TRIGONOMETRY

Charles P. McKeague
Cuesta College

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Preface to the Instructor

This trigonometry book is a standard right triangle approach to trigonometry. Each section is written so that it can be discussed in a 45 to 50 minute class session. The text covers all the material usually taught in trigonometry. In addition, there is an appendix on logarithms.

The emphasis of the book is on understanding the definitions and principles of trigonometry and their application to problem solving. However, when memorization is necessary, I say so.

Identities are introduced early in the book in Chapter 1. They are reviewed often and are then covered in more detail in Chapter 5. Also, exact values of the trigonometric functions are emphasized throughout the book. Although tables of trigonometric functions in both decimal degrees and degrees and minutes are used in the book, there are numerous calculator notes placed throughout the text that indicate how the examples and problems could be solved using a calculator instead of a table.

Organization of the Text The book begins with a preface to the student explaining what study habits are necessary to ensure success in mathematics.

The rest of the book is divided into chapters. Each chapter is organized as follows:

1. **Chapter Preface** Each chapter begins with a preface that explains in a very general way what the student can expect to find in the chapter and some of the applications associated with topics in the chapter. In most cases, this preface also includes a list of previous material that is used to develop the concepts in the chapter.

2. **Sections** Following the preface to each chapter, the body of the chapter is divided into sections. Each section contains explanations and examples. The explanations are as simple and intuitive as possible. The examples are chosen to clarify the explanations and preview the problems in the problem sets.

   As mentioned earlier, many sections contain calculator notes that explain how the examples in the sections could be worked on a calculator.

3. Problem Sets Following each section of the text is a problem set. There are five main ideas incorporated into each of the problem sets.

a. *Drill:* There are enough problems in each problem set to ensure proficiency with the material once students have completed all the odd-numbered problems.

b. *Progressive Difficulty:* The problems increase in difficulty as the problem set progresses.

c. *Odd–Even Similarities:* Each pair of consecutive problems is similar. The answers to the odd-numbered problems are listed in the back of the book, and in many cases a short note or partial solution accompanies the answers to these problems. This gives the student a chance to check his or her work and then try a similar problem.

d. *Application Problems:* Whenever possible, I have included a few application problems toward the end of each problem set. My experience is that students are always curious about how the trigonometry they are learning can be applied. They are also much more likely to put some time and effort into trying application problems if there are not an overwhelming number of them to work.

e. *Review Problems:* Starting with Chapter 2, each problem set ends with a few review problems. Generally, these review problems cover material that will be used in the next section. I find that assigning the review problems helps prepare my students for the next day's lecture and makes reviewing part of their daily routine.

4. Chapter Summaries Following the last problem set in each chapter is a chapter summary. Each chapter summary lists all the properties and definitions found in the chapter. In the margin of each chapter summary, next to most of the topics being summarized, is an example that illustrates the kind of problem associated with that topic.

5. Chapter Tests Each chapter ends with a chapter test. These tests are designed to give the student an idea of how well he or she has mastered the material in the chapter. All answers for these chapter tests are included in the back of the book.

SUPPLEMENT TO THE TEXT An *Instructor's Resource Manual* is available upon adoption of the text. This manual contains five different forms of each chapter test, answers to the even-numbered problems, and three final exams.

I think you will find that this book and the *Instructor's Resource Manual* that accompanies it form a very flexible package that will assist you in teaching trigonometry.

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Preface to the Student

Trigonometry can be a very enjoyable subject to study. You will find that there are many interesting and useful problems that trigonometry can be used to solve. Many of my students, however, do not enjoy trigonometry as much as they could because they are always worried about how they are doing in the class. They don't look forward to new topics because they are afraid they won't understand them right away. (Are you like that?)

But I have other students who just don't worry about it. They know they can become as proficient at trigonometry as they want (and get whatever grade they want, too). Most of my other students think these people are smart. They may be, but that's not the reason they do well. They are successful because they have learned that topics in mathematics are not always understandable the first time around. They don't worry if they don't understand something right away, because they know that a little confusion is not unusual. While they are waiting for things to get straightened out, they work lots of problems.

The key to success in all mathematics classes and especially in trigonometry is working problems. The more problems you work, the better you become at working problems. It's that simple. Don't worry about being successful or understanding the material, just work problems. Lots of them. The more problems you work, the better you will understand the material.

That's the answer to the big question of how to master this course. Here are the answers to some other questions that are often asked by students.

How much math do I need to know before taking this class?

You should have passed an intermediate algebra class recently, and you should have an understanding of the basic concepts from geometry. If it has been a few years since you have done any mathematics, you may have to put in some extra time at the beginning of the course to get back some of the skills you have lost.

What is the best way to study?

The best way to study is consistently. You must work problems every day. The more time you spend on your homework in the beginning of the book, the easier the rest of the book will seem. The first two chapters in the book contain the major definitions and properties on which the rest of the chapters are built. Do well on these first two chapters and you will have set a successful pace for the rest of the book.

If I understand everything that goes on in class, can I take it easy on my homework?

Not necessarily. There is a big difference between understanding a problem someone else is working and working the same problem yourself. Watching someone else work through a problem can be helpful, but there is no substitute for thinking the problem through yourself. The concepts and properties in trigonometry will be understandable to you only if you yourself work problems involving them.

I'm worried about not understanding trigonometry. I've passed the prerequisites for it, but I'm afraid trigonometry will be harder to understand.

There will probably be few times when you can understand absolutely everything that goes on in class. This is standard with most math classes, and trigonometry is no exception. It doesn't mean you will never understand it. As you read through the book and try problems on your own, you will understand more and more of the material. The idea is to keep working problems while you try to get your questions answered. By the way, reading the book is important, even if it seems difficult. Reading a math book isn't the same as reading a novel. Reading through the book once isn't going to do it for you. You will have to read some sections a number of times before they really sink in.

If you have decided to be successful in trigonometry, here is a list of things you can do that will help you attain that success.

How to Be Successful in Trigonometry

1. Attend class. There is only one way to find out what goes on in class and that is to be there. Missing class and then hoping to get the information from someone who was there is not the same as being there yourself.

2. Read the book and work problems every day. Remember, the key to success in mathematics is working problems. Work all the problems you can get your hands on. If you have assigned problems to do, do them first. If you don't understand the concepts after finishing the assigned problems, keep working problems until you do.

3. Do it on your own. Don't be misled into thinking someone else's work is your own.

4. Don't expect to understand a topic the first time you see it. Sometimes you will understand everything you are doing and sometimes you won't. If you are a little confused, just keep working problems. Keep in mind that worrying about not understanding the material takes time. You can't worry and work problems at the same time. It is best to worry when you worry and work problems when you work problems.

5. Spend whatever amount of time it takes to master the material. There is really no formula for the exact amount of time you have to spend on trigonometry to master it. You will find out as you go along what is or isn't enough time for you. If you end up having to spend two or three hours on each topic to get to the level you are interested in attaining, then that's how much time it takes. Spending less time than that will not work.

6. Relax. It's probably not as difficult as you think.

1 The Six Trigonometric Functions

To the student:

The material in Chapter 1 is some of the most important material in the book. We begin Chapter 1 with a review of some material from geometry (Section 1.1) and algebra (Section 1.2). Section 1.3 contains the definition for the six trigonometric functions. As you will see, this definition is used again and again throughout the book. It is very important that you understand and memorize the definition.

Once we have been introduced to the definition of the six trigonometric functions in Section 1.3, we then move on to study some of the more important consequences of the definition. These consequences take the form of trigonometric identities, the study of which is also important in trigonometry. Our work with trigonometric identities will take up most of Sections 1.4 and 1.5.

You can get a good start in trigonometry by mastering the material in Chapter 1. Any extra time you spend with the definition of the six trigonometric functions and the identities that are derived from it will be well worth it when you proceed on to Chapters 2 and 3.

As you will notice as you proceed through this book, many of the things we do in trigonometry are connected in one way or another to the Pythagorean theorem. (In case it has been a while since you worked with the Pythagorean theorem, we will review it in the first section of this chapter. For now, it is enough to say that the Pythagorean theorem is the theorem that gives the relationship between the sides in a right triangle.) The Pythagorean theorem is named for the Greek mathematician Pythagoras who is credited with having developed the first proof of it around 550 B.C. It is interesting to note that the theorem itself was used by the Babylonians over a thousand years prior to the time of Pythagoras. For people studying mathematics, it is not uncommon for a certain period of time to go by between the point at which they learn a new topic and the point at which they truly understand that topic.

1.1 Angles, Degrees, and Special Triangles

Before we begin our study of trigonometry, there are a few topics from geometry and algebra that we should review. In this section, we will review some essential topics from geometry. Let's begin by looking at some of the terminology associated with angles.

Angles in General

An angle is formed by two rays with the same end point. The common end point is called the *vertex* of the angle, and the rays are called the *sides* of the angle.

In Figure 1, the vertex of angle θ (theta) is labeled O, and A and B are points on each side of θ. Angle θ can also be denoted by AOB, where the letter associated with the vertex is written between the letters associated with the points on each side.

We can think of θ as having been formed by rotating side OA about the vertex to side OB. In this case, we call side OA the *initial side* of θ and side OB the *terminal side* of θ.

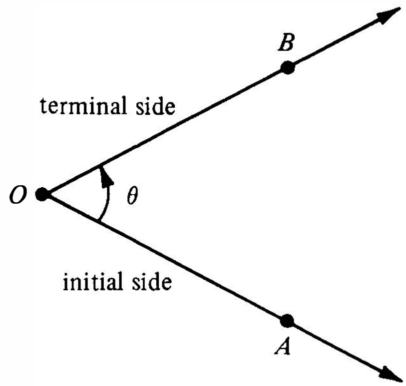


Figure 1

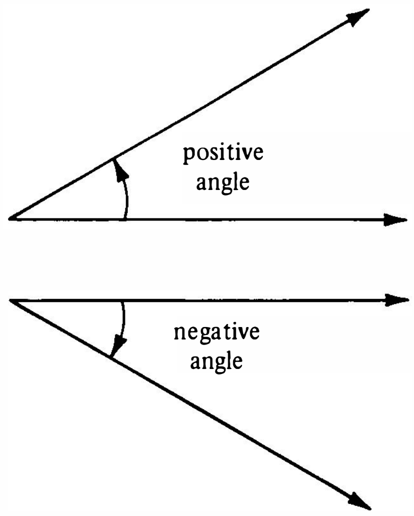


Figure 2

When the rotation from the initial side to the terminal side takes place in a counterclockwise direction, the angle formed is considered a *positive angle*. If the rotation is in a clockwise direction, the angle formed is a *negative angle* (Figure 2).

Degree Measure

One way to measure the size of an angle is with degree measure. The angle formed by rotating a half line through one complete revolution has a measure of 360 degrees, written 360° (Figure 3).

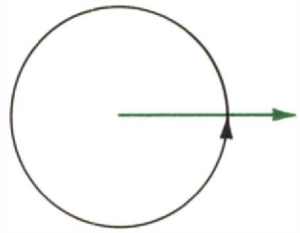


Figure 3

One degree (1°), then, is 1/360 of a full rotation. Likewise, 180° is one-half of a full rotation, and 90° is half of that (or a quarter of a rotation). Angles that measure 90° are called *right angles*. Angles that measure between 0° and 90° are called *acute angles*, while angles that measure between 90° and 180° are called *obtuse angles*.

If two angles have a sum of 90°, then they are called *complementary angles*, and we say each is the *complement* of the other. Two angles with a sum of 180° are called *supplementary angles*.

Note To be precise, we should say "two angles, the sum of the measures of which is 180°, are called supplementary angles" because there is a difference between an angle and its measure. However, in this book, we will not always draw the distinction between an angle and its measure. Many times we will refer to "angle θ" when we actually mean "the measure of angle θ."

▼ **Example 1** Give the complement and the supplement of each angle.

a. 40° **b.** 110° **c.** θ

Solution

a. The complement of 40° is 50° since 40° + 50° = 90°.
The supplement of 40° is 140° since 40° + 140° = 180°.

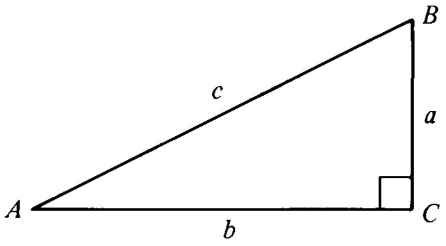
b. The complement of 110° is −20° since 110° + (−20°) = 90°.
The supplement of 110° is 70° since 110° + 70° = 180°.

c. The complement of θ is $90° - \theta$ since $\theta + (90° - \theta) = 90°$.
The supplement of θ is $180° - \theta$ since $\theta + (180° - \theta) = 180°$. ▲

Special Triangles

A *right triangle* is a triangle in which one of the angles is a right angle. In every right triangle, the longest side is called the *hypotenuse* and it is always opposite the right angle. The other two sides are called the *legs* of the right triangle. Since the sum of the angles in any triangle is 180°, the other two angles in a right triangle must be acute angles. Here is an important theorem about right triangles.

Pythagorean Theorem In any right triangle, the square of the length of the longest side (called the hypotenuse) is equal to the sum of the squares of the lengths of the other two sides (called legs).



If $C = 90°$, then $c^2 = a^2 + b^2$

Figure 4

We denote the lengths of the sides of triangle *ABC* in Figure 4 with lowercase letters and the angles or vertices with uppercase letters. It is standard practice in mathematics to label the sides and angles so that a is opposite A, b is opposite B, and c is opposite C.

▼ **Example 2** Solve for x in the right triangle in Figure 5.

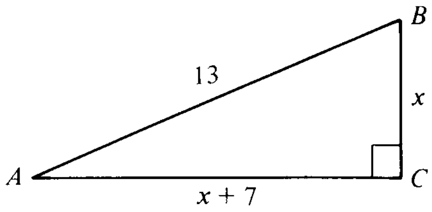


Figure 5

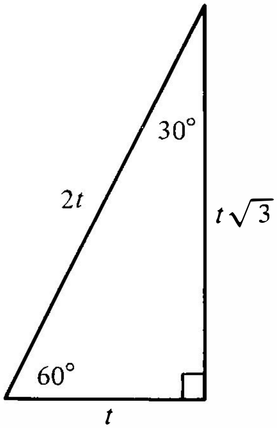
Solution Applying the Pythagorean theorem gives us a quadratic equation to solve.

$(x + 7)^2 + x^2 = 13^2$	
$x^2 + 14x + 49 + x^2 = 169$	Expand $(x + 7)^2$ and 13^2
$2x^2 + 14x + 49 = 169$	Combine similar terms
$2x^2 + 14x - 120 = 0$	Add -169 to both sides
$x^2 + 7x - 60 = 0$	Divide both sides by 2
$(x - 5)(x + 12) = 0$	Factor the left side
$x - 5 = 0$ or $x + 12 = 0$	Set each factor to 0
$x = 5$ or $x = -12$	

Our only solution is $x = 5$. We cannot use $x = -12$ since x is the length of a side of triangle *ABC* and therefore cannot be negative. ▲

The 30°–60°–90° Triangle

In any right triangle in which the two acute angles are 30° and 60°, the longest side (the hypotenuse) is always twice the shortest side (the side opposite the 30° angle), and the side of medium length (the side opposite the 60° angle) is always $\sqrt{3}$ times the shortest side (Figure 6).



30° - 60° - 90°

Figure 6

Note that the shortest side t is opposite the smallest angle 30°. The longest side $2t$ is opposite the largest angle 90°. To verify the relationship between the sides in this triangle we draw an equilateral triangle (one in which all three sides are equal) and label half the base with t (Figure 7).

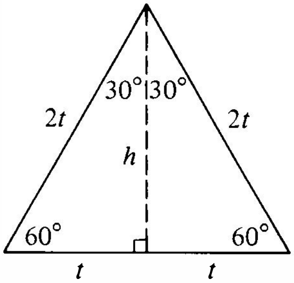


Figure 7

The altitude h (the dotted line) bisects the base. We have two 30°–60°–90° triangles. The longest side in each is $2t$. We find that h is $t\sqrt{3}$ by applying the Pythagorean theorem.

$$\begin{aligned} t^2 + h^2 &= (2t)^2 \\ h &= \sqrt{4t^2 - t^2} \\ &= \sqrt{3t^2} \\ &= t\sqrt{3} \end{aligned}$$

Example 3 If the shortest side of a 30°–60°–90° triangle is 5, find the other two sides.

Solution The longest side is 10 (twice the shortest side), and the side opposite the 60° angle is $5\sqrt{3}$.

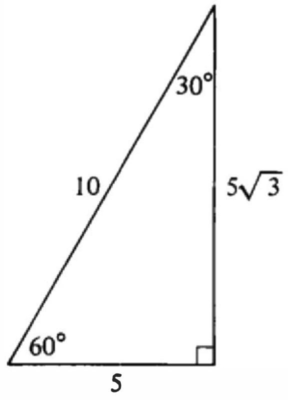


Figure 8

Example 4 A ladder is leaning against a wall. The top of the ladder is 4 feet above the ground and the bottom of the ladder makes an angle of 60° with the ground. How long is the ladder, and how far from the wall is the bottom of the ladder?

Solution The triangle formed by the ladder, the wall, and the ground is a 30°–60°–90° triangle. If we let x represent the distance from the bottom of the ladder to the wall, then the length of the ladder can be represented by $2x$. The distance from the top of the ladder to the ground is $x\sqrt{3}$, since it is opposite the 60° angle, and is also given as 4 feet. Therefore,

$$x\sqrt{3} = 4$$

$$x = \frac{4}{\sqrt{3}}$$

$$= \frac{4\sqrt{3}}{3}$$ Rationalize the denominator by multiplying the numerator and denominator by $\sqrt{3}$.

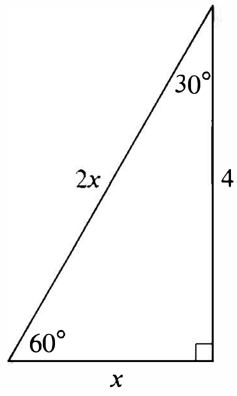


Figure 9

The distance from the bottom of the ladder to the wall, x, is $4\sqrt{3}/3$ feet, so the length of the ladder, $2x$, must be $8\sqrt{3}/3$ feet. Note that these lengths are given in exact values. If we want a decimal approximation for them, we can replace $\sqrt{3}$ with 1.732 to obtain

$$\frac{4\sqrt{3}}{3} \approx \frac{4(1.732)}{3} = 2.309 \text{ feet}$$

$$\frac{8\sqrt{3}}{3} \approx \frac{8(1.732)}{3} = 4.619 \text{ feet}$$

Calculator Note On a calculator with algebraic logic, this last calculation could be done as follows:

8 [×] 3 [√] [÷] 3 [=]

with the answer rounded to three decimal places (that is, three places past the decimal point). ▲

The 45°–45°–90° Triangle

If the two acute angles in a right triangle are both 45°, then the two shorter sides (the legs) are equal and the longest side (the hypotenuse) is $\sqrt{2}$ times as long as the shorter sides. That is, if the shorter sides are of length t, then the longest side has length $t\sqrt{2}$ (Figure 10).

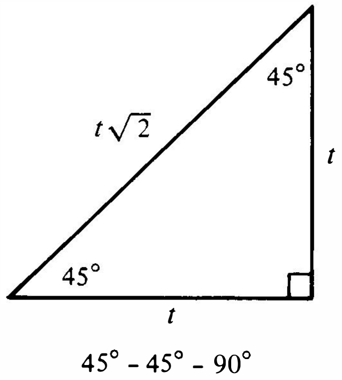


Figure 10

To verify this relationship, we simply note that if the two acute angles are equal, then the sides opposite them are also equal. We apply the Pythagorean theorem to find the length of the hypotenuse.

$$\begin{aligned} \text{hypotenuse} &= \sqrt{t^2 + t^2} \\ &= \sqrt{2t^2} \\ &= t\sqrt{2} \end{aligned}$$

▼ **Example 5** A 10 foot rope connects the top of a tent pole to the ground. If the rope makes an angle of 45° with the ground, find the length of the tent pole.

Solution Assuming that the tent pole forms an angle of 90° with the ground, the triangle formed by the rope, tent pole, and the ground is a 45°–45°–90° triangle.

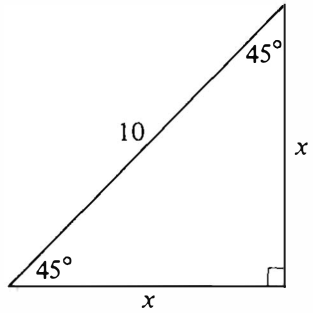


Figure 11

If we let x represent the length of the tent pole, then the length of the rope, in terms of x, is $x\sqrt{2}$ and is also given as 10 feet. Therefore

$$x\sqrt{2} = 10$$

$$x = \frac{10}{\sqrt{2}}$$

$$= 5\sqrt{2}$$

The length of the tent pole is $5\sqrt{2}$ feet. Again, $5\sqrt{2}$ is the exact value of the length of the tent pole. To find a decimal approximation, we replace $\sqrt{2}$ with 1.414 to obtain

$$5\sqrt{2} \approx 5(1.414) = 7.07 \text{ feet}$$ ▲

Problem Set 1.1

Indicate which of the angles below are acute angles and which are obtuse angles. Then give the complement and the supplement of each angle.

1. 10°
2. 50°
3. 45°
4. 90°
5. 120°
6. 160°
7. x
8. y

Problems 9 through 14 refer to Figure 12. (*Remember*: The sum of the three angles in any triangle is always 180°.)

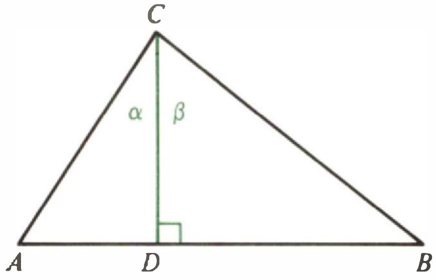


Figure 12

9. Find α if $A = 30°$.

10. Find B if $\beta = 45°$.

11. Find α if $A = \alpha$.

12. Find α if $A = 2\alpha$.

13. Find A if $B = 30°$ and $\alpha + \beta = 100°$.

14. Find B if $\alpha + \beta = 80°$ and $A = 80°$.

15. Through how many degrees does the hour hand of a clock move in 4 hours?

16. It takes the earth 24 hours to make one complete revolution on its axis. Through how many degrees does the earth turn in 12 hours?

Problems 17 through 22 refer to right triangle ABC with $C = 90°$.

17. If $a = 4$ and $b = 3$, find c.

18. If $a = 6$ and $b = 8$, find c.

19. If $a = 8$ and $c = 17$, find b.

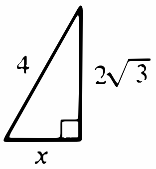
20. If $a = 2$ and $c = 6$, find b.

21. If $b = 12$ and $c = 13$, find a.

22. If $b = 10$ and $c = 26$, find a.

Solve for x in each of the following right triangles:

23.



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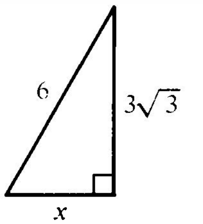

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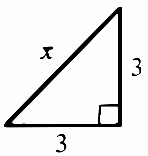

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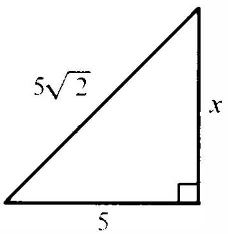

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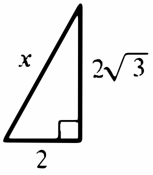

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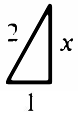

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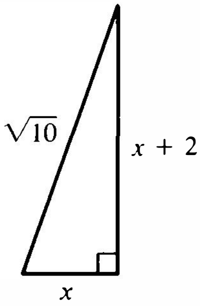

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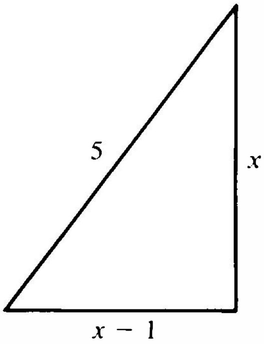
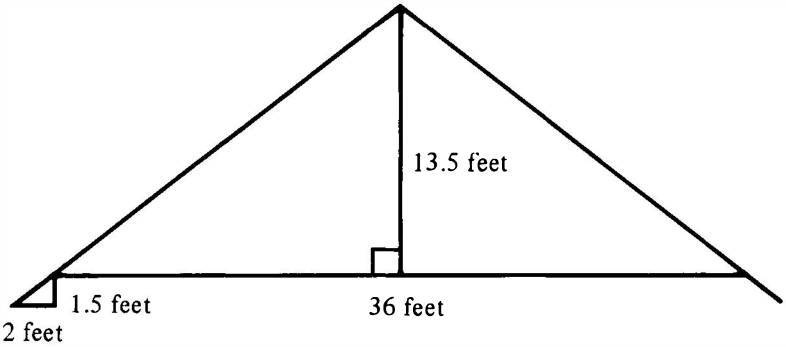


31. The roof of a house is to extend up 13.5 feet above the ceiling, which is 36 feet across. If the edge of the roof is to extend 2 feet beyond the side of the house and 1.5 feet below the ceiling, find the length of one side of the roof.



32. A surveyor is attempting to find the distance across a pond. From a point on one side of the pond he walks 25 yards to the end of the pond and then makes a 90° turn and walks another 60 yards before coming to a point directly across the pond from the point at which he started. What is the distance across the pond?

Find the remaining sides of a 30°–60°–90° triangle if

33. the shortest side is 1.

34. the shortest side is 3.

35. the longest side is 8.

36. the longest side is 5.

37. the side opposite 60° is $3\sqrt{3}$.

38. the side opposite 60° is $2\sqrt{3}$.

39. the side opposite 60° is 6.

40. the side opposite 60° is 4.

41. An escalator in a department store is to carry people a vertical distance of 20 feet between floors. How long is the escalator if it makes an angle of 30° with the ground?

42. A two person tent is to be made so that the height at the center is 4 feet. If the sides of the tent are to meet the ground at an angle of 60°, and the tent is to be 6 feet in length, how many square feet of material will be needed to make the tent? (Assume that the tent has a floor and is closed at both ends, give your answer to the nearest tenth of a square foot.)

Find the remaining sides of a 45°–45°–90° triangle if

43. the shorter sides are each 4/5.

44. the shorter sides are each 1/2.

45. the longest side is $8\sqrt{2}$.

46. the longest side is $5\sqrt{2}$.

47. the longest side is 4.

48. the longest side is 12.

49. A bullet is fired into the air at an angle of 45°. How far does it travel before it is 1,000 feet above the ground? (Assume the bullet travels in a straight line and give your answer to the nearest foot.)

50. If the bullet in Problem 49 is traveling at 2,828 feet per second, how long does it take for the bullet to reach a height of 1,000 feet?

1.2 The Rectangular Coordinate System

In this section we will review some of the concepts developed around the rectangular coordinate system and graphing in two dimensions.

The rectangular (or Cartesian) coordinate system is constructed by drawing two number lines perpendicular to each other.

The horizontal number line is called the x-axis, and the vertical number line is called the y-axis. Their point of intersection is called the origin. The axes divide the plane into four quadrants that are numbered I through IV in a counterclockwise direction (Figure 1).

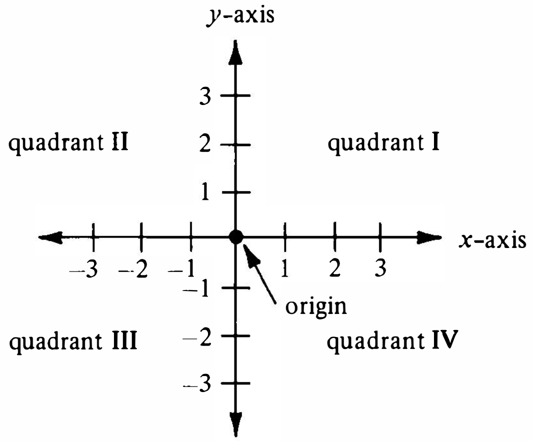


Figure 1

Among other things, the rectangular coordinate system is used to graph ordered pairs (a, b). For the ordered pair (a, b), a is called the x-coordinate (or x-component), and b is called the y-coordinate (or y-component). To graph the ordered pair (a, b) on a rectangular coordinate system, we start at the origin and move a units to the right or left (right if a is positive, and left if a is negative). We then move b units up or down (up if b is positive, and down if b is negative). The point where we end up after these two moves is the graph of the ordered pair (a, b).

Example 1 Graph the ordered pairs $(1, 5)$, $(-2, 4)$, $(-3, -2)$, $(1/2, -4)$, $(3, 0)$, and $(0, -2)$.

Solution To graph the ordered pair $(1, 5)$, we start at the origin and move 1 unit to the right and then 5 units up. We are now at the point whose coordinates are $(1, 5)$. The other ordered pairs are graphed in a similar manner, as in Figure 2.

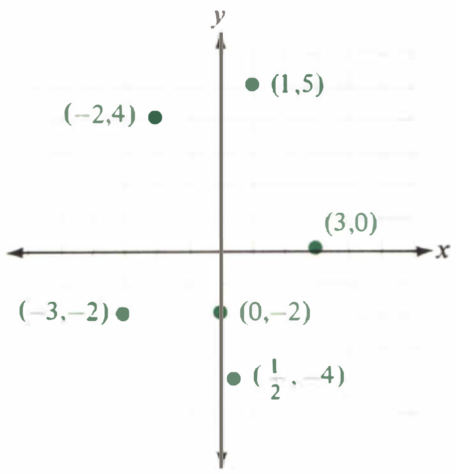


Figure 2

Looking at Figure 2, we see that any point in quadrant I will have both coordinates positive; that is, $(+, +)$. In quadrant II, the form is $(-, +)$. In quadrant III, the form is $(-, -)$ and in quadrant IV it is $(+, -)$. Also, any point on the x-axis will have a y-coordinate of 0 (it has no vertical displacement), and any point on the y-axis will have an x-coordinate of 0 (no horizontal displacement). A summary of this is given in Figure 3.

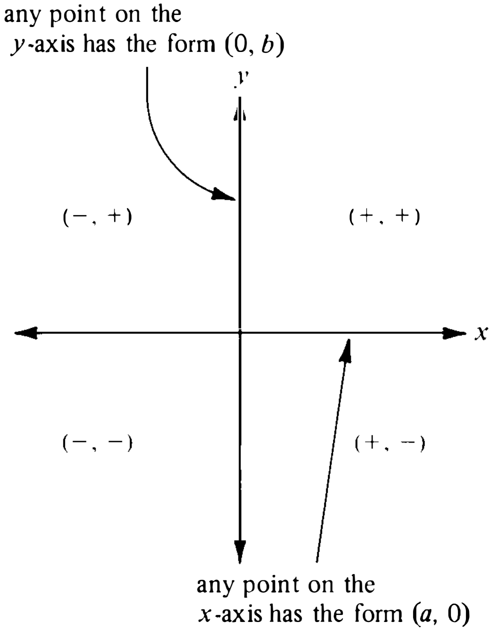


Figure 3

Example 2 Graph the line $2x - 3y = 6$.

Solution Since a line is determined by two points, we can graph the line $2x - 3y = 6$ by first finding two ordered pairs that satisfy its equation. To find ordered pairs that satisfy the equation $2x - 3y = 6$, we substitute any convenient number for either variable, and then solve the equation that results for the corresponding value of the other variable.

If we let	$x = 0$	If	$y = 0$
the equation	$2x - 3y = 6$	the equation	$2x - 3y = 6$
becomes	$2(0) - 3y = 6$	becomes	$2x - 3(0) = 6$
	$-3y = 6$		$2x = 6$
	$y = -2$		$x = 3$

This gives us $(0, -2)$ as one solution to $2x - 3y = 6$.

This gives us $(3, 0)$ as a second solution.

Graphing the points $(0, -2)$ and $(3, 0)$ and then drawing a line through them, we have the graph of $2x - 3y = 6$ (Figure 4).

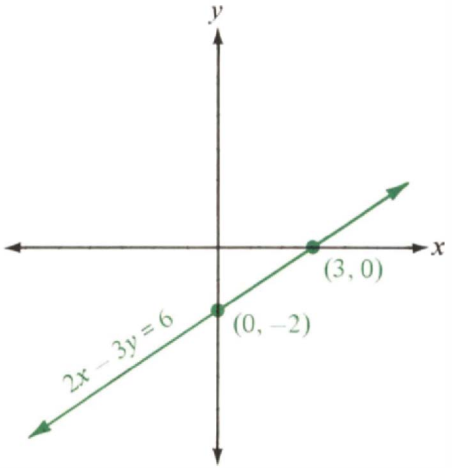


Figure 4

The x-coordinate of the point where our graph crosses the x-axis is called the x-intercept; in this case, it is 3. The y-coordinate of the point where the graph crosses the y-axis is the y-intercept; in this case, −2. ▲

Example 2 illustrates a very important concept in mathematics; the idea that an algebraic expression (an equation) can be associated with a geometric figure (in this case, a straight line). The study of the relationship between equations in algebra and geometric figures is called *analytic geometry*. The foundation of analytic geometry is the rectangular coordinate system.

It is interesting to note that the invention of the rectangular coordinate system is a fairly recent event in the history of mathematics, occurring around 1640. The idea of constructing a coordinate system with two perpendicular lines is credited to the French mathematician and philosopher René Descartes (1596–1650). It has been said that the idea came to him one morning while he was in bed watching a fly crawl on a corner section of his bedroom ceiling. He imagined that he could describe the path of the fly if he knew the relationship (equation) that would give the fly's distance from the two perpendicular walls (axes) at any point in time. The Latin translation of René Descartes is *Renatus Cartesius*, and that is why rectangular coordinates are sometimes referred to as *Cartesian Coordinates*.

The Distance Formula

The distance between any two points (x_1, y_1) and (x_2, y_2) in a rectangular coordinate system is given by the formula

$$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The distance formula can be derived by applying the Pythagorean theorem to the right triangle in Figure 5.

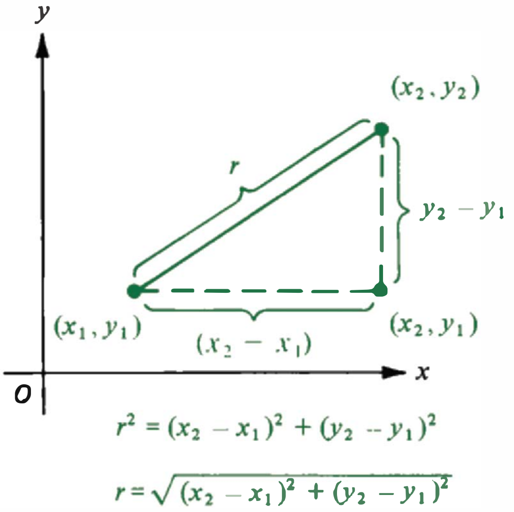


Figure 5

▼ **Example 3** Find the distance between the points (2, 1) and (−1, 5).

Solution It makes no difference which of the points we call (x_1, y_1) and which we call (x_2, y_2), because this distance will be the same between the two points regardless.

$$\begin{aligned} r &= \sqrt{(2 + 1)^2 + (1 - 5)^2} \\ &= \sqrt{3^2 + (-4)^2} \\ &= \sqrt{9 + 16} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

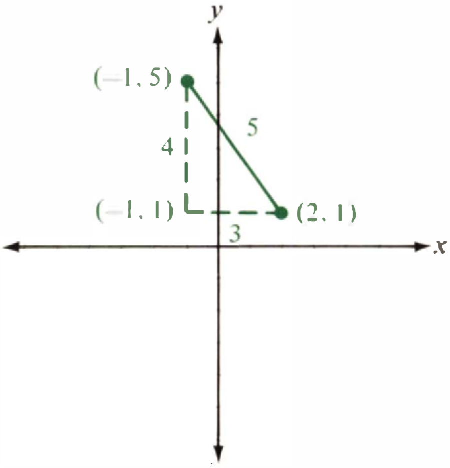


Figure 6 ▲

Example 4 Find the distance from the origin to the point (x, y).

Solution The coordinates of the origin are (0, 0). Applying the distance formula, we have

$$r = \sqrt{(x-0)^2 + (y-0)^2}$$
$$= \sqrt{x^2 + y^2}$$

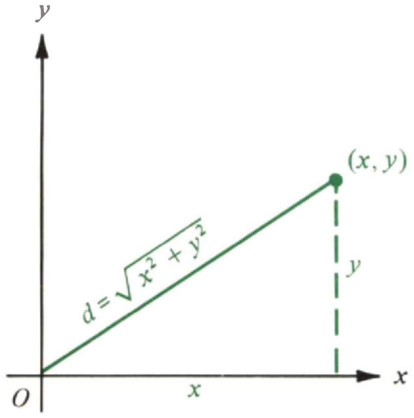


Figure 7

Angles in Standard Position

Definition An angle is said to be in *standard position* if its initial side is along the positive x-axis and its vertex is at the origin.

Example 5 Draw an angle of 45° in standard position and find a point on the terminal side.

Solution If we draw 45° in standard position we see that the terminal side is along the line $y = x$ in quadrant I (Figure 8).

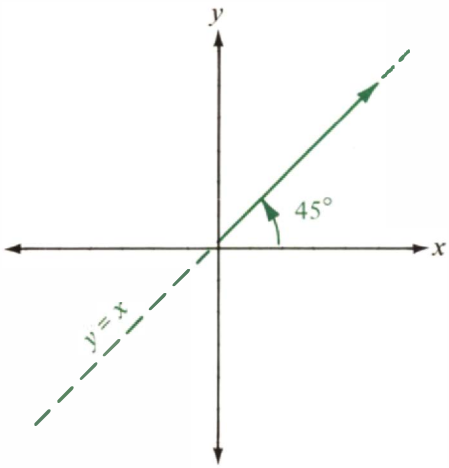


Figure 8

Since the terminal side of 45° lies along the line $y = x$ in the first quadrant, any point on the terminal side will have positive coordinates that satisfy the equation $y = x$. Here are some of the points that do just that.

$$(1, 1) \quad (2, 2) \quad (3, 3) \quad (\sqrt{2}, \sqrt{2}) \quad \left(\frac{1}{2}, \frac{1}{2}\right) \quad \left(\frac{7}{8}, \frac{7}{8}\right)$$ ▲

Vocabulary If an angle, say θ, is in standard position and the terminal side of θ lies in quadrant I, then we say θ lies in quadrant I and abbreviate it like this

$$\theta \in \text{QI}$$

Likewise, $\theta \in$ QII means θ is in standard position with its terminal side in quadrant II.

If the terminal side of an angle in standard position lies along one of the axes, then that angle is called a *quadrantal angle*. For example, an angle of 90° drawn in standard position would be a quadrantal angle, since the terminal side would lie along the positive y-axis. Likewise, 270° in standard position is a quadrantal angle because the terminal side would lie along the negative y-axis (Figure 9).

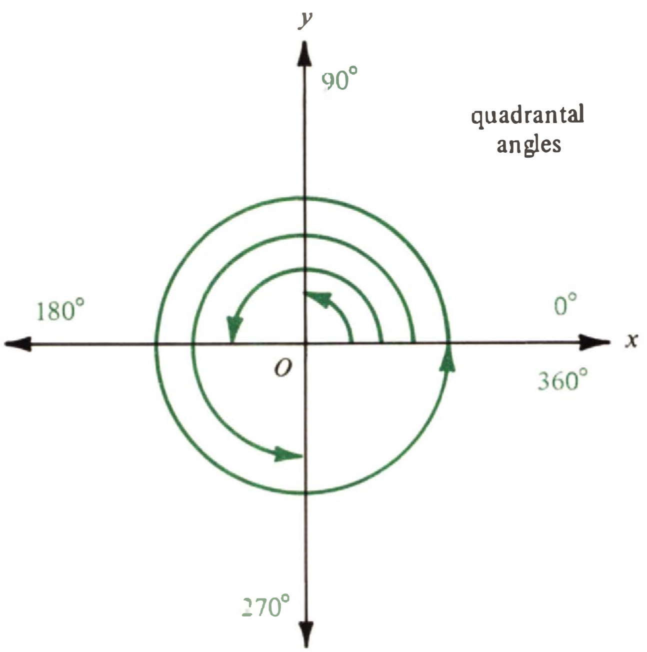


Figure 9

Two angles in standard position with the same terminal side are called *coterminal angles*. Figure 10 shows that 60° and −300° are coterminal angles when they are in standard position.

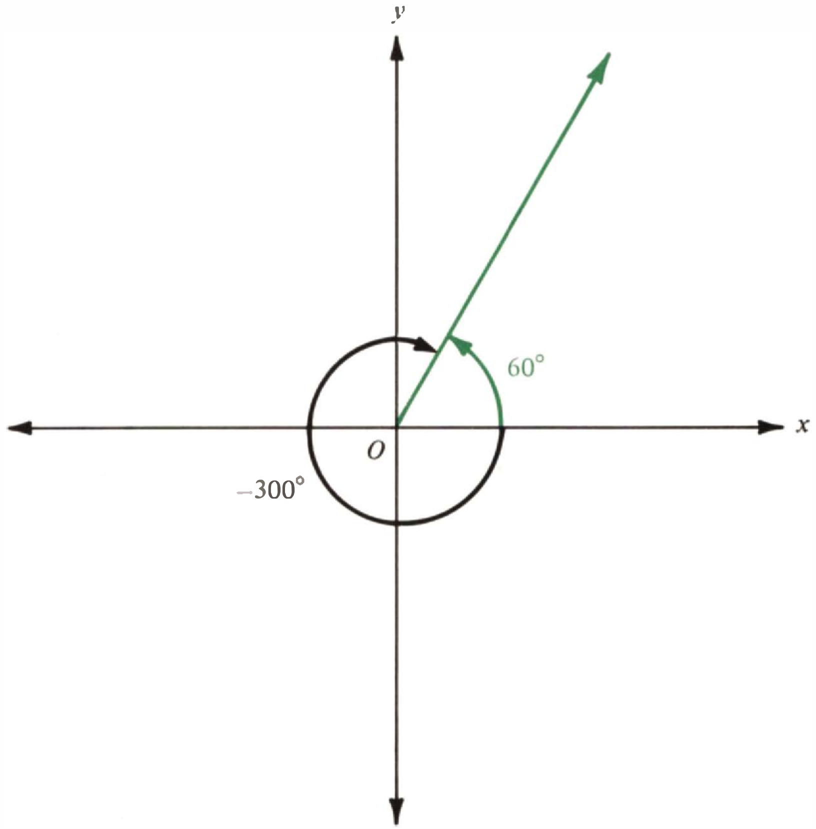


Figure 10

▼ **Example 6** Draw −90° in standard position, name some of the points on the terminal side, and find an angle between 0° and 360° that is coterminal with −90°.

Solution Figure 11 shows −90° in standard position. Since the terminal side is along the negative y-axis, the points $(0, -1)$, $(0, -2)$, $(0, -5/2)$, and, in general, $(0, b)$ where b is a negative number, are all on the terminal side. The angle between 0° and 360° that is coterminal with −90° is 270°.

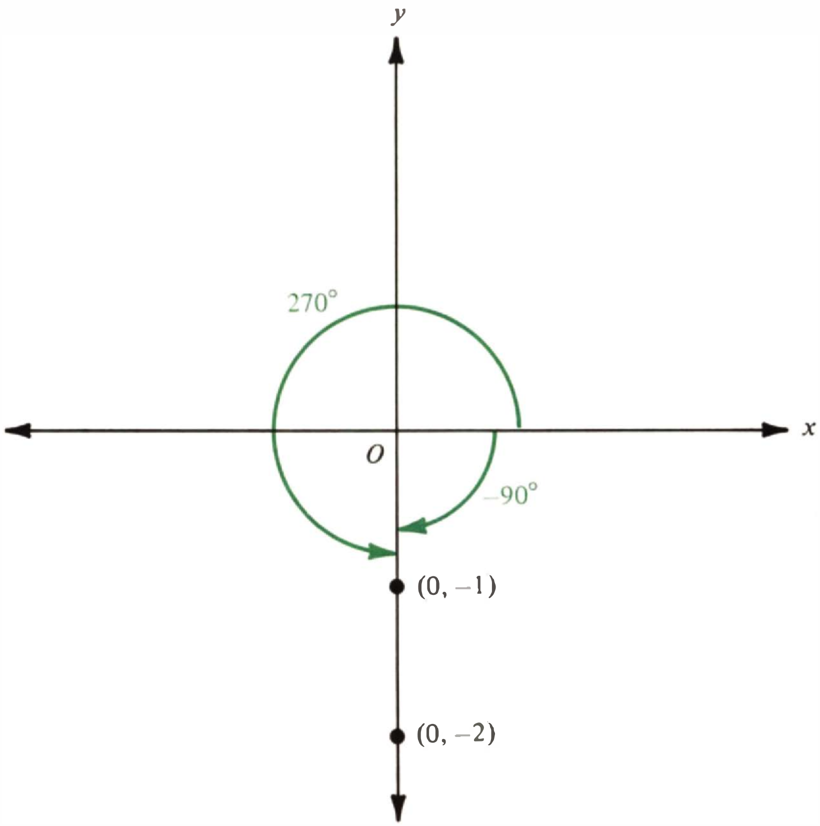


Figure 11

Problem Set 1.2

Graph each of the following ordered pairs on a rectangular coordinate system:

1. $(2, 4)$ **2.** $(2, -4)$
3. $(-2, 4)$ **4.** $(-2, -4)$
5. $(-3, -4)$ **6.** $(-4, -2)$
7. $(5, 0)$ **8.** $(-3, 0)$
9. $(0, -3)$ **10.** $(0, 2)$
11. $\left(-5, \frac{1}{2}\right)$ **12.** $\left(-5, \frac{1}{2}\right)$

Graph each of the following lines:

13. $3x + 2y = 6$ **14.** $3x - 2y = 6$
15. $y = 2x - 1$ **16.** $y = 2x + 3$

17. $y = \frac{1}{2}x$

18. $y = -\frac{1}{2}x$

19. $x = 3$

20. $y = -2$

21. The graph of $x^2 + y^2 = 1$ is a circle with its center at the origin. Graph this circle by finding a few convenient points that satisfy its equation. The circle you will graph is called the *unit circle*. Tell why that is an appropriate name for it.

22. Graph the circle $x^2 + y^2 = 4$.

Find the distance between the following points:

23. (3, 7), (6, 3)

24. (4, 7), (8, 1)

25. (0, 12), (5, 0)

26. (−3, 0), (0, 4)

27. (3, −5), (−2, 1)

28. (−8, 9), (−3, −2)

29. (−1, −2), (−10, 5)

30. (−3, 8), (−1, 6)

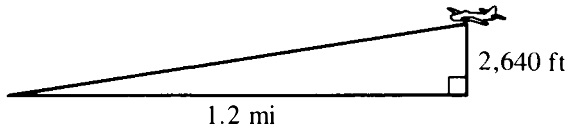
31. Find the distance from the origin out to the point (3, −4).

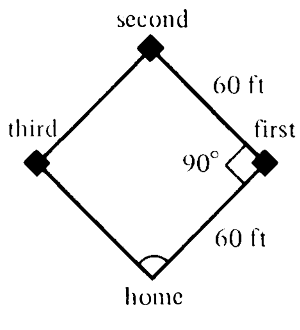
32. Find the distance from the origin out to the point (12, −5).

33. Find x so the distance between $(x, 2)$ and (1, 5) is $\sqrt{13}$.

34. Find y so the distance between $(7, y)$ and (8, 3) is 1.

35. An airplane is approaching Los Angeles International Airport at an altitude of 2,640 feet. If the plane is 1.2 miles from the runway (this is the horizontal distance to the runway), use the Pythagorean theorem to find the distance from the plane to the runway. (5,280 feet equals 1 mile.)





36. In softball, the distance from home plate to first base is 60 feet, as is the distance from first base to second base. If the lines joining home plate to first base and first base to second base form a right angle, how far does a catcher standing on home plate have to throw the ball so that it reaches the shortstop standing on second base?

37. If a coordinate system is superimposed on the softball diamond in Problem 36 with the x-axis along the line from home plate to first base and the y-axis on the line from home plate to third base, what would be the coordinates of home plate, first base, second base, and third base?

38. If a coordinate system is superimposed on the softball diamond in Problem 36 with the origin on home plate and the positive x-axis along the line joining home plate to second base, what would be the coordinates of first base and third base?

Draw each of the following angles in standard position, find a point on terminal side, and name one other angle that is coterminal with it.

39. 135°

40. 45°

41. 225°
42. 315°
43. 90°
44. 360°
45. −45°
46. −90°
47. Draw an angle of 30° in standard position. Then find a if the point $(a, 1)$ is on the terminal side of 30°.
48. Draw 60° in standard position. Then find b if the point $(2, b)$ is on the terminal side of 60°.
49. Draw an angle in standard position whose terminal side contains the point $(3, -2)$. Find the distance from the origin to this point.
50. Draw an angle in standard position whose terminal side contains the point $(2, -3)$. Find the distance from the origin to this point.
51. Plot the points $(0, 0)$, $(5, 0)$, and $(5, 12)$ and show that, when connected, they are the vertices of a right triangle.
52. Plot the points $(0, 2)$, $(-3, 2)$, and $(-3, -2)$ and show that they form the vertices of a right triangle.

1.3 Definition I: Trigonometric Functions

We begin this section with the definition of the trigonometric functions that we will use throughout the book. It is an extremely important definition. You must memorize it.

DEFINITION I If θ is an angle in standard position, and the point (x, y) is any point on the terminal side of θ other than the origin, then the six trigonometric functions of angle θ are defined as follows:

Function		Abbreviation		Definition
The sine of θ	=	$\sin\theta$	=	$\frac{y}{r}$
The cosine of θ	=	$\cos\theta$	=	$\frac{x}{r}$
The tangent of θ	=	$\tan\theta$	=	$\frac{y}{x}$
The cotangent of θ	=	$\cot\theta$	=	$\frac{x}{y}$
The secant of θ	=	$\sec\theta$	=	$\frac{r}{x}$
The cosecant of θ	=	$\csc\theta$	=	$\frac{r}{y}$

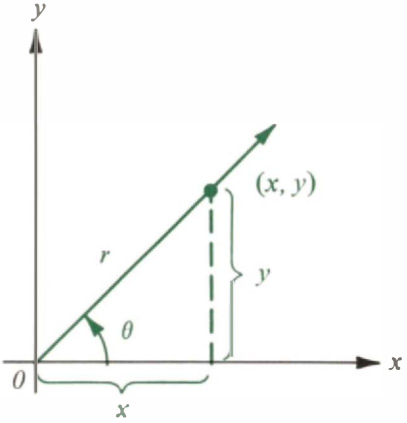


Figure 1

Where $x^2 + y^2 = r^2$, or $r = \sqrt{x^2 + y^2}$.
That is, r is the distance from the origin to (x, y).

As you can see, the six trigonometric functions are simply names given to the six possible ratios that can be made from the numbers x, y, and r as shown in Figure 1. Interestingly, although trigonometry has been studied in one form or another since as early as 1550 B.C., the abbreviations for the six trigonometric functions as we use them today were not introduced until 1662. In that year, the mathematician Albert Girard published a paper on trigonometry in which he used the abbreviations sin, tan, and sec for sine, tangent, and secant. (The first book in which the trigonometric functions were defined in terms of angles was published in 1551. At that time, sine was referred to as *perpendiculum* and cosine as *basis*. Do you see the connection between these words and the above definition?)

Example 1 Find the six trigonometric functions of θ if θ is in standard position and the point $(-2, 3)$ is on the terminal side of θ.

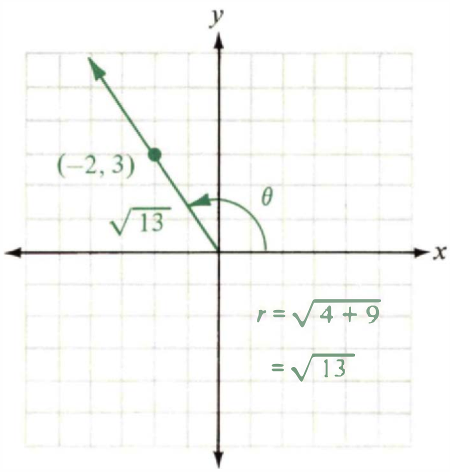


Figure 2

Applying the definition for the six trigonometric functions using the values $x = -2$, $y = 3$ and $r = \sqrt{13}$ we have

$$\sin\theta = \frac{y}{r} = \frac{3}{\sqrt{13}} \qquad \csc\theta = \frac{r}{y} = \frac{\sqrt{13}}{3}$$

$$\cos\theta = \frac{x}{r} = -\frac{2}{\sqrt{13}} \qquad \sec\theta = \frac{r}{x} = -\frac{\sqrt{13}}{2}$$

$$\tan\theta = \frac{y}{x} = -\frac{3}{2} \qquad \cot\theta = \frac{x}{y} = -\frac{2}{3}$$

▲

Note In algebra, when we encounter expressions like $3/\sqrt{13}$ that contain a radical in the denominator, we usually rationalize the denominator; in this case, by multiplying the numerator and the denominator by $\sqrt{13}$.

$$\frac{3}{\sqrt{13}} = \frac{3}{\sqrt{13}} \cdot \frac{\sqrt{13}}{\sqrt{13}} = \frac{3\sqrt{13}}{13}$$

In trigonometry, it is sometimes convenient to use $3\sqrt{13}/13$, and at other times it is easier to use $3/\sqrt{13}$. For now, let's agree not to rationalize any denominators unless we are told to.

Calculator Note You can use a calculator to see the equivalence of $3\sqrt{13}/13$ and $3/\sqrt{13}$. Here is the sequence of keys to press on your calculator to obtain decimal approximations for each expression.

For $\frac{3}{\sqrt{13}}$: 3 [÷] 13 [√] [=]

For $\frac{3\sqrt{13}}{13}$: 3 [×] 13 [√] [÷] 13 [=]

In each case the result should be 0.832050294.

▼ **Example 2** Find the sine and cosine of 45°.

Solution According to the definition above, we can find sin 45° and cos 45° if we know a point (x, y) on the terminal side of 45°, when 45° is in standard position. Figure 3 is a diagram of 45° in standard position. It shows that a convenient point on the terminal side of 45° is the point (1, 1). (We say it is a convenient point because the coordinates are easy to work with.)

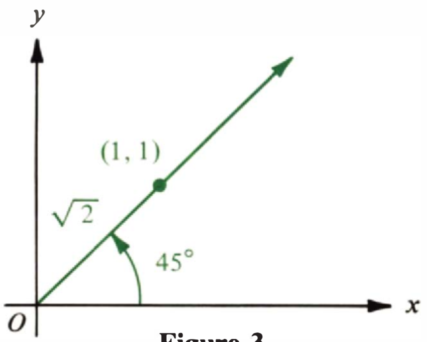


Figure 3

Since $x = 1$ and $y = 1$ and $r = \sqrt{x^2 + y^2}$, we have

$$r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Substituting these values for x, y, and r into our definition for sine and cosine we have

$$\sin 45° = \frac{y}{r} = \frac{1}{\sqrt{2}} \quad \text{and} \quad \cos 45° = \frac{x}{r} = \frac{1}{\sqrt{2}}$$

▲

▼ **Example 3** Find the six trigonometric functions of 270°.

Solution Again, we need to find a point on the terminal side of 270°. From Figure 4, we see that the terminal side of 270° lies along the negative y-axis.

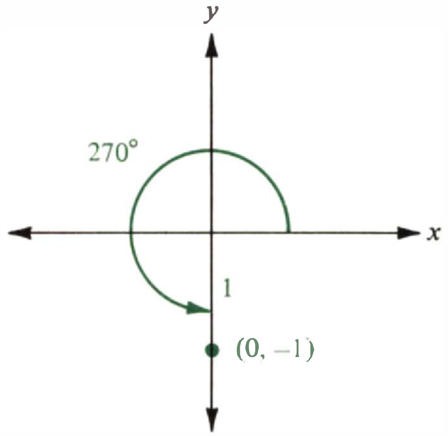


Figure 4

A convenient point on the terminal side of 270° is $(0, -1)$. Therefore,

$$r = \sqrt{x^2 + y^2} = \sqrt{0^2 + (-1)^2} = \sqrt{1} = 1$$

We have $x = 0$, $y = -1$, and $r = 1$. Here are the six trigonometric ratios for $\theta = 270°$.

$$\sin 270° = \frac{y}{r} = \frac{-1}{1} = -1 \qquad \csc 270° = \frac{r}{y} = \frac{1}{-1} = -1$$

$$\cos 270° = \frac{x}{r} = \frac{0}{1} = 0 \qquad \sec 270° = \frac{r}{x} = \frac{1}{0} = \text{undefined}$$

$$\tan 270° = \frac{y}{x} = \frac{-1}{0} = \text{undefined} \qquad \cot 270° = \frac{x}{y} = \frac{0}{-1} = 0$$

Note that tan 270° and sec 270° are undefined since division by 0 is undefined. ▲

Algebraic Signs of Trigonometric Functions

The algebraic sign, + or −, of each of the six trigonometric functions will depend on the quadrant in which θ terminates. For example, in quadrant I all six trigonometric functions are positive since x, y, and r are all positive. On the other hand, in quadrant II, only sin θ and csc θ are positive since x and r are positive and y is negative. Table 1 shows the signs of all the ratios in each of the four quadrants.

Table 1

	QI	QII	QIII	QIV
$\sin\theta = \frac{y}{r}$ and $\csc\theta = \frac{r}{y}$	+	+	−	−
$\cos\theta = \frac{x}{r}$ and $\sec\theta = \frac{r}{x}$	+	−	−	+
$\tan\theta = \frac{y}{x}$ and $\cot\theta = \frac{x}{y}$	+	−	+	−

▼ **Example 4** If $\sin\theta = -5/13$, and θ terminates in quadrant III, find $\cos\theta$ and $\tan\theta$.

Solution Since $\sin\theta = -5/13$, we know the ratio of y to r, or y/r, is $-5/13$. We can let y be -5 and r be 13 and use these values of y and r to find x.

Note We are not saying that if $y/r = -5/13$ that y *must* be -5 and r *must* be 13. We know from algebra that there are many pairs of numbers whose ratio is $-5/13$, not just -5 and 13. Our definition for sine and cosine, however, indicates we can choose *any* point on the terminal side of θ to find $\sin\theta$ and $\cos\theta$.

To find x we use the fact that $x^2 + y^2 = r^2$.

$$\begin{aligned} x^2 + y^2 &= r^2 \\ x^2 + (-5)^2 &= 13^2 \\ x^2 + 25 &= 169 \\ x^2 &= 144 \\ x &= \pm 12 \end{aligned}$$

Is x the number -12 or $+12$?

Since θ terminates in quadrant III, we know any point on its terminal side will have a negative x-coordinate, therefore,

$$x = -12$$

Using $x = -12$, $y = -5$, and $r = 13$ in our original definition we have

$$\cos \theta = \frac{x}{r} = \frac{-12}{13} = -\frac{12}{13}$$

and

$$\tan \theta = \frac{y}{x} = \frac{-12}{-5} = \frac{12}{5}$$

▲

As a final note, we should mention that the trigonometric functions of an angle are independent of the choice of the point (x, y) on the terminal side of the angle. Figure 5 shows an angle θ in standard position.

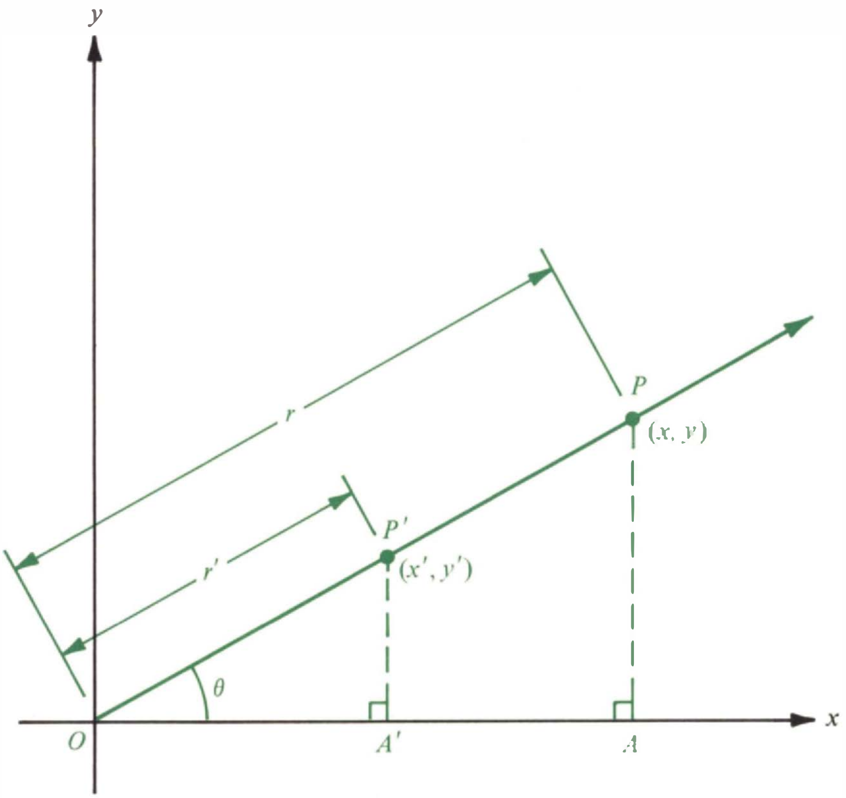


Figure 5

Points $P(x, y)$ and $P'(x', y')$ are both points on the terminal side of θ. Since triangles $P'OA'$ and POA are similar triangles, their corresponding sides are proportional. That is,

$$\sin \theta = \frac{y'}{r'} = \frac{y}{r} \qquad \cos \theta = \frac{x'}{r'} = \frac{x}{r} \qquad \tan \theta = \frac{y'}{x'} = \frac{y}{x}$$

Problem Set 1.3

Draw each of the following angles in standard position, find a point on the terminal side, and then find the sine, cosine, and tangent of each angle:

1. 135° **2.** 225°
3. 90° **4.** 180°
5. −45° **6.** −90°

Indicate the quadrants in which the terminal side of θ must lie in order that

7. $\cos \theta$ is positive **8.** $\cos \theta$ is negative
9. $\sin \theta$ is negative **10.** $\sin \theta$ is positive
11. $\tan \theta$ is positive **12.** $\tan \theta$ is negative

13. $\sin \theta$ is negative and $\tan \theta$ is positive
14. $\sin \theta$ is positive and $\cos \theta$ is negative
15. In which quadrant must the terminal side of θ lie if $\sin \theta$ and $\tan \theta$ are to have the same sign?
16. In which quadrant must the terminal side of θ lie if $\cos \theta$ and $\cot \theta$ are to have the same sign?

Find all six trigonometric functions of θ if the given point is on the terminal side of θ.

17. (3, 4) **18.** (−3, −4)
19. (−3, 4) **20.** (3, −4)
21. (−5, 12) **22.** (−12, 5)
23. (−1, −2) **24.** (1, 2)
25. (a, b) **26.** (m, n)

Find the remaining trigonometric functions of θ if

27. $\sin \theta = 12/13$ and θ terminates in QI
28. $\sin \theta = 12/13$ and θ terminates in QII
29. $\cos \theta = -3/5$ and θ is not in QII
30. $\tan \theta = 3/4$ and θ is not in QIII
31. $\sec \theta = 13/5$ and $\sin \theta < 0$
32. $\csc \theta = 13/5$ and $\cos \theta < 0$
33. $\cot \theta = 1/2$ and $\cos \theta > 0$
34. $\tan \theta = -1/2$ and $\sin \theta > 0$

35. Find $\sin \theta$ and $\cos \theta$ if the terminal side of θ lies along the line $y = 2x$ in quadrant I.
36. Find $\sin \theta$ and $\cos \theta$ if the terminal side of θ lies along the line $y = 2x$ in quadrant III.
37. Find $\sin \theta$ and $\tan \theta$ if the terminal side of θ lies along the line $y = -3x$ in quadrant II.
38. Find $\sin \theta$ and $\tan \theta$ if the terminal side of θ lies along the line $y = -3x$ in quadrant IV.
39. Draw $45°$ and $-45°$ in standard position and then show that $\cos(-45°) = \cos 45°$.
40. Draw $45°$ and $-45°$ in standard position and then show that $\sin(-45°) = -\sin 45°$.
41. Find x if the point $(x, -6)$ is on the terminal side of θ and $\sin \theta = -3/5$.
42. Find y if the point $(10, y)$ is on the terminal side of θ and $\cos \theta = 5/13$.
43. Find all angles between $0°$ and $360°$ for which $\sin \theta = 0$.
44. Find all angles between $0°$ and $360°$ for which $\sin \theta = 1$.
45. Find all angles between $0°$ and $360°$ for which $\cos \theta = 0$.
46. Find all angles between $0°$ and $360°$ for which $\cos \theta = 1$.
47. For which values of θ between $0°$ and $360°$ is $\tan \theta$ undefined?
48. For which values of θ between $0°$ and $360°$ is $\sec \theta$ undefined?

1.4 Introduction To Identities

Before we begin our introduction to identities we need to review some concepts from arithmetic and algebra.

In mathematics, we say that two numbers are *reciprocals* if their product is 1. The reciprocal of 3 is 1/3 since $3(1/3) = 1$. Likewise, the reciprocal of x is $1/x$ since $x(1/x) = 1$ for all numbers x except $x = 0$. (Zero is the only real number without a reciprocal.) Table 1 gives some additional examples of reciprocals.

In algebra, statements such as $2x = x + x$, $x^3 = x \cdot x \cdot x$, and $x/4x = 1/4$ are called identities. They are identities because they are true for all replacements of the variable for which they are defined.

Note $x/4x$ is not equal to $1/4$ when x is 0. The statement $x/4x = 1/4$ is still an identity, however, since it is true for all values of x for which $x/4x$ *is* defined.

The six basic trigonometric identities we will work with in this section are all derived from our definition of the trigonometric functions. Since many trigonometric identities have more than one form, we will list the basic identity first and then give the most common equivalent forms of that identity.

Table 1

Number	Reciprocal	
5	$\frac{1}{5}$	because $5\left(\frac{1}{5}\right) = 1$
-4	$-\frac{1}{4}$	because $-4\left(-\frac{1}{4}\right) = 1$
3	$\frac{1}{3}$	because $3\left(\frac{1}{3}\right) = 1$
a	$\frac{1}{a}$	because $a\left(\frac{1}{a}\right) = 1$
$\frac{1}{5}$	5	because $\left(\frac{1}{5}\right)5 = 1$
$\frac{3}{4}$	$\frac{4}{3}$	because $\left(\frac{3}{4}\right)\left(\frac{4}{3}\right) = 1$
$\frac{a}{b}$	$\frac{b}{a}$	because $\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = 1$

Reciprocal Identities

Our definition for the sine and cosecant functions indicates that they are reciprocals, that is,

$$\csc\theta = \frac{1}{\sin\theta}$$

because

$$\frac{1}{\sin\theta} = \frac{1}{y/r} = \frac{r}{y} = \csc\theta$$

Note that we can also write this same relationship between $\sin\theta$ and $\csc\theta$ in two other forms. Like this

$$\sin\theta = \frac{1}{\csc\theta} \qquad \text{because } \frac{1}{\csc\theta} = \frac{1}{r/y} = \frac{y}{r} = \sin\theta$$

or this

$$\sin\theta\csc\theta = 1 \qquad \text{because } \sin\theta\csc\theta = \frac{y}{r}\cdot\frac{r}{y} = 1$$

The first identity we wrote, $\csc\theta = 1/\sin\theta$, is the basic identity. The second two identities are equivalent forms of the first.

From the discussion above, and from the definition of cos θ, sec θ, tan θ, and cot θ, it is apparent that sec θ is the reciprocal of cos θ, and cot θ is the reciprocal of tan θ. Table 2 lists three basic reciprocal identities and their most common equivalent forms.

Table 2

Reciprocal Identities	Equivalent Forms	
$\csc\theta = \dfrac{1}{\sin\theta}$	$\sin\theta = \dfrac{1}{\csc\theta}$	$\sin\theta\csc\theta = 1$
$\sec\theta = \dfrac{1}{\cos\theta}$	$\cos\theta = \dfrac{1}{\sec\theta}$	$\cos\theta\sec\theta = 1$
$\cot\theta = \dfrac{1}{\tan\theta}$	$\tan\theta = \dfrac{1}{\cot\theta}$	$\tan\theta\cot\theta = 1$

The examples that follow show some of the ways in which we use these reciprocal identities.

▼ Examples

1. If $\sin\theta = \dfrac{3}{5}$, then $\csc\theta = \dfrac{5}{3}$, because

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{3/5} = \frac{5}{3}$$

2. If $\cos\theta = -\dfrac{\sqrt{3}}{2}$, then $\sec\theta = -\dfrac{2}{\sqrt{3}}$.

   (*Remember*: Reciprocals always have the same algebraic sign.)

3. If $\tan\theta = 2$, then $\cot\theta = \dfrac{1}{2}$.

4. If $\csc\theta = a$, then $\sin\theta = \dfrac{1}{a}$.

5. If $\sec\theta = 1$, then $\cos\theta = 1$ (1 is its own reciprocal).
6. If $\cot\theta = -1$, then $\tan\theta = -1$. ▲

Ratio Identities

There are two ratio identities, one for tan θ and one for cot θ. Unlike the reciprocal identities, the ratio identities do not have any common equivalent forms.

Table 3

Ratio Identity	
$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	because $\dfrac{\sin\theta}{\cos\theta} = \dfrac{y/r}{x/r} = \dfrac{y}{x} = \tan\theta$
$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$	because $\dfrac{\cos\theta}{\sin\theta} = \dfrac{x/r}{y/r} = \dfrac{x}{y} = \cot\theta$

▼ **Example 7** If $\sin\theta = -3/5$ and $\cos\theta = 4/5$, find $\tan\theta$ and $\cot\theta$.

Solution Using the ratio identities we have

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-3/5}{4/5} = -\frac{3}{4}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{4/5}{-3/5} = -\frac{4}{3}$$

Note that, once we found $\tan\theta$, we could have used a reciprocal identity to find $\cot\theta$.

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{-3/4} = -\frac{4}{3}$$ ▲

Notation The notation $\sin^2\theta$ is a shorthand notation for $(\sin\theta)^2$. It indicates we are to square the number that is the sine of θ.

▼ **Examples**

8. If $\sin\theta = \frac{3}{5}$, then $\sin^2\theta = \left(\frac{3}{5}\right)^2 = \frac{9}{25}$.

9. If $\cos\theta = -\frac{1}{2}$, then $\cos^3\theta = \left(-\frac{1}{2}\right)^3 = -\frac{1}{8}$.

10. If $\tan\theta = 3$, then $1 + 2\tan^2\theta = 1 + 2(3)^2 = 1 + 2(9) = 19$.

11. If $\sec\theta = -2$, then $3\sec^4\theta - 1 = 3(-2)^4 - 1 = 3(16) - 1 = 47$. ▲

Pythagorean Identity

To derive our Pythagorean identity we start with the relationship between x, y, and r as given in the definition of $\sin\theta$ and $\cos\theta$.

$$x^2 + y^2 = r^2$$

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1 \quad \text{Divide through by } r^2$$

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1 \quad \text{Property of exponents}$$

$$(\cos\theta)^2 + (\sin\theta)^2 = 1 \quad \text{Definition of } \sin\theta \text{ and } \cos\theta$$

$$\cos^2\theta + \sin^2\theta = 1 \quad \text{Notation}$$

This last line is our Pythagorean identity. We will use it many times throughout the book. It states that, for any angle θ, the sum of the squares of $\sin\theta$ and $\cos\theta$ is *always* 1. (There are actually two additional Pythagorean identities that involve the other four trigonometric functions. We will study them when we study identities in more detail in later chapters.)

There are two very useful equivalent forms of the Pythagorean identity. One form occurs when we solve $\cos^2\theta + \sin^2\theta = 1$ for $\cos\theta$, and the other form is the result of solving for $\sin\theta$.

Solving for $\cos\theta$ we have

$$\cos^2\theta + \sin^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta \quad \text{Add } -\sin^2\theta \text{ to both sides}$$

$$\cos\theta = \pm\sqrt{1 - \sin^2\theta} \quad \text{Take the square root of both sides}$$

Similarly, solving for $\sin\theta$ gives us

$$\sin\theta = \pm\sqrt{1 - \cos^2\theta}$$

We summarize this data in Table 4.

Table 4

Pythagorean Identity	Equivalent Forms
$\cos^2\theta + \sin^2\theta = 1$	$\cos\theta = \pm\sqrt{1 - \sin^2\theta}$ $\sin\theta = \pm\sqrt{1 - \cos^2\theta}$

Notice the $\pm$ sign in the equivalent forms. It occurs as part of the process of taking the square root of both sides of the preceding equation. (*Remember*: We would obtain a similar result in algebra if we solved the equation $x^2 = 9$ to get $x = \pm 3$.) In Example 12 we will see how to deal with the $\pm$ sign.

▼ **Example 12** If $\sin\theta = 3/5$ and θ terminates in quadrant II, find $\cos\theta$ and $\tan\theta$.

Solution We begin by using the identity $\cos\theta = \pm\sqrt{1 - \sin^2\theta}$

$$\begin{aligned}
&\text{If} && \sin\theta = \frac{3}{5} \\
&\text{the identity} && \cos\theta = \pm\sqrt{1 - \sin^2\theta} \\
&\text{becomes} && \cos\theta = \pm\sqrt{1 - \left(\frac{3}{5}\right)^2} \\
& && = \pm\sqrt{1 - \frac{9}{25}} \\
& && = \pm\sqrt{\frac{16}{25}} \\
& && = \pm\frac{4}{5}
\end{aligned}$$

Now we know that $\cos\theta$ is either $+4/5$ or $-4/5$. Looking back to the original statement of the problem, however, we see that θ terminates in quadrant II; therefore, $\cos\theta$ must be negative.

$$\cos\theta = -\frac{4}{5}$$

To find $\tan\theta$, we use a ratio identity.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{3/5}{-4/5} = -\frac{3}{4}$$ ▲

As a final note, we should mention that the six basic identities we have derived here, along with their equivalent forms, are very important in the study of trigonometry. It is essential that you memorize them. It may be a good idea to practice writing them from memory until you can write each of the six, and their equivalent forms, perfectly. As time goes by, we will increase our list of identities, so you will want to keep up with them as we go along.

Problem Set 1.4

Give the reciprocal of each number.

1. 7
2. 4
3. 3/5
4. 5/4
5. −2/3
6. −5/13
7. $-1/\sqrt{2}$
8. $-\sqrt{3}/2$
9. x
10. $1/a$
11. $1/(a + b)$
12. $x + y$

13. If $\sin \theta = 4/5$, find $\csc \theta$.
14. If $\cos \theta = \sqrt{3}/2$, find $\sec \theta$.
15. If $\sec \theta = -2$, find $\cos \theta$.
16. If $\csc \theta = -13/12$, find $\sin \theta$.
17. If $\tan \theta = a$ $(a \neq 0)$, find $\cot \theta$.
18. If $\cot \theta = -b$ $(b \neq 0)$, find $\tan \theta$.

For Problems 19 through 24, recall that $\sin^2\theta$ means $(\sin \theta)^2$.

19. If $\sin \theta = 1/\sqrt{2}$, find $1 - \sin^2\theta$.
20. If $\cos \theta = 1/3$, find $1 - \cos^2\theta$.
21. If $\tan \theta = 2$, find $3 \tan^3\theta - 3$.
22. If $\sec \theta = -3$, find $2 \sec^3\theta + 2$.
23. If $\cos \theta = 0.5$, find $2 \cos^2\theta - 1$.
24. If $\sin \theta = 0.6$, find $1 - 2 \sin^2\theta$.

