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Management Science

218

Paul van Loon

A Dynamic Theory of the Firm:
Production,
Finance and Investment



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PREFACE

This volume is the final result of the research project "Micro growth model", that was sponsored by the Central Research Pool of Tilburg University, the Netherlands. Apart from the University Council for this important financial support, I owe Prof. Dr. Piet Verheyen very much for the way in which he introduced me into scientific circles and for the way in which he supervised and stimulated my work. Dr. Jan de Jong and Peter Janssen C.E., Technical University of Eindhoven, piloted me safely through the mathematics of optimal control theory and removed some technical barriers. Their help was indispensable for the success of this project. I would also like to mention the kind support of Prof. Dr. Jack Kleijnen, who gave me many valuable hints on how to present the results of this project. In this way I was able to contact with several researchers inside and outside the Netherlands. Most grateful I am to Prof. Dr. Charles Tapiero, Jerusalem University, who commented on important parts of this book in a constructive way and who suggested many subjects for further research. Also Mr. Geert-Jan van Schijndel, Tilburg University, should be mentioned here, because he closely read the work and I appreciated his remarks and corrections very much. Many colleagues have contributed to the results of this research project in a direct or indirect way. Especially I should like to mention my contacts with Prof. Dr. Jacques Lesourne and Dr. Raymond Leban, Conservatoire National des Arts et Métiers in Paris, with Prof. Dr. Gustav Feichtinger and Dr. Richard Hartl, Technische Hochschule in Vienna, with Dr. Jacques Thépot, European Institute for Advanced Studies in Management in Brussels, with Dr. Elon Ekman, Stockholm School of Economics, with Prof. Dr. Piet Moerland, State University of Groningen, with Dr. H. Kogels, Union of Dutch Enterprises (VNO), and with my colleagues of the scientific staff of the Tilburg University, in particular Mr. Piet Slangen and Dr. Jan van Lieshout, who witnessed the start of this project and discussed the first proposals in depth. Also Mr. Joop Roes, from Coopers & Lybrand in Rotterdam, is to be mentioned, because of his meaningful remarks on the fiscal aspects of my model.

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For my wife, Jeannette, this project provided a regularly absent(minded) husband, dealing with an incomprehensible kind of research. I admire the patience that she has shown and I hope to have still a positive capital value for her at the end of this production process.

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CHAPTER 1. INTRODUCTION.

1. Scope of the book.

The principal aim of this essay is to refine the description of the links between the optimal production, finance and investment policy of the firm by introducing activity analysis into the dynamic theory of the firm. Up to now, production in dynamic models, dealing with the production policy of the firm, was described by means of a continuous production function, implying an infinite number of production possibilities. Activity analysis however, assumes, as distinct from a continuous production function, that the firm can choose among only a limited number of production possibilities. This more realistic assumption will be shown to improve the role of depth investments in the firm's life cycle. Thus, it will be concluded that substituting activity analysis for a continuous production function implies the change from a firm that continuously adapts its way of production to a firm for which depth investments are intermediate stages in its life cycle, succeeded by growth stages or other stationary stages. Further, by introducing activity analysis we can analyse influences on the production policy of the firm more precisely, and in concordance with managerial behaviour. For this reason, this work provides some grounds for bridging the gap between the theory of the firm and its practical implications for decision making.

A second purpose of the book is to extend the description of governmental influences on the firm's policy by introducing investment grants beside corporation profit tax. Whereas corporation profit tax will always slow down corporate growth, investment grants, intending to stimulate employment, will increase corporate activity, but may lead to decreases in employment.

The third purpose of the book is to expand the analysis of the dynamic theory of the firm by comparing the findings with well known results of the static models, and by deriving three general laws of motion (the so-called: optimal decision rules) underlying all the resulting expansion patterns of the firm and, finally, by doing comparative dynamic analysis (or: sensitivity analysis) tracing the influence of government, financial states and wage demands on the firm's policy.

The perspective of this book is imbedded in the realm of optimal control models dealing with the dynamic theory of the firm. To provide the reader with past and traditional research in this field, we discuss the approaches that other authors have used in structuring and explaining the firm's agents such as: employees, shareholders and government.

A more technical contribution of this work is our design of an iterative procedure to construct optimal solutions of optimal control models containing state and control constraints. This result is more general than current procedures.

2. Outline of the book.

The remaining sections of this chapter will be devoted to some observations about dynamic and analytical aspects of the theory of the firm.

In chapter 2 we shall demonstrate how several authors have modelled different aspects of the dynamics of the firm. These aspects are organized on the basis of different sets of agents that are connected to the firm, such as shareholders, employees, competitors and government. In this way, the reader may get an overview of the main themes studied in dynamic theories of the firm as well as of the ingenuity by which the relevant authors have succeeded in formulating different relationships inside the firm and between the firm and its environment and make it manageable in a dynamic analysis.

After some familiarization with the quantitative formulations in chapter 2, we study optimal solutions in chapter 3. This is done on the basis of four dynamic models of the firm that are presented in great detail (together with the relevant optimal solutions). We selected those models, as they can be conceived as predecessors of the main model of the thesis, presented in chapter 4. Further, each of these models contains some new features that are important in understanding the analysis in the following chapters.

In order to facilitate the reading of chapter 3, we have presented the mathematical derivations in appendices 1 and 2. In appendix 1, the reader is introduced into the realm of the Maximum Principle. This Principle yields conditions for optimal solutions of dynamic models. Much attention is paid to a careful description of the effects of control and state constraints. These constraints are inevitable in dynamic models of the firm, as they deal with the pervasive problems of scarcity. They complicate the optimality conditions and the search for an optimal solution, however. For this kind of models, the formulation of the Maximum Principle as presented by Russak, 1970, is suitable. On the basis of this formulation we have designed an iterative solution procedure in appendix 2. Up to now, such a procedure has not been published elsewhere. This procedure has been applied to the relevant models of chapter 3 in order to produce the optimal solutions dealt with in that chapter.

After this more mathematical excursion, the reader returns to chapter 4, in which we present an extended dynamic model of the firm. In this model we have incorporated financing, production, investment and the firm's dividend policy. The production process is described by means of activity analysis which, although well known, is newly

incorporated in a dynamic model of the firm. Activity analysis is closely related to the way in which management often solves production planning problems in reality. A second advantage of this formulation over, say, a continuous production function, will be outlined in our study of depth investments in chapter 5. Another new and important feature of our study is the twofold influence of government on the firm's policy. Both corporation profit taxes and investment grants are incorporated. Since our model contains production as well as financing, we can also study how the direct impact of investment grants on the financial position of the firm will indirectly influence the employment policy of that firm. This is of course essential to investigate the way in which investment grants may raise employment. Links between well known financial records and the model presented are also discussed.

In appendix 3, we describe how the optimal solution of the above model can be found with the help of the new procedure as presented in appendix 2.

Chapters 5 and 6 contain a description and further analysis of this optimal solution. By properly handling constraints on the parameters, we get four different sets of optimal solutions (chapter 5). Each of them can be represented by a "master trajectory" of the firm that holds under given constraints on the parameters. Moreover, the sequence in the presentation is such that each new master trajectory contains new features, compared with the previous master trajectories. We will discuss features such as: stationary and growth stages, initial conditions on the state of the firm, depth investments and consolidation. The links with traditional static theory of the firm are pointed out as well.

Finally, in chapter 6, we discuss in more detail the sensitivity of the optimal solution to the parameters of the model. For that purpose, three optimal policy rules are derived, concerning production, finance and investment. For each policy rule, the impact of the relevant parameters is studied. Further, six different ways of influencing the optimal solution are derived, for example dealing with the rate of growth and with several threshold values of output. For each such way we will also discuss the impact of the relevant parameters. Finally, we will study the global influence of three distinct sets of parameters (governmental, financial and social) on the master trajectories. A sensitivity analysis as presented in chapter 6, is not commonly used in the literature and clearly shows its importance for a better understanding of the model and its solution and therefore for the understanding of the firm.

3. Relevance of dynamic theories of the firm.

The dynamic theory of the firm is motivated by three issues: the need for policies, the contribution of deductive analysis and the need to incorporate time. The need to study policies of firms hardly needs any explanation, because firms are an important

group of economic agents having much influence on society.

We can distinguish between two ways of studying the firm: inductive studies, deriving laws from inquiries and statistical data, and deductive studies, deriving laws from a set of assumptions (mostly through constructing models and analysing solutions). This book follows the latter method in trying to build "sounder methodological bridges" (Vickers, 1968) from micro economics to business economics. This kind of bridges is an indispensable instrument to improve management's understanding of favorable policies, to enable government to assess the impact of its policy on the firm and to provide academic teachers with an instrument to outline the essentials of the firm (Lesourne & Leban, 1981).

Finally, the effects of time and the interrelatedness of economic states are useful in the theory of the firm. Wildsmith, for example, while quoting a statement of Hicks, argued:

In mechanics, statics is concerned with rest, dynamics with motion, but no economic system is ever at rest in anything like the mechanical sense (Wildsmith, 1973, 31).

Furthermore, Tapiero stated:

Managers typically reach decisions in a perspective of time and in the light of temporal criteria (Tapiero, 1978).

So, time is obviously essential to the policy of the firm and introduction of time increases our understanding of the firm. Further it can lead to insights that may not be obtained through other methods such as static analysis.

4. Origins of dynamic theories of the firm.

In the last two decades, new insights regarding the theory of the firm have been gained, due to new instruments, including the Maximum Principle (Pontryagin e.a., 1962) and Dynamic Programming (Bellman, 1957). Prior to these innovations, the theory of the firm dealt with time by means of comparative statics and so-called equilibrium growth models (Baumol, 1962) or steady state growth models (Marris, 1963). These theories may be styled as theories of mechanical growth processes (Albach, 1976).

The new mathematical instruments, mentioned above, enabled economists to describe the growth of a firm in a more satisfactory way, as a stimulus originating from a decisionmaking process within the firm (Ludwig, 1978, 14). This stimulus is constrained by the structure of the firm (for example: its production frontiers) and by the environ-

ment in which the firm operates (for example: governmental regulations and capital rationing).

Recent surveys and text books, such as: Tapiero, 1977, Sethi & Thompson, 1981, and Kamien & Schwartz, 1981, show the great progress of dynamic analysis in the theory of the firm during the past two decades.

5. Nature of the theory of the firm and of this book.

In the dynamic theory of the firm, the emphasis lays on general laws of motion on an aggregate level which, in spite of their general nature, leave room for differences between single firms. These differences are told to stem from the different circumstances with which each firm has to cope, apart from general principles that hold for all of them. The same underlying process may thus lead to different types of growth.

As a consequence of this preference for general laws, the economists are required to keep models as simple as possible so that analytical solutions can be derived. This raises the question of which aspects should be incorporated in the model and which not. Publications in the dynamic theory of the firm are usually dealing with a few aspects of the firm's policy. For example, there is some research in financing (e.g.: Ludwig, 1978, Sethi, 1978, Verheyen, 1981, Van Loon, 1981), advertising (Sethi, 1977, Tapiero, 1978), employment policy (Leban, 1982), research and development (Feichtinger, 1982), inflation (Lesourne & Leban, 1977) and interaction between competitors (Levine & Thépot, 1982). We will discuss these themes in greater detail in the next chapter. Based on the solution procedure discussed in appendix 2, we can deal with more complex models, so we are able to increase the number of aspects of the firm's policy put into a single model. Beside this extension of the "span of control" of the theory of the firm, we will lower the level of aggregation as far as production is concerned by introducing activity analysis. In this way, we take a step forward on Vicker's bridge from the theory of the firm towards business economics.

We will finish this section by presenting some ways of describing growth and the policy of the firm in reality, so that the reader will have a fair idea of the complexity of dynamics of the firm in reality. For, in the theory of the firm, which we will enter in the next chapter, the growth process will be described in only one or two dimensions. In reality, growth and the dynamic policy of the firm are much more complex phenomena. Ludwig, for example, mentioned the following alternative ways of describing growth (and contraction) processes (Ludwig, 1978, 23):

Strategy	: expansion, diversification, contraction.
Market structure	: market penetration, market development, product development, reduction of the range of products.

Direction of growth : horizontal, vertical.
 Kind of growth : internal, external.
 Instruments : own initiatives, cartels, licences, buying, merging.

Further, the complexity of the dynamic policy of the firm in the range of financial growth patterns has been described by Huret, who uses annual reports of 522 French firms (Huret, 1975):

Industrial expansion : considerable expansion of fixed assets financed by long term debt.
 Commercial expansion: decreasing portion of fixed assets, compensated by increasing liquid assets, considerable expansion of short-term creditors.
 External expansion : growth of interests in associated companies equals growth of fixed assets.
 Equilibrium growth : conserving an invariable structure of the balance.
 Defensive strategy : increase of current liabilities without a corresponding change of the structure of assets.
 Decline : stagnating level of liabilities and a decreasing level of fixed assets.

The last example shows a classification of growth processes based on strategic policies (Kieser, e.a., 1977):

Market penetration : price policy and sales stimulating activities under constant demand.
 Market expansion : unlocking new markets through variations of the product mix or regional expansion.
 Vertical diversification : extending the product range through products (and/or services) upward or downward situated in the production column.
 Horizontal diversification : extending the product range through products allied to the present range.
 Concentrical diversification: extending the product range through products that are only weakly related to the present range in a technical or commercial sense.
 Conglomerate or portfolio diversification : extending the product range through products which are not related to the present range, neither in a technical nor in a commercial sense.

As contrasted with these less aggregated descriptions, in the dynamic theory of the firm, growth is mostly measured in terms of an increase (or decrease) of money capital, assets and/or employment.

6. Summary.

This chapter is meant as an introduction to dynamic analysis in general and to growth in particular. Apart from the intentions of the author and an outline of the book, the contribution has been discussed of a dynamic, analytical and theoretical treatment of the policy of the firm. Optimal control theory appears to enable research in this way but, like every instrument for economic analysis, it has its limitations, mainly in the area of the complexity of the models considered. After dealing with the nature of the theory of the firm, this chapter ends with the presentation of some descriptive studies in order to give an idea of the complexity of growth processes of the firm in reality.

CHAPTER 2. A SURVEY OF DYNAMIC THEORIES OF THE FIRM.

1. Introduction.

In section 5 of the introduction chapter we have pointed out several aspects of the dynamics of the firm that have been studied. Each aspect has its own merits and it seems useful to consider the whole area briefly, before concentrating on the subject-matter of this research: the relation between investment, financing and production policies.

We owe much to earlier surveys such as: Ludwig, 1978, Nickell, 1978, Jørgensen, 1980, Lesourne & Leban, 1982 and: Ekman, 1978. Our viewpoint, however, is different from all of them: we will present the subjects covered by research in the field of dynamics of the firm mainly in the light of the parties concerned. The interactions between these interest groups are the origin of the dynamics of the firm. The behaviour of each such group can be put into a dynamic model of the firm in different ways: in the goal function, or as a constraint put upon the firm's policy or as a (dynamic) relationship between some entities that are important to the firm's position. Below we present the subjects that we will discuss with the section numbers to match:

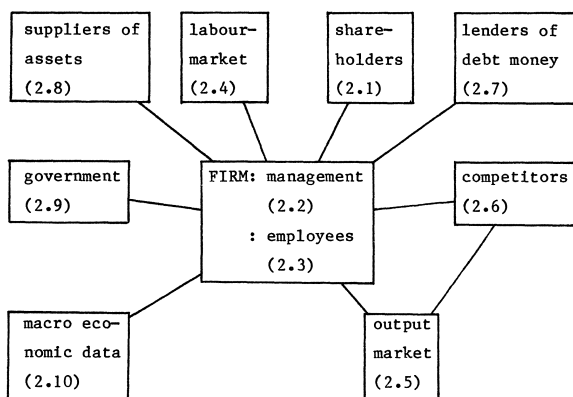


Figure 2.1. Scheme used for surveying research into the dynamics of the firm.

The following is no attempt to cover all material published, because we only intend to show examples of dealing with the different aspects of the theory of the firm.

2. Shareholders.

In the part of the theory of the firm concerned with financial problems of the firm, shareholders often act as dominant goal setters. In this case the firm is supposed to act as if it maximizes its value as conceived by its shareholders. The firm's value is mostly defined as the capital value of the dividend flow (Lesourne, 1976) or the capital value of the cash flow (Jorgenson, 1973) over an infinite period of time. When a finite planning horizon is introduced, the discounted value of the firm at the end of the planning horizon stands for all future returns to equity. This salvage value may be a function of the value of final equity (Krouse & Lee, 1973, Sethi, 1978), or, more specific, the discounted value of final equity (Ludwig, 1978):

$$\text{maximize: } V(0) = \int_0^z e^{-iT} D(T) \, dT + e^{-iz} X(z)$$

in which: D(T): dividend

V(0) : value of the firm (for the shareholders)

X(T) : equity

T : time, $0 \leq T \leq z$

i : time preference rate of the shareholders

z : planning horizon

Furthermore there are publications concerning changes in the group of shareholders. In these publications, the objective of the firm is defined as: maximizing the value of the firm as conceived by the present shareholders. Issuing new shares may be free of charge (Elton & Gruber, 1977) or subject to floatation costs (Senchack, 1975). In the latter case, the value of the firm is formulated as follows:

$$\text{maximize: } V(0) = \int_0^{\infty} e^{-iT} \{ D(T) - (1-n)M(T) \} \, dT$$

in which: V(0) : value of the firm for shareholders present
at T = 0

M(T) : new equity issued

n : floatation costs per unit new equity

Note that the value is maximized over an infinite time period.

3. Management.

Managers are generally supposed to pursue power, prestige, income etc. If they are the dominant party within the firm, and supposed that they are not the owners, the firm will try to maximize growth, mostly in terms of discounted sales. This objective, however, mostly appears in combination with a restriction on the minimal amount of dividends to be paid out, or on the minimal profit level per unit equity to be main-

tained (Leland, 1972):

$$\text{maximize } V(0) = \int_0^{\infty} e^{-i'T} R(T) \, dT$$

$$\text{subject to: } R(T) - wL(T) - p_{\min} K(T) \geq 0$$

in which: $L(T)$: labour

$K(T)$: stock of capital goods

$R(T)$: return on sales

i' : time preference rate of management

p_{\min} : minimum revenue per unit capital invested

w : wage rate

The reason for this is, that managers must safeguard the flow of profits to finance further growth. Moreover, maintaining a certain market value is important to avoid take-overs, possibly ending the realm of the present management in the firm.

Another way of formulating the tension between management and shareholders is the introduction of a utility function which contains (discounted) sales and flow of profits as arguments (Ekman, 1982).

Also in studies of profit maximizing firms we may find a role played by the management. In this case it concerns the limited capacity of the staff to plan and execute expansion investments. This limited span of control may be formulated in several ways, for example, as an upperbound level of the growth of assets (Krouse & Lee, 1973):

$$\frac{\dot{K}}{K} \leq \omega, \text{ in which: } \dot{K} \equiv \frac{\partial K}{\partial T} \text{ and } \omega \text{ is constant.}$$

Note that ω is independent of the level of assets, which is in conformity with Gibrat's law of proportionate effect. Another formulation has been given in: Söderström, 1977, in terms of the division of labour in the firm (conceived as a "production team") between production and team formation. The task of the "team formation department" is to supply sufficient new labour forces, in order to catch the decrease of labour and labour productivity:

$$Q(T) = Q(L(T) - L_f(T))$$

$$\dot{L} = a_1 L_f(T) - a_2 L(T)$$

in which: $L(T)$: stock of labour
 $L_f(T)$: labour assigned to the "team formation department"
 $Q(T)$: level of production
 a_1 : production intensity of the team formation department
 a_2 : productivity decay and quit rate.

Jorgenson gives a similar formulation in terms of the division of capital good services between production of output and installation of investment goods (Jorgenson, 1973).

Finally, the restricted management capacity has been formulated as a decrease of production capacity through internal adjustment costs (Treadway, 1970):

$$Q(T) = Q(K, L, \dot{K}) \text{ with: } \frac{\partial Q}{\partial \dot{K}} < 0$$

in which: $Q(T)$: production capacity

The idea behind this formulation is that growth demands planning capacity, which is drawn from the production planning capacity of the staff.

Note that the above production functions are no longer production functions in the sense of the technical relationship between output and input of production factors. At the firm's level, production capacity appears to be determined not only by technical relations but also by the organisation structure that enables production. So, the production function of a firm is a behavioural relationship (Jones, 1973, 183).

4. Employees.

Employees are staged in two different roles in the relevant literature: as one of the input factors and/or as the dominant participating party in the firm's decisions. When labour is represented as a separate input factor (most publications suppose output, or revenue, to be a function of assets only) the production technology of the firm is described by a neo classical production function (Wong, 1975) or by activity analysis (Van Loon, 1982). In both cases, labour is perfectly adaptable while changes in the amount of capital goods are restricted for technical (depreciation) and financial reasons. Lesourne & Leban introduced labour as an input factor of the quasi-fixed type, due to a restriction on the firing rate (Lesourne & Leban, 1978):

$$\dot{L} \geq -nL, \text{ in which: } n = \text{maximal quit rate, based for example on an agreement with the unions.}$$

Many authors have studied the other way of including labour: as the ruling party in so called "labour managed" firms. In this type of firms, "labour receives the residual

revenue after the other input factors, including capital, have received their predetermined remuneration" (Ekman, 1980). In this kind of models, the firm maximizes income per employee (Jørgensen, 1982):

$$\text{maximize: } V(0) = \int_0^{\infty} e^{-i''T} \left[\frac{S(T) - aK(T)}{L(T)} \right] dT$$

in which: $V(0)$: value of the firm for employees
 i'' : time preference rate of employees
 a : fixed remuneration of capital

If one allows for changes in the group of employees, working in the firm during the period under consideration, the same problems arise as in the case of the shareholders-managed firm.

5. Labour market.

Most publications postulate a perfect labour market, which implies a constant wage rate and perfect adaptability:

$$w = p \frac{\partial Q}{\partial L}, \text{ where } w \quad : \text{ wage rate (constant)}$$

$$p \quad : \text{ selling price}$$

$$Q(L) : \text{ output (= sales volume)}$$

Imperfections in labour markets were mostly explored by French scholars. Beside imperfections due to a restriction on a firing policy, mentioned in the previous section, Leban has studied, for instance, the employment policy of the firm in the tradition of Salop (Salop, 1973). There, the supply of labour and the natural quit rate of employees out of the firm are supposed to depend on the wage rate. This wage rate may be exogenously given or it may be a policy variable of the firm (Leban, 1982):

$$\dot{L} = \{Z(T) - E(T) - N(W)\} L(T)$$

$$Z(T) \leq U(W)$$

in which: $E(T)$: firing rate

$N(W)$: natural quit rate, with $\frac{\partial N}{\partial W} < 0$

$U(W)$: applying rate of labour,

$$\text{with } \frac{\partial U}{\partial W} > 0 \text{ and } \frac{\partial^2 U}{\partial W^2} \leq 0$$

$W(T)$: wage rate

$Z(T)$: recruiting rate

In this model, Leban further supposes a linear firing cost function and a linear hiring cost function.

6. Output market.

The descriptions of the output market can be divided into two categories. To the first category belong publications describing the output market as behaving in accordance with a fixed supply-price function. The firm offers a certain amount of output and receives a price that comes about through the price setting mechanism. The market may be a perfect one, i.e. the price does not change when the amount of output of the firm varies (Takayama, 1974, 685) or an imperfect one, having a decreasing price-sales function (Nickell, 1974).

Publications in the second category focus on the influence of the marketing instruments of the firm on the demand for the firm's output. In marketing as well as in economics, there is a long tradition of studies concerning such subjects as advertising, pricing, product policy, distribution, sales force etc. (see Horsky & Sen, 1980). Optimal Control models of the firm's marketing policy mostly unite these instruments by introducing the notion of advertising expenses, "including sales promotion, product improvement, product quality, or, in general, most of the firm's internal expenses that shift the demand curve of the firm" (Ekman, 1980). We shall briefly instance advertising models. In the class of so called sales-response, or diffusion, models, these advertising expenses work in a direct way on the volume of sales (given a constant selling price). The first publication in this tradition is Vidale & Wolfe, 1957:

$$\dot{S} = a_1 A(T) \left\{ 1 - \frac{S(T)}{d} \right\} - a_2 S(T)$$

in which: A(T): advertising expenses

S(T): sales (volume)

a_1 : response parameter

a_2 : decay parameter

d : total demand of the market

The above formulation implies a saturation effect: when the firm approaches a market share of 100% (so S(T) approaches d), the effectiveness of advertising expenses falls down. The decay parameter a_2 reflects the decreasing effectiveness of an advertising expenditure in course of time. Luptacik and Feichtinger solved a sales response model in which the firm has two marketing instruments: advertising expenses and selling price (related to the average price on the market): Luptacik, 1980, and: Feichtinger, 1980.

The second way of describing the influence of advertising expenses on demand uses a carry over effect through the increase of goodwill. These models are called: advertising capital models. The first model in this tradition is: Nerlove & Arrow, 1962:

$$\dot{B} = A(T) - a_3 B(T)$$

$$S = S(P(T), B(T)) \text{ or: } P = P(Q(T), B(T))$$

in which: $B(T)$: goodwill of the firm

a_3 : decay parameter of goodwill

In this model, the firm also has two instruments to influence demand: advertising expenses and selling price. Advertising expenses raise goodwill and so, ceteris paribus, sales. But through a_3 an opposite effect is built in, based on forgetting by consumers. Tapiero has introduced probabilistic aspects of advertising and forgetting into the Nerlove & Arrow model (Tapiero, 1977). The attitude of the firm towards risk becomes a new element in determining the optimal policies of the firm.

A third group of publications to be mentioned here deal with (uncertainty in) the changing demand expectations due to business fluctuations. They study its impact on the policies of the firm concerning investments (Nickell, 1978, Thépot, 1980) and employment (Leban & Lesourne, 1980). It is worth while to consider one of the conclusions of the last study mentioned, in which optimal anticipation policies are described to realize the importance of dynamical analysis: "an (optimal) anticipation of a recession may be sufficient to generate a recession", indeed, a striking conclusion.

7. Competitors.

In this section we will deal with models that describe competitors, reacting through their own marketing policies and instruments. We do not consider competitive behaviour through price-supply functions, which we have already discussed in the first part of the previous section. In optimal control models of the firm, more detailed descriptions of competitor's behaviour are introduced by extending marketing models. The idea behind this is, that the effectiveness of the advertising and pricing policy of the firm is affected by the advertising and pricing policy of its competitors. Although, for example, Tapiero, 1979, dealt with a multi-firm situation, most models still deal with a duopoly situation: two firms operate on the same market. The eldest formulation stems from Kimball and is in the tradition of the sales response models (Kimball, 1957):

$$\dot{S}_1 = a_{11} A_1(T) S_1(T) - a_{12} A_2(T) S_1(T)$$

$$\dot{S}_2 = a_2 \cdot A_2(T) \cdot S_1(T) - a_1 \cdot A_2(T) \cdot S_2(T)$$

$$S_1(T) + S_2(T) = d$$

in which: $A_j(T)$: advertising expenses of firm j

$S_j(T)$: sales of firm j

a_j : interaction parameters, $a_j > 0$

d : maximum sales potential

The first term of the dynamic sales equations explains that sales will increase when advertising expenses are increased and that the effectiveness is greater according as the market share of the opponent is greater. The second term states that it is hard to keep up sales level if one has already attained a high market share (decay component) and if the competitor is advertising in an intensive way (competitive pressure). The third equation deals with the limit of total market demand and so introduces diminishing returns to advertising.

A formulation of the above problem in the tradition of advertising capital models may be found in Thépot, 1981.

Besides differences in describing the influence of advertising expenses on sales, we should mention differences in reacting to the activities of competitors. We can distinguish between open- and closed-loop reaction patterns and between cooperative and non-cooperative situations.

In an open-loop situation, both firms are completely informed about each other. The question is to find the optimal policy, given the policy of the competitor over the whole period, for instance:

$$A_j^* = A_j^*(T)$$

in which: A_j^* : optimal advertising policy of firm j

There are publications that deal with open-loop solutions of non-cooperative situations (Sethi, 1975) and of cooperative situations (Leitmann, 1974).

In a closed-loop situation, competitors know each others present position and have to make their decisions based on this partial information:

$$A_j^* = A_j^*(S_j(T), S_i(T), P_j(T), P_i(T)), \quad i \neq j$$

One can imagine that this description does not fit cooperative situations, in which exchange of information is supposed. An example of a solution of the non-cooperative, closed-loop type is given by Case (Case, 1979, 198).

Levine & Thépot studied open-loop as well as closed-loop solutions in a joint investment policy and price setting model (Levine & Thépot, 1982). Finally, Tapiero has presented an example of uncertainty introduced in a competitive advertising model (Tapiero, 1979).

All the above publications deal with firms competing on the output market. Jørgensen deals with a situation where two firms compete in the labour market by offering different ways of paying for labour: a profit maximizing firm offers a fixed reward to labour while a labour managed firm offers a reward based on profit per unit labour after payment of a fixed price to capital services (Jørgensen, 1982).

So far, we have dealt with equilibrium patterns or movements towards an equilibrium, that rarely imply the exit of one of both firms. Feichtinger has dealt with a kill or cure situation in which both competitors are developing the same new product (Feichtinger, 1982). The firm that enters the market first, will carry off the loot. Which firm will be successful, depends on its intensity of research efforts, in relation to the competitor's intensity.

8. Lenders of debt money.

Lenders of debt money do not participate actively in the management of the firm. They plead their interests by making conditions on loans in such a way as to minimize risk or by claiming rewards proportional to their risk bearing. These two formulations turn up in publications where borrowing is treated as a means of financing the activities of the firm.

The former formulation mostly yields a fixed interest charge and an upperbound to the amount of debt money available to the firm. This upperbound may be on new debt as a function of the cash flow (Lesourne, 1973, 222) or of the investment expenditures (Ludwig, 1978, 92), or it may be formulated as an upperbound to the total amount of debt as a (linear) function of equity, implying a maximum leverage (Lesourne, 1973, 206).

In the models mentioned above, the firm is not allowed to exceed this upperbound and so to pass into another risk class (that is: a subset of firms having the same risk as judged by investors with its related (higher) interest rate). In the second formulation, the firm is allowed to invest in such a way that its risk profile changes. Authors dealing with this assumption formulate the demanded interest rate as a function of the leverage (Senchak, 1975) or as a function of the total amount of debt (Hochman e.a., 1973).

9. Suppliers of assets.

In this section we will restrict ourselves to the market of fixed assets. Most authors suppose a perfect market of capital goods, where the firm can buy its assets at fixed prices. To facilitate the analysis the value of a capital good is fixed on one unit of money. If one further supposes absence of inflation and of technical progress and if one equalizes technical deterioration rate and depreciation rate, then the value of the amount of capital goods in the firm equals the number of capital goods. This simplifies the joining of investments and production capacity (Treadway, 1970).

Several authors have studied the case of an imperfect market of capital goods in the framework of so called adjustment cost models. Each firm is supposed to have an optimal size, but a firm does not immediately adopt that size because of costs inherent to the adjustment process. These costs are divided into external adjustment costs (investment expenditures) and internal adjustment costs (seize on available productive inputs) (Söderström, 1976):

$$W(T) = p \cdot Q(K(T), L(T), I(T)) - w \cdot L(T) - C(I(T))$$

$$I(T) = \dot{K} - a \cdot K(T)$$

$$\text{with: } \frac{\partial Q}{\partial I} < 0 \text{ (internal adjustment costs)}$$

$$\frac{\partial C}{\partial I} > 0 ; \frac{\partial^2 C}{\partial I^2} < 0 \text{ (external adjustment costs)}$$

in which: $W(T)$: cash flow

$I(T)$: gross investments (capital goods)

$C(T)$: cost of investment

a : depreciation rate

If C'' ($\equiv \partial^2 C / \partial I^2$) equals zero, we are on a perfect market of capital goods. A positive C'' (the case of so-called "convex adjustment costs") applies to a monopsonistic market of capital goods: if the firm wants to increase its rate of growth it will be confronted with increasing prices on the market because of its increased demand of

capital goods (Intriligator, 1971, 202). A negative C'' may represent the case of a low investment level. Then, "economies of growth" (Penrose, 1959, 99) may appear when buying more (or bigger) capital goods.

Another important aspect of the supply of capital goods is technical progress. In economics a distinction is made between embodied and disembodied technical progress. Disembodied progress "applies equally and alike to all resources of men and machines in current use" (Allen, 1968, 236). Embodied technical progress "applies... only to certain tranches of capital equipment, usually machines produced and installed currently, together with the associated labour crews" (Ibidem).

In the theory of the firm embodied technical progress has mostly been introduced in the framework of maintenance models, concerning optimal maintenance and scrapping of capital goods (Bensoussan e.a., 1974, 107). In Nickell, 1978, 127, output is related to investment and maintenance policy under embodied technical progress. Variable (more specifically: increasing) labour productivity is dealt with by Virmany who left maintenance costs out of consideration but combined embodied and disembodied technical progress in the following way (Virmany, 1976):

$$V(0) = \int_0^{\infty} e^{-iT} \{pQ(T) - wL(T) - cI(T)\} dT$$

$$\dot{Q} = H(I(T), Z(T), T) - (x \cdot a_1 - a_4)Q(T)$$

$$\dot{L} = Z(T) - a_1 L(T)$$

in which: $H(I, Z, T)$: plant-choice function
 $Z(T)$: recruitment of labour
 x : degree of homogeneity of H
 a_1 : depreciation rate = rate of technical decay
 a_4 : disembodied technical progress
 c : price of a capital good

The plant-choice function represents the range of techniques and sizes of plants from which the firm can choose at a given point of time. Through the argument T , embodied technical progress is incorporated in the production function. Once the techniques have been chosen and incorporated into a plant of the firm, they remain fixed for the life of the plant (a so-called putty-clay model (Nickell, 1978, 247)). Disembodied technical progress is assumed to manifest itself as an increase of output over time at a constant exponential rate a_4 , so:

$$Q_J(T) = e^{a_5 T} \cdot H\{I(J)e^{-a_6 T}, Z(J)e^{-a_6 T}, J\}$$

$$\begin{aligned} \text{in which: } a_5 &= a_4(T-J) \\ a_6 &= a_1(T-J) \end{aligned}$$

from which the above Q-formula may be derived. Note that the use of a certain vintage decreases over time due to the obsolescence of machines (a_1) and labour productivity increases due to the disembodied technical progress (a_4), which may contain a learning-effect.

10. Government.

The influence of government on the policy of the firm is mostly studied through analysing the influence of corporation profit taxes. Emphasis is frequently laid on the "neutrality" of the tax systems: is there a change in the (supposed efficient) allocation of factor inputs if the government introduces a certain corporation tax system. Or, within the context of dynamic models of the firm: does a certain corporation tax system influence the capital accumulation process and the relative inputs of labour and capital?

The relevant features of tax systems are: the level of the corporation tax rate, tax treatment of depreciation and interest payments, and investment incentives (including investment allowances as well as initial allowances as a special kind of accelerated depreciation). Boadway has studied them for the input factor capital in the optimum equilibrium state of the firm (Boadway, 1980). Muzondo also studied the influence on the policy of the firm towards that equilibrium and on the input of labour (Muzondo, 1979). In his model, adjustment costs are introduced, thus reflecting the quasi-fixedness of capital goods in the short run:

$$\text{maximize: } V(0) = \int_0^{\infty} e^{-iT} W(T) dT$$

$$\text{with: } W(T) = (1-f) \{ pQ(T) - wL(T) \} - c[I(T) - fa'K'(T)] + frY(T)$$

$$I(T) = \dot{K}'(T) + a'K'(T) = \dot{K}(T) + aK(T)$$

$$Q(T) = Q(K(T), L(T), \dot{K})$$

in which: I(T) : investments (capital goods)
 K(T) : economic level of capital stock (capital goods)
 K'(T) : accounting level of capital stock (capital goods)
 W(T) : cash flow after tax
 a : economic depreciation rate

- a' : depreciation rate permitted by the tax law
 (accounting depreciation rate)
 f : corporation tax rate
 p, w, c : given functions of T

In stead of taxing the profit of a firm, the government may tax the cash flow of a firm. This tax system "has been widely recommended as a form of taxing corporate income which is neutral with respect to investment decisions" (Sandmo, 1979). This author points out that this is the case only if the tax rate is constant over time.

Finally, we should mention of research regarding the influence of personal tax systems on the value of the firm as conceived by the shareholders. The relevant value, neglecting differences in corporation tax rates on retained earnings and on distributed profits, may be formulated as follows (Ylä-Liendenpohja, 1978):

$$V(0) = \frac{(1+a_1)(1-a_2)}{1-a_3} \int_0^{\infty} e^{-i'T} D(T) dT$$

$$k = - \frac{i+a_4}{1-a_3}$$

- in which: a_1 : rate of credit for shareholders under the imputation system
 a_2 : marginal income tax rate
 a_3 : effective tax rate for capital gains on an accrual basis
 a_4 : marginal tax rate of personal wealth
 i' : discount rate of the "shareholders-managed" firm
 i : shareholders discount rate (after taxes)

Under the imputation system, shareholders are taxed for the amount $(1+a_1)D(T)$. But the amount $a_1D(T)$ is considered to have been paid by the firm, thus the shareholders pay a tax of $a_2(1-a_1)D(T) - a_1D(T)$, so dividends, net of tax, amount to $(1+a_1)(1-a_2)D(T)$.

The conclusions that can be drawn from the above formula's are, that parameters which determine the degree of double taxation of dividends (a_1 and a_2) do not influence the optimal policy of the firm and, that if the firm takes into account the personal taxes imposed on its shareholders, it will raise its discount rate.

11. Macro-economic data.

Two kinds of macro-economic data have been discussed in previous sections. In section 2.5 we have mentioned the influence of business fluctuations in the framework of studies on the output market of the firm. In section 2.9 we have reviewed research on tax systems.

In addition we can mention sensitivity analysis on parameters such as the discount rate (Oniki, 1973).

The influence of inflation has been studied by several authors. Inflation brings up the problem of the valuation of stocks and the depreciation policy. Lesourne & Leban have incorporated inflation in the next way (Lesourne & Leban, 1977):

$$\dot{X} = (1-f) \{ P(T) Q(K'(T)) - aK(T) - rY(T) + a_1 S(T) \} - D(T)$$

$$\text{with: } P(T) = e^{a_1 T} \cdot p$$

$$S(T) = e^{a_1 T} \cdot p \cdot (a_2 K'(T))$$

$$\dot{K} + ak(T) = (K' + aK') e^{a_1 T}$$

$$K(0) = p C(0)$$

in which: $K(T)$: accounting value of capital
 $K'(T)$: capital stock (units of capital good)
 $S(T)$: value of stocks
 a_1 : inflation rate
 a_2 : assumed fixed proportion between production capacity and stocks.

The authors assume depreciation allowances to be based upon historical costs and full taxation of inflationary gains on stocks. So, the tax collector's office does not support the ideas of replacement value theory. Boadway further studied the impact of replacement cost depreciation and features of neutral tax systems in the case of inflation (Boadway, 1980).

Finally, we mention Lesourne's publication, dealing with a growing environment (Lesourne, 1976). His assumptions were: decreasing returns to investment in a stagnant economy and the appearance of new investment possibilities with the growth of the economy:

$$W(T) = W'(I') \cdot I(T), \text{ with: } \frac{\partial W'}{\partial I'} < 0$$

$$I'(T) = e^{-a_1 T} \cdot I(T)$$

in which: $W'(I')$: average cash flow per unit net investment

$I(T)$: investments of the firm

$I'(T)$: relative investments of the firm (relative to the macro-economic growth rate)

a_1 : macro-economic growth rate of investments

So, in the case of a constant I , thus decreasing I' due to the fact that the firm's investments fall behind the macro growth rate a_1 , marginal cash flow increases over time.

