\omega(t,x),\vartheta)\operatorname{div}_x\mathbf{u}^\omega + p_\omega(\varrho^\omega(t,x),\vartheta)\operatorname{div}_x\mathbf{u}^\omega(t,x)\\
&\quad - \epsilon\delta G_\omega(t,x) + \frac{\delta}{\vartheta^2+\omega^2} - \varepsilon\theta_\omega^5(\vartheta) - \varrho^\omega\mathcal{Q}_\omega(t,x)\Big].
\end{aligned}
\tag{3.84}
$$

In accordance with the properties of mollifiers recalled in Section 10.1 in Appendix, a_{ij}, b, ψ satisfy assumptions of Theorem 10.24 from Section 10.15. Therefore, problem (3.81) admits a (unique) solution $\vartheta = \vartheta_\omega$ which belongs to class

$$
\vartheta_\omega \in C([0,T];C^{2,\nu}(\overline{\Omega})) \cap C^1([0,T]\times\overline{\Omega}), \quad \partial_t\vartheta_\omega \in C^{0,\nu/2}([0,T];C(\overline{\Omega})).
$$

(iv) The proofs of Lemma 3.2, Corollary 3.2 and Lemma 3.3 apply with minor modifications to system (3.81), yielding the uniform bounds

$$\|\frac{1}{\vartheta_\omega}\|_{L^\infty((0,T)\times\Omega)} + \|\vartheta_\omega\|_{L^\infty((0,T)\times\Omega)} \leq c,$$

$$\text{ess} \sup_{t\in(0,T)} \|\vartheta_\omega\|^2_{W^{1,2}(\Omega)} + \int_0^T \Big(\|\partial_t\vartheta_\omega\|^2_{L^2(\Omega)} + \|\Delta_x\mathcal{K}_\delta(\vartheta_\omega)\|^2_{L^2(\Omega)}\Big)\mathrm{d}t \leq c$$

with respect to $\omega \to 0+$. With these bounds and the properties of mollifiers recalled in Section 10.1 at hand, the limit passage from system (3.81) to (3.55–3.57) is an easy exercise.

The results achieved in this section can be stated as follows.

■ APPROXIMATE INTERNAL ENERGY EQUATION:

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$, $\nu \in (0,1)$. Let $\mathbf{u} \in C([0,T];X_n)$ be a given vector field and let $\varrho = \varrho_{\mathbf{u}}$ be the unique solution of the approximate problem* (3.45–3.47) *constructed in Lemma* 3.1. *Further*

(i) *let the initial datum $\vartheta_{0,\delta} \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ be bounded below away from zero as stated in hypothesis* (3.58) *and the source term $\mathcal{Q}_\delta$ satisfies* (3.60)*;*

(ii) *let the constitutive functions p, e, s and the transport coefficients μ, η, κ obey the structural assumptions* (3.7–3.23)*.*

Then problem (3.55–3.57)*, with e_δ, $\mathcal{K}_\delta$ defined in* (3.59) *and $\mathbf{u}$, $\varrho_{\mathbf{u}}$ fixed, possesses a unique strong solution $\vartheta = \vartheta_{\mathbf{u}}$ belonging to the regularity class*

$$Y = \left\{ \begin{array}{c} \partial_t\vartheta \in L^2((0,T)\times\Omega),\ \Delta_x\mathcal{K}_\delta(\vartheta) \in L^2((0,T)\times\Omega), \\ \vartheta \in L^\infty(0,T;W^{1,2}(\Omega)\cap L^\infty(\Omega)), \quad \frac{1}{\vartheta} \in L^\infty((0,T)\times\Omega). \end{array} \right\} \tag{3.85}$$

Moreover, the mapping $\mathbf{u} \to \vartheta_{\mathbf{u}}$ maps bounded sets in $C([0,T];X_n)$ into bounded sets in Y and is continuous with values in $L^2(0,T;W^{1,2}(\Omega))$.

3.4.3 Local solvability of the approximate problem

At this stage, we are ready to show the existence of approximate solutions on a possibly short time interval $(0,T_{\max})$. In accordance with (3.50), X_n is a finite-dimensional subspace of $L^2(\Omega,\mathbb{R}^3)$ endowed with the Hilbert structure induced by $L^2(\Omega;\mathbb{R}^3)$. We denote by P_n the orthogonal projection of $L^2(\Omega,\mathbb{R}^3)$ onto X_n. Furthermore, we set

$$\mathbf{u}_{0,\delta} = \frac{(\varrho\mathbf{u})_0}{\varrho_{0,\delta}}, \quad \mathbf{u}_{0,\delta,n} = P_n[\mathbf{u}_{0,\delta}]. \tag{3.86}$$

We start rewriting (3.49) as a fixed point problem:

$$\mathbf{u}(\tau) = J\Big[\varrho(\tau), \int_0^\tau M(t,\varrho(t),\vartheta(t),\mathbf{u}(t))\mathrm{d}t + (\varrho\mathbf{u})^*_0\Big] \equiv S[\mathbf{u}](\tau),\ \tau \in [0,T], \tag{3.87}$$

where we have written

$$(\varrho\mathbf{u})_0^* \in X_n^*,\ \langle(\varrho\mathbf{u})_0^*;\varphi\rangle \equiv \int_\Omega (\varrho\mathbf{u})_0\cdot\varphi\ \mathrm{d}x \text{ for all } \varphi\in X_n,$$
$$M(t,\varrho,\vartheta,\mathbf{u})\in X_n^*,$$
$$\langle M(t,\varrho,\vartheta,\mathbf{u});\varphi\rangle = \int_\Omega \Big(\varrho[\mathbf{u}\otimes\mathbf{u}]:\nabla_x\varphi + (p+\delta(\varrho^\Gamma+\varrho^2))\mathrm{div}_x\varphi\Big)\ \mathrm{d}x$$
$$-\int_\Omega\Big(\varepsilon(\nabla_x\varrho\nabla_x\mathbf{u})\cdot\varphi + \mathbb{S}_\delta:\nabla_x\varphi - \varrho\mathbf{f}_\delta(t)\cdot\varphi\Big)\ \mathrm{d}x \text{ for all } \varphi\in X_n^*,$$

and

$$J[\varrho,\cdot]: X_n^*\to X_n,\ \int_\Omega \varrho J[\varrho,\chi]\cdot\varphi\ \mathrm{d}x = \langle\chi;\varphi\rangle\ \text{ for all } \chi\in X_n^*,\ \varphi\in X_n.$$

Note that

$$\|\ J[\varrho,\chi]\ \|_{X_n} \le \frac{1}{A}\ \|\ \chi\ \|_{X_n^*},\quad A = \inf_{(t,x)\in(0,T)\times\Omega}\varrho(t,x) \tag{3.88}$$

and

$$\begin{aligned}\|\ J[\varrho_1,\chi]-J[\varrho_2,\chi]\ \|_{X_n} &\le \frac{c}{A_1A_2}\|\varrho_1-\varrho_2\|_{L^\infty(\Omega)}\|\chi\|_{X_n^*},\\ A_i &= \inf_{(t,x)\in(0,T)\times\Omega}\varrho_i(t,x),\ i=1,2,\end{aligned} \tag{3.89}$$

where $c>0$ depends solely on n; in particular, it is independent of the data specified in (2.41) and the parameters ε, δ, Γ.

Given $\mathbf{u}\in C([0,T];X_n)$, the density $\varrho=\varrho_{\mathbf{u}}$ can be identified as the unique (classical) solution of the parabolic problem (3.45–3.48), the existence of which is guaranteed by Lemma 3.1. In particular, the (approximate) density $\varrho_{\mathbf{u}}$ remains bounded below away from zero as soon as we can control $\mathrm{div}_x\mathbf{u}$. Note that, at this level of approximation, the norm of $\mathrm{div}_x\mathbf{u}$ is dominated by that of $\mathbf{u}$ as the dimension of X_n is finite.

With $\mathbf{u}$, $\varrho_{\mathbf{u}}$ at hand, the temperature $\vartheta=\vartheta_{\mathbf{u}}$ can be determined as the unique solution of problem (3.55–3.57) constructed by means of Lemma 3.4, in particular, ϑ is strictly positive with a lower bound in terms of the data, see Corollary 3.2.

If $\|\mathbf{u}\|_{C([0,T];X_n)}\le R$, then

$$\begin{aligned}&\|J[\varrho(\tau),\int_0^\tau M(t,\varrho(t),\mathbf{u}(t),\vartheta(t))\,\mathrm{d}t + (\varrho\mathbf{u})_0^*\|_{X_n}\\ &\quad\le c_0\frac{\overline{\varrho}_0}{\underline{\varrho}_0}\exp(2R\tau)\|\mathbf{u}_{0,\delta,n}\|_{X_n} + \tau h(R) \text{ for all } \tau\in[0,T],\end{aligned} \tag{3.90}$$

where we have used Lemmas 3.1, 3.4, specifically, bounds (3.62), (3.85). The constant c_0, determined in terms of equivalence of norms on X_n, depends solely on n and h is a positive function bounded on bounded sets.

Consequently, if

$$R > 2c_0 \frac{\overline{\varrho}_0}{\underline{\varrho}_0} \|\mathbf{u}_{0,\delta,n}\|_{X_n}, \tag{3.91}$$

the operator $\mathbf{u} \mapsto S[\mathbf{u}]$ determined through (3.87) maps the ball

$$B_{R,\tau_0} = \left\{ \mathbf{u} \in C([0,\tau_0], X_n) \,\middle|\, \|\mathbf{u}\|_{C([0,\tau_0];X_n)} \leq R, \ \mathbf{u}(0) = \mathbf{u}_{0,\delta,n} \right\} \tag{3.92}$$

into itself as soon as τ_0 is small enough.

Moreover, as a consequence of (3.89) and smoothness of ϱ, the image of B_{R,τ_0} consists of uniformly Lipschitz functions on $[0, \tau_0]$, in particular, it belongs to a compact set in $C([0, \tau_0]; X_n)$. Thus a direct application of the Leray-Schauder fixed point theorem yields existence of a solution $\{\varrho, \mathbf{u}, \vartheta\}$ of the approximate problem (3.45–3.57) defined on a (possibly short) time interval $[0, T(n)]$. Finally, taking advantage of Lemma 3.1, we deduce from (3.87) that

$$\mathbf{u} \in C^1([0, T(n)]; X_n). \tag{3.93}$$

The above procedure can be iterated as many times as necessary to reach $T(n) = T$ as long as there is a bound on $\mathbf{u}$ independent of $T(n)$. The existence of such a bound is the main topic discussed in the next section.

3.4.4 Uniform estimates and global existence

Let $\{\varrho, \mathbf{u}, \vartheta\}$ be an approximate solution of problem (3.45–3.57) defined on a time interval $[0, T_{\max}]$, $T_{\max} \leq T$. The last step in the proof of Proposition 3.1 is to establish a uniform (in time) bound on the norm $\|\mathbf{u}(t)\|_{X_n}$ for $t \in [0, T_{\max}]$ *independent* of $T_{\max}$. The existence of such a bound allows us to iterate the local construction described in the previous section in order to obtain an approximate solution defined on the full time interval $[0, T]$. To this end, the *a priori* estimates derived in Section 2.2 will be adapted in order to accommodate the extra terms arising at the actual level of approximation.

First of all, it follows from (3.45), (3.46) that the total mass remains constant in time, specifically,

$$\int_\Omega \varrho(t) \, \mathrm{d}x = \int_\Omega \varrho_{0,\delta} \, \mathrm{d}x = M_{0,\delta} \text{ for all } t \in [0, T_{\max}]. \tag{3.94}$$

The next observation is that the quantity $\psi\mathbf{u}$, with $\psi = \psi(t)$, $\psi \in C^1_c[0, T_{\max})$, can be taken as a test function in the variational formulation of the momentum equation (3.49) to obtain

$$\int_\Omega \Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + \delta(\frac{\varrho^\Gamma}{\Gamma-1} + \varrho^2)\Big)(\tau)\ \mathrm{d}x + \varepsilon\delta\int_0^\tau\int_\Omega |\nabla_x\varrho|^2(\Gamma\varrho^{\Gamma-2}+2)\ \mathrm{d}x\ \mathrm{d}t$$
$$= \int_\Omega \Big(\frac{1}{2}(\varrho\mathbf{u})_0\mathbf{u}(0) + \delta(\frac{\varrho_{0,\delta}^\Gamma}{\Gamma-1} + \varrho_{0,\delta}^2)\Big)\ \mathrm{d}x + \int_0^\tau\int_\Omega \Big(p\mathrm{div}_x\mathbf{u} - \mathbb{S}_\delta : \nabla_x\mathbf{u}\Big)\ \mathrm{d}x\ \mathrm{d}t$$
$$+ \int_0^\tau\int_\Omega \varrho\mathbf{f}_\delta\cdot\mathbf{u}\ \mathrm{d}x\ \mathrm{d}t, \tag{3.95}$$

which, combined with (3.55), gives rise to the approximate energy balance

$$\int_\Omega \Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e_\delta(\varrho,\vartheta) + \delta(\frac{\varrho^\Gamma}{\Gamma-1} + \varrho^2)\Big)(\tau)\ \mathrm{d}x \tag{3.96}$$
$$= \int_\Omega \Big(\frac{1}{2}(\varrho\mathbf{u})_0\mathbf{u}(0) + \varrho_{0,\delta}e_\delta(\varrho_{0,\delta},\vartheta_{0,\delta}) + \delta(\frac{\varrho_{0,\delta}^\Gamma}{\Gamma-1} + \varrho_{0,\delta}^2)\Big)\ \mathrm{d}x$$
$$+ \int_0^\tau\int_\Omega \Big(\varrho\mathbf{f}_\delta\cdot\mathbf{u} + \varrho\mathcal{Q}_\delta + \delta\frac{1}{\vartheta^2} - \varepsilon\vartheta^5\Big)\ \mathrm{d}x\ \mathrm{d}t \text{ for all } \tau\in[0,T_{\max}].$$

Moreover, dividing the approximate internal energy equation (3.55) on ϑ, we obtain, after a straightforward manipulation, an approximate entropy production equation in the form

$$\partial_t(\varrho s_\delta(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s_\delta(\varrho,\vartheta)\mathbf{u}) - \mathrm{div}_x\Big[\Big(\frac{\kappa(\vartheta)}{\vartheta} + \delta(\vartheta^{\Gamma-1} + \frac{1}{\vartheta^2})\Big)\nabla_x\vartheta\Big]$$
$$= \frac{1}{\vartheta}\Big[\mathbb{S}_\delta : \nabla_x\mathbf{u} + \Big(\frac{\kappa(\vartheta)}{\vartheta} + \delta(\vartheta^{\Gamma-1} + \frac{1}{\vartheta^2})\Big)|\nabla_x\vartheta|^2 + \delta\frac{1}{\vartheta^2}\Big] + \frac{\varepsilon\delta}{\vartheta}(\Gamma\varrho^{\Gamma-2}+2)|\nabla_x\varrho|^2$$
$$+ \varepsilon\frac{\Delta_x\varrho}{\vartheta}\Big(\vartheta s_\delta(\varrho,\vartheta) - e_\delta(\varrho,\vartheta) - \frac{p(\varrho,\vartheta)}{\varrho}\Big) - \varepsilon\vartheta^4 + \frac{\varrho}{\vartheta}\mathcal{Q}_\delta \tag{3.97}$$

satisfied a.a. in $(0,T_{\max})\times\Omega$, where

$$s_\delta(\varrho,\vartheta) = s(\varrho,\vartheta) + \delta\log\vartheta, \tag{3.98}$$

and

$$\vartheta s_\delta(\varrho,\vartheta) - e_\delta(\varrho,\vartheta) - \frac{p(\varrho,\vartheta)}{\varrho} = \vartheta s_{M,\delta}(\varrho,\vartheta) - e_{M,\delta}(\varrho,\vartheta) - \frac{p_M(\varrho,\vartheta)}{\varrho}. \tag{3.99}$$

Relations (3.96), (3.97) give rise to uniform estimates similar to those obtained in Section 2.2.3. Indeed, multiplying (3.97) on $\overline{\vartheta}$, where $\overline{\vartheta}$ is an arbitrary positive constant, integrating over Ω, and subtracting the resulting expression from

(3.97), we get

$$\begin{aligned}
&\int_\Omega \Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + H_{\delta,\overline{\vartheta}}(\varrho,\vartheta) + \delta(\frac{\varrho^\Gamma}{\Gamma-1} + \varrho^2)\Big)(\tau)\ \mathrm{d}x \\
&\quad + \overline{\vartheta}\int_0^\tau\int_\Omega \frac{1}{\vartheta}\Big[\mathbb{S}_\delta : \nabla_x\mathbf{u} + \Big(\frac{\kappa(\vartheta)}{\vartheta} + \delta(\vartheta^{\Gamma-1} + \frac{1}{\vartheta^2})\Big)|\nabla_x\vartheta|^2 + \delta\frac{1}{\vartheta^2} \\
&\quad + \varepsilon\delta(\Gamma\varrho^{\Gamma-2} + 2)|\nabla_x\varrho|^2\Big]\ \mathrm{d}x\ \mathrm{d}t + \int_0^\tau\int_\Omega \varepsilon\vartheta^5\ \mathrm{d}x\mathrm{d}t \\
&= \int_\Omega \Big(\frac{1}{2}\varrho_{0,\delta}|\mathbf{u}_{0,\delta}|^2 + H_{\delta,\overline{\vartheta}}(\varrho_{0,\delta},\vartheta_{0,\delta}) + \delta(\frac{\varrho_{0,\delta}^\Gamma}{\Gamma-1} + \varrho_{0,\delta}^2)\Big)\ \mathrm{d}x \\
&\quad + \int_0^\tau\int_\Omega \Big(\varrho\mathbf{f}_\delta\cdot\mathbf{u} + \varrho\Big(1 - \frac{\overline{\vartheta}}{\vartheta}\Big)\mathcal{Q}_\delta + \frac{\delta}{\vartheta^2} + \varepsilon\overline{\vartheta}\vartheta^4\Big)\ \mathrm{d}x\ \mathrm{d}t \\
&\quad - \varepsilon\overline{\vartheta}\int_0^\tau\int_\Omega \frac{\Delta_x\varrho}{\vartheta}\Big(\vartheta s_\delta(\varrho,\vartheta) - e_\delta(\varrho,\vartheta) - \frac{p(\varrho,\vartheta)}{\varrho}\Big)\ \mathrm{d}x\ \mathrm{d}t \\
&\qquad \text{for all } \tau\in[0,T_{\max}],
\end{aligned} \tag{3.100}$$

where $H_{\delta,\overline{\vartheta}}$ is an analogue of the Helmholtz function introduced in (2.48), specifically,

$$H_{\delta,\overline{\vartheta}}(\varrho,\vartheta) = \varrho e_\delta(\varrho,\vartheta) - \overline{\vartheta}\varrho s_\delta(\varrho,\vartheta) = H_{\overline{\vartheta}}(\varrho,\vartheta) + \delta\varrho(\vartheta - \overline{\vartheta}\log\vartheta). \tag{3.101}$$

Here, in accordance with (3.99),

$$\begin{aligned}
&\int_0^\tau\int_\Omega \frac{\Delta_x\varrho}{\vartheta}\Big(\vartheta s_\delta(\varrho,\vartheta) - e_\delta(\varrho,\vartheta) - \frac{p(\varrho,\vartheta)}{\varrho}\Big)\ \mathrm{d}x\ \mathrm{d}t \\
&= -\int_0^\tau\int_\Omega \frac{\partial}{\partial\varrho}\Big(\vartheta s_M(\varrho,\vartheta) - e_M(\varrho,\vartheta) - \frac{p_M(\varrho,\vartheta)}{\varrho}\Big)\frac{|\nabla_x\varrho|^2}{\vartheta}\ \mathrm{d}x\ \mathrm{d}t \\
&\quad - \int_0^\tau\int_\Omega \frac{\partial}{\partial\vartheta}\Big(s_{M,\delta}(\varrho,\vartheta) - \frac{e_{M,\delta}(\varrho,\vartheta)}{\vartheta} - \frac{p_M(\varrho,\vartheta)}{\varrho\vartheta}\Big)\nabla_x\varrho\cdot\nabla_x\vartheta\ \mathrm{d}x\ \mathrm{d}t,
\end{aligned} \tag{3.102}$$

where, by virtue of Gibbs' relation (3.7),

$$\frac{\partial}{\partial\varrho}\Big(\vartheta s_M(\varrho,\vartheta) - e_M(\varrho,\vartheta) - \frac{p_M(\varrho,\vartheta)}{\varrho}\Big) = -\frac{1}{\varrho}\frac{\partial p_M}{\partial\varrho}(\varrho,\vartheta), \tag{3.103}$$

$$\frac{\partial}{\partial\vartheta}\Big(s_{M,\delta}(\varrho,\vartheta) - \frac{e_{M,\delta}(\varrho,\vartheta)}{\vartheta} - \frac{p_M(\varrho,\vartheta)}{\varrho\vartheta}\Big) = \frac{1}{\vartheta^2}\Big(e_{M,\delta}(\varrho,\vartheta) + \varrho\frac{\partial e_M(\varrho,\vartheta)}{\partial\varrho}\Big). \tag{3.104}$$

Equality (3.100) therefore transforms to

$$\begin{aligned}
&\int_\Omega \Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + H_{\delta,\overline{\vartheta}}(\varrho,\vartheta) + \delta(\frac{\varrho^\Gamma}{\Gamma-1} + \varrho^2)\Big)(\tau)\ \mathrm{d}x \\
&\quad + \overline{\vartheta}\int_0^\tau\int_\Omega \sigma_{\varepsilon,\delta}\ \mathrm{d}x\ \mathrm{d}t + \int_0^\tau\int_\Omega \varepsilon\vartheta^5\ \mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \Big(\frac{1}{2} \varrho_{0,\delta} |\mathbf{u}_{0,\delta}|^2 + H_{\delta,\overline{\vartheta}}(\varrho_{0,\delta}, \vartheta_{0,\delta}) + \delta\big(\frac{\varrho_{0,\delta}^\Gamma}{\Gamma - 1} + \varrho_{0,\delta}^2 \big) \Big) \ \mathrm{d}x \\
&+ \int_0^\tau \int_\Omega \Big(\varrho \mathbf{f}_\delta \cdot \mathbf{u} + \varrho \Big(1 - \frac{\overline{\vartheta}}{\vartheta} \Big) \mathcal{Q}_\delta + \frac{\delta}{\vartheta^2} + \varepsilon \overline{\vartheta} \vartheta^4 \Big) \ \mathrm{d}x \ \mathrm{d}t \\
&+ \varepsilon \int_0^\tau \int_\Omega \frac{\overline{\vartheta}}{\vartheta^2} \Big(e_{M,\delta}(\varrho, \vartheta) + \varrho \frac{\partial e_M}{\partial \varrho}(\varrho, \vartheta) \Big) \nabla_x \varrho \nabla_x \vartheta \, \mathrm{d}x \mathrm{d}t
\end{aligned}
$$

for all $\tau \in [0, T_{\max}]$, (3.105)

where

$$
\begin{aligned}
\sigma_{\varepsilon,\delta} = \frac{1}{\vartheta} \Big[\mathbb{S}_\delta : \nabla_x \mathbf{u} + \Big(\frac{\kappa(\vartheta)}{\vartheta} + \delta(\vartheta^{\Gamma-1} + \frac{1}{\vartheta^2}) \Big) |\nabla_x \vartheta|^2 + \delta \frac{1}{\vartheta^2} \\
+ \frac{\varepsilon \delta}{\vartheta} (\Gamma \varrho^{\Gamma-2} + 2) |\nabla_x \varrho|^2 + \varepsilon \frac{\overline{\vartheta}}{\varrho \vartheta} \frac{\partial p_M}{\partial \varrho}(\varrho, \vartheta) |\nabla_x \varrho|^2. \qquad (3.106)
\end{aligned}
$$

Similarly to Section 2.2.3, relation (3.105) provides all the necessary uniform estimates as soon as we check that the terms on the right-hand side can be controlled by the positive quantities on the left-hand side. In order to see that, observe that the term δ/ϑ^2 on the right-hand side of (3.105) is dominated by its counterpart δ/ϑ^3 in the entropy production term $\sigma_{\varepsilon,\delta}$. Analogously, the quantity $\varepsilon \overline{\vartheta} \vartheta^4$ at the right-hand side is "absorbed" by the term $\varepsilon \vartheta^5$ at the left-hand side of (3.105). Finally, the term $\varrho(1 - \frac{\overline{\vartheta}}{\vartheta}) Q_\delta$ can be written as a sum $\varrho(1 - \frac{\overline{\vartheta}}{\vartheta}) Q_\delta 1_{\{\vartheta \le \overline{\vartheta}\}} + \varrho(1 - \frac{\overline{\vartheta}}{\vartheta}) Q_\delta 1_{\{\vartheta > \overline{\vartheta}\}}$, where $\int_0^\tau \int_\Omega \varrho(1 - \frac{\overline{\vartheta}}{\vartheta}) Q_\delta 1_{\{\vartheta \le \overline{\vartheta}\}} \, \mathrm{d}x \, \mathrm{d}t \le 0$, while $| \int_0^\tau \int_\Omega \varrho(1 - \frac{\overline{\vartheta}}{\vartheta}) Q_\delta 1_{\{\vartheta > \overline{\vartheta}\}} \, \mathrm{d}x \, \mathrm{d}t|$ is bounded by $\overline{\varrho} T |\Omega| \|Q_\delta\|_{L^\infty((0,T)\times\Omega)}$.

Consequently, it remains to handle the quantity

$$
\varepsilon \int_\Omega \frac{1}{\vartheta^2} \Big(e_M(\varrho, \vartheta) + \varrho \frac{\partial e_M(\varrho, \vartheta)}{\partial \varrho} \Big) \nabla_x \varrho \cdot \nabla_x \vartheta \ \mathrm{d}x
$$

appearing on the right-hand side of (3.105). To this end, we first use hypothesis (3.13), together with (3.30), in order to obtain

$$
\Big| \frac{1}{\vartheta^2} \Big(e_M(\varrho, \vartheta) + \varrho \frac{\partial e_M(\varrho, \vartheta)}{\partial \varrho} \Big) \nabla_x \varrho \cdot \nabla_x \vartheta \Big| \le c \Big(\frac{\varrho^{\frac{2}{3}} + \vartheta}{\vartheta^2} \Big) |\nabla_x \varrho| |\nabla_x \vartheta|,
$$

where, furthermore,

$$
\frac{|\nabla_x \varrho| |\nabla_x \vartheta|}{\vartheta} \le \omega \frac{|\nabla_x \varrho|^2}{\vartheta} + c(\omega) \frac{|\nabla_x \vartheta|^2}{\vartheta} \text{ for any } \omega > 0,
$$

and, similarly,

$$
\frac{\varrho^{\frac{2}{3}} |\nabla_x \varrho| |\nabla_x \vartheta|}{\vartheta^2} \le \omega \frac{\varrho^{\frac{4}{3}} |\nabla_x \varrho|^2}{\vartheta} + c(\omega) \frac{|\nabla_x \vartheta|^2}{\vartheta^3}.
$$

Thus we infer that

$$
\begin{aligned}
\varepsilon \int_{\Omega} \frac{1}{\vartheta^2} \Big| e_M(\varrho,\vartheta) + \varrho \frac{\partial e_M(\varrho,\vartheta)}{\partial \varrho} \Big| |\nabla_x \varrho| |\nabla_x \vartheta| \ \mathrm{d}x \\
\leq \frac{1}{2} \int_{\Omega} \Big[\delta \Big(\vartheta^{\Gamma-2} + \frac{1}{\vartheta^3} \Big) |\nabla_x \vartheta|^2 + \frac{\varepsilon\delta}{\vartheta} \Big(\Gamma \varrho^{\Gamma-2} + 2 \Big) |\nabla_x \varrho|^2 \Big] \ \mathrm{d}x
\end{aligned}
\tag{3.107}
$$

provided $\varepsilon = \varepsilon(\delta) > 0$ is small enough.

Taking into account the properties of the function $H_{\delta,\overline{\vartheta}}$ (see (2.49–2.50) in Section 2.2.3), we are ready to summarize the so-far obtained estimates as follows:

$$
\left\{
\begin{array}{r}
\mathrm{ess}\sup_{t\in(0,T_{\max})} \int_{\Omega} \Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + H_{\delta,\overline{\vartheta}}(\varrho,\vartheta) + \delta(\frac{\varrho^\Gamma}{\Gamma-1} + \varrho^2) \Big) \ \mathrm{d}x \leq c, \\
\int_0^{T_{\max}} \int_{\Omega} \frac{1}{\vartheta} \Big[\mathbb{S}_\delta(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \Big] \ \mathrm{d}x \ \mathrm{d}t \leq c, \\
\int_0^{T_{\max}} \int_{\Omega} \frac{1}{\vartheta} \Big(\frac{\kappa(\vartheta)}{\vartheta} + \delta(\vartheta^{\Gamma-1} + \frac{1}{\vartheta^2}) \Big) |\nabla_x \vartheta|^2 \Big) \ \mathrm{d}x \ \mathrm{d}t \leq c, \\
\varepsilon \int_0^{T_{\max}} \int_{\Omega} \Big(\delta \frac{1}{\vartheta^3} + \vartheta^5 \Big) \ \mathrm{d}x \ \mathrm{d}t \leq c, \\
\varepsilon\delta \int_0^{T_{\max}} \int_{\Omega} \frac{1}{\vartheta} (\Gamma \varrho^{\Gamma-2} + 2) |\nabla_x \varrho|^2 \ \mathrm{d}x \ \mathrm{d}t \leq c, \\
\int_0^{T_{\max}} \int_{\Omega} \varepsilon \frac{\overline{\vartheta}}{\varrho\vartheta} \frac{\partial p_M}{\partial \varrho}(\varrho,\vartheta) |\nabla_x \varrho|^2 \ \mathrm{d}x \ \mathrm{d}t \leq c,
\end{array}
\right\}
\tag{3.108}
$$

where c is a positive constant depending on the data specified in (2.41) but independent of $T_{\max}$, n, ε, and δ.

At this stage, following the line of arguments presented in Section 2.2.3, we can use the bounds listed in (3.108) in order to deduce uniform estimates on the approximate solutions defined on the time interval $[0, T_{\max}]$ independent of $T_{\max}$. Indeed it follows from (3.108) that

$$
\mathrm{ess} \sup_{t\in(0,T_{\max})} \|\sqrt{\varrho}\mathbf{u}\|^2_{L^2(\Omega;\mathbb{R}^3)} + \int_0^{T_{\max}} \int_{\Omega} \frac{1}{\vartheta} \mathbb{S}_\delta(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x \ \mathrm{d}t \leq c(\mathrm{data}, \varepsilon, \delta),
\tag{3.109}
$$

in particular, by means of hypothesis (3.53) and Proposition 2.1,

$$
\int_0^{T_{\max}} \int_{\Omega} \Big(|\mathbf{u}|^2 + |\nabla_x \mathbf{u}|^2 \Big) \ \mathrm{d}x \ \mathrm{d}t \leq c(\mathrm{data}, \varepsilon, \delta).
$$

Consequently, by virtue of (3.63), the density ϱ is bounded below away from zero uniformly on $[0, T_{\max}]$, and we conclude

$$
\sup_{[0,T_{\max}]} \|\mathbf{u}\|_{X_n} \leq c(\mathrm{data}, \varepsilon, \delta).
\tag{3.110}
$$

As already pointed out, bound (3.110) and the local construction described in the previous section give rise to an approximate solution $\{\varrho, \mathbf{u}, \vartheta\}$ defined on $[0, T]$. We have proved Proposition 3.1.

3.5 Faedo-Galerkin limit

In the previous section, we constructed a family of approximate solutions to the Navier-Stokes-Fourier system satisfying (3.45–3.60), see Proposition 3.1. Our goal in the remaining part of this chapter is to examine successively the asymptotic limit for $n \to \infty$, $\varepsilon \to 0$, and, finally, $\delta \to 0$. The first step of this rather long procedure consists in performing the limit $n \to \infty$.

We recall that the spaces X_n introduced in Section 3.3.1 are formed by sufficiently smooth functions φ (belonging at least to $C^{2,\nu}(\overline{\Omega})$) satisfying either the complete slip boundary condition (3.51) or the no-slip boundary conditions (3.52) as the case may be. Clearly, the approximate velocity field $\mathbf{u} \in C^1([0,T]; X_n)$ belongs to the same class for each fixed $t \in [0,T]$. In the remaining part of the chapter, we make an extra hypothesis that the vector space X,

$$X \equiv \cup_{n=1}^{\infty} X_n \text{ is dense in } W^{1,p}_{\mathbf{n}}(\Omega;\mathbb{R}^3),\ W^{1,p}_0(\Omega;\mathbb{R}^3),\ \text{respectively},$$

for any $1 \leq p < \infty$, where

$$W^{1,p}_{\mathbf{n}}(\Omega;\mathbb{R}^3) = \Big\{ \mathbf{v} \ \Big| \mathbf{v} \in L^p(\Omega;\mathbb{R}^3),\ \nabla_x \mathbf{v} \in L^p(\Omega;\mathbb{R}^{3\times 3}), \\ \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0 \text{ in the sense of traces} \Big\},$$

$$W^{1,p}_0(\Omega;\mathbb{R}^3) = \Big\{ \mathbf{v} \ \Big| \mathbf{v} \in L^p(\Omega;\mathbb{R}^3),\ \nabla_x \mathbf{v} \in L^p(\Omega;\mathbb{R}^{3\times 3}), \\ \mathbf{v}|_{\partial\Omega} = 0 \text{ in the sense of traces} \Big\}.$$

Such a choice of X_n is possible provided Ω belongs to the regularity class $C^{2,\nu}$ required by Theorem 3.1. The interested reader may consult Section 10.7 in Appendix for technical details.

3.5.1 Estimates independent of the dimension of Faedo-Galerkin approximations

For $\varepsilon > 0$, $\delta > 0$ fixed, let $\{\varrho_n, \mathbf{u}_n, \vartheta_n\}_{n=1}^{\infty}$ be a sequence of approximate solutions constructed in Section 3.4. In accordance with (3.108), this sequence admits the following uniform estimates:

$$\operatorname*{ess\,sup}_{t\in(0,T)} \int_\Omega \Big(\frac{1}{2}\varrho_n |\mathbf{u}_n|^2 + H_{\delta,\overline{\vartheta}}(\varrho_n,\vartheta_n) + \delta\big(\frac{\varrho_n^\Gamma}{\Gamma-1} + \varrho_n^2\big)\Big)(t)\ {\rm d}x \leq c, \tag{3.111}$$

$$\int_0^T \int_\Omega \Big\{ \frac{1}{\vartheta_n}\Big[\mathbb{S}_\delta(\vartheta_n, \nabla_x \mathbf{u}_n) : \nabla_x \mathbf{u}_n \\ + \Big(\frac{\kappa(\vartheta_n)}{\vartheta_n} + \delta\big(\vartheta_n^{\Gamma-1} + \frac{1}{\vartheta_n^2}\big)\Big) |\nabla_x \vartheta_n|^2 \Big] + \delta \frac{1}{\vartheta_n^3} + \varepsilon \vartheta_n^5 \Big\}\ {\rm d}x\ {\rm d}t \leq c, \tag{3.112}$$

$$\varepsilon\delta \int_0^T \int_\Omega \frac{1}{\vartheta_n}(\Gamma \varrho_n^{\Gamma-2} + 2)|\nabla_x \varrho_n|^2\ {\rm d}x\ {\rm d}t \leq c, \tag{3.113}$$

and

$$\int_0^T \int_\Omega \varepsilon \frac{\overline{\vartheta}}{\varrho_n \vartheta_n} \frac{\partial p_M}{\partial \varrho}(\varrho_n, \vartheta_n) |\nabla_x \varrho_n|^2 \, \mathrm{d}x \, \mathrm{d}t \leq c, \tag{3.114}$$

where c denotes a generic constant depending only on the data specified in (2.41), in particular, c is independent of the parameters n, ε, and δ.

By virtue of the coercivity properties of $H_{\delta,\overline{\vartheta}}$ established in (2.49), (2.50), the uniform bound (3.111) implies that

$$\{\varrho_n\}_{n=1}^\infty \text{ is bounded in } L^\infty(0,T;L^\Gamma(\Omega)), \tag{3.115}$$

therefore we can assume

$$\varrho_n \to \varrho \text{ weakly-(*) in } L^\infty(0,T;L^\Gamma(\Omega)). \tag{3.116}$$

On the other hand, estimate (3.112), together with hypothesis (3.53) and Proposition 2.1, yield

$$\{\mathbf{u}_n\}_{n=1}^\infty \text{ bounded in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)), \tag{3.117}$$

in particular

$$\mathbf{u}_n \to \mathbf{u} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)), \tag{3.118}$$

at least for a suitable subsequence.

At this point it is worth noting that the limit density ϱ is still a *non-negative* quantity albeit not necessarily strictly positive as this important property stated in (3.63) is definitely lost in the limit passage due to the lack of suitable uniform estimates for $\mathrm{div}_x \mathbf{u}_n$. The fact that the class of weak solutions admits cavities (vacuum regions) seems rather embarrassing from the point of view of the model derived for non-dilute fluids, but still physically acceptable.

Convergence (3.116) can be improved to

$$\varrho_n \to \varrho \quad \text{in } C_{\text{weak}}([0,T];L^\Gamma(\Omega)) \tag{3.119}$$

as ϱ_n, $\mathbf{u}_n$ solve equation (3.45). Indeed we check easily that for all $\varphi \in C_c^\infty(\Omega)$, the functions $t \to [\int_\Omega \varrho_n \varphi \, \mathrm{d}x](t)$ form a bounded and equi-continuous sequence in $C[0,T]$. Consequently, the standard Arzelà-Ascoli theorem (Theorem 0.1) yields

$$\int_\Omega \varrho_n \varphi \, \mathrm{d}x \to \int_\Omega \varrho \varphi \, \mathrm{d}x \quad \text{in } C[0,T] \text{ for any } \varphi \in C_c^\infty(\Omega).$$

Since ϱ_n satisfy (3.115), the convergence extends easily to each $\varphi \in L^{\Gamma'}(\Omega)$ via density.

In order to deduce uniform estimates on the approximate temperature ϑ_n, we exploit the structural properties of the Helmholtz function $H_{\overline{\vartheta}}$. Note that these follow directly from the *hypothesis of thermodynamic stability* and as such may be viewed as a direct consequence of natural physical principles. The following assertion will be amply used in future considerations.

■ COERCIVITY OF THE HELMHOLTZ FUNCTION:

Proposition 3.2. *Let the functions p, e, and s be interrelated through Gibbs' equation* (1.2)*, where p and e comply with the hypothesis of thermodynamic stability* (1.44)*.*

Then for any fixed $\overline{\varrho} > 0$, $\overline{\vartheta} > 0$, *the Helmholtz function*

$$H_{\overline{\vartheta}}(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \overline{\vartheta} \varrho s(\varrho, \vartheta)$$

satisfies

$$H_{\overline{\vartheta}}(\varrho, \vartheta) \geq \frac{1}{4}\Big(\varrho e(\varrho, \vartheta) + \overline{\vartheta} \varrho |s(\varrho, \vartheta)|\Big) - \Big|(\varrho - \overline{\varrho})\frac{\partial H_{2\overline{\vartheta}}}{\partial \varrho}(\overline{\varrho}, 2\overline{\vartheta}) + H_{2\overline{\vartheta}}(\overline{\varrho}, 2\overline{\vartheta})\Big|$$

for all positive ϱ, ϑ.

Proof. As the result obviously holds if $s(\varrho, \vartheta) \leq 0$, we focus on the case $s(\varrho, \vartheta) > 0$. It follows from (2.49), (2.50) that

$$H_{2\overline{\vartheta}}(\varrho, \vartheta) \geq (\varrho - \overline{\varrho})\frac{\partial H_{2\overline{\vartheta}}}{\partial \varrho}(\overline{\varrho}, 2\overline{\vartheta}) + H_{2\overline{\vartheta}}(\overline{\varrho}, 2\overline{\vartheta});$$

whence

$$\begin{aligned} H_{\overline{\vartheta}}(\varrho, \vartheta) = \frac{1}{2}\varrho e(\varrho, \vartheta) + \frac{1}{2}H_{2\overline{\vartheta}}(\varrho, \vartheta) &\geq \frac{1}{2}\varrho e(\varrho, \vartheta) \\ &+ \frac{1}{2}\Big((\varrho - \overline{\varrho})\frac{\partial H_{2\overline{\vartheta}}}{\partial \varrho}(\overline{\varrho}, 2\overline{\vartheta}) + H_{2\overline{\vartheta}}(\overline{\varrho}, 2\overline{\vartheta})\Big), \end{aligned}$$

and, similarly,

$$\begin{aligned} H_{\overline{\vartheta}}(\varrho, \vartheta) = \overline{\vartheta}\varrho s(\varrho, \vartheta) + H_{2\overline{\vartheta}}(\varrho, \vartheta) &\geq \overline{\vartheta}\varrho s(\varrho, \vartheta) \\ &+ (\varrho - \overline{\varrho})\frac{\partial H_{2\overline{\vartheta}}}{\partial \varrho}(\overline{\varrho}, 2\overline{\vartheta}) + H_{2\overline{\vartheta}}(\overline{\varrho}, 2\overline{\vartheta}). \end{aligned}$$

Summing up the last two inequalities we obtain the desired conclusion. □

On the basis of Proposition 3.2, we can deduce from hypothesis (3.9) and the total energy estimate (3.111) that

$$\{\vartheta_n\}_{n=1}^{\infty} \text{ is bounded in } L^\infty(0, T; L^4(\Omega)), \tag{3.120}$$

therefore we may assume

$$\vartheta_n \to \vartheta \text{ weakly-(*) in } L^\infty(0, T; L^4(\Omega)). \tag{3.121}$$

In addition, using boundedness of the entropy production rate stated in (3.112) we get

$$\{\nabla_x \vartheta_n^{\frac{\Gamma}{2}}\}_{n=1}^{\infty},\ \Big\{\nabla_x\Big(\frac{1}{\sqrt{\vartheta_n}}\Big)\Big\}_{n=1}^{\infty} \text{ bounded in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)). \tag{3.122}$$

Estimates (3.120), (3.122), together with Poincaré's inequality formulated in terms of Proposition 2.2, yield

$$\{\vartheta_n\}_{n=1}^{\infty},\ \{\vartheta_n^{\frac{\Gamma}{2}}\}_{n+1}^{\infty} \text{ bounded in } L^2(0,T;W^{1,2}(\Omega)), \tag{3.123}$$

in particular,

$$\vartheta_n \to \vartheta \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)). \tag{3.124}$$

Moreover, by virtue of estimate (3.112), we have

$$\int_0^T \int_\Omega \frac{1}{\vartheta_n^3}\ \mathrm{d}x\ \mathrm{d}t \le c, \tag{3.125}$$

notably the limit function ϑ is positive almost everywhere in $(0,T)\times\Omega$ and satisfies

$$\int_0^T \int_\Omega \frac{1}{\vartheta^3}\ \mathrm{d}x\ \mathrm{d}t \le \liminf_{n\to\infty} \int_0^T \int_\Omega \frac{1}{\vartheta_n^3}\ \mathrm{d}x\ \mathrm{d}t, \tag{3.126}$$

where we have used convexity of the function $z \mapsto z^{-3}$ on $(0,\infty)$, see Theorem 10.20 in Appendix.

Finally, the standard embedding relation $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, together with (3.122), can be used in order to derive higher integrability estimates of ϑ_n, namely

$$\{\vartheta_n\}_{n=1}^{\infty} \text{ bounded in } L^{\Gamma}(0,T;L^{3\Gamma}(\Omega)). \tag{3.127}$$

Note that, as a byproduct of (3.126), (3.127),

$$\{\log(\vartheta_n)\}_{n=1}^{\infty} \text{ is bounded in } L^q((0,T)\times\Omega) \text{ for any finite } q \ge 1. \tag{3.128}$$

3.5.2 Limit passage in the approximate continuity equation

At this stage, we are ready to show strong (pointwise) convergence of the approximate densities and to let $n \to \infty$ in equation (3.45). To this end, we need to control the term $p\mathrm{div}_x\mathbf{u}$ in the approximate energy balance (3.95).

A direct application of (3.32) yields

$$\Big|\int_0^T \int_\Omega p(\varrho_n,\vartheta_n)\mathrm{div}_x\mathbf{u}\ \mathrm{d}x\ \mathrm{d}t\Big| \le c\int_0^T \int_\Omega (\varrho_n^{\frac{5}{3}} + \vartheta_n^{\frac{5}{2}} + \vartheta_n^4)|\mathrm{div}_x\mathbf{u}_n|\ \mathrm{d}x\ \mathrm{d}t,$$

where, by virtue of (3.115), (3.117), (3.120), and (3.127), the last integral is bounded provided $\Gamma > 5$. Accordingly, relation (3.95) gives rise to

$$\varepsilon\delta \int_0^T \int_\Omega (\Gamma \varrho_n^{\Gamma-2} + 2)|\nabla_x \varrho_n|^2 \ \mathrm{d}x \ \mathrm{d}t \le c, \tag{3.129}$$

with c independent of n. Applying the Poincaré inequality (see Proposition 2.2) we get

$$\{\varrho_n\}_{n=1}^\infty, \ \{\varrho_n^{\frac{\Gamma}{2}}\}_{n=1}^\infty \ \text{ bounded in } L^2(0,T;W^{1,2}(\Omega)), \tag{3.130}$$

and

$$\{\varrho_n\}_{n=1}^\infty \text{ bounded in } L^\Gamma(0,T;L^{3\Gamma}(\Omega)). \tag{3.131}$$

The next step is to obtain uniform estimates on $\partial_t \varrho_n$, $\Delta \varrho_n$. This is a delicate task as

$$(\partial_t - \varepsilon\Delta)[\varrho_n] = -\nabla_x \varrho_n \cdot \mathbf{u}_n - \varrho_n \mathrm{div}_x \mathbf{u}_n,$$

where, in accordance with (3.117), (3.130), $\nabla_x \varrho_n \cdot \mathbf{u}_n$ is bounded in $L^1(0,T;L^{\frac{3}{2}}(\Omega))$, notably this quantity is merely integrable with respect to time. To overcome this difficulty, multiply equation (3.45) on $G'(\varrho_n)$ and integrate by parts to obtain

$$\partial_t \int_\Omega G(\varrho_n) \ \mathrm{d}x + \varepsilon \int_\Omega G''(\varrho_n)|\nabla_x \varrho_n|^2 \ \mathrm{d}x = \int_\Omega \Big(G(\varrho_n) - G'(\varrho_n)\varrho_n \Big) \mathrm{div}_x \mathbf{u}_n \ \mathrm{d}x. \tag{3.132}$$

This is of course nothing other than an integrated "parabolic" version of the renormalized continuity equation (2.2). Taking $G(\varrho_n) = \varrho_n \log(\varrho_n)$ we easily deduce

$$\varepsilon \int_0^T \int_\Omega \frac{|\nabla_x \varrho_n|^2}{\varrho_n} \ \mathrm{d}x \ \mathrm{d}t \le c. \tag{3.133}$$

As a consequence of (3.111), the kinetic energy is bounded, specifically,

$$\text{ess sup}_{t\in(0,T)} \int_\Omega \varrho_n |\mathbf{u}_n|^2 \ \mathrm{d}x \ \mathrm{d}t \le c; \tag{3.134}$$

whence estimate (3.133) can be used to obtain

$$\|\nabla_x \varrho_n \cdot \mathbf{u}_n\|_{L^1(\Omega)} \le \left\| \frac{\nabla_x \varrho_n}{\sqrt{\varrho_n}} \right\|_{L^2(\Omega;\mathbb{R}^3)} \|\sqrt{\varrho_n}\mathbf{u}_n\|_{L^2(\Omega;\mathbb{R}^3)},$$

where the product on the right-hand side is bounded in $L^2(0,T)$. Then a standard interpolation argument implies

$$\left\{ \begin{array}{l} \{\nabla_x \varrho_n \cdot \mathbf{u}_n\}_{n=1}^\infty \text{ bounded in } L^q(0,T;L^p(\Omega)) \\ \text{for any } p \in (1, \frac{3}{2}), \quad \text{where } q = q(p) \in (1,2). \end{array} \right\} \tag{3.135}$$

Applying the $L^p - L^q$ theory to the parabolic equation (3.45) (see Section 10.14 in Appendix) we conclude that

$$\begin{gathered}\{\partial_t \varrho_n\}_{n=1}^\infty,\ \{\partial_{x_i}\partial_{x_j}\varrho_n\}_{n=1}^\infty,\ i,j=1,\dots,3 \\ \text{are bounded in } L^q(0,T;L^p(\Omega)) \\ \text{for any } p\in\left(1,\frac{3}{2}\right),\quad \text{where } q=q(p)\in(1,2).\end{gathered} \tag{3.136}$$

Now we are ready to carry out the limit passage $n\to\infty$ in the approximate continuity equation (3.45). To begin, the uniform bounds established (3.136), together with the standard compactness embedding relations for Sobolev spaces, imply

$$\varrho_n \to \varrho \text{ a.a. in } (0,T)\times\Omega. \tag{3.137}$$

Moreover, in view of (3.100), (3.118), (3.135), (3.136), and (3.137), it is easy to let $n\to\infty$ in the approximate continuity equation (3.45) to obtain

$$\partial_t\varrho + \mathrm{div}_x(\varrho\mathbf{u}) = \varepsilon\Delta\varrho \text{ a.a. in } (0,T)\times\Omega, \tag{3.138}$$

where ϱ is a non-negative function satisfying

$$\nabla_x\varrho(t,\cdot)\cdot\mathbf{n}|_{\partial\Omega} = 0 \text{ for a.a. } t\in(0,T) \text{ in the sense of traces,} \tag{3.139}$$

together with the initial condition

$$\varrho(0,\cdot) = \varrho_{0,\delta}, \tag{3.140}$$

where $\varrho_{0,\delta}$ has been specified in (3.48).

Our next goal is to show strong convergence of the gradients $\nabla_x\varrho_n$. To this end, we use the "renormalized" identity (3.132) with $G(z)=z^2$, together with the pointwise convergence established in (3.137), to deduce

$$\int_\Omega \varrho_n^2(\tau)\ \mathrm{d}x + 2\varepsilon\int_0^\tau\int_\Omega |\nabla_x\varrho_n|^2\ \mathrm{d}x\ \mathrm{d}t \to \int_\Omega \varrho_{0,\delta}^2\ \mathrm{d}x - \int_0^\tau\int_\Omega \varrho^2\mathrm{div}_x\mathbf{u}\ \mathrm{d}x\ \mathrm{d}t$$

for any $0<\tau\le T$. On the other hand, multiplying equation (3.138) on ϱ and integrating by parts yields

$$\int_\Omega \varrho_{0,\delta}^2\ \mathrm{d}x - \int_0^\tau\int_\Omega \varrho^2\mathrm{div}_x\mathbf{u}\ \mathrm{d}x\ \mathrm{d}t = \int_\Omega \varrho^2(\tau)\ \mathrm{d}x + 2\varepsilon\int_0^\tau\int_\Omega |\nabla_x\varrho|^2\ \mathrm{d}x\ \mathrm{d}t;$$

whence

$$\nabla_x\varrho_n \to \nabla_x\varrho \text{ (strongly) in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)). \tag{3.141}$$

3.5.3 Strong convergence of the approximate temperatures and the limit in the entropy equation

Strong convergence of the approximate temperatures. The next step is to perform the limit in the approximate entropy balance (3.97). Here the main problem is to show strong (pointwise) convergence of the temperature. Indeed all estimates on $\{\vartheta_n\}_{n=1}^{\infty}$ established above concern only the spatial derivatives leaving open the question of possible time oscillations. Probably the most elegant way to overcome this difficulty is based on the celebrated *Div-Curl lemma* discovered by Tartar [187].

■ DIV-CURL LEMMA:

Proposition 3.3. *Let $Q \subset \mathbb{R}^N$ be an open set. Assume*

$$\mathbf{U}_n \to \mathbf{U} \text{ weakly in } L^p(Q;\mathbb{R}^N),$$
$$\mathbf{V}_n \to \mathbf{V} \text{ weakly in } L^q(Q;\mathbb{R}^N),$$

where

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1.$$

In addition, let

$$\left.\begin{array}{r} \mathrm{div}\ \mathbf{U}_n \equiv \nabla \cdot \mathbf{U}_n, \\ \mathbf{curl}\ \mathbf{V}_n \equiv (\nabla \mathbf{V}_n - \nabla^T \mathbf{V}_n) \end{array}\right\} \text{ be precompact in } \left\{ \begin{array}{l} W^{-1,s}(Q), \\ W^{-1,s}(Q,\mathbb{R}^{N\times N}), \end{array}\right.$$

for a certain $s > 1$.

Then

$$\mathbf{U}_n \cdot \mathbf{V}_n \to \mathbf{U} \cdot \mathbf{V} \text{ weakly in } L^r(Q).$$

Prop. 3.3 is proved in Sect. 10.13 in the Appendix for reader's convenience. □

The basic idea is to apply Proposition 3.3 to the pair of functions

$$\begin{aligned} \mathbf{U}_n &= [\varrho_n s_\delta(\varrho_n, \vartheta_n), \mathbf{r}_n^{(1)}], \\ \mathbf{V}_n &= [\vartheta_n, 0, 0, 0], \end{aligned} \tag{3.142}$$

defined on the set $Q = (0,T) \times \Omega \subset \mathbb{R}^4$, where the term $\mathbf{r}_n^{(1)}$, together with the necessary piece of information concerning $\mathrm{div}_{t,x}\mathbf{U}_n$, are provided by equation (3.97).

To see this, we observe first that the only problematic term on the right-hand side of (3.97) can be handled as

$$\frac{\Delta_x \varrho_n}{\vartheta_n}\Big(\vartheta_n s_\delta(\varrho_n,\vartheta_n) - e_\delta(\varrho_n,\vartheta_n) - \frac{p(\varrho_n,\vartheta_n)}{\varrho_n}\Big) \tag{3.143}$$
$$= \mathrm{div}_x\Big[\Big(\vartheta_n s_{M,\delta}(\varrho_n,\vartheta_n) - e_{M,\delta}(\varrho_n,\vartheta_n) - \frac{p_M(\varrho_n,\vartheta_n)}{\varrho_n}\Big)\frac{\nabla_x\varrho_n}{\vartheta_n}\Big]$$
$$+ \frac{\partial p_M}{\partial \varrho}(\varrho_n,\vartheta_n)\frac{|\nabla_x\varrho_n|^2}{\varrho_n\vartheta_n} - \Big(e_{M,\delta}(\varrho_n,\vartheta_n) + \varrho_n\frac{\partial e_M}{\partial\varrho}(\varrho_n,\vartheta_n)\Big)\frac{\nabla_x\varrho_n\cdot\nabla_x\vartheta_n}{\vartheta_n^2}$$

(cf. (3.102–3.104)). Indeed, in accordance with the uniform estimates (3.107), (3.112), the approximate entropy balance equation (3.97) can be now written in the form

$$\partial_t(\varrho_n s_\delta(\varrho_n,\vartheta_n)) + \mathrm{div}_x(\mathbf{r}_n^{(1)}) = r_n^{(2)} + r_n^{(3)}, \tag{3.144}$$

where

$$\mathbf{r}_n^{(1)} = \varrho_n s_\delta(\varrho_n,\vartheta_n)\mathbf{u}_n - \frac{\kappa_\delta(\vartheta_n)}{\vartheta_n}\nabla_x\vartheta_n$$
$$- \varepsilon\Big(\vartheta_n s_{M,\delta}(\varrho_n,\vartheta_n) - e_{M,\delta}(\varrho_n,\vartheta_n) - \frac{p_M(\varrho_n,\vartheta_n)}{\varrho_n}\Big)\frac{\nabla_x\varrho_n}{\vartheta_n},$$
$$r_n^{(2)} = \frac{1}{\vartheta_n}\Big[\mathbb{S}_\delta(\vartheta_n,\nabla_x\mathbf{u}_n):\nabla_x\mathbf{u}_n + \Big(\frac{\kappa(\vartheta_n)}{\vartheta_n} + \delta(\vartheta_n^{\Gamma-1} + \frac{1}{\vartheta_n^2})\Big)|\nabla_x\vartheta_n|^2 + \delta\frac{1}{{\vartheta_n}^2}\Big]$$
$$+ \frac{\varepsilon\delta}{\vartheta_n}(\Gamma\varrho_n^{\Gamma-2} + 2)|\nabla_x\varrho_n|^2 + \varepsilon\frac{1}{\varrho_n\vartheta_n}\frac{\partial p_M}{\partial\varrho}(\varrho_n,\vartheta_n)|\nabla_x\varrho_n|^2 \geq 0,$$

and

$$r_n^{(3)} = -\varepsilon\Big(e_{M,\delta}(\varrho_n,\vartheta_n) + \varrho_n\frac{\partial e_M}{\partial\varrho}(\varrho_n,\vartheta_n)\Big)\frac{\nabla_x\varrho_n\cdot\nabla_x\vartheta_n}{\vartheta_n^2} - \varepsilon\vartheta_n^4 + \frac{\varrho_n}{\vartheta_n}\mathcal{Q}_\delta.$$

Hence, by virtue of the uniform estimates (3.107), (3.112–3.114), and (3.120),

$$\mathrm{div}_{t,x}\mathbf{U}_n = r_n^{(2)} + r_n^{(3)}$$

is bounded in $L^1((0,T)\times\Omega)$, therefore precompact in $W^{-1,s}((0,T)\times\Omega)$ provided $s\in[1,\frac{4}{3})$ (cf. Section 0.7). On the other hand, due to (3.117), $\mathbf{curl}_{t,x}\mathbf{V}_n$ is obviously bounded in $L^2((0,T)\times\Omega;\mathbb{R}^4)$ which is compactly embedded into $W^{-1,2}((0,T)\times\Omega;\mathbb{R}^4)$. Let us remark that the "space-time" operator $\mathbf{curl}_{t,x}$ applied to the vector field $[\vartheta_n,0,0,0]$ involves only the partial derivatives in the *spatial* variable x.

Consequently, in order to apply Proposition 3.3 in the situation described in (3.142), we have to show that $\varrho_n s(\varrho_n,\vartheta_n)$ and $\mathbf{r}_n^{(1)}$ are bounded in a Lebesgue space "better" than only L^1.

To this end, write

$$\varrho s_\delta(\varrho, \vartheta) = \frac{4}{3} a \vartheta^3 + \varrho s_M(\varrho, \vartheta) + \delta \varrho \log(\vartheta),$$

where $\varrho_n s_M(\varrho_n, \vartheta_n)$ satisfies (3.39), therefore

$$\varrho_n |s_\delta(\varrho_n, \vartheta_n)| \leq c(\varrho_n + \vartheta_n^3 + \varrho_n |\log \varrho_n| + \varrho_n |\log \vartheta_n|).$$

Consequently, thanks to estimates (3.128), (3.130),

$$\{\varrho_n s_\delta(\varrho_n, \vartheta_n)\}_{n=1}^\infty \text{ is bounded in } L^{\frac{\Gamma}{3}}((0,T) \times \Omega),$$
$$\{\varrho_n s_\delta(\varrho_n, \vartheta_n)\mathbf{u}_n\}_{n=1}^\infty \text{ is bounded in } L^p((0,T) \times \Omega),\ \tfrac{1}{p} = \tfrac{1}{2} + \tfrac{3}{\Gamma} \text{ provided } \Gamma > 6. \tag{3.145}$$

Next we observe that (3.112) implies in the way explained in (2.58) that

$$\{\nabla \log(\vartheta_n)\}_{n=1}^\infty \text{ is bounded in } L^2((0,T) \times \Omega; \mathbb{R}^3).$$

Furthermore, it follows from (3.112) that

$$\left\{ \frac{\sqrt{\kappa_\delta(\vartheta_n)}}{\vartheta_n} \nabla_x \vartheta_n \right\}_{n=1}^\infty \text{ is bounded in } L^2((0,T) \times \Omega; \mathbb{R}^3).$$

Moreover, estimates (3.125), (3.127) and (3.120) combined with a simple interpolation yield

$$\{\sqrt{\kappa_\delta(\vartheta_n)}\}_{n=1}^\infty \text{ is bounded in } L^p((0,T) \times \Omega) \text{ for a certain } p > 2,$$

on condition that $\Gamma > 6$. From the last two estimates, we deduce that

$$\left\{ \frac{\kappa_\delta(\vartheta_n)}{\vartheta_n} \nabla_x \vartheta_n \right\}_{n=1}^\infty \text{ is bounded in } L^p((0,T) \times \Omega; R^3) \text{ for a certain } p > 1. \tag{3.146}$$

Finally, the ε-dependent quantity contained in $\mathbf{r}_n^{(1)}$ can be handled in the following way:

- Similarly to the proof of formula (3.145), we conclude, by help of estimates (3.127), (3.128), (3.133), that

  $$\{s_\delta(\varrho_n, \vartheta_n) \nabla \varrho_n\}_{n=1}^\infty \text{ is bounded in } L^{\frac{2\Gamma}{\Gamma+6}}((0,T) \times \Omega) \tag{3.147}$$

  provided $\Gamma > 6$.
- Since the specific internal energy e_M satisfies (3.30), we have

  $$\left| \frac{e_M(\varrho_n, \vartheta_n)}{\vartheta_n} \nabla_x \varrho_n \right| \leq c(1 + \frac{\varrho_n^{\frac{2}{3}}}{\vartheta_n}) |\nabla_x \varrho_n|;$$

  whence, in accordance with estimates (3.115), (3.125), and (3.130),

  $$\left\{ \frac{e_M(\varrho_n, \vartheta_n)}{\vartheta_n} \nabla_x \varrho_n \right\}_{n=1}^\infty \text{ is bounded in } L^{\frac{6\Gamma}{5\Gamma+4}}((0,T) \times \Omega; \mathbb{R}^3). \tag{3.148}$$

- By virtue of (3.31) and (3.32),

$$\left|\frac{p_M(\varrho_n,\vartheta_n)}{\varrho_n\vartheta_n}\nabla_x\varrho_n\right| \le c|\nabla_x\varrho_n|\Big(1+\frac{\varrho_n^{\frac{2}{3}}}{\vartheta_n}\Big), \tag{3.149}$$

where the right-hand side can be controlled exactly as in (3.148).

Having verified the hypotheses of Proposition 3.3 for the vector fields $\mathbf{U}_n$, $\mathbf{V}_n$ specified in (3.142), we are allowed to conclude that

$$\overline{\varrho s_\delta(\varrho,\vartheta)\vartheta} = \overline{\varrho s_\delta(\varrho,\vartheta)}\,\vartheta \tag{3.150}$$

provided $\Gamma > 6$. In formula (3.150) and hereafter, the symbol $\overline{F(\mathbf{U})}$ denotes a weak L^1-limit of the sequence of composed functions $\{F(\mathbf{U}_n)\}_{n=1}^\infty$ (cf. Section 0.8).

Since the entropy is an increasing function of the absolute temperature, relation (3.150) can be used to deduce strong (pointwise) convergence of the sequence $\{\vartheta_n\}_{n=1}^\infty$.

To begin, we recall (3.98), namely

$$\varrho s_\delta(\varrho,\vartheta) = \varrho s_M(\varrho,\vartheta) + \delta\varrho\log(\vartheta) + \frac{4}{3}a\vartheta^3.$$

As all three components of the entropy are increasing in ϑ, we observe that

$$\overline{\varrho s_M(\varrho,\vartheta)\vartheta} \ge \overline{\varrho s_M(\varrho,\vartheta)}\vartheta,\ \ \overline{\varrho\log(\vartheta)\vartheta} \ge \overline{\varrho\log(\vartheta)}\vartheta,\ \text{and}\ \overline{\vartheta^4} \ge \overline{\vartheta^3}\vartheta. \tag{3.151}$$

Indeed, as $\{\varrho_n\}_{n=1}^\infty$ converges strongly (see (3.137)) we have

$$\overline{\varrho s_M(\varrho,\vartheta)\vartheta} = \varrho\overline{s_M(\varrho,\vartheta)\vartheta},\ \ \overline{\varrho s_M(\varrho,\vartheta)} = \varrho\overline{s_M(\varrho,\vartheta)},$$

where, as a direct consequence of monotonicity of s_M in ϑ,

$$\overline{s_M(\varrho,\vartheta)\vartheta} \ge \overline{s_M(\varrho,\vartheta)}\vartheta,$$

see Theorem 10.19 in Appendix. Here, we have used (3.124), (3.137) yielding

$$s_M(\varrho_n,\vartheta)(\vartheta_n-\vartheta) \to 0 \text{ weakly in } L^1((0,T)\times\Omega).$$

The remaining two inequalities in (3.151) can be shown in a similar way.

Combining (3.150), (3.151) we infer that

$$\overline{\vartheta^4} = \overline{\vartheta^3}\vartheta,$$

in particular, at least for a suitable subsequence, we have

$$\vartheta_n \to \vartheta \quad \text{a.e. in } ((0,T)\times\Omega) \tag{3.152}$$

(cf. Theorems 10.19, 10.20 in Appendix).

Limit in the approximate entropy equation. Our ultimate goal in this section is to let $n \to \infty$ in the approximate entropy equation (3.144).

First of all, we estimate the term

$$\varepsilon\Big(e_{M,\delta}(\varrho_n,\vartheta_n) + \varrho_n \frac{\partial e_M}{\partial \varrho}(\varrho_n,\vartheta_n)\Big)\frac{\nabla_x \varrho_n \cdot \nabla_x \vartheta_n}{\vartheta_n^2}$$

in the same way as in (3.107) transforming (3.144) to inequality

$$\begin{aligned}
&\partial_t(\varrho_n s_\delta(\varrho_n,\vartheta_n)) + \mathrm{div}_x\Big(\varrho_n s_\delta(\varrho_n,\vartheta_n)\mathbf{u}_n - \frac{\kappa_\delta(\vartheta_n)}{\vartheta_n}\nabla_x\vartheta_n\Big) \qquad (3.153)\\
&\quad - \varepsilon \mathrm{div}_x\Big[\Big(\vartheta_n s_{M,\delta}(\varrho_n,\vartheta_n) - e_{M,\delta}(\varrho_n,\vartheta_n) - \frac{p_M(\varrho_n,\vartheta_n)}{\varrho_n}\Big)\frac{\nabla_x\varrho_n}{\vartheta_n}\Big]\\
&\geq \frac{1}{\vartheta_n}\Big[\mathbb{S}_\delta(\vartheta_n,\nabla_x\mathbf{u}_n) : \nabla_x\mathbf{u}_n + \Big(\frac{\kappa(\vartheta_n)}{\vartheta_n} + \frac{\delta}{2}(\vartheta_n^{\Gamma-1} + \frac{1}{\vartheta_n^2})\Big)|\nabla_x\vartheta_n|^2 + \delta\frac{1}{{\vartheta_n}^2}\Big]\\
&\quad + \frac{\varepsilon\delta}{2\vartheta_n}(\Gamma\varrho_n^{\Gamma-2} + 2)|\nabla_x\varrho_n|^2 + \varepsilon\frac{1}{\varrho_n\vartheta_n}\frac{\partial p_M}{\partial\varrho}(\varrho_n,\vartheta_n)|\nabla_x\varrho_n|^2 - \varepsilon\vartheta_n^4 + \frac{\varrho_n}{\vartheta_n}\mathcal{Q}_\delta.
\end{aligned}$$

As a consequence of (3.137), (3.145), (3.152),

$$\varrho_n s_\delta(\varrho_n,\vartheta_n) \to \varrho s_\delta(\varrho,\vartheta) \text{ (strongly) in } L^2((0,T)\times\Omega), \tag{3.154}$$

and, in accordance with (3.117),

$$\varrho_n s_\delta(\varrho_n,\vartheta_n)\mathbf{u}_n \to \varrho s_\delta(\varrho,\vartheta)\mathbf{u} \text{ weakly in } L^1((0,T)\times\Omega;\mathbb{R}^3). \tag{3.155}$$

Since the sequence $\{\vartheta_n\}_{n=1}^\infty$ converges a.a. in $(0,T)\times\Omega$, we can use hypotheses (3.21), (3.22), together with estimates (3.120), (3.123), (3.125), (3.127), to get

$$\frac{\kappa(\vartheta_n)}{\vartheta_n} \to \frac{\kappa(\vartheta)}{\vartheta} \quad \text{(strongly) in } L^2((0,T)\times\Omega)$$

yielding, in combination with (3.124),

$$\frac{\kappa(\vartheta_n)}{\vartheta_n}\nabla_x\vartheta_n \to \frac{\kappa(\vartheta)}{\vartheta}\nabla_x\vartheta \quad \text{weakly in } L^1((0,T)\times\Omega;\mathbb{R}^3). \tag{3.156}$$

On the other hand, by virtue of relations (3.122), (3.125), (3.127),

$$\begin{aligned}
&\Big(\vartheta_n^{\Gamma-1} + \frac{1}{\vartheta_n^2}\Big)\nabla_x\vartheta_n = \frac{1}{\Gamma}\nabla_x(\vartheta_n^\Gamma) - \nabla_x(1/\vartheta_n) \qquad (3.157)\\
&\to \frac{1}{\Gamma}\nabla_x(\overline{\vartheta^\Gamma}) - \nabla_x\overline{1/\vartheta} \ \text{ weakly in } L^p((0,T)\times\Omega) \text{ for some } p>1,
\end{aligned}$$

where, according to (3.152),

$$\frac{1}{\Gamma}\nabla_x(\overline{\vartheta^\Gamma}) - \nabla_x\overline{1/\vartheta} = \frac{1}{\Gamma}\nabla_x(\vartheta^\Gamma) - \nabla_x 1/\vartheta = \vartheta^{\Gamma-1}\nabla_x\vartheta + \frac{1}{\vartheta^2}\nabla_x\vartheta. \tag{3.158}$$

In order to control the ε-term on the left-hand side of (3.153), we first observe that

$$\Big|\frac{1}{\vartheta}\Big(\vartheta s_{M,\delta}(\varrho,\vartheta)-e_{M,\delta}(\varrho,\vartheta)-\frac{p_M(\varrho,\vartheta)}{\varrho}\Big)\nabla\varrho\Big|\le c(|\log\vartheta|+|\log\varrho|+\frac{\varrho^{2/3}}{\vartheta}+1)|\nabla\varrho|,$$

where we have used (3.31), (3.32), (3.39).

As a next step, we apply relations (3.123), (3.130), and (3.133), together with the arguments leading to (3.148), in order to deduce boundedness of the quantity

$$\frac{1}{\vartheta_n}\Big(\vartheta s_{M,\delta}(\varrho_n,\vartheta_n)-e_{M,\delta}(\varrho_n,\vartheta_n)-\frac{p_M(\varrho_n,\vartheta_n)}{\varrho_n}\Big)\nabla\varrho_n$$

$$\text{in } L^p((0,T)\times\Omega;\mathbb{R}^3) \text{ for some } p>1.$$

In particular, by virtue of (3.137), (3.141), (3.152), we obtain

$$\begin{aligned}&\frac{1}{\vartheta_n}\Big(\vartheta_n s_{M,\delta}(\varrho_n,\vartheta_n)-e_{M,\delta}(\varrho_n,\vartheta_n)-\frac{p_M(\varrho_n,\vartheta_n)}{\varrho_n}\Big)\nabla\varrho_n\\ &\to\frac{1}{\vartheta}\Big(\vartheta s_{M,\delta}(\varrho,\vartheta)-e_{M,\delta}(\varrho,\vartheta)-\frac{p_M(\varrho,\vartheta)}{\varrho}\Big)\nabla\varrho \quad \text{weakly in } L^1((0,T)\times\Omega;\mathbb{R}^3).\end{aligned} \tag{3.159}$$

Finally, we identify the asymptotic limit for $n\to\infty$ of the approximate entropy production rate represented through the quantities on the right-hand side of (3.153). In accordance with (3.112), we have

$$\left\{\sqrt{((\frac{\mu(\vartheta_n)}{\vartheta_n}+\delta\Big)}\Big(\nabla_x\mathbf{u}_n+\nabla_x^T\mathbf{u}_n-\frac{2}{3}\mathrm{div}_x\mathbf{u}_n\Big)\right\}_{n=1}^{\infty},\ \left\{\sqrt{\frac{\eta(\vartheta_n)}{\vartheta_n}}\mathrm{div}_x\mathbf{u}_n\right\}_{n=1}^{\infty}$$

bounded in $L^2((0,T)\times\Omega;\mathbb{R}^{3\times3})$, and in $L^2((0,T)\times\Omega)$, respectively. In particular,

$$\begin{aligned}&\sqrt{\Big(\frac{\mu(\vartheta_n)}{\vartheta_n}+\delta\Big)}\Big(\nabla_x\mathbf{u}_n+\nabla_x^T\mathbf{u}_n-\frac{2}{3}\mathrm{div}_x\mathbf{u}_n\Big)\\ &\to\sqrt{\Big(\frac{\mu(\vartheta)}{\vartheta}+\delta\Big)}\Big(\nabla_x\mathbf{u}+\nabla_x^T\mathbf{u}-\frac{2}{3}\mathrm{div}_x\mathbf{u}\Big) \text{ weakly in } L^2((0,T)\times\Omega;\mathbb{R}^{3\times3}),\end{aligned} \tag{3.160}$$

where we have used (3.118) and (3.152).

Similarly,

$$\sqrt{\frac{\eta(\vartheta_n)}{\vartheta_n}}\mathrm{div}_x\mathbf{u}_n\to\sqrt{\frac{\eta(\vartheta)}{\vartheta}}\mathrm{div}_x\mathbf{u} \text{ weakly in } L^2((0,T)\times\Omega), \tag{3.161}$$

and, by virtue of (3.112), (3.124) and (3.152),

$$\frac{\sqrt{\kappa_\delta(\vartheta_n)}}{\vartheta_n}\nabla_x\vartheta_n\to\frac{\sqrt{\kappa_\delta(\vartheta)}}{\vartheta}\nabla_x\vartheta \text{ weakly in } L^2((0,T)\times\Omega;\mathbb{R}^3). \tag{3.162}$$

By the same token, due to (3.113), (3.137), (3.141),

$$\sqrt{\Big(\frac{\Gamma\varrho_n^{\Gamma-2}+2}{\vartheta_n}\Big)}\,\nabla_x\varrho_n \to \sqrt{\Big(\frac{\Gamma\varrho^{\Gamma-2}+2}{\vartheta}\Big)}\,\nabla_x\varrho \quad \text{weakly in } L^2((0,T)\times\Omega;\mathbb{R}^3), \tag{3.163}$$

while, by virtue of (3.114), (3.137), (3.141), (3.152),

$$\begin{aligned}&\frac{1}{\sqrt{\varrho_n\vartheta_n}}\sqrt{\frac{\partial p_M}{\partial\varrho}(\varrho_n,\vartheta_n)}\,\nabla_x\varrho_n\\ &\to \frac{1}{\sqrt{\varrho\vartheta}}\sqrt{\frac{\partial p_M}{\partial\varrho}(\varrho,\vartheta)}\,\nabla_x\varrho \quad \text{weakly in } L^2((0,T)\times\Omega;\mathbb{R}^3).\end{aligned} \tag{3.164}$$

Finally, as a consequence of (3.137), (3.152), and the bounds established in (3.125), (3.127), (3.131), we have

$$\varepsilon\vartheta_n^4 - \frac{\varrho_n}{\vartheta_n}\mathcal{Q}_\delta \to \varepsilon\vartheta^4 - \frac{\varrho}{\vartheta}\mathcal{Q}_\delta \text{ in } L^p((0,T)\times\Omega) \text{ for some } p>1. \tag{3.165}$$

The convergence results just established are sufficient in order to perform the weak limit for $n\to\infty$ in the approximate entropy balance (3.153). Although we are not able to show strong convergence of the gradients of ϱ, ϑ, and $\mathbf{u}$, the inequality sign in (3.153) is preserved under the weak limit because of lower semicontinuity of convex superposition operators (cf. Theorem 10.20 in Appendix). Consequently, we are allowed to conclude that

$$\begin{aligned}&\int_0^T\int_\Omega \varrho s_\delta(\varrho,\vartheta)\Big(\partial_t\varphi + \mathbf{u}\cdot\nabla_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t + \int_0^T\int_\Omega \Big(\frac{\mathbf{q}_\delta}{\vartheta} + \varepsilon\mathbf{r}_\epsilon\Big)\cdot\nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t\\ &+ \int_0^T\int_\Omega \sigma_{\varepsilon,\delta}\varphi\ \mathrm{d}x\ \mathrm{d}t \le -\int_\Omega (\varrho s)_{0,\delta}\varphi(0,\cdot)\ \mathrm{d}x + \int_0^T\int_\Omega \Big(\varepsilon\vartheta^4 - \frac{\varrho}{\vartheta}\mathcal{Q}_\delta\Big)\varphi\ \mathrm{d}x\ \mathrm{d}t,\\ &\text{for any } \varphi\in C_c^\infty([0,T)\times\overline{\Omega}),\quad \varphi\ge 0,\end{aligned} \tag{3.166}$$

where we have set

$$\begin{aligned}\mathbf{q}_\delta = \mathbf{q}_\delta(\vartheta,\nabla\vartheta) = \kappa_\delta(\vartheta)\nabla_x\vartheta,\ \kappa_\delta(\vartheta) = \kappa(\vartheta) + \delta\Big(\vartheta^\Gamma + \frac{1}{\vartheta}\Big),\\ s_\delta(\varrho,\vartheta) = s(\varrho,\vartheta) + \delta\log\vartheta,\end{aligned} \tag{3.167}$$

and

$$\begin{aligned}\sigma_{\varepsilon,\delta} &= \frac{1}{\vartheta}\Big[\mathbb{S}_\delta : \nabla_x\mathbf{u} + \Big(\frac{\kappa(\vartheta)}{\vartheta} + \frac{\delta}{2}(\vartheta^{\Gamma-1} + \frac{1}{\vartheta^2})\Big)|\nabla_x\vartheta|^2 + \delta\frac{1}{\vartheta^2}\Big]\\ &\quad + \frac{\varepsilon\delta}{2\vartheta}(\Gamma\varrho^{\Gamma-2}+2)|\nabla_x\varrho|^2 + \varepsilon\frac{\partial p_M}{\partial\varrho}(\varrho,\vartheta)\frac{|\nabla_x\varrho|^2}{\varrho\vartheta},\\ \mathbf{r}_\varepsilon &= -\Big(\vartheta s_{M,\delta}(\varrho,\vartheta) - e_{M,\delta}(\varrho,\vartheta) - \frac{p_M(\varrho,\vartheta)}{\varrho}\Big)\frac{\nabla_x\varrho}{\vartheta}.\end{aligned} \tag{3.168}$$

3.5.4 Limit in the approximate momentum equation

With regard to formulas (3.32), (3.53), estimates (3.115), (3.117), (3.120), (3.127), (3.130), (3.131), and the asymptotic limits established in (3.118), (3.137), (3.141), (3.152), it is easy to identify the limit for $n \to \infty$ in all quantities appearing in the approximate momentum equation (3.49) for a fixed test function φ, with the exception of the convective term. Note that, even at this level of approximation, we have already lost compactness of the velocity field in the time variable because of the hypothetical presence of vacuum zones.

To begin, observe that

$$\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n \to \overline{\varrho \mathbf{u} \otimes \mathbf{u}} \text{ weakly in } L^q((0,T) \times \Omega; \mathbb{R}^{3\times 3}) \text{ for a certain } q > 1,$$

where we have used the uniform bounds (3.111), (3.117). Thus we have to show

$$\overline{\varrho \mathbf{u} \otimes \mathbf{u}} = \varrho \mathbf{u} \otimes \mathbf{u}. \tag{3.169}$$

To this end, observe first that

$$\varrho_n \mathbf{u}_n \to \varrho \mathbf{u} \text{ weakly-(*) in } L^\infty(0,T; L^{\frac{5}{4}}(\Omega; \mathbb{R}^3))$$

as a direct consequence of estimates (3.115), (3.134), and strong convergence of the density established in (3.137).

Moreover, it can be deduced from the approximate momentum equation (3.49) that the functions

$$\left\{ t \mapsto \int_\Omega \varrho_n \mathbf{u}_n \cdot \phi \, \mathrm{d}x \right\} \text{ are equi-continuous and bounded in } C([0,T]) \tag{3.170}$$

for any fixed $\phi \in \cup_{n=1}^\infty X_n$. Since the set $\cup_{n=1}^\infty X_n$ is dense in $L^5(\Omega; \mathbb{R}^3)$ we obtain, by means of the Arzelà-Ascoli theorem, that

$$\varrho_n \mathbf{u}_n \to \varrho \mathbf{u} \text{ in } C_{\text{weak}}([0,T]; L^{5/4}(\Omega)).$$

On the other hand, as the Lebesgue space $L^{5/4}(\Omega)$ is compactly embedded into the dual $W^{-1,2}(\Omega)$, we infer that

$$\varrho_n \mathbf{u}_n \to \varrho \mathbf{u} \text{ (strongly) in } C_{\text{weak}}([0,T]; W^{-1,2}(\Omega; \mathbb{R}^3)). \tag{3.171}$$

Relation (3.171), together with the weak convergence of the velocities in the space $L^2(0,T; W^{1,2}(\Omega; \mathbb{R}^3))$ established in (3.118), give rise to (3.169).

3.5.5 The limit system resulting from the Faedo-Galerkin approximation

Having completed the necessary preliminary steps, in particular, the strong convergence of the density in (3.141), and the strong convergence of the temperature in (3.152), we can let $n \to \infty$ in the approximate system (3.45–3.60) to deduce that the limit quantities $\{\varrho, \mathbf{u}, \vartheta\}$ satisfy:

(i) Approximate continuity equation:

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = \varepsilon \Delta \varrho \text{ a.a. in } (0,T) \times \Omega, \tag{3.172}$$

together with the homogeneous Neumann boundary condition

$$\nabla_x \varrho \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{3.173}$$

and the initial condition

$$\varrho(0,\cdot) = \varrho_{0,\delta}. \tag{3.174}$$

(ii) Approximate balance of momentum:

$$\begin{aligned} &\int_0^T \int_\Omega \Big(\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho [\mathbf{u} \otimes \mathbf{u}] : \nabla_x \varphi + \Big(p + \delta(\varrho^\Gamma + \varrho^2) \Big) \operatorname{div}_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t \\ &= \int_0^T \int_\Omega \Big(\varepsilon (\nabla_x \varrho \nabla_x \mathbf{u}) \cdot \varphi + \mathbb{S}_\delta : \nabla_x \varphi - \varrho \mathbf{f}_\delta \cdot \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t - \int_\Omega (\varrho \mathbf{u})_0 \cdot \varphi \ \mathrm{d}x, \end{aligned} \tag{3.175}$$

satisfied for any test function $\varphi \in C^\infty_c([0,T) \times \overline{\Omega}; \mathbb{R}^3)$, where either

$$\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0 \text{ in the case of the complete slip boundary conditions,} \tag{3.176}$$

or

$$\varphi|_{\partial\Omega} = 0 \text{ in the case of the no-slip boundary conditions,} \tag{3.177}$$

and where we have set

$$\mathbb{S}_\delta = \mathbb{S}_\delta(\vartheta, \nabla_x \mathbf{u}) = (\mu(\vartheta) + \delta\vartheta) \Big(\nabla_x \mathbf{u} + \nabla_x^\perp \mathbf{u} - \frac{2}{3} \operatorname{div}_x \mathbf{u} \ \mathbb{I} \Big) + \eta(\vartheta) \operatorname{div}_x \mathbf{u} \ \mathbb{I}. \tag{3.178}$$

(iii) Approximate entropy inequality:

$$\begin{aligned} &\int_0^T \int_\Omega \varrho s_\delta(\varrho,\vartheta) \Big(\partial_t \varphi + \mathbf{u} \cdot \nabla_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t + \int_0^T \int_\Omega \Big(\frac{\kappa_\delta(\vartheta) \nabla_x \vartheta}{\vartheta} + \varepsilon \mathbf{r} \Big) \cdot \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t \\ &+ \int_0^T \int_\Omega \sigma_{\varepsilon,\delta} \varphi \ \mathrm{d}x \ \mathrm{d}t \le - \int_\Omega (\varrho s)_{0,\delta} \varphi(0,\cdot) \ \mathrm{d}x + \int_0^T \int_\Omega \Big(\varepsilon \vartheta^4 - \frac{\varrho}{\vartheta} \mathcal{Q}_\delta \Big) \varphi \ \mathrm{d}x \ \mathrm{d}t \end{aligned} \tag{3.179}$$

for any test function $\varphi \in C^\infty_c([0,T) \times \overline{\Omega})$, $\varphi \ge 0$, where we have set

$$s_\delta(\varrho,\vartheta) = s(\varrho,\vartheta) + \delta \log \vartheta, \ \ \kappa_\delta(\vartheta) = \kappa(\vartheta) + \delta \Big(\vartheta^\Gamma + \frac{1}{\vartheta} \Big), \tag{3.180}$$

and

$$\begin{aligned} \sigma_{\varepsilon,\delta} &= \frac{1}{\vartheta} \Big[\mathbb{S}_\delta : \nabla_x \mathbf{u} + \Big(\frac{\kappa(\vartheta)}{\vartheta} + \frac{\delta}{2} (\vartheta^{\Gamma-1} + \frac{1}{\vartheta^2}) \Big) |\nabla_x \vartheta|^2 + \delta \frac{1}{\vartheta^2} \Big] \\ &\quad + \frac{\varepsilon\delta}{2\vartheta} (\Gamma \varrho^{\Gamma-2} + 2) |\nabla_x \varrho|^2 + \varepsilon \frac{\partial p_M}{\partial \varrho}(\varrho,\vartheta) \frac{|\nabla_x \varrho|^2}{\varrho \vartheta}, \\ \mathbf{r} &= - \Big(\vartheta s_{M,\delta}(\varrho,\vartheta) - e_{M,\delta}(\varrho,\vartheta) - \frac{p_M(\varrho,\vartheta)}{\varrho} \Big) \frac{\nabla_x \varrho}{\vartheta}. \end{aligned} \tag{3.181}$$

(iv) Approximate total energy balance:

$$\int_\Omega \Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e_\delta(\varrho,\vartheta) + \delta(\frac{\varrho^\Gamma}{\Gamma-1} + \varrho^2)\Big)(\tau)\ \mathrm{d}x \tag{3.182}$$
$$= \int_\Omega \Big(\frac{1}{2}\frac{|(\varrho\mathbf{u})_{0,\delta}|^2}{\varrho_{0,\delta}} + \varrho_{0,\delta}e_{0,\delta} + \delta(\frac{\varrho_{0,\delta}^\Gamma}{\Gamma-1} + \varrho_{0,\delta}^2)\Big)\ \mathrm{d}x$$
$$+ \int_0^\tau \int_\Omega \Big(\varrho\mathbf{f}_\delta\cdot\mathbf{u} + \varrho\mathcal{Q}_\delta + \delta\frac{1}{\vartheta^2} - \varepsilon\vartheta^5\Big)\ \mathrm{d}x\ \mathrm{d}t \text{ for a.a. } \tau\in[0,T],$$

where

$$e_\delta(\varrho,\vartheta) = e(\varrho,\vartheta) + \delta\vartheta. \tag{3.183}$$

3.5.6 The entropy production rate represented by a positive measure

In accordance with the general ideas discussed in Section 1.2, the entropy inequality can be interpreted as a weak formulation of a balance law with the production rate represented by a positive measure. More specifically, writing (3.179) in the form

$$\int_\Omega (\varrho s)_{0,\delta}\varphi(0,\cdot)\ \mathrm{d}x - \int_0^T\int_\Omega \Big(\varepsilon\vartheta^4 - \frac{\varrho}{\vartheta}\mathcal{Q}_\delta\Big)\varphi\ \mathrm{d}x\ \mathrm{d}t$$
$$- \int_0^T\int_\Omega \varrho s_\delta(\varrho,\vartheta)\Big(\partial_t\varphi + \mathbf{u}\cdot\nabla_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t$$
$$+ \int_0^T\int_\Omega \Big(\frac{\kappa_\delta(\vartheta)\nabla_x\vartheta}{\vartheta} + \varepsilon\mathbf{r}\Big)\cdot\nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t \geq \int_0^T\int_\Omega \sigma_{\varepsilon,\delta}\varphi\ \mathrm{d}x\ \mathrm{d}t$$

for any $\varphi\in C_c^\infty([0,T)\times\overline{\Omega})$, $\varphi\geq 0$, the left-hand side can be understood as a non-negative linear form defined on the space of smooth functions with compact support in $[0,T)\times\overline{\Omega}$.

Consequently, by means of the classical *Riesz representation theorem*, there exists a regular, non-negative Borel measure $\Sigma_{\varepsilon,\delta}$ on the set $[0,T)\times\overline{\Omega}$, that can be trivially extended on the compact set $[0,T]\times\overline{\Omega}$ such that

$$\int_0^T\int_\Omega \varrho s_\delta(\varrho,\vartheta)\Big(\partial_t\varphi + \mathbf{u}\cdot\nabla_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t + \int_0^T\int_\Omega \Big(\frac{\kappa_\delta(\vartheta)\nabla_x\vartheta}{\vartheta} + \varepsilon\mathbf{r}\Big)\cdot\nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t$$
$$+ \langle\Sigma_{\varepsilon,\delta};\varphi\rangle_{[\mathcal{M};C]([0,T]\times\overline{\Omega})} = -\int_\Omega (\varrho s)_{0,\delta}\varphi(0,\cdot)\ \mathrm{d}x + \int_0^T\int_\Omega \Big(\varepsilon\vartheta^4 - \frac{\varrho}{\vartheta}\mathcal{Q}_\delta\Big)\varphi\ \mathrm{d}x\ \mathrm{d}t \tag{3.184}$$

for any $\varphi\in C_c^\infty([0,T)\times\overline{\Omega})$. Moreover,

$$\Sigma_{\varepsilon,\delta} \geq \sigma_{\varepsilon,\delta}, \tag{3.185}$$

where we have identified the function $\sigma_{\varepsilon,\delta}\in L^1((0,T)\times\Omega)$ with a non-negative measure, see (1.13–1.17) for more details.

3.6 Artificial diffusion limit

The next step in the proof of Theorem 3.1 is to let $\varepsilon \to 0$ in the approximate system (3.172–3.182) in order to eliminate the artificial diffusion term in (3.172) as well as the other ε-dependent quantities in the remaining equations. Such a step is not straightforward, as we lose the uniform bound on $\nabla_x \varrho$; whence compactness of ϱ with respect to the space variable becomes an issue. In particular, the lack of pointwise convergence of the densities has to be taken into account in the proof of pointwise convergence of the approximate temperatures; accordingly, the procedure described in the previous section relating formulas (3.151), (3.152) has to be considerably modified. Apart from these principal new difficulties a number of other rather technical issues has to be addressed. In particular, uniform bounds must be established in order to show that all ε-dependent quantities in the approximate continuity equation (3.172), momentum equation (3.175), and energy balance (3.182) vanish in the asymptotic limit $\varepsilon \to 0$. Similarly, the non-negative quantities appearing in the approximate entropy production rate $\sigma_{\varepsilon,\delta}$ are used to obtain uniform bounds in order to eliminate the "artificial" entropy flux $\mathbf{r}$ in (3.179).

In order to show pointwise convergence of the approximate temperatures, we take advantage of certain general properties of weak convergence of composed functions expressed conveniently in terms of parameterized (Young) measures (see Section 3.6.2). On the other hand, similarly to the recently developed existence theory for compressible viscous fluids, we use the extra regularity properties of the quantity $\Pi := p(\varrho, \vartheta) - (\frac{4}{3}\mu(\vartheta) + \eta(\vartheta))\mathrm{div}_x \mathbf{u}$, called *effective viscous flux*, in order to establish pointwise convergence of the approximate densities. Such an approach requires a proper description of possible oscillations of the densities provided by the renormalized continuity equation (cf. Section 10.18 in Appendix).

3.6.1 Uniform estimates and limit in the approximate continuity equation

Let $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ be a family of solutions to the approximate system (3.172–3.182) constructed in Section 3.5. Similarly to Section 2.2.3, the total energy balance (3.182), together with the entropy inequality represented through (3.184), give rise to the *dissipation balance*

$$\int_\Omega \Big(\frac{1}{2}\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + H_{\delta,\overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon) + \delta(\frac{\varrho_\varepsilon^\Gamma}{\Gamma - 1} + \varrho_\varepsilon^2)\Big)(\tau)\ \mathrm{d}x \tag{3.186}$$
$$+ \overline{\vartheta}\Sigma_{\varepsilon,\delta}\Big[[0,\tau] \times \overline{\Omega}\Big] + \int_0^\tau \int_\Omega \varepsilon \vartheta^5\ \mathrm{d}x\ \mathrm{d}t$$
$$= \int_\Omega \Big(\frac{1}{2}\frac{|(\varrho\mathbf{u})_0|^2}{\varrho_{0,\delta}} + H_{\delta,\overline{\vartheta}}(\varrho_{0,\delta}, \vartheta_{0,\delta}) + \delta(\frac{\varrho_{0,\delta}^\Gamma}{\Gamma - 1} + \varrho_{0,\delta}^2)\Big)\ \mathrm{d}x$$
$$+ \int_0^\tau \int_\Omega \Big(\varrho_\varepsilon \mathbf{f}_\delta \cdot \mathbf{u}_\varepsilon + \varrho\Big(1 - \frac{\overline{\vartheta}}{\vartheta_\varepsilon}\Big)\mathcal{Q}_\delta + \frac{\delta}{\vartheta_\varepsilon^2} + \varepsilon\overline{\vartheta}\vartheta^4\Big)\ \mathrm{d}x\ \mathrm{d}t \text{ for a.a. } \tau \in [0,T],$$

where $\Sigma_{\varepsilon,\delta} \in \mathcal{M}^+([0,T]\times\overline{\Omega})$ is the entropy production rate introduced in Section 3.5.6, and the "approximate Helmholtz function" $H_{\delta,\overline{\vartheta}}$ is given through (3.101).

Repeating the arguments used after formula (3.105) we obtain

$$\sup_{\varepsilon>0}\left\{ \text{ess}\sup_{t\in(0,T)} \int_\Omega \Big(\frac{1}{2}\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + H_{\delta,\overline{\vartheta}}(\varrho_\varepsilon,\vartheta_\varepsilon) + \delta\big(\frac{\varrho_\varepsilon^\Gamma}{\Gamma-1} + \varrho_\varepsilon^2\big)\Big)(t) \ \mathrm{d}x \right\} < \infty, \tag{3.187}$$

together with

$$\sup_{\varepsilon>0}\left\{ \Sigma_{\varepsilon,\delta}\Big[[0,T]\times\overline{\Omega}\Big] + \int_0^T\int_\Omega \varepsilon\vartheta_\varepsilon^5 \ \mathrm{d}x \ \mathrm{d}t \right\} < \infty, \tag{3.188}$$

where, in accordance with (3.181), (3.185), estimate (3.188) further implies

$$\sup_{\varepsilon>0}\left\{ \int_0^T\int_\Omega \Big\{ \frac{1}{\vartheta_\varepsilon}\Big[\mathbb{S}_\delta(\vartheta_\varepsilon,\nabla_x\mathbf{u}_\varepsilon) : \nabla_x\mathbf{u}_\varepsilon \right. \tag{3.189}$$

$$\left. + \Big(\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} + \delta\big(\vartheta_\varepsilon^{\Gamma-1} + \frac{1}{\vartheta_\varepsilon^2}\big)\Big)|\nabla_x\vartheta_\varepsilon|^2\Big] + \delta\frac{1}{\vartheta_\varepsilon^3} + \varepsilon\vartheta_\varepsilon^5 \Big\} \ \mathrm{d}x \ \mathrm{d}t \right\} < \infty,$$

$$\sup_{\varepsilon>0}\left\{ \varepsilon\delta\int_0^T\int_\Omega \frac{1}{\vartheta_\varepsilon}(\Gamma\varrho_\varepsilon^{\Gamma-2}+2)|\nabla_x\varrho_\varepsilon|^2 \ \mathrm{d}x \ \mathrm{d}t \right\} < \infty, \tag{3.190}$$

and

$$\sup_{\varepsilon>0}\left\{ \int_0^T\int_\Omega \varepsilon \frac{\overline{\vartheta}}{\varrho_\varepsilon\vartheta_\varepsilon}\frac{\partial p_M}{\partial\varrho}(\varrho_\varepsilon,\vartheta_\varepsilon)|\nabla_x\varrho_\varepsilon|^2 \ \mathrm{d}x \ \mathrm{d}t \right\} < \infty. \tag{3.191}$$

Exactly as in Section 3.5, the above estimates can be used to deduce that

$$\varrho_\varepsilon \to \varrho \text{ weakly-(*) in } L^\infty(0,T;L^\Gamma(\Omega)), \tag{3.192}$$

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)), \tag{3.193}$$

and

$$\vartheta_\varepsilon \to \vartheta \text{ weakly-(*) in } L^\infty(0,T;L^4(\Omega)), \tag{3.194}$$

at least for suitable subsequences. Moreover, we have $\mathbf{u}(t,\cdot) \in W^{1,2}_{\mathbf{n}}(\Omega;\mathbb{R}^3)$ for a.a. $t\in(0,T)$ in the case of the complete slip boundary conditions, while $\mathbf{u}(t,\cdot) \in W^{1,2}_0(\Omega;\mathbb{R}^3)$ for a.a. $t\in(0,T)$, if the no-slip boundary conditions are imposed.

Multiplying equation (3.172) by ϱ_ε and integrating by parts we get

$$\frac{1}{2}\int_\Omega \varrho_\varepsilon^2(\tau) \ \mathrm{d}x + \varepsilon\int_0^\tau\int_\Omega |\nabla_x\varrho_\varepsilon|^2 \ \mathrm{d}x \ \mathrm{d}t$$
$$= \frac{1}{2}\int_\Omega \varrho_{0,\delta}^2 \ \mathrm{d}x - \frac{1}{2}\int_0^\tau\int_\Omega \varrho_\varepsilon^2 \mathrm{div}_x\mathbf{u}_\varepsilon \ \mathrm{d}x \ \mathrm{d}t;$$

whence, taking (3.192–3.194) into account, we can see that

$$\{\sqrt{\varepsilon}\nabla_x \varrho_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)),$$

in particular,

$$\varepsilon\nabla_x \varrho_\varepsilon \to 0 \text{ in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)). \tag{3.195}$$

As the time derivative $\partial_t \varrho_\varepsilon$ can be expressed by means of equation (3.172), convergence in (3.192) can be, similarly to (3.119), strengthened to

$$\varrho_\varepsilon \to \varrho \text{ in } C_{\text{weak}}([0,T];L^\Gamma(\Omega)). \tag{3.196}$$

Relation (3.196), combined with (3.193) and boundedness of the kinetic energy, yields

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \varrho\mathbf{u} \text{ weakly-}(*) \text{ in } L^\infty(0,T;L^{\frac{2\Gamma}{\Gamma+1}}(\Omega;\mathbb{R}^3)). \tag{3.197}$$

Thus we conclude that the limit functions ϱ, $\mathbf{u}$ satisfy the integral identity

$$\int_0^T \int_\Omega \Big(\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t + \int_\Omega \varrho_{0,\delta}\varphi(0,\cdot)\ \mathrm{d}x = 0 \tag{3.198}$$

for any test function $\varphi \in C_c^\infty([0,T)\times\overline{\Omega})$. Moreover, since the boundary $\partial\Omega$ is regular (Lipschitz) we can extend continuously the velocity field $\mathbf{u}$ outside Ω in such a way that the resulting vector field belongs to $W^{1,2}(\mathbb{R}^3;\mathbb{R}^3)$. (In the case of no-slip boundary conditions one can take trivial extension, where $\mathbf{u}=0$ outside Ω.) Accordingly, setting $\varrho\equiv 0$ in $\mathbb{R}^3\setminus\Omega$ we can assume that ϱ, $\mathbf{u}$ solve the equation of continuity

$$\partial_t\varrho + \operatorname{div}_x(\varrho\mathbf{u}) = 0 \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^3). \tag{3.199}$$

3.6.2 Entropy balance and strong convergence of the approximate temperatures

Our principal objective is to show strong (pointwise) convergence of the family $\{\vartheta_\varepsilon\}_{\varepsilon>0}$. Following the same strategy as in Section 3.5.3, we divide the proof into three steps:

(i) Div-Curl lemma (Proposition 3.3) is applied to show that

$$\overline{\varrho s_\delta(\varrho,\vartheta)G(\vartheta)} = \overline{\varrho s_\delta(\varrho,\vartheta)}\ \overline{G(\vartheta)}$$

for any $G\in W^{1,\infty}(0,\infty)$. This relation is reminiscent of formula (3.150); the quantity G playing a role of a cut-off function is necessary because of the low integrability of ϑ. The proof uses the same arguments as in Section 3.5.3.

(ii) Although strong convergence of the densities is no longer available at this stage, we can still show that

$$\overline{b(\varrho)G(\vartheta)} = \overline{b(\varrho)}\ \overline{G(\vartheta)}, \tag{3.200}$$

where $b \in C([0,\infty)) \cap L^\infty((0,\infty))$, and G is the same as in the previous step. In order to prove this identity, we use the properties of *renormalized solutions* to the approximate continuity equation (cf. Section 10.18 in Appendix). Very roughly indeed, we can say that possible oscillations in the sequence of approximate densities and temperatures take place in orthogonal directions of the space-time.

(iii) The simple monotonicity argument used in formula (3.151) has to be replaced by a more sophisticated tool. Here, the desired relation

$$s_M(\varrho_\varepsilon, t, x)(G(\vartheta_\varepsilon) - \overline{G(\vartheta)}) \to 0$$

is shown to follow directly from (3.200) by means of a general argument borrowed from the theory of parameterized (Young) measures. An elementary alternative proof of this step involving a compactness argument based on the renormalized continuity equation (more precisely on Theorem 10.30 in Appendix) is shown in Section 3.7.3.

In the remaining part of this section, we develop the ideas delineated in the above program in a more specific way.

Uniform estimates. Seeing that the sequence $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ admits the bounds obtained in (3.189), we infer that $\{\vartheta_\varepsilon\}_{\varepsilon>0}$ satisfies the estimates stated in (3.122–3.128), namely

$$\begin{aligned} \{\vartheta_\varepsilon\}_{\varepsilon>0},\ \{\vartheta_\varepsilon^{\Gamma/2}\}_{\varepsilon>0} \text{ are bounded in } & L^2(0,T;W^{1,2}(\Omega)), \\ \{\nabla(\vartheta_\varepsilon^{-1/2})\}_{\varepsilon>0} \text{ is bounded in } & L^2((0,T)\times\Omega;\mathbb{R}^3), \\ \{\vartheta_\varepsilon^{-1}\}_{\varepsilon>0} \text{ is bounded in } & L^3((0,T)\times\Omega), \\ \{\log\vartheta_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } & L^2(0,T;W^{1,2}(\Omega)) \cap L^\Gamma(0,T;L^{3\Gamma}(\Omega)). \end{aligned} \tag{3.201}$$

Moreover, relations (3.129), (3.130) imply that

$$\{\sqrt{\varepsilon}\varrho_\varepsilon\}_{\varepsilon>)},\ \{\sqrt{\varepsilon}\varrho_\varepsilon^{\frac{\Gamma}{2}}\}_{\varepsilon>0} \quad \text{are bounded in } L^2(0,T;W^{1,2}(\Omega)). \tag{3.202}$$

Application of the Div-Curl lemma. Now we rewrite the approximate entropy balance (3.184) in the form

$$\partial_t\Big(\varrho_\varepsilon s_\delta(\varrho_\varepsilon,\vartheta_\varepsilon)\Big) + \operatorname{div}_x\Big(\varrho_\varepsilon s_\delta(\varrho_\varepsilon,\vartheta_\varepsilon)\mathbf{u}_\varepsilon + \frac{\kappa_\delta(\vartheta_\varepsilon)\nabla_x\vartheta_\varepsilon}{\vartheta_\varepsilon} + \varepsilon\mathbf{r}_\varepsilon\Big) = \Sigma_{\varepsilon,\delta} + \frac{\varrho_\varepsilon}{\vartheta_\varepsilon}\mathcal{Q}_\delta - \varepsilon\vartheta_\varepsilon^4$$

to be understood in the weak sense specified in Sections 1.2, 3.5.6.

Similarly to Section 3.5.3, we intend to apply the Div-Curl lemma (Proposition 3.3) to the four-component vector fields

$$\mathbf{U}_\varepsilon := \left[\varrho_\varepsilon s_\delta(\varrho_\varepsilon,\vartheta_\varepsilon), \varrho_\varepsilon s_\delta(\varrho_\varepsilon,\vartheta_\varepsilon)\mathbf{u}_\varepsilon + \frac{\kappa_\delta(\vartheta_\varepsilon)\nabla_x\vartheta_\varepsilon}{\vartheta} + \varepsilon\mathbf{r}_\varepsilon\right], \tag{3.203}$$

$$\mathbf{V}_\varepsilon := [G(\vartheta_\varepsilon), 0, 0, 0], \tag{3.204}$$

where G is a bounded globally Lipschitz function on $[0,\infty)$.

First observe that the families

$$\mathrm{div}_{t,x}\mathbf{U}_\varepsilon = \Sigma_{\varepsilon,\delta} + \frac{\varrho_\varepsilon}{\vartheta_\varepsilon}\mathcal{Q}_\delta - \varepsilon\vartheta_\varepsilon^4, \quad \mathrm{curl}_{t,x}\mathbf{V}_\varepsilon = G'(\vartheta_\varepsilon)\begin{pmatrix} 0 & \nabla\vartheta_\varepsilon \\ \nabla^T\vartheta_\varepsilon & \mathbf{0} \end{pmatrix}$$

are relatively compact in $W^{-1,s}((0,T)\times\Omega))$, $W^{-1,s}((0,T)\times\Omega;\mathbb{R}^{4\times 4})$ for $s\in[1,\frac{4}{3})$, respectively. Indeed, it is enough to use estimates (3.188), (3.192), (3.194), (3.201), and compactness of the embeddings $\mathcal{M}^+([0,T]\times\overline{\Omega})\hookrightarrow W^{-1,s}((0,T)\times\Omega))$, $L^1((0,T)\times\Omega))\hookrightarrow W^{-1,s}((0,T)\times\Omega))$. Notice that we have, in particular,

$$\varepsilon\vartheta_\varepsilon^4 \to 0 \text{ in } L^1((0,T)\times\Omega) \tag{3.205}$$

as a direct consequence of (3.194).

As the sequence $\{G(\vartheta_\varepsilon)\}_{\varepsilon>0}$ is bounded in $L^\infty((0,T)\times\Omega)$, it is enough to show boundedness of the family $\{\mathbf{U}_\varepsilon\}_{\varepsilon>0}$ in $L^p((0,T)\times\Omega;\mathbb{R}^4)$ for a certain $1<p<\infty$. Combining the arguments already used in (3.145), (3.146) with the bounds (3.192), (3.201), we infer that

$$\{\varrho_\varepsilon s_\delta(\varrho_\varepsilon,\vartheta_\varepsilon)\}_{\varepsilon>0} \text{ is bounded in } L^p((0,T)\times\Omega) \text{ for a certain } p>2, \tag{3.206}$$

while

$$\{\varrho_\varepsilon s_\delta(\varrho_\varepsilon,\vartheta_\varepsilon)\mathbf{u}_\varepsilon\}_{\varepsilon>0},\ \left\{\frac{\kappa_\delta(\vartheta_\varepsilon)}{\vartheta_\varepsilon}\nabla\vartheta_\varepsilon\right\}_{\varepsilon>0} \text{ are bounded in } L^q((0,T)\times\Omega;\mathbb{R}^3)$$
$$\text{for a certain } q>1 \text{ provided } \Gamma>6. \tag{3.207}$$

Finally, following the reasoning of (3.147–3.149), we use (3.201) and (3.202) to obtain

$$\varepsilon\mathbf{r}_\varepsilon \to 0 \text{ in } L^p((0,T)\times\Omega;\mathbb{R}^3)) \text{ for a certain } p>1. \tag{3.208}$$

Having verified all hypotheses of Proposition 3.3 we conclude that

$$\overline{\varrho s_\delta(\varrho,\vartheta)G(\vartheta)} = \overline{\varrho s_\delta(\varrho,\vartheta)}\ \overline{G(\vartheta)} \tag{3.209}$$

for any bounded and continuous function G.

Monotonicity of the entropy and strong convergence of the approximate temperatures – application of the theory of parametrized (Young) measures.
Similarly to Section 3.5.3, relation (3.209) can be used to show strong (pointwise) convergence of $\{\vartheta_\varepsilon\}_{\varepsilon>0}$. Decomposing

$$\varrho s_\delta(\varrho,\vartheta) = \varrho s_M(\varrho,\vartheta) + \delta\varrho\log(\vartheta) + \frac{4}{3}a\vartheta^3,$$

we have to show that

$$\overline{\varrho s_M(\varrho,\vartheta)G(\vartheta)} \ge \overline{\varrho s_M(\varrho,\vartheta)}\ \overline{G(\vartheta)}, \quad \overline{\varrho\log(\vartheta)G(\vartheta)} \ge \overline{\varrho\log(\vartheta)}\ \overline{G(\vartheta)},$$
$$\overline{\vartheta^3 G(\vartheta)} \ge \overline{\vartheta^3}\ \overline{G(\vartheta)} \tag{3.210}$$

for any continuous and increasing G chosen in such a way that all the weak limits exist at least in L^1. Indeed, relations (3.210) combined with (3.209) imply

$$\overline{\vartheta^3 G(\vartheta)} = \overline{\vartheta^3}\ \overline{G(\vartheta)}; \text{ whence } \overline{\vartheta^4} = \overline{\vartheta^3}\vartheta \tag{3.211}$$

yielding, in particular, the desired conclusion

$$\vartheta_\varepsilon \to \vartheta \text{ a.a. in } (0,T) \times \Omega. \tag{3.212}$$

In order to see (3.210), write

$$\begin{aligned} 0 \leq & \left(\varrho_\varepsilon s_M \Big(\varrho_\varepsilon, G^{-1}(G(\vartheta_\varepsilon)) \Big) - \varrho_\varepsilon s_M \Big(\varrho_\varepsilon, G^{-1}(\overline{G(\vartheta)}) \Big) \right) \Big(G(\vartheta_\varepsilon) - \overline{G(\vartheta)} \Big) \\ & = \varrho_\varepsilon s_M(\varrho_\varepsilon, \vartheta_\varepsilon) \Big(G(\vartheta_\varepsilon) - \overline{G(\vartheta)} \Big) - \varrho_\varepsilon s_M \Big(\varrho_\varepsilon, G^{-1}(\overline{G(\vartheta)}) \Big) \Big(G(\vartheta_\varepsilon) - \overline{G(\vartheta)} \Big). \end{aligned}$$

Consequently, the first inequality in (3.210) follows as soon as we can show that

$$\varrho_\varepsilon s_M \Big(\varrho_\varepsilon, G^{-1}(\overline{G(\vartheta)}) \Big) \Big(G(\vartheta_\varepsilon) - \overline{G(\vartheta)} \Big) \to 0 \text{ weakly in } L^1((0,T) \times \Omega). \tag{3.213}$$

The quantity

$$\varrho_\varepsilon s_M \Big(\varrho_\varepsilon, \Big[G^{-1}(\overline{G(\vartheta)}) \Big](t,x) \Big) = \psi(t,x,\varrho_\varepsilon)$$

may be regarded as a superposition of a Carathéodory function with a weakly convergent sequence. In such a situation, a general argument of the theory of parameterized (Young) measures asserts that (3.213) follows as soon as we show that

$$\overline{b(\varrho) G(\vartheta)} = \overline{b(\varrho)}\ \overline{G(\vartheta)} \tag{3.214}$$

for arbitrary smooth and bounded functions b and G (see Theorem 0.10).

Indeed, if $\nu_{(t,x)}^{(\varrho,\vartheta)}$, $\nu_{(t,x)}^{\varrho}$ and $\nu_{(t,x)}^{\vartheta}$ are families of parametrized Young measures associated to sequences $\{(\varrho_\varepsilon, \vartheta_\varepsilon)\}_{\varepsilon>0}$, $\{\varrho_\varepsilon\}_{\varepsilon>0}$ and$\{\vartheta_\varepsilon\}_{\varepsilon>0}$, respectively, then (3.214) implies

$$\int_{\mathbb{R}^2} b(\lambda) G(\mu) \, \mathrm{d}\nu_{(t,x)}^{(\varrho,\vartheta)}(\lambda,\mu) = \int_{\mathbb{R}} b(\lambda) \, \mathrm{d}\nu_{(t,x)}^{\varrho}(\lambda) \times \int_{\mathbb{R}} G(\mu) \, \mathrm{d}\nu_{(t,x)}^{\vartheta}(\mu).$$

This evidently yields a decomposition

$$\nu_{(t,x)}^{(\varrho,\vartheta)}(A \times B) = \nu_{(t,x)}^{\varrho}(A) \nu_{(t,x)}^{\vartheta}(B),$$

where A, B are open subsets in $\mathbb{R}$. Consequently, for any Carathéodory function $\psi(t,x,\lambda)$ and a continuous function $G(\vartheta)$, such that sequences $\psi(\cdot,\cdot,\varrho_n) G(\vartheta_n)$ and

$\psi(\cdot,\cdot,\varrho_n)$, $G(\vartheta_n)$ are weakly convergent in $L^1((0,T)\times\Omega;\mathbb{R}^2)$ and $L^1((0,T)\times\Omega)$, respectively, we have

$$\begin{aligned}[\overline{\psi(\cdot,\cdot,\varrho)G(\vartheta)}](t,x) &= \int_{\mathbb{R}^2} \psi(t,x,\lambda)G(\mu)\,\mathrm{d}\nu^{(\varrho,\vartheta)}_{(t,x)}(\lambda,\mu)\\ &= \int_{\mathbb{R}^2} \psi(t,x,\lambda)G(\mu)\,\mathrm{d}\nu^{\varrho}_{(t,x)}(\lambda)\mathrm{d}\nu^{\vartheta}_{(t,x)}(\mu)\\ &= [\overline{\psi(\cdot,\cdot,\varrho)}\;\overline{G(\vartheta)}](t,x)\end{aligned}$$

which is nothing other than (3.213).

In order to verify (3.214), multiply the approximate continuity equation (3.172) by $b'(\varrho)\varphi$, $\varphi\in C^\infty_c(\Omega)$, and integrate over Ω to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega b(\varrho)\varphi\mathrm{d}x - \int_\Omega b(\varrho)\mathbf{u}\cdot\nabla_x\varphi\mathrm{d}x + \varepsilon\int_\Omega b''(\varrho)|\nabla_x\varrho|^2\varphi\mathrm{d}x\\ + \varepsilon\int_\Omega b'(\varrho)\nabla_x\varrho\cdot\nabla_x\varphi\mathrm{d}x + \int_\Omega(\varrho b'(\varrho)-b(\varrho))\mathrm{div}_x\mathbf{u}\varphi\mathrm{d}x = 0.\end{aligned}\tag{3.215}$$

Consequently, the sequence $\{t\mapsto\int_\Omega b(\varrho_\varepsilon)\varphi\}_{\varepsilon>0}$ is uniformly bounded and equi-continuous in $C([0,T])$; whence

$$b(\varrho_\varepsilon)\to\overline{b(\varrho)} \text{ in } C_{\mathrm{weak}}([0,T];L^\Gamma(\Omega))\tag{3.216}$$

at least for any smooth function b with bounded second derivative.

Now, we use compactness of the embedding $L^\Gamma(\Omega)\hookrightarrow W^{-1,2}(\Omega)$ to deduce

$$b(\varrho_\varepsilon)\to\overline{b(\varrho)} \text{ in } C([0,T];W^{-1,2}(\Omega)).\tag{3.217}$$

On the other hand, in accordance with the uniform bounds established in (3.201),

$$G(\vartheta_\varepsilon)\to\overline{G(\vartheta)} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega));\tag{3.218}$$

whence (3.214) follows from (3.217), (3.218).

In addition to (3.212), the limit temperature field ϑ is positive a.a. on the set $(0,T)\times\Omega$, more precisely, we have

$$\vartheta^{-3}\in L^1((0,T)\times\Omega).\tag{3.219}$$

Indeed, (3.219) follows from the uniform bounds (3.201), the pointwise convergence of $\{\vartheta_\varepsilon\}_{\varepsilon>0}$ established in (3.212), and the property of weak lower semi-continuity of convex functionals (see Theorem 10.20 in Appendix).

Asymptotic limit in the entropy balance. At this stage, we are ready to let $\varepsilon\to 0$ in the approximated entropy equality (3.184). Using relations (3.201–3.212) we obtain, in the same way as in (3.156), (3.157),

$$\frac{\kappa_\delta(\vartheta_\varepsilon)}{\vartheta_\varepsilon}\nabla_x\vartheta_\varepsilon\to\frac{\kappa_\delta(\vartheta)}{\vartheta}\nabla_x\vartheta$$
$$\text{weakly in } L^p((0,T)\times\Omega;\mathbb{R}^3) \text{ for some } p>1.$$

Furthermore, in accordance with (3.192), (3.212), we get

$$\frac{\varrho_\varepsilon}{\vartheta_\varepsilon}\mathcal{Q}_\delta \to \frac{\varrho}{\vartheta}\mathcal{Q}_\delta \text{ weakly in } L^p((0,T)\times\Omega) \text{ for some } p>1.$$

Applying the Div-Curl lemma (Proposition 3.3) to the sequence $\{\mathbf{U}_\varepsilon\}_{\varepsilon>0}$ defined in (3.203) and $\{\mathbf{V}_\varepsilon\}_{\varepsilon>0}$,

$$\mathbf{V}_\varepsilon = [(u_\varepsilon)_i, 0,0,0],\ i=1,2,3,$$

we deduce

$$\varrho_\varepsilon s_\delta(\varrho_\varepsilon,\vartheta_\varepsilon)\mathbf{u}_\varepsilon \to \overline{\varrho s_\delta(\varrho,\vartheta)\mathbf{u}} \text{ weakly in } L^p((0,T)\times\Omega;\mathbb{R}^3) \text{ for a certain } p>1.$$

The terms $\frac{1}{\vartheta_\varepsilon}S_\delta(\vartheta_\varepsilon,\mathbf{u}_\varepsilon):\nabla\mathbf{u}_\varepsilon$, $\frac{\kappa_\delta(\vartheta_\varepsilon)}{\vartheta_\varepsilon}|\nabla\vartheta_\varepsilon|^2$ appearing in $\sigma_{\varepsilon,\delta}$ are weakly lower semi-continuous as we have already observed in (3.160–3.165), while the remaining ε-dependent quantities in $\sigma_{\varepsilon,\delta}$ are non-negative. Finally, by virtue of (3.188), we may assume

$$\Sigma_{\varepsilon,\delta} \to \sigma_\delta \in \text{ weakly-(*) in } \mathcal{M}([0,T]\times\overline{\Omega}), \text{ where } \sigma_\delta\in\mathcal{M}^+([0,T]\times\overline{\Omega}).$$

Recalling the limits (3.205) and (3.208), we let $\varepsilon\to 0$ in (3.184) to obtain

$$\int_0^T\int_\Omega \overline{\varrho s_\delta(\varrho,\vartheta)}\Big(\partial_t\varphi + \mathbf{u}\cdot\nabla_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{3.220}$$
$$+\int_0^T\int_\Omega \frac{\kappa_\delta(\vartheta)\nabla_x\vartheta}{\vartheta}\cdot\nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t + \langle\sigma_\delta;\varphi\rangle_{[C;\mathcal{M}]([0,T]\times\overline{\Omega})}$$
$$= -\int_\Omega(\varrho s)_{0,\delta}\varphi(0,\cdot)\ \mathrm{d}x - \int_0^T\int_\Omega\frac{\varrho}{\vartheta}\mathcal{Q}_\delta\varphi\ \mathrm{d}x\ \mathrm{d}t, \text{ for all } \varphi\in C_c^\infty([0,T)\times\overline{\Omega}),$$

where

$$\sigma_\delta \ge \frac{1}{\vartheta}\mathbb{S}_\delta(\vartheta,\nabla_x\mathbf{u}):\nabla_x\mathbf{u} + \frac{\kappa_\delta(\vartheta)}{\vartheta}|\nabla\vartheta|^2.$$

Consequently, in order to perform the limit $\varepsilon\to 0$ in the remaining equations of the approximate system (3.172–3.182), we have to show

- **(i)** uniform pressure estimates analogous to those established in Section 2.2.5, or, alternatively, in Section 2.2.6,
- **(ii)** strong (pointwise) convergence of the approximate densities.

3.6.3 Uniform pressure estimates

The pressure estimates are derived in the same way as in Section 2.2.5, namely we use the quantities

$$\varphi = \psi\phi,\ \psi\in C_c^\infty(0,T),\ \phi = \mathcal{B}[\varrho_\varepsilon - \overline{\varrho}] \tag{3.221}$$

as test functions in the approximate momentum equation (3.175), where

$$\overline{\varrho} = \frac{1}{|\Omega|}\int_\Omega \varrho_\varepsilon\ \mathrm{d}x,$$

and $\mathcal{B} \approx \operatorname{div}_x^{-1}$ is the *Bogovskii operator* introduced in Section 2.2.5 and investigated in Section 10.5 in Appendix.

Since ϱ_ε satisfies the approximate continuity equation (3.172), we have

$$\partial_t \phi = -\mathcal{B}\left[\operatorname{div}_x(\varrho\mathbf{u} - \varepsilon\nabla_x\varrho)\right]. \tag{3.222}$$

Consequently, by virtue of the basic properties of the operator $\mathcal{B}$ listed in Section 2.2.5,

$$\|\phi(t,\cdot)\|_{W^{1,p}(\Omega;\mathbb{R}^3)} \leq c(p,\Omega)\|\varrho_\varepsilon(t,\cdot)\|_{L^p(\Omega)} \text{ for a.a. t} \in (0,T), \tag{3.223}$$

and

$$\|\partial_t\phi(t,\cdot)\|_{L^p(\Omega;\mathbb{R}^3)} \leq c(p,\Omega)\left\|\varrho_\varepsilon\mathbf{u}_\varepsilon(t,\cdot) + \varepsilon\nabla_x\varrho_\varepsilon(t,\cdot)\right\|_{L^p(\Omega;\mathbb{R}^3)} \text{for a.a. } t \in (0,T) \tag{3.224}$$

for any $1 < p < \infty$.

The last two estimates, together with those previously established in (3.192–3.197), (3.201), and (3.202), render the test functions (3.221) admissible in (3.175) provided, say, $\Gamma \geq 4$. Note that, unlike in Section 2.2.5, the argument of the operator $\mathcal{B}$ is an affine function of ϱ_ε, whereas the necessary uniform estimate on $\{\varrho_\varepsilon\}_{\varepsilon>0}$ in $L^\infty(0,T;L^\Gamma(\Omega))$ is provided by the extra pressure term $\delta\varrho^\Gamma$.

In view of these arguments, we can write, similarly to (2.94),

$$\int_0^T \left[\psi\int_\Omega \Big(p(\varrho_\varepsilon,\vartheta_\varepsilon) + \delta(\varrho_\varepsilon^\Gamma + \varrho_\varepsilon^2)\Big)\varrho_\varepsilon \ \mathrm{d}x\right] \mathrm{d}t = \sum_{j=1}^7 I_j, \tag{3.225}$$

where

$$I_1 = \int_0^T \left[\psi\overline{\varrho}\int_\Omega \Big(p(\varrho_\varepsilon,\vartheta_\varepsilon + \delta(\varrho_\varepsilon^\Gamma + \varrho_\varepsilon^2)\Big) \ \mathrm{d}x\right] \mathrm{d}t,$$

$$I_2 = -\int_0^T \left[\psi\int_\Omega \varrho_\varepsilon\mathbf{u}_\varepsilon \cdot \partial_t\phi \ \mathrm{d}x\right] \mathrm{d}t,$$

$$I_3 = -\int_0^T \left[\psi\int_\Omega \varrho_\varepsilon\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x\phi \ \mathrm{d}x\right] \mathrm{d}t,$$

$$I_4 = \int_0^T \left[\psi\int_\Omega \mathbb{S}_\delta(\mathbf{u}_\varepsilon,\vartheta_\varepsilon) : \nabla_x\phi \ \mathrm{d}x\right] \mathrm{d}t,$$

$$I_5 = -\int_0^T \left[\psi\int_\Omega \varrho_\varepsilon\mathbf{f} \cdot \phi \ \mathrm{d}x\right] \mathrm{d}t,$$

$$I_6 = -\int_0^T \left[\psi'\int_\Omega \varrho_\varepsilon\mathbf{u}_\varepsilon \cdot \phi \ \mathrm{d}x\right] \mathrm{d}t,$$

and

$$I_7 = \int_0^T \psi\Big[\int_\Omega \varepsilon\nabla_x\varrho_\varepsilon\nabla_x\mathbf{u}_\varepsilon \cdot \phi \ \mathrm{d}x\Big] \mathrm{d}t.$$

The simple form of I_7 conditioned by the specific form of the test function φ, where the argument of $\mathcal{B}$ is an affine function of ϱ_ε, is the only technical reason why the limit processes for $\varepsilon \to 0$ and $\delta \to 0$ must be separated.

The integral identity (3.225) can be used to obtain uniform bounds on the pressure independent of ε. Exactly as in Section 2.2.5, we deduce that

$$\|\varrho_\varepsilon\|_{L^{\Gamma+1}((0,T)\times\Omega)} \leq c(\text{data}, \delta), \tag{3.226}$$

and

$$\|p_M(\varrho_\varepsilon, \vartheta_\varepsilon)\|_{L^p((0,T)\times\Omega)} \leq c(\text{data}, \delta) \quad \text{for a certain } p > 1. \tag{3.227}$$

Indeed, these bounds can be obtained by dominating the integrals $I_1 - I_7$ in the spirit of Section 2.2.5, specifically, by means of estimates (3.223), (3.224), (3.192–3.197), and (3.201), provided $\Gamma \geq 4$. In particular, by virtue of (3.193), (3.195),

$$\varepsilon \nabla_x \varrho_\varepsilon \nabla_x \mathbf{u}_\varepsilon \to 0 \text{ in } L^1((0,T) \times \Omega; \mathbb{R}^3)) \tag{3.228}$$

yielding boundedness of integral I_7.