For Problems 25 through 32, let $\sin \theta = -12/13$, and $\cos \theta = -5/13$, and find

25. $\tan \theta$
26. $\cot \theta$
27. $\sec \theta$
28. $\csc \theta$
29. $\sin^2\theta$
30. $\cos^2\theta$
31. $\sin^2\theta + \cos^2\theta$
32. the quadrant in which θ lies

33. If $\sin \theta = -4/5$ and θ terminates in QIII, find $\cos \theta$.
34. If $\sin \theta = -4/5$ and θ terminates in QIV, find $\cos \theta$.
35. If $\cos \theta = \sqrt{3}/2$ and θ terminates in QI, find $\sin \theta$.
36. If $\cos \theta = -1/2$ and θ terminates in QII, find $\sin \theta$.
37. Find $\tan \theta$ if $\sin \theta = 1/3$ and θ terminates in QI.
38. Find $\cot \theta$ if $\sin \theta = 2/3$ and θ terminates in QII.

Find the remaining trigonometric ratios of θ if

39. $\cos \theta = 12/13$ and θ terminates in QI
40. $\sin \theta = 12/13$ and θ terminates in QI
41. $\sin \theta = -1/2$ and θ is not in quadrant III
42. $\cos \theta = -1/3$ and θ is not in quadrant II
43. $\sec \theta = 2$ and $\sin \theta$ is positive
44. $\csc \theta = 2$ and $\cos \theta$ is negative
45. $\csc \theta = a$ and θ terminates in quadrant I
46. $\sec \theta = b$ and θ terminates in quadrant I

47. If $\tan \theta = \sqrt{3}$ and $\csc \theta = -2/\sqrt{3}$, find $\cos \theta$.
48. If $\cot \theta = -\sqrt{7}/3$ and $\csc \theta = 4/3$, find $\cos \theta$.

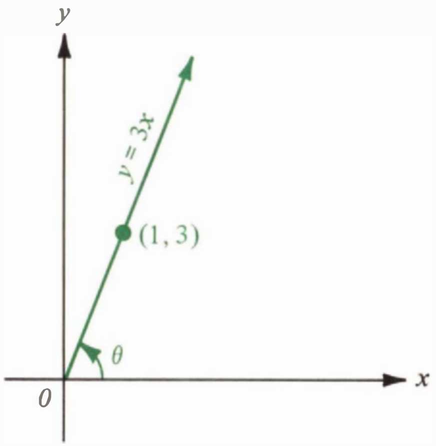
Recall from algebra that the slope of the line through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

It is the change in the y-coordinates divided by the change in the x-coordinates.

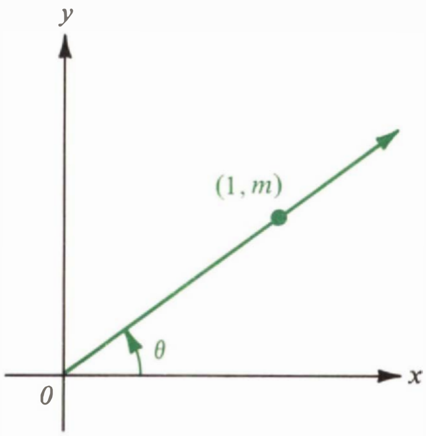
49. The line $y = 3x$ passes through the points (0, 0) and (1, 3). Find its slope.

50. Suppose the angle formed by the line $y = 3x$ and the positive x-axis is θ. Find the tangent of θ.



51. Find the slope of the line $y = mx$. [It passes through the origin and the point $(1, m)$.]

52. Find $\tan \theta$, if θ is the angle formed by the line $y = mx$ and the positive x-axis.



1.5 More on Identities

The topics we will cover in this section are an extension of the work we did with identities in Section 1.4. The first topic involves writing any of the six trigonometric functions in terms of any of the others. Let's look at an example.

Example 1 Write $\tan\theta$ in terms of $\sin\theta$.

Solution When we say we want $\tan\theta$ written in terms of $\sin\theta$, we mean that we want to write an expression that is equivalent to $\tan\theta$ but involves no trigonometric function other than $\sin\theta$. Let's begin by using a ratio identity to write $\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Now we need to replace $\cos\theta$ with an expression involving only $\sin\theta$. Since $\cos\theta = \pm\sqrt{1-\sin^2\theta}$ we have

$$\begin{aligned}\tan\theta &= \frac{\sin\theta}{\cos\theta}\\ &= \frac{\sin\theta}{\pm\sqrt{1-\sin^2\theta}}\\ &= \pm\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\end{aligned}$$

This last expression is equivalent to $\tan\theta$ and is written in terms of $\sin\theta$ only. (In a problem like this it is okay to include numbers and algebraic symbols with $\sin\theta$.)

Here is another example.

Example 2 Write $\sec\theta\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify.

Solution Since $\sec\theta = 1/\cos\theta$ and $\tan\theta = \sin\theta/\cos\theta$, we have

$$\begin{aligned}\sec\theta\tan\theta &= \frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta}\\ &= \frac{\sin\theta}{\cos^2\theta}\end{aligned}$$

The next examples show how we manipulate trigonometric expressions using algebraic techniques.

Example 3 Add $\dfrac{1}{\sin\theta} + \dfrac{1}{\cos\theta}$.

Solution We can add these two expressions in the same way we would add 1/3 and 1/4; by first finding a least common denominator (LCD), and then writing each expression again with the LCD for its denominator.

$$\frac{1}{\sin\theta} + \frac{1}{\cos\theta} = \frac{1}{\sin\theta}\cdot\frac{\cos\theta}{\cos\theta} + \frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\sin\theta}$$ The LCD is $\sin\theta\cos\theta$

$$= \frac{\cos\theta}{\sin\theta\cos\theta} + \frac{\sin\theta}{\cos\theta\sin\theta}$$

$$= \frac{\cos\theta + \sin\theta}{\sin\theta\cos\theta}$$

▲

▼ **Example 4** Multiply $(\sin\theta + 2)(\sin\theta - 5)$.

Solution We multiply these two expressions in the same way we would multiply $(x + 2)(x - 5)$. (In some algebra books this kind of multiplication is accomplished using the FOIL method.)

$$\begin{aligned}(\sin\theta + 2)(\sin\theta - 5) &= \sin\theta\sin\theta - 5\sin\theta + 2\sin\theta - 10\\ &= \sin^2\theta - 3\sin\theta - 10\end{aligned}$$

▲

In the examples that follow, we want to use the six basic identities we developed in Section 1.4, along with some techniques from algebra, to show that some more-complicated identities are true.

▼ **Example 5** Show that the following statement is true by transforming the left side into the right side.

$$\cos\theta\tan\theta = \sin\theta$$

Solution We begin by writing everything on the left side in terms of $\sin\theta$ and $\cos\theta$.

$$\cos\theta\tan\theta = \cos\theta\cdot\frac{\sin\theta}{\cos\theta}$$

$$= \frac{\cos\theta\sin\theta}{\cos\theta}$$

$$= \sin\theta$$ Divide out the $\cos\theta$ common to the numerator and denominator

Since we have succeeded in transforming the left side into the right side, we have shown that the statement $\cos\theta\tan\theta = \sin\theta$ is an identity. ▲

Note You may be wondering at this point, what good all of this is. As we progress through the book, you will see that identities play a very important part in trigonometry (and also in classes for which trigonometry is a prerequisite). This is an introduction to proving identities. In Chapter 5, we will introduce a more formal method of proving identities. For now, we are simply practicing changing trigonometric expressions into other, equivalent, trigonometric expressions.

Example 6 Prove the identity

$$(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$$

Solution Let's agree to prove the identities in this section, and the problem set that follows, by transforming the left side into the right side. In this case, we begin by expanding $(\sin\theta + \cos\theta)^2$. (Remember from algebra, $(a + b)^2 = a^2 + 2ab + b^2$.)

$$(\sin\theta + \cos\theta)^2 = \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta$$

Now we can rearrange the terms on the right side to get $\sin^2\theta$ and $\cos^2\theta$ together.

$$= (\sin^2\theta + \cos^2\theta) + 2\sin\theta\cos\theta$$
$$= 1 + 2\sin\theta\cos\theta$$

As a final note, we should mention that the ability to prove identities in trigonometry is not always obtained immediately. It usually requires a lot of practice. The more you work at it, the better you will become at it. In the meantime, if you are having trouble, check first to see that you have memorized the six basic identities—reciprocal, ratio, Pythagorean, and their equivalent forms, as given in Section 1.4.

Problem Set 1.5

Write each of the following in terms of $\sin\theta$ only:

1. $\cos\theta$
2. $\csc\theta$
3. $\cot\theta$
4. $\sec\theta$

Write each of the following in terms of $\cos\theta$ only:

5. $\sec\theta$
6. $\sin\theta$
7. $\tan\theta$
8. $\csc\theta$

Write each of the following in terms of $\sin\theta$ and $\cos\theta$, and then simplify if possible:

9. $\csc\theta\cot\theta$
10. $\sec\theta\cot\theta$
11. $\csc\theta\tan\theta$
12. $\sec\theta\tan\theta\csc\theta$
13. $\dfrac{\sec\theta}{\csc\theta}$
14. $\dfrac{\csc\theta}{\sec\theta}$

15. $\dfrac{\sec\theta}{\tan\theta}$

16. $\dfrac{\csc\theta}{\cot\theta}$

17. $\dfrac{\tan\theta}{\cot\theta}$

18. $\dfrac{\cot\theta}{\tan\theta}$

19. $\dfrac{\sin\theta}{\csc\theta}$

20. $\dfrac{\cos\theta}{\sec\theta}$

21. $\tan\theta + \sec\theta$

22. $\cot\theta - \csc\theta$

23. $\sin\theta\cot\theta + \cos\theta$

24. $\cos\theta\tan\theta + \sin\theta$

25. $\sec\theta - \tan\theta\sin\theta$

26. $\csc\theta - \cot\theta\cos\theta$

Perform the indicated operations and then simplify your answers if possible. Leave all answers in terms of $\sin\theta$ and/or $\cos\theta$.

27. Add $\dfrac{\sin\theta}{\cos\theta} + \dfrac{1}{\sin\theta}$

28. Add $\dfrac{\cos\theta}{\sin\theta} + \dfrac{\sin\theta}{\cos\theta}$

29. Add $\sin\theta + \dfrac{1}{\cos\theta}$

30. Add $\cos\theta + \dfrac{1}{\sin\theta}$

31. Subtract $\dfrac{1}{\sin\theta} - \sin\theta$

32. Subtract $\dfrac{1}{\cos\theta} - \cos\theta$

33. Multiply $(\sin\theta + 4)(\sin\theta + 3)$

34. Multiply $(\cos\theta + 2)(\cos\theta - 5)$

35. Multiply $(1 - \sin\theta)(1 + \sin\theta)$

36. Multiply $(1 - \cos\theta)(1 + \cos\theta)$

37. Expand $(\sin\theta - \cos\theta)^2$

38. Expand $(\sin\theta - 4)^2$

Show that each of the following statements is an identity by transforming the left side of each one into the right side:

39. $\cos\theta\tan\theta = \sin\theta$

40. $\sin\theta\cot\theta = \cos\theta$

41. $\sin\theta\sec\theta\tan\theta = 1$

42. $\cos\theta\csc\theta\tan\theta = 1$

43. $\dfrac{\csc\theta}{\cot\theta} = \sec\theta$

44. $\dfrac{\sec\theta}{\tan\theta} = \csc\theta$

45. $\dfrac{\sec\theta}{\csc\theta} = \tan\theta$

46. $\dfrac{\csc\theta}{\sec\theta} = \cot\theta$

47. $\sin\theta\tan\theta + \cos\theta = \sec\theta$

48. $\cos\theta\cot\theta + \sin\theta = \csc\theta$

49. $\tan\theta + \cot\theta = \sec\theta\csc\theta$

50. $\tan^2\theta + 1 = \sec^2\theta$

51. $\csc\theta - \sin\theta = \dfrac{\cos^2\theta}{\sin\theta}$

52. $\sec\theta - \cos\theta = \dfrac{\sin^2\theta}{\cos\theta}$

53. $1 - \sin^2\theta = \cos^2\theta$

54. $1 - \cos^2\theta = \sin^2\theta$

55. $(1 - \cos\theta)(1 + \cos\theta) = \sin^2\theta$

56. $(1 + \sin\theta)(1 - \sin\theta) = \cos^2\theta$

57. $(\sin\theta - \cos\theta)^2 - 1 = -2\sin\theta\cos\theta$
58. $(\cos\theta + \sin\theta)^2 - 1 = 2\sin\theta\cos\theta$
59. $\sin\theta(\sec\theta + \csc\theta) = \tan\theta + 1$
60. $\sec\theta(\sin\theta + \cos\theta) = \tan\theta + 1$

Chapter 1 Summary and Review

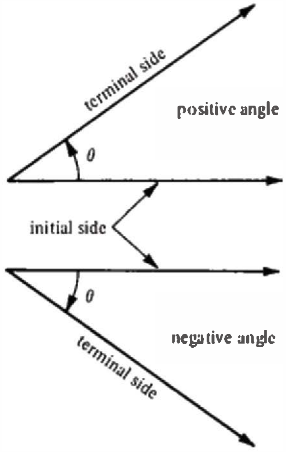
Examples: We will use the margin to give examples of the topics being reviewed, whenever it is appropriate.

The number in brackets next to each heading indicates the section in which that topic is discussed.

ANGLES [1.1]

An angle is formed by two half lines (rays) with a common end point. The common end point is called the *vertex*, and the half lines are called *sides*, of the angle. If we think of an angle as being formed by rotating the initial side about the vertex to the terminal side, then a counterclockwise rotation gives a positive angle, and a clockwise rotation gives a negative angle.

1.

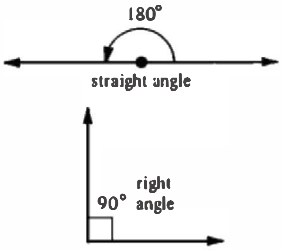


DEGREE MEASURE [1.1]

There are 360° in a full rotation. This means that 1° is 1/360 of a full rotation.

An angle that measures 90° is a *right angle*. An angle that measures 180° is a *straight angle*. Angles that measure between 0° and 90° are called *acute* angles, while angles that measure between 90° and 180° are called *obtuse* angles.

2.

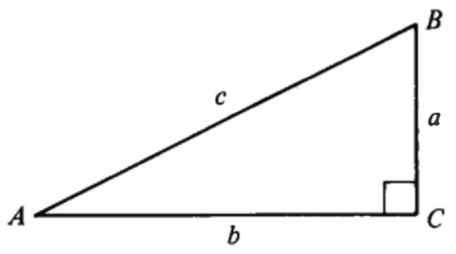


PYTHAGOREAN THEOREM [1.1]

In any right triangle, the square of the length of the longest side (the hypotenuse) is equal to the sum of the squares of the lengths of the other two sides (legs).

3. If ABC is a right triangle with $C = 90°$, and if $a = 4$ and $c = 5$, then

$$4^2 + b^2 = 5^2$$
$$16 + b^2 = 25$$
$$b^2 = 9$$
$$b = 3$$



$$c^2 = a^2 + b^2$$

DISTANCE FORMULA [1.2]

The distance r between the points (x_1, y_1) and (x_2, y_2) is given by the formula

$$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

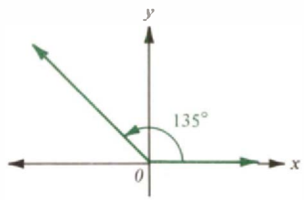
4. The distance between (2, 7) and (−1, 3) is

$$\begin{aligned} r &= \sqrt{(2 + 1)^2 + (7 - 3)^2} \\ &= \sqrt{9 + 16} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

STANDARD POSITION FOR ANGLES [1.2]

An angle is said to be in standard position if its vertex is at the origin and its initial side is along the positive x-axis.

5. 135° in standard position is



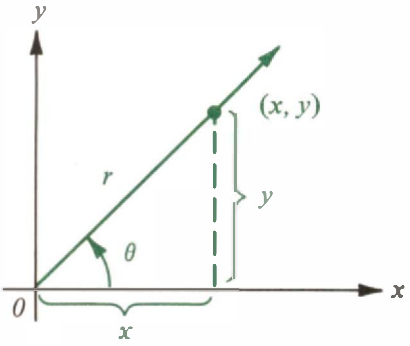
TRIGONOMETRIC FUNCTIONS [1.3]

If θ is an angle in standard position and (x, y) is any point on the terminal side of θ (other than the origin) then

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$



Where $x^2 + y^2 = r^2$, or $r = \sqrt{x^2 + y^2}$. That is, r is the distance from the origin to (x, y).

6. If (−3, 4) is on the terminal side of θ, then

$$r = \sqrt{9 + 16} = 5$$

and

$$\sin\theta = \frac{4}{5} \qquad \csc\theta = \frac{5}{4}$$

$$\cos\theta = -\frac{3}{5} \qquad \sec\theta = -\frac{5}{3}$$

$$\tan\theta = -\frac{4}{3} \qquad \cot\theta = -\frac{3}{4}$$

7. If $\sin \theta > 0$, and $\cos \theta > 0$, then θ must terminate in QI.
If $\sin \theta > 0$, and $\cos \theta < 0$, then θ must terminate in QII.
If $\sin \theta < 0$, and $\cos \theta < 0$, then θ must terminate in QIII.
If $\sin \theta < 0$, and $\cos \theta > 0$, then θ must terminate in QIV.

8. If $\sin \theta = 1/2$ and θ terminates in QI, then

$$\cos \theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - 1/4} = \frac{\sqrt{3}}{2}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$$

$$\cot \theta = \frac{1}{\tan \theta} = \sqrt{3}$$

$$\sec \theta = \frac{1}{\cos \theta} = \frac{2}{\sqrt{3}}$$

$$\csc \theta = \frac{1}{\sin \theta} = 2$$

SIGNS OF THE TRIGONOMETRIC FUNCTIONS [1.3]

The algebraic signs, + or −, of the six trigonometric functions depend on the quadrant in which θ terminates.

	QI	QII	QIII	QIV
sin θ and csc θ	+	+	−	−
cos θ and sec θ	+	−	−	+
tan θ and cot θ	+	−	+	−

BASIC IDENTITIES [1.4]

Reciprocal

$$\csc \theta = \frac{1}{\sin \theta} \qquad \sec \theta = \frac{1}{\cos \theta} \qquad \cot \theta = \frac{1}{\tan \theta}$$

Ratio

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \qquad \cot \theta = \frac{\cos \theta}{\sin \theta}$$

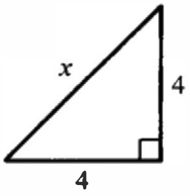
Pythagorean

$$\cos^2\theta + \sin^2\theta = 1$$

Chapter 1 Test

Solve for x in each of the following right triangles:

1.



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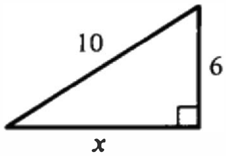

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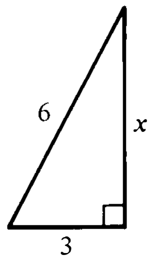

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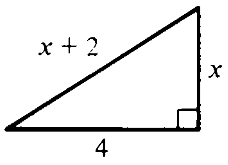


Graph each of the following lines:

5. $2x + 3y = 6$

6. $y = 3x - 1$

7. Find the distance between the points $(4, -2)$ and $(-1, 10)$.
8. Find the distance from the origin to the point (a, b).
9. Find x so that the distance between $(-2, 3)$ and $(x, 1)$ is $\sqrt{13}$.
10. Draw 135° in standard position and find the point on the terminal side with x-coordinate -1.

Find $\sin\theta$, $\cos\theta$, and $\tan\theta$ for each of the following values of θ:

11. 90°

12. −45°

In which quadrant will θ lie if

13. $\sin\theta < 0$ and $\cos\theta > 0$

14. $\csc\theta > 0$ and $\cos\theta < 0$

Find all six trigonometric functions for θ if the given point lies on the terminal side of θ in standard position.

15. $(-6, 8)$

16. $(-3, -1)$

Find the remaining trigonometric functions of θ if

17. $\sin\theta = 1/2$ and θ terminates in QII
18. $\tan\theta = 12/5$ and θ terminates in QIII
19. Find $\sin\theta$ and $\cos\theta$ if the terminal side of θ lies along the line $y = -2x$ in quadrant IV.
20. Find θ if $\sin\theta = 0$, and $0° \leq \theta \leq 360°$
21. If $\sin\theta = -3/4$, find $\csc\theta$.
22. If $\sec\theta = -2$, find $\cos\theta$.

If $\sin\theta = 1/3$ with θ in QI, find each of the following:

23. $\cos\theta$ and $\sec\theta$

24. $\tan\theta$ and $\cot\theta$

25. $1 + \cot^2\theta$ and $\csc^2\theta$

26. $1 + \tan^2\theta$ and $\sec^2\theta$

27. $1 + 27\sin^3\theta$

28. $2\csc^3\theta - 1$

29. Express $\sec\theta\tan\theta$ in terms of $\cos\theta$.
30. Multiply $(\sin\theta + 3)(\sin\theta - 7)$.

31. Expand and simplify $(\cos\theta - \sin\theta)^2$.

32. Subtract $\dfrac{1}{\sin\theta} - \sin\theta$.

Show that each of the following statements is an identity by transforming the left side of each one into the right side:

33. $\sin\theta\cot\theta\sec\theta = 1$

34. $\dfrac{\cot\theta}{\csc\theta} = \cos\theta$

35. $\cot\theta + \tan\theta = \csc\theta\sec\theta$

36. $\cos\theta(\csc\theta + \sec\theta) = \cot\theta + 1$

37. $(1 - \sin\theta)(1 + \sin\theta) = \cos^2\theta$

38. $\sin\theta(\csc\theta + \cot\theta) = 1 + \cos\theta$

Right Triangle Trigonometry

To the student:

We are going to begin this chapter with a second definition for the six trigonometric functions. This second definition will be given in terms of right triangles. As you will see, it will not conflict with our original definition. Since this new definition defines the six trigonometric functions in terms of the sides of a right triangle, there are a number of interesting applications. One of the first applications will be to find the six trigonometric functions of 30°, 45°, and 60°, since these angles can be found in the special triangles we introduced in Chapter 1.

After we have worked with the new definition for the trigonometric functions we will show how to use a table to find approximations of trigonometric functions of angles between 0° and 90°. Since you may be in a class in which calculators are used instead of tables, we will also include notes on how to use a calculator with sin, cos, and tan keys to accomplish the same results.

In Section 2.3 we will use the new definition and tables of trigonometric functions (or a calculator) to find missing parts for some right triangles. Again, we will include notes on using a calculator where appropriate.

In Sections 2.4 and 2.5 we will look at some special applications of the right triangle trigonometry we develop in the first three sections of the chapter.

Finally, at the end of each problem set in this chapter, you will find a set of review problems. These review problems are intended to refresh your memory on topics we have covered previously and, at times, to get you ready for the section that follows. We will continue to place review problems at the end of the Problem Sets throughout the remainder of the book.

2.1 Trigonometric Functions of an Acute Angle

In Chapter 1 we gave a definition for the six trigonometric functions for any angle in standard position. We begin this chapter with a second definition for the six trigonometric functions. This new definition defines the trigonometric functions in terms of the sides and angles of a right triangle. As you will see, this new definition does not conflict with the definition we gave in Chapter 1.

Definition II

If triangle ABC is a right triangle with $C = 90°$, then the s[illegible]c functions for A are defined as follows:

$$\sin A = \frac{\text{side opposite } A}{\text{hypotenuse}} = \frac{a}{c}$$

$$\cos A = \frac{\text{side adjacent } A}{\text{hypotenuse}} = \frac{b}{c}$$

$$\tan A = \frac{\text{side opposite } A}{\text{side adjacent } A} = \frac{a}{b}$$

$$\cot A = \frac{\text{side adjacent } A}{\text{side opposite } A} = \frac{b}{a}$$

$$\sec A = \frac{\text{hypotenuse}}{\text{side adjacent } A} = \frac{c}{b}$$

$$\csc A = \frac{\text{hypotenuse}}{\text{side opposite } A} = \frac{c}{a}$$

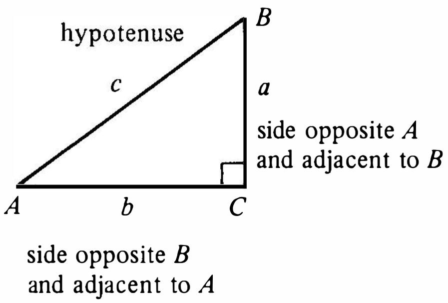


Figure 1

Historically, the first reference to the trigonometric functions, as we have defined them here, is found in the Ahmes Papyrus, which is dated at approximately 1550 B.C. Contained therein are five problems relating to the measurement of pyramids. In four of these problems a reference is made to what is currently known as the cotangent function. For comparison, the first written reference to the sine function did not occur until around 500 A.D.

▼ **Example 1**

Triangle ABC is a right triangle with $C = 90°$. If $a = 6$ [illegible] the six trigonometric functions of A.

Solution We begin by making a diagram of ABC, and then use the given information and the Pythagorean theorem to solve for b.

$$\begin{aligned} b &= \sqrt{c^2 - a^2} \\ &= \sqrt{100 - 36} \\ &= \sqrt{64} \\ &= 8 \end{aligned}$$

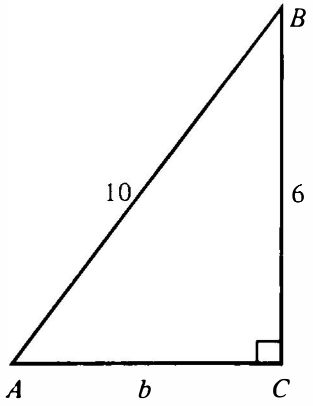


Figure 2

Now we write the six trigonometric functions of A using $a = 6$, $b = 8$, and $c = 10$.

$$\sin A = \frac{a}{c} = \frac{6}{10} = \frac{3}{5} \qquad \csc A = \frac{c}{a} = \frac{5}{3}$$

$$\cos A = \frac{b}{c} = \frac{8}{10} = \frac{4}{5} \qquad \sec A = \frac{c}{b} = \frac{5}{4}$$

$$\tan A = \frac{a}{b} = \frac{6}{8} = \frac{3}{4} \qquad \cot A = \frac{b}{a} = \frac{4}{3}$$

▲

Now that we have done an example using our new definition, let's see how our new definition compares to Definition I from the previous chapter. We can place right triangle ABC on a rectangular coordinate system so that A is in standard position. We then note that a point on the terminal side of A is (b, a).

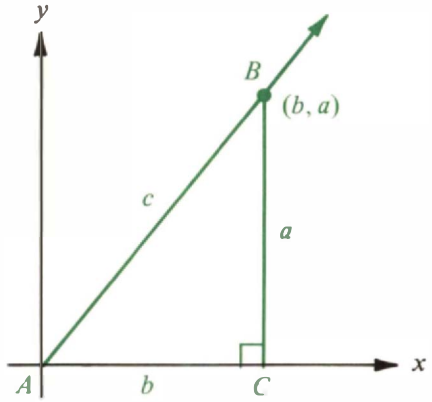


Figure 3

From Definition I in Chapter 1, we have	From Definition II in this chapter we have
$\sin A = \dfrac{a}{c}$	$\sin A = \dfrac{a}{c}$
$\cos A = \dfrac{b}{c}$	$\cos A = \dfrac{b}{c}$
$\tan A = \dfrac{a}{b}$	$\tan A = \dfrac{a}{b}$

The two definitions agree as long as A is an acute angle. If A is not an acute angle, then Definition II does not apply, since in right triangle ABC, A must be an acute angle.

Here is another definition that we will need before we can take Definition II any further.

Definition Sine and *co*sine are *co*functions, as are tangent and *co*-tangent, and secant and *co*secant. We say sine is the cofunction of cosine, and cosine is the cofunction of sine.

Now let's see what happens when we apply Definition II to B in right triangle ABC.

$$\sin B = \frac{\text{side opposite } B}{\text{hypotenuse}} = \frac{b}{c} = \cos A$$

$$\cos B = \frac{\text{side adjacent } B}{\text{hypotenuse}} = \frac{a}{c} = \sin A$$

$$\tan B = \frac{\text{side opposite } B}{\text{side adjacent } B} = \frac{b}{a} = \cot B$$

$$\cot B = \frac{\text{side adjacent } B}{\text{side opposite } B} = \frac{a}{b} = \tan A$$

$$\sec B = \frac{\text{hypotenuse}}{\text{side adjacent } B} = \frac{c}{a} = \csc A$$

$$\csc B = \frac{\text{hypotenuse}}{\text{side opposite } B} = \frac{c}{b} = \sec A$$

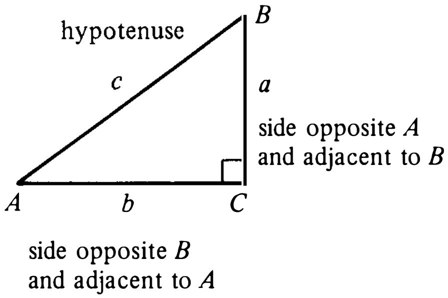


Figure 4

We can summarize the information above with the following theorem:

Cofunction Theorem A trigonometric function of an angle is always equal to the cofunction of the complement of the angle. That is, if x is an acute angle, then, since x and $90° - x$ are complementary angles, it must be true that

$$\begin{aligned} \sin x &= \cos(90° - x) \\ \cos x &= \sin(90° - x) \\ \tan x &= \cot(90° - x) \\ \cot x &= \tan(90° - x) \\ \sec x &= \csc(90° - x) \\ \csc x &= \sec(90° - x) \end{aligned}$$

▼ **Example 2** Fill in the blanks so that each expression becomes a true statement.

a. sin ____ = cos 30° **b.** tan y = cot ____ **c.** sec 75° = csc ____

Solution Using the theorem on cofunctions of complementary angles, we fill in the blanks as follows:

a. $\sin \underline{60°} = \cos 30°$ since sine and cosine are cofunctions and $60° + 30° = 90°$

b. $\tan y = \cot \underline{(90° - y)}$ since tangent and cotangent are cofunctions and $y + (90° - y) = 90°$

c. $\sec 75° = \csc \underline{15°}$ since secant and cosecant are cofunctions and $75° + 15° = 90°$ ▲

For our next application of Definition II we need to recall the two special triangles we introduced in Chapter 1. They are the 30°–60°–90° triangle, and the 45°–45°–90° triangle.

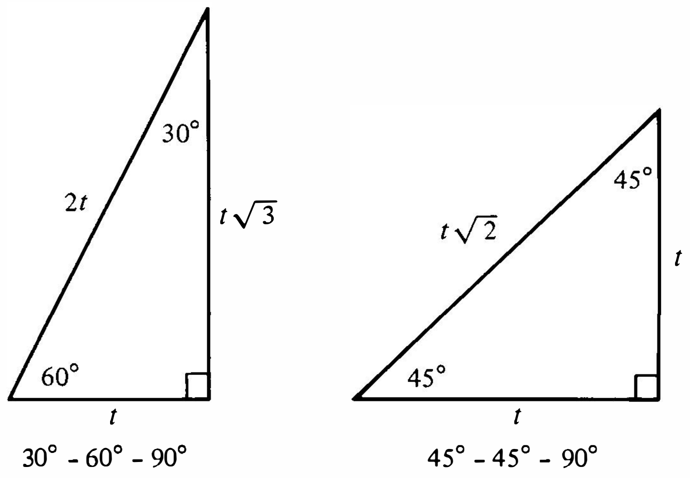


Figure 5

We can use these two special triangles to find the trigonometric functions of 30°, 45°, and 60°. For example,

$$\sin 60° = \frac{\text{side opposite } 60°}{\text{hypotenuse}} = \frac{t\sqrt{3}}{2t} = \frac{\sqrt{3}}{2}$$

$$\tan 30° = \frac{\text{side opposite } 30°}{\text{side adjacent } 30°} = \frac{t}{t\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Dividing out the common factor t

We could go on in this manner—using Definition II and the two special triangles—to find the six trigonometric ratios for 30°, 45°, and 60°. Instead, let us vary things a little and use the information just obtained for sin 60° and tan 30° and the theorem on cofunctions of complementary angles to find cos 30° and cot 60°.

$$\cos 30° = \sin 60° = \frac{\sqrt{3}}{2}$$

$$\cot 60° = \tan 30° = \frac{1}{\sqrt{3}}$$

Cofunction theorem

To vary things even more, we can use some reciprocal identities to find csc 60° and cot 30°.

$$\csc 60° = \frac{1}{\sin 60°} = \frac{2}{\sqrt{3}} \quad \text{or} \quad \frac{2\sqrt{3}}{3}$$ If denominator is rationalized

$$\cot 30° = \frac{1}{\tan 30°} = \sqrt{3}$$

The idea behind using the different methods listed above is not to make things confusing. We have a number of tools at hand and it does not hurt to show the different ways they can be used.

If we were to continue finding sine, cosine, and tangent for these special angles, we would obtain the results summarized in Table 1.

Table 1 Exact Values

θ	30°	45°	60°
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$ or $\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Table 1 is called a table of exact values to distinguish it from a table of approximate values. Later in this chapter we will work with tables of approximate values.

Calculator Note To use a calculator to find the sine, cosine, or tangent of an angle, you first enter the angle (with the calculator in the degree mode) and then press the sin, cos, or tan key. For example, to find sin 30° you would use the following sequence

30 [sin]

The calculator will then display 0.5. The values the calculator gives for the trigonometric functions of an angle are approximations, except for a few cases such as $\sin 30° = 0.5$. If you use a calculator to find sin 60°, the calculator will display 0.866025403, which is a nine digit decimal approximation of $\sqrt{3}/2$. You can check this by using your calculator to find an approximation to $\sqrt{3}/2$ to see that it agrees with the calculator value of sin 60°. In the meantime, remember that if you are asked to find the exact value of a trigonometric function, you must use the values given in the table of exact values.

▼ **Example 3** Use the exact values from the table to show that the following are true.

a. $\cos^2 30° + \sin^2 30° = 1$ **b.** $\cos^2 45° + \sin^2 45° = 1$

Solution

a. $\cos^2 30° + \sin^2 30° = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{4} = 1$

b. $\cos^2 45° + \sin^2 45° = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$ ▲

▼ **Example 4** Let $x = 30°$ and $y = 45°$ in each of the expressions that follow, and then simplify each expression as much as possible.

a. $2 \sin x$ **b.** $\sin 2y$ **c.** $4 \sin(3x - 90°)$

Solution

a. $2 \sin x = 2 \sin 30° = 2\left(\frac{1}{2}\right) = 1$

b. $\sin 2y = \sin 2(45°) = \sin 90° = 1$

c. $4 \sin(3x - 90°) = 4 \sin[3(30°) - 90°] = 4 \sin 0° = 4(0) = 0$ ▲

Problem Set 2.1

Problems 1 through 10 refer to right triangle *ABC* with $C = 90°$. In each case, use the given information to find the six trigonometric functions of *A*.

1. $b = 3, c = 5$

2. $b = 5, c = 13$

3. $a = 2, b = 1$

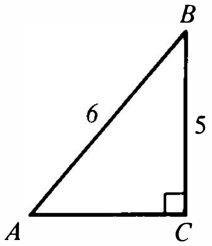
4. $a = 3, b = 2$

5. $a = 2, b = \sqrt{5}$

6. $a = 3, b = \sqrt{7}$

In each triangle below, find sin *A*, cos *A*, tan *A*, and sin *B*, cos *B*, tan *B*.

7.



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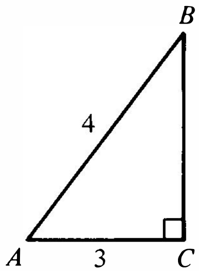

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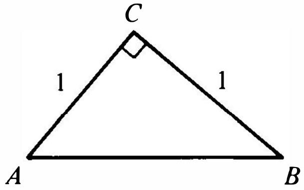

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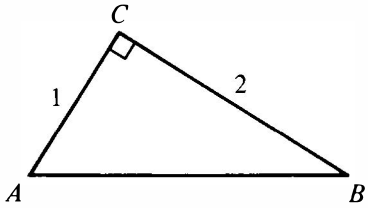

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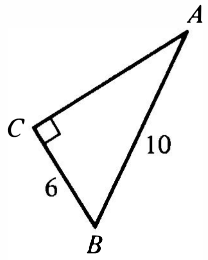

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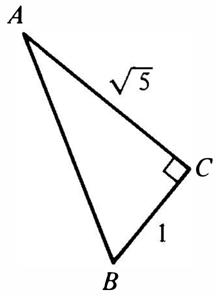

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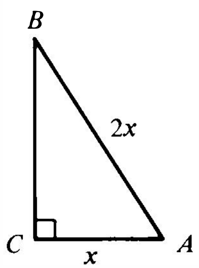

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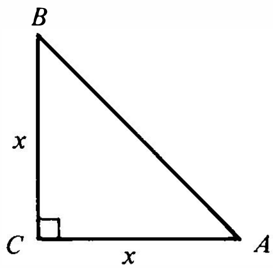


Use the cofunction theorem to fill in the blanks so each expression becomes a true statement.

15. $\sin 10° = \cos$ _____
16. $\cos 40° = \sin$ _____
17. $\tan 8° = \cot$ _____
18. $\cot 12° = \tan$ _____
19. $\sec 73° = \csc$ _____
20. $\csc 63° = \sec$ _____
21. $\sin x = \cos$ _____
22. $\sin y = \cos$ _____
23. $\tan (90° - x) = \cot$ _____
24. $\tan (90° - y) = \cot$ _____

Simplify each expression by first substituting values from the table of exact values and then simplifying the resulting expression.

25. $4 \sin 30°$
26. $5 \sin^2 30°$
27. $(2 \cos 30°)^2$
28. $\sin^3 30°$
29. $\sin 30° \cos 45°$
30. $\sin 30° + \cos 45°$
31. $(\sin 60° + \cos 60°)^2$
32. $\sin^2 60° + \cos^2 60°$

33. $\sin^2 45° - 2 \sin 45° \cos 45° + \cos^2 45°$
34. $(\sin 45° - \cos 45°)^2$

35. $(\tan 45° + \tan 60°)^2$
36. $\tan^2 45° + \tan^2 60°$

For each expression that follows, replace x with 30°, y with 45°, and z with 60°, and then simplify as much as possible.

37. $2 \sin x$
38. $4 \cos y$
39. $4 \cos(z - 30°)$
40. $-2 \sin(y + 45°)$
41. $-3 \sin 2x$
42. $3 \sin 2y$
43. $2 \cos(3x - 45°)$
44. $2 \sin(90° - z)$

Find exact values for each of the following:

45. $\sec 30°$
46. $\csc 30°$
47. $\csc 60°$
48. $\sec 60°$
49. $\cot 45°$
50. $\cot 30°$
51. $\sec 45°$
52. $\csc 45°$

Use a calculator to find an approximation to each of the following expressions:

53. $\sin 60°$
54. $\cos 30°$
55. $\sqrt{3}/2$
56. $\sqrt{2}/2$
57. $\sin 45°$
58. $\cos 45°$
59. $1/\sqrt{2}$
60. $1/\sqrt{3}$
61. $\tan 45°$
62. $\tan 30°$

Review Problems From here on, each problem set will end with a series of review problems. In mathematics, it is very important to review. The more you review, the better you will understand the topics we cover and the longer you will remember them. Also, there will be times when material that seemed confusing earlier will be less confusing the second time around.

The problems that follow review material we covered in Section 1.2.

Find the distance between each pair of points.

63. (5, 1) (2, 5)

64. (3, −2) (−1, −4)

65. Find x so that the distance between $(x, 2)$ and $(1, 5)$ is $\sqrt{13}$.

66. Graph the line $2x - 3y = 6$.

Draw each angle in standard position and name a point on the terminal side.

67. 45°

68. 135°

2.2 Tables and Calculators for Trigonometric Functions of an Acute Angle

We previously defined 1 degree (1°) to be 1/360 of a full rotation. A degree itself can be broken down further. If we divide 1° into 60 equal parts, each one of the parts is called 1 minute, denoted 1′. One minute is 1/60 of a degree; in other words, there are 60 minutes in every degree. The next smaller unit of angle measure is a second. One second, 1″, is 1/60 of a minute. There are 60 seconds in every minute.

$$1^\circ = 60' \quad \text{or} \quad 1' = \left(\frac{1}{60}\right)^\circ$$

$$1' = 60'' \quad \text{or} \quad 1'' = \left(\frac{1}{60}\right)'$$

Table 1 shows how to read angles written in degree measure.

Table 1

The Expression	Is Read
52° 10′	52 degrees, 10 minutes
5° 27′ 30″	5 degrees, 27 minutes, 30 seconds
13° 24′ 75″	13 degrees, 24 minutes, 75 seconds

▼ **Example 1** Add 48° 49′ and 72° 26′.

Solution We can add in columns with degrees in the first column and minutes in the second column.

$$\begin{array}{r} 48^\circ\ 49' \\ +\ 72^\circ\ 26' \\ \hline 120^\circ\ 75' \end{array}$$

Since 60 minutes is equal to 1 degree, we can carry 1 degree from the minutes column to the degrees column.

$$120° \; 75' = 121° \; 15'$$

▲

▼ **Example 2** Subtract 24° 14′ from 90°.

Solution In order to subtract 24° 14′ from 90° we will have to "borrow" 1° and write that 1° as 60′.

$$\begin{array}{r} 90° \\ -\;24°\;14' \\ \hline \end{array} \quad = \quad \begin{array}{r} 89°\;60' \\ -\;24°\;14' \\ \hline 65°\;46' \end{array} \qquad \text{(Still 90°.)}$$

▲

Decimal Degrees

An alternative to using minutes and seconds to break down degrees into smaller units is decimal degrees. For example, 30.5°, 101.75°, and 62.831° are the measures of angles written in decimal degrees.

To convert from decimal degrees to degrees and minutes, we simply multiply the fractional part (the part to the right of the decimal point) of the angle by 60 to convert it to minutes.

▼ **Example 3** Change 27.25° to degrees and minutes.

Solution Multiplying 0.25 by 60 we have the number of minutes equivalent to 0.25°.

$$\begin{aligned} 27.25° &= 27° + 0.25° \\ &= 27° + 0.25(60') \\ &= 27° + 15' \\ &= 27°\;15' \end{aligned}$$

Of course in actual practice, we would not show all of these steps. They are shown here simply to indicate why we multiply only the decimal part of the decimal degree by 60 to change to degrees and minutes. ▲

Calculator Note Some scientific calculators have a key that automatically converts angles given in decimal degrees to degrees and minutes. Consult the manual that came with your calculator to see if yours has this key.

▼ **Example 4** Change 10° 45′ to decimal degrees.

Solution We have to reverse the process we used in Example 3. To change 45′ to a decimal we must divide by 60.

$$\begin{aligned} 10°\,45' &= 10° + 45' \\ &= 10° + 45\left(\frac{1}{60}\right)^{\circ} \\ &= 10° + \frac{45°}{60} \\ &= 10° + 0.75° \\ &= 10.75° \end{aligned}$$

▲

Calculator Note On a calculator, the result given in Example 4 is accomplished as follows:

45 [÷] 60 [+] 10 [=]

The process of converting back and forth between decimal degrees and degrees and minutes can become more complicated when we use decimal numbers with more digits or when we convert to degrees, minutes, and seconds. In this book, most of the angles written in decimal degrees will be written to the nearest tenth or, at most, the nearest hundredth. The angles written in degrees, minutes, and seconds will rarely go beyond the minutes column.

Table 2 lists the most common conversions between decimal degrees and minutes.

Table 2

Decimal Degree	Minutes
0.1°	6′
0.2°	12′
0.3°	18′
0.4°	24′
0.5°	30′
0.6°	36′
0.7°	42′
0.8°	48′
0.9°	54′
1.0°	60′

Tables for Acute Angles

Up until now, the only angles we have been able to determine trigonometric functions for have been angles for which we could find a point on the terminal side or that were part of special triangles. We can find decimal approximations for trigonometric functions of any acute angle by using a calculator with keys for sine, cosine, and tangent or by using the tables at the back of the book.

Table II in the TABLES section in the back of the book gives the values for the six trigonometric functions of any acute angle given in decimal degrees—to the nearest tenth of a degree. Table III gives all six trigonometric functions of acute angles written in degrees and minutes in increments of 10′. Angles between 0° and 45° are listed in the left-hand column of each table and angles between 45° and 90° are listed in the right-hand column. The main thing you have to know to read the table is that angles in the left-hand column correspond to the trigonometric functions listed at the top of the table; likewise, angles in the right-hand column correspond to the trigonometric functions at the bottom of the table.

Calculator Note The examples that follow illustrate how the tables of trigonometric functions are used. If you are using a calculator instead of the tables, check to see that you can obtain the results given in the examples on your calculator. Keep in mind that your calculator will interpret the angles you enter as being in decimal degrees. To use a calculator to find trigonometric functions of angles given in degrees and minutes you must first convert to decimal degrees.

Example 5 Use Table II to find cos 37.8°.

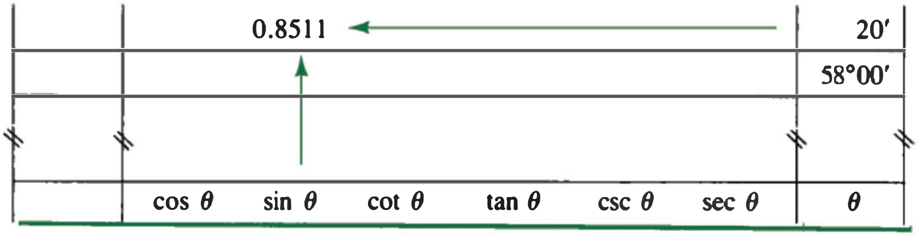
Solution We locate 37.8° in the *left-hand column* and then read across until we are in the column labeled cos θ at the *top*.

θ	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ	
		↓					
37.0							
.8	⟶	0.7902					

The number 0.7902 is just an approximation of cos 37.8°. Cos 37.8° is actually an irrational number, as are the trigonometric functions of most of the other angles listed in the table. ▲

Example 6 Use Table III to find sin 58° 20′.

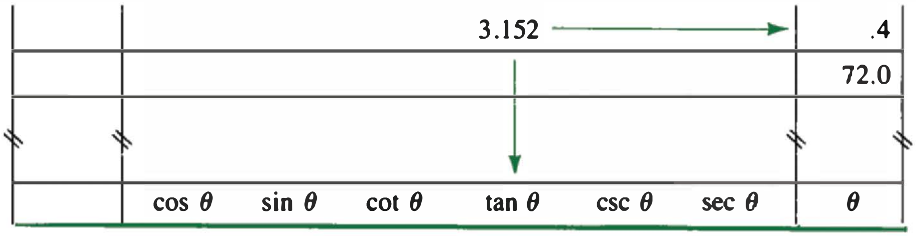
Solution We locate 58° 20′ in the *right-hand column* of Table III and then read across until we are in the column labeled sin θ at the *bottom*.



sin 58° 20′ = 0.8511

Example 7 Use Table II to find θ if $\tan\theta = 3.152$.

Solution We want to use the table in the reverse direction from Examples 5 and 6. Now we are given the tangent of an angle and asked to find the angle. Locating 3.152 in the body of the table, we see that it is in the column with the heading tan θ at the bottom.

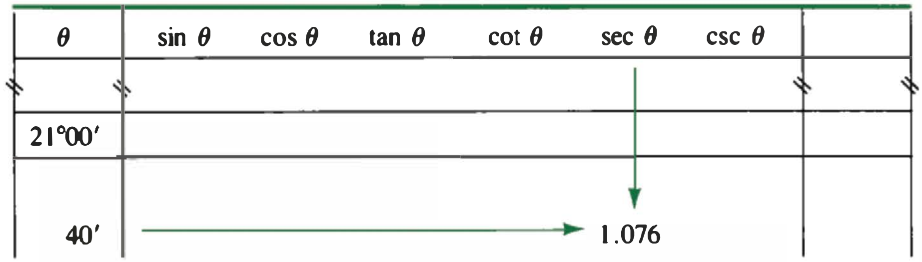


If $\tan\theta = 3.152$, then $\theta = 72.4°$

Calculator Note To work Example 7 on a calculator, you must use the $\tan^{-1}$ or arctan key. This is usually accomplished by first pressing the inv or arc key. (Check your manual to see how your calculator does this. In the index, look under inverse trigonometric functions.) We will explain the notation used on your calculator later in the book. For now, think of $\tan^{-1}$ as meaning *the angle whose tangent is*. When you enter a number and then press this key (or keys) your calculator will find the acute angle with a tangent equal to the number you have entered. We should also mention that the notation $\tan^{-1}$ does not mean 1/tan. It is the notation used in connection with inverse functions, not reciprocals.

Example 8 Find θ to the nearest ten minutes if $\sec\theta = 1.076$.

Solution Since we are asked to find θ to the nearest ten minutes, we must use Table III.



If sec θ = 1.076, then θ = 21° 40′ ▲

Calculator Note To do Example 8 on a calculator, you must use the fact that secant is the reciprocal of cosine. Therefore, since

$$\sec\theta = \frac{1}{\cos\theta}\text{, then, if } \sec\theta = 1.076,\ \cos\theta = \frac{1}{1.076}$$

Using the reasoning given above, to find θ on your calculator you would follow these steps:

1.076 [1/x] [$\cos^{-1}$]

Remember also that your calculator will give you the angle in decimal degrees, 21.66348427 in this case. To compare the answer to Example 8 with the answer your calculator gives you, you will have to convert your calculator answer to degrees and minutes and then round to the nearest 10 minutes.

Problem Set 2.2

Add and subtract as indicated.

1. (37° 45′) + (26° 24′)
2. (41° 20′) + (32° 16′)
3. (51° 55′) + (37° 45′)
4. (63° 38′) + (24° 52′)
5. (61° 33′) + (45° 16′)
6. (77° 21′) + (23° 16′)
7. 90° − (34° 12′)
8. 90° − (62° 25′)
9. 180° − (120° 17′)
10. 180° − (112° 19′)
11. (76° 24′) − (22° 34′)
12. (89° 38′) − (28° 58′)
13. (70° 40′) − (30° 50′)
14. (80° 50′) − (50° 56′)

Convert each of the following to degrees and minutes.

15. 35.4°
16. 63.2°
17. 16.25°
18. 18.75°
19. 92.55°
•20. 34.45°
21. 19.9°
22. 18.8°
23. 28.35°
24. 76.85°

Change each of the following to decimal degrees.

25. $45° 12'$ **26.** $74° 18'$
27. $62° 36'$ **28.** $21° 48'$
29. $17° 20'$ **30.** $29° 40'$
31. $48° 27'$ **32.** $78° 21'$

Use Table II or a calculator to find each of the following:

33. $\sin 26.4°$ **34.** $\cos 24.8°$
35. $\cos 18.3°$ **36.** $\sin 41.9°$
37. $\sin 73.7°$ **38.** $\cos 59.1°$
39. $\tan 87.1°$ **40.** $\cot 78.5°$
41. $\cot 53.6°$ **42.** $\tan 81.4°$

Use Table III or a calculator to find each of the following. (*Remember*: If you are using a calculator, you must convert to decimal degrees before you use the sin, cos, or tan keys.)

43. $\cos 24° 30'$ **44.** $\sin 35° 10'$
45. $\tan 42° 10'$ **46.** $\cot 19° 40'$
47. $\sin 56° 40'$ **48.** $\cos 66° 40'$
49. $\cos 70° 20'$ **50.** $\sin 80° 50'$
51. $\cot 88° 50'$ **52.** $\tan 50° 10'$

Use Table II to find θ if θ is between $0°$ and $90°$ and

53. $\cos \theta = 0.8290$ **54.** $\sin \theta = 0.8290$
55. $\tan \theta = 1.111$ **56.** $\cot \theta = 1.111$
57. $\cos \theta = 0.9348$ **58.** $\sin \theta = 0.9348$
59. $\sin \theta = 0.7108$ **60.** $\cos \theta = 0.7108$
61. $\tan \theta = 0.7536$ **62.** $\cot \theta = 0.7536$

For each statement below, use Table III to find θ. (*Note*: If you are using a calculator, you will have to convert the answers your calculator gives you to degrees and minutes before you can check your answers with the answers in the back of the book.)

63. $\sin \theta = 0.1392$ **64.** $\cos \theta = 0.2250$
65. $\cos \theta = 0.0843$ **66.** $\sin \theta = 0.9964$
67. $\sec \theta = 12.30$ **68.** $\csc \theta = 1.017$
69. $\tan \theta = 0.7954$ **70.** $\cot \theta = 1.611$
71. $\csc \theta = 1.398$ **72.** $\sec \theta = 31.26$

73. What is the largest value of $\sin \theta$ found in Table II?
74. What is the smallest value of $\sin \theta$ found in Table II?
75. If $\sin \theta$ is between 0 and 1 for θ between $0°$ and $90°$, what are the values of $\csc \theta$ for θ between $0°$ and $90°$?
76. If $\sin \theta$ is between 0 and 1 for θ between $0°$ and $90°$, what are the values of $\sin^2\theta$ for θ between $0°$ and $90°$?

Review Problems The problems that follow review material we covered in Section 1.3.

Find sin θ, cos θ, and tan θ for each value of θ. (Do not use tables or calculators.)

77. $90°$ **78.** $135°$

Find the remaining trigonometric functions of θ if

79. $\cos\theta = -5/13$ and θ terminates in QIII
80. $\tan\theta = -3/4$ and θ terminates in QII

In which quadrant must the terminal side of θ lie if

81. $\sin\theta > 0$ and $\cos\theta < 0$ **82.** $\tan\theta > 0$ and $\sec\theta < 0$

2.3 Solving Right Triangles

In this section we will use Definition II for trigonometric functions of an acute angle, along with our tables (or calculators), to find the missing parts to some right triangles. Before we begin, however, we need to talk about significant digits.

DEFINITION The number of *significant digits* (or figures) in a number with digits to the right of the decimal point is found by counting the number of digits from left to right beginning with the first nonzero digit on the left. (In this case, we assume all digits are significant, with the exception of leading zeros—leading zeros being zeros that occur on the left.)

If the number in question is an integer with no digits to the right of the decimal point, then we use the same procedure except that we disregard trailing zeros—trailing zeros being any zeros after the last nonzero digit on the right.

▼ **Example 1** Give the number of significant digits in the side given in each of the following triangles.

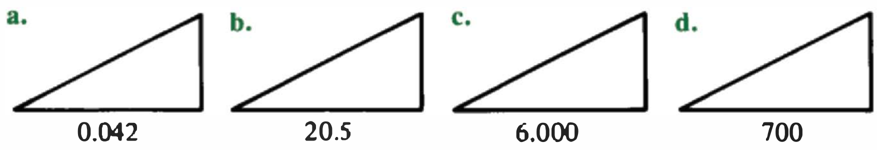


Figure 1

Solution

a. 0.042 has two significant digits.
b. 20.5 has three significant digits.

c. 6.000 has four significant digits.

d. 700 has one significant digit. (The number 700 is an integer with no zeros to the right of the decimal point, so we disregard the zeros on the right. ▲

The relationship between the accuracy of the sides of a triangle and the accuracy of the angles in the same triangle is shown in the following table.

Accuracy of Sides	Accuracy of Angles
Two significant digits	Nearest degree
Three significant digits	Nearest 10 minutes or tenth of a degree
Four significant digits	Nearest minute or hundredth of a degree

▼ **Example 2** For each triangle, round the number in parentheses so that its accuracy corresponds with the accuracy of the other numbers in the triangle.

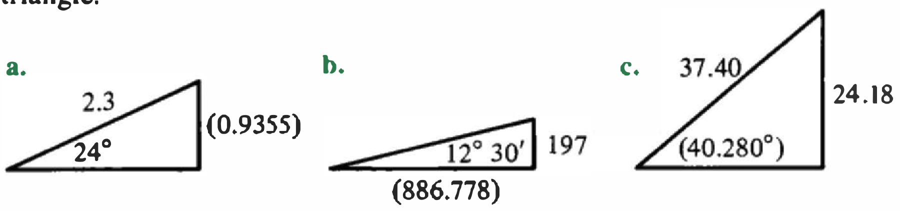


Figure 2

Solution

a. Since 2.3 is accurate to two significant digits, we must round 0.9355 to that accuracy also.

$$0.9355 = 0.94 \text{ to two significant digits}$$

(*Remember*: We do not count leading zeros as significant.)

b. Since 197 has three significant digits, we must round 886.778 to that accuracy also.

$$886.778 = 887 \text{ to three significant digits.}$$

c. Since both of the given sides, 37.40 and 24.18, are accurate to four significant figures, we must round 40.280° to the nearest hundredth of a degree.

$$40.280° = 40.28° \text{ to the nearest hundredth of a degree.}$$ ▲

We are now ready to use Definition II, along with our tables (or calculators), to solve some right triangles. We solve a right triangle by using the information given about it to find all of the missing sides and angles. In Example 3, we are given the values of one side and one of the acute angles and asked to find the remaining two sides and the other acute angle. In all of the examples and in the Problem Set that follows, we will assume C is the right angle in all of our right triangles.

▼ Example 3 In right triangle ABC, $A = 40°$ and $c = 12$ centimeters. Find a, b, and B.

Solution We begin by making a diagram of the situation.

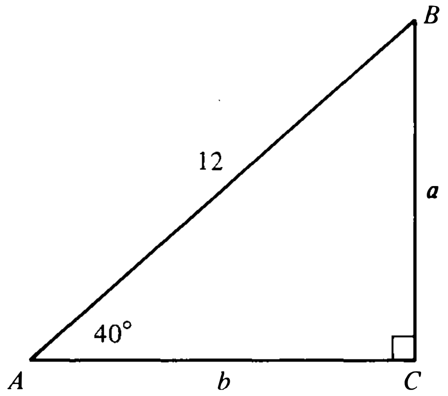


Figure 3

To find B, we use the fact that the sum of the two acute angles in any right triangle is 90°.

$$\begin{aligned} B &= 90° - A \\ &= 90° - 40° \\ B &= 50° \end{aligned}$$

To find a, we can use the formula for $\sin A$.

$$\sin A = \frac{a}{c}$$

Multiplying both sides of this formula by c and then substituting in our given values for A and c we have

$$\begin{aligned} a &= c \sin A \\ &= 12 \sin 40° \\ &= 12(0.6428) && \sin 40° = 0.6428 \text{ from Tables II or III} \\ a &= 7.7 && \text{Answer rounded to nearest tenth} \end{aligned}$$

Calculator Note On a calculator, we would use the sin key to find a as follows:

12 [×] 40 [sin] [=]

There is more than one way to find b.

Using $\cos A = \dfrac{b}{c}$ we have

$$\begin{aligned} b &= c \cos A \\ &= 12 \cos 40° \\ &= 12(0.7660) \\ b &= 9.2 \end{aligned}$$

Using the Pythagorean theorem we have

$$\begin{aligned} c^2 &= a^2 + b^2 \\ b &= \sqrt{c^2 - a^2} \\ &= \sqrt{12^2 - (7.7)^2} \\ &= \sqrt{144 - 59.29} \\ &= \sqrt{84.71} \\ b &= 9.2 \end{aligned}$$

Calculator Note To find b using the Pythagorean theorem and a calculator, we would follow this sequence:

12 [x^2] [−] 7.7 [x^2] [=] [$\sqrt{\ }$]

Note that we rounded the sides we found in this example to two significant digits. We did so because A was given to the nearest degree, dictating that we use two significant digits for the sides of our triangle. ▲

In Example 4, we are given two sides and asked to find the remaining parts of a right triangle.

▼ **Example 4** In right triangle ABC, $a = 2.73$ and $b = 3.41$. Find the remaining side and angles.

Solution Here is a diagram of the triangle.

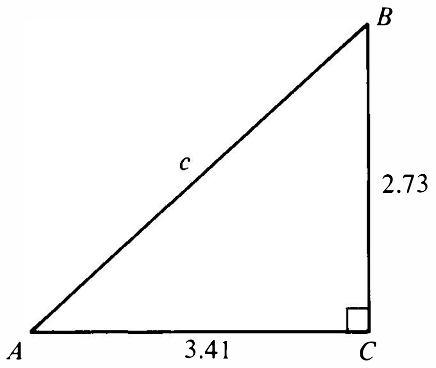


Figure 4

We can find A by using the formula for tan A

$$\tan A = \frac{a}{b}$$

$$= \frac{2.73}{3.41}$$

$$\tan A = 0.8006$$

Now, to find A, we look for the angle whose tangent is closest to 0.8006 using our table or a calculator. If we use Table II, we obtain

$$A = 38.7°$$

Next we find B.

$$\begin{aligned} B &= 90° - A \\ &= 90° - 38.7° \\ B &= 51.3° \end{aligned}$$

Notice we are rounding each angle to the nearest tenth of a degree since the sides we were originally given have three significant digits.

We can find c using the Pythagorean theorem or one of our trigonometric functions. Let's start with a trigonometric function.

$$\text{If } \sin A = \frac{a}{c}$$

$$\text{then} \quad c = \frac{a}{\sin A}$$

$$= \frac{2.73}{\sin 38.7°}$$

$$= \frac{2.73}{0.6252}$$

$$= 4.37 \qquad \text{To three significant digits}$$

Using the Pythagorean theorem, we obtain the same result.

$$\begin{aligned} \text{If} \quad c^2 &= a^2 + b^2 \\ \text{then } c &= \sqrt{a^2 + b^2} \\ &= \sqrt{(2.73)^2 + (3.41)^2} \\ &= \sqrt{19.081} \\ &= 4.37 \end{aligned}$$

▲

Problem Set 2.3

Give the number of significant digits in each of the following numbers:

1. 42.3
2. 2.05
3. 0.76
4. 0.84
5. 3.5723
6. 4.189
7. 2.400
8. 4010
9. 7900
10. 0.0076

If each of the following angles were an angle in a triangle, how many significant digits would a side in the triangle contain?

11. 24°
12. 4.3°
13. 76.2°
14. 23.45°
15. 36° 24′
16. 23° 10′
17. 41.22°
18. 45.7°

Problems 19 through 26 refer to right triangle *ABC* with $C = 90°$. (Round all angles to the nearest degree and all sides to the nearest tenth.)

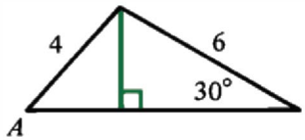
19. If $A = 40°$ and $c = 15$, find a.
20. If $A = 20°$ and $c = 12$, find a.
21. If $B = 20°$ and $c = 12$, find b.
22. If $B = 20°$ and $a = 10$, find b.
23. If $a = 6$ and $b = 10$, find A.
24. If $a = 6$ and $b = 10$, find B.
25. If $a = 12$ and $c = 20$, find B.
26. If $a = 15$ and $c = 25$, find A.

Problems 27 through 42 refer to right triangle *ABC* with $C = 90°$. In each case, solve for all the missing parts using the given information. For problems 27 through 42, use Table II when you find the missing angles.

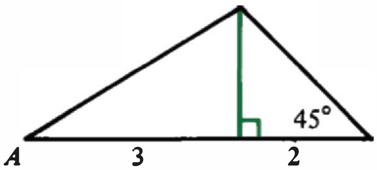
27. $A = 25°$, $c = 24$
28. $A = 41°$, $c = 36$
29. $A = 32.6°$, $a = 43.4$
30. $A = 48.3°$, $a = 3.48$
31. $A = 10° 40'$, $b = 5.93$
32. $A = 66° 50'$, $b = 28.2$
33. $B = 76°$, $c = 5.8$
34. $B = 21°$, $c = 4.2$
35. $B = 26° 20'$, $b = 324$
36. $B = 53° 30'$, $b = 725$
37. $a = 37$, $b = 87$
38. $a = 91$, $b = 85$
39. $a = 2.75$, $c = 4.05$
40. $a = 62.3$, $c = 73.6$
41. $b = 152$, $c = 258$
42. $b = 421$, $c = 555$

In Problems 43 through 46, use the information given in the diagram to find *A* to the nearest degree.

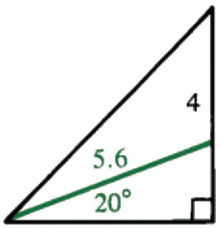
43.



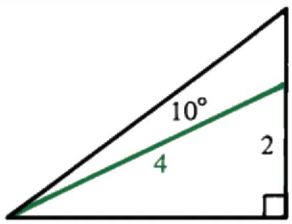
44.



45.



46.



Review Problems The problems below review material we covered in Section 1.4.

If $\sin\theta = -12/13$ and $\cos\theta = 5/13$, find

47. $\tan\theta$

48. $\cot\theta$

49. $\sec\theta$

50. $\csc\theta$

51. If $\sin\theta = 1/3$ and θ terminates in QI, find $\cos\theta$.

52. If $\cos\theta = -2/3$ and θ terminates in QIII, find $\sin\theta$.

2.4 Applications

We are now ready to put our knowledge of solving right triangles to work to solve some application problems.

Example 1 The two equal sides of an isosceles triangle are each 24 centimeters. If each of the two equal angles measures 52°, find the length of the base and the altitude.

Solution An isosceles triangle is any triangle with two equal sides. The angles opposite the two equal sides are called the base angles, and they are always equal. Here is a picture of our isosceles triangle.

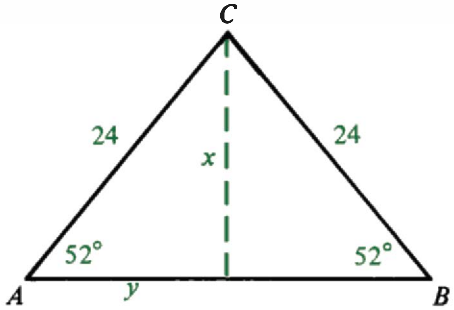


Figure 1

We have labeled the altitude x. We can solve for x using a sine ratio.

$$\text{If } \sin 52° = \frac{x}{24}$$

$$\text{then } x = 24 \sin 52°$$
$$= 24(0.7880)$$
$$= 19 \text{ centimeters} \qquad \text{Rounded to 2 significant digits}$$

We have labeled half the base with y. To solve for y, we can use a cosine ratio.

$$\text{If } \cos 52° = \frac{y}{24}$$

$$\text{then } y = 24 \cos 52°$$
$$= 24(0.6157)$$
$$= 14.7768$$

The base is $2y = 2(14.7768) = 29.5536 = 30$ centimeters to the nearest centimeter. ▲

For our next applications, we need the following definition.

DEFINITION An angle measured from the horizontal up is called an ***angle of elevation***. An angle measured from the horizontal down is called an ***angle of depression***.

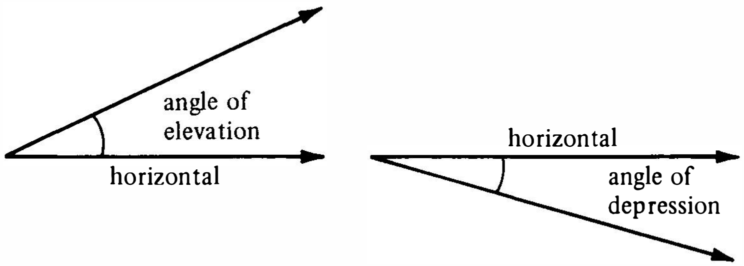


Figure 2

These angles of elevation and depression are always considered positive angles. Also, the nonhorizontal side of each angle is sometimes called the *line of sight* of the observer.

▼ **Example 2** If a 75.0 foot flag pole casts a shadow 43.0 feet long, to the nearest 10 minutes, what is the angle of elevation of the sun from the tip of the shadow?

Solution We begin by making a diagram of the situation.

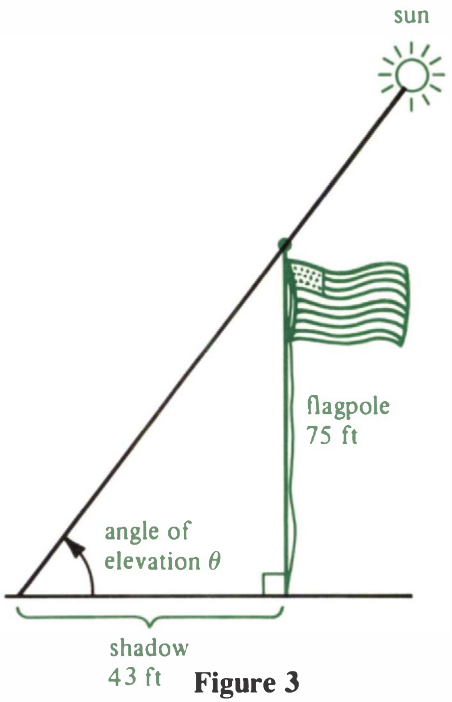


Figure 3

If we let θ = the angle of elevation of the sun then

$$\tan\theta = \frac{75}{43}$$

$$\tan\theta = 1.744$$

which means $\theta = 60°\ 10'$ to the nearest 10 minutes ▲

Example 3 A man climbs 213 meters up the side of a pyramid and finds that the angle of depression to his starting point is 52.6°. How high off the ground is he?

Solution Again, we begin by making a diagram of the situation.

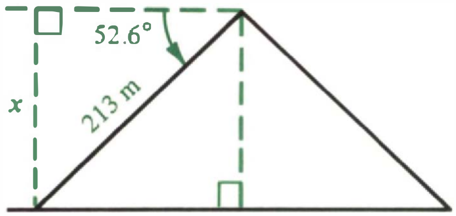


Figure 4

If x is his height above the ground, we can solve for x using a sine ratio.

$$\text{If } \sin 52.6° = \frac{x}{213}$$

$$\begin{aligned} \text{then} \quad x &= 213 \sin 52.6° \\ &= 213(0.7944) \\ &= 169 \text{ meters to three significant digits} \end{aligned}$$

The man is 169 meters above the ground.

Our next applications are concerned with what is called the bearing of a line. It is used in navigation and surveying.

DEFINITION The *bearing of a line l* is the acute angle formed by the north-south line and the line *l*. The notation used to designate the bearing of a line begins with N or S (for north or south), followed by the number of degrees in the angle, and ends with E or W (for east or west). Here are some examples.

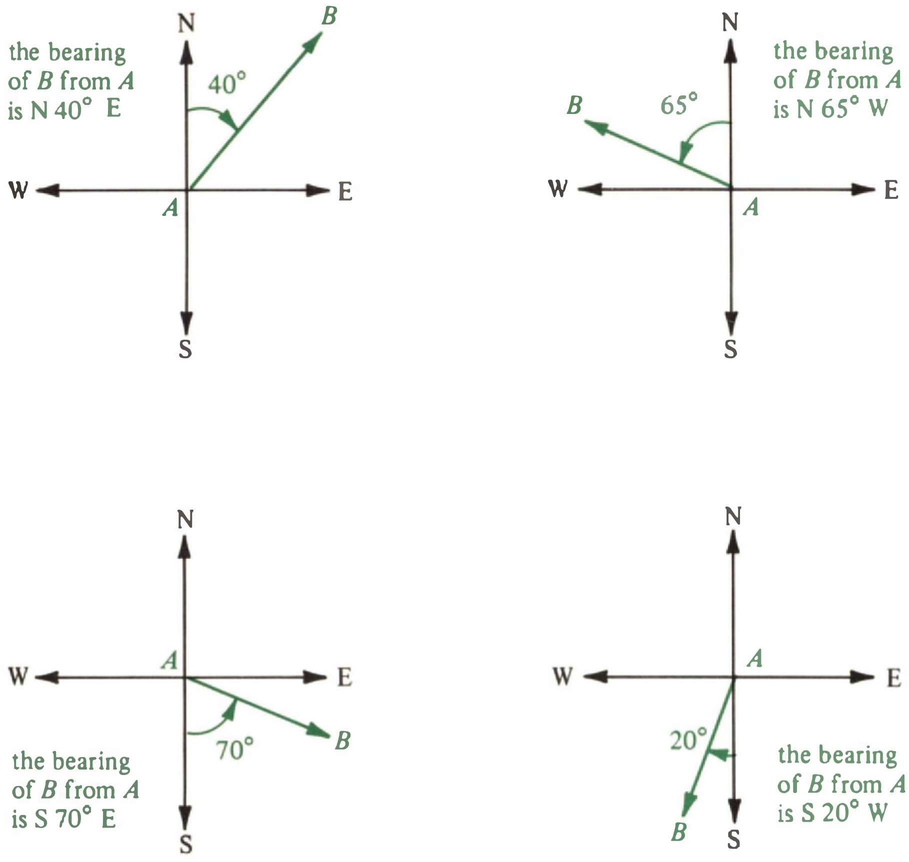
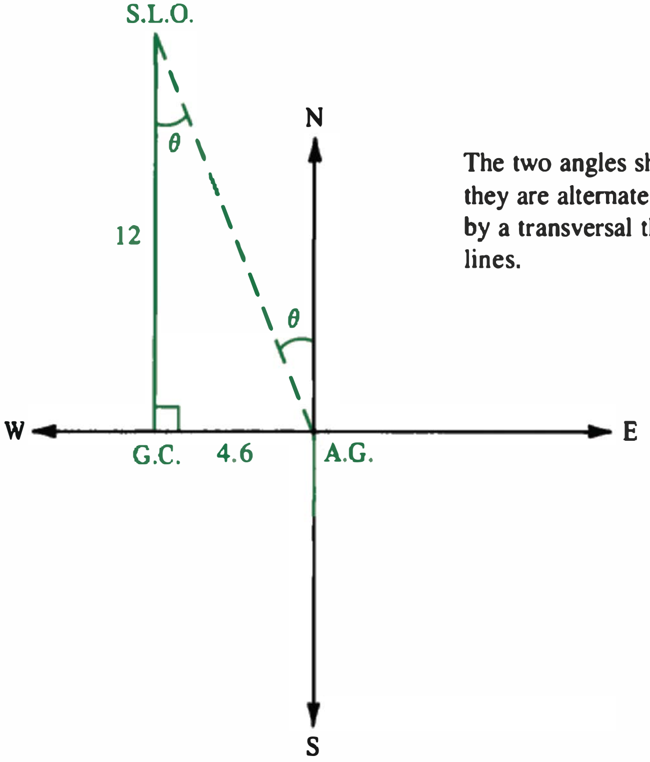


Figure 5

▼ **Example 4** San Luis Obispo, California, is 12 miles due north of Grover City. If Arroyo Grande is 4.6 miles due east of Grover City, what is the bearing of San Luis Obispo from Arroyo Grande?

Solution Since we are looking for the bearing of San Luis Obispo *from* Arroyo Grande, we will put our N-S-E-W system on Arroyo Grande.



The two angles shown are equal because they are alternate interior angles formed by a transversal that cuts two parallel lines.

Figure 6

We solve for θ using the tangent ratio.

$$\tan\theta = \frac{4.6}{12}$$

$$\tan\theta = 0.3833$$

$$\text{which means } \theta = 21^\circ \text{ to the nearest degree}$$

The bearing of San Luis Obispo from Arroyo Grande is N 21° W. ▲

▼ **Example 5** A boat travels on a course of bearing N 52° 40′ E for a distance of 238 miles. How many miles north and how many miles east has the boat traveled?

Solution In the diagram of the situation we put our N-S-E-W system at the boat's starting point.

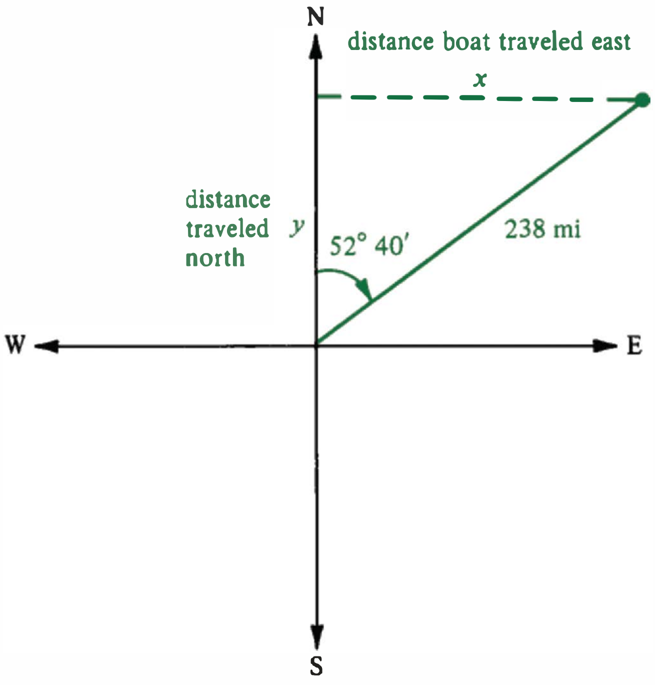


Figure 7

Solving for x with a sine ratio and y with a cosine ratio and rounding our answers to three significant digits, we have

If $\sin 52° 40' = \dfrac{x}{238}$	If $\cos 52° 40' = \dfrac{y}{238}$
then $x = 238(0.7951)$	then $y = 238(0.6065)$
$= 189$ miles	$= 144$ miles

Traveling 238 miles on a line that is N 52° 40′ E will get you to the same place as traveling 144 miles north and then 189 miles east. ▲

Problem Set 2.4

Solve each of the following problems. In each case, be sure to make a diagram of the situation with all the given information labeled.