12. Summary.

In this chapter we surveyed aspects of firm's policy that have been studied within the field of the dynamic theory of the firm. The angle of incidence was: the behaviour of the parties that have an interest in the firm. Inside the firm we discussed the management and the employees. Outside the firm we dwelled upon the suppliers of assets, labour, equity, debt and public services, upon colleagues of the firm and upon buyers of the firm's output. Finally, we have touched upon macro-economic data that influence the policy of the firm. The descriptions of the behaviour of the relevant parties are expressed through the formulation of the maximand as well as of the restrictions and technical relations of the models concerned.

CHAPTER 3. SOME PREDECESSORS.

1. Introduction.

In this chapter we will deal in greater detail with some models that have been considered by other authors and that will be expanded in the next chapters. This may familiarize the reader with alternative ways of modelling the salient features of the firm and with their impact on the optimal trajectories.

We shall first outline the classical model of Jorgenson, 1967, and then we will present a model of Leland, 1972, who included first aspects of production as well as of financing. Furthermore we will present the models of Ludwig, 1978, and Lesourne & Leban, 1978, as examples of the more sophisticated models, published recently.

Emphasis in this chapter is put on modelling and on the model's impact on optimal solutions, and not on a detailed economic analysis of the optimal solution. This approach is common to most publications within the relevant field of research. We will leave that tradition when we deal with our own model.

We suggest for the reader unfamiliar with Optimal Control Theory, to read first appendix 1. In this appendix, conditions for the type of problems with which we will deal in the rest of this book, are described in a narrative way. In appendix 2, the reader can find in greater detail how the following models can be solved by means of an iterative procedure that we have designed, based on the Maximum Principle. We have separated this part from the main text in order to bother as little as possible those readers who are not interested in the mathematical foundations of the economic theories presented here.

2. Investments and depreciation (Jorgenson).

In fact, investment policies can only be described realistically in a dynamic way. It is the process of sacrificing purchasing power now to revenues later on ("breeding effect"). Jorgenson (in: Jorgenson, 1963, and 1967) was among the first economists to present this problem in a dynamic framework. He describes a firm, maximizing its revenue over an infinite period of time. To compare revenue flows of different intervals within the planning period, they are discounted at a rate i , representing the time preference rate of the owners of the firm.

The firm produces one kind of product and sells it on a perfect market, so the selling price is constant. Further, the firm uses two kinds of inputs: labour and capital goods. Both are obtained on perfect markets, so the wage rate and price of capital goods are fixed, too. This results in the next formulation (we drop obvious argu-

ments):

$$\text{maximize}_{I, L} : \int_0^{\infty} e^{-iT} R(K, I, L) dT \quad (1)$$

in which : $R() = p Q(K, L) - wL(T) - cI(T) = \text{revenue flow}$

$Q()$: output = sales volume

$I(T)$: (gross) investments

$K(T)$: stock of capital goods

$L(T)$: employment level

T : time

c : price of capital good

i : discount rate of the shareholders

p : selling price of output

w : wage rate

The impact of investments on the production structure is described by the, now generally used, formulation of net investments:

$$\dot{K} = I - aK \quad (2)$$

in which : a : depreciation rate.

The assumption that current depreciation requirements depend only on the current level of the stock of capital goods in a proportional way holds, for example, if the stock of capital goods is depreciating at an exponential rate and the stock of capital goods is constant or increases at a constant rate (Jorgenson, 1967). Although most dynamic models of the firm yield non-constant growth rates of the capital stock, formula (2) is still used because of its simplicity. But we will see, that the Jorgenson model results in a stationary level of the capital stock, so the conditions for (2) are fulfilled, assuming that the firm depreciates its capital goods at an exponential rate.

Jorgenson further assumes that the production is an increasing, concave function, which implies decreasing returns to scale:

$$\frac{\partial Q}{\partial K} > 0 ; \frac{\partial Q}{\partial L} > 0 ; \frac{\partial^2 Q}{\partial K^2} < 0 ; \frac{\partial^2 Q}{\partial L^2} < 0 \text{ and } \frac{\partial^2 Q}{\partial K^2} \cdot \frac{\partial^2 Q}{\partial L^2} > \left[\frac{\partial^2 Q}{\partial K \partial L} \right]^2 \quad (3)$$

See, for example, Chiang, 1974, page 351.

The last assumption to be stated here, is not mentioned explicitly by Jorgenson. It is quite obvious to assume that it must be profitable at least to start production, so marginal revenue must exceed marginal costs of both inputs used to produce the first unit of output:

$$p \frac{\partial Q}{\partial K} > c(i + a) \text{ and } : p \frac{\partial Q}{\partial L} > w \text{ when } Q = K = L = 0 \quad (4)$$

The formulation of Jorgenson is now presented in (1) through (4). The problem with this formulation is, that the resulting optimal solution dictates an instantaneous adjustment of the stock of capital goods to the level with maximum revenue (see appendix 2):

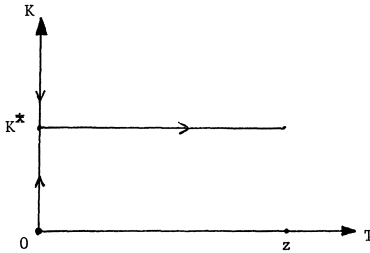


Figure 3.1. Optimal trajectory of the capital stock in the model of Jorgenson.

If the selling price is constant, this level is fixed by:

$$p \frac{\partial Q}{\partial K} = c(i + a) \text{ when } K = K^* \quad (5)$$

From (5) can be concluded that the marginal revenue per capital good just balances the financial obligations and the depreciation of a capital good.

The amount of labour appears to adapt itself perfectly all the time, because it holds continuously that:

$$\frac{\partial Q}{\partial L} = \frac{w}{p} \quad (6)$$

which can be interpreted (after multiplying both sides of (6) by the selling price p) in the same way as (5): marginal revenue to labour equals marginal costs of labour.

After the initial investment (or divestment) to reach the optimal level fixed by (5), the firm will keep the capital stock constant. Due to depreciation, it thus has to replenish continuously and so, investments remain on the replacement level:

$$\dot{K} = \dot{K}^* + \dot{K} = 0 + I = aK^* \quad (7)$$

In order to get rid of the unrealistic immediate adjustment at $T = 0$, two ways in particular have been proposed to amend the above model. The first way is the introduction of adjustment costs, representing the scarcity of inputs and/or the costs of productive capacity caused by the adjustment process. We have discussed this already in sections 3 and 9 of chapter 2. There, we quoted Jones (1973), who stated that the introduction of adjustment costs implies the transformation of the production function from a technical relationship into a behavioural relationship, containing technical constraints as well as organizational and other constraints. This mixing together is a disadvantage if one aims at describing the dynamics of the firm at the lowest level of aggregation in order to link micro economics to business economics.

The second way of getting a smoothed adjustment pattern is the introduction of financing as another aspect governing the dynamics of the firm. In fact, the revenue flow in the model of Jorgenson only serves as a performance index. And so, for example, a beginning entrepreneur having no equity may at once acquire an amount of K^* of capital goods without any financing problems: although the revenue flow at $T = 0$ may have a very large negative value due to the adjustment investments, it does not harm the total performance, because this loss only holds for a neglectable small period of time.

In the static micro economic theory of the firm, Vickers, 1968, was the first to couple the real aspects of production with the financial aspects of the policy of the firm. As far as we know, Leland, 1972, was the first author to couple production and finance in a real dynamic model of the firm.

3. Production and finance I (Leland).

Leland assumes a managerial firm, maximizing total discounted sales over a finite planning horizon plus the final amount of equity. As equity is supposed to be the value of the stock of capital goods reduced by the amount of debt money, Leland formulates the following goal function:

$$\text{maximize}_{L, B} : \int_0^z e^{-i'T} p Q(K, L) dT + e^{-i'z} S(K(z) - Y(z)) \quad (8)$$

in which : $\dot{Y} = B(T)$ (9)

$B(T)$: inflow of debt
 $Y(T)$: total amount of debt
 $Q()$: production function
 $S()$: weighting function of net terminal assets
 i' : time preference rate of the managers.

From (8) and (9) can be derived that in Leland's model, the management runs the firm on the basis of an employment and borrowing policy. The investment policy consists of two parts. In the first place, Leland assumes that the firm retains a fixed portion of the cash flow ("profit" in his terms) for reinvestments. In the second place, debt yields, at a decreasing efficiency, new capital goods:

$$\dot{K} = m G + C(B) \quad (10)$$

in which : $G = Q - wL - rY$: cash flow
 $C(B)$: relation between the borrowing inflow and the investments in capital goods
 m : fixed portion of retained cash flow.

In this way, a behavioural relation is introduced, reflecting imperfections of the debt market. In order to satisfy the shareholders, the management is assumed to keep return (i.e.: cash flow) on net invested capital on or above the time preference rate of the shareholders (i):

$$\frac{Q - wL - rY}{K - Y} > i \quad (11)$$

Initial state conditions are omitted and finally the assumption is introduced that:

$$r = i \quad (12)$$

This is based on Leland's assumption that stockholders may be in a position to lend to the firm, too. In that case they would not accept a return on their equity (i), less than the borrowing rate (r). But, the weak point of the assumption is, that it is not a sufficient reason to imply that the discount rate (i) exactly equals the borrowing rate. Therefore the concept of a perfect capital market has to be introduced. And that concept is contrary to the decreasing efficiency of debt, compared with constant efficiency of retained earnings, in (10).

Another imperfection of the model is that it inevitably results in an ever increasing K, due to the fixed retaining rate and the ever positive cash flow, resulting from the

constant selling price together with (11) and (12):

$$Q - wL - rY > i(K - Y) \quad (13)$$

In spite of these remarks we discussed Leland's model, because it was the first dynamic model that dealt with production and finance simultaneously.

A year after Leland's publication, J. Lesourne published a comprehensive treatise on dynamic models of the firm, solved by means of the Calculus of Variations (Lesourne, 1973). This book contains, among others, distinct models that deal with financial constraints in situations of self-financing with and without issuing new shares, combined with situations with and without borrowing. Several relations in the relevant models stem from financial records as used in practice and this means a real step forward in describing financial constraints on the policy of the firm.

In the same year, Krouse and Lee also published a purely financial dynamic model of the firm (Krouse & Lee, 1973) that, in spite of (or: due to) its shortcomings (see: Sethi, 1978) stimulated many authors to explore the field of dynamic theories of finance. From that flow of publications, we chose Ludwig's dissertation (1978, in German language). Ludwig improved and extended the often quoted, but incorrect dynamic financing model presented in chapter 4 in: Bensoussan et al., 1974, and his description of the solution procedure inspired the design of the solution procedure in appendix 2 of this book.

4. Finance and the value of the firm (Ludwig).

Ludwig deals with a shareholders owned firm, and assumes a finite planning horizon. This results in the next goal function as discussed in section (2.1):

$$\underset{B, I}{\text{maximize}} : \int_0^z e^{-iT} D(T) dT + e^{-iz} X(z) \quad (14)$$

Furthermore, he uses the same state equation of capital goods as Jorgenson did:

$$\dot{K} = I - aK \quad (15)$$

and amends Leland's state equation of debt by introducing a fixed redemption rate b:

$$\dot{Y} = B - bY \quad (16)$$

Assuming that the only assets of the firm are capital goods, we get the balance equation:

$$K = X + Y \quad (17)$$

which implies:

$$\dot{K} = \dot{X} + \dot{Y} \quad (18)$$

Now, Ludwig assumes without any motivation, that:

$$a = b \quad (19)$$

which results, together with (15), (16) and (18) into the state equation of equity:

$$\dot{X} = I - aX - B \quad (20)$$

Earnings are used to issue dividend or to increase the value of equity through retained earnings, so:

$$E = \dot{X} + D = R(K) - aK - rY \quad (21)$$

in which : $E(T)$: earnings
 $R(K)$: return on sales

Note that corporate tax is not considered in (21). Subsequently Ludwig introduces the assumption that at least a certain portion of the earnings will be issued to the shareholders and, moreover, that the firm will only accept situations in which earnings are positive:

$$D > (1-m) E > 0 \quad (22)$$

in which : m : maximum retaining rate

Together with (20) and (21) this results in:

$$I < mE + aX + B \quad (23)$$

So, as contrasted with Leland's findings, investments now appear to be limited, due to the introduction of the financial aspects of the firm's policy.

Ludwig does not allow for divestments so, investments are "irreversible":

$$I > 0 \quad (24)$$

From (15) can be concluded that still a decrease of the capital good stock K is possible. Finally, the inflow of debt is limited by the total amount of new investments:

$$0 < B < hI \quad (25)$$

in which: h : maximum borrowing rate, $0 < h < 1$

On page 58 of his book, Ludwig represents an interesting summary of alternative ways to formulate the limits of borrowing, as presented in literature. The above formulation is defined in terms of flows. Another formulation, that will be used in the next section and in our model is in terms of stocks:

$$Y < kX \quad (26)$$

in which: k : maximum debt-equity rate

Before describing the optimal solution, we will summarize the model of Ludwig:

$$\text{maximize}_{B, I} : \int_0^z e^{-iT} (R(K) - (a+r)Y - I + B) dT + e^{-iz} X(z) \quad (27)$$

$$\text{subject to} : \dot{X} = I - aX - B \quad (28)$$

$$\dot{Y} = B - aY \quad (29)$$

$$0 < I < m(R(K) - aK - rY) + aX + B \quad (30)$$

$$0 < B < hI \quad (+ I > 0) \quad (31)$$

$$X(T) > 0, Y(T) > 0, 0 < T < z \quad (32)$$

$$X(0) = x_0, Y(0) = y_0 \quad (33)$$

$$0 < m < 1, 0 < h < 1 \quad (34)$$

Ludwig derives two distinct optimal trajectories of the firm from the optimality conditions (see appendix 2). Both patterns consist of several paths, representing dis-

tinct stages of the development of the firm. The main features of those paths are put into the next table:

path. nr.	I	B	D	\dot{X} \dot{Y}	K	
1	aK_y^*	max		$\frac{+}{-}$ $\frac{-}{+}$	K_y^*	stationary state when $i > r$
2	aK_{XY}^*	max	min	+ -	K_{XY}^*	consolidation
3	max	max	min	+ +	$\dot{K} > 0$	maximum growth
4	aK_X^*	0		+ -	K_X^*	stationary state when $i < r$
5	max	0	min	+ -	$\dot{K} > 0$	self-financed growth
6	0	0		+ -	$\dot{K} < 0$	contraction

Table 3.1. Characteristics of the feasible paths.

$$\text{in which : } K = K_Y^* \text{ if } \frac{\partial R}{\partial K} - a = (1-h)i + hr \quad (35)$$

$$K = K_{XY}^* \text{ if } \frac{\partial R}{\partial K} - a = r \quad (36)$$

$$K = K_X^* \text{ if } \frac{\partial R}{\partial K} - a = i \quad (37)$$

So, the stationary K^* -values are all characterized by a fixed value of the marginal net revenue, given on the left hand side of the expressions. The relevant values are the financing costs in the case of (from the top downward) debt financing at a rate h , pure debt financing and: pure self-financing.

Which of both trajectories is feasible, depends on whether the discount rate i exceeds or is smaller than the interest rate on debt money, r . The simplest pattern holds in the case of cheap debt money (see figure 3.2).

In combining table 3.1 and figure 3.2 we can get insight into this pattern. In figure 3.2, the optimal patterns for three different sets of initial states are drawn:

$$(a) K(0) > K_y^*$$

$$(b) K(0) = K_y^*$$

$$(c) K(0) < K_y^*$$

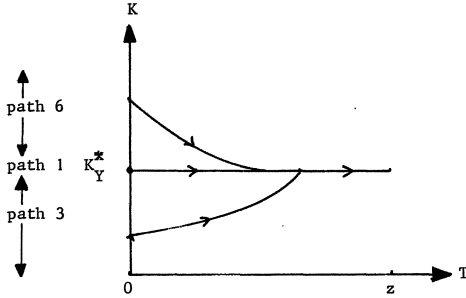


Figure 3.2. Optimal trajectories of the capital stock when $i > r$.

Each pattern has path 1 as final path. So $K = K_Y^*$ appears to be the state of bliss. This is quite natural, because in that case marginal net revenue equals the marginal cost of the cheapest way to finance capital goods, i.e. of debt financing at a rate h . So, the profit flow is then at its maximum and it enables the firm to reach the highest attainable value of the firm.

Furthermore we can derive that, if the firm's capital stock is below the desired level K_Y^* , it borrows as much as possible and issues a minimum dividend in order to grow at the highest speed. Ludwig also has derived a value for the initial leverage below which the viability of the firm is guaranteed (see appendix 2):

$$\frac{Y(0)}{X(0)} < \frac{br + ha}{a(1-h)} \quad (38)$$

The maximum debt-equity rate, h , will be attained only some time after entering the stationary state of path 1.

In the capital stock is above the desired level K_Y^* in figure 3.2, the firm cuts down this stock at the maximum rate that is allowed for, i.e. at the depreciation rate a , for, due to the non negativity constraints on investments, the firm cannot divest. All profits are paid out to the shareholders.

Finally, the reader can imagine that a firm, having an initial stock just on the optimal level K_Y^* , will keep to that level during the whole planning period by investing at such a level as necessary only to replace obsoleted capital goods. The remaining profit is issued to the shareholders.

In the case of expensive debt money, so $i < r$, we get next optimal trajectories of the capital stock:

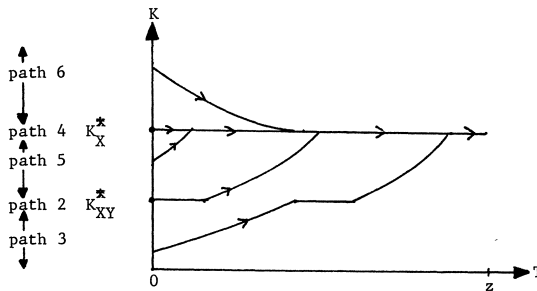


Figure 3.3. Optimal K-trajectories when $i < r$.

The optimal level of the capital stock now equals K_X^* . Due to the cheapness of equity compared to debt, pure self-financing is the cheapest way of financing the capital stock. On the level K_X^* these financing costs equal marginal net revenue on capital goods and so the relevant level yields the maximum profit flow.

Apart from the distinct financial structure, the trajectories starting on or above K_X^* in figure 3.3 have the same meaning as in figure 3.2. The trajectories starting at K_{YX}^* or between K_{YX}^* and K_X^* can be conceived as subtrajectories of the trajectory starting below K_{YX}^* , differing only as far as the initial state constraints are concerned. So, we may restrict the discussion to the pattern, starting below K_{YX}^* in figure 3.3, having the initial state constraint:

$$K(0) < K_{YX}^*$$

for which Ludwig concluded that the viability of the firm is at least ensured if

$$\frac{Y(0)}{X(0)} < \frac{br + ha}{a(1-h)} \quad (39)$$

In order to get a better idea of the relevant pattern, we take this pattern from figure 3.3 and add the trajectory of the other state variable, debt capital (Y):

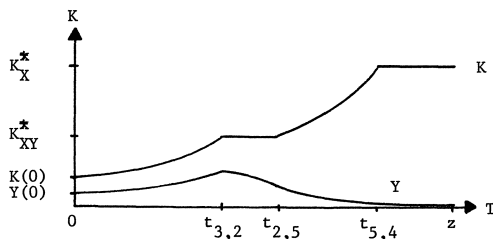


Figure 3.4. Optimal K- and Y trajectory when $i < r$ and $K(0) < K_{YY}^*$.

One can derive from table 3.1 and figure 3.4 that it appears to be profitable to start growing with maximum debt financing. This is because it enables the firm "to benefit more by existing growth changes", as Ludwig stated quite vaguely (page 115 o.c.). We will describe this more precisely in section 4 of chapter 5 with the help of our own model.

As soon as the capital stock reaches the level K_{YY}^* , on which marginal net revenue equals the interest rate on debt money, it is no longer profitable to extend the amount of debt and the firm wants to get rid of it. But, due to the large amount of debt, it still needs more new debt to realise the plan of redemption:

$$m(R - aK - rY) < aY \quad (40)$$

which can be derived from the optimality conditions holding on path 2.

The firm now keeps its capital stock on the level K_{YY}^* , its dividend pay-out on the lower bound and spends all means to diminish the amount of debt up to the level on which no more new debt is needed, so the maximum redemption rate is attained:

$$\dot{Y} = -aY \quad (41)$$

This holds on path 5. On this path, the firm needs less and less money to pay back debt capital at the maximum allowable rate a . It still pays out only the minimum amount of dividend, thus maximizing the rate of growth towards the state of bliss, K_X^* . Having reached that state, the firm continues to pay back debt at the maximum rate and pays out all earnings that remain after replacement investments to the sharehol-

ders.

Ludwig also analyses the case in which $i = r$. In that case, the results concerning the stationary state are indifferent to the financial structure, as one may expect.

We will add one more remark to the way debt is dealt with in Ludwig's model. By (16) the firm is forced to keep a certain amount of debt all the time, due to fixed redemption rate. To Ludwig, the continuous presence of debt money in the firm is a realistic aspect of his model. But we wonder, whether the origin of it, i.e. the infinite pay off period, is such a realistic feature.

5. Production and finance II (Lesourne & Leban).

The last model that we wish to present here describes changes in the financial structure as well as in the production structure of the firm. The relevant model, Lesourne & Leban, 1978, has been published in French. The solution was based on the Calculus of Variations. In appendix 2 we have solved the model by means of the Maximum Principle. This results in different initial conditions to the optimal trajectories than published by Lesourne & Leban. Lesourne & Leban define these conditions in terms of different (exogeneous) values of the interest rate for a given initial value of the capital good stock. We will formulate them in the tradition of the former models as different values of the initial capital good stock.

In the first place, Lesourne & Leban introduce the balance equation as it was presented already in the model of Ludwig:

$$X + Y = K. \quad (42)$$

The state equation of capital goods, too, fits well into this tradition:

$$\dot{K} = I - aK. \quad (43)$$

The extension is in the definition of earnings (see (21)). Lesourne & Leban introduce corporation profit tax and the input of labour next to the input of capital goods:

$$E = \dot{X} + D = (1-f)(R(Q) - wL - aK - rY) \quad (44)$$

in which : f : corporation profit tax rate.

In the second place, Lesourne & Leban introduce a fixed output level, \bar{Q} , above which capital and labour are substitutes and below which the inputs are complementary in the optimal solution. Therefore they assume a special type of production function and

introduce some assumptions concerning R. The production function must belong to a specific class of Cobb-Douglas type functions:

$$Q = K^\alpha L^{1-\alpha}, \alpha < 1 \quad (45)$$

The relevant function is linearly homogeneous. This means that, if both inputs change at a rate g, the resulting output will change at that rate too:

$$Q(gK, gL) = (gK)^\alpha (gL)^{1-\alpha} = g^\alpha K^\alpha L^{1-\alpha} = gQ(K, L) \quad (46)$$

Furthermore the exponent of each input variable can be interpreted as the partial elasticity of output with respect to that input. For example, the partial elasticity of output with respect to capital goods is:

$$\varepsilon_{QK} \equiv \frac{\frac{\partial Q}{\partial K}}{\frac{Q}{K}} = \frac{\alpha \frac{Q}{K}}{\frac{Q}{K}} = \alpha \quad (47)$$

As far as the return function $R(Q)$ is concerned, they introduce the usual assumptions of strict concavity, twice differentiability, a strict increasing function with decreasing marginal returns to scale. Above that, they assume that the function $Q \frac{\partial R}{\partial Q}$ has a unique maximum for $0 < Q < \infty$. Together with (45) and the optimality conditions, these assumptions on R will result in the above mentioned critical value \bar{Q} (see appendix 2).

The firm is of the owner-managed type, maximizing its dividend stream over an infinite planning period. The control variables are: dividend pay-out D, gross investments I and the level of employment L:

$$\text{maximize}_{D, I, L} : \int_0^{\infty} e^{-iT} D(T) dT \quad (49)$$

Dividends and debt are assumed to be non negative. Above that, the total amount of debt available is restricted by the amount of equity:

$$D > 0 \quad (50)$$

$$0 < Y < kX \quad (51)$$

The above model results in nearly the same type of control trajectories for capital goods K, equity X and debt Y as the model of Ludwig. Assumed that

$$i \neq (1-f)r, \quad (52)$$

two cases are to be distinguished: the case of cheap debt capital and the case of cheap equity.

In the case of cheap debt capital, i.e. $i > (1-f)r$, the firm will expand at the maximum rate ($D = 0$) with maximum debt financing ($Y = kX$), till the stationary state is reached, defined by:

$$K = K_Y^* \text{ if } \frac{\partial R}{\partial K} - a = \frac{1}{1+k} \left(kr + \frac{i}{1-f} \right) \quad (53)$$

In order to be able to compare (53) with definition (35) of the model of Ludwig, we must replace the maximum debt to equity rate k in (53) by the maximum debt to capital rate h . Because of

$$\frac{k}{1+k} = h, \quad (54)$$

equation (53) changes into:

$$\frac{\partial R}{\partial K} - a = (1-h) \frac{i}{1-f} + hr \quad (55)$$

The comparison between (55) and (35) reveals the increase of the cost of equity due to corporation profit tax if the net marginal return on equity remains i .

In the case of cheap equity, the firm starts growing with maximum debt financing till the consolidation level is reached, defined by:

$$K = K_{YX}^* \text{ if } \frac{\partial R}{\partial K} - a = r \quad (56)$$

which is the same definition as given in (36) of Ludwig's model. But now, the firm passes off all its debt and starts the second growth stage with only equity financing till the stationary state is reached, defined by:

$$K = K_X^* \text{ if } \frac{\partial R}{\partial K} - a = \frac{i}{1-f} \quad (57)$$

which is the same definition as in (37) of Ludwig's model, with the exception of the influence of the tax rate.

So, the main difference with the model of Ludwig, as far as the financial structure is concerned, is the absence of debt money in the second stage of growth when equity is

cheap.

The initial conditions of the distinct patterns can be formulated in the same way as presented in the model of Ludwig by distinguishing cases in which the value of the initial capital good stock is less than, equal to or bigger than the relevant critical values are defined in (55) through (57).

Furthermore, Lesourne & Leban extend Ludwig's results by introducing labour as a second input into the production process and by analysing the changes of employment in the firm in the course of time. Whether employment is strictly increasing or will start decreasing after a certain point of time, depends on the value of \bar{Q} , compared with the relevant Q^* -values. For example, if $i > (1-f)r$, we may get the following patterns of L :

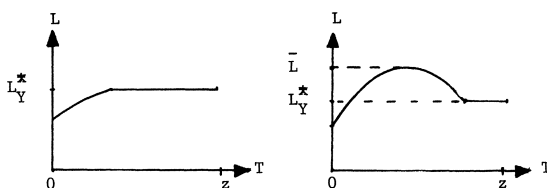


Figure 3.5. The optimal employment trajectories if $i > (1-f)r$ and $\bar{Q} > Q_y^*$ (left) or $\bar{Q} < Q_y^*$ (right).

in which : \bar{L} : employment level when $Q = \bar{Q}$
 L_Y^* : employment level when $Q = Q_y^*$.

Although Lesourne & Leban did some sensitivity analysis in their article, they didn't explain what economic reasons could persuade the firm to start depth investments and when such reasons do apply. They only state sufficient (mathematical) conditions to the appearance of such a case of depth investments. In the presentation of our own model we will say more about the economic aspects of depth investments.

6. Summary.

Some trend setters towards our own model have been reviewed in the above sections. With the help of Jorgenson's model we dealt with the dynamic nature of investments and the necessity to incorporate more aspects of the limits of growth into the model in order to avoid immediate adjustment of the firm to the state of bliss. Leland's model has been presented as a first attempt to incorporate financial aspects as limiting

forces. Ludwig concentrates fully on the financing problem of growth. His model results in two distinct optimal trajectories, one on which the firm always borrows at the maximum rate and the other one on which the firm may start borrowing, but, having reached a certain size (measured by its output level), it starts paying off debt. Finally, the model of Lesourne & Leban combines the aspects of allocating labour and capital with that of financing the growth of the firm. They find conditions under which a decrease in the employment level of a growing firm may occur.

CHAPTER 4. A DYNAMIC MODEL OF THE FIRM.

1. Introduction.

The model to be presented in this chapter analyses the dynamic relations between the firm's production, financing and investment policy and studies the influence of changes in some important parameters, reflecting governmental influence, social policy and the impact of financial institutions on these dynamic relationships.

The model differs from the tradition of Jorgenson - Lesourne & Leban, presented in the previous chapter. First, the allocation of capital goods and labour is formulated in a different way so as to derive meaningful economic notions from the distinct stages in the optimal trajectories of the firm. We will achieve this by describing the production process by means of activity analysis.

Compared with our predecessors, we have further rendered explicit the effects of governmental influence on corporate policy. Beside corporation profit tax, we will introduce investment grants. This is an instrument by which the government intends to influence the investment policy of the firm. The aim is to raise the national employment level indirectly by stimulating investments. Furthermore, in the Dutch case, the government gives additional grants for specific investments, for example in less favourable regions.

In this chapter we will present the relevant model and we will show the links with several areas of economic theory and practice. There after, we will present the main features of the optimal solution. The solution procedure itself is described in appendix 3.

2. Production

We assume that the firm produces a homogeneous output by means of two homogeneous inputs: labour and capital goods. Most publications dealing with the allocation of labour and capital in a dynamic theory of the firm, assume a continuous production function. This implies the assumption that the firm can choose at each moment in time between an infinite number of production possibilities. This does not seem a realistic concept, because, in a realistic situation the management of the firm always choose between a limited number of production possibilities. So, although the continuous production function may be a useful relation on an aggregated level, one may doubt its suitability for the case of a single firm.

We therefore introduce activity analysis as a way to describe the link between the inputs of labour and capital and the output of the firm (see for example: Henderson &

Quandt, 1971, 335). We assume that the firm can choose at each moment in time between only a limited number of (linear) production activities. Each of such production-activities represents a process by which the output is produced by the application of labour and capital goods in a fixed proportion. Moreover, we will restrict ourselves to only two available production activities: a capital-intensive and a labour-intensive production activity:

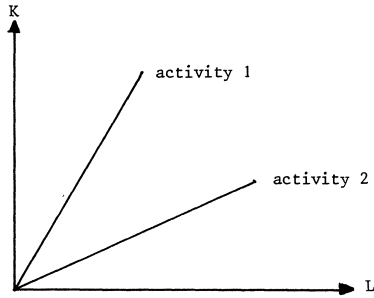


Figure 4.1. The available production possibilities.

This restriction will not affect the quality of the model nor the tenor of the solution, because adding more production activities appears not to imply adding new features to the optimal allocation of labour and capital. If we further assume constant returns to scale and a fixed technology during the planning period, we can write:

$$Q(T) = q_1 K_1(T) + q_2 K_2(T) \quad (1)$$

$$L(T) = \ell_1 K_1(T) + \ell_2 K_2(T) \quad (2)$$

$$K(T) = K_1(T) + K_2(T) \quad (3)$$

in which : K : amount of capital goods available to the firm
 K_j : amount of capital goods assigned to activity j
 L : employment level of the firm
 Q : output level of the firm
 T : time, $0 < T < z$
 ℓ_j : labour to capital ratio of activity j
 q_j : productivity of capital goods assigned to activity j
 z : planning horizon

We have chosen the above formulation with K_j as explanatory variables to the output Q and employment L , because these variables will belong to the set of variables controlled by the firm such as to realise an optimal policy. Equation (3) states that there is no idle capacity within the firm. Both activities are assumed to be efficient, which means that none of them is inferior to the other. If we further conceive activity 1 as the capital-intensive one, then it follows that:

$$q_1 < q_2; \frac{q_1}{l_1} > \frac{q_2}{l_2} \rightarrow l_1 < l_2 \quad (4)$$

3. Sales and operating income.

As far as the output market is concerned, output and sales are assumed to equal each other, so the stock level of final products is constant and independent of the output level. We further assume that the firm is operating under decreasing returns to scale. This decrease may be caused by an imperfect output market or, if we introduce other kinds of costs apart from production costs, by increasing marginal costs of organizing the production due to the increasing scale of the firm:

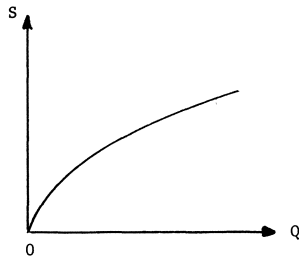


Figure 4.2. The sales-output function.

This leads to some well known concavity properties of the sales function:

$$S(Q) = P(Q) \cdot Q$$

$$\frac{dS}{dQ} > 0; \frac{d^2S}{dQ^2} < 0; S(Q) > 0 \text{ when } Q > 0 \quad (5)$$

in which : S : sales (value)

P : (net) selling price

To facilitate analysis later on, we introduce the notion of operating income O . For this we need three more assumptions concerning labour costs and capital costs. Wages are assumed to be proportional to the amount of labour input L , depreciation is assumed to be proportional to capital goods K (see formula (3.2)). Finally we assume that the price of a capital good equals one unit of money value:

$$O(K_1, K_2) = (q_1^P - w_1)K_1 + (q_2^P - w_2)K_2 - aK \quad (6)$$

in which : O : operating income

K : amount and book value of capital goods

a : depreciation rate

w : wage rate

4. Financing and government.

First we will present three relations that are based on well known financial records of the firm: the balance sheet, the income statement and the cash account. As far as the balance sheet is concerned, we assume that the firm has only one type of assets: capital goods, and two types of money capital: equity and debt, so:

BALANCE SHEET	
assets $K(T)$	$X(T)$ equity
	$Y(T)$ debt

From the balance sheet we derive that:

$$K(T) = X(T) + Y(T) \quad (7)$$

Together with (3) this enables us to construct a first link between the mode of production and the financial structure of the firm:

$$K = K_1 + K_2 = X + Y \quad (8)$$

Second we assume that the firm can raise its equity not only by retained earnings but also by acquiring investment grants. This is a new feature in the dynamic theory of the firm. We further assume investment grants are proportional to the amount of gross investments:

$$\dot{X} = E + gI(T) \quad (9)$$

in which : I : gross investment
 E : retained earnings
 •
 X : increase of equity
 g : investment grant rate

The amount of retained earnings can be derived from the income statement. In order to construct this statement we introduce the following assumptions:

- corporation profit tax is proportional to profit,
- no transaction costs are incurred when borrowing or paying off debt capital,
- taxes are paid at once, grants are received immediately.

These assumptions result in the next income statement:

INCOME STATEMENT

sales S(Q)	wL(T)	wages
	aK(T)	depreciation
	rY(T)	interest on debt
	F(K ₁ ,K ₂ ,Y)	corporation profit tax
	D(T)	dividend pay-out
	E	retained earnings

in which : $F = f(S - wL - aK - rY)$
 f : corporation profit tax rate

Together with (6) and (9) the income statement results in the following state equation of equity:

$$\dot{X} = (1-f)(O-rY) - D + gI \quad (10)$$

The third financial record to be presented here is the cash account. For this we need no further assumptions:

CASH ACCOUNT

sales S(Q)	wL(T)	wages
investment grant gI(T)	rY(T)	interest on debt
increase of debt \dot{Y}	I(T)	gross investment
	F(K ₁ ,K ₂ ,Y)	corporation profit tax
	D(T)	dividend pay-out

From (8) we derive that:

$$\dot{K} = \dot{X} + \dot{Y} \quad (11)$$

Together with (6) and the cash account, this results in the dynamic equation of capital stock as discussed already in section 3.2.:

$$\dot{K} = I - aK \quad (12)$$

Finally, we introduce, in the tradition of Lesourne & Leban, an upperbound on debt in terms of a maximum debt to equity rate:

$$Y < kX \quad (13)$$

in which : k : maximum debt to equity rate

We have already discussed this constraint in section 2.7. Together with the interest rate r , (13) is a way to deal with uncertainty within the framework of a deterministic model. Because the level of r is an indication of the risk-class to which the firm belongs, (13) may be conceived as a condition on the financial structure of the firm that must be fulfilled in order to stay in the relevant risk-class (see: Ludwig, 1978, 51).

5. Policy of the firm.

We further assume that the firm maximizes the shareholder's value of the firm:

$$\text{maximize : } \int_0^z e^{-iT} D \, dT + e^{-iz} \{X(z) - gK(z)\} \quad (14)$$

This hypothesis is not supposed to imply that the firm is really able to and wants to maximize this value, but it results, in our opinion, in a useful abstract representation of the regularities of the firm's policy (Jones, 1973, 2). Note that the final value of the firm equals the value of equity minus investment grants to be paid back due to stopping corporate activity.

As far as its dividend policy is concerned, we assume that the firm is allowed to pay no dividend, so:

$$D > 0 \quad (15)$$

As we shall discuss in section 2 of chapter 5 this condition may be replaced by a more restrictive one, requiring a certain positive dividend flow during the whole planning period, without affecting the tenor of the conclusions that result from the optimal solution.

6. The model.

We have now discussed all the features of the model. In this section we will present the model in its final form.

$$\underset{D, I, K_1, K_2, Y}{\text{maximize}} \quad : \int_0^z e^{-iT} D \, dT + e^{-iz} \{X(z) - gK(z)\} \quad (16)$$

$$\text{subject to} : \dot{X} = (1-f)(O - rY) - D + gI \quad (17)$$

$$\dot{K} = I - aK \quad (18)$$

$$K = K_1 + K_2 \quad (19)$$

$$K = X + Y \quad (20)$$

$$0 < Y < kX \quad (21)$$

$$D > 0, K_1 > 0, K_2 > 0 \quad (22)$$

$$\text{in which} : 0 = (q_1 P(Q) - w_1)K_1 + (q_2 P(Q) - w_2)K_2 - aK \quad (23)$$

$$Q = q_1 K_1 + q_2 K_2 \quad (24)$$

$$q_1 < q_2, q_1/l_1 > q_2/l_2 \rightarrow l_1 < l_2 \quad (25)$$

$$\frac{dS}{dQ} > 0, \frac{d^2S}{dQ^2} < 0, S := P(Q) \cdot Q, S > 0 \text{ when } Q > 0 \quad (26)$$

$$f, g, i, r \quad : \text{constant and between 0 and 1} \quad (27)$$

$$k, l_j, q_j, w, z \quad : \text{constant and non-negative} \quad (28)$$

Note that the non-negativeness of X is ensured by (20) and (21) and the fact that $K > 0$ due to (19) and (22).

The state variables, as defined in appendix 1, are the amount of equity X , and the capital good stock K . The values of these variables thus represent the state of the firm at each moment of time. The firm will try to realize its goal, as defined in (16), through the available control variables: the amount of capital goods assigned to each of the production activities K_1 and K_2 , the size of debt Y , the investment activities I and its dividend policy D . The trajectories of these variables during the planning period represent the firm's policy. The restrictions on this policy and the effects of this policy on the state of the firm and on its performance level are described in the relations (16) through (22).

Here ends the description of the main features of the model. Before introducing the reader to the properties of the optimal solution, we will discuss briefly the assumptions on which the solution is based.

7. Further assumptions.

As we shall see later on, we will have to distinguish between different cases, depending on the mode of production and the prevailing financial structure. In each case, the unit cost of a product, including its part of the cost of capital, can be calculated. Let us denote the relevant unit cost by:

$$c_{jN}, \quad j = 1, 2, 21; \quad N = X, Y, YX$$

in which : j : activity performed by the firm ($j = 21$ means that both activities are performed),

N : index of financial structure:

$N = X$: self-financing case

$N = Y$: maximum debt financing case

$N = YX$: intermediate debt financing case.

Due to later assumptions, $j = 21$ and $N = YX$ cannot occur at the same time. Moreover c_{21X} will appear to equal c_{21Y} and therefore we will write briefly: c_{21} . So there remain seven unit cost levels to be distinguished. Their formulation in terms of the parameters of the model will be presented in the next chapter.