3.6.4 Limit in the approximate momentum equation and in the energy balance

In accordance with estimates (3.226), (3.227), together with (3.194), (3.201), and (3.212),

$$\begin{gathered} p_\delta(\varrho_\varepsilon, \vartheta_\varepsilon) \to \overline{p_\delta(\varrho, \vartheta)} = \overline{p_M(\varrho, \vartheta)} + \tfrac{a}{4}\vartheta^4 + \delta(\overline{\varrho^\Gamma} + \overline{\varrho^2}) \\ \text{weakly in } L^p((0,T) \times \Omega) \text{ for a certain } p > 1, \end{gathered} \tag{3.229}$$

where we have written

$$p_\delta(\varrho, \vartheta) = p_M(\varrho, \vartheta) + \frac{a}{4}\vartheta^4 + \delta(\varrho^\Gamma + \varrho^2). \tag{3.230}$$

On the other hand, by virtue of (3.17), (3.23), (3.194), and (3.212),

$$\mu(\vartheta_\varepsilon) \to \mu(\vartheta),\ \eta(\vartheta_\varepsilon) \to \eta(\vartheta) \text{ (strongly) in } L^p((0,T) \times \Omega) \text{ for any } 1 \leq p < 4.$$

Moreover, since $\mathbb{S}_\delta$ takes the form specified in (3.53), we can use (3.193) in order to deduce

$$\mathbb{S}_\delta(\vartheta_\varepsilon, \nabla_x \mathbf{u}_\varepsilon) \to \mathbb{S}_\delta(\vartheta, \mathbf{u}) \text{ weakly in } L^p((0,T) \times \Omega), \text{ for a certain } p > 1. \tag{3.231}$$

As the limits of the families $\varrho_\varepsilon \mathbf{f}$, $\varrho_\varepsilon \mathbf{u}_\varepsilon$, and $\varepsilon \nabla \varrho_\varepsilon \nabla \mathbf{u}_\varepsilon$ have already been identified through (3.192), (3.197) and (3.228), we are left with the convective term $\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon$. Following the arguments of Section 3.5.4 we observe that

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \varrho \mathbf{u} \text{ in } C_{\text{weak}}([0,T]; L^{\frac{2\Gamma}{\Gamma+1}}(\Omega; \mathbb{R}^3)). \tag{3.232}$$

Consequently, because of compact embedding $L^s(\Omega) \hookrightarrow W^{-1,2}(\Omega)$, $s > \frac{6}{5}$,

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \varrho \mathbf{u} \,\text{(strongly) in } L^p(0,T;W^{-1,2}(\Omega;\mathbb{R}^3))$$

for any $1 \leq p < \infty$. In accordance with (3.193),

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \to \overline{\varrho \mathbf{u} \otimes \mathbf{u}} \,\text{weakly in } L^p((0,T)\times\Omega) \text{ for a certain } p > 1. \tag{3.233}$$

Letting $\varepsilon \to 0$ in the approximate momentum equation (3.175) we get

$$\int_0^T \int_\Omega \Big(\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho [\mathbf{u} \otimes \mathbf{u}] : \nabla_x \varphi + \overline{p_\delta(\varrho,\vartheta)} \mathrm{div}_x \varphi \Big) \,\mathrm{d}x \,\mathrm{d}t \tag{3.234}$$
$$= \int_0^T \int_\Omega \Big(\mathbb{S}_\delta(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \varphi - \varrho \mathbf{f}_\delta \cdot \varphi \Big) \,\mathrm{d}x \,\mathrm{d}t - \int_\Omega (\varrho \mathbf{u})_0 \cdot \varphi \,\mathrm{d}x,$$

for any test function $\varphi \in C_c^\infty([0,T) \times \overline{\Omega}; R^3))$ such that either

$$\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0 \text{ in the case of the complete slip boundary conditions,}$$

or

$$\varphi|_{\partial\Omega} = 0 \text{ in the case of the no-slip boundary conditions.}$$

Finally, as the sequence $\{\varrho_\varepsilon e_\delta(\varrho_\varepsilon, \vartheta_\varepsilon)\}_{\varepsilon>0}$ is bounded in $L^p((0,T)\times\Omega)$ (see (3.30), (3.192–3.194), (3.201)), we are allowed to let $\varepsilon \to 0$ in the approximate energy balance (3.182) to obtain

$$\int_\Omega \Big(\frac{1}{2}\varrho |\mathbf{u}|^2 + \overline{\varrho e_\delta(\varrho,\vartheta)} + \delta\big(\frac{\overline{\varrho^\Gamma}}{\Gamma - 1} + \overline{\varrho^2}\big)\Big)(\tau) \,\mathrm{d}x \tag{3.235}$$
$$= \int_\Omega \Big(\frac{1}{2} \frac{|(\varrho\mathbf{u})_{0,\delta}|^2}{\varrho_{0,\delta}} + \varrho_{0,\delta} e_{0,\delta} + \delta\big(\frac{\varrho_{0,\delta}^\Gamma}{\Gamma-1} + \varrho_{0,\delta}^2\big)\Big) \,\mathrm{d}x$$
$$+ \int_0^\tau \int_\Omega \Big(\varrho \mathbf{f}_\delta \cdot \mathbf{u} + \varrho \mathcal{Q}_\delta + \delta \frac{1}{\vartheta^2} - \varepsilon \vartheta^5 \Big) \,\mathrm{d}x \,\mathrm{d}t \text{ for a.a. } \tau \in [0,T].$$

3.6.5 Strong convergence of the densities

In order to show strong (pointwise) convergence of $\{\varrho_\varepsilon\}_{\varepsilon>0}$, we adapt the method introduced in the context of *barotropic fluids* with constant viscosity coefficients by P.-L.Lions [140], and further developed in [80] in order to accommodate the variable transport coefficients.

Similarly to Section 2.2.6, we use the quantities

$$\varphi(t,x) = \psi(t)\zeta(x)\phi, \ \phi = (\nabla_x \Delta_x^{-1})[1_\Omega \varrho_\varepsilon], \ \psi \in C_c^\infty((0,T)), \, \zeta \in C_c^\infty(\Omega), \tag{3.236}$$

as test functions in the approximate momentum equation (3.175), where the symbol Δ_x^{-1} stands for the inverse Laplace operator considered on the whole space

$\mathbb{R}^3$ introduced in (2.100). The operator $\nabla_x\Delta_x^{-1}$ is investigated in Section 10.16 in Appendix.

Since $\varrho_\varepsilon\mathbf{u}_\varepsilon$ and $\nabla\varrho_\varepsilon$ possess zero normal traces, the approximate continuity equation (3.172) can be extended to the whole $\mathbb{R}^3$, specifically,

$$\partial_t(1_\Omega\varrho_\varepsilon) + \mathrm{div}_x(1_\Omega\varrho_\varepsilon\mathbf{u}_\varepsilon) - \varepsilon\mathrm{div}_x(1_\Omega\nabla\varrho_\varepsilon) = 0 \ \text{ a.e. in } (0,T)\times\mathbb{R}^3. \tag{3.237}$$

Accordingly, we have

$$\partial_t\phi = -(\nabla_x\Delta_x^{-1})\left[\mathrm{div}_x(1_\Omega\varrho_\varepsilon\mathbf{u}_\varepsilon - \varepsilon 1_\Omega\nabla_x\varrho)\right], \tag{3.238}$$

cf. Theorem 10.26 in Appendix.

Now, exactly as in Section 2.2.6, we can use the uniform estimates (3.192–3.197), (3.201), and (3.202), in order to observe that φ defined through (3.236) is admissible in the integral identity (3.175) as soon as $\Gamma \geq 4$. Thus we get

$$\int_0^T\int_\Omega \psi\zeta\Big(p_\delta(\varrho_\varepsilon,\vartheta_\varepsilon)\varrho_\varepsilon - \mathbb{S}_\delta(\vartheta_\varepsilon,\nabla_x\mathbf{u}_\varepsilon):\mathcal{R}[1_\Omega\varrho_\varepsilon]\Big)\,\mathrm{d}x\,\mathrm{d}t = \sum_{j=1}^{8} I_{j,\varepsilon}, \tag{3.239}$$

where

$$I_{1,\varepsilon} = \int_0^T\int_\Omega \psi\zeta\Big(\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\mathcal{R}[1_\Omega\varrho_\varepsilon\mathbf{u}_\varepsilon] - (\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon):\mathcal{R}[1_\Omega\varrho_\varepsilon]\Big)\,\mathrm{d}x\,\mathrm{d}t,$$

$$I_{2,\varepsilon} = -\varepsilon\int_0^T\int_\Omega \psi\zeta\ \varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla_x\Delta^{-1}[\mathrm{div}_x(1_\Omega\nabla_x\varrho_\varepsilon)]\,\mathrm{d}x\,\mathrm{d}t,$$

$$I_{3,\varepsilon} = -\int_0^T\int_\Omega \psi\zeta\varrho_\varepsilon\mathbf{f}_\delta\cdot\nabla_x\Delta_x^{-1}[1_\Omega\varrho_\varepsilon]\,\mathrm{d}x\,\mathrm{d}t,$$

$$I_{4,\varepsilon} = -\int_0^T\int_\Omega \psi p_\delta(\varrho_\varepsilon,\vartheta_\varepsilon)\nabla_x\zeta\cdot\nabla_x\Delta_x^{-1}[1_\Omega\varrho_\varepsilon]\,\mathrm{d}x\,\mathrm{d}t,$$

$$I_{5,\varepsilon} = \int_0^T\int_\Omega \psi\mathbb{S}_\delta(\vartheta_\varepsilon,\nabla_x\mathbf{u}_\varepsilon):\nabla_x\zeta\otimes\nabla_x\Delta_x^{-1}[1_\Omega\varrho_\varepsilon]\,\mathrm{d}x\,\mathrm{d}t,$$

$$I_{6,\varepsilon} = -\int_0^T\int_\Omega \psi(\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon):\nabla_x\zeta\otimes\nabla_x\Delta_x^{-1}[1_\Omega\varrho_\varepsilon]\,\mathrm{d}x\,\mathrm{d}t,$$

$$I_{7,\varepsilon} = -\int_0^T\int_\Omega \partial_t\psi\ \zeta\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla_x\Delta_x^{-1}[1_\Omega\varrho_\varepsilon]\,\mathrm{d}x\,\mathrm{d}t,$$

and

$$I_{8,\varepsilon} = \varepsilon\int_0^T\int_\Omega \nabla_x\varrho_\varepsilon\nabla_x\mathbf{u}_\varepsilon\cdot(\nabla_x\Delta_x^{-1})[1_\Omega\varrho_\varepsilon]\,\mathrm{d}x\,\mathrm{d}t.$$

Here, the symbol $\mathcal{R}$ stands for the *double Riesz transform*, defined componentwise as $\mathcal{R}_{i,j} = \partial_{x_i}\Delta_x^{-1}\partial_{x_j}$, introduced in (2.101).

Repeating the same procedure we use the quantities

$$\varphi(t,x) = \psi(t)\zeta(x)(\nabla_x \Delta_x^{-1})[1_\Omega \varrho], \ \psi \in C_c^\infty(0,T), \ \zeta \in C_c^\infty(\Omega),$$

as test functions in the limit momentum equation (3.234) in order to obtain

$$\int_0^T \int_\Omega \psi\zeta \Big(\overline{p_\delta(\varrho,\vartheta)}\varrho - \mathbb{S}_\delta(\vartheta, \nabla_x \mathbf{u}) : \mathcal{R}[1_\Omega \varrho] \Big) \, \mathrm{d}x \, \mathrm{d}t = \sum_{j=1}^6 I_j, \tag{3.240}$$

where

$$\begin{aligned}
I_1 &= \int_0^T \int_\Omega \psi\zeta \Big(\varrho\mathbf{u} \cdot \mathcal{R}[1_\Omega \varrho\mathbf{u}] - (\varrho\mathbf{u} \otimes \mathbf{u}) : \mathcal{R}[1_\Omega \varrho] \Big) \, \mathrm{d}x \, \mathrm{d}t, \\
I_2 &= -\int_0^T \int_\Omega \psi\zeta\varrho\mathbf{f}_\delta \cdot \nabla_x \Delta_x^{-1}[1_\Omega \varrho_\varepsilon] \, \mathrm{d}x \, \mathrm{d}t, \\
I_3 &= -\int_0^T \int_\Omega \psi \overline{p_\delta(\varrho,\vartheta)} \nabla_x \zeta \cdot \nabla_x \Delta_x^{-1}[1_\Omega \varrho] \, \mathrm{d}x \, \mathrm{d}t, \\
I_4 &= \int_0^T \int_\Omega \psi \mathbb{S}_\delta(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \zeta \otimes \nabla_x \Delta_x^{-1}[1_\Omega \varrho] \, \mathrm{d}x \, \mathrm{d}t, \\
I_5 &= -\int_0^T \int_\Omega \psi (\varrho\mathbf{u} \otimes \mathbf{u}) : \nabla_x \zeta \otimes \nabla_x \Delta_x^{-1}[1_\Omega \varrho] \, \mathrm{d}x \, \mathrm{d}t,
\end{aligned}$$

and

$$I_6 = -\int_0^T \int_\Omega \partial_t \psi \ \zeta \varrho\mathbf{u} \cdot \nabla_x \Delta_x^{-1}[1_\Omega \varrho] \, \mathrm{d}x \, \mathrm{d}t.$$

Combining (3.192) with (3.216) we get

$$\varrho_\varepsilon \to \varrho \text{ in } C_{\text{weak}}([0,T]; L^\Gamma(\Omega)).$$

In accordance with the standard theory of elliptic problems, the pseudodifferential operator $(\nabla_x \Delta_x^{-1})$ "gains" one spatial derivative, in particular, by virtue of the embedding $W^{1,\Gamma}(\Omega) \hookrightarrow C(\overline{\Omega})$, we get

$$(\nabla_x \Delta^{-1})[1_\Omega \varrho_\varepsilon] \to (\nabla_x \Delta^{-1})[1_\Omega \varrho] \text{ in } C([0,T] \times \overline{\Omega}; \mathbb{R}^3)$$

provided $\Gamma > 3$ (see Theorem 10.26 in Appendix). Consequently, we can use relations (3.192), (3.197), (3.229–3.233) in order to see that (i) $I_{2,\varepsilon}, I_{8,\varepsilon} \to 0$, while (ii) the integrals $I_{j,\varepsilon}, j = 3, \ldots, 7$, converge for $\varepsilon \to 0$ to their counterparts in (3.240).

We infer that

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \psi\zeta\Big(p_\delta(\varrho_\varepsilon,\vartheta_\varepsilon)\varrho_\varepsilon - \mathbb{S}_\delta(\vartheta_\varepsilon,\nabla_x\mathbf{u}_\varepsilon):\mathcal{R}[1_\Omega\varrho_\varepsilon]\Big)\,\mathrm{d}x\,\mathrm{d}t \tag{3.241}$$
$$= \int_0^T\int_\Omega \psi\zeta\Big(\overline{p_\delta(\varrho,\vartheta)}\varrho - \mathbb{S}_\delta(\vartheta,\nabla_x\mathbf{u}):\mathcal{R}[1_\Omega\varrho]\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \psi\zeta\Big(\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\mathcal{R}[1_\Omega\varrho_\varepsilon\mathbf{u}_\varepsilon] - (\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon):\mathcal{R}[1_\Omega\varrho_\varepsilon]\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$-\int_0^T\int_\Omega \psi\zeta\Big(\varrho\mathbf{u}\cdot\mathcal{R}[1_\Omega\varrho\mathbf{u}] - (\varrho\mathbf{u}\otimes\mathbf{u}):\mathcal{R}[1_\Omega\varrho]\Big)\,\mathrm{d}x\,\mathrm{d}t.$$

Now, the crucial observation is that the difference of the two right-most quantities in (3.241) vanishes. In order to see this, we need the following assertion (Theorem 10.27 in Appendix) that can be viewed as a straightforward consequence of the Div-Curl lemma.

Lemma 3.5. *Let*

$$\mathbf{U}_\varepsilon \to \mathbf{U} \text{ weakly in } L^p(\mathbb{R}^3;\mathbb{R}^3),$$
$$\mathbf{V}_\varepsilon \to \mathbf{V} \text{ weakly in } L^q(\mathbb{R}^3;\mathbb{R}^3),$$

where

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{r}<1.$$

Then

$$\mathbf{U}_\varepsilon\cdot\mathcal{R}[\mathbf{V}_\varepsilon]-\mathcal{R}[\mathbf{U}_\varepsilon]\cdot\mathbf{V}_\varepsilon \to \mathbf{U}\cdot\mathcal{R}[\mathbf{V}]-\mathcal{R}[\mathbf{U}]\cdot\mathbf{V} \text{ weakly in } L^r(\mathbb{R}^3).$$

This statement provides the following corollary:

Corollary 3.3. *Let*

$$\mathbf{V}_\varepsilon \to \mathbf{V} \text{ weakly in } L^p(\mathbb{R}^3;\mathbb{R}^3),$$
$$r_\varepsilon \to r \text{ weakly in } L^q(\mathbb{R}^3),$$

where

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{s}<1.$$

Then

$$r_\varepsilon\mathcal{R}[\mathbf{V}_\varepsilon]-\mathcal{R}[r_\varepsilon]\mathbf{V}_\varepsilon \to r\mathcal{R}[\mathbf{V}]-\mathcal{R}[r]\mathbf{V} \text{ weakly in } L^s(\mathbb{R}^3;\mathbb{R}^3).$$

Hereafter, we shall use Corollary 3.3 to show that

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \psi\zeta\mathbf{u}_\varepsilon\cdot\Big(\varrho_\varepsilon\mathcal{R}[1_\Omega\varrho_\varepsilon\mathbf{u}_\varepsilon]-\mathcal{R}[1_\Omega\varrho_\varepsilon]\varrho_\varepsilon\mathbf{u}_\varepsilon\Big)\,\mathrm{d}x\,\mathrm{d}t \tag{3.242}$$
$$=\int_0^T\int_\Omega \psi\zeta\mathbf{u}\cdot\Big(\varrho\mathcal{R}[1_\Omega\varrho\mathbf{u}]-\mathcal{R}[1_\Omega\varrho]\varrho\mathbf{u}\Big)\,\mathrm{d}x\,\mathrm{d}t,$$

where, recall, $\mathcal{R}[\mathbf{v}]$ is a vector field with i-th component $\sum_{j=1}^{3} \mathcal{R}_{i,j}[v_j]$ while $\mathcal{R}[a]\mathbf{v}$ is a vector field with i-th component $\sum_{j=1}^{3} \mathcal{R}_{i,j}[a]v_j$.

As shown in (3.196), (3.232),

$$\left\{ \begin{array}{c} \varrho_\varepsilon(t,\cdot) \to \varrho(t,\cdot) \text{ weakly in } L^\Gamma(\Omega), \\ (\varrho_\varepsilon \mathbf{u}_\varepsilon)(t,\cdot) \to (\varrho\mathbf{u})(t,\cdot) \text{ weakly in } L^{\frac{2\Gamma}{\Gamma+1}}(\Omega;\mathbb{R}^3) \end{array} \right\} \text{ for all } t \in [0,T].$$

Applying Corollary 3.3 to $r_\varepsilon = \varrho_\varepsilon(t,\cdot)$, $\mathbf{U}_\varepsilon = \varrho_\varepsilon\mathbf{u}_\varepsilon(t,\cdot)$ (extended by 0 outside Ω), we obtain

$$\begin{gathered} \left(\varrho_\varepsilon \mathcal{R}[1_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon] - \mathcal{R}[1_\Omega \varrho_\varepsilon]\varrho_\varepsilon \mathbf{u}_\varepsilon\right)(t,\cdot) \to \left(\varrho \mathcal{R}[1_\Omega \varrho \mathbf{u}] - \mathcal{R}[1_\Omega \varrho]\varrho\mathbf{u}\right)(t,\cdot)) \\ \text{weakly in } L^{\frac{2\Gamma}{\Gamma+3}}(\Omega), \text{ provided } \Gamma > \tfrac{9}{2} \end{gathered}$$

for all $t \in [0,T]$.

As the embedding $L^{\frac{2\Gamma}{\Gamma+3}}(\Omega) \hookrightarrow W^{-1,2}(\Omega)$ is compact for $\Gamma > 9/2$, we conclude that

$$\begin{gathered} \varrho_\varepsilon \mathcal{R}[1_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon] - \mathcal{R}[1_\Omega \varrho_\varepsilon]\varrho_\varepsilon \mathbf{u}_\varepsilon \to \varrho \mathcal{R}[1_\Omega \varrho \mathbf{u}] - \mathcal{R}[1_\Omega \varrho]\varrho\mathbf{u} \\ \text{in } L^q(0,T;W^{-1,2}(\Omega;\mathbb{R}^3)) \text{ for any } q \geq 1, \end{gathered} \tag{3.243}$$

which, together with (3.193), yields (3.242). Consequently, (3.241) reduces to

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_0^T \int_\Omega \psi\zeta\Big(p_\delta(\varrho_\varepsilon,\vartheta_\varepsilon)\varrho_\varepsilon - \mathbb{S}_\delta(\vartheta_\varepsilon,\nabla_x \mathbf{u}_\varepsilon) : \mathcal{R}[1_\Omega \varrho_\varepsilon]\Big) \, \mathrm{d}x \, \mathrm{d}t \\ = \int_0^T \int_\Omega \psi\zeta\Big(\overline{p_\delta(\varrho,\vartheta)\varrho} - \mathbb{S}_\delta(\vartheta,\nabla_x \mathbf{u}) : \mathcal{R}[1_\Omega \varrho]\Big) \, \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{3.244}$$

Our next goal is to replace in (3.244) the quantity $\mathbb{S}_\delta(\vartheta_\varepsilon,\nabla_x\mathbf{u}_\varepsilon) : \mathcal{R}[1_\Omega\varrho_\varepsilon]$ by $\varrho_\varepsilon\Big(\frac{4}{3}\mu_\delta(\vartheta_\varepsilon) + \eta(\vartheta_\varepsilon)\Big)\mathrm{div}_x\mathbf{u}_\varepsilon$, and, similarly, $\mathbb{S}_\delta(\vartheta,\nabla_x\mathbf{u}) : \mathcal{R}[1_\Omega\varrho]$ by the expression $\varrho\Big(\frac{4}{3}\mu_\delta(\vartheta) + \eta(\vartheta)\Big)\mathrm{div}_x\mathbf{u}$ in (3.244), where $\mu_\delta(\vartheta) = \mu(\vartheta) + \delta\vartheta$.

To this end write

$$\begin{aligned} \int_0^T \int_\Omega \psi\zeta\mu_\delta(\vartheta_\varepsilon)\Big(\nabla_x\mathbf{u}_\varepsilon + \nabla_x^T\mathbf{u}_\varepsilon\Big) : \mathcal{R}[1_\Omega\varrho_\varepsilon] \, \mathrm{d}x \, \mathrm{d}t \\ = \int_0^T \int_\Omega \psi\mathcal{R} : \Big[\zeta\mu_\delta(\vartheta_\varepsilon)\Big(\nabla_x\mathbf{u}_\varepsilon + \nabla_x^T\mathbf{u}_\varepsilon\Big)\Big]\varrho_\varepsilon \, \mathrm{d}x \, \mathrm{d}t, \end{aligned}$$

and

$$\begin{aligned} \int_0^T \int_\Omega \psi\zeta\mu_\delta(\vartheta)\Big(\nabla_x\mathbf{u} + \nabla_x^T\mathbf{u}\Big) : \mathcal{R}[1_\Omega\varrho] \, \mathrm{d}x \, \mathrm{d}t \\ = \int_0^T \int_\Omega \psi\mathcal{R} : \Big[\zeta\mu_\delta(\vartheta)\Big(\nabla_x\mathbf{u} + \nabla_x^T\mathbf{u}\Big)\Big]\varrho \, \mathrm{d}x \, \mathrm{d}t, \end{aligned}$$

where we have used the evident properties of the double Riesz transform recalled in Section 10.16 in Appendix.

Furthermore,

$$\mathcal{R} : \Big[\zeta\mu_\delta(\vartheta_\varepsilon)\Big(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^T \mathbf{u}_\varepsilon\Big)\Big] = 2\zeta\mu_\delta(\vartheta_\varepsilon)\mathrm{div}_x \mathbf{u}_\varepsilon + \omega(\vartheta_\varepsilon, \mathbf{u}_\varepsilon),$$

and

$$\mathcal{R} : \Big[\zeta\mu_\delta(\vartheta)\Big(\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u}\Big)\Big] = 2\zeta\mu_\delta(\vartheta)\mathrm{div}_x \mathbf{u} + \omega(\vartheta, \mathbf{u}),$$

with the commutator

$$\omega(\vartheta, \mathbf{u}) = \mathcal{R} : \Big[\zeta\mu_\delta(\vartheta)\Big(\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u}\Big)\Big] - \zeta\mu_\delta(\vartheta)\mathcal{R} : \Big[\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u}\Big].$$

In order to proceed, we report the following result in the spirit of Coifman and Meyer [49] proved as Theorem 10.28 in Appendix.

■ COMMUTATOR LEMMA:

Lemma 3.6. *Let $w \in W^{1,2}(\mathbb{R}^3)$ and $\mathbf{Z} \in L^p(\mathbb{R}^3; \mathbb{R}^3)$ be given, where $\frac{6}{5} < p < \infty$. Then for any $1 < s < \frac{6p}{6+p}$,*

$$\Big\| \mathcal{R}[w\mathbf{Z}] - w\mathcal{R}[\mathbf{Z}] \Big\|_{W^{\beta,s}(\mathbb{R}^3;\mathbb{R}^3)} \leq c\|w\|_{W^{1,2}(\mathbb{R}^3)} \|\mathbf{Z}\|_{L^p(\mathbb{R}^3;\mathbb{R}^3)},$$

where $0 < \beta = \frac{3}{s} - \frac{6+p}{6p} < 1$ and $c = c(p, s)$ is a positive constant.

Applying Lemma 3.6 to $w = w_\varepsilon = \zeta(\mu(\vartheta_\varepsilon) + \delta\vartheta_\varepsilon)$, $\mathbf{Z} = \mathbf{Z}_\varepsilon = [Z_{\varepsilon,1}, Z_{\varepsilon,2}, Z_{\varepsilon,3}]$, with $Z_{\varepsilon,i} = \partial_{x_i} u_{\varepsilon,j} + \partial_{x_j} u_{\varepsilon,i}$, $j = 1, 2, 3$, where $\{w_\varepsilon\}_{\varepsilon>0}$, $\{\mathbf{Z}_\varepsilon\}_{\varepsilon>0}$ are bounded in $L^2(0, T; W^{1,2}(\Omega))$ and $L^2((0, T) \times \Omega; \mathbb{R}^3)$, respectively, cf. (3.193), (3.201), (3.17–3.18), we deduce that

$$\{\omega(\vartheta_\varepsilon, \mathbf{u}_\varepsilon)\}_{\varepsilon>0} \text{ is bounded in } L^1(0, T; W^{\beta,s}(\Omega)) \tag{3.245}$$

for certain $1 < s < \frac{3}{2}$, $0 < \beta = \frac{3-2s}{s} < 1$.

Next we claim that

$$\omega(\vartheta_\varepsilon, \mathbf{u}_\varepsilon)\varrho_\varepsilon \to \overline{\omega(\vartheta, \mathbf{u})}\varrho \text{ weakly in } L^1((0, T) \times \Omega), \tag{3.246}$$

where, in accordance with relations (3.17), (3.23), (3.193), (3.194), and (3.212),

$$\overline{\omega(\vartheta, \mathbf{u})} = \omega(\vartheta, \mathbf{u}). \tag{3.247}$$

In order to show (3.246), we apply the Div-Curl lemma (see Proposition 3.3) to the four-component vector fields

$$\mathbf{U}_\varepsilon = [\varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon],\ \mathbf{V}_\varepsilon = [\omega(\vartheta_\varepsilon, \mathbf{u}_\varepsilon), 0, 0, 0].$$

In view of relations (3.172), (3.195), (3.245) yielding the sequences $\{\mathrm{div}_{t,x}\mathbf{U}_\varepsilon\}_{\varepsilon>0}$ and $\{\mathbf{curl}_{t,x}\mathbf{V}_\varepsilon\}_{\varepsilon>0}$ compact in $W^{-1,s}((0,T)\times\Omega)$ and $W^{-1,s}((0,T)\times\Omega;\mathbb{R}^{3\times 3})$ for a certain $s>1$, it is enough to observe that

$$\left.\begin{array}{l}\{\mathbf{U}_\varepsilon\}_{\varepsilon>0}\\ \{\mathbf{V}_\varepsilon\}_{\varepsilon>0}\end{array}\right\} \text{ are bounded in } \left\{\begin{array}{l} L^q((0,T)\times\Omega;\mathbb{R}^4),\\ L^r((0,T)\times\Omega;\mathbb{R}^4),\end{array}\right. \quad \text{respectively,}$$

with $1/r+1/q<1$. This is certainly true provided Γ is large enough.

Relations (3.244), (3.246), (3.247) give rise to a remarkable identity.

■ Weak Compactness Identity for Effective Pressure (Level ε) :

$$\overline{p_\delta(\varrho,\vartheta)\varrho} - \left(\frac{4}{3}\mu(\vartheta)+\frac{4}{3}\delta\vartheta+\eta(\vartheta)\right)\overline{\varrho\mathrm{div}_x\mathbf{u}} \tag{3.248}$$
$$= \overline{p_\delta(\varrho,\vartheta)}\varrho - \left(\frac{4}{3}\mu(\vartheta)+\frac{4}{3}\delta\vartheta+\eta(\vartheta)\right)\varrho\mathrm{div}_x\mathbf{u},$$

---

where the quantity $p-(\frac{4}{3}\mu+\eta)\mathrm{div}_x\mathbf{u}$ is usually termed *effective viscous flux* or *effective pressure*. As we will see below, the quantity

$$\overline{\varrho\mathrm{div}_x\mathbf{u}} - \varrho\mathrm{div}_x\mathbf{u}$$

plays a role of a "defect" measure of the density oscillations described through the (renormalized) equation of continuity. Relation (3.248) enables us to relate these oscillations to the changes in the pressure.

In order to exploit (3.248), we multiply the approximate continuity equation (3.172) on $G'(\varrho_\varepsilon)$, where G is a smooth convex function, integrate by parts, and let $\varepsilon\to 0$ to obtain

$$\int_\Omega \overline{G(\varrho)}(\tau)\ \mathrm{d}x + \int_0^\tau\int_\Omega \overline{\Big(G'(\varrho)\varrho - G(\varrho)\Big)\mathrm{div}_x\mathbf{u}}\ \mathrm{d}x\ \mathrm{d}t \le \int_\Omega G(\varrho_{0,\delta})\ \mathrm{d}x \tag{3.249}$$

from which we easily deduce that

$$\int_\Omega \overline{\varrho\log(\varrho)}(\tau)\ \mathrm{d}x + \int_0^\tau\int_\Omega \overline{\varrho\mathrm{div}_x\mathbf{u}}\ \mathrm{d}x\ \mathrm{d}t = \int_\Omega \varrho_{0,\delta}\log(\varrho_{0,\delta})\ \mathrm{d}x \tag{3.250}$$

for a.a. $\tau\in(0,T)$.

To derive a relation similar to (3.250) for the limit functions ϱ, $\mathbf{u}$, we need the renormalized continuity equation introduced in (1.20). Note that we have already shown that the quantities ϱ, $\mathbf{u}$ solve the continuity equation (3.198) in $(0,T)\times\mathbb{R}^3$. On the other hand, the general theory of transport equations developed by DiPerna-Lions asserts that any solution of (3.198) is automatically a renormalized one as soon as, roughly speaking, the quantity $\varrho\mathrm{div}_x\mathbf{u}$ is integrable.

More precisely, we report the following result proved in Section 10.18 in Appendix.

Lemma 3.7. *Assume that* $\varrho \in L^2((0,T) \times \mathbb{R}^3)$, $\mathbf{u} \in L^2(0,T;W^{1,2}(\mathbb{R}^3))$ *solve the equation of continuity* (3.198) *in* $\mathcal{D}'((0,T) \times \mathbb{R}^3))$.

Then ϱ, $\mathbf{u}$ *represent a renormalized solution in the sense specified in* (2.2).

As a consequence of Lemma 3.7 (see also Theorem 10.29 and Lemma 10.13 for more details), we deduce

$$\int_\Omega \varrho \log(\varrho)(\tau)\ \mathrm{d}x + \int_0^\tau \int_\Omega \varrho \mathrm{div}_x \mathbf{u}\ \mathrm{d}x\ \mathrm{d}t \le \int_\Omega \varrho_{0,\delta} \log(\varrho_{0,\delta})\ \mathrm{d}x. \tag{3.251}$$

Since the pressure p_δ is non-decreasing with respect to ϱ and we already know that $\vartheta_\varepsilon \to \vartheta$ strongly in $L^1((0,T) \times \Omega)$, we have

$$\overline{p_\delta(\varrho,\vartheta)\varrho} \ge \overline{p_\delta(\varrho,\vartheta)}\varrho.$$

Indeed,

$$\begin{aligned} &\lim_{n\to\infty} \int_B \Big(p_\delta(\varrho_n,\vartheta_n)\varrho_n - p_\delta(\varrho_n,\vartheta_n)\varrho \Big) \mathrm{d}x\ \mathrm{d}t \\ &= \lim_{n\to\infty} \int_B \Big(p_\delta(\varrho_n,\vartheta_n) - p_\delta(\varrho,\vartheta_n) \Big)(\varrho_n - \varrho) \mathrm{d}x\ \mathrm{d}t \\ &\quad + \lim_{n\to\infty} \int_B p_\delta(\varrho,\vartheta_n)(\varrho_n - \varrho) \mathrm{d}x\ \mathrm{d}t, \end{aligned}$$

where the first term is non-negative, and the second term tends to zero by virtue of the asymptotic limits established in (3.192), (3.212), the bounds (3.194), (3.201), (3.226), (3.227), and the structural properties of p_δ stated in (3.230).

Consequently, relation (3.248) yields

$$\overline{\varrho \mathrm{div}_x \mathbf{u}} \ge \varrho\ \mathrm{div}_x \mathbf{u};$$

whence (3.250) together with (3.251) imply the desired conclusion

$$\overline{\varrho \log(\varrho)} = \varrho \log(\varrho).$$

As $z \mapsto z \log(z)$ is a strictly convex function, we may infer that

$$\varrho_\varepsilon \to \varrho \text{ a.a. in } (0,T) \times \Omega, \tag{3.252}$$

in agreement with Theorem 10.20 in Appendix.

3.6.6 Artificial diffusion asymptotic limit

Strong convergence of the sequence of approximate densities established in (3.252) completes the second step in the proof of Theorem 3.1, eliminating completely the ε-dependent terms in the approximate system. For any $\delta > 0$, we have constructed a trio $\{\varrho, \mathbf{u}, \vartheta\}$ solving the following problem:

(i) Renormalized continuity equation:

$$\int_0^T \int_\Omega \varrho B(\varrho)\Big(\partial_t \varphi + \mathbf{u}\cdot\nabla_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{3.253}$$
$$= \int_0^T \int_\Omega b(\varrho)\mathrm{div}_x\mathbf{u}\varphi\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega \varrho_{0,\delta}B(\varrho_{0,\delta})\varphi(0,\cdot)\ \mathrm{d}x$$

for any

$$b \in L^\infty \cap C[0,\infty),\ B(\varrho) = B(1) + \int_1^\varrho \frac{b(z)}{z^2}\ \mathrm{d}z,$$

and any test function

$$\varphi \in C_c^\infty([0,T)\times\overline{\Omega}).$$

(ii) Approximate balance of momentum:

$$\int_0^T \int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho[\mathbf{u}\otimes\mathbf{u}] : \nabla_x\varphi + \Big(p + \delta(\varrho^\Gamma + \varrho^2)\Big)\mathrm{div}_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{3.254}$$
$$= \int_0^T \int_\Omega \Big(\mathbb{S}_\delta : \nabla_x\varphi - \varrho\mathbf{f}_\delta\cdot\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega (\varrho\mathbf{u})_0\cdot\varphi\ \mathrm{d}x,$$

for any test function $\varphi \in C_c^\infty([0,T)\times\overline{\Omega};\mathbb{R}^3)$, where either

$$\varphi\cdot\mathbf{n}|_{\partial\Omega} = 0 \text{ in the case of the complete slip boundary conditions,} \tag{3.255}$$

or

$$\varphi|_{\partial\Omega} = 0 \text{ in the case of the no-slip boundary conditions.} \tag{3.256}$$

Furthermore,

$$\mathbb{S}_\delta = (\mu(\vartheta) + \delta\vartheta)\Big(\nabla_x\mathbf{u} + \nabla_x^\perp\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\ \mathbb{I}\Big) + \eta(\vartheta)\mathrm{div}_x\mathbf{u}\ \mathbb{I}. \tag{3.257}$$

(iii) Approximate entropy balance:

$$\int_0^T \int_\Omega \varrho s_\delta(\varrho,\vartheta)\Big(\partial_t\varphi + \mathbf{u}\cdot\nabla_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{3.258}$$
$$+ \int_0^T \int_\Omega \frac{\kappa_\delta(\vartheta)\nabla_x\vartheta}{\vartheta}\cdot\nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t + \langle\sigma_\delta;\varphi\rangle_{[\mathcal{M},C]([0,T]\times\overline{\Omega})}$$
$$= -\int_\Omega (\varrho s)_{0,\delta}\varphi(0,\cdot)\ \mathrm{d}x - \int_0^T \int_\Omega \frac{\varrho}{\vartheta}\mathcal{Q}_\delta\varphi\ \mathrm{d}x\ \mathrm{d}t$$

for all $\varphi \in C_c^\infty([0,T)\times\overline{\Omega})$, where $\sigma_\delta \in \mathcal{M}^+([0,T]\times\overline{\Omega})$ satisfies

$$\sigma_\delta \geq \frac{1}{\vartheta}\Big[\mathbb{S}_\delta : \nabla_x\mathbf{u} + \Big(\frac{\kappa(\vartheta)}{\vartheta} + \frac{\delta}{2}(\vartheta^{\Gamma-1} + \frac{1}{\vartheta^2})\Big)|\nabla_x\vartheta|^2 + \delta\frac{1}{\vartheta^2}\Big], \tag{3.259}$$

and where we have set

$$s_\delta(\varrho,\vartheta) = s(\varrho,\vartheta) + \delta\log(\vartheta),\ \ \kappa_\delta(\vartheta) = \kappa(\vartheta) + \delta\Big(\vartheta^\Gamma + \frac{1}{\vartheta}\Big). \tag{3.260}$$

(iv) Approximate energy balance:

$$\int_\Omega \Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) + \delta(\frac{\varrho^\Gamma}{\Gamma-1} + \varrho^2)\Big)(\tau)\ \mathrm{d}x \tag{3.261}$$
$$= \int_\Omega \Big(\frac{1}{2}\frac{|(\varrho\mathbf{u})_0|^2}{\varrho_{0,\delta}} + \varrho_{0,\delta}e_{0,\delta} + \delta(\frac{\varrho_{0,\delta}^\Gamma}{\Gamma-1} + \varrho_{0,\delta}^2)\Big)\ \mathrm{d}x$$
$$+ \int_0^\tau\int_\Omega \Big(\varrho\mathbf{f}_\delta\cdot\mathbf{u} + \varrho\mathcal{Q}_\delta + \delta\frac{1}{\vartheta^2}\Big)\ \mathrm{d}x\ \mathrm{d}t \text{ for a.a. } \tau\in[0,T].$$

3.7 Vanishing artificial pressure

The last and probably the most illuminative step in the proof of Theorem 3.1 is to let $\delta\to 0$ in the approximate system (3.253–3.261). Although many arguments are almost identical or mimic closely those discussed in the previous text, there are still some new ingredients coming into play. Notably, we introduce a concept of *oscillation defect measure* in order to control the density oscillations beyond the theory of DiPerna and Lions. Moreover, weighted estimates of this quantity are used in order to accommodate the physically realistic growth restrictions on the transport coefficients imposed through hypotheses (3.17), (3.23).

3.7.1 Uniform estimates

From now on, let $\{\varrho_\delta,\mathbf{u}_\delta,\vartheta_\delta\}_{\delta>0}$ be a family of approximate solutions satisfying (3.253–3.261). To begin, we recall that the total mass is a constant of motion, specifically,

$$\int_\Omega \varrho_\delta(t,\cdot)\ \mathrm{d}x = \int_\Omega \varrho_{0,\delta}\ \mathrm{d}x \text{ for any } t\in[0,T]. \tag{3.262}$$

Since we assume that

$$\varrho_{0,\delta}\to\varrho_0 \text{ in } L^1(\Omega), \tag{3.263}$$

the bound (3.262) is uniform for $\delta\to 0$.

The next step is the *dissipation balance*

$$\int_\Omega \Big(\frac{1}{2}\varrho_\delta|\mathbf{u}_\delta|^2(\tau) + H_{\overline{\vartheta}}(\varrho_\delta,\vartheta_\delta)(\tau) + \delta(\frac{1}{\Gamma-1}\varrho_\delta^\Gamma + \varrho_\delta^2)(\tau)\Big)\ \mathrm{d}x + \overline{\vartheta}\ \sigma_\delta\Big[[0,\tau]\times\overline{\Omega}\Big]$$
$$= \int_\Omega \Big(\frac{1}{2}\frac{|(\varrho\mathbf{u})_0|^2}{\varrho_{0,\delta}} + H_{\overline{\vartheta}}(\varrho_{0,\delta},\vartheta_{0,\delta}) + \delta(\frac{\varrho_{0,\delta}^\Gamma}{\Gamma-1} + \varrho_{0,\delta}^2)\Big)\ \mathrm{d}x$$
$$+ \int_0^\tau\int_\Omega \Big(\varrho_\delta\mathbf{f}_\delta\cdot\mathbf{u}_\delta + \varrho_\delta(1-\frac{\overline{\vartheta}}{\vartheta_\delta})\mathcal{Q}_\delta + \delta\frac{1}{{\vartheta_\delta}^2}\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{3.264}$$

satisfied for a.a. $\tau \in [0, T]$, which can be deduced from (3.258), (3.261), with the Helmholtz function $H_{\overline{\vartheta}}$ introduced in (2.48). Accordingly, in order to get uniform estimates, we have to take

$$\begin{gathered} \{\mathbf{f}_\delta\}_{\delta>0} \text{ bounded in } L^\infty((0,T)\times\Omega;\mathbb{R}^3), \\ \mathcal{Q}_\delta \geq 0,\ \{\mathcal{Q}_\delta\}_{\delta>0} \text{ bounded in } L^\infty((0,T)\times\Omega) \end{gathered} \tag{3.265}$$

as well as

$$\int_\Omega \Big(\frac{1}{2}\frac{|(\varrho\mathbf{u})_0|^2}{\varrho_{0,\delta}} + H_{\overline{\vartheta}}(\varrho_{0,\delta},\vartheta_{0,\delta}) + \delta(\frac{\varrho_{0,\delta}^\Gamma}{\Gamma-1} + \varrho_{0,\delta}^2)\Big)\ \mathrm{d}x \leq c \tag{3.266}$$

uniformly for $\delta \to 0$.

As the term $\delta/\vartheta_\delta^2$ is "absorbed" by its counterpart in the entropy production σ_δ satisfying (3.259), the dissipation balance (3.264) gives rise, exactly as in Section 3.6.1, to the following uniform estimates:

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\sqrt{\varrho_\delta}\mathbf{u}_\delta(t)\|_{L^2(\Omega;\mathbb{R}^3)} \leq c, \tag{3.267}$$

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\varrho_\delta(t)\|_{L^{\frac{5}{3}}(\Omega)} \leq c, \tag{3.268}$$

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\varrho_\delta(t)\|_{L^\Gamma(\Omega)} \leq \delta^{-\frac{1}{\Gamma}} c, \tag{3.269}$$

and

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\vartheta_\delta(t)\|_{L^4(\Omega)} \leq c. \tag{3.270}$$

In addition, we have

$$\sigma_\delta\Big[[0,T]\times\overline{\Omega}\Big] \leq c, \tag{3.271}$$

and, as a consequence of (3.259),

$$\int_0^T\int_\Omega |\nabla_x \log(\vartheta_\delta)|^2\ \mathrm{d}x\ \mathrm{d}t \leq c, \tag{3.272}$$

$$\int_0^T\int_\Omega |\nabla_x \vartheta_\delta^{\frac{3}{2}}|^2\ \mathrm{d}x\ \mathrm{d}t \leq c, \tag{3.273}$$

and

$$\delta\int_0^T\int_\Omega \frac{1}{\vartheta_\delta^3}\ \mathrm{d}x\ \mathrm{d}t \leq c, \tag{3.274}$$

$$\delta\int_0^T\int_\Omega \Big(\vartheta_\delta^{\Gamma-2} + \frac{1}{\vartheta_\delta^3}\Big)|\nabla_x\vartheta_\delta|^2\ \mathrm{d}x\ \mathrm{d}t \leq c. \tag{3.275}$$

Finally, making use of Korn's inequality established in Proposition 2.1 we deduce, exactly as in (2.65), (2.66), that

$$\| \mathbf{u}_\delta \|_{L^2(0,T;W^{1,p}(\Omega;\mathbb{R}^3))} \leq c \text{ for } p = \frac{8}{5-\alpha}, \tag{3.276}$$

and

$$\| \mathbf{u}_\delta \|_{L^q(0,T;W^{1,s}(\Omega;\mathbb{R}^3))} \leq c \text{ for } q = \frac{6}{4-\alpha},\ \ s = \frac{18}{10-\alpha}, \tag{3.277}$$

where α was introduced in hypotheses (3.17–3.23). Moreover,

$$\delta \int_0^T \int_\Omega \left| \nabla_x \mathbf{u}_\delta + \nabla_x^T \mathbf{u}_\delta - \frac{2}{3}\mathbb{I} \right|^2 \mathrm{d}x \ \mathrm{d}t \leq c. \tag{3.278}$$

Note that estimates (3.270–3.273) yield

$$\{\vartheta_\delta^\beta\}_{\delta>0} \text{ bounded in } L^2(0,T;W^{1,2}(\Omega)) \text{ for any } 1 \leq \beta \leq \frac{3}{2}, \tag{3.279}$$

while (3.276), (3.277), together with hypotheses (3.17), (3.19), and (3.23), imply that

$$\{\mathbb{S}_\delta\}_{\delta>0} \text{ is bounded in } L^q((0,T)\times\Omega;\mathbb{R}^{3\times 3})) \text{ for a certain } q > 1, \tag{3.280}$$

(cf. estimate (2.68)).

Now, *positivity* of the absolute temperature can be shown by help of Proposition 2.2 and Lemma 2.1, exactly as in Section 2.2.4. In particular, estimate (3.272) can be strengthened to

$$\int_0^T \int_\Omega \Big(|\log \vartheta_\delta|^2 + |\nabla_x \log \vartheta_\delta|^2 \Big) \ \mathrm{d}x \ \mathrm{d}t \leq c. \tag{3.281}$$

In order to complete our list of uniform bounds, we evoke the pressure estimates obtained in Section 2.2.5. In the present context, relation (2.95) reads

$$\int_0^T \int_\Omega \Big(\delta \varrho_\delta^\Gamma + p_\delta(\varrho_\delta, \vartheta_\delta) \Big) \varrho_\delta^\nu \ \mathrm{d}x \ \mathrm{d}t \leq c(\text{data}), \tag{3.282}$$

where $\nu > 0$ is a constant exponent.

3.7.2 Asymptotic limit for vanishing artificial pressure

The piece of information provided by the uniform bounds established in the previous section is sufficient for taking $\delta \to 0$ in the approximate system of equations (3.253–3.261).

Due to the structural properties of the molecular pressure p_M derived in (3.32), and because of (3.230), estimates (3.268), (3.270), and (3.276), (3.277) imply that

$$\varrho_\delta \to \varrho \text{ weakly-(*) in } L^\infty(0,T;L^{\frac{5}{3}}(\Omega)), \tag{3.283}$$

$$\vartheta_\delta \to \vartheta \text{ weakly-(*) in } L^\infty(0,T;L^4(\Omega)), \tag{3.284}$$

and

$$\mathbf{u}_\delta \to \mathbf{u} \left\{ \begin{array}{l} \text{weakly in } L^2(0,T;W^{1,p}(\Omega;\mathbb{R}^3)),\ p = \frac{8}{5-\alpha}, \\ \text{weakly in } L^q(0,T;W^{1,s}(\Omega;\mathbb{R}^3)),\ q = \frac{6}{4-\alpha},\ \ s = \frac{18}{10-\alpha}, \end{array} \right\} \tag{3.285}$$

at least for suitable subsequences.

Taking $b \equiv 0$ in the renormalized equation (3.253), we deduce, in view of the previous estimates, that

$$\varrho_\delta \to \varrho \text{ in } C_{\text{weak}}([0,T];L^{\frac{5}{3}}(\Omega)). \tag{3.286}$$

On the other hand, as the Lebesgue space $L^{\frac{5}{3}}(\Omega)$ is compactly embedded into the dual $W^{-1,p'}(\Omega)$, $p' = 8/(3+\alpha)$ as soon as $\alpha \in (2/5,1]$, we conclude, taking (3.283) together with (3.267), (3.268) into account, that

$$\varrho_\delta \mathbf{u}_\delta \to \varrho \mathbf{u} \text{ weakly-(*) in } L^\infty(0,T;L^{\frac{5}{4}}(\Omega;\mathbb{R}^3)). \tag{3.287}$$

A similar argument in the case when the time derivative of the momentum $\varrho_\delta \mathbf{u}_\delta$ is expressed via the approximate momentum equation (3.254) gives rise to

$$\varrho_\delta \mathbf{u}_\delta \to \varrho \mathbf{u} \text{ in } C_{\text{weak}}([0,T];L^{\frac{5}{4}}(\Omega;\mathbb{R}^3)). \tag{3.288}$$

Since

$$W^{1,s}(\Omega) \text{ is compactly embedded into } L^5(\Omega) \text{ for } s = \frac{18}{10-\alpha}, \tag{3.289}$$

we can use (3.285) to conclude that

$$\varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta \to \varrho \mathbf{u} \otimes \mathbf{u} \text{ weakly in } L^q(0,T;L^q(\Omega;\mathbb{R}^{3\times 3})) \text{ for a certain } q > 1. \tag{3.290}$$

In order to handle the approximate pressure in the momentum equation (3.254), we first observe that, as a direct consequence of (3.282),

$$\delta \varrho_\delta \to 0 \text{ in } L^1((0,T)\times\Omega). \tag{3.291}$$

Moreover, writing

$$p(\varrho_\delta,\vartheta_\delta) = p_M(\varrho_\delta,\vartheta_\delta) + \frac{a}{3}\vartheta_\delta^4,$$

and interpolating estimates (3.270), (3.279), we have

$$\vartheta_\delta^4 \to \overline{\vartheta^4} \text{ weakly in } L^q((0,T)\times\Omega) \text{ for a certain } q>1. \tag{3.292}$$

In accordance with hypotheses (3.15), (3.16), the asymptotic structure of p_M derived in (3.32), and in agreement with (3.282), (3.292),

$$p(\varrho_\delta,\vartheta_\delta) = p_M(\varrho_\delta,\vartheta_\delta) + \frac{a}{3}\vartheta_\delta^4 \to \overline{p_M(\varrho,\vartheta)} + \frac{a}{3}\overline{\vartheta^4} \text{ weakly in } L^1((0,T)\times\Omega). \tag{3.293}$$

At this stage, it is possible to let $\delta \to 0$ in equations (3.253), (3.254) to obtain

$$\int_0^T\int_\Omega \Big(\overline{\varrho B(\varrho)}\partial_t\varphi + \overline{\varrho B(\varrho)}\mathbf{u}\cdot\nabla_x\varphi - \overline{b(\varrho)\mathrm{div}_x\mathbf{u}}\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t = -\int_\Omega \varrho_0 B(\varrho_0)\varphi(0,\cdot)\ \mathrm{d}x \tag{3.294}$$

for any test function $\varphi \in C_c^\infty([0,T)\times\overline{\Omega})$ and any

$$b \in L^\infty \cap C[0,\infty),\ B(\varrho) = B(1) + \int_1^\varrho \frac{b(z)}{z}\ \mathrm{d}z.$$

Similarly, we get

$$\begin{aligned}\int_0^T\int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi + (\overline{p_M(\varrho,\vartheta)} + \frac{a}{3}\overline{\vartheta^4})\mathrm{div}_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t \\ = \int_0^T\int_\Omega \Big(\overline{\mathbb{S}(\vartheta,\nabla_x\mathbf{u})} : \nabla_x\varphi - \varrho\mathbf{f}\cdot\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega (\varrho\mathbf{u})_0\cdot\varphi(0,\cdot)\ \mathrm{d}x\end{aligned} \tag{3.295}$$

for any test function $\varphi \in C_c^\infty([0,T)\times\overline{\Omega};\mathbb{R}^3)$ satisfying $\varphi\cdot\mathbf{n}|_{\partial\Omega} = 0$, or, in addition, $\varphi|_{\partial\Omega} = 0$ in the case of the no-slip boundary conditions. Here we have set

$$\overline{\mathbb{S}(\vartheta,\nabla_x\mathbf{u})} = \overline{\mu(\vartheta)\Big(\nabla_x\mathbf{u} + \nabla_x^\perp\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\Big)} + \overline{\eta(\vartheta)\mathrm{div}_x\mathbf{u}\mathbb{I}}. \tag{3.296}$$

Finally, letting $\delta \to 0$ in the approximate total energy balance (3.261) we conclude

$$\begin{aligned}&\int_\Omega \Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + \overline{\varrho e(\varrho,\vartheta)}\Big)(\tau)\ \mathrm{d}x \\ &= \int_\Omega \Big(\frac{1}{2}\frac{|(\varrho\mathbf{u})_0|^2}{\varrho_0} + \varrho_0 e(\varrho_0,\vartheta_0)\Big)\ \mathrm{d}x \\ &\quad + \int_0^\tau\int_\Omega \Big(\varrho\mathbf{f}\cdot\mathbf{u} + \varrho\mathcal{Q}\Big)\ \mathrm{d}x\ \mathrm{d}t \text{ for a.a. } \tau\in(0,T),\end{aligned} \tag{3.297}$$

where we have used estimate (3.274) in order to eliminate the singular term $\delta/\vartheta_\delta^2$. Moreover, we have assumed strong convergence (a.a.) of the approximate data $\mathbf{f}_\delta$, $\varrho_{0,\delta}$, $\vartheta_{0,\delta}$, and $\mathcal{Q}_\delta$.

3.7.3 Entropy balance and pointwise convergence of the temperature

Similarly to the preceding parts, specifically Section 3.6.2, our aim is to use Div-Curl lemma (Proposition 3.3), together with the monotonicity of the entropy, in order to show

$$\vartheta_\delta \to \vartheta \text{ a.a. on } (0,T)\times\Omega. \tag{3.298}$$

Uniform estimates. We have to show that all terms appearing on the left-hand side of the approximate entropy balance (3.258) are either non-negative or belong to an L^p-space, with $p>1$.

To this end, we use the structural properties of the specific entropy s stated in (3.34), (3.39), together with the uniform estimates (3.268), (3.270), (3.281), to deduce that

$$\varrho_\delta s(\varrho_\delta,\vartheta_\delta) \to \overline{\varrho s(\varrho,\vartheta)} \text{ weakly in } L^p((0,T)\times\Omega) \text{ for a certain } p>1. \tag{3.299}$$

Similarly, we have

$$|\varrho_\delta s(\varrho_\delta,\vartheta_\delta)\mathbf{u}_\delta| \le c\Big(|\vartheta_\delta|^3|\mathbf{u}_\delta| + \varrho_\delta|\log(\varrho_\delta)||\mathbf{u}_\delta| + |\mathbf{u}_\delta| + \varrho_\delta|\log(\vartheta_\delta)||\mathbf{u}_\delta|\Big);$$

whence, by virtue of (3.289), combined with estimates (3.283–3.285), there is $p>1$ such that

$$\Big\{|\vartheta_\delta|^3|\mathbf{u}_\delta| + \varrho_\delta|\log(\varrho_\delta)||\mathbf{u}_\delta| + |\mathbf{u}_\delta|\Big\}_{\delta>0} \text{ is bounded in } L^p((0,T)\times\Omega). \tag{3.300}$$

In addition, relations (3.278), (3.291) give rise to

$$\{\varrho_\delta\log(\vartheta_\delta)\mathbf{u}_\delta\}_{\delta>0} \text{ bounded in } L^p((0,T)\times\Omega;\mathbb{R}^3) \text{ for a certain } p>1. \tag{3.301}$$

The entropy flux can be handled by means of the uniform estimates established in (3.270), (3.279). Indeed, writing

$$\frac{\kappa(\vartheta_\delta)}{\vartheta_\delta}|\nabla_x\vartheta_\delta| \le c\Big(|\nabla_x\log(\vartheta_\delta)| + \vartheta_\delta^{\frac{3}{2}}|\nabla_x\vartheta_\delta^{\frac{3}{2}}|\Big)$$

we observe easily that

$$\Big\{\frac{\kappa(\vartheta_\delta)}{\vartheta_\delta}\nabla_x\vartheta_\delta\Big\}_{\delta>0} \text{ is bounded in } L^p((0,T)\times\Omega;\mathbb{R}^3) \tag{3.302}$$

for a suitable $p>1$.

Finally, relations (3.270), (3.275), (3.281) can be used to obtain

$$\left\{\begin{array}{c} \delta\int_0^T \|\vartheta_\delta^{\frac{\Gamma}{2}}(t,\cdot)\|^2_{W^{1,2}(\Omega)}\,\mathrm{d}t \le c, \\ \delta\int_0^T \|\vartheta_\delta^{-\frac{1}{2}}(t,\cdot)\|^2_{W^{1,2}(\Omega)}\,\mathrm{d}t \le c, \end{array}\right\} \tag{3.303}$$

uniformly for $\delta \to 0$. Consequently, seeing that

$$\delta \vartheta_\delta^{\Gamma-1} \nabla_x \vartheta_\delta = \delta \frac{\Gamma}{2} \vartheta_\delta^{\frac{\Gamma}{2}} \nabla_x \vartheta_\delta^{\frac{\Gamma}{2}} = \delta \frac{\Gamma}{2} \vartheta_\delta^{\frac{1}{4}} \vartheta_\delta^{\frac{\Gamma}{2}-\frac{1}{4}} \nabla_x \vartheta_\delta^{\frac{\Gamma}{2}},$$

we can use (3.284), (3.303), together with Hölder's inequality and the embedding relation $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, in order to conclude that

$$\delta \vartheta_\delta^{\Gamma-1} \nabla_x \vartheta_\delta \to 0 \text{ in } L^p((0,T) \times \Omega; \mathbb{R}^3)) \text{ for } \delta \to 0 \text{ and a certain } p > 1. \quad (3.304)$$

Similarly, by the same token,

$$\frac{\delta}{\vartheta_\delta^2} \nabla_x \vartheta_\delta \to 0 \text{ in } L^p((0,T) \times \Omega; \mathbb{R}^3), \text{ where } p > 1. \quad (3.305)$$

Strong convergence of temperature via the Young measures. Having established all necessary estimates we can proceed as in Section 3.6.2.

By virtue of (3.281),

$$\delta \log(\vartheta_\delta)\, G(\vartheta_\delta) \to 0 \text{ in } L^1((0,T) \times \Omega). \quad (3.306)$$

We can apply the Div-Curl lemma (Proposition 3.3) in order to obtain identity

$$\overline{\varrho s(\varrho, \vartheta) G(\vartheta)} = \overline{\varrho s(\varrho, \vartheta)}\; \overline{G(\vartheta)}. \quad (3.307)$$

Consequently, employing Theorem 10.30, we show identity (3.214). Now we apply Theorem 0.10 in the same way as in Section 3.6.2 and conclude that

$$\overline{\varrho s_M(\varrho, \vartheta) G(\vartheta)} \geq \overline{\varrho s_M(\varrho, \vartheta)}\; \overline{G(\vartheta)}. \quad (3.308)$$

We also observe that, according to Theorem 10.19,

$$\overline{\vartheta^3 G(\vartheta)} \geq \overline{\vartheta^3}\; \overline{G(\vartheta)}. \quad (3.309)$$

The symbol G in the last four formulas denotes an arbitrary nondecreasing and continuous function on $[0, \infty)$, chosen in such a way that all the L^1-weak limits in the above formulas exist.

Relations (3.308–3.309) combined with identity (3.307) yield (3.211). The latter identity implies the pointwise convergence (3.298).

Strong convergence of temperature – an alternative proof. The departure point is formula (3.307) with $G(\vartheta) = T_k(\vartheta)$, where the truncation function T_k are defined by formula (3.317) below. The goal is to show the inequality (3.308) by using more elementary arguments than in the previous section. Once this is done, (3.307) and Theorem 10.19 yield

$$\overline{\vartheta^3 T_k(\vartheta)} = \overline{\vartheta^3}\; \overline{T_k(\vartheta)}.$$

Since the sequence ϑ_δ is bounded in $L^\infty(0,T;L^4(\Omega)) \cap L^2(0,T;L^6(\Omega))$, the last inequality and Corollary 10.2 in Appendix, imply

$$\overline{\vartheta^4} = \overline{\vartheta^3}\ \vartheta$$

which proves (3.298).

Accordingly, it is enough to show

$$\overline{\varrho s_M(\varrho,\vartheta) T_k(\vartheta)} \geq \overline{\varrho s_M(\varrho,\vartheta)}\ \overline{T_k(\vartheta)}. \tag{3.310}$$

Due to Corollary 10.2 and property (3.39), we have

$$\sup_{\varepsilon>0} \| \varrho_\delta s_M(\varrho_\delta,\vartheta_\delta) T_k(\vartheta_\delta) - \varrho_\delta s_M(\varrho_\delta, T_k(\vartheta_\delta)) T_k(\vartheta_\delta) \|_{L^1((0,T)\times\Omega)} \to 0$$

and

$$\sup_{\varepsilon>0} \| \varrho_\delta s_M(\varrho_\delta,\vartheta_\delta) \overline{T_k(\vartheta)} - \varrho_\delta s_M(\varrho_\delta, T_k(\vartheta_\delta)) \overline{T_k(\vartheta)} \|_{L^1((0,T)\times\Omega)} \to 0$$

as $k \to \infty$. It is therefore sufficient to prove

$$\overline{\varrho s_M(\varrho,T_k(\vartheta)) T_k(\vartheta)} \geq \overline{\varrho s_M(\varrho,T_k(\vartheta))}\ \overline{T_k(\vartheta)}. \tag{3.311}$$

Due to the monotonicity of function $\vartheta \mapsto s_M(\varrho,\vartheta)$, we have

$$\Big(\varrho_\delta s_M(\varrho_\delta, T_k(\vartheta_\delta)) - \varrho_\delta s_M(\varrho_\delta, \overline{T_k(\vartheta)}) \Big) \Big(T_k(\vartheta_\delta) - \overline{T_k(\vartheta)} \Big) \geq 0.$$

Therefore, (3.311) will be verified if we show that

$$\int_B \varrho_\delta s_M(\varrho_\delta, \overline{T_k(\vartheta)}) \Big(T_k(\vartheta_\delta) - \overline{T_k(\vartheta)} \Big)\, \mathrm{d}x\mathrm{d}t \to 0 \text{ as } \varepsilon \to 0+, \tag{3.312}$$

where B is an arbitrary ball in $(0,T)\times\Omega$.

Since log is a concave function, we have $\overline{\log(T_k(\vartheta))} \leq \log(\overline{T_k(\vartheta)})$. Moreover, the sequence $\{\log(\vartheta_\delta)\}_{\delta>0}$ is bounded in $L^2(0,T;W^{1,2}(\Omega))$ and the same holds for $\{\log(T_k(\vartheta_\delta))\}_{\delta>0}$. Consequently,

$$\log(\overline{T_k(\vartheta)})\ 1_{\{\overline{T_k(\vartheta)} \leq 1\}} \in L^2(0,T;L^6(\Omega)),$$

$$0 < \log(\overline{T_k(\vartheta)})\ 1_{\{\overline{T_k(\vartheta)} > 1\}} \leq \overline{T_k(\vartheta)} \in L^2(0,T;L^6(\Omega)),$$

therefore $\log(\overline{T_k(\vartheta)})$ belongs to the space $L^2(0,T;L^6(\Omega))$. In particular, there exists $z_\epsilon \in C^1([0,T]\times\overline{\Omega})$ such that

$$\| z_\epsilon - \log(\overline{T_k(\vartheta)}) \|_{L^2(0,T;L^6(\Omega))} < \varepsilon$$

where $\varepsilon > 0$ is a parameter that can be taken arbitrarily small. Setting $\Theta = \exp(z_\varepsilon)$ we have

$$\Theta \in C^1([0,T] \times \overline{\Omega}), \quad \min_{(t,x)\in[0,T]\times\overline{\Omega}} \Theta(t,x) > 0.$$

Now, we write

$$\begin{aligned}
&\int_B \varrho_\delta s_M(\varrho_\delta, \overline{T_k(\vartheta)})\Big(T_k(\vartheta_\delta) - \overline{T_k(\vartheta)}\Big)\,\mathrm{d}x\ \mathrm{d}t \\
&= \int_B \Big(\varrho_\delta s_M(\varrho_\delta, \overline{T_k(\vartheta)}) - \varrho_\delta s_M(\varrho_\delta, \Theta)\Big)\Big(T_k(\vartheta_\delta) - \overline{T_k(\vartheta)}\Big)\,\mathrm{d}x\ \mathrm{d}t \\
&\quad + \int_B \varrho_\delta s_M(\varrho_\delta, \Theta)\Big(T_k(\vartheta_\delta) - \overline{T_k(\vartheta)}\Big)\,\mathrm{d}x\ \mathrm{d}t.
\end{aligned} \tag{3.313}$$

We may use (3.11), (3.34) to verify that

$$\begin{aligned}
&\Big|\varrho_\delta s_M(\varrho_\delta, \overline{T_k(\vartheta)}) - \varrho_\delta s_M(\varrho_\delta, \Theta)\Big| \\
&\quad = \varrho_\delta\Big|\int_{\overline{T_k(\vartheta)}}^{\Theta} \frac{1}{r}\frac{\partial e_M}{\partial \vartheta}(\varrho_\delta, r)\,\mathrm{d}r\Big| \le c\varrho_\delta\Big|\log(\overline{T_k(\vartheta)}) - \log(\Theta)\Big|.
\end{aligned}$$

Since ϱ_δ is bounded in $L^\infty(0,T;L^{\frac{5}{3}}(\Omega))$, we infer that

$$\sup_{\delta>0}\Big\|\varrho_\delta\Big(\log(\overline{T_k(\vartheta)}) - \log(\Theta)\Big)\Big\|_{L^2(0,T;L^{\frac{30}{23}}(\Omega))} \le c\varepsilon;$$

whence the first integral on the right-hand side of (3.312) tends to 0 as $\varepsilon \to 0+$.

As a consequence of (3.39), the sequence $B(t,x,\varrho_\delta) = \varrho_\delta s_M(\varrho_\delta, \Theta(t,x))$ satisfies hypothesis (10.127) of Theorem 10.30 in Appendix. We can therefore conclude that

$$\{\varrho_\delta s_M(\varrho_\delta, \Theta)\}_{\delta>0} \text{ is precompact in } L^2(0,T;W^{-1,2}(\Omega)),$$

which, together with the fact that $T_k(\vartheta_\delta) \to \overline{T_k(\vartheta)}$ weakly in $L^2(0,T;W^{1,2}(\Omega))$, concludes the proof of inequality (3.310).

Asymptotic limit in the entropy balance. Using weak lower semicontinuity of convex functionals, we can let $\delta \to 0$ in the approximate entropy balance (3.258) to conclude that

$$\begin{aligned}
&\int_0^T \int_\Omega \overline{\varrho s(\varrho,\vartheta)}\Big(\partial_t\varphi + \mathbf{u}\cdot\nabla_x\varphi\Big)\,\mathrm{d}x\ \mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \frac{\mathbf{q}}{\vartheta}\cdot\nabla_x\varphi\,\mathrm{d}x\ \mathrm{d}t + \langle\sigma;\varphi\rangle_{[\mathcal{M};C]([0,T]\times\overline{\Omega})} \\
&= -\int_\Omega (\varrho s)_0\varphi(0,\cdot)\,\mathrm{d}x - \int_0^T\int_\Omega \frac{\varrho}{\vartheta}\mathcal{Q}\varphi\,\mathrm{d}x\ \mathrm{d}t,
\end{aligned} \tag{3.314}$$

for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$. In this equation

$$\mathbf{q} = -\kappa(\vartheta)\nabla_x\vartheta, \tag{3.315}$$

and $\sigma \in \mathcal{M}^+([0,T]\times\overline{\Omega})$ is a weak-(*) limit in $\mathcal{M}([0,T]\times\overline{\Omega})$ of the sequence σ_δ that exists at least for a chosen subsequence due to estimate (3.271). Employing (3.259), (3.271), (3.285), (3.298) and lower weak semicontinuity of convex functionals, using the fact that all δ-dependent quantities in the entropy production rate at the right-hand side of (3.259) are non-negative, we show that

$$\sigma \geq \frac{1}{\vartheta}\Big(\mathbb{S}(\vartheta,\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} + \frac{\kappa(\vartheta)}{\vartheta}|\nabla\vartheta|^2\Big). \tag{3.316}$$

For more details see the similar reasoning between formulas (3.159–3.161) in Section 3.5.3.

Consequently, in order to complete the proof of Theorem 3.1, we have to show pointwise convergence of the densities. This will be done in the next section.

3.7.4 Pointwise convergence of the densities

We follow the same strategy as in Section 3.6.5, however, some essential steps have to be considerably modified due to lower L^p-integrability available for $\{\varrho_\delta\}_{\delta>0}$, $\{\mathbf{u}_\delta\}_{\delta>0}$.

To begin, we introduce a family of cut-off functions

$$T_k(z) = kT\Big(\frac{z}{k}\Big),\ z \geq 0,\ k \geq 1, \tag{3.317}$$

where $T \in C^\infty[0,\infty)$,

$$T(z) = \begin{cases} z \text{ for } 0 \leq z \leq 1, \\ \text{concave on } [0,\infty), \\ 2 \text{ for } z \geq 3. \end{cases} \tag{3.318}$$

Similarly to Sections 2.2.6, 3.6.5, we use the quantities

$$\varphi(t,x) = \psi(t)\zeta(x)(\nabla_x\Delta_x^{-1})[1_\Omega T_k(\varrho_\delta)],\ \psi \in C_c^\infty(0,T),\ \zeta \in C_c^\infty(\Omega),$$

with the operators $(\nabla_x\Delta_x^{-1})$ introduced in (2.100), as test functions in the approximate momentum equation (3.254) to deduce

$$\int_0^T\int_\Omega \psi\zeta\Big[\Big(p(\varrho_\delta,\vartheta_\delta) + \delta(\varrho_\delta^\Gamma + \varrho_\delta^2)\Big)T_k(\varrho_\delta) - \mathbb{S}_\delta : \mathcal{R}[1_\Omega T_k(\varrho_\delta)]\Big]\ \mathrm{d}x\ \mathrm{d}t = \sum_{j=1}^7 I_{j,\delta}, \tag{3.319}$$

where $\mathbb{S}_\delta := \mathbb{S}_\delta(\vartheta_\delta, \nabla_x \mathbf{u}_\delta)$ and

$$I_{1,\delta} = \int_0^T \int_\Omega \psi\zeta\Big(\varrho_\delta \mathbf{u}_\delta \cdot \mathcal{R}[1_\Omega T_k(\varrho_\delta)\mathbf{u}_\delta] - (\varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta) : \mathcal{R}[1_\Omega T_k(\varrho_\delta)]\Big) \ \mathrm{d}x \ \mathrm{d}t,$$

$$I_{2,\delta} = -\int_0^T \int_\Omega \psi\zeta \ \varrho_\delta \mathbf{u}_\delta \cdot \nabla_x \Delta_x^{-1}\Big[1_\Omega (T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta)\mathrm{div}_x \mathbf{u}_\delta\Big] \ \mathrm{d}x \ \mathrm{d}t,$$

$$I_{3,\delta} = -\int_0^T \int_\Omega \psi\zeta\varrho_\delta \mathbf{f}_\delta \cdot \nabla_x \Delta_x^{-1}[1_\Omega T_k(\varrho_\delta)] \ \mathrm{d}x \ \mathrm{d}t,$$

$$I_{4,\delta} = -\int_0^T \int_\Omega \psi\Big(p(\varrho_\delta, \vartheta_\delta) + \delta(\varrho_\delta^\Gamma + \varrho_\delta^2)\Big)\nabla_x\zeta \cdot \nabla_x \Delta_x^{-1}[1_\Omega T_k(\varrho_\delta)] \ \mathrm{d}x \ \mathrm{d}t,$$

$$I_{5,\delta} = \int_0^T \int_\Omega \psi\mathbb{S}_\delta : \nabla_x\zeta \otimes \nabla_x \Delta_x^{-1}[1_\Omega T_k(\varrho_\delta)] \ \mathrm{d}x \ \mathrm{d}t,$$

$$I_{6,\delta} = -\int_0^T \int_\Omega \psi(\varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta) : \nabla_x\zeta \otimes \nabla_x \Delta_x^{-1}[1_\Omega T_k(\varrho_\delta)] \ \mathrm{d}x \ \mathrm{d}t,$$

and

$$I_{7,\delta} = -\int_0^T \int_\Omega \partial_t\psi \ \zeta\varrho_\delta \mathbf{u}_\delta \cdot \nabla_x \Delta_x^{-1}[1_\Omega T_k(\varrho_\delta)] \ \mathrm{d}x \ \mathrm{d}t.$$

Now, mimicking the strategy of Section 3.6.5, we use

$$\varphi(t,x) = \psi(t)\zeta(x)(\nabla_x \Delta_x^{-1})[1_\Omega \overline{T_k(\varrho)}], \ \psi \in C_c^\infty(0,T), \ \zeta \in C_c^\infty(\Omega)$$

as test functions in the limit momentum balance (3.295) to obtain

$$\int_0^T \int_\Omega \psi\zeta\Big[\Big(\overline{p_M(\varrho,\vartheta)} + \frac{a}{4}\vartheta^4\Big)\overline{T_k(\varrho)} - \mathbb{S} : \mathcal{R}[1_\Omega \overline{T_k(\varrho)}]\Big] \ \mathrm{d}x \ \mathrm{d}t = \sum_{j=1}^7 I_j, \quad (3.320)$$

where

$$I_1 = \int_0^T \int_\Omega \psi\zeta\Big(\varrho\mathbf{u} \cdot \mathcal{R}[1_\Omega \overline{T_k(\varrho)}\mathbf{u}] - (\varrho\mathbf{u} \otimes \mathbf{u}) : \mathcal{R}[1_\Omega \overline{T_k(\varrho)}]\Big) \ \mathrm{d}x \ \mathrm{d}t,$$

$$I_2 = -\int_0^T \int_\Omega \psi\zeta \ \varrho\mathbf{u} \cdot \nabla_x \Delta_x^{-1}\Big[1_\Omega \overline{(T_k(\varrho) - T_k'(\varrho)\varrho)\mathrm{div}_x \mathbf{u}}\Big] \ \mathrm{d}x \ \mathrm{d}t,$$

$$I_3 = -\int_0^T \int_\Omega \psi\zeta\varrho\mathbf{f} \cdot \nabla_x \Delta_x^{-1}[1_\Omega \overline{T_k(\varrho)}] \ \mathrm{d}x \ \mathrm{d}t,$$

$$I_4 = -\int_0^T \int_\Omega \psi\overline{p(\varrho,\vartheta)}\nabla_x\zeta \cdot \nabla_x \Delta_x^{-1}[1_\Omega \overline{T_k(\varrho)}] \ \mathrm{d}x \ \mathrm{d}t,$$

$$I_5 = \int_0^T \int_\Omega \psi\mathbb{S} : \nabla_x\zeta \otimes \nabla_x \Delta_x^{-1}[1_\Omega \overline{T_k(\varrho)}] \ \mathrm{d}x \ \mathrm{d}t,$$

$$I_6 = -\int_0^T \int_\Omega \psi(\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla_x\zeta\otimes\nabla_x\Delta_x^{-1}[1_\Omega\overline{T_k(\varrho)}]\,\mathrm{d}x\,\mathrm{d}t,$$

and

$$I_7 = -\int_0^T \int_\Omega \partial_t\psi\ \zeta\varrho\mathbf{u}\cdot\nabla_x\Delta_x^{-1}[1_\Omega\overline{T_k(\varrho)}]\,\mathrm{d}x\,\mathrm{d}t.$$

We recall that $\mathcal{R} = \mathcal{R}_{i,j}$ is the double Riesz transform introduced in Section 0.5.

To get formula (3.320) we have used (3.285), (3.298) to identify $\overline{\vartheta^4}$ with ϑ^4 and $\overline{\mathbb{S}(\vartheta,\nabla_x\mathbf{u})}$ with $\mathbb{S} := \mathbb{S}(\vartheta,\nabla_x\mathbf{u})$. We also recall that $\mathcal{R} = \mathcal{R}_{i,j}$ is the double Riesz transform introduced in Section 0.5.

Now, letting $\delta\to 0+$ in (3.319), we get

$$\int_0^T \int_\Omega \psi\zeta\left[\overline{p_M(\varrho,\vartheta)\,T_k(\varrho)} + a\vartheta^4\overline{T_k(\varrho)} - \overline{\mathbb{S}:\mathcal{R}[1_\Omega\,T_k(\varrho)]}\right]\mathrm{d}x\mathrm{d}t = \sum_{j=1}^7 I_j, \quad (3.321)$$

where the right-hand side is the same as the right-hand side in (3.320). Here, we have used the commutator lemma in form of Corollary 3.3 with $r_\delta = 1_\Omega\,T_k(\varrho_\delta)$ and $\mathbf{V}_\delta = 1_\Omega\varrho_\delta\mathbf{u}_\delta$ to show that

$$I_{1,\delta}\to I_1 \text{ as } \delta\to 0+,$$

exactly in the same way as explained in detail in Section 3.6.5. We have also employed the pointwise convergence (3.298) to verify that $\overline{\vartheta^4} = \vartheta^4$ and that $\overline{\vartheta^4\,T_k(\varrho)} = \vartheta^4\overline{T_k(\varrho)}$.

Combining (3.320) and (3.321), we get identity

$$\int_0^T \int_\Omega \psi\zeta\left[\overline{p_M(\varrho,\vartheta)\,T_k(\varrho)} - \overline{p_M(\varrho,\vartheta)}\ \overline{T_k(\varrho)}\right]\mathrm{d}x\mathrm{d}t$$
$$= \int_0^T \int_\Omega \psi\zeta\left[\overline{\mathbb{S}:\mathcal{R}[1_\Omega T_k(\varrho)]} - \mathbb{S}:\mathcal{R}[1_\Omega\overline{T_k(\varrho)}]\right]\mathrm{d}x\mathrm{d}t.$$

We again follow the great lines of Section 3.6.5. Employing the evident properties of the Riesz transform evoked in formulas (10.103), we may write

$$\int_0^T \int_\Omega \psi\zeta\overline{\mathbb{S}:\mathcal{R}[1_\Omega\,T_k(\varrho)]}\,\mathrm{d}x\mathrm{d}t$$
$$= \lim_{\delta\to 0+}\int_0^T \int_\Omega \psi\omega(\vartheta_\delta,\mathbf{u}_\delta)\,T_k(\varrho_\delta)\,\mathrm{d}x\mathrm{d}t$$
$$+ \lim_{\delta\to 0+}\int_0^T \int_\Omega \psi\zeta(\frac{4}{3}\mu(\vartheta_\delta) + \eta(\vartheta_\delta))\mathrm{div}_x\mathbf{u}\,T_k(\varrho_\delta)\,\mathrm{d}x\mathrm{d}t$$

and

$$\int_0^T \int_\Omega \psi\zeta\mathbb{S} : \mathcal{R}[1_\Omega \overline{T_k(\varrho)}] \,\mathrm{d}x\mathrm{d}t$$
$$= \int_0^T \int_\Omega \psi\omega(\vartheta, \mathbf{u}) \, \overline{T_k(\varrho)} \,\mathrm{d}x\mathrm{d}t$$
$$+ \int_0^T \int_\Omega \psi\zeta(\frac{4}{3}\mu(\vartheta) + \eta(\vartheta))\mathrm{div}_x\mathbf{u} \, \overline{T_k(\varrho)} \,\mathrm{d}x\mathrm{d}t,$$

where

$$\omega(\vartheta, \mathbf{u}) = \mathcal{R} : \Big[\zeta\mu(\vartheta)\Big(\nabla_x\mathbf{u} + \nabla_x^T\mathbf{u}\Big)\Big] - \zeta\mu(\vartheta)\mathcal{R} : \Big[\nabla_x\mathbf{u} + \nabla_x^T\mathbf{u}\Big].$$

Applying Lemma 3.6 to $w = \zeta\mu(\vartheta_\delta)$, $\mathbf{Z} = [\partial_{x_i} u_{\varepsilon,j} + \partial_{x_j} u_{\varepsilon,i}]_{i=1}^3$, $j \in \{1, 2, 3\}$ fixed, where, according to (3.17–3.18), (3.279), (3.276), the sequences w, $\mathbf{Z}$ are bounded in $L^2(0, T; W^{1,2}(\Omega))$ and $L^{8/(5-\alpha)}((0, T) \times \Omega)$, respectively, we deduce that

$$\{\omega(\vartheta_\delta, \mathbf{u}_\delta)\}_{\delta>0} \text{ is bounded in } L^1(0, T; W^{\beta,s}(\Omega)) \text{ for certain } \beta \in (0, 1),\ s > 1. \tag{3.322}$$

Now, we consider four-dimensional vector fields

$$\mathbf{U}_\delta = [T_k(\varrho_\delta), T_k(\varrho_\delta)\mathbf{u}_\delta],\ \mathbf{V}_\delta = [\omega(\varrho_\delta, \vartheta_\delta), 0, 0, 0]$$

and take advantage of relations (3.253), (3.267), (3.268), (3.270), (3.279), (3.276) and (3.322) in order to show that $\mathbf{U}_\delta$, $\mathbf{V}_\delta$ verify all hypotheses of the Div-Curl lemma stated in Proposition 3.3. Using this proposition, we may conclude that

$$\omega(\vartheta_\delta, \mathbf{u}_\delta)T_k(\varrho_\delta) \to \overline{\omega(\vartheta, \mathbf{u})}\ \overline{T_k(\varrho)} = \omega(\vartheta, \mathbf{u})\ \overline{T_k(\varrho)} \text{ weakly in } L^1((0, T) \times \Omega), \tag{3.323}$$

where we have used (3.285), (3.298) to identify $\overline{\omega(\vartheta, \mathbf{u})}$ with $\omega(\vartheta, \mathbf{u})$.

We thus discover on this level of approximation again the weak compactness identity for the effective pressure

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■ WEAK COMPACTNESS IDENTITY FOR EFFECTIVE PRESSURE (LEVEL δ):

$$\overline{p_M(\varrho, \vartheta)T_k(\varrho)} - \Big(\frac{4}{3}\mu(\vartheta) + \eta(\vartheta)\Big)\ \overline{T_k(\varrho)\mathrm{div}_x\mathbf{u}}$$
$$= \overline{p_M(\varrho, \vartheta)}\ \overline{T_k(\varrho)} - \Big(\frac{4}{3}\mu(\vartheta) + \eta(\vartheta)\Big)\overline{T_k(\varrho)}\ \mathrm{div}_x\mathbf{u}. \tag{3.324}$$

Thus our ultimate goal is to use relation (3.324) in order to show pointwise convergence of the family of approximate densities $\{\varrho_\delta\}_{\delta>0}$. To this end, we revoke the "renormalized" limit equation (3.294) yielding

$$\int_0^T \int_\Omega \Big(\overline{\varrho L_k(\varrho)} \partial_t \varphi + \overline{\varrho L_k(\varrho)} \mathbf{u} \cdot \nabla_x \varphi - \overline{T_k(\varrho) \mathrm{div}_x \mathbf{u}} \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t$$
$$= - \int_\Omega \varrho_0 L_k(\varrho_0) \varphi(0, \cdot) \ \mathrm{d}x \tag{3.325}$$

for any test function $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$, where we have set

$$L_k(\varrho) = \int_1^\varrho \frac{T_k(z)}{z^2} \ \mathrm{d}z.$$

Assume, for a moment, that the limit functions ϱ, $\mathbf{u}$ also satisfy the equation of continuity in the sense of renormalized solutions, in particular,

$$\int_0^T \int_\Omega \Big(\varrho L_k(\varrho) \partial_t \varphi + \varrho L_k(\varrho) \mathbf{u} \cdot \nabla_x \varphi - T_k(\varrho) \mathrm{div}_x \mathbf{u} \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t$$
$$= - \int_\Omega \varrho_0 L_k(\varrho_0) \varphi(0, \cdot) \ \mathrm{d}x \tag{3.326}$$

for any test function $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$.

Now, relations (3.325), (3.326) give rise to

$$\int_\Omega \Big(\overline{\varrho L_k(\varrho)} - \varrho L_k(\varrho) \Big)(\tau) \ \mathrm{d}x + \int_0^\tau \int_\Omega \Big(\overline{T_k(\varrho) \mathrm{div}_x \mathbf{u}} - \overline{T_k(\varrho)} \mathrm{div}_x \mathbf{u} \Big) \ \mathrm{d}x \ \mathrm{d}t$$
$$= \int_0^\tau \int_\Omega \Big(T_k(\varrho) \mathrm{div}_x \mathbf{u} - \overline{T_k(\varrho)} \mathrm{div}_x \mathbf{u} \ \mathrm{d}x \ \mathrm{d}t \text{ for any } \tau \in [0,T]. \tag{3.327}$$

As $\{\vartheta_\delta\}_{\delta > 0}$ converges strongly in L^1 and p_M is a non-decreasing function of ϱ, we can use relation (3.324) to obtain

$$\overline{T_k(\varrho) \mathrm{div}_x \mathbf{u}} - \overline{T_k(\varrho)} \mathrm{div}_x \mathbf{u} \geq 0.$$

Letting $k \to \infty$ in (3.327) we obtain

$$\overline{\varrho \log(\varrho)} = \varrho \log(\varrho) \text{ a.a. on } (0,T) \times \Omega, \tag{3.328}$$

as soon as we are able to show that

$$\int_0^\tau \int_\Omega \Big(T_k(\varrho) \mathrm{div}_x \mathbf{u} - \overline{T_k(\varrho)} \mathrm{div}_x \mathbf{u} \Big) \ \mathrm{d}x \ \mathrm{d}t \to 0 \text{ for } k \to \infty. \tag{3.329}$$

Relation (3.328) yields

$$\varrho_\delta \to \varrho \text{ in } L^1((0,T) \times \Omega), \tag{3.330}$$

see Theorem 10.20 in Appendix. This completes the proof of Theorem 3.1.

Note, however, that two fundamental issues have been left open in the preceding discussion, namely

- the validity of the renormalized equation (3.326),
- relation (3.329).

These two intimately related questions will be addressed in the following section.

3.7.5 Oscillations defect measure

The oscillations defect measure introduced in [87] represents a basic tool for studying density oscillations. Given a family $\{\varrho_\delta\}_{\delta>0}$, a set Q, and $q \geq 1$, we introduce:

■ OSCILLATIONS DEFECT MEASURE:

$$\mathbf{osc}_q[\varrho_\delta \to \varrho](Q) = \sup_{k\geq 1} \Big(\limsup_{\delta\to 0+} \int_Q \Big| T_k(\varrho_\delta) - T_k(\varrho) \Big|^q \, \mathrm{d}x \, \mathrm{d}t \Big), \tag{3.331}$$

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where T_k are the cut-off functions introduced in (3.317).

Assume that

$$\mathrm{div}_x \mathbf{u} \in L^r((0,T)\times\Omega), \ \mathbf{osc}_q[\varrho_\delta \to \varrho]((0,T)\times\Omega) < \infty, \ \text{with } \frac{1}{r} + \frac{1}{q} < 1. \tag{3.332}$$

Seeing that

$$T_k(\varrho) \to \varrho, \ \overline{T_k(\varrho)} \to \varrho \text{ in } L^1((0,T)\times\Omega) \text{ for } k \to \infty,$$

we conclude easily that (3.332) implies (3.329).

A less obvious statement is the following assertion.

Lemma 3.8. *Let $Q \subset \mathbb{R}^4$ be an open set. Suppose that*

$$\varrho_\delta \to \varrho \text{ weakly in } L^1(Q),$$

$$\mathbf{u}_\delta \to \mathbf{u} \text{ weakly in } L^r(Q;\mathbb{R}^3), \tag{3.333}$$

$$\nabla_x \mathbf{u}_\delta \to \nabla_x \mathbf{u} \text{ weakly in } L^r(Q;\mathbb{R}^{3\times 3}), \ r>1, \tag{3.334}$$

and

$$\mathbf{osc}_q[\varrho_\delta \to \varrho](Q) < \infty \text{ for } \frac{1}{q} + \frac{1}{r} < 1, \tag{3.335}$$

where ϱ_δ, $\mathbf{u}_\delta$ solve the renormalized equation (2.2) *in $\mathcal{D}'(Q)$.*

Then the limit functions ϱ, $\mathbf{u}$ solve the renormalized equation (2.2) *in $\mathcal{D}'(Q)$.*

Proof. Clearly, it is enough to show the result on the set $J \times K$, where J is a bounded time interval and K is a ball such that $\overline{J\times K} \subset Q$. Since ϱ_δ is a renormalized solution of (2.2), we get

$$T_k(\varrho_\delta) \to \overline{T_k(\varrho)} \text{ in } C_{\mathrm{weak}}(\overline{J}; L^\beta(K)) \text{ for any } 1 \leq \beta < \infty;$$

whence, by virtue of hypotheses (3.333), (3.334),

$$T_k(\varrho_\delta)\mathbf{u}_\delta \to \overline{T_k(\varrho)}\mathbf{u} \text{ weakly in } L^r(J\times K;\mathbb{R}^3).$$

Consequently, we deduce

$$\partial_t \overline{T_k(\varrho)} + \operatorname{div}_x \Big(\overline{T_k(\varrho)} \mathbf{u} \Big) + \overline{\Big(T_k'(\varrho)\varrho - T_k(\varrho) \Big) \operatorname{div}_x \mathbf{u}} = 0 \text{ in } \mathcal{D}'(J \times K). \quad (3.336)$$

Since $\overline{T_k(\varrho)}$ are bounded, we can apply the regularization technique introduced by DiPerna and Lions [65] (Theorem 10.29), already used in Lemma 3.7, in order to deduce

$$\partial_t h(\overline{T_k(\varrho)}) + \operatorname{div}_x \Big(h(\overline{T_k(\varrho)}) \mathbf{u} \Big) + \Big(h'(\overline{T_k(\varrho)})\ \overline{T_k(\varrho)} - \overline{T_k(\varrho)} \Big) \operatorname{div}_x \mathbf{u}$$
$$= h'(\overline{T_k(\varrho)}) \overline{\Big(T_k(\varrho) - T_k'(\varrho)\varrho \Big) \operatorname{div}_x \mathbf{u}} \text{ in } \mathcal{D}'(J \times K),$$

where h is a continuously differentiable function such that $h'(z) = 0$ for all z large enough, say, $z \geq M$.

Consequently, it is enough to show

$$h'(\overline{T_k(\varrho)}) \overline{\Big(T_k(\varrho) - T_k'(\varrho)\varrho \Big) \operatorname{div}_x \mathbf{u}} \to 0 \text{ in } L^1(J \times K) \text{ for } k \to \infty. \quad (3.337)$$

To this end, denote

$$Q_{k,M} = \{(t,x) \in J \times K \mid |\overline{T_k(\varrho)}| \leq M\}.$$

Consequently,

$$\Big\| h'(\overline{T_k(\varrho)}) \overline{\Big(T_k(\varrho) - T_k'(\varrho)\varrho \Big) \operatorname{div}_x \mathbf{u}} \Big\|_{L^1(J\times K)} \quad (3.338)$$
$$\leq \Big(\sup_{0 \leq z \leq M} |h'(z)| \Big) \Big(\sup_{\delta > 0} \|\operatorname{div}_x \mathbf{u}_\delta\|_{L^r(J\times K)} \Big) \liminf_{\delta \to 0} \|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\|_{L^{r'}(Q_{k,M})},$$

where $1/r + 1/r' = 1$.

Furthermore, a simple interpolation argument yields

$$\|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\|_{L^{r'}(Q_{k,M})} \quad (3.339)$$
$$\leq \|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\|^{\beta}_{L^1(J\times K)} \|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\|^{1-\beta}_{L^q(Q_{k,M})},$$

with $\beta \in (0,1)$.

As the family $\{\varrho_\delta\}_{\delta>0}$ is equi-integrable, we deduce

$$\sup_{\delta>0} \Big\{ \|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\|_{L^1(J\times K)} \Big\} \to 0 \text{ for } k \to \infty. \quad (3.340)$$

Finally, seeing that $|T_k'(\varrho_\delta)\varrho_\delta| \leq T_k(\varrho_\delta)$, we conclude

$$\|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\|_{L^q(Q_{k,M})}$$
$$\leq 2\Big(\|T_k(\varrho_\delta) - T_k(\varrho)\|_{L^q(J\times K)} + \|T_k(\varrho) - \overline{T_k(\varrho)}\|_{L^q(J\times K)} + \|\overline{T_k(\varrho)}\|_{L^q(Q_{k,M})} \Big)$$
$$\leq 2\Big(\|T_k(\varrho_\delta) - T_k(\varrho)\|_{L^q(J\times K)} + \mathbf{osc}_q[\varrho_\delta \to \varrho](J \times K + |J \times K|^{\frac{1}{q}};$$

whence

$$\limsup_{\delta\to 0} \|T_k(\varrho_\delta) - T_k'(\varrho_\delta)\varrho_\delta\|_{L^q(Q_{k,M})} \le 4\mathbf{osc}_q[\varrho_\delta \to \varrho](J\times K) + 2M|J\times K|^{\frac{1}{q}}. \tag{3.341}$$

Clearly, relation (3.337) follows from (3.338–3.341). □

In order to apply Lemma 3.8, we need to establish suitable bounds on $\mathbf{osc}_q[\varrho_\delta \to \varrho]$. To this end, revoking (3.42–3.44) we write

$$p_M(\varrho,\vartheta) = d\varrho^{\frac{5}{3}} + p_m(\varrho,\vartheta) + p_b(\varrho,\vartheta),\ d>0, \tag{3.342}$$

where

$$\frac{\partial p_m(\varrho,\vartheta)}{\partial\varrho} \ge 0, \tag{3.343}$$

and

$$|p_b(\varrho,\vartheta)| \le c(1+\vartheta^{\frac{5}{2}}) \tag{3.344}$$

for all ϱ, $\vartheta > 0$.

Consequently,

$$\begin{aligned} d\limsup_{\delta\to 0+}&\int_0^T\int_\Omega \varphi|T_k(\varrho_\delta) - T_k(\varrho)|^{\frac{8}{3}}\ \mathrm{d}x\ \mathrm{d}t \\ &\le d\int_0^T\int_\Omega \varphi\Big(\overline{\varrho^{\frac{5}{3}}T_k(\varrho)} - \overline{\varrho^{\frac{5}{3}}}\ \overline{T_k(\varrho)}\Big)\ \mathrm{d}x\ \mathrm{d}t \\ &\quad + d\int_0^T\int_\Omega \varphi\Big(\varrho^{\frac{5}{3}} - \overline{\varrho^{\frac{5}{3}}}\Big)\Big(T_k(\varrho) - \overline{T_k(\varrho)}\Big)\ \mathrm{d}x\ \mathrm{d}t \\ &\le \int_0^T\int_\Omega \varphi\Big(\overline{p_M(\varrho,\vartheta)T_k(\varrho)} - \overline{p_M(\varrho,\vartheta)}\ \overline{T_k(\varrho)}\Big)\ \mathrm{d}x \\ &\quad + \left|\int_0^T\int_\Omega \varphi\Big(\overline{p_b(\varrho,\vartheta)T_k(\varrho)} - \overline{p_b(\varrho,\vartheta)}\ \overline{T_k(\varrho)}\Big)\ \mathrm{d}x\ \mathrm{d}t\right| \end{aligned}$$

for any $\varphi \in C_c^\infty((0,T)\times\Omega)$, $\varphi \ge 0$, where we have used (3.343), convexity of $\varrho \mapsto \varrho^{\frac{5}{3}}$, and concavity of T_k on $[0,\infty)$.

In accordance with the uniform bound (3.270) and (3.344), we have

$$\begin{aligned} &\left|\int_0^T\int_\Omega \varphi\Big(\overline{p_b(\varrho,\vartheta)T_k(\varrho)} - \overline{p_b(\varrho,\vartheta)}\ \overline{T_k(\varrho)}\Big)\ \mathrm{d}x\ \mathrm{d}t\right| \qquad (3.345) \\ &\quad\le c_1\Big(1 + \sup_{\delta>0}\|\vartheta_\delta^{\frac{5}{2}}\|_{L^{\frac{8}{5}}((0,T)\times\Omega)}\Big)\Big(\int_0^T\int_\Omega \varphi|T_k(\varrho_\delta) - T_k(\varrho)|^{\frac{8}{3}}\ \mathrm{d}x\ \mathrm{d}t\Big)^{\frac{3}{8}} \\ &\quad\le c_2\limsup_{\delta\to 0}\Big(\int_0^T\int_\Omega \varphi|T_k(\varrho_\delta) - T_k(\varrho)|^{\frac{8}{3}}\ \mathrm{d}x\ \mathrm{d}t\Big)^{\frac{3}{8}}. \end{aligned}$$

Furthermore, introducing a Carathéodory function

$$G_k(t,x,z) = |T_k(z) - T_k(\varrho(t,x))|^{\frac{8}{3}}$$

we get, in accordance with (3.345),

$$\overline{G_k(\cdot,\cdot,\varrho)} \leq c\Big(1 + \overline{p_M(\varrho,\vartheta)T_k(\varrho)} - \overline{p_M(\varrho,\vartheta)}\ \overline{T_k(\varrho)}\Big), \text{ with } c \text{ independent of } k \geq 1.$$

Thus, evoking once more (3.324) we infer that

$$\overline{G_k(\cdot,\cdot,\varrho)} \leq c\Big(1 + (\frac{4}{3}\mu(\vartheta) + \eta(\vartheta))(\overline{\mathrm{div}_x \mathbf{u}\ T_k(\varrho)} - \mathrm{div}_x \mathbf{u}\ \overline{T_k(\varrho)})\Big) \text{ for all } k \geq 1. \tag{3.346}$$

On the other hand, by virtue of hypothesis (3.17) and estimate (3.276), we get

$$\int_0^T \int_\Omega (1+\vartheta)^{-\alpha}\overline{G_k(t,x,\varrho)}\ \mathrm{d}x\ \mathrm{d}t \tag{3.347}$$
$$\leq c\Big(1 + \sup_{\delta>0} \|\mathrm{div}_x \mathbf{u}_\delta\|_{L^{\frac{8}{5-\alpha}}((0,T)\times\Omega)} \limsup_{\delta\to 0+} \|T_k(\varrho_\delta) - T_k(\varrho)\|_{L^{\frac{8}{3+\alpha}}((0,T)\times\Omega)}\Big)$$
$$\leq c\Big(1 + \limsup_{\delta\to 0+} \|T_k(\varrho_\delta) - T_k(\varrho)\|_{L^{\frac{8}{3+\alpha}}((0,T)\times\Omega)}\Big).$$

Taking

$$\frac{8}{3+\alpha} < q < \frac{8}{3},\ \beta = \frac{3q\alpha}{8}$$

and using Hölder's inequality, we obtain

$$\int_0^T \int_\Omega |T_k(\varrho_\delta) - T_k(\varrho)|^q\ \mathrm{d}x\ \mathrm{d}t \tag{3.348}$$
$$= \int_0^T \int_\Omega (1+\vartheta)^{-\beta}(1+\vartheta)^{\beta}|T_k(\varrho_\delta) - T_k(\varrho)|^q\ \mathrm{d}x\ \mathrm{d}t$$
$$\leq c\Big(\int_0^T \int_\Omega (1+\vartheta)^{-\alpha}|T_k(\varrho_\delta) - T_k(\varrho)|^{\frac{8}{3}}\ \mathrm{d}x\ \mathrm{d}t + \int_0^T \int_\Omega (1+\vartheta)^{\frac{3\alpha q}{8-3q}}\ \mathrm{d}x\ \mathrm{d}t\Big).$$

Finally, choosing q such that

$$\frac{8}{3+\alpha} < q \leq \frac{32}{3\alpha+12},\ \text{meaning } \frac{3\alpha q}{8-3q} \leq 4,$$

we can combine relations (3.347), (3.348), together with estimate (3.270), in order to conclude that

$$\mathbf{osc}_q[\varrho_\delta \to \varrho]((0,T)\times\Omega) < \infty \text{ for a certain } q > \frac{8}{3+\alpha}. \tag{3.349}$$

Relation (3.349) together with (3.276) allow us to apply Lemma 3.8 in order to conclude that

- the limit functions ϱ $\mathbf{u}$ satisfy the renormalized equation (3.326),
- relation (3.329) holds.

Thus we have rigorously justified the strong convergence of $\{\varrho_\delta\}_{\delta>0}$ claimed in (3.330). The proof of Theorem 3.1 is complete.

3.8 Regularity properties of the weak solutions

The reader will have noticed that the weak solutions constructed in the course of the proof of Theorem 3.1 enjoy slightly better regularity and integrability properties than those required in Section 2.1. As a matter of fact, the uniform bounds obtained above can be verified for *any* weak solution of the NAVIER-STOKES-FOURIER SYSTEM in the sense of Section 2.1 and not only for the specific one resulting from our approximation procedure. Similarly, the restrictions on the geometry of the spatial domain can be considerably relaxed and other types of domains, for instance, the periodic slab, can be handled.

■ REGULARITY OF THE WEAK SOLUTIONS:

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Assume the data ϱ_0, $(\varrho\mathbf{u})_0$, E_0, $(\varrho s)_0$, the source terms $\mathbf{f}$, $\mathcal{Q}$, the thermodynamic functions p, e, s, and the transport coefficients μ, η, κ satisfy the structural hypotheses* (3.1–3.23) *listed in Section* 3.1. *Let $\{\varrho, \mathbf{u}, \vartheta\}$ be a weak solution to the Navier-Stokes-Fourier system on $(0,T) \times \Omega$ in the sense specified in Section* 2.1.

Then, in addition to the minimal integrability and regularity properties required in (2.5–2.6), (2.13–2.15), (2.30–2.31), *there holds:*

(i)
$$\begin{gathered} \varrho \in C_{\rm weak}([0,T]; L^{\frac{5}{3}}(\Omega)) \cap C([0,T]; L^1(\Omega)), \\ \varrho\mathbf{u} \in C_{\rm weak}([0,T]; L^{\frac{5}{4}}(\Omega)), \end{gathered} \tag{3.350}$$

$$\begin{gathered} \vartheta \in L^2(0,T; W^{1,2}(\Omega)) \cap L^\infty(0,T; L^4(\Omega)), \\ \log\vartheta \in L^2(0,T; W^{1,2}(\Omega)), \end{gathered} \tag{3.351}$$

$$\left\{ \begin{gathered} \mathbb{S}(\vartheta, \nabla_x\mathbf{u}) \in L^q((0,T)\times\Omega; \mathbb{R}^{3\times 3}) \quad \textit{for a certain } q > 1, \\ \mathbf{u} \in L^q(0,T; W^{1,p}(\Omega;\mathbb{R}^3)) \textit{ for } q = \tfrac{6}{4-\alpha}, p = \tfrac{18}{10-\alpha}, \end{gathered} \right\} \tag{3.352}$$

$$\left\{ \begin{gathered} \varrho \in L^q((0,T)\times\Omega) \quad \textit{for a certain } q > \tfrac{5}{3}, \\ p(\varrho,\vartheta) \in L^q((0,T)\times\Omega) \quad \textit{for a certain } q > 1. \end{gathered} \right\} \tag{3.353}$$

(ii) *The total kinetic energy* $\int_\Omega \frac{|\varrho\mathbf{u}|^2}{\varrho} 1_{\{\varrho>0\}}\,\mathrm{d}x$ *is lower semicontinuous on* $(0,T)$*, left lower semicontinuous at* T *and right lower semicontinuous at* 0*; in particular*

$$\liminf_{t\to 0+}\Big(\int_\Omega \frac{|\varrho\mathbf{u}|^2}{\varrho}1_{\{\varrho>0\}}\,\mathrm{d}x\Big)(t) \geq \int_\Omega \frac{|(\varrho\mathbf{u})_0|^2}{\varrho_0}1_{\{\varrho_0>0\}}\,\mathrm{d}x. \tag{3.354}$$

(iii) *The entropy satisfies*

$$\left\{\begin{array}{c} \mathrm{ess}\lim_{t\to 0+}\int_\Omega \varrho s(\varrho,\vartheta)(t,\cdot)\varphi\,\mathrm{d}x \ \geq \int_\Omega \varrho_0 s(\varrho_0,\vartheta_0)\varphi\,\mathrm{d}x \\ \textit{for any } \varphi\in C_c^\infty(\overline{\Omega}),\ \ \varphi\geq 0. \end{array}\right\} \tag{3.355}$$

If, in addition, $\vartheta_0\in W^{1,\infty}(\Omega)$*, then*

$$\mathrm{ess}\lim_{t\to 0+}\int_\Omega \varrho s(\varrho,\vartheta)(t,\cdot)\varphi\,\mathrm{d}x = \int_\Omega \varrho_0 s(\varrho_0,\vartheta_0)\varphi\,\mathrm{d}x,\ \textit{for all } \varphi\in C_c^\infty(\overline{\Omega}). \tag{3.356}$$

Proof. Step 1. Unlike the proof of *existence* based on the classical theory of parabolic equations requiring Ω to be a regular domain, the *integrability properties* (3.350–3.353) of the weak solutions follow directly from the total dissipation balance (2.52) and the space-time pressure estimates obtained by means of the operator $\mathcal{B}\approx \mathrm{div}_x^{-1}$ introduced in Section 2.2.5; for more details, see estimates (2.40), (2.46), (2.66), (2.68), (2.73), (2.96) and (2.98). In particular, it is enough to assume $\Omega\subset\mathbb{R}^3$ to be a bounded *Lipschitz* domain.

Step 2. Strong continuity in time of the density claimed in (3.350) is a general property of the renormalized solutions that follows from the DiPerna and Lions transport theory [65], see Lemma 10.14 in Appendix. Once $\varrho\in C([0,T];L^1(\Omega))\cap C_{\mathrm{weak}}([0,T];L^{\frac{5}{3}}(\Omega))$, we deduce from the momentum equation (2.9) and estimates (3.351–3.353) that one may take a representative of $\mathbf{u}\in L^q(0,T;W^{1,p}(\Omega))$ such that $\mathbf{m}:=\varrho\mathbf{u}\in C_{\mathrm{weak}}([0,T];L^{\frac{5}{4}}(\Omega;\mathbb{R}^3))$. In addition, we may infer from the inequality

$$\|\mathbf{m}(t)\|^2_{L^{\frac{5}{4}}(\Omega)} \leq \|\varrho(t)\|_{L^{\frac{5}{3}}(\Omega)}\|\varrho(t)|\mathbf{u}(t)|^2\|_{L^\infty(0,T;L^1(\Omega))},\ t\in[0,T]$$

that $\mathbf{m}(t)$ vanishes almost anywhere on the set $\{x\in\Omega\,|\,\varrho(t)=0\}$. The expression $\frac{|\mathbf{m}(t)|^2}{\varrho(t)}1_{\{\varrho(t)>0\}}$ is therefore defined for all $t\in[0,T]$ and is equal to $\varrho|\mathbf{u}|^2(t)$ a.a. on $(0,T)$.

Since $\int_\Omega \frac{|\mathbf{m}(t)|^2}{\varrho(t)+\varepsilon}\,\mathrm{d}x \leq \|\varrho\mathbf{u}\|_{L^\infty(0,T;L^1(\Omega))}$ uniformly with $\varepsilon\to 0+$, we deduce by the Beppo-Lévi monotone convergence theorem that

$$\int_\Omega \frac{|\mathbf{m}(t)|^2}{\varrho(t)+\varepsilon}\,\mathrm{d}x \to \int_\Omega \frac{|\mathbf{m}(t)|^2}{\varrho(t)}1_{\{\varrho(t)>0\}}\,\mathrm{d}x<\infty \text{ for all } t\in[0,T].$$

This information together with (3.350) guarantees $\mathbf{m}(t)/\sqrt{\varrho(t)+\varepsilon} \in C_{\rm weak}([0,T]; L^2(\Omega))$. Therefore, for any $\alpha > 0$ and sufficiently small $0 < \varepsilon < \varepsilon(\alpha)$, and for any $\tau \in [0,T)$,

$$\int_\Omega \frac{|\mathbf{m}(\tau)|^2}{\varrho(\tau)} 1_{\{\varrho(\tau)>0\}}\,{\rm d}x - \alpha \le \int_\Omega \frac{|\mathbf{m}(\tau)|^2}{\varrho(\tau)+\varepsilon}\,{\rm d}x$$
$$\le \liminf_{t\to\tau+} \int_\Omega \frac{|\mathbf{m}(t)|^2}{\varrho(t)} 1_{\{\varrho(t)>\varepsilon\}}\,{\rm d}x \le \liminf_{t\to\tau+} \int_\Omega \frac{|\mathbf{m}(t)|^2}{\varrho(t)} 1_{\{\varrho(t)>0\}}\,{\rm d}x,$$

where, to justify the inequality in the middle, we have used (3.350) and the lower weak semicontinuity of convex functionals discussed in Theorem 10.20 in Appendix. We have completed the proof of lower semicontinuity in time of the total kinetic energy, and, in particular, formula (3.354).

Step 3. In agreement with formulas (1.11–1.12), we deduce from the entropy balance (2.27) that

$$[\varrho s(\varrho,\vartheta)](\tau+) \in \mathcal{M}^+(\overline{\Omega}),\ \tau \in [0,T), \quad [\varrho s(\varrho,\vartheta)](\tau-) \in \mathcal{M}^+(\overline{\Omega}),\ \tau \in (0,T],$$
$$[\varrho s(\varrho,\vartheta)](\tau+) \ge [\varrho s(\varrho,\vartheta)](\tau-),\ \tau \in (0,T),$$

where the measures $[\varrho s(\varrho,\vartheta)](\tau+)$, $\tau \in [0,T)$ and $[\varrho s(\varrho,\vartheta)](\tau-)$, $\tau \in (0,T]$ are defined in the following way:

$$\langle [\varrho s(\varrho,\vartheta)](\tau\pm);\zeta\rangle_{[\mathcal{M};C](\overline{\Omega})} := \lim_{\delta\to 0+} \int_{I^\pm_{\tau,\delta}} \int_\Omega [\varrho s(\varrho,\vartheta)](t)[\psi_\delta^{\tau,\pm}]'(t)\zeta\,{\rm d}x\,{\rm d}t$$
$$= \int_\Omega \varrho_0 s(\varrho_0,\vartheta_0)\zeta\,{\rm d}x + \lim_{\delta\to 0+} \left\langle \sigma;\psi_\delta^{(\tau,\pm)}\zeta\right\rangle_{[\mathcal{M};C]([0,T]\times\overline{\Omega})} \tag{3.357}$$
$$+ \int_\Omega \varrho_0 s(\varrho_0,\vartheta_0)\zeta\,{\rm d}x + \int_0^\tau \int_\Omega \left(\varrho s(\varrho,\vartheta)\mathbf{u} + \frac{\mathbf{q}}{\vartheta}\right)\cdot\nabla_x\zeta\,{\rm d}x + \int_0^\tau \int_\Omega \frac{\mathcal{Q}}{\vartheta}\zeta\,{\rm d}x.$$

In this formula, $\zeta \in C(\overline{\Omega})$, $I^+_{\tau,\delta} = (\tau,\tau+\delta)$, $I^-_{\tau,\delta} = (\tau-\delta,\tau)$ and $\psi_\delta^{(\tau,\pm)} \in C^1(\mathbb{R})$ are non-increasing functions such that

$$\psi_\delta^{(\tau,+)}(t) = \left\{ \begin{array}{c} 1 \text{ if } t \in (-\infty,\tau), \\ 0 \text{ if } t \in [\tau+\delta,\infty), \end{array} \right\},\ \psi_\delta^{(\tau,-)}(t) = \left\{ \begin{array}{c} 1 \text{ if } t \in (-\infty,\tau-\delta), \\ 0 \text{ if } t \in [\tau,\infty), \end{array} \right\}.$$

According to the theorem about the Lebesgue points applied to function $\varrho s(\varrho,\vartheta)$ (belonging to $L^\infty(0,T;L^1(\Omega))$), we may infer

$$\langle [\varrho s(\varrho,\vartheta)](\tau-);\zeta\rangle_{[\mathcal{M};C](\overline{\Omega})} = \langle [\varrho s(\varrho,\vartheta)](\tau+);\zeta\rangle_{[\mathcal{M};C](\overline{\Omega})} \tag{3.358}$$
$$= \int_\Omega [\varrho s(\varrho,\vartheta)](\tau)\zeta\,{\rm d}x,\ \zeta \in C^\infty_c(\overline{\Omega}),\ \zeta \ge 0 \text{ for a.a. } \tau \in (0,T).$$

Letting $\delta \to 0+$ in (3.357), we obtain

$$\int_\Omega [\varrho s(\varrho,\vartheta)](\tau+)\zeta \,\mathrm{d}x - \langle \sigma;\zeta\rangle_{[\mathcal{M};C]([0,\tau]\times\overline{\Omega})} \tag{3.359}$$
$$= \int_\Omega \varrho_0 s(\varrho_0,\vartheta_0)\zeta\,\mathrm{d}x + \int_0^\tau\int_\Omega \left(\varrho\frac{\mathcal{Q}}{\vartheta}\zeta + (\varrho s(\varrho,\vartheta)\mathbf{u} + \frac{\mathbf{q}}{\vartheta})\cdot\nabla_x\zeta\right)\mathrm{d}x.$$

In the remaining part of the proof, we shall show that

$$\mathrm{ess}\lim_{\tau\to 0+}\langle\sigma;\zeta\rangle_{[\mathcal{M};C]([0,\tau]\times\overline{\Omega})} = 0. \tag{3.360}$$

Step 4. To this end we employ in the entropy balance (2.27) the test function $\varphi(t,x) = \psi_\delta^{(\tau,+)}(t)\vartheta_0(x)$, $\tau\in(0,T)$, which is admissible provided $\vartheta_0\in W^{1,\infty}(\Omega)$. Using additionally (3.358), we get

$$\int_\Omega \left([\varrho s(\varrho,\vartheta)](\tau) - \varrho_0 s(\varrho_0,\vartheta_0)\right)\vartheta_0\,\mathrm{d}x \tag{3.361}$$
$$= \langle\sigma;\vartheta_0\rangle_{[\mathcal{M};C]([0,\tau]\times\overline{\Omega})} + \int_0^\tau\int_\Omega\left(\varrho s(\varrho,\vartheta)\mathbf{u}+\frac{\mathbf{q}}{\vartheta}\right)\cdot\nabla_x\vartheta_0\,\mathrm{d}x + \int_0^\tau\int_\Omega\frac{\mathcal{Q}}{\vartheta}\vartheta_0\,\mathrm{d}x$$

for a.a. $\tau\in(0,T)$. On the other hand, the total energy balance (2.22) with the test function $\psi = \psi_\delta^{(\tau,+)}$ yields

$$\int_\Omega\left([\frac{1}{2\varrho}|\varrho\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)](\tau) - [\frac{1}{2\varrho_0}|\varrho_0\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0,\vartheta_0)]\right)\mathrm{d}x$$
$$= \int_0^\tau\int_\Omega(\varrho f\mathbf{u} + \varrho\mathcal{Q})\,\mathrm{d}x\,\mathrm{d}t \tag{3.362}$$

for a.a. $\tau\in(0,T)$. Now, we introduce the Helmholtz function

$$H_{\vartheta_0}(\varrho,\vartheta) = \varrho e(\varrho,\vartheta) - \vartheta_0\varrho s(\varrho,\vartheta)$$

and combine (3.361–3.362) to get

$$\int_\Omega\left([\frac{1}{2\varrho}|\varrho\mathbf{u}|^2](\tau) - \frac{1}{2\varrho_0}|\varrho_0\mathbf{u}_0|^2\right)\mathrm{d}x + \int_\Omega[H_{\vartheta_0}(\varrho,\vartheta) - H_{\vartheta_0}(\varrho,\vartheta_0)](\tau)\,\mathrm{d}x$$
$$+\int_\Omega\left(H_{\vartheta_0}(\varrho(\tau),\vartheta) - H_{\vartheta_0}(\varrho_0,\vartheta_0) - (\varrho(\tau)-\varrho_0)\frac{\partial H_{\vartheta_0}}{\partial\varrho}(\varrho_0,\vartheta_0)\right)\mathrm{d}x$$
$$+\int_\Omega(\varrho(\tau)-\varrho_0)\frac{\partial H_{\vartheta_0}}{\partial\varrho}(\varrho_0,\vartheta_0)\,\mathrm{d}x + \langle\sigma;\vartheta_0\rangle_{[\mathcal{M};C]([0,\tau]\times\overline{\Omega})}$$
$$= \int_0^\tau\int_\Omega\left(\varrho f\mathbf{u} + \varrho\mathcal{Q}\left(1-\frac{\vartheta_0}{\vartheta}\right) - \left(\varrho s(\varrho,\vartheta)\mathbf{u}+\frac{\mathbf{q}}{\vartheta}\right)\cdot\nabla\vartheta_0\right)\mathrm{d}x\,\mathrm{d}t \tag{3.363}$$

for a.a. τ in $(0,T)$.

It follows from the thermodynamic stability hypothesis (1.44) that $\varrho \mapsto H_{\vartheta_0}(\varrho, \vartheta_0)$ is strictly convex for any fixed ϑ_0 and that $\vartheta \mapsto H_{\vartheta_0}(\varrho, \vartheta)$ attains its global minimum at ϑ_0, see Section 2.2.3 for more details. Consequently,

$$
\begin{aligned}
H_{\vartheta_0}(\varrho, \vartheta) - H_{\vartheta_0}(\varrho, \vartheta_0) &\geq 0, \\
H_{\vartheta_0}(\varrho, \vartheta) - H_{\vartheta_0}(\varrho_0, \vartheta_0) - (\varrho - \varrho_0)\frac{\partial H_{\vartheta_0}}{\partial \varrho}(\varrho_0, \vartheta_0) &\geq 0.
\end{aligned}
$$

Moreover, due to the strong continuity of density with respect to time stated in (3.350), we show

$$
\lim_{\tau \to 0+} \int_\Omega (\varrho(\tau) - \varrho_0)\frac{\partial H_{\vartheta_0}}{\partial \varrho}(\varrho_0, \vartheta_0)\,\mathrm{d}x = 0,
$$

while the last integral at the right-hand side of (3.363) tends to 0 as $\tau \to 0+$ since the integrand belongs to $L^1((0,T)\times\Omega)$. Thus, relation (3.363) reduces in the limit $\tau \to 0+$ to

$$
\mathrm{ess}\lim_{\tau \to 0+} \langle \sigma; \vartheta_0 \rangle_{[\mathcal{M};C]([0,\tau]\times\overline{\Omega})},
$$

whence $\mathrm{ess}\lim_{\tau\to 0+}\sigma\left[[0,\tau]\times\overline{\Omega}\right] = 0$ and (3.360) follows. Having in mind identity (3.358), statement (3.356) now follows by letting $\tau \to 0+$ in (3.359) (evidently, the right-hand side in (3.359) tends to zero as the integrand belongs to $L^1((0,T)\times\Omega)$).

Theorem 3.2 is proved. □

Chapter 4

Asymptotic Analysis – An Introduction

The extreme generality of the full Navier-Stokes-Fourier system whereby the equations describe the entire spectrum of possible motions – ranging from sound waves, cyclone waves in the atmosphere, to models of gaseous stars in astrophysics – constitutes a serious defect of the equations from the point of view of applications. Eliminating unwanted or unimportant modes of motion, and building in the essential balances between flow fields, allow the investigator to better focus on a particular class of phenomena and to potentially achieve a deeper understanding of the problem. Scaling and asymptotic analysis play an important role in this approach. By scaling the equations, meaning by choosing appropriately the system of the reference units, the parameters determining the behavior of the system become explicit. Asymptotic analysis provides a useful tool in the situations when certain of these parameters called *characteristic numbers* vanish or become infinite.

The main goal of many studies devoted to asymptotic analysis of various physical systems is to derive a simplified set of equations solvable either analytically or at least with less numerical effort. Classical textbooks and research monographs (see Gill [97], Pedlosky [171], Zeytounian [204], [206], among others) focus mainly on the way the scaling arguments together with other characteristic features of the data may be used in order to obtain, mostly in a very formal way, a simplified system, leaving aside the mathematical aspects of the problem. In particular, the *existence* of classical solutions is always tacitly anticipated, while exact results in this respect are usually in short supply. In fact, not only has the problem not been solved, it is not clear that in general smooth solutions exist. This concerns both the primitive Navier-Stokes-Fourier system and the target systems resulting from the asymptotic analysis. Notice that even for the "simple" *incompressible* Navier-Stokes system, the existence of regular solutions represents an outstanding open problem (see Fefferman [77]). Consequently, given the recent state of the art, a rigorous mathematical treatment without any unnecessary restrictions on the size of the observed data as well as the length of the time interval must be based on the concept of *weak solutions* defined in the spirit of Chapter 2. Although suitability of this framework might be questionable because of possible loss of information due to its generality, we show that this class of solutions is sufficiently robust to perform various asymptotic limits and to recover a number of standard models in mathematical fluid mechanics (see Sections 4.2–4.4). Accordingly, the results presented in this book can be viewed as another piece of evidence in support of the mathematical theory based on the concept of *weak solutions*.

In the following chapters, we provide a mathematical justification of several up to now mostly formal procedures, hope to shed some light on the way the simplified *target problems* can be derived from the *primitive system* under suitable scaling, and, last but not least, contribute to further development of the related numerical methods. We point out that *formal* asymptotic analysis performed with respect to a small (large) parameter tells us only that certain quantities may be negligible in certain regimes because they represent higher order terms in the (formal) asymptotic expansion. However, the specific way, i.e., *how* they are filtered off is very often more important than the limit problem itself. A typical example is the high frequency *acoustic waves* in meteorological models that may cause the failure of certain numerical schemes. An intuitive argument states that such sizeable elastic perturbations cannot establish themselves permanently in the atmosphere as the fast acoustic waves rapidly redistribute the associated energy and lead to an equilibrium state void of acoustic modes. Such an idea anticipates the existence of an unbounded physical space with a dominating dispersion effect. However any real physical as well as computational domain is necessarily bounded and the interaction of the acoustic waves with its boundary represents a serious problem from both analytical and numerical points of view, unless the domain is large enough with respect to the characteristic speed of sound in the fluid. Relevant discussion of these issues appears formally in Section 4.4, and, at a rigorous mathematical level, in Chapters 7, 8 below. As we shall see, the problem involves an effective interaction of two different time scales, namely the slow time motion of the background incompressible flow interacting with the fast time propagation of acoustic waves through the convective term in the momentum equation. This is an intrinsic *physical* feature that requires the use of adequate mathematical techniques in order to handle the fast time oscillations (see Chapter 9). In particular, such a problem lies beyond the scope of the standard methods based on formal asymptotic expansions.

The key idea pursued in this book is that besides justifying a number of important models, the asymptotic analysis carried out in a rigorous way provides a considerably improved insight into their structure. The purpose of this introductory chapter is to identify some of the basic problems arising in the asymptotic analysis of the complete Navier-Stokes-Fourier system along with the relevant limit equations. To begin, we introduce a rescaled system expressed in terms of dimensionless quantities and identify a sample of *characteristic numbers*. The central issue addressed in this study is the passage from *compressible* to *incompressible* fluid models. In particular, we always assume that the speed of sound dominates the characteristic speed of the fluid, the former approaching infinity in the asymptotic limit (see Chapter 5). In addition, we study the effect of strong stratification that is particularly relevant in some models arising in astrophysics (see Chapter 6). Related problems concerning the propagation of acoustic waves in large domains and their interaction with the physical boundary are discussed in Chapters 7, 8. Last but not least, we did not fail to notice a close relation between the asymptotic analysis performed in this book and the method of *acoustic analogies* used in engineering problems (see Chapter 9).

4.1 Scaling and scaled equations

For the physical systems studied in this book we recognize four fundamental dimensions: Time, Length, Mass, and Temperature. Each physical quantity that appears in the governing equations can be measured in units expressed as a product of fundamental ones.

The field equations of the Navier-Stokes-Fourier system in the form introduced in Chapter 1 do not reveal anything more than the balance laws of certain quantities characterizing the instantaneous state of a fluid. In order to get a somewhat deeper insight into the structure of possible solutions, we can identify *characteristic values* of relevant physical quantities: the *reference time* $T_{\rm ref}$, the *reference length* $L_{\rm ref}$, the *reference density* $\varrho_{\rm ref}$, the *reference temperature* $\vartheta_{\rm ref}$, together with the *reference velocity* $U_{\rm ref}$, and the characteristic values of other composed quantities $p_{\rm ref}$, $e_{\rm ref}$, $\mu_{\rm ref}$, $\eta_{\rm ref}$, $\kappa_{\rm ref}$, and the source terms $f_{\rm ref}$, $Q_{\rm ref}$. Introducing new independent and dependent variables $X' = X/X_{\rm ref}$ and omitting the primes in the resulting equations, we arrive at the following scaled system.