1. The two equal sides of an isosceles triangle are each 42 centimeters. If the base measures 30 centimeters, find the height and the measure of the two equal angles.

2. An equilateral triangle (one with all sides the same length) has an altitude of 4.3 inches. Find the length of the sides.
3. How long should an escalator be if it is to make an angle of 33° with the floor and carry people a vertical distance of 21 feet between floors?
4. A road up a hill makes an angle of 5° with the horizontal. If the road from the bottom of the hill to the top of the hill is 2.5 miles long, how high is the hill?
5. A 72.5 foot rope from the top of a circus tent pole is anchored to the ground 43.2 feet from the bottom of the pole. What angle does the rope make with the pole?
6. A ladder is leaning against the top of a 7.0 foot wall. If the bottom of the ladder is 4.5 feet from the wall, what is the angle between the ladder and the wall?
7. If a 73.0 foot flag pole casts a shadow 51.0 feet long, what is the angle of elevation of the sun (to the nearest tenth of a degree)?
8. If the angle of inclination of the sun is 63.4° when a building casts a shadow of 37.5 feet, what is the height of the building?
9. A person standing 5 feet from a mirror notices that the angle of depression from his eyes to the bottom of the mirror is 12°, while the angle of elevation to the top of the mirror is 11°. Find the vertical dimension of the mirror to the nearest foot.
10. From a point on the ground, the angle of inclination to the top of a radio antenna on the top of a building is 47° 30′. Moving 33.0 feet farther from the building the angle of inclination changes to 42° 10′. How high off the ground is the top of the radio antenna?
11. From a plane 3,000 feet in the air, the angle of depression to the beginning of the runway is 3° 50′. What is the horizontal distance between the plane and runway? (That is, if the plane flew at the same altitude, how far would it be to the runway?) Give your answer to the nearest tenth of a mile. (5,280 feet = 1 mile)
12. From the edge of a platform 25.1 meters high, a diver notices the angles of depression to each end of the pool are 14.7° and 78.9°. What is the length of the pool?
13. Lompoc, California, is 18 miles due south of Nipomo. Buellton, California, is due east of Lompoc and S 65° E from Nipomo. How far is Lompoc from Buellton?
14. A tree on one side of a river is due west of a rock on the other side of the river. From a stake 21 yards north of the rock, the bearing of the tree is S 18.2° W. How far is it from the rock to the tree?
15. A boat travels on a course of bearing N 37° 10′ W for 79.5 miles. How many miles north and how many miles west has the boat traveled?
16. A boat travels on a course of bearing S 63° 50′ E for 100 miles. How many miles south and how many miles east has the boat traveled?

Review Problems The problems below review material we covered in Section 1.5.

17. Expand and simplify $(\sin\theta - \cos\theta)^2$

18. Add $\sin\theta + \dfrac{1}{\cos\theta}$

Show that each of the following statements is true by transforming the left side of each one into the right side.

19. $\sin\theta \cot\theta = \cos\theta$

20. $\cos\theta \csc\theta \tan\theta = 1$

21. $\dfrac{\sec\theta}{\tan\theta} = \csc\theta$

22. $(1 - \cos\theta)(1 + \cos\theta) = \sin^2\theta$

2.5 Vectors

Many of the quantities that describe the world around us have both magnitude and direction, while others have only magnitude. Quantities that have magnitude and direction are called *vector quantities*, while quantities with magnitude only are called *scalars*. Some examples of vector quantities are force, velocity, and acceleration. For example, a car traveling 50 miles per hour due south has a different velocity then another car traveling due north at 50 miles per hour, while a third car traveling at 25 miles per hour due north has a velocity that is different from either of the first two.

One way to represent vector quantities is with arrows. The direction of the arrow represents the direction of the vector quantity and the length of the arrow corresponds to the magnitude. For example, the velocities of the three cars we mentioned above could be represented like this

Figure 1

Notation To distinguish between vectors and scalars we will write the letters used to represent vectors with bold face type, like **U** or **V**. (When you write them on paper, put an arrow above them like this: $\overrightarrow{U}$ or $\overrightarrow{V}$.) The magnitude of a vector is represented with absolute value symbols. For example, the magnitude of **V** is written $|\mathbf{V}|$. Table 1 illustrates further.

Table 1

Notation	The Quantity Is
V	a vector
$\vec{V}$	a vector
$\overrightarrow{AB}$	a vector
x	a scalar
$\lvert \mathbf{V} \rvert$	the magnitude of vector **V**, a scalar

Equality for Vectors

The position of a vector in space is unimportant. Two vectors are equivalent if they have the same magnitude and direction.

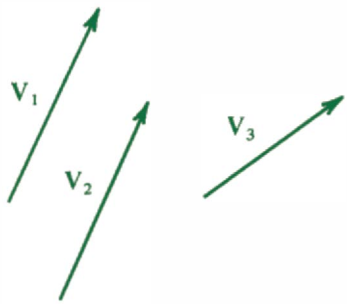


Figure 2

In Figure 2, $\mathbf{V}_1 = \mathbf{V}_2 \neq \mathbf{V}_3$. The vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ are equivalent because they have the same magnitude and the same direction.

Addition of Vectors

The sum of the vectors **U** and **V**, written **U** + **V** and sometimes called the ***resultant vector***, is the vector that extends from the tail of **U** to the tip of **V** when the tail of **V** coincides with the tip of **U**. Figure 3 illustrates.

Figure 3

The vector sum **U** + **V** can also be represented by the diagonal of the parallelogram, the adjacent sides of which are formed by putting the tails of **U** and **V** together (Figure 4).

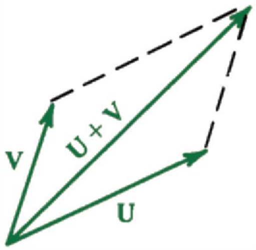
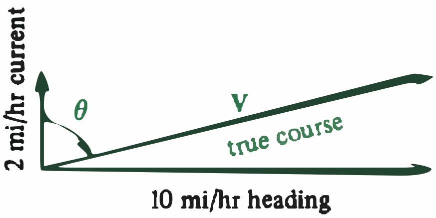


Figure 4

Example 1 illustrates how vectors can be used to solve motion problems.

Example 1 A boat is crossing a river that runs due north. The boat is pointed due east and is moving through the water at 10.0 miles per hour. If the current of the river is a constant 2.0 miles per hour, find the true course of the boat through the water to three significant digits.

Solution Problems like this are a little difficult to read the first time they are encountered. Even though the boat is "headed" due east, the current is pushing it a little toward the north, so it is actually on a course that will take it east and a little north. By representing the heading of the boat and the current of the water with vectors, we can find the true course of the boat.



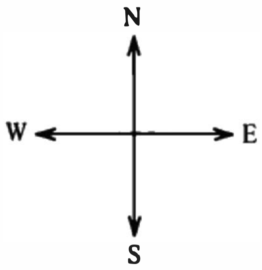


Figure 5

We find θ using a tangent ratio

$$\tan\theta = \frac{10}{2} = 5$$

$$\text{so} \quad \theta = 78.7° \text{ to the nearest tenth}$$

If we let **V** represent the true course of the boat, then we can find the magnitude of **V** using the Pythagorean theorem or a trigonometric ratio. Using the sine ratio, we have

$$\sin\theta = \frac{10}{|\mathbf{V}|}$$

$$|\mathbf{V}| = \frac{10}{\sin\theta}$$

$$= \frac{10}{\sin 78.7°}$$

$$= 10.2$$

The true course of the boat is 10.2 miles per hour at N 78.7° E. That is, the vector **V**, which represents the motion of the boat with respect to the banks of the river, has magnitude of 10.2 miles per hour and a direction N 78.7° E. ▲

Many times it is convenient to write vectors in terms of their horizontal and vertical components. To do so, we first superimpose a coordinate system on the vector in question so that the tail of the vector is at the origin. Figure 6 shows a vector with magnitude 10 making an angle of 52° with the horizontal.

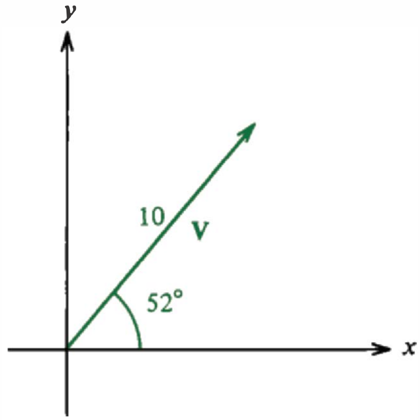


Figure 6

The horizontal and vertical components of vector **V** in Figure 6 are the horizontal and vertical vectors whose sum is **V**. They are shown in Figure 7.

Note that in Figure 7, we labeled the horizontal component of **V** as $\mathbf{V}_x$ and

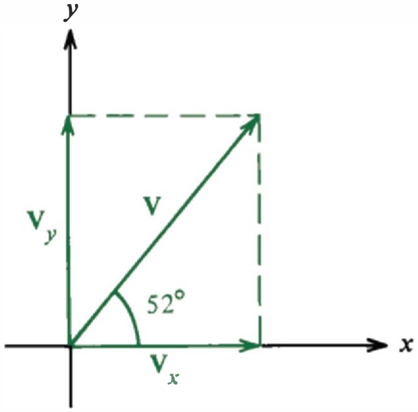


Figure 7

the vertical component as $\mathbf{V}_y$. We can find the magnitudes of the components by using sine and cosine ratios.

$$|\mathbf{V}_x| = |\mathbf{V}| \cos 52° = 10(0.6157) = 6.2 \text{ to the nearest tenth}$$
$$|\mathbf{V}_y| = |\mathbf{V}| \sin 52° = 10(0.7880) = 7.9 \text{ to the nearest tenth}$$

▼ **Example 2** A bullet is fired into the air with an initial velocity of 1500 feet per second at an angle of 30° from the horizontal. Find the horizontal and vertical components of the velocity vector.

Solution Figure 8 is a diagram of the situation.

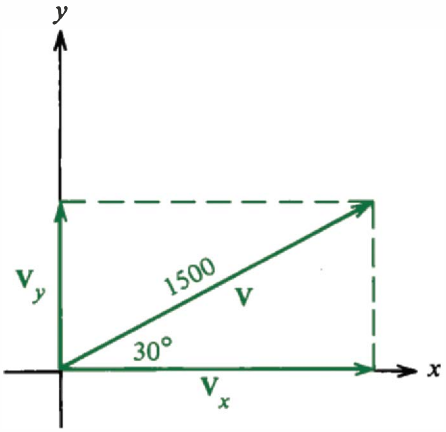


Figure 8

The magnitudes of $\mathbf{V}_x$ and $\mathbf{V}_y$ from Figure 8 to two significant digits are as follows:

$$|\mathbf{V}_x| = 1500 \cos 30° = 1300 \text{ feet per second}$$
$$|\mathbf{V}_y| = 1500 \sin 30° = 750 \text{ feet per second}$$

The bullet has a horizontal velocity of 1300 feet per second and a vertical velocity of 750 feet per second. ▲

The magnitude of a vector can be written in terms of the magnitude of its horizontal and vertical components.

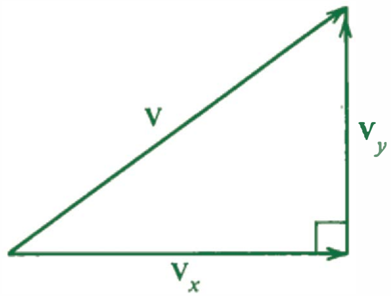


Figure 9

By the Pythagorean theorem we have,

$$|\mathbf{V}| = \sqrt{|\mathbf{V}_x|^2 + |\mathbf{V}_y|^2}$$

▼ **Example 3** An arrow is shot into the air so that its horizontal velocity is 25 feet per second and its vertical velocity is 15 feet per second. Find the velocity of the arrow.

Solution Figure 10 shows the velocity vector and its components along with the angle of inclination of the velocity vector.

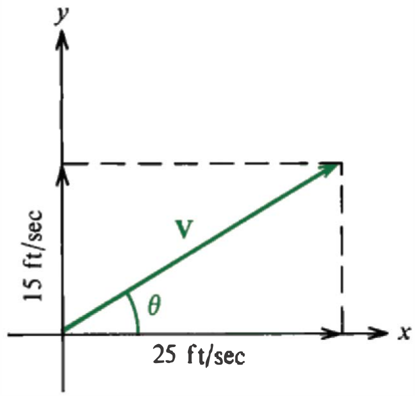


Figure 10

The magnitude of the velocity is given by

$$|\mathbf{V}| = \sqrt{25^2 + 15^2}$$
$$= 29 \text{ feet per second to the nearest whole number}$$

We can find the angle of inclination using a tangent ratio

$$\tan\theta = \frac{|\mathbf{V}_y|}{|\mathbf{V}_x|}$$
$$= \frac{15}{25}$$
$$\tan\theta = 0.6$$
$$\text{so} \qquad \theta = 31° \text{ to the nearest degree}$$

The arrow was shot into the air at 29 feet per second at an angle of inclination of 31°. ▲

Another way to specify the bearing of a moving object is by giving the angle between the north-south line and the vector representing the velocity of the object. Figure 11 shows two vectors that represent the velocities of two ships, one traveling at 14 miles per hour with bearing 120° and another traveling at 12 miles per hour with a bearing of 240°. Note that this method of specifying bearing is different than the method we used previous to this.

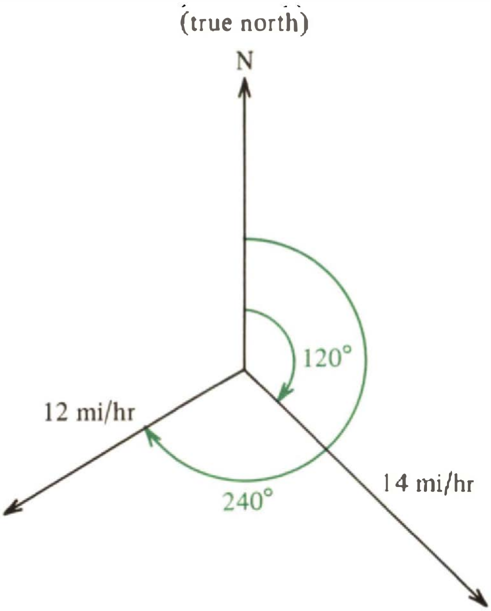


Figure 11

With this method, every course is given with respect to the north-south line, and the bearing angle is always measured clockwise from due north. This new method of giving the direction of a moving object is a little simpler. Under our previous method, we would specify the direction of the ship traveling at 14 miles per hour as S 60° E.

▼ **Example 4** A boat travels for 72 miles on a course with bearing 27° and then changes its course to travel 37 miles on a course with bearing 55°. How far north and how far south has the boat traveled on this 109 mile trip?

Solution We can solve this problem by representing each part of the trip with a vector and then writing each vector in terms of its horizontal and vertical components. Figure 12 shows the vectors that represent the two parts of the trip, along with their horizontal and vertical components.

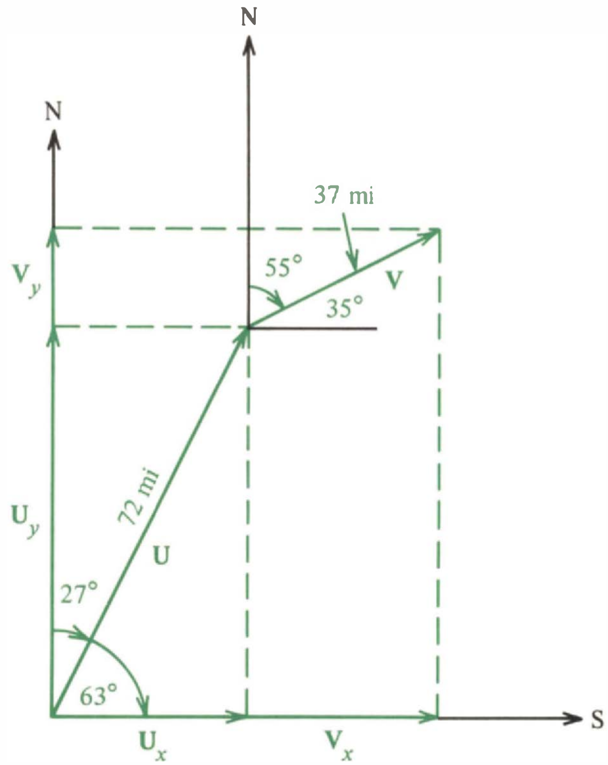


Figure 12

As Figure 12 indicates, the total distance traveled south is given by the sum of the horizontal components, while the total distance traveled north is given by the sum of the vertical components.

$$\text{Total distance traveled south} = |\mathbf{U}_x| + |\mathbf{V}_x|$$
$$= 72 \cos 63° + 37 \cos 35°$$
$$= 63.0 \text{ miles to the nearest tenth}$$

$$\text{Total distance traveled north} = |\mathbf{U}_y| + |\mathbf{V}_y|$$
$$= 72 \sin 63° + 37 \sin 35°$$
$$= 85.4 \text{ miles to the nearest tenth}$$

▲

Problem Set 2.5

Draw vectors representing the following velocities:

1. 30 miles per hour due north
2. 30 miles per hour due south
3. 30 miles per hour due east
4. 30 miles per hour due west
5. 50 centimeters per second N 30° W
6. 50 centimeters per second N 30° E
7. 20 feet per minute S 60° E
8. 20 feet per minute S 60° W

Draw vectors representing the course of a ship that travels

9. 75 miles on a course with bearing 30°
10. 75 miles on a course with bearing 330°
11. 25 miles on a course with bearing 135°
12. 25 miles on a course with bearing 225°

13. A boat is crossing a river that runs due north. The heading of the boat (the direction the boat is pointed) is due east and it is moving through the water at 12.0 miles per hour. If the current of the river is a constant 3 miles per hour, find the true course of the boat to three significant digits.
14. A boat is crossing a river that runs due east. The heading of the boat is due south and its speed is 11.0 feet per second. If the current of the river is 2 feet per second, find the true course of the boat to three significant digits.
15. An airplane is headed N 30.0° E and is traveling at 200 miles per hour through the air. The air currents are moving at a constant 30 miles per hour in the direction N 60.0° W. Find the true course of the plane. (That is, find its speed with respect to the ground in miles per hour and its direction with respect to the ground.) Use three significant digits.
16. An airplane is headed S 50.0° E and is moving through the air at 140 miles per hour. The air currents are moving at a constant 24 miles per hour in the direction S 40.0° W. Find the true course of the plane. Use three significant digits.
17. A ship headed due west is moving through the water at a constant 12 miles per hour. However, the true course of the ship is in the direction N 78° W. If the

current of the water is running due north at a constant rate of speed, find the speed of the current.

18. A plane headed due east is moving through the air at a constant 180 miles per hour. Its true course, however, is in the direction N 65.0° E. If the wind currents are moving due north at a constant rate, find the speed of these currents.

Each problem below refers to a vector **V** with magnitude $|\mathbf{V}|$ that forms an angle θ with the positive x-axis. In each case, give the magnitude of the horizontal and vertical components of **V**.

19. $|\mathbf{V}| = 40, \theta = 30°$
20. $|\mathbf{V}| = 40, \theta = 60°$
21. $|\mathbf{V}| = 13.8, \theta = 24.2°$
22. $|\mathbf{V}| = 17.6, \theta = 67.2°$
23. $|\mathbf{V}| = 420, \theta = 36° 10'$
24. $|\mathbf{V}| = 380, \theta = 16° 40'$
25. $|\mathbf{V}| = 64, \theta = 150°$
26. $|\mathbf{V}| = 48, \theta = 120°$

For each problem below, the magnitude of the horizontal and vertical components of vector **V** are given. In each case find the magnitude of **V**.

27. $|\mathbf{V}_x| = 30, |\mathbf{V}_y| = 40$
28. $|\mathbf{V}_x| = 8, |\mathbf{V}_y| = 6$
29. $|\mathbf{V}_x| = 35.0, |\mathbf{V}_y| = 26.0$
30. $|\mathbf{V}_x| = 45.0, |\mathbf{V}_y| = 15.0$
31. $|\mathbf{V}_x| = 4.5, |\mathbf{V}_y| = 3.8$
32. $|\mathbf{V}_x| = 2.2, |\mathbf{V}_y| = 5.8$

33. A bullet is fired into the air with an initial velocity of 1,200 feet per second at an angle of 45° from the horizontal. Find the magnitude of the horizontal and vertical components of the velocity vector.

34. A bullet is fired into the air with an initial velocity of 1,800 feet per second at an angle of 60° from the horizontal. Find the horizontal and vertical components of the velocity.

35. Use the results of Problem 33 to find the horizontal distance traveled by the bullet in 3 seconds. (Neglect the resistance of air on the bullet.)

36. Use the results of Problem 34 to find the horizontal distance traveled by the bullet in 2 seconds.

37. An arrow is shot into the air so that its horizontal velocity is 35.0 feet per second and its vertical velocity is 15.0 feet per second. Find the velocity of the arrow.

38. The horizontal and vertical components of the velocity of an arrow shot into the air are 15.0 feet per second and 25.0 feet per second, respectively. Find the velocity of the arrow.

39. A ship travels for 135 kilometers on a course with bearing 138°. How far east and how far south has it traveled?

40. A plane flies for 3 hours at 230 kilometers per hour on a course with bearing 215°. How far west and how far south does it travel in the 3 hours?

41. A plane travels for 175 miles on a course with bearing 18° and then changes its course to 49° and travels another 120 miles. Find the total distance traveled north and the total distance traveled east.

42. A ship travels on a course with bearing 168° for 68 miles and then changes its course to 120° and travels for another 112 miles. Find the total distance south and the total distance east that the ship traveled.

Review Problems The problems that follow review material we covered in Section 1.3.

43. Draw 135° in standard position, locate a convenient point on the terminal side, and then find sin 135°, cos 135°, and tan 135°.

44. Draw −270° in standard position, locate a convenient point on the terminal side, and then find sine, cosine, and tangent of −270°.

45. Find sin θ and cos θ if the terminal side of θ lies along the line $y = 2x$ in quadrant I.

46. Find sin θ and cos θ if the terminal side of θ lies along the line $y = -x$ in quadrant II.

47. Find x if the point $(x, -8)$ is on the terminal side of θ and $\sin\theta = -4/5$.

48. Find y if the point $(-6, y)$ is on the terminal side of θ and $\cos\theta = -3/5$.

Chapter 2 Summary and Review

DEFINITION II FOR TRIGONOMETRIC FUNCTIONS [2.1]

If triangle ABC is a right triangle with $C = 90°$, then the six trigonometric functions for angle A are

$$\sin A = \frac{\text{side opposite } A}{\text{hypotenuse}} = \frac{a}{c}$$

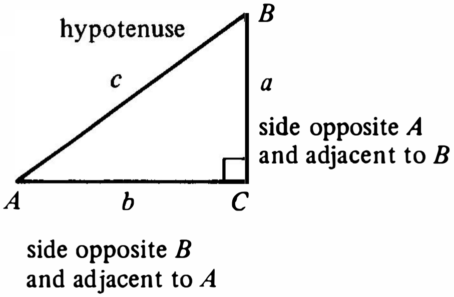
$$\cos A = \frac{\text{side adjacent } A}{\text{hypotenuse}} = \frac{b}{c}$$

$$\tan A = \frac{\text{side opposite } A}{\text{side adjacent } A} = \frac{a}{b}$$

$$\cot A = \frac{\text{side adjacent } A}{\text{side opposite } A} = \frac{b}{a}$$

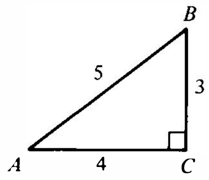
$$\sec A = \frac{\text{hypotenuse}}{\text{side adjacent } A} = \frac{c}{b}$$

$$\csc A = \frac{\text{hypotenuse}}{\text{side opposite } A} = \frac{c}{a}$$



Examples

1.



$$\sin A = \frac{3}{5} = \cos B$$

$$\cos A = \frac{4}{5} = \sin B$$

$$\tan A = \frac{3}{4} = \cot B$$

$$\cot A = \frac{4}{3} = \tan B$$

$$\sec A = \frac{5}{4} = \csc B$$

$$\csc A = \frac{5}{3} = \sec B$$

COFUNCTION THEOREM [2.1]

A trigonometric function of an angle is always equal to the cofunction of its complement. In symbols, since the complement of x is $90° - x$, we have

$$\sin x = \cos(90° - x)$$
$$\cos x = \sin(90° - x)$$
$$\tan x = \cot(90° - x)$$

2. $\sin 3° = \cos 87°$
$\cos 10° = \sin 80°$
$\tan 15° = \cot 75°$
$\cot A = \tan(90° - A)$
$\sec 30° = \csc 60°$
$\csc 45° = \sec 45°$

TRIGONOMETRIC FUNCTIONS OF SPECIAL ANGLES [2.1]

θ	$30°$	$45°$	$60°$
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$ or $\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

3. The values given in the table are called *exact values* because they are not decimal approximations as you would find in Tables II or III.

DEGREES, MINUTES, AND SECONDS [2.2]

There are 360° (degrees) in one complete rotation, 60′ (minutes) in one degree, and 60″ (seconds) in one minute. This is equivalent to saying 1 minute is 1/60 of a degree, and 1 second is 1/60 of a minute.

4.
$$\begin{array}{r} 47°\ 30' \\ +\ 23°\ 50' \\ \hline 70°\ 80' \end{array} = 71°\ 20'$$

CONVERTING TO AND FROM DECIMAL DEGREES [2.2]

To convert from decimal degrees to degrees and minutes, multiply the fractional part of the angle (that which follows the decimal point) by 60 to get minutes.

To convert from degrees and minutes to decimal degrees, divide minutes by 60 to get the fractional part of the angle.

5.
$$\begin{aligned} 74.3° &= 74° + 0.3° \\ &= 70° + 0.3(60') \\ &= 70° + 18' \\ &= 70°\ 18' \end{aligned}$$

$$\begin{aligned} 42°\ 48' &= 42° + \frac{48°}{60} \\ &= 42° + 0.8° \\ &= 42.8° \end{aligned}$$

TABLES OF TRIGONOMETRIC FUNCTIONS [2.2]

Table II gives the values of trigonometric functions of angles written in decimal degrees.

Table III gives the values of trigonometric functions of angles written in degrees and minutes.

When reading either table, notice that the angles in the left-hand column correspond to the headings across the top of the table, and the angles in the right-hand column correspond to the headings across the bottom of the table.

6. From Table II
$\sin 24.8° = 0.4195$
From Table III
$\sin 48° 20' = 0.7470$
From Table II
If $\sin \theta = 0.9755$,
then $\theta = 77.3°$
From Table III
If $\sin \theta = 0.9881$,
then $\theta = 81° 10'$

SIGNIFICANT DIGITS [2.3]

The number of *significant digits* (or figures) in a number that has digits to the right of the decimal point is found by counting the number of digits from left to right, beginning with the first non-zero digit on the left and disregarding the decimal point. If the number is an integer with no digits to the right of the decimal point, we use the same process but disregard all ending zeros unless we have other information.

The relationship between the accuracy of the sides in a triangle and the accuracy of the angles in the same triangle is given below.

Accuracy of Sides	Accuracy of Angles
Two significant digits	Nearest degree
Three significant digits	Nearest 10 minutes or tenth of a degree
Four significant digits	Nearest minute or hundredth of a degree

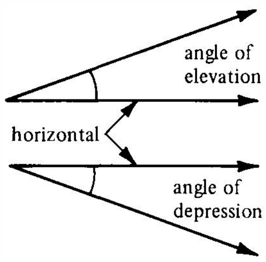
7. These angles and sides correspond in accuracy.

$a = 24$	$A = 39°$
$a = 5.8$	$A = 45°$
$a = 62.3$	$A = 31.3°$
$a = 0.498$	$A = 42.9°$
$a = 2.77$	$A = 37° 10'$
$a = 49.87$	$A = 43° 18'$
$a = 6.932$	$A = 24.81°$

ANGLE OF ELEVATION AND ANGLE OF DEPRESSION [2.4]

An angle measured from the horizontal up is called an *angle of elevation*. An angle measured from the horizontal down is called an *angle of depression*.

8.



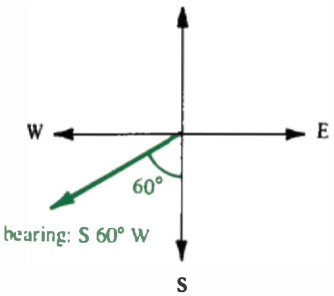
BEARING [2.4]

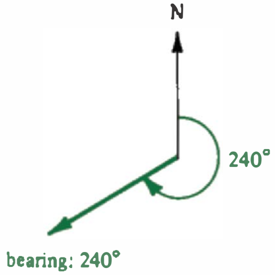
There are two ways to specify the *bearing* of a line or vector.

a. We can specify the bearing of a line by giving the acute angle formed by the north-south line and the line. The notation used to designate the bearing in this way begins with N or S, followed by the number of degrees in the angle, and ends with E or W. As in S 60° W. This type of bearing is used mainly to give the bearing of one object in relation to another.

b. We can also give the bearing of a line or vector by simply stating the angle through which the line or vector has been rotated clockwise from due north, as in a bearing of 240°. This type of bearing is useful in giving the direction of a moving object.

9.

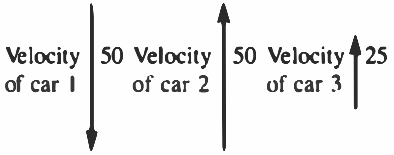




VECTORS [2.5]

Quantities that have both magnitude and direction are called *vector quantities*, while quantities that have only magnitude are called *scalar quantities*. We represent vectors graphically by using arrows. The length of the arrow corresponds to the magnitude of the vector, and the direction of the arrow corresponds to the direction of the vector. In symbols, we denote the magnitude of vector **V** with $|\mathbf{V}|$.

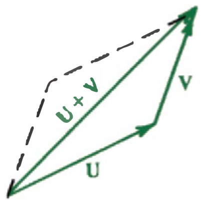
10. If car 1 is traveling at 50 miles per hour due south, car 2 at 50 miles per hour due north, and car 3 at 25 miles per hour due north, then the velocities of the three cars can be represented with vectors.



ADDITION OF VECTORS [2.5]

The sum of the vectors **U** and **V**, written **U** + **V**, is the vector that extends from the tail of **U** to the tip of **V** when the tail of **V** coincides with the tip of **U**.

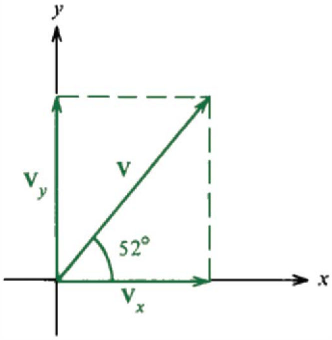
11.



HORIZONTAL AND VERTICAL COMPONENTS OF A VECTOR [2.5]

The horizontal and vertical components of vector **V** are the horizontal and vertical vectors whose sum is **V**. The horizontal component is denoted by $\mathbf{V}_x$, and the vertical component is denoted by $\mathbf{V}_y$.

12.



Chapter 2 Test

Find sin *A*, cos *A*, tan *A*, and sin *B*, cos *B*, and tan *B* in right triangle *ABC*, with $C = 90°$, if

1. $a = 1$ and $b = 2$
2. $b = 3$ and $c = 6$
3. $a = 3$ and $c = 5$
4. $a = 5$ and $b = 12$

Fill in the blanks to make each statement true.

5. sin 14° = cos _____
6. sec _____ = csc 73°

Simplify each expression as much as possible. Do not use tables.

7. $\sin^2 45° + \cos^2 30°$
8. $\tan 45° + \cot 45°$
9. $\sin^2 60° - \cos^2 30°$
10. $\dfrac{1}{\sec 30°}$

11. Add 48° 18′ and 24° 52′.
12. Subtract 15° 32′ from 25° 15′.

Convert to degrees and minutes.

13. 73.2°
14. 16.45°

Convert to decimal degrees.

15. 2° 48′
16. 79° 30′

Use the tables at the back of the book to find the following:

17. sin 24° 20′
18. cos 37.8°
19. tan 63° 50′
20. cot 71° 20′

Use the tables at the back of the book to find θ if

21. $\tan\theta = 0.0816$

22. $\sec\theta = 1.923$

23. $\sin\theta = 0.9465$

24. $\cos\theta = 0.9730$

Give the number of significant digits in each number.

25. 49.35

26. 0.0028

The following problems refer to right triangle *ABC* with $C = 90°$. In each case, find all the missing parts. (Use Table **II** for problems 27 through 28.)

27. $a = 68.0$ and $b = 104$

28. $a = 24.3$ and $c = 48.1$

29. $b = 305$ and $B = 24.9°$

30. $c = 0.462$ and $A = 35° 30'$

31. If the altitude of an isosceles triangle is 25 centimeters and each of the two equal angles measures 17°, how long are the two equal sides?

32. If the angle of elevation of the sun is 75° 30′, how tall is a post that casts a shadow 1.5 feet long?

33. From a 7 foot lifeguard tower located on the long side of a pool, a lifeguard notices the angles of depression to the ends of the pool are 31° and 4°. How long is the pool? Give your answer to the nearest foot.

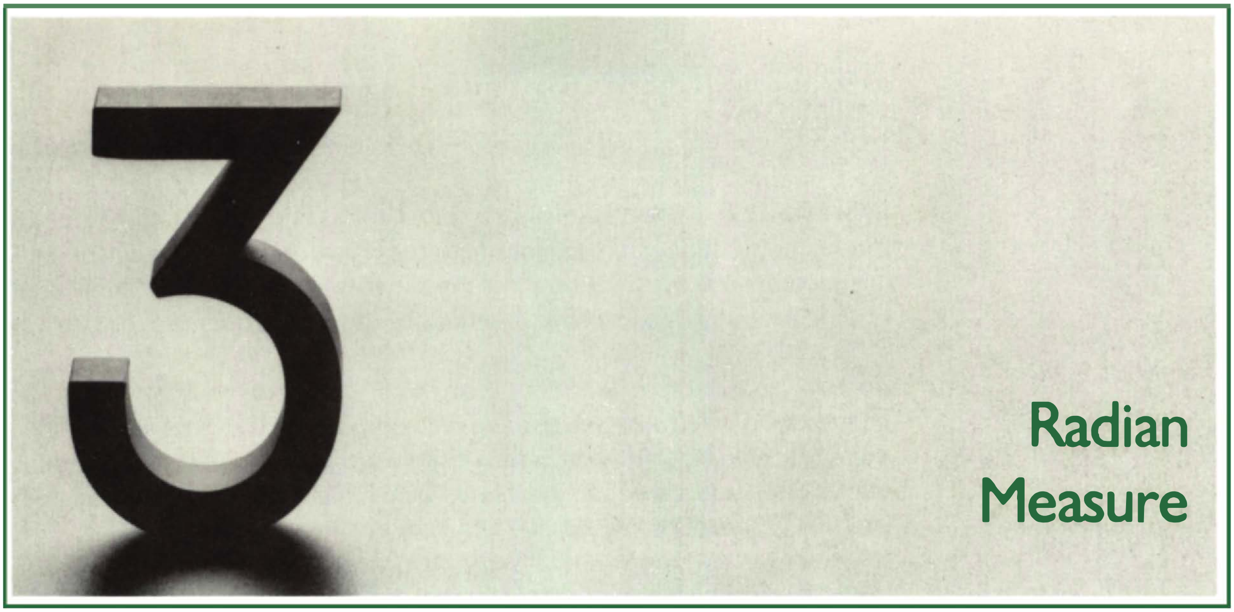
34. A boat travels on a course of bearing S 48° 50′ W for 128 miles. How far south and how far west has the boat traveled?

35. If vector **V** has magnitude 5.0 and makes an angle of 30° with the positive x-axis, find the magnitude of the horizontal and vertical components of **V**.

36. Vector **V** has a horizontal component with magnitude 1 and a vertical component with magnitude 3. What is the angle formed by **V** and the positive x-axis?

37. A bullet is fired into the air with an initial velocity of 800 feet per second at an angle of 60° from the horizontal. Find the magnitude of the horizontal and vertical components of the velocity vector.

38. A ship travels for 120 miles on a course with bearing 120°. How far east and how far south has the ship traveled?



3 Radian Measure

To the student:

In Chapters 1 and 2 we used degree measure exclusively to give the measure of angles. We will begin this chapter with another kind of angle measure called radian measure. Radian measure gives us a way to measure angles with real numbers instead of degrees. As you will see, there are a number of situations that occur in trigonometry for which real numbers are the more appropriate measure for angles.

In Section 3.2, we will show how to convert back and forth between degrees and radians. Once we are able to do these conversions, we can apply all the material we have developed for degrees to radians.

There are some interesting relationships that exist between the trigonometric functions we have defined previously and the points on the unit circle. We cover these relationships and some material on even and odd functions in Section 3.3.

In the last two sections of this chapter, we will apply the material from the first part of the chapter to circles in general. We will begin by deriving formulas for arc length and area and then proceed on to define angular velocity for points in motion on the circumference of a circle.

The most important material in the chapter, from the standpoint of what is needed to continue on through the book, is the material in the first three sections. These are the sections in which you will be introduced to, and become familiar with, radian measure. In Chapter 4, when we graph the different trigonometric functions, we will work almost exclusively in radian measure.

3.1 Reference Angle

In Section 2.2, we used tables or a calculator to find approximate values for trigonometric functions of angles between 0° and 90°. We are going to begin this section with a look at how we can use these same tables to find approximate values for trigonometric functions of *any* angle, not just those between 0° and 90°.

Definition The *reference angle* (sometimes called related angle) for any angle θ in standard position is the positive acute angle between the terminal side of θ and the x-axis. In this book, we will denote the reference angle for θ by $\hat{\theta}$.

Note that, for this definition, $\hat{\theta}$ is always positive and always between 0° and 90°. That is, a reference angle is always an acute angle.

▼ **Example 1** Name the reference angle for each of the following angles.

a. 30° **b.** 135° **c.** 240° **d.** 330°

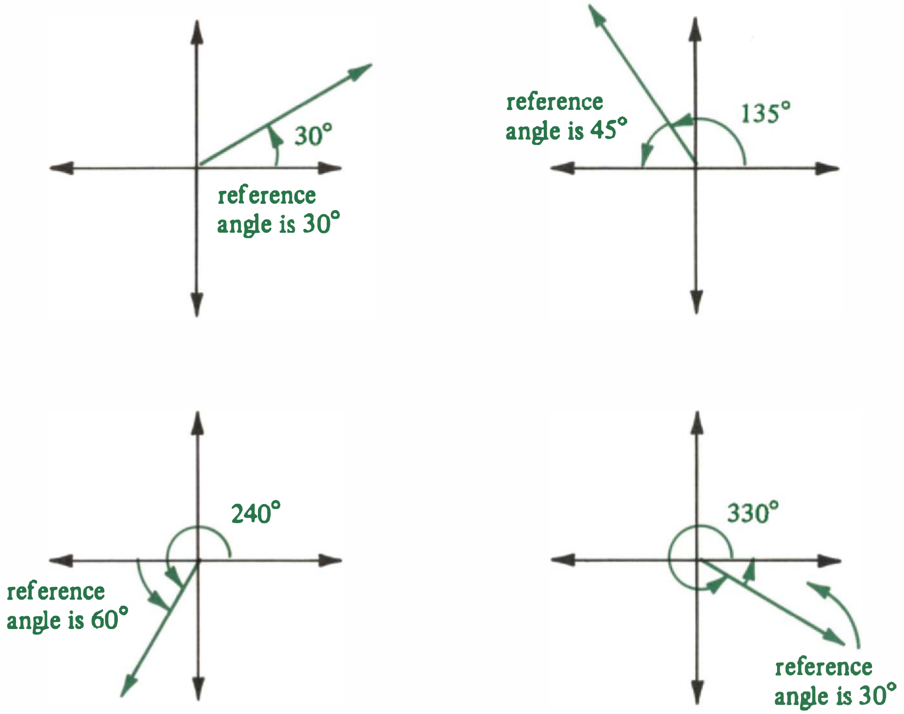


Figure 1

Solution We draw each angle in standard position. The reference angle is the positive acute angle formed by the terminal side of the angle in question and the x-axis. ▲

We can generalize the results of Example 1 as follows: If θ is a positive angle between 0° and 360°, then

$$\text{If } \theta \in \text{QI, then } \hat{\theta} = \theta$$
$$\text{If } \theta \in \text{QII, then } \hat{\theta} = 180^\circ - \theta$$
$$\text{If } \theta \in \text{QIII, then } \hat{\theta} = \theta - 180^\circ$$
$$\text{If } \theta \in \text{QIV, then } \hat{\theta} = 360^\circ - \theta$$

Trigonometric Functions of Angles Between 0° and 360°

We can use our information on reference angles and the signs of the trigonometric functions, to write the following theorem.

Reference Angle Theorem A trigonometric function of an angle and its reference angle differ at most in sign.

We will not give a detailed proof of this theorem, but rather, justify it by example. Let's look at the sines of all the angles between 0° and 360° that have a reference angle of 30°. These angles are 30°, 150°, 210°, and 330°.

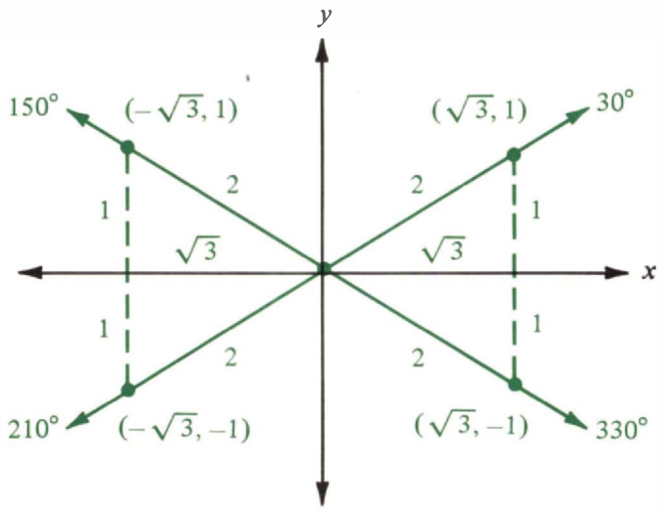


Figure 2

$$\sin 150^\circ = \sin 30^\circ = \frac{1}{2}$$

$$\sin 210^\circ = \sin 330^\circ = -\frac{1}{2}$$

They differ in sign only.

As you can see, any angle with a reference angle of 30° will have a sine of 1/2 or −1/2. The sign, + or −, will depend on the quadrant in which the angle terminates. Using this discussion as justification, we write the following steps used to find trigonometric functions of angles between 0° and 360°.

Step 1. Find $\hat{\theta}$, the reference angle.
Step 2. Determine the sign of the trigonometric function based on the quadrant in which θ terminates.
Step 3. Write the original trigonometric function of θ in terms of the same trigonometric function of $\hat{\theta}$.
Step 4. Use a table or calculator to find the trigonometric function of $\hat{\theta}$.

▼ **Example 2** Find sin 205° 30′.

Solution For this first example, we will list the steps given above as we use them. Figure 3 is a diagram of the situation.

Step 1. We find $\hat{\theta}$ by subtracting 180° from θ.

$$205°\ 30' - 180° = 25°\ 30'$$

Step 2. Since θ terminates in quadrant III, and the sine function is negative in quadrant III, our answer will be negative. That is, in this case, $\sin\theta = -\sin\hat{\theta}$.

Step 3. Using the results of Steps 1 and 2 we write

$$\sin 205°\ 30' = -\sin 25°\ 30'$$

Step 4. Using a calculator or table we finish by finding sin 25° 30′

$$\sin 205°\ 30' = -0.4305$$

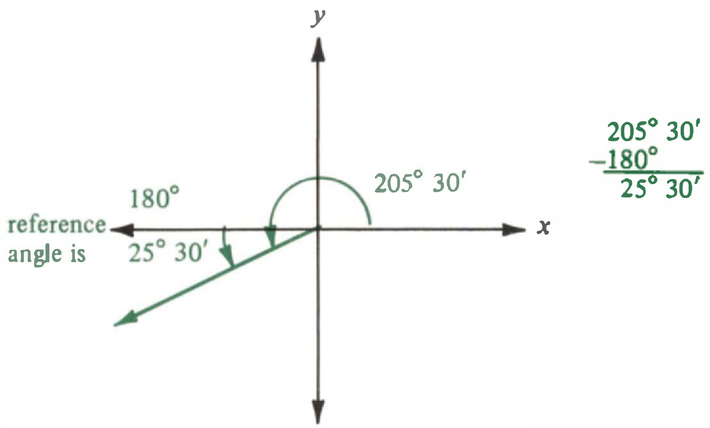


Figure 3 ▲

▼ **Example 3** Find tan 318.4°.

Solution The reference angle is $360° - 318.4° = 41.6°$. Since 318.4° terminates in quadrant IV, its tangent will be negative.

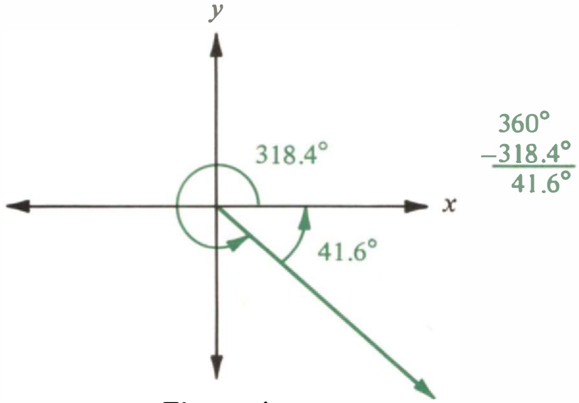


Figure 4

$\tan 318.4° = -\tan 41.6°$ Because tangent is negative in QIV

$= -0.8878$ ▲

Calculator Note Although the answers to Examples 2 and 3 can be found on a calculator simply by entering the angle and then pressing the appropriate function key, it is a good idea to follow the steps shown instead. The concept of reference angle is an important one and we need to practice working with it. There will be topics covered later in the book in which you will need to understand the relationship between an angle and its reference angle.

Angles Larger Than 360° or Smaller Than 0°

Recall from Section 1.2 that coterminal angles always differ from each other by multiples of 360°. For example, 10° and 370° are coterminal, as are −45° and 315°.

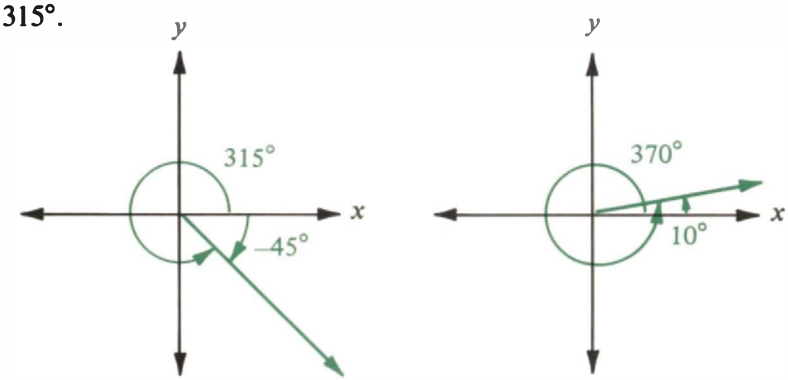


Figure 5

The trigonometric functions of an angle and any angle coterminal to it are always equal. For sine and cosine, we can write this in symbols as follows:

for any integer k,
$\sin(\theta + 360°k) = \sin\theta$ and $\cos(\theta + 360°k) = \cos\theta$

To find values of trigonometric functions for an angle larger than 360° or smaller than 0°, we simply find an angle between 0° and 360° that is coterminal to it. and then use the steps outlined in Examples 2 and 3.

Example 4 Find cos 500°.

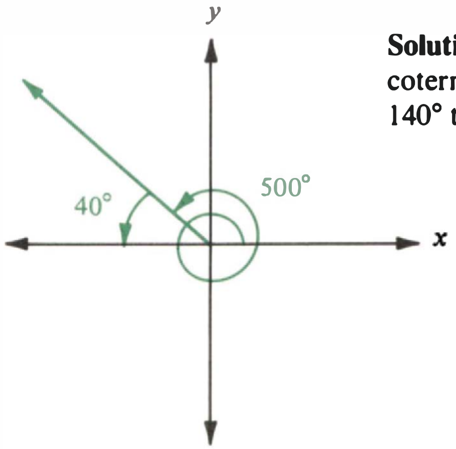


Figure 6

Solution By subtracting 360° from 500°, we obtain 140°, which is coterminal to 500°. The reference angle for 140° is 40°. Since 500° and 140° terminate in quadrant II, their cosine is negative.

$\cos 500°$	$= \cos 140°$	500° and 140° are coterminal
	$= -\cos 40°$	In QII $\cos\theta = -\cos\hat{\theta}$
	$= -0.7660$	Tables or calculator

Example 5 Find tan(−135°).

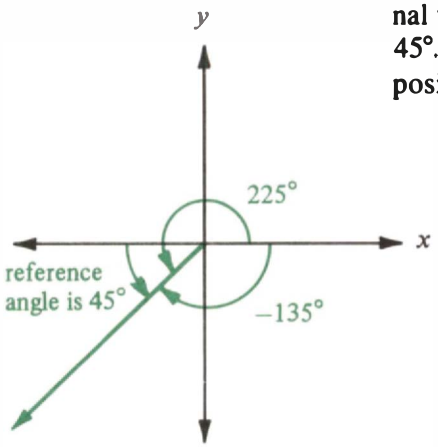


Figure 7

Solution By adding 360° to −135° we obtain 225°, which is coterminal to −135° and between 0° and 360°. The reference angle for 225° is 45°. Since −135° and 225° terminate in quadrant III, their tangent is positive.

$\tan(-135°)$	$= \tan 225°$	Coterminal angles
	$= \tan 45°$	In QIII $\tan\theta = \tan\hat{\theta}$
	$= 1$	Tables or calculator

Example 6 Find csc 750° 30′.

Solution The reference angle for 750° 30′ is 30° 30′, which is obtained by subtracting $2 \cdot 360° = 720°$ from 750° 30′. Since 750° 30′ terminates in quadrant I, its cosecant is positive.

$$\begin{aligned} \csc 750°\,30' &= \csc 30°\,30' \\ &= 1.970 \end{aligned}$$

Calculator Note To find csc 30° 30′ on a calculator, we first change to decimal degrees, and then use the fact that cosecant and sine are reciprocals.

30.5 [sin] [1/*x*]

Example 7 Find θ if $\sin\theta = -0.5592$ and θ terminates in QIII with $0° < \theta < 360°$.

Solution In this example, we must use our tables in the reverse direction from the way we used them in the previous examples. We look in the sine column until we find 0.5592. Reading across we find that the angle whose sine is 0.5592 is 34°. This is our reference angle, $\hat{\theta}$. The angle in quadrant III whose reference angle is 34° is $\theta = 180° + 34° = 214°$.

$$\begin{aligned} \text{If } \sin\theta &= -0.5592 \text{ and } \theta \text{ terminates in QIII,} \\ \text{then } \theta &= 180° + 34° \\ &= 214° \end{aligned}$$

If we wanted to list *all* the angles that terminate in quadrant III and have a sine of −0.5592, we would write

$$\theta = 214° + 360°k \qquad \text{where } k = \text{an integer.}$$

This gives us all angles coterminal with 214°.

Calculator Note If you were to try Example 7 on your calculator by simply displaying −0.5592 and then pressing the $\sin^{-1}$ key, you would not obtain 214° for your answer. Instead, you would get approximately −34° for the answer, which is wrong. To see why this happens you will have to wait until we cover inverse trigonometric functions. In the meantime, if you want to use a calculator on this kind of problem, use it to find the reference angle and then proceed as we did in Example 7. That is, you would display 0.5592 and then press $\sin^{-1}$ to obtain approximately 34°, to which you would add 180°, since, in this case, we know the angle terminates in quadrant III.

Example 8 Find θ to the nearest 10 minutes if $\tan\theta = -0.8541$ and θ terminates in QIV with $0° < \theta < 360°$.

Solution Looking in the column marked tan, we find 0.8541 across from 40° 30′. This is the reference angle $\hat{\theta}$. The angle in QIV with a reference angle of 40° 30′ is

$$\theta = 360° - 40°\,30' = 319°\,30'$$

Again, if we wanted to list *all* angles in quadrant IV with a tangent of −0.8541, we would write

$$\theta = 319°\,30' + 360°k \qquad k = \text{an integer}$$

to include not only 319° 30′ but all angles coterminal with it.

Exact Values

So far, we have worked through all of the examples in this section using tables of approximate values of trigonometric functions. We can use our table of exact values from Section 2.1 to find exact values of angles that are multiples of 30° or 45° (Table 1).

Table 1 Table of Exact Values

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Example 9 Find the exact value of cos 135°.

Solution Since 135° terminates in quadrant II, its cosine is negative. The reference angle is $180° - 135° = 45°$.

$$\cos 135° = -\cos 45°$$
$$= -\frac{1}{\sqrt{2}}$$

Example 10 Find the exact value of csc 300°.

Solution Since 300° terminates in quadrant IV, its cosecant will be negative. The reference angle is 360° − 300° = 60°. To find the exact value of csc 60°, we use the fact that cosecant and sine are reciprocals.

$$\begin{aligned} \csc 300^\circ &= -\csc 60^\circ \\ &= -\frac{1}{\sin 60^\circ} \\ &= -\frac{1}{\sqrt{3}/2} \\ &= -\frac{2}{\sqrt{3}} \end{aligned}$$

▲

▼ **Example 11** Find the exact value of sin 495°.

Solution If we subtract 360° we obtain 495° − 360° = 135°, which terminates in quadrant II with a reference angle of 45°.

$$\begin{aligned} \sin 495^\circ &= \sin 135^\circ \\ &= \sin 45^\circ \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

▲

▼ **Example 12** Find θ if $\sin \theta = -1/2$ and θ terminates in QIII with $0^\circ < \theta < 360^\circ$.

Solution From the table of exact values, we find the reference angle to be 30°. The angle in QIII with a reference angle of 30° is 180° + 30° = 210°. ▲

Problem Set 3.1

Draw each of the following angles in standard position and then name the reference angle:

1. 210° **2.** 150°
3. 143.4° **4.** 253.8°
5. 311.7° **6.** 93.2°
7. 195° 10′ **8.** 171° 40′
9. 331° 20′ **10.** 252° 50′
11. −300° **12.** −330°
13. −120° **14.** −150°

Use Table II or a calculator to find the following. If you use a calculator, use it only to find the trigonometric functions of the reference angle. Remember, we are learning the relationships that exist between an angle, its reference angle, and the trigonometric functions of both. Following the steps in the examples in this section will help us understand these relationships.

15. cos 347°
16. cos 238°
17. cos 101.8°
18. sin 166.7°
19. tan 143.4°
20. tan 253.8°
21. sec 311.7°
22. cos 93.2°
23. cot 390°
24. cot 420°
25. cos 575.4°
26. sin 590.9°

Use Table **III** or a calculator to find the following. (Again, find the reference angle first, even if you are using a calculator. Also, remember, if you use a calculator, you must first convert to decimal degrees before you use the trigonometric function keys.)

27. sin 210°
28. cos 150°
29. cos(−315°)
30. sin(−225°)
31. tan 195° 10′
32. tan 171° 40′
33. sec 314° 40′
34. csc 670° 20′
35. csc 410° 10′
36. sec 380° 50′
37. sin(−120°)
38. cos(−150°)
39. cot(−300° 20′)
40. cot(−330° 30′)

Use either Table **II** or Table **III** or a calculator to find θ, if $0° < \theta < 360°$ and

41. $\sin\theta = -0.3090$ with θ in QIII
42. $\sin\theta = -0.3090$ with θ in QIV
43. $\cos\theta = -0.7660$ with θ in QII
44. $\cos\theta = -0.7660$ with θ in QIII

Find θ in both decimal degrees (to the nearest tenth of a degree) and degrees and minutes (to the nearest ten minutes) if θ is between 0° and 360° and

45. $\tan\theta = 0.5890$ with θ in QIII
46. $\tan\theta = 0.5890$ with θ in QI
47. $\cos\theta = 0.2644$ with θ in QI
48. $\cos\theta = 0.2644$ with θ in QIV
49. $\sin\theta = 0.9652$ with θ in QII
50. $\sin\theta = 0.9652$ with θ in QI

Find exact values for each of the following:

51. sin 120°
52. sin 210°
53. tan 135°
54. tan 315°
55. cos 240°
56. cos 150°
57. csc 330°
58. sec 330°
59. sec 300°
60. csc 300°
61. sin 390°
62. cos 420°
63. cot 480°
64. cot 510°

Find θ, if $0° < \theta < 360°$ and

65. $\sin\theta = -\dfrac{\sqrt{3}}{2}$ and θ in QIII
66. $\sin\theta = -\dfrac{1}{\sqrt{2}}$ and θ in QIII

67. $\cos\theta = -\dfrac{1}{\sqrt{2}}$ and θ in QII

68. $\cos\theta = -\dfrac{\sqrt{3}}{2}$ and θ in QIII

69. $\sin\theta = -\dfrac{\sqrt{3}}{2}$ and θ in QIV

70. $\sin\theta = \dfrac{1}{\sqrt{2}}$ and θ in QII

71. $\tan\theta = \sqrt{3}$ and θ in QIII

72. $\tan\theta = \dfrac{1}{\sqrt{3}}$ and θ in QIII

73. Use a calculator to find θ if $\sec\theta = 2.3931$ and θ is an acute angle. (Give your answer to the nearest tenth of a degree.)

74. Use a calculator to find θ if $\csc\theta = 1.1164$ and θ is an acute angle. (Give your answer to the nearest tenth of a degree.)

75. We know that tan 90° is undefined. If we try to find tan 90° on a calculator, we get an error message. To begin to get an idea of why this happens, we can use a calculator to find the tangent of angles close to 90°. Use a calculator to find $\tan\theta$, if θ takes on values of 85°, 87°, 89°, 89.9°, and 89.99°.

76. Follow the instructions given in Problem 75 but use the secant function instead of the tangent function. (This will require that you use the cos key and the $1/x$ key on your calculator.)

Review Problems The problems that follow review material we covered in Sections 1.1. and 2.1.

Give the complement and supplement of each angle.

77. 70°

78. 120°

79. x

80. $90° - y$

81. If the longest side in a 30°–60°–90° angle is 10, find the length of the other two sides.

82. If the two shorter sides of a 45°–45°–90° triangle are both 3/4, find the length of the hypotenuse.

Simplify each expression by substituting values from the table of exact values and then simplifying the resulting equation.

83. sin 30° cos 60°

84. 4 sin 60° − 2 cos 30°

85. $\sin^2 45° + \cos^2 45°$

86. $(\sin 45° + \cos 45°)^2$

3.2 Radians and Degrees

We begin this section with the definition for the radian measure of an angle. As you will see, specifying the measure of an angle with radian measure gives us a way to associate the measure of an angle with real numbers, rather than degrees. To understand the definition for radian measure, we have to

recall from geometry that a central angle is an angle with its vertex at the center of a circle.

Definition In a circle, a central angle that cuts off an arc equal in length to the radius of the circle has a measure of 1 radian. The following diagram illustrates.

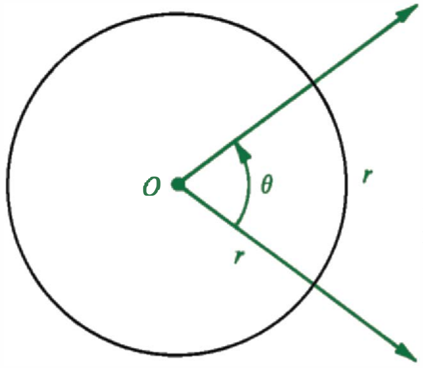


Figure 1

To find the radian measure of any central angle, we have to find out how many radii are in the arc it cuts off. If a central angle θ, in a circle of radius r, cuts off an arc of length s, then the measure of θ, in radians, is given by

$$\theta \text{ (in radians)} = \frac{s}{r}$$

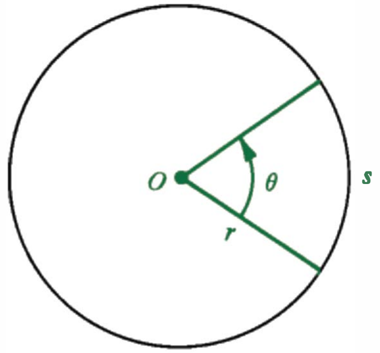


Figure 2

▼ **Example 1** A central angle θ in a circle of radius 3 centimeters cuts off an arc of length 6 centimeters. What is the radian measure of θ?

Solution We have $r = 3$ centimeters and $s = 6$ centimeters, therefore,

$$\theta \text{ (in radians)} = \frac{s}{r} = \frac{6 \text{ centimeters}}{3 \text{ centimeters}} = 2$$

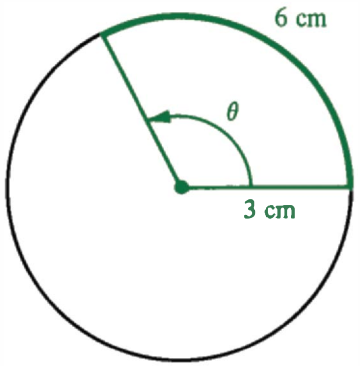


Figure 3

We say the radian measure of θ is 2 or $\theta = 2$ radians. ▲

Note It is common practice to omit the word radian when using radian measure. If no units are showing, an angle is understood to be measured in radians; with degree measure, the degree symbol, °, must be written.

$\theta = 2$ means the measure of θ is 2 radians
$\theta = 2°$ means the measure of θ is 2 degrees

To see the relationship between degrees and radians, we can compare the number of degrees and the number of radians in one full rotation.

The angle formed by one full rotation about the center of a circle of radius r will cut off an arc equal to the circumference of the circle. Since the circumference of a circle of radius r is $2\pi r$, we have

θ measures one full rotation	$\theta = \frac{2\pi r}{r} = 2\pi$	The measure of θ in radians is 2π

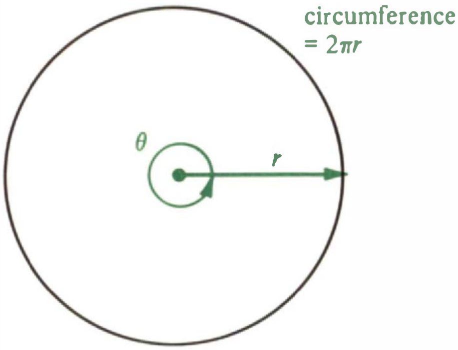


Figure 4

Since one full rotation in degrees is 360°, we have the relationship between radians and degrees.

$$360° = 2\pi \text{ radians}$$

dividing both sides by 2 we have

$$180° = \pi \text{ radians}$$

To obtain conversion factors that will allow us to change back and forth between degrees and radians, we divide both sides of this last equation alternately by 180 and by π.

$$180° = \pi \text{ radians}$$

Divide both sides by 180

$$1° = \frac{\pi}{180} \text{ radians}$$

Divide both sides by π

$$\left(\frac{180}{\pi}\right)^\circ = 1 \text{ radian}$$

To gain some insight into the relationship between degrees and radians, we can approximate π with 3.14 to obtain the approximate number of degrees in 1 radian.

$$\begin{aligned} 1 \text{ radian} &= 1\left(\frac{180}{\pi}\right)^\circ \\ &\approx 1\left(\frac{180}{3.14}\right)^\circ \\ &= 57.3° \end{aligned}$$

We see that 1 radian is approximately 57°. A radian is much larger than a degree. Figure 5 illustrates the relationship between 20° and 20 radians.

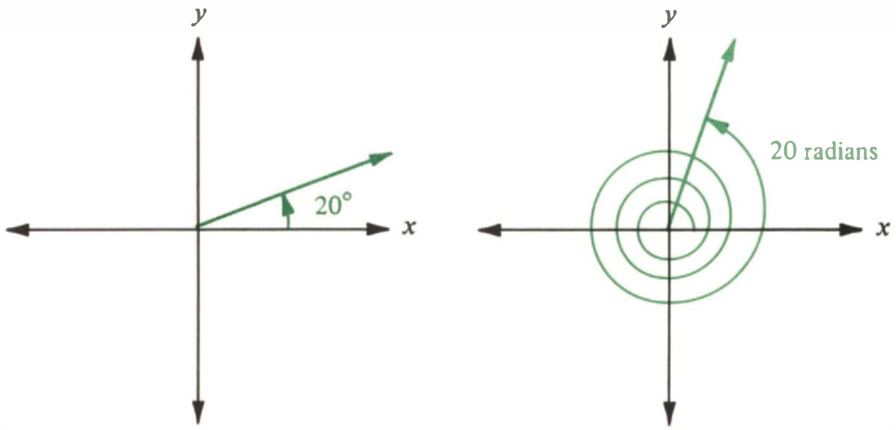


Figure 5

Here are some further conversions between degrees and radians.

Example 2 Convert 45° to radians.

Converting from Degrees to Radians

Solution Since $1° = \frac{\pi}{180}$ radians, and 45° is the same as 45(1°), we have

$$45° = 45\left(\frac{\pi}{180}\right) \text{ radians } = \frac{\pi}{4} \text{ radians}$$

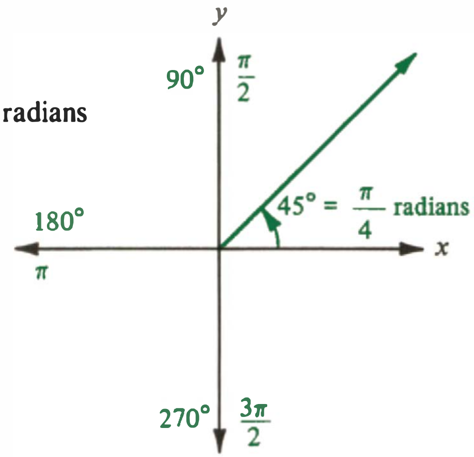


Figure 6

When we leave our answer in terms of π, as in $\pi/4$, we are writing an exact value. If we wanted a decimal approximation we would substitute 3.14 for π.

$$\text{Exact value} \quad \frac{\pi}{4} \approx \frac{3.14}{4} = 0.785 \quad \text{Approximate value}$$

Note also, that if we wanted the radian equivalent of 90°, we could simply multiply $\pi/4$ by 2, since $90° = 2 \times 45°$.

$$90° = 2 \times \frac{\pi}{4} = \frac{\pi}{2}$$

Example 3 Convert 450° to radians.

Solution Multiplying by $\pi/180$ we have

$$450° = 450\left(\frac{\pi}{180}\right) = \frac{5\pi}{2} \text{ radians}$$

Again, $5\pi/2$ is the exact value. If we want a decimal approximation we would substitute 3.14 for π.

$$\text{Exact value} \quad \frac{5\pi}{2} \approx \frac{5(3.14)}{2} = 7.85 \quad \text{Approximate value}$$

Converting from Radians to Degrees

▼ **Example 4** Convert $\pi/6$ to degrees.

Solution To convert from radians to degrees, we multiply by $180/\pi$.

$$\frac{\pi}{6} \text{ (radians)} = \frac{\pi}{6}\left(\frac{180}{\pi}\right)^{\circ}$$
$$= 30^{\circ}$$

Note that 60° is twice 30°, so $2(\pi/6) = \pi/3$ must be the radian equivalent of 60°. ▲

▼ **Example 5** Convert $4\pi/3$ to degrees.

Solution Multiplying by $180/\pi$ we have

$$\frac{4\pi}{3} \text{ (radians)} = \frac{4\pi}{3}\left(\frac{180}{\pi}\right)^{\circ}$$
$$= 240^{\circ}$$ ▲

As is apparent from the preceding examples, changing from degrees to radians and radians to degrees is simply a matter of multiplying by the appropriate conversion factors.

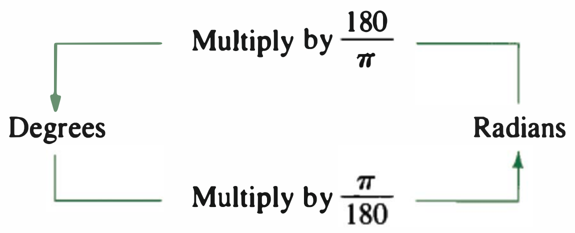


Table 1 (opposite page) shows the most common angles written in both degrees and radians. In each case the radian measure is given in exact values and approximations accurate to the nearest hundredth of a radian.

▼ **Example 6** Find $\sin \frac{\pi}{6}$.

Solution Since $\pi/6$ and 30° are equivalent, so are their sines.

$$\sin \frac{\pi}{6} = \frac{1}{2}$$ ▲

Calculator Note To work this problem on a calculator, we must first put the calculator in radian mode. (Consult the manual that came with your calculator to see how to do this.) If your calculator does not have a key labeled π, use 3.1416. Here is the sequence to key in your calculator to work the problem given in Example 6.

[Rad] 3.1416 [÷] 6 [=] [sin]

Table 1

Degrees	Radians: Exact values	Radians: Approximations
0°	0	0
30°	$\frac{\pi}{6}$	0.52
45°	$\frac{\pi}{4}$	0.79
60°	$\frac{\pi}{3}$	1.05
90°	$\frac{\pi}{2}$	1.57
180°	π	3.14
270°	$\frac{3\pi}{2}$	4.71
360°	2π	6.28

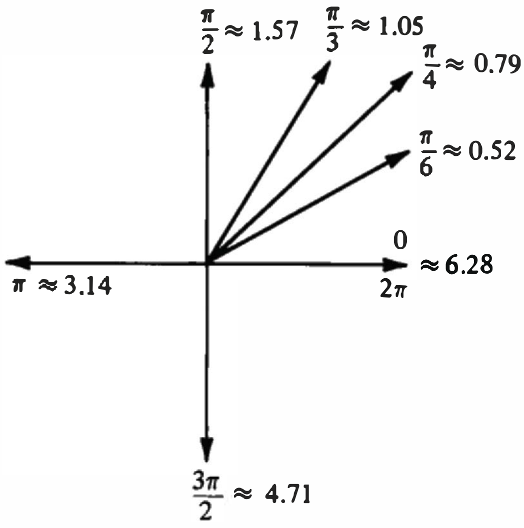


Figure 7

Example 7 Find $4 \sin \frac{7\pi}{6}$.

Solution Since $7\pi/6$ terminates in QIII, its sine will be negative. The reference angle is $7\pi/6 - \pi = \pi/6$.

$$4 \sin \frac{7\pi}{6} = -4 \sin \frac{\pi}{6}$$
$$= -4\left(\frac{1}{2}\right)$$
$$= -2$$

Example 8 Evaluate $4 \sin(2x + \pi)$ when $x = \pi/6$.

Solution Substituting $\pi/6$ for x and simplifying, we have

$$\begin{aligned} 4 \sin\left(2 \cdot \frac{\pi}{6} + \pi\right) &= 4 \sin\left(\frac{\pi}{3} + \pi\right) \\ &= 4 \sin \frac{4\pi}{3} \\ &= 4\left(-\frac{\sqrt{3}}{2}\right) \\ &= -2\sqrt{3} \end{aligned}$$

Problem Set 3.2

Find the radian measure of angle θ, if θ is a central angle in a circle of radius r, and θ cuts off an arc of length s.

1. $r = 3$ centimeters, $s = 9$ centimeters
2. $r = 6$ centimeters, $s = 3$ centimeters
3. $r = 10$ inches, $s = 5$ inches
4. $r = 5$ inches, $s = 10$ inches
5. $r = 4$ inches, $s = 12\pi$ inches
6. $r = 3$ inches, $s = 12$ inches
7. $r = \frac{1}{4}$ centimeter, $s = \frac{1}{2}$ centimeter
8. $r = \frac{1}{4}$ centimeter, $s = \frac{1}{8}$ centimeter
9. Los Angeles and San Francisco are approximately 450 miles apart on the surface of the earth. Assuming that the radius of the earth is 4,000 miles, find the radian measure of the central angle with vertex at the center of the earth that has Los Angeles on one side and San Francisco on the other side.
10. Los Angeles and New York City are approximately 2,500 miles apart on the surface of the earth. Assuming the radius of the earth is 4,000 miles, find the radian measure of the central angle with vertex at the center of the earth that has Los Angeles on one side and New York City on the other side.

Convert each of the following from degree measure to radian measure. Write each answer as an exact value and as an approximation to the nearest hundredth.

11. $30°$
12. $60°$
13. $90°$
14. $270°$
15. $260°$
16. $340°$
17. $-150°$
18. $-210°$
19. $420°$
20. $390°$
21. $-135°$
22. $-120°$

For Problems 23–26, use 3.1416 for π unless your calculator has a key marked π.

23. Use a calculator to convert 120° 40′ to radians. Round your answer to the nearest hundredth. (First convert to decimal degrees, then multiply by the appropriate conversion factor to convert to radians.)

24. Use a calculator to convert 256° 20′ to radians to the nearest hundredth of a radian.

25. Use a calculator to convert 1′ (1 minute) to radians to three significant digits.

26. Use a calculator to convert 1° to radians to three significant digits.

27. If a central angle with its vertex at the center of the earth has a measure of 1′, then the arc on the surface of the earth that is cut off by this angle has a measure of 1 nautical mile. Find the number of regular (statute) miles in 1 nautical mile to the nearest hundredth of a mile. (Use 4,000 miles for the radius of the earth.)

28. If two ships are 20 nautical miles apart on the ocean, how many statute miles apart are they? (Use the results of Problem 27 to do the calculations.)

Convert each of the following from radian measure to degree measure:

29. $\pi/3$

30. $\pi/4$

31. $2\pi/3$

32. $3\pi/4$

33. $-7\pi/6$

34. $-5\pi/6$

35. $10\pi/6$

36. $7\pi/3$

37. 4π

38. 3π

39. $\pi/12$

40. $5\pi/12$

41. $-7\pi/18$

42. $-11\pi/18$

Use a calculator to convert each of the following to degree measure to the nearest tenth of a degree:

43. 1.3

44. 2.4

45. 0.75

46. 0.25

47. 5

48. 6

Give the exact value of each of the following:

49. $\sin \frac{4\pi}{3}$

50. $\cos \frac{4\pi}{3}$

51. $\tan \frac{\pi}{6}$

52. $\cot \frac{\pi}{3}$

53. $\sec \frac{2\pi}{3}$

54. $\csc \frac{3\pi}{2}$

55. $\csc \frac{5\pi}{6}$

56. $\sec \frac{5\pi}{6}$

57. $4\sin\left(-\frac{\pi}{4}\right)$

58. $4\cos\left(-\frac{\pi}{4}\right)$

59. $-\sin\frac{\pi}{4}$

60. $-\cos\frac{\pi}{4}$

61. $2\cos\frac{\pi}{6}$

62. $2\sin\frac{\pi}{6}$

Evaluate each of the following expressions when x is $\pi/6$. In each case, use exact values.

63. $\sin 2x$

64. $\sin 3x$

65. $6\cos 3x$

66. $6\cos 2x$

67. $\sin\left(x + \frac{\pi}{2}\right)$

68. $\sin\left(x - \frac{\pi}{2}\right)$

69. $4\cos\left(2x + \frac{\pi}{3}\right)$

70. $4\cos\left(3x + \frac{\pi}{6}\right)$

Review Problems The problems that follow review material we covered in Section 1.3.

Find all six trigonometric functions of θ, if the given point is on the terminal side of θ.