Our first assumption states that the marginal revenue of the first product to be sold exceeds each of these unit costs:

$$(A1) \quad \frac{dS}{dQ} \Big|_{Q=0} > \text{maximum} \{ c_{jN} \}, \quad j = 1, 2, 21; N = X, Y, YX, \emptyset, (j = 21 \leftrightarrow N = \emptyset)$$

The idea behind this assumption is, that the firm will consider only those alternatives that are profitable from the start. On the basis of this assumption we guarantee that the firm will start investing and producing. On the basis of the following assumption, we will restrict ourselves to cases in which the firm will not continue expanding far beyond profitable limits:

$$(A2) \quad O(K_1, K_2) > 0$$

The next assumption concerns the cost of equity and debt. Beside the problem of the financial structure, we have introduced the problem of the assignment of money-capital to production activities. Therefore, we have to distinguish between the market of equity and the market of debt, for, investors in equity or debt no longer differ only in their risk bearing preferences, but also in their attitude towards management. Investors of equity want to influence management directly (or are the managers themselves) while investors in debt money are only interested in influencing the policy of the firm as far as they will try to reduce their risks. The two types of investors have really different intentions and so, the markets of debt and equity are separated. It will therefore be a coincidence when the prices of equity and debt (after corporate tax) to be paid by the same firm, equal each other:

$$(A3) \quad i \neq (1-f)r$$

The above introduction of the assignment aspect of money capital also implies the acceptance of the possibility that equity is cheaper than debt money for a single firm. For, now we have a real multicriteria situation in which an investor of equity may accept less financial reward for his risk-bearing (compared with the reward of an investor in debt money) due to the attainment of other goals such as being the (a) boss of the firm.

Through the next assumption we will exclude all kinds of degenerated cases in which the firm could choose from an infinite number of equal alternatives:

$$(A4) \quad c_{1N} \neq c_{2N}, \quad N = X, Y, YX$$

We assume further that the firm has a certain initial amount of own capital:

$$(A5) \quad X(0) > 0$$

Finally we assume that the capital stock cannot be financed by debt money and investment grants only, but that at least a certain amount of equity is necessary:

$$(A6) \quad \frac{k}{1+k} + g < 1 + g(1+k) < 1$$

In (6) we have assumed that the price of a capital good equals one unit of money value, and in (13) we have introduced the maximum debt to equity rate k . So, $k/(1+k)$ in (A6) is the maximum amount of debt per capital good. Because g is the investment grant rate, the left hand side of the first expression of (A6) represents the maximum amounts of debt and investment grants that can be attracted per capital good. The right hand side represents the total amount of money capital needed to buy one capital good.

8. Optimal solution.

In the sections 1 through 3 of appendix 3 the reader will find how we derived the optimality conditions of the model (16)-(28), based on the Maximum Principle as described in appendix 1. Just as in the preceding models of chapter 3, the optimal trajectories of the firm can be divided in several stages (called: paths). Each path can be characterized by the boundaries that are active or inactive during the relevant period. In the sections 4 and 5 of appendix 3, the reader will find how we have derived the feasible paths and their relevant properties from the optimality conditions and from the above assumptions. The optimal evolution patterns of our model are built from 12 different paths, presented in table 4.1 on the next page.

The first column of table 4.1 gives a number to each path: this number will be used in the rest of our treatise.

The second column of table 4.1 indicates whether the firm is producing in a capital-intensive way (activity 1) or a labour-intensive way (activity 2) on the relevant path. On paths 3 and 10, the firm is switching from labour-intensive to capital-intensive production.

The third column states the financial structure in which the firm is operating: self-financing (indicated by "X"), maximum debt financing (indicated by "Y") or switching from maximum debt financing to self-financing by paying back debt money (indicated by "YX").

The next column marks paths on which the production level is constant. These stationary values are fixed by the equality of marginal return and the relevant marginal unit cost on that production (= sales) level:

path nr	activity	financial structure	production level	\dot{X} \dot{K}	necessary conditions
1	2	Y		+ +	
2	2	Y	Q_{2Y}^*	0 0	$i > (1-f)r$ $c_{1Y} > c_{2Y}$
3	21	Y	Q_{21}^*	+ +	$c_{1YX} < c_{2YX}$
4	1	Y		+ +	
5	1	Y	Q_{1Y}^*	0 0	$i > (1-f)r$ $c_{1Y} < c_{2Y}$
6	2	YX	Q_{2YX}^*	+ 0	$c_{1YX} > c_{2YX}$
7	2	X		+ +	
8	2	X	Q_{2X}^*	0 0	$i < (1-f)r$ $c_{1YX} > c_{2YX}$ $c_{1X} > c_{2X}$
9	1	YX	Q_{1YX}^*	+ 0	$c_{1YX} < c_{2YX}$
10	21	X	Q_{21}^*	+ +	$c_{1YX} > c_{2YX}$
11	1	X		+ +	
12	1	X	Q_{1X}^*	0 0	$i < (1-f)r$ $c_{1X} < c_{2X}$

Table 4.1. Characteristics of the feasible paths.

$$Q = Q_{jN}^* + \frac{dS}{dQ} = c_{jN}, \quad j = 1, 2, 21; N = X, Y, YX, \emptyset, \quad (j = 21 \rightarrow N = \emptyset) \quad (29)$$

The firm expands its output level on the four paths where the production level is not constant.

The fifth column describes the trajectories of equity X and of the capital good stock K on the relevant paths. The reader can check that in spite of a stationary value of output Q , equity and/or the capital good stock may still be increasing on some paths. On paths 3 and 10 the increase of equity and capital goods is caused by the depth investment process by which more capital goods are needed for the same output level. On paths 6 and 9 equity is increasing while the output level and the capital stock are constant due to the redemption policy of the firm on those paths. Finally paths 2, 5, 8 and 12 remain which represent real stationary states. They will appear to be the final stages of the four different sets of optimal trajectories of the firm.

In the last column necessary conditions for several paths are described, resulting

from the optimality conditions and the assumptions made in the above sections of this chapter. The four final paths appear to have mutually excluding necessary conditions, resulting in the above mentioned four distinct sets of optimal trajectories of the firm. On the remaining paths, the relation between c_{1YX} and c_{2YX} restricts the feasibility. This relation will appear to determine the sequence over time of the different stages of the optimal trajectories.

In the sections 6 through 8 of appendix 3, the reader will find how to construct the optimal trajectories of the firm. Based on different necessary conditions of the four resulting final paths, these trajectories can be classified in four different sets. Within each set each optimal trajectory appears to be part of the "master trajectory" of the relevant set. For example, if $c_{1Y} < c_{2Y}$ and $i > (1-f)r$ we get the following optimal strings:

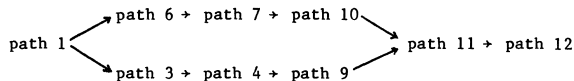
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trajectory 1 :                               path 5
trajectory 2 :                               path 4 + path 5
trajectory 3 :          path 3 + path 4 + path 5
trajectory 4 : path 1 + path 3 + path 4 + path 5 (= master-trajectory)

```

The master trajectory 4 contains all stages of the strings 1 through 3. Which of the above strings is the optimal trajectory depends upon the initial values of equity and of the capital good stock. In general: the lower these initial values, the more stages have to be gone through before the final stage is reached. The relevant conditions on the initial values of the state variables are presented, together with all possible strings, in section 8 of appendix 3. In the next chapter we will describe only the master trajectories, because they are the unions of all the other strings in the relevant sets.

There is one exception. In the set of strings ending with path 12, there appear to be two master patterns, having the same start and finish:



However, both patterns are very close to each other, also in the central part, so that we will present one of them as a variant of the other and not on its own.

9. Summary.

In this chapter we have presented a dynamic model of the firm. The firm operates on an imperfect capital market. It finances its expansion by debt and/or retained earnings but it does not issue new shares. The availability of debt money depends on the amount

of equity. Marginal returns to scale are decreasing. Production is described by means of activity analysis. The financial relations in the model are based upon well known financial records. The government influences the policy of the firm through the corporation profit tax rate and through investment grants.

After a brief discussion of six more assumptions, the main characteristics of the twelve distinct stages, constituting the optimal trajectories of the firm, have been presented and the presentation of the four master trajectories in chapter 5 has been introduced.

CHAPTER 5. OPTIMAL TRAJECTORIES OF THE FIRM.

1. Introduction.

This chapter is devoted to a description of the four different trajectories of the firm, resulting from the optimal solution of the model of chapter 4. The sequence is such that the presentation of each trajectory contains new elements as far as the firm's policy is concerned.

The first trajectory gives information about growth- and stationary stages and forces us at once to analyse the stationary state condition for the most complex case. We will further discuss the meaning of conditions on the initial values of equity and capital goods. The second trajectory deals with depth investments: the switch from labour-intensive to capital-intensive production. The third trajectory describes the process of a firm starting with maximum debt financing and ending as a self-financing producer. The necessary conditions for the last trajectory are such that depth investments as well as consolidation occur in the course of the firm's optimal trajectory.

We will also demonstrate several aspects of the solution procedure that are worthwhile from an economic point of view.

2. Basic trajectory.

We get the simplest master trajectory when debt money is cheap ($i > (1-f)r$) and the unit cost of the labour-intensive activity is smaller than the unit cost of the capital-intensive activity. The firm will always prefer activity 2 and it will finance its investment by means of as much debt money as is allowed for. See figure 5.1, in which:

$$K(z) = K_{2Y}^*, \quad L(z) = \ell_2 K_{2Y}^* \quad \text{and} \quad Y(z) = \frac{k}{1+k} K_{2Y}^*.$$

Figure 5.1 shows that the relevant master trajectory consists of two paths: the growth path 1 and the stationary state path 2. On $T = t_{1,2}$ the firm stops growing and enters the stationary stage. This moment is fixed by the level of output Q_{2Y}^* that is then attained. Below that level it is worthwhile to expand the output capacity because marginal revenue exceeds marginal cost, for, due to diminishing returns to scale and (4.29) it holds on path 1 that:

$$Q < Q_{2Y}^* + \frac{dS}{dQ} > c_{2Y} \quad (1)$$

$$\text{in which : } c_{2Y} = \frac{1}{q_2} \left\{ w \ell_2 + \left(1 - \frac{g}{1-f}\right) a + \frac{k}{1+k} r + \left(1 - \frac{k}{1+k} - g\right) \frac{i}{1-f} \right\}$$

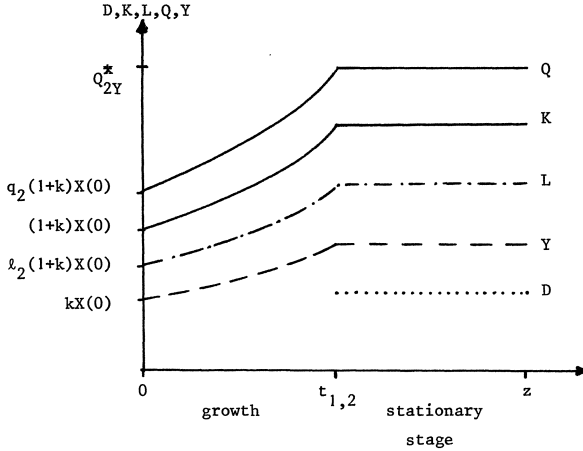


Figure 5.1. Master trajectory if $i > (1-f)r$ and $c_{1Y} > c_{2Y}$.

We will now discuss in more detail the above formulation of c_{2Y} . The part in the main brackets is the cost per capital good assigned to activity 2. It is divided by the output per capital good, q_2 , in order to get the unit cost of activity 2. The cost per capital good consists of four parts:

$$\begin{aligned}
 \text{wages} & : w l_2 \\
 \text{depreciation} & : \left(1 - \frac{g}{1-f}\right) a \\
 \text{interest on debt:} & \frac{k}{1+k} r \\
 \text{cost of equity} & : \left(\frac{1}{1+k} - g\right) \frac{i}{1-f}
 \end{aligned}$$

Wages are simply the wage rate time the amount of labour assigned to each capital good (see: (4.2)). Depreciation appears net from investment grants. These subsidies may be considered as diminishing the price of capital goods at a rate g , resulting in a decrease of depreciation of $a.g$ in the case of absence of corporation profit tax. When corporation profit tax is introduced, we have to reckon with the fact that investment grants are free from corporation profit tax, so the relevant decrease of $a.g$ is then after tax payments and this equals a decrease of depreciation before taxes of $\frac{g}{1-f} a$.

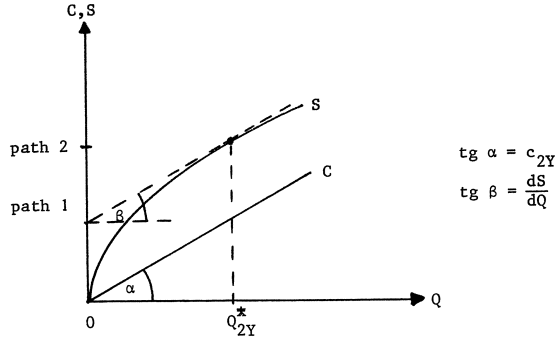


Figure 5.2. Master trajectory of $S(Q)$ if $i > (1-f)r$ and $c_{1Y} > c_{2Y}$.

A further increase of the capital good stock should yield less than i , so the firm will now put down investments to the replacement level (see: (4.18)):

$$Q = Q_{2Y}^* + \dot{K} = 0 \rightarrow I = aK_{2Y}^* \quad (3)$$

in which : $q_2 \cdot K_{2Y}^* = Q_{2Y}^*$,

and pays out the remaining earnings to the shareholders. From (4.17, 20) and (3) we get:

$$\dot{K} = 0 \text{ and } Y = kX \rightarrow \dot{X} = 0 \rightarrow D = (1-f)(0(K_{2Y}^*) - rY) + gaK_{2Y}^* \quad (4)$$

Because earnings are growing on path 1, the rate of growth is increasing in this stage of the trajectory. Later on, when dealing with trajectories consisting of several growth stages, this will appear to be a property of all growth stages in the model. This is in accordance, for example, with results of Singh and Whittington, who found a positive correlation between the sizes of firms and their rates of growth (Singh & Whittington, 1975).

There is another interesting feature in the above basic trajectory. In fact, the firm's policy is based on different (sub)goals during the two stages. The first stage is governed by maximizing the growth rate: alle earnings are used for expansion investments, so, no dividend is paid out. In this way, the "state of bliss" (Das, 1974)

will be attained as quickly as possible. In the final stage, profit is maximal and by retaining earnings only to keep the capital stock at its optimal level, dividend pay-out is maximized. If we should replace the non-negativity constraint on dividend (4.15) by a more restrictive one, requiring a positive dividend outflow during the whole planning period, then the growth to the final stage will be retarded, because part of the financial means can no longer be assigned to the growth of the firm. However, such a constraint will not alter the arrangements of paths of the master trajectories to be discussed in this chapter. So, we have introduced the simple non-negativity constraint in order to avoid needless intricacies.

Due to the properties of the model, the switch from the growth stage to the stationary stage is rather abrupt. But in essence it corresponds with findings such as those of Grabowski & Müller, that mature firms have low yields on retained earnings and that shareholders of mature firms prefer dividend pay-out to retained earnings, while shareholders of younger firms prefer the opposite (Grabowski & Müller, 1975). From a macro economic inductive point of view we get support from Albin and Alcaly, who conclude to two disjunct equilibrium zones in an economy: one containing major firms marked by low growth rates and maximizing their profits, and the other containing relatively fast growing firms showing managerial behaviour such as striving to high growth rates (Albin & Alcaly, 1976).

As we have mentioned in chapter 4, the trajectory described in figure 1 is the "master trajectory" of the relevant set of optimal trajectories. This means that the initial conditions are such that all paths which are feasible in the relevant case are part of the trajectory:

$$X(0) < \frac{1}{q_2(1+k)} Q_{2Y}^* \quad (5)$$

$$K(0) = (1+k)X(0) \quad (6)$$

The initial amount of equity must be less than its stationary value and the firm must start with maximum debt financing. If initial condition (6) is not fulfilled, the firm will attract the missing amount of debt immediately at the start of the pattern and it will invest this amount in capital goods at once. After that, it starts its trajectory on path 1 (supposing condition (5) is still not binding). If the initial condition (5) is violated, the firm will sell the superfluous stock of capital goods (i.e. the stock above the level of Q_{2Y}^*/q_2) and will pay out the resulting revenue to the shareholders at the start of the trajectory too. In this way, the firm is in its optimal stationary state right at the beginning of the optimal trajectory and the optimal trajectory will consist of path 2 only.

Due to these instantaneous adjustments at the start of a trajectory, the firm seems to pass through only (parts of) the master trajectories. One may introduce time lasting adjustment trajectories from non-optimal initial states to the relevant master trajectories by incorporating such retarding features as adjustment costs or lower- and upperbounds on dividend and investments (or divestments). Because we want to emphasize features of the master trajectories themselves in this study, we have not incorporated them for the sake of simplicity and we will further assume that the firm is in such an initial state as to fulfill the initial conditions of the relevant master trajectory.

We will finish this section by a caveat concerning the interpretation of figure 5.1 (and the figures of master trajectories to be presented later on). Only the variables D, K and Y are measured in the same dimension (money), Q is measured in units of output and L in units of labour. So, the ranking of Q and L in this figure is arbitrary. The correct information to be drawn from figure 5.1 is that part of the capital stock is financed by debt money and that the relevant variables exhibit the same proportional growth.

3. Depth investments.

Our firm needs two kinds of input, labour and capital goods, in order to produce its output. It can produce this output in two different ways, one using relatively more capital goods, the other using more labour. Labour is a variable input in this model: the employment level of the firm can be perfectly adapted to the needs of the firm. But capital goods are not freely obtainable, especially not at the start of a trajectory, due to the fact that the firm needs money capital to buy capital goods and money capital is scarce because the firm can get new equity only from retained earnings and because debt capital is rationed. In this situation it may be profitable to start with the labour-intensive activity, even if it results in a higher unit cost than the capital-intensive activity. The reason is, that the firm can attain a higher output level with a given capital good stock by means of the labour-intensive activity than by means of the capital-intensive activity. This higher output brings about a higher sales level, which may compensate for the higher unit cost.

But, due to decreasing returns to scale, this advantage does no longer holds from a certain size on and the firm will then switch to the cheaper capital-intensive activity. This switch is called: depth-investment or intensive investment.

In order to introduce this phenomenon, we have to leave the assumption of section 2, that the labour-intensive activity is the cheapest one. We now assume that $c_{1Y} < c_{2Y}$ and then get the following master trajectory:

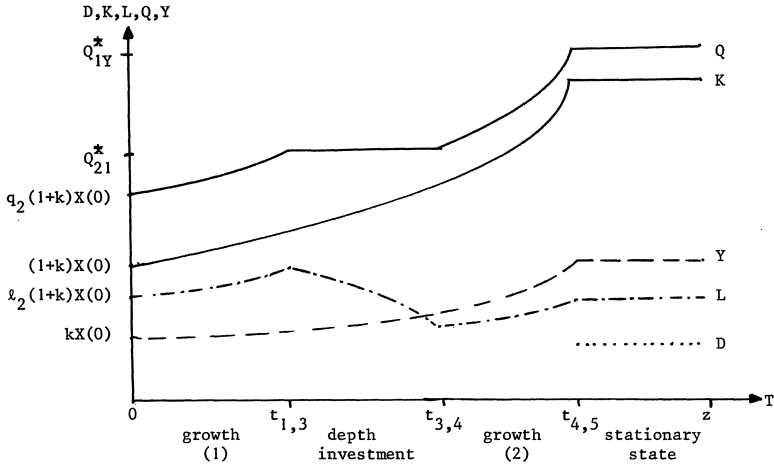


Figure 5.3. The master trajectory if $i > (1-f)r$ and $c_{1Y} < c_{2Y}$.

Compared with the former section, the growth to the final stationary stage is now interrupted by another stationary stage, as far as the output (and thus: sales) volume is concerned, starting at $Q = Q_{21}^*$.

Thus the firm passes through different stages during its optimal trajectory. Models describing the growth of the firm in this way are called: metamorphosis models (see: Kieser e.a., 1977; Albach, 1976).

The depth-investment stage starts when the output level Q_{21}^* is attained. Below this level, the marginal revenue of a capital good assigned to labour-intensity activity 2, exceeds the marginal revenue of a capital good assigned to capital-intensive activity 1:

$$\frac{\partial O}{\partial K_2} > \frac{\partial O}{\partial K_1} \tag{7}$$

By means of the definition of output in equation (4.1) and of operating income in (4.6), this results in:¹⁾

1) From (4.1), (4.6) and (7) we get:

$$q_1 K_1 \frac{\partial P}{\partial K_2} + q_2 P - w_2 + q_2 K_2 \frac{\partial P}{\partial K_2} - a > q_1 P + q_1 K_1 \frac{\partial P}{\partial K_1} - w_1 + q_2 K_2 \frac{\partial P}{\partial K_1} - a$$

(please turn over)

$$\frac{dS}{dQ} > \frac{w(\ell_2 - \ell_1)}{q_2 - q_1} := c_3 \quad (8)$$

So criterion (7) and the critical Q_{21}^* -value are linked through the definition of Q_{21}^* :

$$Q = Q_{21}^* + \frac{dS}{dQ} = c_3 \quad (9)$$

In order to explain this link, we rewrite the third inequality of footnote 1:

$$\frac{\partial S}{\partial K_2} - \frac{\partial S}{\partial K_1} > w\ell_2 - w\ell_1 + (q_2 - q_1) \frac{dS}{dQ} > (\ell_2 - \ell_1)w \quad (10)$$

In (10), inequality (7) is translated into a sales advantage of activity 2 based on its larger capital productivity ($q_2 > q_1$) and into a cost advantage of activity 1 due to its lower labour intensity ($\ell_1 < \ell_2$). Due to the decreasing marginal return dS/dQ and the growth of Q , there will be a moment at which the sign turns into its opposite: the larger output of activity 2 no longer results in a surplus of marginal sales such as to counterbalance the cost disadvantage. Therefore the firm will then switch to the capital-intensive activity 1.

A second remark on formula (10) concerns the absence of depreciation and the cost of capital. From (1) we know that they amount to:

$$\left(1 - \frac{g}{1-f}\right) a + \frac{k}{1+k} r + \left(\frac{1}{1+k} - g\right) \frac{i}{1-f} \quad (11)$$

per capital good. So they are independent of the activity to which the relevant (marginal) capital good will be assigned. Therefore they do not influence the level of Q_{21}^* .

Compared with the model in: Lesourne & Leban, 1978, described in chapter 3, we now have more information about the depth-investment process. We have designed a model yielding conditions to the appearance of depth-investments which have an economic meaning. Moreover, after a period of depth-investment, the firm starts growing again instead of sticking in a stationary state. In this way, depth-investments are described as a mean of reorganizing the production-process in order to enable further

(footnote 1, continued)

$$Q \frac{\partial P}{\partial K_2} + q_2 P - w\ell_2 > Q \frac{\partial P}{\partial K_1} + q_1 P - w\ell_1 + \frac{\partial PQ}{\partial K_2} - w\ell_2 > \frac{\partial PQ}{\partial K_1} - w\ell_1 +$$

$$q_2 \frac{dPQ}{dQ} - w\ell_2 > q_1 \frac{dPQ}{dQ} - w\ell_1 + \frac{dPQ}{dQ} > \frac{w(\ell_2 - \ell_1)}{q_2 - q_1}. \quad (\text{Note that } PQ := S)$$

growth. This fits in with the theory of critical thresholds in the life cycle of a firm, as presented for example in: Clifford, 1976, and: Albach, 1976, in the area of business finance and management.

Our last remark on the article of Lesourne and Leban concerns their definition of the depth-investment process as a substitution between labour and capital. As described in: Pasinetti, 1977, substitution is defined as the process caused by changes in the relative prices of the inputs. In our model, as well as in that of Lesourne & Leban, the change from labour-intensive to capital-intensive production is caused by the diminishing returns to scale. So, we prefer to speak about "reallocation" of labour and capital in this context.

As shown in figure 3, the reallocation process takes some time. This is because the firm needs more capital goods to produce an output Q_{21}^* by means of capital-intensive activity 1. This growth of the capital stock must be financed by means of retained earnings and additional debt money, restricted by the present amount of equity. So, the financial means to buy more capital goods are restricted and this results in the gradual course of the reallocation process.

In figure 5.4, the depth-investment process is shown in the same way as in figure 5.1. During the switch from activity 2 to activity 1, the output is kept constant, so the line $a_1 a_2$ is a so called isoquant. In order to find the slope of this line, we derive from (1)-(3):²⁾

$$Q = \frac{q_2 - q_1}{\ell_2 - \ell_1} L + \frac{q_1 \ell_2 - q_2 \ell_1}{\ell_2 - \ell_1} K \quad (12)$$

2) Conceive K_1 and K_2 as variables, of which the solution values are to be found from (1) and (2) for fixed values of Q and L , and apply Cramer's rule to solve the linear equation system (see for example: Chiang, 1974, 116):

$$\begin{aligned} D_1 &= \begin{vmatrix} Q & q_2 \\ L & \ell_2 \end{vmatrix} = \ell_2 Q - q_2 L ; D_2 = \begin{vmatrix} q_1 & Q \\ \ell_1 & L \end{vmatrix} = q_1 L - \ell_1 Q \\ D_3 &= \begin{vmatrix} q_1 & q_2 \\ \ell_1 & \ell_2 \end{vmatrix} = q_1 \ell_2 - q_2 \ell_1 \\ K &= K_1 + K_2 = \frac{D_1}{D_3} + \frac{D_2}{D_3} = \frac{\ell_2 Q - q_2 L + q_1 L - \ell_1 Q}{q_1 \ell_2 - \ell_1 q_2} \end{aligned}$$

which results in the above formula (see also: Schouten, 1957, for the same results in a slightly different notation).

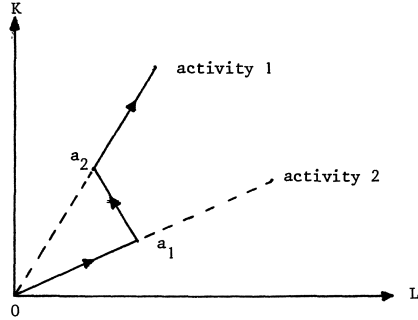


Figure 5.4. The depth investment process
in dynamic activity analysis.

Based on the fact that the isoquant is defined by:

$$dQ = 0 \quad (13)$$

we easily derive from (12) that its slope is:

$$\frac{dK}{dL} = \frac{q_2 - q_1}{q_1^{\ell_2} - q_2^{\ell_1}} < 0 \quad (14)$$

So, the expansion of the capital good stock during the reallocation period is attended with a decrease of labour in the relevant model. We may further derive Q_{12}^* from (12) in an alternative way. It makes sense to switch to the capital-intensive activity as soon as the marginal contribution to the profit stream of labour becomes negative:

$$\frac{\partial S}{\partial L} - w < 0 \rightarrow \frac{dS}{dQ} \frac{\partial Q}{\partial L} - w < 0 \rightarrow \frac{dS}{dQ} < \frac{w(\ell_2 - \ell_1)}{q_2 - q_1} \rightarrow Q > Q_{12}^* \quad (15)$$

The introduction of more (efficient) production activities in this two-inputs case will not yield new information. We then get a larger number of switches during the optimal trajectory towards still more capital-intensive activities, but the rules governing the reallocation process will not be different. Every time only two adjacent activities are compared in the same way as described in this section for the activities 1 and 2. Suppose, for example, that a third activity is introduced, which is more capital-intensive than activity 2, so:

$$q_2 < q_3, q_2/l_2 > q_3/l_3 \quad (16)$$

then the second reallocation process starts as soon as the firm has attained the output level Q_{23}^* , fixed by:

$$Q = Q_{23}^* + \frac{dS}{dQ} = \frac{w(l_3 - l_2)}{q_3 - q_2} \quad (17)$$

(assuming that $Q_{23}^* < Q_{2Y}^*$, of course).

4. Consolidation.

Both previous master trajectories dealt with the case of cheap debt money ($i > (1-f)r$) implying an optimal financial structure with maximum borrowing during the whole trajectory. In this section we will turn to the case of cheap equity ($i < (1-f)r$), yielding the possibility of a change in the optimal financial structure of the firm during its growth process. We further assume, that labour-intensive activity 2 has a smaller unit cost ($c_{1X} > c_{2X}$) which implies, just like in section 2, that the firm will assign all its capital goods to that activity during the whole planning period. In this way we can concentrate on the consolidation process only.

If the firm starts with a sufficiently small amount of equity (see our discussion about initial conditions in section 2), we get the optimal pattern as presented in figure 5.5 on the next page.

Figure 5.5 shows that the firm starts with maximum borrowing in spite of the fact that debt is the expensive way of financing. The reason is that marginal revenue exceeds the cost of debt-financing and so each additional capital good, bought by means of debt money, yields a positive income and so increases the rate of growth:

$$Q < Q_{2YX}^* + \frac{dS}{dQ} > c_{2YX} \quad (18)$$

in which : $c_{2YX} = \frac{1}{q_2} \{ w l_2 + (1 - \frac{g}{1-f})a + (1-g)r \}$

In (18), c_{2YX} is the unit cost of activity 2 if the relevant capital good is financed by debt money only (see the explanation of (1)). Formula (18) can be rewritten as:

$$Q < Q_{2YX}^* + \frac{1}{1-g} \{ (1-f) \frac{\partial O}{\partial K_2} + ga \} > (1-f)r \quad (19)$$

If we compare (19) with (2), we can easily derive that the left hand side of (19) is the marginal revenue of a capital good assigned to activity 2, apart from financing costs. The right hand side is the financing cost net of corporation profit tax if the

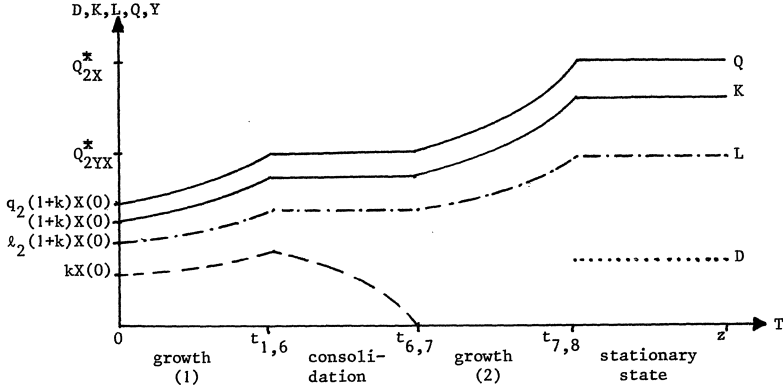


Figure 5.5. The master trajectory if $i < (1-f)r$ and $c_{1X} > c_{2X}$.

relevant capital good is financed by debt money only.

So, the firm will invest all its equity in capital goods and will furthermore attract as much debt money as is possible to invest in order to maximize the flow of earnings. Due to the fact that equity is cheaper than debt money ($i < (1-f)r$), shareholders will waive dividend pay-outs, because retained earnings, invested in the firm again, will yield a revenue larger than $(1-f)r$ and so larger than i , which is the "cut-off" criterion for the growth process to the shareholders, as we have already pointed out in section 2.

The link with the classical leverage formula may be obvious if we define:

$$\frac{1}{1-g} \left\{ (1-f) \frac{\partial Q}{\partial K_2} + ga \right\} = R : \text{marginal return to total capital}$$

$$(1-f)r = C_Y : \text{(marginal) cost of debt capital}$$

$$R_E : \text{marginal return to equity}$$

We then get the next leverage formula:

$$R_E = R + (R - C_Y) \frac{Y}{X} \quad (20)$$

From (20) we can conclude that increasing the leverage factor Y/X results in a higher return to equity if

$$R > C_Y$$

(21)

which is the same condition as for (19), resulting in the range of Q on which (maximum) debt financing is profitable.

As soon as the output level Q_{2YX}^* is attained, the marginal revenue as defined in (19) equals the marginal cost of borrowing. Shareholders can now choose from three different ways of spending the earnings:

1. Accept them as dividend pay-out, resulting in a rate of return of i , when invested elsewhere.
2. Use them for further expansion investments, yielding a rate of return less than $(1-f)r$, due to the decreasing return to scale.
3. Use them to pay back debt money, saving an amount of $(1-f)r$ rent payments.

Because the last possibility is the most attractive one, for: $i < (1-f)r$, the firm will now start to pay back debt money by means of retained earnings. Because rent payments are falling down during this consolidation process, a growing stream of earnings becomes available for redemption and the decrease of debt money accelerates till at $T = t_{6,7}$ in figure 5.5 all debt is paid back.

We can present this change of the financial structure in another way by means of a diagram of the state variables:

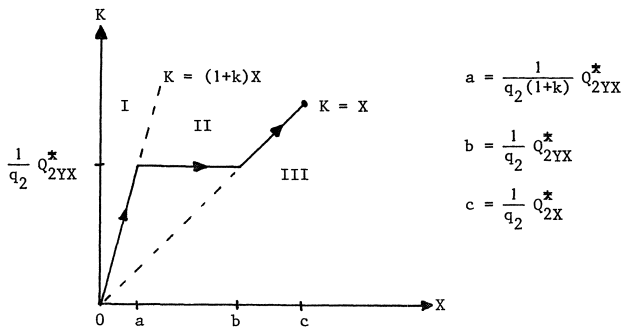


Figure 5.6. The consolidation process.

In figure 5.6, only area II is a feasible region. In area I debt amounts to more than the allowed maximum rate k , in area III there should be equity not invested in capital goods, which is excluded by equation (4.20). The relevant critical X -values can easily be derived from figure 5.5.

After this consolidation phase, the cost of capital has been decreased to such a degree that it is profitable for the firm to start growing again:

$$Q < Q_{2X}^* + \frac{dS}{dQ} > c_{2X} \quad (22)$$

$$\text{in which : } c_{2X} = \frac{1}{q_2} \left\{ w_2 + \left(1 - \frac{g}{1-f}\right)a + (1-g) \frac{i}{1-f} \right\}$$

In (22), c_{2X} is the unit cost of activity 2 if the relevant capital good is financed by equity only. This results in:

$$Q < Q_{2X}^* + \frac{1}{1-g} \left\{ (1-f) \frac{\partial O}{\partial K_2} + ga \right\} > i \quad (23)$$

Here, i is the cost of capital because of the situation of self-financing.

As soon as the firm has paid back its loans, it starts growing at a higher rate on path 7 than it has ended with on path 1.³⁾ The firm will continue this expansion till the output level Q_{2X}^* is attained on $T = t_{7,8}$ in figure 5.5. Then it will stop expanding in order to avoid that the marginal return falls below the critical i -level. The firm will keep investments on the replacement level k_{2X}^* and will pay out the remaining dividend to the shareholders.

Just as on the previous trajectories, we see an accelerating movement during each of the stages of the growth process. Further, the above described trajectory shows in a

3) At the end of path 1 it holds that:

$$\dot{X}(t_{1,6}) = (1-f) \left(0 - \frac{k}{1+k} rK\right) + g \left\{ (1+k) \dot{X}(t_{1,6}) - aK \right\} \quad (23a)$$

At the beginning of path 7, it holds that:

$$\dot{X}(t_{6,7}) = (1-f)0 + g \left\{ \dot{X}(t_{6,7}) - aK \right\} \quad (23b)$$

From (23a) and (23b) it results that:

$$\begin{aligned} \dot{X}(t_{1,6}) < \dot{X}(t_{6,7}) + \frac{1-f}{1-g(1+k)} \left(0 - \frac{k}{1+k} rK\right) - agK < \frac{1-f}{1-g} 0 - agK + \\ \frac{-k}{1-g} 0 < \frac{k}{1+k} rX \end{aligned}$$

which is always true, due to assumption (A2) of chapter 4, the non-negativeness of X and because of (4.27, 28).

simple way the change in the financial structure as it has been observed in several maturing firms (see: Albach, 1976, and: Clifford, 1976).

5. Depth-investments and consolidation.

We now have discussed the main features of our model: the growth- and stationary stages, the reallocation of labour and capital goods and the redemption of debt money. The last two master trajectories to be discussed here, contain all these features simultaneously. They only differ as far as the sequence of the reallocation and the consolidation process is concerned. Because all the relevant features have already been discussed in the previous sections, we only need to point out the differences between both trajectories.

From the above mentioned trajectories we have learnt, that consolidation only occurs if equity is cheaper than debt and that there is a switch to the capital-intensive activity only if this activity yields lower unit costs. To get both changes of the policy of the firm in the same trajectory, we have to assume that:

$$i < (1-f)r \text{ and: } c_{1X} < c_{2X} \quad (24)$$

Which of the relevant changes will occur first, appears to depend on the fact whether:

$$c_{1YX} \gtrless c_{2YX} \quad (25)$$

which can be rewritten into (see (18)):

$$\left\{ \frac{l_2}{q_2} - \frac{l_1}{q_1} \right\} w \gtrless \left\{ \frac{1}{q_1} - \frac{1}{q_2} \right\} \left\{ \left(1 - \frac{g}{1-f}\right)a + (1-g)r \right\} \quad (26)$$

On the left hand side, the part between brackets represents the difference in labour per unit output between both activities. So, the left hand side represents the marginal saving of wage payments per unit output when switching from labour-intensive activity 2 to capital-intensive activity 1. The first part between brackets on the right hand side represents the difference in capital per unit output between both activities. The second part represents the financing cost per capital good in the case of full debt financing. So, the right hand side stands for the increase of the financing cost per unit output when switching from labour-intensive activity 2 to capital-intensive activity 1. If the marginal saving of wage payments is larger than the increase of the financing cost, the firm will first decrease the labour-input by switching to capital-intensive activity 1 and after that it will worry about decreasing the cost of capital through the redemption of debt money:

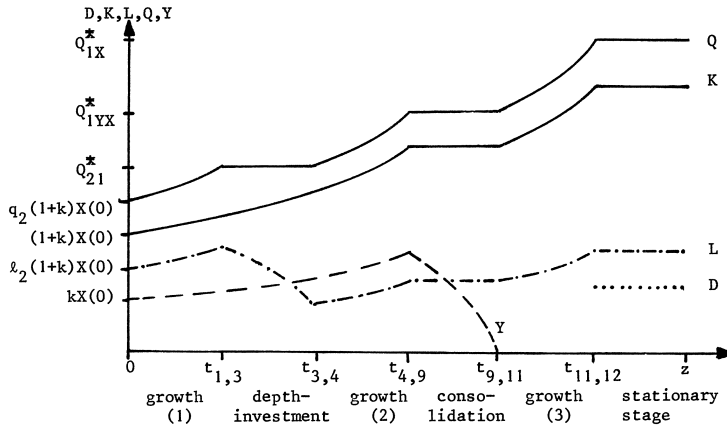


Figure 5.7. The master trajectory if $i < (1-f)r$, $c_{1X} < c_{2X}$ and:
 $c_{1YX} > c_{2YX}$.

The master trajectory of figure 5.7 shows two stages, having a stationary output level before the final stationary stage is attained. The reader may notice that, although the trajectory is more complex, all the relevant features have already been discussed with the help of the previous master trajectories. The possibility that depth investments and the reallocation process coincide, is prohibited by assumption (A4) of the former chapter for the sake of simplicity.

If the marginal saving on wage payments, as defined in (26), is less than the increase of the cost of capital, switching to capital-intensive activity 1 has no sense and the firm will first pay back its debt money and later on it will change to the capital-intensive activity (see figure 5.8 on the next page).

In the same way as in section 2, we can conceive the above trajectories as caused by different (sub)policies of the firm during the successive stages. Both patterns discussed in this section, contain three growth stages, on which managerial (sub)goals may be assumed to be dominating, and two "threshold" stages on which the firm mainly emphasize diminishing the (production or the financing) costs. Finally, in the stationary stage, a policy of guaranteeing maximum dividend pay-out is established.

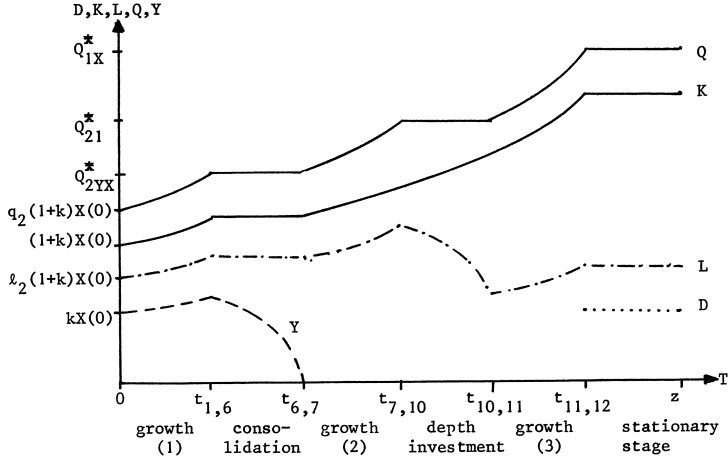


Figure 5.8. The master trajectory if $i < (1-f)r$, $c_{1X} < c_{2X}$ and:
 $c_{1YX} < c_{2YX}$.

6. Summary.

The master trajectories of the four different sets of trajectories, resulting from the optimal solution are presented here. Which of these sets is the optimal one, depends on whether equity or debt is the cheapest mode of money capital and on whether capital-intensive activity 1 or labour-intensive activity 2 brings about the smallest unit cost.

The initial values of the state variables, i.e. equity and the capital good stock, determine whether the whole relevant master trajectory is passed through or only a part of it.

All master trajectories consist of a succession of growth and stationary stages, which agrees with descriptions of the life cycle of firms as described by other authors. The conditions under which depth-investments and/or redemption of debt money may occur are presented and their economic meaning has been analysed.

CHAPTER 6. A FURTHER ANALYSIS

1. Introduction.

In chapter 5 we have described the optimal solution of our model in the same way as done in most publications on dynamics of the firm. Still, a lot of worthwhile economic analyses remain to be done. We will present two more ways of analysis in this chapter. The first way of analysis is a derivation of global decision rules, which together constitute the policy of the firm. The stepping stones in the four master trajectories of chapter 5 are the Q^* -values, of which several are present in more than one trajectory. In chapter 5 we have discussed the factors influencing the level of each Q^* -value. This chapter starts with another way of discussing these Q^* -values, based on the three aspects of the policy of the firm in the relevant model concerning production, finance and investment/dividend.

There after, we will study the influence of environmental changes on six different features of the growth process of the firm. This is a sensitivity analysis concerning parameters that are important in economic analysis: the interest rate r , the discount rate i , the wage rate w , the borrowing rate k , the corporation profit tax rate f and the investment grant rate g .

2. Optimal decision rules.

2.1. Production.

In the model, two types of decisions concerning production can be distinguished: to which activity should the capital goods be assigned? and: to what level should the output be increased? In this first part of section 2, we will restrict ourselves to the former question, as the latter one is in fact within the area of investment and dividend policy.

The firm can assign the available capital goods to capital-intensive activity 1 or to labour-intensive activity 2. In section 3 of chapter 5 we have already shown that the cost of capital is irrelevant in this assignment problem because this cost does not depend on the way in which a capital good will be used. This is in accordance with empirical findings of Gardner & Sheldon, 1975, who found no important financial influence on the capital/output rate of firms. Therefore, the decision rule is based on the marginal return to a capital good, defined as the difference between marginal sales and the marginal cost per capital good:

Assign a capital good to { activity 1 } if : $\frac{\partial S}{\partial K_1} - w_1 \{ \geq \} \frac{\partial S}{\partial K_2} - w_2$

$$\text{if : } \frac{dS}{dQ} \{ \geq \} \frac{w(l_2 - l_1)}{q_2 - q_1} \rightarrow Q \{ \geq \} Q_{21}^* \quad (1)$$

So, financing cost and depreciation are out of consideration in this decision rule. The output level on which the growing firm will switch to the capital-intensive activity 1 appears to depend on the wage rate and some more technical parameters. Note that government has no direct influence on this output level, for, the profit tax rate f and the investment grant rate g are not present in (1). We will use this information for the sensitivity analysis in section 3 of this chapter.

2.2. Financial structure.

The financial structure is characterized by the relative amounts of the two kinds of money capital that are available to the firm: equity and debt. The amount of debt that the firm can attract is restricted by the size of equity. So, the financial structure has two extreme cases: the case that the assets are financed by equity only and the case that the firm is financing by means of the maximal amount of debt that is allowed for. Which of both cases is the optimal one, depends on the marginal return to equity. This return depends, among others, on the chosen activity (which fixes the marginal return to a capital good, as discussed in section (2.1)) and the relevant financial structure (which fixes the cost of capital). In formula (2) of chapter 5, we have presented the marginal return to equity in the case of maximum debt financing and of a labour-intensive way of production (activity 2). From this formula we can derive the following formula of marginal return to equity in the case of maximum debt financing (R_{jY}):

$$R_{jY} = \frac{1}{1 - \frac{k}{1+k} - g} \left\{ (1-f) \left(\frac{\partial O}{\partial K_j} - \frac{1}{1+k} r \right) + ga \right\}, \quad j = 1, 2. \quad (2)$$

in which the suffix j stands for the actual production activity used by the firm. The expression has been explained in chapter 5 already. From (5.21) we can derive the marginal return to equity in the self-financing case (R_{jX}) in the same way:

$$R_{jX} = \frac{1}{1-g} \left\{ (1-f) \frac{\partial O}{\partial K_j} + ga \right\}, \quad j = 1, 2. \quad (3)$$

The firm will now try to realize such a financial structure as to maximize marginal return to equity, so:

choose for { self-financing
maximum debt financing } if: $R_{jX} \{ \geq \} R_{jY}$ "

$$\text{if: } \frac{dS}{dQ} \{ \geq \} \frac{1}{q_j} \{ w_j + (1 - \frac{g}{1-f})a + (1-g)r \} \text{ -- if: } Q \{ \geq \} Q_{jYX}^* \quad (4)$$

From (4) we can derive that the financing decision is influenced by all parameters to be discussed in the sensitivity analysis. Above that, also the choice of the production activity has its impact on the decision through the technical parameters w_j and q_j . The above discussion is a way to explain Q_{jYX}^* , alternative to the discussion in section 4 of chapter 5.

2.3. Investment and dividend.

The last decision rule to be studied in this section concerns the investment and dividend policy of the firm. The firm can spend its earnings in two ways: to pay out dividend or to retain it in the firm in order to invest in capital goods and/or to pay back debt money. The last mentioned decision has implicitly been discussed in the previous part of this section: redemption of debt starts as soon as the firm attains the Q_{jYX}^* -level on which self-financing becomes optimal instead of maximum debt financing. The second possibility, is preferable as long as marginal return to equity exceeds the discount rate of the shareholders i , for the discount rate represents the rate of return that the shareholders can obtain elsewhere. As soon as marginal return to equity falls below i , the firm will pay out dividend instead of going on with expansion investments, as we have discussed already in section 2 of chapter 5. In that case the firm will still invest but only on the replacement level so as to keep the capital good stock (and so: the output) on the optimal level. In this way, the following decision rule can be designed:

don't pay out dividend and spend all earnings on investments
 { make only replacement investments and pay out all remaining earnings } if $R_{jN} \{ \geq \} i$
 decrease the capital good stock and pay out all earnings

$$j = 1, 2 ; N = X, Y \quad (5)$$

The three decision rules as formulated in (1), (4) and (5), cover all the Q^* -values of the master trajectories of chapter 5. These expressions reveal that there is in fact only one policy of the firm, consisting of three decisions rules. The variety of optimal trajectories is caused by differences in the initial state of the firm and by different environmental conditions, represented by different sets of values of the parameters under which it has to operate.

3. Environmental influence on the trajectory of the firm.

In the previous section, we have explained the way in which the relevant Q^* -levels are fixed. Now, we will study how changes in the values of the parameters, enumerated in section 1, influence these values (reallocation, final output and consolidation) and the growth of the firm towards those threshold values (expansion). Moreover we will study changes in the parameters that cause a switch to another master trajectory (substitution and financial substitution effect). In this way we will discuss the environmental influence on the six different features that characterize the shape of the master trajectories.

3.1. Reallocation.

In chapter 5 we discussed the reallocation of labour and capital due to decreasing marginal returns. During the optimal trajectories of the firm, labour and capital are complementary inputs, due to the assumed linear production activities, except in the depth investment stage, in which:

$$Q = Q_{21}^* + \frac{dS}{dQ} = \frac{w(\ell_2 - \ell_1)}{q_2 - q_1} \quad (6)$$

Before going on, we remind that we are dealing with decreasing returns to scale, so changes in the values of the parameters that cause a rise (fall) of the value of the right hand side imply a fall (rise) of the value of Q_{21}^* . From (6) and (4.25) we now can derive that a rise of the wage rate will decrease the output level on which the firm starts the reallocation process. None of the other parameters, mentioned in the beginning of this chapter, appear to influence this level.