■ Scaled Navier-Stokes-Fourier System:

$$\mathrm{Sr}\ \partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{4.1}$$

$$\mathrm{Sr}\ \partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\mathrm{Ma}^2} \nabla_x p = \frac{1}{\mathrm{Re}} \mathrm{div}_x \mathbb{S} + \frac{1}{\mathrm{Fr}^2} \varrho \mathbf{f}, \tag{4.2}$$

$$\mathrm{Sr}\ \partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \frac{1}{\mathrm{Pe}} \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma + \mathrm{Hr} \varrho \frac{\mathcal{Q}}{\vartheta}, \tag{4.3}$$

together with the associated total energy balance

$$\mathrm{Sr}\ \frac{\mathrm{d}}{\mathrm{d}t} \int \left(\frac{\mathrm{Ma}^2}{2} \varrho |\mathbf{u}|^2 + \varrho e \right) \mathrm{d}x = \int \left(\frac{\mathrm{Ma}^2}{\mathrm{Fr}^2} \varrho \mathbf{f} \cdot \mathbf{u} + \mathrm{Hr} \varrho \mathcal{Q} \right) \mathrm{d}x. \tag{4.4}$$

Here, in accordance with the general principles discussed in Chapter 1, the thermodynamic functions $p = p(\varrho, \vartheta)$, $e = e(\varrho, \vartheta)$, and $s = s(\varrho, \vartheta)$ are interrelated through *Gibbs' equation*

$$\vartheta Ds(\varrho, \vartheta) = De(\varrho, \vartheta) + p(\varrho, \vartheta) D\Big(\frac{1}{\varrho}\Big), \tag{4.5}$$

while

$$\mathbb{S} = \mu\Big(\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\Big) + \eta \mathrm{div}_x \mathbf{u}\mathbb{I}, \tag{4.6}$$

$$\mathbf{q} = -\kappa \nabla_x \vartheta, \tag{4.7}$$

and

$$\sigma = \frac{1}{\vartheta}\Big(\frac{\mathrm{Ma}^2}{\mathrm{Re}}\mathbb{S} : \nabla_x \mathbf{u} - \frac{1}{\mathrm{Pe}}\frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\Big). \tag{4.8}$$

Note that relation (4.5) requires satisfaction of a natural compatibility condition

$$p_{\mathrm{ref}} = \varrho_{\mathrm{ref}} e_{\mathrm{ref}}. \tag{4.9}$$

The above procedure gives rise to a sample of dimensionless *characteristic numbers* listed below.

■ Symbol	■ Definition	■ Name
Sr	$L_{\mathrm{ref}}/(T_{\mathrm{ref}} U_{\mathrm{ref}})$	Strouhal number
Ma	$U_{\mathrm{ref}}/\sqrt{p_{\mathrm{ref}}/\varrho_{\mathrm{ref}}}$	Mach number
Re	$\varrho_{\mathrm{ref}} U_{\mathrm{ref}} L_{\mathrm{ref}}/\mu_{\mathrm{ref}}$	Reynolds number
Fr	$U_{\mathrm{ref}}/\sqrt{L_{\mathrm{ref}} f_{\mathrm{ref}}}$	Froude number
Pe	$p_{\mathrm{ref}} L_{\mathrm{ref}} U_{\mathrm{ref}}/(\vartheta_{\mathrm{ref}} \kappa_{\mathrm{ref}})$	Péclet number
Hr	$\varrho_{\mathrm{ref}} Q_{\mathrm{ref}} L_{\mathrm{ref}}/(p_{\mathrm{ref}} U_{\mathrm{ref}})$	Heat release parameter

The set of the chosen characteristic numbers is not unique, however, the maximal number of *independent* ones can be determined by means of Buckingham's Π-theorem (see Curtis et al. [51]).

Much of the subject to be studied in this book is motivated by the situation, where one or more of these parameters approach zero or infinity, and, consequently, the resulting equations contain singular terms. The *Strouhal number* Sr is often set to unity in applications; this implies that the characteristic time scale of flow field evolution equals the convection time scale $L_{\mathrm{ref}}/U_{\mathrm{ref}}$. Large *Reynolds number* characterizes turbulent flows, where the mathematical structure is far less understood than for the "classical" systems. Therefore we concentrate on a sample of interesting and physically relevant cases, with Sr = Re = 1, the characteristic features of which are shortly described in the rest of this chapter.

4.2 Low Mach number limits

In many real world applications, such as atmosphere-ocean flows, fluid flows in engineering devices and astrophysics, velocities are small compared with the speed of sound proportional to $1/\sqrt{\mathrm{Ma}}$ in the *scaled* NAVIER-STOKES-FOURIER SYSTEM. This fact has significant impact on both exact solutions to the governing equations and their numerical approximations. Physically, in the limit of vanishing flow velocity or infinitely fast speed of sound propagation, the elastic features of the fluid become negligible and sound-wave propagation insignificant. The low Mach number regime is particularly interesting when accompanied simultaneously with smallness of other dimensionless parameters such as *Froude*, *Reynolds*, and/or *Péclet numbers*. When the Mach number Ma approaches zero, the pressure is almost constant while the speed of sound tends to infinity. If, simultaneously, the temperature tends to a constant, the fluid is driven to incompressibility. If, in addition, the *Froude number* is small, specifically if $\mathrm{Fr} \approx \sqrt{\mathrm{Ma}}$, a formal asymptotic expansion produces a well-known model – the OBERBECK-BOUSSINESQ APPROXIMATION – probably the most widely used simplification in numerous problems in fluid dynamics (see also the introductory part of Chapter 5). An important consequence of the heating process is the appearance of a driving force in the target system, the size of which is proportional to the temperature.

In most applications, we have

$$\mathbf{f} = \nabla_x F,$$

where $F = F(x)$ is a given potential. Taking $\mathrm{Ma} = \varepsilon$, $\mathrm{Fr} = \sqrt{\varepsilon}$, and keeping all other characteristic numbers of order unity, we formally write

$$\begin{aligned} \varrho &= \overline{\varrho} + \varepsilon \varrho^{(1)} + \varepsilon^2 \varrho^{(2)} + \cdots , \\ \mathbf{u} &= \mathbf{U} + \varepsilon \mathbf{u}^{(1)} + \varepsilon^2 \mathbf{u}^{(2)} + \cdots , \\ \vartheta &= \overline{\vartheta} + \varepsilon \vartheta^{(1)} + \varepsilon^2 \vartheta^{(2)} + \cdots . \end{aligned} \tag{4.10}$$

Regrouping the scaled system with respect to powers of ε, we get, again formally comparing terms of the same order,

$$\nabla_x p(\overline{\varrho}, \overline{\vartheta}) = 0. \tag{4.11}$$

Of course, relation (4.11) *does not* imply that both $\overline{\varrho}$ and $\overline{\vartheta}$ must be constant; however, since we are primarily interested in solutions defined on large time intervals, the necessary uniform estimates on the velocity field have to be obtained from the dissipation equation introduced and discussed in Section 2.2.3. In particular, the entropy production rate $\sigma = \sigma_\varepsilon$ is to be kept small of order $\varepsilon^2 \approx \mathrm{Ma}^2$. Consequently, as seen from (4.7), (4.8), the quantity $\nabla_x \vartheta$ vanishes in the asymptotic limit $\varepsilon \to 0$. It is therefore natural to assume that $\overline{\vartheta}$ is a positive constant; whence, in agreement with (4.11),

$$\overline{\varrho} = \text{const in } \Omega$$

as soon as the pressure is a strictly monotone function of ϱ. The fact that the density ϱ and the temperature ϑ will be always considered in a vicinity of a *thermodynamic equilibrium* $(\overline{\varrho}, \overline{\vartheta})$ is an inevitable hypothesis in our approach to singular limits based on the concept of *weak solution* and energy estimates "in-the-large".

Neglecting all terms of order ε and higher in (4.1–4.4), we arrive at the following system of equations.

■ OBERBECK-BOUSSINESQ APPROXIMATION:

$$\operatorname{div}_x \mathbf{U} = 0, \tag{4.12}$$

$$\overline{\varrho}\Big(\partial_t \mathbf{U} + \operatorname{div}_x(\mathbf{U} \otimes \mathbf{U})\Big) + \nabla_x \Pi = \operatorname{div}_x \left(\mu(\overline{\vartheta})(\nabla_x \mathbf{U} + \nabla_x^T \mathbf{U})\right) + r \nabla_x F, \tag{4.13}$$

$$\overline{\varrho} c_p(\overline{\varrho}, \overline{\vartheta}) \Big(\partial_t \Theta + \operatorname{div}_x(\Theta \mathbf{U})\Big) - \operatorname{div}_x(G\mathbf{U}) - \operatorname{div}_x(\kappa(\overline{\vartheta})\nabla_x \Theta) = 0, \tag{4.14}$$

where

$$G = \overline{\varrho}\, \overline{\vartheta} \alpha(\overline{\varrho}, \overline{\vartheta}) F, \tag{4.15}$$

and

$$r + \overline{\varrho} \alpha(\overline{\varrho}, \overline{\vartheta}) \Theta = 0. \tag{4.16}$$

Here r can be identified with $\varrho^{(1)}$ modulo a multiple of F, while $\Theta = \vartheta^{(1)}$. The *specific heat at constant pressure* c_p is evaluated by means of the standard thermodynamic relation

$$c_p(\varrho, \vartheta) = \frac{\partial e}{\partial \vartheta}(\varrho, \vartheta) + \alpha(\varrho, \vartheta) \frac{\vartheta}{\varrho} \frac{\partial p}{\partial \vartheta}(\varrho, \vartheta), \tag{4.17}$$

where the *coefficient of thermal expansion* α reads

$$\alpha(\varrho, \vartheta) = \frac{1}{\varrho} \frac{\partial_\vartheta p}{\partial_\varrho p}(\varrho, \vartheta). \tag{4.18}$$

An interesting issue is a proper choice of the initial data for the limit system. Note that, in order to obtain a non-trivial dynamics, it is necessary to consider general $\varrho^{(1)}$, $\vartheta^{(1)}$, in particular, the initial values $\varrho^{(1)}(0, \cdot)$, $\vartheta^{(1)}(0, \cdot)$ must be allowed to be large. According to the standard terminology, such a stipulation corresponds to the so-called *ill-prepared initial data* in contrast with the *well-prepared data* for which

$$\frac{\varrho(0, \cdot) - \overline{\varrho}}{\varepsilon} \approx \varrho_0^{(1)}, \ \frac{\vartheta(0, \cdot) - \overline{\vartheta}}{\varepsilon} \approx \vartheta_0^{(1)} \text{ provided } \varepsilon \to 0,$$

where $\varrho_0^{(1)}$, $\vartheta_1^{(1)}$ are related to F through

$$\frac{\partial p}{\partial \varrho}(\overline{\varrho}, \overline{\vartheta}) \varrho_0^{(1)} + \frac{\partial p}{\partial \vartheta}(\overline{\varrho}, \overline{\vartheta}) \vartheta_0^{(1)} = \overline{\varrho} F$$

(cf. Theorem 5.3 in Chapter 5).

Moreover, as we shall see in Chapter 5 below, the initial distribution of the temperature Θ in (4.14) is determined in terms of *both* $\varrho^{(1)}(0,\cdot)$ and $\vartheta^{(1)}(0,\cdot)$. In particular, the knowledge of $\varrho^{(1)}$ – a quantity that "disappears" in the target system – is necessary in order to determine $\Theta \approx \vartheta^{(1)}$. The piece of information provided by the initial distribution of the temperature for the full Navier-Stokes-Fourier system is not transferred entirely on the target problem because of the initial-time *boundary layer*. This phenomenon is apparently related to the problem termed by physicists the *unsteady data adjustment* (see Zeytounian [205]). For further discussion see Section 5.5.

The low Mach number asymptotic limit in the regime of low stratification is studied in Chapter 5.

4.3 Strongly stratified flows

Stratified fluids whose densities, sound speed as well as other parameters are functions of a single depth coordinate occur widely in nature. Several so-called mesoscale regimes in the atmospheric modeling involve flows of strong stable stratification but weak rotation. Numerous observations, numerical experiments as well as purely theoretical studies to explain these phenomena have been recently surveyed in the monograph by Majda [147].

From the point of view of the mathematical theory discussed in Section 4.1, strong stratification corresponds to the choice

$$\mathrm{Ma} = \mathrm{Fr} = \varepsilon.$$

Similarly to the above, we write

$$\begin{aligned}
\varrho &= \tilde{\varrho} + \varepsilon\varrho^{(1)} + \varepsilon^2\varrho^{(2)} + \cdots ,\\
\mathbf{u} &= \mathbf{U} + \varepsilon\mathbf{u}^{(1)} + \varepsilon^2\mathbf{u}^{(2)} + \cdots ,\\
\vartheta &= \overline{\vartheta} + \varepsilon\vartheta^{(1)} + \varepsilon^2\vartheta^{(2)} + \cdots .
\end{aligned}$$

Comparing terms of the same order of the small parameter ε in the Navier-Stokes-Fourier System (4.1–4.4) we deduce

■ HYDROSTATIC BALANCE EQUATION:

$$\nabla_x p(\tilde{\varrho}, \overline{\vartheta}) = \tilde{\varrho}\nabla_x F, \tag{4.19}$$

where we have assumed the driving force in the form $\mathbf{f} = \nabla_x F$, $F = F(x_3)$ depending solely on the depth coordinate x_3. Here the temperature $\overline{\vartheta}$ is still assumed constant, while, in sharp contrast with (4.11), the equilibrium density $\tilde{\varrho}$ depends effectively on the depth (vertical) coordinate x_3.

Accordingly, the standard incompressibility conditions (4.12) has to be replaced by

■ ANELASTIC CONSTRAINT:

$$\mathrm{div}_x(\tilde{\varrho}\mathbf{U}) = 0 \tag{4.20}$$

– a counterpart to the equation of continuity in the asymptotic limit.

In order to identify the asymptotic form of the momentum equation, we assume, for a while, that the pressure p is given by the standard *perfect gas state equation*:

$$p(\varrho, \vartheta) = \mathrm{R}\varrho\vartheta. \tag{4.21}$$

Under these circumstances, the zeroth order terms in (4.2) give rise to

$$\begin{aligned} &\partial_t(\tilde{\varrho}\mathbf{U}) + \mathrm{div}_x(\tilde{\varrho}\mathbf{U}\otimes\mathbf{U}) + \tilde{\varrho}\nabla_x\Pi \\ &\quad = \mu(\overline{\vartheta})\Delta\mathbf{U} + \Big(\frac{1}{3}\mu(\overline{\vartheta}) + \eta(\overline{\vartheta})\Big)\nabla_x\mathrm{div}_x\mathbf{U} - \frac{\vartheta^{(2)}}{\overline{\vartheta}}\tilde{\varrho}\nabla_x F. \end{aligned} \tag{4.22}$$

Note that, similarly to Section 4.2, the quantities $\varrho^{(1)}$, $\vartheta^{(1)}$ satisfy the *Boussinesq relation*

$$\tilde{\varrho}\nabla_x\Big(\frac{\varrho^{(1)}}{\tilde{\varrho}}\Big) + \nabla_x\Big(\frac{\tilde{\varrho}}{\overline{\vartheta}}\vartheta^{(1)}\Big) = 0,$$

which, however, does not seem to be of any practical use here. Instead we have to determine $\vartheta^{(2)}$ by means of the entropy balance (4.3).

In the absence of the source $\mathcal{Q}$, comparing the zeroth order terms in (4.3) yields

$$\mathrm{div}_x(\tilde{\varrho}s(\tilde{\varrho}, \overline{\vartheta})\mathbf{U}) = 0.$$

However, this relation is compatible with (4.20) only if

$$U_3 \equiv 0. \tag{4.23}$$

In such a case, the system of equations (4.20–4.22) coincides with the so-called *layered two-dimensional incompressible flow equations in the limit of strong stratification* studied by Majda [147, Section 6.1]. The flow is layered horizontally, whereas the motion in each layer is governed by the incompressible NAVIER-STOKES EQUATIONS, the vertical stacking of the layers is determined through the hydrostatic balance, and the viscosity induces transfer of horizontal momentum through vertical variations of the horizontal velocity.

An even more complex problem arises when, simultaneously, the *Péclet number* Pe is supposed to be small, specifically, $\mathrm{Pe} = \varepsilon^2$. A direct inspection of the entropy balance equation (4.3) yields, to begin with,

$$\vartheta^{(1)} \equiv 0,$$

and, by comparison of the terms of zeroth order,

$$\tilde{\varrho}\nabla_x F \cdot \mathbf{U} + \kappa(\overline{\vartheta})\Delta\vartheta^{(2)} = 0. \tag{4.24}$$

Equations (4.20–4.22), together with (4.24), form a closed system introduced by Chandrasekhar [43] as a simple alternative to the OBERBECK-BOUSSINESQ APPROXIMATION when both *Froude* and *Péclet numbers* are small. More recently, Ligniéres [135] identified a similar system as a suitable model of flow dynamics in stellar radiative zones. Indeed, under these circumstances, the fluid behaves as a plasma characterized by the following features:

(i) a strong radiative transport predominates the molecular one; therefore the *Péclet number* is expected to be vanishingly small;
(ii) a strong stratification effect due to the enormous gravitational potential of gaseous celestial bodies determines many of the properties of the fluid in the large;
(iii) the convective motions are much slower than the speed rendering the *Mach number* small.

In conclusion, it is worth noting that system (4.20–4.22) represents the so-called ANELASTIC APPROXIMATION introduced by Ogura and Phillipps [167], and Lipps and Hemler [145]. The low Mach number limits for strongly stratified fluids are examined in Chapter 6.

4.4 Acoustic waves

Acoustic waves, as their proper name suggests, are intimately related to *compressibility* of the fluid and as such should definitely disappear in the incompressible limit regime. Accordingly, the impact of the acoustic waves on the fluid motion is neglected in a considerable number of practical applications. On the other hand, a fundamental issue is to understand the way the acoustic waves disappear and to what extent they may influence the motion of the fluid in the course of the asymptotic limit.

4.4.1 Low stratification

The so-called *acoustic equation* provides a useful link between the first-order terms $\varrho^{(1)}$, $\vartheta^{(1)}$, and the zeroth order velocity field $\mathbf{U}$ introduced in the formal asymptotic expansion (4.10). Introducing the fast time variable $\tau = t/\varepsilon$ and neglecting terms of order ε and higher in (4.1–4.3), we deduce

$$\left.\begin{array}{c} \partial_\tau \varrho^{(1)} + \operatorname{div}_x(\overline{\varrho}\mathbf{U}) = 0, \\ \partial_\tau(\overline{\varrho}\mathbf{U}) + \nabla_x\Big[\partial_\varrho p(\overline{\varrho},\overline{\vartheta})\varrho^{(1)} + \partial_\vartheta p(\overline{\varrho},\overline{\vartheta})\vartheta^{(1)} - \overline{\varrho}F\Big] = 0, \\ \partial_\tau\Big[\partial_\varrho s(\overline{\varrho},\overline{\vartheta})\varrho^{(1)} + \partial_\vartheta s(\overline{\varrho},\overline{\vartheta})\vartheta^{(1)}\Big] = 0. \end{array}\right\} \tag{4.25}$$

Thus after a simple manipulation we easily obtain

■ ACOUSTIC EQUATION:

$$\begin{aligned} \partial_\tau r + \mathrm{div}_x \mathbf{V} = 0, \\ \partial_\tau \mathbf{V} + \omega \nabla_x r = 0, \end{aligned} \tag{4.26}$$

where we have set

$$r = \frac{1}{\omega}\Big(\partial_\varrho p(\overline{\varrho}, \overline{\vartheta})\varrho^{(1)} + \partial_\vartheta p(\overline{\varrho}, \overline{\vartheta})\vartheta^{(1)} - \overline{\varrho} F \Big), \quad \mathbf{V} = \overline{\varrho}\mathbf{U},$$

$$\omega = \partial_\varrho p(\overline{\varrho}, \overline{\vartheta}) + \frac{|\partial_\vartheta p(\overline{\varrho}, \overline{\vartheta})|^2}{\overline{\varrho}^2 \partial_\vartheta s(\overline{\varrho}, \overline{\vartheta})}.$$

System (4.26) can be viewed as a wave equation, where the wave speed $\sqrt{\omega}$ is a real number as soon as the hypothesis of thermodynamic stability (1.44) holds. Moreover, the kernel $\mathcal{N}$ of the generator of the associated evolutionary system reads

$$\mathcal{N} = \{ (r, \mathbf{V}) \mid r = \mathrm{const}, \ \mathrm{div}_x \mathbf{V} = 0 \}. \tag{4.27}$$

Consequently, decomposing the vector field $\mathbf{V}$ in the form

$$\mathbf{V} = \underbrace{\mathbf{H}[\mathbf{V}]}_{\text{solenoidal part}} + \underbrace{\mathbf{H}^\perp[\mathbf{V}]}_{\text{gradient part}}, \text{ where } \mathrm{div}_x \mathbf{H}[\mathbf{V}] = 0, \ \mathbf{H}^\perp[\mathbf{V}] = \nabla_x \Psi$$

(cf. Section 10.6 and Theorem 10.12 in Appendix), system (4.26) can be recast as

$$\begin{aligned} \partial_\tau r + \Delta \Psi = 0, \\ \partial_\tau (\nabla_x \Psi) + \omega \nabla_x r = 0. \end{aligned} \tag{4.28}$$

Returning to the original time variable $t = \varepsilon \tau$ we infer that the rapidly oscillating acoustic waves are supported by the gradient part of the fluid velocity, while the time evolution of the solenoidal component of the velocity field remains essentially constant in time, being determined by its initial distribution. In terms of stability of the original system with respect to the parameter $\varepsilon \to 0$, this implies *strong* convergence of the solenoidal part $\mathbf{H}[\mathbf{u}_\varepsilon]$, while the gradient component $\mathbf{H}^\perp[\mathbf{u}_\varepsilon]$ converges, in general, only *weakly* with respect to time. Here and in what follows, the subscript ε refers to quantities satisfying the scaled primitive system (4.1–4.3). The hypothetical oscillations of the gradient part of the velocity field reveal one of the fundamental difficulties in the analysis of asymptotic limits in the present study, namely the problem of "weak compactness" of the convective term $\mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon)$.

Writing

$$\begin{aligned} \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) &\approx \mathrm{div}_x(\overline{\varrho} \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) \\ &= \overline{\varrho} \ \mathrm{div}_x(\mathbf{u}_\varepsilon \otimes \mathbf{H}[\mathbf{u}_\varepsilon]) + \overline{\varrho} \ \mathrm{div}_x(\mathbf{H}[\mathbf{u}_\varepsilon] \otimes \nabla_x \Psi_\varepsilon) \\ &\quad + \frac{1}{2} \overline{\varrho} \nabla_x |\nabla_x \Psi_\varepsilon|^2 + \overline{\varrho} \ \Delta \Psi_\varepsilon \nabla_x \Psi_\varepsilon, \end{aligned}$$

where we have set $\Psi_\varepsilon = \mathbf{H}^\perp[\mathbf{u}_\varepsilon]$, we realize that the only problematic term is $\Delta\Psi_\varepsilon\nabla_x\Psi_\varepsilon$, as the remaining quantities are either weakly pre-compact or can be written as a gradient of a scalar function, therefore irrelevant in the target system (4.12), (4.13), where they can be incorporated in the pressure.

A bit naive approach to solving this problem would be to rewrite the material derivative in (4.13) by means of (4.12) in the form

$$\partial_t(\varrho_\varepsilon\mathbf{u}_\varepsilon) + \mathrm{div}_x(\varrho_\varepsilon\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) = \varrho_\varepsilon\partial_t\mathbf{u}_\varepsilon + \varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla_x\mathbf{u}_\varepsilon \approx \overline{\varrho}\partial_t\mathbf{u}_\varepsilon + \overline{\varrho}\mathbf{u}_\varepsilon\cdot\nabla_x\mathbf{u}_\varepsilon$$
$$\approx \overline{\varrho}\partial_t\mathbf{u}_\varepsilon + \overline{\varrho}\mathbf{u}_\varepsilon\cdot\nabla_x\mathbf{H}[\mathbf{u}_\varepsilon] + \overline{\varrho}\mathbf{H}[\mathbf{u}_\varepsilon]\cdot\nabla_x\mathbf{H}^\perp[\mathbf{u}_\varepsilon] + \overline{\varrho}\frac{1}{2}\nabla_x|\nabla_x\Psi_\varepsilon|^2.$$

Unfortunately, in the framework of the weak solutions, such a step is not allowed at least in a direct fashion.

Alternatively, we can use the acoustic equation (4.28) in order to see that

$$\Delta\Psi_\varepsilon\nabla_x\Psi_\varepsilon = -\partial_\tau(r_\varepsilon\nabla_x\Psi_\varepsilon) - \frac{\omega}{2}\nabla_x r_\varepsilon^2 = -\varepsilon\partial_t(r_\varepsilon\nabla_x\Psi_\varepsilon) - \frac{\omega}{2}\nabla_x r_\varepsilon^2,$$

where the former term on the right-hand side is pre-compact (in the sense of distributions) while the latter is a gradient. This is the heart of the so-called local method developed in the context of isentropic fluid flows by Lions and Masmoudi [141].

4.4.2 Strong stratification

Propagation of the acoustic waves becomes more complex in the case of a strongly stratified fluid discussed in Section 4.3. Similarly to Section 4.4.1, introducing the fast time variable $\tau = t/\varepsilon$ and supposing the pressure in the form $p = \varrho\vartheta$, we deduce the *acoustic equation* in the form

$$\begin{aligned} \partial_\tau r + \frac{1}{\tilde{\varrho}}\mathrm{div}_x\mathbf{V} &= 0, \\ \partial_\tau\mathbf{V} + \overline{\vartheta}\ \tilde{\varrho}\nabla_x r + \nabla_x(\tilde{\varrho}\vartheta^{(1)}) &= 0, \end{aligned} \tag{4.29}$$

where we have set $r = \varrho^{(1)}/\tilde{\varrho}$, $\mathbf{V} = \tilde{\varrho}\mathbf{U}$.

Assuming, in addition, that $\mathrm{Pe} = \varepsilon^2$ we deduce from (4.3) that $\vartheta^{(1)} \equiv 0$; whence equation (4.29) reduces to

■ Stratified Acoustic Equation:

$$\left.\begin{aligned} \partial_\tau r + \frac{1}{\tilde{\varrho}}\mathrm{div}_x\mathbf{V} &= 0, \\ \partial_\tau\mathbf{V} + \overline{\vartheta}\ \tilde{\varrho}\nabla_x r &= 0. \end{aligned}\right\} \tag{4.30}$$

Apparently, in sharp contrast with (4.26), the wave speed in (4.30) depends effectively on the vertical coordinate x_3.

4.4.3 Attenuation of acoustic waves

There are essentially three rather different explanations why the amplitude of the acoustic waves should be negligible.

Well-prepared vs. ill-prepared initial data. For the sake of simplicity, assume that $F = 0$ in (4.25). A proper choice of the initial data for the primitive system can eliminate the effect of acoustic waves as the acoustic equation preserves the norm in the associated energy space. More specifically, taking

$$\varrho^{(1)}(0,\cdot) \approx \vartheta^{(1)}(0,\cdot) \approx 0,\ \mathbf{U}(0,\cdot) \approx \mathbf{V},$$

where $\mathbf{V}$ is a solenoidal function, we easily observe that the amplitude of the acoustic waves remains small uniformly in time. As a matter of fact, the problem is more complex, as the "real" acoustic equation obtained in the course of asymptotic analysis contains forcing terms of order ε, that are not negligible in the "slow" time of the limit system. These issues are discussed in detail in Chapter 9.

Moreover, we point out that, in order to obtain an interesting limit problem, we need

$$\vartheta^{(1)} \approx \Theta$$

to be large (see Section 4.2). Consequently, the initial data for the primitive system considered in this book are always *ill-prepared*, meaning compatible with the presence of large amplitude acoustic waves.

The effect of the kinematic boundary. Although it is sometimes convenient to investigate a fluid confined to an unbounded spatial domain, any *realistic* physical space is necessarily bounded. Accordingly, the interaction of the acoustic waves with the boundary of the domain occupied by the fluid represents an intrinsic feature of any incompressible limit problem.

Viscous fluids adhere completely to the boundary. That means, if the latter is at rest, the associated velocity field $\mathbf{u}$ satisfies the *no-slip boundary condition*

$$\mathbf{u}|_{\partial\Omega} = 0.$$

The no-slip boundary condition, however, is not compatible with free propagation of acoustic waves, unless the boundary is flat or satisfies very particular geometrical constraints. Consequently, a part of the acoustic energy is dissipated through a boundary layer causing a uniform time decay of the amplitude of acoustic waves. Such a situation is discussed in Chapter 7.

Dispersion of the acoustic waves on large domains. As already pointed out, *realistic* physical domains are always bounded. However, it is still reasonable to consider the situation when the diameter of the domain is sufficiently large with respect to the characteristic speed of sound in the fluid. The acoustic waves quickly redistribute the energy and, leaving a fixed bounded subset of the physical space, they will be reflected by the boundary but never come back in a finite lapse of time as the boundary is far away. In practice, such a problem can be equivalently posed on the whole space $\mathbb{R}^3$. Accordingly, the gradient component of the velocity field decays to zero locally in space with growing time. This problem is analyzed in detail in Chapter 8.

4.5 Acoustic analogies

The mathematical simulation of aeroacoustic sound presents in many cases numerous technical problems related to modeling of its generation and propagation. Its importance for diverse industrial applications is nowadays without any doubt in view of various demands in relation to user comfort or environmental regulations. A few examples where aeroacoustics enters into the game are the sounds produced by jet engines of an airliner, the noise produced in high speed trains and cars, wind noise around buildings, ventilator noise in various household appliances, etc.

The departure point of many methods of acoustic simulations (at least those called hybrid methods) is *Lighthill's theory* [133], [134]. The starting point in Lighthill's approach is the system of NAVIER-STOKES EQUATIONS describing the motion of a viscous compressible gas in *isentropic regime*, with unknown functions density ϱ and velocity $\mathbf{u}$. The system of equations reads:

$$\begin{aligned} \partial_t \varrho + \mathrm{div}_x \varrho \mathbf{u} &= 0, \\ \partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p &= \mathrm{div}_x \mathbb{S} + \varrho \mathbf{f}, \end{aligned} \tag{4.31}$$

where $p = p(\varrho)$, and

$$\mathbb{S} = \mu(\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I}) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mu > 0, \ \eta \geq 0.$$

We can rewrite this system in the form

$$\begin{aligned} \partial_t R + \mathrm{div}_x \mathbf{Q} &= 0, \\ \partial_t \mathbf{Q} + \omega \nabla_x R &= \mathbf{F} - \mathrm{div}_x \mathbb{T}, \end{aligned} \tag{4.32}$$

where

$$\mathbf{Q} = \varrho \mathbf{u}, \quad R = \varrho - \overline{\varrho} \tag{4.33}$$

are the momentum and the density fluctuations from the basic constant density distribution $\overline{\varrho}$ of the background flow. Moreover, we have set

$$\begin{aligned} \omega &= \tfrac{\partial p}{\partial \varrho}(\overline{\varrho}) > 0, \quad \mathbf{F} = \varrho \mathbf{f}, \\ \mathbb{T} &= \varrho \mathbf{u} \otimes \mathbf{u} + \Big(p - \omega(\varrho - \overline{\varrho})\Big)\mathbb{I} - \mathbb{S}. \end{aligned} \tag{4.34}$$

The reader will have noticed an apparent similarity of system (4.32) to the *acoustic equation* discussed in the previous part. Condition $\omega > 0$ is an analogue of the hypothesis of thermodynamics stability (3.10) expressing positive compressibility property of the fluid, typically a gas.

Taking the time derivative of the first equation in (4.32) and the divergence of the second one, we convert the system to a "genuine" wave equation

$$\partial_t^2 R - \omega \Delta_x R = -\mathrm{div}_x \mathbf{F} + \mathrm{div}_x(\mathrm{div}_x\ \mathbb{T}), \tag{4.35}$$

with wave speed $\sqrt{\omega}$. The viscous component is often neglected in $\mathbb{T}$ because of the considerable high Reynolds number of typical fluid regimes.

The main idea behind the method of *acoustic analogies* is to view system (4.32), or, equivalently (4.35), as a linear wave equation supplemented with a source term represented by the quantity on the right-hand side. In contrast with the original problem, the source term is assumed to be known or at least it can be determined by solving a kind of simplified problem. Lighthill himself completed system (4.32) adding extra terms corresponding to acoustic sources of different types. The resulting problem in the simplest possible form captures the basic acoustic phenomena in fluids and may be written in the following form.

■ LIGHTHILL'S EQUATION:

$$\begin{aligned} \partial_t R + \mathrm{div}_x \mathbf{Q} &= \Sigma, \\ \partial_t \mathbf{Q} + \omega \nabla_x R &= \mathbf{F} - \mathrm{div}_x \mathbb{T}. \end{aligned} \tag{4.36}$$

According to Lighthill's interpretation, system (4.36) is a non-homogenous wave equation describing the acoustic waves (fluctuations of the density), where the terms on the right-hand side correspond to the *mononopolar* (Σ), *bipolar* ($-\mathbf{F}$), and *quadrupolar* ($\mathrm{div}_x \mathbb{T}$) acoustic sources, respectively. These source terms are considered as known and calculable from the background fluid flow field. The physical meaning of the source terms can be interpreted as follows:

- The first term Σ represents the acoustic sources created by the changes of control volumes due to changes of pressure or displacements of a rigid surface: this source can be schematically described via a particle whose diameter changes (pulsates) creating acoustic waves (density perturbations). It may be interpreted as a non-stationary injection of a fluid mass rate $\partial_t \Sigma$ per unit volume. The acoustic noise of a gun shot is a typical example.
- The second term $\mathbf{F}$ describes the acoustic sources due to external forces (usually resulting from the action of a solid surface on the fluid). These sources are responsible for the most of the acoustic noise in machines and ventilators.
- The third term $\mathrm{div}_x(\mathbb{T})$ is the acoustic source due to the turbulence and viscous effects in the background fluid flow which supports the density oscillations (acoustic waves). The noise of steady or non-steady jets in aero-acoustics is a typical example.

- The tensor $\mathbb{T}$ is called the *Lighthill tensor*. It is composed of three tensors whose physical interpretation is the following: the first term is the Reynolds tensor with components $\varrho u_i u_j$ describing the (nonlinear) turbulence effects, the term $(p-\omega(\varrho-\overline{\varrho}))\mathbb{I}$ expresses the entropy fluctuations and the third one is the viscous stress tensor $\mathbb{S}$.

The method for predicting noise based on *Lighthill's equation* is usually referred to as a hybrid method since noise generation and propagation are treated separately. The first step consists in using data provided by numerical simulations to identify the sound sources. The second step then consists in solving the wave equation (4.36) driven by these source terms to determine the sound radiation. The main advantage of this approach is that most of the conventional flow simulations can be used in the first step.

In practical numerical simulations, the *Lighthill tensor* is calculated from the velocity and density fields obtained by using various direct numerical methods and solvers for compressible Navier-Stokes equations. Then the acoustic effects are evaluated from *Lighthill's equation* by means of diverse direct numerical methods for solving the non-homogenous wave equations (see, e.g., Colonius [50], Mitchell et al. [158], Freud et al. [90], among others). For flows in the low Mach number regimes the direct simulations are costly, unstable, inefficient and non-reliable, essentially due to the presence of rapidly oscillating acoustic waves (with periods proportional to the Mach number) in the equations themselves (see the discussion in the previous part). Thus in the low Mach number regimes the acoustic analogies as *Lighthill's equation*, in combination with the incompressible flow solvers, give more reliable results, see [90].

Acoustic analogies, in particular Lighthill's approach in the low Mach number regime, will be discussed in Chapter 9.

4.6 Initial data

Motivated by the formal asymptotic expansion discussed in the previous sections, we consider the initial data for the scaled Navier-Stokes-Fourier system in the form

$$\varrho(0,\cdot)=\tilde{\varrho}+\varepsilon\varrho^{(1)}_{0,\varepsilon},\ \mathbf{u}(0,\varepsilon)=\mathbf{u}_{0,\varepsilon},\ \vartheta(0,\cdot)=\overline{\vartheta}+\varepsilon\vartheta^{(1)}_{0,\varepsilon},$$

where $\varepsilon=\mathrm{Ma}$, $\varrho^{(1)}_{0,\varepsilon}$, $\mathbf{u}_{0,\varepsilon}$, $\vartheta^{(1)}_{0,\varepsilon}$ are given functions, and $\tilde{\varrho}$, $\overline{\vartheta}$ represent an equilibrium state. Note that the apparent inconsistency in the form of the initial data is a consequence of the fact that smallness of the velocity with respect to the speed of sound is already incorporated in the system by scaling.

The initial data are termed *ill-prepared* if

$$\begin{aligned}\{\varrho^{(1)}_{0,\varepsilon}\}_{\varepsilon>0},\ \{\vartheta^{(1)}_{0,\varepsilon}\}_{\varepsilon>0} &\text{ are bounded in } L^p(\Omega),\\ \{\mathbf{u}_{0,\varepsilon}\}_{\varepsilon>0} &\text{ is bounded in } L^p(\Omega;\mathbb{R}^3)\end{aligned}$$

for a certain $p \geq 1$, typically $p = 2$ or even $p = \infty$. If, in addition,

$$\varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0, \ \vartheta^{(1)}_{0,\varepsilon} \to \vartheta^{(1)}_0, \ \mathbf{H}^\perp[\mathbf{u}_{0,\varepsilon}] \to 0 \text{ a.a. in } \Omega,$$

where $\varrho^{(1)}_0$, $\vartheta^{(1)}_0$ satisfy certain *compatibility conditions*, we say that the data are *well prepared*. For instance, in the situation described in Section 4.2, we require

$$\frac{\partial p}{\partial \varrho}(\overline{\varrho}, \overline{\vartheta})\varrho^{(1)}_0 + \frac{\partial p}{\partial \vartheta}(\overline{\varrho}, \overline{\vartheta})\vartheta^{(1)}_0 = \overline{\varrho} F.$$

In particular, the common definition of the well-prepared data, namely $\varrho^{(1)}_0 = \vartheta^{(1)}_0 = 0$, is recovered as a special case provided $F = 0$.

As observed in Section 4.4, the ill-prepared data are expected to generate large amplitude rapidly oscillating acoustic waves, while the well-prepared data are not. Alternatively, following Lighthill [135], we may say that the well-prepared data may be successfully handled by the linear theory, while the ill-prepared ones require the use of a nonlinear model.

4.7 A general approach to singular limits for the full Navier-Stokes-Fourier system

The overall strategy adopted in this book leans on the concept of *weak solutions* for both the primitive system and the associated asymptotic limit. The starting point is always the complete NAVIER-STOKES-FOURIER SYSTEM introduced in Chapter 1 and discussed in Chapters 2, 3, where one or several characteristic numbers listed in Section 4.1 are proportional to a small parameter. We focus on the passage to incompressible fluid models, therefore the *Mach number* Ma is always of order $\varepsilon \to 0$. On the contrary, the *Strouhal number* Sr as well as the *Reynolds number* Re are assumed to be of order 1. Consequently, the velocity of the fluid in the target system will satisfy a variant of incompressible (viscous) NAVIER-STOKES EQUATIONS coupled with a balance of the internal energy identified through a convenient choice of the characteristic numbers Fr and Pe.

Our theory applies to *dissipative fluid systems* that may be characterized through the following properties.

■ DISSIPATIVE FLUID SYSTEM:

(P1) The total mass of the fluid contained in the physical space Ω is constant at any time.

(P2) In the absence of external sources, the total energy of the fluid is constant or non-increasing in time.

(P3) The system produces entropy, in particular, the total entropy is a non-decreasing function of time. In addition, the system is thermodynamically stable, that means, the maximization of the total entropy over the set of all allowable states with the same total mass and energy delivers a unique equilibrium state provided the system is thermally and mechanically insulated.

The key tool for studying singular limits of dissipative fluid systems is the dissipation balance, or rather inequality, analogous to the corresponding equality introduced in (2.52). Neglecting, for simplicity, the source terms in the scaled system (4.1–4.3), we deduce

■ SCALED DISSIPATION INEQUALITY:

$$\int_\Omega \left(\frac{\mathrm{Ma}^2}{2} \varrho |\mathbf{u}|^2 + H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho}) \frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \right)(\tau, \cdot) \ \mathrm{d}x + \sigma \Big[[0, \tau] \times \overline{\Omega} \Big]$$

$$\leq \int_\Omega \left(\frac{\mathrm{Ma}^2}{2} \frac{|(\varrho \mathbf{u})_0|^2}{\varrho_0} + H_{\overline{\vartheta}}(\varrho_0, \vartheta_0) - (\varrho_0 - \overline{\varrho}) \frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \right) \mathrm{d}x \quad (4.37)$$

for a.a. $\tau \in (0, T)$,

$$\sigma \geq \frac{1}{\vartheta} \left(\frac{\mathrm{Ma}^2}{\mathrm{Re}} \mathbb{S} : \nabla_x \mathbf{u} - \frac{1}{\mathrm{Pe}} \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right), \quad (4.38)$$

where $H_{\overline{\vartheta}} = \varrho e - \overline{\vartheta} \varrho s$ is the Helmholtz function introduced in (2.48). Note that, in accordance with **(P2)**, there is an inequality sign in (4.37) because we admit systems that dissipate energy.

The quantities $\overline{\varrho}$ and $\overline{\vartheta}$ are positive constants characterizing the static distribution of the density and the absolute temperature, respectively. In accordance with **(P1)**, we have

$$\int_\Omega (\varrho(t, \cdot) - \overline{\varrho}) \ \mathrm{d}x = 0 \text{ for any } t \in [0, T],$$

while $\overline{\vartheta}$ is determined by the asymptotic value of the total energy for $t \to \infty$. In accordance with **(P3)**, the static state $(\overline{\varrho}, \overline{\vartheta})$ maximizes the entropy among all states with the same total mass and energy and solves the NAVIER-STOKES-FOURIER SYSTEM with the velocity field $\mathbf{u} = 0$, in other words, $(\overline{\varrho}, \overline{\vartheta})$ is an equilibrium state. In Chapter 6, the constant density equilibrium state $\overline{\varrho}$ is replaced by $\tilde{\varrho} = \tilde{\varrho}(x_3)$.

Basically all available bounds on the family of solutions to the scaled system are provided by (4.37), (4.38). If the Mach number Ma is proportional to a small parameter ε, and, simultaneously $\mathrm{Re} = \mathrm{Pe} \approx 1$, relations (4.37), (4.38) yield a bound on the gradient of the velocity field provided the integral on the right-hand side of (4.37) *divided on* ε^2 remains bounded.

On the other hand, seeing that

$$H_{\overline{\vartheta}}(\varrho_0, \vartheta_0) - (\varrho_0 - \overline{\varrho}) \frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \approx c \left(|\varrho_0 - \overline{\varrho}|^2 + |\vartheta_0 - \overline{\vartheta}|^2 \right)$$

at least in a neighborhood of the static state $(\overline{\varrho}, \overline{\vartheta})$, we conclude, in agreement with the formal asymptotic expansion discussed in Section 4.2, that the quantities

$$\varrho_{0,\varepsilon}^{(1)} = \frac{\varrho(0,\cdot) - \overline{\varrho}}{\varepsilon} \quad \text{and} \quad \vartheta_{0,\varepsilon}^{(1)} = \frac{\vartheta(0,\cdot) - \overline{\vartheta}}{\varepsilon}, \quad \text{and} \quad \mathbf{u}_{0,\varepsilon} = \mathbf{u}(0,\cdot)$$

have to be bounded uniformly for $\varepsilon \to 0$, or, in the terminology introduced in Section 4.6, the initial data must be at least ill-prepared.

As a direct consequence of the structural properties of $H_{\overline{\vartheta}}$ established in Section 2.2.3, it is not difficult to deduce from (4.37) that

$$\varrho^{(1)}(t,\cdot) = \frac{\varrho(t,\cdot) - \overline{\varrho}}{\varepsilon} \quad \text{and} \quad \vartheta^{(1)} = \frac{\vartheta(t,\cdot) - \overline{\vartheta}}{\varepsilon}$$

remain bounded, at least in $L^1(\Omega)$, uniformly for $t \in [0,T]$ and $\varepsilon \to 0$.

Now, we introduce the set of essential values $\mathcal{O}_{\rm ess} \subset (0,\infty)^2$,

$$\mathcal{O}_{\rm ess} := \left\{ (\varrho, \vartheta) \in \mathbb{R}^2 \;\middle|\; \overline{\varrho}/2 < \varrho < 2\overline{\varrho},\ \overline{\vartheta}/2 < \vartheta < 2\overline{\vartheta} \right\}, \tag{4.39}$$

together with its residual counterpart

$$\mathcal{O}_{\rm res} = (0,\infty)^2 \setminus \mathcal{O}_{\rm ess}. \tag{4.40}$$

Let $\{\varrho_\varepsilon\}_{\varepsilon>0}$, $\{\vartheta_\varepsilon\}_{\varepsilon>0}$ be a family of solutions to a scaled Navier-Stokes-Fourier system. In agreement with (4.39), (4.40), we define the *essential set* and *residual set* of points $(t,x) \in (0,T) \times \Omega$ as follows.

■ Essential and Residual Sets:

$$\mathcal{M}^\varepsilon_{\rm ess} \subset (0,T) \times \Omega,$$

$$\mathcal{M}^\varepsilon_{\rm ess} = \{ (t,x) \in (0,T) \times \Omega \mid (\varrho_\varepsilon(t,x), \vartheta_\varepsilon(t,x)) \in \mathcal{O}_{\rm ess} \}, \tag{4.41}$$

$$\mathcal{M}^\varepsilon_{\rm res} = ((0,T) \times \Omega) \setminus \mathcal{M}^\varepsilon_{\rm ess}. \tag{4.42}$$

We point out that $\mathcal{O}_{\rm ess}$, $\mathcal{O}_{\rm res}$ are *fixed* subsets of $(0,\infty)^2$, while $\mathcal{M}^\varepsilon_{\rm ess}$, $\mathcal{M}^\varepsilon_{\rm res}$ are measurable subsets of the time-space cylinder $(0,T) \times \Omega$ *depending* on ϱ_ε, ϑ_ε.

It is also convenient to introduce the "projection" of the set $\mathcal{M}^\varepsilon_{\rm ess}$ for a fixed time $t \in [0,T]$,

$$\mathcal{M}^\varepsilon_{\rm ess}[t] = \{ x \in \Omega \mid (t,x) \in \mathcal{M}^\varepsilon_{\rm ess} \}$$

and

$$\mathcal{M}^\varepsilon_{\rm res}[t] = \Omega \setminus \mathcal{M}^\varepsilon_{\rm ess}[t], \tag{4.43}$$

where both are measurable subsets of Ω for a.a. $t \in (0,T)$.

Finally, each measurable function h can be decomposed as

$$h = [h]_{\rm ess} + [h]_{\rm res}, \tag{4.44}$$

where we set

$$[h]_{\rm ess} = h \ 1_{\mathcal{M}^\varepsilon_{\rm ess}}, \ [h]_{\rm res} = h \ 1_{\mathcal{M}^\varepsilon_{\rm res}} = h - [h]_{\rm ess}. \tag{4.45}$$

Of course, we should always keep in mind that such a decomposition depends on the actual values of ϱ_ε, ϑ_ε.

The specific choice of $\mathcal{O}_{\rm ess}$ is not important. We can take $\mathcal{O}_{\rm ess} = \mathcal{U}$, where $\mathcal{U} \subset \overline{\mathcal{U}} \subset (0,\infty)^2$ is a bounded open neighborhood of the equilibrium state $(\overline{\varrho}, \overline{\vartheta})$. A general idea exploited in this book asserts that the "essential" component $[h]_{\rm ess}$ carries all information necessary in the limit process, while its "residual" counterpart $[h]_{\rm res}$ vanishes in the asymptotic limit for $\varepsilon \to 0$. In particular, the Lebesgue measure of the residual sets $|\mathcal{M}_{\rm res}[t]|$ becomes small uniformly in $t \in (0,T)$ for small values of ε.

Another characteristic feature of our approach is that the entropy production rate σ is small, specifically of order ε^2, in the low Mach number limit. Accordingly, in contrast with the primitive Navier-Stokes-Fourier system, the target problem can be expressed in terms of equations rather than inequalities. The ill-prepared data, for which the perturbation of the equilibrium state is proportional to the Mach number, represent a sufficiently rich scaling leading to non-trivial target problems.

Chapter 5

Singular Limits – Low Stratification

This chapter develops the general ideas discussed in Section 4.2 focusing on the singular limits characterized by the spatially homogeneous (constant) distribution of the limit density. We start with the scaled NAVIER-STOKES-FOURIER SYSTEM introduced in Section 4.1 as a *primitive system*, where we take the *Mach number* Ma proportional to a small parameter ε,

$$\mathrm{Ma} = \varepsilon, \text{ with } \varepsilon \to 0.$$

In addition, we assume that the external sources of mechanical energy are small, in particular,

$$\frac{\mathrm{Ma}}{\mathrm{Fr}} \to 0.$$

Specifically, we focus on the case

$$\mathrm{Fr} = \sqrt{\varepsilon}$$

corresponding to the *low stratification* of the fluid matter provided $\mathbf{f}$ is proportional to the gravitational force. Keeping the remaining characteristic numbers of order unity, we recover the well-known OBERBECK-BOUSSINESQ APPROXIMATION as a target problem in the asymptotic limit $\varepsilon \to 0$. As a byproduct of asymptotic analysis, we discover a variational formulation of *Lighthill's acoustic equation* and discuss the effective form of the acoustic sources in the *low Mach number* regime.

The overall strategy adopted in this chapter is somehow different from the remaining part of the book. We abandon the standard mathematical scheme of theorems followed by proofs and rather concentrate on a general approach, where hypotheses are made when necessary and goals determine the appropriate methods. The final conclusion is then stated in full rigor in Section 5.5. The reader preferring the traditional way of presentation is recommended to consult Section 5.5 first.

In accordance with the general hypotheses discussed above, the scaled NAVIER-STOKES-FOURIER SYSTEM introduced in Section 4.1 can be written in the following form.

■ PRIMITIVE SYSTEM:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{5.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S} + \frac{1}{\varepsilon} \varrho \nabla_x F, \tag{5.2}$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x \Big(\varrho s(\varrho, \vartheta) \mathbf{u} \Big) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma_\varepsilon, \tag{5.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{\varepsilon^2}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varepsilon \varrho F \right) \mathrm{d}x = 0, \tag{5.4}$$

where, similarly to Section 1.4, the viscous stress tensor is given through *Newton's law*

$$\mathbb{S} = \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta) \Big(\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \Big) + \eta(\vartheta) \mathrm{div}_x \mathbf{u} \mathbb{I}, \tag{5.5}$$

the heat flux obeys *Fourier's law*

$$\mathbf{q} = \mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta) \nabla_x \vartheta, \tag{5.6}$$

while the volumetric *entropy production rate* is represented by a non-negative measure σ_ε satisfying

$$\sigma_\varepsilon \geq \frac{1}{\vartheta} \Big(\varepsilon^2 \mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \Big). \tag{5.7}$$

Note that for the total energy balance (5.4) to be compatible with equations (5.1–5.3), system (5.1–5.4) must be supplemented by a suitable set of boundary conditions to be specified below.

Similarly to Section 4.2, we write

$$\varrho = \overline{\varrho} + \varepsilon \varrho^{(1)} + \varepsilon^2 \varrho^{(2)} + \cdots ,$$
$$\mathbf{u} = \mathbf{U} + \varepsilon \mathbf{u}^{(1)} + \varepsilon^2 \mathbf{u}^{(2)} + \cdots ,$$
$$\vartheta = \overline{\vartheta} + \varepsilon \vartheta^{(1)} + \varepsilon^2 \vartheta^{(2)} + \cdots .$$

Grouping equations (5.1–5.4) with respect to powers of ε, and dropping terms containing powers of ε higher than zero in (5.1), (5.2), we formally obtain

$$\mathrm{div}_x \mathbf{U} = 0, \tag{5.8}$$

$$\overline{\varrho} \Big(\partial_t \mathbf{U} + \mathrm{div}_x(\mathbf{U} \otimes \mathbf{U}) \Big) + \nabla_x \Pi = \mathrm{div}_x \left(\mu(\overline{\vartheta})(\nabla_x \mathbf{U} + \nabla_x^\perp \mathbf{U}) \right) + r \nabla_x F \tag{5.9}$$

with a suitable "pressure" or, more correctly, normal stress represented by a scalar function Π, where $r = \varrho^{(1)} + \Phi(F)$ for a continuous function Φ. Note that the component $\Phi(F)\nabla_x F$ can always be incorporated in the pressure gradient $\nabla_x \Pi$.

In order to establish a relation between $\varrho^{(1)}$ and $\vartheta^{(1)}$, we use (5.2) to deduce

$$\nabla_x \Big(\frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \varrho^{(1)} + \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \vartheta^{(1)} \Big) = \overline{\varrho} \nabla_x F,$$

therefore

$$\varrho^{(1)} + \frac{\partial_\vartheta p(\overline{\varrho}, \overline{\vartheta})}{\partial_\varrho p(\overline{\varrho}, \overline{\vartheta})} \vartheta^{(1)} = \frac{\overline{\varrho}}{\partial_\varrho p(\overline{\varrho}, \overline{\vartheta})} F + h(t) \tag{5.10}$$

for a certain spatially homogeneous function h.

In a similar way, the entropy balance equation (5.3) gives rise to

$$\begin{aligned} &\overline{\varrho} \partial_t \left(\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \varrho^{(1)} + \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \vartheta^{(1)} \right) \\ &+ \overline{\varrho} \mathrm{div}_x \left[\left(\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \varrho^{(1)} + \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \vartheta^{(1)} \right) \mathbf{U} \right] - \mathrm{div}_x \left(\frac{\kappa(\overline{\vartheta})}{\overline{\vartheta}} \nabla_x \vartheta^{(1)} \right) = 0. \end{aligned} \tag{5.11}$$

Supposing the "conservative" boundary conditions

$$\mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \nabla_x \vartheta^{(1)} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

we can combine (5.10) with (5.11) to obtain

$$\overline{\varrho} c_p(\overline{\varrho}, \overline{\vartheta}) \Big(\partial_t \Theta + \mathrm{div}_x(\Theta \mathbf{U}) \Big) - \mathrm{div}_x(G\mathbf{U}) - \mathrm{div}_x(\kappa(\overline{\vartheta}) \nabla_x \Theta) = 0, \tag{5.12}$$

where we have set

$$\Theta = \vartheta^{(1)},$$

and

$$G = \overline{\varrho} \ \overline{\vartheta} \alpha(\overline{\varrho}, \overline{\vartheta}) F. \tag{5.13}$$

We recall that the physical constants α, c_p have been introduced in (4.17), (4.18).

Moreover, equality (5.10) takes the form of

■ Boussinesq Relation:

$$r + \overline{\varrho} \alpha(\overline{\varrho}, \overline{\vartheta}) \Theta = 0, \tag{5.14}$$

where r is the same as in equation (5.9).

The system of equations (5.8), (5.9), (5.12), together with (5.14), is the well-known Oberbeck-Boussinesq Approximation having a wide range of applications in geophysical models, meteorology, and astrophysics already discussed in Section 4.1 (see also the survey paper by Zeytounian [205]).

The main goal of the present chapter is to provide a rigorous justification of the formal procedure discussed above in terms of the asymptotic limit of solutions to system (5.1–5.4). Accordingly, there are three main topics to be addressed:

- Identifying a suitable set of physically relevant hypotheses, under which the primitive system (5.1–5.4) possesses a global in time solution $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}$ for any $\varepsilon > 0$ in the spirit of Theorem 3.1.
- Uniform bounds on the quantities

$$\mathbf{u}_\varepsilon, \varrho_\varepsilon, \vartheta_\varepsilon \text{ as well as } \varrho_\varepsilon^{(1)} = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon},\ \vartheta_\varepsilon^{(1)} = \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}$$

  independent of $\varepsilon \to 0$.
- Analysis of oscillations of the acoustic waves represented by the gradient component in the Helmholtz decomposition of the velocity field $\mathbf{u}_\varepsilon$. Since the momentum equation (5.2) contains a singular term proportional to the pressure gradient, we do not expect any uniform estimates on the gradient part of the time derivative $\partial_t(\varrho\mathbf{u})$ not even in a very weak sense.

5.1 Hypotheses and global existence for the primitive system

The existence theory developed in Chapter 3 can be applied to the scaled system (5.1–5.4). In order to avoid unnecessary technical details in the analysis of the asymptotic limit, the hypotheses listed below are far less general than those used in Theorem 3.1.

5.1.1 Hypotheses

We assume that the fluid occupies a bounded domain $\Omega \subset \mathbb{R}^3$. In order to eliminate the effect of a boundary layer on propagation of the acoustic waves, we suppose that the velocity field $\mathbf{u}$ satisfies the *complete slip boundary conditions*

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0,\ \mathbb{S}\mathbf{n} \times \mathbf{n}|_{\partial\Omega} = 0. \tag{5.15}$$

Although such a stipulation may be at odds with practical experience in many models, it is still physically relevant and mathematically convenient. The more realistic *no-slip* boundary conditions are examined in Chapter 7.

In agreement with (5.4), the total energy of the fluid is supposed to be a constant of motion, in particular, the boundary of the physical space is thermally insulated, meaning,

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{5.16}$$

The structural restrictions imposed on the thermodynamic functions p, e, s as well as the transport coefficients μ, η, and κ are motivated by the existence theory established in Chapter 3. Specifically, we set

$$p(\varrho,\vartheta) = p_M(\varrho,\vartheta) + p_R(\vartheta),\ p_M = \vartheta^{\frac{5}{2}} P\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big),\ p_R = \frac{a}{3}\vartheta^4,\ a > 0, \tag{5.17}$$

$$e(\varrho,\vartheta) = e_M(\varrho,\vartheta) + e_R(\varrho,\vartheta),\ e_M = \frac{3}{2}\frac{\vartheta^{\frac{5}{2}}}{\varrho} P\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big),\ e_R = a\frac{\vartheta^4}{\varrho}, \tag{5.18}$$

and

$$s(\varrho,\vartheta) = s_M(\varrho,\vartheta) + s_R(\varrho,\vartheta),\ s_M(\varrho,\vartheta) = S\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big),\ s_R = \frac{4}{3}a\frac{\vartheta^3}{\varrho}, \tag{5.19}$$

where

$$S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z^2} \text{ for all } Z > 0. \tag{5.20}$$

Furthermore, in order to comply with the *hypothesis of thermodynamic stability* formulated in (1.44), we assume $P \in C^1[0,\infty) \cap C^2(0,\infty)$,

$$P(0) = 0,\ P'(Z) > 0 \text{ for all } Z \geq 0, \tag{5.21}$$

$$0 < \frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z} \leq \sup_{z>0} \frac{\frac{5}{3}P(z) - zP'(z)}{z} < \infty, \tag{5.22}$$

and, similarly to (2.44),

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{\frac{5}{3}}} = p_\infty > 0. \tag{5.23}$$

The reader may consult Chapter 1 for the physical background of the above assumptions. As a matter of fact, the presence of the radiative components p_R, e_R, and s_R is not necessary in order to perform the low Mach number limit. On the other hand, the specific form of the molecular pressure p_M, in particular (5.23), provides indispensable uniform bounds and cannot be relaxed. Hypotheses (5.17–5.23) are more restrictive than in Theorem 3.1.

For the sake of simplicity, the transport coefficients μ, η, and κ are assumed to be continuously differentiable functions of the temperature ϑ satisfying the growth restrictions

$$\left.\begin{aligned} 0 < \underline{\mu}(1+\vartheta) \leq \mu(\vartheta) \leq \overline{\mu}(1+\vartheta), \\ 0 \leq \eta(\vartheta) \leq \overline{\eta}(1+\vartheta) \end{aligned}\right\} \text{ for all } \vartheta \geq 0, \tag{5.24}$$

$$0 < \underline{\kappa}(1+\vartheta^3) \leq \kappa(\vartheta) \leq \overline{\kappa}(1+\vartheta^3) \text{ for all } \vartheta \geq 0, \tag{5.25}$$

where $\underline{\mu}$, $\overline{\mu}$, $\overline{\eta}$, $\underline{\kappa}$, and $\overline{\kappa}$ are positive constants. The linear dependence of the viscosity coefficients on ϑ facilitates considerably the analysis and is still physically relevant as the so-called hard sphere model. On the other hand, the theory developed in this chapter can accommodate the whole range of transport coefficients specified in (3.17–3.23).

The initial data are taken in the form

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon} = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon},\ \mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon},\ \vartheta(0,\cdot) = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon}, \tag{5.26}$$

where

$$\overline{\varrho} > 0,\ \overline{\vartheta} > 0,\ \int_\Omega \varrho^{(1)}_{0,\varepsilon}\ \mathrm{d}x = 0 \text{ for all } \varepsilon > 0. \tag{5.27}$$

5.1.2 Global-in-time solutions

The following result may be viewed as a straightforward corollary of Theorem 3.1:

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$. Assume that p, e, s satisfy hypotheses* (5.17–5.23)*, and the transport coefficients μ, η, and κ meet the growth restrictions* (5.24), (5.25)*. Let the initial data be given through* (5.26), (5.27)*, where $\varrho^{(1)}_{0,\varepsilon}$, $\mathbf{u}_{0,\varepsilon}$, $\vartheta^{(1)}_{0,\varepsilon}$ are bounded measurable functions, and let $F \in W^{1,\infty}(\Omega)$.*

Then, for any $\varepsilon > 0$ so small that the initial data $\varrho_{0,\varepsilon}$ and $\vartheta_{0,\varepsilon}$ are strictly positive, there exists a weak solution $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}$ to the Navier-Stokes-Fourier system (5.1–5.7) *on the set $(0,T)\times\Omega$, supplemented with the boundary conditions* (5.15), (5.16)*, and the initial conditions* (5.26)*. More specifically, we have:*

- $$\int_0^T \int_\Omega \varrho_\varepsilon B(\varrho_\varepsilon)\Big(\partial_t \varphi + \mathbf{u}_\varepsilon \cdot \nabla_x \varphi\Big)\ \mathrm{d}x\ \mathrm{d}t = \int_0^T \int_\Omega b(\varrho_\varepsilon)\mathrm{div}_x \mathbf{u}_\varepsilon \varphi\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega \varrho_{0,\varepsilon} B(\varrho_{0,\varepsilon})\varphi(0,\cdot)\ \mathrm{d}x \tag{5.28}$$

  for any b as in (2.3) *and any $\varphi \in C^\infty_c([0,T)\times\overline{\Omega})$;*

- $$\int_0^T \int_\Omega \Big(\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \varphi + \varrho_\varepsilon[\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon] : \nabla_x \varphi + \frac{1}{\varepsilon^2} p(\varrho_\varepsilon, \vartheta_\varepsilon)\mathrm{div}_x \varphi\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{5.29}$$
  $$= \int_0^T \int_\Omega \Big(\mathbb{S}_\varepsilon : \nabla_x \varphi - \frac{1}{\varepsilon}\varrho_\varepsilon \nabla_x F \cdot \varphi\Big)\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega (\varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon})\cdot\varphi(0,\cdot)\ \mathrm{d}x$$

  for any test function $\varphi \in C^\infty_c([0,T)\times\overline{\Omega};\mathbb{R}^3)$, $\varphi\cdot\mathbf{n}|_{\partial\Omega} = 0$;

- $$\int_\Omega \Big(\frac{\varepsilon^2}{2}\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon) - \varepsilon \varrho_\varepsilon F\Big)(t)\ \mathrm{d}x \tag{5.30}$$
  $$= \int_\Omega \Big(\frac{\varepsilon^2}{2}\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2 + \varrho_{0,\varepsilon} e(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) - \varepsilon \varrho_\varepsilon F\Big)\ \mathrm{d}x \text{ for a.a. } t \in (0,T);$$

- $$\int_0^T \int_\Omega \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)\Big(\partial_t \varphi + \mathbf{u}_\varepsilon \cdot \nabla_x \varphi\Big)\ \mathrm{d}x\ \mathrm{d}t + \int_0^T \int_\Omega \frac{\mathbf{q}_\varepsilon}{\vartheta_\varepsilon}\cdot\nabla_x \varphi\ \mathrm{d}x\ \mathrm{d}t$$
  $$+ \langle \sigma_\varepsilon; \varphi \rangle_{[\mathcal{M};C]([0,T]\times\overline{\Omega})} = -\int_\Omega \varrho_{0,\varepsilon} s(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon})\varphi(0,\cdot)\ \mathrm{d}x \tag{5.31}$$

for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$, *with* $\sigma_\varepsilon \in \mathcal{M}^+([0,T] \times \overline{\Omega})$,

$$\sigma_\varepsilon \geq \frac{1}{\vartheta_\varepsilon}\Big(\varepsilon^2 \mathbb{S}_\varepsilon : \nabla_x \mathbf{u}_\varepsilon - \frac{\mathbf{q}_\varepsilon}{\vartheta_\varepsilon} \cdot \nabla_x \vartheta_\varepsilon\Big), \tag{5.32}$$

where

$$\begin{aligned} \mathbb{S}_\varepsilon = \mathbb{S}(\vartheta_\varepsilon, \nabla_x \mathbf{u}_\varepsilon) = \mu(\vartheta_\varepsilon)\Big(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^T \mathbf{u}_\varepsilon - \frac{2}{3}\mathrm{div}_x \mathbf{u}_\varepsilon \ \mathbb{I}\Big) \\ + \eta(\vartheta_\varepsilon)\mathrm{div}_x \mathbf{u}_\varepsilon \ \mathbb{I}, \end{aligned} \tag{5.33}$$

and

$$\mathbf{q}_\varepsilon = \mathbf{q}(\vartheta_\varepsilon, \nabla_x \vartheta_\varepsilon) = -\kappa(\vartheta_\varepsilon)\nabla_x \vartheta_\varepsilon. \tag{5.34}$$

We recall that the weak solution $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}$ enjoys the regularity and integrability properties collected in Theorem 3.2. Let us point out that smallness of the parameter ε is irrelevant in the existence theory and needed here only to ensure that the initial distribution of the density and the temperature is positive.

5.2 Dissipation equation, uniform estimates

A remarkable feature of all asymptotic limits investigated in this book is that the initial values of the *thermostatic* state variables $\varrho_{0,\varepsilon}$, $\vartheta_{0,\varepsilon}$ are close to the *stable equilibrium state* $(\overline{\varrho}, \overline{\vartheta})$. As an inevitable consequence of the *Second law of thermodynamics*, the total entropy of the system is non-decreasing in time approaching its maximal value attained at $(\overline{\varrho}, \overline{\vartheta})$. The total mass and energy of the fluid being constant, the state variables are trapped in a kind of potential well (or rather "cap") in the course of evolution. This is a physical interpretation of the uniform bounds obtained in this section. Mathematically, the same is expressed through the coercivity properties of the *Helmholtz function* $H_{\overline{\vartheta}} = \varrho e - \overline{\vartheta} \varrho s$ discussed in Section 2.2.3. In particular, the uniform bounds established first in Chapter 2 apply to the family $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}$ of solutions of the *primitive system* uniformly for $\varepsilon \to 0$. This observation plays an indispensable role in the analysis of the asymptotic limit.

5.2.1 Conservation of total mass

In accordance with hypothesis (5.27), the total mass

$$M_0 = \int_\Omega \varrho_\varepsilon(t) \ \mathrm{d}x = \overline{\varrho}|\Omega| \tag{5.35}$$

is a constant of motion independent of ε. Note that, by virtue of Theorem 3.2, $\varrho_\varepsilon \in C_{\mathrm{weak}}([0,T]; L^{\frac{5}{3}}(\Omega))$, therefore (5.35) makes sense for *any* $t \in [0,T]$. The case when the total mass of the fluid depends on ε can be accommodated easily by a straightforward modification of the arguments presented below.

5.2.2 Total dissipation balance and related estimates

As observed in Section 4.7, the *total dissipation balance* is the central principle yielding practically all uniform bounds available for the primitive system. Pursuing the ideas of Section 2.2.3 we combine relations (5.30), (5.31) to obtain the *dissipation equality*

$$\int_\Omega \Big(\frac{\varepsilon^2}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + H_{\overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon) - \varepsilon \varrho_\varepsilon F \Big)(t) \ \mathrm{d}x + \overline{\vartheta} \sigma_\varepsilon \Big[[0,t] \times \overline{\Omega} \Big] \tag{5.36}$$
$$= \int_\Omega \Big(\frac{\varepsilon^2}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + H_{\overline{\vartheta}}(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) - \varepsilon \varrho_{0,\varepsilon} F \Big) \ \mathrm{d}x$$

satisfied for a.a. $t \in (0,T)$, where the function $H_{\overline{\vartheta}}$ was introduced in (2.48).

In addition, as the total mass M_0 does not change in time, relation (5.36) can be rewritten in the form

$$\int_\Omega \Big(\frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 - \frac{(\varrho_\varepsilon - \overline{\varrho})}{\varepsilon} F \Big)(t) \ \mathrm{d}x \tag{5.37}$$
$$+ \int_\Omega \frac{1}{\varepsilon^2} \Big(H_{\overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon) - (\varrho_\varepsilon - \overline{\varrho}) \frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \Big)(t) \ \mathrm{d}x + \frac{\overline{\vartheta}}{\varepsilon^2} \sigma_\varepsilon \Big[[0,t] \times \overline{\Omega} \Big]$$
$$= \int_\Omega \Big(\frac{1}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 - \frac{(\varrho_{0,\varepsilon} - \overline{\varrho})}{\varepsilon} F \Big) \ \mathrm{d}x$$
$$+ \int_\Omega \frac{1}{\varepsilon^2} \Big(H_{\overline{\vartheta}}(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) - (\varrho_{0,\varepsilon} - \overline{\varrho}) \frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \Big) \ \mathrm{d}x$$

(cf. (4.37)).

At this stage, we associate to each function h_ε its essential part $[h_\varepsilon]_{\mathrm{ess}}$ and residual part $[h_\varepsilon]_{\mathrm{res}}$ introduced through formulas (4.44), (4.45) in Section 4.7. A common principle adopted in this book asserts that:

- The "residual" components of all ε-dependent quantities appearing in the primitive equations (5.28–5.31) admit uniform bounds that are exactly the same as *a priori* bounds derived in Chapter 2. Moreover, the measure of the "residual" subset $\mathcal{M}_{\mathrm{res}}$ of $(0,T) \times \Omega$ being small, the "residual" parts vanish in the asymptotic limit $\varepsilon \to 0$.
- The decisive piece of information concentrates in the "essential" components, in particular, they determine the limit system of equations. The fact that the "essential" values of ϱ_ε, ϑ_ε are bounded from above as well as from below away from zero facilitates the analysis considerably as all continuously differentiable functions on $\mathbb{R}^2$ are globally Lipschitz when restricted to the range of "essential" quantities.

In order to exploit relation (5.37) we need a piece of information concerning the structural properties of the *Helmholtz function* $H_{\overline{\vartheta}}$. More precisely, we show that the quantity

$$H_{\overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon) - (\varrho_\varepsilon - \overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})$$

is non-negative and strictly coercive, attaining its global minimum zero at the equilibrium state $(\overline{\varrho}, \overline{\vartheta})$. Moreover, it dominates both $\varrho e(\varrho, \vartheta)$ and $s(\varrho, \vartheta)$ whenever (ϱ, ϑ) is far from the equilibrium state. These structural properties utilized in (5.37) yield the desired uniform estimates on ϱ_ε, ϑ_ε as well as on the size of the "residual subset" of $(0, T) \times \Omega$.

Lemma 5.1. *Let $\overline{\varrho} > 0$, $\overline{\vartheta} > 0$ be given constants and let*

$$H_{\overline{\vartheta}}(\varrho, \vartheta) = \varrho\Big(e(\varrho, \vartheta) - \overline{\vartheta}s(\varrho, \vartheta)\Big),$$

where e, s satisfy (5.18–5.23). *Let $\mathcal{O}_{\rm ess}$, $\mathcal{O}_{\rm res}$ be the sets of essential and residual values introduced in* (4.39), (4.40).

Then there exist positive constants $c_i = c_i(\overline{\varrho}, \overline{\vartheta})$, $i = 1, \dots, 4$, such that

(i)
$$c_1\Big(|\varrho - \overline{\varrho}|^2 + |\vartheta - \overline{\vartheta}|^2\Big) \leq H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \leq c_2\Big(|\varrho - \overline{\varrho}|^2 + |\vartheta - \overline{\vartheta}|^2\Big) \tag{5.38}$$

for all $(\varrho, \vartheta) \in O_{\rm ess}$;

(ii)
$$\begin{aligned} &H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \\ &\geq \inf_{(r, \Theta) \in \partial O_{\rm ess}} \left\{ H_{\overline{\vartheta}}(r, \Theta) - (r - \overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \right\} \\ &= c_3(\overline{\varrho}, \overline{\vartheta}) > 0 \end{aligned} \tag{5.39}$$

for all $(\varrho, \vartheta) \in O_{\rm res}$;

(iii)
$$\begin{aligned} &H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \\ &\geq c_4\Big(\varrho e(\varrho, \vartheta) + \varrho|s(\varrho, \vartheta)|\Big) \end{aligned} \tag{5.40}$$

for all $(\varrho, \vartheta) \in O_{\rm res}$.

Proof. To begin, write

$$H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) = \mathcal{F}(\varrho) + \mathcal{G}(\varrho, \vartheta),$$

where

$$\mathcal{F}(\varrho) = H_{\overline{\vartheta}}(\varrho, \overline{\vartheta}) - (\varrho - \overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})$$

and

$$\mathcal{G}(\varrho, \vartheta) = H_{\overline{\vartheta}}(\varrho, \vartheta) - H_{\overline{\vartheta}}(\varrho, \overline{\vartheta}).$$

As already observed in Section 2.2.3, the function $\mathcal{F}$ is strictly convex attaining its global minimum zero at the point $\overline{\varrho}$, while $G(\varrho, \cdot)$ is strictly decreasing for $\vartheta < \overline{\vartheta}$ and strictly increasing for $\vartheta > \overline{\vartheta}$ as a direct consequence of the hypothesis of thermodynamic stability expressed in terms of (5.21), (5.22). In particular, computing the partial derivatives of $H_{\overline{\vartheta}}$ as in (2.49), (2.50) we deduce estimate (5.38). By the same token, the function

$$\vartheta \mapsto H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})$$

is decreasing for $\vartheta < \overline{\vartheta}$ and increasing whenever $\vartheta > \overline{\vartheta}$; whence (5.39) follows.

Finally, as $\mathcal{F}$ is strictly convex, we have

$$H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \geq c(\overline{\varrho}, \overline{\vartheta})\varrho \text{ whenever } \varrho \geq 2\overline{\varrho},$$

and, consequently, estimate (5.40) can be deduced from (5.39) and Proposition 3.2. □

In order to exploit the dissipation balance (5.37), we have to ensure that its right-hand side determined in terms of the initial data is bounded uniformly with respect to $\varepsilon \to 0$. Since the initial data are given by (5.26), (5.27), this can be achieved if

$$\{\sqrt{\varrho_{0,\varepsilon}}\mathbf{u}_{0,\varepsilon}\}_\varepsilon \text{ is bounded in } L^2(\Omega; \mathbb{R}^3), \tag{5.41}$$

and

$$\left\{\varrho^{(1)}_{0,\varepsilon} = \frac{\varrho_{0,\varepsilon} - \overline{\varrho}}{\varepsilon}\right\}_{\varepsilon>0}, \left\{\vartheta^{(1)}_{0,\varepsilon} = \frac{\vartheta_{0,\varepsilon} - \overline{\vartheta}}{\varepsilon}\right\}_{\varepsilon>0} \text{ are bounded in } L^\infty(\Omega). \tag{5.42}$$

Observe that these hypotheses are optimal with respect to the chosen scaling and the desired target problem.

Consequently, using estimate (5.38) we deduce from (5.37) that

$$\text{ess sup}_{t\in(0,T)}\left\| \ [\varrho_\varepsilon - \overline{\varrho}]_{\text{ess}}(t) \ \right\|^2_{L^2(\Omega)} \leq \varepsilon^2 c,$$
$$\text{ess sup}_{t\in(0,T)}\left\| \ [\vartheta_\varepsilon - \overline{\vartheta}]_{\text{ess}}(t) \ \right\|^2_{L^2(\Omega)} \leq \varepsilon^2 c,$$

and, by virtue of (5.40),

$$\text{ess sup}_{t\in(0,T)} \| \, [\varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\rm res} \, \|_{L^1(\Omega)} \leq \varepsilon^2 c, \tag{5.43}$$

$$\text{ess sup}_{t\in(0,T)} \| \, [\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) \,]_{\rm res} \|_{L^1(\Omega)} \leq \varepsilon^2 c. \tag{5.44}$$

Note that, as a consequence of the coercivity properties of the Helmholtz function $H_{\overline{\vartheta}}$ established in Lemma 5.1, the quantity

$$\int_\Omega \frac{(\varrho_\varepsilon - \overline{\varrho})}{\varepsilon} F \, {\rm d}x$$

can be handled as a lower order term.

In addition, we have

$$\text{ess} \sup_{t\in(0,T)} \|\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\|_{L^2(\Omega;R^3)} \leq c, \qquad \sigma_\varepsilon\Big[[0,T]\times\overline{\Omega}\Big] \leq \varepsilon^2 c,$$

and, as a direct consequence of (5.39),

$$\text{ess} \sup_{t\in(0,T)} | \, \mathcal{M}^\varepsilon_{\rm res}[t] \, | \leq \varepsilon^2 c,$$

where the sets $\mathcal{M}^\varepsilon_{\rm res}[t] \subset \Omega$ have been introduced in (4.43). Note that the last estimate reflects the previously vague statement "the measure of the residual set is small".

Since the entropy production rate σ_ε remains small of order ε^2, we deduce from (5.32) that (i) the term $\frac{1}{\vartheta_\varepsilon}\mathbb{S}_\varepsilon : \nabla_x \mathbf{u}_\varepsilon$ is bounded in $L^1((0,T)\times\Omega)$, and, in accordance with hypothesis (5.25), (ii) $\nabla_x(\vartheta_\varepsilon/\varepsilon)$ is bounded in $L^2((0,T)\times\Omega)$. In particular, we observe that $\nabla_x\vartheta_\varepsilon$ vanishes in the asymptotic limit, that is to say ϑ_ε approaches a spatially homogeneous function. As the pressure becomes constant in the *low Mach number* regime, the density is driven to a constant as well. This observation justifies our choice of the initial data. On the other hand, it is intuitively clear that we need a uniform bound on the entropy production rate in order to control the norm of the velocity gradient. In other words, we have to impose the hypothesis of thermodynamic stability (1.44) for the thermostatic variables ϱ_ε, ϑ_ε to remain close to the equilibrium state. We can see again the significant role of *dissipativity* of the system in our approach to singular limits.

5.2.3 Uniform estimates

In this rather technical part, we use the structural properties of thermodynamic functions imposed through the constitutive relations (5.17–5.25) to reformulate the uniform estimates obtained in the previous section in terms of the standard function spaces framework. These estimates or their analogues will be used repeatedly in the future discussion so it is convenient to summarize them in a concise way.

■ Uniform Estimates:

Proposition 5.1. *Let the quantities $e = e(\varrho, \vartheta)$, $s = s(\varrho, \vartheta)$ satisfy hypotheses (5.17–5.23), and let the transport coefficients $\mu = \mu(\vartheta)$, $\eta = \eta(\vartheta)$, and $\kappa = \kappa(\vartheta)$ obey the growth restrictions (5.24), (5.25).*

Then we have:

$$\operatorname{ess}\sup_{t\in(0,T)} \mid \mathcal{M}^{\varepsilon}_{\rm res}[t] \mid \leq \varepsilon^2 c, \tag{5.45}$$

$$\operatorname{ess}\sup\nolimits_{t\in(0,T)} \Big\| \Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\Big]_{\rm ess}(t) \Big\|_{L^2(\Omega)} \leq c, \tag{5.46}$$

$$\operatorname{ess}\sup\nolimits_{t\in(0,T)} \Big\| \Big[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\Big]_{\rm ess}(t) \Big\|_{L^2(\Omega)} \leq c, \tag{5.47}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \int_\Omega \Big([\varrho_\varepsilon]^{\frac{5}{3}}_{\rm res} + [\vartheta_\varepsilon]^4_{\rm res} \Big)(t) \ {\rm d}x \leq \varepsilon^2 c, \tag{5.48}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \|\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\|_{L^2(\Omega;\mathbb{R}^3)} \leq c, \tag{5.49}$$

$$\sigma_\varepsilon\Big[[0,T]\times\overline{\Omega}\Big] \leq \varepsilon^2 c, \tag{5.50}$$

$$\int_0^T \|\mathbf{u}_\varepsilon(t)\|^2_{W^{1,2}(\Omega;\mathbb{R}^3)} \ {\rm d}t \leq c, \tag{5.51}$$

$$\int_0^T \Big\| \Big(\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\Big)(t) \Big\|^2_{W^{1,2}(\Omega))} \ {\rm d}t \leq c, \tag{5.52}$$

$$\int_0^T \Big\| \Big(\frac{\log(\vartheta_\varepsilon) - \log(\overline{\vartheta})}{\varepsilon}\Big)(t) \Big\|^2_{W^{1,2}(\Omega))} \ {\rm d}t \leq c, \tag{5.53}$$

and

$$\int_0^T \Big\| \Big[\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon}\Big]_{\rm res}(t) \Big\|^q_{L^q(\Omega)} \ {\rm d}t \leq c, \tag{5.54}$$

$$\int_0^T \Big\| \Big[\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon}\Big]_{\rm res} \mathbf{u}_\varepsilon(t) \Big\|^q_{L^q(\Omega;\mathbb{R}^3)} \ {\rm d}t \leq c, \tag{5.55}$$

$$\int_0^T \Big\| \Big[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon}\Big]_{\rm res} \Big(\frac{\nabla_x \vartheta_\varepsilon}{\varepsilon}\Big)(t) \Big\|^q_{L^q(\Omega;\mathbb{R}^3)} \ {\rm d}t \leq c \tag{5.56}$$

for a certain $q > 1$, where the generic constant c is independent of $\varepsilon \to 0$.

Proof. **(i)** Estimates (5.45–5.47) and (5.49), (5.50) have been proved in the previous section. Estimate (5.48) follows immediately from (5.18), (5.43), and the structural hypotheses (5.21), (5.23).

(ii) Estimate (5.50) combined with (5.32–5.34) and hypothesis (5.24) gives rise to

$$\int_0^T \|\nabla_x \mathbf{u}_\varepsilon + \nabla_x^T \mathbf{u}_\varepsilon - \frac{2}{3}\mathrm{div}_x \mathbf{u}_\varepsilon \mathbb{I}\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})} \, \mathrm{d}t \leq c. \tag{5.57}$$

On the other hand, we can use estimates (5.49), (5.57), together with (5.45) and Korn's inequality established in Proposition 2.1, in order to obtain (5.51).

(iii) In a similar fashion, we deduce from (5.50) a uniform bound

$$\int_0^T \Big[\Big\| \nabla_x \Big(\frac{\vartheta_\varepsilon}{\varepsilon}\Big) \Big\|^2_{L^2(\Omega)} + \Big\| \nabla_x \Big(\frac{\log(\vartheta_\varepsilon)}{\varepsilon}\Big) \Big\|^2_{L^2(\Omega)} \Big] \, \mathrm{d}t \leq c,$$

which, together with (5.47), (5.45) and Proposition 2.2, gives rise to (5.52), (5.53).

(iv) By virtue of the structural hypotheses (5.21), (5.22), we get

$$|\varrho s(\varrho,\vartheta)| \leq c\Big(1 + \varrho|\log(\varrho)| + \varrho|\log(\vartheta) - \log(\overline{\vartheta})| + \vartheta^3\Big) \tag{5.58}$$

(cf. (3.39)).

On the other hand, it follows from (5.45) that

$$\mathrm{ess}\sup_{t\in(0,T)} \| \, [\frac{1}{\varepsilon}\,]_{\mathrm{res}}(t)\|_{L^2(\Omega)} \leq c, \tag{5.59}$$

while (5.48) yields

$$\mathrm{ess}\sup_{t\in(0,T)} \Big\| \Big[\frac{\varrho_\varepsilon \log(\varrho_\varepsilon)}{\varepsilon}\Big]_{\mathrm{res}} (t) \Big\|_{L^q(\Omega)} \leq c \text{ for any } 1 \leq q < \frac{5}{3}. \tag{5.60}$$

Furthermore, by means of (5.48), (5.53),

$$\int_0^T \Big\| \Big[\frac{\varrho_\varepsilon(\log(\vartheta_\varepsilon) - \log(\overline{\vartheta}))}{\varepsilon}\Big]_{\mathrm{res}} \Big\|^2_{L^p(\Omega)} \mathrm{d}t \leq c \text{ for a certain } p > 1, \tag{5.61}$$

and, finally,

$$\mathrm{ess}\sup_{t\in(0,T)} \Big\| \Big[\frac{\vartheta_\varepsilon^3}{\varepsilon}\Big]_{\mathrm{res}} (t) \Big\|_{L^{\frac{4}{3}}(\Omega)} \leq c\varepsilon, \tag{5.62}$$

where we have used (5.48).

Relations (5.59–5.62), together with (5.58), imply (5.54).

(v) In order to see (5.55), we use estimates (5.49), (5.48), and (5.53) to obtain

$$\Big\{ \Big[\frac{\varrho_\varepsilon(\log(\vartheta_\varepsilon) - \log(\overline{\vartheta}))\mathbf{u}_\varepsilon}{\varepsilon}\Big]_{\mathrm{res}} \Big\}_{\varepsilon>0} \text{ bounded in } L^q(0,T;L^q(\Omega;\mathbb{R}^3))$$

for a certain $q > 1$, which, combined with (5.58–5.62), and (5.45), gives rise to (5.55).

(vi) Finally, in accordance with hypothesis (5.25),

$$\Big[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon}\Big]_{\rm res}\Big|\frac{\nabla_x\vartheta_\varepsilon}{\varepsilon}\Big| \leq c\Big(\Big|\frac{\nabla_x \log(\vartheta)}{\varepsilon}\Big| + [\vartheta_\varepsilon^2]_{\rm res}\Big|\frac{\nabla_x\vartheta_\varepsilon}{\varepsilon}\Big|\Big),$$

where, as a consequence of estimates (5.48), (5.52), and the embedding relation $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$,

$$\{[\vartheta_\varepsilon]_{\rm res}\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0,T;L^2(\Omega)) \cap L^1(0,T;L^3(\Omega)). \tag{5.63}$$

Thus (5.56) follows from (5.52), (5.53) combined with (5.63) and a simple interpolation argument. □

5.3 Convergence

The uniform estimates established in Proposition 5.1 will be used in order to let $\varepsilon \to 0$ in equations (5.28), (5.29), (5.31) and to identify the limit problem. As we have observed in Proposition 5.1, the residual parts of the thermodynamic quantities related to the state variables ϱ, ϑ are small of order ε. In order to handle the essential components, we need the following general result exploited many times in the forthcoming considerations.

Proposition 5.2. *Let $\{\varrho_\varepsilon\}_{\varepsilon>0}$, $\{\vartheta_\varepsilon\}_{\varepsilon>0}$ be two sequences of non-negative measurable functions such that*

$$\left.\begin{array}{l}[\varrho_\varepsilon^{(1)}]_{\rm ess} \to \varrho^{(1)},\\ [\vartheta_\varepsilon^{(1)}]_{\rm ess} \to \vartheta^{(1)}\end{array}\right\} \text{ weakly-(*) in } L^\infty(0,T;L^2(\Omega)) \text{ as } \varepsilon \to 0,$$

where we have denoted

$$\varrho_\varepsilon^{(1)} = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon},\ \vartheta_\varepsilon^{(1)} = \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}.$$

Suppose that

$$\operatorname*{ess\,sup}_{t\in(0,T)} |\mathcal{M}^\varepsilon_{\rm res}[t]| \leq \varepsilon^2 c. \tag{5.64}$$

Let $G \in C^1(\overline{\mathcal{O}_{\rm ess}})$ be a given function, where the sets $\mathcal{M}^\varepsilon_{\rm ess}[t]$, $\mathcal{O}_{\rm ess}$ have been introduced in (4.43), (4.39), *respectively.*

Then

$$\frac{[G(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm ess} - G(\overline{\varrho},\overline{\vartheta})}{\varepsilon} \to \frac{\partial G(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\varrho^{(1)} + \frac{\partial G(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\vartheta^{(1)}$$

weakly-(*) in $L^\infty(0,T;L^2(\Omega))$.

If, in addition, $G \in C^2(\overline{\mathcal{O}_{\rm ess}})$, then

$$\Big\|\frac{[G(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm ess} - G(\overline{\varrho},\overline{\vartheta})}{\varepsilon} - \frac{\partial G(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}[\varrho_\varepsilon^{(1)}]_{\rm ess} + \frac{\partial G(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}[\vartheta_\varepsilon^{(1)}]_{\rm ess}\Big\|_{L^\infty(0,T;L^1(\Omega))} \leq \varepsilon c. \tag{5.65}$$

Remark: *If, in addition, the functions* ϱ_ε, ϑ_ε *satisfy estimate* (5.48), *then* (5.65) *may be replaced by*

$$\left\| \frac{[G(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm ess} - G(\overline{\varrho},\overline{\vartheta})}{\varepsilon} - \frac{\partial G(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\varrho_\varepsilon^{(1)} + \frac{\partial G(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\vartheta_\varepsilon^{(1)} \right\|_{L^\infty(0,T;L^1(\Omega))} \leq \varepsilon c. \tag{5.66}$$

Proof. To begin, by virtue of (5.64),

$$\|\frac{1}{\varepsilon}[G(\overline{\varrho},\overline{\vartheta})]_{\rm res}\|_{L^1(\Omega)} \leq \varepsilon c,\ \|\frac{1}{\varepsilon}[G(\overline{\varrho},\overline{\vartheta})]_{\rm res}\|_{L^2(\Omega)} \leq c,$$

and, consequently, it is enough to show that

$$\Big[\frac{G(\varrho_\varepsilon,\vartheta_\varepsilon) - G(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\Big]_{\rm ess} \to \frac{\partial G(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\varrho^{(1)} + \frac{\partial G(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\vartheta^{(1)} \tag{5.67}$$

weakly-(*) in $L^\infty(0,T;L^2(\Omega))$.

The next step is to observe that (5.67) holds as soon as $G \in C^2(\overline{\mathcal{O}_{\rm ess}})$. Indeed as G is twice continuously differentiable, we have

$$\Big| \Big[\frac{G(\varrho_\varepsilon,\vartheta_\varepsilon) - G(\overline{\varrho},\overline{\vartheta})}{\varepsilon} - \Big(\frac{\partial G(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} + \frac{\partial G(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\Big)\Big]_{\rm ess}\Big|$$
$$\leq \varepsilon\chi_\varepsilon\Big[\Big(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\Big)^2 + \Big(\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\Big)^2\Big],$$

where

$$\|\ [\chi_\varepsilon]_{\rm ess}\|_{L^\infty((0,T)\times\Omega)} \leq c\|G\|_{C^2(\overline{\mathcal{O}_{\rm ess}})}.$$

In particular, we have shown (5.65).

Finally, seeing that

$$\Big|\Big[\frac{G(\varrho_\varepsilon,\vartheta_\varepsilon) - G(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\Big]_{\rm ess}\Big| \leq \|G\|_{C^1(\overline{\mathcal{B}_{\rm ess}})}\Big(\Big|\Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\Big]_{\rm ess}\Big| + \Big|\Big[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\Big]_{\rm ess}\Big|\Big)$$

we complete the proof approximating G by a family of smooth functions uniformly in $C^1(\overline{\mathcal{O}_{\rm ess}})$. □

5.3.1 Equation of continuity

In the low Mach number regime, the equation of continuity (5.28) reduces to the incompressibility constraint (5.8). In order to verify this observation, we first use the uniform estimate (5.51) to deduce

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)) \tag{5.68}$$

passing to a suitable subsequence as the case may be.

Furthermore, we have

$$\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} = \left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\right]_{\rm ess} + \left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\right]_{\rm res},$$

where, in accordance with (5.46),

$$\left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\right]_{\rm ess} \to \varrho^{(1)} \text{ weakly-(*) in } L^\infty(0,T;L^2(\Omega)), \tag{5.69}$$

while estimates (5.45), (5.48) give rise to

$$\left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\right]_{\rm res} \to 0 \text{ in } L^\infty(0,T;L^{\frac{5}{3}}(\Omega)); \tag{5.70}$$

whence

$$\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \to \varrho^{(1)} \text{ weakly-(*) in } L^\infty(0,T;L^{\frac{5}{3}}(\Omega)). \tag{5.71}$$

In particular, (5.71) implies

$$\varrho_\varepsilon \to \overline{\varrho} \text{ in } L^\infty(0,T;L^{\frac{5}{3}}(\Omega)), \tag{5.72}$$

and we can let $\varepsilon \to 0$ in the continuity equation (5.28) in order to conclude that

$$\int_0^T \int_\Omega \mathbf{U} \cdot \nabla_x \varphi \ {\rm d}x \ {\rm d}t = 0$$

for all $\varphi \in C^\infty_c((0,T) \times \overline{\Omega})$. Since the limit velocity field $\mathbf{U}$ belongs to the class $L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$, we have shown

$${\rm div}_x \mathbf{U} = 0 \text{ a.a. in } (0,T)\times\Omega, \ \ \mathbf{U}\cdot\mathbf{n}|_{\partial\Omega} = 0 \text{ in the sense of traces} \tag{5.73}$$

provided the boundary $\partial\Omega$ is at least Lipschitz (cf. Section 10.3 in Appendix).

5.3.2 Entropy balance

With regard to (5.28), we recast the entropy balance (5.31) in the form

$$\begin{aligned} &\int_0^T \int_\Omega \varrho_\varepsilon \Big(\frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \Big) \Big(\partial_t \varphi + \mathbf{u}_\varepsilon \cdot \nabla_x \varphi \Big) \ {\rm d}x \ {\rm d}t \\ &\quad - \int_0^T \int_\Omega \frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \nabla_x \Big(\frac{\vartheta_\varepsilon}{\varepsilon} \Big) \cdot \nabla_x \varphi \ {\rm d}x \ {\rm d}t + \frac{1}{\varepsilon} \langle \sigma_\varepsilon ; \varphi \rangle_{[\mathcal{M}, C]([0,T]\times\overline{\Omega})} \\ &= - \int_\Omega \varrho_{0,\varepsilon} \Big(\frac{s(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \Big) \varphi(0,\cdot) \ {\rm d}x \end{aligned} \tag{5.74}$$

to be satisfied for any $\varphi \in C^\infty_c([0,T) \times \overline{\Omega})$.

Adopting the notation introduced in Proposition 5.2 and using estimate (5.47) we get

$$\left[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\right]_{\rm ess} \to \vartheta^{(1)} \text{ weakly-(*) in } L^\infty(0,T;L^2(\Omega)), \tag{5.75}$$

passing to a suitable subsequence as the case may be. On the other hand, in accordance with (5.52),

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \vartheta^{(1)} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)). \tag{5.76}$$

Note that the limit functions in (5.75), (5.76) coincide since the measure of the "residual" subset of $(0,T)\times\Omega$ tends to zero as claimed in (5.45).

In order to identify the limit problem resulting from (5.74) we proceed by several steps:

(i) Write

$$\varrho_\varepsilon\Big(\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\Big) = [\varrho_\varepsilon]_{\rm ess}\frac{[s(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm ess} - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon}$$
$$+ \left[\frac{\varrho_\varepsilon}{\varepsilon}\right]_{\rm res}\Big([s(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm ess} - s(\overline{\varrho},\overline{\vartheta})\Big) + \left[\frac{\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\right]_{\rm res},$$

where, by virtue of (5.48),

$$\left[\frac{\varrho_\varepsilon}{\varepsilon}\right]_{\rm res}\Big([s(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm ess} - s(\overline{\varrho},\overline{\vartheta})\Big) \to 0 \text{ in } L^\infty(0,T;L^{\frac{5}{3}}(\Omega)), \tag{5.77}$$

and, in accordance with (5.45), (5.54),

$$\left[\frac{\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\right]_{\rm res} \to 0 \text{ in } L^p((0,T)\times\Omega) \text{ for a certain } p>1. \tag{5.78}$$

Similarly, combining (5.45) with (5.55), (5.68), (5.77), we obtain

$$\left[\frac{\varrho_\varepsilon}{\varepsilon}\right]_{\rm res}\Big([s(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm ess} - s(\overline{\varrho},\overline{\vartheta})\Big)\mathbf{u}_\varepsilon \to 0 \text{ in } L^p(0,T;L^p(\Omega;\mathbb{R}^3)), \tag{5.79}$$

and

$$\left[\frac{\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\right]_{\rm res}\mathbf{u}_\varepsilon \to 0 \text{ in } L^p(0,T;L^p(\Omega;\mathbb{R}^3)) \tag{5.80}$$

for a certain $p>1$.

Finally, Proposition 5.2 together with (5.72) yield

$$[\varrho_\varepsilon]_{\rm ess}\frac{[s(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm ess} - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon} \to \overline{\varrho}\Big(\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\varrho^{(1)} + \frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\vartheta^{(1)}\Big) \tag{5.81}$$

weakly-(*) in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$.

(ii) In a similar way, the entropy flux can be written as

$$\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon}\nabla_x\Big(\frac{\vartheta_\varepsilon}{\varepsilon}\Big) = \Big[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon}\Big]_{\rm ess}\nabla_x\Big(\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\Big) + \Big[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon}\Big]_{\rm res}\nabla_x\Big(\frac{\vartheta_\varepsilon}{\varepsilon}\Big),$$

where, as a consequence of (5.75), (5.76),

$$\Big[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon}\Big]_{\rm ess}\nabla_x\Big(\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\Big) \to \frac{\kappa(\overline{\vartheta})}{\overline{\vartheta}}\nabla_x\vartheta^{(1)} \text{ weakly in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{5.82}$$

and, in accordance with (5.45), (5.56),

$$\Big[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon}\Big]_{\rm res}\nabla_x\Big(\frac{\vartheta_\varepsilon}{\varepsilon}\Big) \to 0 \text{ in } L^s(0,T;L^s(\Omega;\mathbb{R}^3)) \text{ for a certain } s>1. \tag{5.83}$$

(iii) Eventually, we have to identify the weak limit $\mathbf{D}$ of the product

$$[\varrho_\varepsilon]_{\rm ess}\frac{[s(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm ess} - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\mathbf{u}_\varepsilon \to \mathbf{D} \text{ weakly in } L^2(0,T;L^{\frac{3}{2}}(\Omega;\mathbb{R}^3)).$$

To this end, we revoke the Div-Curl lemma formulated in Proposition 3.3. Following the notation of Proposition 3.3 we set

$$\begin{aligned}\mathbf{U}_\varepsilon = \Bigg[&[\varrho_\varepsilon]_{\rm ess}\frac{[s(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm ess} - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon},\\ &[\varrho_\varepsilon]_{\rm ess}\frac{[s(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm ess} - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\mathbf{u}_\varepsilon - \Big[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon}\Big]_{\rm ess}\nabla_x\Big(\frac{\vartheta_\varepsilon}{\varepsilon}\Big)\Bigg],\\ \mathbf{V}_\varepsilon = [&G(\mathbf{u}_\varepsilon),0,0,0]\end{aligned}$$

considered as vector fields defined on the set $((0,T)\times\Omega)\subset\mathbb{R}^4$ with values in $\mathbb{R}^4$, for an arbitrary function $G\in W^{1,\infty}(\mathbb{R}^3)$.

Using estimates (5.77–5.83), together with (5.50), we can check that $\mathbf{U}_\varepsilon$, $\mathbf{V}_\varepsilon$ meet all hypotheses of Proposition 3.3; whence, in agreement with (5.81),

$$[\varrho_\varepsilon]_{\rm ess}\frac{[s(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm ess} - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon}G(\mathbf{u}_\varepsilon) \to \overline{\varrho}\Big(\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\varrho^{(1)} + \frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\vartheta^{(1)}\Big)\overline{G(\mathbf{u})}$$

for any G, yielding the desired conclusion

$$[\varrho_\varepsilon]_{\rm ess}\frac{[s(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm ess} - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\mathbf{u}_\varepsilon \to \overline{\varrho}\Big(\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\varrho^{(1)} + \frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\vartheta^{(1)}\Big)\mathbf{U} \tag{5.84}$$

weakly in $L^2(0,T;L^{\frac{3}{2}}(\Omega;\mathbb{R}^3))$.

At this stage, we are ready to let $\varepsilon \to 0$ in the entropy balance equation (5.74) in order to conclude that

$$\int_0^T \int_\Omega \overline{\varrho}\Big(\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial \varrho}\varrho^{(1)} + \frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}\vartheta^{(1)}\Big)\Big(\partial_t\varphi + \mathbf{U}\cdot\nabla_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{5.85}$$
$$- \int_0^T \int_\Omega \frac{\kappa(\overline{\vartheta})}{\overline{\vartheta}}\nabla_x\vartheta^{(1)}\cdot\nabla_x\varphi\ \mathrm{d}x\mathrm{d}t$$
$$= -\int_\Omega \overline{\varrho}\Big(\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial \varrho}\varrho_0^{(1)} + \frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}\vartheta_0^{(1)}\Big)\varphi(0,\cdot)\ \mathrm{d}x$$

for any $\varphi \in C_c^\infty([0,T)\times\overline{\Omega})$, where

$$\varrho_{0,\varepsilon}^{(1)} = \frac{\varrho_{0,\varepsilon} - \overline{\varrho}}{\varepsilon} \to \varrho_0^{(1)} \text{ weakly-(*) in } L^\infty(\Omega), \tag{5.86}$$

and

$$\vartheta_{0,\varepsilon}^{(1)} = \frac{\vartheta_{0,\varepsilon} - \overline{\vartheta}}{\varepsilon} \to \vartheta_0^{(1)} \text{ weakly-(*) in } L^\infty(\Omega). \tag{5.87}$$

A remarkable feature of this process is that the entropy production rate represented by the measure σ_ε disappears in the limit problem (5.85) as a consequence of the uniform bound (5.50). Loosely speaking, the entropy balance "inequality" (5.74) becomes an *equation* (5.85).

To conclude, we deduce from (5.85) that

$$\int_\Omega \overline{\varrho}\Big(\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial \varrho}\varrho^{(1)} + \frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}\vartheta^{(1)}\Big)(t)\ \mathrm{d}x$$
$$= \int_\Omega \overline{\varrho}\Big(\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial \varrho}\varrho_0^{(1)} + \frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}\vartheta_0^{(1)}\Big)\ \mathrm{d}x \text{ for a.a. } t \in (0,T).$$

However, since we have assumed that $\varrho^{(1)}$ has zero mean and the total mass is conserved, this relation reduces to

$$\int_\Omega \vartheta^{(1)}(t)\ \mathrm{d}x = \int_\Omega \vartheta_0^{(1)}\ \mathrm{d}x \text{ for a.a. } t \in (0,T).$$

Assuming, in addition to (5.27), that

$$\int_\Omega \vartheta_{0,\varepsilon}^{(1)}\ \mathrm{d}x = 0 \text{ for all } \varepsilon > 0 \tag{5.88}$$

we conclude

$$\int_\Omega \vartheta^{(1)}(t)\ \mathrm{d}x = 0 \text{ for a.a. } t \in (0,T). \tag{5.89}$$

Clearly, the resulting equation (5.85) should give rise to the heat equation (5.12) in the Oberbeck-Boussinesq approximation as soon as we establish a relation between $\varrho^{(1)}$ and $\vartheta^{(1)}$. This will be done in the next section.

5.3.3 Momentum equation

The asymptotic limit in the momentum equation is one of the most delicate steps as the latter contains the convective term $\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon$, difficult to handle because of possible violent time oscillations of the acoustic waves represented by the gradient component of the velocity.

Incompressible limit. It follows from (5.68), (5.72), combined with the standard embedding relation $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, that

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \overline{\varrho}\mathbf{U} \text{ weakly in } L^2(0,T; L^{\frac{30}{23}}(\Omega;\mathbb{R}^3)). \tag{5.90}$$

Moreover, we deduce from (5.49), (5.72) that

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \overline{\varrho}\mathbf{U} \text{ weakly-(*) in } L^\infty(0,T; L^{\frac{5}{4}}(\Omega;\mathbb{R}^3)), \tag{5.91}$$

which, combined with (5.68), gives rise to

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \to \overline{\varrho \mathbf{U} \otimes \mathbf{U}} \text{ weakly in } L^2(0,T; L^{\frac{30}{29}}(\Omega;\mathbb{R}^{3\times 3})). \tag{5.92}$$

As already noted in Section 4.4, we do not expect to have $\overline{\varrho \mathbf{U} \otimes \mathbf{U}} = \overline{\varrho}\mathbf{U} \otimes \mathbf{U}$ because of possible time oscillations of the gradient component of the velocity field.

Next, as a consequence of (5.48), (5.52),

$$\{\vartheta_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0,T;L^4(\Omega)) \cap L^2(0,T;L^6(\Omega)). \tag{5.93}$$

Thus hypothesis (5.24), together with (5.68), (5.93), and a simple interpolation argument, give rise to

$$\mathbb{S}_\varepsilon \to \mu(\overline{\vartheta})(\nabla_x \mathbf{U} + \nabla_x^T \mathbf{U}) \text{ weakly in } L^q(0,T;L^q(\Omega;\mathbb{R}^3)) \text{ for a certain } q>1. \tag{5.94}$$

Note that, in accordance with (5.73), $\mathrm{div}_x \mathbf{U} = 0$.

Now, it is easy to let $\varepsilon \to 0$ in the momentum equation (5.29) as soon as the test function φ is divergenceless. If this is the case, we get

$$\begin{aligned} &\int_0^T \int_\Omega \Big(\overline{\varrho}\mathbf{U} \cdot \partial_t \varphi + \overline{\varrho \mathbf{U} \otimes \mathbf{U}} : \nabla_x \varphi \Big) \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_0^T \int_\Omega \Big(\mu(\overline{\vartheta})[\nabla_x \mathbf{U} + \nabla_x^T \mathbf{U}] : \nabla_x \varphi - \varrho^{(1)} \nabla_x F \cdot \varphi \Big) \, \mathrm{d}x \, \mathrm{d}t - \int_\Omega (\overline{\varrho}\mathbf{U}_0) \cdot \varphi \, \mathrm{d}x \end{aligned} \tag{5.95}$$

for any test function

$$\varphi \in C_c^\infty([0,T) \times \overline{\Omega}; \mathbb{R}^3), \ \mathrm{div}_x \varphi = 0 \text{ in } \Omega, \ \varphi \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

where we have assumed

$$\mathbf{u}_{0,\varepsilon} \to \mathbf{U}_0 \text{ weakly-(*) in } L^\infty(\Omega;\mathbb{R}^3). \tag{5.96}$$

Note that

$$\int_\Omega \frac{\varrho_\varepsilon}{\varepsilon}\nabla_x F\cdot\varphi\ \mathrm{d}x = \int_\Omega \frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\nabla_x F\cdot\varphi\ \mathrm{d}x$$

as φ is a solenoidal function with vanishing normal trace.

Relation (5.95) together with (5.73) represent a weak formulation of the incompressible NAVIER-STOKES SYSTEM (5.8), (5.9), supplemented with the complete slip boundary condition

$$\mathbf{U}\cdot\mathbf{n}|_{\partial\Omega}=0,\ \big([\nabla_x\mathbf{U}+\nabla_x^T\mathbf{U}]\mathbf{n}\big)\times\mathbf{n}|_{\partial\Omega}=0, \tag{5.97}$$

provided we can replace $\overline{\varrho\mathbf{U}\times\mathbf{U}}$ by $\overline{\varrho}\mathbf{U}\times\mathbf{U}$. Moreover, the function $\mathbf{U}$ satisfies the initial condition

$$\mathbf{U}(0,\cdot)=\mathbf{H}[\mathbf{U}_0], \tag{5.98}$$

where the symbol $\mathbf{H}$ denotes the *Helmoltz projection* onto the space of solenoidal functions (see Section 5.4.1 below and Section 10.6 in Appendix).

The fact that we completely lose control of the pressure term in the asymptotic limit is inevitable for problems with *ill-prepared data*. As a result, the limit process is spoiled by violent oscillations yielding merely the *weak convergence* towards the target problem.

Pressure. The pressure, deliberately eliminated in the previous part, is the key quantity to provide a relation between the limit functions $\varrho^{(1)}$, $\vartheta^{(1)}$. We begin by writing

$$p(\varrho_\varepsilon,\vartheta_\varepsilon)=[p(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm ess}+[p(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm res},$$

where, in accordance with hypotheses (5.21), (5.23),

$$0\le\frac{[p(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm res}}{\varepsilon}\le c\Big(\Big[\frac{1}{\varepsilon}\Big]_{\rm res}+\Big[\frac{\varrho_\varepsilon^{\frac{5}{3}}}{\varepsilon}\Big]_{\rm res}+\Big[\frac{\vartheta_\varepsilon^4}{\varepsilon}\Big]_{\rm res}\Big), \tag{5.99}$$

see also (3.32). Consequently, estimates (5.45), (5.48) imply that

$$\operatorname*{ess\ sup}_{t\in(0,T)}\Big\|\Big[\frac{p(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\Big]_{\rm res}\Big\|_{L^1(\Omega)}\le\varepsilon c. \tag{5.100}$$

Thus by means of Proposition 5.2 and estimate (5.100), we multiply the momentum equation (5.29) on ε and let $\varepsilon\to0$ to obtain

$$\int_0^T\int_\Omega\Big(\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\varrho^{(1)}+\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\vartheta^{(1)}\Big)\mathrm{div}_x\varphi\ \mathrm{d}x\ \mathrm{d}t=-\int_0^T\int_\Omega\overline{\varrho}\nabla_x F\cdot\varphi\ \mathrm{d}x \tag{5.101}$$

for all $\varphi\in C_c^\infty((0,T)\times\Omega;\mathbb{R}^3)$, which is nothing other than (5.10).

If we assume, without loss of generality, that

$$\int_\Omega F\ \mathrm{d}x=0, \tag{5.102}$$

relation (5.101) yields the desired conclusion

$$\varrho^{(1)} = -\frac{\partial_\vartheta p}{\partial_\varrho p}(\overline{\varrho}, \overline{\vartheta})\vartheta^{(1)} + \frac{\overline{\varrho}}{\partial_\varrho p(\overline{\varrho}, \overline{\vartheta})} F. \tag{5.103}$$

Expressing $\varrho^{(1)}$ in (5.85) by means of (5.103) and using Gibbs' equation (2.35), we get

$$\begin{aligned} &\int_0^T \int_\Omega \overline{\varrho} c_p(\overline{\varrho}, \overline{\vartheta})\vartheta^{(1)} \Big(\partial_t \varphi + \mathbf{U} \cdot \nabla_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t \\ &\quad - \int_0^T \int_\Omega \Big(\overline{\varrho} \ \overline{\vartheta} \alpha(\overline{\varrho}, \overline{\vartheta}) F \mathbf{U} \cdot \nabla_x \varphi + \kappa(\overline{\vartheta}) \nabla_x \vartheta^{(1)} \cdot \nabla_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t \\ &= - \int_\Omega \overline{\varrho} \ \overline{\vartheta} \Big(\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \varrho_0^{(1)} + \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \vartheta_0^{(1)} + \alpha(\overline{\varrho}, \overline{\vartheta}) F \Big) \varphi(0, \cdot) \ \mathrm{d}x \end{aligned} \tag{5.104}$$

for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$, where the physical constants c_p, α are determined through (4.17), (4.18). Relation (5.104) represents a weak formulation of equation (5.12) with $\Theta = \vartheta^{(1)}$, supplemented with the homogeneous Neumann boundary condition.

Moreover, it follows from estimate (5.49) combined with (5.68), (5.72) that

$$\sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon \to \sqrt{\overline{\varrho}} \mathbf{U} \text{ weakly in } L^\infty(0,T; L^2(\Omega; \mathbb{R}^3)),$$

in particular,

$$\mathbf{U} \in L^\infty(0,T; L^2(\Omega; \mathbb{R}^3)) \cap L^2(0,T; W^{1,2}(\Omega; \mathbb{R}^3)), \tag{5.105}$$

and, consequently,

$$\mathrm{div}_x(\mathbf{U}\vartheta^{(1)}) = \mathbf{U} \cdot \nabla_x \vartheta^{(1)} \in L^q((0,T) \times \Omega) \text{ for a certain } q > 1.$$

Thus we may use the standard L^2-theory for linear parabolic equations combined with the $L^p - L^q$ estimates reviewed in Section 10.14 of Appendix, in order to conclude that

$$\vartheta^{(1)} \in W^{1,q}(\delta, T; L^q(\Omega)) \cap L^q(\delta, T; W^{2,q}(\Omega)) \cap C([0,T]; L^q(\Omega)) \tag{5.106}$$

for a certain $q > 1$ and any $0 < \delta < T$.

Thus, setting $\Theta = \vartheta^{(1)}$, we obtain

$$\overline{\varrho} c_p(\overline{\varrho}, \overline{\vartheta}) \Big(\partial_t \Theta + \mathbf{U} \cdot \nabla_x \Theta \Big) - \overline{\varrho} \ \overline{\vartheta} \alpha(\overline{\varrho}, \overline{\vartheta}) \mathbf{U} \cdot \nabla_x F - \mathrm{div}_x(\kappa(\overline{\vartheta}) \nabla_x \Theta) = 0 \tag{5.107}$$

for a.a. $(t,x) \in (0,T) \times \Omega$,

$$\nabla_x \Theta \cdot \mathbf{n}|_{\partial\Omega} = 0 \text{ in the sense of traces for a.a. } t \in (0,T), \tag{5.108}$$

and

$$c_p(\overline{\varrho},\overline{\vartheta})\Theta(0,\cdot) = \overline{\vartheta}\Big(\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial \varrho}\varrho_0^{(1)} + \frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}\vartheta_0^{(1)} + \alpha(\overline{\varrho},\overline{\vartheta})F\Big) \text{ a.a. in } \Omega. \tag{5.109}$$

Note that we can take $\delta = 0$ in (5.106) as soon as the initial data in (5.109) are more regular (see Section 10.14 in Appendix).

Finally, we deduce the celebrated *Boussinesq relation*

$$r + \overline{\varrho}\alpha(\overline{\varrho},\overline{\vartheta})\Theta = 0 \tag{5.110}$$

putting

$$r = \varrho^{(1)} - \frac{\overline{\varrho}}{\partial_\varrho p(\overline{\varrho},\overline{\vartheta})}F \tag{5.111}$$

in (5.103). Note that $\varrho^{(1)}$ can be replaced by r in (5.95) as the difference multiplied by $\nabla_x F$ is a gradient, irrelevant in the variational formulation based on solenoidal test functions.

5.4 Convergence of the convective term

So far we have almost completely identified the limit problem for the full NAVIER-STOKES-FOURIER SYSTEM in the regime of the low Mach number and low stratification, specifically,

$$\mathrm{Ma} = \varepsilon,\ \mathrm{Fr} = \sqrt{\varepsilon},\ \varepsilon \to 0.$$

The only missing point is to clarify the relation between the weak limit $\overline{\varrho \mathbf{U}\otimes\mathbf{U}}$ and the product of weak limits $\overline{\varrho}\mathbf{U}\otimes\mathbf{U}$ in the momentum equation (5.95).

As already pointed out in Section 4.4.1, we do not really expect to show that

$$\overline{\varrho \mathbf{U}\otimes\mathbf{U}} = \overline{\varrho}\mathbf{U}\otimes\mathbf{U},$$

however, we may still hope to prove a weaker statement

$$\int_0^T\int_\Omega \overline{\varrho \mathbf{U}\otimes\mathbf{U}} : \nabla_x\varphi \ \mathrm{d}x\ \mathrm{d}t = \int_0^T\int_\Omega [\overline{\varrho}\mathbf{U}\otimes\mathbf{U}] : \nabla_x\varphi \ \mathrm{d}x\ \mathrm{d}t \tag{5.112}$$

for any

$$\varphi \in C_c^\infty((0,T)\times\overline{\Omega};\mathbb{R}^3),\ \mathrm{div}_x\varphi = 0,\ \varphi\cdot\mathbf{n}|_{\partial\Omega} = 0.$$

Relation (5.112) can be interpreted in the way that the difference

$$\mathrm{div}_x\left(\overline{\varrho \mathbf{U}\otimes\mathbf{U}} - \overline{\varrho}\mathbf{U}\otimes\mathbf{U}\right)$$

is proportional to a gradient that may be incorporated into the limit pressure; whence (5.112) is sufficient for replacing $\overline{\varrho \mathbf{U}\otimes\mathbf{U}}$ by $\overline{\varrho}\mathbf{U}\otimes\mathbf{U}$ in (5.95) as required.

The remaining part of this section is devoted to the proof of (5.112). The main ingredients include:

- Helmholtz decomposition of the momentum;
- proof of compactness of the solenoidal part;
- analysis of the acoustic equation governing the time evolution of the gradient component.

5.4.1 Helmholtz decomposition

Before commencing a rigorous analysis, we have to identify the *solenoidal part* (divergenceless, incompressible) and the *gradient part* (acoustic) of a given vector field. The following material is classical and may be found in most of the modern textbooks devoted to mathematical fluid mechanics (see Section 10.6 in Appendix).

■ HELMHOLTZ DECOMPOSITION:

A vector function $\mathbf{v} : \Omega \to \mathbb{R}^3$ is written as

$$\mathbf{v} = \underbrace{\mathbf{H}[\mathbf{v}]}_{\text{solenoidal part}} + \underbrace{\mathbf{H}^{\perp}[\mathbf{v}]}_{\text{gradient part}} ,$$

where

$$\mathbf{H}^{\perp}[\mathbf{v}] = \nabla_x \Psi,$$

$$\Delta \Psi = \operatorname{div}_x \mathbf{v} \text{ in } \Omega, \quad \nabla_x \Psi \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{v} \cdot \mathbf{n}, \quad \int_{\Omega} \Psi \, \mathrm{d}x = 0. \tag{5.113}$$

The standard variational formulation of problem (5.113) reads

$$\int_{\Omega} \nabla_x \Psi \cdot \nabla_x \varphi \, \mathrm{d}x = \int_{\Omega} \mathbf{v} \cdot \nabla_x \varphi \, \mathrm{d}x, \; \int_{\Omega} \Psi \, \mathrm{d}x = 0 \tag{5.114}$$

to be satisfied for any test function $\varphi \in C_c^\infty(\overline{\Omega})$. In particular, as a direct consequence of the standard L^p-theory of elliptic operators (see Section 10.2.1 in Appendix), it can be shown that the *Helmholtz projectors*

$$\mathbf{v} \mapsto \begin{cases} \mathbf{H}[\mathbf{v}], \\ \mathbf{H}^{\perp}[\mathbf{v}] \end{cases}$$

map continuously the spaces $L^p(\Omega;\mathbb{R}^3)$ and $W^{1,p}(\Omega;\mathbb{R}^3)$ into itself for any $1 < p < \infty$ as soon as $\partial\Omega$ is at least of class C^2.

5.4.2 Compactness of the solenoidal part

Keeping in mind our ultimate goal, meaning a rigorous justification of (5.112), we show first that the solenoidal part of the momentum $\mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon]$ does not exhibit oscillations in time; in particular, it converges a.a. in the set $(0,T) \times \Omega$. To this end, take $\mathbf{H}[\varphi]$, $\varphi \in C_c^\infty([0,T) \times \overline{\Omega}; \mathbb{R}^3)$, $\varphi \cdot \mathbf{n} = 0$, as a test function in the variational formulation of the momentum equation (5.29). Note that the normal trace of $\mathbf{H}[\varphi]$ vanishes on $\partial\Omega$ together with that of φ. Consequently, in accordance with the uniform estimates obtained in Section 5.2, notably (5.46), (5.48), and (5.49), we conclude that

$$\mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \to \mathbf{H}[\overline{\varrho}\mathbf{U}] = \overline{\varrho}\mathbf{U} \text{ in } C_{\text{weak}}([0,T]; L^{\frac{5}{4}}(\Omega;\mathbb{R}^3)), \tag{5.115}$$

where we have used (5.73). Note that, similarly to (5.95), the singular terms in equation (5.29) are irrelevant as $\mathrm{div}_x \mathbf{H}[\varphi] = 0$.

In addition, by virtue of (5.71), (5.115), we have

$$\overline{\varrho}\mathbf{H}[\mathbf{u}_\varepsilon] \cdot \mathbf{u}_\varepsilon = \Big(\varepsilon \mathbf{H}[\frac{\overline{\varrho} - \varrho_\varepsilon}{\varepsilon} \mathbf{u}_\varepsilon] + \mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \Big) \cdot \mathbf{u}_\varepsilon \to \overline{\varrho}|\mathbf{U}|^2 \text{ weakly in } L^1(\Omega);$$

in particular,

$$\int_0^T \int_\Omega |\mathbf{H}[\mathbf{u}_\varepsilon]|^2 \, \mathrm{d}x = \int_\Omega \mathbf{H}[\mathbf{u}_\varepsilon] \cdot \mathbf{u}_\varepsilon \, \mathrm{d}x \to \int_\Omega |\mathbf{U}|^2 \, \mathrm{d}x.$$

As $\mathbf{U} = \mathbf{H}[\mathbf{U}]$, the last relation allows us to conclude that

$$\mathbf{H}[\mathbf{u}_\varepsilon] \to \mathbf{U} \text{ in } L^2(0,T; L^2(\Omega;\mathbb{R}^3)). \tag{5.116}$$

Coming back to (5.112) we write

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon = \mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{u}_\varepsilon + \mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}[\mathbf{u}_\varepsilon] + \mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}^\perp[\mathbf{u}_\varepsilon],$$

where, by means of (5.68), (5.115), the compact embedding $W^{1,2}(\Omega) \hookrightarrow L^5(\Omega)$, and the arguments used in (3.232–3.233),

$$\mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{u}_\varepsilon \to \overline{\varrho}\mathbf{U} \otimes \mathbf{U} \text{ weakly in } L^2(0,T; L^{\frac{30}{29}}(\Omega; R^{3\times 3})). \tag{5.117}$$

Moreover, combining (5.116) with (5.90) we infer that

$$\mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}[\mathbf{u}_\varepsilon] \to 0 \text{ weakly in } L^2(0,T; L^{\frac{30}{29}}(\Omega;\mathbb{R}^{3\times 3})). \tag{5.118}$$

In the previous discussion we have repeatedly used the continuity of the Helmholtz projectors on L^p and $W^{1,p}$.

In view of (5.117), (5.118), the proof of relation (5.112) reduces to showing

$$\int_0^T \int_\Omega \mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}^\perp[\mathbf{u}_\varepsilon] : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t \to 0 \tag{5.119}$$

for any

$$\varphi \in C_c^\infty((0,T) \times \overline{\Omega}; \mathbb{R}^3), \ \mathrm{div}_x \varphi = 0, \ \varphi \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

A priori, our uniform estimates do not provide any bound on the time derivative of the gradient part of the velocity. Verification of (5.119) must be therefore based on a detailed knowledge of possible time oscillations and their mutual cancellations in the acoustic waves described by means of $\mathbf{H}^{\perp}[\varrho_\varepsilon \mathbf{u}]$ governed by the acoustic equation introduced in Section 4.4.1. Accordingly, the next three sections are devoted to a detailed deduction of the acoustic equation and the spectral analysis of the corresponding wave operator. The proof of relation (5.119) is postponed to Sections 5.4.6, 5.4.7.

5.4.3 Acoustic equation

A formal derivation of the acoustic equation was given in Section 4.4.1. Here we consider a variational formulation in the spirit of Chapter 2. To this end, we write system (5.28–5.29) in the form:

$$\int_0^T \int_\Omega \Big(\varepsilon \varrho_\varepsilon^{(1)} \partial_t \varphi + \mathbf{V}_\varepsilon \cdot \nabla_x \varphi \Big) \, \mathrm{d}x \, \mathrm{d}t = 0, \tag{5.120}$$

for any $\varphi \in C_c^\infty((0,T) \times \overline{\Omega})$,

$$\begin{aligned} &\int_0^T \int_\Omega \Big(\varepsilon \mathbf{V}_\varepsilon \cdot \partial_t \varphi + \Big[\frac{[p(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\mathrm{ess}} - p(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} - \overline{\varrho} F \Big] \mathrm{div}_x \varphi \Big) \, \mathrm{d}x \, \mathrm{d}t \\ &\quad = \int_0^T \int_\Omega (\overline{\varrho} - \varrho_\varepsilon) \nabla_x F \cdot \varphi \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_\Omega \mathsf{h}_\varepsilon^1 : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t \end{aligned} \tag{5.121}$$

for any $\varphi \in C_c^\infty((0,T) \times \overline{\Omega}; \mathbb{R}^3)$, $\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$, where we have set

$$\varrho_\varepsilon^{(1)} = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}, \ \mathbf{V}_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon, \quad \text{and} \quad \mathsf{h}_\varepsilon^1 = \varepsilon \mathbb{S}_\varepsilon - \varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon - \frac{[p(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\mathrm{res}}}{\varepsilon} \mathbb{I}.$$

Similarly, the entropy balance equation (5.31) can be rewritten with help of (5.28) as

$$\begin{aligned} &\int_0^T \int_\Omega \varepsilon \Big(\varrho_\varepsilon \frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \Big) \partial_t \varphi \, \mathrm{d}x \\ &\quad = \int_0^T \int_\Omega \mathbf{h}_\varepsilon^2 \cdot \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t - \langle \sigma_\varepsilon; \varphi \rangle_{[\mathcal{M};C]([0,T] \times \overline{\Omega})} \end{aligned} \tag{5.122}$$

for any $\varphi \in C_c^\infty((0,T) \times \overline{\Omega})$, where

$$\mathbf{h}_\varepsilon^2 = \frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \nabla_x \vartheta_\varepsilon + \Big(\varrho_\varepsilon s(\overline{\varrho}, \overline{\vartheta}) - \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) \Big) \mathbf{u}_\varepsilon.$$

Following the ideas delineated in Section 4.4.1 we have

$$\frac{[p(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\mathrm{ess}} - p(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} = \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \varrho_\varepsilon^{(1)} + \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \vartheta_\varepsilon^{(1)} + h_\varepsilon^3, \quad \vartheta_\varepsilon^{(1)} = \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon},$$

and, analogously,

$$\begin{aligned}
&\varrho_\varepsilon \frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \\
&= \left[\varrho_\varepsilon \frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \right]_{\rm ess} + \left[\varrho_\varepsilon \frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \right]_{\rm res} \\
&= \overline{\varrho} \left[\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \varrho_\varepsilon^{(1)} + \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \vartheta_\varepsilon^{(1)} \right] + \left[\varrho_\varepsilon \frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \right]_{\rm res} + h_\varepsilon^4,
\end{aligned} \tag{5.123}$$

where, by virtue of Proposition 5.2, specifically (5.65),

$$\operatorname*{ess\ sup}_{t \in (0,T)} \int_\Omega |h_\varepsilon^3(t)| \ {\rm d}x \leq \varepsilon c, \tag{5.124}$$

$$\operatorname*{ess\ sup}_{t \in (0,T)} \int_\Omega |h_\varepsilon^4(t)| \ {\rm d}x \leq \varepsilon c, \tag{5.125}$$

since p and s are twice continuously differentiable on the set $(0, \infty)^2$.