71. $(1, -3)$

72. $(-1, 3)$

73. (m, n)

74. (a, b)

75. Find the remaining trigonometric functions of θ, if $\sin\theta = 1/2$ and θ terminates in QII.

76. Find the remaining trigonometric functions of θ, if $\cos\theta = -1/\sqrt{2}$ and θ terminates in QII.

77. Find all six trigonometric functions of θ, if the terminal side of θ lies along the line $y = 2x$ in QI.

78. Find the six trigonometric functions of θ, if the terminal side of θ lies along the line $y = 2x$ in QIII.

3.3 The Unit Circle and Even and Odd Functions

We will begin this section by using the unit circle and our table of exact values to make a diagram that summarizes what we know about exact values, degrees, and radians.

The unit circle is the circle with its center at the origin and a radius of 1. The equation of the unit circle is $x^2 + y^2 = 1$ (Figure 1).

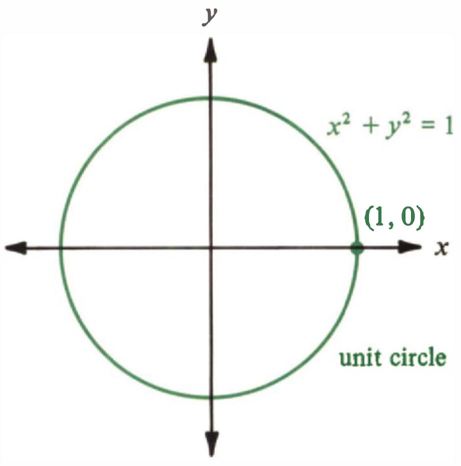


Figure 1

Suppose the terminal side of angle θ, in standard position, intersects the unit circle at point (x, y). Then (x, y) is a point on the terminal side of θ. The distance from the origin to (x, y) is 1, since the radius of the unit circle is 1. Therefore

$$\cos \theta = \frac{x}{r} = \frac{x}{1} = x \qquad \sin \theta = \frac{y}{r} = \frac{y}{1} = y$$

The coordinates of the point (x, y) are $\cos \theta$ and $\sin \theta$ (Figure 2).

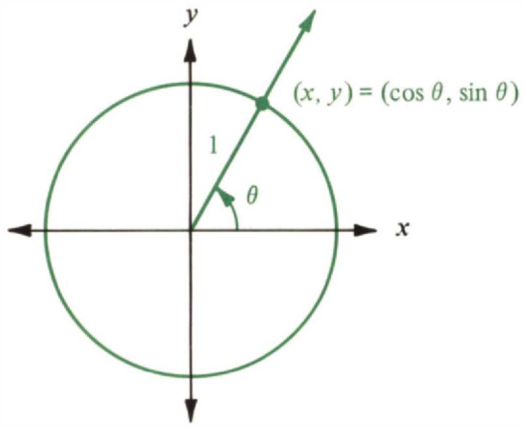


Figure 2

Note This is only true for (x, y) on the *unit* circle. It will not work for points on other circles.

Figure 3 shows the unit circle with multiples of 30° and 45° marked off. Each angle is given in degrees and radians. The cosine and sine of the angle are the x- and y-coordinates, respectively, of the point where the terminal side of each angle intersects the unit circle.

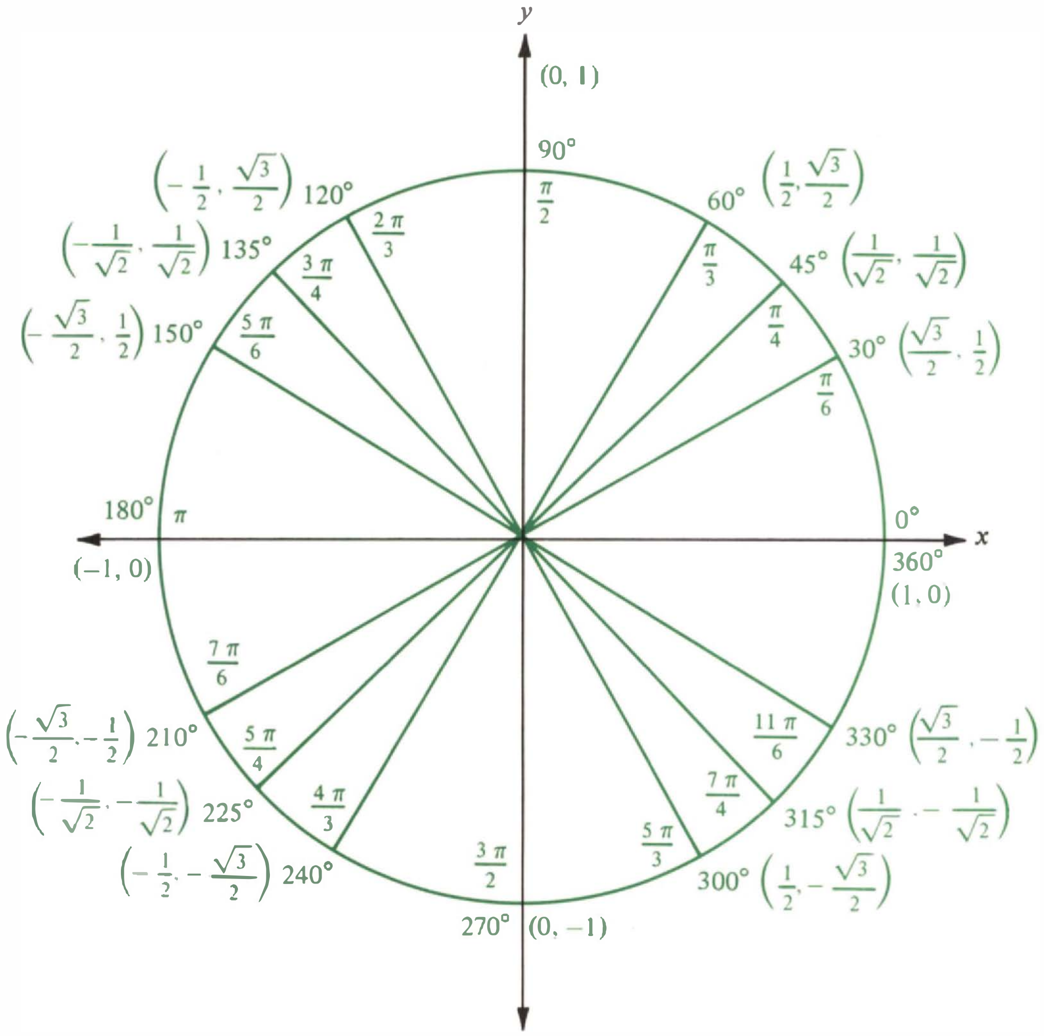


Figure 3

▼ **Example 1** Use Figure 3 to find the six trigonometric functions of $5\pi/6$.

Solution We obtain cosine and sine directly from Figure 3. The other trigonometric functions of $5\pi/6$ are found by using the ratio and reciprocal identities.

$$\sin\frac{5\pi}{6} = \frac{1}{2}$$

$$\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$$

$$\tan\frac{5\pi}{6} = \frac{\sin{}^{5\pi}\!/_6}{\cos{}^{5\pi}\!/_6} = \frac{1/2}{-\sqrt{3}/2} = -\frac{1}{\sqrt{3}}$$

$$\cot\frac{5\pi}{6} = \frac{1}{\tan{}^{5\pi}\!/_6} = \frac{1}{-1/\sqrt{3}} = -\sqrt{3}$$

$$\sec\frac{5\pi}{6} = \frac{1}{\cos{}^{5\pi}\!/_6} = \frac{1}{-\sqrt{3}/2} = -\frac{2}{\sqrt{3}}$$

$$\csc\frac{5\pi}{6} = \frac{1}{\sin{}^{5\pi}\!/_6} = \frac{1}{1/2} = 2$$

▲

Figure 3 is very helpful in visualizing the relationships among the angles shown and the trigonometric functions of those angles. You may want to make a larger copy of this diagram yourself. In the process of doing so you will become more familiar with the relationship between degrees and radians and the exact values of the angles in the diagram.

▼ **Example 2** Use the unit circle to find all values of θ between 0° and 360° for which $\cos\theta = 1/2$.

Solution We look for all ordered pairs on the unit circle with an x-coordinate of $1/2$. The angles associated with these points are the angles for which $\cos\theta = 1/2$. They are $\theta = 60°$ or $\pi/3$ and $\theta = 300°$ or $5\pi/3$. ▲

Even and Odd Functions

Recall from algebra the definitions of even and odd functions.

DEFINITION An *even function* is a function for which

$$f(-x) = f(x) \text{ for all } x \text{ in the domain of } f$$

An even function is a function for which replacing x with $-x$ leaves the equation that defines the function unchanged. If a function is even, then every time the point (x, y) is on the graph, so is the point $(-x, y)$. The function $f(x) = x^2 + 3$ is an even function since

$$f(-x) = (-x)^2 + 3 = x^2 + 3 = f(x)$$

DEFINITION An *odd function* is a function for which

$$f(-x) = -f(x) \text{ for all } x \text{ in the domain of } f$$

An odd function is a function for which replacing x with $-x$ changes the sign of the equation that defines the function. If a function is odd, then every time the point (x, y) is on the graph, so is the point $(-x, -y)$. The function $f(x) = x^3 - x$ is an odd function since

$$f(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) = -f(x)$$

From the unit circle it is apparent that sine is an odd function and cosine is an even function. To begin to see that this is true, we locate $\pi/6$ and $-\pi/6$ ($-\pi/6$ is coterminal with $11\pi/6$) on the unit circle and notice that

$$\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}$$

and

$$\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2} = -\sin\frac{\pi}{6}$$

We can generalize this result by drawing an angle θ and its opposite $-\theta$ in standard position, and then labeling the points where their terminal sides intersect the unit circle with (x, y) and $(x, -y)$, respectively. (Can you see from Figure 4 why we label these two points in this way? That is, does it make sense that if (x, y) is on the terminal side of θ, then $(x, -y)$ must be on the terminal side of $-\theta$?)

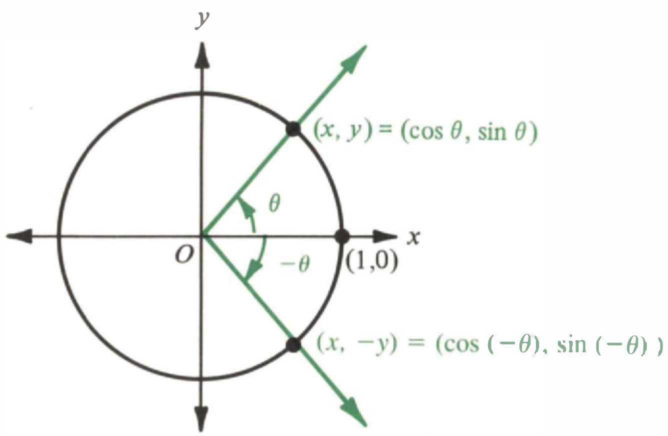


Figure 4

Since, on the unit circle, $\cos\theta = x$ and $\sin\theta = y$, we have

$$\cos(-\theta) = x = \cos\theta \qquad \text{Indicating that cosine is an even function}$$

and

$$\sin(-\theta) = -y = -\sin\theta \qquad \text{Indicating that sine is an odd function}$$

Now that we have established that sine is an odd function and cosine is an even function, we can use our ratio and reciprocal identities to find which of the other trigonometric functions are even or odd. Example 3 shows how this is done for the cosecant function.

Example 3 Show that cosecant is an odd function.

Solution We must prove that $\csc(-\theta) = -\csc\theta$. That is, we must turn $\csc(-\theta)$ into $-\csc\theta$. Here is how it goes

$$\csc(-\theta) = \frac{1}{\sin(-\theta)} \qquad \text{Reciprocal identity}$$

$$= \frac{1}{-\sin\theta} \qquad \text{Sine is a odd function}$$

$$= -\frac{1}{\sin\theta} \qquad \text{Algebra}$$

$$= -\csc\theta \qquad \text{Reciprocal identity}$$

Example 4 Use the even and odd function relationships to find exact values for each of the following.

a. $\sin(-60°)$ **b.** $\cos\left(-\frac{2\pi}{3}\right)$ **c.** $\sec(-225°)$

Solution

a. $\sin(-60°) = -\sin 60°$ Sine is an odd function

$$= -\frac{\sqrt{3}}{2} \qquad \text{Unit circle}$$

b. $\cos\left(-\dfrac{2\pi}{3}\right) = \cos\left(\dfrac{2\pi}{3}\right)$ **Cosine is an even function**

$= -\dfrac{1}{2}$ **Unit circle**

c. $\csc(-225°) = \dfrac{1}{\sin(-225°)}$ **Reciprocal functions**

$= \dfrac{1}{-\sin 225°}$ **Sine is odd function**

$= \dfrac{1}{-(-1/\sqrt{2})}$ **Unit circle**

$= \sqrt{2}$ ▲

An important relationship exists between the radian measure of a central angle in the unit circle and the length of its intercepted arc. Since the radian measure of a central angle is defined as $\theta = s/r$, and the radius of the unit circle is $r = 1$, we have

$$\theta = \frac{s}{r} = \frac{s}{1} = s$$

Thus, the radian measure of a central angle in the unit circle and the length of the arc it cuts off are equal. If the radius of the unit circle were 1 foot, then an angle of 2 radians would cut off an arc of length 2 feet. Likewise, if the radius of the unit circle were 1 centimeter, then an angle of 3 radians would cut off an arc of length 3 centimeters.

There is an interesting diagram we can draw using the ideas given in the preceding paragraph. Figure 5 shows a point $P(x, y)$ that is t units from the point (1, 0) on the circumference of the unit circle. The central angle that cuts off the arc t is in standard position with the point $P(x, y)$ on the terminal side. Therefore, $\cos t = x$ and $\sin t = y$. (Can you see why QB is labeled tan t? It has to do with the fact that tan t is the ratio of sin t to cos t and the similar triangles POA and QOB.)

There are many concepts that can be visualized from Figure 5. One of the more important is the variations that occur in sin t, cos t, and tan t as P travels around the unit circle. To illustrate, imagine P traveling once around the unit circle starting at (1, 0) and ending 2π units later at the same point. As P moves from (1, 0) to (0, 1), t increases from 0 to $\pi/2$. At the same time, sin t increases from 0 to 1, and cos t decreases from 1 down to 0. As we continue around the unit circle, sin t and cos t simply oscillate between -1 and 1, and further, everywhere sin t is 1 or -1, cos t is 0. We will look

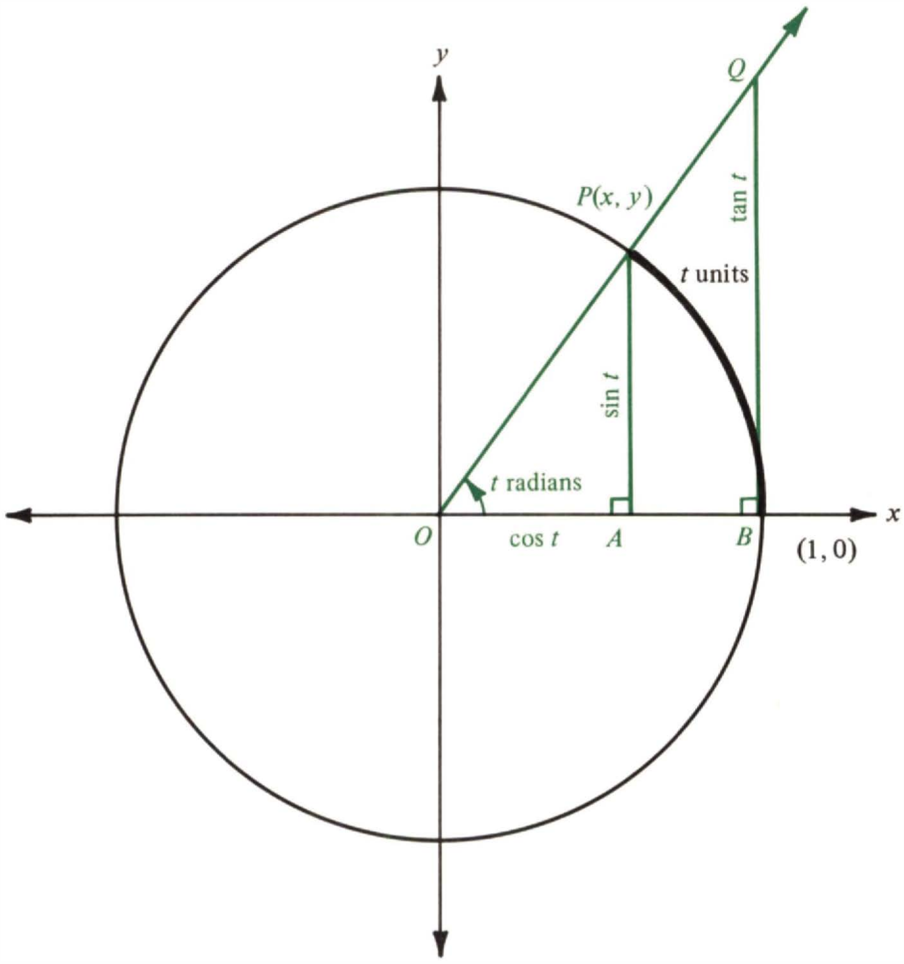


Figure 5

at these oscillations in more detail in Chapter 4. For now, it is enough to notice that the unit circle can be used to visualize the relationships that exist among angles measured in radians, the arcs associated with these angles, and the trigonometric functions of both.

Problem Set 3.3

Use the unit circle to find the six trigonometric functions of each angle.

1. 150°
2. 135°
3. $11\pi/6$
4. $5\pi/3$
5. 180°
6. 270°
7. $3\pi/4$
8. $5\pi/4$

Use the unit circle and the fact that cosine is an even function to find each of the following:

9. $\cos(-60°)$

10. $\cos(-120°)$

11. $\cos\left(-\frac{5\pi}{6}\right)$

12. $\cos\left(-\frac{4\pi}{3}\right)$

Use the unit circle and the fact that sine is an odd function to find each of the following:

13. $\sin(-30°)$

14. $\sin(-90°)$

15. $\sin\left(-\frac{3\pi}{4}\right)$

16. $\sin\left(-\frac{7\pi}{4}\right)$

Use the unit circle to find all values of θ between 0 and 2π for which

17. $\sin\theta = 1/2$

18. $\sin\theta = -1/2$

19. $\cos\theta = -\sqrt{3}/2$

20. $\cos\theta = 0$

21. $\tan\theta = -\sqrt{3}$

22. $\cot\theta = \sqrt{3}$

23. If angle θ is in standard position and intersects the unit circle at $(1/\sqrt{5}, -2/\sqrt{5})$, find $\sin\theta$, $\cos\theta$, and $\tan\theta$.

24. If angle θ is in standard position and intersects the unit circle at $(-1/\sqrt{10}, -3/\sqrt{10})$, find $\sin\theta$, $\cos\theta$, and $\tan\theta$.

25. If θ is an angle in standard position the terminal side of which intersects the unit circle at the point $P(x, y)$, give a definition for $\tan\theta$, $\cot\theta$, $\sec\theta$, and $\csc\theta$ in terms of x and y that is consistant with our original definitions of these functions.

26. Some of the identities we have used in the past are actually easier to derive from the unit circle than from our original definition for the trigonometric functions. For example, since the equation of the unit circle is $x^2 + y^2 = 1$, and, on the unit circle, $x = \cos\theta$ and $y = \sin\theta$, then it follows immediately that $\cos^2\theta + \sin^2\theta = 1$. Use the definitions you gave in Problem 25 for the other four trigonometric functions to derive the basic reciprocal and ratio identities.

27. If $\sin\theta = -1/3$, find $\sin(-\theta)$.

28. If $\cos\theta = -1/3$, find $\cos(-\theta)$.

Make a diagram of the unit circle with an angle θ in quadrant I and its complement $180° - \theta$ in quadrant II. Label the point on the terminal side of θ and the unit circle with (x, y) and the point on the terminal side of $180° - \theta$ and the unit circle with $(-x, y)$. Use this diagram to show that

29. $\sin(180° - \theta) = \sin\theta$

30. $\cos(180° - \theta) = -\cos\theta$

31. Show that tangent is an odd function.

32. Show that cotangent is an odd function.

Prove each identity.

33. $\sin(-\theta)\cot(-\theta) = \cos\theta$

34. $\cos(-\theta)\tan\theta = \sin\theta$

35. $\sin(-\theta)\sec(-\theta)\cot(-\theta) = 1$

36. $\cos(-\theta)\csc(-\theta)\tan(-\theta) = 1$

37. $\csc\theta + \sin(-\theta) = \dfrac{\cos^2\theta}{\sin\theta}$

38. $\sec\theta - \cos(-\theta) = \dfrac{\sin^2\theta}{\cos\theta}$

39. Redraw the diagram in Figure 5 from this section and label the line segment that corresponds to sec t.

40. Make a diagram similar to the diagram in Figure 5 from this section, but instead of labeling the point (1, 0) with B, label the point (0, 1) with B. Then place Q on the line OP and connect Q to B so that QB is perpendicular to the y-axis. Now, if $P(x, y)$ is t units from (1, 0), label the line segments that correspond to sin t, cos t, cot t, and csc t.

Review Problems The problems that follow review material we covered in Section 3.1.

Use Table II or a calculator to find decimal approximations to each of the following:

41. sin 35.5°

42. sin(−35.5°)

43. cos(−43°)

44. cos 43°

45. tan 123.5°

46. tan(−123.5°)

47. sec 214.3°

48. sec(−214.3°)

3.4 Length of Arc and Area of a Sector

In Section 3.2 we found that if a central angle θ, measured in radians, in a circle of radius r, cuts off an arc of length s, then the relationship between s, r, and θ can be written as

$$\theta = \frac{s}{r}$$

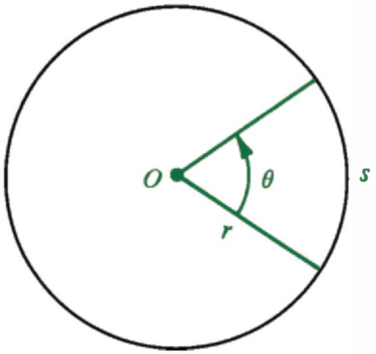


Figure 1

If we multiply both sides of this equation by r, we will obtain the equation that gives arc length s, in terms of r and θ.

$$s = r\theta \qquad (\theta \text{ in radians})$$

Example 1 Give the length of the arc cut off by a central angle of 2 radians in a circle of radius 4.3 inches.

Solution We have $\theta = 2$ and $r = 4.3$ inches. Applying the formula $s = r\theta$ gives us

$$\begin{aligned} s &= r\theta \\ &= 4.3(2) \\ &= 8.6 \text{ inches} \end{aligned}$$

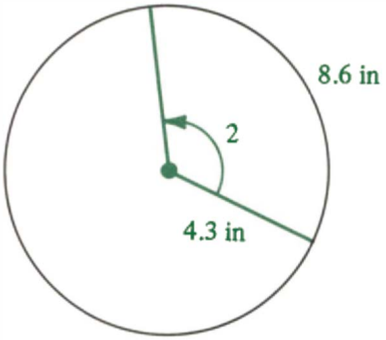


Figure 2

Example 2 Find the length of the arc cut off by an angle of 45° in a circle of radius 2 centimeters.

Solution The formula $s = r\theta$ holds only when θ is measured in radians. If θ is measured in degrees, we must convert to radians by multiplying θ by $\pi/180$.

$$\begin{aligned} s &= r\theta \\ &= 2(45)\left(\frac{\pi}{180}\right) \\ &= \frac{\pi}{2} \text{ centimeters} \end{aligned}$$

To find a decimal approximation for s, we substitute 3.14 for π

$$\begin{aligned} s &\approx \frac{3.14}{2} \text{ centimeters} \\ &= 1.57 \text{ centimeters} \end{aligned}$$

Example 3 The minute hand of a clock is 1.2 centimeters long. To two significant digits, how far does the tip of the minute hand move in 20 minutes?

Solution We have $r = 1.2$ centimeters. Since we are looking for s, we need to find θ. We can use a proportion to find θ. Since one complete rotation is 60 minutes and 2π radians, we say θ is to 2π as 20 minutes is to 60 minutes, or

If $\dfrac{\theta}{2\pi} = \dfrac{20}{60}$

then $\theta = \dfrac{2\pi}{3}$

Now we can find s.

$$\begin{aligned} s &= r\theta \\ &= 1.2\left(\frac{2\pi}{3}\right) \\ &= \frac{2.4\pi}{3} \\ &\approx \frac{2.4(3.14)}{3} \\ &= 2.5 \text{ centimeters} \end{aligned}$$

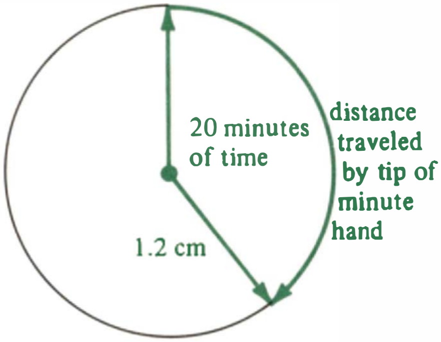


Figure 3

The tip of the minute hand will travel approximately 2.5 centimeters every 20 minutes. ▲

If we are working with relatively small central angles in circles with large radii, we can use the length of the intercepted arc to approximate the length of the associated chord. For example, Figure 4 shows a central angle of 1° in a circle of radius 1,800 feet, along with the arc and chord cut off by 1°. (Figure 4 is not drawn to scale.)

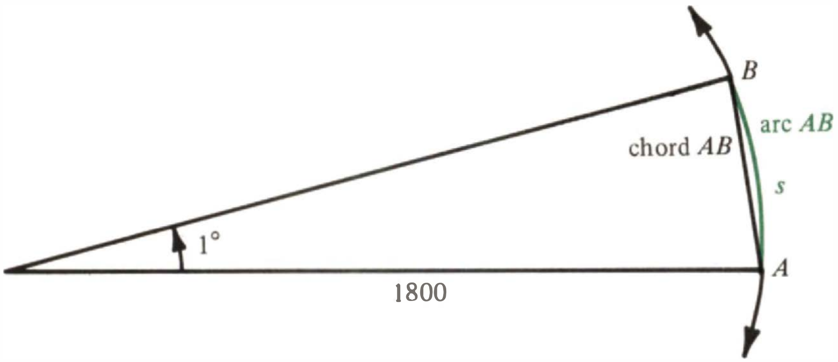


Figure 4

To find the length of arc AB we convert θ to radians by multiplying by $\pi/180°$. Then we apply the formula $s = r\theta$.

$$s = r\theta = 1{,}800(1°)\left(\frac{\pi}{180°}\right) = 10\pi \approx 31.4 \text{ feet}$$

If we were to carry out the calculation of arc AB to six significant digits we would have obtained $s = 31.4159$. The length of the chord AB is 31.4155 to six significant digits (found by using the law of sines which we will cover in Chapter 7). As you can see, the first five digits in each number are the same. It seems reasonable then to approximate the length of chord AB with the length of arc AB.

As our next example illustrates, we can also use the procedure outlined above in the reverse order to find the radius of a circle by approximating arc length with the length of the associated chord.

▼ **Example 4** A person standing on the earth notices that a 747 Jumbo Jet flying overhead subtends an angle of 0.45°. If the length of the jet is 230 feet, find its altitude to the nearest thousand feet.

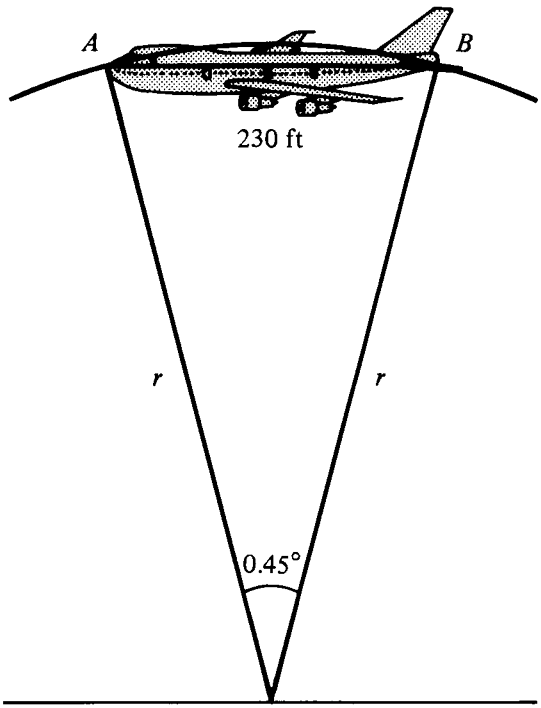


Figure 5

Solution Figure 5 is a diagram of the situation. Since we are working with a relatively small angle in a circle with a large radius, we use the length of the airplane (chord AB in Figure 5) as an approximation of the length of the arc AB.

Since $s = r\theta$, $r = \dfrac{s}{\theta}$

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$$\text{so } r = \frac{}{(0.45^\circ)(\pi/180^\circ)}$$

$$\approx \frac{230(180)}{(0.45)(3.14)} \qquad =$$

$= 29{,}000$ feet to the nearest thousand feet ▲

Area of a Sector

Next we want to derive the formula for the area of the sector formed by a central angle θ (Figure 6).

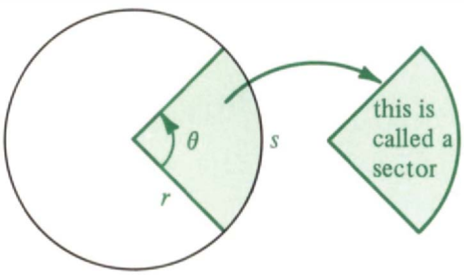


Figure 6

If we let A represent the area of the sector formed by central angle θ, we can find A by setting up a proportion as follows: We say the area A of the sector is to the area of the circle as θ is to one full rotation. That is,

$$\text{Area of sector} \rightarrow \frac{A}{\pi r^2} = \frac{\theta}{2\pi} \leftarrow \text{Central angle } \theta$$
$$\text{Area of circle} \rightarrow \qquad\qquad \leftarrow \text{One full rotation}$$

We solve for A by multiplying both sides of the proportion by πr^2.

$$\pi r^2 \cdot \frac{A}{\pi r^2} = \frac{\theta}{2\pi} \cdot \pi r^2$$

$$A = \frac{1}{2} r^2 \theta$$

Example 5 Find the area of the sector formed by a central angle of 1.4 radians in a circle of radius 2.1 meters.

Solution We have $r = 2.1$ meters and $\theta = 1.4$. Applying the formula for A gives us

$$\begin{aligned} A &= \frac{1}{2} r^2\theta \\ &= \frac{1}{2}(2.1)^2(1.4) \\ &= 3.087 \text{ meters}^2 \end{aligned}$$

Remember Area is measured in square units. When $r = 2.1$ meters, $r^2 = (2.1 \text{ meters})^2 = 4.41 \text{ meters}^2$.

Example 6 If the sector formed by a central angle of 15° has an area of $\pi/3$ centimeters², find the radius of the circle.

Solution We first convert 15° to radians.

$$\theta = 15\left(\frac{\pi}{180}\right) = \frac{\pi}{12}$$

Then we substitute $\theta = \pi/12$ and $A = \pi/3$ into the formula for A, and then solve for r.

$$\begin{aligned} A &= \frac{1}{2} r^2\theta \\ \frac{\pi}{3} &= \frac{1}{2} r^2 \frac{\pi}{12} \\ \frac{\pi}{3} &= \frac{\pi}{24} r^2 \\ r^2 &= \frac{\pi}{3} \cdot \frac{24}{\pi} \\ r^2 &= 8 \\ r &= 2\sqrt{2} \text{ centimeters} \end{aligned}$$

Note that we need only use the positive square root of 8, since we know our radius must be measured with positive units.

Example 7 A lawn sprinkler located at the corner of a yard is set to rotate through 90° and project water out 30 feet. To three significant digits, what area of lawn is watered by the sprinkler?

Solution We have $\theta = 90° = \frac{\pi}{2} \approx 1.57$ radians, and $r = 30$ feet.

$$A = \frac{1}{2} r^2\theta$$

$$\approx \frac{1}{2}(30)^2(1.57)$$

$$= 707 \text{ feet}^2$$

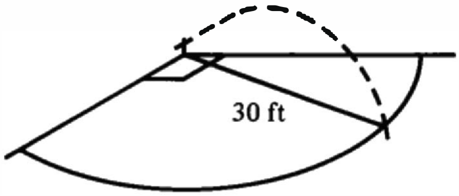


Figure 7

Problem Set 3.4

Unless otherwise stated, all answers in this problem set that need to be rounded should be rounded to three significant digits.

For each problem below, θ is a central angle in a circle of radius r. In each case, find the length of arc s cut off by θ.

1. $\theta = 2$, $r = 3$ inches
2. $\theta = 3$, $r = 2$ inches
3. $\theta = 1.5$, $r = 1.5$ feet
4. $\theta = 2.4$, $r = 1.8$ feet
5. $\theta = \pi/6$, $r = 12$ centimeters
6. $\theta = \pi/3$, $r = 12$ centimeters
7. $\theta = 60°$, $r = 4$ millimeters
8. $\theta = 30°$, $r = 4$ millimeters
9. $\theta = 240°$, $r = 10$ inches
10. $\theta = 315°$, $r = 5$ inches

11. The minute hand of a clock is 2.4 centimeters long. How far does the tip of the minute hand travel in 20 minutes?

12. The minute hand of a clock is 1.2 centimeters long. How far does the tip of the minute hand travel in 40 minutes?

13. A space shuttle 200 miles above the earth is orbiting the earth once every 6 hours. How far does the shuttle travel in 1 hour? (Assume the radius of the earth is 4,000 miles.) Give both the exact value and a three significant digit approximation for your answer.

14. How long, in hours, does it take the space shuttle in Problem 13 to travel 8,400 miles? Give both the exact value and an approximate value for your answer.

15. The pendulum on a grandfather clock swings from side to side once every second. If the length of the pendulum is 4 feet and the angle through which it swings is 20°, how far does the tip of the pendulum travel in one second?

16. Find the total distance traveled in one minute by the tip of the pendulum on the grandfather clock in Problem 15.

17. From the earth, the moon subtends an angle of approximately 0.5°. If the distance to the moon is approximately 240,000 miles, find an approximation for the diameter of the moon accurate to the nearest hundred miles. (See Example 4 and the discussion that preceeds it.)

18. If the distance to the sun is approximately 93 million miles, and, from the earth, the sun subtends an angle of approximately 0.5°, estimate the diameter of the sun to the nearest ten thousand miles.

In each problem below, θ is a central angle that cuts off an arc of length s. In each case, find the radius of the circle.

19. $\theta = 6$, $s = 3$ feet

20. $\theta = 1$, $s = 2$ feet

21. $\theta = 1.4$, $s = 4.2$ inches

22. $\theta = 5.1$, $s = 10.2$ inches

23. $\theta = \pi/4$, $s = \pi$ centimeters

24. $\theta = 3\pi/4$, $s = \pi$ centimeters

25. $\theta = 90°$, $s = \pi/2$ meters

26. $\theta = 180°$, $s = \pi/2$ meters

27. From the ground, a 747 Jumbo Jet flying overhead subtends an angle of 0.42°. If the length of the jet is 230 feet, find its altitude to the nearest thousand feet.

28. From a point on the ground a person notices that a 100 foot antenna on the top of a mountain subtends an angle of 0.35°. If the angle of elevation to the top of the antenna is 13.35°, find the height of the mountain to the nearest hundred feet. First find the distance from the person on the ground to the top of the mountain, then use right triangle trigonometry to find the height of the mountain. Figure 8 illustrates.

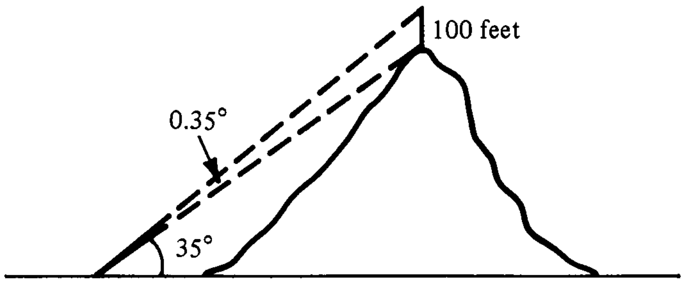


Figure 8

Find the area of the sector formed by central angle θ in a circle of radius r if

29. $\theta = 2$, $r = 3$ centimeters
30. $\theta = 3$, $r = 2$ centimeters
31. $\theta = 2.4$, $r = 4$ inches
32. $\theta = 1.8$, $r = 2$ inches
33. $\theta = \pi/5$, $r = 3$ meters
34. $\theta = 2\pi/5$, $r = 3$ meters
35. $\theta = 15°$, $r = 5$ meters
36. $\theta = 15°$, $r = 10$ meters

37. A central angle of 2 radians cuts off an arc of length 4 inches. Find the area of the sector formed.
38. An arc of length 3 feet is cut off by a central angle of $\pi/4$ radians. Find the area of the sector formed.
39. If the sector formed by a central angle of 30° has an area of $\pi/3$ centimeters^2, find the radius of the circle.
40. What is the length of the arc cut off by angle θ in Problem 39?
41. A sector of area $2\pi/3$ inches^2 is formed by a central angle of 45°. What is the radius of the circle?
42. A sector of area 25 inches^2 is formed by a central angle of 4 radians. Find the radius of the circle.
43. A lawn sprinkler is located at the corner of a yard. The sprinkler is set to rotate through 90° and project water out 60 feet. What is the area of the yard watered by the sprinkler?
44. An automobile windshield wiper 10 inches long rotates through an angle of 60°. If the rubber part of the blade covers only the last 9 inches of the wiper, find the area of the windshield cleaned by the windshield wiper.

Review Problems The problems that follow review material we covered in Section 2.3.

Problems 45 through 48 refer to right triangle ABC with $C = 90°$. In each case, solve for all the missing parts.

45. $A = 40°$, $c = 36$
46. $B = 22°$, $b = 300$
47. $a = 20.5$, $b = 31.4$
48. $a = 16$, $b = 20$

3.5 Velocities

There are two kinds of velocities associated with a point moving on the circumference of a circle. One is called *linear velocity* and is a measure of distance traveled per unit time. The other is called *angular velocity*. Angular velocity is the central angle swept out by the point moving on the circle, divided by time.

DEFINITION If P is a point on a circle of radius r, and P moves a distance

s on the circumference of the circle, in an amount of time t, then the *linear velocity*, v, of P is given by the formula

$$v = \frac{s}{t}$$

▼ **Example 1** A point on a circle travels 5 centimeters in 2 seconds. Find the linear velocity of the point.

Solution Substituting $s = 5$ and $t = 2$ into the equation $v = s/t$ gives us

$$v = \frac{5 \text{ centimeters}}{2 \text{ seconds}}$$

$$= 2.5 \text{ centimeters per second}$$ ▲

Note In all the examples and problems in this section, we are assuming that the point on the circle moves with uniform circular motion. That is, the velocity of the point is constant.

DEFINITION If P is a point moving with uniform circular motion on a circle of radius r, and the line from the center of the circle through P sweeps out a central angle θ, in an amount of time t, then the *angular velocity*, ω, of P is given by the equation

$$\omega = \frac{\theta}{t} \qquad \text{where } \theta \text{ is measured in radians}$$

▼ **Example 2** A point on a circle rotates through $3\pi/4$ radians in 3 seconds. Give the angular velocity of P.

Solution Substituting $\theta = 3\pi/4$ and $t = 3$ into the equation $\omega = \theta/t$ gives us

$$\omega = \frac{3\pi/4 \text{ radians}}{3 \text{ seconds}}$$

$$= \frac{\pi}{4} \text{ radians per second}$$ ▲

▼ **Example 3** A bicycle wheel with a radius of 13 inches turns with an angular velocity of 3 radians per second. Find the distance traveled by a point on the bicycle tire in 1 minute.

Solution We have $\omega = 3$ radians per second, $r = 13$ inches, and $t = 60$ seconds. First we find θ using $\omega = \theta/t$.

$$\text{If} \quad \omega = \frac{\theta}{t}$$

$$\begin{aligned} \text{then } \theta &= \omega t \\ &= 3(60) \\ &= 180 \text{ radians} \end{aligned}$$

To find the distance traveled by the point in 60 seconds, we use the formula $s = r\theta$ from Section 3.4, with $r = 13$ inches, and $\theta = 180$.

$$\begin{aligned} s &= 13(180) \\ &= 2{,}340 \text{ inches} \end{aligned}$$

If we want this result expressed in feet, we divide by 12.

$$\begin{aligned} s &= \frac{2340}{12} \text{ feet} \\ &= 195 \text{ feet} \end{aligned}$$

A point on the tire of the bicycle will travel 195 feet in one minute. If the bicycle was being ridden under these conditions, the rider would travel 195 feet in one minute. ▲

▼ **Example 4** Figure 1 shows a fire truck parked on the shoulder of a freeway next to a long block wall. The red light on the top of the truck is 10 feet from the wall and rotates through one complete revolution every 2 seconds. Find the equations that give the lengths d and l in terms of time t.

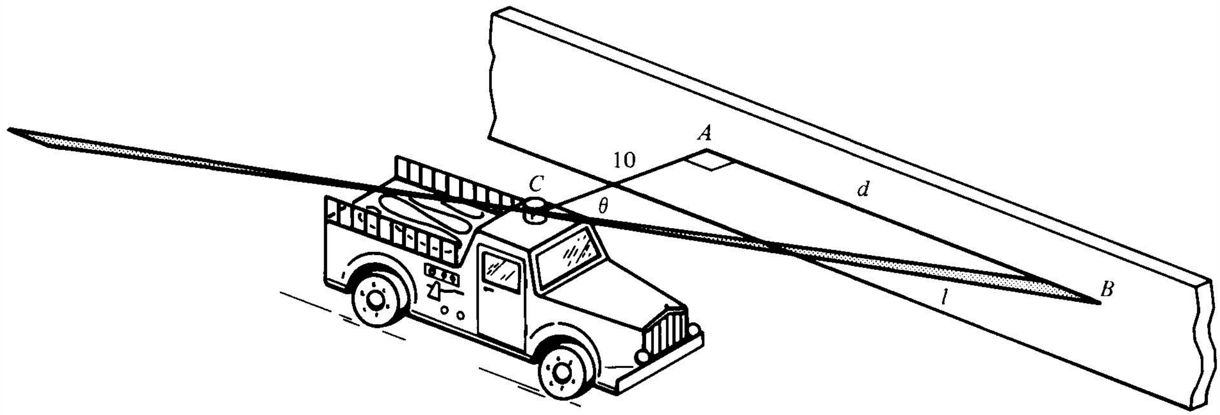


Figure 1

Solution The angular velocity of the rotating red light is

$$\omega = \frac{\theta}{t} = \frac{2\pi \text{ radians}}{2 \text{ seconds}} = \pi \text{ radians per second}$$

From right triangle ABC we have the following relationships

$$\tan\theta = \frac{d}{10} \qquad \text{and} \qquad \sec\theta = \frac{l}{10}$$

$$d = 10\tan\theta \qquad\qquad l = 10\sec\theta$$

Now, these equations give us d and l in terms of θ. To write d and l in terms of t, we solve $\omega = \theta/t$ for θ to obtain $\theta = \omega t = \pi t$. Substituting this for θ in each equation we have d and l expressed in terms of t.

$$d = 10\tan\pi t \qquad l = 10\sec\pi t$$

▲

The Relationship between the Two Velocities

To find the relationship between the two kinds of velocities we have developed so far, we can take the equation that relates arc length and central angle measure, $s = r\theta$, and divide both sides by time, t.

$$\text{If} \quad s = r\theta$$

$$\text{then} \quad \frac{s}{t} = \frac{r\theta}{t}$$

$$\frac{s}{t} = r\frac{\theta}{t}$$

$$v = r\omega$$

Linear velocity is the product of the radius and the angular velocity.

▼ **Example 5** A phonograph record is turning at 45 rpm (revolutions per minute). If the distance from the center of the record to a point on the edge of the record is 3 inches, find the angular velocity and the linear velocity, in feet per minute, of the point.

Solution The quantity 45 rpm is another way of expressing the rate at which the point on the record is moving. We can obtain the angular velocity from it by remembering that one complete revolution is equivalent to 2π radians. Therefore,

$$\begin{aligned}\omega &= 45(2\pi) \text{ radians per minute}\\ &= 90\pi \text{ radians per minute}\end{aligned}$$

To find the linear velocity, we multiply ω by the radius.

$v = r\omega$	
$= 3(90\pi)$	
$= 270\pi$ inches per minute	Exact value
$\approx 270(3.14)$ inches per minute	Approximately
$= 848$ inches per minute	To three significant digits

If we want this last quantity expressed in feet per minute, we divide by 12.

$$v = \frac{848}{12} \text{ feet per minute}$$

$$= 70.7 \text{ feet per minute}$$

▲

Problem Set 3.5

In this problem set, round any answers that need rounding to three significant digits.

Find the linear velocity of a point moving with uniform circular motion, if the point covers a distance s in an amount of time t, where

1. $s = 3$ feet and $t = 2$ minutes
2. $s = 10$ feet and $t = 2$ minutes
3. $s = 12$ centimeters and $t = 4$ seconds
4. $s = 12$ centimeters and $t = 2$ seconds
5. $s = 30$ miles and $t = 2$ hours
6. $s = 100$ miles and $t = 4$ hours

Find the distance s covered by a point moving with linear velocity v for a time t if

7. $v = 20$ feet per second and $t = 4$ seconds
8. $v = 10$ feet per second and $t = 4$ seconds
9. $v = 45$ miles per hour and $t = 1/2$ hour
10. $v = 55$ miles per hour and $t = 1/2$ hour
11. $v = 21$ miles per hour and $t = 20$ minutes
12. $v = 63$ miles per hour and $t = 10$ seconds

Point P sweeps out central angle θ as it rotates on a circle of radius r as given below. In each case, find the angular velocity of point P.

13. $\theta = 2\pi/3$, $t = 5$ seconds
14. $\theta = 3\pi/4$, $t = 5$ seconds
15. $\theta = 12$, $t = 3$ minutes
16. $\theta = 24$, $t = 6$ minutes
17. $\theta = 8\pi$, $t = 3\pi$ seconds
18. $\theta = 12\pi$, $t = 5\pi$ seconds
19. $\theta = 45\pi$, $t = 1.2$ hours
20. $\theta = 24\pi$, $t = 1.8$ hours

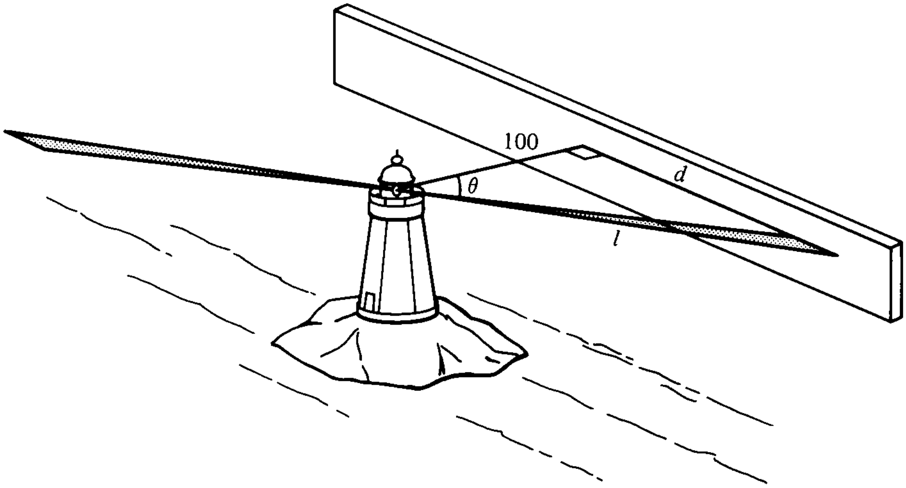


Figure 2

21. Figure 2 shows a lighthouse that is 100 feet from a long straight wall on the beach. The light in the lighthouse rotates through one complete rotation once every 4 seconds. Find an equation that gives the distance d in terms of time t, then find d when t is 1/2 second and 3/2 seconds. What happens when you try $t = 1$ second in the equation? How do you interpret this?

22. Using the diagram in Figure 2, find an equation that expresses l in terms of time t. Find l when t is 0.5 second, 1.0 second, and 1.5 seconds.

In the problems that follow, point P moves with angular velocity ω on a circle of radius r. In each case, find the distance s traveled by the point in time t.

23. $\omega = 4$ radians per second, $r = 2$ inches, $t = 5$ seconds
24. $\omega = 2$ radians per second, $r = 4$ inches, $t = 5$ seconds
25. $\omega = 3\pi/2$ radians per second, $r = 4$ meters, $t = 30$ seconds
26. $\omega = 4\pi/3$ radians per second, $r = 8$ meters, $t = 20$ seconds
27. $\omega = 15$ radians per second, $r = 5$ feet, $t = 1$ minute
28. $\omega = 10$ radians per second, $r = 6$ feet, $t = 2$ minutes

For each of the following problems, find the angular velocity associated with the given rpm's.

29. 10 rpm
30. 20 rpm
31. $33\frac{1}{3}$ rpm
32. $16\frac{2}{3}$ rpm
33. 5.8 rpm
34. 7.2 rpm

For each problem below, a point is rotating with uniform circular motion on a circle of radius r.

35. Find v if $r = 2$ inches and $\omega = 5$ radians per second
36. Find v if $r = 8$ inches and $\omega = 4$ radians per second
37. Find ω if $r = 6$ centimeters and $v = 3$ centimeters per second
38. Find ω if $r = 3$ centimeters and $v = 8$ centimeters per second
39. Find v if $r = 4$ feet and the point rotates at 10 rpm
40. Find v if $r = 1$ foot and the point rotates at 20 rpm
41. The earth rotates through one complete revolution every 24 hours. Since the axis of rotation is perpendicular to the equator, you can think of a person standing on the equator as standing on the edge of a disc that is rotating through one complete revolution every 24 hours. Find the angular velocity of a person standing on the equator.
42. Assuming the radius of the earth is 4,000 miles, use the information from Problem 41 to find the linear velocity of a person standing on the equator.
43. A boy is twirling a model airplane on a string 5 feet long. If he twirls the plane at 0.5 rpm, how far does the plane travel in 2 minutes?
44. A mixing blade on a food processor extends out 3 inches from its center. If the blade is turning at 600 rpm, what is the linear velocity of the tip of the blade in feet per minute?
45. A gasoline driven lawnmower has a blade that extends out 1 foot from its center. The tip of the blade is traveling at the speed of sound, which is 1,100 feet per second. Through how many rpm is the blade turning?
46. An 8 inch floppy disk in a computer (radius = 4 inches) rotates at 300 rpm. Find the linear velocity of a point on the edge of the disk.
47. How far does the tip of a 12 centimeter minute hand on a clock travel in 1 day?
48. How far does the tip of a 10 centimeter hour hand on a clock travel in 1 day?
49. A woman rides a bicycle for 1 hour and travels 16 kilometers (about 10 miles). Find the angular velocity of the wheel if the radius is 30 centimeters.
50. Find the number of rpm for the wheel in Problem 49.

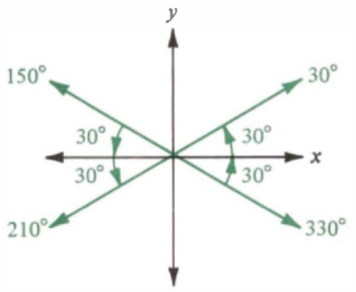
Review Problems The problems that follow review material we covered in Section 2.4.

51. If a 75 foot flag pole casts a shadow 43 feet long, what is the angle of elevation of the sun from the tip of the shadow?
52. A road up a hill makes an angle of 5° with the horizontal. If the road from the bottom of the hill to the top of the hill is 2.5 miles long, how high is the hill?
53. A person standing 5 feet from a mirror notices that the angle of depression from his eyes to the bottom of the mirror is 13°, while the angle of elevation to the top of the mirror is 12°. Find the vertical dimensions of the mirror.
54. A boat travels on a course of bearing S 63° 50′ E for 100 miles. How many miles south and how many miles east has the boat traveled?

Chapter 3 Summary and Review

Examples

1. 30° is the reference angle for 30°, 150°, 210°, and 330°.



REFERENCE ANGLE [3.1]

The *reference angle* $\hat{\theta}$ for any angle θ in standard position is the positive acute angle between the terminal side of θ and the x-axis.

2. $\sin 150° = \sin 30° = \frac{1}{2}$

$\sin 210° = -\sin 30° = -\frac{1}{2}$

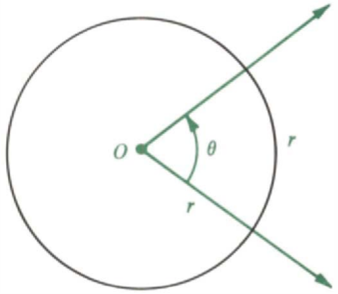
$\sin 330° = -\sin 30° = -\frac{1}{2}$

REFERENCE ANGLE THEOREM [3.1]

A trigonometric function of an angle and its reference angle differ at most in sign.

We find trigonometric functions for angles between 0° and 360° by first finding the reference angle. We then find the value of the trigonometric function of the reference angle and use the quadrant in which the angle terminates to assign the correct sign.

3. angle θ has a measure of 1 radian



the vertex of θ is at the center of the circle: the arc cut off by θ is equal in length to the radius

RADIAN MEASURE [3.2]

In a circle, a central angle that cuts off an arc equal in length to the radius of the circle has a measure of 1 radian.

RADIANS AND DEGREES [3.2]

Changing from degrees to radians and radians to degrees is simply a matter of multiplying by the appropriate conversion factor.

Degrees → Multiply by $\frac{\pi}{180}$ → Radians

Radians → Multiply by $\frac{180}{\pi}$ → Degrees

4. Radians to degrees

$$\frac{4\pi}{3} \text{ radians} = \frac{4\pi}{3}\left(\frac{180}{\pi}\right)^{\circ} = 240^{\circ}$$

Degrees to radians

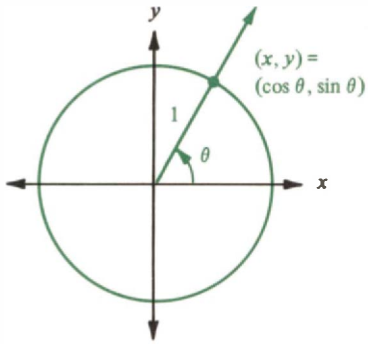
$$450^{\circ} = 450\left(\frac{\pi}{180}\right) = \frac{5\pi}{2} \text{ radians}$$

THE UNIT CIRCLE [3.3]

The unit circle is the circle with its center at the origin and a radius of 1. The equation of the unit circle is $x^2 + y^2 = 1$. Because the radius of the unit circle is 1, any point (x, y) on the circle is such that

$$x = \cos\theta \text{ and } y = \sin\theta$$

5.



EVEN AND ODD FUNCTIONS [3.3]

An *even function* is a function for which

$$f(-x) = f(x) \text{ for all } x \text{ in the domain of } f$$

and an *odd function* is a function for which

$$f(-x) = -f(x) \text{ for all } x \text{ in the domain of } f$$

Cosine is an even function, and sine is an odd function. That is,

$\cos(-\theta) = \cos\theta$ Cosine is an even function

and

$\sin(-\theta) = -\sin\theta$ Sine is an odd function

6. $\tan\theta$ is an odd function.

$$\begin{aligned}\tan(-\theta) &= \frac{\sin(-\theta)}{\cos(-\theta)} \\ &= \frac{-\sin\theta}{\cos\theta} \\ &= -\frac{\sin\theta}{\cos\theta} \\ &= -\tan\theta\end{aligned}$$

ARC LENGTH [3.4]

If s is an arc cut off by a central angle θ, measured in radians, in a circle of radius r, then

$$s = r\theta$$

7. The arc cut off by 2.5 radians in a circle of radius 4 inches is

$$\begin{aligned} s &= 4(2.5) \\ &= 10.0 \text{ inches} \end{aligned}$$

AREA OF A SECTOR [3.4]

The area of the sector formed by a central angle θ in a circle of radius r is

$$A = \frac{1}{2}r^2\theta$$

where θ is measured in radians.

8. The area of the sector formed by a central angle of 2.5 radians in a circle of radius 4 inches is

$$\begin{aligned} A &= \frac{1}{2}(4)^2(2.5) \\ &= 20.0 \text{ inches}^2 \end{aligned}$$

LINEAR VELOCITY [3.5]

If P is a point on a circle of radius r, and P moves a distance s on the circumference of the circle, in an amount of time t, then the *linear velocity*, v, of P is given by the formula

$$v = \frac{s}{t}$$

9. If a point moving at a uniform speed on a circle travels 12 centimeters every 3 seconds, then the linear velocity of the point is

$$\begin{aligned} v &= \frac{12 \text{ centimeters}}{3 \text{ seconds}} \\ &= 4 \text{ centimeters per second} \end{aligned}$$

ANGULAR VELOCITY [3.5]

If P is a point moving with uniform circular motion on a circle of radius r, and the line from the center of the circle through P sweeps out a central angle θ, in an amount of time t, then the *angular velocity*, ω, of P is given by the equation

$$\omega = \frac{\theta}{r} \qquad \text{where } \theta \text{ is measured in radians}$$

The relationship between linear velocity and angular velocity is given by the formula

$$v = r\omega$$

10. If a point moving at uniform speed on a circle of radius 4 inches rotates through $3\pi/4$ radians every 3 seconds, then the angular velocity of the point is

$$\begin{aligned} \omega &= \frac{3\pi/4 \text{ radians}}{3 \text{ seconds}} \\ &= \frac{\pi}{4} \text{ radians per second} \end{aligned}$$

The linear velocity of the same point is given by

$$\begin{aligned} v &= 4\left(\frac{\pi}{4}\right) \\ &= \pi \text{ inches per second} \end{aligned}$$

Chapter 3 Test

Draw each of the following angles in standard position and then name the reference angle:

1. $235°$ **2.** $117.8°$
3. $410° 20'$ **4.** $-225°$

Use the tables in the back of the book or a calculator to find each of the following:

5. $\tan 320°$ **6.** $\tan(-25°)$
7. $\cos(-236.7°)$ **8.** $\sin 322.3°$
9. $\sec 140° 20'$ **10.** $\csc 188° 50'$

Use the tables in the back of the book or a calculator to find θ, if θ is between $0°$ and $360°$ and

11. $\sin \theta = 0.1045$ with θ in QII **12.** $\cos \theta = -0.4772$ with θ in QIII
13. $\cot \theta = 0.9659$ with θ in QIII **14.** $\sec \theta = 1.545$ with θ in QIV

Give the exact value of each of the following:

15. $\sin 225°$ **16.** $\cos 135°$
17. $\tan 330°$ **18.** $\sec 390°$

Convert each of the following to radian measure. Write each answer as an exact value.

19. $250°$ **20.** $-390°$

Convert each of the following to degree measure:

21. $4\pi/3$ **22.** $7\pi/12$

Give the exact value of each of the following:

23. $\sin \frac{2\pi}{3}$ **24.** $\cos \frac{2\pi}{3}$

25. $4 \cos\left(-\frac{3\pi}{4}\right)$ **26.** $2 \cos\left(-\frac{5\pi}{3}\right)$

27. $\sec \frac{5\pi}{6}$ **28.** $\csc \frac{5\pi}{6}$

29. Evaluate $2 \cos\left(3x - \frac{\pi}{2}\right)$ when x is $\frac{\pi}{3}$.

30. Evaluate $4\sin\left(2x + \frac{\pi}{4}\right)$ when x is $\frac{\pi}{4}$.

31. Show that cotangent is an odd function.
32. Prove the identity $\sin(-\theta)\sec(-\theta)\cot(-\theta) = 1$.

For each problem below, θ is a central angle in a circle of radius r. In each case, find the length of arc s cut off by θ.

33. $\theta = \pi/6$, $r = 12$ meters
34. $\theta = 60°$, $r = 6$ feet

In each problem below, θ is a central angle that cuts off an arc of length s. In each case, find the radius of the circle.

35. $\theta = \pi/4$, $s = \pi$ centimeters
36. $\theta = 2\pi/3$, $s = \pi/4$ centimeters

Find the area of the sector formed by central angle θ in a circle of radius r if

37. $\theta = 90°$, $r = 4$ inches
38. $\theta = 2.4$, $r = 3$ centimeters

39. The minute hand of a clock is 2 centimeters long. How far does the tip of the minute hand travel in 30 minutes?
40. A central angle of 4 radians cuts off an arc of length 8 inches. Find the area of the sector formed.

Find the distance s covered by a point moving with linear velocity v for a time t if

41. $v = 30$ feet per second and $t = 3$ seconds
42. $v = 66$ feet per second and $t = 1$ minute

In the problems that follow, point P moves with angular velocity ω on a circle of radius r. In each case, find the distance s traveled by the point in time t.

43. $\omega = 4$ radians per second, $r = 3$ inches, $t = 6$ seconds
44. $\omega = 3\pi/4$ radians per second, $r = 8$ feet, $t = 20$ seconds

For each of the following problems, find the angular velocity associated with the given rpm's.

45. 6 rpm
46. 2 rpm

For each problem below, a point is rotating with uniform circular motion on a circle of radius r.

47. Find ω if $r = 10$ centimeters and $v = 5$ centimeters per second
48. Find ω if $r = 3$ centimeters and $v = 5$ centimeters per second
49. Find v if $r = 2$ feet and the point rotates at 20 rpm
50. Find v if $r = 1$ foot and the point rotates at 10 rpm
51. A belt connects a pulley of radius 8 centimeters to a pulley of radius 6 centimeters. Each point on the belt is traveling at 24 centimeters per second. Find the angular velocity of each pulley (see Figure 1).

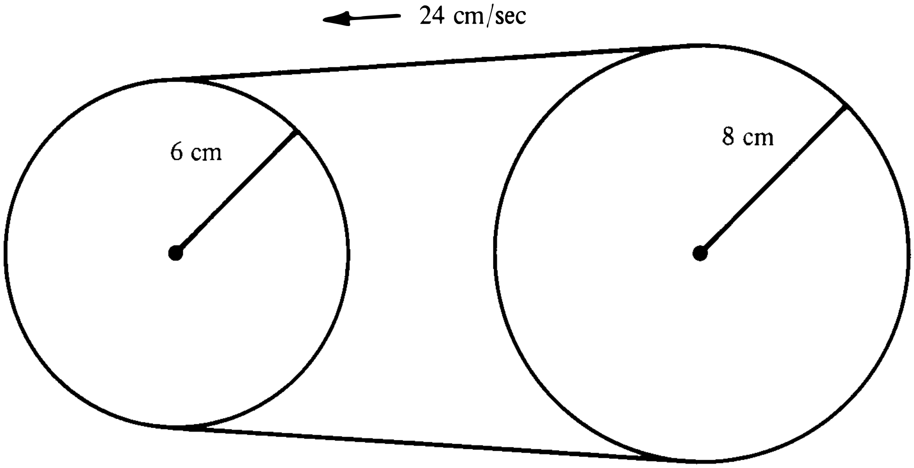


Figure 1

52. A propeller with radius 1.5 feet is rotating at 900 rpm. Find the linear velocity of the tip of the propeller. Give the exact value and a three significant digit approximation.

4 Graphing and Inverse Functions

To the student:

In this chapter we will consider the graphs of the trigonometric functions. We will begin by graphing $y = \sin x$ and $y = \cos x$, and then proceed to more complicated graphs. To begin with, we will make tables of values of x and y that satisfy the equations and then use the information in these tables to sketch the associated graphs. Our ultimate goal, however, is to graph trigonometric functions of the form $y = A \sin(Bx + C)$ and $y = A \cos(Bx + C)$ without the use of tables. Equations of this form have many applications. For instance, they can be used as mathematical models for sound waves and to describe such things as the variations in electrical current in an alternating circuit.

The last two sections of this chapter (Sections 4.5 and 4.6) are concerned with inverse trigonometric relations and inverse trigonometric functions. Since these topics are closely related to the work you did previously with inverse functions in algebra, we will begin Section 4.5 with two examples that review the concepts of inverse relations in general.

4.1 Basic Graphs

The Sine Graph

To graph the equation $y = \sin x$ we begin by making a table of values of x and y that satisfy the equation, and then use the information in the table to sketch the graph. To make it easy on ourselves, we will let x take on values that are multiples of $\pi/4$. As an aid in sketching the graphs, we will approximate $1/\sqrt{2}$ with 0.7.

Table 1

x	$y = \sin x$	(x, y)
0	$y = \sin 0 = 0$	$(0, 0)$
$\frac{\pi}{4}$	$y = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} = 0.7$	$\left(\frac{\pi}{4}, 0.7\right)$
$\frac{\pi}{2}$	$y = \sin \frac{\pi}{2} = 1$	$\left(\frac{\pi}{2}, 1\right)$
$\frac{3\pi}{4}$	$y = \sin \frac{3\pi}{4} = \frac{1}{\sqrt{2}} = 0.7$	$\left(\frac{3\pi}{4}, 0.7\right)$
π	$y = \sin \pi = 0$	$(\pi, 0)$
$\frac{5\pi}{4}$	$y = \sin \frac{5\pi}{4} = -\frac{1}{\sqrt{2}} = -0.7$	$\left(\frac{5\pi}{4}, -0.7\right)$
$\frac{3\pi}{2}$	$y = \sin \frac{3\pi}{2} = -1$	$\left(\frac{3\pi}{2}, -1\right)$
$\frac{7\pi}{4}$	$y = \sin \frac{7\pi}{4} = -\frac{1}{\sqrt{2}} = -0.7$	$\left(\frac{7\pi}{4}, -0.7\right)$
2π	$y = \sin 2\pi = 0$	$(2\pi, 0)$

Graphing each ordered pair and then connecting them with a smooth curve, we obtain the following graph:

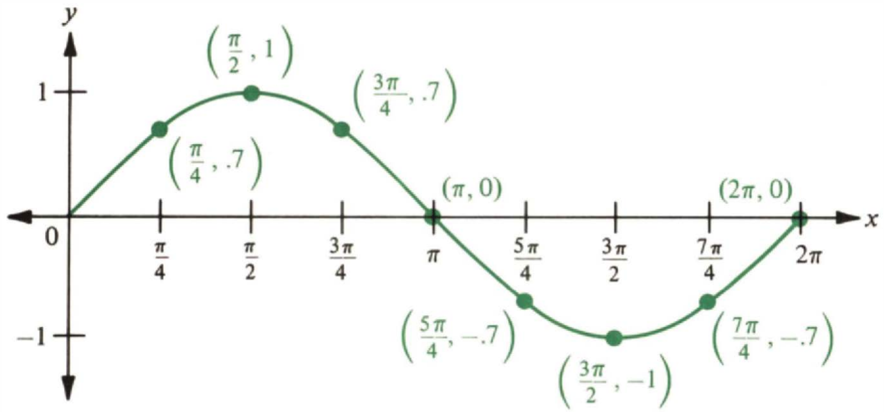


Figure 1

To further justify the graph in Figure 1, we could find additional ordered pairs that satisfy the equation. For example, we could continue our table by letting x take on multiples of $\pi/6$ and $\pi/3$. If we were to do so, we would find that any new ordered pair that satisfied the equation $y = \sin x$ would be such that its graph would lie on the curve in Figure 1. Figure 2 shows the curve in Figure 1 again, but this time with all the ordered pairs with x-coordinates that are multiples of $\pi/6$ or $\pi/3$ and the corresponding y-coordinates that satisfy the equation $y = \sin x$.

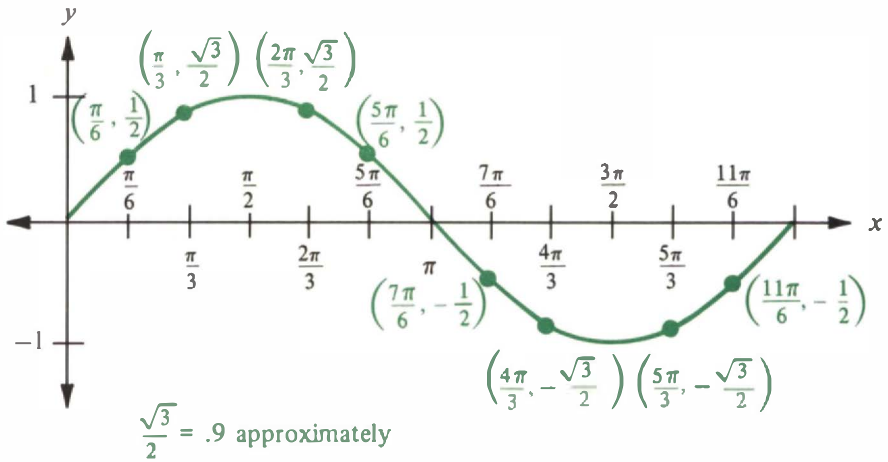


Figure 2

Example 1 Use the graph of $y = \sin x$ in Figure 2 to find all values of x, between 0 and 2π, for which $\sin x = 1/2$.

Solution We locate $1/2$ on the y-axis and draw a horizontal line through it. We follow this line to the points where it intersects the graph of $y = \sin x$. The values of x just below these points of intersection are the values of x for which $\sin x = 1/2$. As Figure 3 indicates, they are $\pi/6$ and $5\pi/6$.

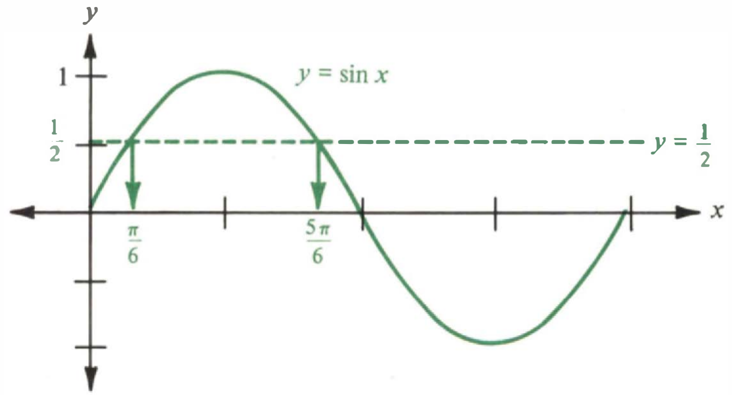
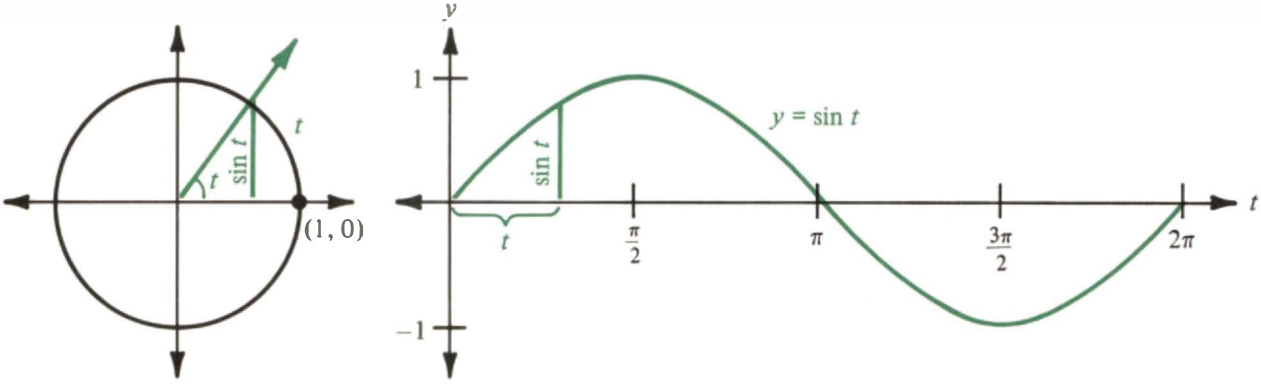


Figure 3

Graphing $y = \sin x$ Using the Unit Circle

We can also obtain the graph of $y = \sin x$ by using the unit circle. If we start at (1, 0) and rotate once around the unit circle (through 2π) we can find the value of y in the equation $y = \sin t$ by simply keeping track of the y-coordinates of the points on the unit circle through which the terminal side of our angle t passes. (In this case, we use the variable t instead of the variable x to represent our angle so as not to confuse it with the x-coordinate of the point (x, y) on the unit circle.)



unit circle

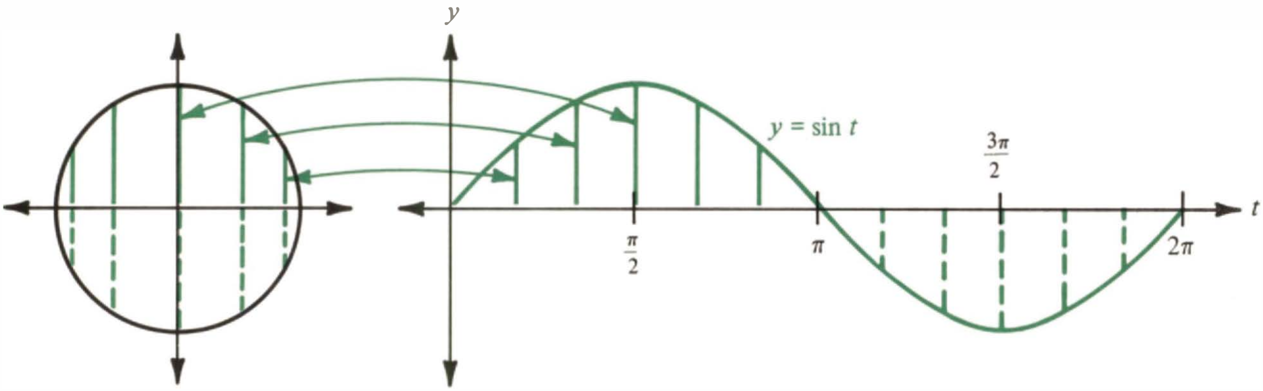


Figure 4

Extending the Sine Graph

Figures 1 through 4 each show one complete cycle of $y = \sin x$. We can extend the graph of $y = \sin x$ to the right of $x = 2\pi$ by realizing that, once we go past $x = 2\pi$, we will begin to name angles that are coterminal with the angles between 0 and 2π. Because of this, we will start to repeat the values of sin x. Likewise, if we let x take on values to the left of $x = 0$, we will simply get the values of sin x between 0 and 2π in the reverse order. Figure 5 shows the graph of $y = \sin x$ extended beyond the interval from $x = 0$ to $x = 2\pi$.

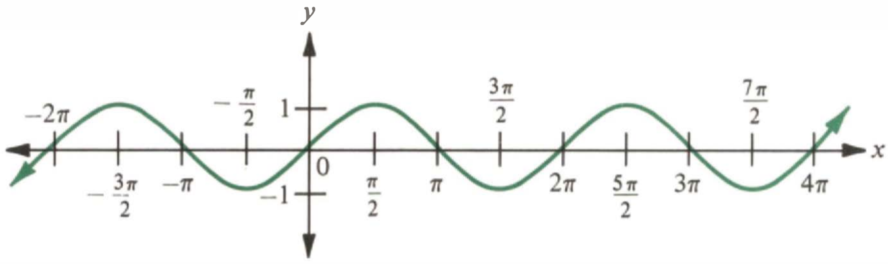


Figure 5

Our graph of $y = \sin x$ never goes above 1 or below -1, and it repeats itself every 2π units on the x-axis. This gives rise to the following two definitions.

DEFINITION (PERIOD) For any function $y = f(x)$, the smallest positive number p for which

$$f(x + p) = f(x) \qquad \text{for all } x$$

is called the *period* of $f(x)$. In the case of $y = \sin x$, the period is 2π since 2π is the smallest positive number for which

$$\sin(x + 2\pi) = \sin x \text{ for all } x$$

DEFINITION (AMPLITUDE) If the greatest value of y is M and the least value of y is m, then the *amplitude* of the graph of y is defined to be

$$A = \frac{1}{2}|M - m|$$

In the case of $y = \sin x$, the amplitude is 1 because $1/2|1 - (-1)| = 1/2(2) = 1$.