3.2. Final output.

In the final stage of the trajectory, the firm has attained the optimal level of output and it yields maximal profit. The level of output (and of employment) depends on the values of the environmental parameters, for, from the previous section we know that this level is fixed by:

$$R_{jN} = i + \quad (7)$$

$$\frac{dS}{dQ} = c_{jX} = \frac{1}{q_j} \left\{ w\ell_j + \left(1 - \frac{g}{1-f}\right)a + (1-g) \frac{i}{1-f} \right\} \text{ when } i < (1-f)r$$

$$\frac{dS}{dQ} = c_{jY} = \frac{1}{q_j} \left\{ w\ell_j + \left(1 - \frac{g}{1-f}\right)a + \frac{k}{1+k} r + \left(1 - \frac{k}{1+k} - g\right) \frac{i}{1-f} \right\}$$

$$\text{when } i > (1-f)r \quad (8)$$

From (8) we conclude that, in the case of $i > (1-f)r$, a rise of the profit tax rate f , of the discount rate i , of the interest rate r and/or of the wage rate w will decrease the final output level and hence the level of employment and the amount of issued dividend. On the other hand, an increase of the borrowing rate k and the investment grant rate g will raise the stationary value of Q . The explanation is quite obvious and will therefore be left to the reader.

In the case of expensive debt money ($i < (1-f)r$), neither r nor k influences the final output level, because the firm does not borrow in its final stage. The remaining parameters f , g , i and w affect the final output level in the same way as in the above case.

3.3. Consolidation.

The third feature of the growth process of the firm to be studied here is the output level at which the firm starts its consolidation. From chapter 5 we know that this level is fixed by:

$$Q = Q_{JYX}^* - \frac{dS}{dQ} = \frac{1}{q_j} \left\{ w k_j + \left(1 - \frac{g}{1-f}\right)a + (1-g)r \right\} \quad (9)$$

In the same way as in the analysis of the final output effect, we can derive directly from (9) that the firm will start paying back its debt at a lower level of output, when the wage rate w or the interest rate i is increasing. A rise of the corporation profit tax rate f and of the investment grant rate g will increase the relevant output level.

3.4. Expansion.

In this part we will discuss the environmental influence on the rate of growth of the firm. We can measure the firm size, and thus its rate of growth, by means of several standards such as sales, employment, assets and equity. Smyth e.a., 1975, and: Shalit & Sankar, 1977, have shown that in empirical research, these standards are not interchangeable without any more and that different conclusions can be drawn, depending on the measure chosen by the analyst. As in our model shareholders wealth is the criterion function, we have chosen equity as a measure of the size of the firm, because this standard is the only one relevant for the shareholders.

Because we have fixed the lower bound of dividend pay out on zero value, the firm does not pay out any dividend before attaining the final stationary stage. So, from (4.17) we derive that before entering the final stage, it holds that:

$$\dot{X} = (1-f)(0 - rY) + gI \quad (10)$$

$$\frac{\partial \dot{X}}{\partial g} = I > 0 \quad (11)$$

In this way, the model confirms that investment grants have a positive influence on the rate of growth of the firm. From (10) and (4.23) we can derive that:

$$\frac{\partial \dot{X}}{\partial w} = - (1-f)(k_1 K_1 + k_2 K_2) < 0, \quad (12)$$

which shows the negative influence of the wage rate on the rate of growth. The same holds for the corporation profit tax rate f , if we assume that:

$$\left(1 - \frac{1}{1+k} - g\right) i > ga \quad (13)$$

The left hand side of (13) is the minimum return to a capital good, necessary to satisfy the shareholders. The part between brackets is the reciprocal of the purchasing power multiplier in (5.2). It represents the amount of equity needed to buy a capital good when it is financed with as much debt and investment grants as possible. The right hand side is the decrease in depreciation caused by the investment grant. So, in (13) we assume that the return of the relevant capital good to the shareholders is not based only on the advantage of investment grants. This assumption is sufficient to derive from (10) that:¹⁾

$$0 - rY > 0 \rightarrow \frac{\partial \dot{X}}{\partial f} < 0 \quad (14)$$

When the firm is borrowing, the interest rate also has a negative influence on the rate of growth, for, from (10) we can derive:

1) From (10), $Y < kX$ and: $X + Y = K$ results that:

$$\frac{\partial \dot{X}}{\partial f} = - (0 - rY) < - 0 + \frac{k}{1+k} r K$$

From the master trajectories of chapter 5 we can derive that, when $i > (1-f)r$:

$$Y > 0 \rightarrow Q < Q_{jY}^* \rightarrow \frac{\partial 0}{\partial K} > \frac{k}{1+k} r + \left(1 - \frac{k}{1+k} - g\right) \frac{i}{1-f} - \frac{g}{1-f} a$$

Due to the concavity of S , and so of 0 , this yields:

$$0 > K \frac{\partial 0}{\partial K} \rightarrow 0 > K \left\{ \frac{k}{1+k} r + \left(1 - \frac{k}{1+k} - g\right) \frac{i}{1-f} - \frac{g}{1-f} a \right\}$$

which results, together with (13), in (14). In the case of $i < (1-f)r$ (in which:

$Y > 0 \rightarrow Q < Q_{jYX}^*$) the same results can be derived.

$$\frac{\partial \dot{X}}{\partial r} = - (1-f) Y < 0 \text{ when } Y > 0 \quad (15)$$

Finally, when the firm is borrowing at the maximum rate, the value of the borrowing rate will influence the rate of growth:

$$Y = kX + K = (1+k)X + X = \frac{1}{1-g(1+k)} \{ (1-f)O + \{ (1+k)a - k(1-f)r \} X \} \quad (16)$$

From (16) we derive that:²⁾

$$\frac{\partial \dot{X}}{\partial k} > 0, \quad (17)$$

which implies that relaxing the borrowing constraint will accelerate the growth process of the firm.

3.5. Substitution.

In this part we will discuss substitution between labour and capital in the final stage of a trajectory, i.e. the change in the relative amounts of both inputs due to a change in their relative prices, at a given level of output. This appears through a switch from one production activity to the other in the final stage. As the output level, and thus total and marginal returns are fixed, the firm will minimize its costs. This agrees with findings in the previous chapter, based on the shape of the master trajectories, that the final activity is

2) Expression (16) results in:

$$\frac{\partial \dot{X}}{\partial k} = \left\{ \frac{1}{1-g(1+k)} \right\}^2 \{ (1-f)gO + [a - (1-g)(1-f)r + (1-g(1+k)) (1-f) \frac{\partial O}{\partial K}] X \}$$

due to the concavity of O and of the fact that $K = (1+k)X$, this implies:

$$\frac{\partial \dot{X}}{\partial k} > \left\{ \frac{1}{1-g(1+k)} \right\}^2 \{ (1-f) \frac{\partial O}{\partial K} + a - (1-g)(1-f)r \} X$$

When $i > (1-f)r$, so $Q < Q_{jY}^*$ one can derive, like in footnote 1, :

$$\frac{\partial \dot{X}}{\partial k} > \left\{ \frac{1}{1-g(1+k)} \right\}^2 \left\{ \left(1 - \frac{k}{1+k} - g \right) (1 - (1-f)r) + (1-g)a \right\} X > 0$$

When $i < (1-f)r$, so $Q < Q_{jYX}^*$, the above inequality results in:

$$\frac{\partial \dot{X}}{\partial k} > \left\{ \frac{1}{1-g(1+k)} \right\}^2 (1-g) a > 0$$

$$\left\{ \begin{array}{l} \text{activity 1} \\ \text{activity 2} \end{array} \right\} \text{ when } c_{1N} \left\{ \begin{array}{l} < \\ > \end{array} \right\} c_{2N}, N = X, Y \leftrightarrow \text{ when } \frac{w}{c_K} \left\{ \begin{array}{l} < \\ > \end{array} \right\} s_{12} \quad (18)$$

in which : c_K : cost of capital in the final stage

$$= \left(1 - \frac{g}{1-f}\right) a + \frac{k}{1+k} r + \left(1 - \frac{k}{1+k} - g\right) \frac{1}{1-f} \text{ when } i > (1-f)r$$

$$= \left(1 - \frac{g}{1-f}\right) a + (1-g) \frac{1}{1-f} \text{ when } i < (1-f)r$$

s_{12} : rate of technical substitution between activity 1 and activity 2

$$= \frac{q_2 - q_1}{k_2 q_1 - k_1 q_2} > 0$$

This is in accordance with the well known analysis in static micro economics, which we will present with the help of the following figure:

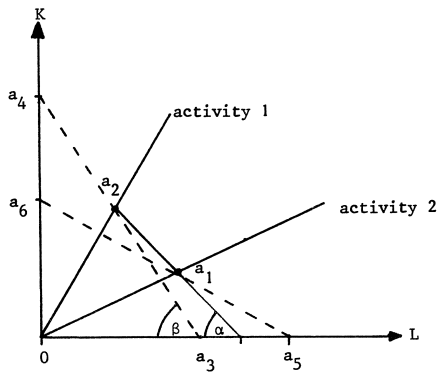


Figure 6.1. Substitution of labour and capital.

The line a_1a_2 is the isoquant as defined in (5.13). In (5.14) we derived its slope, and so $\tan \alpha$:

$$\text{tang } \alpha = - \frac{dK}{dL} = \frac{q_2 - q_1}{q_1^{\lambda_2} - q_2^{\lambda_1}} := s_{12} \quad (19)$$

We assume this slope to be exogenously fixed. Now, consider the so called iso-budget line a_3a_4 , defined by:

$$\text{total costs} = c_K K + w L = y \quad (: \text{ fixed budget}) \quad (20)$$

This line represents all combinations of inputs of labour and capital if the budget of y is spent. The point where the iso-budget line touches the isoquant of the highest output level, represents the combination of inputs of labour and capital that results in the highest output level for a fixed budget y and for fixed prices of labour, w , and capital, c_K . From figure 6.1 one can derive that this point is a_2 (so the firm prefers the capital-intensive activity) if:

$$\text{tang } \beta > \text{tang } \alpha \quad (21)$$

so, from (19) and (20), if:

$$\frac{w}{c_K} > s_{12} \quad (22)$$

This is in accordance with (18). Now, the value of $\text{tang } \beta$ may decrease due to a decrease of the cost of labour, w , and/or a rise of the cost of capital c_K . Then the iso-budget line (we still keep the budget fixed on the level y) switches to a_5a_6 and a_1 will become the optimal combination of inputs.

The improvement of (18) compared to the analysis of figure 6.1 is that, due to the more complex underlying model, we have derived more details of the composition of the cost of capital c_K and so we are able to trace more precisely the influence of separate parameters on the substitution process.

From (18) can be derived that a rise of the wage rate w , the investment grant rate g and the borrowing rate k will stimulate the choice of the capital-intensive production activity 1 and so substitution in a capital-intensive direction. A rise of the interest rate r and of the discount rate i will stimulate substitution in a labour-intensive direction. The same is true for a rise of the corporation profit tax rate f , due to assumption (11) in this section.

3.6. Financial substitution.

From the master trajectories, described in chapter 5, one can conclude that the optimal final financial structure is:

$$\left\{ \begin{array}{l} \text{maximum debt financing} \\ \text{self-financing} \end{array} \right\} \text{ when } \frac{i}{1-f} \left\{ \begin{array}{l} > \\ < \end{array} \right\} r \quad (23)$$

If we call $i/(1-f)$: the price of equity, and r : the price of debt, then we may use the term "financial substitution" to denote a change in the inputs of debt and equity in the final stage due to a change in their (relative) prices. This is analogous to the substitution process as defined in production theory (see the previous part of this section). Let us describe this with the help of the following figure:

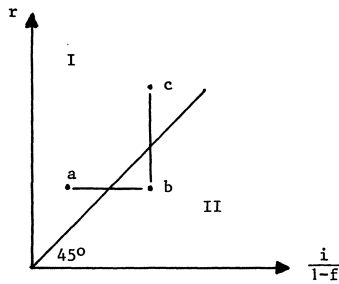


Figure 6.2. Changes that cause financial substitution effects.

If the set of values of the relevant parameters belongs to area I of figure 6.2, the firm will finally finance its equipment only by means of equity. In area II the firm will borrow at the maximum rate in its final stage.

A movement from a to b may be caused by a rise of the time preference rate and/or a rising corporation profit tax level. It provokes an increase of debt at the cost of equity. A movement from b to c may be caused by a rising interest rate. In that case, debt is pushed out and replaced by equity.

In the next table we have summarized the findings of this section:

a rise of	profit tax rate f	investm. grant r. g	time pref. rate i	borrowing rate k	interest rate r	wage rate w
Impact on						
reallocation level	0	0	0	0	0	-
final output level	-	+	-	(+)	(-)	-
consolidation level	+	+	0	0	-	-
expansion rate	-	+	0	+	-	-
substitution	+ L	+ K	+ L	(+ K)	(+ L)	
financial substitution	+ Y	0	+ Y	0	+ X	0

Table 6.1. Impact of the parameters on the main features of the master trajectories.

In which : + : rise of the feature value
 - : fall of the feature value
 0 : no influence on the relevant feature
 + K : substitution in a capital-intensive direction
 + L : substitution in a labour-intensive direction
 + X : substitution towards a self-financing structure
 + Y : substitution towards maximum borrowing
 () : the parameter only influences the feature if $i > (1-f)r$

4. Influence of (sets of) environmental parameters.

In the former section we have studied the main features of the master trajectories and how they are influenced by changes in the values of the environmental parameters. So, we have discussed the separate rows of table 6.1 there. In this section we will discuss the columns of table 6.1 by considering changes in the environmental parameters apart from each other. We will study the over-all influence of each such parameter on the trajectory by putting together its influences on the different features. We will demonstrate this with the help of the first optimal trajectory of chapter 5, section 5. The reader can easily do the same analysis for the other trajectories himself. In the remaining part of this chapter, substitution effects will be left out of consideration because they imply a change to another trajectory.

We will present the figure of the relevant trajectory again before starting the discussion:

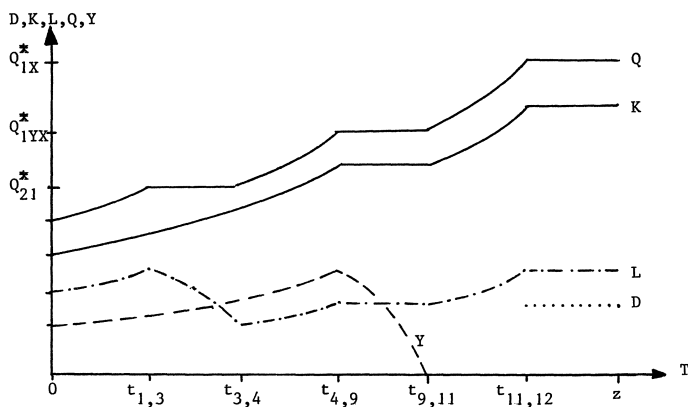


Figure 6.3. The master trajectory if $i < (1-f)r$, $c_{1X} < c_{2X}$ and: $c_{1YX} > c_{2YX}$.

4.1. Corporation profit tax rate.

From table 6.1 can be concluded that the corporation profit tax rate has different, sometimes opposite influences on the optimal trajectory of the firm. As far as the three Q^* -values are concerned, table 6.1 indicates that a rise of the corporation profit tax rate f will result in a decrease of Q_{1X}^* (final output effect), an increase of Q_{1YX}^* (consolidation effect) and an unaltered Q_{21}^* -level. The fact that Q_{1YX}^* rises, meaning that the firm will postpone the consolidation process, stems from decreasing net cost of borrowing when the tax rate rises, due to the tax deduction effect. On the other hand, the rise of the tax rate will decrease earnings after tax payments from which (expansion) investments have to be paid. In this way, the rate of growth falls down. This all may result in the altered shape of the trajectory of figure 6.3 as presented in figure 6.4 on the next page.

The signs on the vertical axis of figure 6.4 indicate an increase (+) or a decrease (-) of the relevant Q^* -value, compared to figure 6.3. The signs on the horizontal axis indicate an increase (+) or a decrease (-) of the relevant period, compared to figure 6.3.

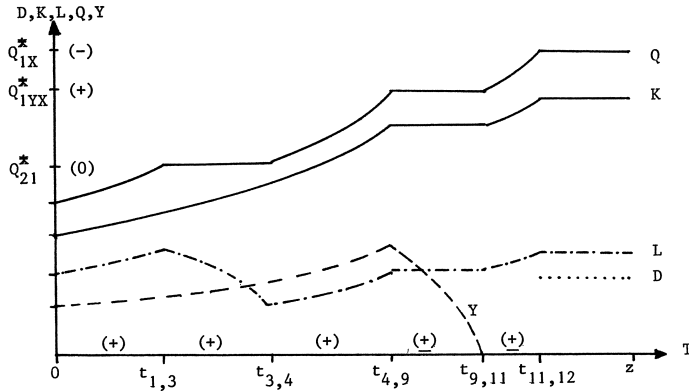


Figure 6.4. Change of the trajectory of figure 6.2 when f increases.

The first two periods increased due to the lower rate of growth and the unaltered Q_{21}^* value. The third period increased because of the lower rate of growth and because of the rise of Q_{1YX}^* . The fourth period increased due to the lower rate of growth and due to the larger amount of debt to be paid back. On the other hand, the increased value of Q_{1YX}^* implies a greater profit volume which will (partly) counter-balance both time lasting effects. In the fifth period two opposite influences of a rise of f hold, too: the fall of the rate of growth due to increased taxes and the smaller distance between Q_{1YX}^* and Q_{1X}^* due to a rise of Q_{1YX}^* and a fall of Q_{1X}^* .

We can conclude that a rise of the corporation profit tax rate f will favour employment in the firm till $T = t_{9,11}$: the depth-investment process is postponed and it will take place in a more moderate tempo. Above that the employment level in the fourth stage will be on a higher level. Further, due to a rise of the corporation profit tax rate, the firm will keep its debt for a longer time period, for, till $T = t_{4,9}$ all periods increase. But, after $T = t_{9,11}$, so when we are talking about mature firms, a rise in the corporation profit tax rate will provoke, beside a decreasing growth rate, also declining profits and also a declining employment level in the final stage.

From the above discussion the enrichment may become clear of economic analysis due to the introduction of dynamics. In static theory only the influence of the corporation

profit tax rate in the final stage can be studied, while dynamic analysis reveals a more complex influence of the corporation profit tax rate, depending on the maturity of the firm.

4.2. Investment grant rate.

A rise of the investment grant rate will cause, according to table 6.1, a rise of Q_{1X}^* as well as of Q_{1YX}^* , while the value of Q_{21}^* will be unaltered. As the rate of growth will also rise due to the additional financial means, we get the following figure:

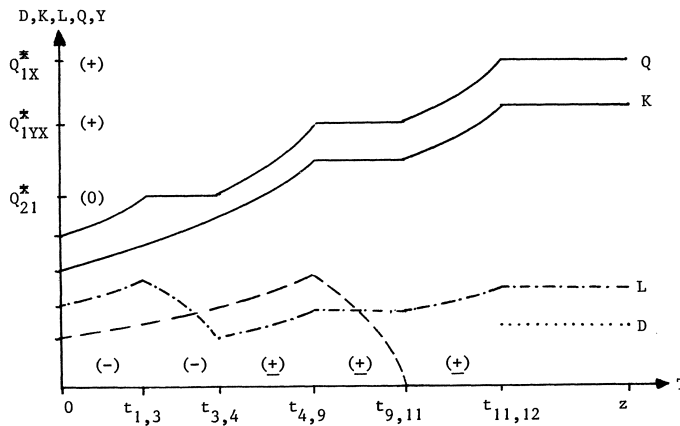


Figure 6.5. Change of the trajectory of figure 6.2 when g increases.

The government of several countries have introduced investment grants mainly to increase employment by stimulating investments. So, it is interesting to see whether employment will in fact increase when g rises. From figure 6.5 can be concluded that, due to the higher growth rate, employment rises more quickly in the first stage. But the reverse of the medal is that the same higher growth rate makes earlier the moment on which depth investments, and thus the decrease in employment, will start (note that the output level Q_{21}^* , on which this process starts, is not altered). After this period of depth investments, investment grants will influence employment in a positive way because of the increase of the growth rate and of the threshold values of Q_{1YX}^* and Q_{1X}^* . So, investment grants may have opposite influences. On the one hand, they lower the cost of capital, thereby freeing financial means to stimulate growth and to attain higher output levels. On the other hand, the decrease of the cost of capital

stimulates firms to depth investments and push out labour, if there is an attractive way of capital-intensive production available.

4.3. Abolishing investment grants.

The rates we have discussed in the previous two parts of this section can be conceived as describing the influence of government on the policy of the firm. The main difference between them is that the corporation profit tax is a global instrument, having a checking influence, while investment grants are awarded to stimulate separate firms. One might wonder what kind of influence should result from coupling both instruments by assuming, for example, that the government may decrease the investment grant rate and uses the financial means saved to decrease the corporation profit tax rate in order to decrease the specific character of its policy. Verhoeven, 1982, found some figures that enable us to link both instruments for the Dutch case. He calculated that, if the government should fully abolish investment grants, the corporation profit tax rate should be decreased from 48% to 22,5%. This is an extreme case of course, but it may clarify the combined effect of diminishing investment grants as well as corporation profit tax.

Consider the investment grant rate g and the corporation profit tax rate f as variables. Then, we can derive from (10) that, before the stationary stage it holds that:

$$\dot{\Delta X} = - (0 - rY) \Delta f + I \Delta g \quad (24)$$

Abolishing investment grants and the above mentioned decrease of corporation profit tax imply:

$$\Delta g = -g \text{ and } \Delta f = 0,225 - 0,48 = -0,255 \quad (25)$$

From (24) and (25) we can derive that the above combination of governmental measures causes an acceleration of the growth of the firm if:

$$\dot{\Delta X} > 0 \rightarrow 0,255 (0 - rY) > gI \quad (26)$$

so, if investment grants received by the firm, gI , are less than 25½% of profit before tax, $0 - rY$.

We further know from the previous parts of this section that Q_{21}^* is insensitive to changes of f and g and that Q_{1YX}^* will decrease when f and/or g is falling. As far as Q_{1X}^* is concerned, we can derive from (3) that:

$$Q = Q_{1X}^* + \frac{\partial Q}{\partial K_1} = \frac{i - g(a+i)}{1-f} \quad (27)$$

Abolishing investment grants, say at $T = t_a$, will increase (decrease) Q_{1X}^*

$$\text{if : } \frac{\partial Q}{\partial K_1} \begin{matrix} > < \\ T < t_a & T > t_a \end{matrix} \quad (28)$$

Because: $T < t_a + f = 0,48$ and: $T > t_a + f = 0,225$, $g = 0$; (28) implies:

$$\text{if : } \frac{i - g(a+i)}{0,52} > < \frac{i}{0,775} \rightarrow \text{if : } i > < \frac{g}{0,33 - g} a \quad (29)$$

The effect thus depends on the time preference rate of the shareholders i , the investment grant rate g and the depreciation rate a . We can present the relationship by means of a table in which the critical i -values appear as a function of the lifetime of the investment project and the average investment grant rate of the relevant project. In this table we have transformed the relevant rates of the continuous model into values of the discrete rates as used in practice:³⁾

average grant% \ lifetime	16%	12%	8%	4%
3 years	38,2%	23,2%	13,0%	5,6%
6 years	17,2%	10,4%	5,8%	2,5%
9 years	11,1%	6,7%	3,8%	1,6%
12 years	8,2%	5,0%	2,8%	1,2%
15 years	6,7%	3,9%	2,2%	0,9%

Table 6.2. Threshold values of i concerning the influence of investment grants on final employment.

3) Say i' is the discount rate used by the shareholders in % per year, than:
 $\ln(1+i') = i$. Further: $a = -\ln\left(1 - \frac{1}{\text{lifetime}}\right)$

From table 6.2 can be concluded that abolishing investment grants will increase profit and employment in the final stage of the optimal trajectory of figure 6.2 for medium and long range investment projects (> 6 years) that do not consist of investments that are granted at the maximum rate ($g < 0,12$). Assume that we are dealing with such a project, then we can put all the above mentioned effects together in the following figure:

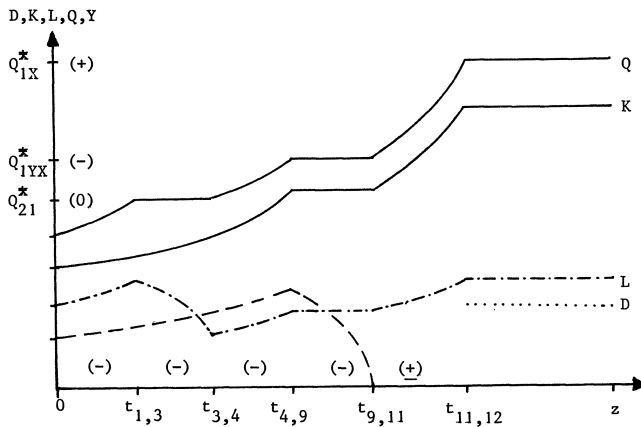


Figure 6.6. Change of the trajectory of figure 6.2 when $g = 0$ and f decreases.

From figure 6.6 can be concluded that, if the government should abolish investment grants in the Dutch case and should lower the profit tax rate, then this should stimulate depth investment and consolidation in younger firms having a high marginal return to sales, and should increase the profit and employment level in mature firms, having a more moderate marginal return to sales, supposed that equity and capital-intensive production are cheap.

4.4. Financial parameters.

If we want to study changes in the time preference rate i , the borrowing rate k and/or the interest rate r , we should be aware of the causality between changes of their values. A change of i and/or r may be caused by autonomous changes in the market, and they may or may not influence the value of k . But, if we conceive the value of k as an indicator of the risk class to which the firm belongs (see chapter 2, section 7), then

a rise of k implies a switch to a class of firms with a higher degree of risk and we can assume that thus a rise of k will cause a rise of r and i . In this section we will study the relation last mentioned in more detail.

From table 6.1 we can conclude that these simultaneous changes will not influence the reallocation level Q_{21}^* and will lower the consolidation level Q_{1YX}^* in figure 6.3.

The influence of the change mentioned above in the values of the three financial parameters on the expansion rate is not clear without anymore, because of the contrary sign of the influence of a rise of the borrowing rate k (+) and of the interest rate r (-). The relationship assumed above can be presented as:

$$r = r(k) \text{ with } : \frac{dr}{dk} > 0 \text{ and } : i = i(k), \text{ so:} \quad (30)$$

$$\dot{X} = \frac{\partial X}{\partial k} dk + \frac{\partial X}{\partial r} dr + \frac{\partial X}{\partial i} di = \frac{\partial X}{\partial r} \frac{\partial r}{\partial k} dk + \frac{\partial X}{\partial r} dr$$

$$\text{having the sign: } \{(-) \times (+) \times (+)\} + \{(-) \times (+)\} = - \quad (31)$$

From (31) results that a rise of k and hence of r (and i) will lower the expansion rate during the period that the firm is borrowing at the maximum rate. After the consolidation period, k nor r appear in the \dot{X} -formula, so they do not longer influence the rate of growth. In fact, during the consolidation period the firm changes to a less risky class, which will decrease i and r . We skip this refinement here, but we refer to the work of Senchak, 1975, mentioned already in section 7 of chapter 2 who introduced the interest rate as a function of the leverage factor. As far as our trajectory is concerned, we assume that the rise of k will only raise r and i during the first four periods, till the consolidation stage is finished. So the final output level Q_{1X}^* will not be influenced. Taking all in all, when k , r and i are rising we get a change of figure 6.3 as presented in figure 6.7.

The firm starts in figure 6.7 at a higher output level, compared to figure 6.2, due to the extended borrowing facility. Together with the unchanged reallocation level Q_{21}^* this will shorten the first period. Opposite influence comes from the decreased expansion rate, so the length of the first period is undetermined. Also in the third period contrary forces influence the length of the period: the decrease of the consolidation level Q_{1YX}^* will shorten the period, while the decreased expansion rate will extend the length of the period. In the consolidation stage, the amount of debt to be paid back is positively influenced by the rise of the value k , while the fall of the value of Q_{1YX}^* has the opposite effect. In the beginning of this stage, the increase of \dot{X} will be lower than in figure 6.3 due to the higher leverage level.

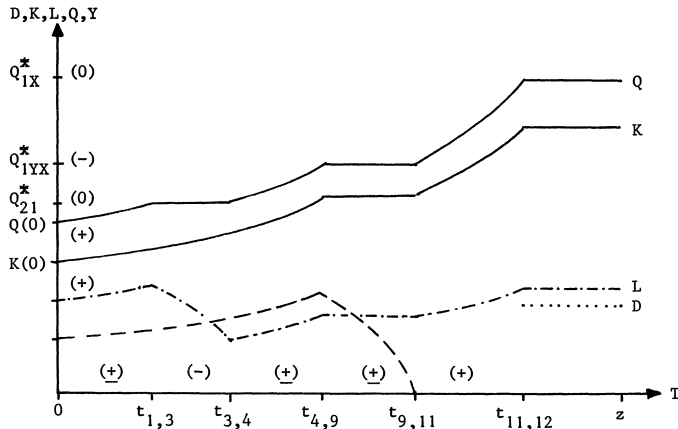


Figure 6.7. Change of the trajectory of figure 6.2 when k , i and r increase.

Finally, in the fifth stage, the length of the period is determined by the greater difference between the values of Q_{1X}^* and Q_{1YX}^* .

Although we could very well trace the changes in the Q^* -values due to a rise of k , i and r , it still remains difficult to give a picture of the change of the whole trajectory because of the decreasing growth of equity. Anyhow, the firm starts at a higher output and employment level and will start its consolidation at a lower output level. Due to the fact that all debt has been paid back before the final stages are entered, and so the time preference rate has taken its value of figure 6.3 again, final profit and employment levels remain unchanged. The decreased growth of equity up to the fourth period (so: in younger firms) may be translated into a flatter change of employment and of output in those stages.

4.5. Wage rate.

From table 6.1. we can conclude that a rise of the wage rate will lower the output levels on which the firm starts its depth investments, its redemption of debt money and its final output stage. This need not to imply that the final stage will be attained earlier, for, the expansion rate will decrease as well. This implies that in the case of a rise (or fall) of the wage rate, the change of the length of any period is

undetermined, unless we have more specific information about the values of the parameters of the model. Anyhow, a rise of the wage rate will decrease output and employment during the whole trajectory of the firm.

5. Summary.

An analytical solution of an optimal control model lends itself very well to all kinds of (marginal) analysis. In this chapter we studied the solution in more detail by means of three presentations. First, we derived decision rules the firm has to apply in order to realize its optimal policy. These decision rules relate to production, finance and investment, the building blocks of our model. After that, we concerned ourselves with the influence of several "environmental" parameters on six distinct characteristics of the master trajectories. This influence may cause changes in the shape of the relevant trajectory or it may cause the firm to change to another master trajectory. Finally we studied the environmental influence through changes in the values of three sets of parameters: the governmental parameters (corporation tax rate and investment grant rate), the financial parameters (borrowing rate, interest rate and time preference rate of the shareholders) and, at the end, a social parameter (the wage rate).

CHAPTER 7. CONCLUSIONS.

In this short concluding chapter, we will look back to the results of the previous three chapters. We have extended the tradition of the dynamic micro-economic theory of the firm by introducing activity analysis and investment grants. Further, we have enriched economic analysis by dealing more intensively with sensitivity analysis.

The analysis is conclusive on the optimal production, investment, financing and dividend policy of the firm. We will summarize the main results in the following twelve statements:

1. If we assume that the sales-function and the properties of the production activities are exogenously fixed, then the following factors will determine the optimal trajectory of the firm during the planning period:
 - a. the relationship between the total costs per unit output of the two activities ($c_1 \gtrless c_2$), as far as the optimal production policy is concerned,
 - b. the relationship between the prices of equity and debt (after corporation profit tax) as far as the optimal financial policy is concerned ($i \lesseqgtr (1-f)r$),
 - c. the size of the initial available amount of equity, $X(0)$, and equipment, $K(0)$, as far as the stage is concerned at which the firm starts on its trajectory.
2. When the price of debt is less than the price of equity, the firm will always borrow at the maximum level on its optimal trajectory.
3. When the total cost per unit output of the capital-intensive activity exceeds that of the labour-intensive one, the firm will never perform the capital-intensive activity.
4. When the price of debt exceeds the price of equity, the firm may still be interested in borrowing. The output level must be so small that marginal return on investment exceeds the price of debt.
5. When the total cost per unit output of the labour-intensive activity exceeds that of the capital-intensive one, it may still be profitable to the firm to perform the labour-intensive activity. The output level must be so small that the sales advantage per unit of a capital good assigned to the labour-intensive activity, surpasses the total cost disadvantage of the latter activity.
6. The firm may perform both available production activities at the same time, but only during a transitory stage of depth investments on its optimal trajectory.

7. Although the firm in our model only maximizes its value in favour of the shareholders, it seems to act like it is pursuing different (sub) goals during the successive stages of its trajectory. So, it may start as a growth maximizer, switching to cost minimization and, after a second or even a third period of growth maximization, it may end as a profit maximizer.
8. The optimal trajectories are based on three decision rules, concerning production (choice of production activities), finance (choice of the financial structure) and the distribution of the financial means between investments and dividend.
9. The influence of changes of the parameter values (i.e.: rates of: corporate profit tax, investment grants, time preference rate of the shareholders, borrowing, interest and wage) on the shape of an optimal trajectory is complicated. This influence can be subdivided into impacts of these changes on:
 - a. the reallocation level: the output level on which the firm switches from labour-intensive to capital-intensive production,
 - b. the final output level: the output level on which the firm stops expanding its output and starts paying out dividend,
 - c. the consolidation level: the output level on which the firm starts switching from maximum debt financing to self-financing,
 - d. the expansion rate: the speed at which the amount of equity increases,
 - e. the substitution effect: the choice of the production activity in the final stage of a trajectory,
 - f. the financial substitution effect: the choice of the financial structure in the final stage of a trajectory.
10. Under certain circumstances, as described in chapter 6, abolishing investment grants combined with a decrease of the corporation profit tax rate, will stimulate depth investments and consolidation in younger firms and will raise profit and employment in mature firms.
11. Under the same circumstances, as mentioned under 10, extension of the borrowing facilities, combined with an increase of the interest rate and of the discount rate, will lower the increase of equity and the output level on which the firm starts paying back debt, but will not affect the final profit and employment level.
12. Under the circumstances, mentioned under 10., a decrease of the wage rate will increase output and employment during the whole trajectory of the firm.

Apart from these results, the study may be the basis for further extensions. It should be interesting, for example, to study the allocation of labour between firms operating on the same market, by introducing competitors in the framework of a dynamic game, as is done in the more limited advertising models by Levine & Thépot, 1982. Another extension that merits consideration, is the introduction of a time dependent sales function, as published for example in: Lesourne & Leban, 1980, through which the impact of business cycles on the allocation of labour and capital can be studied. Further improvement of modelling the production problem could be realized by introducing vintages of machines (Virmany, 1976) and switching costs. Next, one could replace the assumption of an imperfect output market by the assumption of a perfect output market, which implies fixing the selling price. Then, the model becomes a linear optimal control model. This kind of model can be solved by means of linear programming (see: Krener, 1982), which opens the opportunity to build more extensive, detailed models, having a greater value for practitioners. In this way, Vicker's bridge from micro-economics to business economics could be pushed on. Also the dynamic interaction between the firm's development and its investors seems a valuable extension. A further refinement of the description of the tax systems, especially the impact on the investor's income (see: Ylä-Liedenpohja, 1978) will then be necessary. Finally, the introduction of stochastic elements into such aspects as financing, technical progress and demand may enrich the value of the analysis, on the understanding that, in spite of the increased complexity, an analytical analysis remains possible in the way we have done in this thesis.

Many more ideas can be raised here. They all will confirm that the underlying research is done in an area having many opportunities for further, interesting research and holding out projects of important results.

APPENDIX 1. AN INTERPRETATION OF THE MAXIMUM PRINCIPLE.

1. Introduction.

This appendix is meant to be an informal introduction to the Maximum Principle, in the tradition of Dorfman, 1969, and Ludwig, 1978, page 162.¹⁾ First, we will introduce some technical terms. After that the optimality conditions will be presented in three steps. We start with a description of the Maximum Principle in a more general form, together with a discussion of the so called "Hamiltonian (function)" and the "costate variables". Second we will study the impact of constraints on the control variables and introduce a "dynamic Lagrangian (function)". Finally, some ways will be presented in which constraints on the state variables can be dealt with.

This appendix is called an informal introduction, because it concentrates on the main ideas leading to the optimality conditions, without bothering about technical issues such as: continuity, shape of the relevant functions and constraints etc. But after the presentation of the main points, we will state the optimality conditions in a more complete form.

2. Technical terms.

In this section we will use the Jorgenson model of chapter 3 to introduce some technical terms. The problem reads:

$$\begin{array}{l} \text{maximize} \\ I, L \end{array} : \int_0^z e^{-iT} \{ p \cdot Q(K(T), L(T)) - w \cdot L(T) - c \cdot I(T) \} dT \quad (1)$$

$$\text{subject to} : \dot{K} = I(T) - a \cdot K(T) \quad (2)$$

In order to get a proper description of the problem, we have to add the following constraints:

$$I_{\min} < I(T) < I_{\max} \quad (3)$$

1) We refer to the following books for more mathematical, rigorous or more detailed introductions with an economic background: Intriligator, 1971, pp. 292-305 and pp. 344-369: quite an easy introduction but without an explicit discussion of constraints on control and state variables; Takayama, 1974, pp. 600-719: more rigorous, not dealing with constraints on state variables; Sethi & Thompson, 1981: a comprehensive tour through Optimal Control Land, paying much attention to published applications; Kamien & Schwartz, 1981, pp. 11-250; a handsome step by step introduction to the optimality conditions, dealing extensively with constraints on the state variables.

$$K(T) > 0 \quad (4)$$

$$K(0) = k_0 \quad (5)$$

A description of the above problem in the jargon of Optimal Control Theory can be given as follows (Sethi & Thompson, 1981, 2):

The system to be controlled is the firm. The state of the system is measured by the state variable K . The value of this state variable is controlled (directly or indirectly) by the control variables $I(T)$ and $L(T)$. Given the value of the state variable K and the control variable I , the state equation (or: system equation) (2) determines the instantaneous rate of change of the state variable. So, based on the initial value $K(0)$, fixed by the initial state condition (5), and the values of I over the whole planning period (control history), we can integrate (2) over time to get the state trajectory of the firm. The firm wants an investment and employment plan, maximizing the objective function (1). The designer of this plan has to reckon with the laws of motion of the firm as described in (2) and (5), with the state constraint (4) and with the control constraints (3). Any plan, fulfilling these constraints is called a feasible solution.

We shall now turn to the optimality conditions of the Maximum Principle. To facilitate the more general presentation in this appendix, we will now switch to a notation, quite different from the notation in the rest of this work.

3. The Maximum Principle of Pontryagin.

At any date t , the firm has a specific state inherited from its past performance, represented by the state vector $x(t)$. This vector may consist, for example, of the amount of equity, the stock of capital goods, the employment level, the goodwill etc. Based on this state the firm make decisions represented by the control vector $u(t)$, consisting, for example, of investments, dividend pay out etc. These control variables have their impact on the performance level of the firm (profit, sales, employment). This performance level is measured in terms of a rate per unit of time, $f_0(x(t), u(t), t)$ and a valuation of the final state of the firm $s(x(z), z)$. We assume that the firm maximizes this performance level over the planning period $[0, z]$, so:

$$\underset{\tilde{u}}{\text{maximize}} : V(x_0, u, 0) = \int_0^z f_0(x(t), u(t), t) dt + s(x(z), z) \quad (6)$$

in which : $x(t)$: m dimensional vector of state variables.

$x(0) = x_0$: initial state (exogeneously given).

$u(t)$: n dimensional vector of control variables.
 \tilde{u} : entire trajectory of control variables over the planning period.
 t : time, $0 < t < z$.