Now, we rewrite system (5.120–5.122) in terms of new independent variables

$$r_\varepsilon = \frac{1}{\omega} \Big(\omega \varrho_\varepsilon^{(1)} + A \varrho_\varepsilon \frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} - \overline{\varrho} F \Big), \quad \mathbf{V}_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon,$$

where we have set

$$\omega = \partial_\varrho p(\overline{\varrho}, \overline{\vartheta}) + \frac{|\partial_\vartheta p(\overline{\varrho}, \overline{\vartheta})|^2}{\overline{\varrho}^2 \ \partial_\vartheta s(\overline{\varrho}, \overline{\vartheta})} \quad \text{and} \quad A = \frac{\partial_\vartheta p(\overline{\varrho}, \overline{\vartheta})}{\overline{\varrho} \ \partial_\vartheta s(\overline{\varrho}, \overline{\vartheta})}. \tag{5.126}$$

After a bit lengthy but straightforward manipulation we arrive at the system

$$\int_0^T \int_\Omega \Big(\varepsilon r_\varepsilon \partial_t \varphi + \mathbf{V}_\varepsilon \cdot \nabla_x \varphi \Big) \ {\rm d}x \ {\rm d}t = \frac{A}{\omega} \Big[\int_0^T \int_\Omega \mathbf{h}_\varepsilon^2 \cdot \nabla_x \varphi \ {\rm d}x \ {\rm d}t - \langle \sigma_\varepsilon; \varphi \rangle \Big] \tag{5.127}$$

for any $\varphi \in C_c^\infty((0,T) \times \overline{\Omega})$,

$$\begin{aligned}
&\int_0^T \int_\Omega \Big(\varepsilon \mathbf{V}_\varepsilon \cdot \partial_t \varphi + \omega r_\varepsilon {\rm div}_x \varphi \Big) \ {\rm d}x \ {\rm d}t \\
&= \int_0^T \int_\Omega (\overline{\varrho} - \varrho_\varepsilon) \nabla_x F \cdot \varphi \ {\rm d}x \ {\rm d}t + \int_0^T \int_\Omega \Big(\mathsf{h}_\varepsilon^1 : \nabla_x \varphi - h_\varepsilon^3 {\rm div}_x \varphi \Big) \ {\rm d}x \ {\rm d}t \\
&\quad + A \int_0^T \int_\Omega \left(\left[\varrho_\varepsilon \frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \right]_{\rm res} + h_\varepsilon^4 \right) {\rm div}_x \varphi \ {\rm d}x \ {\rm d}t
\end{aligned} \tag{5.128}$$

for any $\varphi \in C_c^\infty((0,T) \times \overline{\Omega}; \mathbb{R}^3)$, $\varphi \cdot \mathbf{n}|_{\partial \Omega} = 0$. System (5.127), (5.128) represents a non-homogeneous variant of the acoustic equation (4.26).

Our ultimate goal in this section is to show that the quantities on the right-hand side of (5.127), (5.128) vanish for $\varepsilon \to 0$. In order to see this, we use first the uniform estimates (5.46), (5.48) to obtain

$$\text{ess} \sup_{t\in(0,T)} \|(\varrho_\varepsilon - \overline{\varrho})\nabla_x F\|_{L^{\frac{5}{3}}(\Omega;\mathbb{R}^3)} = \varepsilon \text{ ess} \sup_{t\in(0,T)} \left\| \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \nabla_x F \right\|_{L^{\frac{5}{3}}(\Omega;\mathbb{R}^3)} \leq \varepsilon c, \tag{5.129}$$

and, by virtue of (5.92), (5.94), (5.100),

$$\|\mathsf{h}^1_\varepsilon\|_{L^q(0,T;L^1(\Omega;\mathbb{R}^{3\times 3}))} \leq \varepsilon c \text{ for a certain } q > 1. \tag{5.130}$$

In a similar way, relation (5.44) together with (5.45), (5.48) give rise to

$$\text{ess} \sup_{t\in(0,T)} \left\| \left[\varrho_\varepsilon \frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \right]_{\text{res}} \right\|_{L^1(\Omega)} \leq \varepsilon c. \tag{5.131}$$

Finally, writing

$$\mathbf{h}^2_\varepsilon = \varepsilon \frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \nabla_x \frac{\vartheta_\varepsilon}{\varepsilon} - \varepsilon \Big(\frac{[\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\text{ess}} - \overline{\varrho} s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \Big) \mathbf{u}_\varepsilon$$
$$- \varepsilon \Big[\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon} \Big]_{\text{res}} \mathbf{u}_\varepsilon + \varepsilon \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} s(\overline{\varrho}, \overline{\vartheta}) \mathbf{u}_\varepsilon$$

we can use estimates (5.51), (5.53), (5.56), and (5.71), together with Proposition 5.2, in order to conclude that

$$\|\mathbf{h}^2_\varepsilon\|_{L^q(0,T;L^q(\Omega;\mathbb{R}^3))} \leq \varepsilon c \text{ for a certain } q > 1. \tag{5.132}$$

Having established all necessary estimates, we can use (5.123–5.125), together with (5.129–5.132), in order to rewrite system (5.127), (5.128) in a more concise form. We should always keep in mind, however, that the resulting problem is nothing other than the primitive NAVIER-STOKES-FOURIER SYSTEM conveniently rearranged in the form of an *acoustic analogy* in the spirit of Lighthill [135] discussed in Section 4.5.

■ SCALED ACOUSTIC EQUATION:

$$\int_0^T \int_\Omega \Big(\varepsilon r_\varepsilon \partial_t \varphi + \mathbf{V}_\varepsilon \cdot \nabla_x \varphi \Big) \, \mathrm{d}x \, \mathrm{d}t$$
$$= \frac{A}{\omega} \left(\int_0^T \int_\Omega \mathbf{h}^2_\varepsilon \cdot \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t - \langle \sigma_\varepsilon; \varphi \rangle_{[\mathcal{M};C]([0,T]\times\overline{\Omega})} \right) \tag{5.133}$$

for any $\varphi \in C^\infty_c((0,T) \times \overline{\Omega})$,

$$\int_0^T \int_\Omega \Big(\varepsilon \mathbf{V}_\varepsilon \cdot \partial_t \varphi + \omega r_\varepsilon \mathrm{div}_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t$$
$$= \int_0^T \int_\Omega \Big(\mathsf{h}^5_\varepsilon : \nabla_x \varphi + \mathbf{h}^6_\varepsilon \cdot \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t \tag{5.134}$$

for any $\varphi \in C^\infty_c((0,T) \times \overline{\Omega}; \mathbb{R}^3)$, $\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$.

In accordance with the previous estimates, the functions $\mathbf{h}^2_\varepsilon$, h^5_ε, and $\mathbf{h}^6$ satisfy

$$\left\{ \begin{array}{c} \|\mathbf{h}^2_\varepsilon\|_{L^q(0,T;L^1(\Omega;\mathbb{R}^3))} + \|\mathbf{h}^6_\varepsilon\|_{L^q(0,T;L^1(\Omega;\mathbb{R}^3))} \leq \varepsilon c, \\ \|\mathsf{h}^5_\varepsilon\|_{L^q(0,T;L^1(\Omega;\mathbb{R}^{3\times 3}))} \leq \varepsilon c \end{array} \right\} \tag{5.135}$$

for a certain $q > 1$, and, in accordance with (5.50), $\sigma_\varepsilon \in \mathcal{M}^+([0,T] \times \overline{\Omega})$ is a non-negative measure such that

$$| \langle \sigma_\varepsilon ; \varphi \rangle_{[\mathcal{M};C]([0,T]\times\overline{\Omega})} | \leq \varepsilon^2 c \|\varphi\|_{C([0,T]\times\overline{\Omega})}. \tag{5.136}$$

5.4.4 Formal analysis of the acoustic equation

In view of estimates (5.135), (5.136), the right-hand side of system (5.133), (5.134) is small of order ε. In particular, these terms are negligible in the fast time scaling $\tau \approx t/\varepsilon$ as we have observed in Section 4.4.1. In order to get a better insight into the complexity of the wave phenomena described by the *acoustic equation*, we perform a formal asymptotic analysis of (5.133), (5.134) in real time t, keeping all quantities of order ε and lower. Such a procedure leads formally to the system

$$\varepsilon \partial_t r_\varepsilon + \mathrm{div}_x \mathbf{V}_\varepsilon = \varepsilon G^1_\varepsilon, \tag{5.137}$$
$$\varepsilon \partial_t \mathbf{V}_\varepsilon + \omega \nabla_x r_\varepsilon = \varepsilon \mathbf{G}^2_\varepsilon, \tag{5.138}$$

with

$$G^1_\varepsilon = \frac{A}{\varepsilon\omega} \Big(\mathrm{div}_x \mathbf{h}^2_\varepsilon + \langle \sigma_\varepsilon ; \varphi \rangle_{[\mathcal{M};C]([0,T]\times\overline{\Omega})} \Big), \tag{5.139}$$
$$\mathbf{G}^2_\varepsilon = \frac{1}{\varepsilon} \left(\mathrm{div}_x \mathsf{h}^5_\varepsilon - \mathbf{h}^6_\varepsilon \right). \tag{5.140}$$

Solutions of the linear system (5.137), (5.138) can be expressed by means of *Duhamel's formula*

$$\left[\begin{array}{c} r_\varepsilon(t) \\ \mathbf{V}_\varepsilon(t) \end{array} \right] = S\left(\frac{t}{\varepsilon}\right) \left[\begin{array}{c} r_\varepsilon(0) \\ \mathbf{V}_\varepsilon(0) \end{array} \right] + \int_0^t \left(S\left(\frac{t-s}{\varepsilon}\right) \left[\begin{array}{c} G^1_\varepsilon(s) \\ \mathbf{G}^2_\varepsilon(s) \end{array} \right] \right) \ \mathrm{d}s,$$

where

$$S(t) \left[\begin{array}{c} R_0 \\ \mathbf{Q}_0 \end{array} \right] = \left[\begin{array}{c} R(t) \\ \mathbf{Q}(t) \end{array} \right] \tag{5.141}$$

is the solution group of the homogeneous problem

$$\partial_t R + \mathrm{div}_x \mathbf{Q} = 0,\ \partial_t \mathbf{Q} + \omega \nabla_x R,\ \mathbf{Q}\cdot\mathbf{n}|_{\partial\Omega} = 0,\ R(0) = R_0,\ \mathbf{Q}(0) = \mathbf{Q}_0.$$

We easily deduce the energy equality

$$\int_\Omega \frac{1}{2}\left(\omega R^2 + |\mathbf{Q}|^2\right)(t)\ \mathrm{d}x = \int_\Omega \frac{1}{2}\left(\omega R_0^2 + |\mathbf{Q}_0|^2\right)\ \mathrm{d}x$$

satisfied for any $t \in \mathbb{R}$. In particular, the mapping $t \mapsto S(t)$ represents a group of isomorphisms on the Hilbert space $L^2(\Omega) \times L^2(\Omega; \mathbb{R}^3)$.

For the sake of simplicity, assume that G_ε^1, $\mathbf{G}_\varepsilon^2$ are more regular in the x-variable than guaranteed by (5.135), (5.136), namely

$$\|G_\varepsilon^1\|_{L^q(0,T;W^{1,2}(\Omega))} \le c,\ \|\mathbf{G}_\varepsilon^2\|_{L^q(0,T;W^{1,2}(\Omega;\mathbb{R}^3))} \le c \text{ for a certain } q > 1$$

uniformly with respect to ε.

Writing

$$\left[\begin{array}{c} r_\varepsilon(t) \\ \mathbf{V}_\varepsilon(t) \end{array}\right] = S\left(\frac{t}{\varepsilon}\right)\left[\left[\begin{array}{c} r_\varepsilon(0) \\ \mathbf{V}_\varepsilon(0) \end{array}\right] + \int_0^t \left(S\left(\frac{-s}{\varepsilon}\right)\left[\begin{array}{c} G_\varepsilon^1(s) \\ \mathbf{G}_\varepsilon^2(s) \end{array}\right]\right)\ \mathrm{d}s\right]$$

we observe that the family of mappings

$$t \in [0,T] \mapsto \left[\left[\begin{array}{c} r_\varepsilon(0) \\ \mathbf{V}_\varepsilon(0) \end{array}\right] + \int_0^t \left(S\left(\frac{-s}{\varepsilon}\right)\left[\begin{array}{c} G_\varepsilon^1(s) \\ \mathbf{G}_\varepsilon^2(s) \end{array}\right]\right)\ \mathrm{d}s\right]$$

is *precompact* in the space $C([0,T]; L^2(\Omega) \times L^2(\Omega; \mathbb{R}^3))$ provided

$$r_\varepsilon(0) \to r_0 \text{ in } L^2(\Omega),\ \mathbf{V}_\varepsilon(0) \to \mathbf{V}_0 \text{ in } L^2(\Omega; \mathbb{R}^3).$$

Consequently, we have

$$\left.\begin{array}{r} \sup_{t\in[0,T]} \|r_\varepsilon(t) - R_\varepsilon(t)\|_{L^2(\Omega)} \to 0, \\ \sup_{t\in[0,T]} \|\mathbf{V}_\varepsilon(t) - \mathbf{Q}_\varepsilon(t)\|_{L^2(\Omega;R^3)} \to 0 \end{array}\right\} \text{ for } \varepsilon \to 0,$$

where

$$\left[\begin{array}{c} R_\varepsilon(t) \\ \mathbf{Q}_\varepsilon(t) \end{array}\right] = S\left(\frac{t}{\varepsilon}\right)\left[\left[\begin{array}{c} r_0 \\ \mathbf{V}_0 \end{array}\right] + \int_0^t \left[\begin{array}{c} F^1(s) \\ \mathbf{F}^2(s) \end{array}\right]\ \mathrm{d}s\right], \tag{5.142}$$

and where $[F^1, \mathbf{F}^2]$ denote a weak limit of

$$S\left(-\frac{s}{\varepsilon}\right)\left[\begin{array}{c} G_\varepsilon^1(s) \\ \mathbf{G}_\varepsilon(s) \end{array}\right]$$

in $L^q(0,T; L^2(\Omega) \times L^2(\Omega; \mathbb{R}^3))$.

Finally, (5.142) can be written in the form

$$\varepsilon \partial_t R_\varepsilon + \mathrm{div}_x \mathbf{Q}_\varepsilon = \varepsilon H^1_\varepsilon, \tag{5.143}$$

$$\varepsilon \partial_t \mathbf{Q}_\varepsilon + \omega \nabla_x R_\varepsilon = \varepsilon \mathbf{H}^2_\varepsilon, \tag{5.144}$$

with

$$\begin{bmatrix} H^1_\varepsilon(t) \\ \mathbf{H}^2_\varepsilon(t) \end{bmatrix} = S\left(\frac{t}{\varepsilon}\right) \begin{bmatrix} F^1(t) \\ \mathbf{F}^2(t) \end{bmatrix}.$$

System (5.143), (5.144) may be regarded as a scaled variant of *Lighthill's equation* (4.36) discussed in Section 4.5, where the acoustic source terms can be determined in terms of *fixed* functions F^1, $\mathbf{F}^2$. For a fixed $\varepsilon > 0$, system (5.143), (5.144) provides a reasonable approximation of propagation of the acoustic waves in the *low Mach number* regime.

In practice the functions F^1, $\mathbf{F}^2$, or even their oscillating counterparts H^1_ε, $\mathbf{H}^2_\varepsilon$, should be fixed by experiments. This is the basis of the so-called hybrid methods in numerical analysis, where the source terms in the acoustic equation are determined by means of a suitable hydrodynamic approximation. Very often, the limit passage is performed at the first step, where the functions G^1_ε, $\mathbf{G}^2_\varepsilon$ are being replaced by their formal incompressible limits for $\varepsilon \to 0$ (see Golanski et al. [101]). As we have seen, however, the effective form of the acoustic sources has to be deduced as a kind of time average of highly oscillating quantities on which we do not have any control in the low Mach number limit. This observation indicates certain limitations of the hybrid methods used in numerical simulations. Indeed as we show in Chapter 9, any method based on *linear acoustic analogy* can be effective only for problems with well-prepared data. This is due to the fact that the wave operator used in (5.143), (5.144) is *linear* thus applicable only to small perturbations of the limit problem. In the case of ill-prepared data, the non-linear character of the equations, in particular of the convective term, must be taken into account in order to obtain reliable results. These topics will be further elaborated in Chapter 9.

The purpose of the previous discussion was to motivate the following steps in the analysis of the low Mach number limit. In particular, we shall reduce the problem to a finite number of modes represented by the eigenfunctions of the wave operator in (5.133), (5.134) that are smooth in the x-variable. Inspired by the formal approach delineated above, we show that the non-vanishing oscillatory part of the convective term can be represented by a gradient of a scalar function irrelevant in the incompressible limit.

5.4.5 Spectral analysis of the wave operator

We consider the following *eigenvalue problem* associated to the operator on the left-hand side of (5.133), (5.134):

$$\mathrm{div}_x \mathbf{w} = \lambda q, \ \omega \nabla_x q = \lambda \mathbf{w} \text{ in } \Omega, \ \mathbf{w} \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{5.145}$$

System (5.145) can be reformulated as a homogeneous Neumann problem

$$-\Delta_x q = \Lambda q \text{ in } \Omega,\ \nabla_x q \cdot \mathbf{n}|_{\partial\Omega} = 0,\ -\Lambda = \frac{\lambda^2}{\omega}. \tag{5.146}$$

As is well known, problem (5.146) admits a countable system of eigenvalues

$$0 = \Lambda_0 < \Lambda_1 \leq \Lambda_2 \leq \cdots$$

with the associated system of (real) eigenfunctions $\{q_n\}_{n=0}^\infty$ forming an orthogonal basis of the Hilbert spaces $L^2(\Omega)$ (see Theorem 10.7 in Appendix). The corresponding (complex) eigenfunctions $\mathbf{w}_{\pm n}$ are determined through (5.145) as

$$\mathbf{w}_{\pm n} = \pm \mathrm{i}\sqrt{\frac{\omega}{\Lambda_n}}\nabla_x q_n,\ n = 1, 2, \ldots .$$

Furthermore, the Hilbert space $L^2(\Omega;\mathbb{R}^3)$ admits an orthogonal decomposition

$$L^2(\Omega;\mathbb{R}^3) = L^2_\sigma(\Omega;\mathbb{R}^3) \oplus L^2_g(\Omega;\mathbb{R}^3), \tag{5.147}$$

where

$$L^2_g(\Omega;\mathbb{R}^3) = \mathrm{closure}_{L^2}\Big\{\mathrm{span}\ \{\frac{-\mathrm{i}}{\sqrt{\omega}}\ \mathbf{w}_n\}_{n=1}^\infty\Big\},$$

and where the symbol L^2_σ denotes the subspace of divergenceless functions

$$L^2_\sigma(\Omega;\mathbb{R}^3) = \mathrm{closure}_{L^2}\{\mathbf{v} \in C^\infty_c(\Omega;\mathbb{R}^3)\ |\ \mathrm{div}_x\mathbf{v} = 0 \text{ in } \Omega\}$$

(see Sections 10.6, 10.2.2 in Appendix).

5.4.6 Reduction to a finite number of modes

Having collected the necessary material, we go back to problem (5.119). To begin, we make use of spatial compactness of $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ in order to reduce the problem to a finite number of modes associated to the eigenfunctions $\{\mathrm{i}\ \mathbf{w}_n\}_{n=1}^\infty$ introduced in (5.147). To this end, let

$$\mathbf{P}_M : L^2(\Omega;\mathbb{R}^3) \to \mathrm{span}\{\frac{-\mathrm{i}}{\sqrt{\omega}}\ \mathbf{w}_n\}_{n\leq M},\ M = 1, 2, \ldots$$

denote the corresponding orthogonal projectors. Note that $\mathbf{P}_M$ commutes with $\mathbf{H}^\perp$ and set

$$\mathbf{H}^\perp_M[\mathbf{v}] = \mathbf{P}_M\mathbf{H}^\perp[\mathbf{v}] = \mathbf{H}^\perp[\mathbf{P}_M\mathbf{v}].$$

For any function $\mathbf{v} \in L^1(\Omega;\mathbb{R}^3)$ we introduce the Fourier coefficients

$$a_n[\mathbf{v}] = \frac{-\mathrm{i}}{\sqrt{\omega}}\int_\Omega \mathbf{v}\cdot\mathbf{w}_n\ \mathrm{d}x \tag{5.148}$$

along with a scale of Hilbert spaces $H_g^\alpha(\Omega;\mathbb{R}^3) \subset L_g^2(\Omega;\mathbb{R}^3)$ endowed with the norm

$$\|\mathbf{v}\|^2_{H_g^\alpha} = \sum_{n=1}^{\infty} \Lambda_n^\alpha |a_n[\mathbf{v}]|^2,$$

where $\{\Lambda_n\}_{n=0}^\infty$ is the family of eigenvalues associated to the Neumann problem (5.146). It is easy to check that $\|\cdot\|_{H_g^1(\Omega;\mathbb{R}^3)}$ is equivalent to the standard Sobolev norm $\|\cdot\|_{W^{1,2}(\Omega;\mathbb{R}^3)}$ restricted to the space $H_g^1(\Omega;\mathbb{R}^3)$.

A straightforward computation yields

$$\|\mathbf{H}^\perp[\mathbf{v}] - \mathbf{H}^\perp_M[\mathbf{v}]\|^2_{H_g^{\alpha_1}} = \sum_{n=M}^{\infty} \Lambda_n^{\alpha_1}|a_n[\mathbf{v}]|^2 \leq {\Lambda_M}^{\alpha_1-\alpha_2} \sum_{n=M}^{\infty} \Lambda_n^{\alpha_2}|a_n[\mathbf{v}]|^2 \tag{5.149}$$
$$= {\Lambda_M}^{\alpha_1-\alpha_2} \|\mathbf{H}^\perp[\mathbf{v}] - \mathbf{H}^\perp_M[\mathbf{v}]\|^2_{H_g^{\alpha_2}(\Omega;\mathbb{R}^3)} \text{ for } \alpha_2 \geq \alpha_1.$$

Moreover, since $H_g^0 = L_g^2$ and $H_g^1(\Omega;\mathbb{R}^3) \hookrightarrow L^6(\Omega;\mathbb{R}^3)$, a simple interpolation argument yields

$$H_g^\alpha(\Omega;\mathbb{R}^3) \hookrightarrow L^5(\Omega;\mathbb{R}^3) \text{ whenever } \alpha \geq \frac{9}{10}. \tag{5.150}$$

Consequently, writing the quantity (5.119) in the form

$$\int_0^T \int_\Omega \mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}^\perp[\mathbf{u}_\varepsilon] : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t$$
$$= \int_0^T \int_\Omega \mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}^\perp_M[\mathbf{u}_\varepsilon] : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t$$
$$+ \int_0^T \int_\Omega \mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes (\mathbf{H}^\perp[\mathbf{u}_\varepsilon] - \mathbf{H}^\perp_M[\mathbf{u}_\varepsilon]) : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t$$

we can use the uniform estimate (5.91), together with (5.149), (5.150), in order to conclude that

$$\left| \int_0^T \int_\Omega \mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes (\mathbf{H}^\perp[\mathbf{u}_\varepsilon] - \mathbf{H}^\perp_M[\mathbf{u}_\varepsilon]) : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t \right| \leq c {\Lambda_M}^{-\frac{1}{20}}$$

uniformly with respect to $\varepsilon \to 0$ for any fixed φ, where $\Lambda_M \to \infty$ for $M \to \infty$.

Similarly, by means of (5.150) and a simple duality argument,

$$\| \mathbf{H}^\perp[\mathbf{v}] - \mathbf{H}^\perp_M[\mathbf{v}] \|^2_{[W^{1,2}(\Omega;\mathbb{R}^3)]^*} \leq c M^{-\frac{1}{10}} \|\mathbf{v}\|^2_{L^{\frac{5}{4}}(\Omega;\mathbb{R}^3)},$$

where we have identified $\mathbf{v}$ with a bounded linear form on $W^{1,2}(\Omega;\mathbb{R}^3)$ via the standard Riesz formula

$$\langle \mathbf{v}; \varphi \rangle_{[W^{1,2}]^*;W^{1,2}} = \int_\Omega \mathbf{v} \cdot \varphi \, \mathrm{d}x.$$

In view of these arguments, the proof of (5.119) reduces to showing

$$\int_0^T \int_\Omega \mathbf{H}_M^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}_M^\perp[\mathbf{u}_\varepsilon] : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t \to 0,$$

or, equivalently, in agreement with (5.72),

$$\int_0^T \int_\Omega \mathbf{H}_M^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}_M^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t \to 0, \tag{5.151}$$

for any fixed $M \geq 1$ and any $\varphi \in C_c^\infty((0,T) \times \overline{\Omega}; \mathbb{R}^3)$, $\mathrm{div}_x \varphi = 0$, $\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$.

In principle, the operator $\mathbf{H}_M^\perp$ in (5.151) could have been replaced by any *smoothing operator*, for instance, a spatial convolution with a suitable family of regularizing kernels. The advantage of our choice based on the spectral decomposition of the wave operator is that the problem is reduced to a *finite number* of ordinary differential equations.

5.4.7 Weak limit of the convective term – time lifting

The analysis of the asymptotic limit of system (5.1–5.4) will be completed as soon as we establish (5.151). To this end, we exploit the fact that the time oscillations of the quantities $\mathbf{H}_M^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon]$ are completely determined by means of the scaled acoustic equation (5.133), (5.134).

We start noticing that equation (5.133) contains the measure σ_ε as a forcing term. In particular, the corresponding solutions of the acoustic equation (5.133), (5.134) may not be continuous with respect to time. In order to eliminate this rather unpleasant phenomenon, we use the method of time-lifting introducing the "primitive" Σ_ε through formula

$$\langle \Sigma_\varepsilon; \varphi \rangle_{[\mathcal{M};C]([0,T]\times\overline{\Omega})} = \langle \sigma_\varepsilon; I[\varphi] \rangle_{[\mathcal{M};C]([0,T]\times\overline{\Omega})},$$

where we set

$$I[\varphi](\tau, x) = \int_0^\tau \varphi(t,x) \, \mathrm{d}t \text{ for all } \varphi \in C([0,T] \times \overline{\Omega}).$$

Accordingly, system (5.133), (5.134) can be rewritten in the form

$$\int_0^T \int_\Omega \Big(\varepsilon Z_\varepsilon \partial_t \varphi + \mathbf{V}_\varepsilon \cdot \nabla_x \varphi \Big) \, \mathrm{d}x \, \mathrm{d}t = \frac{A}{\omega} \int_0^T \int_\Omega \mathbf{h}_\varepsilon^2 \cdot \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t \tag{5.152}$$

for any $\varphi \in C_c^\infty((0,T) \times \overline{\Omega})$,

$$\int_0^T \int_\Omega \Big(\varepsilon \mathbf{V}_\varepsilon \cdot \partial_t \varphi + \omega Z_\varepsilon \mathrm{div}_x \varphi \Big) \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega \Big(\mathsf{h}_\varepsilon^7 : \nabla_x \varphi + \mathbf{h}_\varepsilon^6 \cdot \varphi \Big) \, \mathrm{d}x \, \mathrm{d}t \tag{5.153}$$

for any $\varphi \in C_c^\infty((0,T)\times\overline{\Omega};\mathbb{R}^3)$, $\varphi\cdot\mathbf{n}|_{\partial\Omega}=0$, where we have set

$$Z_\varepsilon = \frac{1}{\omega}\left(\omega\varrho_\varepsilon^{(1)} + A\varrho_\varepsilon\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(\overline{\varrho},\overline{\vartheta})}{\varepsilon} - \overline{\varrho}F + A\frac{\Sigma_\varepsilon}{\varepsilon}\right), \quad \mathbf{V}_\varepsilon = \varrho_\varepsilon\mathbf{u}_\varepsilon,$$

$$\mathsf{h}_\varepsilon^7 = \mathsf{h}_\varepsilon^5 + \frac{A}{\varepsilon\omega}\Sigma_\varepsilon\mathbb{I}.$$

Note that, by virtue of the standard representation theorem (Theorem 0.2), the quantity Σ_ε can be viewed as a bounded (non-negative) linear form on the Banach space $L^1(0,T;C(\overline{\Omega}))$ that can be identified with an element of the dual space $L^\infty(0,T;\mathcal{M}^+(\overline{\Omega}))$. As a matter of fact, it is easy to check that

$$\langle\Sigma_\varepsilon(\tau);\varphi\rangle_{[\mathcal{M};C](\overline{\Omega})} = \lim_{\delta\to 0+}\langle\sigma_\varepsilon;\psi_\delta\varphi\rangle_{[\mathcal{M};C]([0,T]\times\overline{\Omega})},\ \varphi\in C(\overline{\Omega}), \tag{5.154}$$

for *any* $\tau\in[0,T)$, where

$$\psi_\delta(t) = \left\{\begin{array}{l} 1 \text{ if } t\le\tau, \\ \frac{1}{\delta}(t-\tau) \text{ for } t\in(\tau,\tau+\delta), \\ 0 \text{ if } t\ge\tau+\delta. \end{array}\right.$$

In particular, as a direct consequence of the uniform bound established in (5.50), we get

$$\operatorname*{ess\ sup}_{t\in(0,T)}\|\Sigma_\varepsilon(t)\|_{\mathcal{M}^+(\overline{\Omega})} \le \|\sigma_\varepsilon\|_{\mathcal{M}^+([0,T]\times\overline{\Omega})} \le \varepsilon^2 c. \tag{5.155}$$

Accordingly, we have identified

$$\int_\Omega \Sigma_\varepsilon\varphi\ \mathrm{d}x := \langle\Sigma_\varepsilon;\varphi\rangle_{[\mathcal{M};C](\overline{\Omega})} \tag{5.156}$$

everywhere in (5.152), (5.153).

Loosely speaking, possible instantaneous changes of Σ_ε are matched by those of the entropy density so that the quantity Z_ε remains continuous in time. Note that the wave equation (5.152), (5.153) is well posed even for measure-valued initial data as the space of measures can be identified with a subspace of a Sobolev space of negative order.

For q_n, Λ_n solving the eigenvalue problem (5.146), we take

$$\varphi(t,x) = \psi(t)q_n(x),\ \psi\in C_c^\infty(0,T)$$

as a test function in (5.152), and

$$\varphi(t,x) = \psi(t)\frac{1}{\sqrt{\Lambda_n}}\nabla_x q_n,\ \psi\in C_c^\infty(0,T),$$

as a test function in (5.153). After a straightforward manipulation, we deduce a system of ordinary differential equations

$$\left\{\begin{array}{c} \varepsilon\partial_t b_n[Z_\varepsilon] - \sqrt{\Lambda_n}a_n[\mathbf{V}_\varepsilon] = \chi^1_{\varepsilon,n}, \\ \varepsilon\partial_t a_n[\mathbf{V}_\varepsilon] + \omega\sqrt{\Lambda_n}b_n[Z_\varepsilon] = \chi^2_{\varepsilon,n},\ n=1,\dots \end{array}\right\} \tag{5.157}$$

to be satisfied by the Fourier coefficients a_n defined in (5.148), and

$$b_n[Z_\varepsilon] = \int_\Omega Z_\varepsilon q_n \ \mathrm{d}x,$$

with convention (5.156). Here, in agreement with (5.135), (5.155),

$$\|\chi^1_{\varepsilon,n}\|_{L^1(0,T)} + \|\chi^2_{\varepsilon,n}\|_{L^1(0,T)} \leq \varepsilon c \text{ for any fixed } n = 1, \dots . \tag{5.158}$$

Moreover, it is convenient to rewrite (5.157) in terms of the Helmholtz projectors, namely

$$\left\{ \begin{array}{c} \varepsilon\partial_t [Z_\varepsilon]_M + \mathrm{div}_x(\mathbf{H}^\perp_M[\varrho_\varepsilon \mathbf{u}_\varepsilon]) = \tilde{\chi}^1_{\varepsilon,M}, \\ \varepsilon\partial_t \mathbf{H}^\perp_M[\varrho_\varepsilon \mathbf{u}_\varepsilon] + \omega\nabla_x [Z_\varepsilon]_M = \tilde{\chi}^2_{\varepsilon,M}, \end{array} \right\} \tag{5.159}$$

where we have set

$$[Z_\varepsilon]_M = \sum_{n=1}^{M} b_n[Z_\varepsilon] q_n,$$

and where, by virtue of (5.158),

$$\|\tilde{\chi}^1_{\varepsilon,M}\|_{L^1((0,T)\times\Omega)} + \|\tilde{\chi}^2_{\varepsilon,M}\|_{L^1((0,T)\times\Omega;\mathbb{R}^3)} \leq \varepsilon c \tag{5.160}$$

for any fixed $M \geq 1$. Let us remark that both $[Z_\varepsilon]_M$ and $\mathbf{H}^\perp_M[\varrho_\varepsilon \mathbf{u}_\varepsilon]$ are at least twice continuously differentiable with respect to x and absolutely continuous in t so that (5.159) makes sense.

Now we are in the situation discussed in Section 4.4.1. Introducing the potential $\Psi_{\varepsilon,M}$,

$$\nabla_x \Psi_{\varepsilon,M} = \mathbf{H}^\perp_M[\varrho_\varepsilon \mathbf{u}_\varepsilon], \ \int_\Omega \Psi_{\varepsilon,M} \ \mathrm{d}x = 0,$$

we can rewrite the integral in (5.151) as

$$\int_0^T \int_\Omega \mathbf{H}^\perp_M[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}^\perp_M[\varrho_\varepsilon \mathbf{u}_\varepsilon] : \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t = - \int_0^T \int_\Omega \Delta_x \Psi_{\varepsilon,M} \nabla_x \Psi_{\varepsilon,M} \cdot \varphi \ \mathrm{d}x \ \mathrm{d}t$$

provided

$$\varphi \in C^\infty_c((0,T) \times \overline{\Omega}; \mathbb{R}^3), \ \mathrm{div}_x \varphi = 0, \ \varphi \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

Furthermore, keeping in mind that φ is a solenoidal function with vanishing normal trace, meaning orthogonal to gradients, we can use equation (5.159) in order to obtain

$$\begin{aligned} &\int_0^T \int_\Omega \Delta \Psi_{\varepsilon,M} \nabla_x \Psi_{\varepsilon,M} \cdot \varphi \ \mathrm{d}x \ \mathrm{d}t \\ &= \varepsilon \int_0^T \int_\Omega [Z_\varepsilon]_M \nabla_x \Psi_{\varepsilon,M} \cdot \partial_t \varphi \ \mathrm{d}x \ \mathrm{d}t \\ &\quad + \int_0^T \int_\Omega \Big(\tilde{\chi}^1_{\varepsilon,M} \mathbf{H}^\perp_M[\varrho_\varepsilon \mathbf{u}_\varepsilon] \cdot \varphi + [Z_\varepsilon]_M \tilde{\chi}^2_{\varepsilon,M} \cdot \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t, \end{aligned}$$

where, in accordance with the uniform bounds established in (5.160), the right-hand side tends to zero as $\varepsilon \to 0$ for any fixed φ (cf. the formal arguments in Section 4.4.1). Thus we have shown (5.151) yielding the desired conclusion (5.112). Accordingly, the term $\overline{\varrho \mathbf{U} \otimes \mathbf{U}}$ can be replaced by $\varrho \mathbf{U} \otimes \mathbf{U}$ in the momentum equation (5.95), which, together with (5.73), (5.107), represent a weak formulation of the OBERBECK-BOUSSINESQ APPROXIMATION. We shall summarize our results in the next section.

5.5 Conclusion – main result

In this chapter, we have performed the asymptotic limit in the primitive NAVIER-STOKES-FOURIER SYSTEM in the case of *low Mach number* and *low stratification*. We have identified the limit (target) problem as OBERBECK-BOUSSINESQ APPROXIMATION. In the remaining part, we recall a weak formulation of the target problem and state our convergence result in a rigorous way. In addition, we discuss validity of the energy inequality for the target system and the problem of a proper choice of the initial data (data adjustment). The fact that the weak formulation of the limit momentum equation is based on *solenoidal* test functions should be viewed as the weakest point of this framework, based on the ideas of Leray [132] and Hopf [115]. The reader will have noticed that the pressure or rather the normal stress Π is in fact absent in the weak formulation of the limit problem and may be recovered by the methods described in Caffarelli et al. [37]. Apparently, we were not able to establish any relation between Π and the asymptotic limit of the thermodynamic pressure $p(\varrho_\varepsilon, \vartheta_\varepsilon)$.

5.5.1 Weak formulation of the target problem

We say that functions $\mathbf{U}$ and Θ represent a *weak solution* to the OBERBECK-BOUSSINESQ APPROXIMATION if

$$\mathbf{U} \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)) \cap L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)),$$

$$\Theta \in W^{1,q}_{\rm loc}((0,T];L^q(\Omega)) \cap L^q_{\rm loc}((0,T];W^{2,q}(\Omega)) \cap C([0,T];L^q(\Omega))$$

for a certain $q > 1$, and the following holds:

(i) Incompressibility and impermeability of the boundary:

$$\operatorname{div}_x \mathbf{U} = 0 \text{ a.a. on } (0,T) \times \Omega, \ \ \mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0 \text{ in the sense of traces.} \tag{5.161}$$

(ii) Incompressible Navier-Stokes system with complete slip on the boundary:

$$\int_0^T \int_\Omega \Big(\overline{\varrho} \mathbf{U} \cdot \partial_t \varphi + \overline{\varrho} \mathbf{U} \otimes \mathbf{U} : \nabla_x \varphi \Big) \, {\rm d}x \, {\rm d}t \tag{5.162}$$

$$= \int_0^T \int_\Omega \Big(\mu(\overline{\vartheta}) [\nabla_x \mathbf{U} + \nabla_x^T \mathbf{U}] : \nabla_x \varphi - r \nabla_x F \cdot \varphi \Big) \, {\rm d}x \, {\rm d}t - \int_\Omega (\overline{\varrho} \mathbf{U}_0) \cdot \varphi \, {\rm d}x$$

for any test function

$$\varphi \in C_c^\infty([0,T) \times \overline{\Omega}; \mathbb{R}^3),\ \mathrm{div}_x \varphi = 0 \text{ in } \Omega,\ \varphi \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

(iii) Heat equation with insulated boundary:

$$\begin{aligned} &\overline{\varrho} c_p(\overline{\varrho}, \overline{\vartheta}) \Big(\partial_t \Theta + \mathbf{U} \cdot \nabla_x \Theta) \Big) - \mathrm{div}_x (\kappa(\overline{\vartheta}) \nabla_x \Theta) \\ &= \overline{\varrho}\ \overline{\vartheta} \alpha(\overline{\varrho}, \overline{\vartheta}) \mathbf{U} \cdot \nabla_x F \text{ a.a. in } (0,T) \times \Omega, \end{aligned} \tag{5.163}$$

$$\nabla_x \Theta \cdot \mathbf{n}|_{\partial\Omega} = 0 \text{ in the sense of traces for a.a. } t \in (0,T), \tag{5.164}$$

$$\Theta(0, \cdot) = \Theta_0 \text{ a.a. in } \Omega. \tag{5.165}$$

(iv) Boussinesq relation:

$$r + \overline{\varrho} \alpha(\overline{\varrho}, \overline{\vartheta}) \Theta = 0. \tag{5.166}$$

The integral identity (5.162), together with the incompressibility constraint imposed through (5.161), represent the standard *weak formulation* of the incompressible Navier-Stokes system (5.8), (5.9), supplemented with the *complete slip boundary conditions*

$$\mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0,\ [\nabla_x \mathbf{U} + \nabla_x^T \mathbf{U}] \mathbf{n} \times \mathbf{n}|_{\partial\Omega} = 0. \tag{5.167}$$

Moreover, it is easy to check that $\mathbf{U} \in C_{\mathrm{weak}}([0,T]; L^2(\Omega; \mathbb{R}^3))$ and

$$\int_\Omega \mathbf{U}(0, \cdot) \cdot \varphi \ \mathrm{d}x = \int_\Omega \mathbf{U}_0 \cdot \varphi \ \mathrm{d}x \text{ for all } \varphi \in C_c^\infty(\overline{\Omega}),\ \mathrm{div}_x \varphi = 0,\ \varphi \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

in other words,

$$\mathbf{U}(0, \cdot) = \mathbf{H}[\mathbf{U}_0],$$

where $\mathbf{H}$ is the Helmholtz projection onto the space of solenoidal functions. This fact can be interpreted in terms of the asymptotic limit performed in this chapter in the sense that the piece of information provided by the gradient component $\mathbf{H}^\perp[\mathbf{u}_{0,\varepsilon}]$ is lost in the limit problem because of the process of *acoustic filtering* removing the rapidly oscillating acoustic waves from the system. This rather unpleasant but inevitable feature obviously disappears if we consider *well-prepared data*, specifically,

$$\mathbf{H}^\perp[\mathbf{u}_{0,\varepsilon}] \to 0 \text{ in } L^2(\Omega)$$

(cf. Section 4.6). A similar problem connected with the initial distribution Θ_0 of the limit temperature will be discussed in the remaining part of this chapter. We would like to point out, however, that considering *well-prepared data* in all state variables would eliminate completely the heat equation, giving rise to a system with constant temperature.

5.5.2 Main result

We are in a position to state the main result concerning the asymptotic limit of solutions to the complete NAVIER-STOKES-FOURIER SYSTEM in the regime of low *Mach number* Ma $= \varepsilon$ and under low stratification – the *Froude number* Fr$= \sqrt{\varepsilon}$. We recall that the underlying physical system is energetically isolated, in particular, the normal component of the heat flux vanishes on the boundary. The boundary is assumed to be acoustically hard, meaning, the complete slip boundary conditions are imposed on the fluid velocity. The initial state of the system is determined through a family of *ill-prepared data*. The system is driven by a potential force $\nabla_x F$, where F is a regular time independent function.

■ LOW MACH NUMBER LIMIT – LOW STRATIFICATION:

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$. Assume that p, e, s satisfy hypotheses* (5.17–5.23)*, with $P \in C^1[0,\infty) \cap C^2(0,\infty)$, the transport coefficients μ, η, and κ meet the growth restrictions* (5.24), (5.25)*, and the driving force is determined by a scalar potential $F = F(x)$ such that*

$$F \in W^{1,\infty}(\Omega), \ \int_\Omega F \ \mathrm{d}x = 0.$$

Let $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ be a family of weak solutions to the scaled Navier-Stokes-Fourier system (5.1–5.7) *on the set $(0,T) \times \Omega$, supplemented with the boundary conditions* (5.15), (5.16)*, and the initial data*

$$\varrho_\varepsilon(0,\cdot) = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon}, \ \mathbf{u}_\varepsilon(0,\cdot) = \mathbf{u}_{0,\varepsilon}, \ \vartheta_\varepsilon(0,\cdot) = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon},$$

where

$$\overline{\varrho} > 0, \ \overline{\vartheta} > 0$$

are constant, and

$$\int_\Omega \varrho^{(1)}_{0,\varepsilon} \ \mathrm{d}x = \int_\Omega \vartheta^{(1)}_{0,\varepsilon} \ \mathrm{d}x = 0 \textit{ for all } \varepsilon > 0.$$

Moreover, assume that

$$\left\{ \begin{array}{c} \varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0 \ \textit{weakly-}(*) \ \textit{in} \ L^\infty(\Omega), \\ \mathbf{u}_{0,\varepsilon} \to \mathbf{U}_0 \ \textit{weakly-}(*) \ \textit{in} \ L^\infty(\Omega;\mathbb{R}^3), \\ \vartheta^{(1)}_{0,\varepsilon} \to \vartheta^{(1)}_0 \ \textit{weakly-}(*) \ \textit{in} \ L^\infty(\Omega). \end{array} \right\} \tag{5.168}$$

Then

$$\operatorname{ess}\sup_{t\in(0,T)} \|\varrho_\varepsilon(t) - \overline{\varrho}\|_{L^{\frac{5}{3}}(\Omega)} \leq \varepsilon c,$$

and, at least for a suitable subsequence,

$$\mathbf{u}_\varepsilon \to \mathbf{U} \ \textit{weakly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)),$$
$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} = \vartheta_\varepsilon^{(1)} \to \Theta \ \textit{weakly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)),$$

where $\mathbf{U}$ *and* Θ *solve the Oberbeck-Boussinesq approximation in the sense specified in Section* 5.5.1*, where the initial distribution of the temperature* Θ_0 *can be determined in terms of* $\varrho_0^{(1)}$*,* $\vartheta_0^{(1)}$*, and* F*, specifically,*

$$\Theta(0,\cdot) = \Theta_0 \equiv \frac{\overline{\vartheta}}{c_p(\overline{\varrho},\overline{\vartheta})}\Big(\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial \varrho}\varrho_0^{(1)} + \frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}\vartheta_0^{(1)} + \alpha(\overline{\varrho},\overline{\vartheta})F\Big). \tag{5.169}$$

5.5.3 Determining the initial temperature distribution

As indicated by formula (5.169), determining the initial distribution of the temperature represents a rather delicate issue. Note that the initial state of the primitive NAVIER-STOKES-FOURIER SYSTEM is uniquely determined by *three* state variables $\{\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}, \vartheta_{0,\varepsilon}\}$, while the limit OBERBECK-BOUSSINESQ approximation contains only *two* independent state functions, namely $\mathbf{U}$ and Θ. On the other hand, determining the initial distribution of Θ in (5.163) requires the knowledge of $\varrho_0^{(1)}$ – a meaningless quantity for the limit problem! Here, similarly to Section 5.5.1, an extra hypothesis imposed on the data may save the game, for instance,

$$\varrho_{0,\varepsilon}^{(1)} = \frac{\varrho_{0,\varepsilon} - \overline{\varrho}}{\varepsilon} \to 0 \text{ in } L^2(\Omega) \text{ for } \varepsilon \to 0. \tag{5.170}$$

An alternative choice of data will be discussed in the next section.

Obviously, the above mentioned problems are intimately related to the existence of an initial-time boundary layer resulting from the presence of rapidly oscillating acoustic waves discussed in Section 5.4.

5.5.4 Energy inequality for the limit system

It is interesting to see if the resulting OBERBECK-BOUSSINESQ SYSTEM specified in Section 5.5.1 satisfies some form of the kinetic energy balance. Formally, taking the scalar product of the momentum equation (5.9) with $\mathbf{U}$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega \overline{\varrho}|\mathbf{U}|^2 \ \mathrm{d}x + \frac{\mu(\overline{\vartheta})}{2}\int_\Omega |\nabla_x \mathbf{U} + \nabla_x^T \mathbf{U}|^2 \ \mathrm{d}x = \int_\Omega r\nabla_x F \mathbf{U} \ \mathrm{d}x, \tag{5.171}$$

where r obeys *Boussinesq's relation* (5.166). However, for the reasons explained in detail in the introductory part of Chapter 2, the weak solutions are not expected to satisfy (5.171) but rather a considerably weaker *energy inequality*

$$\frac{1}{2}\int_\Omega \overline{\varrho}|\mathbf{U}|^2(\tau)\ \mathrm{d}x + \frac{\mu(\overline{\vartheta})}{2}\int_0^\tau \int_\Omega |\nabla_x \mathbf{U} + \nabla_x^T \mathbf{U}|^2\ \mathrm{d}x\ \mathrm{d}t \tag{5.172}$$
$$\leq \frac{1}{2}\int_\Omega \overline{\varrho}|\mathbf{U}_0|^2\ \mathrm{d}x + \int_0^\tau \int_\Omega r\nabla_x F\cdot \mathbf{U}\ \mathrm{d}x\ \mathrm{d}t$$

for a.a. $\tau \in [0,T]$. The weak solutions of the incompressible Navier-Stokes system satisfying, in addition, the energy inequality (5.172) were termed "turbulent" solutions in the seminal paper of Leray [132].

A natural idea is to derive (5.172) directly from the dissipation equality (5.37). To this end, however, supplementary assumptions have to be imposed on the family of initial data. In addition to the hypotheses of Theorem 5.2, suppose that

$$\left\{ \begin{array}{l} \varrho_{0,\varepsilon}^{(1)} \to \varrho_0^{(1)}, \\ \mathbf{u}_{0,\varepsilon} \to \mathbf{U}_0, \\ \vartheta_{0,\varepsilon}^{(1)} \to \vartheta_0^{(1)} \end{array} \right\} \text{a.a. in } \Omega, \tag{5.173}$$

in particular, by virtue of hypothesis (5.168), the data converge strongly in $L^p(\Omega)$ for any $1 \leq p < \infty$. Still relation (5.173) *does not* imply that the initial data are well prepared.

We recall that, in accordance with (5.71), (5.76),

$$\varrho_\varepsilon^{(1)} = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \to \varrho^{(1)} \text{ weakly-(*) in } L^\infty(0,T;L^{5/3}(\Omega)),$$
$$\vartheta_\varepsilon^{(1)} = \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \vartheta^{(1)} \equiv \Theta \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)),$$

where the limit quantities are interrelated through

$$\frac{\partial p}{\partial \varrho}(\overline{\varrho},\overline{\vartheta})\varrho^{(1)} + \frac{\partial p}{\partial \vartheta}(\overline{\varrho},\overline{\vartheta})\vartheta^{(1)} = \overline{\varrho}F \tag{5.174}$$

(see (5.103)).

Asymptotic form of the dissipation balance. Rewriting the dissipation equality (5.37) by help of (5.32) we get

$$-\int_0^T \int_\Omega \left(\frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 - \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}F\right)\partial_t\psi\ \mathrm{d}x\ \mathrm{d}t \tag{5.175}$$
$$-\int_0^T \int_\Omega \frac{1}{\varepsilon^2}\left(H_{\overline{\vartheta}}(\varrho_\varepsilon,\vartheta_\varepsilon) - (\varrho_\varepsilon - \overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})\right)\partial_t\psi\ \mathrm{d}x\ \mathrm{d}t$$
$$+\int_0^T \int_\Omega \frac{\overline{\vartheta}}{\vartheta_\varepsilon}\left(\mathbb{S}_\varepsilon : \nabla_x \mathbf{u}_\varepsilon + \frac{\kappa(\vartheta_\varepsilon)|\nabla_x\vartheta_\varepsilon|^2}{\varepsilon^2\vartheta_\varepsilon}\right)\psi\ \mathrm{d}x\ \mathrm{d}t$$

$$\leq \int_\Omega \left(\frac{1}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 - \frac{(\varrho_{0,\varepsilon} - \overline{\varrho})}{\varepsilon} F \right) \, \mathrm{d}x$$
$$+ \int_\Omega \frac{1}{\varepsilon^2} \left(H_{\overline{\vartheta}}(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) - (\varrho_{0,\varepsilon} - \overline{\varrho}) \frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \right) \, \mathrm{d}x$$

for any function $\psi \in C^1[0,T]$ such that

$$\psi(0) = 1, \ \psi(T) = 0, \ \partial_t \psi \leq 0 \text{ in } [0,T].$$

Assume, for simplicity, that $H_{\overline{\vartheta}} = H_{\overline{\vartheta}}(\varrho, \vartheta)$ is three times continuously differentiable in an open neighborhood O of the equilibrium state $(\overline{\varrho}, \overline{\vartheta})$. As $H_{\overline{\vartheta}}$ is determined in terms of the function P, it is enough that P be in $C^3(0,\infty)$. Under this extra hypothesis, since $\partial_\vartheta H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) = 0$, we have

$$H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho}) \frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \tag{5.176}$$
$$= \frac{1}{2} \left(\frac{\partial^2 H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho^2} (\varrho - \overline{\varrho})^2 + 2 \frac{\partial^2 H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho \partial \vartheta} (\varrho - \overline{\varrho})(\vartheta - \overline{\vartheta}) + \frac{\partial^2 H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta^2} (\vartheta - \overline{\vartheta})^2 \right)$$
$$+ \chi(\varrho, \vartheta),$$

with

$$|\chi(\varrho, \vartheta)| \leq c \left(|\varrho - \overline{\varrho}|^3 + |\vartheta - \overline{\vartheta}|^3 \right) \text{ as soon as } (\varrho, \vartheta) \in \mathcal{O}_{\rm ess},$$

where $\mathcal{O}_{\rm ess}$ is the set of essential values introduced in (4.39).

Note that, in accordance with (2.49), (2.50),

$$\frac{\partial^2 H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho^2} = \frac{1}{\overline{\varrho}} \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho}, \ \frac{\partial^2 H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta^2} = \frac{\overline{\varrho}}{\overline{\vartheta}} \frac{\partial e(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta}, \text{ and } \frac{\partial^2 H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho \partial \vartheta} = 0.$$

Consequently, by virtue of hypotheses (5.168), (5.173), the expression on the right-hand side of inequality (5.175) tends to

$$\int_\Omega \left(\frac{1}{2} \overline{\varrho} |\mathbf{U}_0|^2 + \frac{1}{2\overline{\varrho}} \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} |\varrho_0^{(1)}|^2 + \frac{\overline{\varrho}}{2\overline{\vartheta}} \frac{\partial e(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} |\vartheta_0^{(1)}|^2 - \varrho_0^{(1)} F \right) \, \mathrm{d}x. \tag{5.177}$$

Next, we have

$$\int_0^T \int_\Omega \frac{\overline{\vartheta}}{\vartheta_\varepsilon} \left(\mathbb{S}_\varepsilon : \nabla_x \mathbf{u}_\varepsilon + \frac{\kappa(\vartheta_\varepsilon) |\nabla_x \vartheta_\varepsilon|^2}{\varepsilon^2 \vartheta_\varepsilon} \right) \psi \, \mathrm{d}x \, \mathrm{d}t$$
$$\geq \int_0^T \int_\Omega \left[\frac{\overline{\vartheta} \mu(\vartheta_\varepsilon)}{2\vartheta_\varepsilon} \right]_{\rm ess} \left| \nabla_x \mathbf{u}_\varepsilon + \nabla_x^T \mathbf{u}_\varepsilon - \frac{2}{3} {\rm div}_x \mathbf{u}_\varepsilon \mathbb{I} \right|^2 \psi \, \mathrm{d}x \, \mathrm{d}t$$
$$+ \int_0^T \int_\Omega \left[\frac{\overline{\vartheta} \kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon^2} \right]_{\rm ess} \left| \nabla_x \left(\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right) \right|^2 \psi \, \mathrm{d}x \, \mathrm{d}t,$$

therefore

$$\liminf_{\varepsilon\to 0}\int_0^T\int_\Omega \frac{\overline{\vartheta}}{\vartheta_\varepsilon}\left(\mathbb{S}_\varepsilon : \nabla_x \mathbf{u}_\varepsilon + \frac{\kappa(\vartheta_\varepsilon)|\nabla_x\vartheta_\varepsilon|^2}{\varepsilon^2\vartheta_\varepsilon}\right)\psi \, \mathrm{d}x \, \mathrm{d}t \tag{5.178}$$
$$\geq \int_0^T\int_\Omega \left(\frac{\mu(\overline{\vartheta})}{2}\left|\nabla_x\mathbf{U}+\nabla_x^T\mathbf{U}\right|^2 + \frac{\kappa(\overline{\vartheta})}{\overline{\vartheta}}\left|\nabla_x\Theta\right|^2\right)\psi \, \mathrm{d}x \, \mathrm{d}t.$$

Moreover,

$$-\liminf_{\varepsilon\to 0}\int_0^T\int_\Omega\left(\frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 - \frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}F\right)\partial_t\psi \, \mathrm{d}x \, \mathrm{d}t \tag{5.179}$$
$$\geq -\int_0^T\int_\Omega\left(\frac{1}{2}\overline{\varrho}|\mathbf{U}|^2 + \varrho^{(1)}F\right)\partial_t\psi \, \mathrm{d}x \, \mathrm{d}t,$$

where, similarly to (5.178), we have used weak lower semi-continuity of convex functionals, see Theorem 10.20 in Appendix.

Writing

$$H_{\overline{\vartheta}}(\varrho_\varepsilon,\vartheta_\varepsilon) - (\varrho_\varepsilon-\overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})}{\partial\varrho} - H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})$$
$$= H_{\overline{\vartheta}}(\varrho_\varepsilon,\overline{\vartheta}) - (\varrho_\varepsilon-\overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})}{\partial\varrho} - H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta}) + H(\varrho_\varepsilon,\vartheta_\varepsilon) - H(\varrho_\varepsilon,\overline{\vartheta})$$

we observe easily that the function

$$\varrho\mapsto h(\varrho) = H_{\overline{\vartheta}}(\varrho,\overline{\vartheta}) - (\varrho-\overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})}{\partial\varrho} - H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})$$

is strictly convex, attaining its global minimum at $\varrho=\overline{\varrho}$. Moreover, in agreement with (2.49),

$$\frac{\partial^2 h(\varrho)}{\partial\varrho^2} = \frac{1}{\varrho}\frac{\partial p(\varrho,\overline{\vartheta})}{\partial\varrho}.$$

Our goal is to show that

$$\liminf_{\varepsilon\to 0}\int_0^T\int_\Omega\frac{1}{\varepsilon^2}h(\varrho_\varepsilon)\varphi \, \mathrm{d}x \, \mathrm{d}t \geq \frac{1}{2\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\int_0^T\int_\Omega|\varrho^{(1)}|^2\varphi \, \mathrm{d}x \, \mathrm{d}t \tag{5.180}$$

for any non-negative $\varphi\in C_c^\infty([0,T]\times\overline{\Omega})$. To this end, we first observe that

$$\int_0^T\int_\Omega\frac{1}{\varepsilon^2}h(\varrho_\varepsilon)\varphi \, \mathrm{d}x \, \mathrm{d}t \geq \int_{\{\varrho_\varepsilon \ | \ |\varrho_\varepsilon-\overline{\varrho}|<D\}}\frac{1}{\varepsilon^2}h(\varrho_\varepsilon)\varphi \, \mathrm{d}x \, \mathrm{d}t$$
$$\geq \frac{1}{2}\inf\{\partial^2_\varrho h(\varrho) \ | \ |\varrho-\overline{\varrho}|<D\}\int_0^T\int_\Omega 1_{\{\varrho_\varepsilon \ | \ |\varrho_\varepsilon-\overline{\varrho}|<D\}}|\varrho_\varepsilon^{(1)}|^2\varphi \, \mathrm{d}x \, \mathrm{d}t$$

for any $D > 0$ small enough. By virtue of (5.45), we have

$$1_{\{\varrho_\varepsilon \ | \ |\varrho_\varepsilon - \overline{\varrho}| < D\}} \to 1 \text{ as } \varepsilon \to 0 \text{ a.a. in } (0,T) \times \Omega;$$

whence, using (5.69), we conclude that

$$\liminf_{\varepsilon\to 0} \int_0^T \int_\Omega \frac{1}{\varepsilon^2} h(\varrho_\varepsilon)\varphi \, \mathrm{d}x \, \mathrm{d}t \geq \frac{1}{2} \inf\{\partial^2_\varrho h(\varrho) \mid |\varrho - \overline{\varrho}| < D\} \int_0^T \int_\Omega |\varrho^{(1)}|^2 \varphi \, \mathrm{d}x \, \mathrm{d}t$$

for any $D > 0$ small enough. Thus, letting $D \to 0$ we get (5.180).

In accordance with (5.180),

$$\begin{aligned}
- \liminf_{\varepsilon\to 0} \int_0^T \int_\Omega \frac{1}{\varepsilon^2} \left(H_{\overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon) - (\varrho_\varepsilon - \overline{\varrho}) \frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \right) \partial_t \psi \, \mathrm{d}x \, \mathrm{d}t \\
\geq - \int_0^T \int_\Omega \frac{1}{2\overline{\varrho}} \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} |\varrho^{(1)}|^2 \partial_t \psi \, \mathrm{d}x \, \mathrm{d}t \\
- \liminf_{\varepsilon\to 0} \int_0^T \int_\Omega \left(\frac{H_{\overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon) - H_{\overline{\vartheta}}(\varrho_\varepsilon, \overline{\vartheta})}{\varepsilon^2} \right) \partial_t \psi \, \mathrm{d}x \, \mathrm{d}t, \qquad (5.181)
\end{aligned}$$

where, similarly to (5.176),

$$H_{\overline{\vartheta}}(\varrho, \vartheta) - H_{\overline{\vartheta}}(\varrho, \overline{\vartheta}) = \frac{(\vartheta - \overline{\vartheta})^2}{2} \varrho \frac{\partial s(\varrho, \overline{\vartheta})}{\partial \vartheta} + \chi(\varrho, \vartheta),$$

with

$$|\chi(\varrho, \vartheta)| \leq c|\vartheta - \overline{\vartheta}|^3 \text{ for all } (\varrho, \vartheta) \in \mathcal{O}_{\mathrm{ess}}.$$

It follows from the uniform bounds established in (5.46), (5.52), and (5.62) that

$$\left\{ \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right\}_{\varepsilon > 0} \text{ is bounded in } L^q((0,T) \times \Omega) \text{ for a certain } q > 2.$$

Consequently, going back to (5.181) we infer that

$$\begin{aligned}
- \liminf_{\varepsilon\to 0} \int_0^T \int_\Omega \frac{1}{\varepsilon^2} \left(H_{\overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon) - (\varrho_\varepsilon - \overline{\varrho}) \frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \right) \partial_t \psi \, \mathrm{d}x \, \mathrm{d}t \\
\geq - \int_0^T \int_\Omega \frac{1}{2\overline{\varrho}} \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} |\varrho^{(1)}|^2 \partial_t \psi \, \mathrm{d}x \, \mathrm{d}t \\
- \liminf_{\varepsilon\to 0} \int_0^T \int_\Omega \left[\frac{H_{\overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon) - H_{\overline{\vartheta}}(\varrho_\varepsilon, \overline{\vartheta})}{\varepsilon^2} \right]_{\mathrm{ess}} \partial_t \psi \, \mathrm{d}x \, \mathrm{d}t, \\
\geq - \int_0^T \int_\Omega \frac{1}{2} \left(\frac{1}{\overline{\varrho}} \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} |\varrho^{(1)}|^2 + \overline{\varrho} \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} |\Theta|^2 \right) \partial_t \psi \, \mathrm{d}x \, \mathrm{d}t. \qquad (5.182)
\end{aligned}$$

Summing up relations (5.175–5.182), we derive the asymptotic form of the dissipation inequality

$$\int_\Omega \left(\frac{1}{2}\overline{\varrho}|\mathbf{U}|^2 + \frac{1}{2\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \varrho}|\varrho^{(1)}|^2 + \frac{\overline{\varrho}}{2\overline{\vartheta}}\frac{\partial e(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}|\Theta|^2 - \varrho^{(1)}F \right)(\tau,\cdot)\ \mathrm{d}x \tag{5.183}$$
$$+ \int_0^\tau \int_\Omega \left(\frac{\mu(\overline{\vartheta})}{2}\left|\nabla_x \mathbf{U} + \nabla_x^T \mathbf{U}\right|^2 + \frac{\kappa(\overline{\vartheta})}{\overline{\vartheta}}|\nabla_x \Theta|^2 \right)\ \mathrm{d}x\ \mathrm{d}t$$
$$\leq \int_\Omega \left(\frac{1}{2}\overline{\varrho}|\mathbf{U}_0|^2 + \frac{1}{2\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \varrho}|\varrho_0^{(1)}|^2 + \frac{\overline{\varrho}}{2\overline{\vartheta}}\frac{\partial e(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}|\vartheta_0^{(1)}|^2 - \varrho_0^{(1)}F \right)\ \mathrm{d}x$$

for a.a. $\tau \in (0,T)$.

Asymptotic thermal energy balance. Our next goal is to compare (5.183) with the associated thermal energy balance computed by means of equation (5.163). To this end, we need to multiply (5.163) on Θ and integrate over Ω. Although equation (5.163) is satisfied in the strong sense (a.a. in $(0,T)\times\Omega$), the regularity of the limit temperature field Θ is not sufficient to justify this step. Instead we multiply (5.163) on $H'(\Theta)$, where H is a smooth bounded function with two bounded derivatives. After a straightforward manipulation, we obtain

$$\overline{\varrho}c_p(\overline{\varrho},\overline{\vartheta})\int_\Omega H(\Theta)(\tau,\cdot)\ \mathrm{d}x + \int_\delta^\tau \int_\Omega \kappa(\overline{\vartheta})H''(\Theta)|\nabla_x\Theta|^2\ \mathrm{d}x\ \mathrm{d}t$$
$$= \overline{\varrho}c_p(\overline{\varrho},\overline{\vartheta})\int_\Omega H(\Theta(\delta,\cdot))\ \mathrm{d}x + \overline{\varrho}\overline{\vartheta}\alpha(\overline{\varrho},\overline{\vartheta})\int_\delta^\tau \int_\Omega H'(\Theta)\nabla_x F\cdot\mathbf{U}\ \mathrm{d}x\ \mathrm{d}t$$

for any $0<\delta<\tau\leq T$. Moreover, since $\Theta \in C([0,T];L^q(\Omega))$ for a certain $q>1$, we can let $\delta \to 0$ to deduce

$$\overline{\varrho}c_p(\overline{\varrho},\overline{\vartheta})\int_\Omega H(\Theta)(\tau,\cdot)\ \mathrm{d}x + \int_0^\tau \int_\Omega \kappa(\overline{\vartheta})H''(\Theta)|\nabla_x\Theta|^2\ \mathrm{d}x\ \mathrm{d}t$$
$$= \overline{\varrho}c_p(\overline{\varrho},\overline{\vartheta})\int_\Omega H(\Theta_0)\ \mathrm{d}x + \overline{\varrho}\overline{\vartheta}\alpha(\overline{\varrho},\overline{\vartheta})\int_0^\tau \int_\Omega H'(\Theta)\nabla_x F\cdot\mathbf{U}\ \mathrm{d}x\ \mathrm{d}t.$$

Now, approximating $H(z)\approx z^2$ we can use the Lebesgue convergence theorem to conclude

$$\overline{\varrho}c_p(\overline{\varrho},\overline{\vartheta})\int_\Omega |\Theta|^2(\tau,\cdot)\ \mathrm{d}x + 2\int_0^\tau \int_\Omega \kappa(\overline{\vartheta})|\nabla_x\Theta|^2\ \mathrm{d}x\ \mathrm{d}t \tag{5.184}$$
$$= \overline{\varrho}c_p(\overline{\varrho},\overline{\vartheta})\int_\Omega |\Theta_0|^2\ \mathrm{d}x + 2\overline{\varrho}\overline{\vartheta}\alpha(\overline{\varrho},\overline{\vartheta})\int_0^\tau \int_\Omega \Theta\nabla_x F\cdot\mathbf{U}\ \mathrm{d}x\ \mathrm{d}t$$

for any $\tau \in [0,T]$.

Computing F by means of (5.174), we combine (5.183), (5.184) to obtain, after a bit of tedious but straightforward manipulation,

$$\begin{aligned}
&\int_\Omega \frac{1}{2}\overline{\varrho}|\mathbf{U}|^2(\tau,\cdot)\ \mathrm{d}x + \int_0^\tau \int_\Omega \frac{\mu(\overline{\vartheta})}{2}|\nabla_x \mathbf{U} + \nabla_x^T \mathbf{U}|^2\ \mathrm{d}x\ \mathrm{d}t \qquad (5.185)\\
&\quad \leq \int_\Omega \frac{1}{2}\overline{\varrho}|\mathbf{U}_0|^2\ \mathrm{d}x + \int_0^\tau \int_\Omega r\nabla_x F\cdot \mathbf{U}\ \mathrm{d}x\ \mathrm{d}t + \frac{\overline{\varrho}}{2\partial_\varrho p(\overline{\varrho},\overline{\vartheta})}\int_\Omega F^2\ \mathrm{d}x\\
&\quad + \int_\Omega \left(\frac{1}{2\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \varrho}|\varrho_0^{(1)}|^2 + \frac{\overline{\varrho}}{2\overline{\vartheta}}\frac{\partial e(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}|\vartheta_0^{(1)}|^2 - \varrho_0^{(1)}F \right)\ \mathrm{d}x\\
&\quad - \int_\Omega \frac{\overline{\varrho}c_p(\overline{\varrho},\overline{\vartheta})}{2\overline{\vartheta}}|\Theta_0|^2\ \mathrm{d}x.
\end{aligned}$$

Initial data adjustment. Our ultimate goal is to fix the initial distribution of $\varrho_0^{(1)}$ in such a way that the last two integrals in (5.185) vanish. To this end, we assume that $\varrho_0^{(1)}$, $\vartheta_0^{(1)}$ are chosen to satisfy

$$\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \varrho}\varrho_0^{(1)} + \frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}\vartheta_0^{(1)} = \overline{\varrho}F. \qquad (5.186)$$

Relation (5.186) can be regarded as a kind of *pressure compatibility condition* to be compared with (5.174).

If this is the case, we easily check that

(i) Θ_0 given through formula (5.169) *coincides* with $\vartheta_0^{(1)}$,

and, moreover,

(ii) we obtain the desired conclusion

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{2\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \varrho}|\varrho_0^{(1)}|^2 + \frac{\overline{\varrho}}{2\overline{\vartheta}}\frac{\partial e(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}|\vartheta_0^{(1)}|^2 - \varrho_0^{(1)}F \right)\ \mathrm{d}x\\
&\qquad + \frac{\overline{\varrho}}{2\partial_\varrho p(\overline{\varrho},\overline{\vartheta})}\int_\Omega F^2\ \mathrm{d}x = \int_\Omega \frac{\overline{\varrho}c_p(\overline{\varrho},\overline{\vartheta})}{2\overline{\vartheta}}|\Theta_0|^2\ \mathrm{d}x,
\end{aligned}$$

in particular, relation (5.185) gives rise to the kinetic energy inequality (5.172).

In the light of the previous arguments, it is relation (5.186) rather than (5.170) that yields the correct data adjustment for the limit problem. Note that (5.186) coincides with our definition of *well-prepared data* introduced in Section 4.6.

As for the energy balance for the limit problem, we have shown the following result.

■ KINETIC ENERGY INEQUALITY:

Theorem 5.3. *In addition to the hypotheses of Theorem* 5.2, *let* $P \in C^1[0,\infty) \cap C^3(0,\infty)$, *and let*

$$\left\{ \begin{array}{l} \varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0, \\ \mathbf{u}_{0,\varepsilon} \to \mathbf{U}_0, \\ \vartheta^{(1)}_{0,\varepsilon} \to \vartheta^{(1)}_0 \end{array} \right\} \text{ a.a. in } \Omega,$$

where $\varrho^{(1)}_0$, $\vartheta^{(1)}_0$ *satisfy the pressure compatibility condition* (5.186).

Then the limit quantities $\mathbf{U}$, Θ *identified in Theorem* 5.2 *satisfy the kinetic energy inequality*

$$\frac{1}{2}\int_\Omega \overline{\varrho}|\mathbf{U}|^2(\tau)\ \mathrm{d}x + \frac{\mu(\overline{\vartheta})}{2}\int_0^\tau\int_\Omega |\nabla_x\mathbf{U} + \nabla_x^T\mathbf{U}|^2\ \mathrm{d}x\ \mathrm{d}t$$
$$\leq \frac{1}{2}\int_\Omega \overline{\varrho}|\mathbf{U}_0|^2\ \mathrm{d}x + \int_0^\tau\int_\Omega r\nabla_x F\cdot\mathbf{U}\ \mathrm{d}x\ \mathrm{d}t$$

for a.a. $\tau \in (0,T)$.

Since r and Θ are interrelated through *Boussinesq equation* (5.166), we can use (5.184) to deduce the total energy inequality for the OBERBECK-BOUSSINESQ APPROXIMATION in the form

$$\frac{1}{2}\int_\Omega \Big(\overline{\varrho}|\mathbf{U}|^2 + \overline{\varrho}c_p(\overline{\varrho},\overline{\vartheta})|\Theta|^2\Big)(\tau)\ \mathrm{d}x \tag{5.187}$$
$$+ \int_0^\tau\int_\Omega \left(\frac{\mu(\overline{\vartheta})}{2}|\nabla_x\mathbf{U} + \nabla_x^T\mathbf{U}|^2 + \kappa(\overline{\vartheta})|\nabla_x\Theta|^2\right)\ \mathrm{d}x\ \mathrm{d}t$$
$$\leq \frac{1}{2}\int_\Omega \Big(\overline{\varrho}|\mathbf{U}_0|^2 + \overline{\varrho}c_p(\overline{\varrho},\overline{\vartheta})|\Theta_0|^2\Big)\ \mathrm{d}x \text{ for a.a. } \tau \in [0,T].$$

Chapter 6

Stratified Fluids

We expand the methods developed in the previous chapter in order to handle the *strongly stratified systems* discussed briefly in Section 4.3. In comparison with the previous considerations, a new aspect arises, namely the thermodynamic state variables ϱ and ϑ undergo a scaling procedure similar to that of kinematic quantities like velocity, length, and time. In particular, both thermal and caloric equations of state modify their form reflecting substantial changes of the *material properties* of the fluid.

6.1 Motivation

Many recent applications of mathematical fluid mechanics are motivated by problems arising in *astrophysics*. However, investigations in this field are hampered by both theoretical and observational problems. The vast range of different scales extending in the case of stars from the stellar radius to 10^2 m or even less entirely prevents a complex numerical solution. Progress in this field therefore calls for a combination of physical intuition with rigorous analysis of highly simplified mathematical models.

A typical example is the flow dynamics in stellar radiative zones representing a major challenge of the current theory of stellar interiors. Under these circumstances, the fluid is a plasma characterized by the following specific features:

- A strong *radiative transport* predominates the molecular one. This is due to extremely hot and energetic radiation fields prevailing in the plasma. The Péclet number is therefore expected to be vanishingly small.
- Strong *stratification effects*, because of the enormous gravitational potential of gaseous celestial bodies, determine many of the properties of the fluid in the large.

- The convective motions are much slower than the speed of sound yielding a small Mach number. The fluid is therefore almost *incompressible*, whereas the density variations can be simulated via the anelastic approximation (see also Gough [104], Gilman and Glatzmaier [99], [98]).

Motivated by the previous discussion and in accordance with the general philosophy of this book, we assume that the time evolution of the fluid we deal with is governed by the NAVIER-STOKES-FOURIER SYSTEM subjected to an appropriate scaling. Similarly to the preceding chapters we suppose the Mach number is small, specifically,

$$\mathrm{Ma} = \varepsilon, \ \varepsilon \to 0.$$

Unlike the situation studied in Chapter 5, the strongly stratified fluids are characterized by the Froude number Fr proportional to Ma,

$$\mathrm{Fr} = \varepsilon.$$

Finally, the transport coefficients enhanced by radiation give rise to a small Péclet number, specifically,

$$\mathrm{Pe} = \varepsilon^2.$$

As a matter of fact, there are several possibilities of different scaling leading to the above values of the characteristic numbers Ma, Fr, and Pe. The subsequent analysis is based on a proper choice of constitutive equations reflecting the relevant physical properties of the fluid in the high temperature regime. In particular, the radiation component of the pressure, specific internal energy, and entropy as well as the heat conductivity coefficient augmented by a radiation part will be taken into account and play a significant role in the analysis of the asymptotic limit.

6.2 Primitive system

6.2.1 Field equations

A suitable but still highly simplified platform for studying fluids under the specific circumstances required by models in astrophysics is represented by the NAVIER-STOKES-FOURIER SYSTEM (balance of mass, momentum, and entropy) introduced in Section 1 that may be stated in a concise form:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{6.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S} - \varrho g \mathbf{j}, \tag{6.2}$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma, \tag{6.3}$$

where $g > 0$ is the gravitational constant and $\mathbf{j} = (0, 0, 1)$.

In order to develop the essential ideas without becoming entangled in the complexities of shapes of the underlying physical space, we confine ourselves to a very simple geometry expressed by means of the Cartesian coordinates $x = (x_1, x_2, x_3)$, where (x_1, x_2) denotes the *horizontal* directions, while x_3 stands for the *depth* pointing downward parallel to the gravitational force $g\mathbf{j}$. In addition, the periodic boundary conditions are imposed in the horizontal plane, the physical domain $\Omega \subset \mathbb{R}^3$ being an infinite slab bounded above and below by two parallel plates. Such a stipulation can be written in a concise form

$$\Omega = \mathcal{T}^2 \times (0,1), \tag{6.4}$$

where

$$\mathcal{T}^2 = \Big([0,1]|_{\{0,1\}} \Big)^2 \text{ is a two-dimensional flat torus.}$$

Similarly to the preceding chapters, the physical boundary is assumed to be impermeable and mechanically "smooth" (acoustically hard) so that the fluid velocity satisfies the complete slip boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0. \tag{6.5}$$

The bottom part of the boundary is thermally insulated:

$$\mathbf{q} \cdot \mathbf{n}|_{\{x_3 = 0\}} = 0, \tag{6.6}$$

while a radiative condition

$$\mathbf{q} \cdot \mathbf{n} = \beta(\vartheta)(\vartheta - \overline{\vartheta})|_{\{x_3 = 1\}} \tag{6.7}$$

is imposed on the upper part of the boundary.

Accordingly, the *total energy balance* takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \Big(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) + \varrho g x_3 \Big) \, \mathrm{d}x = \int_{\{x_3=1\}} \beta(\vartheta)(\overline{\vartheta} - \vartheta) \, \mathrm{d}S_x. \tag{6.8}$$

Obviously, condition (6.7), frequently used in astrophysical models, has a strong stabilizing effect driving the system to the state of the reference temperature $\overline{\vartheta}$.

6.2.2 Constitutive relations

A pivoting preliminary idea of how to obtain a simplified model under the circumstances described in Section 6.1 asserts that the characteristic temperature of the system is very large. This fact, in combination with physically relevant constitutive equations, gives rise to a tentative scaling to be incorporated in the values of the characteristic numbers Ma, Fr, and Pe.

Similarly to Chapter 5, the thermodynamic functions p, e, and s are determined through a single function P and a scalar parameter a as follows:

$$p(\varrho, \vartheta) = p_M(\varrho, \vartheta) + p_R(\vartheta), \ p_M = \vartheta^{\frac{5}{2}} P\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}} \Big), \ p_R = \frac{a}{3} \vartheta^4, \ a > 0, \tag{6.9}$$

$$e(\varrho,\vartheta) = e_M(\varrho,\vartheta) + e_R(\varrho,\vartheta),\ e_M = \frac{3}{2}\frac{\vartheta^{\frac{5}{2}}}{\varrho}P\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big),\ e_R = a\frac{\vartheta^4}{\varrho}, \tag{6.10}$$

and

$$s(\varrho,\vartheta) = s_M(\varrho,\vartheta) + s_R(\varrho,\vartheta),\ s_M(\varrho,\vartheta) = S\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big),\ s_R = \frac{4}{3}a\frac{\vartheta^3}{\varrho}, \tag{6.11}$$

where

$$S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z^2} \text{ for all } Z > 0. \tag{6.12}$$

Moreover, the *hypothesis of thermodynamic stability* requires

$$P'(Z) > 0,\ \frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z} > 0 \text{ for all } Z > 0. \tag{6.13}$$

Here we assume $P \in C^2[0,\infty)$ such that

$$P(0) = 0,\ P'(0) = p_0 > 0, \tag{6.14}$$

and, similarly to hypotheses (5.22), (5.23),

$$\sup_{Z>0} \frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z} < \infty,\ \lim_{Z\to\infty} \frac{P(Z)}{Z^{\frac{5}{3}}} = p_\infty > 0. \tag{6.15}$$

The viscous stress tensor $\mathbb{S}$ obeys the classical Newton's law

$$\mathbb{S}[\vartheta,\nabla_x\mathbf{u}] = \mu(\vartheta)\Big(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\Big), \tag{6.16}$$

where we have deliberately omitted the bulk viscosity component assumed to be zero for the plasma. The heat flux $\mathbf{q}$ is given by Fourier's law

$$\mathbf{q}[\vartheta,\nabla_x\vartheta] = -\kappa(\vartheta)\nabla_x\vartheta. \tag{6.17}$$

In order to avoid unnecessary technicalities, we simply assume that μ is an affine function of the absolute temperature, specifically,

$$\mu(\vartheta) = \mu_0 + \mu_1\vartheta, \quad \text{with } \mu_0,\mu_1 > 0. \tag{6.18}$$

Similarly,

$$\kappa(\vartheta) = \kappa_M(\vartheta) + d\vartheta^3,\ \kappa_M(\vartheta) = \kappa_0 + \kappa_1\vartheta, \quad \text{with } d,\kappa_0,\kappa_1 > 0, \tag{6.19}$$

and

$$\beta(\vartheta) = \beta_1\vartheta,\ \beta_1 > 0. \tag{6.20}$$

As already pointed out in Section 1.4.3, the extra cubic term in (6.19) is responsible for the fast transport of heat due to radiation.

6.2.3 Scaling

Keeping in mind the characteristic features of the underlying physical system discussed in Section 6.1, we introduce a tentative scaling as follows:

- the characteristic temperature of the system is large, specifically of order $\varepsilon^{-2\alpha/3}$, where ε is a small positive parameter, and $2 < \alpha < 3$;
- the radiative constants satisfy $a \approx \varepsilon^{2\alpha+1}$, $d \approx \varepsilon^{4\alpha/3-2}$;
- the characteristic velocity is of order $\varepsilon^{1-\alpha/3}$, the characteristic length of order $\varepsilon^{-1-\alpha/3}$, the reference time is of order ε^{-2} so that the Strouhal number equals 1;
- the gravitational constant g is of order $\varepsilon^{1-\alpha/3}$;
- $\beta_1 \approx \varepsilon^{\alpha/3}$.

The reader may consult Section 1.4.4 for typical values of the physical constants appearing above.

Under these circumstances, the re-scaled system (6.1–6.3), (6.8) reads as follows:

■ Scaled Navier-Stokes-Fourier System:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{6.21}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p_\varepsilon(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}_\varepsilon[\vartheta, \nabla_x \mathbf{u}] - \frac{1}{\varepsilon^2} \varrho g \mathbf{j}, \tag{6.22}$$

$$\partial_t(\varrho s_\varepsilon(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s_\varepsilon(\varrho, \vartheta)\mathbf{u}) + \frac{1}{\varepsilon^2} \mathrm{div}_x \left(\frac{\mathbf{q}_\varepsilon[\vartheta, \nabla_x \vartheta]}{\vartheta} \right) = \sigma_\varepsilon, \tag{6.23}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{\varepsilon^2}{2} \varrho |\mathbf{u}|^2 + \varrho e_\varepsilon(\varrho, \vartheta) + \varrho g x_3 \right) \mathrm{d}x = \int_{\{x_3 = 1\}} \beta_1 \vartheta \frac{\overline{\vartheta} - \vartheta}{\varepsilon} \, \mathrm{d}S_x, \tag{6.24}$$

---

supplemented with

■ Scaled Equations of State:

$$p_\varepsilon(\varrho, \vartheta) = \frac{\vartheta^{\frac{5}{2}}}{\varepsilon^\alpha} P\left(\varepsilon^\alpha \frac{\varrho}{\vartheta^{\frac{3}{2}}} \right) + \varepsilon \frac{a}{3} \vartheta^4, \tag{6.25}$$

$$e_\varepsilon(\varrho, \vartheta) = \frac{3}{2\varrho} \frac{\vartheta^{\frac{5}{2}}}{\varepsilon^\alpha} P\left(\varepsilon^\alpha \frac{\varrho}{\vartheta^{\frac{3}{2}}} \right) + \varepsilon a \frac{\vartheta^4}{\varrho}, \tag{6.26}$$

$$s_\varepsilon(\varrho, \vartheta) = S\left(\varepsilon^\alpha \frac{\varrho}{\vartheta^{\frac{3}{2}}} \right) - S(\varepsilon^\alpha) + \varepsilon \frac{4a}{3} \frac{\vartheta^3}{\varrho}. \tag{6.27}$$

Accordingly, the viscous stress tensor $\mathbb{S}_\varepsilon$ is given as

$$\mathbb{S}_\varepsilon(\vartheta, \nabla_x \mathbf{u}) = (\varepsilon^{2\alpha/3}\mu_0 + \mu_1\vartheta)\Big(\nabla_x \mathbf{u} + \nabla_x^\perp \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\Big), \tag{6.28}$$

while the heat flux $\mathbf{q}_\varepsilon$ reads

$$\mathbf{q}_\varepsilon(\vartheta, \nabla_x\vartheta) = -\Big(\varepsilon^{2+2\alpha/3}\kappa_0 + \varepsilon^2\kappa_1\vartheta + d\vartheta^3\Big)\nabla_x\vartheta. \tag{6.29}$$

We recall that in the framework of weak solutions considered in this book, the entropy production rate σ_ε is a non-negative measure on the set $[0, T] \times \overline{\Omega}$ satisfying

$$\sigma_\varepsilon \geq \frac{1}{\vartheta}\Big(\varepsilon^2 \mathbb{S}_\varepsilon(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{1}{\varepsilon^2}\frac{\mathbf{q}_\varepsilon(\vartheta, \nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\Big), \tag{6.30}$$

where

$$\frac{1}{\vartheta}\Big(\varepsilon^2 \mathbb{S}_\varepsilon(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{1}{\varepsilon^2}\frac{\mathbf{q}_\varepsilon(\vartheta, \nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\Big) \tag{6.31}$$

$$\geq \frac{\varepsilon^2}{2}\mu_1\Big|\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\Big|^2 + \varepsilon^{2\alpha/3}\kappa_0|\nabla_x \log(\vartheta)|^2 + \frac{\kappa_1}{\vartheta}|\nabla_x\vartheta|^2 + \frac{d}{\varepsilon^2}\vartheta|\nabla_x\vartheta|^2.$$

The homogeneous boundary conditions (6.5), (6.6) remain unaffected by the scaling, while the radiative condition (6.7) is converted to

$$\frac{1}{\varepsilon^2}\mathbf{q}_\varepsilon(\vartheta, \nabla_x\vartheta)\cdot\mathbf{n} = \beta_1\vartheta\frac{\vartheta - \overline{\vartheta}}{\varepsilon}|_{\{x_3=1\}}. \tag{6.32}$$

Thus, at least formally, system (6.21–6.24) corresponds to system (4.1–4.4) with the values of the characteristic numbers

$$\mathrm{Ma} = \mathrm{Fr} = \varepsilon,\ \mathrm{Pe} = \varepsilon^2.$$

A fundamentally new feature of the present problem is the fact that the material properties of the fluid change during the scaling process. In this context, it is interesting to note that the state equation for the pressure approaches the standard perfect gas law in the asymptotic limit $\varepsilon \to 0$, namely

$$p_\varepsilon(\varrho, \vartheta) \to p_0\varrho\vartheta \text{ as } \varepsilon \to 0.$$

This is in good agreement with the well-founded observation that any monoatomic gas obeys approximately the perfect gas state equation in the non-degenerate area of high temperatures and moderate values of the density. This remarkable property plays a significant role in the asymptotic analysis of the system for $\varepsilon \to 0$ eliminating artificial pressure components in the so-called *anelastic limit* discussed below.

6.3 Asymptotic limit

6.3.1 Static states

Static states are solutions of system (6.21–6.24) with vanishing velocity field. In the present setting, the temperature corresponding to any static state is constant, specifically, $\vartheta = \overline{\vartheta}$. Accordingly, the density ϱ must satisfy

$$\nabla_x p_\varepsilon(\varrho, \overline{\vartheta}) + \varrho g \mathbf{j} = 0 \text{ in } \Omega, \ \ \varrho \geq 0,$$

where ϱ is uniquely determined by the total mass

$$M_0 = \int_\Omega \varrho \ {\rm d}x.$$

Note that, in general, any static solution ϱ may and indeed does depend on ε.

For future analysis, it seems more convenient to approximate the pressure by its linearization, namely

$$p_\varepsilon(\varrho, \vartheta) \approx p_0 \varrho \vartheta,$$

and to solve the corresponding linear problem

$$p_0 \overline{\vartheta} \nabla_x \tilde{\varrho} + \tilde{\varrho} g \mathbf{j} = 0 \text{ in } \Omega, \ \int_\Omega \tilde{\varrho} \ {\rm d}x = M_0. \tag{6.33}$$

It is easy to check that (6.33) admits a unique (non-negative) solution in the form

$$\tilde{\varrho} = \tilde{\varrho}(x_3) = c(M_0) \exp\Big(- \frac{g x_3}{p_0 \overline{\vartheta}} \Big).$$

In agreement with our previous discussion, the density distribution given by $\tilde{\varrho}$ is a very good approximation of the static state provided ε is small enough.

6.3.2 Solutions to the primitive system

Analogously as in Chapter 5, we prescribe the initial data in the form:

$$\varrho(0, \cdot) = \varrho_{0,\varepsilon} = \tilde{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon}, \ \mathbf{u}(0, \cdot) = \mathbf{u}_{0,\varepsilon}, \ \vartheta(0, \cdot) = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon}, \tag{6.34}$$

where $\tilde{\varrho} = \tilde{\varrho}(x_3)$ solves (6.33), $\overline{\vartheta}$ is the equilibrium temperature introduced in (6.32), and

$$\int_\Omega \varrho^{(1)}_{0,\varepsilon} \ {\rm d}x = 0. \tag{6.35}$$

Given $\varepsilon > 0$, we suppose that the scaled NAVIER-STOKES-FOURIER SYSTEM (6.21–6.30), supplemented with the boundary conditions (6.5), (6.6), (6.32), and the initial conditions (6.34) admits a weak solution $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}$ on the set $(0,T) \times \Omega$

in the sense specified in Section 2.1. As a matter of fact, the main existence result established in Theorem 3.1 does not cover the case of the radiative boundary condition (6.32). On the other hand, however, the abstract theory developed in Chapter 3 can be easily modified in order to accommodate more general boundary conditions including (6.32).

In accordance with Theorems 3.1, 3.2, we assume that

$$\left\{ \begin{array}{c} \varrho_\varepsilon \geq 0, \ \varrho_\varepsilon \in L^\infty(0,T;L^{\frac{5}{3}}(\Omega)), \\ \mathbf{u}_\varepsilon \in L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)), \ \mathbf{u}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0, \\ \vartheta_\varepsilon > 0 \text{ a.a. in } (0,T)\times\Omega, \ \vartheta_\varepsilon \in L^\infty(0,T;L^4(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega)), \end{array} \right\} \tag{6.36}$$

and the following integral identities hold:

(i) Renormalized equation of continuity:

$$\int_0^T \int_\Omega \varrho_\varepsilon B(\varrho_\varepsilon)\Big(\partial_t\varphi + \mathbf{u}_\varepsilon \cdot \nabla_x\varphi\Big) \ \mathrm{d}x \ \mathrm{d}t \tag{6.37}$$
$$= \int_0^T \int_\Omega b(\varrho_\varepsilon)\mathrm{div}_x\mathbf{u}_\varepsilon\varphi \ \mathrm{d}x \ \mathrm{d}t - \int_\Omega \varrho_{0,\varepsilon}B(\varrho_{0,\varepsilon})\varphi(0,\cdot) \ \mathrm{d}x$$

for any test function $\varphi \in C_c^\infty([0,T)\times\overline{\Omega})$, and any b as in (2.3);

(ii) Momentum equation:

$$\int_0^T \int_\Omega \Big(\varrho_\varepsilon\mathbf{u}_\varepsilon \cdot \partial_t\varphi + \varrho_\varepsilon\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x\varphi + \frac{1}{\varepsilon^2}p_\varepsilon(\varrho_\varepsilon,\vartheta_\varepsilon)\mathrm{div}_x\varphi\Big) \ \mathrm{d}x \ \mathrm{d}t \tag{6.38}$$
$$= \int_0^T \int_\Omega \Big(\mathbb{S}_\varepsilon(\vartheta_\varepsilon,\nabla_x\mathbf{u}_\varepsilon) : \nabla_x\varphi + \frac{1}{\varepsilon^2}\varrho g\varphi_3\Big) \ \mathrm{d}x \ \mathrm{d}t - \int_\Omega \varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon}\varphi(0,\cdot) \ \mathrm{d}x,$$

for any test function $\varphi \in C_c^\infty([0,T)\times\overline{\Omega};\mathbb{R}^3)$, $\varphi\cdot\mathbf{n}|_{\partial\Omega} = 0$, where $\mathbb{S}_\varepsilon$ is given by (6.28);

(iii) Entropy balance equation:

$$\int_0^T \int_\Omega \Big[\varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon,\vartheta_\varepsilon)\Big(\partial_t\varphi + \mathbf{u}_\varepsilon\cdot\nabla_x\varphi\Big) + \frac{1}{\varepsilon^2}\frac{\mathbf{q}_\varepsilon(\vartheta_\varepsilon,\nabla_x\vartheta_\varepsilon)}{\vartheta_\varepsilon}\cdot\nabla_x\varphi\Big] \ \mathrm{d}x \ \mathrm{d}t \tag{6.39}$$
$$+ \langle\sigma_\varepsilon;\varphi\rangle_{[\mathcal{M};C]([0,T]\times\overline{\Omega})} - \int_0^T \int_{\{x_3=1\}} \beta_1\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\varphi \ \mathrm{d}S_x \ \mathrm{d}t$$
$$= -\int_\Omega \varrho_{0,\varepsilon}s_\varepsilon(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon})\varphi(0,\cdot) \ \mathrm{d}x$$

for any $\varphi \in C_c^\infty([0,T)\times\overline{\Omega})$, where $\mathbf{q}_\varepsilon$ is given by (6.29), and σ_ε is a non-negative measure satisfying inequality (6.30);

(iv) Total energy balance:

$$\int_\Omega \Big(\frac{\varepsilon^2}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \varrho_\varepsilon e_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon) + \varrho_\varepsilon g x_3 \Big)(\tau) \ \mathrm{d}x \tag{6.40}$$
$$= \int_\Omega \Big(\frac{\varepsilon^2}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + \varrho_{0,\varepsilon} e_\varepsilon(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) + \varrho_{0,\varepsilon} g x_3 \Big) \ \mathrm{d}x$$
$$+ \int_0^\tau \int_{\{x_3=1\}} \beta_1 \vartheta_\varepsilon \frac{\overline{\vartheta} - \vartheta_\varepsilon}{\varepsilon} \ \mathrm{d}S_x \ \mathrm{d}t$$

for a.a. $\tau \in (0,T)$.

6.3.3 Main result

The limit problem has been formally identified in Section 4.3. It consists of the following set of equations.

■ HYDROSTATIC BALANCE EQUATION:

$$p_0 \overline{\vartheta} \nabla_x \tilde{\varrho} + \tilde{\varrho} g \mathbf{j} = 0; \tag{6.41}$$

■ ANELASTIC CONSTRAINT:

$$\mathrm{div}_x (\tilde{\varrho} \mathbf{U}) = 0; \tag{6.42}$$

■ MOMENTUM EQUATION:

$$\partial_t (\tilde{\varrho} \mathbf{U}) + \mathrm{div}_x (\tilde{\varrho} \mathbf{U} \otimes \mathbf{U}) + \tilde{\varrho} \nabla_x \Pi = \mu_1 \overline{\vartheta} \Delta \mathbf{U} + \frac{1}{3} \mu_1 \overline{\vartheta} \nabla_x \mathrm{div}_x \mathbf{U} + \frac{\vartheta^{(2)}}{\overline{\vartheta}} \tilde{\varrho} g \mathbf{j}, \tag{6.43}$$

where $\mathbf{U}$ satisfies the complete slip boundary conditions

$$\mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \Big[\mu_1 \overline{\vartheta} \Big(\nabla_x \mathbf{U} + \nabla_x^T \mathbf{U} \Big) \mathbf{n} \Big] \times \mathbf{n}|_{\partial\Omega} = 0, \tag{6.44}$$

and $\vartheta^{(2)}$ is related to the vertical component of the velocity through

$$\tilde{\varrho} g U_3 = d \overline{\vartheta}^3 \Delta \vartheta^{(2)} \text{ in } \Omega, \ \nabla_x \vartheta^{(2)} \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{6.45}$$

A suitable *weak formulation* of the momentum equation (6.43), supplemented with the anelastic constraint (6.42), and the complete slip boundary conditions (6.44), reads:

$$\int_0^T \int_\Omega \Big(\tilde{\varrho} \mathbf{U} \cdot \varphi + \tilde{\varrho} \mathbf{U} \otimes \mathbf{U} : \nabla_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t \tag{6.46}$$
$$= \int_0^T \int_\Omega \mu_1 \overline{\vartheta} \Big(\nabla_x \mathbf{U} + \nabla_x^T \mathbf{U} - \frac{2}{3} \mathrm{div}_x \mathbf{U} \mathbb{I} \Big) : \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t$$
$$- \int_0^T \int_\Omega \frac{\vartheta^{(2)}}{\overline{\vartheta}} \tilde{\varrho} g \varphi_3 \ \mathrm{d}x \ \mathrm{d}t - \int_\Omega \tilde{\varrho} \mathbf{U}_0 \cdot \varphi(0, \cdot) \ \mathrm{d}x$$

to be satisfied for any test function

$$\varphi \in C_c^\infty([0,T) \times \overline{\Omega}; \mathbb{R}^3), \ \varphi \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \ \mathrm{div}_x(\tilde{\varrho}\varphi) = 0.$$

Formula (6.46) suggests that the standard concept of Helmholtz projectors introduced in Section 5.4.1 has to be modified in order to handle the anelastic approximation. To this end, any vector function $\mathbf{v} : \Omega \to \mathbb{R}^3$ is now decomposed as

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■ WEIGHTED HELMHOLTZ DECOMPOSITION:

$$\mathbf{v} = \underbrace{\mathbf{H}_{\tilde{\varrho}}[\mathbf{v}]}_{\text{solenoidal part}} + \underbrace{\mathbf{H}^{\perp}_{\tilde{\varrho}}[\mathbf{v}]}_{\text{weighted gradient part}}, \tag{6.47}$$

with the weighted gradient part given through formula

$$\mathbf{H}^{\perp}_{\tilde{\varrho}}[\mathbf{v}] = \tilde{\varrho}\nabla_x\Psi,$$

---

where the scalar potential Ψ is determined as a unique solution to the Neumann problem:

$$\mathrm{div}_x(\tilde{\varrho}\nabla_x\Psi) = \mathrm{div}_x\mathbf{v} \text{ in } \Omega, \ \tilde{\varrho}\nabla_x\Psi \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{v}\cdot\mathbf{n}, \ \int_\Omega \Psi \ \mathrm{d}x = 0. \tag{6.48}$$

A weak (variational) formulation of (6.48) can be written in the form

$$\int_\Omega \tilde{\varrho}\nabla_x\Psi \cdot \nabla_x\varphi \ \mathrm{d}x = \int_\Omega \mathbf{v}\cdot\nabla_x\varphi \ \mathrm{d}x, \ \int_\Omega \Psi \ \mathrm{d}x = 0 \tag{6.49}$$

to be satisfied for any test function $\varphi \in C_c^\infty(\overline{\Omega})$. Since the function $\tilde{\varrho}$ is regular and bounded below on Ω away from zero, the mappings $\mathbf{H}_{\tilde{\varrho}}$, $\mathbf{H}^{\perp}_{\tilde{\varrho}}$ enjoy the same continuity properties as the standard Helmholtz projectors, in particular, they are bounded on $W^{1,p}(\Omega;\mathbb{R}^3)$ as well as on $L^p(\Omega;\mathbb{R}^3)$ provided $1 < p < \infty$ (see Section 10.6 in Appendix).

Having collected the preliminary material we are in a position to state the main result to be proved in the remaining part of this chapter. The resulting problem, arising as a simultaneous singular limit of the Mach, Froude, and Péclet numbers, can be viewed as a simple model of the *fluid motion in stellar radiative zones.*

■ LOW MACH NUMBER LIMIT – STRONG STRATIFICATION:

Theorem 6.1. *Let* $\Omega = \mathcal{T}^2 \times (0,1)$. *Suppose that* $P \in C^2[0,\infty)$ *satisfies hypotheses* (6.13–6.15). *Let* $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ *be a family of weak solutions to the rescaled Navier-Stokes-Fourier system* (6.21–6.30) *on* $(0,T)\times\Omega$ *in the sense specified in Section* 6.3.2, *with the parameter* $\alpha \in (2,3)$, *supplemented with the boundary conditions* (6.5), (6.6), (6.32), *and the initial conditions*

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon} = \tilde{\varrho} + \varepsilon\varrho^{(1)}_{0,\varepsilon}, \ \mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon}, \ \vartheta(0,\cdot) = \overline{\vartheta} + \varepsilon\vartheta^{(1)}_{0,\varepsilon},$$

where $\tilde\varrho$ *solves the linearized static problem* (6.33), $\varrho^{(1)}_{0,\varepsilon}$ *satisfies* (6.35), *and*

$$\left\{ \begin{array}{c} \{\varrho^{(1)}_{0,\varepsilon}\}_{\varepsilon>0},\ \{\vartheta^{(1)}_{0,\varepsilon}\}_{\varepsilon>0} \text{ are bounded in } L^\infty(\Omega), \\ \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ weakly-(*) in } L^\infty(\Omega;\mathbb{R}^3). \end{array} \right\}$$

Then, at least for suitable subsequences, we have

$$\varrho_\varepsilon \to \tilde\varrho \text{ in } C([0,T];L^q(\Omega)) \text{ for any } 1 \le q < \frac{5}{3},$$
$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)),$$
$$\vartheta_\varepsilon \to \overline{\vartheta} \text{ in } L^2(0,T;W^{1,2}(\Omega)),$$

and

$$\nabla_x\Big(\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon^2}\Big) \to \nabla_x\vartheta^{(2)} \text{ weakly in } L^1(0,T;L^1(\Omega;\mathbb{R}^3)),$$

where $\tilde\varrho$, $\overline{\vartheta}$, $\mathbf{U}$, $\vartheta^{(2)}$ *is a weak solution to problem* (6.41–6.45), *supplemented with the initial condition*

$$\tilde\varrho\mathbf{U}_0 = \mathbf{H}_{\tilde\varrho}[\tilde\varrho\mathbf{u}_0].$$

Remark: *The same result can be shown provided that* $\Omega \subset \mathbb{R}^3$ *is a bounded regular domain, the driving force of the form* $\mathbf{f} = \nabla_x F$, *where* $F \in W^{1,\infty}(\Omega)$, *and the boundary condition* (6.7) *imposed on the whole* $\partial\Omega$.

At a purely conceptual level, the principal ideas of the proof of Theorem 5.1 are identical to those introduced in Chapter 4 and further developed in Chapter 5. In particular, each function h defined on the set $(0,T)\times\Omega$ will be decomposed as

$$h = [h]_{\rm ess} + [h]_{\rm res},$$

where, similarly to (4.44), (4.45),

$$[h]_{\rm ess} = h\ 1_{\mathcal{M}^\varepsilon_{\rm ess}},\ [h]_{\rm res} = h\ 1_{\mathcal{M}^\varepsilon_{\rm res}},$$
$$\mathcal{M}^\varepsilon_{\rm ess} = \{(t,x)\in(0,T)\times\Omega \mid \underline{\varrho}/2 < \varrho_\varepsilon(t,x) < 2\overline{\varrho},\ \overline{\vartheta}/2 < \vartheta_\varepsilon(t,x) < 2\overline{\vartheta}\},$$
$$\mathcal{M}^\varepsilon_{\rm res} = ((0,T)\times\Omega)\setminus\mathcal{M}^\varepsilon_{\rm ess},$$

where the constants $\underline{\varrho}$, $\overline{\varrho}$ have been fixed in such a way that

$$0 < \underline{\varrho} < \inf_{x\in\Omega}\tilde\varrho(x) \le \sup_{x\in\Omega}\tilde\varrho(x) < \overline{\varrho}. \tag{6.50}$$

As already pointed out in Chapter 5, the "residual" parts are expected to vanish for $\varepsilon\to 0$, while the total information on the asymptotic limit is carried by the "essential" components.

A significant new aspect of the problem arises in the analysis of propagation of the acoustic waves. In agreement with the formal arguments discussed in Section 4.4.2, the speed of the sound waves in a highly stratified fluid changes effectively with the depth (vertical) coordinate. Consequently, the spectral analysis of the wave operator must be considerably modified, the basic modes being orthogonal in a *weighted* space reflecting the anisotropy in the system.

6.4 Uniform estimates

Although the uniform bounds deduced below are of the same nature as in Section 5.2, a rigorous analysis becomes more technical as the structural properties of the thermodynamic functions depend on the parameter ε.

6.4.1 Dissipation equation, energy estimates

To begin, observe that the total mass is a constant of motion, specifically,

$$\int_\Omega \varrho_\varepsilon(t,\cdot)\ \mathrm{d}x = \int_\Omega \tilde{\varrho}\ \mathrm{d}x = M_0 \text{ for all } t \in [0,T]. \tag{6.51}$$

Exactly as in Chapter 5, combining the entropy production equation (6.39) with the total energy balance (6.40) we arrive at the *total dissipation balance*:

$$\begin{aligned} &\int_\Omega \left[\frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\Big(H^\varepsilon_{\overline{\vartheta}}(\varrho_\varepsilon,\vartheta_\varepsilon) + \varrho_\varepsilon g x_3\Big)\right](\tau,\cdot)\ \mathrm{d}x \\ &\qquad + \frac{\overline{\vartheta}}{\varepsilon^2}\sigma_\varepsilon\Big[[0,\tau]\times\overline{\Omega}\Big] + \int_0^\tau \int_{\{x_3=1\}} \beta_1 \frac{(\vartheta_\varepsilon - \overline{\vartheta})^2}{\varepsilon^3}\ \mathrm{d}S_x\ \mathrm{d}t \\ &= \int_\Omega \left[\frac{1}{2}\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2}\Big(H^\varepsilon_{\overline{\vartheta}}(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon}) + \varrho_{0,\varepsilon} g x_3\Big)\right]\ \mathrm{d}x \text{ for } a.a.\ \tau \in [0,T], \end{aligned} \tag{6.52}$$

where we have set

$$H^\varepsilon_{\overline{\vartheta}}(\varrho,\vartheta) = \varrho e_\varepsilon(\varrho,\vartheta) - \overline{\vartheta}\varrho s_\varepsilon(\varrho,\vartheta).$$

Since the functions p_ε, e_ε, and s_ε satisfy *Gibbs' equation* (1.2) for any fixed $\varepsilon > 0$, we easily compute

$$\frac{\partial^2 H^\varepsilon_{\overline{\vartheta}}(\varrho,\overline{\vartheta})}{\partial\varrho^2} = \frac{1}{\varrho}\frac{\partial p_\varepsilon(\varrho,\overline{\vartheta})}{\partial\varrho} = \frac{\overline{\vartheta}}{\varrho}P'\Big(\varepsilon^\alpha \frac{\varrho}{\overline{\vartheta}^{\frac{3}{2}}}\Big); \tag{6.53}$$

whence

$$\frac{\partial H^\varepsilon_{\overline{\vartheta}}(\varrho,\overline{\vartheta})}{\partial\varrho} = \int_1^\varrho \frac{1}{z}\frac{\partial p_\varepsilon(z,\overline{\vartheta})}{\partial\varrho}\ \mathrm{d}z + \text{const},$$

in particular,

$$\frac{\partial H^\varepsilon_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon,\overline{\vartheta})}{\partial\varrho} + g x_3 = \text{const}, \tag{6.54}$$

where $\tilde{\varrho}_\varepsilon$ is the solution of the "exact" static problem

$$\nabla_x p_\varepsilon(\tilde{\varrho}_\varepsilon,\overline{\vartheta}) + \tilde{\varrho}_\varepsilon g\mathbf{j} = 0,\ \int_\Omega \tilde{\varrho}_\varepsilon\ \mathrm{d}x = M_0. \tag{6.55}$$

In accordance with (6.54), relation (6.52) can be rewritten in the form

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega \varrho_\varepsilon|\mathbf{u}_\varepsilon|^2(\tau,\cdot)\,\mathrm{d}x + \frac{1}{\varepsilon^2}\int_\Omega \Big(H^\varepsilon_{\overline{\vartheta}}(\varrho_\varepsilon,\vartheta_\varepsilon) - H^\varepsilon_{\overline{\vartheta}}(\varrho_\varepsilon,\overline{\vartheta})\Big)(\tau,\cdot)\,\mathrm{d}x \qquad (6.56)\\
&\quad + \frac{1}{\varepsilon^2}\int_\Omega \Big(H^\varepsilon_{\overline{\vartheta}}(\varrho_\varepsilon,\overline{\vartheta}) - (\varrho_\varepsilon - \tilde{\varrho}_\varepsilon)\frac{\partial H^\varepsilon_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon,\overline{\vartheta})}{\partial\varrho} - H^\varepsilon_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon,\overline{\vartheta})\Big)(\tau,\cdot)\,\mathrm{d}x\\
&\quad + \frac{\overline{\vartheta}}{\varepsilon^2}\sigma_\varepsilon\Big[[0,\tau]\times\overline{\Omega}\Big] + \int_0^\tau\int_{\{x_3=1\}} \beta_1\frac{(\vartheta_\varepsilon-\overline{\vartheta})^2}{\varepsilon^3}\,\mathrm{d}S_x\,\mathrm{d}t\\
&= \frac{1}{2}\int_\Omega \varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2(\tau,\cdot)\,\mathrm{d}x + \frac{1}{\varepsilon^2}\int_\Omega \Big(H^\varepsilon_{\overline{\vartheta}}(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon}) - H^\varepsilon_{\overline{\vartheta}}(\varrho_{0,\varepsilon},\overline{\vartheta})\Big)(\tau,\cdot)\,\mathrm{d}x\\
&\quad + \frac{1}{\varepsilon^2}\int_\Omega \Big(H^\varepsilon_{\overline{\vartheta}}(\varrho_{0,\varepsilon},\overline{\vartheta}) - (\varrho_{0,\varepsilon} - \tilde{\varrho}_\varepsilon)\frac{\partial H^\varepsilon_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon,\overline{\vartheta})}{\partial\varrho} - H^\varepsilon_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon,\overline{\vartheta})\Big)(\tau,\cdot)\,\mathrm{d}x
\end{aligned}
$$

for a.a $\tau \in [0,T]$.

The following assertion shows that the "exact" static solution $\tilde{\varrho}_\varepsilon$ and the "limit" static solution $\tilde{\varrho}$ are close as soon as ε is small enough.

Lemma 6.1. *Let $\tilde{\varrho}$ be the solution of problem* (6.33)*, while $\tilde{\varrho}_\varepsilon$ satisfies* (6.55)*.*

Then

$$\sup_{x\in\Omega}|\tilde{\varrho}_\varepsilon(x) - \tilde{\varrho}(x)| \le c\varepsilon^\alpha, \qquad (6.57)$$

where the constant c is independent of ε, and α has been introduced in Section 6.2.3.

Proof. Obviously both $\tilde{\varrho}_\varepsilon$ and $\tilde{\varrho}$ depend solely on the vertical coordinate x_3, and, in addition,

$$\int_0^1 \Big(\tilde{\varrho}_\varepsilon(x_3) - \tilde{\varrho}(x_3)\Big)\,\mathrm{d}x_3 = 0. \qquad (6.58)$$

Moreover, as a consequence of hypothesis (6.14), there exist positive constants $\underline{\varrho}$, $\overline{\varrho}$ such that

$$0 < \underline{\varrho} < \inf_{x\in\Omega}\tilde{\varrho}_\varepsilon(x) \le \sup_{x\in\Omega}\tilde{\varrho}_\varepsilon(x) < \overline{\varrho} \qquad (6.59)$$

uniformly for $\varepsilon \to 0$.

Finally, as $P \in C^2[0,\infty)$, a direct inspection of (6.25), (6.33), (6.55) yields

$$\Big|\frac{\mathrm{d}}{\mathrm{d}x_3}(\log(\tilde{\varrho}_\varepsilon) - \log(\tilde{\varrho}))\Big| \le \varepsilon^\alpha c,$$

which, combined with (6.58), (6.59), implies (6.57). □

In order to exploit the total dissipation balance (6.56) for obtaining uniform estimates, we first observe that the expression on the right-hand side is bounded, in terms of the initial data, uniformly for $\varepsilon \to 0$. To this end, we use Gibbs' equation (1.2) to obtain

$$\frac{\partial H^{\varepsilon}_{\overline{\vartheta}}(\varrho,\vartheta)}{\partial \vartheta} = \varrho(\vartheta - \overline{\vartheta})\frac{\partial s_{\varepsilon}(\varrho,\vartheta)}{\partial \vartheta}, \tag{6.60}$$

in particular,

$$\frac{1}{\varepsilon^2}\Big|H^{\varepsilon}_{\overline{\vartheta}}(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon}) - H^{\varepsilon}_{\overline{\vartheta}}(\varrho_{0,\varepsilon},\overline{\vartheta})\Big| \leq c_1 \Big|\frac{\vartheta_{0,\varepsilon} - \overline{\vartheta}}{\varepsilon}\Big|^2 = c_1|\vartheta^{(1)}_{0,\varepsilon}|^2 \leq c_2.$$

Indeed a direct computation yields

$$\frac{\partial s_{\varepsilon}(\varrho,\vartheta)}{\partial \vartheta} = -\frac{3}{2\vartheta}S'(Z)Z + \varepsilon\frac{4a}{\varrho}\vartheta^2 \text{ for } Z = \varepsilon^{\alpha}\frac{\varrho}{\vartheta^{\frac{3}{2}}}; \tag{6.61}$$

whence the desired bound follows from hypothesis (6.15).

Similarly, in accordance with (6.53), the function $H^{\varepsilon}_{\overline{\vartheta}}$ is twice continuously differentiable in ϱ, in particular,

$$\begin{aligned}\frac{1}{\varepsilon^2}\Big|H^{\varepsilon}_{\overline{\vartheta}}(\varrho_{0,\varepsilon},\overline{\vartheta}) - (\varrho_{0,\varepsilon} - \tilde{\varrho}_{\varepsilon})\frac{\partial H^{\varepsilon}_{\overline{\vartheta}}(\tilde{\varrho}_{\varepsilon},\overline{\vartheta})}{\partial \varrho} - H^{\varepsilon}_{\overline{\vartheta}}(\tilde{\varrho}_{\varepsilon},\overline{\vartheta})\Big| \leq c_1\Big|\frac{\varrho_{0,\varepsilon} - \tilde{\varrho}_{\varepsilon}}{\varepsilon}\Big|^2 \\ \leq c_2\left(\Big|\frac{\varrho_{0,\varepsilon} - \tilde{\varrho}}{\varepsilon}\Big|^2 + \Big|\frac{\tilde{\varrho} - \tilde{\varrho}_{\varepsilon}}{\varepsilon}\Big|^2\right) = c_2\left(|\varrho^{(1)}_{0,\varepsilon}|^2 + \Big|\frac{\tilde{\varrho} - \tilde{\varrho}_{\varepsilon}}{\varepsilon}\Big|^2\right);\end{aligned}$$

whence the desired uniform bound is provided by Lemma 6.1, where $\alpha \in (2,3)$.

The hypothesis of thermodynamic stability expressed through (6.13), together with (6.53), (6.60), imply that all integrated quantities on the left-hand side of the total dissipation balance (6.56) are non-negative, and, consequently, we deduce immediately the following uniform estimates:

$$\operatorname{ess}\sup_{t\in(0,T)} \int_{\Omega} \varrho_{\varepsilon}|\mathbf{u}_{\varepsilon}|^2 \, \mathrm{d}x \leq c, \tag{6.62}$$

$$\|\sigma_{\varepsilon}\|_{\mathcal{M}^+([0,T]\times\overline{\Omega})} \leq \varepsilon^2 c, \tag{6.63}$$

and, by virtue of hypothesis (6.20),

$$\int_0^T \int_{\{x_3=1\}} \Big|\frac{\vartheta_{\varepsilon} - \overline{\vartheta}}{\varepsilon}\Big|^2 \, \mathrm{d}S_x \, \mathrm{d}t \leq \varepsilon c. \tag{6.64}$$

Note that $\vartheta_{\varepsilon} \in L^2(0,T;W^{1,2}(\Omega))$ possesses a well-defined trace on $\partial\Omega$ for a.a. $t \in (0,T)$.

As for the integrals containing the function $H^{\varepsilon}_{\overline{\vartheta}}$, observe first that

$$H^{\varepsilon}_{\overline{\vartheta}}(\varrho_{\varepsilon},\vartheta_{\varepsilon}) - H^{\varepsilon}_{\overline{\vartheta}}(\varrho_{\varepsilon},\overline{\vartheta}) \geq c|\vartheta_{\varepsilon} - \overline{\vartheta}|^2$$

as soon as

$$\underline{\varrho}/2 < \varrho_\varepsilon < 2\overline{\varrho},\ \overline{\vartheta}/2 < \vartheta_\varepsilon < 2\overline{\vartheta},$$

where, as a direct consequence (6.13), (6.60), and (6.61), the constant c is independent of ε. In particular, we have obtained

$$\operatorname{ess}\sup_{t\in(0,T)} \left\| \left[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\right]_{\rm ess} \right\|_{L^2(\Omega)} \leq c. \tag{6.65}$$

Furthermore, it follows from hypotheses (6.13–6.15) that

$$P'(Z) \geq c(1 + Z^{\frac{2}{3}}) > 0 \text{ for all } Z \geq 0, \tag{6.66}$$

in particular,

$$\frac{\partial^2 H^\varepsilon_{\overline{\vartheta}}(\varrho, \overline{\vartheta})}{\partial \varrho^2} = \frac{\overline{\vartheta}}{\varrho} P'\Big(\varepsilon^\alpha \frac{\varrho}{\overline{\vartheta}^{\frac{3}{2}}}\Big) \geq \frac{c}{\varrho}. \tag{6.67}$$

Consequently, boundedness of the third integral in (6.56) gives rise to

$$\operatorname{ess}\sup_{t\in(0,T)} \left\| \left[\frac{\varrho_\varepsilon - \tilde{\varrho}_\varepsilon}{\varepsilon}\right]_{\rm ess} \right\|_{L^2(\Omega)} \leq c;$$

whence, by virtue of Lemma 6.1,

$$\operatorname{ess}\sup_{t\in(0,T)} \left\| \left[\frac{\varrho_\varepsilon - \tilde{\varrho}}{\varepsilon}\right]_{\rm ess} \right\|_{L^2(\Omega)} \leq c. \tag{6.68}$$

Next, it follows from (6.53), (6.60), and (6.61) that

$$\inf_{(\varrho,\vartheta)\in\mathcal{M}_{\rm res}} \Big(H^\varepsilon_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \tilde{\varrho}_\varepsilon) \frac{\partial H^\varepsilon_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon, \overline{\vartheta})}{\partial \varrho} - H^\varepsilon_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon, \overline{\vartheta}) \Big) \tag{6.69}$$

$$= \inf_{(\varrho,\vartheta)\in\partial\mathcal{M}_{\rm ess}} \Big(H^\varepsilon_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \tilde{\varrho}_\varepsilon) \frac{\partial H^\varepsilon_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon, \overline{\vartheta})}{\partial \varrho} - H^\varepsilon_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon, \overline{\vartheta}) \Big) \geq c > 0,$$

where, by virtue of Lemma 6.1, the constant c is independent of ε, $\tilde{\varrho}_\varepsilon$. Thus we infer, exactly as in Chapter 5, that

$$\operatorname{ess}\sup_{t\in(0,T)} |\mathcal{M}^\varepsilon_{\rm res}[t]| \leq \varepsilon^2 c, \tag{6.70}$$

where, similarly to (4.43), we have set

$$\mathcal{M}^\varepsilon_{\rm res}[t] = \mathcal{M}^\varepsilon_{\rm res}|_{\{t\}\times\Omega} \subset \Omega.$$

In other words, the measure of the residual set is small and vanishes with $\varepsilon \to 0$. In addition, by virtue of (6.67), (6.70),

$$\operatorname{ess}\sup_{t\in(0,T)} \int_\Omega |\ [\varrho_\varepsilon \log(\varrho_\varepsilon)]_{\rm res}\ |\ {\rm d}x \leq \varepsilon^2 c. \tag{6.71}$$

As a direct consequence of estimates (6.70), (6.71), we deduce that the residual component of any *affine* function of ϱ_ε divided on ε^2 is bounded in the space $L^\infty(0,T;L^1(\Omega))$. On the other hand, by virtue of Proposition 3.2,

$$H^\varepsilon_{\overline{\vartheta}}(\varrho,\vartheta) \geq \frac{1}{4}\Big(\varrho e_\varepsilon(\varrho,\vartheta) + \overline{\vartheta}\varrho|s_\varepsilon(\varrho,\vartheta)|\Big) - \Big|(\varrho-\overline{\varrho})\frac{\partial H^\varepsilon_{2\overline{\vartheta}}}{\partial\varrho}(\overline{\varrho},2\overline{\vartheta}) + H^\varepsilon_{2\overline{\vartheta}}(\overline{\varrho},2\overline{\vartheta})\Big|$$

for any ϱ, ϑ, therefore we can use again relation (6.56) in order to conclude that

$$\operatorname*{ess\ sup}_{t\in(0,T)} \int_\Omega [\varrho_\varepsilon e_\varepsilon(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm res}\ {\rm d}x \leq \varepsilon^2 c, \tag{6.72}$$

and

$$\operatorname*{ess\ sup}_{t\in(0,T)} \int_\Omega |\ [\varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm res}\ |\ {\rm d}x \leq \varepsilon^2 c. \tag{6.73}$$

Note that, as a consequence of (6.27) and hypothesis (6.15), both $\frac{\partial H^\varepsilon_{2\overline{\vartheta}}}{\partial\varrho}(\overline{\varrho},2\overline{\vartheta})$ and $H^\varepsilon_{2\overline{\vartheta}}(\overline{\varrho},2\overline{\vartheta})$ are uniformly bounded for $\varepsilon\to 0$.

In accordance with hypothesis (6.26) and (6.66),

$$\operatorname*{ess\ sup}_{t\in(0,T)} \int_\Omega [\varrho_\varepsilon\vartheta_\varepsilon]_{\rm res}\ {\rm d}x \leq \varepsilon^2 c, \tag{6.74}$$

$$\operatorname*{ess\ sup}_{t\in(0,T)} \int_\Omega [\vartheta_\varepsilon]^4_{\rm res}\ {\rm d}x \leq \varepsilon c, \tag{6.75}$$

and

$$\operatorname*{ess\ sup}_{t\in(0,T)} \int_\Omega [\varrho_\varepsilon]^{\frac{5}{3}}_{\rm res}\ {\rm d}x \leq \varepsilon^{2-2\alpha/3} c. \tag{6.76}$$

Note that $2-2\alpha/3>0$ as $\alpha\in(2,3)$.

To conclude, we exploit the piece of information provided by the uniform bound (6.63). In accordance with (6.28–6.30), we deduce immediately that

$$\int_0^T\int_\Omega |\nabla_x\mathbf{u}_\varepsilon + \nabla_x^T\mathbf{u}_\varepsilon - \frac{2}{3}{\rm div}_x\mathbf{u}_\varepsilon\mathbb{I}|^2\ {\rm d}x\ {\rm d}t \leq c, \tag{6.77}$$

$$\int_0^T\int_\Omega \vartheta_\varepsilon\Big|\nabla_x\Big(\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon^2}\Big)\Big|^2\ {\rm d}x\ {\rm d}t \leq c, \tag{6.78}$$

$$\int_0^T\int_\Omega \frac{1}{\vartheta_\varepsilon}\Big|\nabla_x\Big(\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\Big)\Big|^2\ {\rm d}x\ {\rm d}t \leq c, \tag{6.79}$$

and

$$\int_0^T\int_\Omega \Big|\nabla_x\Big(\log(\vartheta_\varepsilon)-\log(\overline{\vartheta})\Big)\Big|^2\ {\rm d}x\ {\rm d}t \leq \varepsilon^{2-2\alpha/3}c. \tag{6.80}$$

Note that (6.80) implies $\nabla_x\log(\vartheta_\varepsilon)\approx 0$ in the asymptotic limit as $\alpha<3$.

Combining estimates (6.62), (6.70), (6.77) we get, by help of a variant of Korn's inequality established in Proposition 2.1,

$$\{\mathbf{u}_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2(0,T;W^{1,2}(\Omega)). \tag{6.81}$$

Similarly, by means of Proposition 2.2, relations (6.70), (6.74), together with (6.78), (6.79), (6.80) yield

$$\Big\{\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\Big\}_{\varepsilon>0} \quad \text{bounded in} \quad L^2(0,T;W^{1,2}(\Omega)), \tag{6.82}$$

$$\Big\{\frac{\sqrt{\vartheta_\varepsilon} - \sqrt{\overline{\vartheta}}}{\varepsilon}\Big\}_{\varepsilon>0} \quad \text{bounded in} \quad L^2(0,T;W^{1,2}(\Omega)), \tag{6.83}$$

and

$$\|\log(\vartheta_\varepsilon) - \log(\overline{\vartheta})\|_{L^2(0,T;W^{1,2}(\Omega))} \leq \varepsilon^{1-\alpha/3}c. \tag{6.84}$$

6.4.2 Pressure estimates

The upper bound (6.76) on the residual component of the density is considerably weaker than its counterpart (5.48) established in Chapter 5. This is an inevitable consequence of the scaling that preserves only the linear part of the pressure yielding merely the "logarithmic" estimate (6.71). Deeper considerations, based on the pressure estimates discussed in Section 2.2.5, are necessary in order to provide better bounds required later in the limit passage.

Following the leading idea of Section 2.2.5, we define the quantities

$$\varphi(t,x) = \psi(t)\mathcal{B}\Big[b(\varrho_\varepsilon) - \frac{1}{|\Omega|}\int_\Omega b(\varrho_\varepsilon)\ \mathrm{d}x\Big],\ \psi \in C_c^\infty(0,T)$$

to be used as test functions in the variational formulation of the momentum equation (6.38). Here the symbol $\mathcal{B}$ stands for the *Bogovskii operator* on the domain Ω introduced in Section 10.5 in Appendix.