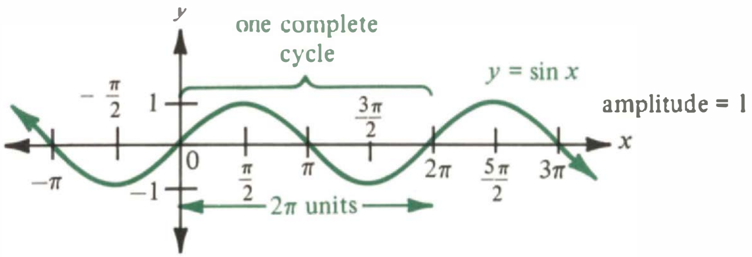


Figure 6

The Cosine Curve

The graph of $y = \cos x$ has the same general shape as the graph of $y = \sin x$.

Example 2 Sketch the graph of $y = \cos x$.

Solution We can arrive at the graph by making a table of convenient values of x and y or by making an appropriate diagram of the unit circle. Since the values of cosine will repeat themselves every 2π units on the x-axis, we use our table or the unit circle to sketch the graph of $y = \cos x$ from 0 to 2π and then continue this graph to the right of 2π and to the left of 0.

Table 2

x	$y = \cos x$	(x, y)
0	$y = \cos 0 = 1$	$(0, 1)$
$\frac{\pi}{4}$	$y = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$	$\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$
$\frac{\pi}{2}$	$y = \cos \frac{\pi}{2} = 0$	$\left(\frac{\pi}{2}, 0\right)$
$\frac{3\pi}{4}$	$y = \cos \frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$	$\left(\frac{3\pi}{4}, -\frac{1}{\sqrt{2}}\right)$
π	$y = \cos \pi = -1$	$(\pi, -1)$
$\frac{5\pi}{4}$	$y = \cos \frac{5\pi}{4} = -\frac{1}{\sqrt{2}}$	$\left(\frac{5\pi}{4}, -\frac{1}{\sqrt{2}}\right)$
$\frac{3\pi}{2}$	$y = \cos \frac{3\pi}{2} = 0$	$\left(\frac{3\pi}{2}, 0\right)$
$\frac{7\pi}{4}$	$y = \cos \frac{7\pi}{4} = \frac{1}{\sqrt{2}}$	$\left(\frac{7\pi}{4}, \frac{1}{\sqrt{2}}\right)$
2π	$y = \cos 2\pi = 1$	$(2\pi, 1)$

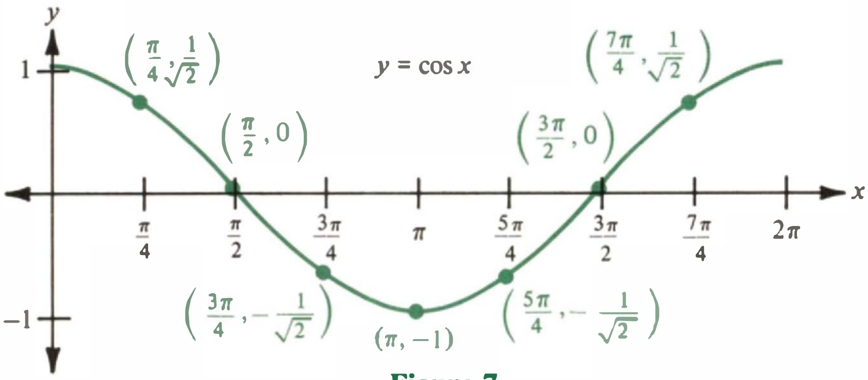


Figure 7

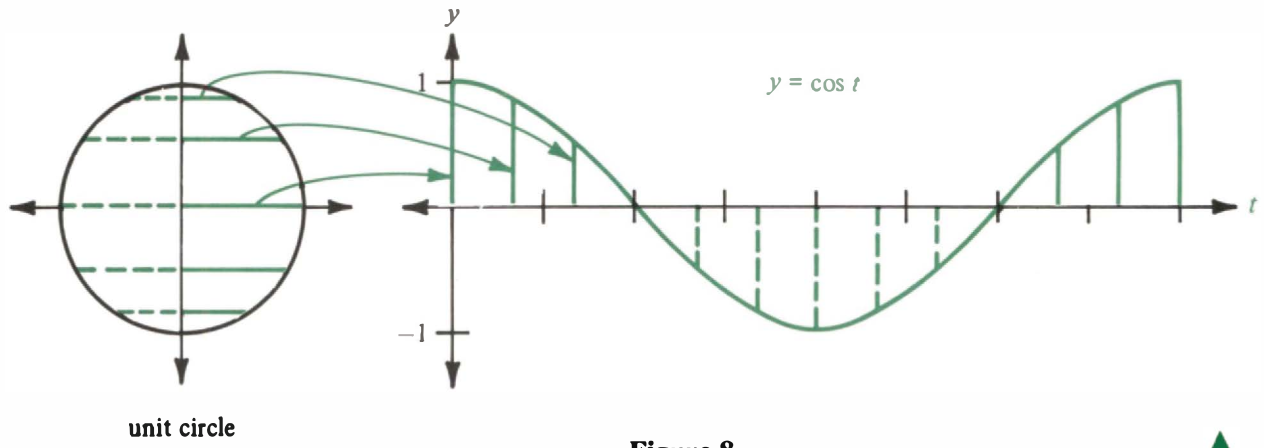


Figure 8

Example 3 Find all values of x for which $\cos x = -1$.

Solution We draw a horizontal line through $y = -1$ and notice where it intersects the graph of $y = \cos x$. The x-coordinates of those points are solutions to $\cos x = -1$.

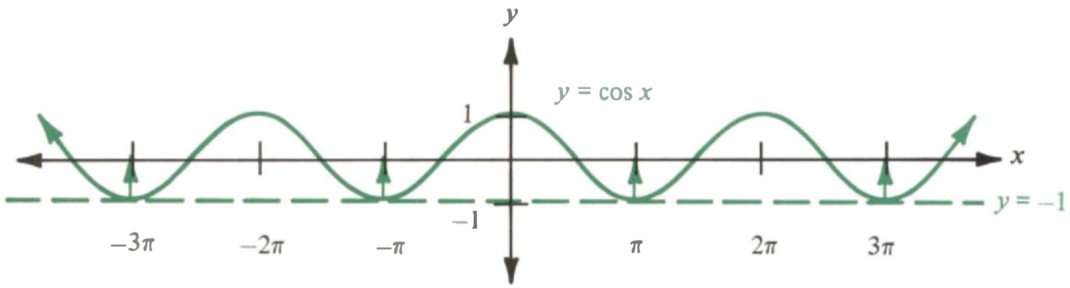


Figure 9

Figure 9 indicates that all solutions to $\cos x = -1$ are

$$x = \ldots -3\pi, -\pi, \pi, 3\pi, \ldots$$

Since each pair of consecutive solutions differ by 2π, we can write the solutions in a more compact form as

$$x = \pi + 2k\pi \qquad \text{where } k \text{ is an integer}$$

The Tangent Graph

Table 3 lists some solutions to $y = \tan x$ between $x = 0$ and $x = 2\pi$. Note that, at $\pi/2$ and $3\pi/2$, tangent is undefined. If we think of $\tan x$ as $(\sin x)/(\cos x)$, it must be undefined at these values of x since $\cos x$ is 0 at multiples of $\pi/2$ and division by 0 is undefined.

Table 3

x	$\tan x$
0	0
$\frac{\pi}{4}$	1
$\frac{\pi}{3}$	$\sqrt{3} \approx 1.7$
$\frac{\pi}{2}$	undefined
$\frac{2\pi}{3}$	$-\sqrt{3} \approx -1.7$
$\frac{3\pi}{4}$	-1
π	0
$\frac{5\pi}{4}$	1
$\frac{4\pi}{3}$	$\sqrt{3} \approx 1.7$
$\frac{3\pi}{2}$	undefined
$\frac{5\pi}{3}$	$-\sqrt{3} \approx -1.7$
$\frac{7\pi}{4}$	-1
2π	0

The entries in Table 3 for which $y = \tan x$ is undefined correspond to values of x for which there is no corresponding value of y. That is, there will be no point on the graph with an x-coordinate of $\pi/2$ or $3\pi/2$. To help us remember this, we have drawn dotted vertical lines through $x = \pi/2$ and $x = 3\pi/2$. These vertical lines are called *asymptotes*. Our graph will never cross or touch these lines. Figure 10 shows the information we have so far. If we were to use a calculator or table to find other values of tan x close to the asymptotes in Figure 10, we would find that tan x would become very large as we got close to the left side of an asymptote and very small as we got close to the right side of an asymptote. For example, if we were to think of the numbers on the x-axis as degrees rather than radians, $x = 85°$ would

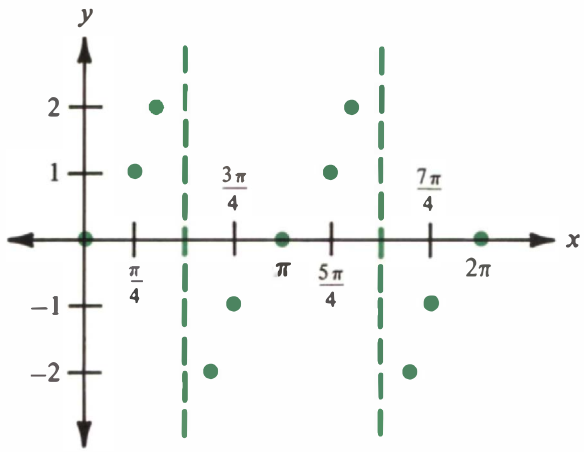


Figure 10

be found just to the left of the asymptote at $\pi/2$ and tan 85° would be approximately 11. Likewise tan 89° would be approximately 57, and tan 89.9° would be about 573. As you can see, as x moves closer to $\pi/2$ from the left side of $\pi/2$, tan x gets larger and larger. In Figure 11 we connect the points

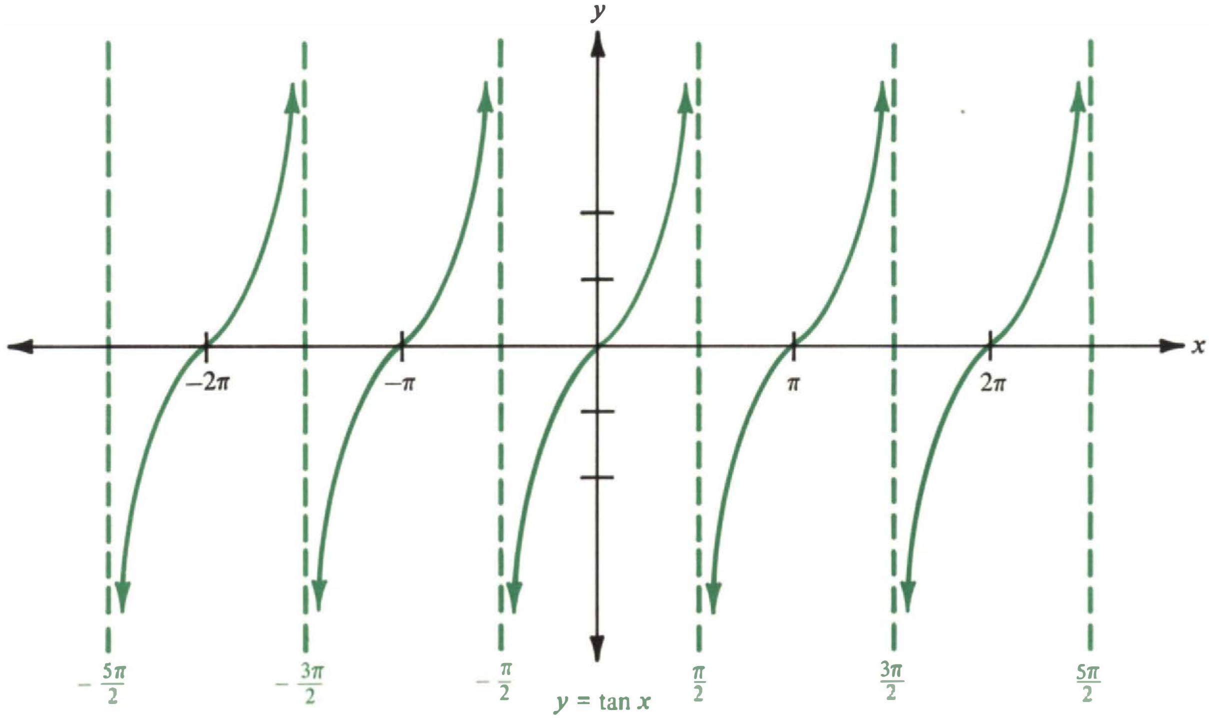


Figure 11

from Table 3 in the manner appropriate to this discussion and then extend this graph above 2π and below 0 as we did with our sine and cosine curves.

As Figure 11 indicates, the period of $y = \tan x$ is π. The tangent function has no amplitude since there is no largest or smallest value of y on the graph of $y = \tan x$.

Example 4

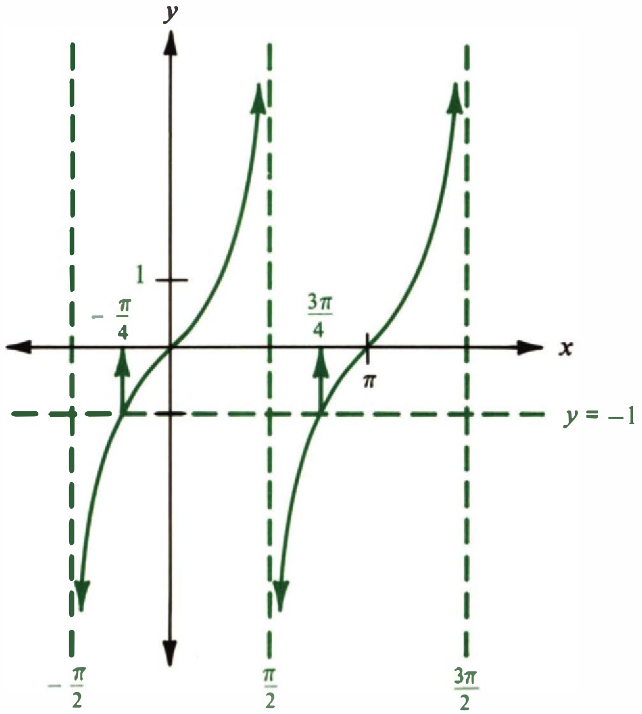


Figure 12

The values of x between $x = -\pi/2$ and $x = 3\pi/2$ that satisfy the equation $\tan x = -1$ are $x = -\pi/4$ and $3\pi/4$.

Example 5 Sketch the graph of $y = \csc x$.

The Cosecant Graph

Solution Instead of making a table of values to help us graph $y = \csc x$, we can use the fact that $\csc x$ and $\sin x$ are reciprocals.

When sin x is	csc x will be
1	1
$\frac{1}{2}$	2
$\frac{1}{3}$	3
$\frac{1}{4}$	4
0	undefined
$-\frac{1}{4}$	-4
$-\frac{1}{3}$	-3
$-\frac{1}{2}$	-2
-1	-1

Figure 13 shows how this looks graphically.

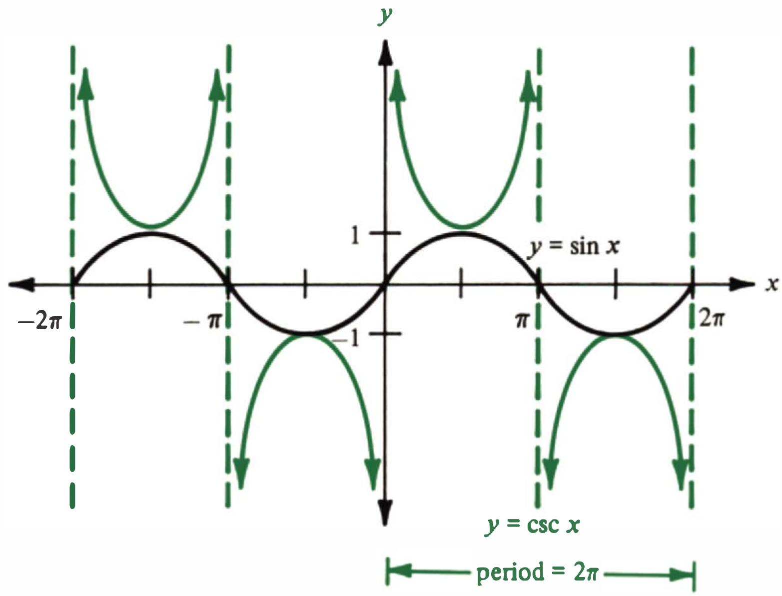


Figure 13

From the graph, we see that the period of $y = \csc x$ is 2π and, as was the case with $y = \tan x$, there is no amplitude. ▲

The Cotangent and Secant Graphs

In Problem Set 4.1 you will be asked to graph $y = \cot x$ and $y = \sec x$. These graphs are shown in Figures 14 and 15 for reference.

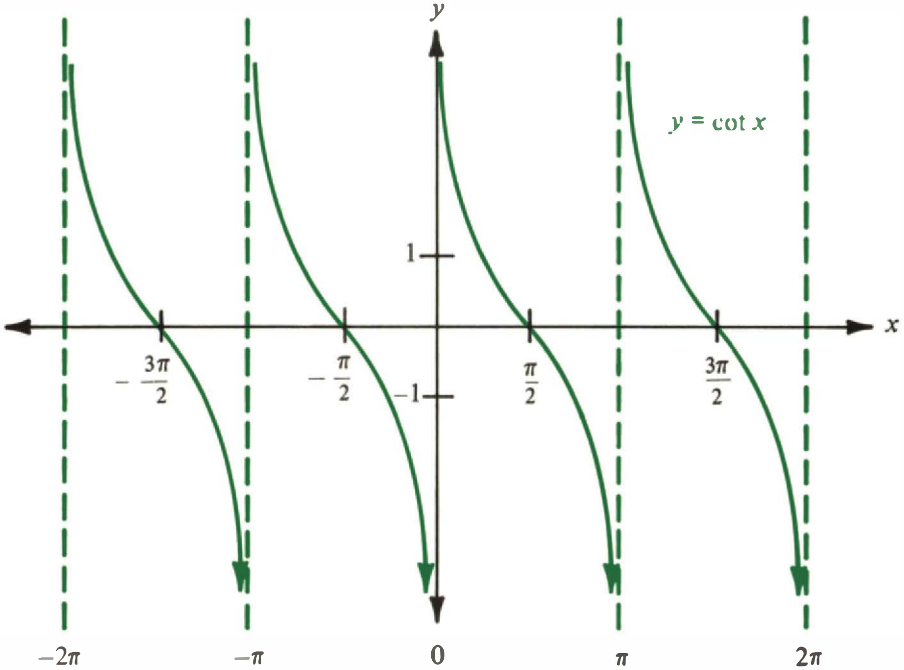


Figure 14

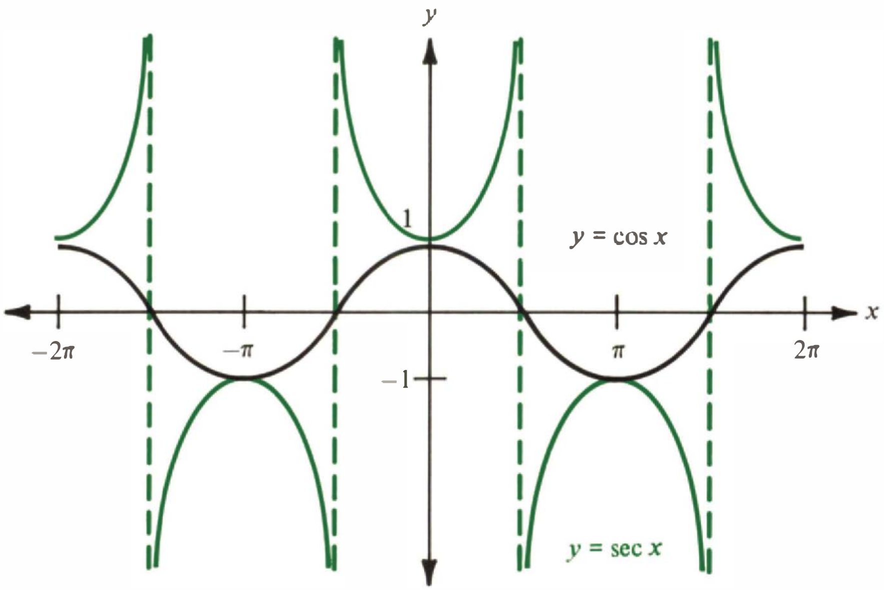


Figure 15

To end this section, we list some of the important facts we know about the graphs of the six trigonometric functions. (Recall from algebra that the domain of a function $y = f(x)$ is the set of values that the variable x can assume, while the range is the set of values that y assumes.)

Functions	Domain	Range	Amplitude	Period
$y = \sin x$ and $y = \cos x$	all real numbers	$-1 \le y \le 1$	1	2π
$y = \tan x$ and $y = \cot x$	all real numbers except where k is an integer	all real numbers	none	π
$y = \sec x$ and $y = \csc x$	all real numbers except $x = k\pi$ where k is an integer	$x \le -1$ or $x \ge 1$	none	2π

Problem Set 4.1

Although we have worked a number of the problems below in the section previous to this, working them again yourself will help you become more familiar with the graphs of the six basic trigonometric functions. In each case, make a table of values for each function using multiples of $\pi/4$ for x. Then use the entries in the table to sketch the graph of each function for x between 0 and 2π.

1. $y = \cos x$
2. $y = \cot x$
3. $y = \csc x$
4. $y = \sin x$
5. $y = \tan x$
6. $y = \sec x$

Use the graphs you found in Problems 1 through 6 to find all values of x between 0 and 2π for which each of the following is true:

7. $\cos x = 1/2$
8. $\sin x = 1/2$
9. $\csc x = -1$
10. $\sec x = 1$
11. $\cos x = -1/\sqrt{2}$
12. $\cot x = -1$
13. $\tan x = 1$
14. $\sin x = -\sqrt{3}/2$
15. $\cos x = \sqrt{3}/2$
16. $\sin x = 1/\sqrt{2}$

Sketch the graphs of each of the following between $x = -4\pi$ and $x = 4\pi$ by extending the graphs you made in Problems 1 through 6:

17. $y = \sin x$
18. $y = \cos x$
19. $y = \sec x$
20. $y = \csc x$
21. $y = \cot x$
22. $y = \tan x$

Use the graphs you found in Problems 17 through 22 to find all values of x between $x = 0$ and $x = 4\pi$ for which each of the following is true:

23. $\sin x = \sqrt{3}/2$

24. $\cos x = 1/\sqrt{2}$

25. $\sec x = -1$

26. $\csc x = 1$

27. $\sin x = -1/2$

28. $\cos x = -1/2$

29. $\cot x = 1$

30. $\tan x = -1$

31. $\sin x = 1/\sqrt{2}$

32. $\cos x = -\sqrt{3}/2$

Find all values of x for which the following are true:

33. $\cos x = 0$

34. $\sin x = 0$

35. $\sin x = 1$

36. $\cos x = 1$

37. $\tan x = 0$

38. $\cot x = 0$

39. Figure 16 is another diagram of the unit circle with the line segment corresponding to $\tan t$ showing. Make a diagram similar to the diagrams in Figures 4 and 8 from this section that shows how the unit circle can be used to obtain the graph of $y = \tan t$ from $t = -\pi/2$ to $t = \pi/2$.

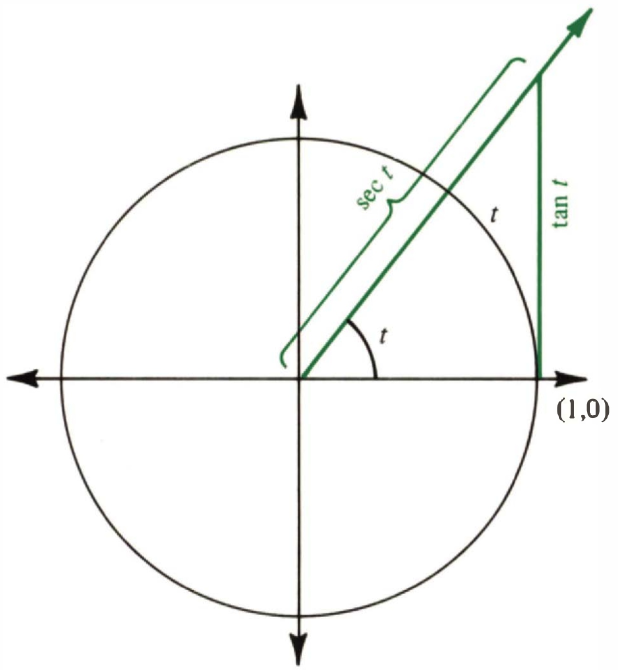
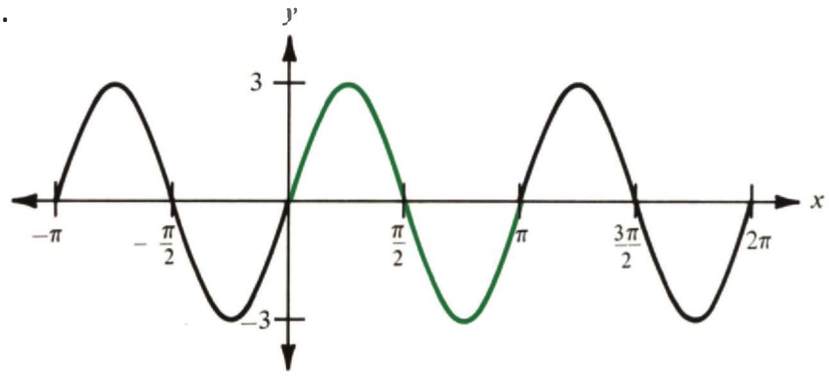


Figure 16

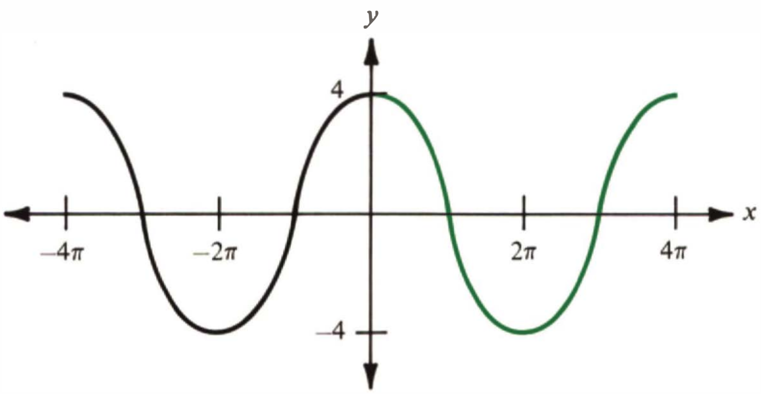
40. Following the instructions in Problem 39, use Figure 16 to sketch the graph of $y = \sec t$ from $t = 0$ to $t = \pi$.

Give the amplitude and period of each of the following graphs:

41.



42.



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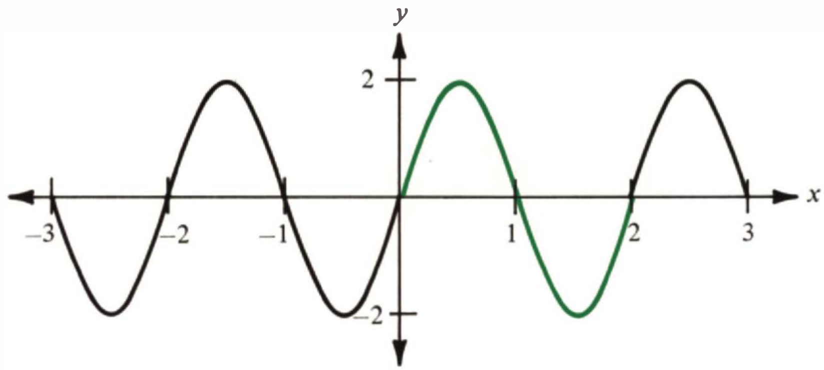

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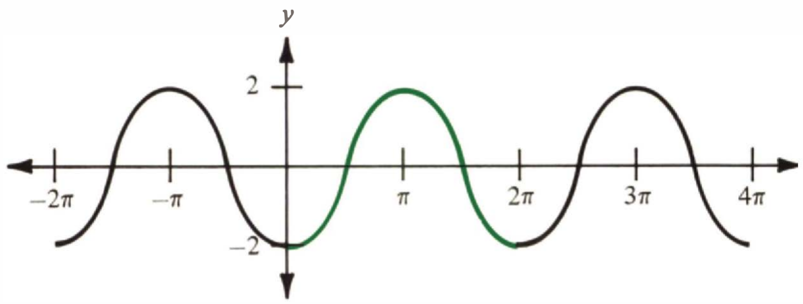
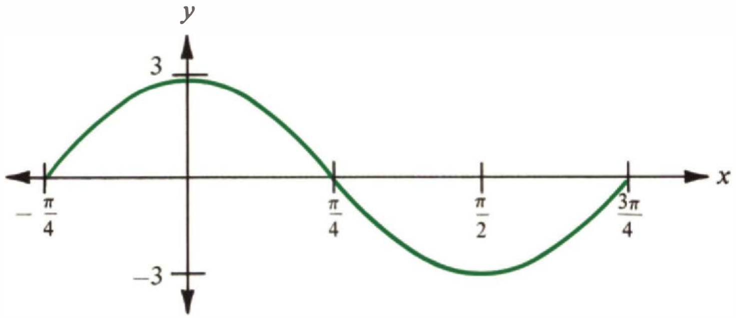
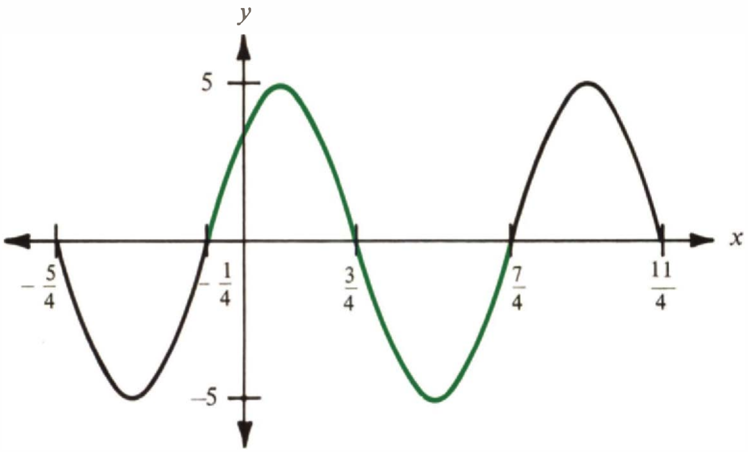


45.



46.



47. Sketch the graph of $y = 2 \sin x$ from $x = 0$ to $x = 2\pi$ by making a table using multiples of $\pi/2$ for x. What is the amplitude of the graph you obtain?

48. Sketch the graph of $y = (1/2)\cos x$ from $x = 0$ to $x = 2\pi$ by making a table using multiples of $\pi/2$ for x. What is the amplitude of the graph you obtain?

49. Make a table using multiples of $\pi/4$ for x to sketch the graph of $y = \sin 2x$

from $x = 0$ to $x = 2\pi$. After you have obtained the graph, state the number of complete cycles your graph goes through between 0 and 2π.

50. Make a table using multiples of $\pi/6$ and $\pi/3$ to sketch the graph of $y = \sin 3x$ from $x = 0$ to $x = 2\pi$. After you have obtained the graph, state the number of complete cycles your graph goes through between 0 and 2π.

Review Problems The problems that follow review material we covered in Sections 1.5 and 3.3.

Prove the following identities.

51. $\cos\theta \tan\theta = \sin\theta$

52. $\sin\theta \tan\theta + \cos\theta = \sec\theta$

53. $(1 + \sin\theta)(1 - \sin\theta) = \cos^2\theta$

54. $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$

55. $\csc\theta + \sin(-\theta) = \dfrac{\cos^2\theta}{\sin\theta}$

56. $\sec\theta - \cos(-\theta) = \dfrac{\sin^2\theta}{\cos\theta}$

4.2 Amplitude and Period

In Section 4.1 the graphs of $y = \sin x$ and $y = \cos x$ were shown to have a period of 2π and an amplitude of 1. In this section we will extend our work with graphing to include a more detailed look at amplitude and period.

Example 1 Sketch the graph of $y = 2\sin x$, if $0 \le x \le 2\pi$.

Solution The coefficient 2 on the right side of the equation will simply multiply each value of $\sin x$ by a factor of 2. Therefore, the values of y in $y = 2\sin x$ should all be twice the corresponding values of y in $y = \sin x$. Table 1 contains some values for $y = 2\sin x$. Figure 1 shows the graphs of $y = \sin x$ and $y = 2\sin x$. (We are including the graph of $y = \sin x$ simply for reference and comparison. With both graphs to look at, it is easier to see what change is brought about by the coefficient 2.)

Table 1

x	$y = 2\sin x$	(x, y)
0	$y = 2\sin 0 = 2(0) = 0$	$(0, 0)$
$\dfrac{\pi}{2}$	$y = 2\sin\dfrac{\pi}{2} = 2(1) = 2$	$\left(\dfrac{\pi}{2}, 2\right)$
π	$y = 2\sin\pi = 2(0) = 0$	$(\pi, 0)$
$\dfrac{3\pi}{2}$	$y = 2\sin\dfrac{3\pi}{2} = 2(-1) = -2$	$\left(\dfrac{3\pi}{2}, -2\right)$
2π	$y = 2\sin 2\pi = 2(0) = 0$	$(2\pi, 0)$

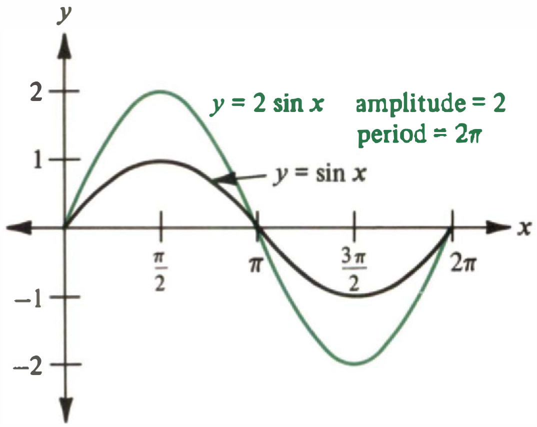


Figure 1

The coefficient 2 in $y = 2 \sin x$ changes the amplitude from 1 to 2, but does not affect the period. That is, we can think of the graph of $y = 2 \sin x$ as if it were the graph of $y = \sin x$ with the amplitude extended to 2 instead of 1. ▲

▼ **Example 2** Sketch one complete cycle of the graph of $y = \frac{1}{2} \cos x$.

Solution Table 2 gives us some points on the curve $y = (1/2)\cos x$. Figure 2 shows the graphs of both $y = (1/2)\cos x$ and $y = \cos x$ on the same set of axes, from $x = 0$ to $x = 2\pi$.

Table 2

x	$y = \frac{1}{2} \cos x$	(x, y)
0	$y = \frac{1}{2} \cos 0 = \frac{1}{2}(1) = \frac{1}{2}$	$\left(0, \frac{1}{2}\right)$
$\frac{\pi}{2}$	$y = \frac{1}{2} \cos \frac{\pi}{2} = \frac{1}{2}(0) = 0$	$\left(\frac{\pi}{2}, 0\right)$
π	$y = \frac{1}{2} \cos \pi = \frac{1}{2}(-1) = -\frac{1}{2}$	$\left(\pi, -\frac{1}{2}\right)$
$\frac{3\pi}{2}$	$y = \frac{1}{2} \cos \frac{3\pi}{2} = \frac{1}{2}(0) = 0$	$\left(\frac{3\pi}{2}, 0\right)$
2π	$y = \frac{1}{2} \cos 2\pi = \frac{1}{2}(1) = \frac{1}{2}$	$\left(2\pi, \frac{1}{2}\right)$

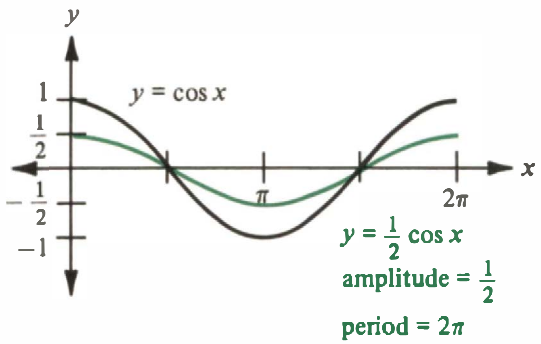


Figure 2

The coefficient 1/2 in $y = (1/2)\cos x$ determines the amplitude of the graph.

Generalizing the results of these first two examples, we can say that if A is a positive number, then the graph of $y = A \sin x$ and $y = A \cos x$ will have amplitude A.

Example 3 Graph $y = 3 \cos x$ and $y = \frac{1}{4} \sin x$, if $0 \leq x \leq 2\pi$.

Solution The amplitude for $y = 3 \cos x$ is 3, while the amplitude for $y = (1/4)\sin x$ is 1/4. The graphs are shown in Figure 3.

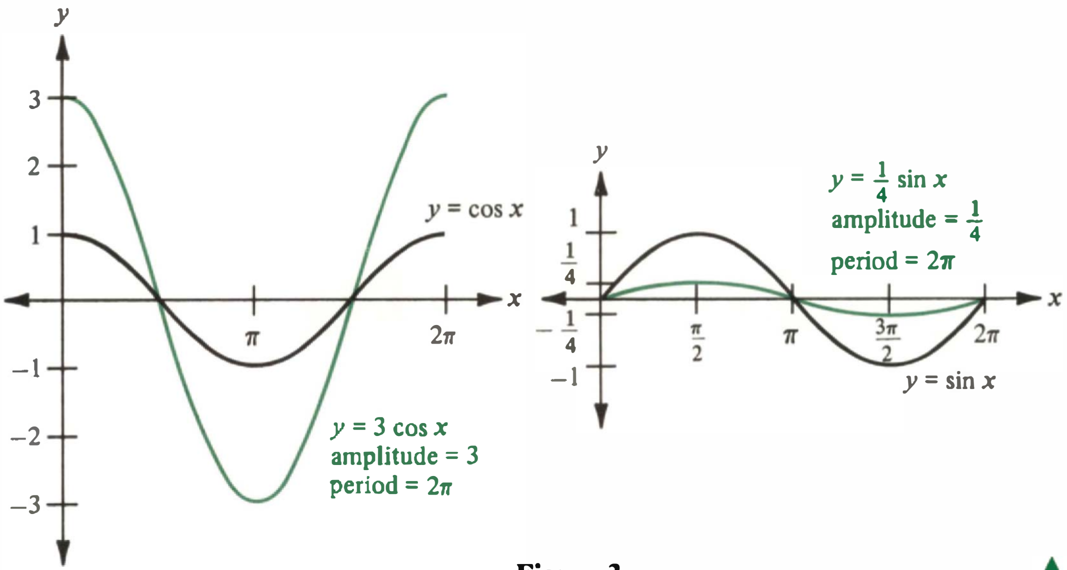


Figure 3

Example 4 Graph $y = \sin 2x$, if $0 \le x \le 2\pi$.

Solution To see how the coefficient 2 in $y = \sin 2x$ affects the graph, we can make a table in which the values of x are multiples of $\pi/4$. (Multiples of $\pi/4$ are convenient because the coefficient 2 divides the 4 in $\pi/4$ exactly.) Table 3 shows the values of x and y, while Figure 4 contains the graphs of $y = \sin x$ and $y = \sin 2x$.

Table 3

x	$y = \sin 2x$	(x, y)
0	$y = \sin 2 \cdot 0 = 0$	$(0, 0)$
$\frac{\pi}{4}$	$y = \sin 2 \cdot \frac{\pi}{4} = \sin \frac{\pi}{2} = 1$	$\left(\frac{\pi}{4}, 1\right)$
$\frac{\pi}{2}$	$y = \sin 2 \cdot \frac{\pi}{2} = \sin \pi = 0$	$\left(\frac{\pi}{2}, 0\right)$
$\frac{3\pi}{4}$	$y = \sin 2 \cdot \frac{3\pi}{4} = \sin \frac{3\pi}{2} = -1$	$\left(\frac{3\pi}{4}, -1\right)$
π	$y = \sin 2\pi = 0$	$(\pi, 0)$
$\frac{5\pi}{4}$	$y = \sin 2 \cdot \frac{5\pi}{4} = \sin \frac{5\pi}{2} = 1$	$\left(\frac{5\pi}{4}, 1\right)$
$\frac{3\pi}{2}$	$y = \sin 2 \cdot \frac{3\pi}{2} = \sin 3\pi = 0$	$\left(\frac{3\pi}{2}, 0\right)$
$\frac{7\pi}{4}$	$y = \sin 2 \cdot \frac{7\pi}{4} = \sin \frac{7\pi}{2} = -1$	$\left(\frac{7\pi}{4}, -1\right)$
2π	$y = \sin 2 \cdot 2\pi = \sin 4\pi = 0$	$(2\pi, 0)$

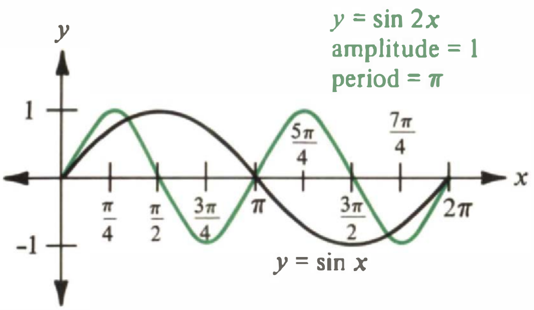


Figure 4

The graph of $y = \sin 2x$ has a period of π. It goes through two complete cycles in 2π units on the x-axis.

Example 5 Graph $y = \sin 3x$ from $x = 0$ to $x = 2\pi$.

Solution To see the effect of the coefficient 3 on the graph, it is convenient to use a table in which the values of x are multiples of $\pi/6$, because 3 divides 6 exactly.

Table 4

x	$y = \sin 3x$	(x, y)
0	$y = \sin 3 \cdot 0 = \sin 0 = 0$	$(0, 0)$
$\frac{\pi}{6}$	$y = \sin 3 \cdot \frac{\pi}{6} = \sin \frac{\pi}{2} = 1$	$\left(\frac{\pi}{6}, 1\right)$
$\frac{\pi}{3}$	$y = \sin 3 \cdot \frac{\pi}{3} = \sin \pi = 0$	$\left(\frac{\pi}{3}, 0\right)$
$\frac{\pi}{2}$	$y = \sin 3 \cdot \frac{\pi}{2} = \sin \frac{3\pi}{2} = -1$	$\left(\frac{\pi}{2}, -1\right)$
$\frac{2\pi}{3}$	$y = \sin 3 \cdot \frac{2\pi}{3} = \sin 2\pi = 0$	$\left(\frac{2\pi}{3}, 0\right)$

The information in Table 4 indicates the period of $y = \sin 3x$ is $2\pi/3$. The graph will go through three complete cycles in 2π units on the x-axis. Figure 5 shows the graph of $y = \sin 3x$ and the graph of $y = \sin x$, on the interval $0 \le x \le 2\pi$.

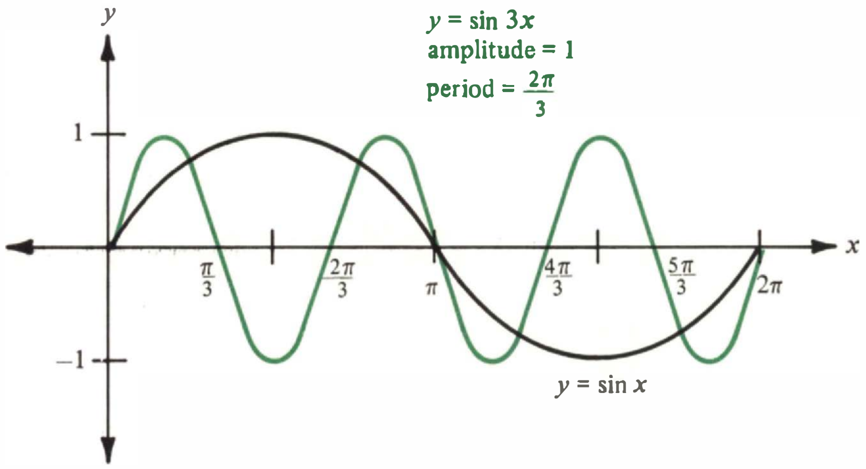


Figure 5

Table 5 summarizes the information obtained from Examples 4 and 5.

Table 5

Equation	Number of Cycles every 2π units	Period
$y = \sin x$	1	2π
$y = \sin 2x$	2	π
$y = \sin 3x$	3	$\frac{2\pi}{3}$
$y = \sin Bx$	B	$\frac{2\pi}{B}$ B is positive

Example 6 Graph one complete cycle of $y = \cos \frac{1}{2} x$.

Solution The coefficient of x is 1/2. The graph will go through 1/2 of a complete cycle every 2π units. The period will be

$$\text{Period} = \frac{2\pi}{1/2} = 4\pi$$

Figure 6 shows the graph.

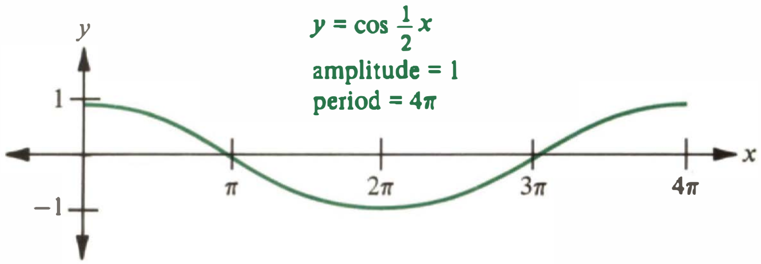


Figure 6

Continuing to generalize from the examples we have worked so far in this section, we can say that the graphs of $y = A \sin Bx$ and $y = A \cos Bx$, where A and B are positive numbers, will have amplitude A and period $2\pi/B$.

In the next three examples, we use this information about amplitude and period to graph one complete cycle of some sine and cosine curves, then we extend these graphs to cover more than one complete cycle.

Examples Graph one complete cycle of each of the following equations and then extend the graph to cover the given interval.

7. $y = 3 \sin 2x,\ -\pi \le x \le 2\pi$

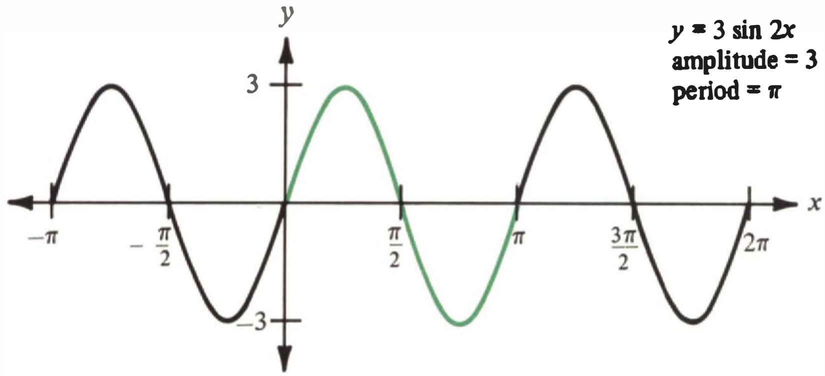


Figure 7

8. $y = 4 \cos \frac{1}{2} x,\ -4\pi \le x \le 4\pi$

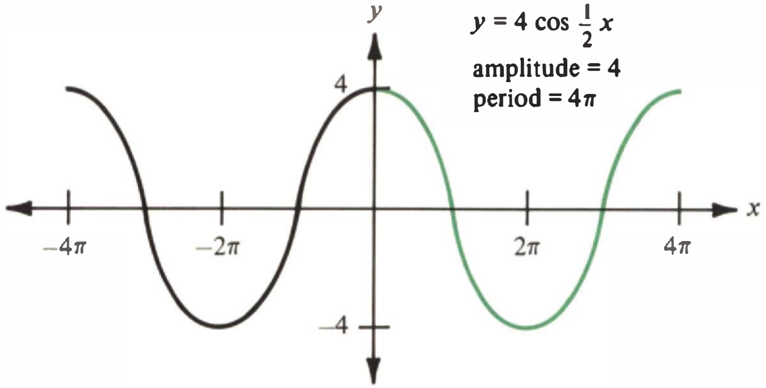


Figure 8

9. $y = 2 \sin \pi x,\ -3 \le x \le 3$

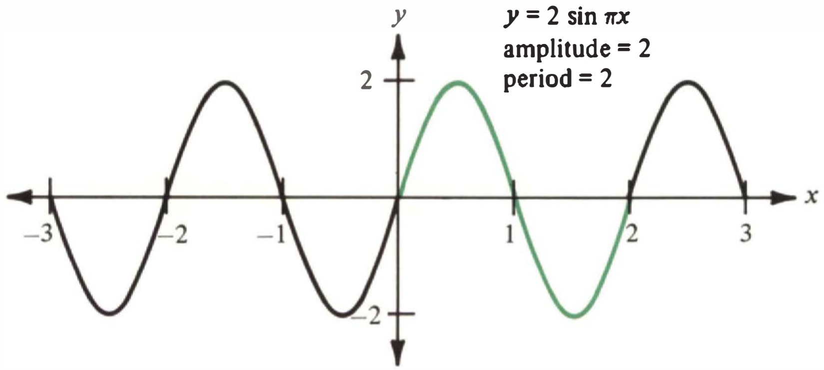


Figure 9

Note that, on the graphs in Figures 7, 8, and 9, the axes have not been labeled proportionally. Instead, they are labeled so that the amplitude and period are easy to read. As you can see, once we have the graph of one complete cycle of a curve, it is easy to extend the curve to any interval of interest.

So far in this section, all of the coefficients A and B we have encountered have been positive. If we are given an equation to graph in which B is negative, we can use the properties of even and odd functions to rewrite the equation with B positive. For example,

$y = 3 \sin(-2x)$ is equivalent to $y = -3 \sin 2x$
because sine is an odd function.
$y = 3 \cos(-2x)$ is equivalent to $y = 3 \cos 2x$
because cosine is an even function.

So we do not need to worry about negative values of B; we simply make them positive by using the properties of even and odd functions and then graph as usual. To see how a negative value of A affects graphing, we will graph $y = -2 \cos x$.

Example 10 Graph $y = -2 \cos x$, from $x = -2\pi$ to $x = 4\pi$.

Solution Each value of y on the graph of $y = -2 \cos x$ will be the opposite of the corresponding value of y on the graph of $y = 2 \cos x$. The result is that the graph of $y = -2 \cos x$ is the reflection of the graph of $y = 2 \cos x$ about the x-axis. Figure 10 shows the extension of one complete cycle of $y = -2 \cos x$ to the interval $-2\pi \le x \le 4\pi$.

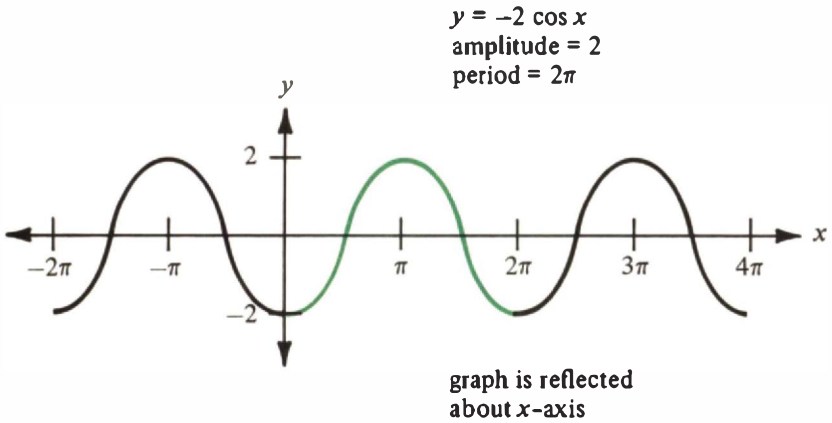


Figure 10

Summary **The graphs of $y = A \sin Bx$ and $y = A \cos Bx$, where B is a positive number, will have**

$$\text{Amplitude} = |A|$$

$$\text{Period} = \frac{2\pi}{B}$$

They will be reflected about the x-axis if A is negative.

We conclude this section with a look at the graph of one of the equations we found in Example 4 of Section 3.5. Example 11 gives the main facts from that example.

▼ **Example 11** Figure 11 shows a fire truck parked on the shoulder of a freeway next to a long block wall. The red light on the top of the truck is 10 feet from the wall and rotates through one complete revolution every 2 seconds. Graph the equation that gives the length d in terms of time t from $t = 0$ to $t = 2$.

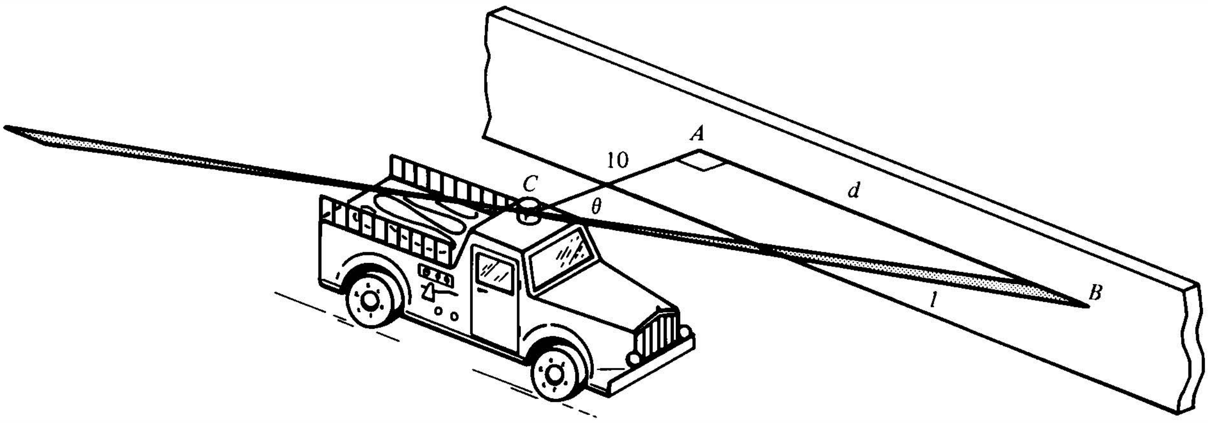


Figure 11

Solution From Example 4 in Section 3.5 we know that

$$d = 10 \tan \pi t$$

To graph this equation between $t = 0$ and $t = 2$, we construct a table of values in which t assumes all multiples of 1/4 from $t = 0$ to $t = 2$.

Plotting the points given in Table **6** and then connecting them with a smooth tangent curve we have the graph in Figure 12.

Table 6

t	$d = 10 \tan \pi t$	d
0	$d = 10 \tan \pi \cdot 0 = 10 \tan 0 = 0$	0
$\frac{1}{4}$	$d = 10 \tan \pi \cdot \frac{1}{4} = 10 \tan \frac{\pi}{4} = 10$	10
$\frac{1}{2}$	$d = 10 \tan \pi \cdot \frac{1}{2} = 10 \tan \frac{\pi}{2}$	undefined
$\frac{3}{4}$	$d = 10 \tan \pi \cdot \frac{3}{4} = 10 \tan \frac{3\pi}{4} = -10$	-10
1	$d = 10 \tan \pi \cdot 1 = 10 \tan \pi = 0$	0
$\frac{5}{4}$	$d = 10 \tan \pi \cdot \frac{5}{4} = 10 \tan \frac{5\pi}{4} = 10$	10
$\frac{3}{2}$	$d = 10 \tan \pi \cdot \frac{3}{2} = 10 \tan \frac{3\pi}{2}$	undefined
$\frac{7}{4}$	$d = 10 \tan \pi \cdot \frac{7}{4} = 10 \tan \frac{7\pi}{4} = -10$	-10
2	$d = 10 \tan \pi \cdot 2 = 10 \tan 2\pi = 0$	0

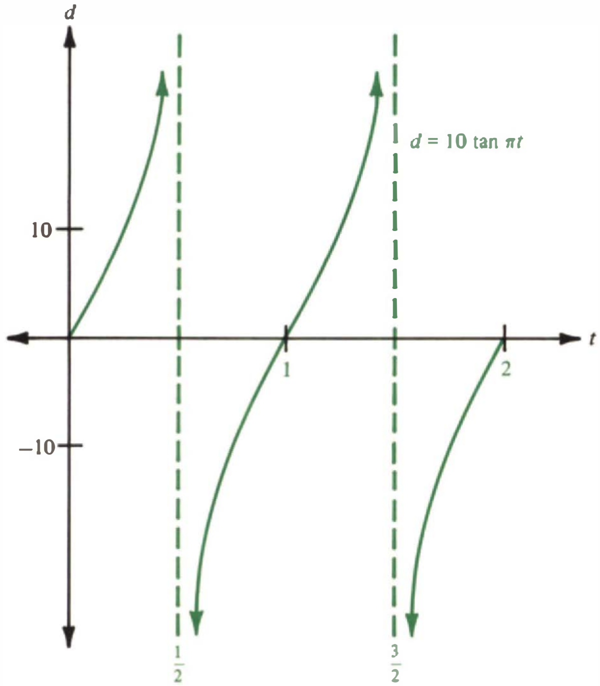


Figure 12

Note that the period of the function $d = 10 \tan \pi t$ is 1. Since the period of $y = \tan x$ is π, we can conclude that the new period 1 must come from dividing the normal period π by π to get 1. Also, the number 10 in the equation $d = 10 \tan \pi t$ causes the tangent graph to rise faster above the horizontal axis and fall faster below it. ▲

We can generalize the results of Example 11 to conclude that, if A and B are positive numbers, the graphs of $y = A \tan Bx$ and $y = A \cot Bx$ will have period π / B. Each graph will rise and fall at a faster rate than the corresponding graphs of $y = \tan x$ and $y = \cot x$ if A is greater than 1 and at a slower rate if A is between 0 and 1.

Problem Set 4.2

Graph one complete cycle of each of the following. In each case label the axes accurately and identify the amplitude and period for each graph.

1. $y = 6 \sin x$

2. $y = 6 \cos x$

3. $y = \sin 2x$

4. $y = \sin \frac{1}{2} x$

5. $y = \cos \frac{1}{3} x$

6. $y = \cos 3x$

7. $y = \frac{1}{3} \sin x$

8. $y = \frac{1}{2} \cos x$

9. $y = \sin \pi x$

10. $y = \cos \pi x$

11. $y = \sin \frac{\pi}{2} x$

12. $y = \cos \frac{\pi}{2} x$

Graph one complete cycle for each of the following. In each case label the axes so that the amplitude and period are easy to read.

13. $y = 4 \sin 2x$

14. $y = 2 \sin 4x$

15. $y = 2 \cos 4x$

16. $y = 3 \cos 2x$

17. $y = 3 \sin \frac{1}{2} x$

18. $y = 2 \sin \frac{1}{3} x$

19. $y = \frac{1}{2} \cos 3x$

20. $y = \frac{1}{2} \sin 3x$

21. $y = \frac{1}{2} \sin \frac{\pi}{2} x$

22. $y = 2 \sin \frac{\pi}{2} x$

Graph each of the following over the given interval. Label the axes so that the amplitude and period are easy to read.

23. $y = 2 \sin \pi x, \ -4 \le x \le 4$

24. $y = 3 \cos \pi x, \ -2 \le x \le 4$

25. $y = 3 \sin 2x,\ -\pi \le x \le 2\pi$

26. $y = -3 \sin 2x,\ -2\pi \le x \le 2\pi$

27. $y = -3 \cos \frac{1}{2} x,\ -2\pi \le x \le 6\pi$

28. $y = 3 \cos \frac{1}{2} x,\ -4\pi \le x \le 4\pi$

29. $y = -2 \sin(-3x),\ 0 \le x \le 2\pi$

30. $y = -2 \cos(-3x),\ 0 \le x \le 2\pi$

31. The current in an alternating circuit varies in intensity with time. If I represents the intensity of the current and t represents time, then the relationship between I and t is given by

$$I = 20 \sin 120\pi t$$

where I is measured in amperes and t is measured in seconds. Find the maximum value of I and the time it takes for I to go through one complete cycle.

32. A weight is hung from a spring and set in motion so that it moves up and down continuously. The velocity v of the weight at any time t is given by the equation

$$v = 3.5 \cos 2\pi t$$

where v is measured in meters and t is measured in seconds. Find the maximum velocity of the weight and the amount of time it takes for the weight to move from its lowest position to its highest position.

33. Since the period for $y = \tan x$ is π, the graph of $y = \tan x$ will go through one complete cycle every π units. Through how many cycles will the graph of $y = \tan 2x$ go every π units? What is the period of $y = \tan 2x$? Sketch the graph of $y = \tan 2x$, from $x = -\pi/4$ to $x = 3\pi/4$.

34. Through how many complete cycles will the graph of $y = \tan (1/2)x$ go every π units? What is the period of this graph? Sketch the graph from $x = -\pi$ to $x = 3\pi$.

35. Figure 13 shows a lighthouse that is 100 feet from a long straight wall on the beach. The light in the lighthouse rotates through one complete rotation once every 4 seconds. In Problem 21 of Problem Set 3.5 you found the equation that gives d in terms of t to be $d = 100 \tan (\pi/2)t$. Graph this equation by making a table in which t assumes all multiples of 1/2 from $t = 0$ to $t = 4$.

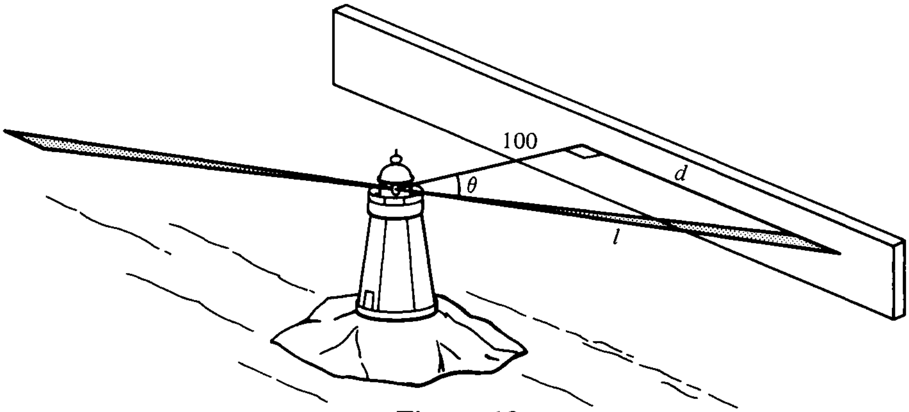


Figure 13

36. Sketch the graph of $y = -\tan x$, for $-\pi/2 \le x \le 3\pi/2$.
37. Give the period of $y = \cot 3x$. Sketch the graph from $x = 0$ to $x = \pi$.
38. Sketch the graph of $y = \cot \pi x$ from $x = -1$ to $x = 2$.
39. Graph $y = 2 \csc x$ by making a table in which x assumes multiples of $\pi/4$ starting at $x = 0$ and ending at $x = 2\pi$. (Use 1.4 as an approximation for $\sqrt{2}$.) Does the graph obtained by connecting the points from the table agree with the fact that $2 \csc x = 2(1/\sin x)$? That is, does your graph seem reasonable when viewed in terms of reciprocal functions?
40. Graph $y = (1/2)\sec x$ between $x = -\pi/2$ and $x = 3\pi/2$.
41. Referring to Example 11 and Figure 11 from this section, the equation that gives l in terms of time t is $l = 10 \sec \pi t$. Graph this equation by making a table of values in which t takes on multiples of 1/4 starting at $t = 0$ and ending at $t = 2$.
42. In Figure 13 above, the equation that gives l in terms of t is $l = 100 \sec (\pi/2)t$. Graph this equation from $t = 0$ to $t = 4$.
43. The period of $y = \csc 2x$ is $2\pi/2 = \pi$. Graph $y = \csc 2x$ from $x = 0$ to $x = 2\pi$.
44. Graph $y = \sec 3x$ from $x = 0$ to $x = 2\pi$.
45. Graph one complete cycle of $y = 3 \csc 2x$.
46. Graph one complete cycle of $y = 2 \csc 3x$.

Review Problems The problems that follow review material we covered in Section 3.2. Reviewing these problems will help you with the next section.

Evaluate each of the following if x is $\pi/2$ and y is $\pi/6$.

47. $\sin\left(x + \frac{\pi}{2}\right)$

48. $\sin\left(x - \frac{\pi}{2}\right)$

49. $\cos\left(y - \frac{\pi}{6}\right)$

50. $\cos\left(y + \frac{\pi}{6}\right)$

51. $\sin(x + y)$

52. $\cos(x + y)$

53. $\sin x + \sin y$

54. $\cos x + \cos y$

4.3 Phase Shift

In this section we will consider equations of the form

$$y = A \sin(Bx + C) \text{ and } y = A \cos(Bx + C) \text{ where } B > 0$$

We already know how the coefficients A and B affect the graphs of these equations. The only thing we have left to do is discover what effect C has on the graphs. We will start our investigation with a couple of equations in which A and B are equal to 1.

Example 1 Graph $y = \sin\left(x + \frac{\pi}{2}\right)$, if $-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$.

Solution Since we have not graphed an equation of this form before, it is a good idea to begin by making a table (Table 1). In this case, multiples of $\pi/2$ will be the most convenient replacements for x in the table. Also, if we start with $x = -\pi/2$, our first value of y will be 0.

Table 1

x	$y = \sin\left(x + \frac{\pi}{2}\right)$	(x, y)
$-\frac{\pi}{2}$	$y = \sin\left(-\frac{\pi}{2} + \frac{\pi}{2}\right) = \sin 0 = 0$	$\left(-\frac{\pi}{2}, 0\right)$
0	$y = \sin\left(0 + \frac{\pi}{2}\right) = \sin \frac{\pi}{2} = 1$	$(0, 1)$
$\frac{\pi}{2}$	$y = \sin\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \sin \pi = 0$	$\left(\frac{\pi}{2}, 0\right)$
π	$y = \sin\left(\pi + \frac{\pi}{2}\right) = \sin \frac{3\pi}{2} = -1$	$(\pi, -1)$
$\frac{3\pi}{2}$	$y = \sin\left(\frac{3\pi}{2} + \frac{\pi}{2}\right) = \sin 2\pi = 0$	$\left(\frac{3\pi}{2}, 0\right)$

Graphing these points and then drawing the sine curve that connects them, gives us the graph of $y = \sin(x + \pi/2)$ as shown in Figure 1. Figure 1 also includes the graph of $y = \sin x$ for reference; we are trying to discover how the graph of $y = \sin(x + \pi/2)$ and $y = \sin x$ differ.

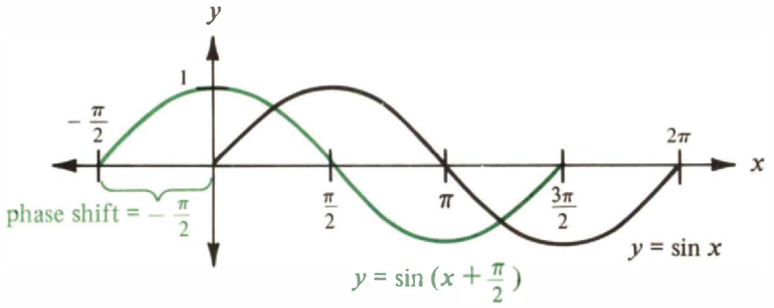


Figure 1

It seems that the graph of $y = \sin(x + \pi/2)$ is shifted $\pi/2$ units to the left of the graph of $y = \sin x$. We say the graph of $y = \sin(x + \pi/2)$ has a *phase shift* of $-\pi/2$, where the negative sign indicates the shift is to the left (in the negative direction). ▲

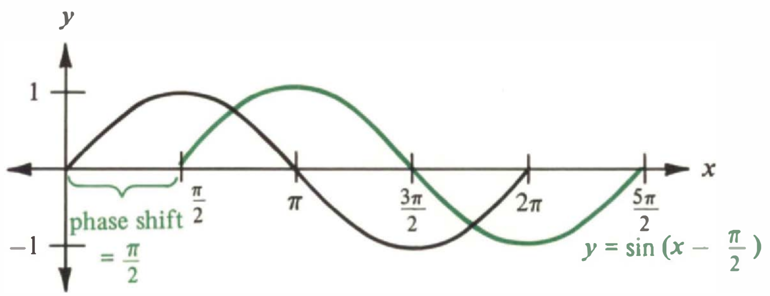
From the results in Example 1, we would expect the graph of $y = \sin(x - \pi/2)$ to have a phase shift of $+\pi/2$. That is, we expect the graph of $y = \sin(x - \pi/2)$ to be shifted $\pi/2$ units to the right of the graph of $y = \sin x$.

▼ **Example 2** Graph one complete cycle of $y = \sin\left(x - \frac{\pi}{2}\right)$

Solution Proceeding as we did in Example 1, we make a table (Table 2) using multiples of $\pi/2$ for x, and then use the information in the table to sketch the graph. In this example, we start with $x = \pi/2$, since this value of x will give us $y = 0$.

Table 2

x	$y = \sin\left(x - \frac{\pi}{2}\right)$	(x, y)
$\frac{\pi}{2}$	$y = \sin\left(\frac{\pi}{2} - \frac{\pi}{2}\right) = \sin 0 = 0$	$\left(\frac{\pi}{2}, 0\right)$
π	$y = \sin\left(\pi - \frac{\pi}{2}\right) = \sin \frac{\pi}{2} = 1$	$(\pi, 1)$
$\frac{3\pi}{2}$	$y = \sin\left(\frac{3\pi}{2} - \frac{\pi}{2}\right) = \sin \pi = 0$	$\left(\frac{3\pi}{2}, 0\right)$
2π	$y = \sin\left(2\pi - \frac{\pi}{2}\right) = \sin \frac{3\pi}{2} = -1$	$(2\pi, -1)$
$\frac{5\pi}{2}$	$y = \sin\left(\frac{5\pi}{2} - \frac{\pi}{2}\right) = \sin 2\pi = 0$	$\left(\frac{5\pi}{2}, 0\right)$



The graph of $y = \sin(x - \pi/2)$, as we expected, is a sine curve shifted $\pi/2$ units to the right of the graph of $y = \sin x$ (Figure 2). The phase shift, in this case, is $+\pi/2$. ▲

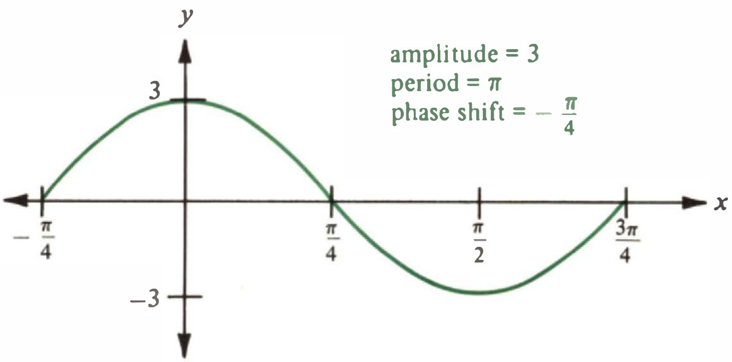
Before we write any conclusions about phase shift, we should look at another example in which A and B are not 1.

▼ **Example 3** Graph $y = 3\sin\left(2x + \frac{\pi}{2}\right)$, if $-\frac{\pi}{4} \le x \le \frac{3\pi}{4}$.

Solution We know the coefficient $B = 2$ will change the period from 2π to π. Because the period is smaller (π instead of 2π), we should use values of x in our table (Table 3) that are closer together, like multiples of $\pi/4$ instead of $\pi/2$.

Table 3

x	$y = 3\sin\left(2x + \frac{\pi}{2}\right)$	(x, y)
$-\frac{\pi}{4}$	$y = 3\sin\left(2 \cdot -\frac{\pi}{4} + \frac{\pi}{2}\right) = 3\sin 0 = 0$	$\left(-\frac{\pi}{4}, 0\right)$
0	$y = 3\sin\left(2 \cdot 0 + \frac{\pi}{2}\right) = 3\sin\frac{\pi}{2} = 3$	$(0, 3)$
$\frac{\pi}{4}$	$y = 3\sin\left(2 \cdot \frac{\pi}{4} + \frac{\pi}{2}\right) = 3\sin \pi = 0$	$\left(\frac{\pi}{4}, 0\right)$
$\frac{\pi}{2}$	$y = 3\sin\left(2 \cdot \frac{\pi}{2} + \frac{\pi}{2}\right) = 3\sin\frac{3\pi}{2} = -3$	$\left(\frac{\pi}{2}, -3\right)$
$\frac{3\pi}{4}$	$y = 3\sin\left(2 \cdot \frac{3\pi}{4} + \frac{\pi}{2}\right) = 3\sin 2\pi = 0$	$\left(\frac{3\pi}{4}, 0\right)$



The amplitude and period are as we would expect. The phase shift, however, is half of $-\pi/2$. The phase shift, $-\pi/4$, comes from the ratio $-C/B$ or in this case

$$\frac{-\pi/2}{2} = -\frac{\pi}{4}$$

The phase shift in the equation $y = A \sin(Bx + C)$ depends on both B and C. In this example, the period is half of the period of $y = \sin x$ and the phase shift is half of the phase shift of $y = \sin(x + \pi/2)$ that was found in Example 1 (see Figure 3). ▲

Although all of the examples we have completed so far in this section have been sine equations, the results also apply to cosine equations. Here is a summary.

Summary The graphs of $y = A \sin(Bx + C)$ and $y = A \cos(Bx + C)$, where $B > 0$, will have the following characteristics:

1. Amplitude $= |A|$
2. Period $= \dfrac{2\pi}{B}$
3. Phase shift $= -\dfrac{C}{B}$

If $A < 0$, the graphs will be reflected about the x-axis.

The information on amplitude, period, and phase shift allows us to sketch sine and cosine curves without having to make tables.

▼ **Example 4** Graph one complete cycle of $y = 2 \sin(3x + \pi)$.

Solution Here is a detailed list of steps to use in graphing sine and cosine curves for which B is positive.

Step 1. Use A, B, and C to find the amplitude, period, and phase shift.

$$\text{Amplitude} = |A| = 2$$

$$\text{Period} = \frac{2\pi}{B} = \frac{2\pi}{3}$$

$$\text{Phase shift} = -\frac{C}{B} = -\frac{\pi}{3}$$

Step 2. On the x-axis, label the starting point, ending point, and the point halfway between them for each cycle of the curve in question. The starting point is the phase shift. The ending point is the phase shift plus the period. (In many cases, it is also a good idea to label the points one-fourth and three-fourths of the way between the starting point and ending point, unless you do not have room on the graph to do so.)

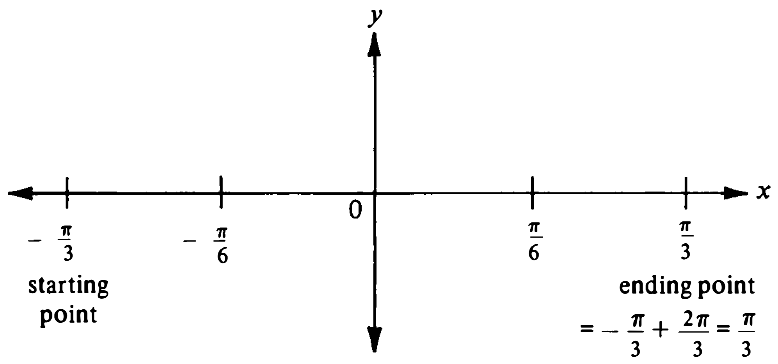


Figure 4

Step 3. Label the y-axis with the amplitude and the opposite of the amplitude. It is okay if the units on the x-axis and the y-axis are not proportional. That is, one unit on the y-axis can be a different length than one unit on the x-axis. The idea is to make the graph easy to read.