 $s(x(z), z)$: (discounted) salvage value of the firm at the end of the planning period.

Above that, the relevant controls influence the rate of change of the state. This rate also depends on the present state and date:

$$\dot{x}_i = f_i(x(t), u(t), t), \quad i = 1, \dots, m \quad (7)$$

For the time being we assume that the present state does not restrict the decision possibilities of the firm.

Now, the problem is to select a decision history \tilde{u} (i.e.: the policy of the firm during the planning period), that maximizes the result V , assumed that the firm initially is in the state x_0 . The complexity is caused by the twofold effect of a decision history: an immediate effect on the result through f_0 and a carry-over effect through its impact on the state of the firm and thus on future results. The Maximum Principle handles this problem by reducing the optimization over the whole planning period to the (sub) optimization over successive short time intervals.

More specifically, to explain the Maximum Principle we must study the problem for an arbitrary short time interval within the planning period, say $[t, t+\Delta t]$. Within this interval, the firm cannot change its control history $u(t)$. The performance level that can be reached by the firm from t onwards may then be described by:

$$V(x, u, t) = f_0(x(t), u(t), t) \Delta + \int_{t+\Delta t}^z f_0(x(\tau), u(\tau), \tau) d\tau + s(x(z), z) \quad (8)$$

The first part of the right hand side represents the immediate effect, the second part is the same function as in (6), but starting at $t+\Delta t$.

We are now going to study the decision problem of the firm in the interval $[t, t+\Delta t]$, assuming that the firm will act in an optimal way after that period. Therefore we introduce the symbol V^* , representing the performance level if the firm carries out the best policy. Thus at date t we may write:

$$V^*(x(t), t) = \underset{\tilde{u}}{\text{maximum}} V(x(t), \tilde{u}, t) \quad (9)$$

Assume that the firm realizes this best policy from $t+\Delta t$ on. The performance of (8), then turns into:

$$W(x(t), u(t), t) = f_0(x(t), u(t), t) \cdot \Delta + V^*(x(t+\Delta t), t+\Delta t) \quad (10)$$

The policy of the firm over the interval $[t, t+\Delta t]$ is thus optimal if:

$$\begin{aligned} W(x(t), u(t), t) &= V^*(x(t), t) = \\ &= \underset{\tilde{u}}{\text{maximum}} \left\{ \int_t^{t+\Delta t} f_0(x(\tau), u(\tau), \tau) d\tau + V^*(x(t+\Delta t), t+\Delta t) \right\} \quad (11) \end{aligned}$$

The reduction of the overall optimization problem to an incremental optimization is now presented in (8), (10) and (11), assuming that these equations hold for all t , $0 < t < z$. The relevant optimization problem has two aspects: the choice of the control $u(t)$ and the (indirect) choice of the state $x(t)$.

In order to derive from (8), (10) and (11) the three optimality conditions of the Maximum Principle to be discussed in this section, we will first introduce the function $\psi_i(t)$, to denote the marginal contribution of the state variable $x_i(t)$ to the performance level under the optimal policy, and the symbol $\psi(t)$ to denote the m dimensional vector of values of ψ_i at date t , so:

$$\psi(t) := \frac{\partial}{\partial x} V^*(x(t), t) \quad (12)$$

From (11) and (12) can be derived, that the optimal value of the control history u^* , has to fulfill the so-called "Hamiltonian-Jacobi-Bellman equation":

$$\begin{aligned} \underset{\tilde{u}}{\text{maximum}} \left\{ f_0(x^*(t), u(t), t) + \sum_{i=1}^m \psi_i(t) f_i(x^*(t), u(t), t) + \right. \\ \left. + \frac{\partial}{\partial t} V^*(x^*(t), t) \right\} = 0 \text{ for each } t, 0 < t < z. \quad (13) \end{aligned}$$

(see, for example, Bryson & Ho, 1969, 131-135, Sethi & Thompson, 1981, 27-29). Because $\partial V^* / \partial t$ does not depend on u , the same optimal value of the control results from:

$$\underset{\tilde{u}}{\text{maximize}} H(x^*(t), u(t), \psi(t), t) := f_0(x^*(t), u(t), t) + \sum_{i=1}^m \psi_i(t) f_i(x^*(t), u(t), t)$$

for each t , $0 < t < z$ (14)

In order to understand the meaning of (14), we have to go back to (10) and see how the application of $u(t)$ influences the value of the performance level, $V^*(x(t), t)$, in two ways. The first way of influencing concerns the instantaneous, direct influence of the application of u on V^* . This contribution can be formulated as:

$$f_0(x^*(t), u(t), t) \Delta t \quad (15)$$

The second way in which V^* will be influenced is an indirect one, and is caused by the change of the state variable at $t+\Delta t$, Δx , due to the application of the control $u(t)$. This contribution can be formulated as:

$$\frac{\partial}{\partial x} V^*(x(t+\Delta t), t+\Delta t) \Delta x, \text{ in which } : \Delta x = \Delta \cdot x(t) = \Delta \cdot f_1(x^*(t), u(t), t) \quad (16)$$

Adding (15) and (16) results in the total influence on the performance level of the application of the control $u(t)$ during a short interval $[t, t+\Delta t]$:

$$\{f_0(x^*(t), u(t), t) + \Sigma \psi(t) \cdot f_1(x^*(t), u(t), t)\} \Delta t = H(x^*(t), u(t), \psi(t), t) \Delta t \quad (17)$$

From (17) and (13) we can derive that the value of the function H in (14) represents an approximation of the performance stream during a very small period of time. It is a function of the control vector $u(t)$. The optimal adjustment of the state variables is implicitly considered through the vector $\psi(t)$. The function H is called "Hamiltonian function" and derives its name from the Irish physicist and mathematician William R. Hamilton, 1806-1865. The "marginal values" of the state variables $\psi_1(t)$, discussed before as "marginal contributions to the performance level", are called: "costate variables".

Related to condition (14), a second optimality condition can be derived from (13), by considering $x(t)$ instead of $u(t)$ as argument to be fixed on its optimal value (see also: Bensoussan e.a., 1974, 17). Suppose small perturbations of the state variables around the optimal value $x^*(t)$ on a fixed point of time t . If we keep the control history on its optimal value u^* , then for $x = x^*(t)$ the following first order condition has to hold in order to satisfy (13):

$$\frac{\partial}{\partial x} \{H(x^*(t), u^*(t), \psi(t)) + \frac{\partial}{\partial t} V^*(x^*(t), t)\} = 0 \quad (18)$$

This leads to the so called "Euler-Lagrange equation":

$$-\dot{\psi}_j = \frac{\partial H}{\partial x_j} = \frac{\partial f_0}{\partial x_j} + \sum_{i=1}^m \psi_i \frac{\partial f_i}{\partial x_j}, \quad j = 1, \dots, m$$

$$i = 1, \dots, m \quad (19)$$

(see: Sethi & Thompson, 1981, 29-31. Bryson & Ho, 1964, 138, derived (19) in a different way). This equation asserts that, in the case of the optimal policy of the firm, the marginal value of the state decreases at a rate proportional to its direct contribution to the performance level plus its more lasting contribution through the increase of (the value of) the state. In other words: the state loses value or depreciates as time passes, at the rate at which its potential contribution to the performance level becomes its past contribution (Dorfman, 1969, 821). The relevant contribution may have a negative value, such as, for example, the contribution of debt money in Ludwig's model of chapter 3.

A third optimality condition to be dealt with in this section concerns a special case: the marginal contribution of the state at the end of the planning period: $\psi(z)$.

From (6), (9) and (12) we can derive that:

$$\psi(z) = \frac{\partial}{\partial x} V^*(x(z), z) = \frac{\partial}{\partial x} \left\{ \int_z^z f_0(x(t), u(t), t) dt + s(x(z), z) \right\} +$$

$$\psi(z) = \frac{\partial}{\partial x} s(x(z), z) \quad (20)$$

This condition is called "transversality condition". It states that the marginal contribution only consists of the discounted marginal salvage value. For, at the final time point of the planning period, $t = z$, the state cannot generate a performance stream that still increases the performance level within the planning period.

Taking the final value $\psi_i(z)$ from (20) and integrating (19) results in another expression of $\psi_i(t)$:

$$\psi_i(t) = \int_t^z \frac{\partial H}{\partial x_i} dt + \frac{\partial}{\partial x_i} s(x(z), z), \quad i = 1, \dots, m \quad (21)$$

So, in the optimal solution, the marginal value of a state variable (capital good, equity, goodwill) equals its total future marginal contribution to the performance level, including its marginal salvage value.

The transition from the overall approach of (6) to the incremental approach of (11) has changed the dynamic optimization problem into an infinite number of static optimization problems, coupled through (19) and (20).

These are the main results of the Maximum Principle to be discussed in this section. We will now summarize them, and add some mathematical details about sufficiency conditions and uniqueness of the optimal solution without further discussion.

Problem 1.

$$\underset{\tilde{u}}{\text{maximize}} : V(x_0, \tilde{u}, 0) = \int_0^z f_0(x(\tau), u(\tau), \tau) d\tau + s(x(z), z) \quad (22)$$

$$\text{subject to : } \dot{x}_i = f_i(x(t), u(t), t) \quad (23)$$

$$x_i(0) = x_{i0} \quad , \quad i = 1, \dots, m \quad (24)$$

in which : t : time, $0 < t < z$

z : planning horizon (fixed)

u(t) : n dimensional vector of control variables, piecewise continuous. So, a finite number of jumps in the control variables is allowed for.

Further we assume:

$$u(t) \in U, U \text{ a given set in } R^n \quad (25)$$

\tilde{u} : control history over the whole planning period

x(t) : n dimensional vector of state variables

x_{i0} : exogenously fixed initial state of the firm

$f_0(\cdot)$: instantaneous performance stream, continuously differentiable in (x, u, t)

$f_i(\cdot)$: rate of change of state variable x_i , continuously differentiable in (x, u, t)

s(\cdot) : salvage value of the firm, continuously differentiable in (x(z), z).

Theorem 1. (necessity)

Define the Hamiltonian:

$$H(x, u, \psi, t) = f_0(x, u, t) + \sum_{i=1}^m \psi_i \cdot f_i(x, u, t) \quad (26)$$

then, for an optimal history $u^*(t)$ of the above problem 1 and the resulting state trajectory $x^*(t)$, it is necessary that there is a continuous, non zero vector function $\psi(t) = (\psi_1(t), \dots, \psi_m(t))$ such that:

$$\begin{aligned} H_{\text{optimal}} &:= \{H(x^*(t), u^*(t), \psi(t), t)\} \\ &= \underset{u(t)}{\text{maximum}} \{H(x^*(t), u(t), \psi(t), t)\} \text{ for each } t, 0 < t < z \end{aligned} \quad (27)$$

and, except at points of discontinuity of $u^*(t)$, that:

$$\psi_i(t) = -\frac{\partial}{\partial x_i} H(x^*(t), u^*(t), \psi(t), t), \quad i = 1, \dots, m \quad (28)$$

$$\psi_i(z) = \frac{\partial}{\partial x_i} s(x^*(z), z), \quad i = 1, \dots, m \quad (29)$$

The above conditions are necessary but not sufficient for an optimal solution. The Norwegians Seierstad and Sydsaeter published several formulations of sufficiency conditions for different optimal control problems (Seierstad & Sydsaeter, 1977). For the above problem they presented the following alternative systems of sufficiency conditions (o.c., page 370):

Theorem 2. (sufficiency)

Suppose $(x^*(t), u^*(t))$ is a feasible solution of problem 1, satisfying the conditions of theorem 1. Then this solution is optimal to problem 1

if : H_{optimal} , as defined in (27), is concave in x , $x \in \mathbb{R}^n$, or:

if : the Hamiltonian, as defined in (24), is jointly concave in x and u , and U , as defined in (23), is convex, for all x and $u \in \{U \times \mathbb{R}^n\}$.

Finally, we borrow from Van Long & Vousden the following uniqueness theorem (Van Long & Vousden, 1977, 30):

Theorem 3. (uniqueness)

Let $(x^*(t), u^*(t))$ be an optimal solution to problem 1, satisfying sufficiency theorem 2. If H_{optimal} is strictly concave in x , than any other optimal solution $(\bar{x}(t), \bar{u}(t)) \neq (x^*(t), u^*(t))$ must satisfy $\bar{x}(t) = x^*(t)$.

4. Mixed control constraints.

In the former section we only dealt implicitly with a feasible region of the control variables $u_j(t)$. These controls must lie in some set U , being the entire n dimensional Euclidian space or a proper subset of it. In this section we will specify this region more precisely and we will study the consequences for the optimality conditions. In fact, we introduce the assumption that the decisions of the firm are limited in absolute terms or by the actual state of the firm. For example: the production level is non-negative and may be restricted by the amount of available machinery, or: the amount of loans that can be attracted is limited by the size of equity:

$$Q > 0 \text{ and } Q = Q(K(t), L(t), t) \quad (30)$$

$$kX(t) - Y(t) > 0$$

with state variables : K (amount of machinery) and X (amount of equity),
 control variables: L (employment level), Y (amount of debt),
 k : maximum borrowing rate
 $Q(\cdot)$: production function
 t : time.

More generally:

$$g_k(x(t), u(t), t) > 0, k = 1, \dots, r \quad (31)$$

in which the functions g_k are assumed to be continuously differentiable in (x, u, t) space. Further we assume that each function g_k contains at least one control variable u_j as an argument.

In order to study the impact of this kind of restrictions on the optimal policy of the firm, we return to the incremental decision problem as presented in (8) through (11) and (14). We have to change the definition in (9) of the performance level of the best overall policy, V^* into:

$$V^*(x(t), t) = \underset{\tilde{u}}{\text{maximum}} V(x(t), \tilde{u}, t)$$

$$\text{for all } \tilde{u} \text{ with values } u(t) \text{ satisfying: } g_k(x, u, t) > 0, k = 1, \dots, r \quad (32)$$

The incremental description of the optimal policy in (14) now changes into:

$$\begin{aligned} \text{maximize} \quad & : H(x^*(t), u(t), \psi(t), t) \text{ for each } t, 0 < t < z \\ & u(t) \\ \\ \text{subject to} \quad & : g_k(x^*(t), u(t), t) > 0 \end{aligned} \quad (33)$$

Note that $x^*(t)$ has a fixed value and $\psi(t)$ is a given vector valued function of $u(t)$ if t is fixed. So, the available policies $u(t)$ are limited by the actual state of the firm. The optimization problem is thus transformed into an infinite number of constrained optimization problems of the form:

$$\begin{aligned} \text{maximize} \quad & : Z(u(t), t) \text{ for each } t, 0 < t < z \\ & u(t) \\ \\ \text{subject to} \quad & : z_k(u(t), t) > 0, k = 1, \dots, r \end{aligned} \quad (34)$$

The solutions to these problems, together constituting the optimal control history over the whole planning period, are derived through the well-known Method of Lagrange Multipliers (see for instance: Intriligator, 1971, 28, Takayama, 1974, 373). For each restriction, we introduce a Lagrange multiplier $\lambda_k(t)$, representing its "shadow price". That is the impact of a marginal relaxation of the restriction on the instantaneous increase of the performance stream at date t , $H(x^*(t), u^*(t), t)$. The shadow price will be zero if the relevant restriction is not binding in the optimal solution, because the relaxation of such a restriction will not yield a better performance:

$$\lambda_k(t) \cdot g_k(x^*(t), u^*(t), t) = 0 \quad (35)$$

Shadow prices can only be non negative because relaxation of restrictions cannot lead to a fall of the maximum performance level:

$$\lambda_k(t) > 0 \quad (36)$$

In the optimal situation, the marginal contribution of the j -th control variable equals its marginal costs. These costs are the units of capacity of the relevant restrictions, needed to realise a marginal rise of the control variable, weighed by the relevant shadow prices:

$$\frac{\partial}{\partial u_j} H(u^*(t), t) = - \sum_{k=1}^r \lambda_k(\bar{t}) \frac{\partial}{\partial u_j} g_k(u^*(t), t) \quad (37)$$

The restrictions on the control variables also have an impact on the optimal state trajectories and so on the conditions, coupling the above subproblems (34). From (19) we know that in the optimal solution the depreciation rate of the state equals its contribution to the performance level. This contribution consisted of two parts, an instantaneous increase of the performance level and an increase of the state valued by the costate variables $\psi_i(t)$, representing future performance streams. Now, we have to add a third term. For, changes in the state variables cause changes in the feasible control region, due to the assumption in (31) that boundaries depend on the actual state of the firm. In the optimal solution we may value these changes through the above introduced shadow prices $\lambda_k(t)$. So, (19) turns into:

$$-\dot{\psi}_i(t) = \frac{\partial H}{\partial x_i} + \sum_{k=1}^r \lambda_k(t) \frac{\partial g_k}{\partial x_i}, \quad i = 1, \dots, m$$

$$\text{in which : } g_k = g_k(x^*(t), u^*(t), t) \quad (38)$$

As stated, (35) through (38) are based on the Method of Lagrange Multipliers used in constrained programming problems. We can simplify the notation of (37) and (38) by introducing the Lagrangian function (or: "extended Hamiltonian"):

$$L(x, u, \psi, \lambda, t) = H(x(t), u(t), \psi(t), t) + \sum_{k=1}^r \lambda_k(t) \cdot g_k(x(t), u(t), t) \quad (39)$$

So (37) and (38) may be rewritten as:

$$-\dot{\psi}_i(t) = \frac{\partial L}{\partial x_i}, \quad i = 1, \dots, m \quad (40)$$

$$\frac{\partial L}{\partial u_j} = 0, \quad j = 1, \dots, n \quad (41)$$

Integration of (40) backwards, starting from the obviously unchanged transversality condition (20), results in the new version of (21):

$$\psi_i(t) = \int_t^z \frac{\partial L}{\partial x_i} dt + \frac{\partial}{\partial x_i} s(x(z), z), \quad i = 1, \dots, m \quad (42)$$

In the optimal solution, the marginal value of the state variable equals its total future marginal contribution to the performance level plus its valued contribution to relieve (or restrict) the decision possibilities of the firm.

From (33) and (39) can be concluded that the value of the Lagrangian equals the value of the Hamiltonian. But, the difference comes up as soon as we start studying the influence of control and state variables through the relevant partial derivatives. Then the derivatives of the Lagrangian appear to contain the impact of changes in the boundaries of the control region on the instantaneous optimal performance H.

To summarize the discussion of this section, we present the following problem and theorems:

Problem 2.

$$\underset{\tilde{u}}{\text{maximize}} : V(x_0, \tilde{u}, 0) = \int_0^z f_0(x(\tau), u(\tau), \tau) d\tau + s(x(z), z) \quad (43)$$

$$\text{subject to} : \dot{x}_i = f_i(x(t), u(t), t), \quad i = 1, \dots, m \quad (44)$$

$$g_k(x(t), u(t), t) > 0, \quad k = 1, \dots, r \quad (45)$$

$$x_i(0) = x_{i0} \quad (46)$$

in which all the above variables and functions have the same characteristics as in Problem 1 and further: $g_k(\cdot)$ is continuously differentiable in (x, u) .

Theorem 4. (necessity)

Let the Hamiltonian be:

$$H(x, u, \psi, t) = f_0(x, u, t) + \sum_{i=1}^m \psi_i \cdot f_i(x, u, t) \quad (47)$$

and define the Lagrangian:

$$L(x, u, \psi, \lambda, t) = H(x, u, \psi, t) + \sum_{k=1}^r \lambda_k(t) \cdot g_k(x, u, t) \quad (48)$$

then, for an optimal control history u^* of the above problem 2 and the resulting state trajectory $x^*(t)$ to be optimal, it is necessary that there are functions $\psi(t) = (\psi_1(t), \dots, \psi_m(t))$ and $\lambda(t) = (\lambda_1(t), \dots, \lambda_r(t)) > 0$, in which $\psi(t)$ is continuous and $\dot{\psi}(t)$ and $\lambda(t)$ are piecewise continuous, such that:

$$\begin{aligned} H_{\text{optimal}} &:= H(x^*(t), u^*(t), \psi(t), t) \\ &= \underset{\tilde{u}}{\text{maximum}} H(x^*(t), u(t), \psi(t), t) \text{ for each } t, 0 < t < z \end{aligned} \quad (49)$$

and, except at points of discontinuity of $u^*(t)$ and $\lambda(t)$, that:

$$\dot{\psi}_i(t) = -\frac{\partial}{\partial x_i} L(x^*(t), u^*(t), \psi(t), x(t), t), \quad i = 1, \dots, m \quad (50)$$

$$\frac{\partial}{\partial u_j} L(x^*(t), u^*(t), \psi(t), \lambda(t), t) = 0, \quad j = 1, \dots, n \quad (51)$$

$$\lambda_k(t) \cdot g_k(x^*(t), u^*(t), t) = 0, \quad k = 1, \dots, r \quad (52)$$

$$\psi_i(z) = \frac{\partial}{\partial x_i} s(x^*(z), z) \quad (53)$$

The above conditions are necessary for the optimal solution. The following theorems deal with sufficiency conditions (Seierstad & Sydsaeter, 1977, 374-377):

Theorem 5. (sufficiency)

Suppose $(x^*(t), u^*(t))$ is a feasible solution of problem 2, satisfying the conditions of Theorem 4. Then this solution is optimal to problem 2 if $H(x, u, \psi, t)$ as defined in (26) is concave in (x, u) and $g_k(x, u, t)$ is quasi concave in (x, u) .

In the second sufficiency theorem, to be presented here, the above mentioned concavity requirement on the Hamiltonian is relaxed and the quasi-concavity of g_k is not assumed but the functions g_k must satisfy a "constraint qualification", guaranteeing a well shaped feasible region without so called "cusp points":

Constraint qualification.

Let $Z(t) = \{z : g_z(x^*(t), u^*(t), t) = 0\}$ be the set of indices of active constraints. In this case the constraint qualification is satisfied if the number of indices of $Z(t)$ equals the rank of the

$$\text{matrix } \left\{ \frac{\partial}{\partial u_j} (g_z(x^*(t), u^*(t), t)) \right\} \text{ with } z \in Z(t) \text{ and } j = 1, \dots, n \quad (54)$$

This condition asserts that the number of active constraints is less than or equal to the number of control variables, although the total number of constraints may exceed the number of control variables. An extensive discussion of the constraint qualification can be found in Takayama (1974), pag. 86-108. Further we have to define:

$$A(t) := \{x : g(x, u, t) > 0 \text{ for some } u\} \quad (55)$$

Theorem 6. (sufficiency)

Suppose $(x^*(t), u^*(t))$ is a feasible solution of problem 2, satisfying the conditions of theorem 4 and the constraint qualification (54). In this case, the solution is optimal to problem 2 if:

$A(t)$ is convex, and:

H_{optimal} as defined in (49), is a concave function of x on $A(t)$.

The following uniqueness theorems stem from Van Long & Vousden, 1977, page 30 and Seierstad & Sydsaeter, 1975, page 376:

Theorem 7. (uniqueness)

Let $(x^*(t), u^*(t))$ be an optimal solution of problem 2, satisfying sufficiency theorems 5 and/or 6, then this solution is the only optimal one if L , as defined in (48), fulfills:

$$\frac{\partial^2 L}{\partial u^2} < 0 \text{ (Seierstad \& Sydsaeter).}$$

If H_{optimal} is strictly concave in x , then any other optimal solution $(\bar{x}, \bar{u}) \neq (x^*, u^*)$ must satisfy $\bar{x}(t) = x^*(t)$. (Van Long & Vousden).

5. State constraints.

The interpretation of the Maximum Principle in section 3 can be found in several publications on applications of the Maximum Principle to economic theory. Amplifications when control constraints are introduced, are not explicitly dealt with. But often the reader is referred to the interpretation of Lagrange multipliers in static constrained optimization problems, which implies omitting the interpretation of the real dynamic formula (38). In the case of state constraints, we found nowhere any attempt to give an interpretation. So we have to steer with the help of our own compass.

Apparently there are three ways of defining the relevant optimality conditions. The definitions with the easiest interpretation will be presented first. It has discontinuous costate variables, a fact that is difficult for solution procedures. The definition with the nicest technical characteristics will be presented last, because it has a less obvious interpretation of the Hamiltonian. This definition will be used to solve the relevant models in this book. The remaining method will be used as an intermediate one, facilitating the interpretation of the last way of defining the optimality conditions.

We drop the control constraints as discussed in the former section for a while and concentrate on dealing with state constraints only. So, consider the problem:

$$\text{maximize} \quad : \int_0^z f_0(x, u, t) dt + s(x(z), z) \quad (56)$$

$$\text{subject to} \quad : \dot{x}_i = f_i(x, u, t), \quad i = 1, \dots, m \quad (57)$$

$$h_\lambda(x, t) > 0, \quad \lambda = 1, \dots, s \quad (58)$$

$$x_i(0) = x_{i0} \quad (59)$$

The first method deals with the constraints of (58) in the same way as with the control constraints (31) by introducing dynamic Lagrangian parameters, say $\mu(t)$, and defining the optimality conditions in the usual way (we drop obvious arguments) :

Form the Hamiltonian of the above problem:

$$H(x, u, \psi, t) = f_0 + \sum_i \psi_i f_i \quad (60)$$

and the Lagrangian:

$$L(x, u, \psi, \mu, t) = H + \sum_{\ell} \mu_{\ell}(t) \cdot h_{\ell} \quad (61)$$

Then, necessary conditions to an optimal solution are:

$$\dot{\psi}_i = - \frac{\partial L}{\partial x_i} \quad (62)$$

$$\frac{\partial L}{\partial u_j} = 0 \quad (63)$$

$$\mu_{\ell} \cdot h_{\ell} = 0 \quad (64)$$

$$\mu_{\ell} > 0 \quad (65)$$

$$\psi_i(z) = \frac{\partial s}{\partial x_i} \quad (66)$$

$$i = 1, \dots, m ; j = 1, \dots, n ; \ell = 1, \dots, s.$$

The interpretation of the dynamic Lagrangian multipliers of the conditions (62) through (66) is the same as in the former section. So, the extension of the problem by introduction of state constraints seems to give no new difficulties. But, this is only true in the case where the first total derivative of the state constraints, dh/dx , contains one or more control variables. Then the system can be steered by means of the controls in such a way as to approach or to leave the boundaries in an arbitrarily smoothed way. If the first total derivative of any state constraint does not contain a control variable, which is the case in our models, then we get discontinuities in the costate variables. The less controllable motion of the system may cause bumps against the state constraint boundaries at certain points in time, say t_p . The decreasing marginal value of the state, as described for an "non bumping" case in (19) and (38), will then depend on the instantaneous restricting state constraint(s) only. If we denote by $\beta_{p \ell}(t_p)$: the marginal harm to the instantaneous performance level due to the state constraint h_{ℓ} on time t_p , then the next conditions have to be added to the above conditions:

$$\psi_i(t_p^-) - \psi_i(t_p^+) = \sum_{\ell=1}^s \beta_{p \ell} \frac{\partial h_{\ell}}{\partial x_i} \quad (67)$$

$$\beta_{p \ell} \cdot h_{\ell} = 0 \quad (68)$$

$$\beta_{p \ell} > 0 \quad (69)$$

$$0 < t_p^- < \dots < t_p^+ < z : \text{discontinuity points of } \psi_i \quad (70)$$

In which t_p^- and t_p^+ are the left- and right-hand side limit of t_p . Equation (62) now only holds at moments when ψ_i exists and (63) only applies when u_j are continuous.

Although the above formulation is easy to interpret, the discontinuity of ψ_i is a nasty characteristic when one is looking for solution procedures. In this book, we will use another formulation published in: Russak, 1970. In order to understand the main ideas, we will first present another formulation used, for example, by Arrow and Kurz to handle (non negativity) constraints on the state variables (Arrow & Kurz, 1970, 41). In the relevant formulation, the state constraints (58) are replaced by:

$$\phi_{\ell}(x, u, t) > 0 \text{ whenever } h_{\ell}(x, t) = 0 \quad (71)$$

$$\phi_{\ell} := \frac{dh_{\ell}}{dt} = \frac{\partial h_{\ell}}{\partial t} + \frac{\partial h_{\ell}}{\partial x} \cdot x \quad (72)$$

Transgressing boundaries is thus prevented by requiring a move along or away from the boundary when the system is on the relevant boundary. The state constraint is therefore replaced by a control constraint. If the first derivative to time has no control variable as an argument, then one has to take the derivative of the smallest degree that has a control variable as an argument. We suppose furthermore that the first derivative is a function of u .

We can apply the optimality conditions of the former section to the problem (56), (57), (59), (71), (72):

Let the Hamiltonian and the Lagrangian be:

$$\bar{H}(x, u, \bar{\psi}, t) = f_0 + \sum_i \bar{\psi}_i \cdot f_i \quad (73)$$

$$\bar{L}(x, u, \bar{\psi}, \bar{u}, t) = H + \sum_{\ell} \bar{u}_{\ell}(t) \phi_{\ell} \quad (74)$$

Then, necessary conditions to an optimal solution are:

$$\dot{\bar{\psi}} = - \frac{\partial \bar{L}}{\partial \mathbf{x}} \quad (75)$$

$$\frac{\partial \bar{L}}{\partial \mathbf{u}} = 0 \quad (76)$$

$$\bar{\mu}_\ell(t) \cdot \phi_\ell = 0 \text{ when } h_\ell = 0 \quad (77)$$

$$\bar{\mu}(t) > 0 \quad (78)$$

$$\dot{\bar{\mu}}(t) < 0 \quad (79)$$

$$\bar{\psi}(z) = \frac{\partial s}{\partial \mathbf{x}} \quad (80)$$

The link between the auxiliary variables of the conditions (62)-(66) and of (73)-(80) is that:

$$\psi = \bar{\psi} + \bar{\mu} \frac{\partial h}{\partial \mathbf{x}} \quad (81)$$

$$\mu = - \dot{\bar{\mu}} \quad (82)$$

The transformation of the state constraint into a control constraint yielded a dynamic Lagrangian parameter $\bar{\mu}$ from which we have additional information through (79). But still, $\bar{\psi}$ is discontinuous. This is solved in the third formulation to be discussed here by defining $\bar{\mu}$ in such a way that (Seierstad & Sydsaeter, 1975, 388):

$$\bar{\mu}_\ell(t_p^-) - \bar{\mu}_\ell(t_p^+) = \beta_{p\ell}, \ell = 1, \dots, s; p = 1, \dots, v. \quad (83)$$

with the same conditions as stated in (67) through (70). Then $\bar{\psi}$, as defined in (79), becomes continuous on all discontinuous points of ψ . Above that, according to Seierstad & Sydsaeter, it can be proved that the above formulation (73) - (80), (83), yields the same results as defining the Hamiltonian as:

$$\bar{H}(\mathbf{x}, \mathbf{u}, \bar{\psi}, \bar{\mu}, t) = f_0 + \sum_i \bar{\psi}_i \cdot f_i + \sum_\ell \bar{\mu}_\ell \cdot \phi_\ell \quad (84)$$

This is the formulation presented in: Russak, 1970. Note that the Hamiltonian in (84) has not the same meaning as the preceding Hamiltonians. The latter are formulations of

the instantaneous performance flow only, while to the former some valuation of the boundaries of the state space has been added.

In order to avoid overlap, we will now present the theorems concerning the general problem, having control constraints as well as state constraints.

Problem 3.

$$\underset{\tilde{u}}{\text{maximize}} : V(x_0, u, 0) = \int_0^z f_0(x(\tau), u(\tau), \tau) d\tau + s(x(z), z) \quad (85)$$

$$\text{subject to : } \dot{x}_i = f_i(x(t), u(t), t) : \text{state equations} \quad (86)$$

$$g_k(x(t), u(t), t) > 0 : \text{control constraints} \quad (87)$$

$$h_\lambda(x(t), t) > 0 : \text{state constraints} \quad (88)$$

$$x_i(0) = x_{i0} : \text{initial state constraints} \quad (89)$$

$i = 1, \dots, m$; $j = 1, \dots, n$; $k = 1, \dots, r$; $\lambda = 1, \dots, s$

x_i : continuous

u_j : piecewise continuous

$f_0(\cdot)$, $f_i(\cdot)$, $g_k(\cdot)$, $h_\lambda(\cdot)$, and $s(\cdot)$: all continuously differentiable in

their own arguments.

t : time, $0 < t < z$

z : planning horizon (fixed)

\tilde{u} : control history over the whole planning period

x_{i0} : exogeneously fixed initial state of the firm

The following theorems are derived from Russak, 1974, Seierstad & Sydsaeter, 1977; and: Van Long & Vousden, 1977. The non negativity of the functions μ and the concerning transversality conditions are derived in: Peter Janssen, 1980, at that time working at the Technical University of Eindhoven with Prof. Hautus. In Russak, 1976, a

"reversed" characterization of μ is derived.

Theorem 8. (necessity)

Let the Hamiltonian be:

$$H(x, u, \psi, \mu, t) = f_0 + \sum_{i=1}^m \psi_i(t) \cdot f_i + \sum_{\ell=1}^s \mu_\ell(t) \cdot \phi_\ell \quad (90)$$

and the Lagrangian be defined as:

$$L(x, u, \psi, \mu, \lambda, t) = H + \sum_{k=1}^r \lambda_k(t) \cdot g_k \quad (91)$$

in which:

$$\phi_\ell(x, u, t) := \frac{dh_\ell}{dt} = \frac{\partial h_\ell}{\partial t} + \frac{\partial h_\ell}{\partial x} f(x, u, t) \quad (92)$$

and suppose each ϕ_ℓ contains at least one control variable as an argument. Then, for an optimal control history $u^*(t)$ of the above problem 3 and the resulting state trajectory $x^*(t)$ to be optimal, it is necessary that the constraint qualification (54) holds and that there are functions $\psi(t) = (\psi_1(t), \dots, \psi_m(t))$, $\lambda(t) = (\lambda_1(t), \dots, \lambda_r(t))$ and $\mu(t) = (\mu_1(t), \dots, \mu_s(t))$ such that:

$$\begin{aligned} H_{\text{optimal}} &:= H(x^*(t), u^*(t), \psi(t), \lambda(t), t) \\ &= \underset{u(t)}{\text{maximum}} \{ H(x^*(t), u(t), \psi(t), \lambda(t), t) \} \text{ for each } t, 0 < t < z \quad (93) \end{aligned}$$

and, except at points of discontinuity of $u^*(t)$:

$$\dot{\psi}_1(t) = - \frac{\partial}{\partial x_1} L(x^*, u^*, \psi, \mu, \lambda, t) : \text{costate equations} \quad (94)$$

$$\frac{\partial}{\partial u_j} L(x^*, u^*, \psi, \mu, \lambda, t) : \text{first order conditions} \quad (95)$$

$$\lambda_k(t) \cdot g_k(x^*, u^*, t) = 0 : \text{complementary slackness conditions} \quad (96)$$

$$\mu_\ell \cdot h_\ell(x^*, t) = 0 : \text{complementary slackness conditions} \quad (97)$$

$$\dot{\mu}_\ell < 0 \quad (98)$$

$$\lambda_k(t), \mu_\ell(t) > 0 \quad : \text{ non negativity restrictions} \quad (99)$$

$$\psi_1(z) = \frac{\partial}{\partial x_1} s(x^*(z), z) \quad : \text{ transversality conditions} \quad (100)$$

$$\mu_\ell(z) \cdot h_\ell(x^*(z), z) \quad : \text{ transversality conditions} \quad (101)$$

Further it must hold that:

$$\psi(t) \quad : \text{ continuous with piecewise continuous derivatives}$$

$$\dot{\psi}(t) \text{ and } \lambda(t) \quad : \text{ continuous on intervals of continuity of } u(t)$$

$$\mu(t) \quad : \text{ continuous on intervals of continuity of } u \text{ and when } \phi(x, u, t) \text{ is discontinuous.}$$

More over, on points of discontinuity of u , say t_p , it holds that

$$\mu_\ell(t_p^-) - \mu_\ell(t_p^+) = \beta_{p \ell} \quad (102)$$

$$\beta_{p \ell} \cdot h_\ell = 0 \quad (103)$$

$$\beta_{p \ell} : \text{ non negative numbers, } p = 1, \dots, v ; \ell = 1, \dots, s \quad (104)$$

Theorem 9. (sufficiency)

Suppose $(x^*(t), u^*(t))$ is a feasible solution of problem 3, satisfying the conditions of theorem 8. Then this solution is optimal to problem 3 if $A(t)$, as defined in (55), is convex and H_{optimal} , as defined in (93), is a concave function of x on $A(t)$.

Theorem 10. (uniqueness)

Let $(x^*(t), u^*(t))$ be an optimal solution of problem 3, satisfying sufficiency theorem 9, then this is the only optimal solution if L , as defined in (91), fulfills:

$$\frac{\partial^2 L}{\partial u^2} < 0.$$

If H_{optimal} is strictly concave in x on $A(t)$, then any other optimal solution $(\bar{x}, \bar{u}) \neq (x^*, u^*)$ must satisfy $\bar{x}(t) = x^*(t)$.

6. Summary.

After the introduction of some technical terms, by which the main features of optimal control models are usually described, the optimality conditions of a non-constrained optimal control problem are presented, based on an incremental approach. The meaning of the Hamiltonian and the costate variables is dealt with. At the end of this section, sufficiency and uniqueness conditions are added. In the next section, constraints on control variables are introduced. The Hamiltonian function is extended to a dynamic Lagrangian function and the meaning of dynamic Lagrangian multipliers is discussed. This section, too, ends with a statement of the relevant sufficiency and uniqueness conditions. Finally, the impact of constraints on the state variables are discussed. The reader is ushered into the optimality conditions as defined by Russak (1970). These optimality conditions have nice (continuity) properties which make them superior to other formulations. The optimality conditions of the general problem, containing control and state constraints, and the relevant sufficiency and uniqueness conditions conclude this appendix.

APPENDIX 2. SOLUTIONS OF THE MODELS OF CHAPTER 3.

1. Introduction.

The Maximum Principle, as presented in the preceding appendix, results in a set of conditions to be fulfilled by the optimal solution of an optimal control model, but not in the optimal solution itself. In order to find the optimal solution, we have to solve the system of optimality conditions. The usual procedure to solve is a trial and error procedure.

In this and the next appendix, we will use a systematic way of searching for optimal solutions. The procedure has been developed to reduce the heuristics of the solution stage as much as possible. Having developed this procedure, we could shorten solving time substantially and, more over, it enabled us to solve more complex models such as that of chapter 4.

This procedure may have a more general applicability. Therefore we will present its principles first, before applying it to the models of chapter 3.

2. A general solution procedure.

To facilitate the discussion, we will first dwell upon the nature of an optimal solution. The firm, which is the system to be controlled in this book, must be guided in such a way as to maximize some performance level without violating fixed restrictions. The set of active restrictions may change over time, due to changes in the shapes of the restrictions and due to changes in the optimal policy of the firm. Now, conceive the development of the firm over time as a succession of stages that can be distinguished from each other by differences in the set of active constraints. With this idea as basis, we will first derive which stages (called: paths) are feasible and what are the (distinguishing) features of each of them (see figure A2.1 on the next page). After that we will string them to complete patterns, and these strings are the very optimal solutions of the model. The systematic way in which to deal with stringing paths is the new feature of our solution procedure.

We go back to appendix 1 in order to point out how to distinguish between paths. Because we are looking for optimal control patterns, we are mainly interested in studying changes in the set of active constraints. From the complementary slackness conditions (96) and (98) in appendix 1 we can derive a way of describing distinct paths through their sets of zero-valued and positive valued lagrangian parameters λ_k , and of zero-valued or negative derivatives of the auxiliary variables μ_k . Positive respectively negative values indicate that the concerning restrictions are active. The first stage of the solution procedure, i.e. finding feasible paths and their characteris-

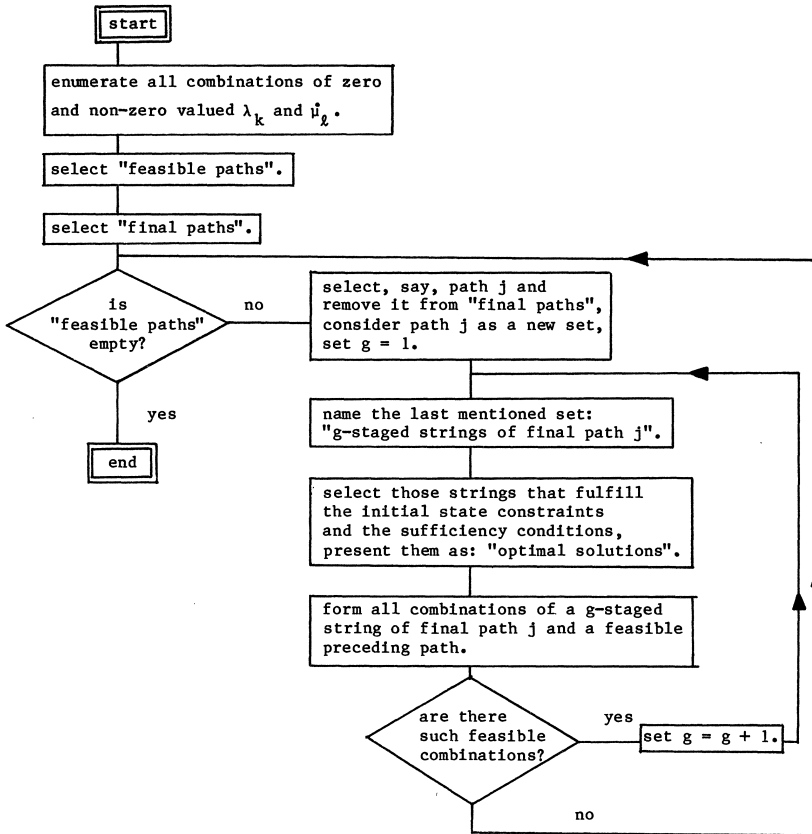


Figure A2.1. Scheme of the solution procedure.

tics, consists of enumerating all combinations of zero and non-zero-valued lagrangian parameters and derivatives of the auxiliary variables, and deriving characteristics for each combination as such. Some combinations can be left out without much study (for example: a control variable cannot be on an upper boundary and at the same time on a lower boundary if these boundaries have different values, so the relevant lagrangian parameters cannot be positive at the same time. Thus, all combinations in which both parameters are positive can be left out). Other combinations appear to be infea-

sible only after the derivation of its characteristics. This analysis thus yields a set of feasible paths and information about the characteristics of the distinct paths.

The second stage of the solution procedure is the coupling of paths to strings, covering the optimal policy of the firm over the whole planning period. This is done by starting at the time horizon z , and going backwards in time. Based on the transversality conditions (100) and (101) of appendix 1 we can select those paths that may be final paths, i.e. paths feasible at $T = z$. For each such final path we will then select feasible preceding paths. Therefore we test for each path whether coupling with the relevant final path will or will not violate the (necessary) continuity properties of the state variables and the auxiliary variables ψ_i , λ_k and μ_l as prescribed by Theorem 8 of appendix 1. In this testing procedure we will often use characteristics as derived in the former stage of the solution procedure.