After a bit tedious but rather straightforward manipulation, which is completely analogous to that one performed and rigorously justified in Section 2.2.5, we arrive at the following relation:

$$\begin{aligned} &\frac{1}{\varepsilon^2}\int_0^T\int_\Omega \psi p_\varepsilon(\varrho_\varepsilon,\vartheta_\varepsilon)b(\varrho_\varepsilon)\ \mathrm{d}x\ \mathrm{d}t \qquad (6.85)\\ &= \frac{1}{\varepsilon^2|\Omega|}\int_0^T\int_\Omega \psi p_\varepsilon(\varrho_\varepsilon,\vartheta_\varepsilon)\ \mathrm{d}x\Big(\int_\Omega b(\varrho_\varepsilon)\ \mathrm{d}x\Big)\mathrm{d}t\\ &\quad + \frac{1}{\varepsilon^2}\int_0^T\int_\Omega g\psi\varrho_\varepsilon\mathbf{j}\cdot\mathcal{B}\Big[b(\varrho_\varepsilon) - \frac{1}{|\Omega|}\int_\Omega b(\varrho_\varepsilon)\ \mathrm{d}x\Big]\ \mathrm{d}x\ \mathrm{d}t + I_\varepsilon, \end{aligned}$$

where we have set

$$
\begin{aligned}
I_\varepsilon = &\int_0^T \int_\Omega \psi \mathbb{S}_\varepsilon : \nabla_x \mathcal{B}\Big[b(\varrho_\varepsilon) - \frac{1}{|\Omega|}\int_\Omega b(\varrho_\varepsilon)\ \mathrm{d}x \Big]\ \mathrm{d}x\ \mathrm{d}t \\
&- \int_0^T \int_\Omega \psi \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \mathcal{B}\Big[b(\varrho_\varepsilon) - \frac{1}{|\Omega|}\int_\Omega b(\varrho_\varepsilon)\ \mathrm{d}x \Big]\Big)\ \mathrm{d}x\ \mathrm{d}t \\
&- \int_0^T \int_\Omega \partial_t \psi \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{B}\Big[b(\varrho_\varepsilon) - \frac{1}{|\Omega|}\int_\Omega b(\varrho_\varepsilon)\ \mathrm{d}x \Big]\ \mathrm{d}x\ \mathrm{d}t \\
&+ \int_0^T \int_\Omega \psi \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{B}[\mathrm{div}_x (b(\varrho_\varepsilon)\mathbf{u}_\varepsilon)]\ \mathrm{d}x\ \mathrm{d}t \\
&+ \int_0^T \psi \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{B}\Big[(\varrho_\varepsilon b'(\varrho_\varepsilon) - b(\varrho_\varepsilon))\mathrm{div}_x \mathbf{u}_\varepsilon \\
&- \frac{1}{|\Omega|}\int_\Omega (b(\varrho_\varepsilon) - b'(\varrho_\varepsilon)\varrho_\varepsilon)\mathrm{div}_x \mathbf{u}_\varepsilon\ \mathrm{d}x \Big]\ \mathrm{d}x\ \mathrm{d}t.
\end{aligned}
$$

Taking the uniform estimates established in Section 6.4.1 into account we can show, exactly as in Section 2.2.5, that all integrals contained in I_ε are bounded uniformly for $\varepsilon \to 0$ as soon as

$$
|b(\varrho)| + |\varrho b'(\varrho)| \le c\varrho^\gamma \text{ for } 0 < \gamma < 1 \text{ small enough.} \tag{6.86}
$$

In order to comply with (6.86), let us take $b \in C^\infty[0,\infty)$ such that

$$
b(\varrho) = \begin{cases} 0 \text{ for } 0 \le \varrho \le 2\overline{\varrho}, \\ \in [0, \varrho^\gamma] \text{ for } 2\overline{\varrho} < \varrho \le 3\overline{\varrho}, \\ \varrho^\gamma \text{ if } \varrho > 3\overline{\varrho}, \end{cases} \tag{6.87}
$$

with $\gamma > 0$ sufficiently small to be specified below. In particular, we have

$$
b(\varrho_\varepsilon) = b([\varrho_\varepsilon]_{\mathrm{res}});
$$

whence, in accordance with (6.71),

$$
\operatorname*{ess\ sup}_{t \in (0,T)} \int_\Omega b(\varrho_\varepsilon)\ \mathrm{d}x \le c\varepsilon^2 \tag{6.88}
$$

as soon as $0 < \gamma < 1$. Consequently, the first integral at the right-hand side of (6.85) is bounded.

In order to control the second term, we use the fact that $\tilde{\varrho}$, $\overline{\vartheta}$ solve the static problem (6.41).

Accordingly, we get

$$
\begin{aligned}
&\frac{1}{\varepsilon^2}\int_\Omega \varrho_\varepsilon g\mathbf{j}\cdot\mathcal{B}\Big[b(\varrho_\varepsilon)-\frac{1}{|\Omega|}\int_\Omega b(\varrho_\varepsilon)\ \mathrm{d}x\Big]\ \mathrm{d}x \qquad (6.89)\\
&=\frac{1}{\varepsilon}\int_\Omega \Big[\frac{\varrho_\varepsilon-\tilde\varrho}{\varepsilon}\Big]_{\rm ess} g\mathbf{j}\cdot\mathcal{B}\Big[b(\varrho_\varepsilon)-\frac{1}{|\Omega|}\int_\Omega b(\varrho_\varepsilon)\ \mathrm{d}x\Big]\ \mathrm{d}x\\
&\quad+\int_\Omega \Big[\frac{\varrho_\varepsilon-\tilde\varrho}{\varepsilon^2}\Big]_{\rm res} g\mathbf{j}\cdot\mathcal{B}\Big[b(\varrho_\varepsilon)-\frac{1}{|\Omega|}\int_\Omega b(\varrho_\varepsilon)\ \mathrm{d}x\Big]\ \mathrm{d}x\\
&\quad-\frac{p_0}{\varepsilon^2}\int_\Omega \tilde\varrho\,\overline{\vartheta}\Big(b(\varrho_\varepsilon)-\frac{1}{|\Omega|}\int_\Omega b(\varrho_\varepsilon)\ \mathrm{d}x\Big)\ \mathrm{d}x,
\end{aligned}
$$

where the last integral is uniformly bounded because of (6.88).

On the other hand, by virtue of the L^p-estimates for $\mathcal{B}$ (see Theorem 10.11 in Appendix),

$$
\begin{aligned}
&\frac{1}{\varepsilon}\Big|\int_\Omega \Big[\frac{\varrho_\varepsilon-\tilde\varrho}{\varepsilon}\Big]_{\rm ess}\mathbf{j}\cdot\mathcal{B}\Big[b(\varrho_\varepsilon)-\frac{1}{|\Omega|}\int_\Omega b(\varrho_\varepsilon)\ \mathrm{d}x\Big]\ \mathrm{d}x\Big| \qquad (6.90)\\
&\quad\le\frac{c}{\varepsilon}\mathrm{ess}\sup_{t\in(0,T)}\Big\|\Big[\frac{\varrho_\varepsilon-\tilde\varrho}{\varepsilon}\Big]_{\rm ess}\Big\|_{L^2(\Omega)}\mathrm{ess}\sup_{t\in(0,T)}\Big\|\ b(\varrho_\varepsilon)-\frac{1}{|\Omega|}\int_\Omega b(\varrho_\varepsilon)\ \mathrm{d}x\Big\|_{L^{\frac{6}{5}}(\Omega)},
\end{aligned}
$$

and, by the same token,

$$
\begin{aligned}
&\Big|\int_\Omega \Big[\frac{\varrho_\varepsilon-\tilde\varrho}{\varepsilon^2}\Big]_{\rm res}\mathbf{j}\cdot\mathcal{B}\Big[b(\varrho_\varepsilon)-\frac{1}{|\Omega|}\int_\Omega b(\varrho_\varepsilon)\ \mathrm{d}x\Big]\ \mathrm{d}x\Big| \qquad (6.91)\\
&\quad\le\mathrm{ess}\sup_{t\in(0,T)}\Big\|\Big[\frac{\varrho_\varepsilon-\tilde\varrho}{\varepsilon^2}\Big]_{\rm res}\Big\|_{L^1(\Omega)}\mathrm{ess}\sup_{t\in(0,T)}\Big\|\ b(\varrho_\varepsilon)-\frac{1}{|\Omega|}\int_\Omega b(\varrho_\varepsilon)\ \mathrm{d}x\Big\|_{L^4(\Omega)}.
\end{aligned}
$$

Finally, in accordance with (6.71), (6.88),

$$
\int_\Omega |b(\varrho_\varepsilon)|^q\ \mathrm{d}x\le\int_\Omega[\varrho_\varepsilon]_{\rm res}^{\gamma q}\ \mathrm{d}x\le\int_\Omega[\varrho_\varepsilon\log(\varrho_\varepsilon)]_{\rm res}\ \mathrm{d}x\le c\varepsilon^2 \qquad (6.92)
$$

as soon as $\gamma\le 1/q$.

Estimates (6.89–6.92) yield a uniform bound on the second term at the right-hand side of (6.85). The remaining integrals grouped in I_ε are bounded by virtue of the estimates established in the previous part exactly as in Section 2.2.5. Consequently, we conclude that

$$
\int_0^T\int_\Omega p_\varepsilon(\varrho_\varepsilon,\vartheta_\varepsilon)b(\varrho_\varepsilon)\ \mathrm{d}x\ \mathrm{d}t\le\varepsilon^2 c, \qquad (6.93)
$$

provided b is given by (6.87), with $0<\gamma<1/4$.

6.5 Convergence towards the target system

The uniform estimates deduced in the preceding section enable us to pass to the limit in the family $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$.

Specifically, by virtue of (6.68), (6.70), (6.76), we have

$$\varrho_\varepsilon \to \tilde{\varrho} \text{ in } L^\infty(0,T; L^{\frac{5}{3}}(\Omega)) \cap C([0,T]; L^q(\Omega)) \text{ for any } 1 \leq q < \frac{5}{3}. \tag{6.94}$$

Moreover, in accordance with (6.81), we may assume

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T; W^{1,2}(\Omega; \mathbb{R}^3)), \tag{6.95}$$

passing to a subsequence as the case may be, where

$$\mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0 \text{ in the sense of traces.} \tag{6.96}$$

Finally, it follows from (6.82) that

$$\vartheta_\varepsilon \to \overline{\vartheta} \text{ in } L^2(0,T; W^{1,2}(\Omega)). \tag{6.97}$$

Our goal in the remaining part of this section is to identify the limit system of equations governing the time evolution of the velocity $\mathbf{U}$.

6.5.1 Anelastic constraint

Combining (6.94), (6.95) we let $\varepsilon \to 0$ in the equation of continuity expressed through the integral identity (6.37) in order to obtain

$$\mathrm{div}_x(\tilde{\varrho}\mathbf{U}) = 0 \text{ a.a. in } (0,T) \times \Omega. \tag{6.98}$$

This is the so-called *anelastic approximation* discussed in Section 4.3 characterizing the strong stratification of the fluid in the vertical direction.

6.5.2 Determining the pressure

As already pointed out in Section 4.3, a successful analysis of the anelastic limit in the *isothermal* regime is conditioned by the fact that the thermal equation of state relating the pressure to the density and the temperature is that of a perfect gas, namely $p = p_0\varrho\vartheta$.

Let us examine the quantity

$$\frac{1}{\varepsilon^2}\left(p_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon) - p_0\varrho_\varepsilon\vartheta_\varepsilon - \varepsilon\frac{a}{3}\overline{\vartheta}^4\right) \tag{6.99}$$

$$= \frac{1}{\varepsilon^2}\left(\frac{\vartheta_\varepsilon^{\frac{5}{2}}}{\varepsilon^\alpha}P\left(\varepsilon^\alpha\frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}}\right) - \frac{\vartheta_\varepsilon^{\frac{5}{2}}}{\varepsilon^\alpha}\varepsilon^\alpha P'(0)\frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}}\right) + \frac{a}{3}\frac{\vartheta_\varepsilon^4 - \overline{\vartheta}^4}{\varepsilon}$$

$$= \frac{1}{\varepsilon^2}\Big[\frac{\vartheta_\varepsilon^{\frac{5}{2}}}{\varepsilon^\alpha}P\Big(\varepsilon^\alpha\frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}}\Big) - \frac{\vartheta_\varepsilon^{\frac{5}{2}}}{\varepsilon^\alpha}\varepsilon^\alpha P'(0)\frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}}\Big]_{\rm ess}$$
$$+ \frac{1}{\varepsilon^2}\Big[\frac{\vartheta_\varepsilon^{\frac{5}{2}}}{\varepsilon^\alpha}P\Big(\varepsilon^\alpha\frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}}\Big) - \frac{\vartheta_\varepsilon^{\frac{5}{2}}}{\varepsilon^\alpha}\varepsilon^\alpha P'(0)\frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}}\Big]_{\rm res} + \frac{a}{3}\frac{\vartheta_\varepsilon^4 - \overline{\vartheta}^4}{\varepsilon}.$$

To begin, since P is twice continuously differentiable, we deduce

$$\frac{1}{\varepsilon^2}\Big|\Big[\frac{\vartheta_\varepsilon^{\frac{5}{2}}}{\varepsilon^\alpha}P\Big(\varepsilon^\alpha\frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}}\Big) - \frac{\vartheta_\varepsilon^{\frac{5}{2}}}{\varepsilon^\alpha}\varepsilon^\alpha P'(0)\frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}}\Big]_{\rm ess}\Big| \le c\varepsilon^{\alpha-2}\Big[\frac{\varrho_\varepsilon^2}{\vartheta_\varepsilon^{\frac{1}{2}}}\Big]_{\rm ess}, \tag{6.100}$$

where the expression on the right-hand side tends to zero for $\varepsilon \to 0$ uniformly on $(0,T)\times\Omega$ as soon as $\alpha > 2$.

Next, by virtue of hypothesis (6.15),

$$\frac{1}{\varepsilon^2}\Big|\Big[\frac{\vartheta_\varepsilon^{\frac{5}{2}}}{\varepsilon^\alpha}P\Big(\varepsilon^\alpha\frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}}\Big) - \frac{\vartheta_\varepsilon^{\frac{5}{2}}}{\varepsilon^\alpha}\varepsilon^\alpha P'(0)\frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}}\Big]_{\rm res}\Big| \le c\frac{\varepsilon^{2\alpha/3}}{\varepsilon^2}[\varrho_\varepsilon]_{\rm res}^{5/3}. \tag{6.101}$$

On the other hand, it follows from the refined pressure estimates (6.93) that

$$\frac{1}{\varepsilon^2}\int_0^T\int_\Omega [\varrho_\varepsilon]_{\rm res}^{5/3+\gamma}\ {\rm d}x\ {\rm d}t \le c \text{ for a certain } \gamma > 0. \tag{6.102}$$

Thus writing

$$\frac{\varepsilon^{2\alpha/3}}{\varepsilon^2}\int_0^T\int_\Omega [\varrho_\varepsilon]_{\rm res}^{5/3}\ {\rm d}x$$
$$= \varepsilon^{2\alpha/3}\frac{1}{\varepsilon^2}\int\int_{\{0\le\varrho_\varepsilon\le K\}}[\varrho_\varepsilon]_{\rm res}^{5/3}\ {\rm d}x\ {\rm d}t + \frac{\varepsilon^{2\alpha/3}}{\varepsilon^2}\int\int_{\{\varrho_\varepsilon > K\}}[\varrho_\varepsilon]_{\rm res}^{5/3}\ {\rm d}x\ {\rm d}t$$

we have, by means of (6.70),

$$\varepsilon^{2\alpha/3}\frac{1}{\varepsilon^2}\int\int_{\{0\le\varrho_\varepsilon\le K\}}[\varrho_\varepsilon]_{\rm res}^{5/3}\ {\rm d}x\ {\rm d}t \le c\varepsilon^{2\alpha/3}K^{5/3},$$

while, in accordance with (6.102),

$$\frac{\varepsilon^{2\alpha/3}}{\varepsilon^2}\int\int_{\{\varrho_\varepsilon > K\}}[\varrho_\varepsilon]_{\rm res}^{5/3}\ {\rm d}x\ {\rm d}t \le cK^{-\gamma}.$$

Consequently, we conclude that

$$\frac{\varepsilon^{2\alpha/3}}{\varepsilon^2}[\varrho_\varepsilon]_{\rm res}^{5/3} \to 0 \text{ in } L^1((0,T)\times\Omega) \text{ for } \varepsilon \to 0. \tag{6.103}$$

Finally, the radiation pressure can be decomposed as

$$\vartheta_\varepsilon^4 - \overline{\vartheta}^4 = [\vartheta_\varepsilon^4 - \overline{\vartheta}^4]_{\rm res} + [\vartheta_\varepsilon^4 - \overline{\vartheta}^4]_{\rm ess},$$

where, by virtue of the uniform estimates (6.70), (6.75), and (6.82),

$$\int_0^T \int_\Omega |[\vartheta_\varepsilon^4 - \overline{\vartheta}^4]_{\rm res}| \ {\rm d}x \ {\rm d}t \le c \int_0^T \int_\Omega |\vartheta_\varepsilon - \overline{\vartheta}|([\vartheta_\varepsilon]^3_{\rm res} + [\overline{\vartheta}]^3_{\rm res})| \ {\rm d}x \ {\rm d}t \tag{6.104}$$
$$\le c\|\vartheta_\varepsilon - \overline{\vartheta}\|_{L^2(0,T;L^4(\Omega))} \text{ess} \sup_{t\in(0,T)} \Big(\|[\vartheta_\varepsilon]^3_{\rm res}\|_{L^{\frac{4}{3}}(\Omega)} + \|[\overline{\vartheta}]^3_{\rm res}\|_{L^{\frac{4}{3}}(\Omega)} \Big) \le c\varepsilon^{\frac{7}{4}}.$$

In order to control the essential component of the radiation pressure, we first recall a variant of *Poincaré's inequality*

$$\|\vartheta_\varepsilon^{\frac{3}{2}} - \overline{\vartheta}^{\frac{3}{2}}\|^2_{L^2((0,T)\times\Omega)}$$
$$\le c\Big[\int_0^T \int_\Omega \vartheta_\varepsilon |\nabla_x \vartheta_\varepsilon|^2 \ {\rm d}x + \Big(\int_0^T \int_{\{x_3=1\}} |\vartheta_\varepsilon^{\frac{3}{2}} - \overline{\vartheta}^{\frac{3}{2}}| \ {\rm d}S_x \ {\rm d}t \Big)^2 \Big],$$

where

$$\Big(\int_0^T \int_{\{x_3=1\}} |\vartheta_\varepsilon^{\frac{3}{2}} - \overline{\vartheta}^{\frac{3}{2}}| \ {\rm d}S_x \ {\rm d}t \Big)^2$$
$$\le c \int_0^T \int_{\{x_3=1\}} |\vartheta_\varepsilon - \overline{\vartheta}|^2 \ {\rm d}S_x \ {\rm d}t \int_0^T \int_{\{x_3=1\}} (\vartheta_\varepsilon + \overline{\vartheta}) {\rm d}S_x \ {\rm d}t.$$

Here and hereafter, we have used that

$$c_1 |[\vartheta_\varepsilon - \overline{\vartheta}]_{\rm ess}| \le |[\vartheta_\varepsilon^p - \overline{\vartheta}^p]_{\rm ess}| \le c_2 |[\vartheta_\varepsilon - \overline{\vartheta}]_{\rm ess}|, \ p > 0.$$

Thus the uniform estimates (6.64), (6.78) imply that

$$\|[\vartheta_\varepsilon^p - \overline{\vartheta}^p]_{\rm ess}\|_{L^2((0,T)\times\Omega)} \le c\|[\vartheta_\varepsilon^{\frac{3}{2}} - \overline{\vartheta}^{\frac{3}{2}}]_{\rm ess}\|_{L^2((0,T)\times\Omega)} \le c(p)\varepsilon^{\frac{3}{2}} \text{ for any } p > 0. \tag{6.105}$$

Moreover, from (6.104), (6.105) we infer that

$$\Big\| \frac{\vartheta_\varepsilon^4 - \overline{\vartheta}^4}{\varepsilon} \Big\|_{L^1((0,T)\times\Omega)} \le c\varepsilon^{\frac{1}{2}}. \tag{6.106}$$

Summing up the estimates (6.101), (6.103–6.106) we conclude that

$$\frac{1}{\varepsilon^2} \Big(p_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon) - p_0 \varrho_\varepsilon \vartheta_\varepsilon - \varepsilon \frac{a}{3} \overline{\vartheta}^4 \Big) \to 0 \text{ in } L^1((0,T)\times\Omega). \tag{6.107}$$

In other words $(1/\varepsilon^2)\nabla_x p_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon) \approx (p_0/\varepsilon^2)\nabla_x(\varrho_\varepsilon \vartheta_\varepsilon)$ in the asymptotic limit $\varepsilon \to 0$.

6.5.3 Driving force

Our next goal is to determine the asymptotic limit of the driving force acting on the fluid through the momentum equation (6.38). In accordance with (6.107), the thermal equation of state reduces to that of a perfect gas, therefore it is enough to examine the quantity

$$\frac{1}{\varepsilon^2}\int_0^T\int_\Omega \Big(p_0\varrho_\varepsilon\vartheta_\varepsilon \mathrm{div}_x\varphi - \varrho_\varepsilon g\varphi_3\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{6.108}$$
$$= \frac{p_0}{\varepsilon^2}\int_0^T\int_\Omega \frac{\overline{\vartheta}}{\tilde{\varrho}}\varrho_\varepsilon \mathrm{div}_x(\tilde{\varrho}\varphi)\ \mathrm{d}x\ \mathrm{d}t + p_0\int_0^T\int_\Omega \varrho_\varepsilon\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon^2}\mathrm{div}_x\varphi\ \mathrm{d}x\ \mathrm{d}t,$$

where we have exploited the fact that $\tilde{\varrho}$ solves the linearized static problem (6.41).

In order to handle the latter term on the right-hand side of (6.108), we first write

$$\int_0^T\int_\Omega \varrho_\varepsilon\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon^2}\mathrm{div}_x\varphi\ \mathrm{d}x\ \mathrm{d}t \tag{6.109}$$
$$= \frac{1}{\varepsilon^2}\int_0^T\int_\Omega (\varrho_\varepsilon-\tilde{\varrho})(\vartheta_\varepsilon-\overline{\vartheta})\mathrm{div}_x\varphi\ \mathrm{d}x\ \mathrm{d}t + \int_0^T\int_\Omega \tilde{\varrho}\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon^2}\mathrm{div}_x\varphi\ \mathrm{d}x\ \mathrm{d}t,$$

and, furthermore,

$$\frac{1}{\varepsilon^2}(\varrho_\varepsilon-\tilde{\varrho})(\vartheta_\varepsilon-\overline{\vartheta}) = \Big[\frac{\varrho_\varepsilon-\tilde{\varrho}}{\varepsilon}\Big]_{\mathrm{ess}}\Big[\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\Big]_{\mathrm{ess}} + \Big[\frac{\varrho_\varepsilon-\tilde{\varrho}}{\varepsilon}\Big]_{\mathrm{res}}\Big(\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\Big),$$

where, as a straightforward consequence of the uniform estimates (6.68), (6.105),

$$\Big\|\Big[\frac{\varrho_\varepsilon-\tilde{\varrho}}{\varepsilon}\Big]_{\mathrm{ess}}\Big[\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\Big]_{\mathrm{ess}}\Big\|_{L^1((0,T)\times\Omega)} \le \sqrt{\varepsilon}c \to 0.$$

In addition, using (6.82) in combination with the standard embedding relation $W^{1,2}(\Omega)\hookrightarrow L^6(\Omega)$, we obtain

$$\Big\|\Big[\frac{\varrho_\varepsilon-\tilde{\varrho}}{\varepsilon}\Big]_{\mathrm{res}}\Big(\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\Big)\Big\|_{L^1((0,T)\times\Omega)} \le c\Big\|\Big[\frac{\varrho_\varepsilon-\tilde{\varrho}}{\varepsilon}\Big]_{\mathrm{res}}\Big\|_{L^2(0,T;L^{\frac{6}{5}}(\Omega))}.$$

Moreover, by a simple interpolation argument,

$$\Big\|\Big[\frac{\varrho_\varepsilon-\tilde{\varrho}}{\varepsilon}\Big]_{\mathrm{res}}\Big\|_{L^{\frac{6}{5}}(\Omega)} \le \Big\|\Big[\frac{\varrho_\varepsilon-\tilde{\varrho}}{\varepsilon}\Big]_{\mathrm{res}}\Big\|_{L^1(\Omega)}^{\frac{7}{12}}\Big\|\Big[\frac{\varrho_\varepsilon-\tilde{\varrho}}{\varepsilon}\Big]_{\mathrm{res}}\Big\|_{L^{\frac{5}{3}}(\Omega)}^{\frac{5}{12}},$$

where, in accordance with the bounds (6.70), (6.71), (6.76),

$$\operatorname*{ess\ sup}_{t\in(0,T)}\Big\|\Big[\frac{\varrho_\varepsilon-\tilde{\varrho}}{\varepsilon}\Big]_{\mathrm{res}}\Big\|_{L^1(\Omega)}^{\frac{7}{12}}\Big\|\Big[\frac{\varrho_\varepsilon-\tilde{\varrho}}{\varepsilon}\Big]_{\mathrm{res}}\Big\|_{L^{\frac{5}{3}}(\Omega)}^{\frac{5}{12}} \le \varepsilon^{\frac{1}{6}}c \to 0.$$

Resuming the previous considerations we may infer that

$$\frac{1}{\varepsilon^2}(\varrho_\varepsilon - \tilde{\varrho})(\vartheta_\varepsilon - \overline{\vartheta}) \to 0 \text{ in } L^1((0,T)\times\Omega), \tag{6.110}$$

therefore it is enough to find a suitable uniform bound on the family $\{(\vartheta_\varepsilon - \overline{\vartheta})/\varepsilon^2\}_{\varepsilon>0}$. To this end, write

$$\sqrt{\overline{\vartheta}}\nabla_x \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon^2} = \frac{\sqrt{\overline{\vartheta}} - \sqrt{\vartheta_\varepsilon}}{\varepsilon}\nabla_x \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} + \sqrt{\vartheta_\varepsilon}\nabla_x \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon^2},$$

where, by virtue of (6.65), (6.75), (6.83), and the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$,

$$\left\{\frac{\sqrt{\overline{\vartheta}} - \sqrt{\vartheta_\varepsilon}}{\varepsilon}\right\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0,T;L^1(\Omega)) \cap L^2(0,T;L^6(\Omega)).$$

Consequently, by means of (6.78), (6.82), and a simple interpolation argument, we get

$$\left\{\nabla_x\left(\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon^2}\right)\right\}_{\varepsilon>0} \text{ bounded in } L^q(0,T;L^q(\Omega;\mathbb{R}^3)) \text{ for a certain } q>1. \tag{6.111}$$

Thus, finally,

$$\int_0^T \int_\Omega \tilde{\varrho}\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon^2}\mathrm{div}_x\varphi \ \mathrm{d}x\ \mathrm{d}t$$
$$= \int_0^T \int_\Omega \tilde{\varrho}\vartheta_\varepsilon^{(2)}\mathrm{div}_x\varphi \ \mathrm{d}x\ \mathrm{d}t + \frac{1}{|\Omega|}\int_0^T \left(\int_\Omega \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon^2}\ \mathrm{d}x\right)\int_\Omega \tilde{\varrho}\mathrm{div}_x\varphi \ \mathrm{d}x\ \mathrm{d}t,$$

where we have set

$$\vartheta_\varepsilon^{(2)} = \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon^2} - \frac{1}{|\Omega|}\int_\Omega \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon^2}\ \mathrm{d}x. \tag{6.112}$$

In accordance with (6.111),

$$\vartheta_\varepsilon^{(2)} \to \vartheta^{(2)} \text{ weakly in } L^q(0,T;W^{1,q}(\Omega)) \text{ for a certain } q>1. \tag{6.113}$$

Furthermore, after a simple manipulation, we observe that

$$\int_\Omega \tilde{\varrho}\mathrm{div}_x\varphi\ \mathrm{d}x = \int_\Omega \Big(1+\log(\tilde{\varrho})\Big)\mathrm{div}_x(\tilde{\varrho}\varphi)\ \mathrm{d}x. \tag{6.114}$$

Putting together relations (6.108–6.114) we conclude that

$$\frac{1}{\varepsilon^2}\int_0^T\int_\Omega \Big(p_0\varrho_\varepsilon\vartheta_\varepsilon\mathrm{div}_x\varphi - \varrho_\varepsilon g\varphi_3\Big)\ \mathrm{d}x\ \mathrm{d}t = \frac{p_0}{\varepsilon^2}\int_0^T\int_\Omega \frac{\overline{\vartheta}}{\tilde{\varrho}}\varrho_\varepsilon\mathrm{div}_x(\tilde{\varrho}\varphi)\ \mathrm{d}x\ \mathrm{d}t$$
$$+ \frac{p_0}{|\Omega|}\int_0^T\left(\int_\Omega \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon^2}\ \mathrm{d}x\right)\int_\Omega \Big(1+\log(\tilde{\varrho})\Big)\mathrm{div}_x(\tilde{\varrho}\varphi)\ \mathrm{d}x\ \mathrm{d}t$$
$$+ p_0\int_0^T\int_\Omega \tilde{\varrho}\vartheta_\varepsilon^{(2)}\mathrm{div}_x\varphi\ \mathrm{d}x\ \mathrm{d}t + \int_0^T\int_\Omega \chi_\varepsilon\mathrm{div}_x\varphi\ \mathrm{d}x\ \mathrm{d}t, \tag{6.115}$$

where

$$\chi_\varepsilon \to 0 \text{ in } L^1((0,T)\times\Omega).$$

Note that the terms containing $\mathrm{div}_x(\tilde{\varrho}\varphi)$ are irrelevant in the limit $\varepsilon \to 0$ as the admissible test functions in (6.46) obey the anelastic constraint $\mathrm{div}_x(\tilde{\varrho}\varphi) = 0$.

6.5.4 Momentum equation

At this stage, we can use the limits obtained in Section 6.5 in combination with (6.107), (6.115), in order to let $\varepsilon \to 0$ in the momentum equation (6.38). We thereby obtain

$$\int_0^T \int_\Omega \Big(\tilde{\varrho}\mathbf{U}\cdot\partial_t\varphi + \overline{\varrho\mathbf{U}\otimes\mathbf{U}} : \nabla_x\varphi \Big)\ \mathrm{d}x\ \mathrm{d}t \tag{6.116}$$
$$= \int_0^T \int_\Omega \Big(\mathbb{S} : \nabla_x\varphi - p_0\tilde{\varrho}\vartheta^{(2)}\mathrm{div}_x\varphi \Big)\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega \tilde{\varrho}\mathbf{u}_0\varphi(0,\cdot)\ \mathrm{d}x,$$

for any test function

$$\varphi \in C_c^\infty([0,T)\times\overline{\Omega};\mathbb{R}^3),\ \ \varphi\cdot\mathbf{n}|_{\partial\Omega} = 0,\ \ \mathrm{div}_x(\tilde{\varrho}\varphi) = 0,$$

where

$$\mathbb{S} = \mu_1\overline{\vartheta}\Big(\nabla_x\mathbf{U} + \nabla_x^T\mathbf{U} - \frac{2}{3}\mathrm{div}_x\mathbf{U}\mathbb{I}\Big), \tag{6.117}$$

and the symbol $\overline{\varrho\mathbf{U}\otimes\mathbf{U}}$ denotes a weak limit of $\{\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon\}_{\varepsilon>0}$. Moreover, since $\tilde{\varrho}$ satisfies (6.41), we have

$$p_0\int_0^T \int_\Omega \tilde{\varrho}\vartheta^{(2)}\mathrm{div}_x\varphi\ \mathrm{d}x\ \mathrm{d}t = \int_0^T \int_\Omega \frac{\vartheta^{(2)}}{\overline{\vartheta}}\tilde{\varrho}g\varphi_3\ \mathrm{d}x\ \mathrm{d}t$$

in agreement with (6.46).

Consequently, in order to complete the proof of Theorem 6.1, we must verify:

(i) identity

$$\int_0^T \int_\Omega \overline{\varrho\mathbf{U}\otimes\mathbf{U}} : \nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t = \int_0^T \int_\Omega \tilde{\varrho}\mathbf{U}\otimes\mathbf{U} : \nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t \tag{6.118}$$

for any admissible test function in (6.116);

(ii) equation (6.45) relating the temperature $\vartheta^{(2)}$ to the vertical component of the velocity U_3. These are the main topics to be discussed in the next two sections.

6.6 Analysis of acoustic waves

As already pointed out in Section 4.4.2, the acoustic equation describing the time evolution of the gradient part of the velocity in strongly stratified fluids exhibits a wave speed varying with direction, in particular, with the vertical (depth) coordinate. A typical example of a highly anisotropic wave system due to the presence of internal gravity waves arises in the singular limit problem discussed in this chapter.

6.6.1 Acoustic equation

Formally, the equation of continuity (6.21) can be written in the form

$$\varepsilon\partial_t\left(\frac{\varrho_\varepsilon-\tilde{\varrho}}{\varepsilon\tilde{\varrho}}\right)+\frac{1}{\tilde{\varrho}}\mathrm{div}_x(\varrho_\varepsilon\mathbf{u}_\varepsilon)=0. \tag{6.119}$$

Similarly, by means of the identity,

$$p_0\overline{\vartheta}\nabla_x\varrho_\varepsilon+\varrho_\varepsilon g\mathbf{j}=p_0\overline{\vartheta}\tilde{\varrho}\nabla_x\left(\frac{\varrho_\varepsilon-\tilde{\varrho}}{\tilde{\varrho}}\right),$$

momentum equation (6.22) reads

$$\begin{aligned}&\varepsilon\partial_t(\varrho_\varepsilon\vartheta_\varepsilon)+p_0\overline{\vartheta}\tilde{\varrho}\nabla_x\left(\frac{\varrho_\varepsilon-\tilde{\varrho}}{\varepsilon\tilde{\varrho}}\right) \qquad (6.120)\\ &\quad=\frac{1}{\varepsilon}\nabla_x\left(p_0\varrho_\varepsilon\vartheta_\varepsilon-p(\varrho_\varepsilon,\vartheta_\varepsilon)-p_0\varrho_\varepsilon(\vartheta_\varepsilon-\overline{\vartheta})\right)+\varepsilon\mathrm{div}_x\left(\mathbb{S}_\varepsilon-\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon\right).\end{aligned}$$

System (6.119), (6.120) may be regarded as a classical formulation of the acoustic equation discussed in Section 4.4.2.

In terms of the weak solutions, the previous formal arguments can be justified in the following manner. Taking $\varphi/\tilde{\varrho}$ as a test function in (6.37) we obtain

$$\int_0^T\int_\Omega\left(\varepsilon\frac{\varrho_\varepsilon-\tilde{\varrho}}{\varepsilon\tilde{\varrho}}\partial_t\varphi+\tilde{\varrho}\frac{\varrho_\varepsilon\mathbf{u}_\varepsilon}{\tilde{\varrho}}\cdot\nabla_x\frac{\varphi}{\tilde{\varrho}}\right)\,\mathrm{d}x\,\mathrm{d}t=-\int_\Omega\varepsilon\frac{\varrho_{0,\varepsilon}-\tilde{\varrho}}{\varepsilon\tilde{\varrho}}\varphi(0,\cdot)\,\mathrm{d}x \tag{6.121}$$

to be satisfied for any $\varphi\in C_c^\infty([0,T)\times\overline{\Omega})$. In a similar fashion, the momentum equation (6.38) gives rise to

$$\begin{aligned}&\int_0^T\int_\Omega\left(\varepsilon\frac{\varrho_\varepsilon\mathbf{u}_\varepsilon}{\tilde{\varrho}}\cdot\partial_t\varphi+p_0\overline{\vartheta}\frac{\varrho_\varepsilon-\tilde{\varrho}}{\varepsilon\tilde{\varrho}}\mathrm{div}_x\varphi\right)\,\mathrm{d}x\,\mathrm{d}t \qquad (6.122)\\ &=-\int_\Omega\varepsilon\frac{\varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon}}{\tilde{\varrho}}\cdot\varphi(0,\cdot)\,\mathrm{d}x\\ &\quad+\int_0^T\int_\Omega\left(\varepsilon h_\varepsilon\mathrm{div}_x\frac{\varphi}{\tilde{\varrho}}+\varepsilon\mathbb{G}_\varepsilon:\nabla_x\frac{\varphi}{\tilde{\varrho}}+p_0\tilde{\varrho}\frac{\overline{\vartheta}-\vartheta_\varepsilon}{\varepsilon}\mathrm{div}_x\frac{\varphi}{\tilde{\varrho}}\right)\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega}; \mathbb{R}^3)$, $\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$, where

$$h_\varepsilon = \frac{1}{\varepsilon^2}\Big(p_0 \varrho_\varepsilon \vartheta_\varepsilon - p_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon)\Big) + p_0 \left(\frac{\tilde{\varrho} - \varrho_\varepsilon}{\varepsilon}\right)\left(\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\right),$$

and

$$\mathbb{G}_\varepsilon = \mathbb{S}_\varepsilon - \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon.$$

In accordance with the uniform bounds (6.107), (6.110),

$$h_\varepsilon \to 0 \text{ in } L^1((0,T) \times \Omega),$$

while, by virtue of (6.62), (6.75), (6.76), and (6.81),

$$\{\mathbb{G}_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^q(0,T; L^q(\Omega; \mathbb{R}^{3\times 3})) \text{ for a certain } q > 1.$$

In addition, relation (6.105) implies

$$\left\| \left[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\right]_{\rm ess} \right\|_{L^2((0,T)\times\Omega)} \leq \sqrt{\varepsilon} c,$$

and (6.70), together with (6.82), give rise to

$$\left\| \left[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\right]_{\rm res} \right\|_{L^1((0,T)\times\Omega)} \leq \varepsilon c.$$

Consequently, introducing the quantities

$$r_\varepsilon = \frac{\varrho_\varepsilon - \tilde{\varrho}}{\varepsilon \tilde{\varrho}}, \ \mathbf{V}_\varepsilon = \frac{\varrho_\varepsilon \mathbf{u}_\varepsilon}{\tilde{\varrho}},$$

we can rewrite system (6.121), (6.122) in a concise form as

■ Stratified Acoustic Equation:

$$\int_0^T \int_\Omega \left(\varepsilon r_\varepsilon \partial_t \varphi + \tilde{\varrho} \mathbf{V}_\varepsilon \cdot \nabla_x \left(\frac{\varphi}{\tilde{\varrho}}\right)\right) \, {\rm d}x \, {\rm d}t = - \int_\Omega \varepsilon r_\varepsilon(0,\cdot) \varphi(0,\cdot) \, {\rm d}x \tag{6.123}$$

for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$,

$$\int_0^T \int_\Omega \Big(\varepsilon \mathbf{V}_\varepsilon \cdot \partial_t \varphi + p_0 \overline{\vartheta} r_\varepsilon {\rm div}_x \varphi \Big) \, {\rm d}x \, {\rm d}t = - \int_\Omega \varepsilon \mathbf{V}_\varepsilon(0,\cdot) \cdot \varphi(0,\cdot) \, {\rm d}x + \sqrt{\varepsilon} \int_0^T \int_\Omega \mathbb{H}_\varepsilon : \nabla_x \frac{\varphi}{\tilde{\varrho}} \, {\rm d}x \, {\rm d}t \tag{6.124}$$

for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega}; \mathbb{R}^3)$, $\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$,

where

$$\{\mathbb{H}_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^1((0,T) \times \Omega; \mathbb{R}^{3\times 3}). \tag{6.125}$$

We recall that the left-hand side of (6.123), (6.124) can be understood as a weak formulation of the wave operator introduced in (4.30).

Two characteristic features of the wave equation (6.123), (6.124) can be easily identified:

- the wave speed depends effectively on the vertical (depth) coordinate x_3,
- the right-hand side is "large" of order $\sqrt{\varepsilon}$ in comparison with the frequency of the characteristic wavelength proportional to ε.

6.6.2 Spectral analysis of the wave operator

We consider the eigenvalue problem associated to the differential operator in (6.123), (6.124), namely

$$\tilde{\varrho}\nabla_x\left(\frac{q}{\tilde{\varrho}}\right) = \lambda \mathbf{w},\ p_0\overline{\vartheta}\mathrm{div}_x\mathbf{w} = \lambda q \text{ in } \Omega, \tag{6.126}$$

supplemented with the boundary condition

$$\mathbf{w}\cdot\mathbf{n}|_{\partial\Omega} = 0. \tag{6.127}$$

Equivalently, it is enough to solve

$$-\mathrm{div}_x\left[\tilde{\varrho}\nabla_x\left(\frac{q}{\tilde{\varrho}}\right)\right] = \Lambda\tilde{\varrho}\left(\frac{q}{\tilde{\varrho}}\right) \text{ in } \Omega, \tag{6.128}$$

with

$$\nabla_x\left(\frac{q}{\tilde{\varrho}}\right)\cdot\mathbf{n}|_{\partial\Omega} = 0, \tag{6.129}$$

where

$$\lambda^2 = -\Lambda p_0\overline{\vartheta}. \tag{6.130}$$

It is a routine matter to check that problem (6.128), (6.129) admits a complete system of real eigenfunctions $\{q_{j,m}\}_{j=0,m=1}^{\infty,m_j}$, together with the associated eigenvalues $\Lambda_{j,m}$ such that

$$\left\{ \begin{array}{c} m_0 = 1,\ \Lambda_{0,1} = 0,\ q_{0,1} = \tilde{\varrho}, \\ 0 < \Lambda_{1,1} = \cdots = \Lambda_{1,m_1}(=\Lambda_1) < \Lambda_{2,1} = \cdots = \Lambda_{2,m_2}(=\Lambda_2) < \cdots, \end{array} \right\} \tag{6.131}$$

where m_j stands for the multiplicity of Λ_j. Moreover, it can be shown that the system of functions $\{q_{j,m}\}_{j=0,m=1}^{\infty,m_j}$ forms an orthonormal basis of the weighted Lebesgue space $L^2_{1/\tilde{\varrho}}(\Omega)$ endowed with the scalar product

$$\langle v; w\rangle_{L^2_{1/\tilde{\varrho}}(\Omega)} = \int_\Omega vw\frac{\mathrm{d}x}{\tilde{\varrho}}$$

(see Section 10.2.2 in Appendix and also Chapter 3 in the monograph by Wilcox [202]).

Consequently, any solution of (6.126), (6.127) can be written in the form

$$\left\{ \begin{array}{c} \lambda = \lambda_{\pm j} = \pm \mathrm{i}\sqrt{p_0\overline{\vartheta}\Lambda_j},\ q = q_{j,m},\ \mathbf{w} = \mathbf{w}_{\pm j,m}, \\ \mathbf{w}_{\pm j,m} = \mp \mathrm{i}(\sqrt{p_0\overline{\vartheta}\Lambda_j})^{-1}\tilde{\varrho}\nabla_x \frac{q_{j,m}}{\tilde{\varrho}}, \\ \text{for } j = 1,\dots,m,\ m = 1,\dots,m_j, \end{array} \right\} \tag{6.132}$$

where a direct computation yields

$$\int_\Omega \mathbf{w}_{j,m} \cdot \mathbf{w}_{k,l} \ \frac{\mathrm{d}x}{\tilde{\varrho}} = -\frac{1}{p_0\overline{\vartheta}} \int_\Omega q_{j,m} q_{k,l} \ \frac{\mathrm{d}x}{\tilde{\varrho}}. \tag{6.133}$$

In addition, the eigenspace corresponding to the eigenvalue $\lambda_0 = \Lambda_{0,1} = 0$ coincides with

$$\mathcal{N} = \left\{ (c\tilde{\varrho}, \mathbf{w}) \ \Big| \ c = \text{const},\ \mathbf{w} \in L^2_{\sigma,1/\tilde{\varrho}}(\Omega;\mathbb{R}^3) \right\},$$

where the symbol $L^2_{\sigma,1/\tilde{\varrho}}(\Omega;\mathbb{R}^3)$ stands for the space of solenoidal functions

$$L^2_{\sigma,1/\tilde{\varrho}}(\Omega;\mathbb{R}^3) = \text{closure}_{L^2_{1/\tilde{\varrho}}}\{\mathbf{w} \in C^\infty_c(\Omega;\mathbb{R}^3) \mid \mathrm{div}_x \mathbf{w} = 0\} = L^2_\sigma(\Omega,\mathbb{R}^3).$$

Accordingly, the Hilbert space $L^2_{1/\tilde{\varrho}}(\Omega;\mathbb{R}^3)$ admits an orthogonal decomposition

$$L^2_{1/\tilde{\varrho}}(\Omega;\mathbb{R}^3) = L^2_{\sigma,1/\tilde{\varrho}}(\Omega;\mathbb{R}^3) \oplus \text{closure}_{L^2_{1/\tilde{\varrho}}}\{\text{span}\{\mathrm{i}\mathbf{w}_{j,m}\}_{j=1,m=1}^{\infty,m_j}\},$$

with the corresponding orthogonal projections represented by the Helmholtz projectors $\mathbf{H}_{\tilde{\varrho}}$, $\mathbf{H}^\perp_{\tilde{\varrho}}$ introduced in (6.47).

Finally, taking $\varphi = \psi_1(t)q_{j,m}$ as a test function in (6.123), and $\varphi = \psi_2\mathbf{w}_{j,m}$ in (6.124), with $\psi_1,\psi_2 \in C^\infty_c(0,T)$, we arrive at an infinite system of ordinary differential equations in the form:

$$\left\{ \begin{array}{c} \varepsilon\partial_t b_{j,m}[r_\varepsilon] - \omega\sqrt{\Lambda_j}\ a_{j,m}[\mathbf{V}_\varepsilon] = 0, \\ \varepsilon\partial_t a_{j,m}[\mathbf{V}_\varepsilon] + \sqrt{\Lambda_j}\ b_{j,m}[r_\varepsilon] = \sqrt{\varepsilon} H^{j,m}_\varepsilon \end{array} \right\} \tag{6.134}$$

for $j = 1,2,\dots,$ and $m = 1,\dots,m_j$, where we have introduced the "Fourier coefficients"

$$b_{j,m}[r_\varepsilon] = \int_\Omega r_\varepsilon q_{j,m}\ \mathrm{d}x,\ a_{j,m}[\mathbf{V}] = \frac{\mathrm{i}}{\sqrt{\omega}} \int_\Omega \mathbf{V}_\varepsilon \cdot \mathbf{w}_{j,m}\ \mathrm{d}x, \text{ and } \omega = p_0\overline{\vartheta}. \tag{6.135}$$

In accordance with (6.125),

$$\{H^{j,m}_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^1(0,T) \text{ for any fixed } j,m. \tag{6.136}$$

6.6.3 Convergence of the convective term

The description of the time oscillations of the acoustic modes provided by (6.134) is sufficient in order to identify the asymptotic limit of the convective term in the momentum equation (6.38). More precisely, our aim is to show that

$$\int_0^T \int_\Omega \varrho_\varepsilon [\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon] : \nabla_x \Big(\frac{\varphi}{\tilde\varrho} \Big) \ \mathrm{d}x \ \mathrm{d}t \to \int_0^T \int_\Omega \tilde\varrho [\mathbf{U} \otimes \mathbf{U}] : \nabla_x \Big(\frac{\varphi}{\tilde\varrho} \Big) \ \mathrm{d}x \ \mathrm{d}t \quad (6.137)$$

for any function φ such that

$$\varphi \in C_c^\infty((0,T) \times \overline{\Omega}; \mathbb{R}^3), \ \mathrm{div}_x \varphi = 0, \ \varphi \cdot \mathbf{n}|_{\partial\Omega} = 0. \quad (6.138)$$

If this is the case, the limit equation (6.116) gives rise to (6.46).

In order to see (6.137), we follow formally the approach used in Section 5.4.6, that means, we reduce (6.137) to a finite number of modes that can be explicitly expressed by help of (6.134).

Strong convergence of the solenoidal part. We claim that

$$\mathbf{H}_{\tilde\varrho}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \to \mathbf{H}_{\tilde\varrho}[\tilde\varrho \mathbf{U}] = \tilde\varrho \mathbf{U} \text{ in } L^1((0,T) \times \Omega; \mathbb{R}^3). \quad (6.139)$$

To this end, we take

$$\varphi(t,x) = \frac{\psi(t)}{\tilde\varrho} \mathbf{H}_{\tilde\varrho}[\tilde\varrho \mathbf{\Phi}], \ \mathbf{\Phi} \in C_c^\infty(\overline{\Omega}; \mathbb{R}^3), \ \mathbf{\Phi} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \psi \in C_c^\infty(0,T)$$

as a test function in the momentum equation (6.38). Seeing that

$$\int_0^T \partial_t \psi \int_\Omega \mathbf{H}_{\tilde\varrho}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \cdot \mathbf{\Phi} \ \mathrm{d}x \ \mathrm{d}t = \int_0^T \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \varphi \ \mathrm{d}x \ \mathrm{d}t,$$

and taking into account relations (6.107), (6.115), together with the uniform estimates obtained in Section 6.4.1, we conclude that the mappings

$$t \in [0,T] \mapsto \int_\Omega \mathbf{H}_{\tilde\varrho}[\varrho_\varepsilon \mathbf{u}_\varepsilon](t) \cdot \mathbf{\Phi} \ \mathrm{d}x$$

are precompact in $C[0,T]$, in other words,

$$\mathbf{H}_{\tilde\varrho}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \to \mathbf{H}_{\tilde\varrho}[\tilde\varrho \mathbf{U}] = \tilde\varrho \mathbf{U} \text{ in } C_{\mathrm{weak}}([0,T]; L^{\frac{5}{4}}(\Omega; \mathbb{R}^3)), \quad (6.140)$$

where we have used (6.62), (6.94), and compactness of the embedding $L^{\frac{5}{4}}(\Omega) \hookrightarrow [W^{1,2}(\Omega)]^*$.

On the other hand, as $\mathbf{H}_{\tilde\varrho}$, $\mathbf{H}^{\perp}_{\tilde\varrho}$ are orthogonal in the weighted space $L^2_{1/\tilde\varrho}$, and (6.95) holds, we can use (6.140) in order to obtain

$$\begin{aligned}\int_0^T \Big(\int_\Omega \mathbf{H}_{\tilde\varrho}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \cdot \mathbf{H}_{\tilde\varrho}[\tilde\varrho \mathbf{u}_\varepsilon] \frac{\mathrm{d}x}{\tilde\varrho} \Big) \ \mathrm{d}t \\ = \int_0^T \int_\Omega \mathbf{H}_{\tilde\varrho}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \cdot \mathbf{u}_\varepsilon \ \mathrm{d}t \to \int_0^T \int_\Omega \mathbf{H}_{\tilde\varrho}[\tilde\varrho \mathbf{U}] \cdot \mathbf{U} \ \mathrm{d}x \ \mathrm{d}t \\ = \int_0^T \Big(\int_\Omega \tilde\varrho^2 |\mathbf{U}|^2 \frac{\mathrm{d}x}{\tilde\varrho} \Big) \ \mathrm{d}t. \end{aligned} \tag{6.141}$$

In accordance with (6.94),

$$\operatorname{ess}\sup_{t\in(0,T)} \|\varrho_\varepsilon(t) - \tilde\varrho\|_{L^{\frac{5}{3}}(\Omega)} \to 0,$$

and we may infer from (6.141) that

$$\mathbf{H}_{\tilde\varrho}[\tilde\varrho \mathbf{u}_\varepsilon] \to \tilde\varrho \mathbf{U} \text{ in } L^2((0,T)\times\Omega;\mathbb{R}^3), \tag{6.142}$$

which, by the same token, gives rise to (6.139).

Time oscillations of the gradient part. Initially, we write

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon = \frac{1}{\tilde\varrho} \mathbf{H}_{\tilde\varrho}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \tilde\varrho \mathbf{u}_\varepsilon + \frac{1}{\tilde\varrho} \mathbf{H}^{\perp}_{\tilde\varrho}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}_{\tilde\varrho}[\tilde\varrho \mathbf{u}_\varepsilon] + \frac{1}{\tilde\varrho} \mathbf{H}^{\perp}_{\tilde\varrho}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}^{\perp}_{\tilde\varrho}[\tilde\varrho \mathbf{u}_\varepsilon].$$

Since both $\mathbf{H}_{\tilde\varrho}$ and $\mathbf{H}^{\perp}_{\tilde\varrho}$ are continuous in $L^p(\Omega;\mathbb{R}^3)$ for any $1 < p < \infty$ (see Section 10.2.1 in Appendix), we have

$$\mathbf{H}^{\perp}_{\tilde\varrho}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \to 0 \text{ weakly-(*) in } L^\infty(0,T;L^{\frac{5}{4}}(\Omega;\mathbb{R}^3)). \tag{6.143}$$

Consequently, we can use (6.139), (6.142) to reduce (6.137) to showing

$$\int_0^T \int_\Omega \Big(\mathbf{H}^{\perp}_{\tilde\varrho}[\tilde\varrho \mathbf{V}_\varepsilon] \otimes \mathbf{H}^{\perp}_{\tilde\varrho}[\tilde\varrho \mathbf{u}_\varepsilon] \Big) : \nabla_x \Big(\frac{\varphi}{\tilde\varrho} \Big) \ \frac{\mathrm{d}x}{\tilde\varrho} \ \mathrm{d}t \to 0 \text{ for } \varepsilon \to 0 \tag{6.144}$$

for any φ satisfying (6.137), where $\mathbf{V}_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon / \tilde\varrho$ is the quantity appearing in the acoustic equation (6.123), (6.124).

We proceed in two steps:

(i) To begin, we reduce (6.144) to a finite number of modes. Similarly to (6.135), we introduce the "Fourier coefficients"

$$a_{j,m}[\mathbf{Z}] = \frac{\mathrm{i}}{\sqrt{\omega}} \int_\Omega \mathbf{Z} \cdot \mathbf{w}_{j,m} \ \mathrm{d}x \text{ for any } \mathbf{Z} \in L^1(\Omega;\mathbb{R}^3).$$

Moreover, similarly to Section 5.4.6, we set

$$\mathbf{H}^{\perp}_{\tilde\varrho,M}[\tilde\varrho\mathbf{Z}] = \frac{-\mathrm{i}}{\sqrt{\omega}} \sum_{j,0<\Lambda_j\le M} \sum_{m=1}^{m_j} a_{j,m}[\mathbf{Z}]\mathbf{w}_{j,m}. \tag{6.145}$$

Now a straightforward manipulation yields

$$\left\{ \begin{array}{c} \mathbf{H}^{\perp}_{\tilde\varrho}[\tilde\varrho\mathbf{V}_\varepsilon] \otimes \mathbf{H}^{\perp}_{\tilde\varrho}[\tilde\varrho\mathbf{u}_\varepsilon] \\ = \Big[\mathbf{H}^{\perp}_{\tilde\varrho,M}[\tilde\varrho\mathbf{V}_\varepsilon] + \Big(\mathbf{H}^{\perp}_{\tilde\varrho}[\tilde\varrho\mathbf{V}_\varepsilon] - \mathbf{H}^{\perp}_{\tilde\varrho,M}[\tilde\varrho\mathbf{V}_\varepsilon]\Big)\Big] \\ \otimes \\ \Big[\mathbf{H}^{\perp}_{\tilde\varrho,M}[\tilde\varrho\mathbf{u}_\varepsilon] + \Big(\mathbf{H}^{\perp}_{\tilde\varrho}[\tilde\varrho\mathbf{u}_\varepsilon] - \mathbf{H}^{\perp}_{\tilde\varrho,M}[\tilde\varrho\mathbf{u}_\varepsilon]\Big)\Big], \end{array} \right\} \tag{6.146}$$

where we can write

$$\begin{aligned} &\mathbf{H}^{\perp}_{\tilde\varrho}[\tilde\varrho\mathbf{V}_\varepsilon] - \mathbf{H}^{\perp}_{\tilde\varrho,M}[\tilde\varrho\mathbf{V}_\varepsilon] \\ &\quad = \mathbf{H}^{\perp}_{\tilde\varrho}[(\varrho_\varepsilon - \tilde\varrho)\mathbf{u}_\varepsilon] - \mathbf{H}^{\perp}_{\tilde\varrho,M}[(\varrho_\varepsilon - \tilde\varrho)\mathbf{u}_\varepsilon] + \mathbf{H}^{\perp}_{\tilde\varrho}[\tilde\varrho\mathbf{u}_\varepsilon] - \mathbf{H}^{\perp}_{\tilde\varrho,M}[\tilde\varrho\mathbf{u}_\varepsilon]. \end{aligned}$$

Using relations (6.94), (6.95) we obtain

$$\mathbf{H}^{\perp}_{\tilde\varrho}[(\varrho_\varepsilon - \tilde\varrho)\mathbf{u}_\varepsilon] - \mathbf{H}^{\perp}_{\tilde\varrho,M}[(\varrho_\varepsilon - \tilde\varrho)\mathbf{u}_\varepsilon] \to 0 \text{ in } L^1((0,T)\times\Omega;\mathbb{R}^3) \text{ as } \varepsilon \to 0 \tag{6.147}$$

for any fixed M.

On the other hand, using orthogonality of the functions $\{q_{j,m}\}$, together with Parseval's identity with respect to the scalar product of $L^2_{1/\tilde\varrho}(\Omega)$ and relation (6.132), we get

$$\|\mathrm{div}_x(\tilde\varrho\mathbf{u}_\varepsilon)\|^2_{L^2_{1/\tilde\varrho}(\Omega)} = \sum_{j=1}^{\infty}\sum_{m=1}^{m_j} \Lambda_j a^2_{j,m}[\mathbf{u}_\varepsilon].$$

Moreover, in accordance with (6.133),

$$\begin{aligned} &\|\mathbf{H}^{\perp}_{\tilde\varrho}[\tilde\varrho\mathbf{u}_\varepsilon] - \mathbf{H}^{\perp}_{\tilde\varrho,M}[\tilde\varrho\mathbf{u}_\varepsilon]\|^2_{L^2_{1/\tilde\varrho}(\Omega;\mathbb{R}^3)} \\ &\qquad = \sum_{j;\Lambda_j>M}\sum_{m=1}^{m_j} a^2_{j,m}[\mathbf{u}_\varepsilon] \le \frac{1}{M}\|\mathrm{div}_x(\tilde\varrho\mathbf{u}_\varepsilon)\|^2_{L^2_{1/\tilde\varrho}(\Omega)}. \end{aligned}$$

Thus, by virtue of (6.81), we are allowed to conclude that

$$\mathbf{H}^{\perp}_{\tilde\varrho}[\tilde\varrho\mathbf{u}_\varepsilon] - \mathbf{H}^{\perp}_{\tilde\varrho,M}[\tilde\varrho\mathbf{u}_\varepsilon] \to 0 \text{ in } L^2(0,T;L^2_{1/\tilde\varrho}(\Omega;\mathbb{R}^3)) \text{ as } M \to \infty \tag{6.148}$$

uniformly with respect to $\varepsilon \to 0$.

In view of relations (6.147), (6.148), the proof of (6.137) simplifies considerably, being reduced to showing

$$\int_0^T \int_\Omega \Big(\mathbf{H}^{\perp}_{\tilde\varrho,M}[\tilde\varrho\mathbf{V}_\varepsilon] \otimes \mathbf{H}^{\perp}_{\tilde\varrho,M}[\tilde\varrho\mathbf{u}_\varepsilon]\Big) : \nabla_x\Big(\frac{\varphi}{\tilde\varrho}\Big)\frac{\mathrm{d}x}{\tilde\varrho}\ \mathrm{d}t \to 0$$

or, equivalently, by virtue of (6.94),

$$\int_0^T \int_\Omega \Big(\mathbf{H}^\perp_{\tilde\varrho,M}[\tilde\varrho \mathbf{V}_\varepsilon] \otimes \mathbf{H}^\perp_{\tilde\varrho,M}[\tilde\varrho \mathbf{V}_\varepsilon] \Big) : \nabla_x \Big(\frac{\varphi}{\tilde\varrho} \Big) \frac{\mathrm{d}x}{\tilde\varrho} \ \mathrm{d}t \to 0 \tag{6.149}$$

for any test function φ satisfying (6.138) and any fixed M.

(ii) In order to see (6.149), we first observe that

$$\int_0^T \int_\Omega \Big(\mathbf{H}^\perp_{\tilde\varrho,M}[\tilde\varrho \mathbf{V}_\varepsilon] \otimes \mathbf{H}^\perp_{\tilde\varrho,M}[\tilde\varrho \mathbf{V}_\varepsilon] \Big) : \nabla_x \Big(\frac{\varphi}{\tilde\varrho} \Big) \frac{\mathrm{d}x}{\tilde\varrho} \ \mathrm{d}t$$
$$= \int_0^T \int_\Omega (\tilde\varrho \nabla_x \Psi_\varepsilon \otimes \nabla_x \Psi_\varepsilon) : \nabla_x \Big(\frac{\varphi}{\tilde\varrho} \Big) \ \mathrm{d}x \ \mathrm{d}t,$$

where, by means of (6.145),

$$\Psi_\varepsilon = \frac{1}{\omega} \sum_{j \le M} \sum_{m=1}^{m_j} \frac{a_{j,m}[\mathbf{V}_\varepsilon]}{\sqrt{\Lambda_j}} \Big(\frac{q_{j,m}}{\tilde\varrho} \Big). \tag{6.150}$$

First, integrating the above expression by parts and making use of the fact that $\mathrm{div}_x \varphi = 0$, we get

$$\int_0^T \int_\Omega (\tilde\varrho \nabla_x \Psi_\varepsilon \otimes \nabla_x \Psi_\varepsilon) : \nabla_x \Big(\frac{\varphi}{\tilde\varrho} \Big) \ \mathrm{d}x \ \mathrm{d}t$$
$$= - \int_0^T \int_\Omega \mathrm{div}_x \Big(\tilde\varrho \nabla_x \Psi_\varepsilon \Big) \nabla_x \Psi_\varepsilon \cdot \Big(\frac{\varphi}{\tilde\varrho} \Big) \ \mathrm{d}x \ \mathrm{d}t,$$

where, in agreement with (6.128),

$$-\mathrm{div}_x (\tilde\varrho \nabla_x \Psi_\varepsilon) = \frac{1}{\omega} \sum_{j \le M} \sum_{m=1}^{m_j} \sqrt{\Lambda_j} a_{j,m}[\mathbf{V}_\varepsilon] q_{j,m}.$$

The next step is to use system (6.134) in order to obtain

$$- \int_0^T \int_\Omega \mathrm{div}_x \Big(\tilde\varrho \nabla_x \Psi_\varepsilon \Big) \nabla_x \Psi_\varepsilon \cdot \Big(\frac{\varphi}{\tilde\varrho} \Big) \ \mathrm{d}x \ \mathrm{d}t$$
$$= \frac{\varepsilon}{\omega^2} \int_0^T \int_\Omega \sum_{j \le M} \sum_{m=1}^{m_j} \partial_t b_{j,m}[r_\varepsilon] \frac{q_{j,m}}{\tilde\varrho} \nabla_x \Psi_\varepsilon \cdot \varphi \ \mathrm{d}x \ \mathrm{d}t$$
$$= - \frac{\varepsilon}{\omega^2} \int_0^T \int_\Omega \sum_{j \le M} \sum_{m=1}^{m_j} b_{j,m}[r_\varepsilon] \frac{q_{j,m}}{\tilde\varrho} \nabla_x \Psi_\varepsilon \cdot \partial_t \varphi \ \mathrm{d}x \ \mathrm{d}t$$
$$= \frac{\varepsilon}{\omega^2} \int_0^T \int_\Omega \sum_{j \le M} \sum_{m=1}^{m_j} b_{j,m}[r_\varepsilon] \frac{q_{j,m}}{\tilde\varrho} \partial_t \nabla_x \Psi_\varepsilon \cdot \varphi \ \mathrm{d}x \ \mathrm{d}t.$$

We see immediately that the first integral on the right-hand side of the above equality tends to zero for $\varepsilon \to 0$, therefore the proof of (6.149) will be complete as soon as we are able to verify that the amplitude of $\partial_t \nabla_x \Psi_\varepsilon \cdot \varphi$ grows at most as ε^{-k} for a certain $k < 1$. Thus it is enough to show that

$$\Big| \int_0^T \int_\Omega \sum_{j \leq M} \sum_{m=1}^{m_j} b_{j,m}[r_\varepsilon] \frac{q_{j,m}}{\tilde{\varrho}} \partial_t \nabla_x \Psi_\varepsilon \cdot \varphi \, \mathrm{d}x \, \mathrm{d}t \Big| \leq \frac{c}{\sqrt{\varepsilon}}. \tag{6.151}$$

In order to see (6.151), we make use of the second equation in (6.134), and (6.150) to express

$$\partial_t \nabla_x \Psi_\varepsilon = -\frac{1}{\varepsilon \omega} \sum_{j \leq M} \sum_{m=1}^{m_j} b_{j,m}[r_\varepsilon] \nabla_x \Big(\frac{q_{j,m}}{\tilde{\varrho}} \Big) + \frac{1}{\sqrt{\varepsilon} \omega} \sum_{j \leq M} \sum_{m=1}^{m_j} \frac{1}{\sqrt{\Lambda_j}} H_\varepsilon^{j,m} \nabla_x \Big(\frac{q_{j,m}}{\tilde{\varrho}} \Big),$$

where $H_\varepsilon^{j,m}$ are bounded in $L^1(0,T)$ as stated in (6.136).

Finally, we observe that the expression

$$\begin{aligned} &\Big(\sum_{j \leq M} \sum_{m=1}^{m_j} b_{j,m}[r_\varepsilon] \frac{q_{j,m}}{\tilde{\varrho}} \Big) \sum_{j \leq M} \sum_{m=1}^{m_j} b_{j,m}[r_\varepsilon] \nabla_x \Big(\frac{q_{j,m}}{\tilde{\varrho}} \Big) \\ &= \frac{1}{2} \nabla_x \Big(\sum_{j \leq M} \sum_{m=1}^{m_j} b_{j,m}[r_\varepsilon] \frac{q_{j,m}}{\tilde{\varrho}} \Big)^2 \end{aligned}$$

is a perfect gradient, in particular

$$\int_0^T \int_\Omega \Big(\sum_{j \leq M} \sum_{m=1}^{m_j} b_{j,m}[r_\varepsilon] \frac{q_{j,m}}{\tilde{\varrho}} \Big) \sum_{j \leq M} \sum_{m=1}^{m_j} b_{j,m}[r_\varepsilon] \nabla_x \Big(\frac{q_{j,m}}{\tilde{\varrho}} \Big) \cdot \varphi \, \mathrm{d}x \, \mathrm{d}t = 0$$

as $\mathrm{div}_x \varphi = 0$, $\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$. Consequently, we have verified (6.151); whence (6.149).

Thus we conclude that (6.118) holds, notably the integral identity (6.116) coincides with (6.46). Consequently, in order to complete the proof of Theorem 6.1, we have to check that $\vartheta^{(2)}$ identified in (6.113) is related to the vertical component U_3 through (6.45). This is the objective of the last section.

6.7 Asymptotic limit in entropy balance

In contrast with Chapter 5, the analysis of the entropy equation (6.39) is rather simple. To begin, we get

$$\langle \sigma_\varepsilon; \varphi \rangle_{[\mathcal{M};C]([0,T]\times\overline{\Omega})} - \int_0^T \int_{\{x_3=1\}} \beta_1 \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \varphi \, \mathrm{d}S_x \, \mathrm{d}t \to 0 \text{ as } \varepsilon \to 0 \tag{6.152}$$

for any fixed $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$ as a direct consequence of the uniform estimates (6.63), (6.64).

Similarly, by virtue of (6.82), (6.84),

$$\varepsilon^{2\alpha/3}\kappa_0\nabla_x \log(\vartheta_\varepsilon) + \kappa_1\nabla_x\vartheta \to 0 \text{ in } L^2((0,T)\times\Omega;\mathbb{R}^3),$$

and, consequently,

$$\begin{aligned}
-\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \frac{1}{\varepsilon^2}\frac{\mathbf{q}_\varepsilon}{\vartheta_\varepsilon}\cdot\nabla_x\varphi \ \mathrm{d}x\ \mathrm{d}t &= \lim_{\varepsilon\to 0}\int_0^T\int_\Omega d\vartheta_\varepsilon^2\frac{\nabla_x\vartheta_\varepsilon}{\varepsilon^2}\cdot\nabla_x\varphi \ \mathrm{d}x\ \mathrm{d}t \\
&= \lim_{\varepsilon\to 0}\int_0^T\int_\Omega d\vartheta_\varepsilon^2\nabla_x\vartheta_\varepsilon^{(2)}\cdot\nabla_x\varphi \ \mathrm{d}x\ \mathrm{d}t,
\end{aligned}$$

where the quantities $\vartheta_\varepsilon^{(2)}$ have been introduced in (6.112).

Furthermore, writing

$$\begin{aligned}
&\int_0^T\int_\Omega d\vartheta_\varepsilon^2\nabla_x\vartheta_\varepsilon^{(2)}\cdot\nabla_x\varphi \ \mathrm{d}x\ \mathrm{d}t \\
&= \int_0^T\int_\Omega d[\vartheta_\varepsilon]^2_{\mathrm{ess}}\nabla_x\vartheta_\varepsilon^{(2)}\cdot\nabla_x\varphi \ \mathrm{d}x\ \mathrm{d}t + \int_0^T\int_\Omega d[\vartheta_\varepsilon]^{3/2}_{\mathrm{res}}\sqrt{\vartheta_\varepsilon}\nabla_x\left(\frac{\vartheta_\varepsilon}{\varepsilon^2}\right)\cdot\nabla_x\varphi \ \mathrm{d}x\ \mathrm{d}t,
\end{aligned}$$

we can use (6.97), (6.113) to deduce

$$\int_0^T\int_\Omega d[\vartheta_\varepsilon]^2_{\mathrm{ess}}\nabla_x\vartheta_\varepsilon^{(2)}\cdot\nabla_x\varphi \ \mathrm{d}x\ \mathrm{d}t \to \int_0^T\int_\Omega d\overline{\vartheta}^2\nabla_x\vartheta^{(2)}\cdot\nabla_x\varphi \ \mathrm{d}x\ \mathrm{d}t,$$

while the uniform estimates (6.75), (6.78) give rise to

$$\int_0^T\int_\Omega d[\vartheta_\varepsilon]^{3/2}_{\mathrm{res}}\sqrt{\vartheta_\varepsilon}\nabla_x\left(\frac{\vartheta_\varepsilon}{\varepsilon^2}\right)\cdot\nabla_x\varphi \ \mathrm{d}x\ \mathrm{d}t \to 0.$$

Thus, we conclude that

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \frac{1}{\varepsilon^2}\frac{\mathbf{q}_\varepsilon}{\vartheta_\varepsilon}\cdot\nabla_x\varphi \ \mathrm{d}x\ \mathrm{d}t = -d\overline{\vartheta}^2\int_0^T\int_\Omega \nabla_x\vartheta^{(2)}\cdot\nabla_x\varphi \ \mathrm{d}x\ \mathrm{d}t \tag{6.153}$$

for any $\varphi\in C_c^\infty([0,T)\times\overline{\Omega})$.

Finally, in order to handle the convective term in (6.39), we write

$$\varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon,\vartheta_\varepsilon) = [\varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\mathrm{ess}} + [\varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\mathrm{res}},$$

where, in accordance with (6.72),

$$[\varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\mathrm{res}} \to 0 \text{ in } L^1((0,T)\times\Omega). \tag{6.154}$$

Now, similarly to (6.11), we can decompose

$$\varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon,\vartheta_\varepsilon) = \varrho_\varepsilon s_{M,\varepsilon}(\varrho_\varepsilon,\vartheta_\varepsilon) + \varrho_\varepsilon s_{R,\varepsilon}(\varrho_\varepsilon,\vartheta_\varepsilon),$$

where, by virtue of (6.75),

$$\varrho_\varepsilon s_{R,\varepsilon}(\varrho_\varepsilon,\vartheta_\varepsilon) = \varepsilon\frac{4a}{3}\vartheta_\varepsilon^3 \to 0 \text{ in } L^\infty(0,T;L^{\frac{4}{3}}(\Omega)), \tag{6.155}$$

in particular

$$\varrho_\varepsilon s_{R,\varepsilon}(\varrho_\varepsilon,\vartheta_\varepsilon)\mathbf{u}_\varepsilon \to 0 \text{ in } L^2(0,T;L^{\frac{12}{11}}(\Omega;\mathbb{R}^3)). \tag{6.156}$$

On the other hand, due to (6.11), (6.12),

$$\varrho_\varepsilon s_{M,\varepsilon}(\varrho_\varepsilon,\vartheta_\varepsilon) = \varrho_\varepsilon\left(S\Big(\varepsilon^\alpha\frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{3/2}}\Big) - S(\varepsilon^\alpha)\right),$$

where, in accordance with hypothesis (6.15),

$$\left|S\Big(\varepsilon^\alpha\frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{3/2}}\Big) - S(\varepsilon^\alpha)\right| \le \left|\int_{\varepsilon^\alpha}^{\varepsilon^\alpha\frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{3/2}}} S'(Z)\ \mathrm{d}Z\right| \le c(|\log(\varrho_\varepsilon)| + |\log(\vartheta_\varepsilon)|).$$

Consequently, using the uniform bounds established in (6.62), (6.71), (6.76), and (6.84), we obtain

$$[\varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm res}\mathbf{u}_\varepsilon \to 0 \text{ in } L^q((0,T)\times\Omega;\mathbb{R}^3) \text{ for a certain } q>1. \tag{6.157}$$

Thus in order to complete our analysis, we have to determine the asymptotic limit of the "essential" component of the entropy $[\varrho_\varepsilon s_{M,\varepsilon}(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm ess}$. To this end, write

$$S(Z) = -\log(Z) + \tilde{S}(Z),$$

where

$$\tilde{S}'(Z) = -\frac{3}{2}\frac{\frac{5}{3}(P(Z)-p_0Z) - (P'(Z)-p_0)Z}{Z^2}.$$

As P is twice continuously differentiable on $[0,\infty)$, and, in addition, satisfies (6.15), we have

$$|\tilde{S}'(Z)| \le c \text{ for all } Z>0.$$

Consequently, we obtain

$$[\varrho_\varepsilon s_{M,\varepsilon}(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm ess} \to p_0\tilde{\varrho}\Big(\frac{3}{2}\log(\overline{\vartheta}) - \log(\tilde{\varrho})\Big) \text{ in } L^q((0,T)\times\Omega) \tag{6.158}$$

for any $1\le q<\infty$.

Going back to (6.39) and resuming relations (6.152–6.158) we conclude that

$$-d\overline{\vartheta}^2\int_0^T\int_\Omega \nabla_x\vartheta^{(2)}\cdot\nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t = p_0\int_0^T\int_\Omega \tilde{\varrho}\log(\tilde{\varrho})\mathbf{U}\cdot\nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t \tag{6.159}$$

for any test function $\varphi\in C_c^\infty((0,T)\times\overline{\Omega})$, where we have used the *anelastic constraint* (6.98) and (6.95). Since $\tilde{\varrho}$ solves the static problem (6.41), relation (6.159) is nothing other than a variational formulation of (6.45). Theorem 6.1 has been proved.

Chapter 7

Interaction of Acoustic Waves with Boundary

As we have seen in the previous chapters, one of the most delicate issues in the analysis of singular limits for the NAVIER-STOKES-FOURIER SYSTEM in the low Mach number regime is the influence of the acoustic waves. If the physical domain is bounded, the acoustic waves, being reflected by the boundary, inevitably develop high frequency oscillations resulting in the *weak* convergence of the velocity field. This rather unpleasant phenomenon creates additional problems when handling the convective term in the momentum equation (cf. Sections 5.4.7, 6.6.3 above). In this chapter, we focus on the mechanisms so-far neglected by which the acoustic energy is dissipated into heat, and the ways in which the dissipation may be used in order to show *strong* (pointwise) convergence of the velocity.

The principal mechanism of dissipation in the NAVIER-STOKES-FOURIER SYSTEM is of course *viscosity*, here imposed through Newton's rheological law. At a first glance, the presence of the viscous stress $\mathbb{S}$ in the momentum equation does not seem to play any significant role in the analysis of acoustic waves. In the situation described in Section 4.4.1, the acoustic equation can be written in the form

$$\left\{ \begin{array}{c} \varepsilon\partial_t r_\varepsilon + \mathrm{div}_x(\mathbf{V}_\varepsilon) = \text{“small terms”}, \\ \varepsilon\partial_t \mathbf{V}_\varepsilon + \omega\nabla_x r_\varepsilon = \varepsilon \mathrm{div}_x \mathbb{S}_\varepsilon + \text{“small terms”}. \end{array} \right\} \tag{7.1}$$

Replacing for simplicity $\mathrm{div}_x \mathbb{S}_\varepsilon$ by $\Delta \mathbf{V}_\varepsilon$, we examine the associated eigenvalue problem:

$$\begin{aligned} \mathrm{div}_x \mathbf{w} &= \lambda r, \\ \omega\nabla_x r - \varepsilon\Delta_x \mathbf{w} &= \lambda \mathbf{w}. \end{aligned} \tag{7.2}$$

Applying the divergence operator to the second equation and using the first one to express all quantities in terms of r, we arrive at the eigenvalue problem

$$-\Delta_x r = \lambda^2 r / (\varepsilon\lambda - \omega).$$

Under the periodic boundary conditions, meaning $\Omega = \mathcal{T}^3$, the corresponding eigenvalues are given as

$$\frac{\lambda_n^2}{\varepsilon\lambda_n - \omega} = \Lambda_n,$$

where Λ_n are the (real non-negative) eigenvalues of the Laplace operator supplemented with the periodic boundary conditions. It is easy to check that

$$\lambda_n = \frac{\varepsilon\Lambda_n \pm \mathrm{i}\sqrt{4\omega\Lambda_n - \varepsilon^2\Lambda_n^2}}{2}.$$

Moreover, the corresponding eigenfunctions read

$$\{r_n, \mathbf{w}_n\},\ \mathbf{w}_n = \frac{\omega - \varepsilon\lambda_n}{\lambda_n}\nabla_x r_n,$$

where r_n are the eigenfunctions of the Laplacian supplemented with the periodic boundary conditions.

The same result is obtained provided the velocity field satisfies the complete slip boundary conditions (1.19), (1.27) leading to the Neumann boundary conditions for r, namely

$$\mathbf{w}\cdot\mathbf{n}|_{\partial\Omega} = \nabla_x r\cdot\mathbf{n}|_{\partial\Omega} = 0.$$

In particular, the eigenfunctions differ from those of the limit problem with $\varepsilon = 0$ only by a multiplicative constant approaching 1 for $\varepsilon \to 0$.

Physically speaking, the complete slip boundary conditions correspond to the ideal *mechanically smooth* boundary of the physical space. As suggested by the previous arguments, the effect of viscosity in this rather hypothetical situation does not change significantly the asymptotic analysis in the low Mach number limit.

■ CONJECTURE I (NEGATIVE):

The dissipation of acoustic energy caused by viscosity in domains with mechanically smooth boundaries is irrelevant in the low Mach number regime. The decay of acoustic waves is exponential with a rate independent of ε.

On the other hand, the decay rate of the acoustic waves may change substantially if the fluid interacts with the boundary, meaning, if some kind of "dissipative" boundary conditions is imposed on the velocity field. Thus, for instance, the no-slip boundary conditions (1.28) give rise to

$$\mathbf{w}|_{\partial\Omega} = 0. \tag{7.3}$$

Accordingly, system (7.2), supplemented with (7.3), becomes a *singularly perturbed* eigenvalue problem. In particular, if the (overdetermined) limit problem

$$\mathrm{div}_x\mathbf{w} = \lambda r,\ \omega\nabla_x r = \lambda\mathbf{w},\ \mathbf{w}|_{\partial\Omega} = 0 \tag{7.4}$$

admits only the trivial solution for $\lambda \neq 0$, we can expect that a boundary layer is created in the limit process $\varepsilon \to 0$ resulting in a faster decay of the acoustic waves. This can be seen by means of the following heuristic argument. Suppose that problem (7.2), (7.3) admits a family of eigenfunctions $\{r_\varepsilon, \mathbf{w}_\varepsilon\}_{\varepsilon>0}$ with the associated set of eigenvalues $\{\lambda_\varepsilon\}_{\varepsilon>0}$. Multiplying (7.2) on $\overline{r}_\varepsilon$, $\overline{\mathbf{w}}_\varepsilon$, where the bar stands for the complex conjugate, integrating the resulting expression over Ω, and using (7.3), we obtain

$$\varepsilon \int_\Omega |\nabla_x \mathbf{w}_\varepsilon|^2 \, \mathrm{d}x = (1+\omega)\mathrm{Re}[\lambda_\varepsilon] \int_\Omega \Big(|r_\varepsilon|^2 + |\mathbf{w}_\varepsilon|^2 \Big) \, \mathrm{d}x,$$

where Re denotes the real part of a complex number. Normalizing $\{r_\varepsilon, \mathbf{w}_\varepsilon\}_{\varepsilon>0}$ in $L^2(\Omega) \times L^2(\Omega; \mathbb{R}^3)$ we easily observe that

$$\frac{\mathrm{Re}[\lambda_\varepsilon]}{\varepsilon} \to \infty,$$

since otherwise $\{\mathbf{w}_\varepsilon\}_{\varepsilon>0}$ would be bounded in $W^{1,2}(\Omega; \mathbb{R}^3)$ and any weak accumulation point $(r, \mathbf{w})$ of $\{r_\varepsilon, \mathbf{w}_\varepsilon\}_{\varepsilon>0}$ would represent a nontrivial solution of the overdetermined limit system (7.4).

■ CONJECTURE II (POSITIVE):

Sticky boundaries in combination with viscous effects may produce a decay rate of acoustic waves that is considerably faster than their frequency in the low Mach number regime. In particular, mechanical energy is converted into heat and the acoustic waves are annihilated at a time approaching zero in the low Mach number limit.

7.1 Problem formulation

Motivated by the previous discussion, we examine the low Mach number limit for the NAVIER-STOKES-FOURIER SYSTEM supplemented with the no-stick boundary condition for the velocity. The fact that the fluid adheres completely to the wall of the physical space imposes additional restrictions on the propagation of the acoustic waves. Our goal is to identify the geometrical properties of the domain, for which this implies strong convergence of the velocity field in the asymptotic limit.

7.1.1 Field equations

We consider the same scaling of the field equations as in Chapter 5. Specifically, we set

$$\mathrm{Ma} = \varepsilon, \ \mathrm{Fr} = \sqrt{\varepsilon}$$

obtaining

■ Scaled Navier-Stokes-Fourier system:

$$\partial_t \varrho_\varepsilon + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0, \tag{7.5}$$

$$\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \frac{1}{\varepsilon^2}\nabla_x p(\varrho_\varepsilon, \vartheta_\varepsilon) = \mathrm{div}_x \mathbb{S}_\varepsilon + \frac{1}{\varepsilon}\varrho_\varepsilon \nabla_x F, \tag{7.6}$$

$$\partial_t(\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)) + \mathrm{div}_x(\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)\mathbf{u}_\varepsilon) + \mathrm{div}_x\Big(\frac{\mathbf{q}_\varepsilon}{\vartheta}\Big) = \sigma_\varepsilon, \tag{7.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Big(\frac{\varepsilon^2}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon) - \varepsilon\varrho_\varepsilon F\Big)\,\mathrm{d}x = 0, \tag{7.8}$$

where

$$\sigma_\varepsilon \geq \frac{1}{\vartheta_\varepsilon}\Big(\varepsilon^2 \mathbb{S}_\varepsilon : \nabla_x \mathbf{u}_\varepsilon - \frac{\mathbf{q}_\varepsilon \cdot \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon}\Big). \tag{7.9}$$

System (7.5–7.8) is supplemented, exactly as in Chapter 5, with the constitutive relations:

$$\mathbb{S}_\varepsilon = \mathbb{S}(\vartheta_\varepsilon, \nabla_x \mathbf{u}_\varepsilon) = \mu(\vartheta_\varepsilon)\Big(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^T \mathbf{u}_\varepsilon - \frac{2}{3}\mathrm{div}_x \mathbf{u}_\varepsilon \mathbb{I}\Big), \tag{7.10}$$

$$\mathbf{q}_\varepsilon = \mathbf{q}(\vartheta_\varepsilon, \nabla_x \vartheta_\varepsilon) = -\kappa(\vartheta_\varepsilon)\nabla_x \vartheta_\varepsilon, \tag{7.11}$$

and

$$p(\varrho_\varepsilon, \vartheta_\varepsilon) = p_M(\varrho_\varepsilon, \vartheta_\varepsilon) + p_R(\vartheta_\varepsilon),\ p_M = \vartheta_\varepsilon^{\frac{5}{2}} P\Big(\frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}}\Big),\ p_R = \frac{a}{3}\vartheta_\varepsilon^4, \tag{7.12}$$

$$e(\varrho_\varepsilon, \vartheta_\varepsilon) = e_M(\varrho_\varepsilon, \vartheta_\varepsilon) + e_R(\varrho_\varepsilon, \vartheta_\varepsilon),\ e_M = \frac{3}{2}\frac{\vartheta_\varepsilon^{\frac{5}{2}}}{\varrho_\varepsilon} P\Big(\frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}}\Big),\ e_R = a\frac{\vartheta_\varepsilon^4}{\varrho_\varepsilon}, \tag{7.13}$$

$$s(\varrho_\varepsilon, \vartheta_\varepsilon) = s_M(\varrho_\varepsilon, \vartheta_\varepsilon) + s_R(\varrho_\varepsilon, \vartheta_\varepsilon),\quad s_M(\varrho_\varepsilon, \vartheta_\varepsilon) = S\Big(\frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{\frac{3}{2}}}\Big),\ s_R = \frac{4}{3}a\frac{\vartheta_\varepsilon^3}{\varrho_\varepsilon}, \tag{7.14}$$

where

$$S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z^2} \text{ for all } Z > 0. \tag{7.15}$$

The reader will have noticed that the bulk viscosity has been neglected in (7.10) for the sake of simplicity.

As always in this book, equations (7.5–7.8) are interpreted in the weak sense specified in Chapter 1 (see Section 7.2 below). We recall that the technical restrictions imposed on the constitutive functions are dictated by the existence theory developed in Chapter 3 and could be relaxed, to a certain extent, as far as the singular limit passage is concerned.

7.1.2 Physical domain and boundary conditions

As indicated in the introductory part, the geometry of the physical domain plays a crucial role in the study of propagation of the acoustic waves. As already pointed out, the existence of an effective mechanism of dissipation of the acoustic waves is intimately linked to solvability of the (overdetermined) system (7.4) that can be written in a more concise form as

$$-\Delta_x r = \Lambda r \text{ in } \Omega,\ \frac{\lambda^2}{\omega} = -\Lambda,\ \nabla_x r|_{\partial\Omega} = 0. \tag{7.16}$$

The problem of existence of a non-trivial, meaning non-constant, solution to (7.16) is directly related to the so-called *Pompeiu property* of the domain Ω. A remarkable result of Williams [203] asserts that if (7.16) possesses a non-constant solution in a domain in $\mathbb{R}^N$ whose boundary is homeomorphic to the unit sphere, then, necessarily, $\partial\Omega$ must admit a description by a system of charts that are *real analytic*. The celebrated *Schiffer's conjecture* claims that (7.16) admits a non-trivial solution in the aforementioned class of domains only if Ω is a ball.

In order to avoid the unsurmountable difficulties mentioned above, we restrict ourselves to a very simple geometry of the physical space. Similarly to Chapter 6, we assume the motion of the fluid is 2π-periodic in the horizontal variables (x_1, x_2), and the domain Ω is an infinite slab determined by the graphs of two given functions B_{bottom}, B_{top},

$$\Omega = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in \mathcal{T}^2,\ B_{\text{bottom}}(x_1, x_2) < x_3 < B_{\text{top}}(x_1, x_2)\}, \tag{7.17}$$

where $\mathcal{T}^2$ denotes the flat torus,

$$\mathcal{T}^2 = \left([-\pi, \pi]|_{\{-\pi,\pi\}}\right)^2.$$

Although the specific length of the period is not essential, this convention simplifies considerably the notation used in the remaining part of this chapter.

In the simple geometry described by (7.17), it is easy to see that problem (7.16) admits a non-trivial solution, namely $r = \cos(x_3)$ as soon as the boundary is flat, more precisely, if $B_{\text{bottom}} = -\pi$, $B_{\text{top}} = 0$. On the other hand, we claim that problem (7.16) possesses only the trivial solution in *domains with variable bottoms* as stated in the following assertion.

Proposition 7.1. *Let Ω be given through* (7.17), *with*

$$\left\{ \begin{array}{c} B_{\text{bottom}} = -\pi - h(x_1, x_2),\ B_{\text{top}} = 0, \\ \textit{where} \\ h \in C(\mathcal{T}^2),\ |h| < \pi \textit{ for all } (x_1, x_2) \in \mathcal{T}^2. \end{array} \right\} \tag{7.18}$$

Assume there is a function $r \neq$ const solving the eigenvalue problem (7.16) *for a certain Λ.*

Then $h \equiv$ constant.

Proof. Since r is constant on the top part, specifically $r(x_1, x_2, 0) = r_0$, the function

$$V(x_1, x_2, x_3) = r(x_1, x_2, x_3) - r_0 \cos(\sqrt{\Lambda} x_3)$$

satisfies

$$-\Delta_x V = \Lambda V \text{ in } \Omega, \text{ and, in addition, } \nabla_x V|_{B_{\rm top}} = V|_{B_{\rm top}} = 0.$$

Accordingly, the function V extended to be zero in the upper half plane $\{x_3 > 0\}$ solves the eigenvalue problem (7.16) in $\Omega \cup \{x_3 \geq 0\}$. Consequently, by virtue of the unique continuation property of the elliptic operator $\Delta_x + \Lambda I$ (analyticity of solutions to elliptic problems discussed in Section 10.2.1 in Appendix), we get $V \equiv 0$, in other words,

$$r = r_0 \cos(\sqrt{\Lambda} x_3) \text{ in } \Omega.$$

However, as r must be constant on the bottom part $\{x_3 = -\pi - h(x_1, x_2)\}$, we conclude that $h \equiv$ const. □

Our future considerations will be concerned with fluids confined to domains described through (7.17), with flat "tops" and variables "bottoms" as in (7.18) with $h \not\equiv$ const. We impose the *no-slip* boundary conditions for the velocity field

$$\mathbf{u}_\varepsilon|_{\partial\Omega} = 0, \tag{7.19}$$

together with the no-flux boundary condition for the temperature

$$\mathbf{q}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{7.20}$$

Accordingly, the system is energetically insulated in agreement with (7.8).

As we shall see, our approach applies to any bounded sufficiently smooth spatial domain $\Omega \subset \mathbb{R}^3$, on which the overdetermined problem (7.16) admits only the trivial (constant) solution r. In particular, the arguments in the proof of Proposition 7.1 can be used provided a part of the boundary is flat and the latter is connected.

7.2 Main result

7.2.1 Preliminaries – global existence

Exactly as in Chapter 5, we consider the initial data in the form

$$\left\{ \begin{array}{c} \varrho_\varepsilon(0, \cdot) = \varrho_{0,\varepsilon} = \overline{\varrho} + \varepsilon \varrho_{0,\varepsilon}^{(1)}, \\ \mathbf{u}_\varepsilon(0, \cdot) = \mathbf{u}_{0,\varepsilon}, \\ \vartheta_\varepsilon(0, \cdot) = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon \vartheta_{0,\varepsilon}^{(1)}, \end{array} \right\} \tag{7.21}$$

where

$$\left\{ \begin{array}{c} \int_\Omega \varrho^{(1)}_{0,\varepsilon}\ \mathrm{d}x = 0,\ \varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)} \text{ weakly-(*) in } L^\infty(\Omega), \\ \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ weakly-(*) in } L^\infty(\Omega;\mathbb{R}^3), \\ \int_\Omega \vartheta^{(1)}_{0,\varepsilon}\ \mathrm{d}x = 0,\ \vartheta^{(1)}_{0,\varepsilon} \to \vartheta^{(1)}_0 \text{ weakly in } L^\infty(\Omega), \end{array} \right\} \tag{7.22}$$

with positive constants $\overline{\varrho}$, $\overline{\vartheta}$.

For readers' convenience, we recall the list of hypotheses, under which system (7.5–7.15), supplemented with the boundary conditions (7.19), (7.20), and the initial conditions (7.21), possesses a weak solution defined on an arbitrary time interval $(0,T)$. To begin, we need the *hypothesis of thermodynamic stability* (1.44) expressed in terms of the function P as

$$P \in C^1[0,\infty) \cap C^2(0,\infty),\ P(0) = 0,\ P'(Z) > 0 \text{ for all } Z \geq 0, \tag{7.23}$$

$$0 < \frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z} \leq \sup_{z>0} \frac{\frac{5}{3}P(z) - zP'(z)}{z} < \infty, \tag{7.24}$$

together with the coercivity assumption

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{\frac{5}{3}}} = p_\infty > 0. \tag{7.25}$$

Similarly to Chapter 5, the transport coefficients μ, η, and κ are assumed to be continuously differentiable functions of the temperature ϑ satisfying the growth restrictions

$$0 < \underline{\mu}(1+\vartheta) \leq \mu(\vartheta) \leq \overline{\mu}(1+\vartheta) \text{ for all } \vartheta \geq 0,\ \mu' \text{ bounded in } [0,\infty), \tag{7.26}$$

and

$$0 < \underline{\kappa}(1+\vartheta^3) \leq \kappa(\vartheta) \leq \overline{\kappa}(1+\vartheta^3) \text{ for all } \vartheta \geq 0, \tag{7.27}$$

where $\underline{\mu}$, $\overline{\mu}$, $\underline{\kappa}$, and $\overline{\kappa}$ are positive constants.

Now, as a direct consequence of the abstract existence result established in Theorem 3.1, we claim that for any $\varepsilon > 0$, the scaled Navier-Stokes-Fourier system (7.5–7.9), supplemented with the boundary conditions (7.19–7.20), and the initial conditions (7.21), possesses a weak solution $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ on the set $(0,T)\times\Omega$ such that

$$\varrho_\varepsilon \in L^\infty(0,T;L^{\frac{5}{3}}(\Omega)),\ \mathbf{u}_\varepsilon \in L^2(0,T;W^{1,2}_0(\Omega;\mathbb{R}^3)),\ \vartheta_\varepsilon \in L^2(0,T;W^{1,2}(\Omega)).$$

More specifically, we have:

(i) Renormalized equation of continuity:

$$\begin{aligned} \int_0^T \int_\Omega \varrho_\varepsilon B(\varrho_\varepsilon)\Big(\partial_t\varphi + \mathbf{u}_\varepsilon\cdot\nabla_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t \\ = \int_0^T \int_\Omega b(\varrho_\varepsilon)\mathrm{div}_x\mathbf{u}_\varepsilon\varphi\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega \varrho_{0,\varepsilon}B(\varrho_{0,\varepsilon})\varphi(0,\cdot)\ \mathrm{d}x \end{aligned} \tag{7.28}$$

for any b as in (2.3) and any $\varphi \in C^\infty_c([0,T)\times\overline{\Omega})$;

(ii) Momentum equation:

$$\int_0^T \int_\Omega \Big(\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \varphi + \varrho_\varepsilon [\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon] : \nabla_x \varphi + \frac{1}{\varepsilon^2} p(\varrho_\varepsilon, \vartheta_\varepsilon) \mathrm{div}_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t \tag{7.29}$$
$$= \int_0^T \int_\Omega \Big(\mathbb{S}_\varepsilon : \nabla_x \varphi - \frac{1}{\varepsilon} \varrho_\varepsilon \nabla_x F \cdot \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t - \int_\Omega (\varrho_{0,\varepsilon} \mathbf{u}_{0,\varepsilon}) \cdot \varphi \ \mathrm{d}x$$

for any test function

$$\varphi \in C_c^\infty([0,T) \times \Omega; \mathbb{R}^3);$$

(iii) Total energy balance:

$$\int_\Omega \Big(\frac{\varepsilon^2}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon) - \varepsilon \varrho_\varepsilon F \Big)(t) \ \mathrm{d}x \tag{7.30}$$
$$= \int_\Omega \Big(\frac{\varepsilon^2}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + \varrho_{0,\varepsilon} e(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) - \varepsilon \varrho_\varepsilon F \Big) \ \mathrm{d}x \text{ for a.a. } t \in (0,T);$$

(iv) Entropy balance:

$$\int_0^T \int_\Omega \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) \Big(\partial_t \varphi + \mathbf{u}_\varepsilon \cdot \nabla_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t$$
$$+ \int_0^T \int_\Omega \frac{\mathbf{q}_\varepsilon}{\vartheta_\varepsilon} \cdot \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t + \langle \sigma_\varepsilon ; \varphi \rangle_{[\mathcal{M};C]([0,T]\times\overline{\Omega})}$$
$$= - \int_\Omega \varrho_{0,\varepsilon} s(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) \varphi(0, \cdot) \ \mathrm{d}x \tag{7.31}$$

for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$, where $\sigma_\varepsilon \in \mathcal{M}^+([0,T] \times \overline{\Omega})$ satisfies (7.9).

Note that the satisfaction of the no-slip boundary conditions is ensured by the fact that the velocity field $\mathbf{u}_\varepsilon(t,\cdot)$ belongs to the Sobolev space $W_0^{1,2}(\Omega;\mathbb{R}^3)$ defined as a completion of $C_c^\infty(\Omega;\mathbb{R}^3)$ with respect to the $W^{1,2}$-norm. Accordingly, the test functions in the momentum equation (7.29) must be compactly supported in Ω, in particular, the Helmholtz projection $\mathbf{H}[\varphi]$ is no longer an admissible test function in (7.29).

7.2.2 Compactness of the family of velocities

In order to avoid confusion, let us point out that the principal result to be shown in this chapter is *pointwise compactness of the family of velocity fields* $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$. Indeed following step by step the analysis presented in Chapter 5 we can show that the limit system obtained by letting $\varepsilon \to 0$ is the same as in Theorem 5.2, specifically, the Oberbeck-Boussinesq approximation (5.161–5.166).

Thus the main result of this chapter reads as follows:

■ COMPACTNESS OF VELOCITIES ON DOMAINS WITH VARIABLE BOTTOMS:

Theorem 7.1. *Let Ω be the infinite slab introduced in* (7.17), (7.18), *where the "bottom" part of the boundary is given by a function h satisfying*

$$h \in C^3(\mathcal{T}^2),\ |h| < \pi,\ h \not\equiv \text{const}. \tag{7.32}$$

Assume that $\mathbb{S}_\varepsilon$, $\mathbf{q}_\varepsilon$ as well as the thermodynamic functions p, e, and s are given by (7.10–7.15), *where P meets the structural hypotheses* (7.23–7.25), *while the transport coefficients μ and κ satisfy* (7.26), (7.27). *Finally, let $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ be a family of weak solutions to the Navier-Stokes-Fourier system satisfying* (7.28–7.31), *where the initial data are given by* (7.21), (7.22).

Then, at least for a suitable subsequence,

$$\mathbf{u}_\varepsilon \to \mathbf{U} \ \text{in}\ L^2((0,T) \times \Omega; \mathbb{R}^3), \tag{7.33}$$

where $\mathbf{U} \in L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^3))$, $\operatorname{div}_x \mathbf{U} = 0$.

The remaining part of this chapter is devoted to the proof of Theorem 7.1 which is tedious and rather technical. It is based on careful analysis of the singular eigenvalue problem (7.2), (7.3) in a boundary layer by means of the abstract method proposed by Vishik and Ljusternik [198] and later adapted by Desjardins et al. [61] to the low Mach number limit problems in the context of isentropic fluid flows. In contrast with [61], we "save" one degree of approximation – a fact that simplifies considerably the analysis and makes the proof relatively transparent and easily applicable to other choices of boundary conditions (see [83]).

7.3 Uniform estimates

We begin the proof of Theorem 7.1 by recalling the uniform estimates that can be obtained exactly as in Chapter 5. Thus we focus only on the principal ideas, referring to the corresponding parts of Section 5.2 for all technical details.

As the initial distribution of the density is a zero mean perturbation of the constant state $\overline{\varrho}$, we have

$$\int_\Omega \varrho_\varepsilon(t)\ \mathrm{d}x = \int_\Omega \varrho_{0,\varepsilon}\ \mathrm{d}x = \overline{\varrho}|\Omega|,$$

in particular,

$$\int_\Omega (\varrho_\varepsilon(t) - \overline{\varrho})\ \mathrm{d}x = 0 \text{ for all } t \in [0,T]. \tag{7.34}$$

To obtain further estimates, we combine (7.30), (7.31) to deduce the dissipation balance equality in the form

$$\int_\Omega \left[\frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2} \Big(H_{\overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon) - \varepsilon \varrho_\varepsilon F \Big) \right](\tau) \ \mathrm{d}x + \frac{\overline{\vartheta}}{\varepsilon} \sigma_\varepsilon \Big[[0, \tau] \times \overline{\Omega} \Big] \tag{7.35}$$
$$= \int_\Omega \left[\frac{1}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2} \Big(H_{\overline{\vartheta}}(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) - \varepsilon \varrho_\varepsilon F \Big) \right] \ \mathrm{d}x \ \text{for a.a.} \ \tau \in [0, T],$$

where $H_{\overline{\vartheta}}$ is the Helmholtz function introduced in (2.48).

As we have observed in (2.49), (2.50), the hypothesis of thermodynamic stability $\partial_\varrho p > 0$, $\partial_\vartheta e > 0$, expressed in terms of (7.23), (7.24), implies that

$$\varrho \mapsto H_{\overline{\vartheta}}(\varrho, \overline{\vartheta}) \ \text{is a strictly convex function,}$$

while

$$\vartheta \mapsto H_{\overline{\vartheta}}(\varrho, \vartheta) \ \text{attains its strict minimum at} \ \overline{\vartheta} \ \text{for any fixed} \ \varrho.$$

Consequently, subtracting a suitable affine function of ϱ from both sides of (7.35), and using the coercivity properties of $H_{\overline{\vartheta}}$ stated in Lemma 5.1, we deduce the following list of uniform estimates:

- **Energy estimates:**

$$\operatorname*{ess\,sup}_{t \in (0,T)} \| \sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon \|_{L^2(\Omega;\mathbb{R}^3)} \leq c \ [\text{cf. } (5.49)], \tag{7.36}$$

$$\operatorname*{ess\,sup}_{t \in (0,T)} \left\| \left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right]_{\rm ess} \right\|_{L^2(\Omega)} \leq c \ \ [\text{cf. } (5.46)], \tag{7.37}$$

$$\operatorname*{ess\,sup}_{t \in (0,T)} \left\| \left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right]_{\rm res} \right\|_{L^{\frac{5}{3}}(\Omega)} \leq \varepsilon^{\frac{1}{5}} c \ \ [\text{cf. } (5.45), (5.48)], \tag{7.38}$$

$$\operatorname*{ess\,sup}_{t \in (0,T)} \left\| \left[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right]_{\rm ess} \right\|_{L^2(\Omega)} \leq c \ \ [\text{cf. } (5.47)], \tag{7.39}$$

$$\operatorname*{ess\,sup}_{t \in (0,T)} \left\| [\vartheta_\varepsilon]_{\rm res} \right\|_{L^4(\Omega)} \leq \varepsilon^{\frac{1}{2}} c \ \ [\text{cf. } (5.48)], \tag{7.40}$$

$$\operatorname*{ess\,sup}_{t \in (0,T)} \left\| \left[\frac{p(\varrho_\varepsilon, \vartheta_\varepsilon) - p(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \right]_{\rm res} \right\|_{L^1(\Omega)} \leq \varepsilon c \ \ [\text{cf. } (5.45), (5.100)]. \tag{7.41}$$

- **Estimates based on energy dissipation:**

$$\| \sigma_\varepsilon \|_{\mathcal{M}^+([0,T] \times \overline{\Omega})} \leq \varepsilon^2 c \ \ [\text{cf. } (5.50)], \tag{7.42}$$

$$\int_0^T \| \mathbf{u}_\varepsilon \|^2_{W_0^{1,2}(\Omega;\mathbb{R}^3)} \ \mathrm{d}t \leq c \ \ [\text{cf. } (5.51)], \tag{7.43}$$

$$\int_0^T \left\| \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right\|^2_{W^{1,2}(\Omega)} \ \mathrm{d}t \leq c \ \ [\text{cf. } (5.52)], \tag{7.44}$$

$$\int_0^T \left\| \frac{\log(\vartheta_\varepsilon) - \log(\overline{\vartheta})}{\varepsilon} \right\|^2_{W^{1,2}(\Omega)} \ \mathrm{d}t \leq c \ \ [\text{cf. } (5.53)]. \tag{7.45}$$

- **Entropy estimates:**

$$\operatorname*{ess\,sup}_{t\in(0,T)} \left\| \left[\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon} \right]_{\rm res} \right\|_{L^1(\Omega)} {\rm d}t \leq \varepsilon c \quad \text{[cf. (5.44)]}, \tag{7.46}$$

$$\int_0^T \left\| \left[\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon} \right]_{\rm res} \right\|_{L^q(\Omega)}^q {\rm d}t \leq c \text{ for a certain } q > 1 \quad \text{[cf. (5.54)]}, \tag{7.47}$$

$$\int_0^T \left\| \left[\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon} \mathbf{u}_\varepsilon \right]_{\rm res} \right\|_{L^q(\Omega;\mathbb{R}^3)}^q {\rm d}t \leq c \text{ for a certain } q > 1 \quad \text{[cf. (5.55)]}, \tag{7.48}$$

$$\int_0^T \left\| \left[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \right]_{\rm res} \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon} \right\|_{L^q(\Omega;\mathbb{R}^3)}^q {\rm d}t \to 0 \text{ for a certain } q > 1 \quad \text{[cf. (5.56)]}. \tag{7.49}$$

Let us recall that the "essential" component $[h]_{\rm ess}$ of a function h and its "residual" counterpart $[h]_{\rm res}$ have been introduced in (4.44), (4.45).

We conclude with the estimate on the "measure of the residual set" established in (5.46), specifically,

$$\operatorname*{ess\,sup}_{t\in(0,T)} |\mathcal{M}^\varepsilon_{\rm res}[t]| \leq \varepsilon^2 c, \tag{7.50}$$

with $\mathcal{M}^\varepsilon_{\rm res}[t] \subset \Omega$ introduced in (4.43).

7.4 Analysis of acoustic waves

7.4.1 Acoustic equation

The acoustic equation governing the time oscillations of the gradient part of the velocity field is essentially the same as in Chapter 5. However, a refined analysis to be performed below requires a more elaborate description of the "small" terms as well as the knowledge of the precise rate of convergence of these quantities toward zero.

We start rewriting the equation of continuity (7.5) in the form

$$\int_0^T \int_\Omega \left(\varepsilon \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \partial_t \varphi + \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \varphi \right) {\rm d}x \ {\rm d}t = - \int_\Omega \varepsilon \frac{\varrho_{0,\varepsilon} - \overline{\varrho}}{\varepsilon} \ {\rm d}x \tag{7.51}$$

for any $\varphi \in C^\infty_c([0,T) \times \overline{\Omega})$.

Similarly, the momentum equation (7.29) can be written as

$$
\begin{aligned}
&\int_0^T \int_\Omega \varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \varphi \ \mathrm{d}x \ \mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \Big(\frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big]_{\mathrm{ess}} + \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \Big[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \Big]_{\mathrm{ess}} - \overline{\varrho} F \Big) \mathrm{div}_x \varphi \ \mathrm{d}x \ \mathrm{d}t \\
&\quad - \int_0^T \int_\Omega \varepsilon \mathbb{S}_\varepsilon : \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t \\
&= -\varepsilon \int_\Omega \varrho_{0,\varepsilon} \mathbf{u}_{0,\varepsilon} \cdot \varphi \ \mathrm{d}x + \varepsilon \int_0^T \int_\Omega \mathbb{G}_1^\varepsilon : \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t + \varepsilon \int_0^T \int_\Omega \mathbf{G}_\varepsilon^2 \cdot \varphi \ \mathrm{d}x \ \mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \Big(G_\varepsilon^3 + G_\varepsilon^4 \Big) \mathrm{div}_x \varphi \ \mathrm{d}x \ \mathrm{d}t,
\end{aligned}
\tag{7.52}
$$

for any $\varphi \in C_c^\infty([0,T) \times \Omega; \mathbb{R}^3)$, where we have set

$$
\mathbb{G}_\varepsilon^1 = -\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon, \ \ \mathbf{G}_\varepsilon^2 = \frac{\overline{\varrho} - \varrho_\varepsilon}{\varepsilon} \nabla_x F, \tag{7.53}
$$

$$
G_\varepsilon^3 = -\frac{[p(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\mathrm{res}}}{\varepsilon}, \tag{7.54}
$$

and

$$
G_\varepsilon^4 = \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big]_{\mathrm{ess}} + \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \Big[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \Big]_{\mathrm{ess}} - \Big(\frac{[p(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\mathrm{ess}} - p(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \Big). \tag{7.55}
$$

It is important to notice that validity of (7.52) can be extended to the class of test functions satisfying

$$
\varphi \in C_c^\infty([0,T] \times \overline{\Omega}; \mathbb{R}^3), \ \ \varphi|_{\partial\Omega} = 0 \tag{7.56}
$$

by means of a simple density argument. Indeed, in accordance with the integrability properties of the weak solutions established in Theorem 3.2, it is enough to use the density of $C_c^\infty(\Omega)$ in $W_0^{1,p}(\Omega)$ for any finite p.

Since $\mathbf{u}_\varepsilon \in L^2(0,T; W_0^{1,2}(\Omega; \mathbb{R}^3))$, in particular, the trace of $\mathbf{u}_\varepsilon$ vanishes on the boundary, we are allowed to use the Gauss-Green theorem to obtain

$$
\begin{aligned}
&\int_0^T \int_\Omega \varepsilon \mathbb{S}_\varepsilon : \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t \\
&= -\varepsilon \int_0^T \int_\Omega \frac{2\mu(\overline{\vartheta})}{\overline{\varrho}} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathrm{div}_x [[\nabla_x \varphi]] \ \mathrm{d}x \ \mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \frac{2\varepsilon\mu(\overline{\vartheta})}{\overline{\varrho}} (\varrho_\varepsilon - \overline{\varrho}) \mathbf{u}_\varepsilon \cdot \mathrm{div}_x [[\nabla_x \varphi]] \ \mathrm{d}x \ \mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \varepsilon \Big(\mu(\vartheta_\varepsilon) - \mu(\overline{\vartheta}) \Big) \Big(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^\perp \mathbf{u}_\varepsilon - \frac{2}{3} \mathrm{div}_x \mathbf{u}_\varepsilon \mathbb{I} \Big) : \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t
\end{aligned}
\tag{7.57}
$$

for any φ as in (7.56), where we have introduced the notation

$$[[\mathbb{M}]] = \frac{1}{2}\Big[\mathbb{M} + \mathbb{M}^T - \frac{2}{3}\text{trace}[\mathbb{M}]\ \mathbb{I}\Big].$$

In a similar fashion, the entropy balance (7.31) can be rewritten as

$$\int_0^T \int_\Omega \varepsilon\Big(\frac{\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon) - \varrho_\varepsilon s(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\Big)\partial_t\varphi \ \mathrm{d}x\ \mathrm{d}t \tag{7.58}$$
$$= - \int_\Omega \varepsilon\Big(\frac{\varrho_{0,\varepsilon} s(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon}) - \varrho_{0,\varepsilon} s(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\Big)\varphi(0,\cdot)\ \mathrm{d}x - \langle \sigma_\varepsilon;\varphi\rangle_{[\mathcal{M};C]([0,T]\times\overline{\Omega})}$$
$$+ \int_0^T \int_\Omega \Big(\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon}\nabla_x\vartheta_\varepsilon + \Big(\varrho_\varepsilon s(\overline{\varrho},\overline{\vartheta}) - \varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)\Big)\mathbf{u}_\varepsilon\Big)\cdot\nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t$$

for any $\varphi \in C_c^\infty([0,T)\times\overline{\Omega})$.

Summing up relations (7.51–7.58) we obtain, exactly as in Section 5.4.3, a linear hyperbolic equation describing the propagation of acoustic waves.

■ ACOUSTIC EQUATION:

$$\int_0^T \int_\Omega \Big(\varepsilon r_\varepsilon \partial_t\varphi + \mathbf{V}_\varepsilon\cdot\nabla_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{7.59}$$
$$= - \int_\Omega \varepsilon r_{0,\varepsilon}\varphi(0,\cdot)\ \mathrm{d}x + \frac{A}{\omega}\Big(\int_0^T\int_\Omega \mathbf{G}_5^\varepsilon\cdot\nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t - \langle\sigma_\varepsilon,\varphi\rangle\Big)$$

for any $\varphi \in C_c^\infty([0,T)\times\overline{\Omega})$,

$$\int_0^T \int_\Omega \Big(\varepsilon\mathbf{V}_\varepsilon\cdot\partial_t\varphi + \omega r_\varepsilon \mathrm{div}_x\varphi + \varepsilon D\mathbf{V}_\varepsilon\cdot\mathrm{div}_x[[\nabla_x\varphi]]\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{7.60}$$
$$= - \int_\Omega \varepsilon\mathbf{V}_{0,\varepsilon}\cdot\varphi(0,\cdot)\ \mathrm{d}x$$
$$+ \int_0^T \int_\Omega \Big(\mathbf{G}_6^\varepsilon\cdot\mathrm{div}_x[[\nabla_x\varphi]] + \mathbb{G}_7^\varepsilon : \nabla_x\varphi + G_8^\varepsilon \mathrm{div}_x\varphi + \mathbf{G}_9^\varepsilon\cdot\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t$$

for any $\varphi \in C_c^\infty([0,T)\times\mathbb{R}^3;\mathbb{R}^3)$, $\varphi|_{\partial\Omega} = 0$,

where we have set

$$r_\varepsilon = \frac{1}{\omega}\Big(\omega\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} + A\varrho_\varepsilon\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon} - \overline{\varrho}F\Big), \mathbf{V}_\varepsilon = \varrho_\varepsilon\mathbf{u}_\varepsilon, \tag{7.61}$$

$$r_{0,\varepsilon} = \frac{1}{\omega}\Big(\omega\frac{\varrho_{0,\varepsilon} - \overline{\varrho}}{\varepsilon} + A\varrho_{0,\varepsilon}\frac{s(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon}) - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon} - \overline{\varrho}F\Big), \mathbf{V}_{0,\varepsilon} = \varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon}, \tag{7.62}$$

with

$$\omega = \partial_\varrho p(\overline{\varrho}, \overline{\vartheta}) + \frac{|\partial_\vartheta p(\overline{\varrho}, \overline{\vartheta})|^2}{\overline{\varrho}^2 \, \partial_\vartheta s(\overline{\varrho}, \overline{\vartheta})}, \ A = \frac{\partial_\vartheta p(\overline{\varrho}, \overline{\vartheta})}{\overline{\varrho} \, \partial_\vartheta s(\overline{\varrho}, \overline{\vartheta})}, \ D = \frac{2\mu(\overline{\vartheta})}{\overline{\varrho}}. \tag{7.63}$$

Note that the integral identities (7.59), (7.60) represent a weak formulation of equation (7.1), where the "small" terms read as follows:

$$\mathbf{G}_5^\varepsilon = \frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \nabla_x \vartheta_\varepsilon + \Big(\varrho_\varepsilon s(\overline{\varrho}, \overline{\vartheta}) - \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) \Big) \mathbf{u}_\varepsilon, \tag{7.64}$$

$$\mathbf{G}_6^\varepsilon = \varepsilon D(\varrho_\varepsilon - \overline{\varrho}) \mathbf{u}_\varepsilon, \tag{7.65}$$

$$\mathbb{G}_7^\varepsilon = 2\varepsilon(\mu(\vartheta_\varepsilon) - \mu(\overline{\vartheta}))[[\nabla_x \mathbf{u}_\varepsilon]] - \varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon, \tag{7.66}$$

$$\begin{aligned} G_8^\varepsilon = {} & A\varrho_\varepsilon \Big[\frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \Big]_{\rm res} - \Big[\frac{p(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon} \Big]_{\rm res} \\ & + A \Big\{ \Big[\varrho_\varepsilon \frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \Big]_{\rm ess} \\ & - \overline{\varrho} \Big(\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big]_{\rm ess} + \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \Big[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \Big]_{\rm ess} \Big) \Big\} \\ & - \Big\{ \frac{[p(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\rm ess} - p(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} - \Big(\frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big]_{\rm ess} \\ & + \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \Big[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \Big]_{\rm ess} \Big) \Big\} + \omega \Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big]_{\rm res}, \end{aligned} \tag{7.67}$$

and

$$\mathbf{G}_9^\varepsilon = (\overline{\varrho} - \varrho_\varepsilon) \nabla_x F. \tag{7.68}$$

7.4.2 Spectral analysis of the acoustic operator

In this part, we are concerned with the spectral analysis of the linear operator associated to problem (7.59), (7.60), namely we examine the differential operator

$$\left[\begin{array}{c} v \\ \mathbf{w} \end{array} \right] \mapsto \mathcal{A} \left[\begin{array}{c} v \\ \mathbf{w} \end{array} \right] + \varepsilon \mathcal{B} \left[\begin{array}{c} v \\ \mathbf{w} \end{array} \right], \tag{7.69}$$

with

$$\mathcal{A} \left[\begin{array}{c} v \\ \mathbf{w} \end{array} \right] = \left[\begin{array}{c} \omega {\rm div}_x \mathbf{w} \\ \nabla_x v \end{array} \right], \ \mathcal{B} \left[\begin{array}{c} v \\ \mathbf{w} \end{array} \right] = \left[\begin{array}{c} 0 \\ D {\rm div}_x [[\nabla_x \mathbf{w}]] \end{array} \right]$$

that can be viewed as the formal adjoint of the generator in (7.59), (7.60). In accordance with (7.19), we impose the homogeneous Dirichlet boundary condition for $\mathbf{w}$,

$$\mathbf{w}|_{\partial\Omega} = 0. \tag{7.70}$$

Let us start with the limit eigenvalue problem

$$\mathcal{A}\left[\begin{array}{c} v \\ \mathbf{w} \end{array}\right] = \lambda \left[\begin{array}{c} v \\ \mathbf{w} \end{array}\right], \text{ meaning } \left\{\begin{array}{c} \omega \mathrm{div}_x \mathbf{w} = \lambda v, \\ \nabla_x v = \lambda \mathbf{w} \end{array}\right\} \tag{7.71}$$

which can be *equivalently* reformulated as

$$-\Delta_x v = \Lambda v, \ \Lambda = -\frac{\lambda^2}{\omega}, \tag{7.72}$$

where the boundary condition (7.70) transforms to $\nabla_x v|_{\partial\Omega} = 0$, in particular,

$$\mathbf{w} \cdot \mathbf{n}|_{\partial\Omega} = \nabla_x v \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{7.73}$$

Note that the null space (kernel) of $\mathcal{A}$ is

$$\begin{aligned} \mathrm{Ker}[\mathcal{A}] &= \left[\begin{array}{c} \mathrm{span}\{1\} \\ L^2_\sigma(\Omega; \mathbb{R}^3) \end{array}\right] \qquad (7.74) \\ &= \{(v, \mathbf{w}) \mid v = \mathrm{const}, \ \mathbf{w} \in L^2(\Omega; \mathbb{R}^3), \ \mathrm{div}_x \mathbf{w} = 0, \ \mathbf{w} \cdot \mathbf{n}|_{\partial\Omega} = 0\}. \end{aligned}$$

As is well known, the Neumann problem (7.72), (7.73) admits a countable set of real eigenvalues $\{\Lambda_n\}_{n=0}^\infty$,

$$0 = \Lambda_0 < \Lambda_1 < \Lambda_2 \cdots ,$$

where the associated family of real eigenfunctions $\{v_{n,m}\}_{n=0,m=1}^{\infty,m_n}$ forms an orthonormal basis of the Hilbert space $L^2(\Omega)$. Moreover, we denote by

$$E_n = \mathrm{span}\{v_{n,m}\}_{m=1}^{m_n}, \ n = 0, 1, \ldots$$

the eigenspace corresponding to the eigenvalue Λ_n of multiplicity m_n. In particular, $m_0 = 1$, $E_0 = \mathrm{span}\{1\}$ (see Theorem 10.7 in Appendix).

Under hypothesis (7.32), Proposition 7.1 implies that $v_0 = 1/\sqrt{|\Omega|}$ is the only eigenfunction that satisfies the supplementary boundary condition $\nabla_x v_0|_{\partial\Omega} = 0$. Thus the term $\varepsilon\mathcal{B}$, together with (7.70), may be viewed as a *singular perturbation* of the operator $\mathcal{A}$.

Accordingly, the eigenvalue problem (7.71), (7.73) admits a system of eigenvalues

$$\lambda_{\pm n} = \pm \mathrm{i}\sqrt{\omega \Lambda_n}, \ n = 0, 1, \ldots$$

lying on the imaginary axis. The associated eigenspaces are

$$\left\{\begin{array}{c} \mathrm{span}\{1\} \times L^2_\sigma(\Omega; \mathbb{R}^3) \text{ for } n = 0, \\ \mathrm{span}\left\{(v_{n,m}, \mathbf{w}_{\pm n,m}) = \frac{1}{\lambda_{\pm n}} \nabla_x v_{n,m}\right\}_{m=1}^{m_n} \text{ for } n = 1, 2, \ldots . \end{array}\right\}$$

In the remaining part of this chapter, we fix $n > 0$ and set

$$\lambda = \lambda_n = \mathrm{i}\sqrt{\omega \Lambda_n},\ v = v_{n,1},\ \mathbf{w} = \mathbf{w}_{n,1} = \frac{1}{\lambda_n} \nabla_x v_{n,1}, \tag{7.75}$$

together with

$$E = E_n = \mathrm{span}\{v_{(1)}, \dots, v_{(m)}\},\ v_{(j)} = v_{n,j},\ m = m_n. \tag{7.76}$$

In order to match the incompatibility of the boundary conditions (7.70), (7.73), we look for "approximate" eigenfunctions of the perturbed problem (7.80), (7.82) in the form

$$v_\varepsilon = (v^{\mathrm{int},0} + v^{\mathrm{bl},0}) + \sqrt{\varepsilon}(v^{\mathrm{int},1} + v^{\mathrm{bl},1}), \tag{7.77}$$

$$\mathbf{w}_\varepsilon = (\mathbf{w}^{\mathrm{int},0} + \mathbf{w}^{\mathrm{bl},0}) + \sqrt{\varepsilon}(\mathbf{w}^{\mathrm{int},1} + \mathbf{w}^{\mathrm{bl},1}), \tag{7.78}$$

where we set

$$v^{\mathrm{int},0} = v,\ \mathbf{w}^{\mathrm{int},0} = \mathbf{w}. \tag{7.79}$$

The functions v_ε, $\mathbf{w}_\varepsilon$ are determined as solutions to the following approximate problem.

■ Approximate Eigenvalue Problem:

$$\mathcal{A}\begin{bmatrix} v_\varepsilon \\ \mathbf{w}_\varepsilon \end{bmatrix} + \varepsilon \mathcal{B} \begin{bmatrix} v_\varepsilon \\ \mathbf{w}_\varepsilon \end{bmatrix} = \lambda_\varepsilon \begin{bmatrix} v_\varepsilon \\ \mathbf{w}_\varepsilon \end{bmatrix} + \sqrt{\varepsilon} \begin{bmatrix} s_\varepsilon^1 \\ \mathbf{s}_\varepsilon^2 \end{bmatrix},$$

meaning,

$$\left\{ \begin{array}{c} \omega \mathrm{div}_x \mathbf{w}_\varepsilon = \lambda_\varepsilon v_\varepsilon + \sqrt{\varepsilon} s_\varepsilon^1, \\ \nabla_x v_\varepsilon + \varepsilon D \mathrm{div}_x[[\nabla_x \mathbf{w}_\varepsilon]] = \lambda_\varepsilon \mathbf{w}_\varepsilon + \sqrt{\varepsilon} \mathbf{s}_\varepsilon^2, \end{array} \right\} \tag{7.80}$$

where

$$\lambda_\varepsilon = \lambda^0 + \sqrt{\varepsilon}\lambda^1,\ \text{with } \lambda^0 = \lambda, \tag{7.81}$$

supplemented with the homogeneous Dirichlet boundary condition

$$\mathbf{w}_\varepsilon|_{\partial\Omega} = 0. \tag{7.82}$$

There is a vast amount of literature, in particular in applied mathematics, devoted to formal asymptotic analysis of singularly perturbed problems based on the so-called WKB (Wentzel-Kramers-Brilbuin) expansions for boundary layers similar to (7.77), (7.78). An excellent introduction to the mathematical aspects of the theory is the book by Métivier [155]. The "interior" functions $v^{\mathrm{int},k} = v^{\mathrm{int},k}(x)$, $\mathbf{w}^{\mathrm{int},k} = \mathbf{w}^{\mathrm{int},k}(x)$ depend only on $x \in \Omega$, while the "boundary layer" functions

$v^{\mathrm{bl},k}(x,Z) = v^{\mathrm{bl},k}(x,Z)$, $\mathbf{w}^{\mathrm{bl},k} = \mathbf{w}^{\mathrm{bl},k}(x,Z)$ depend on x and the fast variable $Z = d(x)/\sqrt{\varepsilon}$, where d is a generalized distance function to $\partial\Omega$,

$$d \in C^3(\overline{\Omega}),\ d(x) = \begin{cases} \mathrm{dist}[x,\partial\Omega] \text{ for all } x \in \overline{\Omega} \text{ such that } \mathrm{dist}[x,\partial\Omega] \le \delta, \\ d(x) \ge \delta \text{ otherwise.} \end{cases} \tag{7.83}$$

Note that the distance function enjoys the same regularity as the boundary $\partial\Omega$, namely as the function h appearing in hypothesis (7.32).

The rest of this section is devoted to identifying all terms in the asymptotic expansions (7.77), (7.78), the remainders s^1_ε, $\mathbf{s}^2_\varepsilon$, and the value of λ^1. In accordance with the heuristic arguments in the introductory part of this chapter, we expect to recover $\lambda^1 \neq 0$, specifically, $\mathrm{Re}[\lambda^1] < 0$ yielding the desired exponential decay rate of order $\sqrt{\varepsilon}$ (no contradiction with the sign of $\mathrm{Re}[\lambda_\varepsilon]$ in the introductory section as the elliptic part of problem (7.80–7.82) has negative spectrum!). This rather tedious task is accomplished in several steps.