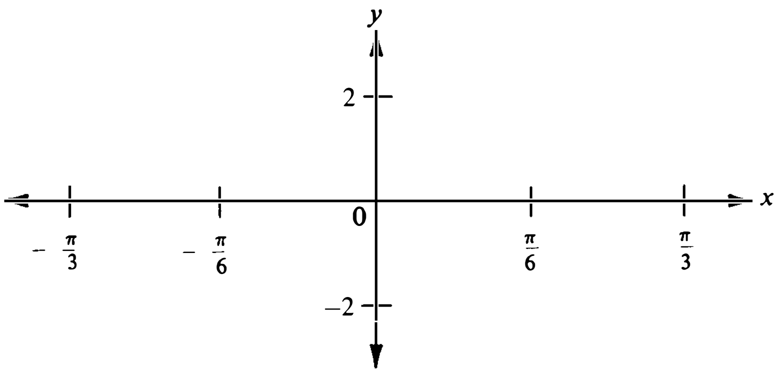


Figure 5

Step 4. Sketch in the curve in question, keeping in mind that the graph will be reflected about the x-axis if A is negative. In this case, we want a sine curve that will be 0 at the starting point and 0 at the ending point.

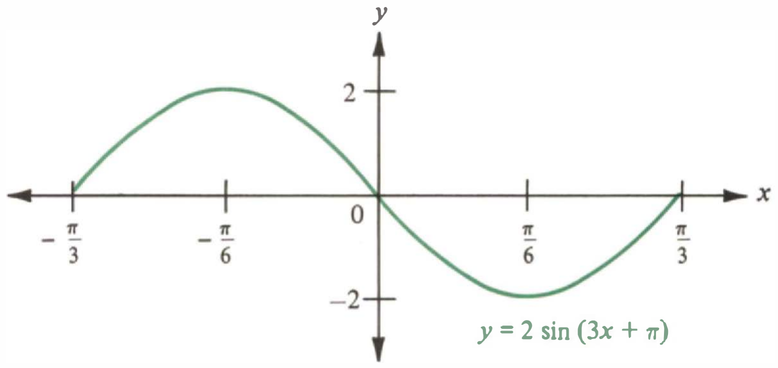


Figure 6 ▲

The steps listed in Example 4 may seem complicated at first. With a little practice they do not take much time at all. Especially when compared with the time it would take to make a table. Also, once we have graphed one complete cycle of the curve, it would be fairly easy to extend the graph in either direction.

▼ **Example 5** Graph $y = 2\cos(3x + \pi)$ from $x = -2\pi/3$ to $x = 2\pi/3$.

Solution A, B, and C are the same here as they were in Example 4. We use the same labeling on the axes as we used in Example 4, but we draw in a cosine curve instead of a sine curve and then extend it to cover the interval $-2\pi/3 \leq x \leq 2\pi/3$.

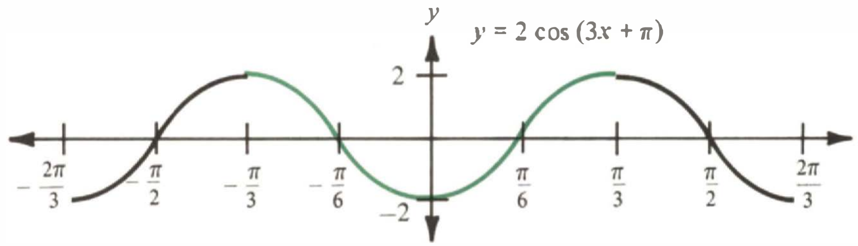


Figure 7 ▲

▼ **Example 6** Graph $y = -2\cos(3x + \pi)$ for $-2\pi/3 \leq x \leq 2\pi/3$.

Solution The graph will be the graph found in Example 5 reflected about the x-axis, since the only difference in the two equations is that A is negative here, where it was positive in Example 5.

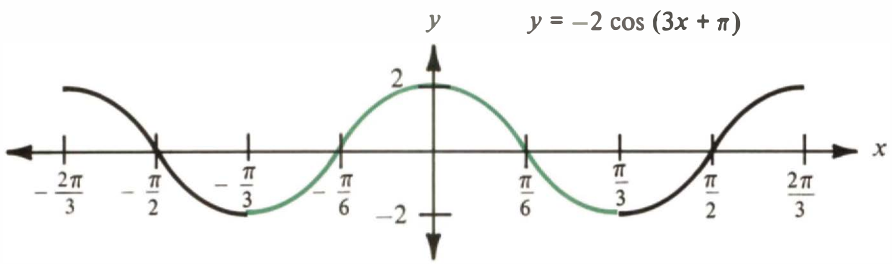


Figure 8

Example 7 Graph $y = -4\cos\left(2x - \frac{3\pi}{2}\right)$ from $x = 0$ to $x = 2\pi$.

Solution We use A, B, and C to help us sketch one complete cycle of the curve, then we extend the resulting graph to cover the interval $0 \leq x \leq 2\pi$.

Amplitude = 4 Period $= \frac{2\pi}{2} = \pi$ Phase shift $= -\frac{-3\pi/2}{2} = \frac{3\pi}{4}$

One complete cycle will start at $3\pi/4$ on the x-axis. It will end π units later at $3\pi/4 + \pi = 7\pi/4$. Since A is negative, the graph is reflected about the x-axis.

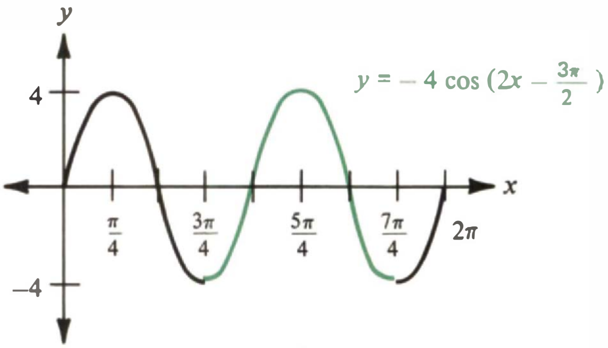


Figure 9

Example 8 Graph $y = 5\sin\left(\pi x + \frac{\pi}{4}\right)$, $-\frac{5}{4} \leq x \leq \frac{11}{4}$

Solution In this example, $A = 5$, $B = \pi$, and $C = \pi/4$. The graph will have the following characteristics:

Amplitude = 5, Period $= \frac{2\pi}{\pi} = 2$, and Phase shift $= -\frac{\pi/4}{\pi} = -\frac{1}{4}$

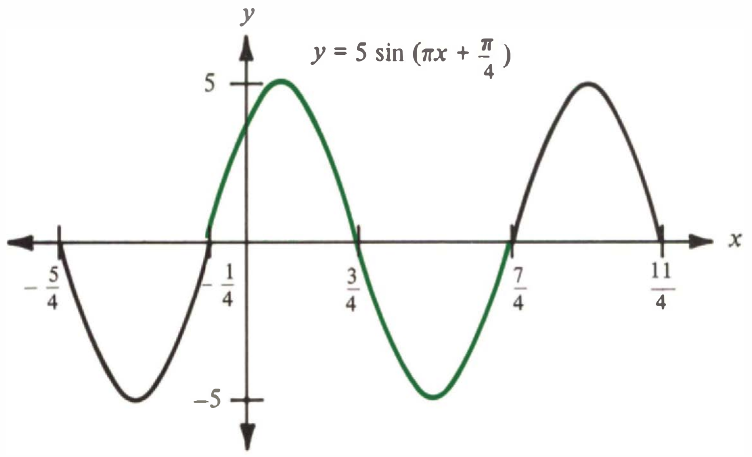


Figure 10

Problem Set 4.3

For each equation, first identify the phase shift and then sketch one complete cycle of the graph. In each case, graph $y = \sin x$ on the same coordinate system.

1. $y = \sin\left(x + \frac{\pi}{4}\right)$

2. $y = \sin\left(x + \frac{\pi}{6}\right)$

3. $y = \sin\left(x - \frac{\pi}{4}\right)$

4. $y = \sin\left(x - \frac{\pi}{6}\right)$

5. $y = \sin\left(x + \frac{\pi}{3}\right)$

6. $y = \sin\left(x - \frac{\pi}{3}\right)$

For each equation, identify the phase shift and then sketch one complete cycle of the graph. In each case, graph $y = \cos x$ on the same coordinate system.

7. $y = \cos\left(x - \frac{\pi}{2}\right)$

8. $y = \cos\left(x + \frac{\pi}{2}\right)$

9. $y = \cos\left(x + \frac{\pi}{3}\right)$

10. $y = \cos\left(x - \frac{\pi}{4}\right)$

For each equation, identify the amplitude, period, and phase shift. Then label the axes accordingly and sketch one complete cycle of the curve.

11. $y = \sin(2x - \pi)$

12. $y = \sin(2x + \pi)$

13. $y = \sin\left(\pi x + \frac{\pi}{2}\right)$

14. $y = \sin\left(\pi x - \frac{\pi}{2}\right)$

15. $y = -\cos\left(2x + \frac{\pi}{2}\right)$

16. $y = -\cos\left(2x - \frac{\pi}{2}\right)$

17. $y = 2\sin\left(\frac{1}{2}x + \frac{\pi}{2}\right)$

18. $y = 3\cos\left(\frac{1}{2}x + \frac{\pi}{3}\right)$

19. $y = \frac{1}{2}\cos\left(3x - \frac{\pi}{2}\right)$

20. $y = \frac{4}{3}\cos\left(3x + \frac{\pi}{2}\right)$

21. $y = 3\sin\left(\frac{\pi}{3}x - \frac{\pi}{3}\right)$

22. $y = 3\cos\left(\frac{\pi}{3}x - \frac{\pi}{3}\right)$

Graph each of the following equations over the given interval. In each case, be sure to label the axes so that the amplitude, period, and phase shift are easy to read.

23. $y = 4\cos\left(2x - \frac{\pi}{2}\right), \ -\frac{\pi}{4} \le x \le \frac{3\pi}{2}$

24. $y = 3\sin\left(2x - \frac{\pi}{3}\right), \ -\frac{5\pi}{6} \le x \le \frac{7\pi}{6}$

25. $y = -4\cos\left(2x - \frac{\pi}{2}\right), \ -\frac{\pi}{4} \le x \le \frac{3\pi}{2}$

26. $y = -3\sin\left(2x - \frac{\pi}{3}\right), \ -\frac{5\pi}{6} \le x \le \frac{7\pi}{6}$

27. $y = \frac{2}{3}\sin\left(3x + \frac{\pi}{2}\right), \ -\pi \le x \le \pi$

28. $y = \frac{3}{4}\sin\left(3x - \frac{\pi}{2}\right), \ -\frac{\pi}{2} \le x \le \frac{3\pi}{2}$

29. $y = -\frac{2}{3}\sin\left(3x + \frac{\pi}{2}\right), \ -\pi \le x \le \pi$

30. $y = -\frac{3}{4}\sin\left(3x - \frac{\pi}{2}\right), \ -\frac{\pi}{2} \le x \le \frac{3\pi}{2}$

Sketch one complete cycle of each of the following by first graphing the appropriate sine or cosine curve and then using the reciprocal relationships. In each case, be sure to include the asymptotes on your graph.

31. $y = \csc\left(x + \frac{\pi}{4}\right)$

32. $y = \sec\left(x + \frac{\pi}{4}\right)$

33. $y = \sec\left(2x - \frac{\pi}{2}\right)$

34. $y = \csc\left(2x - \frac{\pi}{2}\right)$

35. $y = \csc\left(2x + \frac{\pi}{3}\right)$

36. $y = \sec\left(2x - \frac{\pi}{3}\right)$

The periods for $y = \tan x$ and $y = \cot x$ are π. The graphs of $y = \tan(Bx + C)$ and $y = \cot(Bx + C)$ will have periods $= \pi/B$ and phase shifts $= -C/B$, for $B > 0$. Sketch the graph of each equation below. Don't limit your graphs to one complete cycle and be sure to show the asymptotes with each graph.

37. $y = \tan\left(x + \frac{\pi}{4}\right)$

38. $y = \tan\left(x - \frac{\pi}{4}\right)$

39. $y = \cot\left(x - \frac{\pi}{4}\right)$

40. $y = \cot\left(x + \frac{\pi}{4}\right)$

41. $y = \tan\left(2x - \frac{\pi}{2}\right)$

42. $y = \tan\left(2x + \frac{\pi}{2}\right)$

Review Problems The problems below review material we covered in Section 3.4.

43. Find the length of arc cut off by a central angle of $\pi/6$ radians in a circle of radius 10 centimeters.

44. The minute hand of a clock is 2.6 centimeters long. How far does the tip of the minute hand travel in 30 minutes?

45. Find the radius of a circle if a central angle of 6 radians cuts off an arc of length 4 feet.

46. Find the area of the sector formed by a central angle of 45° in a circle of radius 8 inches.

4.4 Graphing Combinations of Functions

In this section we will graph equations of the form $y = y_1 + y_2$ where y_1 and y_2 are algebraic or trigonometric functions of x. For instance, the equation $y = 1 + \sin x$ can be thought of as the sum of the two functions $y_1 = 1$ and $y_2 = \sin x$. That is,

$$\text{if} \quad y_1 = 1 \text{ and } y_2 = \sin x,$$
$$\text{then } y = y_1 + y_2$$

Using this kind of reasoning, the graph of $y = 1 + \sin x$ is obtained by adding each value of y_2 in $y_2 = \sin x$ to the corresponding value of y_1 in $y_1 = 1$. Graphically, we can show this by adding the values of y from the graph of y_2 to the corresponding values of y from the graph of y_1 (Figure 1).

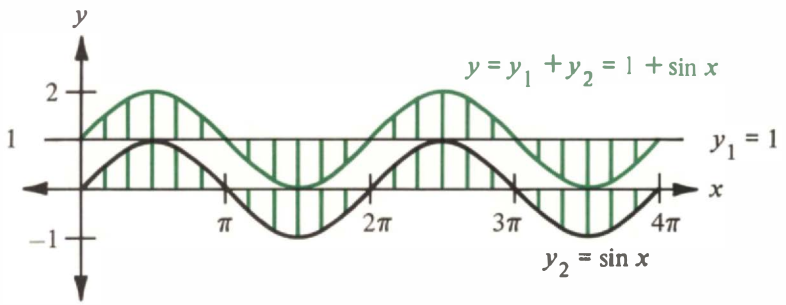


Figure 1

Although in actual practice you may not draw in the little vertical lines we have shown here, they do serve the purpose of allowing us to visualize the idea of adding the y-coordinates on one graph to the corresponding y-coordinates on another graph.

Example 1 Graph $y = \frac{1}{3}x - \sin x$ between $x = 0$ and $x = 4\pi$.

Solution We can think of the equation $y = (1/3)x - \sin x$ as the sum of the equations $y_1 = (1/3)x$ and $y_2 = -\sin x$. Graphing each of these two equations on the same set of axes and then adding the values of y_2 to the corresponding values of y_1, we have the graph shown in Figure 2.

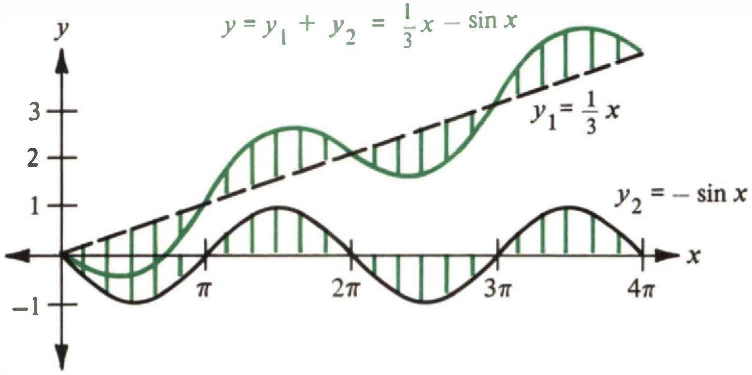


Figure 2

For the rest of the examples in this section, we will not show the vertical lines used to visualize the process of adding y-coordinates. Sometimes the graphs become too confusing to read when the vertical lines are included. It is the idea behind the vertical lines that is important, not the lines themselves. (And remember, the alternative to graphing these types of equations by adding y-coordinates is to make a table. If you try using a table on some of the examples that follow, you will see that the method being presented here is much faster.)

Example 2 Graph $y = 2 \sin x + \cos 2x$ for x between 0 and 4π.

Solution We can think of y as the sum of y_1 and y_2, where

$$y_1 = 2 \sin x \qquad \text{(amplitude 2, period } 2\pi)$$

and

$$y_2 = \cos 2x \qquad \text{(amplitude 1, period } \pi)$$

The graphs of y_1, y_2 and $y = y_1 + y_2$ are shown in Figure 3.

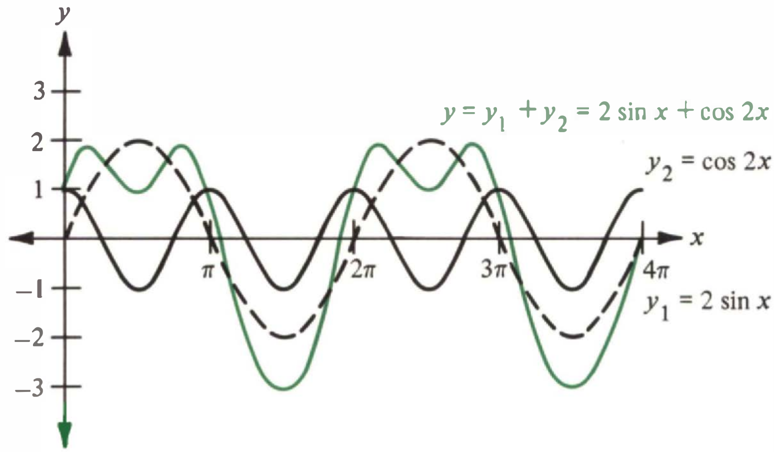


Figure 3

▼ **Example 3** Graph $y = \cos x + \cos 2x$ for $0 \leq x \leq 4\pi$.

Solution We let $y = y_1 + y_2$, where

$$y_1 = \cos x \qquad \text{(amplitude 1, period } 2\pi)$$

and

$$y_2 = \cos 2x \qquad \text{(amplitude 1, period } \pi)$$

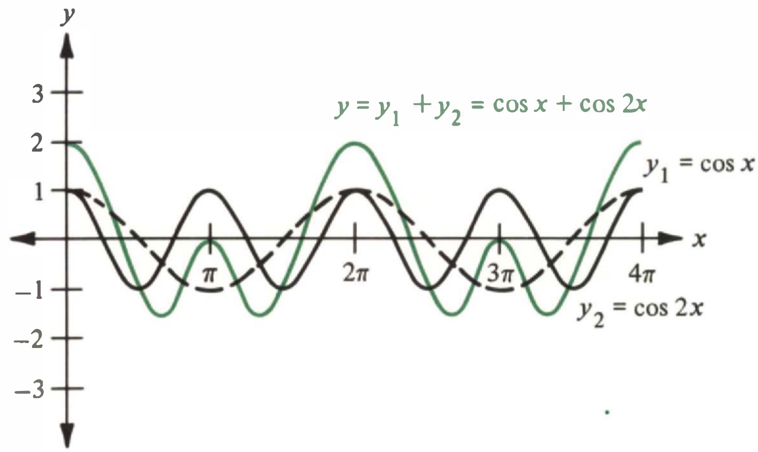


Figure 4

▼ **Example 4** Graph $y = \sin x + \cos x$ for x between 0 and 4π.

Solution We let $y_1 = \sin x$ and $y_2 = \cos x$ and graph y_1, y_2, and $y = y_1 + y_2$.

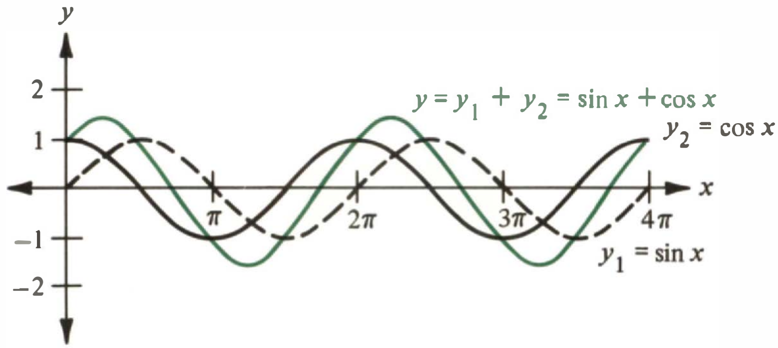


Figure 5

The graph of $y = \sin x + \cos x$ has amplitude $\sqrt{2}$. If we were to extend the graph to the left, we would find it crossed the x-axis at $-\pi/4$. It would then be apparent that the graph of $y = \sin x + \cos x$ is the same as the graph of $y = \sqrt{2}\sin(x + \pi/4)$; both are the sine curves with amplitude $\sqrt{2}$ and phase shift $-\pi/4$. ▲

Problem Set 4.4

Use addition of y-coordinates to sketch the graph of each of the following between $x = 0$ and $x = 4\pi$.

1. $y = 1 + \sin x$
2. $y = 1 + \cos x$
3. $y = 2 - \cos x$
4. $y = 2 - \sin x$
5. $y = 4 + 2\sin x$
6. $y = 4 + 2\cos x$
7. $y = \frac{1}{3}x - \cos x$
8. $y = \frac{1}{2}x - \sin x$
9. $y = \frac{1}{2}x - \cos x$
10. $y = \frac{1}{3}x + \cos x$

Sketch the graph of each equation from $x = 0$ to $x = 8$.

11. $y = x + \sin \pi x$
12. $y = x + \cos \pi x$

Sketch the graph from $x = 0$ to $x = 4\pi$.

13. $y = 3\sin x + \cos 2x$
14. $y = 3\cos x + \sin 2x$
15. $y = 2\sin x - \cos 2x$
16. $y = 2\cos x - \sin 2x$
17. $y = \sin x + \sin \frac{x}{2}$
18. $y = \cos x + \cos \frac{x}{2}$

19. $y = \sin x + \sin 2x$

20. $y = \cos x + \cos 2x$

21. $y = \cos x + \frac{1}{2}\sin 2x$

22. $y = \sin x + \frac{1}{2}\cos 2x$

23. $y = \sin x - \cos x$

24. $y = \cos x - \sin x$

25. Make a table using multiples of $\pi/2$ for x between 0 and 4π to help sketch the graph of $y = x \sin x$.

26. Sketch the graph of $y = x \cos x$.

Review Problems The problems below review material we covered in Section 3.5.

27. A point moving on the circumference of a circle covers 5 feet every 20 seconds. Find the linear velocity of the point.

28. A point is moving with a linear velocity of 20 feet per second on the circumference of a circle. How far does the point move in 1 minute?

29. A point is moving with an angular velocity of 3 radians per second on a circle of radius 6 meters. How far does the point travel in 10 seconds?

30. A point is rotating at 5 rpm on a circle of radius 6 inches. What is the linear velocity of the point?

4.5 Inverse Trigonometric Relations

In this section we will introduce and develop the inverse relations for our six trigonometric functions. Before we do, however, we will review a couple of topics from inverse functions in general.

There are two main topics from inverse functions that we need to review. The first is the process by which you obtain the equation of the inverse of a function from the function itself, and the second is the relationship between the graph of a function and the graph of its inverse. These two topics are reviewed in Examples 1 and 2.

Example 1 Write an equation for the inverse of the function $y = x^2 - 4$ and then graph both the function and its inverse.

Solution To find the inverse of $y = x^2 - 4$, we exchange x and y in the equation and then solve for y.

The inverse of $y = x^2 - 4$	
is $x = y^2 - 4$	Exchange x and y
or $y^2 - 4 = x$	
$y^2 = x + 4$	Add 4 to both sides
$y = \pm\sqrt{x + 4}$	Take the square root of both sides

The inverse of the function $y = x^2 - 4$ is given by the equation $y = \pm\sqrt{x + 4}$. If we were using function notation, we would write: The inverse of $f(x) = x^2 - 4$ is $f^{-1}(x) = \pm\sqrt{x + 4}$. The graph of $y = x^2 - 4$ is a parabola that crosses the x-axis at -2 and 2 and has its vertex at $(0, -4)$. To graph its inverse, we reflect the graph of $y = x^2 - 4$ about the line $y = x$. Figure 1 shows both graphs.

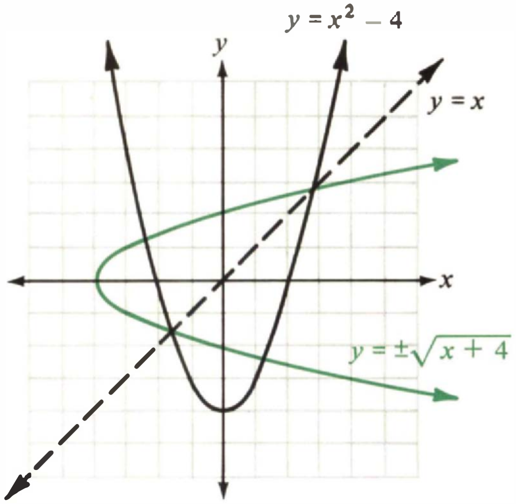


Figure 1

From Figure 1 we see that $y = \pm\sqrt{x + 4}$ is not a function because we can find a vertical line that will cross the graph of $y = \pm\sqrt{x + 4}$ in more than one place, indicating that some values of x correspond to more than one value of y. ▲

The three important points about inverse functions illustrated in Example 1 are:

1. The equation of the inverse is found by exchanging x and y in the equation of the function.
2. The graph of the inverse can be found by reflecting the graph of the function about the line $y = x$.
3. A graph is not the graph of a function if a vertical line crosses it in more than one place.

Note Again, this discussion is meant to be a review. If it is not making any sense, then you should read up on functions and their inverses and then come back and try it again.

The next example also illustrates the three points outlined in Example 1 and shows what we have done in the past (in algebra) when we were unable to solve the equation of the inverse of a function explicitly for y.

Example 2 Find the equation of the inverse of the exponential function $y = 2^x$ and graph both $y = 2^x$ and its inverse.

Solution We can find an equation that represents the inverse of $y = 2^x$ by simply exchanging x and y to get

$$x = 2^y$$

But it is impossible to solve this for y in terms of x by algebraic methods. The solution to this problem is to invent a new notation and simply give it the meaning we want it to have. In this case, we say the expression $x = 2^y$ is equivalent to

$$y = \log_2 x$$

and we have an equation for the inverse of $y = 2^x$ that gives y in terms of x. (If you do not think you did this before, look back to your intermediate algebra book, the chances are very good that your book simply invented the notation $y = \log_2 x$ as a way of expressing the inverse of $y = 2^x$.) If we were using function notation, we would say: The inverse of the function $f(x) = 2^x$ is $f^{-1}(x) = \log_2 x$. Figure 2 shows the graphs of $y = 2^x$ and its inverse $y = \log_2 x$.

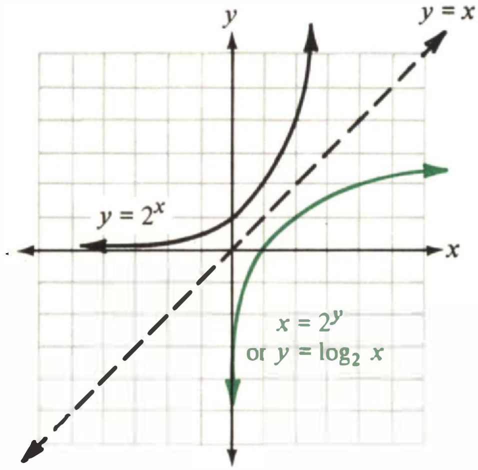


Figure 2

As you can see from Figure 2, both $y = 2^x$ and $y = \log_2 x$ are functions since no vertical line can be found that will cross either graph in more than one place. ▲

The Inverse Sine Relation

To find the inverse of $y = \sin x$, we interchange x and y to obtain

$$x = \sin y$$

This is the equation of the inverse sine relation. Since we would like to write this relationship with y in terms of x, we take the notation

$$y = \sin^{-1}x \text{ or } y = \arcsin x$$

to mean $x = \sin y$. Both the expressions $y = \sin^{-1}x$ and $y = \arcsin x$ are used to indicate the inverse of $y = \sin x$. In each case, we read the expression as "y is the angle whose sine is x." The notation $y = \sin^{-1}x$ should not be interpreted as meaning the reciprocal of $\sin x$. That is,

$$\sin^{-1}x \neq \frac{1}{\sin x}$$

If we want the reciprocal of $\sin x$, we use $\csc x$ or $(\sin x)^{-1}$, but never $\sin^{-1}x$.

To graph $y = \sin^{-1}x$, we simply reflect the graph of $y = \sin x$ about the line $y = x$, as shown in Figure 3.

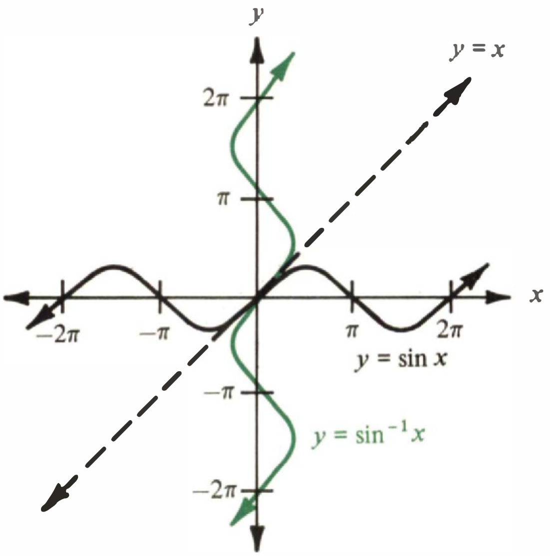


Figure 3

As you can see from the graph, $y = \sin^{-1}x$ is a relation but not a function. For every value of x in the domain, there are many values of y.

Example 3 Find all values of y in degrees that satisfy each of the following expressions.

a. $y = \sin^{-1}\frac{1}{2}$ **b.** $y = \arcsin\left(-\frac{\sqrt{3}}{2}\right)$ **c.** $y = \sin^{-1}0$

Solution

a. If $y = \sin^{-1}(1/2)$, then $\sin y = 1/2$. This means y is an angle with a sine of $1/2$. Therefore,

$$y = 30° + 360°k \quad \text{or} \quad 150° + 360°k, \text{ where } k = \text{an integer}$$

b. If $y = \arcsin(-\sqrt{3}/2)$, then $\sin y = -\sqrt{3}/2$, meaning y is an angle with a sine of $-\sqrt{3}/2$.

$$y = 240° + 360°k \quad \text{or} \quad 300° + 360°k, \text{ where } k = \text{an integer}$$

c. If $y = \sin^{-1}0$, then $\sin y = 0$, indicating that y is an angle whose sine is 0. Hence, y must be an integer multiple of 180°.

$$y = 180°k, \text{ where } k = \text{an integer}$$

To summarize what we have developed thus far, and extend it to the other five trigonometric functions, we have the following definition.

Definition (Inverse Trigonometric Relations)

The inverse of	Is written	And is equivalent to
$y = \sin x$	$y = \sin^{-1}x$ or $y = \arcsin x$	$x = \sin y$
$y = \cos x$	$y = \cos^{-1}x$ or $y = \arccos x$	$x = \cos y$
$y = \tan x$	$y = \tan^{-1}x$ or $y = \arctan x$	$x = \tan y$
$y = \cot x$	$y = \cot^{-1}x$ or $y = \text{arccot } x$	$x = \cot y$
$y = \sec x$	$y = \sec^{-1}x$ or $y = \text{arcsec } x$	$x = \sec y$
$y = \csc x$	$y = \csc^{-1}x$ or $y = \text{arccsc } x$	$x = \csc y$

Example 4 Evaluate and give the results in radians.

a. $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ **b.** $\text{arccot}(-1)$ **c.** $\sin^{-1}(3)$

Solution In each case, if we set y equal to the expression in question, then we can use the definitions directly.

a. If $y = \cos^{-1}(1/\sqrt{2})$, then $\cos y = 1/\sqrt{2}$. Therefore,

$$y = \frac{\pi}{4} + 2k\pi \quad \text{or} \quad \frac{7\pi}{4} + 2k\pi$$

where k is an integer.

b. If $y = \text{arccot}(-1)$, then $\cot y = -1$. Therefore,

$$y = \frac{3\pi}{4} + k\pi$$

Note that, since the period of the cotangent function is π, we find all solutions within one period, and then we add on multiples of the period. That is why we add $k\pi$ instead of $2k\pi$ to $3\pi/4$.

c. If $y = \sin^{-1}(3)$, then $\sin y = 3$, which is impossible. There is no angle with a sine of 3, since for any y, $-1 \leq \sin y \leq 1$. ▲

Problem Set 4.5

For each equation below, write an equation for its inverse and then sketch the graph of the equation and its inverse on the same coordinate system.

1. $y = x^2 + 4$
2. $y = x^2 - 3$
3. $y = 2x^2$
4. $y = \frac{1}{2}x^2$
5. $y = 3x - 2$
6. $y = 2x + 3$
7. $x^2 + y^2 = 9$
8. $x^2 + y^2 = 4$
9. $y = 3^x$
10. $y = 4^x$
11. Graph $y = \cos x$ between -2π and 2π and then reflect the graph about the line $y = x$ to obtain the graph of $y = \arccos x$.
12. Graph $y = \sin x$ between $-\pi/2$ and $\pi/2$ and then reflect the graph about the line $y = x$ to obtain the graph of $y = \sin^{-1}x$ between $-\pi/2$ and $\pi/2$.
13. Graph $y = \tan x$ for x between $-3\pi/2$ and $3\pi/2$ and then reflect the graph about the line $y = x$ to obtain the graph of $y = \arctan x$.
14. Graph $y = \cot x$ for x between 0 and 2π and then reflect the graph about the line $y = x$ to obtain the graph of $y = \cot^{-1}x$.

The domain of a relation is the set of all first coordinates (the set of all values that x can assume). The range of a relation is the set of all second coordinates (all the values that y can assume).

15. What is the domain of the relation $y = \sin^{-1}x$?

16. What is the range of the relation $y = \sin^{-1} x$?
17. What is the range of $y = \arccos x$?
18. What is the domain of $y = \text{arccsc}\, x$?

Find all values of y, in degrees, that satisfy each of the following expressions:

19. $y = \sin^{-1}\frac{\sqrt{3}}{2}$

20. $y = \sin^{-1}\frac{1}{\sqrt{2}}$

21. $y = \cos^{-1}\frac{1}{2}$

22. $y = \cos^{-1}\frac{\sqrt{3}}{2}$

23. $y = \sin^{-1}\left(-\frac{1}{2}\right)$

24. $y = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

25. $y = \arccos(0)$

26. $y = \arccos(-1)$

27. $y = \arcsin(-1)$

28. $y = \arcsin(1)$

29. $y = \tan^{-1}(1)$

30. $y = \cot^{-1}(1)$

31. $y = \sec^{-1}(2)$

32. $y = \csc^{-1}(2)$

Evaluate and give each result in radians.

33. $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

34. $\cos^{-1}\left(-\frac{1}{2}\right)$

35. $\arctan(\sqrt{3})$

36. $\text{arccot}\left(\frac{1}{\sqrt{3}}\right)$

37. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

38. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

39. $\text{arcsec}(\sqrt{2})$

40. $\text{arccsc}(\sqrt{2})$

41. $\csc^{-1}(1)$

42. $\sec^{-1}(1)$

43. $\cot^{-1}(0)$

44. $\tan^{-1}(1)$

45. $\arccos(1)$

46. $\arcsin(0)$

Review Problems The problems that follow review material we covered in Section 4.2.

Graph each of the following equations over the indicated interval. Be sure to label the x- and y-axes so that the amplitude and period are easy to see.

47. $y = 4 \sin 2x$, one complete cycle
48. $y = 2 \sin 4x$, one complete cycle
49. $y = 2 \sin \pi x, \quad -4 \le x \le 4$
50. $y = 3 \cos \pi x, \quad -2 \le x \le 4$
51. $y = -3 \cos \frac{1}{2} x, \quad -2\pi \le x \le 6\pi$
52. $y = -3 \sin 2x, \quad -2\pi \le x \le 2\pi$

4.6
Inverse Trigonometric Functions

In order that the function $y = \sin x$ have an inverse that is also a function, it is necessary to restrict the values that y can assume in $y = \sin^{-1}x$. Although there are many intervals over which $y = \sin^{-1}x$ can be restricted and still assume all values of y exactly once, the interval we will restrict it to is the interval $-\pi/2 \leq y \leq \pi/2$. Figure 1 contains the graph of $y = \sin^{-1}x$ with the restricted interval showing.

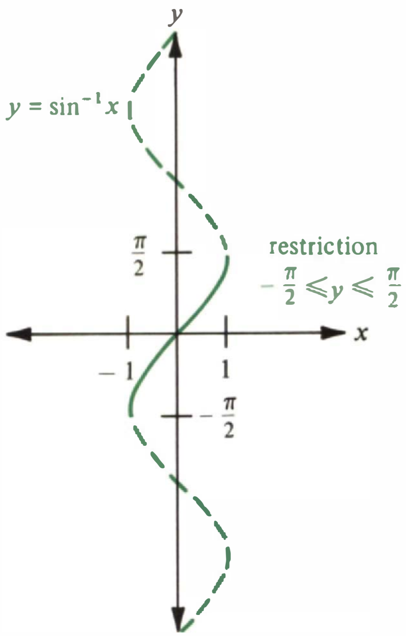


Figure 1

It is apparent from Figure 1 that if $y = \sin^{-1}x$ is restricted to the interval $-\pi/2 \leq y \leq \pi/2$, then each value of x is associated with exactly one value of y and we have a function rather than just a relation. (That is, on the interval $-\pi/2 \leq y \leq \pi/2$, the graph of $y = \sin^{-1}x$ is such that no vertical line crosses it in more than one place.)

NOTATION The notation used to indicate the difference between inverse sine relations and inverse sine functions is as follows:

	Notation	Meaning
A relation	$y = \sin^{-1}x$ or $y = \arcsin x$	$x = \sin y$
A function	$y = \text{Sin}^{-1}x$ or $y = \text{Arcsin } x$	$x = \sin y$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

In this section we will limit our discussion of inverse trigonometric functions to the inverses of the three major functions: sine, cosine, and tangent. The other three inverse trigonometric functions can be handled with the use of the reciprocal identities.

Figure 2 shows the graphs of $y = \cos^{-1}x$ and $y = \tan^{-1}x$ and the restrictions that will allow them to become functions instead of just relations.

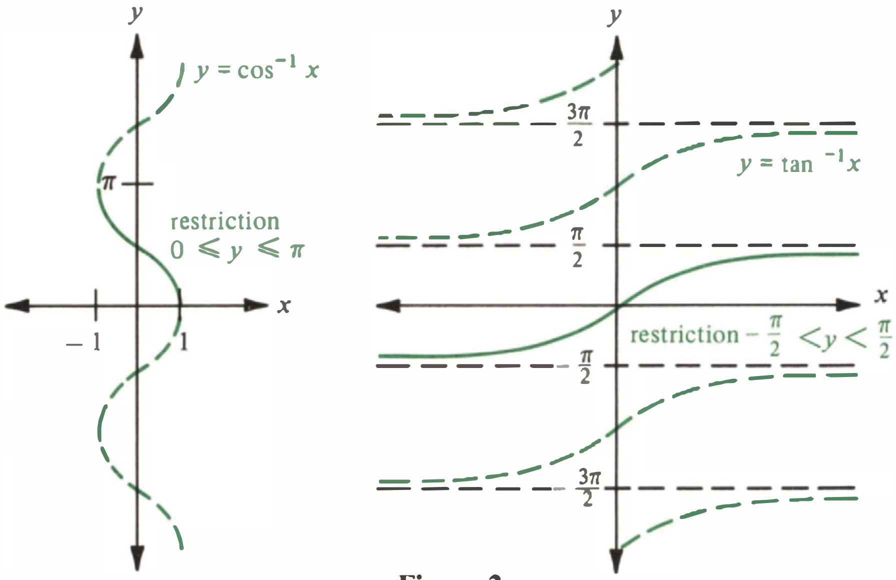


Figure 2

To summarize, here is the definition for the three major inverse trigonometric functions.

DEFINITION (INVERSE TRIGONOMETRIC FUNCTIONS) The inverse functions for $y = \sin x$, $y = \cos x$, and $y = \tan x$ are as follows:

Inverse function	Meaning
$y = \text{Sin}^{-1}x$ or $y = \text{Arcsin } x$	$x = \sin y$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
In words: y is the angle between $-\pi/2$ and $\pi/2$ whose sine is x.	
$y = \text{Cos}^{-1}x$ or $y = \text{Arccos } x$	$x = \cos y$ and $0 \leq y \leq \pi$
In words: y is the angle between 0 and π whose cosine is x.	
$y = \text{Tan}^{-1}x$ or y $= Arctan\ x$	$x = \tan y$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$
In words: y is the angle between $-\pi/2$ and $\pi/2$ whose tangent is x.	

▼ **Example 1** Evaluate in radians without using a calculator or tables.

a. $\text{Sin}^{-1}\frac{1}{2}$ **b.** $\text{Arccos}\left(-\frac{\sqrt{3}}{2}\right)$ **c.** $\text{Tan}^{-1}(-1)$

Solution

a. The angle between $-\pi/2$ and $\pi/2$ whose sine is $1/2$ is $\pi/6$.

$$\text{Sin}^{-1}\frac{1}{2} = \frac{\pi}{6}$$

b. The angle between 0 and π with a cosine of $-\sqrt{3}/2$ is $5\pi/6$.

$$\text{Arccos}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$$

c. The angle between $-\pi/2$ and $\pi/2$ the tangent of which is -1 is $-\pi/4$.

$$\text{Tan}^{-1}(-1) = -\frac{\pi}{4}$$ ▲

Note In part c of Example 1, it would be incorrect to give the answer as $7\pi/4$. It is true that $\tan 7\pi/4 = -1$, but $7\pi/4$ is not between $-\pi/2$ and $\pi/2$. There is a difference.

▼ **Example 2** Use a calculator or tables to evaluate each expression to the nearest tenth of a degree.

a. Arcsin(0.5075) **b.** Arcsin(−0.5075)
c. $\text{Cos}^{-1}(0.6428)$ **d.** $\text{Cos}^{-1}(-0.6428)$
e. Arctan(4.474) **f.** Arctan(−4.474)

Solution The easiest method of evaluating these expressions is to use a calculator. Make sure the calculator is set to the degree mode and then enter the number and push the appropriate button. Scientific calculators are programmed so that the restrictions on the inverse trigonometric functions are automatic. We just push the buttons and the rest is taken care of. If a table is used, we have to be careful to place the angle in question in the appropriate quadrant and name it so that it falls within the restriction for that particular inverse function.

a. $\text{Arcsin}(0.5075) = 30.5°$
b. $\text{Arcsin}(-0.5075) = -30.5°$

Reference angle 30.5°

c. $\text{Cos}^{-1}(0.6428) = 50.0°$
d. $\text{Cos}^{-1}(-0.6428) = 130.0°$ } Reference angle 50°
e. $\text{Arctan}(4.474) = 77.4°$
f. $\text{Arctan}(-4.474) = -77.4°$ } Reference angle 77.4° ▲

▼ **Example 3** Evaluate $\sin(\text{Tan}^{-1}3/4)$ without using a calculator or tables.

Solution We begin by letting $\theta = \text{Tan}^{-1}(3/4)$. (Remember, $\text{Tan}^{-1}x$ is the angle whose tangent is x.) Then we have:

$$\text{If } \theta = \text{Tan}^{-1}\frac{3}{4}\text{, then } \tan\theta = \frac{3}{4} \text{ and } 90° < \theta < 90°$$

We can draw a triangle in which one of the acute angles is θ. Since $\tan\theta = 3/4$, we label the side opposite θ with 3 and the side adjacent θ with 4. The hypotenuse is found by applying the Pythagorean theorem.

$$\begin{aligned}\text{hypotenuse} &= \sqrt{3^2 + 4^2}\\ &= \sqrt{25}\\ &= 5\end{aligned}$$

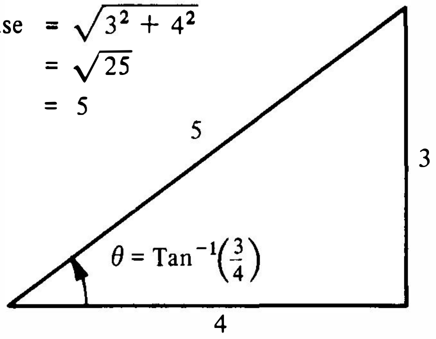


Figure 3

From Figure 3 we find $\sin\theta$ using the ratio of the side opposite θ to the hypotenuse.

$$\sin\left(\text{Tan}^{-1}\frac{3}{4}\right) = \sin\theta = \frac{3}{5}$$ ▲

Calculator Note If we were to do the same problem with the aid of a calculator, the sequence would look like this:

3 [÷] 4 [=] [tan⁻¹] [sin]

The display would read 0.6 which is 3/5.

Although it is a lot easier to use a calculator on problems like the one in Example 3, solving it without a calculator will be of more use to you in the future.

▼ **Example 4** Write the expression $\sin(\text{Cos}^{-1}x)$ as an equivalent expression in x only. (Assume x is positive.)

Solution We let $\theta = \text{Cos}^{-1}x$, then $\cos\theta = x$. To help visualize the problem we draw a right triangle with an acute angle of θ and label it so that $\cos\theta = x$. This is accomplished by labeling the side adjacent to θ with x and the hypotenuse with 1. That way, the ratio of the side adjacent θ to the hypotenuse is $x/1 = x$. The side opposite θ is found by applying the Pythagorean theorem.

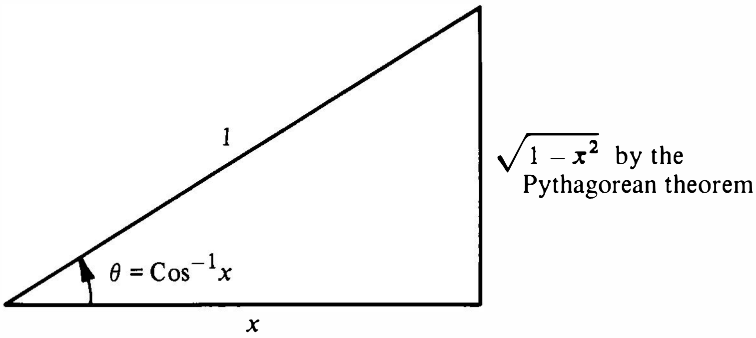


Figure 4

Finding $\sin\theta$ is simply a matter of taking the ratio of the opposite to the hypotenuse.

$$\sin(\text{Cos}^{-1}x) = \sin\theta = \frac{\sqrt{1 - x^2}}{1} = \sqrt{1 - x^2}$$

Note that, since we assumed x was positive, $\text{Cos}^{-1}x$ is between 0° and 90°, so that $\sin(\text{Cos}^{-1}x)$ is also positive. Without the restriction that x is positive, we would need a ± sign in front of the square root in our answer. ▲

Problem Set 4.6

Evaluate each expression without using a calculator or tables and write your answers in radians.

1. $\text{Sin}^{-1}\left(\frac{\sqrt{3}}{2}\right)$
2. $\text{Cos}^{-1}\left(\frac{1}{2}\right)$
3. $\text{Cos}^{-1}(-1)$
4. $\text{Cos}^{-1}(0)$
5. $\text{Tan}^{-1}(1)$
6. $\text{Tan}^{-1}(0)$
7. $\text{Arccos}\left(-\frac{1}{\sqrt{2}}\right)$
8. $\text{Arccos}(1)$

9. $\text{Sin}^{-1}\left(-\frac{1}{2}\right)$

10. $\text{Sin}^{-1}\left(\frac{1}{\sqrt{2}}\right)$

11. $\text{Arctan}(\sqrt{3})$

12. $\text{Arctan}\left(\frac{1}{\sqrt{3}}\right)$

13. $\text{Arcsin}(0)$

14. $\text{Arcsin}\left(-\frac{\sqrt{3}}{2}\right)$

15. $\text{Tan}^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

16. $\text{Tan}^{-1}(-\sqrt{3})$

17. $\text{Cos}^{-1}\left(-\frac{1}{2}\right)$

18. $\text{Sin}^{-1}(1)$

19. $\text{Arccos}\left(\frac{\sqrt{3}}{2}\right)$

20. $\text{Arcsin}(-1)$

Use a calculator or tables to evaluate each expression to the nearest tenth of a degree.

21. $\text{Sin}^{-1}(0.1702)$

22. $\text{Sin}^{-1}(-0.1702)$

23. $\text{Cos}^{-1}(-0.8425)$

24. $\text{Cos}^{-1}(0.8425)$

25. $\text{Tan}^{-1}(0.3799)$

26. $\text{Tan}^{-1}(-0.3799)$

27. $\text{Arcsin}(0.9627)$

28. $\text{Arccos}(0.9627)$

29. $\text{Cos}^{-1}(-0.4664)$

30. $\text{Sin}^{-1}(-0.4664)$

31. $\text{Arctan}(-2.748)$

32. $\text{Arctan}(-0.3640)$

33. $\text{Sin}^{-1}(-0.7660)$

34. $\text{Cos}^{-1}(-0.7660)$

Evaluate without using a calculator.

35. $\cos\left(\text{Tan}^{-1}\frac{3}{4}\right)$

36. $\csc\left(\text{Tan}^{-1}\frac{3}{4}\right)$

37. $\tan\left(\text{Sin}^{-1}\frac{3}{5}\right)$

38. $\tan\left(\text{Cos}^{-1}\frac{3}{5}\right)$

39. $\sec\left(\text{Cos}^{-1}\frac{1}{\sqrt{5}}\right)$

40. $\sin\left(\text{Cos}^{-1}\frac{1}{\sqrt{5}}\right)$

41. $\sin\left(\text{Cos}^{-1}\frac{1}{2}\right)$

42. $\cos\left(\text{Sin}^{-1}\frac{1}{2}\right)$

43. $\cot\left(\text{Tan}^{-1}\frac{1}{2}\right)$

44. $\cot\left(\text{Tan}^{-1}\frac{1}{3}\right)$

Evaluate without using a calculator.

45. $\sin\left(\text{Sin}^{-1}\frac{3}{5}\right)$

46. $\cos\left(\text{Cos}^{-1}\frac{3}{5}\right)$

47. $\cos\left(\text{Cos}^{-1}\frac{1}{2}\right)$

48. $\sin\left(\text{Sin}^{-1}\frac{1}{\sqrt{2}}\right)$

49. $\tan\left(\text{Tan}^{-1}\frac{1}{2}\right)$

50. $\tan\left(\text{Tan}^{-1}\frac{3}{4}\right)$

For each expression below, write an equivalent expression that involves x only. (Assume x is positive.)

51. $\cos(\text{Cos}^{-1}x)$

52. $\sin(\text{Sin}^{-1}x)$

53. $\cos(\text{Sin}^{-1}x)$

54. $\tan(\text{Cos}^{-1}x)$

55. $\sin(\text{Tan}^{-1}x)$

56. $\cos(\text{Tan}^{-1}x)$

57. $\sin\left(\text{Cos}^{-1}\frac{1}{x}\right)$

58. $\cos\left(\text{Sin}^{-1}\frac{1}{x}\right)$

59. $\sec\left(\text{Cos}^{-1}\frac{1}{x}\right)$

60. $\csc\left(\text{Sin}^{-1}\frac{1}{x}\right)$

Review Problems The problems that follow review material we covered in Section 4.3.

Graph one complete cycle of each of the following equations. Be sure to label the x- and y-axes so that the amplitude, period, and phase shift for each graph is easy to see.

61. $y = \sin\left(x - \frac{\pi}{4}\right)$

62. $y = \sin\left(x + \frac{\pi}{6}\right)$

63. $y = \cos\left(2x + \frac{\pi}{2}\right)$

64. $y = \sin(2x + \pi)$

65. $y = 3\sin\left(2x - \frac{\pi}{3}\right)$

66. $y = 3\cos\left(2x - \frac{\pi}{3}\right)$

Chapter 4 Summary and Review

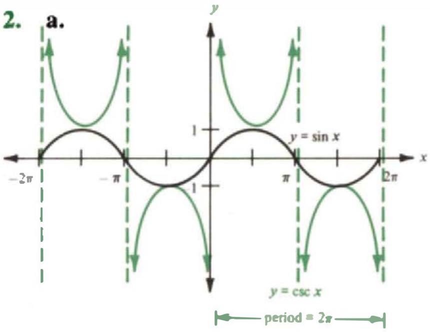
PERIODIC FUNCTIONS [4.1]

A function $y = f(x)$ is said to be periodic with period p if p is the smallest positive number such that $f(x + p) = f(x)$ for all x in the domain of f.

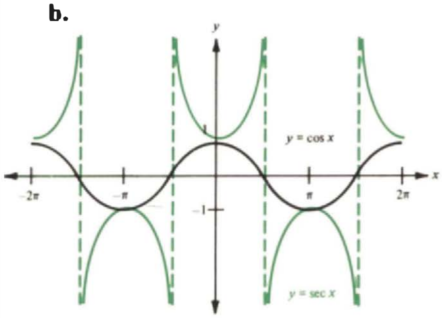
Examples

1. Since $\sin(x + 2\pi) = \sin x$, the function $y = \sin x$ is periodic with period 2π. Likewise, since $\tan(x + \pi) = \tan x$, the function $y = \tan x$ is periodic with period π.

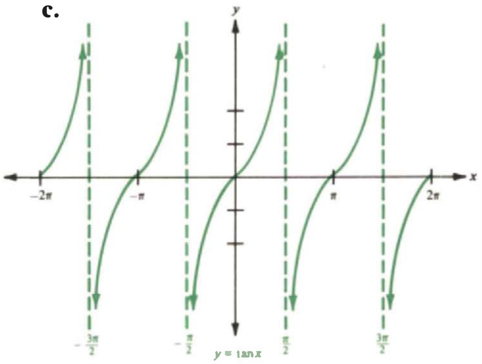
2. a.



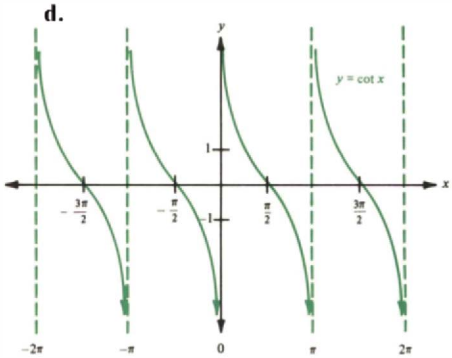
b.



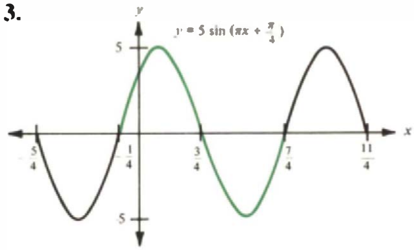
c.



d.



3.



$A = \frac{1}{2}|5 - (-5)| = \frac{1}{2}(10) = 5$

BASIC GRAPHS [4.1]

The graphs of $y = \sin x$ and $y = \cos x$ are both periodic with period 2π. The amplitude of each graph is 1. The sine curve passes through 0 on the y-axis, while the cosine curve passes through 1 on the y-axis.

The graphs of $y = \csc x$ and $y = \sec x$ are also periodic with period 2π. We graph them by using the fact that they are reciprocals of sine and cosine. Since there is no largest or smallest value of y, we say the secant and cosecant curves have no amplitude.

The graphs of $y = \tan x$ and $y = \cot x$ are periodic with period π. The tangent curve passes through the origin, while the cotangent is undefined when x is 0. There is no amplitude for either graph.

AMPLITUDE [4.2]

The *amplitude A* of a curve is half the absolute value of the difference between the largest value of y, denoted by M, and the smallest value of y, denoted by m.

$$A = \frac{1}{2}|M - m|$$

PHASE SHIFT [4.3]

The *phase shift* for a sine or cosine curve, is the distance the curve has moved right or left from the curve $y = \sin x$ or $y = \cos x$. For example, we usually think of the graph of $y = \sin x$ as starting at the origin. If we graph another sine curve that starts at $\pi/4$, then we say this curve has a phase shift of $\pi/4$.

4. The phase shift for the graph in Example 3 is $-1/4$.

GRAPHING SINE AND COSINE CURVES [4.2, 4.3]

The graphs of $y = A\sin(Bx + C)$ and $y = A\cos(Bx + C)$, where $B > 0$, will have the following characteristics:

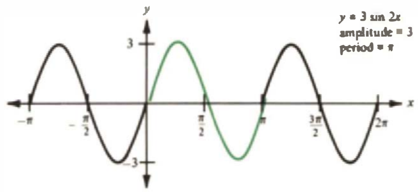
$$\text{Amplitude} = |A|$$

$$\text{Period} = \frac{2\pi}{B}$$

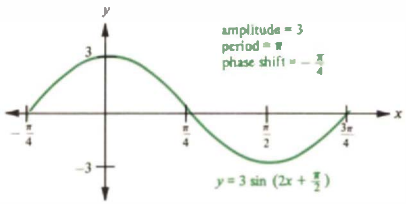
$$\text{Phase shift} = -\frac{C}{B}$$

To graph one of these curves, we first find the phase shift and label that point on the x-axis (this will be our starting point). We then add the period to the phase shift and mark the result on the x-axis (this is our ending point). We mark the y-axis with the amplitude. Finally, we sketch in one complete cycle of the curve in question keeping in mind that, if A is negative, the graph must be reflected about the x-axis.

5. a.



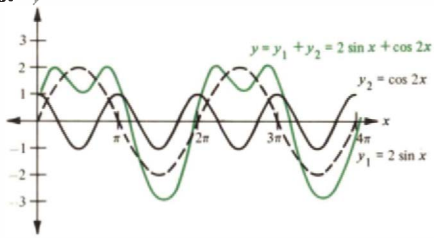
b.



GRAPHING BY ADDITION OF Y-COORDINATES [4.4]

To graph equations of the form $y = y_1 + y_2$, where y_1 and y_2 are algebraic or trigonometric functions of x, we graph y_1 and y_2 separately on the same coordinate system and then add the two graphs to obtain the graph of y.

6.



a. If $y = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$, then $\cos y = \frac{1}{\sqrt{2}}$.

Therefore,

$$y = \frac{\pi}{4} + 2k\pi \quad \text{or} \quad \frac{7\pi}{4} + 2k\pi$$

where k is an integer.

b. If $y = \text{arccot}(-1)$, then $\cot y = -1$. Therefore,

$$y = \frac{3\pi}{4} + 2k\pi \quad \text{or} \quad \frac{7\pi}{4} + 2k\pi$$

where k is an integer.

c. If $y = \sin^{-1}(3)$, then $\sin y = 3$, which is impossible. There is no angle with a sine of 3, since for any y, $-1 \le \sin y \le 1$.

INVERSE TRIGONOMETRIC RELATIONS [4.5]

The inverse of	Written	Is equivalent to
$y = \sin x$	$y = \sin^{-1}x$ or $y = \arcsin x$	$x = \sin y$
$y = \cos x$	$y = \cos^{-1}x$ or $y = \arccos x$	$x = \cos y$
$y = \tan x$	$y = \tan^{-1}x$ or $y = \arctan x$	$x = \tan y$
$y = \cot x$	$y = \cot^{-1}x$ or $y = \text{arccot}\, x$	$x = \cot y$
$y = \sec x$	$y = \sec^{-1}x$ or $y = \text{arcsec}\, x$	$x = \sec y$
$y = \csc x$	$y = \csc^{-1}x$ or $y = \text{arccsc}\, x$	$x = \csc y$

8. Evaluate in radians without using a calculator or tables.

a. $\text{Sin}^{-1}\frac{1}{2}$

The angle between $-\pi/2$ and $\pi/2$ whose sine is $1/2$ is $\pi/6$.

$$\text{Sin}^{-1}\frac{1}{2} = \frac{\pi}{6}$$

b. $\text{Arccos}\left(-\frac{\sqrt{3}}{2}\right)$

The angle between 0 and π with a cosine of $-\sqrt{3}/2$ is $5\pi/6$.

$$\text{Arccos}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$$

c. $\text{Tan}^{-1}(-1)$

The angle between $-\pi/2$ and $\pi/2$ the tangent of which is -1 is $-\pi/4$.

$$\text{Tan}^{-1}(-1) = -\frac{\pi}{4}$$

INVERSE TRIGONOMETRIC FUNCTIONS [4.6]

Inverse function	Meaning
$y = \text{Sin}^{-1}x$ or $y = \text{Arcsin}\, x$	$x = \sin y$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
In words: y is the angle between $-\pi/2$ and $\pi/2$ whose sine is x.	
$y = \text{Cos}^{-1}x$ or $y = \text{Arccos}\, x$	$x = \cos y$ and $0 \le y \le \pi$
In words: y is the angle between 0 and π whose cosine is x.	
$y = \text{Tan}^{-1}x$ or $y = \text{Arctan}\, x$	$x = \tan y$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$
In words: y is the angle between $-\pi/2$ and $\pi/2$ whose tangent is x.	

Chapter 4 Test

Graph each of the following between $x = -4\pi$ and $x = 4\pi$.

1. $y = \sin x$

2. $y = \cos x$

3. $y = \tan x$

4. $y = \sec x$

5. How many complete cycles of the graph of the equation $y = \sin x$ are shown in your answer to Problem 1?

6. How many complete cycles of the graph of the equation $y = \tan x$ are shown in your answer to Problem 3?

7. Use your answer to Problem 4 to find all values of x between -4π and 4π for which $\sec x = -1$.

8. Use your answer to Problem 2 to find all values of x between -4π and 4π for which $\cos x = 1/2$.

For each equation below, first identify the amplitude and period and then use this information to sketch one complete cycle of the graph.

9. $y = \cos \pi x$

10. $y = -3 \cos x$

Graph each of the following on the given interval.

11. $y = 3 \sin 2x, \; -\pi \le x \le 2\pi$

12. $y = 2 \sin \pi x, \; -4 \le x \le 4$

For each equation below identify the amplitude, period, and phase shift and then use this information to sketch one complete cycle of the graph.

13. $y = \sin\left(x + \frac{\pi}{4}\right)$

14. $y = \cos\left(x - \frac{\pi}{2}\right)$

15. $y = 3 \sin\left(2x - \frac{\pi}{3}\right)$

16. $y = 3 \sin\left(\frac{\pi}{3} x - \frac{\pi}{3}\right)$

17. $y = \csc\left(x + \frac{\pi}{4}\right)$

18. $y = \tan\left(2x - \frac{\pi}{2}\right)$

Graph each of the following on the given interval.

19. $y = 2 \sin(3x - \pi), \; -\frac{\pi}{3} \le x \le \frac{5\pi}{3}$

20. $y = 2 \sin\left(\frac{\pi}{2} x - \frac{\pi}{4}\right), \; -\frac{1}{2} \le x \le \frac{13}{2}$

Sketch the following between $x = 0$ and $x = 4\pi$.

21. $y = \frac{1}{2} x - \sin x$

22. $y = \sin x + \cos 2x$

Find all values of y, in degrees, that satisfy each of the following expressions:

23. $y = \sin^{-1} 0$

24. $y = \cos^{-1}(-1)$

25. Graph $y = \mathrm{Cos}^{-1} x$

26. Graph $y = \mathrm{Arcsin}\, x$

Evaluate each expression without using a calculator or tables and write your answer in radians.

27. $\text{Sin}^{-1}\left(\frac{1}{2}\right)$ **28.** $\text{Cos}^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

29. $\text{Arctan}(-1)$ **30.** $\text{Arcsin}(1)$

Use a calculator or tables to evaluate each expression to the nearest tenth of a degree.

31. $\text{Arcsin}(0.5934)$ **32.** $\text{Arctan}(-0.8302)$

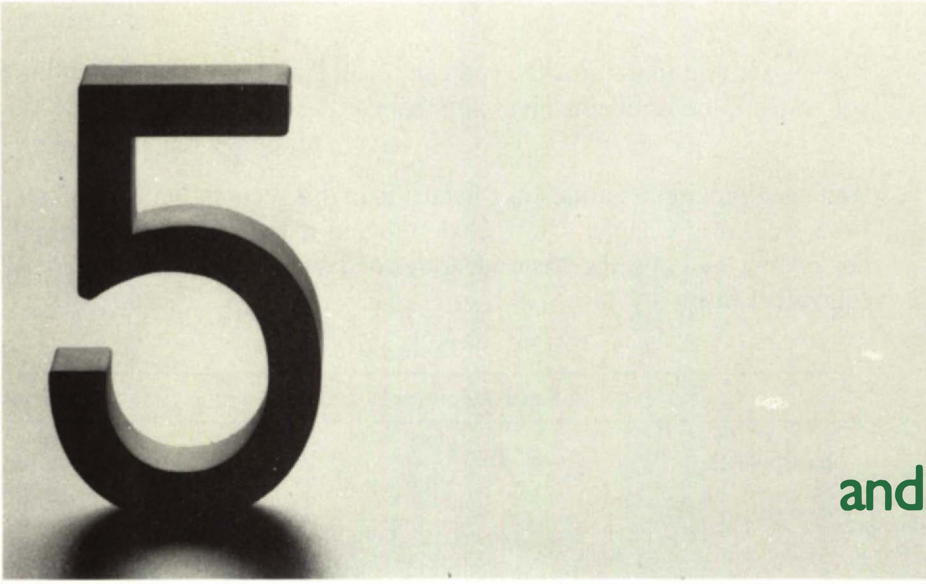
33. $\text{Arccos}(-0.6981)$ **34.** $\text{Arcsin}(-0.2164)$

Evaluate without using a calculator.

35. $\tan\left(\text{Cos}^{-1}\frac{2}{3}\right)$ **36.** $\cos\left(\text{Tan}^{-1}\frac{2}{3}\right)$

For each expression below, write an equivalent expression that involves x only. (Assume x is positive.)

37. $\sin(\text{Cos}^{-1}x)$ **38.** $\tan(\text{Sin}^{-1}x)$



Identities and Formulas

To the student:

Recall that an identity in mathematics is a statement that two quantities are equal for all replacements of the variable for which they are defined. For instance, the statement

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

is a trigonometric identity. We will begin this chapter by reviewing the list of six basic trigonometric identities and their more common equivalent forms.

In Sections 5.2, 5.3, and 5.4 we will use the identities from Section 5.1 to derive formulas for expressions of the form $\sin(A + B)$, $\sin 2A$, and $\sin A/2$. Then we will use these formulas to verify other identities and find exact values for trigonometric functions of angles that are multiples of 15°. We end the chapter with a look at identities involving inverse trigonometric functions and some special identities that allow us to change certain sums and differences to products.

The most important information needed to be successful in this chapter is the material on identities that we developed from the definition of the six trigonometric functions in Chapter 1. You need to have the basic identities and their common equivalent forms memorized. Also important is your knowledge of the exact values of trigonometric functions of 30°, 45°, and 60°. These exact values appear in a number of problems throughout the chapter and the more quickly you can recall them, the easier it will be for you to work the problems involving them.

5.1 Proving Identities

We began proving identities in Chapter 1. In this section, we will extend the work we did in Chapter 1 to include proving more complicated identities. For review, here are the basic identities and some of their more important equivalent forms.

Table 1

	Basic identities	Common equivalent forms
Reciprocal	$\csc\theta = \dfrac{1}{\sin\theta}$	$\sin\theta = \dfrac{1}{\csc\theta}$
	$\sec\theta = \dfrac{1}{\cos\theta}$	$\cos\theta = \dfrac{1}{\sec\theta}$
	$\cot\theta = \dfrac{1}{\tan\theta}$	$\tan\theta = \dfrac{1}{\cot\theta}$
Ratio	$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	
	$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$	
Pythagorean	$\cos^2\theta + \sin^2\theta = 1$	$\sin^2\theta = 1 - \cos^2\theta$ $\sin\theta = \pm\sqrt{1 - \cos^2\theta}$ $\cos^2\theta = 1 - \sin^2\theta$ $\cos\theta = \pm\sqrt{1 - \sin^2\theta}$
	$1 + \tan^2\theta = \sec^2\theta$	
	$1 + \cot^2\theta = \csc^2\theta$	

Recall that an identity in trigonometry is a statement that two expressions are equal for all replacements of the variable for which each statement is defined. To prove (or verify) a trigonometric identity, we use trigonometric substitutions and algebraic manipulations to either

1. Transform the right side of the identity into the left side, or
2. Transform the left side of the identity into the right side.

The main thing to remember in proving identities is to work on each side of the identity separately. We do not want to use properties from algebra that involve both sides of the identity—like the addition property of equality. We prove identities in order to develop the ability to transform one trigonometric expression into another. When we encounter problems in other courses that require the use of the techniques used to verify identities, we usually find that the solution to these problems hinges upon transforming an expression containing trigonometric functions into less complicated expressions. In these cases, we do not usually have an equal sign to work with.

▼ **Example 1** Prove $\sin\theta\cot\theta = \cos\theta$.

Proof To help us remember to work with each side independently, we will separate the two sides with a vertical line. Also, we will list the justification for each step of the proof next to the expressions we have transformed. We place a question mark over the equal sign in the original expression to indicate that it is an equality we are attempting to verify.

	$\sin\theta\cot\theta$	$\stackrel{?}{=} \cos\theta$
Ratio identity	$\sin\theta\cdot\dfrac{\cos\theta}{\sin\theta}$	
Multiply	$\dfrac{\sin\theta\cos\theta}{\sin\theta}$	
Divide out common factor $\sin\theta$	$\cos\theta$	$= \cos\theta$

In this example, we have transformed the left side into the right side. Note that the format of this proof is different from the format used to prove identities in Chapter 1. The underlying idea is the same however. We verify identities by transforming one expression into another. ▲

▼ **Example 2** Prove $\tan x + \cos x = \sin x(\sec x + \cot x)$.

Proof We can begin by applying the distributive property to the right side to multiply through by $\sin x$. Then we can change each expression on the right side to an equivalent expression involving only sines and cosines.

$\tan x + \cos x \stackrel{?}{=}$	$\sin x(\sec x + \cot x)$	
	$\sin x\sec x + \sin x\cot x$	Multiply
	$\sin x\cdot\dfrac{1}{\cos x} + \sin x\cdot\dfrac{\cos x}{\sin x}$	Reciprocal and ratio identities
	$\dfrac{\sin x}{\cos x} + \cos x$	Multiply
$\tan x + \cos x =$	$\tan x + \cos x$	Ratio identity

In this case, we transformed the right side into the left side. ▲

Before we go on to the next example, let's list some guidelines that may be useful in learning how to prove identities.

Guidelines for Proving Identities

1. It is usually best to work on the more complicated side first.
2. Look for trigonometric substitutions involving the basic identities that may help simplify things.
3. Look for algebraic operations, such as adding fractions, the distributive property, or factoring, that may simplify the side you are working with or that will at least lead to an expression that will be easier to simplify.
4. If you cannot think of anything else to do, change everything to sines and cosines and see if that helps.
5. Always keep an eye on the side you are not working with to be sure you are working toward it. There is a certain sense of direction that accompanies a successful proof.

Probably the best advice is to remember that these are simply guidelines. The best way to become proficient at proving trigonometric identities is to practice. The more identities you prove, the more you will be able to prove and the more confident you will become. *Don't be afraid to stop and start over if you don't seem to be getting anywhere*. With most identities, there are a number of different proofs that will lead to the same result. Some of the proofs will be longer than others.

Example 3 Prove $\dfrac{\cos^4 t - \sin^4 t}{\cos^2 t} = 1 - \tan^2 t$.

Proof In this example, factoring the numerator on the left side will reduce the exponents there from 4 to 2.

$$\frac{\cos^4 t - \sin^4 t}{\cos^2 t} \stackrel{?}{=} 1 - \tan^2 t$$

Factor	$\dfrac{(\cos^2 t + \sin^2 t)(\cos^2 t - \sin^2 t)}{\cos^2 t}$	
Pythagorean identity	$\dfrac{1(\cos^2 t - \sin^2 t)}{\cos^2 t}$	
Separate into two fractions	$\dfrac{\cos^2 t}{\cos^2 t} - \dfrac{\sin^2 t}{\cos^2 t}$	
Ratio identity	$1 - \tan^2 t$	$= 1 - \tan^2 t$

▲

Example 4 Prove $1 + \cos\theta = \dfrac{\sin^2\theta}{1 - \cos\theta}$.

Proof We begin this proof by applying an alternate form of the Pythagorean identity to the right side to write $\sin^2\theta$ as $1 - \cos^2\theta$. Then we factor $1 - \cos^2\theta$ as the difference of two squares and reduce to lowest terms.

$$1 + \cos\theta \stackrel{?}{=} \frac{\sin^2\theta}{1 - \cos\theta}$$

$$\frac{1 - \cos^2\theta}{1 - \cos\theta} \qquad \text{Pythagorean identity}$$

$$\frac{(1 - \cos\theta)(1 + \cos\theta)}{1 - \cos\theta} \qquad \text{Factor}$$

$$1 + \cos\theta = 1 + \cos\theta \qquad \text{Reduce}$$

Example 5 Prove $\tan x + \cot x = \sec x \csc x$.

Proof We begin this proof by writing the left side in terms of sines and cosines. Then we simplify the left side by finding a common denominator in order to add the resulting fractions.

$$\tan x + \cot x \stackrel{?}{=} \sec x \csc x$$

Change to sines and cosines $\qquad \dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x}$

LCD $\qquad \dfrac{\sin x}{\cos x} \cdot \dfrac{\sin x}{\sin x} + \dfrac{\cos x}{\sin x} \cdot \dfrac{\cos x}{\cos x}$

Add fractions $\qquad \dfrac{\sin^2 x + \cos^2 x}{\sin x \cos x}$

Pythagorean identity $\qquad \dfrac{1}{\cos x \sin x}$

Write as separate fractions $\qquad \dfrac{1}{\cos x} \cdot \dfrac{1}{\sin x}$

Reciprocal identities $\qquad \sec x \csc x = \sec x \csc x$

Example 6 Prove $\dfrac{\sin\alpha}{1 + \cos\alpha} + \dfrac{1 + \cos\alpha}{\sin\alpha} = 2\csc\alpha$.

Proof The common denominator for the left side is $\sin\alpha(1 + \sin\alpha)$. We multiply the first fraction by $(\sin\alpha)/(\sin\alpha)$ and the second fraction by $(1 + \cos\alpha)/(1 + \cos\alpha)$ to produce two equivalent fractions with the same denominator.

$$\frac{\sin\alpha}{1+\cos\alpha} + \frac{1+\cos\alpha}{\sin\alpha} \stackrel{?}{=} 2\csc\alpha$$

LCD $\quad \frac{\sin\alpha}{\sin\alpha}\cdot\frac{\sin\alpha}{1+\cos\alpha} + \frac{1+\cos\alpha}{\sin\alpha}\cdot\frac{1+\cos\alpha}{1+\cos\alpha}$

Add numerators $\quad \frac{\sin^2\alpha + (1+\cos\alpha)^2}{\sin\alpha(1+\cos\alpha)}$

Expand $(1+\cos\alpha)^2$ $\quad \frac{\sin^2\alpha + 1 + 2\cos\alpha + \cos^2\alpha}{\sin\alpha(1+\cos\alpha)}$

Pythagorean identity $\quad \frac{2 + 2\cos\alpha}{\sin\alpha(1+\cos\alpha)}$

Factor out a 2 $\quad \frac{2(1+\cos\alpha)}{\sin\alpha(1+\cos\alpha)}$

Reduce $\quad \frac{2}{\sin\alpha}$

Reciprocal identity $\quad 2\csc\alpha = 2\csc\alpha$ ▲

▼ **Example 7** Prove $\frac{1+\sin t}{\cos t} = \frac{\cos t}{1-\sin t}$.