If the set of feasible preceding paths appears to be empty, then the relevant final paths are descriptions of the optimal policy of the firm for the whole planning period, supposing they fulfill the initial state constraints (88) and the sufficiency conditions of Theorem 9 in appendix 1.

If the set of feasible preceding paths is not empty, we apply the testing procedure for feasible preceding paths again to each of them.

Depending on when the set of feasible preceding paths becomes empty, we may have to apply the testing procedure again, in this way finding a still longer string of paths, constituting an optimal policy pattern.

The procedure is summarized in figure A2.1.

3. The model of Jorgenson.

Jorgenson solved his model by means of the Calculus of Variations, a technique closely related to the Maximum Principle. The main advantage of the Maximum Principle over the Calculus of Variations is that it is more suitable to handle constraints on control and state variables.

To get a solution of the model by means of the Maximum Principle, we introduce (artificial) boundaries on the variables I : $I_{\min} < 0$ and $I_{\max} > 0$. We assume I_{\min} to be very small and I_{\max} to be very large, so as to describe a nearly instantaneous adjustment of K in:

$$K = I - aK \text{ when } I = I_{\min} \text{ or } I = I_{\max}. \quad (1)$$

Furthermore we improve the model by introducing a non negativity constraint and an initial constraint on K. So we get:

$$\text{maximize}_{I, L} \int_0^{\infty} e^{-iT} \{ p \cdot Q(K(T), L(T)) - w \cdot L(T) - c \cdot I(T) \} dT \quad (2)$$

$$\text{subject to } \dot{K} = I(T) - a \cdot K(T) \quad (3)$$

$$I_{\min} < I(T) < I_{\max} \quad (4)$$

$$K(T) > 0 \quad (5)$$

$$K(0) = k_0 \quad (6)$$

In the above problem, constraints (4) are control constraints and (5) is a state constraint, so Theorems 8 through 10 of appendix 1 apply to this problem. An exception must be made for the transversality condition, because Jorgenson supposes an infinite time horizon, whereas we deal with finite time horizons in the preceding appendix. As the costate variable represents the marginal contribution of the state to the performance function, one can imagine that the transversality condition (100) of appendix 1 changes into:

$$\lim_{T \rightarrow \infty} \psi(T) = 0 \quad (7)$$

For a more detailed discussion of the infinite horizon problem we refer to Sethi & Thompson, 1981, page 85 and Seierstad & Sydsaeter, 1976, page 383. Dropping obvious arguments, we can write:

Let the Hamiltonian be:

$$H = e^{-iT} \{ pQ - wL - cI \} + (\psi + \mu)(I - aK) \quad (8)$$

and the Lagrangian:

$$L = H + \lambda_1(I - I_{\min}) + \lambda_2(I_{\max} - I) \quad (9)$$

then it must hold that:

$$\dot{\psi} = - \frac{\partial L}{\partial K} = - e^{-iT} p \frac{\partial Q}{\partial K} + a(\psi + \mu) \quad (10)$$

$$\frac{\partial L}{\partial I} = -e^{-iT}c + \psi + \mu + \lambda_1 - \lambda_2 = 0 \quad (11)$$

$$\frac{\partial L}{\partial L} = e^{-iT} \left\{ p \frac{\partial Q}{\partial L} - w \right\} = 0 \quad (12)$$

$$\lambda_1 (I - I_{\min}) = 0 \quad (13)$$

$$\lambda_2 (I_{\max} - I) = 0 \quad (14)$$

$$\dot{\mu} < 0 \quad (15)$$

$$\dot{\mu} \cdot K = 0 \quad (16)$$

$$\lambda_1, \lambda_2, \mu > 0 \quad (17)$$

$$\lim_{T \rightarrow \infty} \psi(T) = 0 \quad (18)$$

$$\lim_{T \rightarrow \infty} \mu(T) \cdot k(T) = 0 \quad (19)$$

The continuity properties of ψ , λ , and μ are described in (102) - (104) of appendix 1. We will concentrate on situations in which it is profitable at least to start up production. This is described by the assumption that marginal revenue exceeds the marginal costs of both inputs for the first piece of output to be produced:

$$p \frac{\partial Q}{\partial L} > w \text{ and } p \frac{\partial Q}{\partial K} > c(i + a) \text{ for } Q = K = L = 0 \quad (20)$$

In this case, K will always be positive, so $\dot{\mu} = 0$. Now, from (9), (13) and (14) we can derive that three paths have to be studied (see Table A2.1). The fourth combination, with both lagrangian parameters having a positive value, is not feasible due to the fact that from (13) and (14) it holds:

path. nr.	λ_1	λ_2	I
1	0	0	$I_{\min} < I < I_{\max}$
2	+	0	$I = I_{\min}$
3	0	+	$I = I_{\max}$

Table A2.1. Paths of the model of Jorgenson.

$$\lambda_1 > 0, \lambda_2 > 0 \rightarrow I_{\min} = I = I_{\max} \quad (21)$$

which is contrary to the assumption that $I_{\min} < 0 < I_{\max}$.

Finally we can derive from (12) that:

$$\frac{\partial Q}{\partial L} = \frac{w}{p} \quad (22)$$

This fixed optimal labour productivity implies, due to the concavity of $Q(K,L)$ that Q may be conceived as a concave function of K . So, to each marginal productivity belongs only one value of K :

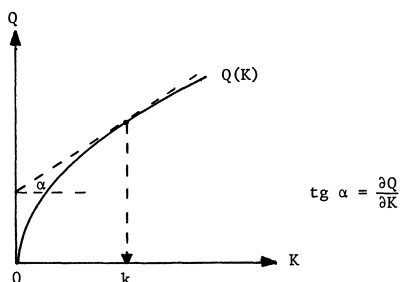


Figure A2.2. Production as a function of K for optimal values of L .

We will now discuss the characteristics of the distinct paths.

Path 1. ($\lambda_1 = \lambda_2 = 0$)

From $\lambda_1 = \lambda_2 = 0$ and (10) and (11) we can derive:

$$\dot{\psi} = -e^{-iT} \frac{w}{p} \frac{\partial Q}{\partial K} + a(\psi + \mu) \quad (23)$$

$$\psi + \mu = e^{-iT} c \quad (24)$$

Differentiating the last equation to time results in:

$$\dot{\psi} + \dot{\mu} = -ie^{-iT}c \quad (25)$$

Combining (23) through (25) yields:

$$e^{-iT} p \frac{\partial Q}{\partial K} = (a + i) c e^{-iT} + \dot{\mu} \quad (26)$$

From (15) and (26) we may conclude:

$$p \frac{\partial Q}{\partial K} < (a + i) c \quad (27)$$

Assumption (20) together with (22) and (27) imply that:

$$Q > 0 + K > 0 + \dot{\mu} = 0 \quad (28)$$

Inserting (28) in (26) delivers:

$$p \frac{\partial Q}{\partial K} = (a + i) c \quad (29)$$

From (22) and (29) the conclusion can be drawn that output and both inputs have a stationary value, say Q^* , K^* and L^* , on path 1. Furthermore we can conclude from (24) that path 1 fulfills the transversality conditions (18) and (19), because from (24):

$$\psi + \mu = 0 \text{ when } T \rightarrow \infty, \text{ which enables:} \quad (30)$$

$$\psi = \mu = 0 \text{ when } T \rightarrow \infty \text{ and } K > 0 \quad (31)$$

Path 2. ($\lambda_1 > 0$, $\lambda_2 = 0$)

There is one interesting feature of path 2 to be presented in this stage of the solution procedure: it cannot be a final path. From (5) and (13) we can derive:

$$I = I_{\min} + \dot{K} < 0 + K > 0 \quad (32)$$

And furthermore (11) yields:

$$\lambda_1 > 0, \lambda_2 = 0 + \psi + \mu = e^{-iT} - \lambda_1 \quad (33)$$

which implies:

$$\text{for } T \rightarrow \infty : \psi + \mu = -\lambda_1 < 0 \quad (34)$$

while from the transversality conditions (18), (19) and from (32) it must hold that:

$$\text{for } T \rightarrow \infty \text{ and } K > 0 : \psi = \mu = 0 \quad (35)$$

which is contrary to (34), so path 2 cannot fulfill the transversality conditions and therefore it cannot be a final path.

Path 3. ($\lambda_1 = 0, \lambda_2 > 0$)

In the same way one can derive that path 3 cannot be a final path. From (14):

$$\lambda_2 > 0 \rightarrow I = I_{\max} \rightarrow \dot{K} > 0 \rightarrow K > 0 \quad (36)$$

Thus (18) and (19) imply:

$$\text{for } T \rightarrow \infty : \psi = \mu = 0 \quad (37)$$

which is contrary to condition (11) that states:

$$\lambda_2 > 0 \rightarrow \psi + \mu = e^{-iT} c + \lambda_2 > 0 \quad (38)$$

so path 3, too, cannot be a final path.

According to the above findings the coupling procedure is quite simple: only path 1 can be a final path and the other two paths can only precede path 1. So, the only "one staged string" as defined in figure A2.1, is: path 1. The question we must check is: can path 1 be an initial path as well? If so, it should obey the initial state condition (6). In combination with the findings of (29) we can conclude that there is a necessary condition for path 1 to be an initial path as well and thus to be a complete string:

$$K(0) = K^* \quad (39)$$

So, only for one initial value of K, namely the stationary value, path 1 is the optimal initial path. In that case, the firm starts on path 1 at $T = 0$ and remains on it during the whole planning period.

This solution satisfies the sufficiency conditions of theorem (A1.9) as well as the uniqueness condition of theorem (A1.10). So it is the unique optimal solution for $K(0) = K^*$.

We now have to check for two staged strings. That is: can paths 2 and 3 precede path 1 and on which conditions?

Path 2 → path 1.

Because K is positive on path 2 as well as on path 1, μ has a fixed value (see: (16)). Together with the continuity of ψ and the fact that $\lambda_2 = 0$, we may thus conclude from (11) that λ_1 is continuous. Therefore, λ_1 has to become zero at the end of path 2. This is possible, only if

$$\dot{\lambda}_1 < 0 \text{ when } \lambda_1 = 0 \text{ on path 2} \quad (40)$$

From (10) and (11), together with $\lambda_2 = 0$ and $\dot{\mu} = 0$ we derive:

$$\dot{\lambda}_1 = -e^{-iT} ic - \dot{\psi} \quad (41)$$

$$= e^{-iT} \left\{ p \frac{\partial Q}{\partial K} - (a + i) c \right\} + a \lambda_1 \quad (42)$$

So, from (40) and (42) follows the necessary condition:

$$p \frac{\partial Q}{\partial K} < (a + i) c \text{ on the coupling time point.} \quad (43)$$

Because of the decrease of K on path 2, the concavity of Q and the fixed labour productivity (see: (22)), (43) results in:

$$K > K^* \text{ on path 2 } \rightarrow K(0) > K^* \quad (44)$$

In this way, we have derived from the continuity properties of ψ and λ_1 a necessary initial state constraint for the relevant string. We leave it to the reader to check the sufficiency and uniqueness conditions.

Path 3 → path 1.

In the same way as in the case of path 2 → path 1, we can derive the continuity of λ_2 on the coupling time point, so it must hold that:

$$\dot{\lambda}_2 < 0 \text{ when } \lambda_2 = 0 \text{ on path 3} \quad (45)$$

and this results in the necessary condition:

$$K(0) < K^* \quad (46)$$

Now we must check whether there are feasible strings of the third generation, containing three paths each. There are four possible combinations:

- (1) path 1 + path 2 + path 1
- (2) path 3 + path 2 + path 1
- (3) path 1 + path 3 + path 1
- (4) path 2 + path 3 + path 1

They all are infeasible. The first combination, for example, requires

$$\lambda_1^* > 0 \text{ when } \lambda_1 = 0 \text{ on path 2} \quad (47)$$

due to the continuity of λ_1 on the first coupling point and due to the fact that $\lambda_1 = 0$ on path 1 and $\lambda_1 > 0$ on path 2. In the same way as derived in (40) - (44), this results in

$$K < K^* \text{ on path 2} \quad (48)$$

And this is contrary to condition (44)

The second combination is infeasible because the succession of path 3 by path 2 demands a necessary condition that is contrary to condition (44), which is necessary, in its turn, for the succession of path 2 by path 1. From (11) we derive:

$$\psi + \mu = \lambda_2 - \lambda_1 + e^{-iT}c \quad (49)$$

Furthermore we have seen:

$\psi + \mu$: continuous

$$\lambda_2 > 0 \text{ on path 3 and } \lambda_2 = 0 \text{ on path 2} \quad (50)$$

$$\lambda_1 = 0 \text{ on path 3 and } \lambda_1 > 0 \text{ on path 2.}$$

Thus the continuity of $(\psi + \mu)$ can only be guaranteed if

$$\lambda_2(t_{3,2}^-) = 0 = \lambda_1(t_{3,2}^+) \quad (51)$$

in which : $t_{i,j}$: point in time on which path j succeeds path i

$\lambda_k(t_{i,j}^-)$: left hand side limit of λ_k on $T = t_{i,j}$

$\lambda_k(t_{i,j}^+)$: right hand side limit of λ_k on $T = t_{i,j}$

It is necessary for meeting (51) that

$$\dot{\lambda}_1 > 0 \text{ when } \lambda_1 = 0 \text{ on path 2,} \quad (52)$$

resulting in:

$$K < K^* \text{ on path 2.}$$

Because of the decrease of K on path 2, this implies:

$$K < K^* \text{ on } t_{2,1} \quad (53)$$

Knowing that $K = K^*$ on path 1, this should imply a jump in K on $t_{2,1}$ which is infeasible. So, (51) seems to prevent the coupling on $t_{2,1}$ and thus the second combination is infeasible.

The infeasibility of the two remaining combinations can be shown along the same lines as presented above. In summary, we found three optimal solutions:

if $K(0) > K^*$: path 2 + path 1

if $K(0) = K^*$: path 1

if $K(0) < K^*$: path 3 + path 1

With the knowledge that K falls as quickly as possible on path 2 ($I = I_{\min}$) and rises as quickly as possible on path 3 ($I = I_{\max}$), we can state that the optimal policy of the firm is: to jump to the stationary state (Q^*, K^*, L^*) at start of the planning period and to remain there till the end.

4. The model of Ludwig.

After having discussed to a great extent the main features of the solution procedure with the help of Jorgenson's model, we will outline the solutions of the other models in this appendix, occasionally illustrated by some details. The interested reader can find more in the relevant publications themselves.

Assume $Y(0) > 0$ and $X(0) > 0$, then from (3.28), (3.29) and (3.31) follows that:

$$X(T) > 0 \text{ and } Y(T) > 0, 0 < T < z \quad (54)$$

Furthermore Ludwig assumes a positive concave net return (defined as: sales after depreciation) function:

$$R(K) - aK > 0, \frac{\partial R}{\partial K} > a \text{ and } \frac{\partial^2 R}{\partial K^2} < 0 \quad (55)$$

Note that, due to $K = X + Y$, it holds that:

$$\frac{dR}{dK} = \frac{\partial R}{\partial X} = \frac{\partial R}{\partial Y} \quad (56)$$

Finally, he introduces an imperfect capital market through:

$$i \neq r \quad (57)$$

Now, the Hamiltonian becomes:

$$H = e^{-iT} \{ R(X+Y) - (a+r) \cdot Y - I + B \} + \psi_1 (I - aX - B) + \psi_2 (B - aY) \quad (58)$$

and the Lagrangian:

$$L = H + \lambda_1 (hI - B) + \lambda_2 \{ m(R(X+Y) - aX - (a+r) \cdot Y) + aX + B - I \} + \lambda_3 B \quad (59)$$

In order to present a uniform treatise of the optimality conditions of all models in this book, we did not introduce discounted lagrangian parameters in the above lagrangian function, as distinct from Ludwig. As the model contains only control constraints, theorems 4 through 7 of appendix 1 apply. The necessary conditions are

$$\dot{\psi}_1 = - \frac{\partial L}{\partial X} = a \psi_1 - e^{-iT} \frac{dR}{dK} - \lambda_2 \left\{ \frac{dR}{dK} \cdot m + (1-d)a \right\} \quad (60)$$

$$\dot{\psi}_2 = - \frac{\partial L}{\partial Y} = a \psi_2 - (e^{-iT} + \lambda_2 m) \left\{ \frac{dR}{dK} - a - r \right\} \quad (61)$$

$$\frac{\partial L}{\partial B} = e^{-iT} - \psi_1 + \psi_2 - \lambda_1 + \lambda_2 + \lambda_3 = 0 \quad (62)$$

$$\frac{\partial L}{\partial I} = \psi_1 - e^{-1T} + h\lambda_1 - \lambda_2 = 0 \quad (63)$$

$$\lambda_1(hI - B) = 0 \quad (64)$$

$$\lambda_2\{(R - (a+r)Y - aX)m + aX + B - I\} = 0 \quad (65)$$

$$\lambda_3 B = 0 \quad (66)$$

$$\psi_1(z) = e^{-1z} ; \psi_2(z) = 0 \quad (67)$$

$$\lambda_1, \lambda_2, \lambda_3 > 0 \quad (68)$$

Again, in order to get a consistent definition for all models, we do not follow Ludwig's definition, but alter it slightly. From (64) through (66) we can form the following paths:

path nr.	λ_1	λ_2	λ_3	path nr.	λ_1	λ_2	λ_3
1	+	0	0	5	0	+	+
2	0	+	0	6	+	0	+
3	+	+	0	7	+	+	+
4	0	0	+	8	0	0	0

Table A2.2. Paths of the model of Ludwig.

Path 7 is infeasible, because from (64) through (66) it results that

$$(R - aK - rY)m + aX = 0 \quad (69)$$

while (3.21) and (3.22) imply:

$$(R - aK - rY)m > 0 \quad (70)$$

and (54):

$$aX > 0 \quad (71)$$

Path 8 is infeasible because we can derive from (60) through (63) that

$$\lambda_1 = \lambda_2 = \lambda_3 = 0 \text{ implies:}$$

$$\frac{dR}{dK} = a + i = a + r + i = r \quad (72)$$

which is contrary to assumption (57).

We will now turn to the characteristics of the remaining paths.

Path 1.

First we derive a stationary value of K on path 1:

$$\lambda_1 > 0 \quad \dot{\psi}_1 = a \psi_1 - e^{-iT} \frac{dR}{dK} \quad (73)$$

$$\lambda_2 = \lambda_3 = 0 \quad \rightarrow \dot{\psi}_2 = a \psi_2 - e^{-iT} \left[\frac{dR}{dK} - a - r \right] \quad (74)$$

(60) - (63)

$$\lambda_1 = e^{-iT} - \psi_1 + \psi_2 \quad (75)$$

$$h \lambda_1 = e^{-iT} - \psi_1 \quad (76)$$

$$(75), (76) \rightarrow (1-h)\lambda_1 = \psi_2 \quad (77)$$

$$(76), (77) \rightarrow \frac{1}{h} (-1e^{-iT} - \psi_1) = \frac{\dot{\psi}_2}{1-h} \quad (78)$$

$$(73), (74), (76) - (78) \rightarrow \frac{dR}{dK} - a = (1-h)i + hr + \dot{K} = 0 \quad (79)$$

Let us indicate this stationary value of K with: K_y^* . The assumed values of the lagrangian parameters result in the following conditions:

$$\lambda_1 > 0 \quad B = hI \quad (80)$$

$$\lambda_2 = \lambda_3 = 0 \quad \rightarrow I < (R - (a+r)Y - aX)m + aX + B \quad (81)$$

$$(64) - (66) \quad B > 0 \quad (82)$$

These expressions lead to the following characteristics:

$$(80), (3.28), (3.29) : \dot{K} = \dot{X} = \dot{Y} = I - a(X + Y) = 0$$

$$\rightarrow \begin{cases} I = aK_Y^* & (83) \\ B = ahK_Y^* & (84) \end{cases}$$

Furthermore we can derive a dynamic equation of the only positive lagrangian parameter, λ_1 :

$$(74), (77), (79) + \dot{\lambda}_1 = a\lambda_1 - (i - r)e^{-iT} \quad (85)$$

Condition (81) can be reformulated as:

$$(81), (83), (84) : (rm + a)Y < mR(K_Y^*) - (m - h)aK_Y^* \quad (86)$$

Finally, we can use the above optimal values of I and B to specify more precisely the state equations of path 1:

$$(3.28), (83), (84) : \dot{X} = (1-h)aK_Y^* - aX \quad (87)$$

$$(3.29), (84) : \dot{Y} = haK_Y^* - aY \quad (88)$$

Taking all in all, we have found the following characteristics of path 1:

$$(79) : K = K_Y^*, \text{ as defined in (79)}$$

$$(83) : I = aK_Y^*$$

$$(84) : B = ahK_Y^*$$

$$(87) : \dot{X} = (1-h)aK_Y^* - aX$$

$$(88) : \dot{Y} = haK_Y^* - aY$$

$$(85) : \dot{\lambda}_1 = a\lambda_1 - (i - r)e^{-iT}$$

$$(68) : \lambda_1 > 0$$

$$(76) : \psi_1 = e^{-iT} - h\lambda_1$$

$$(77) : \psi_2 = (1-h)\lambda_1$$

$$(69) : \lambda_2 = \lambda_3 = 0$$

$$(86) : (a + m)Y < mR(K_Y^*) - (m - h)aK_Y^*$$

In the same way, we can derive the characteristics of the other 5 paths. In order to avoid confusing details, we will present only those characteristics that are needed to demonstrate the coupling procedure and to support the description of the optimal trajectories in the main text of chapter 3.

Path 2 : $\lambda_1 = \lambda_3 = 0, \lambda_2 > 0$

$$K = K_{YX}^* \quad (89)$$

$$\dot{Y} = -m(R - aK - rY) < 0 \quad (90)$$

$$m(R(K_{YX}^*) - aK_{YX}^* - rY) + haK_{YX}^* < aY \quad (91)$$

$$\psi_1 = e^{-iT} + \lambda_2 \quad (92)$$

$$\psi_2 = 0 \quad (93)$$

Path 3 : $\lambda_1 > 0, \lambda_2 > 0, \lambda_3 = 0$

$$\dot{X} = m(R - aK - rY) \quad (94)$$

$$\dot{Y} = \frac{h}{1-h} \{ m(R - aK - rY) + aX \} - aY \quad (95)$$

$$\lambda_2 = \frac{h}{1-h} \left\{ e^{-iT} \left((1-h)i + hr + a - \frac{\partial R}{\partial K} \right) + \lambda_2 m \left(hr - \frac{dR}{dK} + a \right) \right\} \quad (96)$$

$$\psi_1 = e^{-iT} - h\lambda_1 + \lambda_2 \quad (97)$$

$$\psi_2 = (1-h)\lambda_1 \quad (98)$$

Path 4 : $\lambda_1 = \lambda_2 = 0, \lambda_3 > 0$

$$K = K_X^*, \dot{Y} = -aY \quad (99)$$

$$\psi_1 = e^{-iT} \quad (100)$$

$$\psi_2 = -\lambda_3 \quad (101)$$

Path 5 : $\lambda_1 = 0, \lambda_2 > 0, \lambda_3 > 0$

$$\dot{Y} = -aY \quad (102)$$

$$D = (1-m)(R - aK - rY) \quad (103)$$

$$\psi_1 = e^{-iT} + \lambda_2 \quad (104)$$

$$\psi_2 = -\lambda_3 \quad (105)$$

Path 6 : $\lambda_1 > 0, \lambda_2 = 0, \lambda_3 > 0$

$$\dot{\lambda}_3 = a\lambda_3 + e^{-iT} \frac{1}{h} \left(\frac{dR}{dK} - a - (1-h)i - hr \right) \quad (106)$$

$$\psi_1 = e^{-iT} - h\lambda_1 \quad (107)$$

$$\psi_2 = (1-h)\lambda_1 - \lambda_3 \quad (108)$$

The next step in the solution procedure is to select feasible final paths. We will restrict ourselves here to checking path 1, because we intend to present the procedure only for a subset of optimal strings. With the help of that presentation, the interested reader will be able to construct the remaining optimal strings.

The transversality condition (67), together with equation (76) imply:

$$\psi_1(z) = e^{-iz} + \lambda_1(z) = 0 \text{ when } T \rightarrow z \quad (109)$$

This expression implies:

$$\dot{\lambda}_1 < 0 \text{ when } \lambda_1 = 0 \text{ on a final path.} \quad (110)$$

From (85) we derive:

$$i > r \rightarrow i > r \quad (111)$$

due to assumption (57). So, path 1 can only be a feasible final path if the discount rate exceeds the interest rate on debt capital.

In order to find criteria to select feasible preceding paths, we derive from (85), (109) and (111):

$$\lambda_1(t_{j1}^+) > 0 \quad (112)$$

and from (76) and (77):

$$\psi_1(t_{j1}^+) = e^{-it_{j1}^+} - h\lambda_1(t_{j1}^+) \quad (113)$$

$$\psi_2(t_{j1}^+) = (1-h)\lambda_1(t_{j1}^+) > 0 \quad (114)$$

in which : t_{j1} : point in time on which path 1 succeeds path j.

We are now able to select feasible preceding paths:

path 2 : not feasible, because $\psi_2(t_{21}^-) = 0$, so there should be a jump in ψ_2 , while ψ_2 is continuous.

path 3 : allowed if $\lambda_2(t_{31}^-) = 0 + \dot{\lambda}_2 < 0$ when $\lambda_2 = 0 \rightarrow \frac{dR}{dK} - a > (1-h)i + hr$

$$\text{so : } K < K_Y^* \text{ on path 3} \quad (115)$$

Furthermore K must increase on path 3 in order to reach K_Y^* . In the Jorgenson model we handled a similar problem by supposing a sufficiently large marginal revenue when $K = 0$. Ludwig deals with this problem in a different way by deriving a sufficiently large initial value of X in the following way:

$$K = X + Y \rightarrow \dot{K} = \dot{X} + \dot{Y} = \frac{1}{1-h} m (R - aK - rY) + \frac{h}{1-h} aX - aY \quad (116)$$

Due to the concavity of R it holds that:

$$R(K) > K \frac{dR}{dK} \text{ and, while } K < K_Y^*, \text{ so : } \frac{dR}{dK} > a + r :$$

$$R(K) > (r + a)K \quad (117)$$

So \dot{K} is certainly positive if

$$\frac{1}{1-h} m (rK - rY) + \frac{h}{1-h} aX - aY > 0 \rightarrow \frac{Y}{X} < \frac{mr + ha}{(1-h)a} \quad (118)$$

path 4 : infeasible because $\psi_2(t_{41}^-) < 0$

path 5 : infeasible because $\psi_2(t_{51}^-) < 0$

path 6 : feasible if $\lambda_3(t_{61}^-) = 0 + \dot{\lambda}_3 < 0$ when $\lambda_3 = 0$

$$\frac{dR}{dK} - a < (1-h)i + hr$$

$$\text{so : } K > K_Y^* \text{ on path 6.} \quad (119)$$

In the same way, one can find that there are no feasible predecessors of the paths 3 and 6. So we have found both strings, ending with path 1:

if $i > r$ and $K(0) < K_Y^*$ and $\frac{Y(0)}{K(0)} < \frac{mr + ha}{(1-h)a}$: path 3 + path 1

if $i > r$ and $K(0) > K_Y^*$: path 6 + path 1

In the same way, the other patterns presented in chapter 3 can be derived.

5. The model of Lesourne and Leban.

To get a closed control region, we add artificial boundaries on D and I. Furthermore we remove Y by substituting $(K-X)$ for it and so we get:

$$\text{maximize}_{D, I, L} : \int_0^{\infty} D e^{-iT} dT \quad (120)$$

$$\text{subject to} : \dot{X} = (1-f)(R - wL - (r+a)K - rX) - D \quad (121)$$

$$\dot{K} = I - aK \quad (122)$$

$$X < K < (1+k)X \quad (123)$$

$$0 < D < D_{\max} \quad (124)$$

$$I_{\min} < I < I_{\max} \quad (125)$$

$$R = R(K, L), \text{ strictly increasing and concave} \quad (126)$$

This model contains state constraints (123) as well as control constraints (124) and (125), so theorems 8 up to 10 of appendix 1 hold in this case. The Hamiltonian is:

$$\begin{aligned} H = e^{-iT} & D + (\psi_1 - \mu_1 + (1+k)\mu_2)((1-f)\{R - wL - (r+a)K + rX\} - D) + \\ & + (\psi_2 + \mu_1 - \mu_2)(I - aK) \end{aligned} \quad (126)$$

and the Lagrangian:

$$L = H + \lambda_1 D + \lambda_2 (D_{\max} - D) + \lambda_3 (I - I_{\min}) + \lambda_4 (I_{\max} - I) \quad (127)$$

From the Jorgenson model we know that paths, during which the controls are on an artificial boundary, describe adjustments of the initial state of the firm in such a way

as to become a point on an optimal string.¹⁾ Because of their lack of economic meaning, we drop them here, assuming:

$$\lambda_2 = \lambda_3 = \lambda_4 = 0 \quad (128)$$

We now get the following optimality conditions:

$$\dot{\psi}_1 = -(\psi_1 - \mu_1 + (1+k)\mu_2)(1-f)r \quad (129)$$

$$\dot{\psi}_2 = -(\psi_1 - \mu_1 + (1+k)\mu_2)(1-f)\left(\frac{\partial R}{\partial K} - (r+a)\right) \quad (130)$$

$$\psi_1 - \mu_1 + (1+k)\mu_2 = e^{-iT} + \lambda_1 \quad (131)$$

$$\psi_2 = \mu_2 - \mu_1 \quad (132)$$

$$(\psi_1 - \mu_1 + (1+k)\mu_2)(1-f)\left(\frac{\partial R}{\partial L} - w\right) = 0 \quad (133)$$

$$\lambda_1 D = 0 \quad (134)$$

$$\dot{\mu}_1 < 0, \dot{\mu}_2 < 0 \quad (135)$$

$$\dot{\mu}_1(K - X) = 0, \dot{\mu}_2((1+k)X - K) = 0 \quad (136)$$

$$\lambda_1, \mu_1, \mu_2 > 0 \quad (137)$$

$$\lim_{T \rightarrow \infty} \psi_1(T) = 0, \lim_{T \rightarrow \infty} \psi_2(T) = 0 \quad (138)$$

$$\lim_{T \rightarrow \infty} \mu_1(T) \cdot (K - X) = 0, \lim_{T \rightarrow \infty} \mu_2(T) \cdot ((1+k)X - K) = 0 \quad (139)$$

From (131) and (133) the result is that:

$$\frac{\partial R}{\partial L} = w \rightarrow R = R(K) \text{ in the optimal solution} \quad (140)$$

From (134) and (136) we derive six possible paths:

1) Arrow and Kurz have proved, more in general, that jumps in the state variables can occur only at the beginning of a planning period if the Hamiltonian in the optimal solution is strictly concave in the state variables (Arrow & Kurz (1970), page 56).

path nr.	λ_1	$\dot{\mu}_1$	$\dot{\mu}_2$	path nr.	λ_1	$\dot{\mu}_1$	$\dot{\mu}_2$
1	0	-	0	4	+	-	0
2	0	0	-	5	+	0	-
3	0	0	0	6	+	0	0

Table A2.3. Paths of the model of Lesourne & Leban.

On the paths 1 through 3 it holds that $\lambda_1 = 0$. Due to (131) and (132) this results in:

$$\dot{\psi}_1 + \dot{\psi}_2 + ie^{-iT} = -k \dot{\mu}_2 \quad (141)$$

$$\dot{\psi}_1 + (1+k)\dot{\psi}_2 + ie^{-iT} = -k \dot{\mu}_1 \quad (142)$$

and by inserting (129) and (130):

$$e^{-iT}(1-f) \left\{ \frac{\partial G}{\partial K} - a - \frac{i}{1-f} \right\} = k \dot{\mu}_2 \quad (143)$$

$$e^{-iT}(1-f) \left\{ \frac{\partial G}{\partial K} - a - \frac{1}{1+k} \left(kr + \frac{i}{1-f} \right) \right\} = \frac{k}{1+k} \dot{\mu}_1 \quad (144)$$

Path 1.

$$(136) : \dot{\mu}_1 < 0, \dot{\mu}_2 = 0 \rightarrow Y = 0 \text{ and } X = K < (1+k)X \quad (145)$$

$$(144) : \dot{\mu}_1 < 0 \rightarrow \frac{\partial R}{\partial K} - a < \frac{1}{1+k} \left(kr + \frac{i}{1-f} \right) \quad (146)$$

$$(143) : \dot{\mu}_2 = 0 \rightarrow \frac{\partial R}{\partial K} - a = \frac{i}{1-f} \rightarrow K = K_X^* \quad (147)$$

$$(146), (147) : \frac{i}{1-f} < \frac{1}{1+k} \left(kr + \frac{i}{1-f} \right) \rightarrow i < (1-f)r \quad (148)$$

So, this path can appear only if equity is cheaper than debt capital.

Path 2.

$$(136) : \dot{\mu}_1 = 0, \dot{\mu}_2 < 0 \rightarrow X < K = (1+k)X \quad (149)$$

$$(144) : \dot{\mu}_1 = 0 \rightarrow \frac{\partial R}{\partial K} - a = \frac{1}{1+k} \left(kr + \frac{i}{1-f} \right) \rightarrow K = K_Y^* \quad (150)$$

$$(143) : \dot{\mu}_2 < 0 \rightarrow \frac{\partial R}{\partial K} - a < \frac{i}{1-f} \quad (151)$$

(150),

$$(151) : i > (1-f)r \text{ is necessary to enable path 2} \quad (152)$$

Path 3.

$$(136) : \dot{\mu}_1 = \dot{\mu}_2 = 0 \rightarrow X < K < (1+k)X \quad (153)$$

In the same way as described above, this yields : $i = (1-f)r$, which is excluded by assumption (3.52). So, two paths remain: the pure equity financing case and the maximum debt financing case. On both paths, the capital stock has a stationary value and so, due to the fixed financial structure:

$$\dot{K} = 0 \rightarrow \dot{X} = 0 \quad (154)$$

which yields an expression for D and I:

$$(121), (154), Y = 0 : D = (1-f) \{ R(K_X^*) - wL - aK_X^* \} \quad (155)$$

$$I = aK_X^* \quad (156)$$

$$(121), (158), Y = kX : D = (1-f) \{ R(K_Y^*) - wL - \left(a + \frac{k}{1+k} r \right) K_Y^* \} \quad (157)$$

$$I = aK_Y^* \quad (158)$$

From (131), (132), (138) and (139) one can conclude that paths 1 and 2 can fulfill the transversality conditions if all auxiliary variables vanish when T approaches ∞ .

In the same way as for paths 1 through 3, we can derive from (131) and (132) for paths 4 through 6:

$$\dot{\lambda}_1 = \dot{\psi}_1 + \dot{\psi}_2 + k \dot{\mu}_2 + i e^{-iT} \quad (159)$$

$$\dot{\lambda}_1^* = \dot{\psi}_1 + (1+k)\dot{\psi}_2 + k \dot{\mu}_1 + i e^{-iT} \quad (160)$$

These expressions will be used for the coupling procedure later on.
Furthermore, (130) through (132) result in:

$$\dot{\psi}_2 = \dot{\mu}_2 - \dot{\mu}_1 = - (e^{-iT} + \lambda_1)(1-f)\left(\frac{\partial R}{\partial K} - (r+a)\right) \quad (161)$$

Path 4.

$$(136) : \dot{\mu}_1 < 0, \dot{\mu}_2 = 0 \rightarrow X = K < (1+k) X \quad (162)$$

$$(161), \quad (162) : \frac{\partial R}{\partial K} - a < r \rightarrow K > K_{XY}^* \quad (163)$$

Path 5.

$$(136) : \dot{\mu}_1 = 0, \dot{\mu}_2 < 0 \rightarrow X < K < (1+k) X \quad (164)$$

$$(161), \quad (164) : \frac{\partial R}{\partial K} - a > r \rightarrow K < K_{XY}^* \quad (165)$$

From (140), (163) and the strict increasing property of R, we can conclude that:

$$R - wL - (r+a)K > 0 \rightarrow \dot{X} > 0 \quad (166)$$

Path 6.

$$(136) : \dot{\mu}_1 = \dot{\mu}_2 = 0 \rightarrow X < K < (1+k) X \quad (167)$$

$$(161), \quad (167) : \frac{\partial R}{\partial K} - a = r \rightarrow K = K_{XY}^* \quad (168)$$

From (140), (168) and the concavity of R we can conclude that on path 6
(R - wL - (a+r)K) is on its maximum, so:

$$R - wL - (a+r)K > 0 \quad (169)$$

(121), (168), (169) :

$$\dot{X} > 0 \rightarrow \dot{Y} < 0, \text{ due to } \dot{K} = 0 \quad (170)$$

So, the firm is paying back debt capital, keeping its capital stock and its money capital on a fixed level.

The fact that λ_1 is positive on paths 4 through 6, prevents that the transversality condition can hold, for, due to (131) and (132):

$$\psi_2(T) = 0 + \mu_2 = \mu_1 + \psi_1 - k \mu_2 = e^{-iT} + \lambda_1 > 0 \quad (171)$$

So, for $T \rightarrow \infty$ and $\psi_1 = 0$ (see: (138)), it must hold that

$$\lim_{T \rightarrow \infty} \mu_2(T) > 0 \quad (172)$$

due to the transversality condition (139). This is path 5 when λ_1 is positive. On that path, K is strictly increasing, so, we can assume that it surpasses K_{YX}^* while T is still far from infinity, and thus it will change into another path before that time. None of the other two paths is a feasible final path, due to (172), so paths 1 and 2 are the only final paths.

This brings us to the coupling procedure. Paths 1 and 2 are the only final paths and, just like in the former models, may form a whole string if the initial state is exactly the relevant stationary state, so:

if : $i < (1-f)r$ and $X(0) = K_X^*$, then the optimal trajectory is:

$$\begin{aligned} \text{path 1, with : } K(T) &= K_X^* \\ Y(T) &= 0 \\ I(T) &= aK_X^* \\ D(T) &= (1-f)\{r(K_X^*) - wL - aK_X^*\} \end{aligned} \quad (173)$$

if : $i > (1-f)r$ and $X(0) = \frac{1}{1+k} K_Y^*$, then the optimal trajectory is:

$$\begin{aligned} \text{path 2 with: } K(T) &= K_Y^* \\ Y(T) &= \frac{k}{1+k} K_Y^* \\ I(T) &= a K_Y^* \\ D(T) &= (1-f)\{R(K_Y^*) - wL - (a + \frac{k}{1+k} r)K_Y^*\} \end{aligned} \quad (174)$$

Strings consisting of two stages are formed by coupling paths 4 through 6 before path 1 or 2. From (104) in appendix 1 one can derive the continuity of μ_1 and μ_2 , even when

the control variables I and D are jumping, for in that case, $\phi(x,u,t)$, as defined in (92) of appendix 1, will be discontinuous and so, μ_1 and μ_2 are continuous. This nice property implies, through (131), the continuity of λ_1 , so it is necessary for paths preceding paths 1 and 2 that:

$$\dot{\lambda}_1 < 0 \text{ when } \lambda_1 = 0 \text{ on a preceding path.} \quad (175)$$

Let us consider the relevant three paths apart from each other:

Path 4.

From (129), (130), (131), (159) and $\dot{\mu}_2 = 0$ one can derive that (175) implies:

$$\dot{\lambda}_1 = \dot{\psi}_1 + \dot{\psi}_2 + ie^{-iT} = -e^{-iT}(1-f)\left\{\frac{\partial R}{\partial K} - a - \frac{i}{1-f}\right\} < 0 \text{ when } \lambda_1 = 0 \quad (176)$$

$$\rightarrow \frac{\partial R}{\partial K} - a > \frac{i}{1-f} \rightarrow K < K_X^* \quad (177)$$

Together with the necessary condition (163), this results in another necessary condition:

$$i < (1-f)r \quad (178)$$

So, we get:

if $i < (1-f)r$ and $K_{YX}^* < X(0) < K_X^*$, then the optimal trajectory is:

$$\begin{array}{ll} \text{path 4} & \rightarrow \quad \text{path 1} \\ K_{YX}^* < K < K_X^* & K = K_X^* \\ \dot{K} > 0 & \\ Y = 0 & Y = 0 \\ D = 0 & D = (1-f)\{R(K_X^*) - wL - ak_X^*\} \end{array} \quad (179)$$

which is the case of pure equity financed growth.

Path 5.

From (165) and (166) result two possibilities of ending this path:

$$(5.1) : \lambda_1 = 0 \text{ and then a change to path 1 or path 2}$$

$$(5.2) : K = K_{YX}^* \text{ before } \lambda_1 = 0$$

(5.1) $\lambda_1 = 0$ and then a change to path 1 or path 2.

On path 5 it holds that $\dot{\mu}_1 = 0$, so through (129) - (131) and (160) we can translate (175) into:

$$\dot{\lambda}_1 = -e^{-tT} (1-f)(1+k) \left\{ \frac{\partial R}{\partial K} - a - \frac{1}{1+k} \left(kr + \frac{i}{1-f} \right) \right\} < 0 \quad (180)$$

$$\rightarrow \frac{\partial R}{\partial K} - a > \frac{1}{1+k} \left(kr + \frac{i}{1-f} \right) \rightarrow K < K_Y^* \quad (181)$$

Due to the continuity of X and K, this path must be connected with path 2, the case of maximum debt financing. Due to its necessary condition (152) we get:

if $i > (1-f)r$ and $X(0) < \frac{1}{1+k} K_Y^*$, then the optimal trajectory is:

path 5 + path 2

$$K < K_Y^* \quad K = K_Y^*$$

$$\dot{K} > 0$$

$$Y = \frac{k}{1+k} K \quad Y = \frac{k}{1+k} K_Y^*$$

$$D = 0 \quad D = (1-f) \left\{ R(K_Y^*) - wL - \left(a + \frac{k}{1+k} r \right) K_Y^* \right\} \quad (182)$$

which is the case of maximum debt financed growth.

(5.2) $K = K_{YX}^*$ before $\lambda_1 = 0$.

We must check the following extensions:

path 1 : not allowed for, because then a jump in λ_1 should occur, which is prohibited by the optimality conditions, as shown in relation to (175).

path 4 : infeasible, too, because then it must hold for the left side limit of Y on the coupling point of time:

$$\dot{Y} = b\dot{X} > 0 \quad (183)$$

and for the right side limit:

$$\dot{Y} = 0 \quad (184)$$

so we get on the coupling time point:

$$\dot{X} + \dot{Y} = \kappa = \dot{X} \quad (185)$$

which implies a jump in X, and this is contrary to the continuity of X.

path 6 : feasible, starting with $Y = 0$. It may end with:

$$0 < Y < kX \text{ and } \lambda_1 = 0 \text{ or: } Y = 0 \text{ and } \lambda_1 > 0. \quad (186)$$

In the same way as above, we can check that only the last situation results in a feasible successor, i.e. path 4, and that after transgression of this path, till $K = K_X^*$, path 1 is entered.

Combining all necessary conditions, we can derive the following optimal patterns from the above analysis:

If $i < (1-f)$ and $X(0) < \frac{1}{1+k} K_{XY}^*$, then the optimal trajectory is:

path 6 → path 5 → path 4 → path 1

$$K < K_{YX}^* \quad K = K_{YX}^* \quad K_{YX}^* < K < K_X^* \quad K = K_X^*$$

$$\dot{K} > 0 \quad \dot{X} > 0, \dot{Y} < 0 \quad \dot{K} > 0$$

$$Y = kX \quad 0 < Y < kX \quad Y = 0 \quad Y = 0$$

$$D = 0 \quad D = 0 \quad D = 0 \quad D = (1-f)\{r(K_X^*) - wL - aK_X^*\}$$

If $i < (1-f)r$ and $X(0) = \frac{1}{1+k} K_{YX}^*$, then the optimal trajectory is

path 5 → path 4 → path 1

If $i < (1-f)r$ and $\frac{1}{1+k} K_{YX}^* < X(0) < K_Y^*$, then the optimal trajectory is:

$$\text{path 4} \rightarrow \text{path 1.} \quad (187)$$

We will finish this section with a derivation of the critical Q -value \bar{Q} , discussed in the main text. If the output in the optimal solution is below this level, capital goods and labour are complementary inputs. If the output exceeds this level, both inputs become substitutes to each other.

First, we repeat some information that will be used in the derivation:

$$(126): Q = K^\alpha L^{1-\alpha} + \frac{\partial Q}{\partial K} = \alpha \frac{Q}{K} \text{ and } \frac{\partial Q}{\partial L} = (1-\alpha) \frac{Q}{L} \quad (188)$$

$$(3.48): \Delta(Q) := -\frac{d(Q \frac{dR}{dQ})}{dQ} \text{ is a decreasing function of } Q \text{ with:}$$

$$\Delta(0) > 0 \text{ and } \Delta(\infty) < 0 \quad (189)$$

The strict concavity of $R(Q)$ and the optimality condition (140) give:

$$\frac{dR}{dQ} > 0, \frac{d^2 R}{dQ^2} < 0 \text{ and } : \frac{\partial R}{\partial L} = w \quad (190)$$

We start the derivation by transforming (190) with the help of (188):

$$\frac{\partial R}{\partial L} = \frac{dR}{dQ} \cdot \frac{\partial Q}{\partial L} = (1-\alpha) \frac{Q}{L} \frac{dR}{dQ} = w + (1-\alpha) Q \frac{dR}{dQ} = wL \quad (191)$$

Differentiation of (191) yields:

$$(1-\alpha) \frac{d}{dQ} (Q \frac{dR}{dQ}) dQ = w dL \quad (192)$$

Inserting (188) and (189) in (192) results into:

$$(1-\alpha) \Delta \left\{ \alpha \frac{Q}{K} dK + (1-\alpha) \frac{Q}{L} dL \right\} = w dL \quad (193)$$

Substituting w by (191) and omitting $(1-\alpha)$:

$$\Delta \left\{ \alpha \frac{Q}{K} dK + (1-\alpha) \frac{Q}{L} dL \right\} = \frac{Q}{L} \frac{dR}{dQ} dL + \frac{dL}{dK} = \frac{\alpha \Delta L}{K \left\{ \frac{dR}{dQ} - (1-\alpha) \Delta \right\}} \quad (194)$$

Now, from (189) we can derive that:

$$\Delta = Q \frac{d^2 R}{dQ^2} + \frac{dR}{dQ} \quad (195)$$

Inserting (195) into the denominator of (194) yields:

$$\frac{dL}{dK} = \frac{\alpha \Delta L}{K \left[\alpha \frac{dR}{dQ} - (1-\alpha) Q \frac{d^2 R}{dQ^2} \right]} \quad (196)$$

From (190) we can conclude that the denominator of (196) has a positive value, so (dL/dK) has the same sign as Δ . From (189) we know that there is a value of Q , say \bar{Q} , for which this sign must change from a positive value into a negative value, which thus implies the transition from the case of complementary inputs into the case of substitution.

6. Summary.

As far as we know, none of the publications on dynamic economic models of the firm in the area of Optimal Control contains a description of the way in which the author has found the relevant optimal solution. Discussions with several authors confirm, that the solution in most cases has been arrived at in a heuristic, intuitive way. In this appendix we present a general, iterative procedure to solve Optimal Control models containing mixed control and state constraints, that is based on the optimality conditions as formulated in the previous appendix. The procedure is applied to the three dynamic models of the firm that may be regarded as precursors of the model of chapter 4.

APPENDIX 3. SOLUTION OF THE MODEL OF CHAPTER 4.

1. A reduced form of the model.

In order to simplify the solution procedure, we will first leave out some mathematically superfluous elements in the formulation (4.16) - (4.28). From (4.19) and (4.20) results that, for a given state of the firm, not all control variables can be chosen freely, so the firm has in fact less independent instruments than the number of control variables. This fact will be used to reduce the model by leaving out two endogenous variables. In our formulation, we have selected K_2 and Y to be removed from the model. We can rewrite (4.19) and (4.20) in:

$$K_2 = K - K_1 \quad (1)$$

$$Y = K - X \quad (2)$$

Substitution of the above expressions in (4.16) through (4.24) results in the next reduced form of the model:

$$\text{maximize}_{D, I, K_1} \int_0^z e^{-iT} D \, dT + e^{-iz} \{X(z) - gK(z)\} \quad (3)$$

$$\text{subject to : } \dot{X} = (1-f)(0 + rX - rK) - D + gI \quad (4)$$

$$\dot{K} = I - aK \quad (5)$$

$$K > X \quad (6)$$

$$(1+k)X > K \quad (7)$$

$$K_1 < K \quad (8)$$

$$D > 0, K_1 > 0 \quad (9)$$

$$\text{in which : } 0 = \{q_1 P(Q) - w_1\} K_1 + \{q_2 P(Q) - w_2\} (K - K_1) - aK \quad (10)$$

$$Q = q_1 K_1 + q_2 (K - K_1) = (q_1 - q_2) K_1 + q_2 K \quad (11)$$

So, O is a function of K and K_1 . The non-negativeness of K is ensured by (8) and (9). The non-negativeness of X is ensured by the non-negativeness of K together with (7).

To get a closed control region, we have to put artificial boundaries to D and I in the same way as done in several models of appendix 2:

$$D < D_{\max} \quad (12)$$

$$I_{\min} < I < I_{\max} \quad (13)$$