Differential operators applied to the boundary layer correction functions. To avoid confusion, we shall write $\nabla_x \mathbf{w}^{\mathrm{bl},k}(x, d(x)/\sqrt{\varepsilon})$ for the gradient of the composed function $x \mapsto \mathbf{w}^{\mathrm{bl},k}(x, d(x)/\sqrt{\varepsilon})$, while $\nabla_x \mathbf{w}^{\mathrm{bl},k}(x,Z)$, $\partial_Z \mathbf{w}^{\mathrm{bl},k}(x,Z)$ stand for the differential operators applied to a function of two variables x and Z. It is a routine matter to compute:

$$[[\nabla_x \mathbf{w}^{\mathrm{bl},k}(x, d(x)/\sqrt{\varepsilon})]] = [[\nabla_x \mathbf{w}^{\mathrm{bl},k}(x,Z)]]$$
$$+ \frac{1}{2\sqrt{\varepsilon}} \left[\partial_Z \mathbf{w}^{\mathrm{bl},k}(x,Z) \otimes \nabla_x d + \nabla_x d \otimes \partial_Z \mathbf{w}^{\mathrm{bl},k}(x,Z) - \frac{2}{3} \partial_Z \mathbf{w}^{\mathrm{bl},k}(x,Z) \cdot \nabla_x d\, \mathbb{I} \right].$$

Similarly, we get

$$\mathrm{div}_x [\mathbf{w}^{\mathrm{bl},k}(x, d(x)/\sqrt{\varepsilon})] = \mathrm{div}_x \mathbf{w}^{\mathrm{bl},k}(x,Z) + \frac{1}{\sqrt{\varepsilon}} \partial_Z \mathbf{w}^{\mathrm{bl},k}(x,Z) \cdot \nabla_x d(x),$$
$$\nabla_x [v^{\mathrm{bl},k}(x, d(x)/\sqrt{\varepsilon})] = \nabla_x v^{\mathrm{bl},k}(x,Z) + \frac{1}{\sqrt{\varepsilon}} \partial_Z v^{\mathrm{bl},k}(x,Z) \nabla_x d(x),$$

and

$$\mathrm{div}_x [[\nabla_x \mathbf{w}^{\mathrm{bl},k}(x, d(x)/\sqrt{\varepsilon})]] = \mathrm{div}_x [[\nabla_x \mathbf{w}^{\mathrm{bl},k}(x,Z)]]$$
$$+ \frac{1}{\sqrt{\varepsilon}} \Big\{ [\partial_Z \nabla_x \mathbf{w}^{\mathrm{bl},k}(x,Z)] \nabla_x d(x) + \frac{1}{6} (\partial_Z \mathrm{div}_x \mathbf{w}^{\mathrm{bl},k}(x,Z)) \nabla_x d(x)$$
$$+ \frac{1}{6} [\partial_Z \nabla_x^T \mathbf{w}^{\mathrm{bl},k}(x,Z)] \nabla_x d(x)$$
$$+ \frac{1}{2} \partial_Z \mathbf{w}^{\mathrm{bl},k}(x,Z) \Delta_x d(x) + \frac{1}{6} [\nabla_x^2 d(x)] \partial_Z \mathbf{w}^{\mathrm{bl},k}(x,Z) \Big\}$$
$$+ \frac{1}{2\varepsilon} \Big\{ \partial_Z^2 \mathbf{w}^{\mathrm{bl},k}(x,Z) |\nabla_x d(x)|^2 + \frac{1}{3} \partial_Z^2 \mathbf{w}^{\mathrm{bl},k}(x,Z) \cdot \nabla_x d(x) \nabla_x d(x) \Big\}$$

for $k = 0, 1$, where Z stands for $d(x)/\sqrt{\varepsilon}$.

Consequently, substituting ansatz (7.77), (7.78) in (7.80), (7.81), we arrive at the following system of equations:

$$\omega \mathrm{div}_x \mathbf{w}^{\mathrm{int},1}(x) = \lambda^0 v^{\mathrm{int},1}(x) + \lambda^1 v^{\mathrm{int},0}(x), \tag{7.84}$$

$$\nabla_x v^{\mathrm{int},1}(x) = \lambda^0 \mathbf{w}^{\mathrm{int},1}(x) + \lambda^1 \mathbf{w}^{\mathrm{int},0}(x), \tag{7.85}$$

$$\partial_Z \mathbf{w}^{\mathrm{bl},0}(x,Z) \cdot \nabla_x d(x) = 0, \tag{7.86}$$

$$\omega \left(\mathrm{div}_x \mathbf{w}^{\mathrm{bl},0}(x,Z) + \partial_Z \mathbf{w}^{\mathrm{bl},1}(x,Z) \cdot \nabla_x d(x)\right) = \lambda^0 v^{\mathrm{bl},0}(x,Z), \tag{7.87}$$

$$\partial_Z v^{\mathrm{bl},0}(x,Z) \nabla_x d(x) = 0, \tag{7.88}$$

and

$$\begin{aligned}&\left(\nabla_x v^{\mathrm{bl},0}(x,Z) + \partial_Z v^{\mathrm{bl},1}(x,Z)\nabla_x d(x)\right)\\ &\quad + \frac{D}{2}\left(\partial_Z^2 \mathbf{w}^{\mathrm{bl},0}(x,Z)|\nabla_x d(x)|^2 + \frac{1}{3}\partial_Z^2 \mathbf{w}^{\mathrm{bl},0}(x,Z) \cdot \nabla_x d(x) \nabla_x d(x)\right)\\ &= \lambda^0 \mathbf{w}^{\mathrm{bl},0}(x,Z).\end{aligned} \tag{7.89}$$

Moreover, the remainders s_ε^1, $\mathbf{s}_\varepsilon^2$ are determined by means of (7.80) as

$$\begin{aligned}s_\varepsilon^1 &= \mathrm{div}_x(\mathbf{w}^{\mathrm{bl},1}(x,Z)) - \lambda^0 v^{\mathrm{bl},1}(x,Z)\\ &\quad - \lambda^1 v^{\mathrm{bl},0}(x,Z) - \sqrt{\varepsilon}\lambda^1\left(v^{\mathrm{int},1}(x) + v^{\mathrm{bl},1}(x,Z)\right),\end{aligned} \tag{7.90}$$

$$\begin{aligned}\mathbf{s}_\varepsilon^2 &= D\Big\{[\partial_Z \nabla_x \mathbf{w}^{\mathrm{bl},0}(x,Z)]\nabla_x d(x) + \frac{1}{6}[\partial_Z \mathrm{div}_x \mathbf{w}^{\mathrm{bl},0}(x,Z)]\nabla_x d(x)\\ &\quad + \frac{1}{6}[\partial_Z \nabla_x^T \mathbf{w}^{\mathrm{bl},0}(x,Z)]\nabla_x d(x) + \frac{1}{2}\partial_Z \mathbf{w}^{\mathrm{bl},0}(x,Z)\Delta_x d(x)\\ &\quad + \frac{1}{6}[\nabla_x^2 d(x)]\partial_Z \mathbf{w}^{\mathrm{bl},0}(x,Z) + \frac{1}{2}\partial_Z^2 \mathbf{w}^{\mathrm{bl},1}(x,Z)|\nabla_x d(x)|^2\\ &\quad + \frac{1}{6}\partial_Z^2 \mathbf{w}^{\mathrm{bl},1}(x,Z) \cdot \nabla_x d(x)\nabla_x d(x)\Big\}\\ &\quad + \nabla_x v^{\mathrm{bl},1}(x,Z) - \lambda^0 \mathbf{w}^{\mathrm{bl},1}(x,Z) - \lambda^1 \mathbf{w}^{\mathrm{bl},0}(x,Z)\\ &\quad + \sqrt{\varepsilon}\Big\{D\Big(\mathrm{div}_x[[\nabla_x \mathbf{w}^{\mathrm{int},0}(x)]] + \mathrm{div}_x[[\nabla_x \mathbf{w}^{\mathrm{bl},0}(x,Z)]]\\ &\quad + [\partial_Z \nabla_x \mathbf{w}^{\mathrm{bl},1}(x,Z)]\nabla_x d(x) + \frac{1}{6}[\partial_Z \mathrm{div}_x \mathbf{w}^{\mathrm{bl},1}(x,Z)]\nabla_x d(x)\\ &\quad + \frac{1}{6}[\partial_Z \nabla_x^T \mathbf{w}^{\mathrm{bl},1}(x,Z)]\nabla_x d(x)\\ &\quad + \frac{1}{2}\partial_Z \mathbf{w}^{\mathrm{bl},1}(x,Z)\Delta_x d(x) + \frac{1}{6}[\nabla_x^2 d(x)]\partial_Z \mathbf{w}^{\mathrm{bl},1}(x,Z)\Big)\\ &\quad - \lambda^1 \mathbf{w}^{\mathrm{int},1}(x) - \lambda^1 \mathbf{w}^{\mathrm{bl},1}(x,Z)\Big\}\\ &\quad + \varepsilon\Big\{\mathrm{div}_x[[\nabla_x \mathbf{w}^{\mathrm{int},1}(x)]] + \mathrm{div}_x[[\nabla_x \mathbf{w}^{\mathrm{bl},1}(x,Z)]]\Big\},\end{aligned} \tag{7.91}$$

where $Z = d(x)/\sqrt{\varepsilon}$.

Finally, in agreement with (7.82), we require

$$\mathbf{w}^{\mathrm{bl},k}(x,0) + \mathbf{w}^{\mathrm{int},k}(x,0) = 0 \text{ for all } x \in \partial\Omega,\ k = 0,1. \tag{7.92}$$

Determining the zeroth order terms. System (7.84–7.89) consists of six equations for the unknowns $v^{\mathrm{bl},0}$, $\mathbf{w}^{\mathrm{bl},0}$, $v^{\mathrm{int},1}$, $\mathbf{w}^{\mathrm{int},1}$, and $v^{\mathrm{bl},1}$, $\mathbf{w}^{\mathrm{bl},1}$. Note that, in agreement with (7.79),

$$\begin{gathered}\omega \mathrm{div}_x \mathbf{w}^{\mathrm{int},0} = \lambda^0 v^{\mathrm{int},0},\ \lambda^0 \mathbf{w}^{\mathrm{int},0} = \nabla v^{\mathrm{int},0},\\ \mathbf{w}^{\mathrm{int},0}\cdot \mathbf{n}|_{\partial\Omega} = \nabla_x v^{\mathrm{int},0}\cdot \mathbf{n}|_{\partial\Omega} = 0.\end{gathered} \tag{7.93}$$

Moreover, since the matrix $\{\int_{\partial\Omega} \nabla_x v_{(i)} \cdot \nabla_x \mathbf{v}_{(j)} \ \mathrm{d}S_x\}_{i,j=1}^m$ is diagonalizable, the basis $\{v_{(1)},\dots,v_{(m)}\}$ of the eigenspace E introduced in (7.75), (7.76) may be chosen in such a way that

$$\int_\Omega v_{(i)} v_{(j)} \ \mathrm{d}x = \delta_{i,j},\ \int_{\partial\Omega} \nabla_x v_{(i)} \cdot \nabla_x v_{(j)} \ \mathrm{d}S_x = 0 \text{ for } i \neq j, \tag{7.94}$$

where $v^{\mathrm{int},0} = v_{(1)}$.

Since there are no boundary conditions imposed on the component v, we can take

$$v^{\mathrm{bl},0}(x,Z) \equiv v^{\mathrm{bl},1}(x,Z) \equiv 0, \tag{7.95}$$

in particular, equation (7.88) holds.

Furthermore, equation (7.86) requires the quantity $\mathbf{w}^{\mathrm{bl},0}(x,Z)\cdot\nabla_x d(x)$ to be independent of Z. On the other hand, by virtue of (7.73), (7.92), the function $x \mapsto \mathbf{w}^{\mathrm{bl},0}(x, d(x)/\sqrt{\varepsilon})$ must have zero normal trace on $\partial\Omega$. Since $d(x) = 0$, $\nabla_x d(x) = -\mathbf{n}(x)$ for any $x \in \partial\Omega$, we have to take

$$\mathbf{w}^{\mathrm{bl},0}(x,Z)\cdot \nabla d(x) = 0 \text{ for all } x \in \overline{\Omega}, Z \geq 0. \tag{7.96}$$

Consequently, equation (7.89) reduces to

$$\frac{D}{2}\partial_Z^2 \mathbf{w}^{\mathrm{bl},0}(x,Z)|\nabla_x d(x)|^2 = \lambda^0 \mathbf{w}^{\mathrm{bl},0}(x,Z) \text{ to be satisfied for } Z>0. \tag{7.97}$$

For a fixed $x \in \overline{\Omega}$, relation (7.97) represents an ordinary differential equation of second order in Z, for which the initial conditions $\mathbf{w}^{\mathrm{bl},0}(x,0)$ are uniquely determined by (7.92), namely

$$\mathbf{w}^{\mathrm{bl},0}(x,0) = -\mathbf{w}^{\mathrm{int},0}(x) \text{ for all } x \in \partial\Omega. \tag{7.98}$$

It is easy to check that problem (7.97), (7.98) admits a unique solution that *decays to zero* for $Z \to \infty$, specifically,

$$\mathbf{w}^{\mathrm{bl},0}(x,Z) = -\chi(d(x))\mathbf{w}^{\mathrm{int},0}\left(x - d(x)\nabla_x d(x)\right)\exp(-\Gamma Z), \tag{7.99}$$

where $\chi \in C^\infty[0,\infty)$,

$$\chi(d) = \begin{cases} 1 \text{ for } d \in [0, \delta/2], \\ 0 \text{ if } d > \delta, \end{cases} \tag{7.100}$$

and

$$\Gamma^2 = \frac{2\lambda^0}{D}, \ \mathrm{Re}[\Gamma] > 0. \tag{7.101}$$

It seems worth noting that formula (7.99) is compatible with (7.96) as for $x \in \Omega$ the point $x - \nabla_x d(x)/d(x)$ is the nearest to x on $\partial\Omega$ as soon as $d(x)$ coincides with $\mathrm{dist}[x, \partial\Omega]$.

First order terms. Equation (7.87), together with the ansatz made in (7.95), give rise to

$$\partial_Z \Big(\mathbf{w}^{\mathrm{bl},1}(x, Z) \cdot \nabla_x d(x) \Big) = -\mathrm{div}_x(\mathbf{w}^{\mathrm{bl},0}(x, Z)). \tag{7.102}$$

In view of (7.99), equation (7.102) admits a unique solution with *exponential decay* for $Z \to \infty$ for any fixed $x \in \overline{\Omega}$, namely

$$\mathbf{w}^{\mathrm{bl},1}(x, Z) \cdot \nabla_x d(x) = \frac{1}{\Gamma} \mathrm{div}_x(\mathbf{w}^{\mathrm{bl},0}(x, Z)).$$

Thus we can set

$$\mathbf{w}^{\mathrm{bl},1}(x, Z) = \frac{1}{\Gamma} \mathrm{div}_x(\mathbf{w}^{\mathrm{bl},0}(x, Z)) \nabla_x d(x) + \mathbf{H}(x) \exp(-\Gamma Z), \tag{7.103}$$

for a function $\mathbf{H}$ such that

$$\mathbf{H}(x) \cdot \nabla_x d(x) = 0 \tag{7.104}$$

to be determined below. Note that, in accordance with formula (7.99), $|\nabla_x d(x)| = |\nabla_x \mathrm{dist}[x, \partial\Omega]| = 1$ on the set where $\mathbf{w}^{\mathrm{bl},0} \neq 0$.

Determining λ^1. Our ultimate goal is to identify $v^{\mathrm{int},1}$, $\mathbf{w}^{\mathrm{int},1}$, and, in particular λ^1, by help of equations of (7.84), (7.85). In accordance with (7.92), the normal trace of the quantity $\mathbf{w}^{\mathrm{int},1}(x) + \mathbf{w}^{\mathrm{bl},1}(x, 0)$ must vanish for $x \in \partial\Omega$; whence, by virtue of (7.103),

$$0 = \mathbf{w}^{\mathrm{int},1}(x) \cdot \mathbf{n}(x) + \mathbf{w}^{\mathrm{bl},1}(x, 0) \cdot \mathbf{n}(x) = \mathbf{w}^{\mathrm{int},1}(x) \cdot \mathbf{n}(x) - \frac{1}{\Gamma} \mathrm{div}_x(\mathbf{w}^{\mathrm{bl},0}(x, 0)) \tag{7.105}$$

for any $x \in \partial\Omega$.

As a consequence of (7.93), system (7.84), (7.85) can be rewritten as a second order elliptic equation

$$\Delta_x v^{\mathrm{int},1} + \Lambda v^{\mathrm{int},1} = 2 \frac{\lambda^1 \lambda^0}{\omega} v^{\mathrm{int},0} \text{ in } \Omega, \tag{7.106}$$

where $\Lambda = -(\lambda^0)^2/\omega$. Problem (7.106) is supplemented with the *non-homogeneous* Neumann boundary condition determined by means of (7.93), (7.85), and (7.105), namely

$$\nabla_x v^{\text{int},1} \cdot \mathbf{n}(x) = \frac{\lambda^0}{\Gamma} \text{div}_x(\mathbf{w}^{\text{bl},0}(x,0)) \text{ for all } x \in \partial\Omega. \tag{7.107}$$

According to the standard Fredholm alternative for elliptic problems (see Section 10.2.2 in Appendix), system (7.106), (7.107) is solvable as long as

$$\frac{\omega}{\Gamma}\int_{\partial\Omega} \text{div}_x(\mathbf{w}^{\text{bl},0}(x,0)) v_{(k)} \, \mathrm{d}S_x = 2\lambda^1 \int_{\Omega} v^{\text{int},0} v_{(k)} \, \mathrm{d}x \text{ for } k = 1, \dots, m,$$

where $\{v_{(1)}, \dots, v_{(m)}\}$ is the system of eigenvectors introduced in (7.94). In accordance with our agreement, $v_{(1)} = v^{\text{int},0}$, therefore we set

$$\lambda^1 = \frac{\omega}{2\Gamma}\int_{\partial\Omega} \text{div}_x(\mathbf{w}^{\text{bl},0}(x,0)) v^{\text{int},0} \, \mathrm{d}S_x \tag{7.108}$$

and verify that

$$\int_{\partial\Omega} \text{div}_x(\mathbf{w}^{\text{bl},0}(x,0)) v_{(k)} \, \mathrm{d}S_x = 0 \text{ for } k = 2, \dots, m. \tag{7.109}$$

To this end, use (7.93), (7.99) to compute

$$\begin{aligned} &\text{div}_x(\mathbf{w}^{\text{bl},0}(x,0)) \\ &\quad = -\text{div}_x\left(\mathbf{w}^{\text{int},0}(x - d(x)\nabla_x d(x))\right) \\ &\quad = -\frac{1}{\lambda^0}\text{div}_x\left(\nabla_x v^{\text{int},0}(x - d(x)\nabla_x d(x))\right) \\ &\quad = -\frac{1}{\lambda^0}\nabla_x^2 v^{\text{int},0}\left(x - d(x)\nabla_x d(x)\right) : \Big(\mathbb{I} - \nabla_x d(x) \otimes \nabla_x d(x) - d(x)\nabla^2 d(x)\Big) \end{aligned}$$

whenever $\text{dist}[x, \partial\Omega] < \delta/2$. Consequently,

$$\begin{aligned} \int_{\partial\Omega} \text{div}_x(\mathbf{w}^{\text{bl},0}(x,0)) v_{(k)} \, \mathrm{d}S_x &= -\frac{1}{\lambda^0}\int_{\partial\Omega} \nabla_x^2 v^{\text{int},0} : (\mathbb{I} - \mathbf{n}\otimes\mathbf{n}) v_{(k)} \, \mathrm{d}S_x \\ = \frac{1}{\lambda^0}\int_{\partial\Omega} \Delta_S v^{\text{int},0} v_{(k)} \, \mathrm{d}S_x &= \frac{1}{\lambda^0}\int_{\partial\Omega} \nabla_x v^{\text{int},0} \cdot \nabla_x v_{(k)} \, \mathrm{d}S_x, \end{aligned}$$

where the symbol Δ_S denotes the Laplace-Beltrami operator on the (compact) Riemannian manifold $\partial\Omega$. Indeed expression $\left[\nabla_x^2 v^{\text{int},0} : (\mathbf{n}\otimes\mathbf{n} - \mathbb{I})\right]$ represents the standard "flat" Laplacian of the function $v^{\text{int},0}$ with respect to the tangent plane at each point of $\partial\Omega$ that *coincides* (up to a sign) with the associated Laplace-Beltrami operator on the manifold $\partial\Omega$ applied to the restriction of $v^{\text{int},0}|_{\partial\Omega}$ provided $\nabla_x v^{\text{int},0} \cdot \mathbf{n} = 0$ on $\partial\Omega$ (see Gilbarg and Trudinger [96, Chapter 16]).

In accordance with (7.94), we infer that

$$\int_{\partial\Omega} \nabla_x v^{\mathrm{int},0} \cdot \nabla_x v_{(k)} \, \mathrm{d}S_x = \begin{cases} \int_{\partial\Omega} |\nabla_x v^{\mathrm{int},0}|^2 \, \mathrm{d}S_x \text{ if } k = 1, \\ 0 \text{ for } k = 2, \dots, m. \end{cases}$$

In particular, we get (7.109), and, using (7.72), (7.101),

$$\lambda^1 = -\Gamma \frac{D}{4\Lambda} \int_{\partial\Omega} |\nabla_x v^{\mathrm{int},0}|^2 \, \mathrm{d}S_x.$$

Seeing that $\Lambda > 0$, and, by virtue of (7.101), $\mathrm{Re}[\Gamma] > 0$, we utilize hypothesis (7.32) together with Proposition 7.1 to deduce the desired conclusion

$$\mathrm{Re}[\lambda_1] < 0. \tag{7.110}$$

This is the crucial point of the proof of Theorem 7.1.

Having identified $v^{\mathrm{int},1}$ by means of (7.106), (7.107) we use (7.85) to compute

$$\mathbf{w}^{\mathrm{int},1} = \frac{1}{\lambda^0} \left(\nabla_x v^{\mathrm{int},1} - \lambda^1 \mathbf{w}^{\mathrm{int},0} \right).$$

Finally, in order to meet the boundary conditions (7.92), we set

$$\mathbf{H}(x) = -\chi(d(x)) \Big(\mathbf{w}^{\mathrm{int},1}(x) - (\mathbf{w}^{\mathrm{int},1} \cdot \nabla_x d(x)) \nabla_x d(x) \Big) \text{ for } x \in \overline{\Omega}$$

in (7.103), with χ given by (7.100).

Conclusion. By a direct inspection of (7.90), (7.91), where all quantities are evaluated by means (7.95), (7.99), (7.103), we infer that

$$|s^1_\varepsilon| + |\mathbf{s}^2_\varepsilon| \le c \left(\sqrt{\varepsilon} + \exp\left(-\mathrm{Re}[\Gamma] \frac{d(x)}{\sqrt{\varepsilon}} \right) \right),$$

in particular s^1_ε, $\mathbf{s}^2_\varepsilon$ are uniformly bounded in $\overline{\Omega}$ and tend to zero uniformly on any compact $K \subset \Omega$.

The results obtained in this section are summarized in the following assertion.

Proposition 7.2. *Let* Ω *be given through* (7.17), *with*

$$B_{\mathrm{top}} = 0, \; B_{\mathrm{bottom}} = -\pi - h,$$
$$h \in C^3(\mathcal{T}^2), \; |h| < \pi, \; h \not\equiv \mathrm{const}.$$

Assume that $v^{\mathrm{int},0}$, $\mathbf{w}^{\mathrm{int},0}$, *and* $\lambda^0 \neq 0$ *solve the eigenvalue problem* (7.71), (7.73).

Then the approximate eigenvalue problem (7.80–7.82) *admits a solution in the form* (7.77), (7.78), *where*

- *the functions* $v^{\rm int,1} = v^{\rm int,1}(x)$, $\mathbf{w}^{\rm int,1} = \mathbf{w}^{\rm int,1}(x)$ *belong to the class* $C^2(\overline{\Omega})$*;*
- *the boundary layer corrector functions* $v^{\rm bl,0} = v^{\rm bl,1} = 0$, $\mathbf{w}^{\rm bl,0} = \mathbf{w}^{\rm bl,0}(x,Z)$, $\mathbf{w}^{\rm bl,1} = \mathbf{w}^{\rm bl,1}(x,Z)$ *are all of the form* $\mathbf{h}(x)\exp(-\Gamma Z)$*, where* $\mathbf{h} \in C^2(\overline{\Omega};\mathbb{R}^3)$*, and* ${\rm Re}[\Gamma] > 0$*;*
- *the approximate eigenvalue* λ_ε *is given by* (7.81)*, where*

$$\mathrm{Re}[\lambda^1] < 0; \tag{7.111}$$

- *the remainders* s^1_ε*,* $\mathbf{s}^2_\varepsilon$ *satisfy*

$$s^1_\varepsilon \to 0 \ \textit{in}\ L^q(\Omega),\ \mathbf{s}^2_\varepsilon \to 0 \ \textit{in}\ L^q(\Omega;\mathbb{R}^3)\ \textit{as}\ \varepsilon \to 0\ \textit{for any}\ 1 \le q < \infty. \tag{7.112}$$

7.5 Strong convergence of the velocity field

We are now in a position to establish the main result of this chapter stated in Theorem 3.1, namely

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ strongly in } L^2((0,T)\times\Omega;\mathbb{R}^3). \tag{7.113}$$

We recall that, in accordance with (7.43),

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T;W^{1,2}_0(\Omega;\mathbb{R}^3)); \tag{7.114}$$

at least for a suitable subsequence. Moreover, exactly as in Section 5.3.1, we have

$$\mathrm{div}_x\, \mathbf{U} = 0.$$

Consequently, it remains to control possible oscillations of the velocity field in time. To this end, similarly to Chapter 5, the problem is reduced to a finite number of acoustic modes that can be treated by means of the spectral theory developed in the preceding section.

7.5.1 Compactness of the solenoidal component

It follows from the uniform estimates (7.36–7.38) that

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \overline{\varrho}\mathbf{U} \text{ weakly-(*) in } L^\infty(0,T;L^{\frac{5}{4}}(\Omega;\mathbb{R}^3)). \tag{7.115}$$

Using quantities

$$\varphi(t,x) = \psi(t)\phi(x),\ \psi \in C^\infty_c(0,T),\ \phi \in C^\infty_c(\Omega),\ \mathrm{div}_x \phi = 0$$

as test functions in the momentum equation (7.29) we deduce, by means of the standard Arzelà-Ascoli theorem, that the scalar functions

$$t \mapsto \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \phi \ \mathrm{d}x \text{ are precompact in } C[0,T].$$

Note that

$$\int_\Omega \frac{1}{\varepsilon} \varrho_\varepsilon \nabla_x F \cdot \phi \, \mathrm{d}x = \int_\Omega \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \nabla_x F \phi \, \mathrm{d}x$$

as ϕ is a divergenceless vector field.

Consequently, with the help of (7.115) and a simple density argument, we infer that the family

$$t \mapsto \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathbf{H}[\phi] \, \mathrm{d}x \text{ is precompact in } C[0,T]$$

for any $\phi \in C_c^\infty(\Omega; \mathbb{R}^3)$, where $\mathbf{H}$ denotes the Helmholtz projection introduced in Section 5.4.1. In other words,

$$\mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \to \overline{\varrho} \mathbf{H}[\mathbf{U}] = \overline{\varrho} \mathbf{U} \text{ in } C_{\text{weak}}([0,T]; L^{\frac{5}{4}}(\Omega; \mathbb{R}^3)). \tag{7.116}$$

Let us point out that $\mathbf{H}[\phi]$ *is not* an admissible test function in (7.29), however, it can be approximated in $L^p(\Omega; \mathbb{R}^3)$ by smooth solenoidal functions with compact support for finite p (see Section 10.6 in Appendix).

Thus, combining relations (7.114), (7.116), we infer

$$\int_0^T \int_\Omega \mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \cdot \mathbf{H}[\mathbf{u}_\varepsilon] \, \mathrm{d}x \, \mathrm{d}t \to \overline{\varrho} \int_0^T \int_\Omega |\mathbf{H}[\mathbf{U}]|^2 \, \mathrm{d}x \, \mathrm{d}t,$$

which, together with estimates (7.37), (7.38), gives rise to

$$\int_0^T \int_\Omega |\mathbf{H}[\mathbf{u}_\varepsilon]|^2 \, \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_\Omega |\mathbf{U}|^2 \, \mathrm{d}x \, \mathrm{d}t$$

yielding, finally, the desired conclusion

$$\mathbf{H}[\mathbf{u}_\varepsilon] \to \mathbf{U} \text{ (strongly) in } L^2((0,T) \times \Omega; \mathbb{R}^3). \tag{7.117}$$

7.5.2 Reduction to a finite number of modes

Exactly as in (5.146), we decompose the space L^2 as a sum of the subspace of solenoidal vector fields L^2_σ and gradients L^2_g:

$$L^2(\Omega; \mathbb{R}^3) = L^2_\sigma(\Omega; \mathbb{R}^3) \oplus L^2_g(\Omega; \mathbb{R}^3).$$

Since we already know that the solenoidal components of the velocity field $\mathbf{u}_\varepsilon$ are precompact in L^2, the proof of (7.113) reduces to showing

$$\mathbf{H}^\perp[\mathbf{u}_\varepsilon] \to \mathbf{H}^\perp[\mathbf{U}] = 0 \text{ in } L^2((0,T) \times \Omega; \mathbb{R}^3).$$

Moreover, since the embedding $W_0^{1,2}(\Omega; \mathbb{R}^3) \hookrightarrow L^2(\Omega; \mathbb{R}^3)$ is compact, it is enough to show

$$\left[t \mapsto \int_\Omega \mathbf{u}_\varepsilon \cdot \mathbf{w} \, \mathrm{d}x \right] \to 0 \text{ in } L^2(0,T), \tag{7.118}$$

for any fixed $\mathbf{w} = \frac{1}{\lambda}\nabla_x v$, where v, $\mathbf{w}$, $\lambda \neq 0$ solve the eigenvalue problem (7.71), (7.73) (cf. Section 5.4.6).

In view of (7.37), (7.38), relation (7.118) follows as soon as we show

$$\left[t \mapsto \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathbf{w}\ \mathrm{d}x\right] \to 0 \text{ in } L^2(0,T),$$

where the latter quantity can be expressed by means of the acoustic equation (7.59), (7.60). In addition, since the solutions of the eigenvalue problem (7.71), (7.73) come in pairs $[v, \mathbf{w}, \lambda]$, $[v, -\mathbf{w}, -\lambda]$, it is enough to show

$$\left[t \mapsto \int_\Omega \Big(r_\varepsilon v + \mathbf{V}_\varepsilon \cdot \mathbf{w}\Big)\ \mathrm{d}x\right] \to 0 \text{ in } L^2(0,T) \tag{7.119}$$

for any solution v, $\mathbf{w}$ of (7.71), (7.73) associated to an eigenvalue $\lambda \neq 0$, where r_ε, $\mathbf{V}_\varepsilon$ are given by (7.61).

Finally, in order to exploit the information on the spectrum of the perturbed acoustic operator, we claim that (7.119) can be replaced by

$$\left[t \mapsto \int_\Omega \Big(r_\varepsilon v_\varepsilon + \mathbf{V}_\varepsilon \cdot \mathbf{w}_\varepsilon\Big)\ \mathrm{d}x\right] \to 0 \text{ in } L^2(0,T), \tag{7.120}$$

where v_ε, $\mathbf{w}_\varepsilon$ are the solutions of the approximate eigenvalue problem (7.80), (7.82) constructed in the previous section. Indeed, by virtue of Proposition 7.2, we have

$$v_\varepsilon \to v \text{ in } C(\overline{\Omega}),\ \ \mathbf{w}_\varepsilon \to \mathbf{w} \text{ in } L^q(\Omega;\mathbb{R}^3) \text{ for any } 1 \leq q < \infty.$$

Accordingly, the proof of Theorem 7.1 reduces to showing (7.120). This will be done in the following section.

7.5.3 Strong convergence

In order to complete the proof of Theorem 7.1, our ultimate goal consists in showing (7.120). To this end, we make use of the specific form of the acoustic equation (7.59), (7.60), together with the associated spectral problem (7.80), (7.82). Taking the quantities $\psi(t)v_\varepsilon(x)$, $\psi(t)\mathbf{w}_\varepsilon(x)$, $\psi \in C_c^\infty(0,T)$, as test functions in (7.59), (7.60), respectively, we obtain

$$\int_0^T \Big(\varepsilon\chi_\varepsilon\partial_t\psi + \lambda_\varepsilon\chi_\varepsilon\psi\Big)\ \mathrm{d}t + \sqrt{\varepsilon}\int_0^T \psi\int_\Omega \Big(r_\varepsilon s_\varepsilon^1 + \mathbf{V}_\varepsilon \cdot \mathbf{s}_\varepsilon^2\Big)\ \mathrm{d}x\ \mathrm{d}t = \sum_{m=1}^7 I_m^\varepsilon, \tag{7.121}$$

where we have set

$$\chi_\varepsilon(t) = \int_\Omega \Big(r_\varepsilon(t,\cdot)v_\varepsilon + \mathbf{V}_\varepsilon(t,\cdot)\cdot\mathbf{w}_\varepsilon\Big)\ \mathrm{d}x,$$

and the symbols I^ε_m stand for the "small" terms:

$$I_1^\varepsilon = \frac{A}{\omega}\int_0^T \psi \int_\Omega \Big[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \nabla_x \vartheta_\varepsilon + \Big(\varrho_\varepsilon s(\overline{\varrho}, \overline{\vartheta}) - \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) \Big) \mathbf{u}_\varepsilon \Big] \cdot \nabla_x v_\varepsilon \ {\rm d}x \ {\rm d}t,$$

$$I_2^\varepsilon = -\frac{A}{\omega} < \sigma_\varepsilon ; \psi v_\varepsilon >_{[\mathcal{M};C]([0,T]\times\overline{\Omega})},$$

$$I_3^\varepsilon = D \int_0^T \psi \int_\Omega \varepsilon^2 \Big(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big) \mathbf{u}_\varepsilon \cdot {\rm div}_x [[\nabla_x \mathbf{w}_\varepsilon]] \ {\rm d}x \ {\rm d}t,$$

$$I_4^\varepsilon = \int_0^T \psi \int_\Omega \varepsilon^2 \Big(\frac{\mu(\vartheta_\varepsilon) - \mu(\overline{\vartheta})}{\varepsilon} \Big) [[\nabla_x \mathbf{u}_\varepsilon]] : \nabla_x \mathbf{w}_\varepsilon \ {\rm d}x \ {\rm d}t,$$

$$I_5^\varepsilon = - \int_0^T \psi \int_\Omega \varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \mathbf{w}_\varepsilon \ {\rm d}x \ {\rm d}t,$$

$$I_6^\varepsilon = \int_0^T \psi \int_\Omega \varepsilon \Big(\frac{\overline{\varrho} - \varrho_\varepsilon}{\varepsilon} \Big) \nabla_x F \cdot \mathbf{w}_\varepsilon \ {\rm d}x \ {\rm d}t,$$

and

$$I_7^\varepsilon = \int_0^T \psi \int_\Omega G_8^\varepsilon \ {\rm div}_x \mathbf{w}_\varepsilon \ {\rm d}x \ {\rm d}t,$$

where G_8^ε is given by (7.67).

Our aim is to show that each of the integrals can be written in the form

$$I^\varepsilon \approx \int_0^T \psi(t) \Big(\varepsilon \gamma^\varepsilon(t) + \varepsilon^{1+\beta} \Gamma^\varepsilon(t) \Big) \ {\rm d}t,$$

where

$$\left\{ \begin{array}{c} \{\gamma_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^q(0,T) \text{ for a certain } q > 1, \\ \{\Gamma_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^1(0,T), \text{ and } \beta > 0. \end{array} \right\}$$

This rather tedious task, to be achieved by means of Proposition 7.2 combined with the uniform estimates listed in Section 7.3, consists of several steps as follows:

(i) By virtue of Hölder's inequality, we have

$$\begin{aligned} & \Big| \int_\Omega \Big[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \nabla_x \vartheta_\varepsilon \cdot \nabla_x v_\varepsilon \ {\rm d}x \Big| \qquad (7.122) \\ & \leq \varepsilon \|v_\varepsilon\|_{W^{1,\infty}(\Omega)} \Big| \int_\Omega \Big[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \Big]_{\rm ess} \Big| \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon} \Big| \ {\rm d}x \Big| + \Big| \int_\Omega \Big[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \Big]_{\rm res} \Big| \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon} \Big| \ {\rm d}x \Big| \\ & = \varepsilon \gamma_{1,1}^\varepsilon, \text{ with } \{\gamma_1^\varepsilon\}_{\varepsilon>1} \text{ bounded in } L^q(0,T) \text{ for a certain } q > 1, \end{aligned}$$

where we have used estimates (7.44) and (7.49). Note that, in accordance with Proposition 7.2, both correction terms $v^{\rm bl,0}$, $v^{\rm bl,1}$ vanish identically, in particular,

$$\|v_\varepsilon\|_{W^{1,\infty}(\Omega)} \leq c \text{ uniformly in } \varepsilon. \qquad (7.123)$$

In a similar way,

$$\left| \int_\Omega \Big(\varrho_\varepsilon s(\overline{\varrho}, \overline{\vartheta}) - \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) \Big) \mathbf{u}_\varepsilon \cdot \nabla_x v_\varepsilon \ \mathrm{d}x \right| \tag{7.124}$$

$$\leq \varepsilon \|v_\varepsilon\|_{W^{1,\infty}(\Omega)} \left[\int_\Omega \left| \left[\frac{\varrho_\varepsilon s(\overline{\varrho}, \overline{\vartheta}) - \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon} \right]_{\mathrm{ess}} \right| |\mathbf{u}_\varepsilon| \ \mathrm{d}x \right.$$

$$\left. + \int_\Omega \left| \left[\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon} \right]_{\mathrm{res}} \mathbf{u}_\varepsilon \right| \ \mathrm{d}x \ \mathrm{d}t + |s(\overline{\varrho}, \overline{\vartheta})| \int_\Omega \left[\frac{\varrho_\varepsilon}{\varepsilon} \right]_{\mathrm{res}} |\mathbf{u}_\varepsilon| \ \mathrm{d}x \right] = \varepsilon \gamma^\varepsilon_{1,2}.$$

Thus we can use Proposition 5.2, together with estimates (7.37–7.39), (7.43), (7.48), (7.50), in order to conclude that

$$\{\gamma^\varepsilon_{1,2}\}_{\varepsilon>0} \text{ is bounded in } L^q(0,T) \text{ for a certain } q > 1.$$

Summing up (7.122), (7.124) we infer that

$$I^\varepsilon_1 = \varepsilon \int_0^T \psi(t) \gamma^\varepsilon_1(t) \ \mathrm{d}t, \ \text{ with } \{\gamma^\varepsilon_1\}_{\varepsilon>0} \text{ bounded in } L^q(0,T) \text{ for a certain } q > 1. \tag{7.125}$$

(ii) As a straightforward consequence of estimate (7.42) we obtain

$$I^\varepsilon_2 = \varepsilon^2 \left\langle \Gamma^\varepsilon_2; \psi \right\rangle_{[\mathcal{M};C][0,T]}, \text{ where } \{\Gamma^\varepsilon_2\}_{\varepsilon>0} \text{ is bounded in } \mathcal{M}^+[0,T]. \tag{7.126}$$

(iii) Taking advantage of the form of $\mathbf{w}^{\mathrm{bl},0}$, $\mathbf{w}^{\mathrm{bl},1}$ specified in Proposition 7.2, we obtain

$$\| \varepsilon \mathrm{div}_x [[\nabla_x \mathbf{w}_\varepsilon]] \ \|_{L^\infty(\Omega;\mathbb{R}^3)} \leq c$$

uniformly for $\varepsilon \to 0$. This fact, combined with the uniform bounds established in (7.37), (7.38), (7.43), and the standard embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, gives rise to

$$I^\varepsilon_3 = \varepsilon \int_0^T \psi(t) \gamma^\varepsilon_3(t) \ \mathrm{d}t, \tag{7.127}$$

where

$$\{\gamma^\varepsilon_3\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T).$$

(iv) Similarly to the preceding step, we deduce

$$\| \sqrt{\varepsilon} \mathbf{w}_\varepsilon \|_{W^{1,\infty}(\Omega;\mathbb{R}^3)} \leq c; \tag{7.128}$$

whence, by virtue of (7.40), (7.43), and (7.44),

$$I^\varepsilon_4 = \varepsilon^{3/2} \int_0^T \psi(t) \Gamma^\varepsilon_4(t) \ \mathrm{d}t, \tag{7.129}$$

where

$$\{\Gamma^\varepsilon_4\}_{\varepsilon>0} \text{ is bounded in } L^1(0,T).$$

(v) Probably the most delicate issue is how to handle the integrals in I_5^ε. To this end, we first write

$$
\int_0^T \psi \int_\Omega \varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \mathbf{w}_\varepsilon \, \mathrm{d}x \, \mathrm{d}t
$$
$$
= \int_0^T \psi \int_\Omega \varepsilon^2 \Big(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big) \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \mathbf{w}_\varepsilon \, \mathrm{d}x \, \mathrm{d}t + \overline{\varrho} \int_0^T \psi \int_\Omega \varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \mathbf{w}_\varepsilon \, \mathrm{d}x \, \mathrm{d}t,
$$

where, by virtue of (7.37), (7.38), (7.43), and the gradient estimate established in (7.128),

$$
\int_0^T \psi \int_\Omega \varepsilon^2 \Big(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big) \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \mathbf{w}_\varepsilon \, \mathrm{d}x \, \mathrm{d}t = \varepsilon^{3/2} \int_0^T \psi(t) \Gamma_{5,1}^\varepsilon(t) \, \mathrm{d}t, \qquad (7.130)
$$

with

$$
\{\Gamma_{5,1}^\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^1(0,T).
$$

On the other hand, a direct computation yields

$$
\int_\Omega (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \mathbf{w}_\varepsilon \, \mathrm{d}x = - \int_\Omega \mathrm{div}_x \mathbf{u}_\varepsilon \mathbf{u}_\varepsilon \cdot \mathbf{w}_\varepsilon \, \mathrm{d}x - \int_\Omega (\nabla_x \mathbf{u}_\varepsilon \mathbf{u}_\varepsilon) \cdot \mathbf{w}_\varepsilon \, \mathrm{d}x. \quad (7.131)
$$

Now, we have

$$
\int_\Omega \mathrm{div}_x \mathbf{u}_\varepsilon \mathbf{u}_\varepsilon \cdot \mathbf{w}_\varepsilon \, \mathrm{d}x = \int_\Omega \mathrm{div}_x \mathbf{u}_\varepsilon [\mathbf{u}_\varepsilon]_{\mathrm{ess}} \cdot \mathbf{w}_\varepsilon \, \mathrm{d}x + \int_\Omega \mathrm{div}_x \mathbf{u}_\varepsilon [\mathbf{u}_\varepsilon]_{\mathrm{res}} \cdot \mathbf{w}_\varepsilon \, \mathrm{d}x,
$$

where, in accordance with estimates (7.36), (7.43),

$$
\{\mathrm{div}_x \mathbf{u}_\varepsilon [\mathbf{u}_\varepsilon]_{\mathrm{ess}}\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T; L^1(\Omega;\mathbb{R}^3)),
$$

while

$$
\|\mathrm{div}_x \mathbf{u}_\varepsilon [\mathbf{u}_\varepsilon]_{\mathrm{res}}\|_{L^1(0,T;L^1(\Omega;\mathbb{R}^3))}
\le c\varepsilon^{2/3} \|\nabla_x \mathbf{u}_\varepsilon\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{3\times3}))} \|\mathbf{u}_\varepsilon\|_{L^2(0,T;L^6(\Omega;\mathbb{R}^3))},
$$

where we have used (7.43), the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, and the bound on the measure of the "residual set" established in (7.50).

Applying the same treatment to the latter integral on the right-hand side of (7.131) and adding the result to (7.130) we conclude that

$$
I_5^\varepsilon = \varepsilon^{3/2} \int_0^T \psi(t) \Gamma_{5,1}^\varepsilon \, \mathrm{d}t + \varepsilon \int_0^T \psi(t) \gamma_5^\varepsilon(t) \, \mathrm{d}t + \varepsilon^{5/3} \int_0^T \psi(t) \Gamma_{5,2}^\varepsilon \, \mathrm{d}t, \qquad (7.132)
$$

where

$$
\{\gamma_5^\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T),
$$

and

$$
\{\Gamma_{5,1}^\varepsilon\}_{\varepsilon>0}, \ \{\Gamma_{5,2}^\varepsilon\}_{\varepsilon>0} \text{ are bounded in } L^1(0,T).
$$

(vi) In view of estimates (7.37), (7.38), it is easy to check that

$$I_6^\varepsilon = \varepsilon \int_0^T \psi(t)\gamma_6^\varepsilon(t)\ \mathrm{d}t, \tag{7.133}$$

with

$$\{\gamma_6^\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T).$$

(vii) Finally, in accordance with the first equation in (7.80) and Proposition 7.2,

$$\|\mathrm{div}_x \mathbf{w}_\varepsilon\|_{L^\infty(\Omega)} \leq c;$$

therefore relations (7.38–7.41), (7.46), together with Proposition 5.2, can be used in order to conclude that

$$I_7^\varepsilon = \varepsilon \int_0^T \psi(t)\gamma_7^\varepsilon(t)\ \mathrm{d}t, \tag{7.134}$$

where

$$\{\gamma_7^\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0,T).$$

We are now in a position to use relation (7.121) in order to show (7.120). To begin, we focus on the integral

$$\sqrt{\varepsilon}\int_0^T \psi \int_\Omega \Big(r_\varepsilon s_\varepsilon^1 + \mathbf{V}_\varepsilon \cdot \mathbf{s}_\varepsilon^2 \Big)\ \mathrm{d}x$$

appearing on the left-hand side of (7.121), with r_ε, $\mathbf{V}_\varepsilon$ specified in (7.61). Writing

$$\sqrt{\varepsilon}\int_0^T \psi \int_\Omega \Big(r_\varepsilon s_\varepsilon^1 + \mathbf{V}_\varepsilon \cdot \mathbf{s}_\varepsilon^2 \Big)\ \mathrm{d}x$$
$$= \sqrt{\varepsilon}\int_0^T \psi \int_\Omega \Big([r_\varepsilon]_{\mathrm{ess}} s_\varepsilon^1 + [r_\varepsilon]_{\mathrm{res}} s_\varepsilon^1 + (\varrho_\varepsilon \mathbf{u}_\varepsilon) \cdot \mathbf{s}_\varepsilon^2 \Big)\ \mathrm{d}x$$

we can use the uniform estimates (7.36–7.41), together with pointwise convergence of the remainders established in (7.112), in order to deduce that

$$\sqrt{\varepsilon}\int_0^T \psi \int_\Omega \Big(r_\varepsilon s_\varepsilon^1 + \mathbf{V}_\varepsilon \cdot \mathbf{s}_\varepsilon^2 \Big)\ \mathrm{d}x = \sqrt{\varepsilon}\int_0^T \psi(t)\beta_\varepsilon(t)\ \mathrm{d}t, \tag{7.135}$$

where

$$\beta_\varepsilon \to 0 \text{ in } L^\infty(0,T). \tag{7.136}$$

Next, we use a family of standard regularizing kernels

$$\zeta_\delta(t) = \frac{1}{\delta}\zeta\Big(\frac{t}{\delta}\Big),\ \delta \to 0,$$
$$\zeta \in C_c^\infty(-1,1),\ \zeta \geq 0,\ \int_{-1}^1 \zeta(t)\ \mathrm{d}t = 1$$

in order to handle the "measure-valued" term in (7.121). To this end, we take ζ_δ as a test function in (7.121) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\chi_{\varepsilon,\delta} - \frac{\lambda_\varepsilon}{\varepsilon}\chi_{\varepsilon,\delta} = \sqrt{\varepsilon} h^1_{\varepsilon,\delta} + h^2_{\varepsilon,\delta} + \frac{1}{\sqrt{\varepsilon}} h^3_{\varepsilon,\delta}, \tag{7.137}$$

where we have set

$$\chi_{\varepsilon,\delta}(t) = \int_{\mathbb{R}} \zeta_\varepsilon(t-s)\psi_\delta(s)\ \mathrm{d}s$$

for $t \in (\delta, T-\delta)$.

In accordance with the uniform estimates (7.122–7.134), we have

$$\{h^1_{\varepsilon,\delta}\}_{\varepsilon>0} \text{ bounded in } L^1(0,T),\ \{h^2_{\varepsilon,\delta}\}_{\varepsilon>0} \text{ bounded in } L^p(0,T) \text{ for a certain } p>1, \tag{7.138}$$

uniformly for $\delta \to 0$, where we have used the standard properties of mollifiers recorded in Theorem 10.1 in Appendix. Similarly, by virtue of (7.135), (7.136),

$$\sup_{\delta>0} \|h^3_{\varepsilon,\delta}\|_{L^\infty(0,T)} \le \nu(\varepsilon),\ \ \nu(\varepsilon) \to 0 \text{ for } \varepsilon \to 0. \tag{7.139}$$

Here all functions in (7.138), (7.139) have been extended to be zero outside $(\delta, T-\delta)$.

The standard variation-of-constants formula yields

$$|\chi_{\varepsilon,\delta}(t)| \le \exp\Big(\mathrm{Re}[\lambda_\varepsilon/\varepsilon](t-\delta)\Big)\mathrm{ess}\sup_{s\in(0,T)} |\chi_{\varepsilon,\delta}(s)| + \sqrt{\varepsilon}\int_0^T |h^1_{\varepsilon,\delta}(s)|\ \mathrm{d}s$$

$$+\int_\delta^t \exp\Big(\mathrm{Re}[\lambda_\varepsilon/\varepsilon](t-s)\Big)|h^2_{\varepsilon,\delta}(s)|\ \mathrm{d}s + \int_\delta^t \frac{1}{\sqrt{\varepsilon}}\exp\Big(\mathrm{Re}[\lambda_\varepsilon/\varepsilon](t-s)\Big)|h^3_{\varepsilon,\delta}(s)|\ \mathrm{d}s;$$

therefore letting first $\delta \to 0$ and then $\varepsilon \to 0$ yields the desired conclusion (7.120). Note that, in accordance with (7.111),

$$\mathrm{Re}[\lambda_\varepsilon/\varepsilon] \le -\frac{c}{\sqrt{\varepsilon}} \text{ for a certain } c>0,$$

in particular

$$\int_0^t \frac{1}{\sqrt{\varepsilon}}\exp\Big(\mathrm{Re}[\lambda_\varepsilon/\varepsilon](t-s)\Big)\ \mathrm{d}s < c$$

uniformly for $\varepsilon \to 0$. The proof of Theorem 7.1 is now complete.

Chapter 8

Problems on Large Domains

Many theoretical problems in continuum fluid mechanics are formulated on unbounded physical domains, most frequently on the whole Euclidean space $\mathbb{R}^3$. Although, arguably, any physical but also numerical space is necessarily bounded, the concept of an unbounded domain offers a useful approximation in situations when the influence of the boundary on the behavior of the system can be neglected. The acoustic waves examined in the previous chapters are often ignored in meteorological models, where the underlying ambient space is large when compared with the characteristic speed of the fluid as well as the speed of sound. However, as we have seen in Chapter 5, the way the acoustic waves "disappear" in the asymptotic limit may include fast oscillations in the time variable that may produce undesirable numerical instabilities. In this chapter, we examine the incompressible limit of the NAVIER-STOKES-FOURIER SYSTEM in the situation when the spatial domain is large with respect to the characteristic speed of sound in the fluid. Remarkably, although very large, our physical space is still bounded exactly in the spirit of the leading idea of this book that the notions of "large" and "small" depend on the chosen scale.

8.1 Primitive system

Similarly to the previous chapters, our starting point is the full NAVIER-STOKES-FOURIER SYSTEM, where the *Mach number* is proportional to a small parameter ε, while the remaining characteristic numbers are kept of order unity.

■ SCALED NAVIER-STOKES-FOURIER SYSTEM:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{8.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p = \mathrm{div}_x \mathbb{S}, \tag{8.2}$$

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma, \tag{8.3}$$

with

$$\sigma \geq \frac{1}{\vartheta}\Big(\varepsilon^2\mathbb{S} : \nabla_x\mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\Big), \tag{8.4}$$

---

where the inequality sign in (8.4) is motivated by the existence theory developed in Chapter 3. The viscous stress tensor $\mathbb{S}$ satisfies the standard *Newton's rheological law*

$$\mathbb{S} = \mathbb{S}(\vartheta, \nabla_x\mathbf{u}) = \mu(\vartheta)\Big(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\Big), \tag{8.5}$$

where we have deliberately omitted the contribution of bulk viscosity, while the heat flux $\mathbf{q}$ obeys *Fourier's law*

$$\mathbf{q} = \mathbf{q}(\vartheta, \nabla_x\vartheta) = -\kappa(\vartheta)\nabla_x\vartheta. \tag{8.6}$$

For the sake of simplicity, we ignore the influence of external forces assumed to be zero in the present setting.

System (8.1–8.3) is considered on a family of spatial domains $\{\Omega_\varepsilon\}_{\varepsilon>0}$ large enough in order to eliminate the effect of the boundary on the local behavior of acoustic waves. Seeing that the speed of sound in (8.1–8.3) is proportional to $1/\varepsilon$ we shall assume that the family $\{\Omega_\varepsilon\}_{\varepsilon>0}$ has the following property.

■ PROPERTY (L):

For any $x \in \mathbb{R}^3$, there is $\varepsilon_0 = \varepsilon_0(x)$ such that $x \in \Omega_\varepsilon$ for all $0 < \varepsilon < \varepsilon_0$. Moreover,

$$\varepsilon\mathrm{dist}[x, \partial\Omega_\varepsilon] \to \infty \text{ for } \varepsilon \to 0 \tag{8.7}$$

for any $x \in \mathbb{R}^3$.

In other words, given a fixed bounded cavity $B \subset \Omega_\varepsilon$ in the ambient space, the acoustic waves initiated in B cannot reach the boundary, reflect, and come back during a finite time interval $(0, T)$.

Similarly to Chapter 5, we suppose that the initial distribution of the density and the temperature are close to a spatially homogeneous state, specifically,

$$\varrho(0, \cdot) = \varrho_{0,\varepsilon} = \overline{\varrho} + \varepsilon\varrho^{(1)}_{0,\varepsilon}, \tag{8.8}$$

$$\vartheta(0, \cdot) = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon\vartheta^{(1)}_{0,\varepsilon}, \tag{8.9}$$

where $\overline{\varrho}$, $\overline{\vartheta}$ are positive constants. In addition, we denote

$$\mathbf{u}(0, \cdot) = \mathbf{u}_{0,\varepsilon}. \tag{8.10}$$

Problem formulation. We consider a family $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ of (weak) solutions to problem (8.1–8.6) on a compact time interval $[0,T]$ emanating from the initial state satisfying (8.8–8.10). Our main goal formulated in Theorem 8.1 below is to show that

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ in } L^2((0,T)\times B;\mathbb{R}^3) \text{ for any bounded ball } B \subset \mathbb{R}^3, \tag{8.11}$$

at least for a suitable subsequence $\varepsilon \to 0$, where the limit velocity field complies with the standard incompressibility constraint

$$\mathrm{div}_x \mathbf{u} = 0. \tag{8.12}$$

Thus, in contrast with the case of a bounded domain examined in Chapter 5, we recover *strong* (*pointwise*) *convergence* of the velocity field regardless of the specific shape of the domain and the boundary conditions imposed.

The strong convergence of the velocity is a consequence of the dispersive properties of the *acoustic equation* – waves of different frequencies move in different directions – mathematically formulated in terms of *Strichartz's estimates*. Here we use their local variant discovered by Smith and Sogge [184].

As already pointed out, the considerations should be independent of the behavior of $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ "far away" from the set B, in particular we do not impose any specific boundary conditions. On the other hand, certain restrictions have to be made in order to prevent the energy from being "pumped" into the system at infinity. Specifically, the following hypotheses are required:

- **(i)** The total mass of the fluid contained in Ω_ε is proportional to $|\Omega_\varepsilon|$, in particular the average density is constant.
- **(ii)** The system dissipates energy, specifically, the total energy of the fluid contained in Ω_ε is non-increasing in time.
- **(iii)** The system produces entropy, the total entropy is non-decreasing in time.

The matter in this chapter is organized as follows. Similarly to the preceding part of this book, our analysis is based on the uniform estimates of the family $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ resulting from the dissipation inequality deduced in the same way as in Chapter 5 (see Section 8.2). The time evolution of the velocity field, specifically its gradient component, is governed by a wave equation (acoustic equation) discussed in Section 5.4.3 and here revoked in Section 8.3. Since the acoustic waves propagate with a finite speed proportional to $1/\varepsilon$, the acoustic equation may be considered on the whole of physical spaces $\mathbb{R}^3$, where efficient tools based on Fourier analysis are available (see Section 8.4). In particular, we obtain the desired conclusion stated in (8.11) (see Section 8.5) and reformulated in a rigorous way in Theorem 8.1.

8.2 Uniform estimates

The uniform estimates derived below follow immediately from the general axioms **(i)**–**(iii)** stated in the previous section, combined with the *hypothesis of thermodynamic stability* (see (1.44))

$$\frac{\partial p(\varrho,\vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho,\vartheta)}{\partial \vartheta} > 0, \tag{8.13}$$

where $e = e(\varrho,\vartheta)$ is the specific internal energy interrelated to p and s through *Gibbs' equation* (1.2). We recall that the first condition in (8.13) asserts that the compressibility of the fluid is always positive, while the second one says that the specific heat at constant volume is positive.

Although the estimates deduced below are formally the same as in Chapter 5, we have to pay special attention to the fact that the volume of the ambient space expands for $\varepsilon \to 0$. In particular, the constants associated to various embedding relations may depend on ε. A priori, we do not assume that Ω_ε are bounded, however, the existence theory developed in Chapter 3 relies essentially on boundedness of the underlying physical domain.

8.2.1 Estimates based on the hypothesis of thermodynamic stability

In accordance with assumption **(i)** in Section 8.1, the total mass of the fluid contained in Ω_ε is proportional to $|\Omega_\varepsilon|$. This can be formulated as

$$\int_{\Omega_\varepsilon} \Big(\varrho_\varepsilon(t,\cdot) - \overline{\varrho} \Big) \ {\rm d}x = 0 \text{ for a.a. } t \in (0,T), \tag{8.14}$$

in particular, we take

$$\int_{\Omega_\varepsilon} \varrho^{(1)}_{0,\varepsilon} \ {\rm d}x = 0 \tag{8.15}$$

in (8.8).

Similarly, by virtue of assumption **(ii)**, the total energy is a non-increasing function of time, meaning

$$\int_{\Omega_\varepsilon} \left[\frac{\varepsilon^2}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2(t) + \varrho_\varepsilon e(\varrho_\varepsilon,\vartheta_\varepsilon)(t) - \frac{\varepsilon^2}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2(0) - \varrho_{0,\varepsilon} e(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon}) \right] \ {\rm d}x \leq 0. \tag{8.16}$$

Finally, in accordance with the *Second law of thermodynamics* expressed through assumption **(ii)**, the system produces entropy, in particular,

$$\int_{\Omega_\varepsilon} \Big[\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)(t) - \varrho_{0,\varepsilon} s(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon}) \Big] \ {\rm d}x = \sigma_\varepsilon \Big[[0,t] \times \overline{\Omega}_\varepsilon \Big] \tag{8.17}$$

for a.a. $t \in (0,T)$, where the entropy production rate σ_ε is a non-negative measure satisfying

$$\sigma_\varepsilon \geq \frac{1}{\vartheta_\varepsilon}\Big(\varepsilon^2 \mathbb{S}_\varepsilon : \nabla_x \mathbf{u}_\varepsilon - \frac{\mathbf{q}_\varepsilon \cdot \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon}\Big),\ \mathbb{S}_\varepsilon = \mathbb{S}(\vartheta_\varepsilon, \nabla_x \mathbf{u}_\varepsilon),\ \mathbf{q}_\varepsilon = \mathbf{q}(\vartheta_\varepsilon, \nabla_x \vartheta_\varepsilon). \quad (8.18)$$

As we have observed several times in this book, the previous relations can be combined to obtain:

■ Total Dissipation Inequality:

$$\int_{\Omega_\varepsilon} \Big[\frac{1}{2}\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\Big(H_{\overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon) - \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})(\varrho_\varepsilon - \overline{\varrho}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})\Big)\Big](t)\ \mathrm{d}x$$
$$+ \frac{\overline{\vartheta}}{\varepsilon^2}\sigma_\varepsilon\left[[0,t] \times \overline{\Omega}_\varepsilon\right] \quad (8.19)$$
$$\leq \int_{\Omega_\varepsilon} \Big[\frac{1}{2}\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2}\Big(H_{\overline{\vartheta}}(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) - \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})(\varrho_{0,\varepsilon} - \overline{\varrho}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})\Big)\Big]\ \mathrm{d}x$$

for a.a. $t \in [0,T]$,

where

$$H_{\overline{\vartheta}}(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \overline{\vartheta}\varrho s(\varrho, \vartheta)$$

is the *Helmholtz function* introduced in (2.48). If Ω_ε are bounded domains and the problem is supplemented with the boundary conditions (5.15), (5.16) compatible with the general principles **(i)**–**(iii)**, validity of (8.19) has been verified in Chapter 5.

Since, by virtue of Gibbs' relation (1.2),

$$\frac{\partial^2 H_{\overline{\vartheta}}(\varrho, \overline{\vartheta})}{\partial \varrho^2} = \frac{1}{\varrho}\frac{\partial p(\varrho, \overline{\vartheta})}{\partial \varrho}, \quad \frac{\partial H_{\overline{\vartheta}}(\varrho, \vartheta)}{\partial \vartheta} = \frac{\varrho}{\vartheta}(\vartheta - \overline{\vartheta})\frac{\partial e(\varrho, \vartheta)}{\partial \vartheta},$$

the hypothesis of thermodynamic stability (8.13) implies that

$$\varrho \mapsto H_{\overline{\vartheta}}(\varrho, \overline{\vartheta}) \text{ is strictly convex on } (0, \infty),$$

and

$$\vartheta \mapsto H_{\overline{\vartheta}}(\varrho, \vartheta) \text{ is decreasing for } \vartheta < \overline{\vartheta} \text{ and increasing for } \vartheta > \overline{\vartheta}$$

(see Section 2.2.3).

At first glance, it may seem incorrect to introduce the integral $\int_{\Omega_\varepsilon} H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})\ \mathrm{d}x$ in (8.19) as soon as Ω_ε is unbounded. However, the integrated quantity on the right-hand side of (8.19) is non-negative and the integral converges provided we assume, say,

$$\|\varrho^{(1)}_{0,\varepsilon}\|_{L^1 \cap L^\infty(\Omega_\varepsilon)} + \|\vartheta^{(1)}_{0,\varepsilon}\|_{L^1 \cap L^\infty(\Omega_\varepsilon)} + \|\mathbf{u}_{0,\varepsilon}\|_{L^2 \cap L^\infty(\Omega_\varepsilon;\mathbb{R}^3)} \leq c, \quad (8.20)$$

with c independent of ε.

As a direct consequence of the structural properties of the Helmholtz function observed in Lemma 5.1, boundedness of the left-hand side of (8.19) gives rise to a number of useful uniform estimates. Similarly to Section 5.2, we obtain

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon;\mathbb{R}^3)} \le c, \tag{8.21}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \left\| \left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\right]_{\rm ess} \right\|_{L^2(\Omega_\varepsilon)} \le c, \tag{8.22}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \left\| \left[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\right]_{\rm ess} \right\|_{L^2(\Omega_\varepsilon)} \le c, \tag{8.23}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \| \, [\varrho_\varepsilon e(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm res}\|_{L^1(\Omega_\varepsilon)} \le \varepsilon^2 c, \tag{8.24}$$

and

$$\operatorname*{ess\,sup}_{t\in(0,T)} \| \, [\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm res}\|_{L^1(\Omega_\varepsilon)} \le \varepsilon^2 c, \tag{8.25}$$

where the “essential” and “residual” components have been introduced through (4.44), (4.45). In addition, we control the measure of the “residual set”, specifically,

$$\operatorname*{ess\,sup}_{t\in(0,T)} |\mathcal{M}^\varepsilon_{\rm res}[t]| \le \varepsilon^2 c, \tag{8.26}$$

where $\mathcal{M}^\varepsilon_{\rm res}[t] \subset \Omega$ was introduced in (4.43). Note that estimate (8.26) is particularly important as it says that the measure of the “residual” set, on which the density and the temperature are far away from the equilibrium state $(\overline{\varrho},\overline{\vartheta})$, is small, and, in addition, independent of the measure of the whole set Ω_ε.

Finally, we deduce

$$\|\sigma_\varepsilon\|_{\mathcal{M}^+([0,T]\times\overline{\Omega}_\varepsilon)} \le \varepsilon^2 c, \tag{8.27}$$

therefore,

$$\int_0^T \int_{\Omega_\varepsilon} \frac{1}{\vartheta_\varepsilon}\mathbb{S}_\varepsilon : \nabla_x \mathbf{u}_\varepsilon \ {\rm d}x \ {\rm d}t \le c, \tag{8.28}$$

and

$$\int_0^T \int_{\Omega_\varepsilon} -\frac{\mathbf{q}_\varepsilon \cdot \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon^2} \ {\rm d}x \ {\rm d}t \le \varepsilon^2 c. \tag{8.29}$$

8.2.2 Estimates based on the specific form of constitutive relations

The uniform bounds obtained in the previous section may be viewed as a consequence of the general physical principles postulated through axioms **(i)**–**(iii)** in the introductory section, combined with the hypothesis of thermodynamic stability (8.13). In order to convert them to a more convenient language of the standard function spaces, structural properties of the thermodynamic functions as well as of the transport coefficients must be specified.

Motivated by the general hypotheses of the existence theory developed in Section 3, exactly as in Section 5, we consider the *state equation* for the pressure in the form

$$p(\varrho,\vartheta) = \underbrace{p_M(\varrho,\vartheta)}_{\text{molecular pressure}} + \underbrace{p_R(\vartheta)}_{\text{radiation pressure}}, \ p_M = \vartheta^{\frac{5}{2}} P\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big), \ p_R = \frac{a}{3}\vartheta^4, \ a > 0, \tag{8.30}$$

while the internal energy reads

$$e(\varrho,\vartheta) = e_M(\varrho,\vartheta) + e_R(\varrho,\vartheta), \ e_M = \frac{3}{2}\frac{\vartheta^{\frac{5}{2}}}{\varrho} P\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big), \ e_R = a\frac{\vartheta^4}{\varrho}, \tag{8.31}$$

and, in accordance with Gibbs' relation (1.2),

$$s(\varrho,\vartheta) = s_M(\varrho,\vartheta) + s_R(\varrho,\vartheta), \ s_M(\varrho,\vartheta) = S\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big), \ s_R = \frac{4}{3}a\frac{\vartheta^3}{\varrho}, \tag{8.32}$$

where

$$S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z^2} \text{ for all } Z > 0. \tag{8.33}$$

The *hypothesis of thermodynamic stability* (8.13) reformulated in terms of the structural properties of P requires

$$P \in C^1[0,\infty) \cap C^2(0,\infty), \ P(0) = 0, \ P'(Z) > 0 \text{ for all } Z \geq 0, \tag{8.34}$$

$$0 < \frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z} \leq \sup_{z>0} \frac{\frac{5}{3}P(z) - zP'(z)}{z} < \infty. \tag{8.35}$$

Furthermore, it follows from (8.35) that $P(Z)/Z^{5/3}$ is a decreasing function of Z, and we assume that

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{\frac{5}{3}}} = p_\infty > 0. \tag{8.36}$$

The transport coefficients μ and κ will be continuously differentiable functions of the temperature ϑ satisfying the growth restrictions

$$\left\{ \begin{array}{c} 0 < \underline{\mu}(1+\vartheta) \leq \mu(\vartheta) \leq \overline{\mu}(1+\vartheta), \\ 0 < \underline{\kappa}(1+\vartheta^3) \leq \kappa(\vartheta) \leq \overline{\kappa}(1+\vartheta^3) \text{ for all } \vartheta \geq 0, \end{array} \right\} \tag{8.37}$$

where $\underline{\mu}$, $\overline{\mu}$, $\underline{\kappa}$, and $\overline{\kappa}$ are positive constants.

Having completed the list of hypotheses on constitutive relations, we observe that (8.37), together with estimate (8.28), and Newton's rheological law expressed in terms of (8.5), give rise to

$$\int_0^T \| \nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon - \frac{2}{3}\mathrm{div}_x \mathbf{u}_\varepsilon \mathbb{I} \|^2_{L^2(\Omega_\varepsilon;\mathbb{R}^{3\times 3})} \, \mathrm{d}t \leq c, \tag{8.38}$$

with c independent of $\varepsilon \to 0$.

At this stage, we apply *Korn's inequality* in the form stated in Proposition 2.1 to $r = [\varrho_\varepsilon]_{\rm ess}$, $\mathbf{v} = \mathbf{u}_\varepsilon$ and use the bounds established in (8.26), (8.38) in order to conclude that

$$\int_0^T \| \mathbf{u}_\varepsilon \|^2_{W^{1,2}(\tilde\Omega_\varepsilon;\mathbb{R}^3)} \, {\rm d}t \leq c \text{ uniformly for } \varepsilon \to 0, \tag{8.39}$$

where $\tilde\Omega_\varepsilon \subset \Omega_\varepsilon$ denotes the largest ball centered at zero such that

$$\inf_{x\in\tilde\Omega_\varepsilon} {\rm dist}[x, \partial\Omega_\varepsilon] > 2,$$

specifically,

$$\tilde\Omega_\varepsilon = B(0; r_\varepsilon), \ r_\varepsilon = \inf_{x\in\partial\Omega_\varepsilon} |x| - 2.$$

It is important to observe that the constant c in (8.39) is *independent* of the radius of the ball $\tilde\Omega_\varepsilon$. This can be seen by writing $\tilde\Omega_\varepsilon$ as a union of a finite number of unit cubes with mutually disjoint interiors contained in Ω_ε and applying Proposition 2.1 to each of them, separately. It is easy to see that PROPERTY (L) stated in the introductory part, as well as all the uniform estimates established so far, remain valid if we replace Ω_ε by $\tilde\Omega_\varepsilon$. In general, we *cannot* expect (8.39) to be valid on Ω_ε unless some restrictions are imposed on the boundary $\partial\Omega_\varepsilon$.

In a similar fashion, we can use Fourier's law (8.6) together with (8.29) to obtain

$$\int_0^T \int_{\Omega_\varepsilon} \frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon^2} |\nabla_x \vartheta_\varepsilon|^2 \, {\rm d}x \, {\rm d}t \leq \varepsilon^2 c, \tag{8.40}$$

which, combined with the structural hypotheses (8.37), the uniform bounds established in (8.23), (8.26), and the *Poincaré inequality* stated in Proposition 2.2, yields

$$\int_0^T \|\vartheta_\varepsilon - \overline{\vartheta}\|^2_{W^{1,2}(\tilde\Omega_\varepsilon)} \, {\rm d}t + \int_0^T \| \log(\vartheta_\varepsilon) - \log(\overline{\vartheta})\|^2_{W^{1,2}(\tilde\Omega_\varepsilon)} \, {\rm d}t \leq \varepsilon^2 c. \tag{8.41}$$

Finally, a combination of (8.24), (8.36) yields

$$\text{ess} \sup_{t\in(0,T)} \int_{\Omega_\varepsilon} [\varrho_\varepsilon]^{5/3}_{\rm res} \, {\rm d}x \leq \varepsilon^2 c. \tag{8.42}$$

8.3 Acoustic equation

The *acoustic equation*, introduced in Chapter 4 and thoroughly investigated in various parts of this book, governs the time evolution of the acoustic waves and as such represents a key tool for studying the time oscillations of the velocity field in the incompressible limits for problems endowed with ill-prepared data. It can be viewed as a linearization of system (8.1–8.3) around the static state $\{\overline{\varrho}, 0, \overline{\vartheta}\}$.

If $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ satisfy (8.1–8.3) in the weak sense specified in Chapter 1, we get, exactly as in Section 5.4.3,

$$\int_0^T \int_{\Omega_\varepsilon} \Big[\varepsilon\Big(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\Big)\partial_t\varphi + \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \varphi\Big] \ \mathrm{d}x \ \mathrm{d}t = 0 \tag{8.43}$$

for any test function $\varphi \in C_c^\infty((0,T)\times\Omega_\varepsilon)$;

$$\int_0^T \int_{\Omega_\varepsilon} \varepsilon\varrho_\varepsilon\Big(\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\Big)\partial_t\varphi \ \mathrm{d}x \ \mathrm{d}t \tag{8.44}$$
$$= \int_0^T \int_{\Omega_\varepsilon} \varepsilon\varrho_\varepsilon\Big(\frac{s(\overline{\varrho},\overline{\vartheta}) - s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\Big)\mathbf{u}_\varepsilon \cdot \nabla_x\varphi \ \mathrm{d}x \ \mathrm{d}t$$
$$+ \int_0^T \int_{\Omega_\varepsilon} \frac{\kappa(\vartheta_\varepsilon)\nabla_x\vartheta_\varepsilon}{\vartheta_\varepsilon} \cdot \nabla_x\varphi \ \mathrm{d}x \ \mathrm{d}t - < \sigma_\varepsilon; \varphi >_{[\mathcal{M};C]([0,T]\times\overline{\Omega})}$$

for any test function $\varphi \in C_c^\infty((0,T)\times\Omega_\varepsilon)$; and

$$\int_0^T \int_{\Omega_\varepsilon} \Big[\varepsilon(\varrho_\varepsilon\mathbf{u}_\varepsilon)\cdot\partial_t\varphi + \Big(\frac{p(\varrho_\varepsilon,\vartheta_\varepsilon) - p(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\Big)\mathrm{div}_x\varphi\Big] \ \mathrm{d}x \ \mathrm{d}t \tag{8.45}$$
$$= \int_0^T \int_{\Omega_\varepsilon} \varepsilon\Big(\mathbb{S}_\varepsilon - \varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon\Big) : \nabla_x\varphi \ \mathrm{d}x \ \mathrm{d}t$$

for any test function $\varphi \in C_c^\infty((0,T)\times\Omega_\varepsilon;\mathbb{R}^3)$.

Thus, after a simple manipulation, we obtain

$$\int_0^T \int_{\Omega_\varepsilon} \Big[\varepsilon\omega r_\varepsilon\partial_t\varphi + \omega\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla_x\varphi\Big] \ \mathrm{d}x \ \mathrm{d}t \tag{8.46}$$
$$= A\int_0^T \int_{\Omega_\varepsilon} \varepsilon\varrho_\varepsilon\Big(\frac{s(\overline{\varrho},\overline{\vartheta}) - s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\Big)\mathbf{u}_\varepsilon\cdot\nabla_x\varphi \ \mathrm{d}x \ \mathrm{d}t$$
$$+ A\int_0^T \int_{\Omega_\varepsilon} \frac{\kappa\nabla_x\vartheta_\varepsilon}{\vartheta_\varepsilon}\cdot\nabla_x\varphi \ \mathrm{d}x \ \mathrm{d}t - A\,\langle\sigma_\varepsilon;\varphi\rangle_{[\mathcal{M};C]([0,T]\times\overline{\Omega})}$$

for all $\varphi \in C_c^\infty((0,T)\times\Omega_\varepsilon)$, and

$$\int_0^T \int_{\Omega_\varepsilon} \Big[\varepsilon(\varrho_\varepsilon\mathbf{u}_\varepsilon)\cdot\partial_t\varphi + \omega r_\varepsilon\mathrm{div}_x\varphi\Big] \ \mathrm{d}x \ \mathrm{d}t \tag{8.47}$$
$$= \int_0^T \int_{\Omega_\varepsilon} \Big[\omega r_\varepsilon - \Big(\frac{p(\varrho_\varepsilon,\vartheta_\varepsilon) - p(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\Big)\Big]\mathrm{div}_x\varphi \ \mathrm{d}x \ \mathrm{d}t$$
$$+ \int_0^T \int_{\Omega_\varepsilon} \varepsilon\Big(\mathbb{S}_\varepsilon - \varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon\Big) : \nabla_x\varphi \ \mathrm{d}x \ \mathrm{d}t$$

for any test function $\varphi \in C_c^\infty((0,T) \times \Omega_\varepsilon; \mathbb{R}^3)$, where we have set

$$r_\varepsilon = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} + \frac{A}{\omega}\varrho_\varepsilon\Big(\frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon}\Big), \tag{8.48}$$

with the constants ω, A determined through

$$A\overline{\varrho}\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} = \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta}, \ \omega + A\overline{\varrho}\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} = \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho}. \tag{8.49}$$

As a direct consequence of *Gibbs' equation* (1.2), we have

$$\frac{\partial s}{\partial \varrho} = -\frac{1}{\varrho^2}\frac{\partial p}{\partial \vartheta},$$

in particular, $\omega > 0$ as soon as e, p comply with the *hypothesis of thermodynamic stability* stated in (8.13).

Finally, exactly as in Section 5.4.7, we introduce the "time lifting" Σ_ε of the measure σ_ε as

$$\Sigma_\varepsilon \in L^\infty(0,T; \mathcal{M}^+(\overline{\Omega}_\varepsilon)),$$

$$\langle \Sigma_\varepsilon; \psi \rangle_{[L^\infty(0,T;\mathcal{M}(\overline{\Omega}_\varepsilon));L^1(0,T;C(\overline{\Omega}))]} := \langle \sigma_\varepsilon; I[\varphi] \rangle_{[\mathcal{M};C]([0,T]\times\overline{\Omega}_\varepsilon)}, \tag{8.50}$$

where

$$I[\varphi](t,x) = \int_0^t \varphi(s,x) \ \mathrm{d}s.$$

Consequently, system (8.46), (8.47) can be written in a concise form as

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■ ACOUSTIC EQUATION:

$$\int_0^T \int_{\Omega_\varepsilon} \Big[\varepsilon Z_\varepsilon \partial_t \varphi + \mathbf{V}_\varepsilon \cdot \nabla_x \varphi\Big] \ \mathrm{d}x \ \mathrm{d}t = \int_0^T \int_{\Omega_\varepsilon} \varepsilon \mathbf{F}_\varepsilon^1 \cdot \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t \tag{8.51}$$

for all $\varphi \in C_c^\infty((0,T) \times \Omega_\varepsilon)$,

$$\int_0^T \int_{\Omega_\varepsilon} \Big[\varepsilon \mathbf{V}_\varepsilon \cdot \partial_t \varphi + \omega Z_\varepsilon \mathrm{div}_x \varphi\Big] \ \mathrm{d}x \ \mathrm{d}t = \int_0^T \int_{\Omega_\varepsilon} \Big(\varepsilon \mathbb{F}_\varepsilon^2 : \nabla_x \varphi + \varepsilon F_\varepsilon^3 \mathrm{div}_x \varphi\Big) \ \mathrm{d}x \ \mathrm{d}t$$
$$+ \frac{A}{\varepsilon\omega} \langle \Sigma_\varepsilon; \mathrm{div}_x \varphi \rangle_{[L^\infty(0,T;\mathcal{M}(\overline{\Omega}_\varepsilon));L^1(0,T;C(\overline{\Omega}_\varepsilon))]} \tag{8.52}$$

for all $\varphi \in C_c^\infty((0,T) \times \Omega_\varepsilon; \mathbb{R}^3)$,

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where we have set

$$Z_\varepsilon = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} + \frac{A}{\omega}\varrho_\varepsilon\Big(\frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon}\Big) + \frac{A}{\varepsilon\omega}\Sigma_\varepsilon, \ \mathbf{V}_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon, \tag{8.53}$$

$$\mathbf{F}_\varepsilon^1 = \frac{A}{\omega}\varrho_\varepsilon\Big(\frac{s(\overline{\varrho}, \overline{\vartheta}) - s(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon}\Big)\mathbf{u}_\varepsilon + \frac{A}{\omega}\frac{\kappa \nabla_x \vartheta_\varepsilon}{\varepsilon \vartheta_\varepsilon}, \tag{8.54}$$

$$\mathbb{F}_\varepsilon^2 = \mathbb{S}_\varepsilon - \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon, \tag{8.55}$$

and

$$F_\varepsilon^3 = \omega\left(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon^2}\right) + A\varrho_\varepsilon\Big(\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon^2}\Big) - \Big(\frac{p(\varrho_\varepsilon,\vartheta_\varepsilon) - p(\overline{\varrho},\overline{\vartheta})}{\varepsilon^2}\Big). \tag{8.56}$$

Here, similarly to Chapter 5, we have identified

$$\int_\Omega \Sigma_\varepsilon \varphi \, \mathrm{d}x := \langle \Sigma_\varepsilon; \varphi\rangle_{[\mathcal{M};C](\overline{\Omega})}.$$

8.4 Regularization and extension to $\mathbb{R}^3$

The acoustic equation (8.51), (8.52) provides a suitable platform for studying the time evolution of the velocity field. Since our ultimate goal is to establish the pointwise convergence of the family $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$, and since the latter is bounded in the space $L^2(0,T;W^{1,2}(\tilde{\Omega}_\varepsilon;\mathbb{R}^3))$, where the Sobolev space $W^{1,2}(\tilde{\Omega}_\varepsilon)$ is compactly embedded into $L^2(B;\mathbb{R}^3)$ for any fixed ball $B \subset \mathbb{R}^3$, it is enough to control only the oscillations with respect to time. Consequently, in order to facilitate the future analysis, we regularize equations (8.51), (8.52) in the x-variable and extend them to the whole physical space $\mathbb{R}^3$. By virtue of Property (L), any solution of the extended system will coincide with the original one on any compact B as the speed of sound is proportional to $1/\varepsilon$. Moreover, since $\tilde{\Omega}_\varepsilon$ satisfies Property (L) and all uniform bounds established in the previous part hold on $\tilde{\Omega}_\varepsilon$, we can assume that $\Omega_\varepsilon = \tilde{\Omega}_\varepsilon$.

8.4.1 Uniform estimates

To begin, we establish uniform bounds for all terms appearing on the right-hand side of the acoustic equation (8.51), (8.52).

Writing

$$\varrho_\varepsilon\Big(\frac{s(\overline{\varrho},\overline{\vartheta}) - s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\Big) = [\varrho_\varepsilon]_{\rm ess}\Big(\frac{s(\overline{\varrho},\overline{\vartheta}) - s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\Big) + [\varrho_\varepsilon]_{\rm res}\Big(\frac{s(\overline{\varrho},\overline{\vartheta}) - s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\Big),$$

we can use the uniform estimates (8.22), (8.23) in order to obtain

$$\operatorname{ess}\sup_{t\in(0,T)} \Big\| [\varrho_\varepsilon]_{\rm ess}\Big(\frac{s(\overline{\varrho},\overline{\vartheta}) - s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\Big)\Big\|_{L^2(\Omega_\varepsilon)} \leq c. \tag{8.57}$$

Furthermore, estimate (8.57) combined with (8.39) yields

$$\int_0^T \Big\| [\varrho_\varepsilon]_{\rm ess}\Big(\frac{s(\overline{\varrho},\overline{\vartheta}) - s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\Big)\mathbf{u}_\varepsilon\Big\|^2_{L^1(\Omega_\varepsilon;\mathbb{R}^3)} \leq c, \tag{8.58}$$

where both (8.57) and (8.58) are uniform for $\varepsilon \to 0$.

On the other hand, in accordance with (8.25), (8.26), and (8.42),

$$\operatorname{ess}\sup_{t\in(0,T)} \left\| [\varrho_\varepsilon]_{\rm res} \Big(\frac{s(\overline{\varrho},\overline{\vartheta}) - s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon} \Big) \right\|_{L^1(\Omega_\varepsilon)} \le \varepsilon c. \tag{8.59}$$

Now, it follows from the structural hypotheses (8.33–8.35) that

$$|\varrho s_M(\varrho,\vartheta)| \le c(1 + \varrho|\log(\varrho)| + \varrho|\log(\vartheta)|) \text{ for all positive } \varrho,\ \vartheta.$$

We deduce from (8.26), (8.42) that

$$\operatorname{ess}\sup_{t\in(0,T)} \left\| \frac{[\varrho_\varepsilon]_{\rm res}|\log(\varrho_\varepsilon)|}{\varepsilon} \right\|_{L^{6/5}(\Omega_\varepsilon)} \le c, \tag{8.60}$$

which, together with (8.21), gives rise to the uniform bound

$$\operatorname{ess}\sup_{t\in(0,T)} \left\| \frac{[\varrho_\varepsilon]_{\rm res}|\log(\varrho_\varepsilon)|}{\varepsilon} \mathbf{u}_\varepsilon \right\|^2_{L^1(\Omega_\varepsilon)} \,{\rm d}t \le c. \tag{8.61}$$

We can write

$$\left| \frac{[\varrho_\varepsilon]_{\rm res}|\log(\vartheta_\varepsilon)|\mathbf{u}_\varepsilon}{\varepsilon} \right| \le \sqrt{[\varrho_\varepsilon]_{\rm res}} \frac{|\log(\vartheta_\varepsilon) - \log(\overline{\vartheta})|}{\varepsilon} \sqrt{[\varrho_\varepsilon]_{\rm res}}\, |\mathbf{u}_\varepsilon| + \frac{[\varrho_\varepsilon]_{\rm res}}{\varepsilon} |\mathbf{u}_\varepsilon|\, |\log(\overline{\vartheta})|$$

and use the uniform estimates (8.21), (8.26), (8.41), and (8.42) in order to obtain

$$\int_0^T \left\| \frac{[\varrho_\varepsilon]_{\rm res}|\log(\vartheta_\varepsilon)|}{\varepsilon} \mathbf{u}_\varepsilon \right\|^2_{L^1(\Omega_\varepsilon)} \,{\rm d}t \le c. \tag{8.62}$$

Note that, as $\Omega_\varepsilon = \tilde{\Omega}_\varepsilon$ is a ball of radius tending to infinity for $\varepsilon \to 0$, we have

$$\|v\|_{L^6(\Omega_\varepsilon)} \le c\|v\|_{W^{1,2}(\Omega_\varepsilon)}, \tag{8.63}$$

with c *independent* of ε. Thus we conclude

$$\int_0^T \left\| [\varrho_\varepsilon]_{\rm res} \Big(\frac{s_M(\overline{\varrho},\overline{\vartheta}) - s_M(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon} \Big) \mathbf{u}_\varepsilon \right\|^2_{L^1(\Omega_\varepsilon;\mathbb{R}^3)} \le c. \tag{8.64}$$

As the contribution of the radiation energy in (8.24) gives rise to a bound

$$\operatorname{ess}\sup_{t\in(0,T)} \int_{\Omega_\varepsilon} [\vartheta_\varepsilon]^4_{\rm res} \,{\rm d}x \le \varepsilon^2 c, \tag{8.65}$$

it is easy to check that (8.64) holds also for the radiation component $\varrho_\varepsilon s_R(\varrho_\varepsilon,\vartheta_\varepsilon) \approx \vartheta_\varepsilon^3$; whence we infer

$$\int_0^T \left\| [\varrho_\varepsilon]_{\rm res} \Big(\frac{s(\overline{\varrho},\overline{\vartheta}) - s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon} \Big) \mathbf{u}_\varepsilon \right\|^2_{L^1(\Omega_\varepsilon;\mathbb{R}^3)} \le c. \tag{8.66}$$

Furthermore, using estimates (8.39), (8.40) we get

$$\int_0^T \left(\| \, [\, \mathbb{S}_\varepsilon]_{\rm ess} \|^2_{L^2(\Omega_\varepsilon;\mathbb{R}^{3\times 3})} + \left\| \, [\kappa(\vartheta_\varepsilon)]_{\rm ess} \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon \vartheta_\varepsilon} \right\|^2_{L^2(\Omega_\varepsilon;\mathbb{R}^3)} \right) \, {\rm d}t \le c. \tag{8.67}$$

Finally, estimate (8.65) can be used in combination with (8.28), (8.40) in order to conclude that

$$\int_0^T \left(\| \, [\, \mathbb{S}_\varepsilon]_{\rm res} \|^2_{L^2(\Omega_\varepsilon;\mathbb{R}^{3\times 3})} + \left\| \, [\kappa(\vartheta_\varepsilon)]_{\rm res} \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon \vartheta_\varepsilon} \right\|^2_{L^1(\Omega_\varepsilon;\mathbb{R}^3)} \right) \, {\rm d}t \le c. \tag{8.68}$$

As a matter of fact, it can be shown that the presence of the radiation terms is not necessary provided we content ourselves with a weaker bound

$$\int_0^T \left(\| \, [\, \mathbb{S}_\varepsilon]_{\rm res} \|_{L^2(\Omega_\varepsilon;\mathbb{R}^{3\times 3})} + \left\| \, [\kappa(\vartheta_\varepsilon)]_{\rm res} \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon \vartheta_\varepsilon} \right\|_{L^1(\Omega_\varepsilon;\mathbb{R}^3)} \right) \, {\rm d}t \le c.$$

The above estimates allow us to establish uniform bounds on all quantities appearing in the acoustic equation (8.51), (8.52). To begin, it follows from (8.22), (8.26), (8.27), (8.57), and (8.59) that

$$Z_\varepsilon = Z^1_\varepsilon + Z^2_\varepsilon + Z^3_\varepsilon, \tag{8.69}$$

with

$$\left\{ \begin{array}{l} \{Z^1_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T;L^2(\Omega_\varepsilon)), \\ \{Z^2_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T;L^1(\Omega_\varepsilon)), \\ \{Z^3_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T;\mathcal{M}^+(\overline{\Omega}_\varepsilon)). \end{array} \right\} \tag{8.70}$$

Similarly, using (8.21), (8.26) together with (8.42), we obtain

$$\mathbf{V}_\varepsilon = \mathbf{V}^1_\varepsilon + \mathbf{V}^2_\varepsilon, \tag{8.71}$$

where

$$\left\{ \begin{array}{l} \{\mathbf{V}^1_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0,T;L^2(\Omega_\varepsilon;\mathbb{R}^3)), \\ \{\mathbf{V}^2_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0,T;L^1(\Omega_\varepsilon;\mathbb{R}^3)). \end{array} \right\} \tag{8.72}$$

Furthermore, in accordance with (8.58), (8.66–8.68),

$$\mathbf{F}^1_\varepsilon = \mathbf{F}^{1,1}_\varepsilon + \mathbf{F}^{1,2}_\varepsilon, \tag{8.73}$$

with

$$\left\{ \begin{array}{l} \{\mathbf{F}^{1,1}_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2(0,T;L^2(\Omega_\varepsilon;\mathbb{R}^3)), \\ \{\mathbf{F}^{1,2}_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2(0,T;L^1(\Omega_\varepsilon;\mathbb{R}^3)). \end{array} \right\} \tag{8.74}$$

By the same token, estimate (8.68) yields

$$\mathbb{F}^2_\varepsilon = \mathbb{F}^{2,1}_\varepsilon + \mathbb{F}^{2,2}_\varepsilon, \tag{8.75}$$

where

$$\left\{ \begin{array}{l} \{\mathbb{F}_\varepsilon^{2,1}\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T;L^2(\Omega_\varepsilon;\mathbb{R}^{3\times 3})), \\ \{\mathbb{F}_\varepsilon^{2,2}\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T;L^1(\Omega_\varepsilon;\mathbb{R}^{3\times 3})). \end{array} \right\} \tag{8.76}$$

Finally, by virtue of our choice of the parameters A, B in (8.49), we conclude, with the help of (8.22–8.27), that

$$F_\varepsilon^3 = F_\varepsilon^{3,1} + F_\varepsilon^{3,2}, \tag{8.77}$$

with

$$\left\{ \begin{array}{l} \{F_\varepsilon^{3,1}\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T;L^1(\Omega_\varepsilon)), \\ \{F_\varepsilon^{3,2}\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T;\mathcal{M}^+(\overline{\Omega}_\varepsilon)). \end{array} \right\} \tag{8.78}$$

8.4.2 Regularization

Since the acoustic equation is considered on "large" domains, a suitable regularization is provided by a spatial convolution with a family of regularizing kernels $\{\zeta_\delta\}_{\delta>0}$, namely

$$[\mathbf{v}]^\delta(t,x) = \int_{\mathbb{R}^3} \zeta_\delta(x-y)\mathbf{v}(t,y)\ \mathrm{d}y,$$

where the kernels ζ_δ are specified in Section 10.1 in Appendix.

To begin, in view of (8.39), we can suppose that

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } L^2(0,T;W^{1,2}(B;\mathbb{R}^3)) \tag{8.79}$$

for any bounded domain $B \subset \mathbb{R}^3$. Moreover, in view of the uniform bounds (8.22), (8.42), we can pass to the limit in the continuity equation (8.1) to observe that

$$\mathrm{div}_x \mathbf{u} = 0.$$

Now, we claim that the desired relation (8.11) follows as soon as we are able to show

$$\left\{ \begin{array}{c} [\varrho_\varepsilon \mathbf{u}_\varepsilon]^\delta \to \overline{\varrho}[\mathbf{u}]^\delta \text{ in } L^2((0,T)\times B;\mathbb{R}^3) \ \text{ as } \varepsilon \to 0 \\ \text{for any bounded domain } B \subset \mathbb{R}^3, \text{ and any } \textit{fixed } \delta > 0. \end{array} \right\} \tag{8.80}$$

Indeed relation (8.80), together with the uniform bounds (8.22), (8.39), and (8.42), imply that

$$[\overline{\varrho}\ \mathbf{u}_\varepsilon]^\delta = \varepsilon[\frac{\overline{\varrho} - \varrho_\varepsilon}{\varepsilon}\mathbf{u}_\varepsilon]^\delta + [\varrho_\varepsilon \mathbf{u}_\varepsilon]^\delta \to \overline{\varrho}[\mathbf{u}]^\delta,$$

meaning

$$[\mathbf{u}_\varepsilon]^\delta \to [\mathbf{u}]^\delta \text{ in } L^2((0,T)\times B;\mathbb{R}^3) \text{ for any bounded } B \subset \mathbb{R}^3.$$

On the other hand,

$$\mathbf{v}(x) - [\mathbf{v}]^\delta(x) = \int_{\mathbb{R}^3} \frac{\mathbf{v}(x) - \mathbf{v}(x-y)}{|y|} \zeta_\delta(y)|y| \ \mathrm{d}y, \tag{8.81}$$

where, by virtue of the standard property of functions in $W^{1,2}$ known as Lagrange's formula,

$$\left\| \frac{\mathbf{v}(\cdot) - \mathbf{v}(\cdot - y)}{|y|} \right\|_{L^2(B;\mathbb{R}^3)} \leq c \|\mathbf{v}\|_{W^{1,2}(\Omega_\varepsilon;\mathbb{R}^3)} \text{ for any } \mathbf{v} \in W^{1,2}(\Omega_\varepsilon;\mathbb{R}^3) \text{ for } y \neq 0.$$

Consequently, taking $\mathbf{v} = \mathbf{u}_\varepsilon$ in (8.81) and applying Young's inequality, we obtain

$$\|\mathbf{u}_\varepsilon(t,\cdot) - [\mathbf{u}_\varepsilon(t,\cdot)]^\delta\|_{L^2(B;\mathbb{R}^3)} \leq \delta c \|\mathbf{u}_\varepsilon(t,\cdot)\|_{W^{1,2}(\Omega_\varepsilon;\mathbb{R}^3)} \text{ for a.a. } t \in (0,T).$$

Thus, in view of the uniform bound (8.39), we conclude that (8.80) implies (8.11).

The time evolution of the quantities $[\varrho_\varepsilon \mathbf{u}_\varepsilon]^\delta$ is governed by a system of equations obtained by means of regularization of (8.51), (8.52). Taking the quantities $\varphi(t,y) = \psi(t)\zeta_\delta(x-y)$ as test functions in (8.51), (8.52), we obtain the following system of equations:

■ REGULARIZED ACOUSTIC EQUATION, I:

$$\varepsilon \partial_t [Z_\varepsilon]^\delta + \mathrm{div}_x [\mathbf{V}_\varepsilon]^\delta = \varepsilon \mathrm{div}_x \mathbf{G}_{\varepsilon,\delta}, \tag{8.82}$$

$$\varepsilon \partial_t [\mathbf{V}_\varepsilon]^\delta + \omega \nabla_x [Z_\varepsilon]^\delta = \varepsilon \mathrm{div}_x \mathbb{H}_{\varepsilon,\delta} \tag{8.83}$$

for a.a. $t \in (0,T)$, $x \in \Omega_{\varepsilon,\delta}$,

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where

$$\Omega_{\varepsilon,\delta} = \{x \in \Omega_\varepsilon \mid \mathrm{dist}[x, \partial\Omega_\varepsilon]\} > \delta.$$

In accordance with the uniform estimates established in Section 8.4.1, we have

$$\{\mathbf{G}_{\varepsilon,\delta}\}_{\varepsilon>0} \text{ bounded in } L^2(0,T;W^{k,2}(\Omega_{\varepsilon,\delta};\mathbb{R}^3)), \tag{8.84}$$

$$\{\mathbb{H}_{\varepsilon,\delta}\}_{\varepsilon>0} \text{ bounded in } L^2(0,T;W^{k,2}(\Omega_{\varepsilon,\delta};\mathbb{R}^{3\times 3})), \tag{8.85}$$

$$\{[Z_\varepsilon]^\delta\}_{\varepsilon>0} \text{ bounded in } C([0,T];W^{k,2}(\Omega_{\varepsilon,\delta})), \tag{8.86}$$

and

$$[\mathbf{V}_\varepsilon]^\delta = [\varrho_\varepsilon \mathbf{u}_\varepsilon]^\delta, \ \{[\mathbf{V}_\varepsilon]^\delta\}_{\varepsilon>0} \text{ bounded in } C([0,T];W^{k,2}(\Omega_{\varepsilon,\delta};\mathbb{R}^3)), \tag{8.87}$$

for any $k = 0,1,\ldots$, where we have used the standard properties of the smoothing operators collected in Theorem 10.1 in Appendix. All estimates depend on k, "blow up" if the parameter δ approaches zero, but are uniform for $\varepsilon \to 0$. Note that such a procedure cannot improve the spatial decay of the regularized quantities determined by the "worst" space in (8.74–8.78), namely L^2.

8.4.3 Extension to the whole space $\mathbb{R}^3$

The acoustic equation (8.82), (8.83) enjoys the property of *finite speed of propagation* that is equal to $\sqrt{\omega}/\varepsilon$. In other words, if $Z_i = [Z_\varepsilon]^\delta$, $\mathbf{V}_i = [\mathbf{V}_\varepsilon]^\delta$, $i = 1, 2$, satisfy (8.82), (8.83) for $\mathbf{G}_{\varepsilon,\delta} = \mathbf{G}_i$, $\mathbb{H}_{\varepsilon,\delta} = \mathbb{H}_i$, respectively, and if

$$Z_1(0,\cdot) = Z_2(0,\cdot),\ \mathbf{V}_1(0,\cdot) = \mathbf{V}_2(0,\cdot)$$
$$\text{in } B_{T\sqrt{\omega}/\varepsilon} = \{x \in \mathbb{R}^3,\ \mathrm{dist}[x, B] < T\sqrt{\omega}/\varepsilon\},$$

$$\mathbf{G}_1 = \mathbf{G}_2,\ \mathbb{H}_1 = \mathbb{H}_2 \text{ a.a. in } (0,T) \times B_{T\sqrt{\omega}/\varepsilon},$$

for some ball $B \subset \mathbb{R}^3$, then

$$Z_1 = Z_2,\ \mathbf{V}_1 = \mathbf{V}_2 \text{ in } [0,T] \times B.$$

Indeed the functions $Z = Z_1 - Z_2$, $\mathbf{V} = \mathbf{V}_1 - \mathbf{V}_2$ satisfy the homogeneous equation

$$\partial_t Z + \frac{1}{\varepsilon}\mathrm{div}_x \mathbf{V} = 0,\ \partial_t \mathbf{V} + \frac{\omega}{\varepsilon}\nabla_x Z = 0 \text{ in } (0,T) \times B_{T\sqrt{\omega}/\varepsilon},$$

supplemented with the initial data

$$Z(0,\cdot) = 0,\ \mathbf{V}(0,\cdot) = 0 \text{ in } B_{T\sqrt{\omega}/\varepsilon}.$$

In particular,

$$\partial_t\Big(\omega Z^2 + |\mathbf{V}|^2\Big) + \frac{2\omega}{\varepsilon}\mathrm{div}_x(Z\mathbf{V}) = 0 \text{ in } (0,T) \times B_{T\sqrt{\omega}/\varepsilon}; \tag{8.88}$$

whence we get the desired result

$$Z(\tau,\cdot) = 0,\ \mathbf{V}(\tau,\cdot) = 0 \text{ in } B \text{ for any } \tau \in [0,T]$$

integrating (8.88) over the cone

$$\Big\{(t,x)\ \Big|\ t \in (0,\tau),\ x \in B_{T\sqrt{\omega}/\varepsilon},\ \mathrm{dist}[x, \partial B_{T\sqrt{\omega}/\varepsilon}] > t\sqrt{\omega}/\varepsilon\Big\}.$$

In view of this observation and Property (L), the functions $\mathbf{G}_{\varepsilon,\delta}$, $\mathbb{H}_{\varepsilon,\delta}$ appearing in the acoustic equation (8.82), (8.83) as well as the initial values $[Z_\varepsilon]^\delta(0,\cdot)$, $[\mathbf{V}_\varepsilon]^\delta(0,\cdot)$ can be extended outside Ω_ε in such a way that

$$[\varrho_\varepsilon \mathbf{u}_\varepsilon]^\delta = \mathbf{V}_{\varepsilon,\delta} \text{ in } [0,T] \times B, \tag{8.89}$$

where $Z_{\varepsilon,\delta}$, $\mathbf{V}_{\varepsilon,\delta}$ represent the unique solution of the problem:

■ Regularized Acoustic Equation, II:

$$\varepsilon\partial_t Z_{\varepsilon,\delta} + \mathrm{div}_x \mathbf{V}_{\varepsilon,\delta} = \varepsilon \mathrm{div}_x \mathbf{G}_{\varepsilon,\delta}, \tag{8.90}$$

$$\varepsilon\partial_t \mathbf{V}_{\varepsilon,\delta} + \omega\nabla_x Z_{\varepsilon,\delta} = \varepsilon \mathrm{div}_x \mathbb{H}_{\varepsilon,\delta}, \tag{8.91}$$

for a.a. $t \in (0,T)$, $x \in \mathbb{R}^3$, supplemented with the initial conditions

$$Z_{\varepsilon,\delta}(0,\cdot) = Z_{0,\varepsilon,\delta},\ \mathbf{V}_{\varepsilon,\delta} = \mathbf{V}_{0,\varepsilon,\delta}. \tag{8.92}$$

In accordance with hypothesis (8.20) on integrability of the initial data, relation (8.53), and the uniform estimates (8.84), (8.85), we may assume that

$$\{Z_{0,\varepsilon,\delta}\}_{\varepsilon>0} \text{ is bounded in } W^{k,1}(\mathbb{R}^3), \tag{8.93}$$

$$\{\mathbf{V}_{0,\varepsilon,\delta}\}_{\varepsilon>0} \text{ is bounded in } W^{k,1}(\mathbb{R}^3;\mathbb{R}^3), \tag{8.94}$$

$$\{\mathbf{G}_{\varepsilon,\delta}\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T;W^{k,2}(\mathbb{R}^3;\mathbb{R}^3)), \tag{8.95}$$

and

$$\{\mathbb{H}_{\varepsilon,\delta}\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T;W^{k,2}(\mathbb{R}^3;\mathbb{R}^3)) \tag{8.96}$$

for any $k = 0, 1, \ldots$, and any fixed $\delta > 0$. Let us point out again that the previous estimates depend on k and δ but are independent of $\varepsilon \to 0$.

Moreover, without loss of generality, we may suppose that all functions are compactly supported, and, in addition,

$$\int_{\mathbb{R}^3} Z_{0,\varepsilon,\delta}\ \mathrm{d}x = 0. \tag{8.97}$$

8.5 Dispersive estimates and time decay of the acoustic waves

In view of relations (8.80), (8.89), the proof of strong convergence of the velocities claimed in (8.11) reduces to showing

$$\mathbf{V}_{\varepsilon,\delta} \to \mathbf{V}_\delta \text{ strongly in } L^2((0,T)\times B;\mathbb{R}^3) \text{ as } \varepsilon \to 0 \text{ for any fixed } \delta > 0, \tag{8.98}$$

where $\mathbf{V}_{\varepsilon,\delta}$ solves the acoustic equation (8.90), (8.91), and, in view of (8.79),

$$\mathbf{V}_\delta|_{(0,T)\times B} = \overline{\varrho}[\mathbf{u}]^\delta|_{(0,T)\times B}.$$

Since $\delta > 0$ is fixed, we drop the subscript δ in what follows.

Compactness of the solenoidal component. To begin, observe that

$$\int_{\mathbb{R}^3} Z_\varepsilon(t,\cdot)\,\mathrm{d}x = 0 \tag{8.99}$$

as a direct consequence of (8.90), (8.97).

Analogously as in Chapter 5, we introduce the *Helmholtz decomposition* in the form

$$\mathbf{v} = \mathbf{H}[\mathbf{v}] + \mathbf{H}^\perp[\mathbf{v}],$$

where $\mathbf{H}^\perp \approx \nabla_x \Delta_x^{-1} \mathrm{div}_x$ can be determined in terms of the Fourier symbols as

$$\mathbf{H}^\perp[\mathbf{v}] = \mathcal{F}^{-1}_{\xi\to x}\Big[\frac{\xi\otimes\xi}{|\xi|^2}\mathcal{F}_{x\to\xi}[\mathbf{v}]\Big],$$

where $\mathcal{F}$ denotes the Fourier transform in the x-variable.

Applying $\mathbf{H}$ to equation (8.91) we immediately see that

$$\{\partial_t\left(\mathbf{H}[\mathbf{V}_\varepsilon]\right)\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T;W^{k,2}(\mathbb{R}^3;\mathbb{R}^3)), \tag{8.100}$$

in particular, given the regularity of the initial data stated in (8.93), (8.94), we can assume

$$\mathbf{H}[\mathbf{V}_\varepsilon] \to \mathbf{V} \text{ in } L^2((0,T)\times B;\mathbb{R}^3) \tag{8.101}$$

for a certain (solenoidal) vector field $\mathbf{V} \in L^2((0,T)\times B;\mathbb{R}^3)$. Note that, as a direct consequence of the embedding relation $W^{n,3}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ for $n>3$, we have $W^{m,1}(\mathbb{R}^3) \hookrightarrow W^{k,2}(\mathbb{R}^3)$ as soon as $m > k+3$.

A wave equation for the gradient component. In view of (8.101), the proof of (8.98) reduces to showing strong convergence for the gradient components $\mathbf{H}^\perp[\mathbf{V}_\varepsilon]$. In accordance with (8.99), the acoustic equation (8.90), (8.91) gives rise to

■ LINEAR WAVE EQUATION:

$$\varepsilon\partial_t z_\varepsilon - \Delta\Psi_\varepsilon = \varepsilon g_\varepsilon, \tag{8.102}$$

$$\varepsilon\partial_t \Psi_\varepsilon - \omega z_\varepsilon = \varepsilon h_\varepsilon, \tag{8.103}$$

with the initial condition

$$z_\varepsilon(0,\cdot) = z_{0,\varepsilon},\ \ \Psi_\varepsilon(0,\cdot) = \Psi_{0,\varepsilon}, \tag{8.104}$$

where we have introduced

$$z_\varepsilon = -Z_\varepsilon,\ \ \Psi_\varepsilon = \Delta_x^{-1}\mathrm{div}_x[\mathbf{V}_\varepsilon],\ \ \nabla_x\Psi_\varepsilon = \mathbf{H}^\perp[\mathbf{V}_\varepsilon].$$

In accordance with (8.90–8.97), we have

$$\{z_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } W^{k,1}(\mathbb{R}^3),\ \ \{\Psi_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } W^{k,2}(\mathbb{R}^3), \tag{8.105}$$

together with

$$\{g_\varepsilon\}_{\varepsilon>0},\ \{h_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2(0,T;W^{k,2}(\mathbb{R}^3)). \tag{8.106}$$

Moreover,

$$\int_{\mathbb{R}^3} g_\varepsilon \ \mathrm{d}x = \int_{\mathbb{R}^3} h_\varepsilon \ \mathrm{d}x = 0. \tag{8.107}$$

Here, we have used the fact that

$$\varepsilon\partial_t \nabla_x \Psi_\varepsilon - \omega \nabla_x z_\varepsilon = \varepsilon \nabla_x h_\varepsilon$$

implies (8.103) since the quantities $\partial_t \Psi_\varepsilon$, z_ε, h_ε belong to $L^2((0,T)\times\mathbb{R}^3)$.

Note that, in contrast with $\nabla_x \Psi_{0,\varepsilon}$ belonging to the class (8.94), the potential $\Psi_{0,\varepsilon}$ may not belong to the space $L^1(\mathbb{R}^3)$.

Dispersive estimates and decay for the wave equation. In accordance with (8.101), we have to verify strong convergence of the potential Ψ_ε on the set $(0,T)\times B$. In fact, we show that

$$\nabla_x \Psi_\varepsilon \to 0 \text{ in } L^2((0,T)\times B;\mathbb{R}^3), \tag{8.108}$$

which, in particular, completes the proof of (8.11). To this end, we invoke the dispersive estimates available for the linear wave equation (8.102), (8.103).

To begin, we express solutions of the evolutionary problem (8.102–8.104) by means of *Duhamel's formula*

$$\left[\begin{array}{c} z_\varepsilon \\ \Psi_\varepsilon \end{array}\right](t) = S\Big(\frac{t}{\varepsilon}\Big)\left[\begin{array}{c} z_{0,\varepsilon} \\ \Psi_{0,\varepsilon} \end{array}\right] + \int_0^t S\Big(\frac{t-s}{\varepsilon}\Big)\left[\begin{array}{c} g_\varepsilon(s) \\ h_\varepsilon(s) \end{array}\right] \mathrm{d}s, \tag{8.109}$$

where

$$S(t)\left[\begin{array}{c} z_0 \\ \Psi_0 \end{array}\right] = \left[\begin{array}{c} z(t) \\ \Psi(t) \end{array}\right] \tag{8.110}$$

is the unique solution of the homogeneous problem

$$\partial_t z - \Delta\Psi = 0,\ \partial_t\Psi - \omega z = 0,\ z(0) = z_0,\ \Psi(0) = \Psi_0. \tag{8.111}$$

The spatial Fourier transform is an exceptionally well-suited tool in order to deal with solutions of the homogeneous wave equation (8.111). More specifically, we have

$$\begin{aligned} z(t,x) = {} & \exp(\mathrm{i}\sqrt{-\omega\Delta_x}t)\left[\frac{1}{2}\left(\frac{\mathrm{i}}{\sqrt{\omega}}\sqrt{-\Delta_x}[\Psi_0] + z_0\right)\right] \\ & + \exp(-\mathrm{i}\sqrt{-\omega\Delta_x}t)\left[\frac{1}{2}\left(-\frac{\mathrm{i}}{\sqrt{\omega}}\sqrt{-\Delta_x}[\Psi_0] + z_0\right)\right], \end{aligned} \tag{8.112}$$

$$\begin{aligned} \Psi(t,x) = {} & \exp(\mathrm{i}\sqrt{-\omega\Delta_x}t)\left[\frac{1}{2}\left(\Psi_0 - \mathrm{i}\frac{\sqrt{\omega}}{\sqrt{-\Delta_x}}[z_0]\right)\right] \\ & + \exp(-\mathrm{i}\sqrt{-\omega\Delta_x}t)\left[\frac{1}{2}\left(\Psi_0 + \mathrm{i}\frac{\sqrt{\omega}}{\sqrt{-\Delta_x}}[z_0]\right)\right], \end{aligned} \tag{8.113}$$

or, in terms of the spatial Fourier transform,

$$\begin{aligned}\mathcal{F}_{x\to\xi}[z](t,\xi) &= \frac{1}{2}\left(\frac{\mathrm{i}}{\sqrt{\omega}}|\xi|\mathcal{F}_{x\to\xi}[\Psi_0](\xi) + \mathcal{F}_{x\to\xi}[z_0](\xi)\right)\exp\left(\mathrm{i}\sqrt{\omega}|\xi|t\right)\\ &+ \frac{1}{2}\left(-\frac{\mathrm{i}}{\sqrt{\omega}}|\xi|\mathcal{F}_{x\to\xi}[\Psi_0](\xi) + \mathcal{F}_{x\to\xi}[z_0](\xi)\right)\exp\left(-\mathrm{i}\sqrt{\omega}|\xi|t\right),\\ \mathcal{F}_{x\to\xi}[\Psi](t,\xi) &= \frac{1}{2}\left(\mathcal{F}_{x\to\xi}[\Psi_0](\xi) - \mathrm{i}\frac{\sqrt{\omega}}{|\xi|}|\mathcal{F}_{x\to\xi}[z_0](\xi)\right)\exp\left(\mathrm{i}\sqrt{\omega}|\xi|t\right)\\ &+ \frac{1}{2}\left(\mathcal{F}_{x\to\xi}[\Psi_0](\xi) + \mathrm{i}\frac{\sqrt{\omega}}{|\xi|}|\mathcal{F}_{x\to\xi}[z_0](\xi)\right)\exp\left(-\mathrm{i}\sqrt{\omega}|\xi|t\right).\end{aligned}$$

At this stage, it is convenient to introduce the scale of *homogeneous Sobolev spaces* $H^\alpha(\mathbb{R}^3)$,

$$\begin{aligned}H^\alpha(\mathbb{R}^3) = \Big\{v \in \mathcal{S}'(\mathbb{R}^3) \;\Big|\; &\|v\|_{H^\alpha}\\ &= \|(-\omega\Delta_x)^{\alpha/2}v\|_{L^2(\mathbb{R}^3)} \equiv \int_{\mathbb{R}^3}\omega^\alpha|\xi|^{2\alpha}|\mathcal{F}_{x\to\xi}[v](\xi)|^2\,\mathrm{d}\xi < \infty\Big\},\;\; \alpha\in\mathbb{R},\end{aligned}$$

where the symbol $\mathcal{S}'(\mathbb{R}^3)$ denotes the Schwartz space of tempered distributions on $\mathbb{R}^3$.

Using formulas (8.112), (8.113) we recover the standard energy equality for the homogeneous wave equation (8.110), namely

$$\|\Phi(t,\cdot)\|^2_{H^{\alpha+1}(\mathbb{R}^3)} + \|z(t,\cdot)\|^2_{H^\alpha(\mathbb{R}^3)} = \|\Phi_0\|^2_{H^{\alpha+1}(\mathbb{R}^3)} + \|z_0\|^2_{H^\alpha(\mathbb{R}^3)} \text{ for all } t\in\mathbb{R} \tag{8.114}$$

whenever the right-hand side is finite.

The following result is the key tool for proving (8.108) (cf. Smith and Sogge [184, Lemma 2.2]).

Lemma 8.1. *Let $B \subset \mathbb{R}^3$ be a bounded ball.*

Then

$$\int_{-\infty}^{\infty}\left\|\exp\left(\mathrm{i}\sqrt{-\omega\Delta_x}t\right)[v]\right\|^2_{L^2(B)}\,\mathrm{d}t \le c_B\|v\|^2_{L^2(\mathbb{R}^3)} \tag{8.115}$$

for any $v \in L^2(\mathbb{R}^3)$.

Proof. It is enough to show (8.115) for a smooth function v. Take a non-negative function $\varphi \in C_c^\infty(\mathbb{R}^3)$ such that $\varphi|_B \equiv 1$. It is easy to see that

$$\int_{-\infty}^{\infty}\left\|\exp\left(\mathrm{i}\sqrt{-\omega\Delta_x}t\right)[v]\right\|^2_{L^2(B)}\,\mathrm{d}t \le \int_{\mathbb{R}^4}\left|\varphi\exp\left(\mathrm{i}\sqrt{-\omega\Delta_x}t\right)[v]\right|^2\,\mathrm{d}x\,\mathrm{d}t.$$

Denoting by

$$\hat{w} = \mathcal{F}_{x\to\xi}[w]$$

the *space* Fourier transform, we can compute

$$\begin{aligned}\mathcal{F}_{(t,x)\to(\tau,\xi)}\left[\varphi\exp\left(\mathrm{i}\sqrt{-\omega\Delta_x}t\right)[v]\right] &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^3}\hat{\varphi}(\xi-\eta)\delta(\tau-\sqrt{\omega}|\eta|)\hat{v}(\eta)\ \mathrm{d}\eta\\ &= \frac{1}{\sqrt{2\pi}}\int_{\{\tau=\sqrt{\omega}|\eta|\}}\hat{\varphi}(\xi-\eta)\hat{v}(\eta)\ \mathrm{d}S_\eta,\end{aligned}$$

where δ is the Dirac distribution at zero and $\mathcal{F}_{(t,x)\to(\tau,\xi)}$ denotes the *time-space* Fourier transform. In particular, by virtue of Plancherel's identity,

$$\begin{aligned}&\int_{\mathbb{R}^4}\left|\varphi\exp\left(\mathrm{i}\sqrt{-\omega\Delta_x}t\right)[v]\right|^2\ \mathrm{d}x\ \mathrm{d}t\\ &\quad= \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\left|\int_{\{\tau=\sqrt{\omega}|\eta|\}}\hat{\varphi}(\xi-\eta)\hat{v}(\eta)\ \mathrm{d}S_\eta\right|^2\ \mathrm{d}\xi\ \mathrm{d}\tau.\end{aligned}$$

Furthermore, using the Cauchy-Schwartz inequality in the variable η we get

$$\begin{aligned}&\left|\int_{\{\tau=\sqrt{\omega}|\eta|\}}\hat{\varphi}(\xi-\eta)\hat{v}(\eta)\ \mathrm{d}S_\eta\right|^2\\ &\quad\le \int_{\{\tau=\sqrt{\omega}|\eta|\}}|\hat{\varphi}(\xi-\eta)|\ \mathrm{d}S_\eta\ \int_{\{\tau=\sqrt{\omega}|\eta|\}}|\varphi(\xi-\eta)|\ |\hat{v}(\eta)|^2\ \mathrm{d}S_\eta,\end{aligned}$$

where we exploit the fact that $\varphi\in C_c^\infty(\mathbb{R}^3)$ in order to observe that

$$\sup_{\tau\in R,\ \xi\in\mathbb{R}^3}\left\{\int_{\{\tau=\sqrt{\omega}|\eta|\}}|\hat{\varphi}(\xi-\eta)|\ \mathrm{d}S_\eta\right\}\le c_1(\varphi).$$

Consequently,

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\left|\int_{\{\tau=\sqrt{\omega}|\eta|\}}\hat{\varphi}(\xi-\eta)\hat{v}(\eta)\ \mathrm{d}S_\eta\right|^2\ \mathrm{d}\xi\ \mathrm{d}\tau\\ &\quad\le c_1(\varphi)\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\int_{\{\tau=\sqrt{\omega}|\eta|\}}|\hat{\varphi}(\xi-\eta)||\hat{v}(\eta)|^2\ \mathrm{d}S_\eta\ \mathrm{d}\xi\ \mathrm{d}\tau\\ &\quad= c_1(\varphi)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|\hat{\varphi}(\xi-\eta)||\hat{v}(\eta)|^2\ \mathrm{d}\eta\ \mathrm{d}\xi\\ &\quad\le c_2(\varphi)\int_{\mathbb{R}^3}|\hat{v}(\eta)|^2\ \mathrm{d}\eta = c_2(\varphi)\|v\|^2_{L^2(\mathbb{R}^3)}.\end{aligned}$$

□

Using the explicit formulas (8.112), (8.113), together with Lemma 8.1, we deduce

$$\int_{-\infty}^{\infty} \left\| S(t) \begin{bmatrix} z_0 \\ \Psi_0 \end{bmatrix} \right\|_{E(B)}^2 \mathrm{d}t \leq c \left\| \begin{bmatrix} z_0 \\ \Psi_0 \end{bmatrix} \right\|_{H^0(\mathbb{R}^3)\times H^1(\mathbb{R}^3)}^2, \tag{8.116}$$

where we have introduced the semi-norm

$$\left\| \begin{bmatrix} z \\ \Psi \end{bmatrix} \right\|_{E(B)}^2 = \|z\|_{L^2(B)}^2 + \|\nabla_x \Psi\|_{L^2(B;\mathbb{R}^3)}^2.$$

Rescaling (8.116) in t we get

$$\int_{-\infty}^{\infty} \left\| S\Big(\frac{t}{\varepsilon}\Big) \begin{bmatrix} z_{0,\varepsilon} \\ \Psi_{0,\varepsilon} \end{bmatrix} \right\|_{E(B)}^2 \mathrm{d}t \leq \varepsilon c \left\| \begin{bmatrix} z_{0,\varepsilon} \\ \Psi_{0,\varepsilon} \end{bmatrix} \right\|_{H^0(\mathbb{R}^3)\times H^1(\mathbb{R}^3)}^2. \tag{8.117}$$

Finally, by the same token,

$$\begin{aligned} &\int_0^T \left\| \int_0^t S\Big(\frac{t-s}{\varepsilon}\Big) \begin{bmatrix} g_\varepsilon(s) \\ h_\varepsilon(s) \end{bmatrix} \mathrm{d}s \right\|_{E(B)}^2 \mathrm{d}t \qquad (8.118)\\ &\leq c(T) \int_0^T \int_{-\infty}^{\infty} \left\| S\Big(\frac{t}{\varepsilon}\Big) S\Big(\frac{-s}{\varepsilon}\Big) \begin{bmatrix} g_\varepsilon(s) \\ h_\varepsilon(s) \end{bmatrix} \right\|_{E(B)}^2 \mathrm{d}t\ \mathrm{d}s \\ &\leq \varepsilon c(T) \int_0^T \left\| S\Big(\frac{-s}{\varepsilon}\Big) \begin{bmatrix} g_\varepsilon(s) \\ h_\varepsilon(s) \end{bmatrix} \right\|_{H^0(\mathbb{R}^3)\times H^1(\mathbb{R}^3)}^2 \mathrm{d}s \\ &= \varepsilon c(T) \int_0^T \left\| \begin{bmatrix} g_\varepsilon(s) \\ h_\varepsilon(s) \end{bmatrix} \right\|_{H^0(\mathbb{R}^3)\times H^1(\mathbb{R}^3)}^2 \mathrm{d}s, \end{aligned}$$

where we have used the fact that $(S(t))_{t\in R}$ is a group of isometries on the energy space $H^0(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$.

Revoking formula (8.109), we can use (8.117), (8.118), together with the uniform estimates (8.105), (8.106) in order to obtain the desired relation (8.108).

8.6 Conclusion – main result

The main result of this chapter can be stated in the following form.

■ Local Decay of Acoustic Waves:

Theorem 8.1. *Let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a family of domains in $\mathbb{R}^3$ having* Property (L). *Assume that the thermodynamic functions p, e, s as well as the transport coefficients μ, κ satisfy the structural hypotheses* (8.30–8.37). *Let $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ be a solution of the* Navier-Stokes-Fourier system (8.1–8.6) *in $\mathcal{D}'((0,T)\times\Omega_\varepsilon)$*

satisfying (8.14–8.18)*, where the initial data* (8.8)*,* (8.9)*,* (8.10) *satisfy hypothesis* (8.20)*.*

Then, at least for a suitable subsequence, we have

$$\mathbf{u}_\varepsilon \to \mathbf{u} \textit{ in } L^2((0,T) \times B; \mathbb{R}^3) \textit{ for any bounded ball } B \subset \mathbb{R}^3,$$

where $\mathrm{div}_x \mathbf{u} = 0$*.*

The presence of the radiation terms in the system is not necessary. The same result can be obtained if $a = 0$ in (8.30). Moreover, exactly as in Chapter 5, we can show that the solutions $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ of the complete NAVIER-STOKES-FOURIER SYSTEM tend to the corresponding solution of the OBERBECK-BOUSSINESQ SYSTEM (locally in space) as $\varepsilon \to 0$. The details are left to the reader.

Chapter 9

Acoustic Analogies

We interpret our previous results on the singular limits of the NAVIER-STOKES-FOURIER SYSTEM in terms of the *acoustic analogies* discussed briefly in Chapters 4, 5. Let us recall that an acoustic analogy is represented by a non-homogeneous wave equation supplemented with source terms obtained simply by regrouping the original (primitive) system. In the low Mach number regime, the source terms may be evaluated on the basis of the limit (incompressible) system. This is the principal idea of the so-called *hybrid method* used in numerical analysis. Our goal is to discuss the advantages as well as limitations of this approach in light of the exact mathematical results obtained so far.

As a model problem, we revoke the situation examined in Chapter 5, where the fluid is driven by an external force $\mathbf{f}$ of moderate strength in comparison with the characteristic frequency of the acoustic waves. More precisely, we consider a family of weak solutions $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ to the NAVIER-STOKES-FOURIER SYSTEM:

$$\partial_t \varrho_\varepsilon + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0, \tag{9.1}$$

$$\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \frac{1}{\varepsilon^2}\nabla_x p(\varrho_\varepsilon, \vartheta_\varepsilon) = \mathrm{div}_x \mathbb{S}_\varepsilon + \varrho_\varepsilon \mathbf{f}, \tag{9.2}$$

$$\partial_t(\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)) + \mathrm{div}_x(\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)\mathbf{u}_\varepsilon) + \mathrm{div}_x\left(\frac{\mathbf{q}_\varepsilon}{\vartheta_\varepsilon}\right) = \sigma_\varepsilon, \tag{9.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{\varepsilon^2}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon)\right)\,\mathrm{d}x = \varepsilon^2\int_\Omega \varrho_\varepsilon \mathbf{f}\cdot\mathbf{u}_\varepsilon\,\mathrm{d}x, \tag{9.4}$$

where the thermodynamic functions p, e, and s satisfy hypotheses (5.17–5.23) specified in Section 5.1.

In addition , we suppose that

$$\mathbb{S}_\varepsilon = \mu(\vartheta_\varepsilon)\Big(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^T \mathbf{u}_\varepsilon - \frac{2}{3}\mathrm{div}_x \mathbf{u}_\varepsilon\;\mathbb{I}\Big), \tag{9.5}$$

$$\mathbf{q}_\varepsilon = -\kappa(\vartheta_\varepsilon)\nabla_x \vartheta_\varepsilon, \tag{9.6}$$

and, in agreement with our concept of weak solutions,

$$\sigma_\varepsilon \geq \frac{1}{\vartheta_\varepsilon}\left(\varepsilon^2 \mathbb{S}_\varepsilon : \nabla_x \mathbf{u}_\varepsilon - \frac{\mathbf{q}_\varepsilon \cdot \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon}\right), \tag{9.7}$$

where the transport coefficients μ, κ obey (5.24), (5.25).