Proof The trick to proving that this identity is true requires that we multiply the numerator and denominator on the right side by $1 + \sin t$. (This is similar to rationalizing the denominator.)

$$\frac{1+\sin t}{\cos t} \stackrel{?}{=} \frac{\cos t}{1-\sin t}$$

$\frac{\cos t}{1-\sin t}\cdot\frac{1+\sin t}{1+\sin t}$ $\quad$ Multiply numerator and denominator by $1 + \sin t$

$\frac{\cos t(1+\sin t)}{1-\sin^2 t}$ $\quad$ Multiply out the denominator

$\frac{\cos t(1+\sin t)}{\cos^2 t}$ $\quad$ Pythagorean identity

$\frac{1+\sin t}{\cos t} = \frac{1+\sin t}{\cos t}$ $\quad$ Reduce

Note that it would have been just as easy for us to verify this identity by multiplying the numerator and denominator on the left side by $1 - \sin t$. ▲

Problem Set 5.1

Prove that each of the following identities are true:

1. $\cos\theta \tan\theta = \sin\theta$
2. $\sec\theta \cot\theta = \csc\theta$
3. $\csc\theta \tan\theta = \sec\theta$
4. $\tan\theta \cot\theta = 1$
5. $\dfrac{\tan A}{\sec A} = \sin A$
6. $\dfrac{\cot A}{\csc A} = \cos A$
7. $\sec\theta \cot\theta \sin\theta = 1$
8. $\tan\theta \csc\theta \cos\theta = 1$
9. $\cos x(\csc x + \tan x) = \cot x + \sin x$
10. $\sin x(\sec x + \csc x) = \tan x + 1$
11. $\cot x - 1 = \cos x(\csc x - \sec x)$
12. $\tan x(\cos x + \cot x) = \sin x + 1$
13. $\cos^2 x(1 + \tan^2 x) = 1$
14. $\sin^2 x(\cot^2 x + 1) = 1$
15. $(1 - \sin x)(1 + \sin x) = \cos^2 x$
16. $(1 - \cos x)(1 + \cos x) = \sin^2 x$
17. $\dfrac{\cos^4 t - \sin^4 t}{\sin^2 t} = \cot^2 t - 1$
18. $\dfrac{\sin^4 t - \cos^4 t}{\sin^2 t \cos^2 t} = \sec^2 t - \csc^2 t$
19. $1 + \sin\theta = \dfrac{\cos^2\theta}{1 - \sin\theta}$
20. $1 - \sin\theta = \dfrac{\cos^2\theta}{1 + \sin\theta}$
21. $\dfrac{1 - \sin^4\theta}{1 + \sin^2\theta} = \cos^2\theta$
22. $\dfrac{1 - \cos^4\theta}{1 + \cos^2\theta} = \sin^2\theta$
23. $\sec^2\theta - \tan^2\theta = 1$
24. $\csc^2\theta - \cot^2\theta = 1$
25. $\sec^4\theta - \tan^4\theta = \dfrac{1 + \sin^2\theta}{\cos^2\theta}$
26. $\csc^4\theta - \cot^4\theta = \dfrac{1 + \cos^2\theta}{\sin^2\theta}$
27. $\tan\theta - \cot\theta = \dfrac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta}$
28. $\sec\theta - \csc\theta = \dfrac{\sin\theta - \cos\theta}{\sin\theta\cos\theta}$
29. $\csc B - \sin B = \cot B \cos B$
30. $\sec B - \cos B = \tan B \sin B$
31. $\cot\theta\cos\theta + \sin\theta = \csc\theta$
32. $\tan\theta\sin\theta + \cos\theta = \sec\theta$
33. $\dfrac{\cos x}{1 + \sin x} + \dfrac{1 + \sin x}{\cos x} = 2\sec x$
34. $\dfrac{\cos x}{1 + \sin x} - \dfrac{1 - \sin x}{\cos x} = 0$

35. $\dfrac{1}{1 + \cos x} + \dfrac{1}{1 - \cos x} = 2\csc^2 x$

36. $\dfrac{1}{1 - \sin x} + \dfrac{1}{1 + \sin x} = 2\sec^2 x$

37. $\dfrac{1 - \sec x}{1 + \sec x} = \dfrac{\cos x - 1}{\cos x + 1}$

38. $\dfrac{\csc x - 1}{\csc x + 1} = \dfrac{1 - \sin x}{1 + \sin x}$

39. $\dfrac{\cos t}{1 + \sin t} = \dfrac{1 - \sin t}{\cos t}$

40. $\dfrac{\sin t}{1 + \cos t} = \dfrac{1 - \cos t}{\sin t}$

41. $\dfrac{(1 - \sin t)^2}{\cos^2 t} = \dfrac{1 - \sin t}{1 + \sin t}$

42. $\dfrac{\sin^2 t}{(1 - \cos t)^2} = \dfrac{1 + \cos t}{1 - \cos t}$

43. $\dfrac{\sec \theta + 1}{\tan \theta} = \dfrac{\tan \theta}{\sec \theta - 1}$

44. $\dfrac{\csc \theta - 1}{\cot \theta} = \dfrac{\cot \theta}{\csc \theta + 1}$

45. Show that $\sin(A + B)$ is not, in general, equal to $\sin A + \sin B$ by substituting 30° for A and 60° for B in both expressions and simplifying. (An example like this, that shows that a statement is not true for certain values of the variables, is called a *counterexample* for the statement.)

46. Show that $\sin 2x \neq 2 \sin x$ by substituting 30° for x and then simplifying both sides.

Review Problems The problems that follow review material we covered in Sections 1.4 and 3.2. Reviewing these problems will help you with some of the material in the next section.

47. If $\sin A = 3/5$ and A terminates in quadrant I, find $\cos A$ and $\tan A$.

48. If $\cos B = -5/13$ with B in quadrant III, find $\sin B$ and $\tan B$.

Give the exact value of each of the following:

49. $\sin \dfrac{\pi}{3}$

50. $\cos \dfrac{\pi}{3}$

51. $\cos \dfrac{\pi}{6}$

52. $\sin \dfrac{\pi}{6}$

Convert to degrees.

53. $\pi/12$

54. $5\pi/12$

5.2 Sum and Difference Formulas

The expressions $\sin(A + B)$ and $\cos(A + B)$ occur frequently enough in mathematics that it is necessary to find expressions equivalent to them that involve sines and cosines of single angles. The most obvious question to begin with is

$$\text{Is } \sin(A + B) = \sin A + \sin B?$$

The answer is no. Substituting almost any pair of numbers for A and B in the formula will yield a false statement. As a counterexample, we can let $A = 30°$ and $B = 60°$ in the formula above and then simplify each side. (A counterexample is an example that shows that a statement is not, in general, true.)

$$\sin(30° + 60°) \stackrel{?}{=} \sin 30° + \sin 60°$$

$$\sin 90° \stackrel{?}{=} \frac{1}{2} + \frac{\sqrt{3}}{2}$$

$$1 \neq \frac{1 + \sqrt{3}}{2}$$

The formula just doesn't work. The next question is, what are the formulas for $\sin(A + B)$ and $\cos(A + B)$? The answer to that question is what this section is all about. Let's start by deriving the formula for $\cos(A + B)$.

We begin by drawing angle A in standard position and then adding B and $-B$ to it. Figure 1 shows these angles in relation to the unit circle. Note that the points on the unit circle through which the terminal sides of the angles A, $A + B$, and $-B$ pass have been labeled with the sines and cosines of those angles.

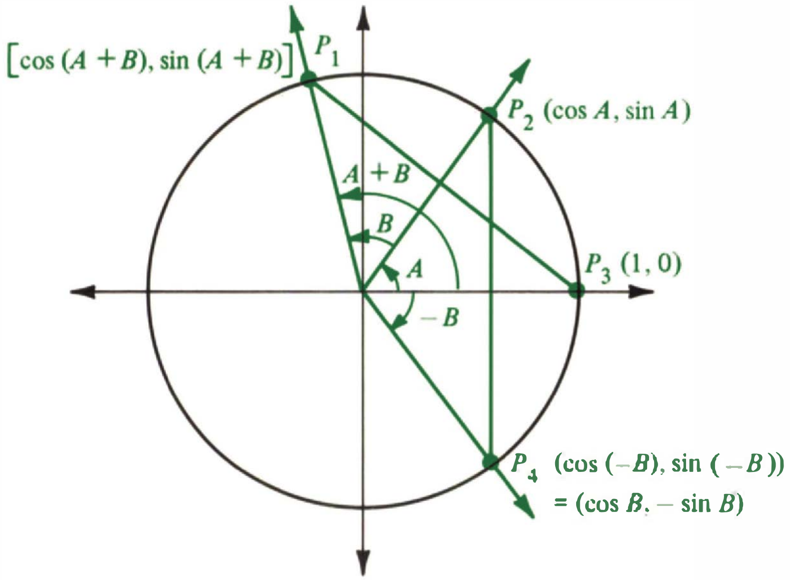


Figure 1

To derive the formula for $\cos(A + B)$, we simply have to see that line segment $\overline{P_1P_3}$ is equal to line segment $\overline{P_2P_4}$. (From geometry, they are chords cut off by equal central angles.)

$$\overline{P_1P_3} = \overline{P_2P_4}$$

Squaring both sides gives us

$$(\overline{P_1P_3})^2 = (\overline{P_2P_4})^2$$

Now, applying the distance formula, we have

$$[\cos(A + B) - 1]^2 + [\sin(A + B) - 0]^2 = (\cos A - \cos B)^2 + (\sin A + \sin B)^2$$

Let's call this Equation 1. Taking the left side of Equation 1, expanding it, and then simplifying by using the Pythagorean identity gives us

Left Side of Equation 1

$\cos^2(A + B) - 2\cos(A + B) + 1 + \sin^2(A + B)$	Expand squares
$= -2\cos(A + B) + 2$	Pythagorean identity

Applying the same two steps to the right side of Equation 1 looks like this

Right Side of Equation 1

$$\cos^2 A - 2\cos A\cos B + \cos^2 B + \sin^2 A + 2\sin A\sin B + \sin^2 B$$
$$= -2\cos A\cos B + 2\sin A\sin B + 2$$

Equating the simplified versions of the left and right sides of Equation 1 we have

$$-2\cos(A + B) + 2 = -2\cos A\cos B + 2\sin A\sin B + 2$$

Adding -2 to both sides and then dividing both sides by -2 gives us the formula we are after

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

This is the first formula in a series of formulas for trigonometric functions of the sum or difference of two angles. It must be memorized. Before we derive the others, let's look at some of the ways we can use our first formula.

▼ **Example 1** Find the exact value for cos 75°.

Solution We write 75° as 45° + 30° and then apply the formula for cos(A + B).

$$\begin{aligned}\cos 75^\circ &= \cos(45^\circ + 30^\circ)\\ &= \cos 45^\circ\cos 30^\circ - \sin 45^\circ\sin 30^\circ\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

▲

Example 2 Show that $\cos(x + 2\pi) = \cos x$.

Solution Applying the formula for $\cos(A + B)$, we have

$$\begin{aligned}\cos(x + 2\pi) &= \cos x \cos 2\pi - \sin x \sin 2\pi \\ &= \cos x \cdot 1 - \sin x \cdot 0 \\ &= \cos x\end{aligned}$$

Notice that this is not a new relationship. We already know that if two angles are coterminal, then their cosines are equal—and $x + 2\pi$ and x are coterminal. What we have done here is shown this to be true with a formula instead of the definition of cosine. ▲

Example 3 Write $\cos 3x \cos 2x - \sin 3x \sin 2x$ as a single cosine.

Solution We apply the formula for $\cos(A + B)$ in the reverse direction from the way we applied it in the first two examples.

$$\begin{aligned}\cos 3x \cos 2x - \sin 3x \sin 2x &= \cos(3x + 2x) \\ &= \cos 5x\end{aligned}$$ ▲

Here is the derivation of the formula for $\cos(A - B)$. It involves the formula for $\cos(A + B)$ and the formulas for even and odd functions.

$$\begin{aligned}\cos(A - B) &= \cos[A + (-B)] && \text{Write } A - B \text{ as a sum} \\ &= \cos A \cos(-B) - \sin A \sin(-B) && \text{Sum formula} \\ &= \cos A \cos B - \sin A(-\sin B) && \text{Cosine is an even function, sine is odd} \\ &= \cos A \cos B + \sin A \sin B\end{aligned}$$

The only difference in the formulas for the expansion of $\cos(A + B)$ and $\cos(A - B)$ is the sign between the two terms. Here are both formulas again.

$$\begin{aligned}\cos(A + B) &= \cos A \cos B - \sin A \sin B \\ \cos(A - B) &= \cos A \cos B + \sin A \sin B\end{aligned}$$

Again, both formulas are important and should be memorized.

Example 4 Show that $\cos(90° - A) = \sin A$.

Solution We will need this formula when we derive the formula for $\sin(A + B)$.

$$\begin{aligned}\cos(90° - A) &= \cos 90° \cos A + \sin 90° \sin A \\ &= 0 \cdot \cos A + 1 \cdot \sin A \\ &= \sin A\end{aligned}$$ ▲

Note that the formula we just derived is not a new formula. The angles $90° - A$ and A are complementary angles and we already know the sine of an angle is always equal to the cosine of its complement. We could also state it this way:

$$\sin(90° - A) = \cos A$$

We can use this information to derive the formula for $\sin(A + B)$. To understand this derivation, you must recognize that $A + B$ and $90° - (A + B)$ are complementary angles.

$\sin(A + B)$	$= \cos[90° - (A + B)]$	The sine of an angle is the cosine of its complement
	$= \cos[90° - A - B]$	Remove parentheses
	$= \cos[(90° - A) - B]$	Regroup within brackets

Now we expand using the formula for the cosine of a difference

$$= \cos(90° - A)\cos B + \sin(90° - A)\sin B$$
$$= \sin A \cos B + \cos A \sin B$$

This is the formula for the sine of a sum. To find the formula for $\sin(A - B)$ we write $A - B$ as $A + (-B)$ and proceed as follows:

$$\begin{aligned}\sin(A - B) &= \sin[A + (-B)]\\ &= \sin A \cos(-B) + \cos A \sin(-B)\\ &= \sin A \cos B - \cos A \sin B\end{aligned}$$

▼ **Example 5** Find the exact value of $\sin \frac{\pi}{12}$.

Solution We have to write $\pi/12$ in terms of two numbers the exact values of which are known. The numbers $\pi/3$ and $\pi/4$ will work since their difference is $\pi/12$.

$$\begin{aligned}\sin \frac{\pi}{12} &= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\\ &= \sin \frac{\pi}{3} \cos \frac{\pi}{4} - \cos \frac{\pi}{3} \sin \frac{\pi}{4}\\ &= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

This is the same answer we obtained in Example 1 when we found the exact value of cos 75°. It should be, though, because $\pi/12 = 15°$, which is the complement of 75°, and the cosine of an angle is equal to the sine of its complement. ▲

▼ **Example 6** If $\sin A = 3/5$ with A in QI and $\cos B = -5/13$ with B in QIII, find $\sin(A + B)$, $\cos(A + B)$, and $\tan(A + B)$.

Solution We have sin A and cos B. We need to find cos A and sin B before we can apply any of our formulas. Some equivalent forms of our Pythagorean identity will help here.

If $\sin A = \dfrac{3}{5}$ with A in QI, then

$$\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$$

If $\cos B = -\dfrac{5}{13}$ with B in QIII, then

$$\sin B = -\sqrt{1 - \left(-\frac{5}{13}\right)^2} = -\frac{12}{13}$$

We have

$$\sin A = \frac{3}{5} \qquad \sin B = -\frac{12}{13}$$

$$\cos A = \frac{4}{5} \qquad \cos B = -\frac{5}{13}$$

Therefore,

$$\begin{aligned}\sin(A + B) &= \sin A \cos B + \cos A \sin B \\ &= \frac{3}{5}\left(-\frac{5}{13}\right) + \frac{4}{5}\left(-\frac{12}{13}\right) \\ &= -\frac{63}{65}\end{aligned}$$

$$\begin{aligned}\cos(A + B) &= \cos A \cos B - \sin A \sin B \\ &= \frac{4}{5}\left(-\frac{5}{13}\right) - \frac{3}{5}\left(-\frac{12}{13}\right) \\ &= \frac{16}{65}\end{aligned}$$

$$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)}$$

$$= \frac{-63/65}{16/65}$$

$$= -\frac{63}{16}$$

▲

Notice also that $A + B$ must terminate in quadrant IV because

$$\sin(A + B) < 0 \text{ and } \cos(A + B) > 0.$$

While working through the last part of Example 6, you may have wondered if there is a separate formula for $\tan(A + B)$. (More likely, you are hoping there isn't.) There is, and it is derived from the formulas we already have.

$$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)}$$

$$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

To be able to write this last line in terms of tangents only, we must divide numerator and denominator by $\cos A \cos B$.

$$= \frac{\dfrac{\sin A \cos B}{\cos A \cos B} + \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}}$$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Since tangent is an odd function, the formula for $\tan(A - B)$ will look like this

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

▼ **Example 7** If $\sin A = 3/5$ with A in QI and $\cos B = -5/13$ with B in QIII, find $\tan(A + B)$ by using the formula

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Solution The angles A and B as given here are the same ones used previously in Example 6. Looking over Example 6 again, we find that

$$\tan A = \frac{3}{4} \text{ and } \tan B = \frac{12}{5}$$

Therefore,

$$\begin{aligned}
\tan(A + B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \\
&= \frac{\frac{3}{4} + \frac{12}{5}}{1 - \frac{3}{4} \cdot \frac{12}{5}} \\
&= \frac{\frac{15}{20} + \frac{48}{20}}{1 - \frac{9}{5}} \\
&= \frac{\frac{63}{20}}{-\frac{4}{5}} \\
&= -\frac{63}{16}
\end{aligned}$$

Which is the same result we obtained previously. ▲

Problem Set 5.2

Find exact values for each of the following:

1. $\sin 15°$

2. $\sin 75°$

3. $\tan 15°$

4. $\tan 75°$

5. $\sin \frac{7\pi}{12}$

6. $\cos \frac{7\pi}{12}$

7. $\cos 195°$

8. $\sin 195°$

9. $\cos \frac{11\pi}{12} \left(\frac{11}{12} = \frac{3}{4} + \frac{1}{6}\right)$

10. $\sin \frac{11\pi}{12}$

Show that each of the following is true:

11. $\sin(x + 2\pi) = \sin x$

12. $\cos(x - 2\pi) = \cos x$

13. $\cos\left(x - \frac{\pi}{2}\right) = \sin x$

14. $\sin\left(x - \frac{\pi}{2}\right) = -\cos x$

15. $\cos(180° - \theta) = -\cos\theta$

16. $\sin(180° - \theta) = \sin\theta$

17. $\sin(90° + \theta) = \cos\theta$

18. $\cos(90° + \theta) = -\sin\theta$

19. $\tan\left(x + \frac{\pi}{4}\right) = \frac{1 + \tan x}{1 - \tan x}$

20. $\tan\left(x - \frac{\pi}{4}\right) = \frac{\tan x - 1}{\tan x + 1}$

Write each expression as a single trigonometric function.

21. $\sin 3x \cos 2x + \cos 3x \sin 2x$

22. $\cos 3x \cos 2x + \sin 3x \sin 2x$

23. $\cos 5x \cos x - \sin 5x \sin x$

24. $\sin 8x \cos x - \cos 8x \sin x$

25. $\sin 45° \cos\theta + \cos 45° \sin\theta$

26. $\cos 60° \cos\theta - \sin 60° \sin\theta$

27. $\cos 30° \sin\theta + \sin 30° \cos\theta$

28. $\cos 60° \sin\theta + \sin 60° \cos\theta$

29. $\cos 15° \cos 75° - \sin 15° \sin 75°$

30. $\cos 15° \cos 75° + \sin 15° \sin 75°$

31. Graph one complete cycle of $y = \sin x \cos \pi/4 + \cos x \sin \pi/4$ by first rewriting the right side in the form $\sin(A + B)$.

32. Graph one complete cycle of $y = \sin x \cos \pi/6 - \cos x \sin \pi/6$ by first rewriting the right side in the form $\sin(A - B)$.

33. Graph one complete cycle of $y = 2(\sin x \cos \pi/3 + \cos x \sin \pi/3)$ by first rewriting the right side in the form $2\sin(A + B)$.

34. Graph one complete cycle of $y = 2(\sin x \cos \pi/3 - \cos x \sin \pi/3)$ by first rewriting the right side in the form $2\sin(A - B)$.

35. Let $\sin A = 3/5$ with A in QII and $\sin B = -5/13$ with B in QIII. Find $\sin(A + B)$, $\cos(A + B)$, and $\tan(A + B)$. In what quadrant does $A + B$ terminate?

36. Let $\cos A = -5/13$ with A in QII and $\sin B = 3/5$ with B in QI. Find $\sin(A - B)$, $\cos(A - B)$, and $\tan(A - B)$. In what quadrant does $A - B$ terminate?

37. If $\sin A = 1/\sqrt{5}$ with A in QI and $\tan B = 3/4$ with B in QI, find $\tan(A + B)$ and $\cot(A + B)$. In what quadrant does $A + B$ terminate?

38. If $\sec A = \sqrt{5}$ with A in QI and $\sec B = \sqrt{10}$ with B in QI, find $\sec(A + B)$. [First find $\cos(A + B)$.]

39. If $\tan(A + B) = 3$ and $\tan B = 1/2$, find $\tan A$.

40. If $\tan(A + B) = 2$ and $\tan B = 1/3$, find $\tan A$.

41. Write a formula for $\sin 2x$ by writing $\sin 2x$ as $\sin(x + x)$ and using the formula for the sine of a sum.

42. Write a formula for $\cos 2x$ by writing $\cos 2x$ as $\cos(x + x)$ and using the formula for the cosine of a sum.

Review Problems The problems that follow review material we covered in Section 4.2.

Graph one complete cycle of each of the following:

43. $y = 4\sin 2x$

44. $y = 2\sin 4x$

45. $y = \frac{1}{2} \cos 3x$

46. $y = \frac{1}{2} \sin 3x$

47. $y = \frac{1}{2} \sin \frac{\pi}{2} x$

48. $y = 2 \sin \frac{\pi}{2} x$

5.3 Double-Angle Formulas

We will begin this section by deriving the formulas for sin 2A and cos 2A using the formulas for sin(A + B) and cos(A + B). The formulas we derive for sin 2A and cos 2A are called *double-angle* formulas. Here is the derivation of the formula for sin 2A.

$$\begin{aligned} \sin 2A &= \sin(A + A) && \text{Write } 2A \text{ as } A + A \\ &= \sin A \cos A + \cos A \sin A && \text{Sum formula} \\ &= \sin A \cos A + \sin A \cos A && \text{Commutative property} \\ &= 2 \sin A \cos A \end{aligned}$$

The first thing to notice about this formula is that it indicates the 2 in sin 2A *cannot* be factored out and written as a coefficient. That is,

$$\sin 2A \neq 2 \sin A$$

For example, if $A = 30°$, $\sin 2 \cdot 30° = \sin 60° = \sqrt{3}/2$, which is not the same as $2 \sin 30° = 2(1/2) = 1$.

Here are some examples of how we can apply the double-angle formula $\sin 2A = 2 \sin A \cos A$.

Example 1 If $\sin A = 3/5$ with A in QII, find sin 2A.

Solution In order to apply the formula for sin 2A we must first find cos A. Since A terminates in QII, cos A is negative.

$$\cos A = -\sqrt{1 - \sin^2 A} = -\sqrt{1 - \left(\frac{3}{5}\right)^2} = -\sqrt{\frac{16}{25}} = -\frac{4}{5}$$

Now we can apply the formula for sin 2A.

$$\begin{aligned} \sin 2A &= 2 \sin A \cos A \\ &= 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right) \\ &= -\frac{24}{25} \end{aligned}$$

We can also use our new formula to expand the work we did previously with identities.

Example 2 Prove $(\sin\theta + \cos\theta)^2 = 1 + \sin 2\theta$.

Proof $(\sin\theta + \cos\theta)^2 \stackrel{?}{=} 1 + \sin 2\theta$

Expand $\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta$

Pythagorean identity $1 + 2\sin\theta\cos\theta$

Double-angle identity $1 + \sin 2\theta = 1 + \sin 2\theta$ ▲

Example 3 Prove $\sin 2x = \dfrac{2\cot x}{1 + \cot^2 x}$.

Proof $\sin 2x \stackrel{?}{=} \dfrac{2\cot x}{1 + \cot^2 x}$

$$\frac{2\cdot\dfrac{\cos x}{\sin x}}{1 + \dfrac{\cos^2 x}{\sin^2 x}}$$ Ratio identity

$$\frac{2\sin x\cos x}{\sin^2 x + \cos^2 x}$$ Multiply numerator and denominator by $\sin^2 x$

$2\sin x\cos x$ Pythagorean identity

$\sin 2x = \sin 2x$ Double-angle identity ▲

There are three forms of the double-angle formula for $\cos 2A$. The first involves both sine and cosine, the second involves only cosine, and the third, just sine. Here is how we obtain the first of these three formulas.

$$\begin{aligned} \cos 2A &= \cos(A + A) && \text{Write } 2A \text{ as } A + A \\ &= \cos A\cos A - \sin A\sin A && \text{Sum formula} \\ &= \cos^2 A - \sin^2 A \end{aligned}$$

To write this last formula in terms of $\cos A$ only, we substitute $1 - \cos^2 A$ for $\sin^2 A$.

$$\begin{aligned} \cos 2A &= \cos^2 A - (1 - \sin^2 A) \\ &= \cos^2 A - 1 + \cos^2 A \\ &= 2\cos^2 A - 1 \end{aligned}$$

To write the formula in terms of sin A only, we substitute $1 - \sin^2 A$ for $\cos^2 A$ in the last line above.

$$\begin{aligned} \cos 2A &= 2\cos^2 A - 1 \\ &= 2(1 - \sin^2 A) - 1 \\ &= 2 - 2\sin^2 A - 1 \\ &= 1 - 2\sin^2 A \end{aligned}$$

Here are the three forms of the double-angle formula for cos $2A$.

Summary

$\cos 2A$	$= \cos^2 A - \sin^2 A$	First form
	$= 2\cos^2 A - 1$	Second form
	$= 1 - 2\sin^2 A$	Third form

▼ **Example 4** If $\sin A = 1/\sqrt{5}$, find $\cos 2A$.

Solution In this case, since we are given sin A, applying the third form of the formula for cos $2A$ will give us the answer more quickly than applying either of the other two forms.

$$\begin{aligned} \cos 2A &= 1 - 2\sin^2 A \\ &= 1 - 2\left(\frac{1}{\sqrt{5}}\right)^2 \\ &= 1 - \frac{2}{5} \\ &= \frac{3}{5} \end{aligned}$$

▲

▼ **Example 5** Prove $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$.

Proof We can write cos $4x$ as cos $2 \cdot 2x$ and apply our double-angle formula. Since the right side is written in terms of cos x only, we will choose the second form of our double-angle formula for cos $2A$.

	$\cos 4x$	$\stackrel{?}{=} 8\cos^4 x - 8\cos^2 x + 1$
	$\cos 2 \cdot 2x$	
Double-angle formula	$2\cos^2 2x - 1$	
Double-angle formula	$2(2\cos^2 x - 1)^2 - 1$	
Square	$2(4\cos^4 x - 4\cos^2 x + 1) - 1$	
Distribute	$8\cos^4 x - 8\cos^2 x + 2 - 1$	
Add	$8\cos^4 x - 8\cos^2 x + 1$	$= 8\cos^4 x - 8\cos^2 x + 1$

▲

Example 6 Write $\cos 150°$ in terms of $\cos 75°$.

Solution We note that $150° = 2 \cdot 75°$ and apply the second form of the formula for $\cos 2A$.

$$\begin{aligned}\cos 150° &= \cos 2 \cdot 75° \\ &= 2\cos^2 75° - 1\end{aligned}$$

Example 7 Prove $\tan\theta = \dfrac{1-\cos 2\theta}{\sin 2\theta}$.

Proof $\tan\theta \stackrel{?}{=} \dfrac{1-\cos 2\theta}{\sin 2\theta}$

$\dfrac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta}$ Double-angle formulas

$\dfrac{2\sin^2\theta}{2\sin\theta\cos\theta}$ Simplify numerator

$\dfrac{\sin\theta}{\cos\theta}$ Divide out common factor $2\sin\theta$

$\tan\theta = \tan\theta$ Ratio identity

Problem Set 5.3

Let $\sin A = -3/5$ with A in QIII and find

1. $\sin 2A$
2. $\cos 2A$
3. $\tan 2A$
4. $\cot 2A$

Let $\cos x = 1/\sqrt{10}$ with x in QIV and find

5. $\cos 2x$
6. $\sin 2x$
7. $\cot 2x$
8. $\tan 2x$

Let $\tan\theta = 5/12$ with θ in QI and find

9. $\sin 2\theta$
10. $\cos 2\theta$
11. $\csc 2\theta$
12. $\sec 2\theta$

Let $\csc t = \sqrt{5}$ with t in QII and find

13. $\cos 2t$
14. $\sin 2t$
15. $\sec 2t$
16. $\csc 2t$

17. Express $\cos 100°$ in terms of $\sin 50°$.
18. Express $\cos 100°$ in terms of $\cos 50°$.

19. Express $\sin \frac{\pi}{5}$ in terms of $\sin \frac{\pi}{10}$ and $\cos \frac{\pi}{10}$.

20. Express $\cos \frac{\pi}{4}$ in terms of $\cos \frac{\pi}{8}$.

Use exact values to show that each of the following is true.

21. $\sin 60° = 2 \sin 30° \cos 30°$

22. $\cos 60° = 1 - 2 \sin^2 30°$

23. $\cos 120° = \cos^2 60° - \sin^2 60°$

24. $\sin 90° = 2 \sin 45° \cos 45°$

25. $\cos \pi = 2 \cos^2 \frac{\pi}{2} - 1$

26. $\sin \pi = 2 \sin \frac{\pi}{2} \cos \frac{\pi}{2}$

27. $\sin \frac{2\pi}{3} = 2 \sin \frac{\pi}{3} \cos \frac{\pi}{3}$

28. $\cos \frac{2\pi}{3} = \cos^2 \frac{\pi}{3} - \sin^2 \frac{\pi}{3}$

29. Find the exact value of $\sin 15°$ from the expression $\cos 30° = 1 - 2 \sin^2 15°$.

30. Find the exact value of $\cos 15°$ from the expression $\cos 30° = 2 \cos^2 15° - 1$.

Prove each of the following identities:

31. $(\sin x - \cos x)^2 = 1 - \sin 2x$

32. $(\cos x - \sin x)(\cos x + \sin x) = \cos 2x$

33. $\cos^2\theta = \dfrac{1 + \cos 2\theta}{2}$

34. $\sin^2\theta = \dfrac{1 - \cos 2\theta}{2}$

35. $\cot \theta = \dfrac{\sin 2\theta}{1 - \cos 2\theta}$

36. $\cos 2\theta = \dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}$

37. $2 \csc 2x = \tan x + \cot x$

38. $2 \cot 2x = \cot x - \tan x$

39. $\sin 3\theta = 3 \sin \theta - 4 \sin^3\theta$

40. $\cos 3\theta = 4 \cos^3\theta - 3 \cos \theta$

41. $\cos^4 x - \sin^4 x = \cos 2x$

42. $\sin^4 x - \cos^4 x = -\cos 2x$

43. $\cot \theta - \tan \theta = \dfrac{\cos 2\theta}{\sin \theta \cos \theta}$

44. $\csc \theta - 2 \sin \theta = \dfrac{\cos 2\theta}{\sin \theta}$

The double-angle formula for $\tan 2A$ is

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

45. Derive the formula above by writing $\tan 2A$ as $\tan(A + A)$ and then applying the formula for the tangent of a sum.

46. Use the formula above and the reciprocal identity for $\tan A$ and $\cot A$ to prove the following double-angle identity for $\cot 2A$.

$$\cot 2A = \frac{\cot^2 A - 1}{2 \cot A}$$

47. If $\tan A = 3/4$, find $\tan 2A$.

48. If $\tan A = -\sqrt{3}$, find $\tan 2A$.

Review Problems The problems that follow review material we covered in Section 4.3.

Graph one complete cycle.

49. $y = \sin\left(x + \frac{\pi}{4}\right)$

50. $y = \sin\left(x - \frac{\pi}{4}\right)$

51. $y = \frac{1}{2}\cos\left(3x - \frac{\pi}{2}\right)$

52. $y = \frac{4}{3}\cos\left(3x + \frac{\pi}{2}\right)$

5.4 Half-Angle Formulas

In this section we will derive formulas for $\sin A/2$ and $\cos A/2$. These formulas are called half-angle formulas and are derived from the double-angle formulas for $\cos 2A$.

In Section 5.3 we developed three ways to write the formula for $\cos 2A$, two of which were

$$\cos 2A = 1 - 2\sin^2 A \quad \text{and} \quad \cos 2A = 2\cos^2 A - 1$$

Since the choice of the letter we use to denote the angles in these formulas is arbitrary, we can use an x instead of A.

$$\cos 2x = 1 - 2\sin^2 x \text{ and } \cos 2x = 2\cos^2 x - 1$$

Let us exchange sides in the first formula and solve for $\sin x$.

$$1 - 2\sin^2 x = \cos 2x \qquad \text{Exchange sides}$$

$$-2\sin^2 x = -1 + \cos 2x \qquad \text{Add } -1 \text{ to both sides}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \text{Divide both sides by } -2$$

$$\sin x = \pm\sqrt{\frac{1 - \cos 2x}{2}} \qquad \text{Take the square root of both sides}$$

Since every value of x can be written as 1/2 of some other number A, we can replace x with $A/2$. This is equivalent to saying $2x = A$.

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{2}}$$

This last expression is the half-angle formula for $\sin A/2$. To find the half-angle formula for $\cos A/2$, we solve $\cos 2x = 2\cos^2 x - 1$ for $\cos x$ and then replace x with $A/2$ (and $2x$ with A). Without showing the steps involved in this process, here is the result

$$\cos \frac{A}{2} = \pm\sqrt{\frac{1 + \cos A}{2}}$$

In both half-angle formulas the sign, + or −, in front of the radical is determined by the quadrant in which $A/2$ terminates.

Example 1 If $\cos A = 3/5$ with A in QIV, find $\sin A/2$, $\cos A/2$, and $\tan A/2$.

Solution First of all, if A terminates in QIV, then $A/2$ terminates in QII. Here is why.

$$A \in \text{QIV} \Rightarrow 270° < A < 360° \Rightarrow \frac{270°}{2} < \frac{A}{2} < \frac{360°}{2}$$

$$\text{or } 135° < \frac{A}{2} < 180° \Rightarrow \frac{A}{2} \in \text{QII}$$

In quadrant II, sine is positive, cosine and tangent are negative.

$$\sin \frac{A}{2} = \sqrt{\frac{1 - \cos A}{2}} = \sqrt{\frac{1 - 3/5}{2}} = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}}$$

$$\cos \frac{A}{2} = -\sqrt{\frac{1 + \cos A}{2}} = -\sqrt{\frac{1 + 3/5}{2}} = -\sqrt{\frac{4}{5}} = -\frac{2}{\sqrt{5}}$$

$$\tan \frac{A}{2} = \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = \frac{\frac{1}{\sqrt{5}}}{-\frac{2}{\sqrt{5}}} = -\frac{1}{2}$$

Example 2 If $\sin A = -12/13$ with A in QIII, find the six trigonometric functions of $A/2$.

Solution To use the half-angle formulas, we need to find $\cos A$.

$$\cos A = -\sqrt{1 - \sin^2 A} = -\sqrt{1 - \left(-\frac{12}{13}\right)^2} = -\sqrt{\frac{25}{169}} = -\frac{5}{13}$$

Also, if A terminates in QIII, then $A/2$ terminates in QII because

$$A \in \text{QIII} \Rightarrow 180° < A < 270°$$

$$\frac{180°}{2} < \frac{A}{2} < \frac{270°}{2}$$

$$90° < \frac{A}{2} < 135° \Rightarrow \frac{A}{2} \in \text{QII}$$

In quadrant II, sine is positive and cosine is negative.

$$\sin\frac{A}{2} = \sqrt{\frac{1 - (-5/13)}{2}} = \sqrt{\frac{9}{13}} = \frac{3}{\sqrt{13}} \qquad \cos\frac{A}{2} = -\sqrt{\frac{1 + (-5/13)}{2}} = -\sqrt{\frac{4}{13}} = -\frac{2}{\sqrt{13}}$$

Now that we have sine and cosine of $A/2$, we can apply the ratio identity for tangent to find $\tan A/2$.

$$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \frac{\frac{3}{\sqrt{13}}}{-\frac{2}{\sqrt{13}}} = -\frac{3}{2}$$

Next we apply our reciprocal identities to find cosecant, secant, and cotangent of $A/2$.

$$\csc\frac{A}{2} = \frac{1}{\sin\frac{A}{2}} = \frac{\sqrt{13}}{3} \qquad \sec\frac{A}{2} = \frac{1}{\cos\frac{A}{2}} = -\frac{\sqrt{13}}{2}$$

$$\cot\frac{A}{2} = \frac{1}{\tan\frac{A}{2}} = -\frac{2}{3}$$

▲

▼ **Example 3** Use a half-angle formula to find the exact value of $\sin 15°$.

$$\sin 15^\circ = \sin \frac{30^\circ}{2} = \sqrt{\frac{1 - \cos 30^\circ}{2}}$$

$$= \sqrt{\frac{1 - \sqrt{3}/2}{2}}$$

$$= \sqrt{\frac{2 - \sqrt{3}}{4}}$$

$$= \frac{\sqrt{2 - \sqrt{3}}}{2}$$

▲

Note We found the exact value of sin 15° previously in Example 5 of Section 5.2. (At that time, we were working in radians and thus called it $\pi/12$. But $\pi/12 = 15^\circ$.) The value of sin 15° from Example 5, Section 5.2 was

$$\sin 15^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}$$

Can you show that the two answers are the same?

▼ **Example 4** Prove $\sin^2 \frac{x}{2} = \frac{\tan x - \sin x}{2 \tan x}$.

Proof We can use a half-angle formula on the left side. In this case, since we have $\sin^2(x/2)$ we write the half-angle formula without the square root sign. After that, we multiply the numerator and denominator on the left side by tan x because the right side has tan x in both the numerator and denominator.

	$\sin^2 \frac{x}{2}$	$\stackrel{?}{=} \frac{\tan x - \sin x}{2 \tan x}$
Square of half-angle formula	$\frac{1 - \cos x}{2}$	
Multiply numerator and denominator by tan x	$\frac{\tan x}{\tan x} \cdot \frac{1 - \cos x}{2}$	
Distributive property	$\frac{\tan x - \tan x \cos x}{2 \tan x}$	
tan x cos x is sin x	$\frac{\tan x - \sin x}{2 \tan x}$	$= \frac{\tan x - \sin x}{2 \tan x}$

▲

Example 5 Prove $\sin^4\theta = \frac{1}{4} - \frac{\cos 2\theta}{2} + \frac{\cos^2 2\theta}{4}$.

Proof We could use double-angle formulas on the right side. Instead, since we are working with half-angle formulas in this section, let's use a half-angle formula on the left side.

$$\sin^4\theta \stackrel{?}{=} \frac{1}{4} - \frac{\cos 2\theta}{2} + \frac{\cos^2 2\theta}{4}$$

Half-angle formula with θ instead of $A/2$: $\left[\pm\sqrt{\frac{1-\cos 2\theta}{2}}\right]^4$

Square the square root: $\left(\frac{1-\cos 2\theta}{2}\right)^2$

Expand: $\frac{1 - 2\cos 2\theta + \cos^2 2\theta}{4}$

Divide each term in numerator by 4: $\frac{1}{4} - \frac{\cos 2\theta}{2} + \frac{\cos^2 2\theta}{4} = \frac{1}{4} - \frac{\cos 2\theta}{2} + \frac{\cos^2 2\theta}{4}$

Notice that we used a half-angle formula in the first step, even though our angle was not $A/2$. It is not the $A/2$ in our half-angle formula that is important. It is the *relationship* between $A/2$ on one side and A on the other. The fact that $A/2$ is half of A is equivalent to saying A is twice $A/2$. The formulas

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$$

and

$$\sin\theta = \pm\sqrt{\frac{1-\cos 2\theta}{2}}$$

and

$$\sin 2x = \pm\sqrt{\frac{1-\cos 4x}{2}}$$

all say the same thing about the functions involved. They are equivalent formulas.

Problem Set 5.4

If $\cos A = 1/2$ with A in QIV, find

1. $\sin\frac{A}{2}$

2. $\cos\frac{A}{2}$

3. $\csc \frac{A}{2}$

4. $\sec \frac{A}{2}$

If $\sin A = -3/5$ with A in QIII, find

5. $\cos \frac{A}{2}$

6. $\sin \frac{A}{2}$

7. $\sec \frac{A}{2}$

8. $\csc \frac{A}{2}$

If $\sin B = -12/13$ with B in QIII, find

9. $\sin \frac{B}{2}$

10. $\csc \frac{B}{2}$

11. $\cos \frac{B}{2}$

12. $\sec \frac{B}{2}$

13. $\tan \frac{B}{2}$

14. $\cot \frac{B}{2}$

If $\sin A = 4/5$ with A in QII, and $\sin B = 3/5$ with B in QI, find

15. $\sin \frac{A}{2}$

16. $\cos \frac{A}{2}$

17. $\cos 2A$

18. $\sin 2A$

19. $\sec 2A$

20. $\csc 2A$

21. $\cos \frac{B}{2}$

22. $\sin \frac{B}{2}$

23. $\sin(A + B)$

24. $\cos(A + B)$

25. $\cos(A - B)$

26. $\sin(A - B)$

If $\cos \frac{\theta}{2} = \frac{12}{13}$ with $\frac{\theta}{2}$ in QI, find

27. $\cos \frac{\theta}{4}$

28. $\sin \frac{\theta}{4}$

29. $\sin \theta$

30. $\cos \theta$

Use half-angle formulas to find exact values for each of the following:

31. $\cos 15°$

32. $\tan 15°$

33. $\sin 75°$

34. $\cos 75°$

35. $\cos 105°$

36. $\sin 105°$

Prove the following identities.

37. $\sin^2 \frac{\theta}{2} = \frac{\csc \theta - \cot \theta}{2 \csc \theta}$

38. $2 \cos^2 \frac{\theta}{2} = \frac{\sin^2\theta}{1 - \cos \theta}$

39. $\cos^2 \frac{\theta}{2} = \frac{\tan \theta + \sin \theta}{2 \tan \theta}$

40. $2 \sin^2 \frac{\theta}{2} = \frac{\sin^2\theta}{1 + \cos \theta}$

41. $\tan \frac{x}{2} = \frac{1 - \cos x}{\sin x}$

42. $\tan \frac{x}{2} = \frac{\sin x}{1 + \cos x}$

43. $\cos^4\theta = \frac{1}{4} + \frac{\cos 2\theta}{2} + \frac{\cos^2 2\theta}{4}$

44. $4 \sin^4\theta = 1 - 2 \sin 2\theta + \cos^2 2\theta$

45. $\sin \frac{t}{2} \cos \frac{t}{2} = 2 \sin t$

46. $\left(\cos \frac{t}{2} - \sin \frac{t}{2}\right)^2 = 1 - \sin t$

Review Problems The problems below review material we covered in Section 4.6. Reviewing these problems will help you with the next section.

Evaluate without using a calculator or tables.

47. $\sin\left(\text{Arcsin } \frac{3}{5}\right)$

48. $\cos\left(\text{Arcsin } \frac{3}{5}\right)$

49. $\cos(\text{Arctan } 2)$

50. $\sin(\text{Arctan } 2)$

Write an equivalent expression that involves x only. (Assume x is positive.)

51. $\sin(\text{Tan}^{-1}x)$

52. $\cos(\text{Tan}^{-1}x)$

53. $\tan(\text{Sin}^{-1}x)$

54. $\tan(\text{Cos}^{-1}x)$

5.5 Additional Identities

There are two main parts to this section, both of which rely on the work we have done previously with identities and formulas. In the first part of this section, we will extend our work on identities to include problems that involve inverse trigonometric functions. In the second part, we will use the formulas we obtained for the sine and cosine of a sum or difference to write some new formulas involving sums and products.

Identities and Formulas Involving Inverse Functions

The solution to our first example combines our knowledge of inverse trigonometric functions with our formula for $\sin(A + B)$.

▼ **Example 1** Evaluate $\sin(\text{Arcsin } 3/5 + \text{Arctan } 2)$ without using a calculator or tables.

Solution We can simplify things somewhat if we let $\alpha = \text{Arcsin } 3/5$ and $\beta = \text{Arctan } 2$.

$$\sin\left(\text{Arcsin}\,\frac{3}{5} + \text{Arctan}\,2\right)$$
$$= \sin(\alpha + \beta)$$
$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Drawing and labeling a triangle for α and another for β we have

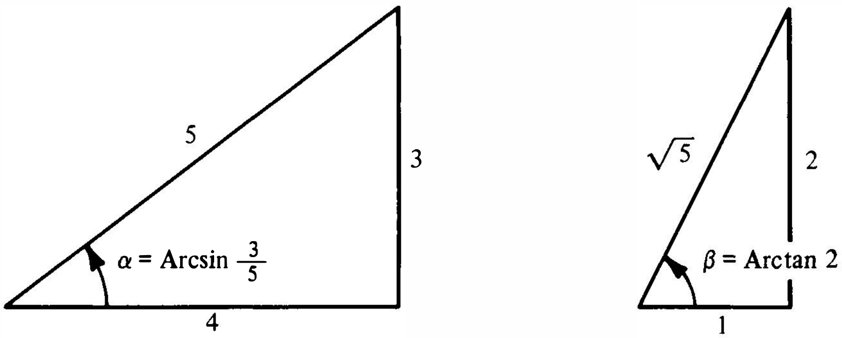


Figure 1

From the triangles in Figure 1 we have

$$\sin\alpha = \frac{3}{5} \qquad \sin\beta = \frac{2}{\sqrt{5}}$$

$$\cos\alpha = \frac{4}{5} \qquad \cos\beta = \frac{1}{\sqrt{5}}$$

Substituting these numbers into

$$\sin\alpha\cos\beta + \cos\alpha\sin\beta$$

gives us,

$$\frac{3}{5}\cdot\frac{1}{\sqrt{5}} + \frac{4}{5}\cdot\frac{2}{\sqrt{5}} = \frac{11}{5\sqrt{5}}$$ ▲

Calculator Note **To work this problem on a calculator, we would use the following sequence:**

0.6 [$\sin^{-1}$] [+] 2 [$\tan^{-1}$] [=] [sin]

The display would show 0.9839 to four decimal places, which is the decimal approximation of $11/5\sqrt{5}$. It is appropriate to check your work on problems like this by using your calculator. The concepts are best understood, however, by working through the problems without using a calculator.

Here is a similar example involving inverse trigonometric functions and a double-angle identity.

Example 2 Write $\sin(2\,\mathrm{Tan}^{-1}x)$ as an equivalent expression involving only x. (Assume x is positive.)

Solution We begin by letting $\theta = \mathrm{Tan}^{-1}x$ and then draw a right triangle with an acute angle of θ. If we label the opposite side with x and the adjacent side with 1, the ratio of the side opposite θ to the side adjacent θ is $x/1 = x$.

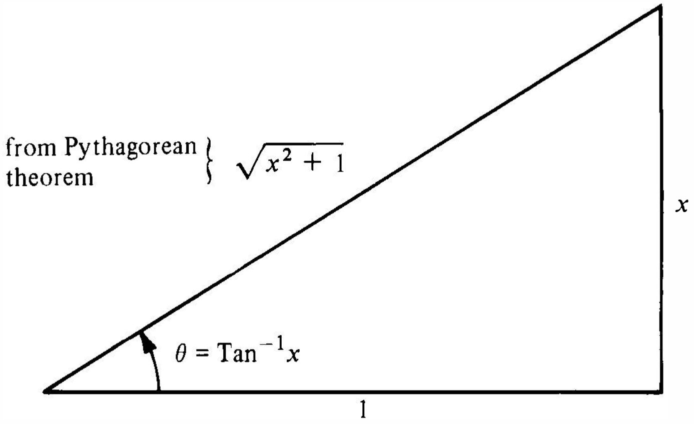


Figure 2

From Figure 2, we have $\sin\theta = x/(\sqrt{x^2+1})$ and $\cos\theta = 1/(\sqrt{x^2+1})$. Therefore,

$$\begin{aligned}
\sin(2\,\mathrm{Tan}^{-1}x) &= \sin 2\theta && \text{Substitute } \theta \text{ for } \mathrm{Tan}^{-1}x \\
&= 2\sin\theta\cos\theta && \text{Double-angle identity} \\
&= 2\cdot\frac{x}{\sqrt{x^2+1}}\cdot\frac{1}{\sqrt{x^2+1}} && \text{From Figure 2} \\
&= \frac{2x}{x^2+1} && \text{Multiplication}
\end{aligned}$$

To conclude our work with identities in this chapter, we will derive some additional formulas that contain sums and products of sines and cosines.

Product to Sum Formulas

If we add the formula for $\sin(A - B)$ to the formula for $\sin(A + B)$, we will eventually arrive at a formula for the product $\sin A \cos B$.

$$\begin{array}{rl} \sin A \cos B + \cos A \sin B &= \sin(A + B) \\ \sin A \cos B - \cos A \sin B &= \sin(A - B) \\ \hline 2 \sin A \cos B &= \sin(A + B) + \sin(A - B) \end{array}$$

Dividing both sides of this result by 2 gives us

$$\sin A \cos B = \frac{1}{2} [\sin(A + B) + \sin(A - B)] \qquad (1)$$

By similar methods, we can derive the formulas that follow.

$$\cos A \sin B = \frac{1}{2} [\sin(A + B) - \sin(A - B)] \qquad (2)$$

$$\cos A \cos B = \frac{1}{2} [\cos(A + B) + \cos(A - B)] \qquad (3)$$

$$\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)] \qquad (4)$$

These four product formulas are of use in calculus. The reason they are useful is that they indicate how we can convert a product into a sum. In calculus, it is sometimes much easier to work with sums of trigonometric functions than it is to work with products.

▼ **Example 3** Verify product formula (3) for $A = 30°$ and $B = 120°$.

Solution Substituting $A = 30°$ and $B = 120°$ into

$$\cos A \cos B = \frac{1}{2} [\cos(A + B) + \cos(A - B)]$$

we have

$$\cos 30° \cos 120° = \frac{1}{2} [\cos 150° + \cos(-90°)]$$

$$\frac{\sqrt{3}}{2} \cdot \left(-\frac{1}{2}\right) = \frac{1}{2}\left(-\frac{\sqrt{3}}{2} + 0\right)$$

$$-\frac{\sqrt{3}}{4} = -\frac{\sqrt{3}}{4} \qquad \text{A true statement}$$ ▲

▼ **Example 4** Write $10 \cos 5x \sin 3x$ as a sum or difference.

$$10 \cos 5x \sin 3x = 10 \cdot \frac{1}{2}[\sin(5x + 3x) - \sin(5x - 3x)]$$

$$= 5(\sin 8x - \sin 2x)$$ ▲

Sum to Product Formulas

By some simple manipulations we can change our product formulas into sum formulas. If we take the formula for $\sin A \cos B$ and exchange sides and then multiply through by 2 we have

$$\sin(A + B) + \sin(A - B) = 2 \sin A \cos B$$

If we let $\alpha = A + B$ and $\beta = A - B$, then we can solve for A by adding the left sides and the right sides.

$$\begin{aligned} A + B &= \alpha \\ A - B &= \beta \\ \hline 2A \quad &= \alpha + \beta \\ A &= \frac{\alpha + \beta}{2} \end{aligned}$$

By subtracting the expression for β from the expression for α, we have

$$B = \frac{\alpha - \beta}{2}$$

Writing the equation $\sin(A + B) + \sin(A - B) = 2 \sin A \cos B$ in terms of α and β gives us our first sum formula

$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} \qquad (5)$$

Similarly, the following sum formulas can be derived from the other product formulas

$$\sin \alpha - \sin \beta = 2 \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2} \qquad (6)$$

$$\cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} \qquad (7)$$

$$\cos \alpha - \cos \beta = -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2} \qquad (8)$$

▼ **Example 5** Verify sum formula (7) for $\alpha = 30°$ and $\beta = 90°$.

Solution We substitute $\alpha = 30°$ and $\beta = 90°$ into sum formula (7) and simplify each side of the resulting equation

$$\cos 30° + \cos 90° = 2 \cos \frac{30° + 90°}{2} \cos \frac{30° - 90°}{2}$$

$$\cos 30° + \cos 90° = 2 \cos 60° \cos(-30°)$$

$$\frac{\sqrt{3}}{2} + 0 = 2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} \quad \text{A true statement}$$

▲

▼ **Example 6** Verify the identity.

$$-\tan x = \frac{\cos 3x - \cos x}{\sin 3x + \sin x}$$

Proof Applying the formulas for $\cos \alpha - \cos \beta$ and $\sin \alpha - \sin \beta$ to the right side and then simplifying, will get us to $-\tan x$.

$$-\tan x \stackrel{?}{=} \frac{\cos 3x - \cos x}{\sin 3x + \sin x}$$

$$\frac{-2 \sin \frac{3x + x}{2} \sin \frac{3x - x}{2}}{2 \sin \frac{3x + x}{2} \cos \frac{3x - x}{2}} \quad \text{Sum to product formulas}$$

$$\frac{-2 \sin 2x \sin x}{2 \sin 2x \cos x} \quad \text{Simplify}$$

$$-\frac{\sin x}{\cos x} \quad \text{Divide out common factors}$$

$$-\tan x = -\tan x \quad \text{Ratio identity}$$

▲

Problem Set 5.5

Evaluate each expression below without using a calculator or tables. (Assume any variables represent positive numbers.)

1. $\sin\left(\text{Arcsin } \frac{3}{5} - \text{Arctan } 2\right)$
2. $\cos\left(\text{Arcsin } \frac{3}{5} - \text{Arctan } 2\right)$
3. $\cos\left(\text{Tan}^{-1}\frac{1}{2} + \text{Sin}^{-1}\frac{1}{2}\right)$
4. $\sin\left(\text{Tan}^{-1}\frac{1}{2} - \text{Sin}^{-1}\frac{1}{2}\right)$
5. $\sin\left(2 \text{ Cos}^{-1}\frac{1}{\sqrt{5}}\right)$
6. $\sin\left(2 \text{ Tan}^{-1}\frac{3}{4}\right)$

7. $\tan(\text{Sin}^{-1}x)$
8. $\tan(\text{Cos}^{-1}x)$
9. $\sin(2\ \text{Sin}^{-1}x)$
10. $\sin(2\ \text{Cos}^{-1}x)$
11. $\cos(2\ \text{Cos}^{-1}x)$
12. $\cos(2\ \text{Sin}^{-1}x)$

13. Verify product formula (4) for $A = 30°$ and $B = 120°$.
14. Verify product formula (1) for $A = 120°$ and $B = 30°$.

Rewrite each expression as a sum or difference, then simplify if possible.

15. $10 \sin 5x \cos 3x$
16. $10 \sin 5x \sin 3x$
17. $\cos 8x \cos 2x$
18. $\cos 2x \sin 8x$
19. $\sin 60° \cos 30°$
20. $\cos 90° \cos 180°$
21. $\sin 4\pi \sin 2\pi$
22. $\cos 3\pi \sin \pi$

23. Verify sum formula (6) for $\alpha = 30°$ and $\beta = 90°$.
24. Verify sum formula (8) for $\alpha = 90°$ and $\beta = 30°$.

Rewrite each expression as a product. Simplify if possible.

25. $\sin 7x + \sin 3x$
26. $\cos 5x - \cos 3x$
27. $\cos 45° + \cos 15°$
28. $\sin 75° - \sin 15°$
29. $\sin \frac{7\pi}{12} - \sin \frac{\pi}{12}$
30. $\cos \frac{\pi}{12} + \cos \frac{7\pi}{12}$

Verify each identity.

31. $-\cot x = \dfrac{\sin 3x + \sin x}{\cos 3x - \cos x}$
32. $\tan x = \dfrac{\cos 3x + \cos x}{\sin 3x - \sin x}$
33. $\cot x = \dfrac{\sin 4x + \sin 6x}{\cos 4x - \cos 6x}$
34. $-\tan 4x = \dfrac{\cos 3x - \cos 5x}{\sin 3x - \sin 5x}$
35. $\tan 4x = \dfrac{\sin 5x + \sin 3x}{\cos 3x + \cos 5x}$
36. $\cot 2x = \dfrac{\sin 3x - \sin x}{\cos x - \cos 3x}$

Review Problems The problems below review material we covered in Section 4.4.

37. Graph $y = 1 + \sin x$ for x between 0 and 4π.
38. Graph $y = x + \sin \pi x$ for x between 0 and 8.
39. Graph $y = \cos x + \frac{1}{2} \sin 2x$ for x between 0 and 4π.
40. Graph $y = \sqrt{3} \sin x + \cos x$, from $x = 0$ to $x = 4\pi$.

Chapter 5 Summary and Review

Examples

BASIC IDENTITIES [5.1]

	Basic identities	Common equivalent forms
Reciprocal	$\csc\theta = \dfrac{1}{\sin\theta}$ $\sec\theta = \dfrac{1}{\cos\theta}$ $\cot\theta = \dfrac{1}{\tan\theta}$	$\sin\theta = \dfrac{1}{\csc\theta}$ $\cos\theta = \dfrac{1}{\sec\theta}$ $\tan\theta = \dfrac{1}{\cot\theta}$
Ratio	$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$	
Pythagorean	$\cos^2\theta + \sin^2\theta = 1$ $1 + \tan^2\theta = \sec^2\theta$ $1 + \cot^2\theta = \csc^2\theta$	$\sin^2\theta = 1 - \cos^2\theta$ $\sin\theta = \pm\sqrt{1 - \cos^2\theta}$ $\cos^2\theta = 1 - \sin^2\theta$ $\cos\theta = \pm\sqrt{1 - \sin^2\theta}$

PROVING IDENTITIES [5.1]

An identity in trigonometry is a statement that two expressions are equal for all replacements of the variable for which each statement is defined. To prove a trigonometric identity, we use trigonometric substitutions and algebraic manipulations to either

1. Transform the right side into the left side, or
2. Transform the left side into the right side.

Remember to work on each side separately. We do not want to use properties from algebra that involve both sides of the identity—like the addition property of equality.

1. To prove

$$\tan x + \cos x = \sin x(\sec x + \cot x),$$

we can multiply through by $\sin x$ on the right side and then change to sines and cosines.

Proof

$$\tan x + \cos x \stackrel{?}{=} \sin x(\sec x + \cot x)$$

$$\sin x \sec x + \sin x \cot x$$

$$\sin x \cdot \frac{1}{\cos x} + \sin x \cdot \frac{\cos x}{\sin x}$$

$$\frac{\sin x}{\cos x} + \cos x$$

$$\tan x + \cos x = \tan x + \cos x$$

2. **a.** Show that $\sin(x + \pi) = -\sin x$

Proof

$$\begin{aligned}\sin(x + \pi) &= \sin x \cos \pi + \cos x \sin \pi \\ &= \sin x(-1) + \cos x(0) \\ &= -\sin x\end{aligned}$$

b. To find the exact value for $\cos 75°$, we write $75°$ as $45° + 30°$ and then apply the formula for $\cos(A + B)$.

$\cos 75°$

$$\begin{aligned}&= \cos(45° + 30°) \\ &= \cos 45° \cos 30° - \sin 45° \sin 30° \\ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

SUM AND DIFFERENCE FORMULAS [5.2]

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

3. **a.** If $\sin A = 3/5$ with A in QII, then

$$\cos 2A = 1 - 2\sin^2 A = 1 - 2\left(\frac{3}{5}\right)^2 = \frac{7}{25}$$

b. Prove $\tan \theta = \dfrac{1 - \cos 2\theta}{\sin 2\theta}$

Proof

$$\tan \theta \stackrel{?}{=} \frac{1 - \cos 2\theta}{\sin 2\theta}$$

$$\frac{1 - (1 - 2\sin^2\theta)}{2 \sin \theta \cos \theta}$$

$$\frac{2\sin^2\theta}{2 \sin \theta \cos \theta}$$

$$\frac{\sin \theta}{\cos \theta}$$

$$\tan \theta = \tan \theta$$

DOUBLE-ANGLE FORMULAS [5.3]

$$\sin 2A = 2 \sin A \cos A$$

$\cos 2A$	$= \cos^2 A - \sin^2 A$	First form
	$= 2\cos^2 A - 1$	Second form
	$= 1 - 2\sin^2 A$	Third form

HALF-ANGLE FORMULAS [5.4]

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$$

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$$

4. We can use a half-angle formula to find the exact value of sin 15° by writing 15° as 30°/2.

$$\sin 15° = \sin\frac{30°}{2} = \sqrt{\frac{1-\cos 30°}{2}}$$

$$= \sqrt{\frac{1-\sqrt{3}/2}{2}}$$

$$= \sqrt{\frac{2-\sqrt{3}}{4}}$$

$$= \frac{\sqrt{2-\sqrt{3}}}{2}$$

PRODUCT TO SUM FORMULAS [5.5]

$$\sin A\cos B = \frac{1}{2}[\sin(A+B)+\sin(A-B)]$$

$$\cos A\sin B = \frac{1}{2}[\sin(A+B)-\sin(A-B)]$$

$$\cos A\cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)]$$

$$\sin A\sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)]$$

5. We can write the product

$$10\cos 5x\sin 3x$$

as a difference by applying the second product to sum formula

$$10\cos 5x\sin 3x$$

$$= 10\cdot\frac{1}{2}[\sin(5x+3x)-\sin(5x-3x)]$$

$$= 5(\sin 8x - \sin 2x)$$

SUM TO PRODUCT FORMULAS [5.5]

$$\sin\alpha+\sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha-\sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\cos\alpha+\cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha-\cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

6. Prove $-\tan x = \dfrac{\cos 3x - \cos x}{\sin 3x + \sin x}$

Proof

$$-\tan x \stackrel{?}{=} \frac{\cos 3x-\cos x}{\sin 3x+\sin x}$$

$$\frac{-2\sin\frac{3x+x}{2}\sin\frac{3x-x}{2}}{2\sin\frac{3x+x}{2}\cos\frac{3x-x}{2}}$$

$$\frac{-2\sin 2x\sin x}{2\sin 2x\cos x}$$

$$-\frac{\sin x}{\cos x}$$

$$-\tan x = -\tan x$$

Chapter 5 Test

Prove each identity.

1. $\tan \theta = \sin \theta \sec \theta$
2. $\dfrac{\cot \theta}{\csc \theta} = \cos \theta$
3. $(\sec x - 1)(\sec x + 1) = \tan^2 x$
4. $\sec \theta - \cos \theta = \tan \theta \sin \theta$
5. $\dfrac{\cos t}{1 - \sin t} = \dfrac{1 + \sin t}{\cos t}$
6. $\dfrac{1}{1 - \sin t} + \dfrac{1}{1 + \sin t} = 2 \sec^2 t$
7. $\sin(\theta - 90°) = -\cos \theta$
8. $\cos\left(\dfrac{\pi}{2} + \theta\right) = -\sin \theta$
9. $\cos^4 A - \sin^4 A = \cos 2A$
10. $\cot A = \dfrac{\sin 2A}{1 - \cos 2A}$
11. $\cot x - \tan x = \dfrac{\cos 2x}{\sin x \cos x}$
12. $\tan \dfrac{x}{2} = \dfrac{\sin x}{1 + \cos x}$

Let $\sin A = -3/5$ with A in QIV and $\sin B = 12/13$ with B in QII and find

13. $\sin(A + B)$
14. $\cos(A - B)$
15. $\cos 2B$
16. $\sin 2B$
17. $\sin \dfrac{A}{2}$
18. $\cos \dfrac{A}{2}$

Find exact values for each of the following:

19. $\sin 75°$
20. $\cos 15°$
21. $\tan \dfrac{\pi}{12}$
22. $\cot \dfrac{\pi}{12}$

Write each expression as a single trigonometric function.

23. $\cos 4x \cos 5x - \sin 4x \sin 5x$
24. $\sin 15° \cos 75° + \cos 15° \sin 75°$
25. If $\sin A = -1/\sqrt{5}$ with A in QIII, find $\cos 2A$ and $\cos A/2$.
26. If $\sec A = \sqrt{10}$ with A in QI, find $\sin 2A$ and $\sin A/2$.
27. Find $\tan A$ if $\tan B = 1/2$ and $\tan(A + B) = 3$.
28. Find $\cos x$ if $\cos 2x = 1/2$.

Evaluate each expression below without using a calculator or tables. (Assume any variables represent positive numbers.)

29. $\cos\left(\text{Arcsin } \frac{4}{5} - \text{Arctan } 2\right)$

30. $\sin\left(\text{Arccos } \frac{4}{5} + \text{Arctan } 2\right)$

31. $\cos(2 \text{ Sin}^{-1}x)$

32. $\sin(2 \text{ Cos}^{-1}x)$

33. Rewrite the product $\sin 6x \sin 4x$ as a sum or difference.

34. Rewrite the sum $\cos 15° + \cos 75°$ as a product and simplify.

6 Equations

To the student:

In this chapter, we will use a number of the techniques and definitions we have covered previously to solve equations that contain trigonometric functions. We begin by solving simple equations that have the form of the linear and quadratic equations you solved in algebra but do not require the use of any identities or trigonometric formulas. In Section 6.2, we extend our work with equations to include equations whose solutions depend on the use of the basic trigonometric identities we covered in Section 5.1. In Section 6.3, we also include equations that contain multiples of angles. These equations are solved with the help of the basic trigonometric identities and the double- and half-angle formulas we derived in Sections 5.3 and 5.4. In Section 6.4, we end the chapter with a look at parametric equations and some additional techniques of graphing.

To be successful in this chapter, you need a good working knowledge of trigonometric identities and formulas presented in Chapter 5. It is also very important that you know the exact values for trigonometric functions of 30°, 45°, and 60°.

6.1 Solving Trigonometric Equations

The solution set for an equation is the set of all numbers which, when used in place of the variable, make the equation a true statement. For example, the solution set for the equation $4x^2 - 9 = 0$ is $\{-3/2, 3/2\}$ since these are the only two numbers which, when used in place of x, turn the equation into a true statement.

In algebra, the first kind of equations you learned to solve were linear (or first-degree) equations in one variable. Solving these equations was accomplished by applying two important properties: *the addition property of equality* and *the multiplication property of equality*. These two properties were stated as follows:

Addition Property of Equality

For any three algebraic expressions A, B, and C

$$\text{If } A = B$$
$$\text{then } A + C = B + C$$

In Words: Adding the same quantity to both sides of an equation will not change the solution set.

Multiplication Property of Equality

For any three algebraic expressions A, B, and C with $C \neq 0$.

$$\text{If } A = B$$
$$\text{then } AC = BC$$

In Words: Multiplying both sides of an equation by the same nonzero quantity will not change the solution set.

Here is an example that shows how we use these two properties to solve a linear equation in one variable.

Example 1 Solve for x: $5x + 7 = 2x - 5$

Solution $5x + 7 = 2x - 5$

$3x + 7 = -5$ Add $-2x$ to each side

$3x = -12$ Add -7 to each side

$x = -4$ Multiply each side by $\frac{1}{3}$

Notice in the last step we could just as easily have divided both sides by 3 instead of multiplying both sides by 1/3. Division by a number and multiplication by its reciprocal are equivalent operations.

The process of solving trigonometric equations is very similar to the process of solving algebraic equations. With trigonometric equations, we look for values of an *angle* that will make the equation into a true statement. We usually begin by solving for a specific trigonometric function of that angle and then use the concepts we have developed earlier to find the angle. Here are some examples that illustrate this procedure.

Example 2 Solve for x: $2 \sin x - 1 = 0$.

Solution We can solve for $\sin x$ using our methods from algebra. We then use our knowledge of trigonometry to find x.

$$2 \sin x - 1 = 0$$
$$2 \sin x = 1$$
$$\sin x = \frac{1}{2}$$

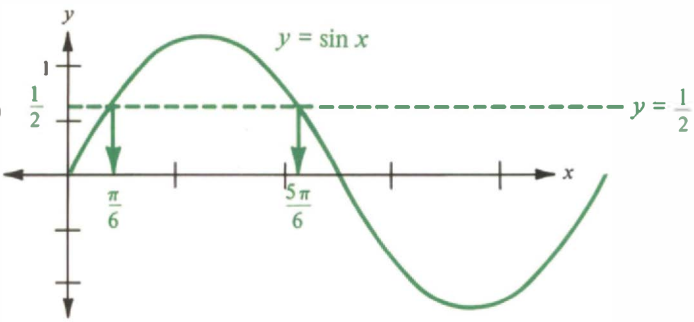


Figure 1

From Figure 1 we can see that if we are looking for real solutions between 0 and 2π, then x is either $\pi/6$ or $5\pi/6$. On the other hand, if we want degree solutions between 0° and 360°, then our solutions will be 30° and 150°. Without the aid of Figure 1, we would reason that, since $\sin x = 1/2$, the reference angle for x is 30°. Then, since 1/2 is a positive number and the sine function is positive in quadrants I and II, x must be 30° or 150°.

Solutions Between 0° and 360°	
In Degrees	*In Radians*
$x = 30°$ or $x = 150°$	$x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$

Since the sine function is periodic with period 2π (or 360°), adding multiples of 2π (or 360°) will give us all solutions.

All Solutions (k is an integer)	
In Degrees	*In Radians*
$x = 30° + 360°k$	$x = \frac{\pi}{6} + 2k\pi$
or $x = 150° + 360°k$	or $x = \frac{5\pi}{6} + 2k\pi$

▲

Note Unless the directions state otherwise, let's agree to solve the rest of the equations in this section for solutions between 0° and 360° (or 0 and 2π) only. Further, let's agree to use decimal degrees rather than degrees and minutes when we write our degree solutions.

▼ **Example 3** Solve $2 \sin \theta - 3 = 0$, if $0° \le \theta \le 360°$.

Solution We begin by solving for $\sin \theta$.

$$2 \sin \theta - 3 = 0$$

$$2 \sin \theta = 3 \qquad \text{Add 3 to both sides}$$

$$\sin \theta = \frac{3}{2} \qquad \text{Divide both sides by 2}$$

Since $\sin \theta$ is between -1 and 1 for all values of θ, $\sin \theta$ can never be 3/2. Therefore, there is no solution to our equation. ▲

▼ **Example 4** Solve $3 \sin \theta - 2 = 7 \sin \theta - 1$, if $0° \le \theta \le 360°$.

Solution We can solve for $\sin \theta$ by collecting all the variable terms on the left side and all the constant terms on the right side.

$$3 \sin \theta - 2 = 7 \sin \theta - 1$$

$$-4 \sin \theta - 2 = -1 \qquad \text{Add } -7 \sin \theta \text{ to each side}$$

$$-4 \sin \theta = 1 \qquad \text{Add 2 to each side}$$

$$\sin \theta = -\frac{1}{4} \qquad \text{Divide each side by } -4$$

Since we have not memorized the angle whose sine is $-1/4$, we must convert $-1/4$ to a decimal and use Table II or a calculator to find the reference angle.

$$\sin \theta = -\frac{1}{4} = -.2500$$

From Table II, we find that the angle whose sine is nearest to 0.2500 is 14.5°. Therefore, the reference angle is 14.5°. Since $\sin \theta$ is negative, θ will terminate in quadrants III or IV.

In quadrant III we have
$$\theta = 180° + 14.5° = 194.5°$$

In quadrant IV we have
$$\theta = 360° - 14.5° = 345.5°$$

Calculator Note Remember, because of the restricted values on your calculator, if you enter −0.2500 and press the $\sin^{-1}$ key, your calculator will display approximately −14.5°, which is incorrect. The best way to proceed is to use your calculator to find the reference angle by entering 0.2500 and pressing the $\sin^{-1}$ key. Then do the rest of the calculations as we have above. ▲

The next kind of trigonometric equation we will solve is quadratic in form. In algebra, the two most common methods of solving quadratic equations are by factoring and by applying the quadratic formula. Here is an example that reviews the factoring method.

▼ **Example 5** Solve $2x^2 - 9x = 5$ for x.

Solution We begin by writing the equation in standard form (0 on one side—decreasing powers of the variable on the other). We then factor the left side and set each factor equal to 0.

$$2x^2 - 9x = 5$$

$$2x^2 - 9x - 5 = 0 \quad \text{Standard form}$$

$$(2x + 1)(x - 5) = 0 \quad \text{Factor}$$

$$2x + 1 = 0 \quad \text{or} \quad x - 5 = 0 \quad \text{Set each factor to 0}$$

$$x = -\frac{1}{2} \quad \text{or} \quad x = 5 \quad \text{Solve resulting equations}$$

The two solutions, $x = -1/2$ and $x = 5$, are the only two numbers that satisfy the original equation. ▲

▼ **Example 6** Solve $2\cos^2 t - 9\cos t = 5$, if $0 \le t \le 2\pi$.

Solution This equation is the equation from Example 5 with $\cos t$ in place of x. The fact that t is between 0 and 2π indicates we are to write our solutions in radians.

$$2\cos^2 t - 9\cos t = 5$$

$$2\cos^2 t - 9\cos t - 5 = 0 \quad \text{Standard form}$$

$$(2\cos t + 1)(\cos t - 5) = 0 \quad \text{Factor}$$

$$2\cos t + 1 = 0 \quad \text{or} \quad \cos t - 5 = 0 \quad \text{Set each factor to 0}$$

$$\cos t = -\frac{1}{2} \quad \text{or} \quad \cos t = 5$$

The first result, $\cos t = -1/2$, gives us $t = 2\pi/3$ or $t = 4\pi/3$. The second result, $\cos t = 5$, has no solution. For any value of t, $\cos t$ must be between -1 and 1. It can never be 5. ▲

▼ **Example 7** Solve $2\sin^2\theta + 2\sin\theta - 1 = 0$, if $0° \le \theta \le 360°$.

Solution The equation is already in standard form. However, if we try to factor the left side, we find it does not factor. We must use the quadratic formula. The quadratic formula states that the solutions to the equation

$$ax^2 + bx + c = 0$$

will be

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In our case the coefficients a, b, and c are

$$a = 2, \quad b = 2, \quad c = -1$$

Using these numbers we can solve for $\sin\theta$ as follows:

$$\sin\theta = \frac{-2 \pm \sqrt{4 - 4(2)(-1)}}{2(2)}$$

$$= \frac{-2 \pm \sqrt{12}}{4}$$

$$= \frac{-2 \pm 2\sqrt{3}}{4}$$

$$= \frac{-1 \pm \sqrt{3}}{2}$$

Using the approximation $\sqrt{3} = 1.7321$, we arrive at the following decimal approximations for $\sin\theta$:

$$\sin\theta = \frac{-1 + 1.7321}{2} \quad \text{or} \quad \sin\theta = \frac{-1 - 1.7321}{2}$$

$$\sin\theta = 0.3661 \qquad \text{or} \quad \sin\theta = -1.3661$$

We will not obtain any solutions from the second expression, $\sin\theta = -1.3661$, since $\sin\theta$ must be between -1 and 1. For $\sin\theta = 0.3661$, we use Table II to find the angle whose sine is nearest to 0.3661. That angle is 21.5°, and it is the reference angle for θ. Since $\sin\theta$ is positive, θ must terminate in quadrants I or II. Therefore,

$$\theta = 21.5° \quad \text{or} \quad \theta = 180° - 21.5° = 158.5°$$

▲

Problem Set 6.1

Solve each equation for θ if $0° \le \theta \le 360°$. Do not use a calculator or tables.

1. $2\sin\theta = 1$

2. $2\cos\theta = 1$

3. $2\cos\theta - \sqrt{3} = 0$

4. $2\cos\theta + \sqrt{3} = 0$

5. $2\tan\theta + 2 = 0$

6. $\sqrt{3}\cot\theta - 1 = 0$

Solve each equation for all solutions between 0 and 2π (including 0 and 2π, if they are solutions). Give all answers as exact values in radians. Do not use a calculator or tables.

7. $4\sin x - \sqrt{3} = 2\sin x$

8. $\sqrt{3} + 5\sin x = 3\sin x$

9. $2\cos t = 6\cos t - \sqrt{12}$

10. $5\cos t + \sqrt{12} = \cos t$

11. $3\sin x + 5 = -2\sin x$

12. $3\sin x + 4 = 4$

Find all solutions between 0° and 360° (including 0° and 360°, if they are solutions). Use Table II or a calculator on the last step and write all answers to the nearest tenth of a degree.

13. $4\sin\theta - 3 = 0$

14. $4\sin\theta + 3 = 0$

15. $2\cos\theta - 5 = 3\cos\theta - 2$

16. $4\cos\theta - 1 = 3\cos\theta + 4$

17. $\sin\theta - 3 = 5\sin\theta$

18. $\sin\theta - 4 = -2\sin\theta$

Solve for x, if $0 \le x \le 2\pi$. Write your answers in exact values only.