Two state variables remain: X and K , but the set of controls has been reduced to $\{D, I, K_1\}$. The reduced model will be used in this appendix. Through (1) and (2) one can easily trace the consequences of the optimal solution for the removed variables.

2. Optimality conditions.

Equations (6) and (7) are constraints on state variables, (8) is a mixed constraint and (9) are pure control constraints, so theorems 8 through 10 of appendix 2 apply to this case. From theorem 8 we derive the following optimality conditions:

Let the Hamiltonian be:

$$\begin{aligned} H = e^{-tT} D + \{ \psi_1 - \mu_1 + (1+k) \mu_2 \} \{ (1-f)[O(K, K_1) + rX - rK] \\ - D + gI \} + (\psi_2 + \mu_1 - \mu_2)(I - aK) \end{aligned} \quad (14)$$

and let the Lagrangian be defined as:

$$\begin{aligned} L = H + \lambda_1(K - K_1) + \lambda_2 K_1 + \lambda_3 D + \lambda_4 (D_{\max} - D) + \\ + \lambda_5 (I - I_{\min}) + \lambda_6 (I_{\max} - I) \end{aligned} \quad (15)$$

Then there must be functions $\psi_1(T)$, $\psi_2(T)$, $\mu_1(T)$, $\mu_2(T)$, $\lambda_1(T)$, $\lambda_2(T)$ and $\lambda_3(T)$, such that:

$$\dot{\psi}_1 = - \frac{\partial L}{\partial X} = - \{ \psi_1 - \mu_1 + (1+k) \mu_2 \} (1-f)r \quad (16)$$

$$\begin{aligned} \dot{\psi}_2 = - \frac{\partial L}{\partial K} = - \{ \psi_1 - \mu_1 + (1+k) \mu_2 \} (1-f) \left[\frac{\partial O}{\partial K} - r \right] + \\ + a(\psi_2 + \mu_1 - \mu_2) - \lambda_1 \end{aligned} \quad (17)$$

$$\frac{\partial L}{\partial D} = e^{-IT} - \psi_1 + \mu_1 - (1+k) \mu_2 + \lambda_3 - \lambda_4 = 0 \quad (18)$$

$$\frac{\partial L}{\partial I} = \{\psi_1 - \mu_1 + (1+k) \mu_2\} g + \psi_2 + \mu_1 - \mu_2 + \lambda_5 - \lambda_6 = 0 \quad (19)$$

$$\frac{\partial L}{\partial K_1} = \{\psi_1 - \mu_1 + (1+k) \mu_2\} (1-f) \frac{\partial O}{\partial K_1} - \lambda_1 + \lambda_2 = 0 \quad (20)$$

$$\lambda_1 (K - K_1) = 0, \dots, \lambda_6 (I_{\max} - I) = 0 \quad (21)$$

$$\dot{\mu}_1 < 0, \dot{\mu}_2 < 0, \dot{\mu}_1 (K - X) = 0, \dot{\mu}_2 \{(1+k)X - K\} = 0 \quad (22)$$

$$\lambda_k > 0, \mu_\ell > 0, k = 1, \dots, 6; \ell = 1, 2. \quad (23)$$

$$\psi_1(z) = e^{-iz}, \psi_2(z) = -ge^{-iz} \quad (24)$$

$$\mu_1(z) \{K(z) - X(z)\} = 0, \mu_2(z) \{(1+k)X(z) - K(z)\} = 0 \quad (25)$$

$$\psi \quad : \text{continuous with piecewise continuous derivatives} \quad (26)$$

$$\dot{\psi} \text{ and } \lambda \quad : \text{continuous on intervals of continuity of } \{D, I, K_1\} \quad (27)$$

$$\mu_1 \quad : \text{continuous on intervals of continuity of } \{D, I, K_1\} \\ \text{or when } (K - X) \text{ is discontinuous} \quad (28)$$

$$\mu_2 \quad : \text{continuous on intervals of continuity of } \{D, I, K_1\} \\ \text{or when } \{(1+k)X - K\} \text{ is discontinuous} \quad (29)$$

3. A reduced form of the optimality conditions.

Before starting to design the optimal solution, we may simplify the optimality conditions. In the first place, we leave out cases in which D and/or I are on their artificial boundaries, so:

$$\lambda_4 = \lambda_5 = \lambda_6 = 0 \quad (30)$$

Further, from the conditions in section 2 we can derive that μ_1 , μ_2 and λ_3 are continuous functions. From (28) follows that μ_1 is continuous

$$\text{if } \{D, I, K_1\} \text{ continuous and/or } (K - X) \text{ discontinuous,} \quad (31)$$

which can be rewritten, using (4) and (5), into:

if $\{D, I, K_1\}$ continuous and/or

$$\{(1-g)I + D - [a - (1-f)r]K - (1-f)rX - (1-f)0\} \text{ is discontinuous} \quad (32)$$

Due to the closed control region, X and K are continuous, so at least one of the control variables must be discontinuous in order to get a discontinuity of the last expression of (32). So, the above conditions (32) are complementary to each other and (31) will always be fulfilled. Therefore we may conclude that μ_1 is continuous in the above optimality conditions.

In the same way one can derive the continuity of μ_2 from (4), (5) and (29). Because of the continuity of μ_1 and μ_2 , and because of (18), (26) and (30) we may further conclude that λ_3 is continuous.

Taking all in all, we can reduce (16) through (29) to the next form:

$$\psi_1 = \mu_1 - (1+k) \mu_2 + e^{-1T} + \lambda_3 \quad (33)$$

$$\psi_2 = \mu_2 - \mu_1 - g(e^{-1T} + \lambda_3) \quad (34)$$

$$\dot{\psi}_1 = - (e^{-1T} + \lambda_3)(1-f)r \quad (35)$$

$$\dot{\psi}_2 = - (e^{-1T} + \lambda_3) \left\{ (1-f) \left(\frac{\partial 0}{\partial K} - r \right) + ag \right\} - \lambda_1 \quad (36)$$

$$(e^{-1T} + \lambda_3)(1-f) \frac{\partial 0}{\partial K_1} = \lambda_1 - \lambda_2 \quad (37)$$

$$\lambda_1(K - K_1) = 0, \lambda_2 K_1 = 0, \lambda_3 D = 0 \quad (38)$$

$$\dot{\mu}_1 < 0, \dot{\mu}_2 < 0 \quad (39)$$

$$\dot{\mu}_1(K - X) = 0, \dot{\mu}_2\{(1+k)X - K\} = 0 \quad (40)$$

$$\lambda_k > 0, \mu_k > 0, k = 1, 2, 3; \ell = 1, 2. \quad (41)$$

$$\psi_1(z) = e^{-1z}, \psi_2(z) = -ge^{-1z} \quad (42)$$

$$\mu_1(z) \{K(z) - X(z)\} = 0, \mu_2(z) \{(1+k)X(z) - K(z)\} = 0 \quad (43)$$

$$\psi_1, \psi_2, \lambda_3, \mu_1 \text{ and } \mu_2 : \text{continuous functions} \quad (44)$$

$$\dot{\psi}_1, \dot{\psi}_2, \dot{\lambda}_1 \text{ and } \dot{\lambda}_2 : \text{continuous on intervals of continuity of } \{D, I, K_1\} \quad (45)$$

Finally we will derive some equations that will be needed later on in this appendix. We know that:

$$Q = q_1 K_1 + q_2 (K - K_1) = (q_1 - q_2) K_1 + q_2 K \quad (46)$$

and from (10) that:

$$0 = S - (w\ell_1 - w\ell_2) K_1 - (w\ell_2 + a) K \quad (47)$$

$$\text{so: } \frac{\partial Q}{\partial K_1} = q_1 - q_2 ; \frac{\partial Q}{\partial K} = q_2 \quad (48)$$

$$\frac{\partial 0}{\partial K_1} = w(\ell_2 - \ell_1) - (q_1 - q_2) \frac{dS}{dQ} \quad (49)$$

$$\frac{\partial 0}{\partial K} = q_2 \frac{dS}{dQ} - w\ell_2 - a \quad (50)$$

Finally, we can easily find, due to assumption A4 in chapter 4, that only two different rankings of the relevant unit costs can occur:

$$c_{1N} < c_{2N} < c_{21} , \text{ or } : c_{21} < c_{2N} < c_{1N} , N = X, Y, YX$$

$$\text{in which : } c_{21} = \frac{w(\ell_2 - \ell_1)}{q_2 - q_1} \quad (51)$$

4. Infeasible paths.

From (38) and (40) one can form 32 different combinations of zero and non-zero valued lagrangian parameters and first derivatives of the auxiliary variables μ_k . Not all of them are feasible paths. In this section we will look for combinations that cannot fulfill the above optimality conditions and the assumptions A1 through A6 as defined in section 7 of chapter 4. The remaining combinations, which are the feasible paths constituting the optimal patterns, will be discussed in more detail in the next section.

$$\text{a. } \dot{\mu}_1 = \dot{\mu}_2 = \dot{\lambda}_3 = 0$$

From (33) results that:

$$e^{-iT} = \psi_1 - \mu_1 + (1+k) \mu_2 + \dot{\psi}_1 = -ie^{-iT} \quad (52)$$

Together with (35) this leads to the necessary condition that $i = (1-f)r$, which is contrary to assumption A3. So, the above combination is infeasible.

b. $\lambda_1 = \lambda_2 = 0$ and $\dot{\mu}_1 = \dot{\mu}_2 = 0$

c. $\lambda_1 = \lambda_2 = 0$ and $\dot{\mu}_2 = \lambda_3 = 0$

d. $\lambda_1 = \lambda_2 = 0$ and $\dot{\mu}_1 = \lambda_3 = 0$

If $\lambda_1 = \lambda_2 = 0$, it results from (37) and (49) that:

$$(1-f) \frac{\partial 0}{\partial K_1} = 0 + \frac{dS}{dQ} = \frac{w(\lambda_2 - \lambda_1)}{q_2 - q_1} = c_{21} \quad (53)$$

Suppose further that:

$$\frac{dS}{dQ} = \frac{1}{q_2} (w\lambda_2 + a + \beta) \quad (54)$$

in which β is a parameter to be specified later on, then it holds that

$$\frac{1}{q_1} (w\lambda_1 + a + \beta) = \frac{1}{q_2} (w\lambda_2 + a + \beta) \quad (55)$$

We will use this theorem to prove that each of the above mentioned three cases conflicts with assumption A4. This assumption excludes the incidental cases in which the unit costs of both activities, defined for different financial structures, equal each other. We start with $\dot{\mu}_1 = \dot{\mu}_2 = 0$, which implies through (33) and (34) that

$$g \dot{\psi}_1 = -\dot{\psi}_2 \quad (56)$$

together with (35), (36), (50) and $\lambda_1 = 0$, this results in:

$$\frac{dS}{dQ} = \frac{1}{q_2} \left\{ w\lambda_2 + \left(1 - \frac{g}{1-f}\right) a + (1-g)r \right\} := c_{2YX} \quad (57)$$

From (53) and (57) we can conclude that

$$c_{1YX} = c_{2YX} \quad (58)$$

which is contrary to assumption A4. So, combination b is infeasible.

In the case of $\dot{\mu}_2 = \lambda_3 = 0$, we get from (33) and (34):

$$\dot{\mu}_1 = \dot{\psi}_1 + ie^{-i\Gamma} \quad (59)$$

$$g \dot{\psi}_1 + \dot{\psi}_2 = -(1-g) \dot{\mu}_1 \quad (60)$$

Combining (59) and (60) with (35), (36) and (50) yields:

$$\frac{dS}{dQ} = \frac{1}{q_2} \left\{ w\lambda_2 + \left(1 - \frac{g}{1-f}\right) a + (1-g) \frac{i}{1-f} \right\} := c_{2X} \quad (61)$$

Due to (53) and (61) a necessary condition to case c should be: $c_{1X} = c_{2X}$, which is excluded by assumption A4, so combination c is infeasible.

When $\dot{\mu}_1 = \lambda_3 = 0$ we can derive from (33) and (34) that:

$$(1+k) \dot{\mu}_2 = -\dot{\psi}_1 - ie^{-i\Gamma} \quad (62)$$

$$g \dot{\psi}_1 + \dot{\psi}_2 = \{1 - g(1+k)\} \dot{\mu}_2 \quad (63)$$

resulting in:

$$\frac{dS}{dQ} = \frac{1}{q_2} \left\{ w\lambda_2 + \left(1 - \frac{g}{1-f}\right) a + \frac{k}{1+k} r + \left(\frac{1}{1+k} - g\right) \frac{i}{1-f} \right\} := c_{2Y} \quad (64)$$

Together with (55) this leads to the necessary condition that: $c_{1Y} = c_{2Y}$, which is, again contrary to assumption A4 and therefore combination d is cancelled.

e. $\dot{\mu}_1 < 0$ and $\dot{\mu}_2 < 0$

f. $\lambda_1 > 0$ and $\lambda_2 > 0$

Due to (40) and the conditions (4) through (8), the first case implies that

$$X = K = 0 \rightarrow K_1 = I = D = 0 \quad (65)$$

The second case results into the same conclusion through (38):

$$K = K_1 = 0 \rightarrow X = I = D = 0 \quad (66)$$

Due to the continuity of X, both cases have the necessary initial condition:

$$X(t_0) = 0 \quad (67)$$

in which t_0 is the moment at which the relevant case starts. But, as we shall see in the next section, none of the remaining paths can end with a zero amount of equity. So, there are no feasible preceding paths for the above cases. Furthermore, assumption A5 of chapter 4 prohibits the above cases to be feasible initial paths, because the initial value of equity must be more than zero. So, there is no place for the cases e and f in any optimal trajectory and therefore they are infeasible.

All in all, the following combinations are infeasible:

case	λ_1	λ_2	λ_3	$\dot{\mu}_1$	$\dot{\mu}_2$
a			0	0	0
b	0	0		0	0
c	0	0	0		0
d	0	0	0	0	
e	+	+			
f				-	-

Table A3.1. The infeasible combinations.

5. Feasible paths.

Based on the complementary slackness conditions (38) and (40) and table 1, we can now list the remaining feasible paths:

path nr	λ_1	λ_2	λ_3	$\dot{\mu}_1$	$\dot{\mu}_2$	path nr	λ_1	λ_2	λ_3	$\dot{\mu}_1$	$\dot{\mu}_2$
1	0	+	+	0	-	7	0	+	+	-	0
2	0	+	0	0	-	8	0	+	0	-	0
3	0	0	+	0	-	9	+	0	+	0	0
4	+	0	+	0	-	10	0	0	+	-	0
5	+	0	0	0	-	11	+	0	+	-	0
6	0	+	+	0	0	12	+	0	0	-	0

Table A3.2. The feasible paths.

In this section we will demonstrate in more detail the main characteristics of these feasible paths. We need this information partly for the coupling procedure and partly for the analysis of the optimal solution. In this section we will apply another numbering of formulas: the first number indicates the path concerned and the second number the rank of the formula in the relevant subsection.

Path 1.

$$\lambda_1 = \dot{\mu}_1 = 0 \quad (1.1)$$

$$\lambda_2 > 0, \lambda_3 > 0, \dot{\mu}_2 < 0 \quad (1.2)$$

$$(38), (40), : K_1 = 0 \quad (1.3)$$

$$(1.2) \quad D = 0 \quad (1.4)$$

$$(1+k)X = K \leftrightarrow Y = kX \quad (1.5)$$

$$(37), (41), : \lambda_2 = - (e^{-1T} + \lambda_3)(1-f) \frac{\partial O}{\partial K_1} + \frac{\partial O}{\partial K_1} < 0 \rightarrow \frac{dS}{dQ} > c_{21} \rightarrow Q < Q_{21}^* \quad (1.6)$$

(49), (51),

(1.1)

$$(33)-(36), : g \dot{\psi}_1 + \dot{\psi}_2 = \{1 - g(1+k)\} \dot{\mu}_2 < 0 +$$

(39), (1.1)

$$(A.6) \quad (e^{-1T} + \lambda_3) \{ - (1-f) \frac{\partial O}{\partial K} + (1-g)(1-f)r - ag \} < 0 +$$

$$\frac{dS}{dQ} > c_{2YX} \rightarrow Q < Q_{2YX}^* \quad (1.7)$$

$$(4), (5), : \dot{X} = (1-f) \left\{ 0 - \frac{k}{1+k} rK \right\} + g(\dot{K} + aK) \leftrightarrow$$

(40), (1.2)

$$\{ 1 - g(1+k) \} \dot{X} = (1-f) \left\{ 0 + \frac{ag}{1-f} - \frac{k}{1+k} r K \right\} \quad (1.8)$$

Due to the concavity of S, (1.4) and (1.7) it holds that:

$$S > Q \frac{dS}{dQ} > K \cdot q_2 \cdot c_{2YX} \quad (1.9)$$

$$(38), (47), : 0 = S - (w\lambda_2 + a)K \quad (1.10)$$

(1.2)

$$(1.8-10), : \{ 1 - g(1+k) \} \dot{X} > (1-f) \left\{ \frac{1 - g(1+k)}{1+k} rK \right\} \rightarrow \dot{X} > \frac{1-f}{1+k} rK \quad (1.11)$$

(59)

$$(1.5, 11) : \dot{X} > 0, \dot{K} > 0, \dot{Y} > 0 \quad (1.12)$$

Due to (1.2), λ_3 is positive on path 1, but its value may approach zero at the beginning or at the end of this path. From the non-negativeness restriction (41), we can derive the following necessary conditions for such cases:

$$\overset{+}{\lambda}_3 = 0 \rightarrow \overset{\cdot}{\lambda}_3 > 0 \text{ when } \lambda_3 = 0 \text{ on path 1} \quad (1.13)$$

$$\overset{+}{\lambda}_3 = 0 \rightarrow \overset{\cdot}{\lambda}_3 < 0 \text{ when } \lambda_3 = 0 \text{ on path 1} \quad (1.14)$$

(An arrow to the right (left) indicates the right (left) side limit of the relevant variable on the relevant time intersection). Now we will translate these conditions in conditions on the parameters and/or on the value of Q.

$$(33)-(36), : \overset{\cdot}{\lambda}_3 = \frac{1}{1 - (1+k)g} \{ \overset{\cdot}{\psi}_1 + (1+k) \overset{\cdot}{\psi}_2 \} + i e^{-iT} \quad (1.1)$$

$$= - \frac{(1+k)(1-f)}{1 - (1+k)g} e^{-iT} q_2 \left\{ \frac{dS}{dQ} - c_{2Y} \right\} \text{ when } \lambda_3 = 0 \quad (1.15)$$

$$(1.13,15) : \overset{+}{\lambda}_3 = 0 \rightarrow \frac{dS}{dQ} < c_{2Y} \quad (1.16)$$

$$(A4),(51), : \overset{+}{\lambda}_3 = 0 \rightarrow c_{21} < c_{2Y} \rightarrow c_{1Y} > c_{2Y} \quad (1.17)$$

$$(1.6,7,16) \quad c_{2YX} < c_{2Y} \rightarrow i > (1-f)r$$

$$(1.14,15) : \overset{+}{\lambda}_3 = 0 \rightarrow \frac{dS}{dQ} > c_{2Y} \rightarrow Q < Q_{2Y}^* \quad (1.18)$$

Path 2.

$$\lambda_1 = \lambda_3 = \mu_1 = 0 \quad (2.1)$$

$$\lambda_2 > 0, \mu_2 < 0 \quad (2.2)$$

$$(38),(40), : K_1 = 0 \rightarrow K = K_2 \quad (2.3)$$

(2.2)

$$(1+k)X = K \rightarrow Y = kX \quad (2.3)$$

$$(33),(35), : \overset{\cdot}{\psi}_1 = - (1+k) \overset{\cdot}{\mu}_2 - i e^{-iT} = - e^{-iT} (1-f)r \quad (2.4)$$

(2.1)

$$(A3),(40), : \overset{\cdot}{\mu}_2 < 0 \rightarrow i > (1-f)r \quad (2.5)$$

(2.4)

In the same way as on path 1, we can derive:

$$(33)-(36), : g \dot{\psi}_1 + \dot{\psi}_2 = \{1 - g(1+k)\} \dot{\mu}_2 + \frac{dS}{dQ} = c_{2Y} \rightarrow Q = Q_{2Y}^* \quad (2.6)$$

(2.1,4)

Analogously to (1.6) of path 1, we get:

$$(37), : \frac{dS}{dQ} > c_{21} \quad (2.7)$$

(2.1,2)

$$(A4),(51), : c_{1Y} > c_{2Y} \quad (2.8)$$

(2.6,7)

Path 3.

$$\lambda_1 = \lambda_2 = \dot{\mu}_1 = 0 \quad (3.1)$$

$$\lambda_3 > 0, \dot{\mu}_2 < 0 \quad (3.2)$$

$$(38),(40), : D = 0 \quad (3.3)$$

(3.2)

$$(1+k)X = K \rightarrow Y = kX$$

Analogous to the previous paths, it holds that:

$$(37),(49), : \frac{\partial Q}{\partial K_1} = 0 + \frac{dS}{dQ} = c_{21} \rightarrow Q = Q_{21}^* \quad (3.4)$$

(3.1)

Just like on path 1 we can derive:

$$(33)-(36), : \frac{dS}{dQ} > c_{2YX} \quad (3.5)$$

(39),(3.1)

$$(A4),(51), : c_{1YX} < c_{2YX} \quad (3.6)$$

(3.4,5)

We will now prove that X increases on path 3.

$$(1.5) : \dot{K} = (1+k)\dot{X} \quad (3.7)$$

$$(4), (5), \quad : \quad \dot{X} = (1-f) \left\{ 0 - \frac{k}{1+k} rK \right\} + gI = \frac{(1-f)}{1-g(1+k)} \left\{ 0 + \left(\frac{ag}{1-f} - \frac{k}{1+k} r \right) K \right\} \quad (3.8)$$

(3.3)

Due to the concavity of S and (3.5) it holds that:

$$S > Q \frac{dS}{dQ} > Q \cdot c_{2YX} \quad (3.9)$$

Together with (46), (59) and (3.6) this results in:

$$S > \ell_1 K_1 w + \ell_2 (K - K_2) w + K \left\{ \left(1 - \frac{g}{1-f} \right) a + (1-g)r \right\} \quad (3.10)$$

$$(47), (3.10): \quad 0 > \left\{ (1-g)r - \frac{g}{1-f} a \right\} K \quad (3.11)$$

$$(3.8, 11) \quad : \quad \dot{X} > \frac{1-f}{1+k} rK > 0 \quad \text{q.e.d.} \quad (3.12)$$

We can further derive from (46) and (3.4):

$$\dot{Q} = 0 + q_2 (1+k) \dot{X} = q_2 \dot{K} = (q_2 - q_1) \dot{K}_1 \quad (3.13)$$

$$(3.12, 13), \quad : \quad \dot{X} > 0, \quad \dot{K}_1 > 0, \quad \dot{K}_2 = \dot{K} - \dot{K}_1 < 0 \quad (3.14)$$

$q_1 < q_2$

Just like on path 1 we can derive that:

$$\dot{\lambda}_3 = - \frac{(1+k)(1-f)}{1 - (1+k)g} e^{-iT} q_2 \left\{ \frac{dS}{dQ} - c_{2Y} \right\} \text{ when } \lambda_3 = 0 \quad (3.15)$$

and combining this with (3.4) and the necessary conditions as mentioned on path 1, we get:

$$\begin{aligned} \dot{\lambda}_3 = 0 & \rightarrow \frac{dS}{dQ} < c_{2Y} \rightarrow c_{1Y} > c_{2Y} \\ \dot{\lambda}_3 = 0 & \rightarrow \frac{dS}{dQ} > c_{2Y} \rightarrow c_{1Y} < c_{2Y} \end{aligned} \quad (3.16)$$

Path 4.

$$\lambda_2 = \mu_1 = 0 \quad (4.1)$$

$$\lambda_1 > 0, \quad \lambda_3 > 0, \quad \dot{\mu}_2 < 0 \quad (4.2)$$

Next results can be found in the same way as on the above paths, so we will only mention them:

$$\begin{aligned}
 (4.2) \quad & : K = K_1 \\
 & D = 0 \\
 & (1+k)X = K \rightarrow Y = kX
 \end{aligned} \tag{4.3}$$

$$\begin{aligned}
 (37), \quad & : \frac{dS}{dQ} < c_{21} \rightarrow Q > Q_{21}^* \\
 (4.1,2) \quad &
 \end{aligned} \tag{4.4}$$

$$\begin{aligned}
 (33)-(36), \quad & : \frac{dS}{dQ} > c_{2YX} \rightarrow Q < Q_{2YX}^* \\
 (4.1,2) \quad &
 \end{aligned} \tag{4.5}$$

$$\begin{aligned}
 (A4), (51), \quad & : c_{1YX} < c_{2YX} \\
 (4.4,5) \quad &
 \end{aligned} \tag{4.6}$$

(4), (4,2-5) and the concavity of S:

$$\dot{X} > 0, \dot{K} > 0, \dot{Y} > 0 \tag{4.7}$$

In the same way as on the previous paths we derive from (33) - (36):

$$\dot{\lambda}_3 = \frac{1}{1-g(1+k)} \{ \dot{\psi}_1 + (1+k) \dot{\psi}_2 \} + i e^{-iT} \tag{4.8}$$

Which now leads, due to (4.1) and (37), to:

$$\dot{\lambda}_3 = - \frac{(1+k)(1-f)}{1-(1+k)g} e^{-iT} q_1 \left\{ \frac{dS}{dQ} - c_{1Y} \right\} \text{ when } \lambda_3 = 0 \tag{4.9}$$

This results, together with (4.6), in the following necessary conditions for zero initial or final values of λ_3 on path 4:

$$\begin{aligned}
 \dot{\lambda}_3 = 0 \rightarrow & \frac{dS}{dQ} < c_{1Y} \rightarrow c_{1YX} < c_{1Y} \rightarrow (1-f)r < i \\
 \dot{\lambda}_3 = 0 \rightarrow & Q < Q_{1Y}^*
 \end{aligned} \tag{4.10}$$

Path 5.

$$\lambda_2 = \lambda_3 = \dot{\mu}_1 = 0 \quad (5.1)$$

$$\lambda_1 > 0, \dot{\mu}_2 < 0 \quad (5.2)$$

$$(5.1,2) : K = K_1 \quad (5.3)$$

$$(1+k)X = K \rightarrow Y = kX \quad (5.4)$$

Like on path 2 it holds, due to (5.1), that:

$$i > (1-f)r \quad (5.5)$$

$$(33)-(36), : g \dot{\psi}_1 + \dot{\psi}_2 = \{1 - g(1+k)\} \dot{\mu}_2 + \frac{dS}{dQ} = c_{1Y} \quad Q = Q_{1Y}^* \quad (5.6)$$

(49), (50),
(5.1,2)

$$(37), (49), : \frac{\partial 0}{\partial K_1} > 0 + \frac{dS}{dQ} < c_{21} \quad (5.7)$$

(5.1,2)

$$(A4), (51), : c_{1Y} < c_{2Y} \quad (5.8)$$

(5.6,7)

Path 6.

$$\lambda_1 = \dot{\mu}_1 = \dot{\mu}_2 = 0 \quad (6.1)$$

$$\lambda_2 > 0, \lambda_3 > 0 \quad (6.3)$$

$$(6.1,2) : K_1 = 0 \rightarrow K = K_2 \quad (6.3)$$

$$D = 0 \quad (6.4)$$

$$(33)-(36), : g \dot{\psi}_1 + \dot{\psi}_2 = 0 + \frac{dS}{dQ} = c_{2YX} \quad Q = Q_{2YX}^* \quad (6.5)$$

(6.1)

$$(37), (49), : \frac{\partial 0}{\partial K_1} < 0 + \frac{dS}{dQ} < c_{21} \quad (6.6)$$

(6.1,2)

$$(A4), (51), : c_{1YX} > c_{2YX} \quad (6.7)$$

(6.5,6)

In the same way as on path 1, the concavity of S, together with (6.5) leads to:

$$\dot{X} > 0 \rightarrow \dot{Y} < 0 \quad (\text{because } \dot{K} = 0 \text{ due to } \dot{Q} = 0) \quad (6.8)$$

$$(33)-(36), : (1-g) \dot{\lambda}_3 = \dot{\psi}_1 + \dot{\psi}_2 + (1-g) i e^{-iT}$$

(58)

$$\dot{\lambda}_3 = \frac{q_2 e^{-iT}}{(1-g)(1-f)} \left\{ c_{2X} - \frac{dS}{dQ} \right\} \text{ when } \lambda_3 = 0 \quad (6.9)$$

Which results together with (6.5) and analogous to path 1 in:

$$\begin{aligned} \dot{\lambda}_3 = 0 \rightarrow Q > Q_{2X}^* \rightarrow c_{2YX} < c_{2X} \rightarrow i > (1-f)r \\ \dot{\lambda}_3 = 0 \rightarrow Q < Q_{2X}^* \rightarrow c_{2YX} > c_{2X} \rightarrow i < (1-f)r \end{aligned} \quad (6.10)$$

Path 7.

$$\lambda_1 = \mu_2 = 0 \quad (7.1)$$

$$\lambda_2 > 0, \lambda_3 > 0, \mu_1 < 0 \quad (7.2)$$

$$(7.1,2) : K_1 = 0 \rightarrow K = K_2$$

$$D = 0 \quad (7.3)$$

$$K = X \rightarrow Y = 0$$

$$(37), (49), : \frac{dS}{dQ} > c_{21} \rightarrow Q < Q_{21}^*$$

(7.1,2) (7.4)

$$(33)-(36), : g \dot{\psi}_1 + \dot{\psi}_2 = - (1-g) \dot{\mu}_1 + \frac{dS}{dQ} < c_{2YX}$$

(39), (7.1,2) (7.5)

$$(A4), (51), : c_{1YX} > c_{2YX} \quad (7.6)$$

(7.4,5)

$$(A2), : \dot{X} = \frac{1-f}{1-g} 0 + \frac{ag}{1-g} K > 0 \rightarrow \dot{K} > 0 \quad (7.7)$$

(7.1-3)

(4)-(6)

Just like on path 6, it holds that:

$$\dot{\lambda}_3 = \frac{1}{1-g} (\dot{\psi}_1 + \dot{\psi}_2) + i e^{-iT} \text{ when } \lambda_3 = 0 \quad (7.8)$$

$$(A4), (51), : \lambda_3 = 0 \rightarrow Q > Q_{2X}^* \rightarrow c_{21} < c_{2X} \rightarrow c_{1X} > c_{2X} \quad (7.9)$$

(7.4,8)

$$\lambda_3 = 0 \rightarrow Q < Q_{2X}^* \rightarrow c_{2X} < c_{2YX} \rightarrow i < (1-f)r \quad (7.10)$$

Path 8.

$$\lambda_1 = \lambda_3 = \dot{\mu}_2 = 0 \quad (8.1)$$

$$\lambda_2 > 0, \dot{\mu}_1 < 0 \quad (8.2)$$

$$(8.1,2) : K_1 = 0 \leftrightarrow K = K_2 \quad (8.3)$$

$$K = X \leftrightarrow Y = 0$$

$$(A3), (33), : \dot{\psi}_1 = \dot{\mu}_1 - i e^{-iT} = e^{-iT} (1-f)r \rightarrow i < (1-f)r \quad (8.4)$$

(35), (39),
(8.1)

$$(33)-(36), : \dot{\psi}_1 + \dot{\psi}_2 = - (1-g) i e^{-iT} + \frac{dS}{dQ} = c_{2X} \leftrightarrow Q = Q_{2X}^* \quad (8.5)$$

(8.1,2,4)

$$(37), : \frac{\partial O}{\partial K_1} < 0 + \frac{dS}{dQ} > c_{21} \quad (8.6)$$

(8.1,2)

$$(A4), (51), : c_{1X} > c_{2X} \quad (8.7)$$

(8.5,6)

Path 9.

$$\lambda_2 = \dot{\mu}_1 = \dot{\mu}_2 = 0 \quad (9.1)$$

$$\lambda_1 > 0, \lambda_3 > 0 \quad (9.2)$$

$$(9.1,2) : K = K_1 \quad (9.3)$$

$$D = 0$$

$$(33)-(37), : g \dot{\psi}_1 = \dot{\psi}_2 = 0 + \frac{dS}{dQ} = c_{1YX} \rightarrow Q = Q_{1YX}^* \quad (9.4)$$

(9.1)

$$(37), : \frac{\partial O}{\partial K_1} > 0 + \frac{dS}{dQ} < c_{21} \quad (9.5)$$

(9.1,2)

$$(A4), (51), : c_{1YX} < c_{2YX} \quad (9.6)$$

(9.4,5)

Due to the concavity of S and due to (9.4) we can derive, analogous to the above paths:

$$\dot{X} > 0 + \dot{Y} < 0 \quad (9.7)$$

$$(33)-(36), : \dot{\lambda}_3 = \frac{q_1 e^{-iT}}{(1-g)(1-f)} \left\{ c_{1X} - \frac{dS}{dQ} \right\} \text{ when } \lambda_3 = 0 \quad (9.8)$$

(49), (50),
(9.1,2)

$$(A3), : \begin{aligned} \dot{\lambda}_3 = 0 + Q > Q_{1X}^* &\rightarrow c_{1YX} < c_{1X} \rightarrow i > (1-f)r \\ \dot{\lambda}_3 = 0 + Q < Q_{1X}^* &\rightarrow c_{1YX} > c_{1X} \rightarrow i < (1-f)r \end{aligned} \quad (9.9)$$

(9.4,8)

Path 10.

$$\lambda_1 = \lambda_2 = \dot{\mu}_2 = 0 \quad (10.1)$$

$$\lambda_3 > 0, \dot{\mu}_1 < 0 \quad (10.2)$$

$$(10.1,2) : D = 0 \quad (10.3)$$

$$K = X + Y = 0 \quad (10.4)$$

$$(37), (49), : \frac{\partial O}{\partial K_1} = 0 + \frac{dS}{dQ} = c_{21} \rightarrow Q = Q_{21}^* \quad (10.5)$$

(10.1)

$$(A2), (4)- : \dot{X} = \frac{1-f}{1-g} O + \frac{ag}{1-g} K > 0 \quad (10.6)$$

(6), (10.3)

$$(10.4,6) : \dot{Q} = 0 \rightarrow \dot{K}_1 > 0, \dot{K}_2 < 0 \quad (\text{see path 3}) \quad (10.7)$$

In the same way as on path 7 we get:

$$(33)-(36) : \frac{dS}{dQ} < c_{2YX} \quad (10.8)$$

$$(39), (10.1,2)$$

$$(A4), (51), : c_{1YX} > c_{2YX} \quad (10.9)$$

$$(10.4,8)$$

Like on path 9 it holds that:

$$\dot{\lambda}_3 = \frac{q_1 e^{-iT}}{(1-g)(1-f)} \left\{ c_{1X} - \frac{dS}{dQ} \right\} \text{ when } \lambda_3 = 0 \quad (10.10)$$

$$(A4), (51), : \dot{\lambda}_3 = 0 \rightarrow Q > Q_{1X}^* \rightarrow c_{1X} > c_{2X} \quad (10.11)$$

$$(10,4,10)$$

$$\dot{\lambda}_3 = 0 \rightarrow Q < Q_{1X}^* \rightarrow c_{1X} < c_{2X}$$

Path 11.

$$\lambda_2 = \mu_2 = 0 \quad (11.1)$$

$$\lambda_1 > 0, \lambda_3 > 0, \mu_1 < 0 \quad (11.2)$$

$$(11.1,2) : K = K_1 \quad (11.3)$$

$$D = 0$$

$$X = K \rightarrow Y = 0$$

$$(37), (49), : \frac{\partial O}{\partial K_1} > 0 \rightarrow \frac{dS}{dQ} < c_{21} + Q > Q_{21}^* \quad (11.4)$$

$$(11.1,2)$$

In the same way as on path 10 we can derive that:

$$\dot{X} > 0 \rightarrow \dot{K} > 0 \rightarrow \dot{Q} > 0 \quad (11.5)$$

Also the necessary conditions for λ_3 to vanish can be found in the same way as on path 10, for the same formula of $\dot{\lambda}_3$ holds on this path:

$$\dot{\lambda}_3 = 0 + \dot{Q} > \dot{Q}_{1X}^* \quad (11.6)$$

$$\dot{\lambda}_3 = 0 + \dot{Q} < \dot{Q}_{1X}^* \rightarrow c_{1X} < c_{1YX} \rightarrow i < (1-f)r \quad (11.7)$$

Path 12.

$$\lambda_2 = \lambda_3 = \dot{\mu}_2 = 0 \quad (12.1)$$

$$\lambda_1 > 0, \dot{\mu}_1 < 0 \quad (12.2)$$

$$(12.1,2) : K = K_1 \quad (12.3)$$

$$K = X - Y = 0$$

Like on path 8, $\lambda_3 = \dot{\mu}_2 = 0$ results in the necessary condition:

$$i < (1-f)r \quad (12.4)$$

$$(34)-(37), : \dot{\psi}_1 + \dot{\psi}_2 = - (1-g) e^{-iT} + \frac{dS}{dQ} = c_{1X} \rightarrow Q = Q_{1X}^* \quad (12.5)$$

(12.1,2)

$$(37),(49), : \frac{\partial O}{\partial K_1} > 0 + \frac{dS}{dQ} < c_{21} \quad (12.6)$$

(12.1,2)

$$(A4),(51), : c_{1X} < c_{2X} \quad (12.7)$$

(12.5,6)

6. Final paths.

Final paths must fulfill the transversality conditions. Together with (33) and (34) we get:

$$\mu_1(z) = (1+k) \mu_2(z) - \lambda_3(z) \quad (68)$$

$$\mu_1(z) = \mu_2(z) - g \lambda_3(z) \quad (69)$$

$$\text{implying: } \mu_2(z) = \frac{1-g}{k} \lambda_3(z) \quad (70)$$

We will now demonstrate in an indirect way that λ_3 has to vanish at the planning horizon. Suppose:

$$\lambda_3(z) > 0 \quad (71)$$

Then, by (70) and (43) it must hold that:

$$\mu_2(z) > 0 \rightarrow K(z) = (1+k)X(z) < X(z) + \mu_1(z) = 0 \quad (72)$$

This results, together with (69) and (70) in:

$$\mu_1(z) = \frac{1 - (1+k)g}{g} \lambda_3(z) = 0 \quad (73)$$

Due to (71) this can only be true if:

$$(1+k)g = 1, \quad (74)$$

which is contrary to assumption A6. So, due to (69), (70) and (74), feasible final paths are characterized by:

$$\mu_1(z) = \mu_2(z) = \lambda_3(z) = 0 \quad (75)$$

This results in the following final paths:

path nr	necessary conditions	main properties of the stationary state
2	$i > (1-f)r$ and $c_{1Y} > c_{2Y}$	$Y = kX$ and $K = K_2$
5	$i > (1-f)r$ and $c_{1Y} < c_{2Y}$	$Y = kX$ and $K = K_1$
8	$i < (1-f)r$ and $c_{1Y} > c_{2Y}$	$Y = 0$ and $K = K_2$
12	$i < (1-f)r$ and $c_{1Y} < c_{2Y}$	$Y = 0$ and $K = K_1$

Table A3.3. The feasible final paths.

As described in figure 1 of appendix 2, we will now start looking for feasible paths, preceding the above mentioned final paths. In order to select them, we derive some coupling conditions from the optimality conditions in the next section.

7. Coupling conditions.

From (2) and from the fact that the variables X and K are continuous, we can derive that Y must be continuous, so:

$$\begin{matrix} + & + \\ Y = Y & \text{on a coupling time point} \end{matrix} \quad (76)$$

We have derived in (44) that:

$$\lambda_3 \text{ is continuous} \quad (77)$$

From the continuity of K and from the fact that activity 1 is the capital-intensive one, so $q_1 < q_2$, it results that:

$$\overset{\leftarrow}{K} = \overset{\leftarrow}{K}_2 \text{ and } Q = Q_{JN}^* + \overset{\leftarrow}{Q} < Q_{JN}^* \text{ on a coupling point of time} \quad (78)$$

in which : $j = 1, 2, 21$; $N = X, Y, YX, \emptyset$ ($j = 21 \leftrightarrow N = \emptyset$)

The above implication is true because:

$$Q = q_1 \overset{\leftarrow}{K}_1 + q_2 (\overset{\leftarrow}{K} - \overset{\leftarrow}{K}_1) < q_2 \overset{\leftarrow}{K} \text{ when } \overset{\leftarrow}{K}_1 > 0, \text{ for: } \overset{\leftarrow}{K} = \overset{\leftarrow}{K} \text{ and } q_1 < q_2.$$

In the same way one can derive that:

$$\overset{\leftarrow}{K} = \overset{\leftarrow}{K}_1 \text{ and } Q = Q_{JN}^* + \overset{\leftarrow}{Q} > Q_{JN}^* \quad (79)$$

Beside the rankings as mentioned in (51), we need the following implication which can easily be derived:

$$i < (1-f)r + c_{1X} > c_{2X} + c_{1Y} > c_{2Y} + c_{1YX} > c_{2YX} \quad (80)$$

8. Coupling procedure.

In this section, we will describe the selection procedure for feasible preceding paths. Only for the first set of strings, all ending with path 2, we will describe this procedure at length. Having all the information on relevant details in this way, the reader is assumed to be able to construct the selection procedure for the remaining sets of strings with the help of the tables, presented later on.

8.1. Strings ending with path 2.

From the coupling conditions and the characteristics of path 2, we can derive that a path, preceding path 2, must have the next properties:

$$\overset{\leftarrow}{Y} = kX ; c_{1Y} > c_{2Y} ; \lambda_3 = 0 \text{ and } i > (1-f)r \quad (81)$$

We will now check which path(s) fulfill(s) these constraints:

path nr	feasible predecessor	reason
1	yes	
2	no	$\lambda_3 > 0$
4	no	$\lambda_3 > 0$ for $Q > Q_{21}^* > Q_{1Y}^*$ due to $c_{1Y} > c_{2Y}$ and $\dot{Q} > 0$
5	no	$c_{1Y} < c_{2Y}$ should be necessary for feasibility
6	no	$\lambda_3 > 0$ because $i > (1-f)r$
7	no	$\lambda_3 > 0$ because $i > (1-f)r$
8	no	$i < (1-f)r$ should be necessary for feasibility
9	no	$\lambda_3 > 0$ because $i > (1-f)r$
10	no	$Y = 0$
11	no	$\lambda_3 > 0$ because $i > (1-f)r$
12	no	$i < (1-f)r$ should be necessary for feasibility

Table A3.4. Selection of paths preceding path 2.

So, only path 1 can precede path 2. On path 1, the amount of capital goods rises and so, because there is no switch of production activity, total output increases. From this fact, from the necessary conditions of the final path 2 and from the characteristics of paths 1 and 2, we can derive the following properties for a feasible path preceding path 1:

$$\begin{array}{ll}
 \text{path 1} & \text{preceding path} \\
 \begin{array}{l}
 \overset{+}{Y} = \overset{+}{kX} \\
 \overset{+}{Q} < \overset{+}{Q_{2Y}^*} \text{ and } \overset{+}{K}_1 = 0 \\
 \overset{+}{\lambda}_3 > 0
 \end{array} &
 \begin{array}{l}
 \overset{+}{Y} = \overset{+}{kX} \\
 \overset{+}{Q} < \overset{+}{Q_{2Y}^*} \\
 \overset{+}{\lambda}_3 > 0
 \end{array}
 \end{array} \quad (82)$$

on both paths: $i > (1-f)r$ and $c_{1Y} > c_{2Y}$

None of the feasible paths can meet all these conditions at the same time:

path nr	feasible predecessor	reason
2	no	$\lambda_3 = 0$
3	no	$Q = Q_{21}^* = q_1 K(t_{31}) + \overset{+}{Q} = q_2 K(t_{31}) = \frac{q_2}{q_1} Q_{21}^* > Q_{2Y}^*$
4	no	$Q_{21}^* < \overset{+}{Q} = q_1 K(t_{41}) + \overset{+}{Q} = q_2 K(t_{41}) > \frac{q_2}{q_1} Q_{21}^* > Q_{2Y}^*$
5	no	$c_{1Y} < c_{2Y}$ should be necessary for feasibility
6	no	$\overset{+}{Y} < kX$
7	no	$\overset{+}{Y} = 0$
8	no	$i < (1-f)r$ should be necessary for feasibility
9	no	$\overset{+}{Y} < kX$
10	no	$\overset{+}{Y} = 0$
11	no	$\overset{+}{Y} = 0$
12	no	$\overset{+}{Y} = 0$

Table A3.5. Selection of paths preceding path 1 + path 2.

By combining the above results with the characteristics of paths 1 and 2, we get two strings of final path 2, a one staged and a two staged string. So, in the case of $i > (1-f)r$, $c_{1Y} > c_{2Y}$ and: $K(0) = (1+k) X(0)$, the following strings and initial conditions hold:

initial conditions	optimal trajectory
$X(0) = a_1$	path 2
$X(0) < a_1$	path 1 + path 2

Table A3.6. Optimal trajectories resulting from tables A3.4 and A3.5.

in which : $a_1 = \frac{1}{(1+k)q_2} Q_{2Y}^*$ (83)

8.2. Strings ending with path 5.

In the same way as above, by combining the properties of the paths and the coupling conditions in selecting feasible preceding paths, one can derive the other trajectories. From these remaining strings, we will only present the conditions to be posed on preceding paths. The selection procedure itself will be left to the reader. For the string ending with path 5, we get the following preceding conditions:

path 5	conditions to precede			
	path 5	path 4	path 3	path 1
$Q = Q_{1Y}^*$	$Q > Q_{1Y}^*$	$Q > Q_{21}^*$ when $K > K_1$	$Q < Q_{21}^*$ when $K_1 > K_1$	$Q < Q_{21}^*$
$Y = kX$	$Y = kX$	$Y = kX$	$Y = kX$	$Y = kX$
$\lambda_3 = 0$	$\lambda_3 = 0$	$\lambda_3 > 0$	$\lambda_3 > 0$	$\lambda_3 > 0$
$i > (1-f)r$	$i > (1-f)r$	$i > (1-f)r$	$i > (1-f)r$	$i > (1-f)r$
$c_{1Y} < c_{2Y}$	$c_{1Y} < c_{2Y}$	$c_{1Y} < c_{2Y}$	$c_{1Y} < c_{2Y}$	$c_{1Y} < c_{2Y}$
preceding path	path 4	path 3	path 1	-

Table A3.7. Preceding conditions of the strings with final path 5.

From table 6 and the characteristics of the paths, we can derive the following strings and relevant initial conditions in the case of $i > (1-f)r$, $c_{1Y} < c_{2Y}$ and:

$$K(0) = (1+k)X(0)$$

initial conditions	optimal trajectory
$X(0) = b_1$	path 5
$b_2 < X(0) < b_1$	path 4 + path 5
$b_3 < X(0) < b_2$	path 3 + path 4 + path 5
$X(0) < b_3$	path 1 + path 3 + path 4 + path 5

Table A3.8. Optimal trajectories resulting from table A3.7.

in which : $b_1 = \frac{1}{(1+k)q_1} Q_{1Y}^*$, $b_2 = \frac{1}{(1+k)q_1} Q_{21}^*$, $b_3 = \frac{1}{(1+k)q_2} Q_{21}^*$ (84)

8.3. Strings ending with path 8.

path 8	conditions to precede			
	path 8	path 7	path 6	path 1
$Q = Q_{2X}^*$	$Q = Q_{2X}^*$	$Q < Q_{21}^*$		$Q < Q_{2YX}^*$
$K_1 = 0$	when $K_1 = 0$			
$Y = 0$	$Y = 0$	$Y = 0$	$Y > 0$	$Y > 0$
$\lambda_3 = 0$	$\lambda_3 = 0$	$\lambda_3 > 0$	$\lambda_3 > 0$	$\lambda_3 > 0$
$i < (1-f)r$	$i < (1-f)r$	$i < (1-f)r$	$i < (1-f)r$	$i < (1-f)r$
$c_{1X} > c_{2X}$	$c_{1X} > c_{2X}$	$c_{1X} > c_{2X}$	$c_{1X} > c_{2X}$	$c_{1X} > c_{2X}$
preceding path	path 7	path 6	path 1	-

Table A3.9. Selection of paths preceding path 2.

Note that:

$i < (1-f)r$ and $c_{1X} > c_{2X} + c_{1Y} > c_{2Y} + c_{1YX} > c_{2YX}$ (85)

So, we can derive the following optimal strings and their relevant initial conditions for the case of $i < (1-f)r$ and $c_{1X} > c_{2X}$:

initial conditions	optimal trajectory
$X(0) = K(0)$ and $c_1 < X(0)$	path 8
$X(0) = K(0)$ and $c_2 < X(0) < c_1$	path 7 + path 8
$K(0) = b_2$ and $c_3 < X(0) < c_2$	path 6 + path 7 + path 8
$K(0) = (1+k)X(0)$ and $X(0) < c_3$	path 1 + path 6 + path 7 + path 8

Table A3.10. Optimal trajectories resulting from table A3.9.

in which : $c_1 = \frac{1}{q_2} Q_{2X}^*$, $c_2 = \frac{1}{q_2} Q_{2YX}^*$ and $c_3 = \frac{1}{(1+k)q_2} Q_{2YX}^*$ (86)

8.4. Strings ending with path 12.

In the selection we will need the following orderings of Q^* -values:

$$\left. \begin{array}{l}
 i < (1-f)r \rightarrow \left\{ \begin{array}{l}
 Q_{1YX}^* < Q_{1Y}^* < Q_{1X}^* \\
 Q_{2YX}^* < Q_{2Y}^* < Q_{2X}^*
 \end{array} \right\} \\
 c_{1X} < c_{2X} \rightarrow \left\{ \begin{array}{l}
 Q_{21}^* < Q_{2X}^* < Q_{1X}^*
 \end{array} \right\}
 \end{array} \right\} Q_{2YX}^* < Q_{2Y}^* < Q_{2X}^* < Q_{1X}^* \quad (87)$$

path 12	conditions to precede	
	path 12	path 11
$Q = Q_{1X}^*$	$Q = Q_{1X}^*$ when $K = K_1$	$Q > Q_{21}^*$ when $K = K_2$
$K = K_1$	$Q > Q_{1X}^*$ when $K_2 > 0$	
$Y = 0$	$Y = 0$	$Y = 0$
$\lambda_3 = 0$	$\lambda_3 = 0$	$\lambda_3 > 0$
$i < (1-f)r$	$i < (1-f)r$	$i < (1-f)r$
$c_{1X} < c_{2X}$	$c_{1X} < c_{2X}$	$c_{1X} < c_{2X}$
preceding path	path 11	path 9 when $c_{1YX} < c_{2YX}$ path 10 when $c_{1YX} > c_{2YX}$

Tabel A3.11. Selection of paths preceding path 12.

From this table can be derived that two distinct strings may precede path 11, one ending with path 9 and the other ending with path 10. In addition, we have found a supplementary condition which indicates, that each string holds under different conditions. We will present the tables of the relevant strings apart from each other. The condition on c_{jYX} results in another ordering of Q^* -values:

$$c_{1YX} \lesssim c_{2YX} \rightarrow Q_{12}^* \lesssim Q_{2YX}^* \lesssim Q_{1YX}^* \quad (88)$$

This ordering will be used in the selection procedure, too.

	path 9	conditions to precede		
		path 4	path 3	path 1
	$Q > Q_{1YX}^* (> Q_{21}^*)$ $Y > 0$ $\lambda_3 > 0$ $i < (1-f)r$ $c_{1YX} < c_{2YX}$	$Q > Q_{1YX}^*$ $Y = kX$ $\lambda_3 > 0$ $i < (1-f)r$ $c_{1YX} < c_{2YX}$	$Q < Q_{21}^*$ $Y = kX$ $\lambda_3 > 0$ $i < (1-f)r$ $c_{1YX} < c_{2YX}$	$Q < Q_{21}^*$ $Y = kX$ $\lambda_3 > 0$ $i < (1-f)r$ $c_{1YX} < c_{2YX}$
preceding path	path 4	path 3	path 1	-

Table A3.12. Selection of paths preceding path 9 when $c_{1YX} < c_{2YX}$.

So we get the next optimal strings and initial conditions to match in the case of $i < (1-f)r$ and $c_{1YX} < c_{2YX}$ ($\rightarrow c_{1X} < c_{2X}$) :

initial conditions :	optimal trajectory
$K(0) = X(0)$ and $d_1 < X(0)$	path 12
$K(0) = X(0)$ and $d_2 < X(0) < d_1$	path 11 + path 12
$K(0) = c_2$ and $d_3 < X(0) < d_2$	path 9 + path 11 + path 12
$K(0) = (1+k)X(0)$ and $d_4 < X(0) < d_3$	path 4 + path 9 + path 11 + path 12
$K(0) = (1+k)X(0)$ and $d_5 < X(0) < d_4$	path 3 + path 4 + path 9 + path 11 + path 12
$K(0) = (1+k)X(0)$ and $X(0) < d_5$	path 1 + path 3 + path 4 + path 9 + path 11 + path 12

Table A3.13. Optimal trajectories resulting from tables A3.11 and A3.12.

In which : $d_1 = \frac{1}{q_1} Q_{1X}^*$, $d_2 = \frac{1}{q_1} Q_{1YX}^*$, $d_3 = \frac{1}{1+k} \frac{1}{q_1} Q_{1YX}^*$,

$$d_4 = \frac{1}{1+k} \frac{1}{q_1} Q_{21}^*$$
, $d_5 = \frac{1}{1+k} \frac{1}{q_2} Q_{21}^*$. (89)

	path 10	conditions to precede		
		path 7	path 6	path 1
	$Q < Q_{21}^*$ when $K_1 = K$ $Q = Q_{21}^*$ when $K_2 = K$ $Y = 0$ $\lambda_3 > 0$ $i < (1-f)r$ $c_{1YX} > c_{2YX}$	$Q < Q_{21}^*$ $Y = 0$ $\lambda_3 > 0$ $i < (1-f)r$ $c_{1YX} > c_{2YX}$	$Q < Q_{2YX}^*$ $Y > 0$ $\lambda_3 > 0$ $i < (1-f)r$ $c_{1YX} > c_{2YX}$	$Q < Q_{2YX}^*$ $Y = kX$ $\lambda_3 > 0$ $i < (1-f)r$ $c_{1YX} > c_{2YX}$
preceding path	path 7	path 6	path 1	-

Table A3.14. Selection of paths preceding path 10 when $c_{1YX} > c_{2YX}$.

From table 13 and the properties of the relevant paths we can derive the following optimal patterns:

initial conditions	optimal trajectory
$K(0) = X(0)$ and $e_1 < X(0)$	path 12
$K(0) = X(0)$ and $e_2 < X(0) < e_1$	path 11 → path 12
$K(0) = X(0)$ and $e_3 < X(0) < e_2$	path 10 → path 11 → path 12
$K(0) = X(0)$ and $e_4 < X(0) < e_3$	path 7 → path 10 → path 11 → path 12
$K(0) = d_4$ and $e_5 < X(0) < e_4$	path 6 → path 7 → path 10 → path 11 → path 12
$K(0) = (1+k)X(0)$ and $X(0) < e_5$	path 1 → path 6 → path 7 → path 10 → path 11 → path 12

Table A3.15. Optimal patterns resulting from tables A3.11 and A3.14.

In which : $e_1 = \frac{1}{q_1} Q_{1X}^*$, $e_2 = \frac{1}{q_1} Q_{21}^*$, $e_3 = \frac{1}{q_2} Q_{21}^*$, $e_4 = \frac{1}{q_2} Q_{2YX}^*$,

$$e_5 = \frac{1}{1+k} \frac{1}{q_2} Q_{2YX}^* \tag{90}$$

9. Summary.

In order to solve the model of chapter 4, it is reduced to its mathematically most condensed form. The reduced model still contains 2 state variables, 3 control variables, 2 state constraints and 3 (mixed) control constraints. The solution procedure, as designed in appendix 2, is then applied to this reduced model and the main features of this process are described.

LIST OF SYMBOLS

(Small letters are constants, capitals are variables)

A : advertising expenditures, age of a capital good.	a : depreciation rate.
B : goodwill, inflow of debt.	a_j : parameter.
C : costs.	b : redemption rate.
D : dividend.	c : price of capital good.
E : firing rate, (retained) earnings.	d : total demand of the output market.
F : corporation profit tax.	f : corporation profit tax rate.
G : gross revenue.	g : investment grant rate.
H : plant choice function.	h : maximum borrowing rate.
I : investments.:	i : discount rate.
J : birthdate of the eldest vintage of machines.	k : maximum debt/equity rate.
K : stock of capital goods (or its value).	l_j : labour to capital rate of activity j.
K_j : capital goods assigned to activity j.	m : retaining rate.
L : labour.	n : floatation costs of new equity issued.
M : new equity issued.	p : price.
N : natural quit rate.	q_j : capital productivity of activity j.
O : operating income.	r : interest rate on debt.
P : price.	s_{ij} : rate of technical substitution between activity i and j.
Q : output (-capacity).	w : wage rate.
R : return (on sales).	x : degree of homogeneity of H.
S : sales (volume or value).	z : planning horizon.
T : time.	
U : supply of labour, stocks other than capital goods.	
V : value of the firm.	
W : wages, cash flow.	
X : equity.	
Y : debt.	
Z : recruitment rate, terminal value of assets.	

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