Exactly as in Chapter 5, the problem is posed on a regular bounded spatial domain $\Omega \subset \mathbb{R}^3$, and supplemented with the conservative boundary conditions

$$\mathbf{u}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0,\ \mathbb{S}_\varepsilon \mathbf{n} \times \mathbf{n}|_{\partial\Omega} = 0,\ \mathbf{q}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{9.8}$$

The initial data are taken in the form

$$\left\{ \begin{array}{c} \varrho_\varepsilon(0,\cdot) = \overline{\varrho} + \varepsilon \varrho_{0,\varepsilon}^{(1)} + \varepsilon^2 \varrho_{0,\varepsilon}^{(2)}, \\ \mathbf{u}_\varepsilon(0,\cdot) = \mathbf{u}_{0,\varepsilon} + \varepsilon \mathbf{u}_{0,\varepsilon}^{(1)}, \\ \vartheta_\varepsilon(0,\cdot) = \overline{\vartheta} + \varepsilon \vartheta_{0,\varepsilon}^{(1)} + \varepsilon^2 \vartheta_{0,\varepsilon}^{(2)}, \end{array} \right\} \tag{9.9}$$

where $\overline{\varrho} > 0$, $\overline{\vartheta} > 0$ are constant, and

$$\int_\Omega \varrho_{0,\varepsilon}^{(j)}\ \mathrm{d}x = \int_\Omega \vartheta_{0,\varepsilon}^{(j)}\ \mathrm{d}x = 0 \text{ for } j = 1, 2, \text{ and } \varepsilon > 0. \tag{9.10}$$

9.1 Asymptotic analysis and the limit system

In accordance with the arguments set forth in Chapter 5, the limit problem can be identified exactly as in Theorem 5.2. Assuming that

$$\mathbf{f} \text{ is a function belonging to } L^\infty((0,T) \times \Omega; \mathbb{R}^3), \tag{9.11}$$

$$\left\{ \begin{array}{c} \{\varrho_{0,\varepsilon}^{(2)}\}_{\varepsilon>0},\ \{\vartheta_{0,\varepsilon}^{(2)}\}_{\varepsilon>0} \text{ are bounded in } L^\infty(\Omega), \\ \{\mathbf{u}_{0,\varepsilon}^{(1)}\}_{\varepsilon>0} \text{ is bounded in } L^\infty(\Omega;\mathbb{R}^3), \end{array} \right\} \tag{9.12}$$

and

$$\left\{ \begin{array}{c} \varrho_{0,\varepsilon}^{(1)} \to \varrho_0^{(1)} \text{ weakly-(*) in } L^\infty(\Omega) \\ \mathbf{u}_{0,\varepsilon} \to \mathbf{U}_0 \text{ weakly in } L^\infty(\Omega;\mathbb{R}^3), \\ \vartheta_{0,\varepsilon}^{(1)} \to \vartheta_0^{(1)} \text{ weakly-(*) in } L^\infty(\Omega), \end{array} \right\} \tag{9.13}$$

we have that, at least for a suitable subsequence,

$$\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} := \varrho_\varepsilon^{(1)} \to \varrho^{(1)} \text{ weakly-(*) in } L^\infty(0,T;L^{5/3}(\Omega)), \tag{9.14}$$

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)), \tag{9.15}$$

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} := \vartheta_\varepsilon^{(1)} \to \Theta \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{9.16}$$

where $\mathbf{U}$, Θ solve the *target problem* in the form

$$\mathrm{div}_x \mathbf{U} = 0, \tag{9.17}$$

$$\overline{\varrho}\Big(\partial_t \mathbf{U} + \mathrm{div}_x(\mathbf{U}\otimes\mathbf{U})\Big) + \nabla_x \Pi = \mathrm{div}_x\Big(\mu(\overline{\vartheta})(\nabla_x \mathbf{U} + \nabla_x^T \mathbf{U})\Big) + \overline{\varrho}\mathbf{f}, \tag{9.18}$$

$$\overline{\varrho}c_p(\overline{\varrho},\overline{\vartheta})\Big(\partial_t \Theta + \mathbf{U}\nabla_x\Theta\Big) - \mathrm{div}_x\Big(\kappa(\overline{\vartheta})\nabla_x\Theta\Big) = 0, \tag{9.19}$$

with the boundary conditions

$$\mathbf{U}\cdot\mathbf{n}|_{\partial\Omega} = 0,\ (\nabla_x\mathbf{U} + \nabla_x^T\mathbf{U})\mathbf{n}\times\mathbf{n}|_{\partial\Omega} = 0,\ \nabla_x\Theta\cdot\mathbf{n} = 0, \tag{9.20}$$

and the initial data

$$\mathbf{U}(0,\cdot) = \mathbf{H}[\mathbf{U}_0],\ \Theta(0,\cdot) = \frac{\overline{\vartheta}}{c_p(\overline{\varrho},\overline{\vartheta})}\Big(\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\varrho_0^{(1)} + \frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\vartheta_0^{(1)}\Big). \tag{9.21}$$

Moreover, by virtue of (5.103),

$$\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\varrho^{(1)} + \frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\Theta = 0. \tag{9.22}$$

The proof is precisely like that of Theorem 5.2, except that we have to deal with a *bounded* driving term $\mathbf{f}$ in place of a *singular* one $\frac{1}{\varepsilon}\nabla_x F$. Accordingly, the fluid part represented by the incompressible NAVIER-STOKES SYSTEM (9.17), (9.18) is completely independent of the limit temperature field Θ. The reader can consult the corresponding parts of Chapter 5 for the weak formulation of both the primitive and the target system as well as for all details concerning the proof. We recall that the specific heat at constant pressure c_p is related to $\overline{\varrho}$, $\overline{\vartheta}$ by (4.17).

9.2 Acoustic equation revisited

The primitive system (9.1–9.3) can be written in the form of a linear wave equation derived in Section 5.4.3, namely

■ SCALED ACOUSTIC EQUATION:

$$\int_0^T\int_\Omega \Big(\varepsilon r_\varepsilon \partial_t\varphi + \mathbf{V}_\varepsilon\cdot\nabla_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{9.23}$$
$$= \varepsilon\frac{A}{\omega}\left(\int_0^T\int_\Omega \mathbf{s}_\varepsilon^1\cdot\nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t - \frac{1}{\varepsilon}\langle\sigma_\varepsilon;\varphi\rangle_{[\mathcal{M};C]([0,T]\times\overline{\Omega})}\right)$$

for any $\varphi\in C_c^\infty((0,T)\times\overline{\Omega})$,

$$\int_0^T \int_\Omega \Big(\varepsilon \mathbf{V}_\varepsilon \cdot \partial_t \varphi + \omega r_\varepsilon \mathrm{div}_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t \tag{9.24}$$
$$= \varepsilon \int_0^T \int_\Omega \Big(\mathbf{s}^2_\varepsilon : \nabla_x \varphi + \mathbf{s}^3_\varepsilon \cdot \varphi + s^4_\varepsilon \mathrm{div}_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t$$

for any $\varphi \in C^\infty_c((0,T) \times \overline{\Omega}; \mathbb{R}^3)$, $\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$,

---

where A, $\omega > 0$ are constants given by (5.126), and the source terms $\mathbf{s}^1_\varepsilon$, $\mathbf{s}^2_\varepsilon$, $\mathbf{s}^3_\varepsilon$, s^4_ε have been identified as follows:

$$\mathbf{s}^1_\varepsilon = \frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \left(\nabla_x \frac{\vartheta_\varepsilon}{\varepsilon} \right) + \varrho_\varepsilon \left(\frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \right) \mathbf{u}_\varepsilon, \tag{9.25}$$

$$\mathbf{s}^2_\varepsilon = \mathbb{S}_\varepsilon - \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon, \tag{9.26}$$

$$\mathbf{s}^3_\varepsilon = -\varrho_\varepsilon \mathbf{f}, \tag{9.27}$$

$$s^4_\varepsilon = \frac{1}{\varepsilon} \left(\frac{p(\overline{\varrho}, \overline{\vartheta}) - p(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon} + A \varrho_\varepsilon \frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} + \omega \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right). \tag{9.28}$$

In addition, we have

$$r_\varepsilon = \frac{1}{\omega} \left(\omega \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} + A \varrho_\varepsilon \frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \right), \ \mathbf{V}_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon. \tag{9.29}$$

In accordance with the uniform bounds established in Section 5.3, specifically (5.44), (5.69), (5.70), and (5.77), we have

$$r_\varepsilon = [r_\varepsilon]_{\mathrm{ess}} + [r_\varepsilon]_{\mathrm{res}},$$

with

$$\{[r_\varepsilon]_{\mathrm{ess}}\}_{\varepsilon > 0} \text{ bounded in } L^\infty(0,T; L^2(\Omega)), \tag{9.30}$$

and

$$[r_\varepsilon]_{\mathrm{res}} \to 0 \text{ in } L^\infty(0,T; L^1(\Omega)). \tag{9.31}$$

Similarly, by virtue of (5.41), (5.45), and (5.48),

$$\mathbf{V}_\varepsilon = [\mathbf{V}_\varepsilon]_{\mathrm{ess}} + [\mathbf{V}_\varepsilon]_{\mathrm{res}},$$

where

$$\{[\mathbf{V}_\varepsilon]_{\mathrm{ess}}\}_{\varepsilon > 0} \text{ is bounded in } L^\infty(0,T; L^2(\Omega; \mathbb{R}^3)), \tag{9.32}$$

while

$$[\mathbf{V}_\varepsilon]_{\mathrm{res}} \to 0 \text{ in } L^\infty(0,T; L^1(\Omega; \mathbb{R}^3)). \tag{9.33}$$

Finally,

$$\left\{ \begin{array}{c} \{\mathbf{s}^1_\varepsilon\}_{\varepsilon > 0}, \ \{\mathbf{s}^3_\varepsilon\}_{\varepsilon > 0} \text{ are bounded in } L^q(0,T; L^1(\Omega; \mathbb{R}^3)), \\ \{s^2_\varepsilon\}_{\varepsilon > 0} \text{ is bounded in } L^q(0,T; L^1(\Omega; \mathbb{R}^{3\times 3})), \\ \{s^4_\varepsilon\}_{\varepsilon > 0} \text{ is bounded in } L^q(0,T; L^1(\Omega)) \end{array} \right\} \tag{9.34}$$

for a certain $q > 1$, and

$$\|\sigma_\varepsilon\|_{\mathcal{M}^+([0,T]\times\overline{\Omega})} \leq \varepsilon^2 c \tag{9.35}$$

as stated in (5.135), (5.136). It is worth noting that these bounds are optimal, in particular, *compactness* of the source terms in the afore-mentioned spaces is not expected. This fact is intimately related to the time oscillations of solutions to the acoustic equation.

We conclude this part by introducing a "lifted" measure, namely $\Sigma_\varepsilon \in L^\infty(0,T;\mathcal{M}^+(\overline{\Omega}))$,

$$\langle \Sigma_\varepsilon; \varphi \rangle_{[L^\infty(0,T;\mathcal{M};L^1(0,T;C(\overline{\Omega}))]} = \langle \sigma_\varepsilon; I[\varphi] \rangle_{[\mathcal{M};C]([0,T]\times\overline{\Omega})},$$
$$I[\varphi](\tau,x) := \int_0^\tau \varphi(t,x)\ \mathrm{d}t \text{ for } \varphi \in\ L^1(0,T;C(\overline{\Omega})),$$

and rewriting system (9.23), (9.24) in the form

$$\int_0^T \int_\Omega \Big(\varepsilon Z_\varepsilon \partial_t \varphi + \mathbf{V}_\varepsilon \cdot \nabla_x \varphi \Big)\ \mathrm{d}x\ \mathrm{d}t = \varepsilon \frac{A}{\omega} \int_0^T \int_\Omega \mathbf{s}^1_\varepsilon \cdot \nabla_x \varphi\ \mathrm{d}x\ \mathrm{d}t \tag{9.36}$$

for any $\varphi \in C^\infty_c((0,T)\times\overline{\Omega})$,

$$\int_0^T \int_\Omega \Big(\varepsilon \mathbf{V}_\varepsilon \cdot \partial_t \varphi + \omega Z_\varepsilon \mathrm{div}_x \varphi \Big)\ \mathrm{d}x\ \mathrm{d}t \tag{9.37}$$
$$= \varepsilon \int_0^T \int_\Omega \Big(\mathbf{s}^2_\varepsilon : \nabla_x \varphi + \mathbf{s}^3_\varepsilon \cdot \varphi + s^4_\varepsilon \mathrm{div}_x \varphi + s^5_\varepsilon \mathrm{div}_x \varphi \Big)\ \mathrm{d}x\ \mathrm{d}t$$

for any $\varphi \in C^\infty_c((0,T)\times\overline{\Omega};\mathbb{R}^3)$, $\varphi\cdot\mathbf{n}|_{\partial\Omega} = 0$, where

$$Z_\varepsilon = r_\varepsilon + \frac{A}{\varepsilon\omega}\Sigma_\varepsilon,\ s^5_\varepsilon = \frac{A}{\varepsilon^2}\Sigma_\varepsilon,$$

where, exactly as in Section 5.4.7, we identify

$$\int_\Omega \Sigma_\varepsilon \varphi\ \mathrm{d}x = \langle \Sigma_\varepsilon; \varphi \rangle_{[\mathcal{M};C](\overline{\Omega})}.$$

Solutions of system (9.36), (9.37) may be written in a more concise form in terms of the *Fourier coefficients*:

$$a^\sigma_n[\mathbf{V}] := \int_\Omega \mathbf{V}\cdot\mathbf{v}_n\ \mathrm{d}x,\ a^g_n[\mathbf{V}] := \frac{1}{\sqrt{\Lambda_n}} \int_\Omega \mathbf{V}\cdot\nabla_x q_n\ \mathrm{d}x,\ n = 1,2,\ldots,$$
$$b_0[Z] = \frac{1}{\sqrt{|\Omega|}} \int_\Omega Z\ \mathrm{d}x,\ b_n[Z] = \int_\Omega r q_n\ \mathrm{d}x,\ n = 1,2,\ldots,$$

where $\{\mathbf{v}_n\}_{n=1}^{\infty}$ is an orthonormal basis of the space $L^2_\sigma(\Omega;\mathbb{R}^3)$ of solenoidal fields with zero normal trace, and $\{q_n\}_{n=0}^{\infty}$ is the complete orthonormal system of eigenfunctions of the homogeneous Neumann problem

$$-\Delta_x q_n = \Lambda_n q_n,\ 0 = \Lambda_0 < \Lambda_1 \leq \Lambda_2 \leq \cdots,\ q_0 = \frac{1}{\sqrt{|\Omega|}}.$$

We start with the homogeneous wave equation

$$\left\{ \begin{array}{l} \partial_t R_t + \mathrm{div}_x \mathbf{Q} = 0, \\ \partial_t \mathbf{Q} + \omega \nabla_x R = 0, \end{array} \right\} \text{ in } (0,T)\times\Omega,\ \mathbf{Q}\cdot\mathbf{n}|_{\partial\Omega} = 0,\ R(0) = \mathbf{R}_0,\ \mathbf{Q}(0) = \mathbf{Q}_0.$$

It is easy to check that the associated solution operator

$$S(t)\left[\begin{array}{c} R_0 \\ \mathbf{Q}_0 \end{array}\right] = \left[\begin{array}{c} R(t) \\ \mathbf{Q}(t) \end{array}\right]$$

can be expressed in terms of the Fourier coefficients as

$$b_0[R(t)] = b_0[R_0], \quad a_n^\sigma[\mathbf{Q}(t)] = a_n^\sigma[\mathbf{Q}_0] \text{ for } n = 1,2,\ldots, \tag{9.38}$$

$$\begin{aligned} b_n[R(t)] = {} & \exp(\mathrm{i}\sqrt{\omega\Lambda_n}t)\left[\frac{1}{2}\left(-\mathrm{i}\frac{1}{\sqrt{\omega}}a_n^g[\mathbf{Q}_0] + b_n[R_0]\right)\right] \\ & + \exp(-\mathrm{i}\sqrt{\omega\Lambda_n}t)\left[\frac{1}{2}\left(\mathrm{i}\frac{1}{\sqrt{\omega}}a_n^g[\mathbf{Q}_0] + b_n[R_0]\right)\right] \text{ for } n = 1,2,\ldots \end{aligned} \tag{9.39}$$

and

$$\begin{aligned} a_n^g[\mathbf{Q}(t)] = {} & \exp(\mathrm{i}\sqrt{\omega\Lambda_n}t)\left[\frac{1}{2}\left(a_n^g[\mathbf{Q}_0] + \mathrm{i}\sqrt{\omega}b_n[R_0]\right)\right] \\ & + \exp(-\mathrm{i}\sqrt{\omega\Lambda_n}t)\left[\frac{1}{2}\left(a_n^g[\mathbf{Q}_0] - \mathrm{i}\sqrt{\omega}b_n[R_0]\right)\right] \text{ for } n = 1,2,\ldots. \end{aligned} \tag{9.40}$$

These formulas are the discrete counterparts to those defined in (8.112), (8.113) by means of the Fourier transform. Accordingly, the solution operator $S(t)$ can be extended to a considerably larger class of initial data, for which the Fourier coefficients a_n, b_n may be defined, in particular, the data may belong to the space $\mathcal{M}$ of measures or distributions of higher order.

Similarly, we can identify solutions of the non-homogeneous problem

$$\left\{ \begin{array}{l} \partial_t R + \mathrm{div}_x \mathbf{Q} = h^1, \\ \partial_t \mathbf{Q} + \omega \nabla_x R = \mathbf{h}^2, \end{array} \right\} \text{ in } (0,T)\times\Omega,\ \mathbf{Q}\cdot\mathbf{n}|_{\partial\Omega} = 0,\ R(0) = \mathbf{R}_0,\ \mathbf{Q}(0) = \mathbf{Q}_0,$$

by means of the standard *Duhamel's formula*

$$\left[\begin{array}{c} R(t) \\ \mathbf{Q}(t) \end{array}\right] = S(t)\left[\begin{array}{c} R_0 \\ \mathbf{Q}_0 \end{array}\right] + \int_0^t S(t-s)\left[\begin{array}{c} h^1(s) \\ \mathbf{h}^2(s) \end{array}\right]\,\mathrm{d}s.$$

Finally, the solutions of the scaled equation

$$\left\{ \begin{array}{l} \varepsilon \partial_t R + \mathrm{div}_x \mathbf{Q} = \varepsilon h^1, \\ \varepsilon \partial_t \mathbf{Q} + \omega \nabla_x R = \varepsilon \mathbf{h}^2, \end{array} \right\} \text{ in } (0,T) \times \Omega, \ \mathbf{Q} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ R(0) = \mathbf{R}_0, \ \mathbf{Q}(0) = \mathbf{Q}_0, \tag{9.41}$$

can be expressed as

$$\left[\begin{array}{c} R(t) \\ \mathbf{Q}(t) \end{array} \right] = S\left(\frac{t}{\varepsilon}\right) \left[\begin{array}{c} R_0 \\ \mathbf{Q}_0 \end{array} \right] + \int_0^t S\left(\frac{t-s}{\varepsilon}\right) \left[\begin{array}{c} h^1(s) \\ \mathbf{h}^2(s) \end{array} \right] \, \mathrm{d}s, \tag{9.42}$$

where again the right-hand side may belong to a suitable class of distributions, in particular, formula (9.42) applies to solutions of the acoustic equation (9.36), (9.37).

9.3 Two-scale convergence

As we have observed several times in the previous chapters, solutions of the scaled acoustic equation (9.23), (9.24) are expected to develop fast time oscillations with the frequency proportional to $1/\varepsilon$. It is therefore natural to investigate the asymptotic behavior of solutions with respect to both the real (slow) time t and the fast time $\tau = t/\varepsilon$. To this end, we adapt the concept of *two-scale convergence* introduced by Allaire [6] and Nguetseng [163] to characterize the limit behavior of oscillating solutions in the theory of homogenization. The reader may consult the review paper by Visintin [200] for more information on the recent development of the two-scale calculus. Here we use the following weak-strong definition of two-scale convergence.

■ TWO-SCALE CONVERGENCE:

We shall say that a sequence $\{w_\varepsilon = w_\varepsilon(t,x)\}_{\varepsilon>0} \subset L^\infty(0,T;L^1(\Omega))$ *two-scale converges to a function* $w = w(\tau,t,x)$, $w \in L^\infty_{\mathrm{loc}}([0,\infty) \times [0,T]; L^1(\Omega))$, *if*

$$\operatorname*{ess\ sup}_{t \in (0,T)} \left| \int_\Omega \left[w_\varepsilon(t,x) - w\left(\frac{t}{\varepsilon}, t, x\right) \right] \varphi(x) \, \mathrm{d}x \right| \to 0 \tag{9.43}$$

for any $\varphi \in C^\infty_c(\Omega)$.

Now, we are ready to formalize the ideas discussed in Section 5.4.4 in terms of the two-scale convergence. The main issue to be discussed here is to investigate the *time oscillations*, as our definition requires only weak convergence in the spatial variable. Unfortunately, the result presented below gives only a very rough description of oscillations in terms of *completely unknown* driving terms.

Theorem 9.1. *Let* $\begin{bmatrix} r_\varepsilon \\ \mathbf{V}_\varepsilon \end{bmatrix}$ *be a family of solutions to the scaled acoustic equations* (9.23), (9.24) *belonging to class* (9.30–9.33), *where the terms on the right-hand side satisfy* (9.34), (9.35).

Then

$$\begin{bmatrix} r_\varepsilon \\ \mathbf{V}_\varepsilon \end{bmatrix} \text{ two-scale converges to } S(\tau) \begin{bmatrix} G_1(t,\cdot) \\ \mathbf{G}_2(t,\cdot) \end{bmatrix}, \ \tau = \frac{t}{\varepsilon},$$

for certain functions

$$G^1 \in C_{\text{weak}} \cap L^\infty([0,T]; L^2(\Omega)), \ \mathbf{G}^2 \in C_{\text{weak}} \cap L^\infty([0,T]; L^2(\Omega; \mathbb{R}^3)),$$

where S is the solution operator defined by means of (9.38–9.40).

Remark: *As solutions of the acoustic equation are almost-periodic, the preceding result implies (it is in fact stronger than) the two-scale convergence on general Besicovitch spaces developed by Casado-Díaz and Gayte* [42].

Proof. **(i)** Seeing that

$$r_\varepsilon = Z_\varepsilon - \frac{A}{\varepsilon\omega}\Sigma_\varepsilon,$$

where, by virtue of (9.35),

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\Sigma_\varepsilon(t)\|_{\mathcal{M}(\overline{\Omega})} \leq \varepsilon^2 c, \tag{9.44}$$

it is enough to show the result for Z_ε, $\mathbf{V}_\varepsilon$ solving system (9.36), (9.37). Moreover, as the two-scale convergence defined through (9.43) is weak with respect to the spatial variable, we have to show the result only for each Fourier mode in (9.38–9.40), separately. More specifically, we write

$$\begin{bmatrix} Z_\varepsilon \\ \mathbf{V}_\varepsilon \end{bmatrix} = S\left(\frac{t}{\varepsilon}\right) \begin{bmatrix} G^1_\varepsilon \\ \mathbf{G}^2_\varepsilon \end{bmatrix} \tag{9.45}$$

and show that

$$b_n[G^1_\varepsilon], \ a^\sigma_n[\mathbf{G}^2_\varepsilon], \ a^g_n[\mathbf{G}^2_\varepsilon] \text{ are precompact in } C[0,T] \text{ for any fixed } n. \tag{9.46}$$

To this end, we associate to the forcing terms in (9.36), (9.37) their Fourier projections

$$b_0[h^1_\varepsilon] = 0, \ b_n[h^1_\varepsilon] = -\frac{A}{\omega}\int_\Omega \mathbf{s}^1_\varepsilon \cdot \nabla_x q_n \ \mathrm{d}x, \ n = 1,2,\ldots,$$

$$a^\sigma_n[\mathbf{h}^2_\varepsilon] = -\int_\Omega \left(\mathbf{s}^2_\varepsilon : \nabla_x \mathbf{v}_n + \mathbf{s}^3_\varepsilon \cdot \mathbf{v}_n\right) \ \mathrm{d}x, \ n = 1,2,\ldots,$$

and

$$a_n^g[\mathbf{h}_\varepsilon^2] = -\frac{1}{\sqrt{\Lambda_n}} \int_\Omega \left(\mathbf{s}_\varepsilon^2 : \nabla_x^2 q_n + \mathbf{s}_\varepsilon^3 \cdot \nabla_x q_n - \Lambda_n (s_\varepsilon^4 + s_\varepsilon^5) q_n \right) \, \mathrm{d}x, \ n = 1, 2, \dots .$$

As a direct consequence of the uniform bounds (9.34), (9.35),

$$\{b_n[h_\varepsilon^1]\}_{\varepsilon>0}, \ \{a_n^\sigma[\mathbf{h}_\varepsilon^2]\}_{\varepsilon>0}, \ \{a_n^g[\mathbf{h}_\varepsilon^2]\}_{\varepsilon>0} \text{ are bounded in } L^q(0,T) \tag{9.47}$$

for a certain $q > 1$.

Using Duhamel's formula (9.42), we obtain

$$b_0[Z_\varepsilon(t)] = b_0[Z_\varepsilon(0)], \ a_n^\sigma[\mathbf{V}_\varepsilon(t)] = a_n^\sigma[\mathbf{V}_\varepsilon(0)] + \int_0^t a_n^\sigma[\mathbf{h}_\varepsilon^2(s)] \, \mathrm{d}s, \ n = 1, 2, \dots .$$

By virtue of (9.30–9.33), together with (9.44), we can assume that

$$Z_\varepsilon(0,\cdot) \to Z_0 \text{ weakly-(*) in } \mathcal{M}(\overline{\Omega}), \ \mathbf{V}_\varepsilon(0,\cdot) \to \mathbf{V}_0 \text{ weakly in } L^1(\Omega), \tag{9.48}$$

with

$$Z_0 \in L^2(\Omega), \ \mathbf{V} \in L^2(\Omega; \mathbb{R}^3).$$

In particular,

$$b_n[Z_\varepsilon(0)] \to b_n[Z_0], \ a_n^\sigma[\mathbf{V}_\varepsilon(0)] \to a_n^\sigma[\mathbf{V}]_0, \text{ and } a_n^g[\mathbf{V}_\varepsilon(0)] \to a_n^g[\mathbf{V}_0] \text{ as } \varepsilon \to 0$$

for any fixed n.

Moreover, it follows from (9.47) that the family

$$\{t \mapsto \int_0^t a_n^\sigma[\mathbf{h}_\varepsilon^2(s)] \, \mathrm{d}s\}_{\varepsilon>0} \text{ is precompact in } C[0,T].$$

Similarly, in accordance with (9.39),

$$\begin{aligned} b_n[Z_\varepsilon(t)] = {} & \exp\left(\mathrm{i}\sqrt{\omega\Lambda_n}\frac{t}{\varepsilon} \right) \\ & \times \left[\frac{1}{2} \left(-\frac{\mathrm{i}}{\sqrt{\omega}} \left(a_n^g[\mathbf{V}_\varepsilon(0)] + \int_0^t \exp\left(-\mathrm{i}\sqrt{\omega\Lambda_n}\frac{s}{\varepsilon} \right) a_n^g[\mathbf{h}_\varepsilon^2(s)] \, \mathrm{d}s \right) \right. \right. \\ & \left. \left. \quad + b_n[Z_\varepsilon(0)] + \int_0^t \exp\left(-\mathrm{i}\sqrt{\omega\Lambda}\frac{s}{\varepsilon} \right) b_n[h_\varepsilon^1(s)] \mathrm{d}s \right) \right] \\ & + \exp\left(-\mathrm{i}\sqrt{\omega\Lambda_n}\frac{t}{\varepsilon} \right) \\ & \times \left[\frac{1}{2} \left(\frac{\mathrm{i}}{\sqrt{\omega}} \left(a_n^g[\mathbf{V}_\varepsilon(0)] + \int_0^t \exp\left(\mathrm{i}\sqrt{\omega\Lambda_n}\frac{s}{\varepsilon} \right) a_n^g[\mathbf{h}_\varepsilon^2(s)] \, \mathrm{d}s \right) \right. \right. \\ & \left. \left. \quad + b_n[Z_\varepsilon(0)] + \int_0^t \exp\left(\mathrm{i}\sqrt{\omega\Lambda}\frac{s}{\varepsilon} \right) b_n[h_\varepsilon^1(s)] \mathrm{d}s \right) \right], \end{aligned}$$

where the family of functions

$$\left\{ t \mapsto \int_0^t \exp\left(\pm \mathrm{i} \sqrt{\omega \Lambda_n} \frac{s}{\varepsilon} \right) a_n^g [\mathbf{h}_\varepsilon^2(s)] \ \mathrm{d}s \right\}, \ t \in [0,T],$$

$$\left\{ t \mapsto \int_0^t \exp\left(\pm \mathrm{i} \sqrt{\omega \Lambda_n} \frac{s}{\varepsilon} \right) b_n [h_\varepsilon^1(s)] \ \mathrm{d}s \right\}, \ t \in [0,T]$$

are precompact in $C[0,T]$.

As the remaining terms can be treated in a similar way, we have shown (9.45), (9.46). Consequently, we may assume that

$$\left. \begin{aligned} b_n[G_\varepsilon^1] &\to b_n[G^1] \\ a_n^\sigma[\mathbf{G}_\varepsilon^2] &\to a_n^\sigma[\mathbf{G}^2] \\ a_g^\sigma[\mathbf{G}_\varepsilon^2] &\to a_n^\sigma[\mathbf{G}^2] \end{aligned} \right\} \text{in } C[0,T] \text{ for any fixed } n,$$

where the limit distributions G^1, $\mathbf{G}_\varepsilon^2$ are uniquely determined by their Fourier coefficients. In other words,

$$\begin{bmatrix} P_M^1[r_\varepsilon] \\ \mathbf{P}_M^2[\mathbf{V}_\varepsilon] \end{bmatrix} \text{ two-scale converges to } S(\tau) \begin{bmatrix} P_M^1[G^1] \\ \mathbf{P}_M^2[\mathbf{G}^2] \end{bmatrix}, \ \tau = \frac{t}{\varepsilon},$$

for any fixed M, where P_M^1, $\mathbf{P}_M^2$ are projections on the first M Fourier modes, specifically,

$$P_M^1[r] = \sum_{n \leq M} b_n[r] q_n, \ \mathbf{P}_M^2[\mathbf{V}] = \sum_{n \leq M} \left(a_n^\sigma[\mathbf{V}] \mathbf{v}_n + a_n^g[\mathbf{V}] \frac{1}{\sqrt{\Lambda_n}} \nabla_x q_n \right).$$

(ii) It remains to show that the quantities G^1, $\mathbf{G}^2$ are bounded in the L^2-norm uniformly in time. To this end, we use the estimates (9.56–9.59) in order to see that

$$\limsup_{\varepsilon \to 0} \left(\operatorname*{ess\ sup}_{t \in (0,T)} \| P_M^1[r_\varepsilon] \|_{L^2(\Omega)} \right) \leq c_1, \tag{9.49}$$

$$\limsup_{\varepsilon \to 0} \left(\operatorname*{ess\ sup}_{t \in (0,T)} \| \mathbf{P}_M^2[\mathbf{V}_\varepsilon] \|_{L^2(\Omega;\mathbb{R}^3)} \right) \leq c_2, \tag{9.50}$$

where the constants c_1, c_2 are independent of M.

On the other hand, since $P_M^1[r_\varepsilon]$, $\mathbf{P}_M^2[\mathbf{V}_\varepsilon]$ two-scale converge, we have

$$\operatorname*{ess\ sup}_{t \in (0,T)} \left\| \begin{bmatrix} P_M^1[r_\varepsilon] \\ \mathbf{P}_M^2[\mathbf{V}_\varepsilon] \end{bmatrix} - S\left(\frac{t}{\varepsilon}\right) \begin{bmatrix} P_M^1[G^1] \\ \mathbf{P}_M^2[\mathbf{G}^2] \end{bmatrix} \right\|_{L^2(\Omega) \times L^2(\Omega;\mathbb{R}^3)} \to 0$$

as $\varepsilon \to 0$ for any fixed M. Since S is an isometry on $L^2(\Omega)\times L^2(\Omega;\mathbb{R}^3)$, we conclude that

$$\begin{aligned}
&\operatorname*{ess\ sup}_{t\in[0,T]} \left\| \begin{bmatrix} P_M^1[G^1] \\ \mathbf{P}_M^2[\mathbf{G}^2] \end{bmatrix} \right\|_{L^2(\Omega)\times L^2(\Omega;\mathbb{R}^3)} \\
&\quad = \operatorname*{ess\ sup}_{t\in[0,T]} \left\| S\left(\frac{t}{\varepsilon}\right) \begin{bmatrix} P_M^1[G^1] \\ \mathbf{P}_M^2[\mathbf{G}^2] \end{bmatrix} \right\|_{L^2(\Omega)\times L^2(\Omega;\mathbb{R}^3)} \\
&\quad \leq \limsup_{\varepsilon\to 0} \left[\operatorname*{ess\ sup}_{t\in(0,T)} \left\| \begin{bmatrix} P_M^1[r_\varepsilon] \\ \mathbf{P}_M^2[\mathbf{V}_\varepsilon] \end{bmatrix} \right\|_{L^2(\Omega)\times L^2(\Omega;\mathbb{R}^3)} \right] \\
&\quad \leq c,
\end{aligned}$$

where, as stated in (9.49), (9.50), the constant is independent of M. □

9.3.1 Approximate methods

We intend to simplify system (9.23), (9.24) by replacing the source terms by their asymptotic limits for $\varepsilon \to 0$. To begin, by virtue of the uniform bound (5.50), we observe that

$$\sigma_\varepsilon/\varepsilon \to 0 \text{ in } \mathcal{M}([0,T]\times\overline{\Omega}).$$

Similarly, by means of the same arguments as in Section 5.3.2,

$$\mathbf{s}_\varepsilon^1 \to \mathbf{s}^1 \text{ weakly in } L^1((0,T)\times\Omega;\mathbb{R}^3),$$

where

$$\mathbf{s}^1 = \frac{\kappa(\overline{\vartheta})}{\overline{\vartheta}}\nabla_x\Theta + \overline{\varrho}\left(\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\varrho^{(1)} + \frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\Theta\right)\mathbf{U}.$$

We simplify further by eliminating completely the temperature fluctuations, supposing that the *initial state* of the primitive system is almost *isentropic*, specifically,

$$\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\varrho_0^{(1)} + \frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\vartheta_0^{(1)} = 0.$$

Consequently, the limit temperature Θ solves the Neumann problem for the heat equation (9.19) with the prescribed initial state $\Theta_0 = 0$. As a straightforward consequence of the heat energy balance established in (5.184), we obtain $\Theta = 0$. Moreover, utilizing relation (9.22), we get $\varrho^{(1)} = 0$; whence $\mathbf{s}^1 = 0$. Thus we have shown that it is reasonable, at least in view of the uniform bounds obtained in Section 5.2, to replace the right-hand side in (9.23) by zero, provided the initial entropy of the primitive system is close to its (maximal) value attained at the equilibrium state $(\overline{\varrho},\overline{\vartheta})$.

A similar treatment applied to the acoustic sources in (9.24) requires more attention. Obviously, we can still replace

$$\mathbb{S}_\varepsilon \approx \mu(\overline{\vartheta})\Big(\nabla_x\mathbf{U} + \nabla_x^T\mathbf{U}\Big), \text{ and } \mathbf{s}_\varepsilon^3 \approx \mathbf{s}^3 := -\overline{\varrho}\mathbf{f}$$

but the asymptotic limit of the convective term $\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon$ is far less obvious as we have already observed in Section 5.4. All we know for sure is

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \to \overline{\varrho}\overline{\mathbf{U} \otimes \mathbf{U}} \text{ weakly in } L^1((0,T) \times \Omega; \mathbb{R}^{3\times 3}),$$

where, in general,

$$\overline{\mathbf{U} \otimes \mathbf{U}} \not\equiv \mathbf{U} \otimes \mathbf{U}$$

unless the velocity fields $\mathbf{u}_\varepsilon$ converge pointwise to $\mathbf{U}$ in $(0,T) \times \Omega$.

A similar problem occurs when dealing with s^4_ε. Note that, in accordance with our choice of the parameters ω, Λ (cf. (5.126)),

$$-\partial_\varrho p(\overline{\varrho}, \overline{\vartheta}) + A\overline{\varrho}\partial_\varrho s(\overline{\varrho}, \overline{\vartheta}) + \omega = 0, \ -\partial_\vartheta p(\overline{\varrho}, \overline{\vartheta}) + A\overline{\varrho}\partial_\vartheta s(\overline{\varrho}, \overline{\vartheta}) = 0;$$

whence, by virtue of the uniform bounds established in Section 5.2,

$$\left\| \frac{p(\overline{\varrho}, \overline{\vartheta}) - p(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon} + A\varrho_\varepsilon \frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} + \omega \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right\|_{L^1((0,T)\times\Omega)} \leq \varepsilon c.$$

However, in order to obtain $s^4_\varepsilon \to 0$ in some sense, we need strong convergence

$$\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} := \varrho^{(1)}_\varepsilon \to \varrho^{(1)} = 0, \ \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} := \vartheta^{(1)} \to \Theta = 0 \text{ pointwise in } (0,T) \times \Omega.$$

In light of the previous arguments, any kind of *linear* acoustic analogy is likely to provide a good approximation of propagation of the acoustic waves only when their amplitude is considerably smaller than the Mach number, or, in the standard terminology, in the case of well-prepared data. We are going to discuss this issue in the next section.

9.4 Lighthill's acoustic analogy in the low Mach number regime

9.4.1 Ill-prepared data

Motivated by the previous discussion, we suppose that solutions r_ε, $\mathbf{V}_\varepsilon$ of the scaled acoustic equation can be approximated by R_ε, $\mathbf{Q}_\varepsilon$ solving a wave equation, where, in the spirit of *Lighthill's acoustic analogy*, the source terms have been evaluated on the basis of the limit system (9.17), (9.18). In addition, we shall assume that the limit solution is smooth so that the weak formulation of the problem may be replaced by the classical one as follows.

■ LIGHTHILL'S EQUATION:

$$\varepsilon\partial_t R_\varepsilon + \mathrm{div}_x \mathbf{Q}_\varepsilon = 0, \tag{9.51}$$

$$\varepsilon\partial_t \mathbf{Q}_\varepsilon + \omega\nabla_x R_\varepsilon = \varepsilon\Big(\mu(\overline{\vartheta})\mathrm{div}_x(\nabla_x\mathbf{U} + \nabla_x^T\mathbf{U}) - \mathrm{div}_x(\overline{\varrho}\mathbf{U}\otimes\mathbf{U}) + \overline{\varrho}\mathbf{f}\Big), \tag{9.52}$$

$$\mathbf{Q}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{9.53}$$

supplemented with the initial conditions

$$R_\varepsilon(0,\cdot) = r_\varepsilon(0,\cdot),\ \mathbf{Q}_\varepsilon(0,\cdot) = \mathbf{V}_\varepsilon(0,\cdot). \tag{9.54}$$

Since $\mathbf{U}$ is a smooth solution of the incompressible NAVIER-STOKES SYSTEM (9.17), (9.18), we can rewrite (9.51), (9.52) in the form

$$\varepsilon\partial_t(R_\varepsilon - \varepsilon(\Pi/\omega)) + \mathrm{div}_x(\mathbf{Q}_\varepsilon - \overline{\varrho}\mathbf{U}) = -\varepsilon^2\partial_t(\Pi/\omega),$$
$$\varepsilon\partial_t(\mathbf{Q}_\varepsilon - \overline{\varrho}\mathbf{U}) + \omega\nabla_x(R_\varepsilon - \varepsilon(\Pi/\omega)) = 0,$$

which can be viewed as another non-homogeneous wave equation with the same wave propagator and with a source of order ε^2. In other words, if the initial data are *ill-prepared*, meaning

$$r_{\varepsilon,0} \approx r_{0,\varepsilon} - \varepsilon(\Pi/\omega),\ \mathbf{H}^\perp[\mathbf{V}_{0,\varepsilon}] \text{ of order } 1,$$

the presence of Lighthill's tensor in (9.52) yields a perturbation of order ε with respect to the homogeneous problem. Consequently, for the ill-prepared data, *Lighthill's equation* can be replaced, with the same degree of "precision", by the homogeneous wave equation

$$\varepsilon\partial_t R_\varepsilon + \mathrm{div}_x \mathbf{Q}_\varepsilon = 0,$$
$$\varepsilon\partial_t \mathbf{Q}_\varepsilon + \omega\nabla_x R_\varepsilon = 0.$$

Thus we conclude, together with Lighthill [135, Chapter 1], that use of a linear theory, for waves of any kind, implies that we consider disturbances so small that in equations of motion we can view them as quantities whose products can be neglected. In particular, the *ill-prepared* data must be handled by the methods of *nonlinear* acoustics (see Enflo and Hedberg [72]).

9.4.2 Well-prepared data

If the initial data are *well prepared*, meaning

$$r_{\varepsilon,0},\ \mathbf{H}^\perp[\mathbf{V}_{0,\varepsilon}] \text{ are of order } \varepsilon,$$

or, in terms of the initial data for the primitive system,

$$\varrho^{(1)}_{0,\varepsilon} = \vartheta^{(1)}_{0,\varepsilon} = 0,\ \mathbf{u}_{0,\varepsilon} = \mathbf{U}_0$$

in (9.9), then, replacing $R_\varepsilon \approx R_\varepsilon/\varepsilon$, $\mathbf{Q}_\varepsilon \approx \mathbf{Q}_\varepsilon/\varepsilon$, we can write *Lighthill's equation* (9.51), (9.52) in the form

$$\varepsilon \partial_t R_\varepsilon + \mathrm{div}_x \mathbf{Q}_\varepsilon = 0, \tag{9.55}$$

$$\varepsilon \partial_t \mathbf{Q}_\varepsilon + \omega \nabla_x R_\varepsilon = \Big(\mu(\overline{\vartheta}) \mathrm{div}_x (\nabla_x \mathbf{U} + \nabla_x^T \mathbf{U}) - \mathrm{div}_x (\overline{\varrho} \mathbf{U} \otimes \mathbf{U}) + \overline{\varrho} \mathbf{f} \Big), \tag{9.56}$$

$$\mathbf{Q}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{9.57}$$

$$R_\varepsilon(0,\cdot) = R_{0,\varepsilon}, \ \mathbf{Q}_\varepsilon(0,\cdot) = \mathbf{Q}_{0,\varepsilon}, \tag{9.58}$$

where the initial data $R_{0,\varepsilon}$, $\mathbf{Q}_{0,\varepsilon}$ are determined by means of the "second-order" terms $\varrho^{(2)}_{0,\varepsilon}$, $\vartheta^{(2)}_{0,\varepsilon}$, and $\mathbf{u}^{(1)}_{0,\varepsilon}$.

For simplicity, assume that $\mathbf{U}$, Π represent a smooth solution of the incompressible Navier-Stokes system (9.17), (9.18), (9.20), satisfying the initial condition

$$\mathbf{U}(0,\cdot) = \mathbf{U}_0,$$

where $\mathbf{U}_0$ solves the *stationary problem*

$$\overline{\varrho} \mathrm{div}_x (\mathbf{U}_0 \otimes \mathbf{U}_0) + \nabla_x \Pi_0 = \mathrm{div}_x \Big(\mu(\overline{\vartheta}) (\nabla_x \mathbf{U}_0 + \nabla_x^T \mathbf{U}_0) \Big) + \overline{\varrho} \mathbf{f}_0, \ \mathrm{div}_x \mathbf{U}_0 = 0 \text{ in } \Omega, \tag{9.59}$$

$$\Pi_0 = \Pi(0,\cdot),$$

supplemented with the boundary conditions (9.20). Here, the driving force $\mathbf{f}_0$ is a function of x only, and the solution $\mathbf{U}_0$, Π_0 is called the *background flow.* We normalize the pressure so that

$$\int_\Omega \Pi(t,\cdot) \ \mathrm{d}x = \int_\Omega \Pi_0 \ \mathrm{d}x = 0$$

for all $t \in [0,T]$.

Our aim is to find a suitable description for the asymptotic limits of R_ε, $\mathbf{Q}_\varepsilon$ when $\varepsilon \to 0$. These quantities, solving the scaled Lighthill's equation (9.55), (9.56), are likely to develop fast oscillations in time that would be completely ignored should we use the standard concept of weak limits. Instead we use again the *two-scale convergence* introduced in the previous section. We claim the following result.

■ Asymptotic Lighthill's Equation:

Theorem 9.2. *Let R_ε, $\mathbf{Q}_\varepsilon$ be the (unique) solution of problem* (9.55–9.58)*, where*

$$R_{0,\varepsilon} \to R_0 \ \textit{in} \ L^2(\Omega); \ \mathbf{Q}_{0,\varepsilon} \to \mathbf{Q}_0 \ \textit{in} \ L^2(\Omega;\mathbb{R}^3), \ \mathbf{H}[\mathbf{Q}_{0,\varepsilon}] = 0,$$

and where $\mathbf{U}$, Π is a smooth solution of problem (9.17), (9.18), (9.20)*, with $\mathbf{U}(0,\cdot) = \mathbf{U}_0$, $\Pi(0,\cdot) = \Pi_0$ satisfying* (9.59).

Then

$$\{R_\varepsilon, \mathbf{Q}_\varepsilon\}_{\varepsilon>0} \text{ two-scale converges to } \{R + \Pi/\omega - \Pi_0/\omega; \mathbf{Q}\},$$

where R, $\mathbf{Q}$ *is the unique solution of the problem* (9.51), (9.52) *in the form*

$$\varepsilon\partial_t R + \mathrm{div}_x \mathbf{Q} = 0,$$
$$\varepsilon\partial_t \mathbf{Q} + \omega\nabla_x R = \Big(\mu(\overline{\vartheta})\mathrm{div}_x(\nabla_x \mathbf{U}_0 + \nabla_x^T \mathbf{U}_0) - \mathrm{div}_x(\overline{\varrho}\mathbf{U}_0 \otimes \mathbf{U}_0) + \overline{\varrho}\mathbf{f}_0\Big),$$
$$\mathbf{Q}\cdot\mathbf{n}|_{\partial\Omega} = 0,$$
$$R(0,\cdot) = R_0,\ \mathbf{Q}(0,\cdot) = \mathbf{Q}_0.$$

Remark: *In particular, solutions* R, $\mathbf{Q}$ *of the limit system can be written in the form* $R = R(t/\varepsilon, t, x)$, $\mathbf{Q} = \mathbf{Q}(t/\varepsilon, t, x)$.

Proof. As all quantities are smooth, it is easy to check that

$$R_\varepsilon = \Pi/\omega + Z_\varepsilon,\ \mathbf{Q}_\varepsilon = \mathbf{Y}_\varepsilon,$$

where Z_ε, $\mathbf{Y}_\varepsilon$ is the unique solution of the problem

$$\varepsilon\partial_t Z_\varepsilon + \mathrm{div}_x \mathbf{Y}_\varepsilon = -\varepsilon\partial_t\Pi/\omega,$$
$$\varepsilon\partial_t \mathbf{Y}_\varepsilon + \omega\nabla_x Z_\varepsilon = 0,$$
$$\mathbf{Y}_\varepsilon\cdot\mathbf{n}|_{\partial\Omega} = 0,$$
$$Z_\varepsilon(0,\cdot) = R_{0,\varepsilon} - \Pi_0/\omega,\ \mathbf{Y}_\varepsilon(0,\cdot) = \mathbf{Q}_{0,\varepsilon}.$$

Similarly to Section 5.4.4, we can write

$$\begin{bmatrix} Z_\varepsilon(t) \\ \mathbf{Y}_\varepsilon(t) \end{bmatrix} = S\left(\frac{t}{\varepsilon}\right)\begin{bmatrix} R_{0,\varepsilon} - \Pi_0/\omega \\ \mathbf{Q}_{0,\varepsilon} \end{bmatrix} - S\left(\frac{t}{\varepsilon}\right)\int_0^t S\left(\frac{-s}{\varepsilon}\right)\begin{bmatrix} \partial_t\Pi/\omega \\ 0 \end{bmatrix}\,\mathrm{d}s,$$

where S is the solution operator associated to the homogeneous problem introduced in Section 9.2.

It is easy to check that $(Z_\varepsilon, \mathbf{Y}_\varepsilon)$ two-scale converges to

$$\begin{bmatrix} Z \\ \mathbf{Y} \end{bmatrix} = S\left(\frac{t}{\varepsilon}\right)\begin{bmatrix} R_0 - \Pi_0/\omega \\ \mathbf{Q}_0 \end{bmatrix},$$

which completes the proof. Indeed since $\int_\Omega \Pi\ \mathrm{d}x = 0$, we get

$$t \mapsto \int_0^t S\left(\frac{-s}{\varepsilon}\right)\begin{bmatrix} \partial_t\Pi/\omega \\ 0 \end{bmatrix}\,\mathrm{d}s \to 0 \text{ in } C([0,T]; L^2(\Omega)), \tag{9.60}$$

as the integrated quantity can be written as a Fourier series with respect to the eigenvectors of the wave operator identified in Section 5.4.5. Since all (non-zero) Fourier modes take the form

$$\exp\left(\pm\mathrm{i}|\Lambda|\frac{s}{\varepsilon}\right)a(s)\begin{bmatrix} q(x) \\ -\frac{\mathrm{i}}{\sqrt{|\Lambda|}}\nabla_x q(x) \end{bmatrix},\ \Lambda \neq 0,$$

relation (9.60) follows. □

In this section, we have deliberately omitted a highly non-trivial issue, namely to what extent Lighthill's equation can be used as a description of the acoustic waves for well-prepared data. Apparently, we need higher order uniform bounds that implicitly imply regularity of solutions of the target system. Moreover, these bounds also imply *existence* of regular solutions for the primitive system provided the data are close to the equilibrium state. Positive results in this direction were obtained by Hagstrom and Lorentz [105].

In order to conclude this section, let us point out that *Lighthill's equation* (9.51–9.54) may indicate completely misleading results when applied on bounded domains with acoustically soft boundary conditions. As we have seen in Chapter 7, the oscillations of the acoustic waves are effectively damped by a boundary layer provided the velocity vanishes on the boundary as soon as the latter satisfies certain geometrical conditions, even for *ill-prepared data.* On the contrary, *Lighthill's equation* predicts violent oscillations of the velocity field with the frequency proportional to $1/\varepsilon$ in the low Mach number limit. Of course, in this case, system (9.51), (9.52) is not even well posed if the boundary condition (9.53) is replaced by $\mathbf{Q}_\varepsilon|_{\partial\Omega} = 0$.

9.5 Concluding remarks

In the course of the previous discussion, we have assumed that the solution $\mathbf{U}$ of the limit incompressible Navier-Stokes system is smooth. Of course, smoothness of solutions should be determined by the initial datum $\mathbf{U}_0$. Unfortunately, in the three-dimensional physical space, it is a major open problem whether solutions to the incompressible Navier-Stokes system emanating from smooth data remain smooth at any positive time. Still there is a large class, although not explicitly known, of the initial data for which the system (9.17), (9.18) admits a smooth solution. In particular, this is true for small perturbations of smooth stationary states.

The problem becomes even more delicate in the framework of the asymptotic limits studied in this book. Although we are able to identify the low Mach number limit as a weak solution of system (9.17), (9.18) emanating from the initial datum $\mathbf{U}_0$, it is still not completely clear if this weak solution coincides with the strong (regular) one provided the latter exists.

Fortunately, such a weak-strong uniqueness result holds provided the weak solution $\mathbf{U}$ of (9.17), (9.18) satisfies the *energy inequality*:

$$\frac{1}{2}\int_\Omega \overline{\varrho}|\mathbf{U}|^2(\tau)\ \mathrm{d}x + \frac{\mu(\overline{\vartheta})}{2}\int_0^\tau\int_\Omega |\nabla_x\mathbf{U} + \nabla_x^T\mathbf{U}|^2\ \mathrm{d}x\ \mathrm{d}t \tag{9.61}$$
$$\leq \frac{1}{2}\int_\Omega \overline{\varrho}|\mathbf{U}_0|^2(\tau)\ \mathrm{d}x + \int_0^\tau\int_\Omega \overline{\varrho}\mathbf{f}\cdot\mathbf{U}\ \mathrm{d}x\ \mathrm{d}t \text{ for a.a. } \tau\in(0,T).$$

As we have shown, the solutions obtained in the low Mach number asymptotic analysis do satisfy (9.61) as soon as the data are "suitably" prepared (see Theorem 5.3).

Now, for the sake of simplicity, assume that $\mathbf{f}$ is independent of t and that $\mathbf{U}_0 = \mathbf{w}$, where $\mathbf{w}$ is a regular stationary solution to the incompressible NAVIER-STOKES SYSTEM, specifically,

$$\mathrm{div}_x \mathbf{w} = 0, \tag{9.62}$$

$$\mathrm{div}_x(\overline{\varrho}\mathbf{w} \otimes \mathbf{w}) + \nabla_x \Pi = \mu(\overline{\vartheta})\mathrm{div}_x\Big(\nabla_x \mathbf{w} + \nabla_x^T \mathbf{w}\Big) + \overline{\varrho}\mathbf{f}, \tag{9.63}$$

satisfying the complete slip boundary conditions

$$\mathbf{w} \cdot \mathbf{n}|_{\partial\Omega} = 0,\ (\nabla_x \mathbf{w} + \nabla_x^T \mathbf{w})\mathbf{n} \times \mathbf{n}|_{\partial\Omega} = 0. \tag{9.64}$$

We claim that $\mathbf{w} = \mathbf{U}$ as soon as $\mathbf{U}$ is a weak solution of (9.17), (9.18), (9.20), with $\mathbf{U}(0,\cdot) = \mathbf{U}_0 = \mathbf{w}$, in the sense specified in Section 5.5.1 provided $\mathbf{U}$ satisfies the energy inequality (9.61). Indeed as $\mathbf{w}$ is smooth and satisfies the boundary conditions (9.64), the quantities $\psi(t)\mathbf{w}$, $\psi \in C^1_c(0,T)$, can be used as test functions in the weak formulation of (9.18), and, conversely, the stationary equation (9.63) can be multiplied on $\mathbf{U}$ and integrated by parts. Thus, after a straightforward manipulation, we obtain

$$\int_\Omega |\mathbf{U}(\tau) - \mathbf{w}|^2 \ \mathrm{d}x + \frac{\mu(\overline{\vartheta})}{\overline{\varrho}} \int_0^\tau \int_\Omega \Big|\nabla_x(\mathbf{U} - \mathbf{w}) + \nabla_x^T(\mathbf{U} - \mathbf{w})\Big|^2 \ \mathrm{d}x \ \mathrm{d}t$$
$$\leq 2\int_0^\tau \int_\Omega \big((\mathbf{U} \otimes \mathbf{U}) : \nabla_x \mathbf{w} + (\mathbf{w} \otimes \mathbf{w}) : \nabla_x \mathbf{U}\big) \ \mathrm{d}x \ \mathrm{d}t \text{ for a.a. } \tau \in (0,T),$$

where, by means of by-parts integration,

$$\int_\Omega \big((\mathbf{U} \otimes \mathbf{U}) : \nabla_x \mathbf{w} + (\mathbf{w} \otimes \mathbf{w}) : \nabla_x \mathbf{U}\big) \ \mathrm{d}x$$
$$= \sum_{i,j=1}^3 \int_\Omega \partial_{x_i} w_j U_j (U_i - w_i) \ \mathrm{d}x$$
$$= \int_\Omega [\nabla_x \mathbf{w}(\mathbf{U} - \mathbf{w})] \cdot (\mathbf{U} - \mathbf{w}) \ \mathrm{d}x + \frac{1}{2}\int_\Omega \nabla_x |\mathbf{w}|^2 \cdot (\mathbf{U} - \mathbf{w}) \ \mathrm{d}x$$
$$= \int_\Omega [\nabla_x \mathbf{w}(\mathbf{U} - \mathbf{w})] \cdot (\mathbf{U} - \mathbf{w}) \ \mathrm{d}x.$$

Consequently, the desired result $\mathbf{U} = \mathbf{w}$ follows directly from Gronwall's lemma.

Chapter 10

Appendix

For readers' convenience, a number of standard results used in the preceding text is summarized in this chapter. Nowadays classical statements are appended with the relevant reference material, while complete proofs are provided in the cases when a compilation of several different techniques is necessary. A significant part of the theory presented below is related to general problems in mathematical fluid mechanics and may be of independent interest.

Throughout this appendix, M denotes a positive integer while $N \in \mathbb{N}$ refers to the space dimension. The space dimension is always taken greater than or equal to 2, if not stated explicitly otherwise.

10.1 Mollifiers

A function $\zeta \in C_c^\infty(\mathbb{R}^M)$ is termed a *regularizing kernel* if

$$\mathrm{supp}[\zeta] \subset (-1,1)^M,\ \zeta(-x) = \zeta(x) \geq 0, \quad \int_{\mathbb{R}^M} \zeta(x)\,dx = 1. \tag{10.1}$$

For a measurable function a defined on $\mathbb{R}^M$ with values in a Banach space X, we denote

$$S_\omega[a] = a^\omega(x) = \zeta_\omega * a = \int_{\mathbb{R}^M} \zeta_\omega(x-y)a(y)\,dy, \tag{10.2}$$

where

$$\zeta_\omega(x) = \frac{1}{\omega^M}\zeta(\frac{x}{\omega}),\ \omega > 0,$$

provided the integral on the right-hand side exists. The operator $S_\omega : a \mapsto a^\omega$ is called a *mollifier*. Note that the above construction easily extends to distributions by setting $a^\omega(x) = \langle a; \zeta_\omega(x - \cdot)\rangle_{[\mathcal{D}';\mathcal{D}](\mathbb{R}^M)}$.

■ MOLLIFIERS:

Theorem 10.1. *Let X be a Banach space. If $a \in L^1_{loc}(\mathbb{R}^M; X)$, then we have $a^\omega \in C^\infty(\mathbb{R}^M; X)$. In addition, the following holds:*

(i) *If $a \in L^p_{\rm loc}(\mathbb{R}^M; X)$, $1 \le p < \infty$, then $a^\omega \in L^p_{\rm loc}(\mathbb{R}^M; X)$, and*

$$a^\omega \to a \ \text{ in } L^p_{\rm loc}(\mathbb{R}^M; X) \text{ as } \omega \to 0.$$

(ii) *If $a \in L^p(\mathbb{R}^M; X)$, $1 \le p < \infty$, then $a^\omega \in L^p(\mathbb{R}^M; X)$,*

$$\|a^\omega\|_{L^p(\mathbb{R}^M;X)} \le \|a\|_{L^p(\mathbb{R}^M;X)}, \text{ and } a^\omega \to a \quad \text{in} \quad L^p(\mathbb{R}^M; X) \text{ as } \omega \to 0.$$

(iii) *If $a \in L^\infty(\mathbb{R}^M; X)$, then $a^\omega \in L^\infty(\mathbb{R}^M; X)$, and*

$$\|a^\omega\|_{L^\infty(\mathbb{R}^M;X)} \le \|a\|_{L^\infty(\mathbb{R}^M;X)}.$$

(iv) *If $a \in C^k(U; X)$, where $U \subset \mathbb{R}^M$ is an (open) ball, then $(\partial^\alpha a)^\omega(x) = \partial^\alpha a^\omega(x)$ for all $x \in U$, $\omega \in (0, \text{dist}[x, \partial U])$ and for any multi-index α, $|\alpha| \le k$. Moreover,*

$$\|a^\omega\|_{C^k(\overline{B};X)} \le \|a\|_{C^k(\overline{V};X)}$$

for any $\omega \in (0, \text{dist}[\partial B, \partial V])$, where B, V are (open) balls in $\mathbb{R}^M$ such that $\overline{B} \subset V \subset \overline{V} \subset U$. Finally,

$$a^\omega \to a \quad \text{in} \quad C^k(\overline{B}; X) \text{ as } \omega \to 0.$$

See Amann [8, Chapter III.4], or Brezis [35, Chapter IV.4]. □

10.2 Basic properties of some elliptic operators

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. We consider a general *elliptic equation* in the divergence form

$$\mathcal{A}(x, u) = -\sum_{i,j=1}^N \partial_{x_i}(a_{i,j}(x)\partial_{x_j} u) + c(x)u = f \text{ for } x \in \Omega, \tag{10.3}$$

supplemented with the boundary condition

$$\delta \mathbf{u} + (\delta - 1)\sum_{j=1}^N a_{i,j}\partial_{x_j} u \; n_j|_{\partial\Omega} = g, \tag{10.4}$$

where $\delta = 0, 1$. We suppose that

$$a_{i,j} = a_{j,i} \in C^1(\overline{\Omega}), \ \sum_{i,j} a_{i,j}\xi_i\xi_j \ge \alpha|\xi|^2 \tag{10.5}$$

for a certain $\alpha > 0$ and all $\xi \in \mathbb{R}^N$, $|\xi| = 1$. The case $\delta = 1$ corresponds to the *Dirichlet problem*, $\delta = 0$ is termed the *Neumann problem*.

In several applications discussed in this book, Ω is also taken in the form

$$\Omega = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in \mathcal{T}^2,\ B_{\text{bottom}}(x_1, x_2) < x_3 < B_{\text{top}}(x_1, x_2)\}, \quad (10.6)$$

where the *horizontal* variable (x_1, x_2) belongs to the *flat torus*

$$\mathcal{T}^2 = \left([-\pi, \pi]|_{\{-\pi,\pi\}}\right)^2.$$

Although all results below are formulated in terms of standard domains, they apply to domains Ω given by (10.6) as well, provided we identify

$$\partial\Omega = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in \mathcal{T}^2,\ x_3 = B_{\text{bottom}}(x_1, x_2)\}$$
$$\cup\{(x_1, x_2, x_3) \mid (x_1, x_2) \in \mathcal{T}^2,\ x_3 = B_{\text{top}}(x_1, x_2)\}.$$

This is due to the fact that all theorems concerning regularity of solutions to elliptic equations are of local character.

10.2.1 A priori estimates

We start with the classical *Schauder estimates.*

■ HÖLDER REGULARITY:

Theorem 10.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{k+2,\nu}$, $k = 0, 1, \dots,$ with $\nu > 0$. Suppose, in addition to* (10.5), *that $a_{i,j} \in C^{k+1,\nu}(\overline{\Omega})$, $i, j = 1, \dots, N$, $c \in C^{k,\nu}(\overline{\Omega})$. Let u be a classical solution of problem* (10.3), (10.4), *where $f \in C^{k,\nu}(\overline{\Omega})$, $g \in C^{k+\delta+1,\nu}(\partial\Omega)$.*

Then

$$\|u\|_{C^{k+2,\nu}(\overline{\Omega})} \le c\left(\|f\|_{C^{k,\nu}(\overline{\Omega})} + \|g\|_{C^{k+1,\nu}(\partial\Omega)} + \|u\|_{C(\overline{\Omega})}\right).$$

See Ladyzhenskaya and Uralceva [130, Theorems 3.1 and 3.2, Chapter 3], Gilbarg and Trudinger [96, Theorem 6.8]. □

Similar bounds can be also obtained in the L^p-framework. We report the celebrated result by Agmon, Douglis, and Nirenberg [2] (see also Lions and Magenes [138]). The hypotheses we use concerning regularity of the boundary and the coefficients $a_{i,j}$, c are not optimal but certainly sufficient in all situations considered in this book.

■ Strong L^p-regularity:

Theorem 10.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class C^2. In addition to* (10.5)*, assume that $c \in C(\overline{\Omega})$. Let $u \in W^{2,p}(\Omega)$, $1 < p < \infty$, be a (strong) solution of problem* (10.3), (10.4)*, with $f \in L^p(\Omega)$, $g \in W^{\delta+1-1/p,p}(\partial\Omega)$.*

Then

$$\|u\|_{W^{2,p}(\Omega)} \leq c\left(\|f\|_{L^p(\Omega)} + \|g\|_{W^{\delta+1-1/p,p}(\partial\Omega)} + \|u\|_{L^p(\Omega)}\right).$$

See Agmon, Douglis and Nirenberg [2]. □

The above estimates can be extrapolated to "negative" spaces. For the sake of simplicity, we set $g = 0$ in the Dirichlet case $\delta = 1$. In order to formulate adequate results, let us introduce the Dirichlet form associated to the operator $\mathcal{A}$, namely

$$[\mathcal{A}u, v] := \int_\Omega a_{i,j}(x)\partial_{x_j} u \partial_{x_i} v + c(x)uv \, \mathrm{d}x.$$

In such a way, the operator $\mathcal{A}$ can be regarded as a continuous linear mapping

$$\mathcal{A} : W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega) \text{ for the Dirichlet boundary condition}$$

or

$$\mathcal{A} : W^{1,p}(\Omega) \to [W^{1,p'}(\Omega)]^* \text{ for the Neumann boundary condition,}$$

where

$$1 < p < \infty, \ \frac{1}{p} + \frac{1}{p'} = 1.$$

■ Weak L^p-regularity:

Theorem 10.4. *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain of class C^2, and $1 < p < \infty$. Let $a_{i,j}$ satisfy* (10.5)*, and let $c \in L^\infty(\Omega)$.*

(i) *If $u \in W_0^{1,p}(\Omega)$ satisfies*

$$[\mathcal{A}u, v] = \langle f, v\rangle_{[W^{-1,p}; W_0^{1,p'}](\Omega)} \text{ for all } v \in W_0^{1,p'}(\Omega)$$

for a certain $f \in W^{-1,p}(\Omega)$, then

$$\|u\|_{W_0^{1,p}(\Omega)} \leq c\left(\|f\|_{W^{-1,p}(\Omega)} + \|u\|_{W^{-1,p}(\Omega)}\right).$$

(ii) *If $u \in W^{1,p}(\Omega)$ satisfies*

$$[\mathcal{A}u, v] = \langle F, v\rangle_{[[W^{1,p'}]^*; W^{1,p'}](\Omega)} \text{ for all } v \in W^{1,p'}(\Omega)$$

for a certain $F \in [W^{1,p'}]^*(\Omega)$, *then*

$$\|u\|_{W^{1,p}(\Omega)} \leq c \left(\|F\|_{[W^{1,p'}]^*(\Omega)} + \|u\|_{[W^{1,p'}]^*(\Omega)} \right).$$

In particular, if

$$[\mathcal{A}u, v] = \int_\Omega f v \, \mathrm{d}x - \int_{\partial\Omega} g v \, \mathrm{d}S_x \text{ for all } v \in W^{1,p'}(\Omega),$$

then

$$\|u\|_{W^{1,p}(\Omega)} \leq c \left(\|f\|_{[W^{1,p'}]^*(\Omega)} + \|g\|_{W^{-1/p,p}(\partial\Omega)} + \|u\|_{[W^{1,p'}]^*(\Omega)} \right).$$

See J.-L. Lions [137], Schechter [177]. □

Remark: *The hypothesis concerning regularity of the boundary can be relaxed to* $C^{0,1}$ *in the case of the Dirichlet boundary condition, and to* $C^{1,1}$ *for the Neumann boundary condition.*

Remark: *The norm containing* u *on the right-hand side of the estimates in Theorems* 10.2–10.4 *is irrelevant and may be omitted, provided that the solution is unique in the given class.*

Remark: *As we have observed, elliptic operators, in general, enjoy the degree of regularity allowed by the data. In particular, the solutions of elliptic problems with constant or* (*real*) analytic coefficients *are* analytic *on any open subset of their domain of definition. For example, if*

$$\Delta u + \mathbf{b} \cdot \nabla_x u + cu = f \text{ in } \Omega \subset \mathbb{R}^N,$$

where $\mathbf{b}$, c *are constant, and* Ω *is a domain, then* $\mathbf{u}$ *is analytic in* Ω *provided that* f *is analytic* (*see John* [118, Chapter VII]) . *The result can be extended to elliptic systems and even up to the boundary, provided the latter is analytic* (*see Morrey and Nirenberg* [159]).

10.2.2 Fredholm alternative

Now, we focus on the problem of *existence*. Given the scope of applications considered in this book, we consider only the Neumann problem, specifically $\delta = 0$ in system (10.3), (10.4). Similar results hold also for the Dirichlet boundary conditions. A useful tool is the *Fredholm alternative* formulated in the following theorem.

■ FREDHOLM ALTERNATIVE:

Theorem 10.5. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain of class* C^2*. In addition to* (10.5)*, assume that* $c \in C(\overline{\Omega})$*,* $1 < p < \infty$*,* $k = 1, 2$*, and* $\delta = 0$*.*

Then either

(i) *Problem* (10.3), (10.4) *possesses a unique solution* $u \in W^{k,p}(\Omega)$ *for any* f, g *belonging to the regularity class*

$$f \in [W^{1,p'}(\Omega)]^*, \ g \in W^{-\frac{1}{p},p}(\partial\Omega) \ \textit{if} \ k = 1, \tag{10.7}$$

$$f \in L^p(\Omega), \ g \in W^{1-\frac{1}{p},p}(\partial\Omega) \ \textit{if} \ k = 2; \tag{10.8}$$

or

(ii) *the null space*

$$\ker[\mathcal{A}] = \{u \in W^{k,p}(\Omega) \mid u \ \textit{solve} \ (10.3), (10.4) \ \textit{with} \ f = g = 0\}$$

is of finite dimension, and problem (10.3), (10.4) *admits a solution for* f, g *belonging to the class* (10.7), (10.8) *only if*

$$\langle f; w \rangle_{[[W^{1,p'}]^*; W^{1,p'}](\Omega)} - \langle g; w \rangle_{[W^{-1/p,p}, W^{1/p,p'}](\partial\Omega)} = 0$$

for all $w \in \ker[\mathcal{A}]$.

See Amann [7, Theorem 9.2], Geymonat and Grisvard [95]. □

In the concrete cases, the Fredholm alternative gives existence of a solution u, while the estimates of u in $W^{k,p}(\Omega)$ in terms of f and g follow from Theorems 10.3 and 10.4 via a uniqueness contradiction argument.

For example, in the sequel, we shall deal with a simple Neumann problem for generalized Laplacian

$$-\mathrm{div}_x \left(\eta \nabla_x \left(\frac{v}{\eta} \right) \right) = f \text{ in } \Omega, \quad \nabla_x \left(\frac{v}{\eta} \right) \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

where η is a sufficiently smooth and positive function on $\overline{\Omega}$ and $f \in L^p(\Omega)$ with a certain $1 < p < \infty$. In this case the Fredholm alternative guarantees existence of $u \in W^{2,p}(\Omega)$ provided $f \in L^p(\Omega)$, $\int_\Omega f \mathrm{d}x = 0$. The solution is unique in the class $u \in W^{2,p}(\Omega)$, $\int_\Omega \frac{u}{\eta} \mathrm{d}x = 0$ and satisfies estimate

$$\|u\|_{W^{2,p}(\Omega)} \le c \|f\|_{L^p(\Omega)}.$$

10.2.3 Spectrum of a generalized Laplacian

We begin by introducing a densely defined (unbounded) linear operator

$$\Delta_{\eta,\mathcal{N}} = \mathrm{div}_x\left(\eta\nabla_x\left(\frac{v}{\eta}\right)\right), \tag{10.9}$$

with the function η to be specified later, acting from $L^p(\Omega)$ to $L^p(\Omega)$ with domain of definition

$$\mathcal{D}(\Delta_{\eta,\mathcal{N}}) = \{u \in W^{2,p}(\Omega) \,|\, \nabla_x\left(\frac{v}{\eta}\right)\cdot \mathbf{n}|_{\partial\Omega} = 0\}. \tag{10.10}$$

Further we denote $\Delta_{\mathcal{N}} = \Delta_{1,\mathcal{N}}$ the classical Laplacian with the homogenous Neumann boundary condition.

We shall apply the results of Sections 10.2.1–10.2.2 to the spectral problem that consists in finding couples (λ, v), $\lambda \in \mathbb{C}$, $v \in \mathcal{D}(\Delta_{\eta,\mathcal{N}})$ that verify

$$-\mathrm{div}_x\left(\eta\nabla_x\left(\frac{v}{\eta}\right)\right) = \lambda v \text{ in } \Omega,\ \nabla_x\left(\frac{v}{\eta}\right)\cdot \mathbf{n}|_{\partial\Omega} = 0.$$

The results announced in the main theorem of this section are based on a general theorem of functional analysis concerning the spectral properties of compact operators.

Let $T : X \to X$ be a linear operator on a Hilbert space X endowed with scalar product $\langle\cdot;\cdot\rangle$. We say that a complex number λ belongs to the *spectrum* of T (one writes $\lambda \in \sigma(T)$) if $\ker(T - \lambda\mathbb{I}) \neq \{0\}$ or if $(T - \lambda\mathbb{I})^{-1} : X \to X$ is not a bounded linear operator (here $\mathbb{I}$ denotes the identity operator). We say that λ is an *eigenvalue* of T or belongs to the *discrete* (*pointwise*) *spectrum* of T (and write $\lambda \in \sigma_p(T) \subset \sigma(T)$) if $\ker(T - \lambda\mathbb{I}) \neq \{0\}$. In the latter case, the non-zero vectors belonging to $\ker(T - \lambda\mathbb{I})$ are called *eigenvectors* and the vector space $\ker(T - \lambda\mathbb{I})$ an *eigenspace*.

■ SPECTRUM OF A COMPACT OPERATOR:

Theorem 10.6. *Let H be an infinite-dimensional Hilbert space and $T : H \to H$ a compact linear operator. Then*

(i) $0 \in \sigma(T)$;

(ii) $\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$;

(iii) $$\begin{cases} \sigma(T) \setminus \{0\} \text{ is finite, or else} \\ \sigma(T) \setminus \{0\} \text{ is a sequence tending to } 0. \end{cases}$$

(iv) *If $\lambda \in \sigma(T) \setminus \{0\}$, then the dimension of the eigenspace* $\ker(T - \lambda\mathbb{I})$ *is finite.*

(v) *If T is a* positive operator, *meaning $\langle Tv; v\rangle \geq 0$, $v \in H$, then $\sigma(T) \subset [0, +\infty)$.*

(vi) *If T is a* symmetric operator*, meaning* $\langle Tv; w\rangle = \langle v; Tw\rangle$, $v, w \in H$*, then* $\sigma(T) \subset \mathbb{R}$*. If in addition H is separable, then H admits an orthonormal basis of eigenvectors that consists of eigenvectors of T.*

See Evans [74, Chapter D, Theorems 6,7]

The main theorem of this section reads:

■ SPECTRUM OF THE GENERALIZED LAPLACIAN WITH NEUMANN BOUNDARY CONDITION:

Theorem 10.7. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class C^2. Let*

$$\eta \in C^1(\overline{\Omega}), \quad \inf_{x\in\Omega} \eta(x) = \underline{\eta} > 0.$$

Then the spectrum of the operator $-\Delta_{\eta,\mathcal{N}}$, where $\Delta_{\eta,\mathcal{N}}$ is defined in (10.9–10.10)*, coincides with the discrete spectrum and the following holds:*

(i) *The spectrum consists of a sequence $\{\lambda_k\}_{k=0}^{\infty}$ of real eigenvalues, where $\lambda_0 = 0$, $0 < \lambda_k < \lambda_{k+1}$, $k = 1, 2, \ldots$, and $\lim_{k\to\infty} \lambda_k = \infty$;*

(ii) $0 < \dim(E_k) < \infty$ *and* $E_0 = \operatorname{span}\{\eta\}$*, where* $E_k = \ker(-\Delta_{\eta,\mathcal{N}} - \lambda_k \mathbb{I})$ *is the eigenspace corresponding to the eigenvalue λ_k;*

(iii) $L^2(\Omega) = \bigoplus_{k=0}^{\infty} E_k$*, where the direct sum is orthogonal with respect to the scalar product*

$$\langle u; v\rangle_{1/\eta} = \int_\Omega u\overline{v}\, \frac{\mathrm{d}x}{\eta}$$

(*here the line over v means the complex conjugate of v*).

Proof. We set

$$T : L^2(\Omega) \to L^2(\Omega), \quad Tf = \begin{cases} -\Delta_{\eta,\mathcal{N}}^{-1} f \text{ if } f \in \dot{L}^2(\Omega), \\ 0 \text{ if } f \in \operatorname{span}\{1\}, \end{cases}$$

$$\Delta_{\eta,\mathcal{N}}^{-1} : \dot{L}^2(\Omega) = \{f \in L^2(\Omega) \mid \int_\Omega f\, \mathrm{d}x = 0\} \mapsto \{u \in L^2(\Omega) \mid \int_\Omega \frac{u}{\eta}\, \mathrm{d}x = 0\},$$

$$-\Delta_{\eta,\mathcal{N}}^{-1} f = u \iff -\Delta_{\eta,\mathcal{N}} u = f.$$

In accordance with the regularity properties of elliptic operators collected in Sections 10.2.1–10.2.2 (see notably Theorems 10.3 and 10.5), the operator T is a compact operator.

A double integration by parts yields

$$-\int_\Omega \operatorname{div}_x \left(\eta \nabla_x \left(\frac{v}{\eta}\right)\right) u\, \frac{\mathrm{d}x}{\eta} = \int_\Omega \eta \nabla_x \left(\frac{v}{\eta}\right) \cdot \nabla_x \left(\frac{u}{\eta}\right) \mathrm{d}x$$
$$= -\int_\Omega \operatorname{div}_x \left(\eta \nabla_x \left(\frac{u}{\eta}\right)\right) v\, \frac{\mathrm{d}x}{\eta}.$$

Taking in the last formula $u = Tf$, $f \in L^2(\Omega)$, $v = Tg$, $g \in L^2(\Omega)$ and recalling that functions $\frac{Tf}{\eta}$, $\frac{Tg}{\eta}$ have zero mean, we deduce that

$$\int_\Omega Tf\, g\, \frac{\mathrm{d}x}{\eta} = \int_\Omega f\, Tg\, \frac{\mathrm{d}x}{\eta} \qquad \text{and} \qquad \int_\Omega Tf\, \overline{f}\, \frac{\mathrm{d}x}{\eta} \geq 0.$$

To resume, we have proved that T is a compact positive linear operator on $L^2(\Omega)$ that is symmetric with respect to the scalar product $\langle \cdot\,;\cdot \rangle_{1/\eta}$. Now, all statements of Theorem 10.7 follow from Theorem 10.6. □

10.3 Normal traces

Let Ω be a bounded domain in $\mathbb{R}^N$. For $1 \leq q, p \leq \infty$, we introduce a Banach space

$$E^{q,p}(\Omega) = \{\mathbf{u} \in L^q(\Omega; \mathbb{R}^N) |\, \mathrm{div}\mathbf{u} \in L^p(\Omega)\}. \tag{10.11}$$

endowed with norm

$$\|\mathbf{u}\|_{E^q(\Omega)} := \|\mathbf{u}\|_{E^q(\Omega;\mathbb{R}^3)} + \|\mathrm{div}\mathbf{u}\|_{L^p(\Omega)}. \tag{10.12}$$

We also define

$$E_0^{q,p}(\Omega) = \mathrm{closure}_{E^{q,p}(\Omega)}\Big\{C_c^\infty(\Omega; \mathbb{R}^N)\Big\}$$

and

$$E^p(\Omega) = E^{p,p}(\Omega), \quad E_0^p(\Omega) = E_0^{p,p}(\Omega).$$

Our goal is to introduce the concept of *normal traces* and to derive a variant of Green's formula for the functions belonging to $E^{q,p}(\Omega)$.

■ NORMAL TRACES:

Theorem 10.8. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain, and let $1 < p < \infty$. Then there exists a unique linear operator $\gamma_\mathbf{n}$ with the following properties:*

(i)
$$\gamma_\mathbf{n} : E^p(\Omega) \mapsto [W^{1-\frac{1}{p'},p'}(\partial\Omega)]^* := W^{-\frac{1}{p},p}(\partial\Omega), \tag{10.13}$$

and

$$\gamma_\mathbf{n}(\mathbf{u}) = \gamma_0(\mathbf{u}) \cdot \mathbf{n} \text{ a.a. on } \partial\Omega \text{ whenever } \mathbf{u} \in C^\infty(\overline{\Omega}; \mathbb{R}^N). \tag{10.14}$$

(ii) *The Stokes formula*

$$\int_\Omega v \mathrm{div}\mathbf{u}\, \mathrm{d}x + \int_\Omega \nabla v \cdot \mathbf{u}\, \mathrm{d}x = \langle \gamma_\mathbf{n}(\mathbf{u})\, ;\, \gamma_0(v) \rangle\,, \tag{10.15}$$

holds for any $\mathbf{u} \in E^p(\Omega)$ and $v \in W^{1,p'}(\Omega)$, where $\langle \cdot\,;\,\cdot \rangle$ denotes the duality pairing between $W^{1-\frac{1}{p'},p'}(\Omega)$ and $W^{-\frac{1}{p},p}(\Omega)$.

(iii)
$$\ker[\gamma_\mathbf{n}] = E_0^p(\Omega). \tag{10.16}$$

(iv) *If $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^N)$, then $\gamma_\mathbf{n}(\mathbf{u})$ in $L^p(\partial\Omega)$, and $\gamma_\mathbf{n}(\mathbf{u}) = \gamma_0(\mathbf{u}) \cdot \mathbf{n}$ a.a. on $\partial\Omega$.*

Proof of Theorem 10.8. As a matter of fact, Theorem 10.8 is a standard result whose proof can be found in Temam [189, Chapter 1]. We give a concise proof based on the following three lemmas that may be of independent interest.

Step 1. We start with a technical result, the proof of which can be found in Galdi [92, Lemma 3.2]. We recall that a domain $Q \subset \mathbb{R}^N$ is said to be *star-shaped* if there exists $a \in Q$ such that $Q = \{x \in \mathbb{R}^N \,|\, |x-a| < h(\frac{x-a}{|x-a|})\}$, where h is a positive continuous function on the unit sphere; it is said to be star-shaped with respect to a ball $B \subset Q$ if it is star-shaped with respect to any of its points.

Lemma 10.1. *Let Ω be a bounded Lipschitz domain.*

Then there exists a finite family of open sets $\{\mathcal{O}_i\}_{i\in I}$ and a family of balls $\{B^{(i)}\}_{i\in I}$ such that each $\Omega_i := \Omega \cap \mathcal{O}_i$ is star-shaped with respect to the ball $B^{(i)}$, and

$$\overline{\Omega} \subset \cup_{i\in I}\mathcal{O}_i.$$

Step 2. The main ingredient of the proof of Theorem 10.8 is the density of smooth functions in the spaces $E^{q,p}(\Omega)$.

Lemma 10.2. *Let Ω be a bounded Lipschitz domain and $1 \le p \le q < \infty$. Then $C^\infty(\overline{\Omega};\mathbb{R}^N) = C_c^\infty(\overline{\Omega})$ is dense in $E^{q,p}(\Omega)$.*

Proof of Lemma 10.2. Hypothesis $q \ge p$ is of technical character and can be relaxed if, for instance, Ω is of class $C^{1,1}$. It ensures that $\mathbf{u}\varphi \in E^{q,p}(\Omega)$ as soon as $\varphi \in C_c^\infty(\Omega)$. Moreover, according to Lemma 10.1, any bounded Lipschitz domain can be decomposed as a finite union of *star-shaped domains* with respect to a ball. Using the corresponding subordinate partition of unity we may assume, without loss of generality, that Ω is a star-shaped domain with respect to a ball centered at the origin of the Cartesian coordinate system.

For $\mathbf{u} \in E^{q,p}(\Omega)$ we write $\mathbf{u}_\tau(x) = \mathbf{u}(\tau x)$, $\tau > 0$, so that if $\tau \in (0,1)$, $\mathbf{u}_\tau \in E^{q,p}(\tau^{-1}\Omega)$ and $\mathrm{div}(\mathbf{u}_\tau) = \tau(\mathrm{div}\mathbf{u})_\tau$ in $\mathcal{D}'(\tau^{-1}\Omega)$, where $\tau^{-1}\Omega = \{x \in \mathbb{R}^N \,|\, \tau x \in \Omega\}$. We therefore have

$$\|\mathrm{div}(\mathbf{u}-\mathbf{u}_\tau)\|_{L^p(\Omega)} \le (1-\tau)\|\mathrm{div}\mathbf{u}\|_{L^p(\Omega)} + \|\mathrm{div}\mathbf{u} - (\mathrm{div}\mathbf{u})_\tau\|_{L^p(\Omega)}. \tag{10.17}$$

Since the translations $\mathbb{R}^N \ni h \to u(\cdot + h) \in L^s(\mathbb{R}^N)$ are continuous for any fixed $u \in L^s(\mathbb{R}^N)$, $1 \le s < \infty$, the right-hand side of formula (10.17) as well as $\|\mathbf{u} - \mathbf{u}_\tau\|_{L^q(\Omega)}$ tend to zero as $\tau \to 1-$. Thus it is enough to prove that $\mathbf{u}_\tau$ can be approximated in $E^{q,p}(\Omega)$ by functions belonging to $C^\infty(\overline{\Omega};\mathbb{R}^N)$.

Since $\overline{\Omega} \subset \tau^{-1}\Omega$, the mollified functions $\zeta_\epsilon * \mathbf{u}_\tau$ belong to $C^\infty(\overline{\Omega};\mathbb{R}^N) \cap E^{q,p}(\Omega)$ provided $0 < \varepsilon < \mathrm{dist}(\Omega, \partial(\tau^{-1}\Omega))$ and tend to $\mathbf{u}_\tau$ in $E^{q,p}(\Omega)$ as $\varepsilon \to 0+$ (see Theorem 10.1). This observation completes the proof of Lemma 10.2. □

Step 3. We are now in a position to define the operator of normal traces. Let Ω be a bounded Lipschitz domain, $1 < p < \infty$, $v \in W^{1-\frac{1}{p'},p'}(\partial\Omega)$, and $\mathbf{u} \in C^\infty(\overline{\Omega};\mathbb{R}^N)$.

According to the trace theorem (see Theorem 0.6), we have

$$\int_{\partial\Omega} v\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}\sigma = \int_{\Omega} \ell(v)\mathrm{div}\mathbf{u}\,\mathrm{d}x + \int_{\Omega} \nabla\ell(v)\cdot\mathbf{u}\,\mathrm{d}x,$$

and

$$\Big|\int_{\partial\Omega} v\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}\sigma\Big| \leq \|\mathbf{u}\|_{E^p(\Omega)}\,\|\ell(v)\|_{W^{1,p'}(\Omega)} \leq c(p,\Omega)\|\mathbf{u}\|_{E^p(\Omega)}\,\|v\|_{W^{1-1/p',p'}(\partial\Omega)},$$

where the first identity is independent of the choice of the lifting operator ℓ. Consequently, the map

$$\gamma_{\mathbf{n}} : \mathbf{u} \to \gamma_0(\mathbf{u})\cdot\mathbf{n} \tag{10.18}$$

is a linear, densely defined (on $C^\infty(\overline{\Omega})$) and continuous operator from $E^p(\Omega)$ to $[W^{1-1/p',p'}(\partial\Omega)]^* = W^{-\frac{1}{p},p}(\partial\Omega)$. Its value at $\mathbf{u}$ is termed the *normal trace* of $\mathbf{u}$ on $\partial\Omega$ and denoted by $\gamma_{\mathbf{n}}(\mathbf{u})$ or $(\mathbf{u}\cdot\mathbf{n})|_{\partial\Omega}$.

Step 4. In order to complete the proof of Theorem 10.8, it remains to show that $\ker[\gamma_{\mathbf{n}}] = E^p_0(\Omega)$. □

Lemma 10.3. *Let Ω be a bounded Lipschitz domain, $1 < p < \infty$, and let $\gamma_{\mathbf{n}}$: $E^p(\Omega) \to W^{-\frac{1}{p},p}(\partial\Omega)$ be the operator defined as a continuous extension of the trace operator introduced in* (10.18). *Then* $\ker[\gamma_{\mathbf{n}}] = E^p_0(\Omega)$.

Proof of Lemma 10.3. Clearly, $C^\infty_c(\Omega) \subset \ker[\gamma_{\mathbf{n}}]$; whence, by continuity of $\gamma_{\mathbf{n}}$, $E^p_0(\Omega) \subset \ker[\gamma_{\mathbf{n}}]$.

Conversely, we set

$$\tilde{\mathbf{u}}(x) = \begin{cases} \mathbf{u}(x) \text{ if } x\in\Omega, \\ 0 \text{ otherwise.}\end{cases}$$

Assumption $\mathbf{u}\in\ker[\gamma_{\mathbf{n}}]$ yields $\int_\Omega v\mathrm{div}\mathbf{u}\,\mathrm{d}x + \int_\Omega \nabla v\cdot\mathbf{u}\,\mathrm{d}x = 0$ for all $v\in C^\infty_c(\mathbb{R}^N)$, meaning that, in the sense the distributions,

$$\mathrm{div}\tilde{\mathbf{u}}(x) = \left\{\begin{array}{c} \mathrm{div}\mathbf{u}(x) \text{ if } x\in\Omega, \\ 0 \text{ otherwise}\end{array}\right\} \in L^p(\mathbb{R}^N),$$

and, finally, $\tilde{\mathbf{u}}\in E^p(\mathbb{R}^N)$.

In agreement with Lemma 10.2, we suppose, without loss of generality, that Ω is star-shaped with respect to the origin of the coordinate system. Similarly to Lemma 10.2, we deduce that $\mathrm{supp}[(\tilde{\mathbf{u}}_{1/\tau})]$ belongs to the set $\overline{\tau\Omega}\subset\Omega$, and, moreover, $\|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_{1/\tau}\|_{E^p(\Omega)} \to 0$ as $\tau\to 1-$.

Consequently, it is enough to approximate $\tilde{\mathbf{u}}_{1/\tau}$ by a suitable function belonging to the set $C^\infty_c(\Omega;\mathbb{R}^N)$. However, according to Theorem 10.1, functions $\zeta_\varepsilon * \mathbf{u}_{1/\tau}$ belong to $C^\infty_c(\Omega)\cap E^p(\Omega)$, provided $0 < \varepsilon < \frac{1}{2}\mathrm{dist}(\tau\Omega,\partial\Omega)$, and $\zeta_\varepsilon * \tilde{\mathbf{u}}_{1/\tau} \to \tilde{\mathbf{u}}_{1/\tau}$ in $E^p(\Omega)$. This completes the proof of Lemma 10.3 as well as that of Theorem 10.8. □

10.4 Singular and weakly singular operators

The *weakly singular integral transforms* are defined through formula

$$[T(f)](x) = \int_{\mathbb{R}^N} K(x, x-y) f(y)\, \mathrm{d}y, \tag{10.19}$$

where

$$K(x,z) = \frac{\theta(x,z)}{|z|^\lambda}, \quad 0 < \lambda < N, \quad \theta \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N). \tag{10.20}$$

A function K satisfying (10.20) is called a *weakly singular kernel.*

The *singular integral transforms* are defined as

$$[T(f)](x) = \lim_{\varepsilon \to 0+} \left(\int_{|x-y| \geq \varepsilon} K(x, x-y) f(y)\, \mathrm{d}y \right) := \text{v.p.} \int_{\mathbb{R}^N} K(x, x-y) f(y)\, \mathrm{d}y, \tag{10.21}$$

where

$$\begin{aligned} K(x,z) = \tfrac{\theta(x, z/|z|)}{|z|^N}, &\quad \theta \in L^\infty(\mathbb{R}^N \times S), \\ S = \{z \in \mathbb{R}^N \mid |z| = 1\}, &\quad \textstyle\int_{|z|=1} \theta(x,z)\, \mathrm{d}S_z = 0. \end{aligned} \tag{10.22}$$

The kernels satisfying (10.22) are called *singular kernels of Calderón-Zygmund type.*

The basic result concerning the weakly singular kernels is the *Sobolev theorem.*

■ WEAKLY SINGULAR INTEGRALS:

Theorem 10.9. *The operator T defined in* (10.19) *with K satisfying* (10.20) *is a bounded linear operator on $L^q(\mathbb{R}^N)$ with values in $L^r(\mathbb{R}^N)$, where $1 < q < \infty$, $\frac{1}{r} = \frac{\lambda}{N} + \frac{1}{q} - 1$. In particular,*

$$\|T(f)\|_{L^r(\mathbb{R}^N)} \leq c \|f\|_{L^q(\mathbb{R}^N)},$$

where the constant c can be expressed in the form $c_0(q, N)\|\theta\|_{L^\infty(\mathbb{R}^N \times \mathbb{R}^N)}$.

See Stein [186, Chapter V, Theorem 1] □

The fundamental result concerning the singular kernels is the *Calderón-Zygmund theorem.*

■ SINGULAR INTEGRALS:

Theorem 10.10. *The operator T defined in* (10.21) *with K satisfying* (10.22) *is a bounded linear operator on $L^q(\mathbb{R}^N)$ for any $1 < q < \infty$. In particular,*

$$\|T(f)\|_{L^q(\mathbb{R}^N)} \leq c \|f\|_{L^q(\mathbb{R}^N)},$$

where the constant c takes the form $c = c_0(q, N)\|\theta\|_{L^\infty(\mathbb{R}^N \times S)}$.

See Calderón-Zygmund [38, Theorem 2], [39, Section 5, Theorem 2]. □

10.5 The inverse of the div-operator (Bogovskii's formula)

We consider the problem

$$\mathrm{div}_x \mathbf{u} = f \text{ in } \Omega, \ \mathbf{u}|_{\partial\Omega} = 0 \tag{10.23}$$

for a given function f, where $\Omega \subset \mathbb{R}^N$ is a bounded domain. Clearly, problem (10.23) admits many solutions that may be constructed in different manners. Here, we adopt the integral formula proposed by Bogovskii [24] and elaborated by Galdi [92]. In such a way, we resolve (10.23) for any smooth f of zero integral mean. In addition, we deduce uniform estimates that allow us to extend solvability of (10.23) to a significantly larger class of right-hand sides f, similarly to Geissert, Heck and Hieber [94]. The main advantage of our construction is that it requires only Lipschitz regularity of the underlying spatial domain. Extensions to other geometries including unbounded domains are possible. We recommend that the interested reader consult the monograph by Galdi [92] or [166, Chapter III] for both positive and negative results in this direction.

Our results are summarized in the following theorem.

■ THE INVERSE OF THE DIV-OPERATOR:

Theorem 10.11. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain.*

(i) *Then there exists a linear mapping $\mathcal{B}$,*

$$\mathcal{B} : \{ f \mid f \in C_c^\infty(\Omega), \ \int_\Omega f \ \mathrm{d}x = 0 \} \to C_c^\infty(\Omega; \mathbb{R}^N),$$

such that $\mathrm{div}_x(\mathcal{B}[f]) = f$, *meaning,* $\mathbf{u} = \mathcal{B}[f]$ *solves* (10.23).

(ii) *We have*

$$\|\mathcal{B}[f]\|_{W^{k+1,p}(\Omega;\mathbb{R}^N)} \leq c\|f\|_{W^{k,p}(\Omega)} \text{ for any } 1 < p < \infty, \ k = 0, 1, \dots, \tag{10.24}$$

in particular, $\mathcal{B}$ can be extended in a unique way as a bounded linear operator

$$\mathcal{B} : \{ f \mid f \in L^p(\Omega), \ \int_\Omega f \ \mathrm{d}x = 0 \} \to W_0^{1,p}(\Omega; \mathbb{R}^N).$$

(iii) *If $f \in L^p(\Omega)$, $\int_\Omega f \ \mathrm{d}x = 0$, and, in addition, $f = \mathrm{div}_x \mathbf{g}$, where $\mathbf{g} \in E_0^{q,p}(\Omega)$, $1 < q < \infty$, then*

$$\|\mathcal{B}[f]\|_{L^q(\Omega;\mathbb{R}^3)} \leq c\|\mathbf{g}\|_{L^q(\Omega;\mathbb{R}^3)}. \tag{10.25}$$

(iv) *$\mathcal{B}$ can be uniquely extended as a bounded linear operator*

$$\mathcal{B} : [\dot{W}^{1,p'}(\Omega)]^* = \{ f \in [W^{1,p'}(\Omega)]^* \mid \langle f; 1 \rangle = 0 \} \to L^p(\Omega; \mathbb{R}^N)$$

in such a way that

$$-\int_\Omega \mathcal{B}[f]\cdot\nabla v \,\mathrm{d}x = \langle f; v\rangle_{\{[W^{1,p'}]^*;W^{1,p'}\}(\Omega)} \text{ for all } v \in W^{1,p'}(\Omega), \quad (10.26)$$

$$\|\mathcal{B}[f]\|_{L^p(\Omega;R^N)} \le c\|f\|_{[W^{1,p'}(\Omega)]^*}. \quad (10.27)$$

Here, a function $f \in C_c^\infty(\Omega)$ *is identified with a linear form in* $[W^{1,p'}(\Omega)]^*$ *via the standard Riesz formula*

$$\langle f; v\rangle_{[W^{1,p'}(\Omega)]^*;W^{1,p'}(\Omega)} = \int_\Omega fv \,\mathrm{d}x \text{ for all } v \in W^{1,p'}(\Omega). \quad (10.28)$$

Remark: *Since* $\mathcal{B}$ *is linear, it is easy to check that*

$$\partial_t\mathcal{B}[f](t,x) = \mathcal{B}[\partial_t f](t,x) \text{ for a.a. } (t,x) \in (0,T)\times\Omega \quad (10.29)$$

provided

$$\partial_t f,\ f \in L^p((0,T)\times\Omega),\ \int_\Omega f(t,\cdot)\,\mathrm{d}x = 0 \text{ for a.a. } t \in (0,T).$$

The proof of Theorem 10.11 is given by means of several steps which may be of independent interest.

Step 1. The first ingredient of the proof is a representation formula for functionals belonging to $[\dot W^{1,p'}(\Omega)]^*$.

Lemma 10.4. *Let* Ω *be a domain in* $\mathbb{R}^N$*, and let* $1 < p \le \infty$.

Then any linear form $f \in [\dot W^{1,p'}(\Omega)]^*$ *admits a representation*

$$\langle\, f;\, v\rangle_{[\dot W^{1,p'}(\Omega)]^*,\dot W^{1,p'}(\Omega)} = \sum_{i=1}^N \int_\Omega w_i\partial_{x_i} v\,\mathrm{d}x,$$

where

$$\mathbf{w} = [w_1,\dots,w_N] \in L^p(\Omega;\mathbb{R}^N) \text{ and } \|f\|_{[\dot W^{1,p'}(\Omega)]^*} = \|\mathbf{w}\|_{L^p(\Omega;\mathbb{R}^N)}.$$

Proof of Lemma 10.4. The operator $I : \dot W^{1,p'}(\Omega) \to L^{p'}(\Omega;\mathbb{R}^N)$, $I(u) = \nabla u$ is an isometric isomorphism mapping $\dot W^{1,p'}(\Omega)$ onto a (closed) subspace $I(\dot W^{1,p'}(\Omega))$ of $L^{p'}(\Omega;\mathbb{R}^N)$. The functional ϕ defined as

$$\langle\,\phi;\nabla u\,\rangle := \langle\, f; u\,\rangle_{[\dot W^{1,p'}(\Omega)]^*,\dot W^{1,p'}(\Omega)}$$

is a linear functional on $I(\dot W^{1,p'}(\Omega))$ satisfying condition

$$\sup\Big\{\,\langle\,\phi;\mathbf{v}\,\rangle \mid \mathbf{v} \in I(\dot W^{1,p'}(\Omega)),\ \|\mathbf{v}\|_{L^{p'}(\Omega;\mathbb{R}^N)} \le 1\Big\} = \|f\|_{[\dot W^{1,p'}(\Omega)]^*}.$$

Therefore by the Hahn-Banach theorem (see, e.g., Brezis [35, Theorem I.1]), there exists a linear functional Φ defined on $L^{p'}(\Omega;\mathbb{R}^N)$ satisfying

$$\langle \Phi; \nabla u \rangle = \langle \phi; \nabla u \rangle, \quad u \in \dot{W}^{1,p'}(\Omega), \quad \|\Phi\|_{[L^{p'}(\Omega;\mathbb{R}^N)]^*} = \|f\|_{[\dot{W}^{1,p'}(\Omega)]^*}.$$

According to the Riesz representation theorem (cf. Remark following Theorem 0.2) there exists a unique $\mathbf{w} \in L^p(\Omega;\mathbb{R}^N)$ such that

$$\langle \Phi; \mathbf{v} \rangle = \int_\Omega \mathbf{w} \cdot \mathbf{v}, \quad \mathbf{v} \in L^{p'}(\Omega;\mathbb{R}^N),$$
$$\|\Phi\|_{[L^{p'}(\Omega;\mathbb{R}^N)]^*} = \|\mathbf{w}\|_{L^p(\Omega;\mathbb{R}^N)}.$$

This yields the statement of Lemma 10.4. □

Step 2. We use Lemma 10.4 to show that $C_c^\infty(\Omega)$ is dense in $[\dot{W}^{1,p'}(\Omega)]^*$.

Lemma 10.5. *Let $\Omega \subset R^N$ be an open set, $1 < p' \le \infty$.*

Then the set $\{C_c^\infty(\Omega) \,|\, \int_\Omega v \,\mathrm{d}x = 0\}$, identified as a subset of $[\dot{W}^{1,p'}(\Omega)]^*$ via (10.28), *is dense in $[\dot{W}^{1,p'}(\Omega)]^*$.*

Proof of Lemma 10.5. Let $\mathbf{w} \in L^p(\Omega;R^N)$ be a representant of $f \in [\dot{W}^{1,p'}(\Omega)]^*$ constructed in Lemma 10.4 and let $\mathbf{w}_n \in C_c^\infty(\Omega;R^N)$ be a sequence converging strongly to $\mathbf{w}$ in $L^p(\Omega;\mathbb{R}^N)$. Then a family of functionals $f_n = \mathrm{div}\mathbf{w}_n \in \{v \in C_c^\infty(\Omega) \,|\, \int_\Omega v \,\mathrm{d}x = 0\}$, defined as $\langle f_n; v \rangle = \int_\Omega \mathbf{w}_n \cdot \nabla v \,\mathrm{d}x = -\int_\Omega \mathrm{div}\mathbf{w}_n v \,\mathrm{d}x$, converges to f in $[\dot{W}^{1,p'}(\Omega)]^*$. This completes the proof. □

Step 3. Having established the preliminary material, we focus on particular solutions to the problem $\mathrm{div}_x u = f$ with a smooth right-hand side f. These solutions have been constructed by Bogovskii [24], and their basic properties are collected in the following lemma.

Lemma 10.6. *Let Ω be a bounded Lipschitz domain.*

Then there exists a linear operator

$$\mathcal{B} : \{f \in C_c^\infty(\Omega) | \int_\Omega f \,\mathrm{d}x = 0\} \mapsto C_c^\infty(\Omega;\mathbb{R}^N) \tag{10.30}$$

such that:

(i)
$$\mathrm{div}_x \mathcal{B}(f) = f, \tag{10.31}$$

and

$$\|\nabla_x \mathcal{B}(f)\|_{W^{k,p}(\Omega;\mathbb{R}^{N\times N})} \le c\|f\|_{W^{k,p}(\Omega)}, \quad 1 < p < \infty,\ k = 0, 1, \ldots, \tag{10.32}$$

where c is a positive constant depending on k, p, diam(Ω) *and the Lipschitz constant associated to the local charts covering $\partial\Omega$.*

(ii) *If* $f = \mathrm{div}_x \mathbf{g}$, *where* $\mathbf{g} \in C_c^\infty(\Omega; \mathbb{R}^N)$, *then*

$$\|\mathcal{B}(f)\|_{L^q(\Omega;\mathbb{R}^{N\times N})} \leq c\|\mathbf{g}\|_{L^q(\Omega;R^3)},\ 1 < q < \infty, \tag{10.33}$$

where c *is a positive constant depending on* q, $\mathrm{diam}(\Omega)$, *and the Lipschitz constant associated to* $\partial\Omega$.

(iii) *If* $f, \partial_t f \in \{v \in C_c^\infty(I \times \Omega) \mid \int_\Omega v(t,x)\,\mathrm{d}x = 0,\, t \in I\}$, *where* I *is an (open) interval, then*

$$\frac{\partial \mathcal{B}(f)}{\partial t}(t,x) = \mathcal{B}\Big(\frac{\partial f}{\partial t}\Big)(t,x) \text{ for all } t \in I,\ x \in \Omega. \tag{10.34}$$

Remark: *In the case of a domain star-shaped with respect to a ball of radius* $\overline{r}$ *and for* $k = 1$, *the estimate of the constants in* (10.6), (10.33) *are given by formula* (10.38) *below. In the case of a Lipschitz domain, it may be evaluated by using* (10.38) *combined with Lemma* 10.1, *and Lemma* 10.7 *below.*

Step 4. Before starting the proof of Lemma 10.6, we observe that it is enough to consider star-shaped domains.

Lemma 10.7. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded Lipschitz domain, and let*

$$f \in C_c^\infty(\Omega), \quad \int_\Omega f\,\mathrm{d}x = 0.$$

Then there exists a family of functions

$$f_i \in C_c^\infty(\Omega_i),\ \int_{\Omega_i} f_i\,\mathrm{d}x = 0,\ \Omega_i = \Omega \cap \mathcal{O}_i \text{ for } i \in I,$$

where $\{\mathcal{O}\}_{i\in I}$ *is the covering of* Ω *constructed in Lemma* 10.1, *and* Ω_i *are star-shaped with respect to a ball. Moreover,*

$$\|f_i\|_{W^{k,p}(\Omega_i)} \leq c\|f\|_{W^{k,p}(\Omega)},\ 1 \leq p \leq \infty,\ k = 0, 1, \dots,$$

where c *is a positive constant dependent solely on* p, k *and* $|\mathcal{O}_i|$, $i \in I$.

Proof of Lemma 10.7. Let $\{\varphi_i\}_{i\in I\cup J}$ be a *partition of unity* subordinate to the covering $\{\mathcal{O}_i\}_{i\in I}$ of $\overline{\Omega}$. We set

$$\Omega_1 = \Omega \cap \mathcal{O}_1,\ \Omega^1 = \cup_{i\in I\setminus\{1\}}\Omega_i, \text{ where } \Omega_i = \mathcal{O}_i \cap \Omega.$$

Next, we introduce

$$f_1 = f\varphi_1 - \kappa_1 \int_{\Omega_1} f\varphi_1\,\mathrm{d}x,\ g = f\phi - \kappa_1 \int_{\Omega^1} f\phi\,\mathrm{d}x,$$

where

$$\kappa_1 \in C_c^\infty(\Omega_1 \cap \Omega^1),\ \int_\Omega \kappa_1\,\mathrm{d}x = 1,\ \phi = \sum_{i\in I\setminus\{1\}} \varphi_i.$$

With this choice,

$$f_1 \in C_c^\infty(\Omega_1),\ \int_{\Omega_1} f_1 \,\mathrm{d}x = 0,\ g \in C_c^\infty(\Omega^1),\ \int_{\Omega^1} g \,\mathrm{d}x = 0,$$

and both f_1 and g satisfy $W^{k,p}$-estimates stated in Lemma 10.7. Applying the above procedure to g in place of f and to Ω^1 in place of Ω, we can proceed by induction and complete the proof after a finite number of steps. □

Step 5.

Proof of Lemma 10.6. In view of Lemma 10.7, it is enough to assume that Ω is a star-shaped domain with respect to a ball $B(0;\overline{r})$, where the latter can be taken of radius $\overline{r}$ centered at the origin of the coordinate system.

In such a case, a possible candidate satisfying all properties stated in Lemma 10.6 is the so-called Bogovskii's solution given by the explicit formula:

$$\mathcal{B}[f](x) = \int_\Omega f(y)\Big[\frac{x-y}{|x-y|^N}\int_{|x-y|}^\infty \zeta_{\overline{r}}\Big(y + s\frac{x-y}{|x-y|}\Big)s^{N-1}\,\mathrm{d}s\Big]\,\mathrm{d}y, \tag{10.35}$$

or, equivalently, after the change of variables $z = x - y$, $r = s/|z|$,

$$\mathcal{B}[f](x) = \int_{\mathbb{R}^N}\Big[f(x-z)z\int_1^\infty \zeta_{\overline{r}}(x - z + rz)r^{N-1}\,\mathrm{d}r\Big]\,\mathrm{d}z, \tag{10.36}$$

where $\zeta_{\overline{r}}$ is a mollifying kernel specified in (10.1–10.2). A detailed inspection of these formulas yields all statements of Lemma 10.6.

Thus, for example, we deduce from (10.36) that $\mathcal{B}[f] \in C^\infty(\Omega)$, and that $\mathrm{supp}[\mathcal{B}[f]] \subset M$ where

$$M = \{z \in \Omega \,|\, z = \lambda z_1 + (1-\lambda)z_2, z_1 \in \mathrm{supp}(f), z_2 \in \overline{B(\overline{r};0)}, \lambda \in [0,1]\}.$$

Since M is closed and contained in Ω, (10.30) follows.

Now we explain how to get (10.6) and estimate (10.6) with $k=1$. Differentiating (10.36) we obtain

$$\begin{aligned}\Big(\partial_i\mathcal{B}_j(f)\Big)(x) &= \int_{\mathbb{R}^N}\frac{\partial f}{\partial x_i}(x-z)z_j\Big[\int_1^\infty \zeta_{\overline{r}}(x-z+rz)r^{N-1}\,\mathrm{d}r\Big]\,\mathrm{d}z\\ &\quad + \int_{\mathbb{R}^N} f(x-z)z_j\Big[\int_1^\infty \frac{\partial\zeta_{\overline{r}}}{\partial x_i}\Big(x-z+rz\Big)r^N\,\mathrm{d}r\Big]\,\mathrm{d}z.\end{aligned}$$

Next, we split the set $\mathbb{R}^N$ in each integral into a ball $B(0;\varepsilon)$ and its complement, realizing that the integrals over $B(0;\varepsilon)$ tend to zero as $\varepsilon \to 0+$. The first of the remaining integrals over $\mathbb{R}^N \setminus B(0;\varepsilon)$ is handled by means of integration by parts. This direct but rather cumbersome calculation leads to

$$\begin{aligned}\Big(\partial_i \mathcal{B}_j[f]\Big)(x) = \lim_{\varepsilon\to 0+}\Bigg\{\int_{|z|\geq\varepsilon} f(x-z)\\ \times\left[\delta_{i,j}\int_1^\infty \zeta_{\overline{r}}(x-z+rz)r^{N-1}\,\mathrm{d}r + z_j\int_1^\infty \frac{\partial\zeta_{\overline{r}}}{\partial_{x_i}}\Big(x-z+rz\Big)r^N\,\mathrm{d}r\right]\mathrm{d}z\\ + \int_{|z|=\varepsilon} f(x-z)\left[z_j\frac{z_i}{|z|}\int_1^\infty \zeta_{\overline{r}}(x-z+rz)r^{N-1}\,\mathrm{d}r\right]\mathrm{d}\sigma_z\Bigg\},\end{aligned}$$

or, equivalently,

$$\begin{aligned}\Big(\partial_i \mathcal{B}_j[f]\Big)(x) = \lim_{\varepsilon\to 0+}\Bigg\{\int_{|y-x|\geq\varepsilon} f(y)\\ \times\left[\frac{\delta_{i,j}}{|x-y|^N}\int_0^\infty \zeta_{\overline{r}}\Big(x+r\frac{x-y}{|x-y|}\Big)(|x-y|+r)^{N-1}\,\mathrm{d}r\right.\\ \left.+\frac{x_j-y_j}{|x-y|^{N+1}}\int_0^\infty \frac{\partial\zeta_{\overline{r}}}{\partial x_i}\Big(x+r\frac{x-y}{|x-y|}\Big)(|x-y|+r)^N\,\mathrm{d}r\right]\mathrm{d}y\Bigg\}\\ + f(x)\lim_{\varepsilon\to 0+}\Bigg\{\int_{|z|=\varepsilon}\left[z_j\frac{z_i}{|z|}\int_1^\infty \zeta_{\overline{r}}(x-z+rz)r^{N-1}\,\mathrm{d}r\right]\mathrm{d}\sigma_z\Bigg\},\end{aligned}$$

where we have used the fact that

$$\lim_{\varepsilon\to 0+}\left\{\int_{|z|=\varepsilon}\left[\Big(f(x-z)-f(x)\Big)z_j\frac{z_i}{|z|}\int_1^\infty \zeta_{\overline{r}}(x-z+rz)r^{N-1}\,\mathrm{d}r\right]\mathrm{d}\sigma_z\right\} = 0.$$

Developing the expressions $(|x-y|+r)^{N-1}$, $(|x-y|+r)^N$ in the volume integral of the above identity by using the binomial formula, we obtain

$$\begin{aligned}\Big(\partial_i \mathcal{B}_j[f]\Big)(x) = \mathrm{v.p.}\Big(\int_\Omega K_{i,j}(x,x-y)f(y)\,\mathrm{d}y\Big)\\ + \int_\Omega G_{i,j}(x,x-y)f(y)\,\mathrm{d}y + f(x)H_{i,j}(x).\end{aligned} \tag{10.37}$$

The terms on the right-hand side have the following properties:

(i) The first kernel reads

$$K_{i,j}(x,z) = \frac{\theta_{i,j}(x,z/|z|)}{|z|^N}$$

with

$$\theta_{i,j}\Big(x,\frac{z}{|z|}\Big) = \delta_{i,j}\int_0^\infty \zeta_{\overline{r}}\Big(x+r\frac{z}{|z|}\Big)r^{N-1}\,\mathrm{d}r + \frac{z_j}{|z|}\int_0^\infty \frac{\partial\zeta_{\overline{r}}}{\partial_{x_i}}\Big(x+r\frac{z}{|z|}\Big)r^N\,\mathrm{d}r.$$

Thus a close inspection shows that

$$\int_{|z|=1} \theta(x,z)\,\mathrm{d}\sigma_z = 0,\ x \in \mathbb{R}^N,$$

$$|\theta(x,z)| \le c(N)\frac{(\mathrm{diam}(\Omega))^N}{\overline{r}^N}\Big(1+\frac{\mathrm{diam}(\Omega)}{\overline{r}}\Big),\ x\in\mathbb{R}^N,\ |z|=1.$$

We infer that $K_{i,j}$ are singular kernels of Calderón-Zygmund type obeying conditions (10.22) that were investigated in Theorem 10.10.

(ii) The second kernel reads

$$G_{i,j}(x,z) = \frac{\theta_{i,j}(x,z)}{|z|^{N-1}},$$

where

$$|\theta_{i,j}(x,z)| \le c(N)\frac{(\mathrm{diam}(\Omega))^N}{\overline{r}^N}\Big(1+\frac{\mathrm{diam}(\Omega)}{\overline{r}}\Big),\ (x,z)\in\mathbb{R}^N\times\mathbb{R}^N.$$

Thus $G_{i,j}$ are weakly singular kernels obeying conditions (10.20) discussed in Theorem 10.9.

(iii) Finally,

$$H_{i,j}(x) = \int_{\mathbb{R}^N}\frac{z_i z_j}{|z|^2}\zeta_{\overline{r}}(x+z)\,\mathrm{d}z,$$

where

$$|H_{i,j}(x)| \le c(N)\frac{(\mathrm{diam}(\Omega))^N}{\overline{r}^N},\ x\in\mathbb{R}^N \qquad \text{and} \qquad \sum_{i=1}^N H_{i,i}(x) = 1.$$

Using these facts together with Theorems 10.9, 10.10 we easily verify estimate (10.6) with $k=1$. We are even able to give an explicit formula for the constant appearing in the estimate, namely

$$c = c_0(p,N)\Big(\frac{\mathrm{diam}(\Omega)}{\overline{r}}\Big)^N\Big(1+\frac{\mathrm{diam}(\Omega)}{\overline{r}}\Big). \tag{10.38}$$

Since

$$\begin{aligned}
&\frac{d}{dr}\Big[\zeta_{\overline{r}}\Big(x+r\frac{x-y}{|x-y|}\Big)(|x-y|+r)^N\Big]\\
&\quad=\sum_{k=1}^N\frac{x_k-y_k}{|x-y|}\frac{\partial\zeta_{\overline{r}}}{\partial x_k}\Big(x+r\frac{x-y}{|x-y|}\Big)(|x-y|+r)^N\\
&\qquad+N\zeta_{\overline{r}}\Big(x+r\frac{x-y}{|x-y|}\Big)(|x-y|+r)^{N-1},
\end{aligned}$$

we have

$$\sum_{i=1}^{N}\Big(\int_{|x-y|\geq\varepsilon} f(y)(K_{i,i}(x,x-y)+G_{i,i}(x,x-y)\Big)\,\mathrm{d}y=\zeta_{\overline{r}}(x)\int_{\Omega} f(y)\,\mathrm{d}y=0.$$

Moreover, evidently,

$$\sum_{i=1}^{N} H_{i,i}(x)=\int_{\Omega}\zeta_{\overline{r}}(y)\,\mathrm{d}y=1;$$

whence (10.6) follows directly from (10.37).

In a similar way, the higher order derivatives of $\mathcal{B}[f]$ can be calculated by means of formula (10.36). Moreover, they can be shown to obey a representation formula of type (10.37), where, however, higher derivatives of f do appear; this leads to estimate (10.6) with an arbitrary positive integer k.

Last but not least, formula (10.36) written in terms of $\mathrm{div}_x\mathbf{g}$ yields, after integration by parts, a representation of $\mathcal{B}[\mathrm{div}_x\mathbf{g}]$ of type (10.37), with f replaced by $\mathbf{g}$. Again, the same reasoning as above yields naturally estimate (10.33).

Finally, property (10.34) is a consequence of the standard result concerning integrals dependent on a parameter.

The proof of Lemma 10.6 is thus complete. □

Step 6.

End of the proof of Theorem 10.11*.* For

$$\langle\, f; v\,\rangle_{[\dot W^{1,p'}(\Omega)]^*,\dot W^{1,p'}(\Omega)}=\int_{\Omega}\mathbf{w}\cdot\nabla v\,\mathrm{d}x,\ \text{with}\ \mathbf{w}\in L^p(\Omega;\mathbb{R}^N),$$

we can take

$$\langle\, f_\varepsilon; v\,\rangle_{[\dot W^{1,p'}(\Omega)]^*,\dot W^{1,p'}(\Omega)}=\int_{\Omega}\mathbf{w}_\varepsilon\cdot\nabla v\,\mathrm{d}x,$$

where $\mathbf{w}_\varepsilon\in C_c^\infty(\Omega;\mathbb{R}^N)$ have been constructed in Lemma 10.5.

Furthermore, let $\mathbf{h}_\varepsilon\in L^p(\Omega;\mathbb{R}^N)$,

$$\textstyle\int_\Omega f_\varepsilon v\,\mathrm{d}x=-\int_\Omega \mathbf{h}_\varepsilon\cdot\nabla v\ \mathrm{d}x\ \text{for all}\ v\in C^\infty(\overline{\Omega}),$$
$$\|f_\varepsilon\|_{[\dot W^{1,p'}(\Omega)]^*,\dot W^{1,p'}(\Omega)}=\|\mathbf{h}_\varepsilon\|_{L^p(\Omega;\mathbb{R}^N)},$$

be a sequence of representants of f_ε introduced in Lemma 10.4. The last formula yields

$$f_\varepsilon=\mathrm{div}\mathbf{h}_\varepsilon,\quad \int_\Omega\Big(v\mathrm{div}\mathbf{h}_\varepsilon+\mathbf{h}_\varepsilon\cdot\nabla v\Big)\,\mathrm{d}x=0,$$

meaning, in particular,

$$\gamma_{\mathbf{n}}(\mathbf{h}_\varepsilon)=0\ \text{and, equivalently,}\ \mathbf{h}_\varepsilon\in E_0^p(\Omega),\ 1<p<\infty$$

(see (10.8) in Theorem 10.8).

In view of the basic properties of the spaces $E^p_0(\Omega)$, we can replace $\mathbf{h}_\varepsilon$ by $\mathbf{g}_\varepsilon \in C^\infty_c(\Omega;\mathbb{R}^N)$ so that

$$\|\mathbf{h}_\varepsilon - \mathbf{g}_\varepsilon\|_{E^p(\Omega)} \to 0.$$

In particular, the sequence $\tilde{f}_\varepsilon$, $< \tilde{f}_\varepsilon; v >_{[\dot W^{1,p'}(\Omega)]^*,\dot W^{1,p'}(\Omega)} = \int_\Omega \mathbf{g}_\varepsilon \cdot \nabla v \,\mathrm{d}x$, converges to f, $\langle f; v\rangle_{[\dot W^{1,p'}(\Omega)]^*,\dot W^{1,p'}(\Omega)} = \int_\Omega \mathbf{w}\cdot\nabla v\,\mathrm{d}x$, strongly in $[\dot W^{1,p'}(\Omega)]^*$.

Due to estimate (10.33), the operator $\mathcal{B}$ is densely defined and continuous from $[\dot W^{1,p'}(\Omega)]^*$ to $L^p(\Omega;\mathbb{R}^N)$, therefore it can be extended by continuity to the whole space $[\dot W^{1,p'}(\Omega)]^*$.

If $\langle f; v\rangle_{[\dot W^{1,p'}(\Omega)]^*,\dot W^{1,p'}(\Omega)} = \int_\Omega wv\,\mathrm{d}x$, with $w = W^{k,p}_0(\Omega)\cap \dot L^p(\Omega)$, we take f_ε such that $\langle f_\varepsilon; v\rangle_{[\dot W^{1,p'}(\Omega)]^*,\dot W^{1,p'}(\Omega)} = \int_\Omega w_\varepsilon v\,\mathrm{d}x$, $w_\varepsilon = \zeta_\varepsilon * w - \kappa\int_\Omega(\zeta_\varepsilon * w)\,\mathrm{d}x$, where $\kappa\in C^\infty_c(\Omega)$, $\int_\Omega \kappa\,\mathrm{d}x = 0$ so that

$$C^\infty_c(\Omega) \ni f_\varepsilon = w_\varepsilon \to f = w \text{ in } W^{k,p}(\Omega).$$

If $\langle f; v\rangle_{[\dot W^{1,p'}(\Omega)]^*,\dot W^{1,p'}(\Omega)} = \int_\Omega \mathbf{w}\cdot\nabla v\,\mathrm{d}x$ with $\mathbf{w}\in E^{q,p}_0(\Omega)$, we take a sequence f_ε such that $\langle f_\varepsilon; v\rangle_{[\dot W^{1,p'}(\Omega)]^*,\dot W^{1,p'}(\Omega)} = \int_\Omega \mathbf{w}_\varepsilon\cdot\nabla v\,\mathrm{d}x$, with $\mathbf{w}\in L^p(\Omega;\mathbb{R}^N)$ $= \int_\Omega \mathrm{div}\mathbf{w}_\varepsilon v\,\mathrm{d}x$, where $\mathbf{w}_\varepsilon\in C^\infty_c(\Omega;\mathbb{R}^N)$, $\mathbf{w}_\varepsilon\to\mathbf{w}$ in $E^{q,p}_0(\Omega)$.

By virtue of estimates (10.6), (10.33), the operator $\mathcal{B}$ is in both cases a densely defined bounded linear operator on $W^{k,p}_0(\Omega)$ ($\hookrightarrow [\dot W^{1,p'}(\Omega)]^*$) ranging in $W^{k+1,p}_0(\Omega)$, and on $E^{q,p}_0(\Omega)$ ($\hookrightarrow [\dot W^{1,p'}(\Omega)]^*$) with values in $L^q(\Omega)\cap W^{1,p}_0(\Omega)$; in particular, it can be continuously extended to $W^{k,p}_0(\Omega)$, and $E^{q,p}_0(\Omega)$, respectively.

This completes the proof of Theorem 10.11. □

10.6 Helmholtz decomposition

Let Ω be a domain in $\mathbb{R}^N$. Set

$$L^p_\sigma(\Omega;\mathbb{R}^N) = \{\mathbf{v}\in L^p(\Omega;\mathbb{R}^N) \mid \mathrm{div}_x\mathbf{v} = 0,\ \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0\}$$

and

$$L^p_{g,\eta}(\Omega;\mathbb{R}^N) = \{\mathbf{v}\in L^p(\Omega;\mathbb{R}^N) \mid \mathbf{v} = \eta\nabla_x\Psi,\ \Psi\in W^{1,p}_{\mathrm{loc}}(\Omega)\},$$

where $\eta\in C(\overline{\Omega})$. The definition and the basic properties of the *Helmholtz decomposition* are collected in the following theorem.

■ Helmholtz Decomposition:

Theorem 10.12. *Let Ω be a bounded domain of class $C^{1,1}$, and let*

$$\eta\in C^1(\overline{\Omega}),\ \inf_{x\in\Omega}\eta(x) = \underline{\eta} > 0.$$

Then the Lebesgue space $L^p(\Omega;\mathbb{R}^N)$ admits a decomposition

$$L^p(\Omega;\mathbb{R}^N) = L^p_\sigma(\Omega;\mathbb{R}^N) \oplus L^p_{g,\eta}(\Omega;\mathbb{R}^N),\ 1 < p < \infty;$$

more precisely,

$$\mathbf{v} = \mathbf{H}_\eta[\mathbf{v}] + \mathbf{H}^\perp_\eta[\mathbf{v}] \text{ for any } \mathbf{v} \in L^p(\Omega;\mathbb{R}^N),$$

with $\mathbf{H}^\perp_\eta[\mathbf{v}] = \eta\nabla_x\Psi$, where $\Psi \in W^{1,p}(\Omega)$ is the unique (weak) solution of the Neumann problem

$$\int_\Omega \eta\nabla_x\Psi\cdot\nabla_x\varphi\ \mathrm{d}x = \int_\Omega \mathbf{v}\cdot\nabla_x\varphi\ \mathrm{d}x \text{ for all } \varphi\in C^\infty(\overline{\Omega}),\ \int_\Omega \Psi\ \mathrm{d}x = 0.$$

In the particular case $p = 2$, the decomposition is orthogonal with respect to the weighted scalar product

$$\langle\mathbf{v};\mathbf{w}\rangle_{1/\eta} = \int_\Omega \mathbf{v}\cdot\mathbf{w}\frac{\mathrm{d}x}{\eta}.$$

Proof. We start the proof with a lemma which is of independent interest.

Lemma 10.8. *Let Ω be a bounded domain of class $C^{0,1}$ and $1 < p < \infty$. Then*

$$L^p_\sigma(\Omega;\mathbb{R}^N) = \mathrm{closure}_{L^p(\Omega;\mathbb{R}^N)} C^\infty_{c,\sigma}(\Omega;\mathbb{R}^N),$$

where

$$C^\infty_{c,\sigma}(\Omega;\mathbb{R}^N) = \{\mathbf{v}\in C^\infty_c(\Omega;\mathbb{R}^N)\ |\ \mathrm{div}_x\mathbf{v} = 0\}.$$

Proof of Lemma 10.8. Let $\mathbf{u} \in L^p_\sigma(\Omega;\mathbb{R}^3)$. Due to Lemma 10.3, there exists a sequence $\mathbf{w}_\varepsilon \in C^\infty_c(\Omega,\mathbb{R}^N)$, such that $\mathbf{w}_\varepsilon \to \mathbf{u}$ in $L^p(\Omega;\mathbb{R}^3)$ and $\mathrm{div}_x\mathbf{w}_\varepsilon \to 0$ in $L^p(\Omega)$ as $\varepsilon \to 0+$. Next we take the sequence $\mathbf{u}_\varepsilon = \mathbf{w}_\varepsilon - \mathcal{B}[\mathrm{div}_x\mathbf{w}_\varepsilon]$, where $\mathcal{B}$ is the Bogovskii operator introduced in Section 10.5. According to Theorem 10.11, the functions $\mathbf{u}_\varepsilon$ belong to $C^\infty_{c,\sigma}(\Omega;\mathbb{R}^N)$ and the sequence $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ converges to $\mathbf{u}$ in $L^p(\Omega;\mathbb{R}^N)$. This completes the proof of Lemma 10.8. □

Existence and uniqueness of Ψ follow from Theorems 10.4, 10.5. Evidently, according to the definition, $\mathbf{H}_\eta[\mathbf{v}] = \mathbf{v} - \eta\nabla_x\Psi \in L^p_\sigma(\Omega;\mathbb{R}^N)$. Finally, we may use density of $C^\infty_{c,\sigma}(\Omega;\mathbb{R}^N)$ in $L^p_\sigma(\Omega;R^N)$ and integration by parts to show that the spaces $L^2_\sigma(\Omega;R^N)$ and $L^2_{g,\eta}(\Omega;\mathbb{R}^N)$ are orthogonal with respect to the scalar product $\langle\cdot;\cdot\rangle_{1/\eta}$. This completes the proof of Theorem 10.12. □

Remark: *In accordance with the regularity properties of the elliptic operators reviewed in Section* 10.2.1*, both $\mathbf{H}_\eta$ and $\mathbf{H}^\perp_\eta$ are continuous linear operators on $L^p(\Omega;\mathbb{R}^N)$ and $W^{1,p}(\Omega;\mathbb{R}^N)$ for any $1 < p < \infty$ provided Ω is of class $C^{1,1}$.*

If $\eta = 1$, we recover the classical Helmholtz decomposition denoted as $\mathbf{H}$, $\mathbf{H}^\perp$ (see, for instance, Galdi [92, Chapter 3]). The result can be extended to a considerably larger class of domains, in particular, it holds for *any* domain $\Omega \subset \mathbb{R}^3$

if $p = 2$. For more details about this issue in the case of arbitrary $1 < p < \infty$ see Farwig, Kozono, Sohr [76] or Simader, Sohr [183], and references quoted therein.

If $\Omega = \mathbb{R}^N$, the operator $\mathbf{H}^\perp$ can be defined by means of the Fourier multiplier

$$\mathbf{H}^\perp[\mathbf{v}](x) = \mathcal{F}^{-1}_{\xi\to x}\left[\frac{\xi\otimes\xi}{|\xi|^2}\mathcal{F}_{x\to\xi}[\mathbf{v}]\right].$$

10.7 Function spaces of hydrodynamics

Let Ω be a domain in $\mathbb{R}^N$. We introduce the following closed subspaces of the Sobolev space $W^{1,p}(\Omega;\mathbb{R}^N)$, $1 \le p \le \infty$:

$$\begin{aligned}
W^{1,p}_{0,\sigma}(\Omega) &= \{\mathbf{v}\in W^{1,p}_0(\Omega;\mathbb{R}^N)\,|\,\mathrm{div}_x\ \mathbf{v}=0\},\\
W^{1,p}_{\mathbf{n}}(\Omega) &= \{\mathbf{v}\in W^{1,p}(\Omega;\mathbb{R}^N)\,|\,\mathbf{v}\cdot\mathbf{n}|_{\partial\Omega}=0\},\\
W^{1,p}_{\mathbf{n},\sigma}(\Omega;\mathbb{R}^N) &= \{\mathbf{v}\in W^{1,p}_{\mathbf{n}}(\Omega)\,|\,\mathrm{div}_x\mathbf{v}=0\}.
\end{aligned}$$

We also consider the vector spaces

$$\begin{aligned}
C^\infty_{c,\sigma}(\Omega;\mathbb{R}^N) &= \{\mathbf{v}\in C^\infty_c(\Omega;\mathbb{R}^N)\,|\,\mathrm{div}\mathbf{v}=0\},\\
C^{k,\nu}_{\mathbf{n}}(\overline{\Omega};\mathbb{R}^N) &= \{\mathbf{v}\in C^{k,\nu}_c(\overline{\Omega};\mathbb{R}^N)\,|\,\mathbf{v}\cdot\mathbf{n}|_{\partial\Omega}=0\},\\
C^{k,\nu}_{\mathbf{n},\sigma}(\overline{\Omega},\mathbb{R}^N) &= \{\mathbf{v}\in C^{k,\nu}_{\mathbf{n}}(\overline{\Omega};\mathbb{R}^N)\,|\,\mathrm{div}_x\mathbf{v}=0\},\\
C^\infty_{\mathbf{n}}(\overline{\Omega};\mathbb{R}^N) &= \cap_{k=1}^\infty C^{k,\nu}_{\mathbf{n}}(\overline{\Omega};\mathbb{R}^N),\quad C^\infty_{\mathbf{n}}(\overline{\Omega};\mathbb{R}^N) = \cap_{k=1}^\infty C^{k,\nu}_{\mathbf{n},\sigma}(\overline{\Omega};\mathbb{R}^N).
\end{aligned}$$

Under certain regularity assumptions on the boundary $\partial\Omega$, these spaces are dense in the afore-mentioned Sobolev spaces, as stated in the following theorem.

■ DENSITY OF SMOOTH FUNCTIONS:

Theorem 10.13. *Suppose that Ω is a bounded domain in $\mathbb{R}^N$, and $1 < p < \infty$. Then we have:*

(i) *If the domain Ω is of class $C^{0,1}$, then the vector space $C^\infty_{c,\sigma}(\Omega;\mathbb{R}^N)$ is dense in $W^{1,p}_{0,\sigma}(\Omega;\mathbb{R}^N)$.*

(ii) *Suppose that Ω is of class $C^{k,\nu}$, $\nu\in(0,1)$, $k = 2,3,\ldots$, then the vector space $C^{k,\nu}_{\mathbf{n},\sigma}(\overline{\Omega};\mathbb{R}^N)$ is dense in $W^{1,p}_{\mathbf{n},\sigma}(\Omega;\mathbb{R}^N)$.*

(iii) *Finally, if Ω is of class $C^{k,\nu}$, $\nu\in(0,1)$, $k = 2,3,\ldots$, then the vector space $C^{k,\nu}_{\mathbf{n}}(\overline{\Omega};\mathbb{R}^N)$ is dense in $W^{1,p}_{\mathbf{n}}(\Omega;\mathbb{R}^N)$.*

Proof. Step 1. In order to show statement (i), we reproduce the proof of Galdi [92, Section II.4.1]. Let $\mathbf{v}\in W^{1,p}_{0,\sigma}(\Omega)\hookrightarrow W^{1,p}_0(\Omega;\mathbb{R}^N)$. There exists a sequence of smooth functions $\mathbf{w}_\varepsilon\in C^\infty_c(\Omega;\mathbb{R}^N)$ such that $\mathbf{w}_\varepsilon\to\mathbf{v}$ in $W^{1,p}(\Omega;\mathbb{R}^N)$, and, obviously, $\mathrm{div}\mathbf{w}_\varepsilon\to 0$ in $L^p(\Omega)$. Let $\mathbf{u}_\varepsilon = \mathcal{B}[\mathrm{div}_x\mathbf{w}_\varepsilon]$, where $\mathcal{B}\approx\mathrm{div}_x^{-1}$ is the operator constructed in Theorem 10.11. In accordance with Theorem 10.11, $\mathbf{u}_\varepsilon \in C_c^\infty(\Omega;\mathbb{R}^N)$, $\mathrm{div}\mathbf{u}_\varepsilon = \mathrm{div}\mathbf{w}_\varepsilon$, and $\|\mathbf{u}_\varepsilon\|_{W^{1,p}(\Omega;\mathbb{R}^N)} \to 0$.

In view of these observations, we have

$$\mathbf{v}_\varepsilon = \mathbf{w}_\varepsilon - \mathbf{u}_\varepsilon \in C_c^\infty(\Omega;\mathbb{R}^N), \quad \mathrm{div}_x\mathbf{v}_\varepsilon = 0,$$
$$\mathbf{v}_\varepsilon \to \mathbf{v} \text{ in } W^{1,p}(\Omega;\mathbb{R}^N)$$

yielding part (i) of Theorem 10.13.

Step 2. Let $\mathbf{v} \in W^{1,p}_{\mathbf{n},\sigma}(\Omega;\mathbb{R}^N) \hookrightarrow W^{1,p}(\Omega;\mathbb{R}^N)$. Take $\mathbf{w}_\varepsilon \in C_c^\infty(\overline{\Omega};\mathbb{R}^N)$ such that $\mathbf{w}_\varepsilon \to \mathbf{v}$ in $W^{1,p}(\Omega;\mathbb{R}^N)$. Obviously, we have

$$\mathrm{div}\mathbf{w}_\varepsilon \to 0 \text{ in } L^p(\Omega),\ \mathbf{w}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} \to 0 \text{ in } W^{1-\frac{1}{p},p}(\partial\Omega).$$

Let $\varphi_\varepsilon \in C_c^{k,\nu}(\overline{\Omega})$, $\int_\Omega \varphi_\varepsilon \,\mathrm{d}x = 0$ be an auxiliary function satisfying

$$\Delta\varphi_\varepsilon = \mathrm{div}\mathbf{w}_\varepsilon, \quad \nabla\varphi_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{w}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega}.$$

Then, in accordance with Theorem 10.2,

$$C^{k,\nu}_{\mathbf{n},\sigma}(\overline{\Omega};\mathbb{R}^N) \ni \mathbf{w}_\varepsilon - \nabla\varphi_\varepsilon \to \mathbf{v} \text{ in } W^{1,p}(\Omega;\mathbb{R}^N).$$

This finishes the proof of part (ii).

Step 3. Let $\mathbf{v} \in W^{1,p}_{\mathbf{n}}(\Omega;\mathbb{R}^N)$. We take $\mathbf{u} = \mathcal{B}(\mathrm{div}_x\mathbf{v})$, where $\mathcal{B}$ is the Bogovskii operator constructed in Theorem 10.11, and set $\mathbf{w} = \mathbf{v} - \mathbf{u}$. Clearly $\mathbf{w} \in W^{1,p}_{\mathbf{n},\sigma}(\Omega;\mathbb{R}^N)$.

In view of statement (ii), there exists a sequence $\mathbf{w}_\varepsilon \in C^{k,\nu}_{\mathbf{n},\sigma}(\overline{\Omega};\mathbb{R}^N)$ such that

$$\mathbf{w}_\varepsilon \to \mathbf{w} \text{ in } W^{1,p}(\Omega;\mathbb{R}^N).$$

On the other hand, for $\mathbf{u}$ belonging to $W_0^{1,p}(\Omega;\mathbb{R}^N)$, there exists a sequence $\mathbf{u}_\varepsilon \in C_c^\infty(\Omega;\mathbb{R}^N)$ such that

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ in } W^{1,p}(\Omega;\mathbb{R}^N).$$

The sequence $\mathbf{v}_\varepsilon = \mathbf{w}_\varepsilon + \mathbf{u}_\varepsilon$ belongs to $C^{k,\nu}_{\mathbf{n}}(\overline{\Omega};\mathbb{R}^N)$ and converges in $W^{1,p}(\Omega;\mathbb{R}^N)$ to $\mathbf{v}$.

This completes the proof of Theorem 10.13 □

The hypotheses concerning regularity of the boundary in statements (ii), (iii) are not optimal but sufficient in all applications treated in this book.

If the domain Ω is of class C^∞, the density of the space $C^\infty_{\mathbf{n}}(\overline{\Omega};\mathbb{R}^N)$ in $W^{1,p}_{\mathbf{n}}(\Omega;\mathbb{R}^N)$ and of $C^\infty_{\mathbf{n},\sigma}(\overline{\Omega};\mathbb{R}^N)$ in $W^{1,p}_{\mathbf{n}}(\Omega;\mathbb{R}^N)$ is a consequence of the theorem.

10.8 Poincaré type inequalities

The Poincaré type inequalities allow us to estimate the L^p-norm of a function by the L^p-norms of its derivatives. The basic result in this direction is stated in the following lemma.

■ POINCARÉ INEQUALITY:

Lemma 10.9. *Let $1 \le p < \infty$, and let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Then the following holds:*

(i) *For any $A \subset \partial\Omega$ with non-zero surface measure, there exists a positive constant $c = c(p, N, A, \Omega)$ such that*

$$\|v\|_{L^p(\Omega)} \le c\left(\|\nabla v\|_{L^p(\Omega;\mathbb{R}^N)} + \int_A |v| \,\mathrm{d}S_x\right) \text{ for any } v \in W^{1,p}(\Omega).$$

(ii) *There exists a positive constant $c = c(p, \Omega)$ such that*

$$\|v - \frac{1}{|\Omega|}\int_\Omega v \,\mathrm{d}x\|_{L^p(\Omega)} \le c\|\nabla v\|_{L^p(\Omega;\mathbb{R}^N)} \text{ for any } v \in W^{1,p}(\Omega).$$

The above lemma can be viewed as a particular case of more general results, for which we refer to Ziemer [207, Chapter 4, Theorem 4.5.1].

Applications in fluid mechanics often require refined versions of Poincaré inequality that are not directly covered by the standard theory. Let us quote Babovski, Padula [11] or [67] as examples of results going in this direction. The following version of the refined Poincaré inequality is sufficiently general to cover all situations treated in this book.

■ GENERALIZED POINCARÉ INEQUALITY:

Theorem 10.14. *Let $1 \le p \le \infty$, $0 < \Gamma < \infty$, $V_0 > 0$, and let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain.*

Then there exists a positive constant $c = c(p, \Gamma, V_0)$ such that

$$\| \ v \ \|_{W^{1,p}(\Omega)} \le c\left[\|\nabla_x v\|_{L^p(\Omega;\mathbb{R}^N)} + \left(\int_V |v|^\Gamma \mathrm{d}x\right)^{\frac{1}{\Gamma}}\right]$$

for any measurable $V \subset \Omega$, $|V| \ge V_0$ and any $v \in W^{1,p}(\Omega)$.

Proof. Fixing the parameters p, Γ, V_0 and arguing by contradiction, we construct sequences $w_n \in W^{1,p}(\Omega)$, $V_n \subset \Omega$ such that

$$\|w_n\|_{L^p(\Omega)} = 1, \quad \|\nabla w_n\|_{W^{1,p}(\Omega;\mathbb{R}^N)} + \left(\int_{V_n} |w_n|^\Gamma \,\mathrm{d}x\right)^{\frac{1}{\Gamma}} < \frac{1}{n}, \tag{10.39}$$

$$|V_n| \ge V_0. \tag{10.40}$$

By virtue of (10.39), we have, at least for a chosen subsequence,

$$w_n \to \overline{w} \text{ in } W^{1,p}(\Omega) \text{ where } \overline{w} = |\Omega|^{-\frac{1}{p}}.$$

Consequently, in particular,

$$\left|\{w_n \leq \frac{\overline{w}}{2}\}\right| \to 0. \tag{10.41}$$

On the other hand, by virtue of (10.39)

$$\left|\{w_n \geq \frac{\overline{w}}{2}\} \cap V_n\right| \leq \left(2/\overline{w}\right)^{\Gamma} \int_{V_n} w_n^{\Gamma} \, \mathrm{d}x \to 0,$$

in contrast to

$$\left|\{w_n \geq \frac{\overline{w}}{2}\} \cap V_n\right| = \left|V_n \setminus \{w_n < \frac{\overline{w}}{2}\}\right| \geq \left|V_n\right| - \left|\{w_n < \frac{\overline{w}}{2}\}\right| \geq V_0,$$

where the last statement follows from (10.40), (10.41). □

Another type of Poincaré inequality concerns norms in the negative Sobolev spaces in the spirit of Nečas [162].

■ POINCARÉ INEQUALITY IN NEGATIVE SPACES:

Lemma 10.10. *Let Ω be a bounded Lipschitz domain, $1 < p < \infty$, and $k = 0, 1, \ldots$. Let $\kappa \in W_0^{k,p'}(\Omega)$, $\int_\Omega \kappa \, \mathrm{d}x = 1$ be a given function.*

(i) *Then we have*

$$\|f\|_{W^{-k,p}(\Omega)} \leq c\Big(\|\nabla_x f\|_{W^{-k-1,p}(\Omega;\mathbb{R}^N)} + \Big|\sum_{|\alpha|\leq k} (-1)^{|\alpha|} \int_\Omega w_\alpha \partial^\alpha \kappa \, dx\Big|\Big)$$
$$\text{for any } f \in W^{-k,p}(\Omega), \tag{10.42}$$

where $\{w_\alpha\}_{|\alpha|\leq k}$, $w_\alpha \in L^p(\Omega)$ is an arbitrary representative of f constructed in Theorem 0.3, and c is a positive constant depending on p, N, Ω.

(ii) *In particular, if $k = 0$, inequality (10.42) reads*

$$\|f\|_{L^p(\Omega)} \leq c\Big(\|\nabla f\|_{W^{-1,p}(\Omega;\mathbb{R}^N)} + \Big|\int_\Omega f\kappa \, \mathrm{d}x\Big|\Big).$$

Proof. Since $C_c^\infty(\Omega)$ is dense in $W^{-k,p}(\Omega)$, it is enough to suppose that f is smooth. By direct calculation, we get

$$\|f\|_{W^{-k,p}(\Omega)} = \sup_{g \in W_0^{k,p'}(\Omega)} \frac{\int_\Omega fg \, \mathrm{d}x}{\|g\|_{W^{k,p'}(\Omega)}}$$

$$\leq \sup_{g\in W_0^{k,p'}(\Omega)} \Big(\frac{\int_\Omega f\ [g-\kappa\int_\Omega g\,\mathrm{d}x]\,\mathrm{d}x}{\|g-\kappa\int_\Omega g\,\mathrm{d}x\|_{W^{k,p'}(\Omega)}} \times \frac{\|g-\kappa\int_\Omega g\,\mathrm{d}x\|_{W^{k,p'}(\Omega)}}{\|g\|_{W^{k,p'}(\Omega)}}\Big)$$

$$+ \sup_{g\in W_0^{k,p'}(\Omega)} \frac{(\int_\Omega g\,\mathrm{d}x)(\int_\Omega f\kappa\,\mathrm{d}x)}{\|g\|_{W^{k,p'}(\Omega)}}$$

$$\leq c(p,\Omega)\Big(\sup_{\mathbf{v}\in W_0^{k+1,p'}(\Omega;\mathbb{R}^N)} \frac{\int_\Omega f \mathrm{div}_x\mathbf{v}\,\mathrm{d}x}{\|\mathbf{v}\|_{W^{k+1,p'}(\Omega;\mathbb{R}^N)}} + \Big|\sum_{|\alpha|\leq k}(-1)^\alpha\int_\Omega w_\alpha\partial^\alpha\kappa\,\mathrm{d}x\Big|\Big),$$

where $\{w_\alpha\}_{\alpha\leq k}$ is any representative of f (see formula (3) in Theorem 0.3), and where the quantity $W_0^{k+1,p'}(\Omega)\ni\mathbf{v}=\mathcal{B}(g-\kappa\int g\,\mathrm{d}x)$ appearing on the last line is a solution of problem

$$\mathrm{div}_x\mathbf{v}=g-\kappa\int_\Omega g\,\mathrm{d}x,\quad \|\mathbf{v}\|_{W^{k+1,p'}\Omega)}\leq c(p,\Omega)\Big\|g-\kappa\int_\Omega g\,\mathrm{d}x\Big\|_{W^{k,p'}(\Omega)}$$

constructed in Theorem 10.11.

The proof of Lemma 10.10 is complete. □

10.9 Korn type inequalities

Korn's inequality has played a central role not only in the development of linear elasticity but also in the analysis of viscous incompressible fluid flows. The reader interested in this topic can consult the review paper of Horgan [116], the recent article of Dain [53], and the relevant references cited therein. While these results rely mostly on the Hilbertian L^2-setting, various applications in the theory of compressible fluid flows require a general L^p-setting and even more.

We start with the standard formulation of Korn's inequality providing a bound of the L^p-norm of the gradient of a vector field in terms of the L^p-norm of its symmetric part.

■ KORN'S INEQUALITY IN L^p:

Theorem 10.15. *Assume that* $1<p<\infty$.

(i) *There exists a positive constant* $c=c(p,N)$ *such that*

$$\|\nabla\mathbf{v}\|_{L^p(\mathbb{R}^N;\mathbb{R}^{N\times N})}\leq c\|\nabla\mathbf{v}+\nabla^T\mathbf{v}\|_{L^p(\mathbb{R}^N;\mathbb{R}^{N\times N})}$$

for any $\mathbf{v}\in W^{1,p}(\mathbb{R}^N;\mathbb{R}^N)$.

(ii) *Let* $\Omega\subset\mathbb{R}^N$ *be a bounded Lipschitz domain. Then there exists a positive constant* $c=c(p,N,\Omega)>0$ *such that*

$$\|\mathbf{v}\|_{W^{1,p}(\Omega;\mathbb{R}^N)}\leq c\Big(\|\nabla\mathbf{v}+\nabla^T\mathbf{v}\|_{L^p(\Omega,\mathbb{R}^{N\times N})}+\int_\Omega|\mathbf{v}|\,\mathrm{d}x\Big)$$

for any $\mathbf{v}\in W^{1,p}(\Omega;\mathbb{R}^N)$.

Proof. *Step* 1. Since $C_c^\infty(\mathbb{R}^N;\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N;\mathbb{R}^N)$, we may suppose that $\mathbf{v}$ is smooth with compact support. We start with the identity

$$\partial_{x_k}\partial_{x_j} v_s = \partial_{x_j} D_{s,k} + \partial_{x_k} D_{s,j} - \partial_{x_s} D_{j,k}, \tag{10.43}$$

where

$$\mathbb{D} = (D_{i,j})_{i,j=1}^N,\ D_{i,j} = \frac{1}{2}(\partial_{x_j} u_i + \partial_{x_i} u_j).$$

Relation (10.43), rewritten in terms of the Fourier transform, reads

$$\xi_k \xi_j \mathcal{F}_{x\to\xi}(v_s) = -\mathrm{i}\Big(\xi_j \mathcal{F}_{x\to\xi}(D_{s,k}) + \xi_k \mathcal{F}_{x\to\xi}(D_{s,j}) - \xi_s \mathcal{F}_{x\to\xi}(D_{j,k})\Big).$$

Consequently,

$$\mathcal{F}_{x\to\xi}(\partial_{x_k} v_s) = \mathcal{F}_{x\to\xi}(D_{s,k}) + \frac{\xi_j\xi_k}{|\xi|^2}\mathcal{F}_{x\to\xi}(D_{s,j}) - \frac{\xi_j\xi_s}{|\xi|^2}\mathcal{F}_{x\to\xi}(D_{j,k}).$$

Thus estimate (i) follows directly from the Hörmander-Mikhlin theorem (Theorem 0.7).

Step 2. Similarly to the previous part, it is enough to consider smooth functions $\mathbf{v}$. Lemma 10.10 applied to formula (10.43) yields

$$\|\nabla\mathbf{v}\|_{L^p(\Omega;\mathbb{R}^{N\times N})} \le c\Big(\|\mathbb{D}\|_{L^p(\Omega;\mathbb{R}^{N\times N})} + \Big|\int_\Omega \nabla\mathbf{v}\kappa\,\mathrm{d}x\Big|\Big),$$

where $\kappa \in C_c^\infty(\Omega)$, $\int_\Omega \kappa\,\mathrm{d}x = 1$. Consequently, estimate (ii) follows. □

In applications to models of *compressible* fluids, it is useful to replace the symmetric gradient in the previous theorem by its *traceless* part. A satisfactory result is stated in the following theorem.

■ GENERALIZED KORN'S INEQUALITY:

Theorem 10.16. *Let* $1 < p < \infty$, *and* $N > 2$.

(i) *There exists a positive constant* $c = c(p, N)$ *such that*

$$\|\nabla\mathbf{v}\|_{L^p(\mathbb{R}^N;\mathbb{R}^{N\times N})} \le c\|\nabla\mathbf{v} + \nabla^T\mathbf{v} - \frac{2}{N}\mathrm{div}\mathbf{v}\mathbb{I}\|_{L^p(\mathbb{R}^N;\mathbb{R}^{N\times N})}$$

for any $\mathbf{v} \in W^{1,p}(\mathbb{R}^N;\mathbb{R}^N)$, *where* $\mathbb{I} = (\delta_{i,j})_{i,j=1}^N$ *is the identity matrix.*

(ii) *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded Lipschitz domain. Then there exists a positive constant* $c = c(p, N, \Omega) > 0$ *such that*

$$\|\mathbf{v}\|_{W^{1,p}(\Omega;\mathbb{R}^N)} \le c\Big(\|\nabla\mathbf{v} + \nabla^T\mathbf{v} - \frac{2}{N}\mathrm{div}\mathbf{v}\mathbb{I}\|_{L^p(\Omega;\mathbb{R}^{N\times N})} + \int_\Omega |\mathbf{v}|\,\mathrm{d}x\Big)$$

for any $\mathbf{v} \in W^{1,p}(\Omega;\mathbb{R}^N)$.

Remark: *As a matter of fact, part* (i) *of Theorem* 10.16 *holds for any* $N \geq 1$. *On the other hand, statement* (ii) *may fail for* $N = 2$ *as shown by Dain* [53].

Proof. Step 1. In order to show (i), we suppose, without loss of generality, that $\mathbf{v}$ is smooth and has a compact support in $\mathbb{R}^N$. Straightforward algebra yields

$$\begin{aligned} \partial_{x_k}\partial_{x_j} v_s &= \partial_{x_j} D_{s,k} + \partial_{x_k} D_{s,j} - \partial_{x_s} D_{j,k} \\ &\quad + \frac{1}{N}\Big(\delta_{s,k}\partial_{x_j}\mathrm{div}_x\mathbf{v} + \delta_{s,j}\partial_{x_k}\mathrm{div}_x\mathbf{v} - \delta_{j,k}\partial_{x_s}\mathrm{div}_x\mathbf{v}\Big), \end{aligned} \tag{10.44}$$

$$(N-2)\partial_{x_s}\mathrm{div}_x\mathbf{v} = 2N\partial_{x_k} D_{s,k} - N\Delta v_s, \tag{10.45}$$

$$\partial_{x_j}(\Delta v_s) = \partial_{x_j}\partial_{x_k} D_{s,k} + \Delta D_{j,s} - \partial_{x_s}\partial_{x_k} D_{j,k} + \frac{1}{N-1}\delta_{j,s}\partial_{x_k}\partial_{x_n} D_{k,n}, \tag{10.46}$$

where $\mathbb{D} = (D_{i,j})_{i,j=1}^N$ denotes the tensor

$$\mathbb{D} = \frac{1}{2}(\nabla_x\mathbf{v} + \nabla_x^T\mathbf{v}) - \frac{1}{N}\mathrm{div}_x\mathbf{v}\mathbb{I}.$$

Moreover, we deduce from (10.44) that

$$\begin{aligned} \mathcal{F}_{x\to\xi}(\partial_{x_k} v_s) &= \mathcal{F}_{x\to\xi}(D_{s,k}) + \frac{\xi_k\xi_j}{|\xi|^2}\mathcal{F}_{x\to\xi}(D_{s,j}) \\ &\quad - \frac{\xi_s\xi_j}{|\xi|^2}\mathcal{F}_{x\to\xi}(D_{j,k}) + \frac{1}{N}\delta_{s,k}\mathcal{F}_{x\to\xi}(\mathrm{div}\mathbf{v}), \end{aligned} \tag{10.47}$$

where, according to (10.45), (10.46),

$$\mathcal{F}_{x\to\xi}(\mathrm{div}\mathbf{v}) = \frac{N}{N-2}\frac{1}{|\xi|^2}\mathcal{F}_{x\to\xi}\big(\partial_s(\Delta v_s)\big) + \frac{2N}{N-2}\frac{\xi_s\xi_j}{|\xi|^2}\mathcal{F}_{x\to\xi}(D_{s,j}),$$

with

$$\frac{1}{|\xi|^2}\mathcal{F}_{x\to\xi}\big(\partial_s(\Delta v_s)\big) = -\Big(\mathcal{F}_{x\to\xi}(D_{s,s}) + \frac{N}{N-1}\frac{\xi_k\xi_n}{|\xi|^2}\mathcal{F}(D_{k,n})\Big).$$

Thus, estimate (i) follows from (10.47) via the Hörmander-Mikhlin multiplier theorem.

Step 2. Similarly to the previous step, it is enough to show (ii) for a smooth $\mathbf{v}$. By virtue of Lemma 10.10, we have

$$\|\partial_{x_k} v_j\|_{L^p(\Omega)} \leq c(p,\Omega)\Big(\|\nabla_x\partial_{x_k} v_j\|_{W^{-1,p}(\Omega;\mathbb{R}^N)} + \Big|\int_\Omega \partial_{x_k} v_j \kappa \,\mathrm{d}x\Big|\Big), \tag{10.48}$$

and

$$\|\Delta v_s\|_{W^{-1,p}(\Omega)} \leq c(p,\Omega)\Big(\|\nabla_x\Delta v_s\|_{W^{-2,p}(\Omega;\mathbb{R}^N)} + \Big|\int_\Omega \Delta v_s\tilde\kappa \,\mathrm{d}x\Big|\Big) \tag{10.49}$$

for any $\kappa \in L^{p'}(\Omega)$, $\int_\Omega \kappa \,\mathrm{d}x = 1$, $\tilde\kappa \in W_0^{1,p'}(\Omega)$, $\int_\Omega \tilde\kappa \,\mathrm{d}x = 1$.

Using the basic properties of the $W^{-1,p}$-norm we deduce from identities (10.44–10.45) that

$$\|\nabla_x \partial_{x_k} v_j\|_{W^{-1,p}(\Omega;\mathbb{R}^N)} \leq c\Big(\|\mathbb{D}\|_{L^p(\Omega;\mathbb{R}^N)} + \|\Delta \mathbf{v}\|_{W^{-1,p}(\Omega;\mathbb{R}^N)}\Big),$$

where the second term at the right-hand side is estimated by help of identity (10.46) and inequality (10.49). Coming back to (10.48) we get

$$\|\partial_{x_k} v_j\|_{L^p(\Omega)} \leq c(p,\Omega)\Big(\|\mathbb{D}\|_{L^p(\Omega;\mathbb{R}^N)} + \Big|\int_\Omega \partial_{x_k} v_j \kappa \,\mathrm{d}x\Big| + \Big|\int_\Omega \Delta v_j \tilde{\kappa}\,\mathrm{d}x\Big|\Big),$$

which, after by-parts integration and with a particular choice $\kappa \in C^1_c(\Omega)$, $\tilde{\kappa} \in C^2_c(\Omega)$, yields estimate (ii). □

We conclude this part with another generalization of the previous results.

■ Generalized Korn-Poincaré Inequality:

Theorem 10.17. *Let $\Omega \subset \mathbb{R}^N$, $N > 2$ be a bounded Lipschitz domain, and let $1 < p < \infty$, $M_0 > 0$, $K > 0$, $\gamma > 0$.*

Then there exists a positive constant $c = c(p, M_0, K, \gamma)$ such that the inequality

$$\|\mathbf{v}\|_{W^{1,p}(\Omega;\mathbb{R}^N)} \leq c\Big(\Big\|\nabla_x \mathbf{v} + \nabla_x^T \mathbf{v} - \frac{2}{N}\mathrm{div}\mathbf{v}\ \mathbb{I}\Big\|_{L^p(\Omega;\mathbb{R}^N)} + \int_\Omega r|\mathbf{v}|\ \mathrm{d}x\Big)$$

holds for any $\mathbf{v} \in W^{1,p}(\Omega;\mathbb{R}^N)$ and any non-negative function r such that

$$0 < M_0 \leq \int_\Omega r\ \mathrm{d}x, \ \int_\Omega r^\gamma\ \mathrm{d}x \leq K \text{ for a certain } \gamma > 1. \tag{10.50}$$

Proof. Without loss of generality, we may assume that $\gamma > \max\{1, \frac{Np}{(N+1)p-N}\}$. Indeed replacing r by $T_k(r)$, where $T_k(z) = \max\{z, k\}$, we can take $k = k(M_0, \gamma)$ large enough. Moreover, it is enough to consider smooth functions $\mathbf{v}$.

Fixing the parameters K, M_0, γ we argue by contradiction. Specifically, we construct a sequence $\mathbf{w}_n \in W^{1,p}(\Omega;\mathbb{R}^N)$ such that

$$\|\mathbf{w}_n\|_{W^{1,p}(\Omega;\mathbb{R}^N)} = 1, \quad \mathbf{w}_n \to \mathbf{w} \text{ weakly in } W^{1,p}(\Omega;\mathbb{R}^N) \tag{10.51}$$

and

$$\Big\|\nabla_x \mathbf{w}_n + \nabla_x^T \mathbf{w}_n - \frac{2}{N}\mathrm{div}_x \mathbf{w}_n\ \mathbb{I}\Big\|_{L^p(\Omega;\mathbb{R}^N)} + \int_\Omega r_n|\mathbf{w}_n|\ \mathrm{d}x < \frac{1}{n} \tag{10.52}$$

for certain

$$r_n \to r \text{ weakly in } L^\gamma(\Omega), \ \int_\Omega r\,\mathrm{d}x \geq M_0 > 0. \tag{10.53}$$

Consequently, due to the compact embedding $W^{1,p}(\Omega)$ into $L^p(\Omega)$, and by virtue of Theorem 10.16,

$$\mathbf{w}_n \to \mathbf{w} \text{ strongly in } W^{1,p}(\Omega; \mathbb{R}^N). \tag{10.54}$$

Moreover, in agreement with (10.51–10.54), the limit $\mathbf{w}$ satisfies the identities

$$\|\mathbf{w}\|_{W^{1,p}(\Omega;\mathbb{R}^N)} = 1, \tag{10.55}$$

$$\nabla \mathbf{w} + \nabla^T \mathbf{w} - \frac{2}{N} \mathrm{div} \mathbf{w} \mathbb{I} = 0, \tag{10.56}$$

$$\int_\Omega r|\mathbf{w}| \, \mathrm{d}x = 0. \tag{10.57}$$

Equation (10.56) which is valid provided $N > 2$, implies that $\Delta \mathrm{div} \mathbf{w} = 0$ and $\Delta \mathbf{w} = \frac{2-N}{N} \mathrm{div} \mathbf{w}$, see (10.45), (10.46). In particular, in agreement with remarks after Theorem 10.2 in Appendix, $\mathbf{w}$ is analytic in Ω. On the other hand, according to (10.57), $\mathbf{w}$ vanishes on the set $\{x \in \Omega \,|\, r(x) > 0\}$ of a non-zero measure; whence $\mathbf{w} \equiv 0$ in Ω in contrast with (10.57).

Theorem 10.17 has been proved. □

10.10 Estimating $\nabla \mathbf{u}$ by means of $\mathrm{div}_x \mathbf{u}$ and $\mathbf{curl}_x \mathbf{u}$

■ ESTIMATING $\nabla \mathbf{u}$ IN TERMS OF $\mathrm{div}_x \mathbf{u}$ AND $\mathbf{curl}_x \mathbf{u}$:

Theorem 10.18. *Assume that* $1 < p < \infty$.

(i) *Then*

$$\|\nabla \mathbf{u}\|_{L^p(\mathbb{R}^N;\mathbb{R}^{N\times N})} \le c(p,N)\Big(\|\mathrm{div}_x \mathbf{u}\|_{L^p(\mathbb{R}^N)} + \|\mathbf{curl}_x \mathbf{u}\|_{L^p(\mathbb{R}^N;\mathbb{R}^{N\times N})}\Big),$$
$$\text{for any } \mathbf{u} \in W^{1,p}(\mathbb{R}^N;\mathbb{R}^N). \tag{10.58}$$

(ii) *If* $\Omega \subset \mathbb{R}^N$ *is a bounded domain, then*

$$\|\nabla \mathbf{u}\|_{L^p(\Omega;\mathbb{R}^{N\times N})} \le c\Big(\|\mathrm{div}_x \mathbf{u}\|_{L^p(\Omega)} + \|\mathbf{curl}_x \mathbf{u}\|_{L^p(\Omega;\mathbb{R}^{N\times N})}\Big),$$
$$\text{for any } \mathbf{u} \in W^{1,p}_0(\Omega;\mathbb{R}^N). \tag{10.59}$$

Proof. To begin, observe that it is enough to show the estimate for

$$\mathbf{u} \in C^\infty_c(\mathbb{R}^N;\mathbb{R}^N).$$

To this end, we write

$$\mathrm{i} \sum_{k=1}^N \xi_k \mathcal{F}_{x\to\xi}(u_k) = \mathcal{F}_{x\to\xi}(\mathrm{div}_x \mathbf{u}),$$

$$\mathrm{i}\Big(\xi_k \mathcal{F}_{x\to\xi}(u_j) - \xi_j \mathcal{F}_{x\to\xi}(u_k)\Big) = \mathcal{F}_{x\to\xi}([\mathrm{curl}]_{j,k}\mathbf{u}), \quad j \ne k.$$

Solving the above system we obtain

$$\mathrm{i}|\xi|^2\mathcal{F}_{x\to\xi}(u_k) = \xi_k\mathcal{F}_{x\to\xi}(\mathrm{div}\mathbf{u}) + \sum_{j\neq k}\xi_j\mathcal{F}_{x\to\xi}([\mathrm{curl}]_{k,j}\mathbf{u}),$$

for $k = 1,\dots,N$. Consequently, we deduce

$$\mathcal{F}_{x\to\xi}(\partial_{x_r}u_k) = \frac{\xi_k\xi_r}{|\xi|^2}\mathcal{F}_{x\to\xi}(\mathrm{div}\mathbf{u}) + \sum_{j\neq k}\frac{\xi_j\xi_r}{|\xi|^2}\mathcal{F}_{x\to\xi}([\mathrm{curl}]_{k,j}\mathbf{u}).$$

Thus estimate (10.58) is obtained as a direct consequence of the Hörmander-Mikhlin theorem on multipliers (Theorem 0.7). □

If the trace of $\mathbf{u}$ does not vanish on $\partial\Omega$, the estimates of type (10.58) depend strongly on the geometrical properties of the domain Ω, namely on the values of its first and second *Betti numbers*.

For example, the estimate

$$\|\nabla\mathbf{u}\|_{L^p(\Omega;\mathbb{R}^{3\times 3})} \leq c(p,N,\Omega)\Big(\|\mathrm{div}_x\mathbf{u}\|_{L^p(\Omega)} + \|\mathbf{curl}_x\mathbf{u}\|_{L^p(\Omega;\mathbb{R}^{3\times 3})}\Big)$$

holds

(i) for any $\mathbf{u}\in W^{1,p}(\Omega;\mathbb{R}^3)$, $\mathbf{u}\times\mathbf{n}|_{\partial\Omega} = 0$, provided Ω is a bounded domain with the boundary of class $C^{1,1}$ and the set $\mathbb{R}^3\setminus\overline{\Omega}$ is (arcwise) connected (meaning $\mathbb{R}^3\setminus\Omega$ does not contain a bounded (arcwise) connected component);

(ii) for any $\mathbf{u}\in W^{1,p}(\Omega;\mathbb{R}^3)$, $\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0$, if Ω is a bounded domain with the boundary of class $C^{1,1}$ whose boundary $\partial\Omega$ is a connected and compact two-dimensional manifold.

The interested reader should consult the papers of von Wahl [201] and Bolik and von Wahl [25] for a detailed treatment of these questions including more general results in the case of non-vanishing tangential and/or normal components of the vector field $\mathbf{u}$.

10.11 Weak convergence and monotone functions

We start with a straightforward consequence of the De la Vallée Poussin criterion of the L^1-weak compactness formulated in Theorem 0.8.

Corollary 10.1. *Let $Q\subset\mathbb{R}^N$ be a domain and let $\{f_n\}_{n=1}^\infty$ be a sequence in $L^1(Q)$ satisfying*

$$\sup_{n>0}\int_Q \Phi(|f_n|)\,\mathrm{d}x < \infty, \tag{10.60}$$

where Φ is a non-negative function continuous on $[0,\infty)$ such that $\lim_{z\to\infty}\Phi(z)/z = \infty$.

Then

$$\sup_{n>0}\left\{\int_{\{|f_n|\geq k\}}|f_n(x)|\mathrm{d}x\right\}\to 0 \quad \textit{as } k\to\infty, \tag{10.61}$$

in particular,

$$k\sup_{n>0}\{|\{|f_n|\geq k\}|\}\to 0 \ \textit{ as } k\to\infty.$$

Typically, $\Phi(z)=z^p$, $p>1$, in which case we have

$$|\{|f_n|\geq k\}|\leq\frac{1}{k}\int_{\{|f_n|\geq k\}}|f_n(x)|\mathrm{d}x\leq\frac{1}{k}\Big(\int_Q|f_n|^p\mathrm{d}x\Big)^{1/p}|\{|f_n|\geq k\}|^{1/p'}.$$

Consequently, we report the following result.

Corollary 10.2. *Let $Q\subset\mathbb{R}^N$ be a domain and let $\{f_n\}_{n=1}^\infty$ be a sequence of functions bounded in $L^p(Q)$, where $p\in[1,\infty)$.*

Then

$$\int_{\{|f_n|\geq k\}}|f_n|^s\mathrm{d}x\leq\frac{1}{k^{p-s}}\sup_{n>0}\Big\{\|f_n\|^p_{L^p(Q)}\Big\},\ s\in[0,p]. \tag{10.62}$$

In particular

$$|\{|f_n|\geq k\}|\leq\frac{1}{k^p}\mathrm{sup}_{n>0}\Big\{\|f_n\|^p_{L^p(Q)}\Big\}. \tag{10.63}$$

In the remaining part of this section, we review some mostly standard material based on monotonicity arguments. There are several variants of these results scattered in the literature, in particular, these arguments have been extensively used in the monographs of P.-L. Lions [140], or [79], [166]. Our aim is to formulate these results at such a level of generality that they may be directly applicable to all relevant situations investigated in this book.

■ Weak Convergence and Monotonicity:

Theorem 10.19. *Let $I\subset\mathbb{R}$ be an interval, $Q\subset\mathbb{R}^N$ a domain, and*

$$(P,G)\in C(I)\times C(I) \quad \textit{a couple of non-decreasing functions.} \tag{10.64}$$

Assume that $\varrho_n\in L^1(Q;I)$ is a sequence such that

$$\left.\begin{cases} P(\varrho_n)\to\overline{P(\varrho)},\\ G(\varrho_n)\to\overline{G(\varrho)},\\ P(\varrho_n)G(\varrho_n)\to\overline{P(\varrho)G(\varrho)}\end{cases}\right\}\quad \textit{weakly in } L^1(Q). \tag{10.65}$$

(i) *Then*

$$\overline{P(\varrho)}\,\overline{G(\varrho)}\leq\overline{P(\varrho)G(\varrho)}. \tag{10.66}$$

(ii) *If, in addition,*

$$
\begin{gathered}
G \in C(\mathbb{R}), \quad G(\mathbb{R}) = \mathbb{R}, \quad G \textit{ is strictly increasing}, \\
P \in C(\mathbb{R}), \quad P \textit{ is non-decreasing},
\end{gathered} \tag{10.67}
$$

and

$$\overline{P(\varrho)G(\varrho)} = \overline{P(\varrho)}\;\overline{G(\varrho)}, \tag{10.68}$$

then

$$\overline{P(\varrho)} = P \circ G^{-1}(\overline{G(\varrho)}). \tag{10.69}$$

(iii) *In particular, if* $G(z) = z$, *then*

$$\overline{P(\varrho)} = P(\varrho). \tag{10.70}$$

Proof. We shall limit ourselves to the case $I = (0, \infty)$ already involving all difficulties encountered in other cases.

Step 1. If P is bounded and G strictly increasing, the proof is straightforward. Indeed, in this case,

$$
\begin{aligned}
0 &\le \lim_{n\to\infty} \int_B \Big[P(\varrho_n) - (P \circ G^{-1})\Big(\overline{G(\varrho)}\Big)\Big]\Big(G(\varrho_n) - \overline{G(\varrho)}\Big)\mathrm{d}x \\
&= \int_B \Big(\overline{P(\varrho)G(\varrho)} - \overline{P(\varrho)}\;\overline{G(\varrho)}\Big)\mathrm{d}x \\
&\quad - \lim_{n\to\infty} \int_B P \circ G^{-1}(\overline{G(\varrho)})\Big(G(\varrho_n) - \overline{G(\varrho)}\Big)\mathrm{d}x,
\end{aligned} \tag{10.71}
$$

where B is a ball in Q and $P \circ G^{-1}(\overline{G(\varrho)}) = \lim_{s\to\overline{G(\varrho)}} P \circ G^{-1}(s)$. By virtue of assumption (10.65), the second term at the right-hand side of the last formula tends to 0; whence the desired inequality (10.66) follows immediately from the standard result on the Lebesgue points.

Step 2. If P is bounded and G non-decreasing, we replace G by a strictly increasing function, say,

$$G_k(z) = G(z) + \frac{1}{k}\arctan(z), \quad k > 0.$$

In accordance with *Step* 1 we obtain

$$\overline{P(\varrho)G(\varrho)} + \frac{1}{k}\overline{P(\varrho)\arctan(\varrho)} \ge \overline{P(\varrho)}\;\overline{G(\varrho)} + \frac{1}{k}\overline{P(\varrho)}\;\overline{\arctan(\varrho)},$$

where we have used the De la Vallé Poussin criterion (Theorem 0.8) to guarantee the existence of the weak limits. Letting $k \to \infty$ in the last formula yields (10.66).

Step 3. If $\lim_{z\to 0+} P(z) \in \mathbb{R}$ and if P is unbounded, we may approximate P by a family of bounded non-decreasing functions,

$$P \circ \mathcal{T}_k, \quad k > 0,$$

where

$$\mathcal{T}_k(z) = k\mathcal{T}(\frac{z}{k}), \quad C^1(\mathbb{R}) \ni \mathcal{T}(z) = \left\{ \begin{array}{c} z \text{ if } z \in [0,1], \\ \text{concave in } (0,\infty), \\ 2 \text{ if } z \geq 3, \\ -\mathcal{T}(-z) \text{ if } z \in (-\infty, 0). \end{array} \right\}. \tag{10.72}$$

Reasoning as in the previous step, we obtain

$$\overline{(P \circ \mathcal{T}_k)(\varrho) G(\varrho)} \geq \overline{(P \circ \mathcal{T}_k)(\varrho)}\ \overline{G(\varrho)}. \tag{10.73}$$

In order to let $k \to \infty$, we observe first that

$$\begin{aligned} &\|\overline{(P \circ \mathcal{T}_k)(\varrho)} - \overline{P(\varrho)}\|_{L^1(Q)} \\ &\quad \leq \liminf_{n\to\infty} \|(P \circ \mathcal{T}_k)(\varrho_n) - P(\varrho_n)\|_{L^1(Q)} \leq 2 \sup_{n\in N} \Big\{ \int_{\{\varrho_n \geq k\}} |P(\varrho_n)| \mathrm{d}x \Big\}, \end{aligned}$$

where the last integral is arbitrarily small provided k is sufficiently large (see Theorem 0.8). Consequently,

$$\overline{(P \circ \mathcal{T}_k)(\varrho)} \to \overline{P(\varrho)} \quad \text{a.e. in } Q.$$

Similarly,

$$\overline{P \circ \mathcal{T}_k(\varrho) G(\varrho)} \to \overline{P(\varrho) G(\varrho)} \quad \text{a.e. in } Q.$$

Thus, letting $k \to \infty$ in (10.73) we obtain again (10.66).

Step 4. Finally, if $\lim_{z\to 0+} P(z) = -\infty$, we approximate P by

$$P_h(z) = \left\{ \begin{array}{ll} P(h) & \text{if } z \in (-\infty, h), \\ P(z) & \text{if } z \geq h \end{array} \right\}, \quad h > 0, \tag{10.74}$$

so that, according to *Step* 3,

$$\overline{P_h(\varrho) G(\varrho)} \geq \overline{P_h(\varrho)}\ \overline{G(\varrho)}, \tag{10.75}$$

As in the previous step, in accordance with Theorem 0.8,

$$\begin{aligned} \|\overline{P_h(\varrho)} - \overline{P(\varrho)}\|_{L^1(Q)} &\leq \liminf_{n\to\infty} \|P_h(\varrho_n) - P(\varrho_n)\|_{L^1(Q)} \\ &\leq 2 \sup_{n\in N} \Big\{ \int_{\{|P(\varrho_n)| \geq |P(h)|\}} |P(\varrho_n)| \mathrm{d}x \Big\} \to 0 \quad \text{as } h \to 0+, \end{aligned} \tag{10.76}$$

and

$$\begin{aligned} &\|\overline{P_h(\varrho) G(\varrho)} - \overline{P(\varrho) G(\varrho)}\|_{L^1(Q)} \\ &\quad \leq 2 \sup_{n\in N} \Big\{ \int_{\{|P(\varrho_n)| \geq |P(h)|\}} |P(\varrho_n) G(\varrho_n)| \mathrm{d}x \Big\} \to 0 \quad \text{as } h \to 0+. \end{aligned} \tag{10.77}$$

Thus we conclude the proof of part **(i)** of Theorem 10.19 by letting $h \to 0+$ in (10.75).

Step 5. Now we are in a position to prove part (ii). We set

$$M_k = \Big\{ x \in B \,|\, \sup_{s\in[-1,1]} G^{-1}\Big(\overline{G(\varrho)} + s\Big)(x) \le k \Big\},$$

where B is a ball in Q, and $k > 0$. Thanks to monotonicity of P and G, we can write

$$\begin{aligned}
0 \le & \int_B 1_{M_k} \Big[P(\varrho_n) - (P \circ G^{-1})\Big(\overline{G(\varrho)} \pm \epsilon\varphi\Big)\Big] \\
& \quad \times \Big(G(\varrho_n) - \overline{G(\varrho)} \mp \epsilon\varphi \Big) \mathrm{d}x \\
= & \int_B 1_{M_k} \Big(P(\varrho_n) G(\varrho_n) - P(\varrho_n)\, \overline{G(\varrho)} \Big) \mathrm{d}x \\
& - \int_B 1_{M_k} \,(P \circ G^{-1})\Big(\overline{G(\varrho)} \pm \epsilon\varphi\Big)\Big(G(\varrho_n) - \overline{G(\varrho)}\Big) \mathrm{d}x \\
& \mp \varepsilon \int_B 1_{M_k} \Big[P(\varrho_n) - (P \circ G^{-1})\Big(\overline{G(\varrho)} \pm \epsilon\varphi\Big)\Big] \varphi \mathrm{d}x,
\end{aligned} \tag{10.78}$$

where $\epsilon > 0$, $\varphi \in C_c^\infty(B)$ and 1_{M_k} is the characteristic function of the set M_k.

For $n \to \infty$ in (10.78), the first integral on the right-hand side tends to zero by virtue of (10.65), (10.68). Recall that $1_{M_k}\overline{G(\varrho)}$ is bounded. On the other hand, the second integral approaches zero by virtue of (10.65). Recall that $1_{M_k}(P \circ G^{-1})\Big(\overline{G(\varrho)} \pm \epsilon\varphi\Big)$ is bounded.

Thus we are left with

$$\int_B 1_{M_k} \Big[\overline{P(\varrho)} - (P \circ G^{-1})\Big(\overline{G(\varrho)} \pm \epsilon\varphi\Big)\Big] \varphi \mathrm{d}x = 0, \quad \varphi \in C_c^\infty(B); \tag{10.79}$$

whence (10.69) follows by sending $\epsilon \to 0+$ and realizing that $\cup_{k>0} M_k = B$. This completes the proof of statement **(ii)**. □

10.12 Weak convergence and convex functions

The idea of monotonicity can be further developed in the framework of *convex functions*. Similarly to the preceding section, the material collected here is standard and may be found in the classical books on convex analysis as, for example, Ekeland and Temam [70], or Azé [10].

Consider a functional

$$F : \mathbb{R}^M \to (-\infty, \infty], \; M \ge 1. \tag{10.80}$$

We say that F is *convex* on a convex set $O \subset \mathbb{R}^M$ if

$$F(tv + (1-t)w) \le tF(v) + (1-t)F(w) \text{ for all } v, w \in O, \; t \in [0,1]; \tag{10.81}$$

F is *strictly convex* on O if the above inequality is strict whenever $v \ne w$.

Compositions of convex functions with weakly converging sequences have a remarkable property of being lower semi-continuous with respect to the weak L^1-topology as shown in the following assertion (cf. similar results in Visintin [199], Balder [13]).

■ WEAK LOWER SEMI-CONTINUITY OF CONVEX FUNCTIONS:

Theorem 10.20. *Let $O \subset \mathbb{R}^N$ be a measurable set and $\{\mathbf{v}_n\}_{n=1}^{\infty}$ a sequence of functions in $L^1(O;\mathbb{R}^M)$ such that*

$$\mathbf{v}_n \to \mathbf{v} \text{ weakly in } L^1(O;\mathbb{R}^M).$$

Let $\Phi : \mathbb{R}^M \to (-\infty, \infty]$ be a lower semi-continuous convex function such that $\Phi(\mathbf{v}_n) \in L^1(O)$ for any n, and

$$\Phi(\mathbf{v}_n) \to \overline{\Phi(\mathbf{v})} \text{ weakly in } L^1(O).$$

Then

$$\Phi(\mathbf{v}) \leq \overline{\Phi(\mathbf{v})} \text{ a.a. on } O. \tag{10.82}$$

If, moreover, Φ is strictly convex on an open convex set $U \subset \mathbb{R}^M$, and

$$\Phi(\mathbf{v}) = \overline{\Phi(\mathbf{v})} \text{ a.a. on } O,$$

then

$$\mathbf{v}_n(\mathbf{y}) \to \mathbf{v}(\mathbf{y}) \text{ for a.a. } \mathbf{y} \in \{\mathbf{y} \in O \mid \mathbf{v}(\mathbf{y}) \in U\} \tag{10.83}$$

extracting a subsequence as the case may be.

Proof. Step 1. Any convex lower semi-continuous function with values in $(-\infty, \infty]$ can be written as a supremum of its affine minorants:

$$\Phi(\mathbf{z}) = \sup\{a(\mathbf{z}) \mid a \text{ an affine function on } \mathbb{R}^M, a \leq \Phi \text{ on } \mathbb{R}^M\} \tag{10.84}$$

(see Theorem 3.1 of Chapter 1 in [70]). Recall that a function is called *affine* if it can be written as a sum of a linear and a constant function.

On the other hand, if $B \subset O$ is a measurable set, we have

$$\int_B \overline{\Phi(\mathbf{v})} \, \mathrm{d}y = \lim_{n\to\infty} \int_B \Phi(\mathbf{v}_n) \, \mathrm{d}y \geq \lim_{n\to\infty} \int_B a(\mathbf{v}_n) \, \mathrm{d}y = \int_B a(\mathbf{v}) \, \mathrm{d}y$$

for any affine function $a \leq \Phi$. Consequently,

$$\overline{\Phi(\mathbf{v})}(\mathbf{y}) \geq a(\mathbf{v})(\mathbf{y})$$

for any $\mathbf{y} \in O$ which is a Lebesgue point of both $\overline{\Phi(\mathbf{v})}$ and $\mathbf{v}$.

Thus formula (10.84) yields (10.82).

Step 2. As any open set $U \subset \mathbb{R}^M$ can be expressed as a countable union of compacts, it is enough to show (10.83) for

$$\mathbf{y} \in M_K \equiv \{\mathbf{y} \in O \mid \mathbf{v}(\mathbf{y}) \in K\},$$

where $K \subset U$ is compact.

Since Φ is strictly convex on U, there exists an open set V such that

$$K \subset V \subset \overline{V} \subset U,$$

and $\Phi : \overline{V} \to R$ is a Lipschitz function (see Corollary 2.4 of Chapter I in [70]). In particular, the subdifferential $\partial\Phi(\mathbf{v})$ is non-empty for each $\mathbf{v} \in K$, and we have

$$\Phi(\mathbf{w}) - \Phi(\mathbf{v}) \geq \underline{\partial}\Phi(\mathbf{v}) \cdot (\mathbf{w} - \mathbf{v}) \text{ for any } \mathbf{w} \in \mathbb{R}^M, \ \mathbf{v} \in K,$$

where $\underline{\partial}\Phi(\mathbf{v})$ denotes the linear form in the subdifferential $\partial\Phi(\mathbf{v}) \subset (\mathbb{R}^M)^*$ with the smallest norm (see Corollary 2.4 of Chapter 1 in [70]).

Next, we shall show the existence of a function ω,

$$\begin{aligned} &\omega \in C[0,\infty), \ \omega(0) = 0, \\ &\omega \text{ non-decreasing on } [0,\infty) \\ &\quad \text{and strictly positive on } (0,\infty), \end{aligned} \tag{10.85}$$

such that

$$\Phi(\mathbf{w}) - \Phi(\mathbf{v}) \geq \underline{\partial}\Phi(\mathbf{v}) \cdot (\mathbf{w} - \mathbf{v}) + \omega(|\mathbf{w} - \mathbf{v}|) \text{ for all } \mathbf{w} \in \overline{V}, \ \mathbf{v} \in K. \tag{10.86}$$

Were (10.86) not true, we would be able to find two sequences $\mathbf{w}_n \in \overline{V}$, $\mathbf{z}_n \in K$ such that

$$\Phi(\mathbf{w}_n) - \Phi(\mathbf{z}_n) - \underline{\partial}\Phi(\mathbf{z}_n) \cdot (\mathbf{w}_n - \mathbf{z}_n) \ \to 0 \text{ for } n \to \infty$$

while

$$|\mathbf{w}_n - \mathbf{z}_n| \geq \delta > 0 \text{ for all } n = 1, 2, \ldots.$$

Moreover, as K is compact, one can assume

$$\mathbf{z}_n \to \mathbf{z} \in K, \ \Phi(\mathbf{z}_n) \to \Phi(\mathbf{z}), \ \mathbf{w}_n \to \mathbf{w} \text{ in } \overline{V}, \ \underline{\partial}\Phi(\mathbf{z}_n) \to L \in \ \mathbb{R}^M,$$

and, consequently,

$$\Phi(\mathbf{y}) - \Phi(\mathbf{z}) \geq L \cdot (\mathbf{y} - \mathbf{z}) \text{ for all } \mathbf{y} \in \mathbb{R}^M,$$

that is $L \in \partial\Phi(\mathbf{z})$.

Now, the function

$$\Psi(\mathbf{y}) \equiv \Phi(\mathbf{y}) - \Phi(\mathbf{z}) - L \cdot (\mathbf{y} - \mathbf{z})$$

is non-negative, convex, and

$$\Psi(\mathbf{z}) = \Psi(\mathbf{w}) = 0, \ |\mathbf{w} - \mathbf{z}| \geq \delta.$$

Consequently, Ψ vanishes on the whole segment $[\mathbf{z}, \mathbf{w}]$, which is impossible as Φ is strictly convex on U.

Seeing that the function

$$a \mapsto \Phi(\mathbf{z} + a\mathbf{y}) - \Phi(\mathbf{z}) - a\underline{\partial}\Phi(\mathbf{z}) \cdot \mathbf{y}$$

is non-negative, convex and non-decreasing for $a \in [0, \infty)$, we infer that the estimate (10.86) holds without the restriction $\mathbf{w} \in \overline{V}$. More precisely, there exists ω as in (10.85) such that

$$\Phi(\mathbf{w}) - \Phi(\mathbf{v}) \geq \underline{\partial}\Phi(\mathbf{v}) \cdot (\mathbf{w} - \mathbf{v}) + \omega(|\mathbf{w} - \mathbf{v}|) \text{ for all } \mathbf{w} \in \mathbb{R}^M, \ \mathbf{v} \in K. \quad (10.87)$$

Taking $\mathbf{w} = \mathbf{v}_n(\mathbf{y})$, $\mathbf{v} = \mathbf{v}(\mathbf{y})$ in (10.87) and integrating over the set M_K we get

$$\int_{M_K} \omega(|\mathbf{v}_n - \mathbf{v}|) \ \mathrm{d}y \leq \int_{M_K} \Phi(\mathbf{v}_n) - \Phi(\mathbf{v}) - \underline{\partial}\Phi(\mathbf{v}) \cdot (\mathbf{v}_n - \mathbf{v}) \ \mathrm{d}y,$$

where the right-hand side tends to zero for $n \to \infty$. Note that the function $\underline{\partial}\Phi(\mathbf{v})$ is bounded measurable on M_k as Φ is Lipschitz on $\overline{V}$, and

$$\underline{\partial}\Phi(\mathbf{v}) = \lim_{\varepsilon \to 0} \nabla\Phi_\varepsilon(\mathbf{v}) \text{ for any } \mathbf{v} \in V,$$

where

$$\Phi_\varepsilon(\mathbf{v}) \equiv \min_{\mathbf{z} \in \mathbb{R}^M} \left\{ \frac{1}{\varepsilon} |\mathbf{z} - \mathbf{v}| + \Phi(\mathbf{z}) \right\} \quad (10.88)$$

is a convex, continuously differentiable function on $\mathbb{R}^M$ (see Propositions 2.6, 2.11 of Chapter 2 in [34]).

Thus

$$\int_{M_K} \omega(|\mathbf{v}_n - \mathbf{v}|) \ \mathrm{d}y \to 0 \text{ for } n \to \infty$$

which yields pointwise convergence (for a subsequence) of $\{\mathbf{v}_n\}_{n=1}^\infty$ to $\mathbf{v}$ a.a. on M_K. □

10.13 Div-Curl lemma

The celebrated Div-Curl lemma of L. Tartar [187] (see also Murat [161]) is a cornerstone of the theory of compensated compactness and became one of the most efficient tools in the analysis of problems with lack of compactness. Here, we recall its L^p-version.

Lemma 10.11. *Let $Q \subset \mathbb{R}^N$ be an open set, and $1 < p < \infty$. Assume*

$$\begin{aligned} \mathbf{U}_n &\to \mathbf{U} \text{ weakly in } L^p(Q; \mathbb{R}^N), \\ \mathbf{V}_n &\to \mathbf{V} \text{ weakly in } L^{p'}(Q; \mathbb{R}^N). \end{aligned} \tag{10.89}$$

In addition, let

$$\left.\begin{aligned} \operatorname{div} \mathbf{U}_n \equiv \nabla \cdot \mathbf{U}_n, \\ \operatorname{curl} \mathbf{V}_n \equiv (\nabla \mathbf{V}_n - \nabla^T \mathbf{V}_n) \end{aligned}\right\} \text{ be precompact in } \left\{\begin{aligned} &W^{-1,p}(Q), \\ &W^{-1,p'}(Q; \mathbb{R}^{N\times N}). \end{aligned}\right. \tag{10.90}$$

Then

$$\mathbf{U}_n \cdot \mathbf{V}_n \to \mathbf{U} \cdot \mathbf{V} \quad \text{in } \mathcal{D}'(Q).$$

Proof. Since the result is local, we can assume that $Q = \mathbb{R}^N$. We have to show that

$$\begin{aligned} &\int_{\mathbb{R}^N} \big(\mathbf{H}[\mathbf{U}_n] + \mathbf{H}^\perp[\mathbf{U}_n]\big) \cdot \big(\mathbf{H}[\mathbf{V}_n] + \mathbf{H}^\perp[\mathbf{V}_n]\big) \varphi \, \mathrm{d}x \\ &\quad \to \int_{\mathbb{R}^N} \big(\mathbf{H}[\mathbf{U}] + \mathbf{H}^\perp[\mathbf{U}]\big) \cdot \big(\mathbf{H}[\mathbf{V}] + \mathbf{H}^\perp[\mathbf{V}]\big) \varphi \, \mathrm{d}x \end{aligned}$$

for any $\varphi \in C_c^\infty(\mathbb{R}^N)$, where $\mathbf{H}$, $\mathbf{H}^\perp$ are the Helmholtz projections introduced in Section 10.6. We have

$$\mathbf{H}^\perp[\mathbf{U}_n] = \nabla \Psi_n^U, \ \mathbf{H}^\perp[\mathbf{V}_n] = \nabla \Psi_n^V,$$

where, in accordance with hypothesis (10.90) and the standard elliptic estimates discussed in Sections 10.2.1, 10.10,

$$\begin{aligned} \nabla \Psi_n^U &\to \nabla \Psi^U = \mathbf{H}^\perp[\mathbf{U}] \text{ in } L^p(B; \mathbb{R}^N), \\ \mathbf{H}[\mathbf{V}_n] &\to \mathbf{H}[\mathbf{V}] \text{ in } L^{p'}(B; \mathbb{R}^N), \end{aligned}$$

and

$$\begin{aligned} \mathbf{H}[\mathbf{U}_n] &\to \mathbf{H}[\mathbf{U}] \text{ weakly in } L^p(B; \mathbb{R}^N), \\ \nabla \Psi_n^V &\to \nabla \Psi^V = \mathbf{H}^\perp[\mathbf{V}] \text{ weakly in } L^{p'}(B; \mathbb{R}^N), \end{aligned}$$

where $B \subset \mathbb{R}^N$ is a ball containing the support of φ.

Consequently, it is enough to handle the term $\mathbf{H}[\mathbf{U}_n] \cdot \nabla_x \Psi_n^V \varphi$. However,

$$\begin{aligned} &\int_{\mathbb{R}^N} \mathbf{H}[\mathbf{U}_n] \cdot \nabla_x \Psi_n^V \varphi \, \mathrm{d}x = - \int_{\mathbb{R}^N} \mathbf{H}[\mathbf{U}_n] \cdot \nabla \varphi \Psi_n^V \, \mathrm{d}x \\ &\to - \int_{\mathbb{R}^N} \mathbf{H}[\mathbf{U}] \cdot \nabla \varphi \Psi^V \, \mathrm{d}x = \int_{\mathbb{R}^N} \mathbf{H}[\mathbf{U}] \cdot \nabla_x \Psi^V \varphi \, \mathrm{d}x. \end{aligned}$$

□

The following variant of the Div-Curl lemma seems more convenient from the perspective of possible applications.

■ DIV-CURL LEMMA:

Theorem 10.21. *Let $Q \subset \mathbb{R}^N$ be an open set. Assume*

$$\begin{aligned} \mathbf{U}_n &\to \mathbf{U} \text{ weakly in } L^p(Q;\mathbb{R}^N), \\ \mathbf{V}_n &\to \mathbf{V} \text{ weakly in } L^q(Q;\mathbb{R}^N), \end{aligned} \tag{10.91}$$

where

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1.$$

In addition, let

$$\left. \begin{aligned} \operatorname{div} \mathbf{U}_n &\equiv \nabla \cdot \mathbf{U}_n, \\ \operatorname{curl} \mathbf{V}_n &\equiv (\nabla \mathbf{V}_n - \nabla^T \mathbf{V}_n) \end{aligned} \right\} \text{ be precompact in } \left\{ \begin{aligned} &W^{-1,s}(Q), \\ &W^{-1,s}(Q;\mathbb{R}^{N\times N}), \end{aligned} \right. \tag{10.92}$$

for a certain $s > 1$. Then

$$\mathbf{U}_n \cdot \mathbf{V}_n \to \mathbf{U} \cdot \mathbf{V} \text{ weakly in } L^r(Q).$$

The proof follows easily from Lemma 10.11 as soon as we observe that precompact sets in $W^{-1,s}$ that are bounded in $W^{-1,p}$ are precompact in $W^{-1,m}$ for any $s < m < p$.

10.14 Maximal regularity for parabolic equations

We consider a parabolic problem:

$$\left\{ \begin{aligned} \partial_t u - \Delta u = f \text{ in } (0,T)\times\Omega, \\ u(0,x) = u_0(x),\ x\in\Omega, \\ \nabla_x u \cdot \mathbf{n} = 0 \text{ in } (0,T)\times\partial\Omega, \end{aligned} \right\} \tag{10.93}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain. In the context of the so-called *strong solutions*, the first equation is satisfied a.e. in $(0,T)\times\Omega$, the initial condition holds a.e. in Ω, and the homogenous Neumann boundary condition is satisfied in the sense of traces.

The following statement holds.

■ MAXIMAL $L^p - L^q$ REGULARITY:

Theorem 10.22. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class C^2, $1 < p, q < \infty$. Suppose that*

$$f \in L^p(0,T; L^q(\Omega)),\ u_0 \in X_{p,q},\ X_{p,q} = \{L^q(\Omega); \mathcal{D}(\Delta_{\mathcal{N}})\}_{1-1/p,p},$$
$$\mathcal{D}(\Delta_{\mathcal{N}}) = \{v \in W^{2,q}(\Omega) \mid \nabla_x v \cdot \mathbf{n}|_{\partial\Omega} = 0\},$$

where $\{\cdot;\cdot\}_{\cdot,\cdot}$ denotes the real interpolation space.

Then problem (10.93) *admits a solution u, unique in the class*

$$u \in L^p(0,T; W^{2,q}(\Omega)),\ \partial_t u \in L^p(0,T; L^q(\Omega)),$$
$$u \in C([0,T]; X_{p,q}).$$

Moreover, there exists a positive constant $c = c(p,q,\Omega,T)$ such that

$$\|u(t)\|_{X_{p,q}} + \|\partial_t u\|_{L^p(0,T;L^q(\Omega))} + \|\Delta u\|_{L^p(0,T;L^q(\Omega))}$$
$$\leq c\big(\|f\|_{L^p(0,T;L^q(\Omega))} + \|u_0\|_{X_{p,q}}\big) \tag{10.94}$$

for any $t \in [0,T]$.

See Amann [8], [7]. □

For the definition of real interpolation spaces see, e.g., Bergh, Löfström [23, Chapter 3]. It is well known that

$$X_{p,q} = \begin{cases} B_{q,p}^{2-\frac{2}{p}}(\Omega) \text{ if } 1 - \frac{2}{p} - \frac{1}{q} < 0, \\ \{u \in B_{q,p}^{2-\frac{2}{p}}(\Omega) \mid \nabla_x u \cdot \mathbf{n}|_{\partial\Omega} = 0\}, \text{ if } 1 - \frac{2}{p} - \frac{1}{q} > 0, \end{cases}$$

see Amann [7]. In the above formula, the symbol $B_{q,p}^s(\Omega)$ refers to the Besov space.

For the definition and properties of the scale of Besov spaces $B_{q,p}^s(\mathbb{R}^N)$ and $B_{q,p}^s(\Omega)$, $s \in \mathbb{R}$, $1 \leq q, p \leq \infty$ see Bergh and Löfström [23, Section 6.2], Triebel [190], [191]. A nice overview can be found in Amann [7, Section5]. Many of the classical spaces are contained as special cases in the Besov scales. It is of interest for the purpose of this book that

$$B_{p,p}^s(\Omega) = W^{s,p}(\Omega),\ s \in (0,\infty) \setminus \mathbb{N},\ 1 \leq p < \infty,$$

where $W^{s,p}(\Omega)$ is the Sobolev-Slobodeckii space.

Extension of Theorem 10.22 to general classes of parabolic equations and systems as well as to different types of boundary conditions are available. For more information concerning the $L^p - L^q$ maximal regularity for parabolic systems with general boundary conditions, we refer to the book of Amann [8] or to the papers by Denk, Hieber and Prüss [58], [57], [106].

Maximal regularity in the classes of smooth functions relies on a classical argument. A result in this direction reads as follows.

■ MAXIMAL HÖLDER REGULARITY:

Theorem 10.23. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{2,\nu}$, $\nu > 0$. Suppose that*

$$f \in C([0,T]; C^{0,\nu}(\overline{\Omega})),\ u_0 \in C^{2,\nu}(\overline{\Omega}),\ \nabla_x u_0 \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

Then problem (10.93) *admits a unique solution*

$$u \in C([0,T]; C^{2,\nu}(\overline{\Omega})),\ \partial_t u \in C([0,T]; C^{0,\nu}(\overline{\Omega})).$$

Moreover, there exists a positive constant $c = c(p,q,\Omega,T)$ such that

$$\|\partial_t u\|_{C([0,T];C^{0,\nu}(\overline{\Omega}))} + \|u\|_{C([0,T];C^{2,\nu}(\Omega))} \leq c\Big(\|u_0\|_{C^{2,\nu}(\overline{\Omega})} + \|f\|_{C([0,T];C^{0,\nu}(\overline{\Omega}))}\Big). \tag{10.95}$$

See Lunardi [146, Theorem 5.1.2] □

Unlike most of the classical existence theorems that can be found in various monographs on parabolic equations (see, e.g., Ladyzhenskaya, Solonnikov, Uralceva [128]), the above results require merely the continuity in time of the right-hand side. This aspect is very convenient for the applications in this book.

10.15 Quasilinear parabolic equations

In this section we review a well-known result in solvability of the quasilinear parabolic problem:

$$\left\{ \begin{array}{c} \partial_t u - \sum_{i,j=1}^N a_{ij}(t,x,u)\partial_{x_i}\partial_{x_j} u + b(t,x,u,\nabla_x u) = 0 \quad \text{in } (0,T)\times\Omega, \\ \sum_{i,j=1}^N n_i\, a_{ij}\partial_{x_j} u + \psi = 0 \quad \text{on } S_T, \\ u(0,\cdot) = u_0, \end{array} \right\} \tag{10.96}$$

where

$$a_{ij} = a_{ij}(t,x,u),\ i,j = 1,\ldots,N, \quad \psi = \psi(t,x),\ b(t,x,u,\mathbf{z}) \text{ and } u_0 = u_0(x)$$

are continuous functions of their arguments $(t,x) \in [0,T]\times\overline{\Omega}$, $u \in \mathbb{R}$, $\mathbf{z} \in \mathbb{R}^N$, $S_T = [0,T]\times\partial\Omega$ and $\mathbf{n} = (n_1,\ldots,n_N)$ is the outer normal to the boundary $\partial\Omega$.

The results stated below are taken over from the classical book by Ladyzhenskaya, Solonnikov and Uralceva [129]. We refer the reader to this work for all details, and also for the further properties of quasilinear parabolic equations and systems.

■ Existence and Uniqueness for the Quasilinear Parabolic Neumann Problem:

Theorem 10.24. *Let $\nu \in (0,1)$ and let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{2,\nu}$. Suppose that*

(i) $u_0 \in C^{2,\nu}(\overline{\Omega})$, $\quad \psi \in C^1([0,T]\times\overline{\Omega})$, $\quad \nabla_x\psi$ *is Hölder continuous in the variables t and x with exponents $\nu/2$ and ν, respectively,*

$$\sum_{i,j=1}^{N} n_i(x)\, a_{ij}\partial_{x_j}(0,x,u_0(x)) + \psi(0,x) = 0,\ x \in \partial\Omega;$$

(ii) $$a_{ij} \in C^1([0,T]\times\overline{\Omega}\times\mathbb{R}),$$

$\nabla_x a_{ij}$, $\partial_u a_{ij}$ *are ν-Hölder continuous in the variable x;*

(iii) $$b \in C^1([0,T]\times\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N),$$

$\nabla_x b$, $\partial_u b$, $\nabla_{\mathbf{z}} b$ *are ν-Hölder continuous in the variable x;*

(iv) *there exist positive constants $\underline{c}$, $\overline{c}$, c_1, c_2 such that*

$$\begin{aligned} 0 \le a_{ij}(t,x,u)\xi_i\xi_j \le \overline{c}|\xi|^2, &\quad (t,x,u,\xi) \in (0,T]\times\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N,\\ a_{ij}(t,x,u)\xi_i\xi_j \ge \underline{c}|\xi|^2, &\quad (t,x,u,\xi) \in S_T\times\Omega\times\mathbb{R}\times\mathbb{R}^N,\\ -ub(t,x,u,\mathbf{z}) \le c_0|\mathbf{z}|^2 + c_1u^2 + c_2, &\quad (t,x,u,\xi) \in [0,T]\times\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N; \end{aligned}$$

(v) *for any $L>0$ there are positive constants $\underline{C}$ and $\overline{C}$ such that*

$$\begin{aligned} &\underline{C}(L)|\xi|^2 \le a_{ij}(t,x,u)\xi_i\xi_j, \quad (t,x,u,\xi) \in [0,T]\times\overline{\Omega}\times[-L,L]\times\mathbb{R}^N,\\ &\Big|b, \partial_t b, \partial_u b, (1+\mathbf{z})\nabla_{\mathbf{z}} b\Big|(t,x,u,\mathbf{z})\\ &\qquad \le \overline{C}(L)(1+|\mathbf{z}|^2), \quad (t,x,u,\mathbf{z}) \in [0,T]\times\overline{\Omega}\times[-L,L]\times\mathbb{R}^N. \end{aligned}$$

Then problem (10.96) *admits a unique classical solution u belonging to the Hölder space $C^{1,\nu/2;2,\nu}([0,T]\times\overline{\Omega})$, where the symbol $C^{1,\nu/2;2,\nu}([0,T]\times\overline{\Omega})$ stands for the Banach space with norm*

$$\begin{aligned} &\|u\|_{C^1([0,T]\times\overline{\Omega})} + \sup_{(t,\tau,x)\in[0,T]^2\times\overline{\Omega}} \frac{|\partial_t u(t,x) - \partial_t u(\tau,x)|}{|t-\tau|^{\nu/2}}\\ &\qquad + \sum_{i,j=1}^{3} \|\partial_{x_i}\partial_{x_j} u\|_{C([0,T]\times\overline{\Omega})}\\ &\qquad + \sum_{i,j=1}^{3} \sup_{(t,x,y)\in[0,T]\times\overline{\Omega}^2} \frac{|\partial_{x_i}\partial_{x_j} u(t,x) - \partial_{x_i}\partial_{x_j} u(t,y)|}{|x-y|^{\nu}}. \end{aligned}$$

See Ladyzhenskaya, Solonnikov, Uralceva [129, Theorems 7.2, 7.3, 7.4]. □

10.16 Basic properties of the Riesz transform and related operators

Various (*pseudo*) *differential operators* used in the book are identified through their Fourier symbols:

- the Riesz transform:

$$\mathcal{R}_j \approx \frac{i\xi_j}{|\xi|}, \ j = 1, \dots, N,$$

  meaning that

$$\mathcal{R}_j[v] = \mathcal{F}^{-1}_{\xi \to x}\Big[\frac{i\xi_j}{|\xi|}\mathcal{F}_{x\to\xi}[v]\Big];$$

- the “double” Riesz transform:

$$\mathcal{R} = \{\mathcal{R}_{k,j}\}_{k,j=1}^N, \ \mathcal{R} = \Delta_x^{-1}\nabla_x \otimes \nabla_x, \ \mathcal{R}_{i,j} \approx \frac{\xi_i\xi_j}{|\xi|^2}, \ i, j = 1, \dots, N,$$

  meaning that

$$\mathcal{R}_{k,j}[v] = \mathcal{F}^{-1}_{\xi \to x}\Big[\frac{\xi_k\xi_j}{|\xi|^2}\mathcal{F}_{x\to\xi}[v]\Big];$$

- the inverse divergence:

$$\mathcal{A} = \{\mathcal{A}_j\}_{j=1}^N, \ \mathcal{A}_j = \partial_{x_j}\Delta_x^{-1} \approx -\frac{\mathrm{i}\xi_j}{|\xi|^2}, \ j = 1, \dots, N,$$

  meaning that

$$\mathcal{A}_j[v] = -\mathcal{F}^{-1}_{\xi \to x}\Big[\frac{i\xi_j}{|\xi|^2}\mathcal{F}_{x\to\xi}[v]\Big];$$

- the inverse Laplacian:

$$(-\Delta)^{-1} \approx \frac{1}{|\xi|^2},$$

  meaning that

$$(-\Delta)^{-1}[v] = \mathcal{F}^{-1}_{\xi \to x}\Big[\frac{1}{|\xi|^2}\mathcal{F}_{x\to\xi}[v]\Big].$$

In the sequel, we shall investigate boundedness of these pseudo- differential operators in various function spaces. The following theorem is an immediate consequence of the Hörmander-Mikhlin theorem (Theorem 0.7).

■ Continuity of the Riesz Operator:

Theorem 10.25. *The operators $\mathcal{R}_k$, $\mathcal{R}_{k,j}$ are continuous linear operators mapping $L^p(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ for any $1 < p < \infty$. In particular, the following estimate holds true:*

$$\|\mathcal{R}[v]\|_{L^p(\mathbb{R}^N)} \le c(N,p)\|v\|_{L^p(\mathbb{R}^N)} \text{ for all } v \in L^p(\mathbb{R}^N), \tag{10.97}$$

where $\mathcal{R}$ stands for $\mathcal{R}_k$ or $\mathcal{R}_{k,j}$.

As a next step, we examine the continuity properties of the inverse divergence operator. To begin, we recall that for Banach spaces X and Y, with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, the sum $X+Y = \{w = u+v \,|\, u \in X,\, v \in Y\}$ and the intersection $X \cap Y$ can be viewed as Banach spaces endowed with norms $\|w\|_{X+Y} = \inf\Big\{ \max\{\|u\|_X, \|v\|_Y\}, \Big| w = u+v \Big\}$ and $\|w\|_{X\cap Y} = \|w\|_X + \|w\|_Y$, respectively.

■ Continuity Properties of the Inverse Divergence:

Theorem 10.26. *Assume that $N > 1$.*

(i) *The operator $\mathcal{A}_k$ is a continuous linear operator mapping $L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, and $L^p(\mathbb{R}^N)$ into $L^{\frac{Np}{N-p}}(\mathbb{R}^N)$ for any $1 < p < N$.*

(ii) *In particular,*

$$\|\mathcal{A}_k[v]\|_{L^\infty(\mathbb{R}^N)+L^2(\mathbb{R}^N)} \le c(N)\|v\|_{L^1(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)} \quad \text{for all } v \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N), \tag{10.98}$$

and

$$\|\mathcal{A}_k[v]\|_{L^{\frac{Np}{N-p}}(\mathbb{R}^N)} \le c(N,p)\|v\|_{L^p(\mathbb{R}^N)} \quad \text{for all } v \in L^p(\mathbb{R}^N),\ 1 < p < N. \tag{10.99}$$

(iii) *If $v, \frac{\partial v}{\partial t} \in L^p(I \times \mathbb{R}^N)$, where I is an (open) interval, then*

$$\frac{\partial \mathcal{A}_k(f)}{\partial t}(t,x) = \mathcal{A}_k\Big(\frac{\partial f}{\partial t}\Big)(t,x) \quad \text{for a. a. } (t,x) \in I \times \mathbb{R}^N. \tag{10.100}$$

Proof. Step 1. We write

$$-\mathcal{A}_k[v] = \mathcal{F}^{-1}_{\xi\to x}\Big[\frac{i\xi_k}{|\xi|^2} 1_{\{|\xi|\le 1\}} \mathcal{F}_{x\to\xi}[v]\Big] + \mathcal{F}^{-1}_{\xi\to x}\Big[\frac{i\xi_k}{|\xi|^2} 1_{\{|\xi|>1\}} \mathcal{F}_{x\to\xi}[v]\Big].$$

Since v belongs to $L^1(\mathbb{R}^N)$, the function $\mathcal{F}_{x\to\xi}[v]$ is uniformly bounded; whence the quantity $\frac{i\xi_k}{|\xi|^2} 1_{\{|\xi|\le 1\}} \mathcal{F}_{x\to\xi}[v]$ is integrable. Similarly, v being square integrable, $\mathcal{F}_{x\to\xi}[v]$ enjoys the same property so that $\frac{i\xi_k}{|\xi|^2} 1_{\{|\xi|>1\}} \mathcal{F}_{x\to\xi}[v]$ is square integrable as well. After these observations, estimate (10.98) follows immediately from the basic properties of the Fourier transform, see Section 0.5.

Step 2. We introduce $\mathcal{E}(x)$ – the fundamental solution of the Laplace operator, specifically,

$$\Delta_x \mathcal{E} = \delta \quad \text{in } \mathcal{D}'(\mathbb{R}^N), \tag{10.101}$$

where δ denotes the Dirac distribution. If $N \geq 2$, $\partial_{x_k}\mathcal{E}$ takes the form

$$\partial_{x_k}\mathcal{E}(x) = \frac{1}{a_N}\frac{1}{|x|^{N-1}}\frac{x_k}{|x|}, \text{ where } a_N = \left\{ \begin{array}{c} 2\pi \text{ if } N = 2, \\ (N-2)\sigma_N \text{ if } N > 2, \end{array} \right\} \tag{10.102}$$

with σ_N being the area of the unit sphere. From (10.101) we easily deduce that

$$\mathcal{F}_{x\to\xi}[\partial_{x_k}\mathcal{E}] = \frac{1}{(2\pi)^{N/2}}\frac{i\xi_k}{|\xi|^2}.$$

Consequently,

$$\partial_{x_k}\mathcal{E} * v = \mathcal{F}^{-1}_{\xi\to x}\Big[\mathcal{F}_{x\to\xi}[\partial_{x_k}\mathcal{E} * v]\Big] = \frac{1}{(2\pi)^{N/2}}\mathcal{F}^{-1}_{\xi\to x}\left[\frac{i\xi_k}{|\xi|^2}\mathcal{F}_{x\to\xi}[v]\right]$$

where the weakly singular operator $v \to \partial_{x_k}\mathcal{E} * v$ is continuous from $L^p(\mathbb{R}^N)$ to $L^r(\mathbb{R}^N)$, $\frac{1}{r} = \frac{N-1}{N} + \frac{1}{p} - 1$, provided $1 < p < N$ as a consequence of the classical results of harmonic analysis stated in Theorem 10.9. This completes the proof of parts (i), (ii).

Step 3. If $v \in C^\infty_c(\overline{I} \times \mathbb{R}^3)$, statement (iii) follows directly from the theorem on differentiation of integrals with respect to a parameter. Its L^p-version can be proved via the density arguments. □

In order to conclude this section, we recall several elementary formulas that can be verified by means of direct computation.

$$\begin{aligned}
\mathcal{R}_{j,k}[f] = \partial_j \mathcal{A}_k[f] &= -\mathcal{R}_j\Big[\mathcal{R}_k[f]\Big], \\
\mathcal{R}_j\Big[\mathcal{R}_k[f]\Big] &= \mathcal{R}_k\Big[\mathcal{R}_j[f]\Big], \\
\sum_{k=1}^N \mathcal{R}_k\Big[\mathcal{R}_k[f]\Big] &= f, \\
\int_\Omega \mathcal{A}_k[f]\overline{g}\,\mathrm{d}x &= -\int_\Omega f\overline{\mathcal{A}_k[g])}\,\mathrm{d}x, \\
\int_\Omega \mathcal{R}_j\Big[\mathcal{R}_k[f]\Big]\overline{g}\,\mathrm{d}x &= \int_\Omega f\overline{\mathcal{R}_j\Big[\mathcal{R}_k[g]\Big]}\,\mathrm{d}x.
\end{aligned} \tag{10.103}$$

These formulas hold for all $f, g \in \mathcal{S}(\mathbb{R}^N)$ and can be extended by density in accordance with Theorems 10.26, 10.25 to $f \in L^p(\mathbb{R}^N)$, $g \in L^{p'}(\mathbb{R}^N)$, $1 < p < \infty$, whenever the left- and right-hand sides make sense. We also notice that functions $\mathcal{A}_k(f)$, $\mathcal{R}_{j,k}(f)$ are real-valued functions provided f is real valued.

10.17 Commutators involving Riesz operators

This section presents two important results involving Riesz operators. The first one represents a keystone in the proof of the weak continuity property of the effective pressure. Its formulation and proof are taken from [78], [87].

■ COMMUTATORS INVOLVING RIESZ OPERATORS, WEAK CONVERGENCE:

Theorem 10.27. *Let*

$$\mathbf{V}_\varepsilon \to \mathbf{V} \text{ weakly in } L^p(\mathbb{R}^N;\mathbb{R}^N),$$
$$\mathbf{U}_\varepsilon \to \mathbf{U} \text{ weakly in } L^q(\mathbb{R}^N;\mathbb{R}^N),$$

where $\frac{1}{p}+\frac{1}{q}=\frac{1}{s}<1$. *Then*

$$\mathbf{U}_\varepsilon\cdot\mathcal{R}[\mathbf{V}_\varepsilon]-\mathcal{R}[\mathbf{U}_\varepsilon]\cdot\mathbf{V}_\varepsilon \to \mathbf{U}\cdot\mathcal{R}[\mathbf{V}]-\mathcal{R}[\mathbf{U}]\cdot\mathbf{V} \text{ weakly in } L^s(\mathbb{R}^N).$$

Proof. Writing

$$\mathbf{U}_\varepsilon\cdot\mathcal{R}[\mathbf{V}_\varepsilon]-\mathbf{V}_\varepsilon\cdot\mathcal{R}[\mathbf{U}_\varepsilon]=\Big(\mathbf{U}_\varepsilon-\mathcal{R}[\mathbf{U}_\varepsilon]\Big)\cdot\mathcal{R}[\mathbf{V}_\varepsilon]-\Big(\mathbf{V}_\varepsilon-\mathcal{R}[\mathbf{V}_\varepsilon]\Big)\cdot\mathcal{R}[\mathbf{U}_\varepsilon]$$

we easily check that

$$\mathrm{div}_x\Big(\mathbf{U}_\varepsilon-\mathcal{R}[\mathbf{U}_\varepsilon]\Big)=\mathrm{div}_x\Big(\mathbf{V}_\varepsilon-\mathcal{R}[\mathbf{V}_\varepsilon]\Big)=0,$$

while $\mathcal{R}[\mathbf{U}_\varepsilon]$, $\mathcal{R}[\mathbf{V}_\varepsilon]$ are gradients, in particular

$$\mathbf{curl}_x\mathcal{R}[\mathbf{U}_\varepsilon]=\mathbf{curl}_x\mathcal{R}[\mathbf{V}_\varepsilon]=0.$$

Thus the desired conclusion follows from the Div-Curl lemma (Theorem 10.21). □

The following result is in the spirit of Coifman, Meyer [49]. The main ideas of the proof are taken over from [67].

■ COMMUTATORS INVOLVING RIESZ OPERATORS,
BOUNDEDNESS IN SOBOLEV-SLOBODECKII SPACES:

Theorem 10.28. *Let* $w\in W^{1,r}(\mathbb{R}^N)$ *and* $\mathbf{V}\in L^p(\mathbb{R}^N;\mathbb{R}^N)$ *be given, where*

$$1<r<N,\quad 1<p<\infty,\quad \frac{1}{r}+\frac{1}{p}-\frac{1}{N}<1.$$

Then for any s satisfying

$$\frac{1}{r}+\frac{1}{p}-\frac{1}{N}<\frac{1}{s}<1,$$

there exists

$$\beta = \beta(s,p,r) \in (0,1), \quad \frac{\beta}{N} = \frac{1}{s} + \frac{1}{N} - \frac{1}{p} - \frac{1}{r}$$

such that

$$\Big\| \mathcal{R}[w\mathbf{V}] - w\mathcal{R}[\mathbf{V}] \Big\|_{W^{\beta,s}(\mathbb{R}^N;\mathbb{R}^N)} \leq c\|w\|_{W^{1,r}(\mathbb{R}^N)} \|\mathbf{V}\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)},$$

where $c = c(s,p,r)$ is a positive constant.

Proof. We may suppose without loss of generality that $w \in C_c^\infty(\mathbb{R}^N)$, $\mathbf{V} \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N)$. First we notice that the norms

$$\|\mathbf{a}\|_{W^{1,m}(\mathbb{R}^N;\mathbb{R}^N)} \text{ and } \|\mathbf{a}\|_{L^m(\mathbb{R}^N;\mathbb{R}^N)} + \|\mathbf{curl}_x\mathbf{a}\|_{L^m(\mathbb{R}^N;\mathbb{R}^N)} + \|\mathrm{div}_x\mathbf{a}\|_{L^m(\mathbb{R}^N)} \tag{10.104}$$

are equivalent for $1 < m < \infty$, see Theorem 10.18. We also verify by a direct calculation that

$$[(\mathrm{curl}_x(\mathcal{R}[w\mathbf{V}])]_{j,k} = 0, \ [\mathrm{curl}_x(w\mathcal{R}[\mathbf{V}])]_{j,k} = \partial_{x_k} w\, \mathcal{R}_{j,s}[V_s] - \partial_{x_j} w\, \mathcal{R}_{k,s}[V_s], \tag{10.105}$$

and

$$\mathrm{div}_x(\mathcal{R}[w\mathbf{V}]) - \mathrm{div}_x\Big(w\mathcal{R}[\mathbf{V}]\Big) = \sum_{j=1}^N \partial_{x_j} w\, V_j - \sum_{i,j=1}^N \partial_{x_i} w\, \mathcal{R}_{i,j}[V_j]. \tag{10.106}$$

Next we observe that for any s, $\frac{1}{r} + \frac{1}{p} - \frac{1}{N} < \frac{1}{s} < 1$ there exist $1 \leq r_1 = r_1(s,p) < r < r_2 = r_2(s,p) < \infty$ such that

$$\frac{1}{r_1} + \frac{1}{p} - \frac{1}{N} = \frac{1}{s} = \frac{1}{r_2} + \frac{1}{p}.$$

Taking advantage of (10.104–10.106) and using Theorem 10.25 together with the Hölder inequality, we may infer that

$$\Big\| \mathcal{R}[w\mathbf{V}] - w\mathcal{R}[\mathbf{V}] \Big\|_{W^{1,s}(\mathbb{R}^N;\mathbb{R}^N)} \leq c\|w\|_{W^{1,r_2}(\mathbb{R}^N)} \|\mathbf{V}\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)}. \tag{10.107}$$

On the other hand, Theorem 10.25 combined with the continuous embedding $W^{1,r_1}(\mathbb{R}^N) \hookrightarrow L^{\frac{Nr_1}{N-r_1}}(\mathbb{R}^N)$, and the Hölder inequality yield

$$\Big\| \mathcal{R}[w\mathbf{V}] - w\mathcal{R}[\mathbf{V}] \Big\|_{L^s(\mathbb{R}^N;\mathbb{R}^N)} \leq c\|w\|_{W^{1,r_1}(\mathbb{R}^N)} \|\mathbf{V}\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)}. \tag{10.108}$$

We thus deduce that, for any fixed $\mathbf{V} \in L^p(\Omega;\mathbb{R}^N)$, the linear operator $w \to \mathcal{R}[w\mathbf{V}] - w\mathcal{R}[\mathbf{V}]$ is a continuous linear operator from $W^{1,r_2}(\Omega)$ to $W^{1,s}(\Omega,\mathbb{R}^N)$ and from $W^{1,r_1}(\Omega)$ to $L^s(\Omega;\mathbb{R}^N)$. Now we conclude by the Riesz-Thorin interpolation theorem (see [190]) that this operator is as well continuous from $W^{1,r}(\Omega)$ to $W^{\beta,s}(\Omega)$, where $\beta \in (0,1)$ verifies the formula $\frac{\beta}{r_1} + \frac{1-\beta}{r_2} = \frac{1}{r}$.

This finishes the proof. □

10.18 Renormalized solutions to the equation of continuity

In this section we explain the main ideas of the regularization technique developed by DiPerna and Lions [65] and discuss the basic properties of the renormalized solutions to the equation of continuity. To begin, we introduce a variant of the classical Friedrichs commutator lemma.

■ FRIEDRICHS' COMMUTATOR LEMMA IN SPACE:

Lemma 10.12. *Let $N \geq 2$, $\beta \in [1,\infty)$, $q \in [1,\infty]$, where $\frac{1}{q} + \frac{1}{\beta} = \frac{1}{r} \in (0,1]$. Suppose that*

$$\varrho \in L^{\beta}_{\rm loc}(R^N),\ \mathbf{u} \in W^{1,q}_{\rm loc}(R^N;R^N).$$

Then

$$\mathrm{div}_x\Big(S_\epsilon[\varrho\mathbf{u}]\Big) - \mathrm{div}_x\Big(S_\epsilon[\varrho]\mathbf{u}\Big) \to 0 \textit{ in } L^r_{\rm loc}(R^N), \tag{10.109}$$

where S_ε is the mollifying operator introduced in (10.1–10.2).

Proof. We have

$$\mathrm{div}_x\Big(S_\epsilon[\varrho\mathbf{u}]\Big) - \mathrm{div}_x\Big(S_\epsilon[\varrho]\mathbf{u}\Big) = I_\varepsilon - S_\varepsilon(\varrho)\mathrm{div}_x\mathbf{u},$$

where

$$I_\varepsilon(x) = \int_{R^N} \varrho(y)[\mathbf{u}(y) - \mathbf{u}(x)] \cdot \nabla_x \zeta_\varepsilon(x-y)\mathrm{d}y. \tag{10.110}$$

According to Theorem 10.1,

$$S_\varepsilon(\varrho)\mathrm{div}_x\mathbf{u} \to \varrho\mathrm{div}_x\mathbf{u} \quad \text{in } L^r_{\rm loc}(R^N);$$

whence it is enough to show that

$$I_\varepsilon \to \varrho\mathrm{div}_x\mathbf{u} \quad \text{in } L^r_{\rm loc}(R^N). \tag{10.111}$$

After a change of variables $y = x + \varepsilon z$, formula (10.110) reads

$$\begin{aligned} I_\varepsilon(x) &= \int_{|z|\leq 1} \varrho(x+\varepsilon z)\frac{\mathbf{u}(x+\varepsilon z) - \mathbf{u}(x)}{\varepsilon} \cdot \nabla_x\zeta(z)\mathrm{d}z \\ &= \int_0^1\int_{|z|\leq 1} \varrho(x+\varepsilon z)\, z\cdot\nabla_x\mathbf{u}(x+\varepsilon tz)\cdot\nabla_x\zeta(z)\mathrm{d}z\,\mathrm{d}t, \end{aligned} \tag{10.112}$$

where we have used the *Lagrange formula*

$$\mathbf{u}(\xi+\varepsilon z) - \mathbf{u}(\xi) = \varepsilon\int_0^1 z\cdot\nabla_x\mathbf{u}(\xi+\varepsilon tz)\mathrm{d}t.$$

From (10.112) we deduce a general estimate

$$\|I_\varepsilon\|_{L^s(B_R)} \leq c(\overline{r}, s, p, q)\, \|\varrho\|_{L^p(B_{\overline{r}+1})}\| \, \|\nabla_x \vec{u}\|_{L^q(B_{\overline{r}+1})}, \tag{10.113}$$

where $B_{\overline{r}}$ is a ball of radius $\overline{r}$ in $\mathbb{R}^N$, and where

$$\left\{ \begin{array}{c} s \text{ is arbitrary in } [1,\infty) \text{ if } p = q = \infty, \\ \frac{1}{s} = \frac{1}{q} + \frac{1}{p} \text{ if } \frac{1}{q} + \frac{1}{p} \in (0,1] \end{array} \right\}.$$

Formula (10.113) can be used with $\varrho_n - \varrho$ and $p = \beta$, q and $s = r$, where $\varrho_n \in C_c(\mathbb{R}^N)$, $\varrho_n \to \varrho$ strongly in $L^\beta_{\rm loc}(\mathbb{R}^N)$, in order to justify that it is enough to show (10.111), with ϱ belonging to $C_c(\mathbb{R}^N)$. For such a ϱ, we evidently have

$$I_\varepsilon(x) \to [\varrho {\rm div}_x \mathbf{u}](x) \text{ a. a. in } \mathbb{R}^N$$

as is easily seen from (10.112). Moreover, formula (10.113) now with $p = \infty$, yields I_ε bounded in $L^s(B_{\overline{r}})$ with $s > r$. This observation allows us obtain the desired conclusion by means of Vitali's convergence theorem. □

In the case of a time dependent scalar field ϱ and a vector field $\mathbf{u}$, Lemma 10.113 gives rise to the following corollary.

■ FRIEDRICHS COMMUTATOR LEMMA IN TIME-SPACE:

Corollary 10.3. *Let $N \geq 2$, $\beta \in [1,\infty)$, $q \in [1,\infty]$, $\frac{1}{q} + \frac{1}{\beta} = \frac{1}{r} \in (0,1]$. Suppose that*

$$\varrho \in L^\beta_{\rm loc}((0,T) \times \mathbb{R}^N), \ \mathbf{u} \in L^q_{\rm loc}(0,T; W^{1,q}_{\rm loc}(\mathbb{R}^N;\mathbb{R}^N)).$$

Then

$${\rm div}_x \Big(S_\epsilon[\varrho \mathbf{u}] \Big) - {\rm div}_x \Big(S_\epsilon[\varrho]\mathbf{u} \Big) \to 0 \text{ in } L^r_{\rm loc}((0,T) \times \mathbb{R}^N), \tag{10.114}$$

where S_ε is the mollifying operator introduced in (10.1–10.2) *acting solely on the space variables.*

With Lemma 10.12 and Corollary 10.3 at hand, we can start to investigate the renormalized solutions to the continuity equation.

■ RENORMALIZED SOLUTIONS OF THE CONTINUITY EQUATION I:

Theorem 10.29. *Let $N \geq 2$, $\beta \in [1,\infty)$, $q \in [1,\infty]$, $\frac{1}{q} + \frac{1}{\beta} \in (0,1]$. Suppose that the functions $(\varrho, \mathbf{u}) \in L^\beta_{\rm loc}((0,T) \times \mathbb{R}^N) \times L^q_{\rm loc}(0,T; W^{1,q}_{\rm loc}(\mathbb{R}^N;\mathbb{R}^N))$, where $\varrho \geq 0$ a.e. in $(0,T) \times \mathbb{R}^N$, satisfy the transport equation*

$$\partial_t \varrho + {\rm div}_x(\varrho \mathbf{u}) = f \text{ in } \mathcal{D}'((0,T) \times \mathbb{R}^N), \tag{10.115}$$

where $f \in L^1_{\rm loc}((0,T) \times \mathbb{R}^N)$.

Then

$$\partial_t b(\varrho) + \mathrm{div}_x\Big((b(\varrho)\mathbf{u}\Big) + \Big(\varrho b'(\varrho) - b(\varrho)\Big)\mathrm{div}_x\mathbf{u} = f b'(\varrho) \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^N) \tag{10.116}$$

for any

$$b \in C^1([0,\infty)) \cap W^{1,\infty}(0,\infty). \tag{10.117}$$

Proof. Taking convolution of (3.199) with ζ_ε (see (10.1–10.2)), that is to say using $\zeta_\varepsilon(x-\cdot)$ as a test function, we obtain

$$\partial_t\Big(S_\varepsilon[\varrho]\Big) + \mathrm{div}_x\Big(S_\varepsilon[\varrho]\mathbf{u}\Big) = \wp_\varepsilon(\varrho,\mathbf{u}), \tag{10.118}$$

where

$$\wp_\varepsilon(\varrho,\mathbf{u}) = \mathrm{div}_x\Big(S_\varepsilon[\varrho]\mathbf{u}\Big) - \mathrm{div}_x S_\varepsilon[\varrho\mathbf{u}] \text{ a.e. in } (0,T)\times\mathbb{R}^N.$$

Equation (10.118) can be multiplied on $b'(S_\varepsilon[\varrho])$, where b is a globally Lipschitz function on $[0,\infty)$; one obtains

$$\begin{aligned}\partial_t b\left(S_\varepsilon[\varrho]\right) + \mathrm{div}_x\left[b\left(S_\varepsilon[\varrho]\right)\mathbf{u}\right] + \left[S_\varepsilon[\varrho]b'\left(S_\varepsilon[\varrho]\right) - b\left(S_\varepsilon[\varrho]\right)\right]\\ = \wp_\varepsilon(\varrho,\mathbf{u})\, b'\left(S_\varepsilon[\varrho]\right).\end{aligned} \tag{10.119}$$

It is easy to check that for $\varepsilon \to 0+$ the left-hand side of (10.119) tends to the desired expression appearing in the renormalized formulation of the continuity equation (10.116). Moreover, the right-hand side tends to zero as a direct consequence of Corollary 10.3. □

Once the renormalized continuity equation is established for any b belonging to (10.117), it is satisfied for any "renormalizing" function b belonging a larger class. This is clarified in the following lemma.

■ Renormalized Solutions of the Continuity Equation II:

Lemma 10.13. *Let $N \geq 2$, $\beta \in [1,\infty)$, $q \in [1,\infty]$, $\frac{1}{q} + \frac{1}{\beta} \in (0,1]$. Suppose that the functions $(\varrho,\mathbf{u}) \in L^\beta_{\rm loc}((0,T)\times\mathbb{R}^N) \times L^q_{\rm loc}(0,T;W^{1,q}_{\rm loc}(\mathbb{R}^N;\mathbb{R}^N))$, where $\varrho \geq 0$ a.e. in $(0,T)\times\mathbb{R}^N$, satisfy the renormalized continuity equation* (10.116) *for any b belonging to the class* (10.117).

Then we have:

(i) *If $f \in L^p_{\rm loc}((0,T)\times\mathbb{R}^N)$ for some $p > 1$, $p'(\frac{\beta}{2}-1) \leq 1$, then equation* (10.116) *holds for any*

$$b \in C^1([0,\infty)),\ |b'(s)| \leq cs^\lambda, \text{ for } s > 1, \text{ where } \lambda \leq \tfrac{\beta}{2} - 1. \tag{10.120}$$

(ii) *If $f = 0$, then equation* (10.116) *holds for any*

$$\begin{gathered} b \in C([0,\infty)) \cap C^1((0,\infty)), \\ \lim_{s\to 0+}\Big(sb'(s) - b(s)\Big) \in \mathbb{R}, \\ |b'(s)| \le cs^{\lambda} \text{ if } s \in (1,\infty) \text{ for a certain } \lambda \le \tfrac{\beta}{2} - 1 \end{gathered} \tag{10.121}$$

(iii) *The function $z \to b(z)$ in any of the above statements* (i)–(ii) *can be replaced by $z \to cz + b(z)$, $c \in \mathbb{R}$, where b satisfies* (10.120) *or* (10.121) *as the case may be.*

(iv) *If $f = 0$, then*

$$\partial_t\Big(\varrho B(\varrho)\Big) + \mathrm{div}_x\Big(\varrho B(\varrho)\mathbf{u}\Big) + b(\varrho)\mathrm{div}_x\mathbf{u} = 0 \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^N) \tag{10.122}$$

for any

$$b \in C([0,\infty)) \cap L^\infty(0,\infty),\ B(\varrho) = B(1) + \int_1^\varrho \frac{b(z)}{z^2}\mathrm{d}z. \tag{10.123}$$

Proof. Statement (i) can be deduced from (10.116) by approximating conveniently the functions b satisfying relation (10.120) by functions belonging to the class $C^1([0,\infty)) \cap W^{1,\infty}(0,\infty)$ and using consequently the Lebesgue dominated or Vitali's and the Beppo-Levi monotone convergence theorems. We can take a sequence $S_{\frac{1}{n}}(b\circ\mathcal{T}_n)$, $n \to \infty$, where $\mathcal{T}_n$ is defined by (10.72), and with the mollifying operator $S_{\frac{1}{n}}$ introduced in (10.1–10.2).

Statement (ii) follows from (i): The renormalized continuity equation (10.117) certainly holds for $b_h(\cdot) := b(h + \cdot)$. Thus we can pass to the limit $h \to 0+$, take advantage of condition $\lim_{s\to 0+}(sb'(s) - b(s)) \in R$, and apply the Lebesgue dominated convergence.

Statement (iii) results from summing the continuity equation with the renormalized continuity equation.

The function $z \to zB(z)$ satisfies assumptions (10.121). Statement (iv) thus follows immediately from (ii). □

Next, we shall investigate the pointwise behavior of renormalized solutions with respect to time.

■ Time Continuity of Renormalized Solutions

Lemma 10.14. *Let $N \ge 2$, $\beta, q \in (1,\infty)$, $\frac{1}{q} + \frac{1}{\beta} \in (0,1]$. Suppose that the functions $(\varrho,\mathbf{u}) \in L^\infty(0,T;L^\beta_{\rm loc}(\mathbb{R}^N)) \times L^q(0,T;W^{1,q}_{\rm loc}(\mathbb{R}^N;\mathbb{R}^N))$, $\varrho \ge 0$ a.a. in $(0,T)\times\mathbb{R}^N$, satisfy continuity equation* (10.115) *with $f \in L^s_{\rm loc}((0,T)\times\Omega)$, $s > 1$, and renormalized continuity equation* (10.116) *for any b belonging to class* (10.117).

Then

$$\varrho \in C_{\rm weak}([0,T];L^{\beta}(O)) \cap C([0,T],L^p(O))$$

with any $1 \le p < \beta$ *and* O *any bounded domain in* $\mathbb{R}^N$.

Proof. According to Lemma 10.13,

$$\partial_t \sigma + {\rm div}_x(\sigma \mathbf{u}) = \frac{1}{2}\sigma {\rm div}_x \mathbf{u} \text{ in } \mathcal{D}'((0,T)\times \mathbb{R}^N),$$

where we have set $\sigma = \sqrt{\varrho}$; we may therefore assume that

$$\sigma \in C_{\rm weak}([0,T];L^{2\beta}(O)) \quad \text{for any bounded domain } O \subset \mathbb{R}^N. \tag{10.124}$$

Regularizing the latter equation over the space variables, we obtain

$$\partial_t \left(S_\varepsilon[\sigma]\right) + {\rm div}_x \left(S_\varepsilon[\sigma]\mathbf{u}\right) = \frac{1}{2} S_\varepsilon \left[\sigma {\rm div}_x \mathbf{u}\right] + \wp_\varepsilon(\sigma,\mathbf{u}) \quad \text{a.a. in } (0,T)\times\mathbb{R}^N,$$

where S_ε and $\wp_\varepsilon$ are the same as in the proof of Theorem 10.29. Now, applying to the last equation Theorem 10.29 and Lemma 10.13, we get

$$\begin{aligned}&\partial_t \left(S_\varepsilon[\sigma]\right)^2 + {\rm div}_x\big(\left(S_\varepsilon[\sigma]\right)^2\mathbf{u}\big) \\ &= S_\varepsilon[\sigma]S_\varepsilon\left(\sigma {\rm div}_x\mathbf{u}\right) + 2S_\varepsilon[\sigma]\wp_\varepsilon(\sigma,\mathbf{u}) - \left(S_\varepsilon[\sigma]\right)^2 {\rm div}_x\mathbf{u} \quad \text{a.a. in } (0,T)\times\mathbb{R}^N.\end{aligned} \tag{10.125}$$

We employ equation (10.125) together with Theorem 10.1 and Corollary 10.3 to verify that the sequence $\{\int_\Omega \left(S_\varepsilon[\sigma]\right)^2 \eta \,{\rm d}x\}_{\varepsilon>0}$, $\eta \in C_c^\infty(\mathbb{R}^N)$ satisfies assumptions of the Arzelà-Ascoli theorem on $C([0,T])$. Combining this information with separability of $L^{\beta'}(O)$ and the density argument, we may infer that

$$\int_O \left(S_\varepsilon[\sigma]\right)^2 \eta \,{\rm d}x \to \int_O \overline{\sigma^2}(t)\eta \,{\rm d}x \quad \text{in } C([0,T]).$$

for any $\eta \in L^{\beta'}(O)$.

On the other hand, Theorem 10.1 yields

$$\left(S_\varepsilon[\sigma]\right)^2(t) \to \sigma^2(t) \quad \text{in } L^\beta(O) \text{ for all } t\in[0,T];$$

therefore $\int_O \overline{\sigma^2}\eta \,{\rm d}x = \int_O \sigma^2\eta \,{\rm d}x$ on $[0,T]$ and

$$\sigma^2 \in C_{\rm weak}([0,T];L^\beta(O)). \tag{10.126}$$

Relations (10.124) and (10.126) yield $\sigma \in C([0,T];L^2(O))$, whence we complete the proof by a simple interpolation argument. □

We conclude this section with a compactness result involving the renormalized continuity equation.

Theorem 10.30. *Let $N \geq 2$, $\beta > \frac{2N}{N+2}$, Ω be a bounded Lipschitz domain in $\mathbb{R}^N$, $T > 0$, and*

$$B \in C([0,T] \times \overline{\Omega} \times [0,\infty)), \quad \sup_{(t,x)\in(0,T)\times\Omega} |B(t,x,s)| \leq c(1+s^p), \tag{10.127}$$

where c is a positive constant, and $0 < p < \frac{N+2}{2N}\beta$ is a fixed number.

Suppose that $\{\varrho_n \geq 0, \mathbf{u}_n\}_{n=1}^{\infty}$ is a sequence with the following properties:

(i)
$$\varrho_n \to \varrho \ \textit{ weakly-(*) in } L^\infty(0,T;L^\beta(\Omega)),$$
$$\mathbf{u}_n \to \mathbf{u} \ \textit{ weakly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^N)); \tag{10.128}$$

(ii)
$$\int_0^T \int_\Omega \Big(a(\varrho_n)\partial_t\varphi + a(\varrho_n)\mathbf{u}_n\cdot\nabla_x\varphi - (\varrho_n a'(\varrho_n) - a(\varrho_n))\mathrm{div}_x\mathbf{u}_n \Big)\,\mathrm{d}x\mathrm{d}t = 0 \tag{10.129}$$

for all $a \in C^1([0,\infty)) \cap W^{1,\infty}((0,\infty))$, and for all $\varphi \in C_c^\infty((0,T)\times\overline{\Omega})$.

Then the sequence $\{B(\cdot,\cdot,\varrho_n)\}_{n=1}^{\infty}$ is precompact in the space $L^s(0,T;W^{-1,2}(\Omega))$ for any $s \in [1,\infty)$.

Proof. Step 1. Due to Corollary 10.2 and in accordance with assumptions (10.127–10.30),

$$\sup_{n\in\mathbb{N}} \|B(\cdot,\cdot,\mathcal{T}_k(\varrho_n)) - B(\cdot,\cdot,\varrho_n)\|_{L^{\frac{2N}{N+2}}(\Omega)} \to 0 \text{ as } k \to \infty,$$

where $\mathcal{T}_k$ is the truncation function introduced in (10.72). Since $L^\beta(\Omega) \hookrightarrow\hookrightarrow W^{-1,2}(\Omega)$ whenever $\beta > \frac{2N}{N+2}$, it is enough to show precompactness of the sequence of composed functions $B(\cdot,\cdot,\mathcal{T}_k(\varrho_n))$.

Step 2. According to the Weierstrass approximation theorem, there exists a polynomial A_ε on $\mathbb{R}^{N+2}$ such that

$$\|A_\varepsilon - B\|_{C([0,T]\times\overline{\Omega}\times[0,2k])} < \varepsilon,$$

where $\varepsilon > 0$. Therefore,

$$\sup_{n\in\mathbb{N}} \|A_\varepsilon(\cdot,\cdot,\mathcal{T}_k(\varrho_n) - B(\cdot,\cdot,\mathcal{T}_k(\varrho_n)\|_{L^\infty((0,T)\times\Omega)} < \varepsilon.$$

Consequently, it is merely enough to show precompactness of any sequence of type $a_1(t)a_2(x)a(\varrho_n)$, where $a_1 \in C^1([0,T])$, $a_2 \in C^1(\overline{\Omega})$, and where a belongs to $C^1([0,\infty)) \cap W^{1,\infty}((0,\infty))$. However, this is equivalent to proving precompactness of the sequence $a(\varrho_n)$, $a \in C^1([0,\infty))$.

Step 3. Since ϱ_n, $\mathbf{u}_n$ solve equation (10.30), we easily check that the functions $t \to [\int_\Omega a(\varrho_n)\varphi\,\mathrm{d}x](t)$ form a bounded and equi-continuous sequence in $C([0,T])$

for all $\varphi \in C_c^\infty(\Omega)$. Consequently, the standard Arzelà-Ascoli theorem combined with the separability of $L^{\beta'}(\Omega)$ yields, via a density argument and a diagonalization procedure, the existence of a function $\overline{a(\varrho)} \in C_{\text{weak}}([0,T]; L^\beta(\Omega))$ satisfying

$$\int_\Omega a(\varrho_n)\varphi \, \mathrm{d}x \to \int_\Omega \overline{a(\varrho)}\varphi \, \mathrm{d}x \text{ in } C([0,T]) \text{ for all } \varphi \in L^{\beta'}(\Omega)$$

at least for a chosen subsequence. Since $L^\beta(\Omega) \hookrightarrow\hookrightarrow W^{-1,2}(\Omega)$, we deduce that

$$a(\varrho_n)(t,\cdot) \to \overline{a(\varrho)}(t,\cdot) \text{ strongly in } W^{-1,2}(\Omega) \text{ for all } t \in [0,T].$$

Thus applying Vitali's theorem to the sequence $\{\|a(\varrho_n)\|_{W^{-1,2}(\Omega)}\}_{n=1}^\infty$, which is bounded in $L^\infty(0,T)$ completes the proof. $\square$

Chapter 11

Bibliographical Remarks

11.1 Fluid flow modeling

The material collected in Chapter 1 is standard. We refer to the classical monographs by Batchelor [18] or Lamb [131] for a full account of the mathematical theory of continuum fluid mechanics. A more recent treatment may be found in Truesdell and Noll [192] or Truesdell and Rajagopal [193]. An excellent introduction to the mathematical theory of waves in fluids is contained in Lighthill's book [135].

The constitutive equations introduced in Section 1.4, in particular, the mechanical effect of thermal radiation, are motivated by the mathematical models in astrophysics (see Battaner [19]). Relevant material may be also found in the monographs by Bose [27], Mihalas and Weibel-Mihalas [157], Müller and Ruggeri [160], or Oxenius [169]. A general introduction to the theory of equations of state is provided by Eliezer et al. [71].

In the present monograph, we focused on thermodynamics of *viscous* compressible fluids. For the treatment of problems related to inviscid fluids as well as more general systems of *hyperbolic* conservation laws, the literature provides several comprehensive seminal works, for instance, Benzoni-Gavage and Serre [22], Bressan [32], Chen and Wang [45], Dafermos [52], and Serre [182].

The weak solutions in this book are considered on large time intervals. There is a vast literature investigating (strong) solutions with "large" regular external data on short time intervals and/or with "small" regular external data on arbitrary large time intervals for both the Navier-Stokes equations in the barotropic regime and for the Navier-Stokes-Fourier system. These studies were originated by the seminal work of Matsumura and Nishida [151], [152], and further developed by many authors: Beirao da Veiga [21], Danchin [54], [55], Hoff [107], [108], [109], [110], [111], [112], [113], Jiang [117] , Matsumura and Padula [153], [164], Padula and Pokorný [170], Salvi and Straškraba [176], Valli and Zajaczkovski [196], among others.

As far as the singular limits in fluid dynamics are concerned, the mathematical literature provides two qualitatively different groups of results. The first one concerns the investigation of singular limits in passage from the microscopic description provided by the kinetic models of Boltzmann's type to the macroscopic one represented by the Euler, Navier-Stokes, and Navier-Stokes-Fourier equations and their modifications. The reader may find it interesting to compare the methods and techniques used in the present monograph to those developed in the context of kinetic equations and their asymptotic limits by Bardos, Golse and Levermore [14], [15], [16], Bardos and Ukai [17], Golse and Saint-Raymond [103], Golse and Levermore [102], P.-L.Lions and Masmoudi [143], [144], see also the review paper by Villani [197] as well as the references therein. The second group of problems concerns the relations between models at the same conceptual level provided by continuum mechanics studied in this book. We refer to Section 11.4 for the corresponding bibliographic remarks.

11.2 Mathematical theory of weak solutions

Variational (weak) solutions represent the most natural framework for a mathematical formulation of the balance laws arising in continuum fluid mechanics, these being originally formulated in the form of integral identities rather than partial differential equations. Since the truly pioneering work of Leray [132], the theory of variational solutions, based on function spaces of Sobolev type and developed in the work of Ladyzhenskaya [127], Temam [188], Caffarelli et al. [37], Antontsev et al. [9], and, more recently P.-L. Lions [139], has become an important part of modern mathematical physics.

Although many of the above cited references concern incompressible fluids, where weak solutions are expected (but still not proved) to be regular at least for smooth data, the theory of compressible and/or compressible and heat conducting fluids supplemented with arbitrarily large data is more likely to rely on the concept of "genuinely weak" solutions incorporating various types of discontinuities and other irregular phenomena as the case may be (for relevant examples see Desjardins [59], Hoff [112], [113], Hoff and Serre [114], Vaigant [194], among others). Pursuing further this direction some authors developed the theory of *measure-valued solutions* in order to handle the rapid oscillations that solutions may develop in a finite time (see DiPerna [63], DiPerna and Majda [66], Málek et al. [148]). The representation of the basic physical principles in terms of conservation laws has been discussed in a recent paper by Chen and Tores [46] devoted to the study of vector fields with divergence measure.

A rigorous mathematical theory of compressible barotropic fluids with large data was presented only recently in the pioneering work by P.-L. Lions [140] (see also a very interesting related result by Vaigant and Kazhikhov [195]). The fundamental idea discussed already by Hoff [111] and Serre [181] is based on a "weak continuity" property of a physical quantity that we call effective viscous pressure, together with a clever use of the renormalized equation of continuity in order to describe possible density oscillations. A survey of the relevant recent results in this direction can be found in the monograph [166].

11.3 Existence theory

The seminal work of P.-L.Lions [140] on existence for compressible viscous barotropic fluids requires certain growth restrictions on the pressure, specifically, the adiabatic exponent $\gamma \geq \frac{9}{5}$ in the nonsteady case, and $\gamma > \frac{5}{3}$ in the steady case. These results have been improved by means of a more precise description of the density oscillations in [78], [87] up to the adiabatic exponents $\gamma > \frac{3}{2}$. Finally, Frehse, Goj, Steinhauer in [89] and Plotnikov, Sokolowski in [173] derived, independently, new estimates, which have been quite recently used, at least in the steady case, to extend the existence theory to smaller adiabatic exponents, see [174] and [33].

The existence theory presented in this book can be viewed as a part of the programme originated in the monograph [79]. In comparison with [79], the present study contains some new material, notably, the constitutive equations are much more realistic, with structural restrictions based on purely physical principles, and the transport coefficients are allowed to depend on the temperature. These new ingredients of the existence theory have been introduced in a series of papers [80], [81], [82], and [85].

Several new ideas related to the existence problem for the full Navier-Stokes-Fourier system with density dependent shear and bulk viscosities satisfying a particular differential relation have been developed recently in a series of papers by Bresch and Desjardins [30], [29]. Making a clever use of the structure of the equations, the authors discovered a new integral identity which allows one to obtain uniform estimates on the density gradient and which may be used to prove existence of global-in-time solutions.

11.4 Analysis of singular limits

Many recent papers and research monographs explain the role of formal scaling arguments in the physical and numerical analysis of complex models arising in mathematical fluid dynamics. This approach has become of particular relevance in meteorology, where the huge scale differences in atmospheric flows give rise to a large variety of qualitatively different models, see the survey papers by Klein et al. [123], Klein [121], [122], the lecture notes of Majda [147], and the monographs by Chemin et al. [44], Zeytounian [206], [205], [204]. The same is true for applications in astrophysics, see the classical book of Chandrasekhar [43], or the more recent treatment by Gilman, Glatzmeier [99], [98], Lignières [136], among others.

The “incompressible limit” $\mathrm{Ma} \to 0$ for various systems arising in mathematical fluid dynamics was rigorously studied in the seminal work by Klainerman and Majda [120] (see also Ebin [68]). One may distinguish two kinds of qualitatively different results based on different techniques. The first approach applies to strong solutions defined on possibly short time intervals, the length of which, however, is independent of the value of the parameter $\mathrm{Ma} \to 0$. In this framework, the most recent achievements for the full Navier-Stokes-Fourier system can be found in the recent papers by Alazard [3], [4] (for earlier results see the survey papers by Danchin [56], Métivier and Schochet [156], Schochet [180], and the references cited therein).

The second group of results is based on a global-in-time existence theory for the weak solutions of the underlying primitive system of equations, asserting convergence towards solutions of the target system on an arbitrary time interval. Results of this type for the isentropic Navier-Stokes system have been obtained by Lions and Masmoudi [141], [142], and later extended by Desjardins et al. [61], Bresch et al. [31]. For a survey of these as well as of many other related results, see the review paper by Masmoudi [149].

The investigation of singular limits for the full Navier-Stokes-Fourier system in the framework of weak variational solutions originated in [84] and [86]. The spectral analysis of acoustic waves in the presence of strong stratification exposed in Chapter 6 follows the book of Wilcox [202], while the weighted Helmholtz decomposition used throughout the chapter has been inspired by [165]. Related results based on the so-called local method were obtained only recently by Masmoudi [150]. The refined analysis of the acoustic waves presented in Chapter 7 is based on the asymptotic expansion technique developed by Vishik and Ljusternik [198] to handle singular perturbations of elliptic operators, later adopted in the pioneering paper of Desjardins et al. [61] to the wave operator framework. Related techniques are presented in the monograph of Métivier [155]. Problems in $\mathbb{R}^3$ were investigated by Desjardins and Grenier [60].

11.5 Propagation of acoustic waves

There is a vast literature concerning acoustics in fluids, in general, and acoustic analogies and equations, in particular. In the study of the low Mach number limits, we profited from the theoretical work by Schochet [178], [179], [180]. A nice introduction to the linear theory of wave propagation is the classical monograph by Lighthill [135]. The nonlinear acoustic phenomena together with the relevant mathematical theory are exposed in the book by Enflo and Hedberg [72].

Lighthill's acoustic analogy in the spirit of Chapter 9 has been used by many authors, let us mention the numerical results obtained by Golanski et al. [100], [101].

Clearly, this topic is closely related to the theory of wave equations both in linear and nonlinear settings. Any comprehensive list of the literature in this area goes beyond the scope of the present monograph, and we give only a representative sample of results: Bahouri and Chemin [12], Burq [36], Christodoulou and Klainerman [48], Smith and Tataru [185].

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