19. $(\sin x - 1)(2\sin x - 1) = 0$

20. $(\cos x - 1)(2\cos x - 1) = 0$

21. $\tan x(\tan x - 1) = 0$

22. $\tan x(\tan x + 1) = 0$

23. $\sin x + 2\sin x\cos x = 0$

24. $\cos x - 2\sin x\cos x = 0$

25. $2\sin^2 x - \sin x - 1 = 0$

26. $2\cos^2 x + \cos x - 1 = 0$

Solve for θ, if $0° \le \theta \le 360°$.

27. $(2\cos\theta + \sqrt{3})(2\cos\theta + 1) = 0$

28. $(2\sin\theta - \sqrt{3})(2\sin\theta - 1) = 0$

29. $\sqrt{3} \tan \theta - 2 \sin \theta \tan \theta = 0$
30. $\tan \theta - 2 \cos \theta \tan \theta = 0$
31. $2 \cos^2\theta + 11 \cos \theta = -5$
32. $2 \sin^2\theta - 7 \sin \theta = -3$

Use the quadratic formula to find all solutions between 0° and 360° to the nearest tenth of a degree.

33. $2 \sin^2\theta - 2 \sin \theta - 1 = 0$
34. $2 \cos^2\theta + 2 \cos \theta - 1 = 0$
35. $\cos^2\theta + \cos \theta - 1 = 0$
36. $\sin^2\theta - \sin \theta - 1 = 0$
37. $4 \sin^2\theta + 1 = 4 \sin \theta$
38. $1 - 4 \cos \theta = -4 \cos^2\theta$

Write expressions representing all solutions to the equations you solved in the problems below.

39. Problem 1
40. Problem 2
41. Problem 7
42. Problem 8
43. Problem 11
44. Problem 12
45. Problem 13
46. Problem 14

If a projectile (such as a bullet) is fired into the air with an initial velocity of v at an angle of inclination θ, then the height h of the projectile at time t is given by

$$h = -16t^2 + vt \sin \theta$$

47. Give the equation for the height, if v is 1500 feet per second and θ is 30°.
48. Give the equation for h, if v is 600 feet per second and θ is 45°. (Leave your answer in exact value form.)
49. Use the equation found in Problem 47 to find the height of the object after 2 seconds.
50. Use the equation found in Problem 48 to find the height of the object after $\sqrt{3}$ seconds.
51. Find the angle of inclination θ of a rifle barrel, if a bullet fired at 1500 feet per second takes 2 seconds to reach a height of 750 feet.
52. Find the angle of inclination of a rifle, if a bullet fired at 1500 feet per second takes 3 seconds to reach a height of 750 feet. Give your answer to the nearest tenth of a degree.

Review Problems The problems that follow review material we covered in Sections 5.2 and 5.3. Reviewing these problems will help you with the next section.

53. Write the double-angle formula for $\sin 2A$.
54. Write $\cos 2A$ in terms of $\sin A$ only.
55. Write $\cos 2A$ in terms of $\cos A$ only.
56. Write $\cos 2A$ in terms of $\sin A$ and $\cos A$.
57. Expand $\sin(\theta + 45°)$ and then simplify.
58. Expand $\sin(\theta + 30°)$ and then simplify.

6.2 More on Trigonometric Equations

In this section we will use our knowledge of identities to replace some parts of the equations we are solving with equivalent expressions that will make the equations easier to solve. Here are some examples.

▼ **Example 1** Solve $2\cos x - 1 = \sec x$, if $0 \le x \le 2\pi$.

Solution To solve this equation as we have solved the equations in the previous section, we must write each term using the same trigonometric function. To do so, we can use a reciprocal identity to write sec x in terms of cos x.

$$2\cos x - 1 = \frac{1}{\cos x}$$

To clear the equation of fractions, we multiply both sides by cos x. (Note that we must assume $\cos x \neq 0$ in order to multiply both sides by it. If we obtain solutions for which $\cos x = 0$, we will have to discard them.)

$$\cos x(2\cos x - 1) = \frac{1}{\cos x} \cdot \cos x$$

$$2\cos^2 x - \cos x = 1$$

We are left with a quadratic equation that we write in standard form and then solve.

$$2\cos^2 x - \cos x - 1 = 0 \qquad \text{Standard form}$$

$$(2\cos x + 1)(\cos x - 1) = 0 \qquad \text{Factor}$$

$$2\cos x + 1 = 0 \quad \text{or} \quad \cos x - 1 = 0 \qquad \text{Set each factor to 0}$$

$$\cos x = -\frac{1}{2} \quad \text{or} \quad \cos x = 1$$

$$x = \frac{2\pi}{3}, \frac{4\pi}{3} \quad \text{or} \quad x = 0, 2\pi$$

The solutions are 0, $2\pi/3$, $4\pi/3$, and 2π. ▲

▼ **Example 2** Solve $\sin 2\theta + \sqrt{2}\cos\theta = 0$, $0° \le \theta \le 360°$.

Solution In order to solve this equation, both trigonometric functions must be functions of the same angle. As the equation stands now, one angle is 2θ, while the other is θ. We can write everything as a function of θ by using the double-angle identity $\sin 2\theta = 2 \sin \theta \cos \theta$.

$$\sin 2\theta + \sqrt{2} \cos \theta = 0$$

$$2 \sin \theta \cos \theta + \sqrt{2} \cos \theta = 0 \qquad \sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos \theta (2 \sin \theta + \sqrt{2}) = 0 \qquad \text{Factor out } \cos \theta$$

$$\cos \theta = 0 \quad \text{or} \quad 2 \sin \theta + \sqrt{2} = 0 \qquad \text{Set each factor to } 0$$

$$\sin \theta = -\frac{\sqrt{2}}{2}$$

$$\theta = 90°, 270° \quad \text{or} \quad \theta = 225°, 315°$$

Example 3 **Solve $\cos 2\theta + 3 \sin \theta - 2 = 0$, if $0° \le \theta \le 360°$.**

Solution We have the same problem with this equation as we did with the equation in Example 2. We must rewrite $\cos 2\theta$ in terms of functions of just θ. Recall that there are three forms of the double-angle identity for $\cos 2\theta$. We choose the double-angle identity that involves $\sin \theta$ only, since the middle term of our equation involves $\sin \theta$ and it is best to have all terms involve the same trigonometric functions.

$$\cos 2\theta + 3 \sin \theta - 2 = 0$$

$$1 - 2 \sin^2\theta + 3 \sin \theta - 2 = 0 \qquad \cos 2\theta = 1 - 2 \sin^2\theta$$

$$-2 \sin^2\theta + 3 \sin \theta - 1 = 0 \qquad \text{Simplify}$$

$$2 \sin^2\theta - 3 \sin \theta + 1 = 0 \qquad \text{Multiply each side by } -1$$

$$(2 \sin \theta - 1)(\sin \theta - 1) = 0 \qquad \text{Factor}$$

$$2 \sin \theta - 1 = 0 \quad \text{or} \quad \sin \theta - 1 = 0 \qquad \text{Set factors to } 0$$

$$\sin \theta = \frac{1}{2} \qquad \sin \theta = 1$$

$$\theta = 30°, 150° \quad \text{or} \quad \theta = 90°$$

Example 4 **Solve $4 \cos^2 x + 4 \sin x - 5 = 0$, $0 \le x \le 2\pi$.**

Solution We cannot factor and solve this quadratic equation until each term involves the same trigonometric function. If we change the $\cos^2 x$ in the first term to $1 - \sin^2 x$, we will obtain an equation that involves the sine function only.

$4\cos^2 x + 4\sin x - 5 = 0$	
$4(1 - \sin^2 x) + 4\sin x - 5 = 0$	$\cos^2 x = 1 - \sin^2 x$
$4 - 4\sin^2 x + 4\sin x - 5 = 0$	Distributive property
$-4\sin^2 x + 4\sin x - 1 = 0$	Add 4 and -5
$4\sin^2 x - 4\sin x + 1 = 0$	Multiply each side by -1
$(2\sin x - 1)^2 = 0$	Factor
$2\sin x - 1 = 0$	Set factor to 0
$\sin x = \frac{1}{2}$	
$x = \frac{\pi}{6}, \frac{5\pi}{6}$	

Example 5 Solve $\sin\theta - \cos\theta = 1$, if $0° \le \theta \le 360°$.

Solution If we separate $\sin\theta$ and $\cos\theta$ on opposite sides of the equal sign, and then square both sides of the equation, we will be able to use an identity to write the equation in terms of one trigonometric function only.

$\sin\theta - \cos\theta = 1$	
$\sin\theta = 1 + \cos\theta$	Add $\cos\theta$ to each side
$\sin^2\theta = (1 + \cos\theta)^2$	Square each side
$\sin^2\theta = 1 + 2\cos\theta + \cos^2\theta$	Expand $(1 - \cos\theta)^2$
$1 - \cos^2\theta = 1 + 2\cos\theta + \cos^2\theta$	$\sin^2\theta = 1 - \cos^2\theta$
$0 = 2\cos\theta + 2\cos^2\theta$	Standard form
$0 = 2\cos\theta(1 + \cos\theta)$	Factor
$2\cos\theta = 0$ or $1 + \cos\theta = 0$	Set factors to 0
$\cos\theta = 0$ $\cos\theta = -1$	
$\theta = 90°, 270°$ or $\theta = 180°$	

We have three possible solutions, some of which may be extraneous since we squared both sides of the equation in Step 2. Any time we raise both sides of an equation to an even power, we have the possibility of introducing extraneous solutions. We must check each possible solution in our original equation.

Checking $\theta = 90°$	*Checking* $\theta = 180°$
$\sin 90° - \cos 90° = 1$	$\sin 180° - \cos 180° = 1$
$1 - 0 = 1$	$0 - (-1) = 1$
$1 = 1$	$1 = 1$
$\theta = 90°$ is a solution	$\theta = 180°$ is a solution

Checking $\theta = 270°$

$$\sin 270° - \cos 270° = 1$$
$$-1 - 0 = 1$$
$$-1 \neq 1$$

$\theta = 270°$ is not a solution

All possible solutions, except $\theta = 270°$, produce true statements when used in place of the variable in the original equation. $\theta = 270°$ is an extraneous solution produced by squaring both sides of the equation. Our solution set is $\{90°, 180°\}$. ▲

There is another method of solving the equation in Example 5 that involves what we might call a mathematical trick. Instead of squaring both sides, we rewrite the left side of our equation as a single sine function. The procedure for doing so is shown in Example 6. What you have to understand as you read through Example 6 is that the method of solving the equation $\sin \theta - \cos \theta = 1$ shown there is not the kind of method you would probably think up on your own. That is why we say it involves a mathematical trick. It is the kind of thing you have to see someone else do first.

▼ **Example 6** Solve $\sin \theta - \cos \theta = 1$ for all solutions between 0° and 360° by first writing the left side as a single sine function.

Solution The left side has the form $a \sin \theta - b \cos \theta$, where $a = 1$ and $b = 1$. We first find $\sqrt{a^2 + b^2}$, and then divide both sides by it. (This is the trick.)

$$\sqrt{a^2 + b^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\frac{\sin \theta - \cos \theta}{\sqrt{2}} = \frac{1}{\sqrt{2}} \qquad \text{Divide each side by } \sqrt{a^2 + b^2} = \sqrt{2}$$

$$\frac{1}{\sqrt{2}} \sin \theta - \frac{1}{\sqrt{2}} \cos \theta = \frac{1}{\sqrt{2}}$$

The key to this method of solution is to notice that if we substitute cos 45° for the first $1/\sqrt{2}$ and sin 45° for the second $1/\sqrt{2}$, we will have

$$\cos 45° \sin \theta - \sin 45° \cos \theta = \frac{1}{\sqrt{2}}$$

$$\text{or} \quad \sin \theta \cos 45° - \cos \theta \sin 45° = \frac{1}{\sqrt{2}}$$

Now the left side of this equation looks like the expansion of $\sin(A - B) = \sin A \cos B - \cos A \sin B$, where A is θ and B is 45°. Therefore, we can write our equation as:

$$\sin(\theta - 45°) = \frac{1}{\sqrt{2}}$$

$$\theta - 45° = 45° \quad \text{or} \quad \theta - 45° = 135°$$

$$\theta = 90° \quad \text{or} \quad \theta = 180° \qquad \text{Add 45° to each side}$$

These are the same two solutions we obtained in Example 5. This method of solution works well on equations of the form $a \sin \theta \pm b \cos \theta = c$. One advantage to solving these equations this way is that we do not have to check for extraneous solutions since we never square both sides of the equation. ▲

▼ **Example 7** Solve $\sin \frac{x}{2} + \cos x = 0$, if $0 \le x \le 2\pi$.

Solution We use a half-angle identity to replace $\sin x/2$ with an expression involving only $\cos x$.

$$\sin \frac{x}{2} + \cos x = 0$$

$$\pm\sqrt{\frac{1 - \cos x}{2}} + \cos x = 0 \qquad \text{Half-angle identity}$$

$$\pm\sqrt{\frac{1 - \cos x}{2}} = -\cos x \qquad \text{Add } -\cos x \text{ to each side}$$

$$\frac{1 - \cos x}{2} = \cos^2 x \qquad \text{Square each side}$$

$$1 - \cos x = 2\cos^2 x \qquad \text{Multiply each side by 2}$$

$$2\cos^2 x + \cos x - 1 = 0 \qquad \text{Standard form}$$

$$(2\cos x - 1)(\cos x + 1) = 0 \qquad \text{Factor}$$

$$2\cos x - 1 = 0 \quad \text{or} \quad \cos x + 1 = 0 \qquad \text{Set factors to 0}$$

$$\cos x = \frac{1}{2} \qquad \cos x = -1$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3} \qquad x = \pi$$

Since we squared both sides in the process of solving the equation, we must check all possible solutions in the original equation.

Checking

$x = \frac{\pi}{3}$	$x = \pi$	$x = \frac{5\pi}{3}$
$\sin\frac{\pi/3}{2} + \cos\frac{\pi}{3} = 0$	$\sin\frac{\pi}{2} + \cos\pi = 0$	$\sin\frac{5\pi/3}{2} + \cos\frac{5\pi}{3} = 0$
$\sin\frac{\pi}{6} + \cos\frac{\pi}{3} = 0$	$1 + (-1) = 0$	$\sin\frac{5\pi}{6} + \cos\frac{5\pi}{3} = 0$
$\frac{1}{2} + \frac{1}{2} = 0$	$0 = 0$	$\frac{1}{2} + \frac{1}{2} = 0$
$1 \neq 0$		$1 \neq 0$

The only solution is $x = \pi$. ▲

Problem Set 6.2

For all equations written in terms of θ, find all degree solutions from $\theta = 0°$ to $\theta = 360°$. If the equation is written in terms of x, write your solutions in radians using exact values for $0 \le x \le 2\pi$.

1. $\sqrt{3}\sec\theta = 2$
2. $\sqrt{2}\csc\theta = 2$
3. $\sqrt{2}\csc x + 5 = 3$
4. $2\sqrt{3}\sec x + 7 = 3$
5. $4\sin x - 2\csc x = 0$
6. $4\cos x - 3\sec x = 0$
7. $\sec\theta - 2\tan\theta = 0$
8. $\csc\theta + 2\cot\theta = 0$
9. $\sin 2\theta - \cos\theta = 0$
10. $2\sin\theta + \sin 2\theta = 0$
11. $2\cos\theta + 1 = \sec\theta$
12. $2\sin\theta - 1 = \csc\theta$
13. $\cos 2x - 3\sin x - 2 = 0$
14. $\cos 2x - \cos x - 2 = 0$
15. $\cos\theta - \cos 2\theta = 0$
16. $\sin\theta = -\cos 2\theta$
17. $2\cos^2 x + \sin x - 1 = 0$
18. $2\sin^2 x - \cos x - 1 = 0$
19. $4\sin^2 x + 4\cos x - 5 = 0$
20. $4\cos^2 x - 4\sin x - 5 = 0$
21. $2\sin x + \cot x - \csc x = 0$
22. $2\cos x + \tan x = \sec x$
23. $\sin\theta + \cos\theta = \sqrt{2}$
24. $\sin\theta - \cos\theta = \sqrt{2}$
25. $\sqrt{3}\sin\theta + \cos\theta = \sqrt{3}$
26. $\sin\theta - \sqrt{3}\cos\theta = \sqrt{3}$
27. $\sqrt{3}\sin\theta - \cos\theta = 1$
28. $\sin\theta - \sqrt{3}\cos\theta = 1$
29. $\sin\frac{x}{2} - \cos x = 0$
30. $\sin\frac{x}{2} + \cos x = 1$
31. $\cos\frac{x}{2} - \cos x = 1$
32. $\cos\frac{x}{2} - \cos x = 0$

Write expressions that give all solutions to the equations you solved in the problems given below.

33. Problem 3
34. Problem 4
35. Problem 23
36. Problem 24
37. Problem 31
38. Problem 32

39. In the human body, the value of θ that makes the following expression zero is the angle at which an artery of radius r will branch off from a larger artery of radius R in order to minimize the energy loss due to friction. Show that the following expression is 0 when $\cos\theta = r^4/R^4$.

$$r^4\csc^2\theta - R^4\csc\theta\cot\theta$$

40. Find the value of θ that makes the expression in Problem 39 zero, if $r = 2$ millimeters and $R = 4$ millimeters. (Give your answer to the nearest tenth of a degree.)

Review Problems The problems that follow review material we covered in Section 4.1. Reviewing these problems will help you with some of the material in the next section.

41. Find all values of θ between 0° and 720° for which $\cos\theta = \sqrt{3}/2$.
42. Find all values of x between 0 and 4π for which $\sin x = 1/\sqrt{2}$.
43. Find all values of x between 0 and 4π for which $\tan x = 1$.
44. Find all values of θ between 0° and 720° for which $\sin\theta = 0$.

6.3 Trigonometric Equations Involving Multiple Angles

In this section, we will consider equations that contain multiple angles. We will use most of the same techniques to solve these equations that we have used in the past. We have to be careful at the last step, however, when our equations contain multiple angles. Here is an example.

▼ **Example 1** Solve $\cos 2\theta = \sqrt{3}/2$, if $0° \le \theta \le 360°$.

Solution The equation cannot be simplified further. Since we are looking for θ between 0° and 360°, we must first find all values of 2θ between 0° and 720° that satisfy the equation, because

$$\text{if } 0° \le \theta \le 360°, \text{ then } 2(0°) \le 2\theta \le 2(360°)$$
$$\text{or } 0° \le 2\theta \le 720°$$

To find all values of 2θ between 0° and 720° that satisfy our original equation, we first find all solutions to $\cos 2\theta = \sqrt{3}/2$ between 0° and

360°. Then we add 360° to each of these solutions to obtain all solutions between 0° and 720°. That is,

$$\text{If } \cos 2\theta = \frac{\sqrt{3}}{2}$$

$$\text{then } 2\theta = 30° \text{ or } 2\theta = 330° \text{ or } 2\theta = \underbrace{390°}_{30° + 360°} \text{ or } 2\theta = \underbrace{690°}_{330° + 360°}$$

Dividing both sides of each equation by 2 gives us all θ between 0° and 360° that satisfy $\cos 2\theta = \sqrt{3}/2$.

$$\theta = 15° \quad \text{or} \quad \theta = 165° \quad \text{or} \quad \theta = 195° \quad \text{or} \quad \theta = 345°$$ ▲

If we had originally been asked to find *all solutions* to the equation in Example 1, instead of only those between 0° and 360°, our work would have been a little less complicated. To find all solutions, we would first find all values of 2θ between 0° and 360° that satisfy the equation and then we would add on multiples of 360°, since the period of the cosine function is 360°. After that, it is just a matter of dividing everything by 2.

$$\text{If } \cos 2\theta = \frac{\sqrt{3}}{2}$$

$$\text{then } 2\theta = 30° + 360°k \quad \text{or} \quad 2\theta = 330° + 360°k \qquad k \text{ an integer}$$

$$\theta = 15° + 180°k \quad \text{or} \quad \theta = 165° + 180°k \qquad \text{divide by 2}$$

The last line gives us *all values* of θ that satisfy $\cos 2\theta = \sqrt{3}/2$. Note that $k = 0$ and $k = 1$ will give us the four solutions between 0° and 360° that we found in Example 1. Note also that we are including negative angles as solutions since k can assume values of -1, -2, -3, and so forth.

▼ **Example 2** Find all solutions to $\tan 3x = 1$, if x is measured in radians with exact values.

Solution First we find all values of $3x$ between 0 and π that satisfy $\tan 3x = 1$, and then we add on multiples of π because the period of the tangent function is π. After that, we simply divide by 3 to solve for x.

$$\text{If } \tan 3x = 1$$

$$\text{then } 3x = \frac{\pi}{4} + k\pi$$

$$x = \frac{\pi}{12} + \frac{k\pi}{3} \qquad \text{Divide by 3}$$

Note that $k = 0$, 1, and 2 will give us all values of x between 0 and π that satisfy $\tan 3x = 1$.

▼ **Example 3** Solve $\sin 2x \cos x + \cos 2x \sin x = 1/\sqrt{2}$, if $0 \leq x \leq 2\pi$.

Solution We can simplify the left side by using the formula for $\sin (A + B)$.

$$\sin 2x \cos x + \cos 2x \sin x = \frac{1}{\sqrt{2}}$$

$$\sin(2x + x) = \frac{1}{\sqrt{2}}$$

$$\sin 3x = \frac{1}{\sqrt{2}}$$

Finding all solutions looks like this

$$3x = \frac{\pi}{4} + 2k\pi \quad \text{or} \quad 3x = \frac{3\pi}{4} + 2k\pi \qquad k \text{ an integer}$$

$$x = \frac{\pi}{12} + \frac{2k\pi}{3} \quad \text{or} \quad x = \frac{\pi}{4} + \frac{2k\pi}{3} \qquad \text{Divide by 3}$$

To find only those solutions that lie between 0 and 2π, we let k take on values of 0, 1, and 2. Doing so results in the following solutions:

$$x = \frac{\pi}{12}, \frac{\pi}{4}, \frac{9\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}, \text{ and } \frac{19\pi}{12}$$ ▲

▼ **Example 4** Find all solutions to $2 \sin^2 3\theta - \sin 3\theta - 1 = 0$, if θ is measured in degrees.

Solution We have an equation that is quadratic in $\sin 3\theta$. We factor and solve as usual.

$$2 \sin^2 3\theta - \sin 3\theta - 1 = 0 \qquad \text{Standard form}$$

$$(2 \sin 3\theta + 1)(\sin 3\theta - 1) = 0 \qquad \text{Factor}$$

$$2 \sin 3\theta + 1 = 0 \quad \text{or} \quad \sin 3\theta - 1 = 0 \qquad \text{Set factors to 0}$$

$$\sin 3\theta = -\frac{1}{2} \quad \text{or} \quad \sin 3\theta = 1$$

$$3\theta = 210° + 360°k \text{ or } 3\theta = 330° + 360°k \text{ or } 3\theta = 90° + 360°k$$
$$\theta = 70° + 120°k \text{ or } \theta = 110° + 120°k \text{ or } \theta = 30° + 120°k$$ ▲

Example 5 Find all solutions, in radians, for $\tan^2 3x = 1$.

Solution Taking the square root of both sides we have

$$\tan^2 3x = 1$$
$$\tan 3x = \pm 1 \qquad \text{Square root of both sides}$$

Since the period of the tangent function is π, we find all solutions to $\tan 3x = 1$ and $\tan 3x = -1$ between 0 and π, and then we add multiples of π to these solutions. Finally, we divide each side of the resulting equations by 3.

$$3x = \frac{\pi}{4} + k\pi \quad \text{or} \quad \frac{3\pi}{4} + k\pi$$

$$x = \frac{\pi}{12} + \frac{k\pi}{3} \quad \text{or} \quad \frac{\pi}{4} + \frac{k\pi}{3}$$

Example 6 Solve $\sin\theta - \cos\theta = 1$ if $0° \le \theta \le 360°$.

Solution We have solved this equation twice previously in Section 6.2. This time we will simply square both sides.

$$\sin\theta - \cos\theta = 1$$
$$(\sin\theta - \cos\theta)^2 = 1^2 \qquad \text{Square both sides}$$
$$\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta = 1 \qquad \text{Expand left side}$$
$$-2\sin\theta\cos\theta + 1 = 1 \qquad \sin^2\theta + \cos^2\theta = 1$$
$$-2\sin\theta\cos\theta = 0 \qquad \text{Add } -1 \text{ to both sides}$$
$$-\sin 2\theta = 0 \qquad \text{Double-angle identity}$$
$$2\theta = 0°, 180°, 360°, 540°, 720°$$
$$\theta = 0°, 90°, 180°, 270°, 360°$$

Since we squared both sides of the equation in Step 2, we must check all the possible solutions to see if they satisfy the original equation. Doing so gives us solutions $x = 90°$ and $180°$. The others are all extraneous.

Problem Set 6.3

Find all solutions if $0° \le \theta \le 360°$.

1. $\sin 2\theta = \sqrt{3}/2$
2. $\sin 2\theta = -\sqrt{3}/2$
3. $\tan 2\theta = -1$
4. $\cot 2\theta = 1$
5. $\cos 3\theta = -1$
6. $\sin 3\theta = -1$

Find all solutions if $0 \le x \le 2\pi$. Use exact values only.

7. $\sin 2x = 1/\sqrt{2}$
8. $\cos 2x = 1/\sqrt{2}$

9. $\sec 3x = -1$
10. $\csc 3x = 1$
11. $\tan 2x = \sqrt{3}$
12. $\tan 2x = -\sqrt{3}$

Find all degree solutions for each of the following:

13. $\sin 2\theta = 1/2$
14. $\sin 2\theta = -\sqrt{3}/2$
15. $\cos 3\theta = 0$
16. $\cos 3\theta = -1$
17. $\sin 10\theta = \sqrt{3}/2$
18. $\cos 8\theta = 1/2$

Find all solutions if $0 \leq x \leq 2\pi$. Use exact values only.

19. $\sin 2x \cos x + \cos 2x \sin x = 1/2$
20. $\sin 2x \cos x + \cos 2x \sin x = -1/2$
21. $\cos 2x \cos x - \sin 2x \sin x = -\sqrt{3}/2$
22. $\cos 2x \cos x - \sin 2x \sin x = 1/\sqrt{2}$

Find all solutions in radians using exact values only.

23. $\sin 3x \cos 2x + \cos 3x \sin 2x = 1$
24. $\sin 2x \cos 3x + \cos 2x \sin 3x = -1$

25. $\sin^2 4x = 1$
26. $\cos^2 4x = 1$
27. $\cos^3 5x = -1$
28. $\sin^3 5x = -1$

Find all degree solutions.

29. $2 \sin^2 3\theta + \sin 3\theta - 1 = 0$
30. $2 \sin^2 3\theta + 3 \sin 3\theta + 1 = 0$
31. $2 \cos^2 2\theta + 3 \cos 2\theta + 1 = 0$
32. $2 \cos^2 2\theta - \cos 2\theta - 1 = 0$
33. $\tan^2 3\theta = 3$
34. $\cot^2 3\theta = 1$

Find all solutions if $0° \leq \theta \leq 360°$.

35. $\cos \theta - \sin \theta = 1$
36. $\sin \theta - \cos \theta = 1$
37. $\sin \theta + \cos \theta = -1$
38. $\cos \theta - \sin \theta = -1$

39. The formula below gives the relationship between the number of sides, n, the radius, r, and the length of each side, l, in a regular polygon. Find n if $l = r$.

$$l = 2r \sin \frac{180°}{n}$$

40. If central angle θ cuts off a chord of length c in a circle of radius r, then the relationship between θ, c, and r is given by

$$2r \sin \frac{\theta}{2} = c$$

Find θ, if $c = \sqrt{3}r$.

41. In Example 4 of Section 3.5, we found the equation that gives d in terms of t in Figure 1 to be $d = 10 \tan \pi t$. If a person is standing against the wall, 10 feet from point A, how long after the light is at point A will the person see the light? (You must find t when d is 10.)

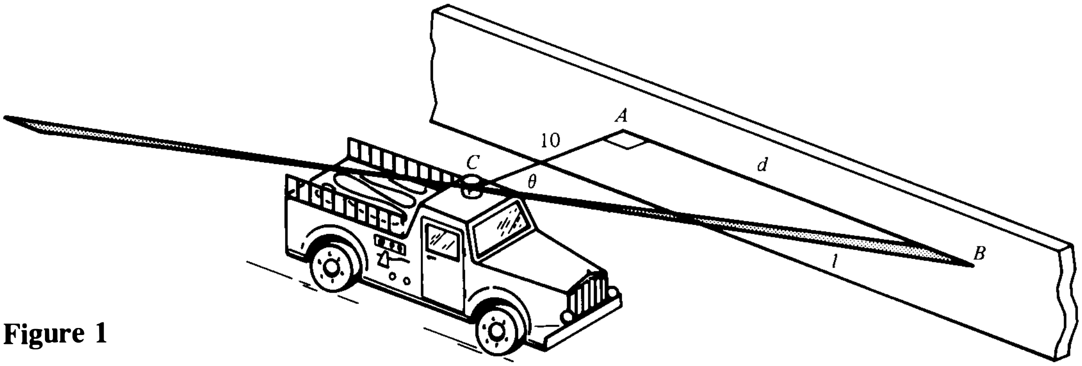


Figure 1

42. In Problem 22 of Problem Set 3.5, you found the equation that gives d in terms of t in Figure 2 to be $d = 100\tan(1/2)\pi t$. Two people are sitting on the wall. One of them is directly opposite the lighthouse, while the other person is 100 feet further down the wall. How long after one of them sees the light does the other one see the light? (There are two solutions depending on who sees the light first.)

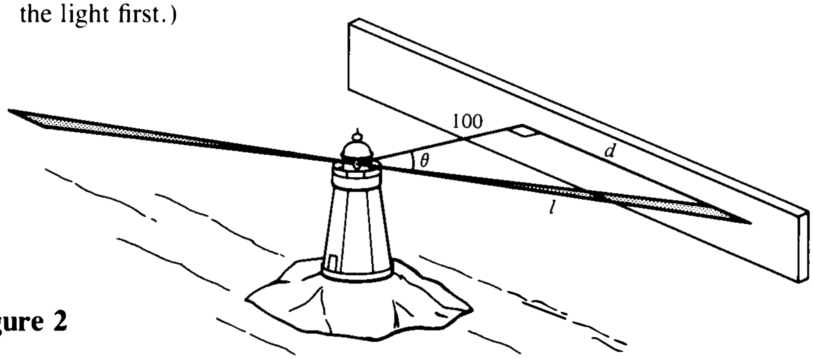


Figure 2

43. Find the smallest positive value of t for which $\sin 2\pi t = 1/2$.
44. Find the smallest positive value of t for which $\sin 2\pi t = 1/\sqrt{2}$.

Review Problems The problems that follow review material we covered in Sections 4.4 and 5.2. Reviewing these problems will help you understand the next section.

Graph each equation between 0 and 4π.

45. $y = \sin x + \cos x$

46. $y = \sin x - \cos x$

Write as a single trigonometric function.

47. $\sin x \cos 45° + \cos x \sin 45°$
48. $\sin x \cos 30° + \cos x \sin 30°$
49. Graph one complete cycle of $y = \sin x \cos \pi/4 + \cos x \sin \pi/4$ by rewriting the right side in the form $\sin(A + B)$.
50. Graph one complete cycle of $y = 2(\sin x \cos \pi/3 - \cos x \sin \pi/3)$ by rewriting the right side in the form $2\sin(A - B)$.

6.4 Parametric Equations and Further Graphing

Many times in mathematics a set of points (x, y) in the plane is described by a pair of equations rather than one equation. For example, $x = \cos t$ and $y = \sin t$, where t is any real number, are a pair of equations that describe a certain curve in the xy-plane. The equations are called *parametric equations* and the variable t is called the *parameter*. The variable t does not appear as part of the graph, but rather, produces values of x and y that do appear as ordered pairs (x, y) on the graph.

Table 1 shows the values of x and y produced by substituting convenient values of t into each equation.

Plotting each ordered pair (x, y) from the last column of Table 1 on a coordinate system, we see that the set of points form the unit circle, starting at $(1, 0)$ when $t = 0$ and ending at $(1, 0)$ when $t = 2\pi$. (See Figure 1.)

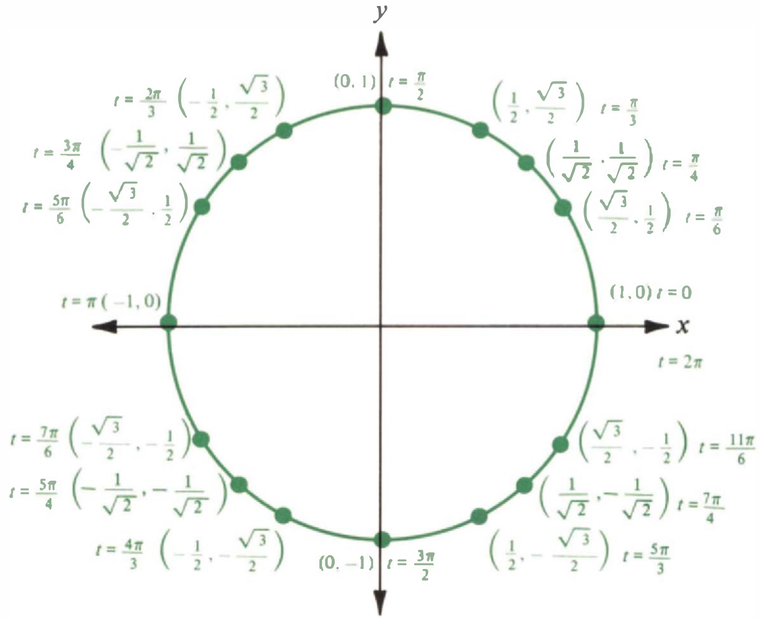


Figure 1

We could have found this graph without going to as much work by using the Pythagorean identity $\cos^2 t + \sin^2 t = 1$. Substituting $x = \cos t$ and $y = \sin t$ into

$$\cos^2 t + \sin^2 t = 1$$

we have

$$x^2 + y^2 = 1$$

which is the equation of the unit circle.

Table 1

t	$x = \cos t$	$y = \sin t$	(x, y)
0	$x = \cos 0 = 1$	$y = \sin 0 = 0$	$(1, 0)$
$\frac{\pi}{6}$	$x = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$	$y = \sin \frac{\pi}{6} = \frac{1}{2}$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\frac{\pi}{4}$	$x = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$	$y = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$	$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
$\frac{\pi}{3}$	$x = \cos \frac{\pi}{3} = \frac{1}{2}$	$y = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\frac{\pi}{2}$	$x = \cos \frac{\pi}{2} = 0$	$y = \sin \frac{\pi}{2} = 1$	$(0, 1)$
$\frac{2\pi}{3}$	$x = \cos \frac{2\pi}{3} = -\frac{1}{2}$	$y = \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$	$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\frac{3\pi}{4}$	$x = \cos \frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$	$y = \sin \frac{3\pi}{4} = \frac{1}{\sqrt{2}}$	$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
$\frac{5\pi}{6}$	$x = \cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$	$y = \sin \frac{5\pi}{6} = \frac{1}{2}$	$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
π	$x = \cos \pi = -1$	$y = \sin \pi = 0$	$(-1, 0)$
$\frac{7\pi}{6}$	$x = \cos \frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$	$y = \sin \frac{7\pi}{6} = -\frac{1}{2}$	$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
$\frac{5\pi}{4}$	$x = \cos \frac{5\pi}{4} = -\frac{1}{\sqrt{2}}$	$y = \sin \frac{5\pi}{4} = -\frac{1}{\sqrt{2}}$	$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$
$\frac{4\pi}{3}$	$x = \cos \frac{4\pi}{3} = -\frac{1}{2}$	$y = \sin \frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$	$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
$\frac{3\pi}{2}$	$x = \cos \frac{3\pi}{2} = 0$	$y = \sin \frac{3\pi}{2} = -1$	$(0, -1)$
$\frac{5\pi}{3}$	$x = \cos \frac{5\pi}{3} = \frac{1}{2}$	$y = \sin \frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$	$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
$\frac{7\pi}{4}$	$x = \cos \frac{7\pi}{4} = \frac{1}{\sqrt{2}}$	$y = \sin \frac{7\pi}{4} = -\frac{1}{\sqrt{2}}$	$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$
$\frac{11\pi}{6}$	$x = \cos \frac{11\pi}{6} = \frac{\sqrt{3}}{2}$	$y = \sin \frac{11\pi}{6} = -\frac{1}{2}$	$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
2π	$x = \cos 2\pi = 1$	$y = \sin 2\pi = 0$	$(1, 0)$

The process of going directly to an equation that contains x and y but not t is called *eliminating the parameter*.

Example 1 Eliminate the parameter t from the parametric equations

$$x = 3 \cos t \qquad y = 2 \sin t$$

Solution Again, we will use the identity $\cos^2 t + \sin^2 t = 1$. Before we do so, however, we must solve the first equation for $\cos t$ and the second equation for $\sin t$.

$$x = 3 \cos t \Rightarrow \cos t = \frac{x}{3}$$

$$y = 2 \sin t \Rightarrow \sin t = \frac{y}{2}$$

Substituting $x/3$ and $y/2$ for $\cos t$ and $\sin t$ into the Pythagorean identity gives us

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

which is the equation of an ellipse. The center is at the origin, the x-intercepts are 3 and -3, and the y-intercepts are 2 and -2. Figure 2 shows the graph.

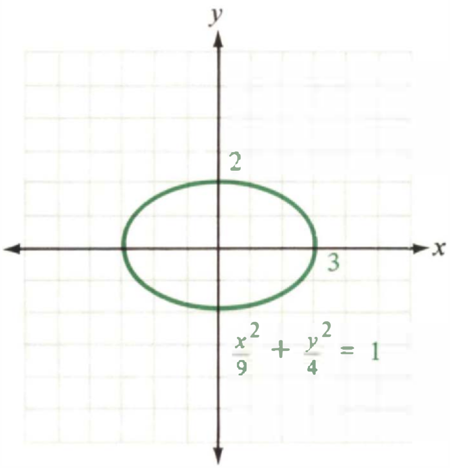


Figure 2

Example 2 Eliminate the parameter t from the equations

$$x = 3 + \sin t \qquad y = \cos t - 2$$

Solution Solving the first equation for $\sin t$ and the second equation for $\cos t$, we have

$$\sin t = x - 3 \quad \text{and} \quad \cos t = y + 2$$

Substituting these expressions into the Pythagorean identity for $\sin t$ and $\cos t$ gives us

$$(x - 3)^2 + (y + 2)^2 = 1$$

which is the equation of a circle with a radius of 1 and center at $(3, -2)$.

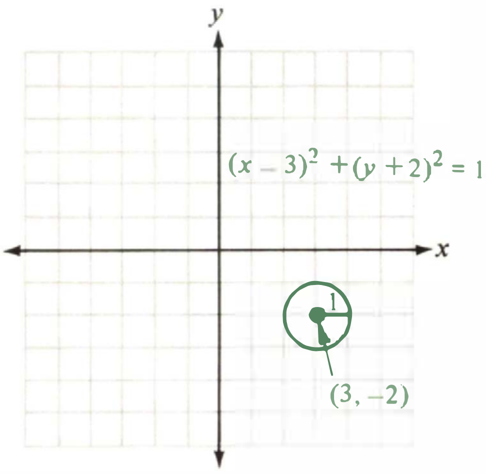


Figure 3

Example 3 Eliminate the parameter t.

$$x = 3 + 2 \sec t \qquad y = 2 + 4 \tan t$$

Solution In this case, we solve for $\sec t$ and $\tan t$ and then use the identity $1 + \tan^2 t = \sec^2 t$.

$$x = 3 + 2 \sec t \Rightarrow \sec t = \frac{x - 3}{2}$$

$$y = 2 + 4 \tan t \Rightarrow \tan t = \frac{y - 2}{4}$$

so

$$1 + \left(\frac{y - 2}{4}\right)^2 = \left(\frac{x - 3}{2}\right)^2$$

or

$$\frac{(x-3)^2}{4} - \frac{(y-2)^2}{16} = 1$$

This is the equation of a hyperbola.

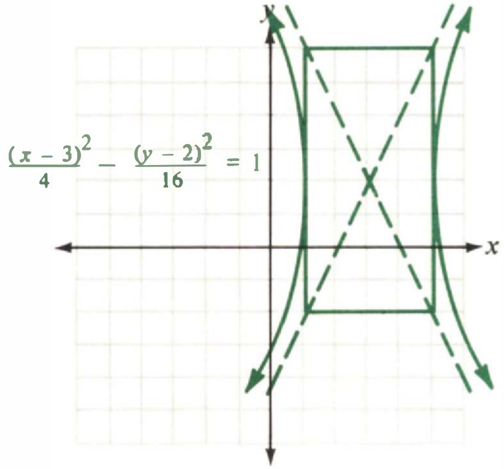


Figure 4

We will end this chapter by looking at one more type of graphing. Recall that in Chapter 4 we graphed the equation $y = \sin x + \cos x$ by graphing $y = \sin x$ and $y = \cos x$ separately and then adding y-coordinates. There is another method of graphing equations of the form $y = a \sin x + b \cos x$ that does not require two separate graphs. Instead, we rewrite the equation so it has the form $y = A \sin(Bx + C)$ so that we can identify the amplitude, period, and phase shift and sketch the graph from them. The key to this new method is multiplying and dividing the right side of $y = a \sin x + b \cos x$ by $\sqrt{a^2 + b^2}$ (we did something similar to this in Example 6 of Section 6.2). That is, we write

$$y = a \sin x + b \cos x$$

as

$$y = \sqrt{a^2 + b^2}\left(\frac{a}{\sqrt{a^2 + b^2}} \sin x + \frac{b}{\sqrt{a^2 + b^2}} \cos x\right)$$

Doing so does not change the equation, because we have simply multiplied by

$$\frac{\sqrt{a^2 + b^2}}{\sqrt{a^2 + b^2}}$$

which is 1.

Example 4 Graph one complete cycle of $y = \sin x + \cos x$.

Solution $y = \sin x + \cos x$ has the form $y = a \sin x + b \cos x$, where a and b are both 1.

$$\sqrt{a^2 + b^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Multiplying and dividing the right side of our original equation by $\sqrt{2}$, gives us

$$y = \sqrt{2}\left(\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x\right)$$

Next we substitute $\cos \pi/4$ for the first $1/\sqrt{2}$ and $\sin \pi/4$ for the second $1/\sqrt{2}$.

$$y = \sqrt{2}\left(\cos \frac{\pi}{4} \sin x + \sin \frac{\pi}{4} \cos x\right)$$

Exchanging the order of the products in each term gives us an expression on the right side that we recognize as an expanded form of the sum formula for $\sin(A + B)$

$$y = \sqrt{2}\left(\sin x \cos \frac{\pi}{4} + \cos x \sin \frac{\pi}{4}\right)$$

Writing the right side in the form $\sin(A + B)$ we have

$$y = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$$

the graph of which is a sine curve with amplitude $\sqrt{2}$, period 2π, and phase shift $-\pi/4$.

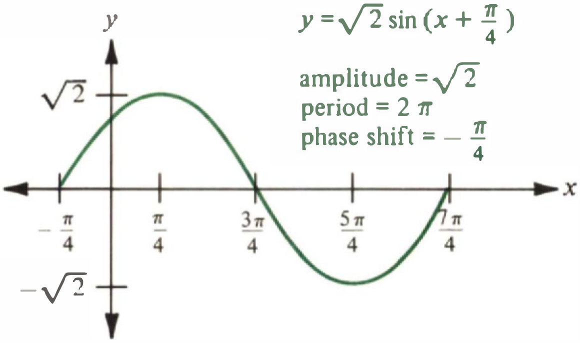


Figure 5

Problem Set 6.4

Eliminate the parameter t from each of the following and then sketch the graph of each:

1. $x = \sin t \quad y = \cos t$
2. $x = -\sin t \quad y = \cos t$
3. $x = 3\cos t \quad y = 3\sin t$
4. $x = 2\cos t \quad y = 2\sin t$
5. $x = 2\sin t \quad y = 4\cos t$
6. $x = 3\sin t \quad y = 4\cos t$
7. $x = 2 + \sin t \quad y = 3 + \cos t$
8. $x = 3 + \sin t \quad y = 2 + \cos t$
9. $x = \sin t - 2 \quad y = \cos t - 3$
10. $x = \cos t - 3 \quad y = \sin t + 2$
11. $x = 3 + 2\sin t \quad y = 1 + 2\cos t$
12. $x = 2 + 3\sin t \quad y = 1 + 3\cos t$
13. $x = 3\cos t - 3 \quad y = 3\sin t + 1$
14. $x = 4\sin t - 5 \quad y = 4\cos t - 3$

Eliminate the parameter t in each of the following:

15. $x = \sec t \quad y = \tan t$
16. $x = \tan t \quad y = \sec t$
17. $x = 3\sec t \quad y = 3\tan t$
18. $x = 3\cot t \quad y = 3\csc t$
19. $x = 2 + 3\tan t \quad y = 4 + 3\sec t$
20. $x = 3 + 5\tan t \quad y = 2 + 5\sec t$
21. $x = \cos 2t \quad y = \sin t$
22. $x = \cos 2t \quad y = \cos t$
23. $x = \sin t \quad y = \sin t$
24. $x = \cos t \quad y = \cos t$
25. $x = 3\sin t \quad y = 2\sin t$
26. $x = 2\sin t \quad y = 3\sin t$

Graph one complete cycle of each of the following by first changing to a single sine function and then using amplitude, period, and phase shift. (See Example 4.)

27. $y = \sin x - \cos x$
28. $y = \sqrt{2}\sin x + \sqrt{2}\cos x$
29. $y = \sin x + \sqrt{3}\cos x$
30. $y = \sin x - \sqrt{3}\cos x$
31. $y = \sqrt{3}\sin x + \cos x$
32. $y = \sqrt{3}\sin x - \cos x$

Review Problems The problems that follow review material we covered in Sections 5.1 and 5.4.

Prove each identity.

33. $\dfrac{\sin x}{1 + \cos x} = \dfrac{1 - \cos x}{\sin x}$
34. $\dfrac{\sin^2 x}{(1 - \cos x)^2} = \dfrac{1 + \cos x}{1 - \cos x}$
35. $\dfrac{1}{1 + \cos t} + \dfrac{1}{1 - \cos t} = 2\csc^2 t$
36. $\dfrac{1}{1 - \sin t} + \dfrac{1}{1 + \sin t} = 2\sec^2 t$
37. $\tan\dfrac{A}{2} = \dfrac{\sin A}{1 + \cos A}$
38. $\tan\dfrac{A}{2} = \dfrac{1 - \cos A}{\sin A}$

Chapter 6 Summary and Review

SOLVING SIMPLE TRIGONOMETRIC EQUATIONS [6.1]

We solve linear equations in trigonometry by applying the properties of equality developed in algebra. The two most important properties from algebra are stated as follows:

Addition Property of Equality

For any three algebraic expressions A, B, and C

$$\text{If } A = B \quad \text{then } A + C = B + C$$

In Words: Adding the same quantity to both sides of an equation will not change the solution set.

Multiplication Property of Equality

For any three algebraic expressions A, B, and C with $C \neq 0$.

$$\text{If } A = B \quad \text{then } AC = BC$$

In Words: Multiplying both sides of an equation by the same nonzero quantity will not change the solution set.

Examples

1. a. Solve for x: $2\cos x - \sqrt{3} = 0$.

$$2\cos x - \sqrt{3} = 0$$
$$2\cos x = \sqrt{3}$$
$$\cos x = \frac{\sqrt{3}}{2}$$

Solutions between 0° and 360°	
In Degrees	*In Radians*
$x = 30°$ or $x = 330°$	$x = \frac{\pi}{6}$ or $x = \frac{11\pi}{6}$

All Solutions (k is an integer)	
In Degrees	*In Radians*
$x = 30° + 360°k$	$x = \frac{\pi}{6} + 2k\pi$
or	or
$x = 330° + 360°k$	$x = \frac{11\pi}{6} + 2k\pi$

To solve a trigonometric equation that is quadratic in $\sin x$ or $\cos x$, we write it in standard form and then factor it or use the quadratic formula.

b. Solve $2\cos^2 t - 9\cos t = 5$, if $0 \leq t \leq 2\pi$

$$2\cos^2 t - 9\cos t = 5$$
$$2\cos^2 t - 9\cos t - 5 = 0$$
$$(2\cos t + 1)(\cos t - 5) = 0$$
$$2\cos t + 1 = 0 \quad \text{or} \quad \cos t - 5 = 0$$
$$\cos t = -\frac{1}{2} \qquad \cos t = 5$$

The first result, $\cos t = -1/2$, gives us $t = 2\pi/3$ or $t = 4\pi/3$. The second result, $\cos t = 5$, has no solution. For any value of t, $\cos t$ must always be between -1 and 1. It will never be 5.

2. Solve $\cos 2\theta + 3 \sin \theta - 2 = 0$, if $0° \leq \theta \leq 360°$.

$$\cos 2\theta + 3 \sin \theta - 2 = 0$$
$$1 - 2 \sin^2\theta + 3 \sin \theta - 2 = 0$$
$$-2 \sin^2\theta + 3 \sin \theta - 1 = 0$$
$$2 \sin^2\theta - 3 \sin \theta + 1 = 0$$
$$(2 \sin \theta - 1)(\sin \theta - 1) = 0$$
$$2 \sin \theta - 1 = 0 \quad \text{or} \quad \sin \theta - 1 = 0$$
$$\sin \theta = \frac{1}{2} \qquad \sin \theta = 1$$
$$\theta = 30°, 150°, 90°$$

USING IDENTITIES IN TRIGONOMETRIC EQUATIONS [6.2]

Sometimes it is necessary to use identities to make trigonometric substitutions when solving equations. Identities are usually required if the equation contains more than one trigonometric function or if there is more than one angle named in the equation. In the example to the left, we begin by replacing $\cos 2\theta$ with $1 - 2 \sin^2\theta$. Doing so gives us a quadratic equation in $\sin \theta$, which we put in standard form and solve by factoring.

3. Find all radian solutions for $\sin 2x \cos x + \cos 2x \sin x = 1/\sqrt{2}$.

Solution

$$\sin 2x \cos x + \cos 2x \sin x = \frac{1}{\sqrt{2}}$$
$$\sin(2x + x) = \frac{1}{\sqrt{2}}$$
$$\sin 3x = \frac{1}{\sqrt{2}}$$
$$3x = \frac{\pi}{4} + 2k\pi \quad \text{or} \quad 3x = \frac{3\pi}{4} + 2k\pi$$
$$x = \frac{\pi}{12} + \frac{2k\pi}{3} \quad \text{or} \quad x = \frac{\pi}{4} + \frac{2k\pi}{3}$$

where k is an integer.

EQUATIONS INVOLVING MULTIPLE ANGLES [6.3]

Sometimes the equations we solve in trigonometry reduce to single equations that contain multiple angles. When this occurs, we have to be careful in the last step that we do not leave out any solutions. For instance, if we are asked to all solutions between $x = 0$ and $x = 2\pi$, and our final equation contains $2x$, we must find all values of $2x$ between 0 and 4π in order that x remain between 0 and 2π.

4. Eliminate the parameter t from the equations

$$x = 3 + \sin t \quad y = \cos t - 2$$

Solving $\sin t$ and $\cos t$ we have

$$\sin t = x - 3 \quad \text{and} \quad \cos t = y + 2$$

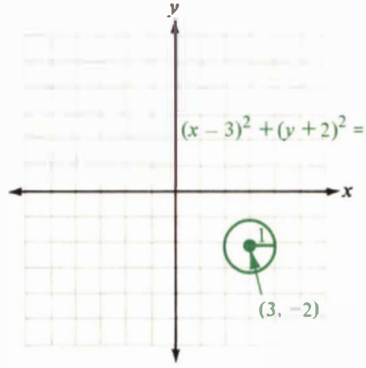
PARAMETRIC EQUATIONS [6.4]

When the coordinates of point (x, y) are described separately by two equations of the form $x = f(t)$ and $y = g(t)$, then the two equations are called *parametric equations* and t is called the parameter. One way to graph a set of points (x, y) that are given in terms of the parameter t is to eliminate the parameter and obtain an equation in just x and y that gives the same set of points (x, y).

Substituting these expressions into the Pythagorean identity, we have

$$(x - 3)^2 + (y + 2)^2 = 1$$

which is the equation of a circle with a radius of 1 and center at $(3, -2)$.



GRAPHING EQUATIONS OF THE FORM $Y = A \sin X + B \cos X$ [6.4]

We graph equations of the form $y = a \sin x + b \cos x$ by multiplying and dividing the right side by $\sqrt{a^2 + b^2}$. Two substitutions are made for $a/\sqrt{a^2 + b^2}$ and $b/\sqrt{a^2 + b^2}$ and the equation that results is then written in the form $y = A \sin(Bx + C)$ using a sum formula.

5. Graph one complete cycle of $y = \sin x + \cos x$.

Multiplying and dividing the right side of our original equation by $\sqrt{2}$, gives us

$$y = \sqrt{2}\left(\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x\right)$$

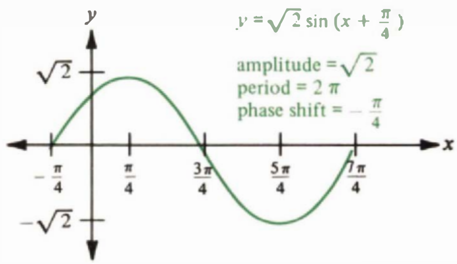
Next we substitute $\cos \pi/4$ for the first $1/\sqrt{2}$ and $\sin \pi/4$ for the second $1/\sqrt{2}$.

$$y = \sqrt{2}\left(\sin x \cos \frac{\pi}{4} + \cos x \sin \frac{\pi}{4}\right)$$

Writing the right side in the form $\sin(A + B)$ we have

$$y = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$$

the graph of which is a sine curve with amplitude $\sqrt{2}$, period 2π, and phase shift $-\pi/4$.



Chapter 6 Test

Find all solutions between 0° and 360°, including 0° and 360° if they are solutions.

1. $2 \sin \theta - 1 = 0$
2. $\sqrt{3} \tan \theta + 1 = 0$
3. $\cos \theta - 2 \sin \theta \cos \theta = 0$
4. $\tan \theta - 2 \cos \theta \tan \theta = 0$
5. $4 \cos \theta - 2 \sec \theta = 0$
6. $2 \sin \theta - \csc \theta = 1$
7. $\sin \dfrac{\theta}{2} + \cos \theta = 0$
8. $\cos \dfrac{\theta}{2} - \cos \theta = 0$
9. $2 \cos^2 2\theta - 3 \cos 2\theta = -1$
10. $4 \sin^2 2\theta - 1 = 0$
11. $\sin \theta + \cos \theta = 1$
12. $\sin \theta - \cos \theta = 1$

Find all solutions for the following equations. Write your answers in radians using exact values.

13. $\cos 2x - 3 \cos x = -2$
14. $\sqrt{3} \sin x - \cos x = 0$
15. $\sin 2x \cos x + \cos 2x \sin x = -1$
16. $\sin^3 4x = 1$

Find all solutions between 0° and 360° to the nearest tenth of a degree, including 0° and 360° if they are solutions.

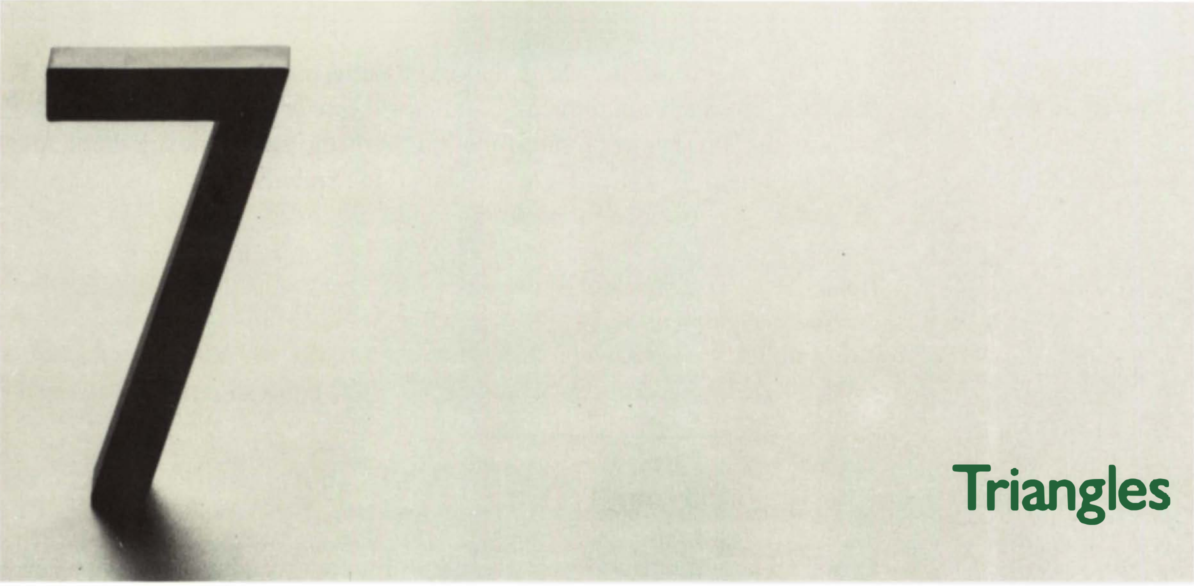
17. $5 \sin^2 \theta - 3 \sin \theta = 2$
18. $4 \cos^2 \theta - 4 \cos \theta = 2$

Eliminate the parameter t from each of the following and then sketch the graph:

19. $x = 3 \cos t \quad y = 3 \sin t$
20. $x = \sec t, \quad y = \tan t$
21. $x = 3 + 2 \sin t \quad y = 1 + 2 \cos t$
22. $x = 3 \cos t - 3 \quad y = 3 \sin t + 1$

Write each equation in terms of a single trigonometric function and then graph using amplitude, period, and phase shift.

23. $y = \sin x - \cos x$
24. $y = \sin x + \sqrt{3} \cos x$



Triangles

To the student:

In Chapter 2 we gave a definition of the six trigonometric functions of an acute angle in a right triangle. Since then, we have used this definition in a number of situations to solve for the different parts of right triangles. In this chapter, we are going to extend our work with triangles to include those that are not necessarily right triangles. We begin in Section 7.1 by deriving a formula called the *law of sines* that gives the relationship between the sides and angles in any triangle. We then use it to solve triangles in which we are given two angles and one side. In Section 7.2, we solve triangles in which we are given two sides and one angle that is not included between the two sides. In some cases, we will find that these triangles can be written in more than one way, and other times we will find that no triangle exists that fits the given information. In Section 7.3, we derive a second formula that relates the sides and angles of any triangle. This formula is called the *law of cosines* and is used to solve for the missing parts of triangles in which we are given two sides and the angle included between them or all three sides. We will end this chapter in Section 7.4 with a number of formulas for finding the area of a triangle.

To be successful in this chapter, you should have a good working knowledge of the material we covered in Chapters 2 and 3 and of the Pythagorean identity we derived in Chapter 5. You also need to be proficient at using a calculator or the tables in the back of the book to find the value of a trigonometric function or to find an angle when given one of its trigonometric functions.

7.1 The Law of Sines

There are many relationships that exist between the sides and angles in a triangle. One such relationship is called ***the law of sines*** and it states that the ratio of the sine of an angle to the length of the side opposite that angle is constant in any triangle. Here it is stated in symbols:

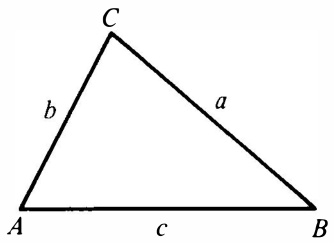


Figure 1

Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

or, equivalently,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Proof The altitude h of the triangle in Figure 2 can be written in terms of sin A or sin B depending on which of the two right triangles we are referring to:

$$\sin A = \frac{h}{b} \qquad \sin B = \frac{h}{a}$$

$$h = b \sin A \qquad h = a \sin B$$

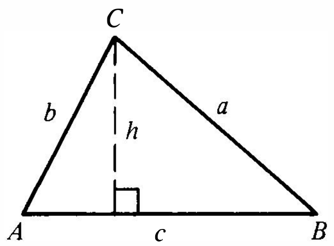


Figure 2

since h is equal to itself, we have

$$h = h$$

$$b \sin A = a \sin B$$

$$\frac{b \sin A}{ab} = \frac{a \sin B}{ab} \quad \text{Divide both sides by } ab$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} \quad \text{Divide out common factors}$$

If we do the same kind of thing with the altitude that extends from A, we will have the third ratio in the law of sines, $(\sin C)/c$, equal to the two ratios above.

Note that the derivation of the law of sines will proceed in the same manner if triangle ABC contains an obtuse angle, as in Figure 3.

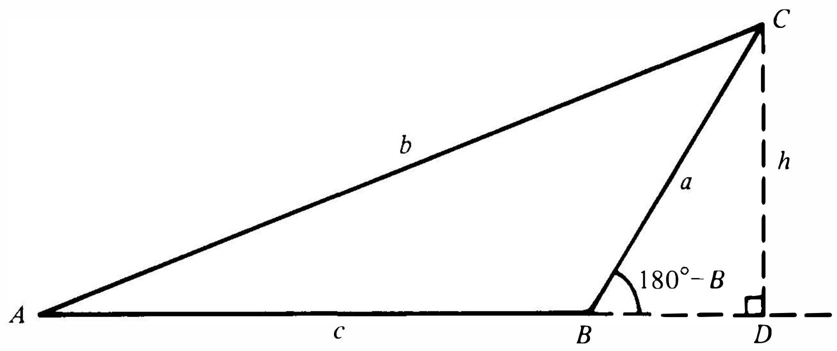


Figure 3

In triangle BDC we have

$$\sin(180° - B) = \frac{h}{a}$$

$$\begin{aligned} \text{but, } \sin(180° - B) &= \sin 180° \cos B - \cos 180° \sin B \\ &= (0)\cos B - (-1)\sin B \\ &= \sin B \end{aligned}$$

So, $\sin B = h/a$, which is the result we obtained previously. Using triangle ADC we have $\sin A = h/b$. As you can see, these are the same two expressions we began with when deriving the law of sines for acute triangle in Figure 2. From this point on, the derivation would match our previous derivation.

We can use the law of sines to find missing parts of triangles in which we are given two angles and a side.

Two Angles and One Side

In our first example, we are given two angles and the side opposite one of them. (You may recall that in geometry these were the parts we needed equal in two triangles in order to prove them congruent using the AAS theorem.)

▼ Example 1 In triangle ABC, $A = 30°$, $B = 70°$, and $a = 8$ centimeters. Find the length of side c.

Solution We begin by drawing a picture of triangle ABC (it does not have to be accurate) and labeling it so that the information we have been given is showing.

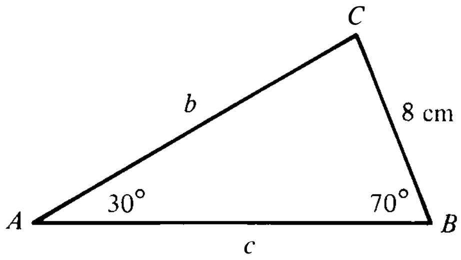


Figure 4

When we use the law of sines, we must have one of the ratios given to us. In this case, since we are given a and A, we have the ratio $a/(\sin A)$. To solve for c, we need to first find angle C. Since the sum of the angles in any triangle is 180°, we have

$$\begin{aligned} C &= 180° - (A + B) \\ &= 180° - (30° + 70°) \\ &= 80° \end{aligned}$$

To find side c, we use the following two ratios given in the law of sines.

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

To solve for c, we multiply both sides by $\sin C$ and then substitute.

$c = \dfrac{a \sin C}{\sin A}$	Multiply both sides by $\sin C$
$= \dfrac{8 \sin 80°}{\sin 30°}$	Substitute in given values
$= \dfrac{8(0.9848)}{0.5000}$	Table or calculator
$= 16$ centimeters	Rounded to the nearest integer ▲

Note The equal sign in the third line above should actually be replaced by the *approximately equal to* symbol, $\cong$, since the decimal 0.9848 is an approximation to sin 80°. (Remember, most of the trigonometric functions we look up in our tables are irrational numbers.) In this chapter, we will use an equal sign in the solutions to the majority of our examples, even when the $\cong$ symbol would be more appropriate, in order to make the examples a little easier to follow.

In our next example, we are given two angles and the side included between them (ASA) and are asked to find all the missing parts.

▼ **Example 2** Solve triangle ABC if $B = 34°$, $C = 82°$, and $a = 5.6$ centimeters.

Solution We begin by finding angle A so that we have one of the ratios in the law of sines completed.

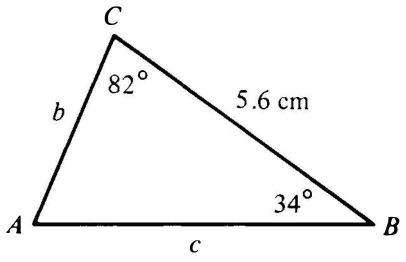


Figure 5

Angle A

$$\begin{aligned} A &= 180° - (B + C) \\ &= 180° - (34° + 82°) \\ &= 64° \end{aligned}$$

Side b

If $\dfrac{b}{\sin B} = \dfrac{a}{\sin A}$

then	$b = \dfrac{a \sin B}{\sin A}$	Multiply both sides by sin B
	$= \dfrac{5.6 \sin 34°}{\sin 64°}$	Substitute in given values
	$= \dfrac{5.6(0.5592)}{0.8988}$	Tables or calculators
	$= 3.5$ centimeters	To the nearest tenth

Side c

If $\dfrac{c}{\sin C} = \dfrac{a}{\sin A}$

then	$c = \dfrac{a \sin C}{\sin A}$	Multiply both sides by sin C
	$= \dfrac{5.6 \sin 82°}{\sin 64°}$	Substitute in given values
	$= \dfrac{5.6(0.9903)}{0.8988}$	Tables or calculators
	$= 6.2$ centimeters	To the nearest tenth ▲

The law of sines, along with some fancy electronic equipment, was used to obtain the results of some of the field events in one of the recent Olympic Games.

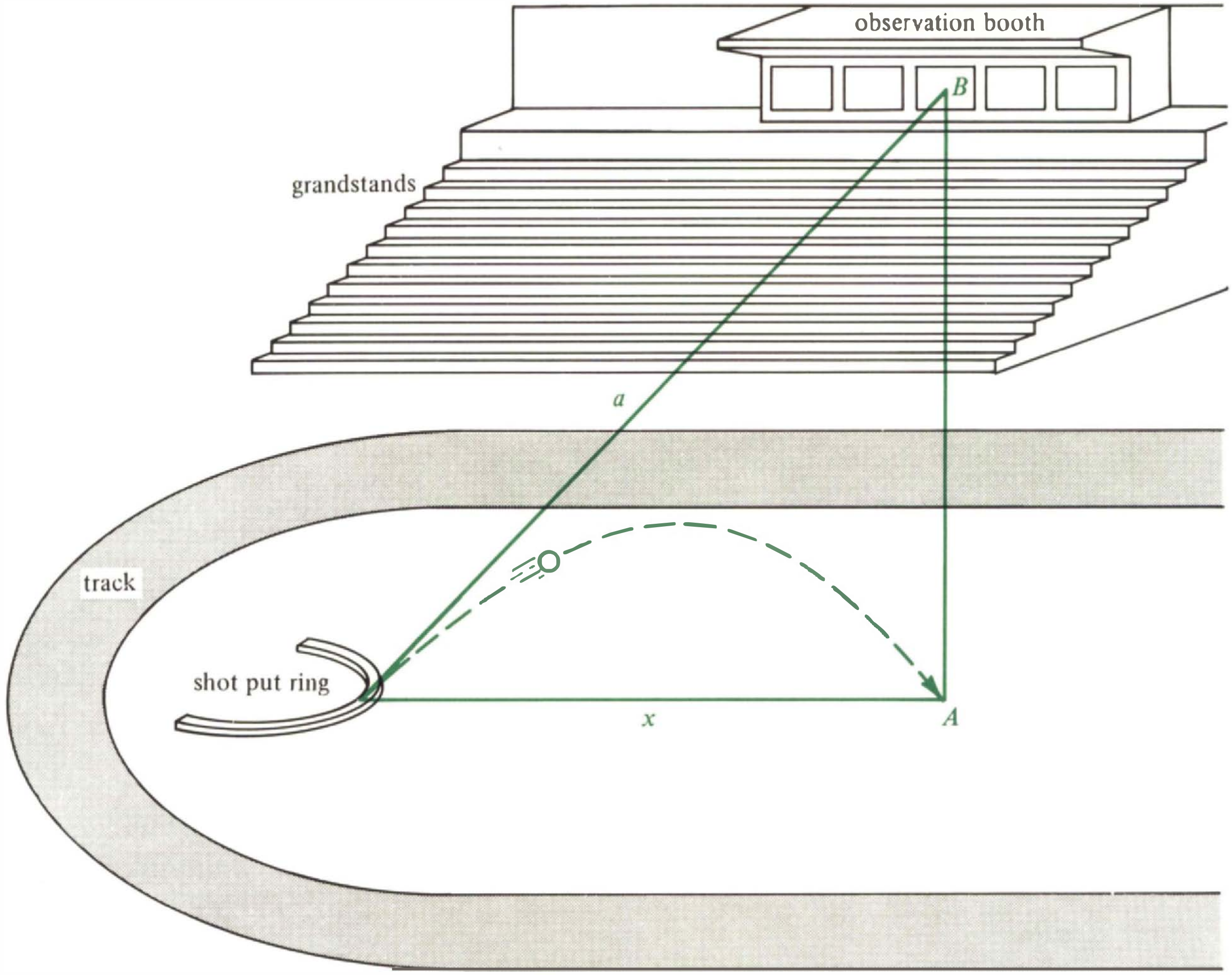


Figure 6

Figure 6 is a diagram of a shot put ring. The shot is tossed (put) from the left and lands at A. A small electronic device is then placed at A (there is usually a dent in the ground where the shot lands, so it is easy to find where to place the device). The device at A sends a signal to a booth in the stands that gives the measures of angles A and B. The distance a is found ahead of time. To find the distance x, the law of sines is used.

$$\frac{x}{\sin B} = \frac{a}{\sin A}$$

$$\text{or} \quad x = \frac{a \sin B}{\sin A}$$

Example 3 Find x in Figure 6 if $a = 562$ feet, $B = 5.7°$, and $A = 85.3°$.

Solution

$$x = \frac{a \sin B}{\sin A}$$

$$= \frac{562 \sin 5.7°}{\sin 85.3°}$$

$$= \frac{562(0.0993)}{0.9966}$$

$$= 56.0 \text{ feet}$$

Problem Set 7.1

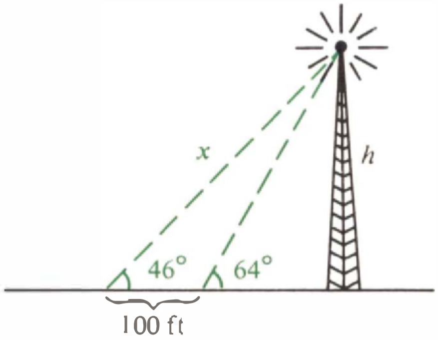
Each problem that follows refers to triangle ABC. Round your answers to the nearest whole number.

1. If $A = 40°$, $B = 60°$, and $a = 12$ centimeters, find b.
2. If $A = 80°$, $B = 30°$, and $b = 14$ centimeters, find a.
3. If $B = 120°$, $C = 20°$, and $c = 28$ inches, find b.
4. If $B = 110°$, $C = 40°$, and $b = 18$ inches, find c.
5. If $A = 10°$, $C = 100°$, and $a = 24$ yards, find c.
6. If $A = 5°$, $C = 125°$, and $c = 510$ yards, find a.
7. If $A = 50°$, $B = 60°$, and $a = 36$ kilometers, find C and then find c.
8. If $B = 40°$, $C = 70°$, and $c = 42$ kilometers, find A and then find a.
9. If $A = 52°$, $B = 48°$, and $c = 14$ centimeters, find C and then find a.
10. If $A = 33°$, $C = 82°$, and $b = 18$ centimeters, find B and then find c.

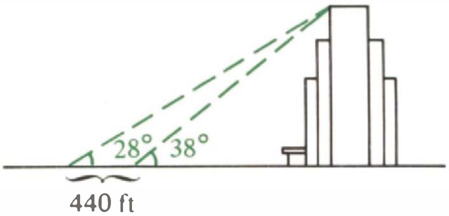
The information below refers to triangle ABC. In each case find all the missing parts.

11. $A = 42.5°$, $B = 71.4°$, $a = 210$ inches
12. $A = 110.4°$, $C = 21.8°$, $c = 240$ inches
13. $A = 46°$, $B = 95°$, $c = 6.8$ meters
14. $B = 57°$, $C = 31°$, $a = 7.3$ meters
15. $A = 43°30'$, $C = 120°30'$, $a = 3.48$ feet
16. $B = 14°20'$, $C = 75°40'$, $b = 2.72$ feet
17. $B = 13.4°$, $C = 24.8°$, $a = 315$ centimeters
18. $A = 105°$, $B = 45°$, $c = 630$ centimeters
19. $A = 27°40'$, $C = 31°20'$, $b = 0.822$ kilometers
20. $B = 124°30'$, $C = 16°30'$, $a = 0.308$ kilometers
21. In triangle ABC, $A = 30°$, $b = 20$ feet, and $a = 2$ feet. Show that it is impossible to solve this triangle by using the law of sines to find $\sin B$.
22. In triangle ABC, $A = 40°$, $b = 20$ feet, and $a = 18$ feet. Use the law of sines to find $\sin B$ and then give two possible values for B.

23. A man standing near a radio station antenna observes that the angle of inclination to the top of the antenna is 64°. He then walks 100 feet further away and observes the angle of inclination to the top of the antenna to be 46°. Find the height of the antenna to the nearest foot. (*Hint*: Find x first.)



24. A person standing on the street looks up to the top of a building and finds the angle of inclination is 38°. She then walks one block further away (440 feet) and finds the angle of inclination to the top of the building is now 28°. How far away from the building is she when she makes her second observation?



25. A man is flying in a hot-air balloon in a straight line at a constant rate of 5 feet per second, while keeping it at a constant altitude. As he approaches the parking lot of a market, he notices the angle of depression from his balloon to a friend's car in the parking lot is 35°. A minute and a half later, after flying directly over his friends car, he looks back to see his friend getting into the car and observes the angle of depression to be 36°. At that time, what is the distance between him and his friend? (Give your answer to the nearest foot.)

26. A woman entering an outside glass elevator on the ground floor of a hotel building glances up to the top of the building across the street and notices the angle of elevation to be 48°. She rides the elevator up three floors (60 feet) and finds the angle of elevation to the top of the building across the street is 32°. How tall is the building across the street? (Give your answer to the nearest foot.)

Review Problems

The problems below review material we covered in Section 3.1.

Find all values of θ between 0° and 360° for which each of the following is true. Write your answer to the nearest tenth of a degree.

27. $\sin\theta = 0.7380$
28. $\sin\theta = 0.7965$
29. $\cos\theta = 0.7380$
30. $\cos\theta = 0.7965$
31. $\sin\theta = 0.9668$
32. $\sin\theta = 0.2351$

7.2 The Ambiguous Case

In this section, we will extend the law of sines to solve triangles in which we are given two sides and the angle opposite one of the given sides.

Example 1 Find angle B in triangle ABC if $a = 2$, $b = 6$, and $A = 30°$.

Solution Applying the law of sines we have

$$\sin B = \frac{b \sin A}{a}$$
$$= \frac{6 \sin 30°}{2}$$
$$= \frac{6(0.5000)}{2}$$
$$= 1.5$$

Since $\sin B$ can never be larger than 1, no triangle exists for which $a = 2$, $b = 6$, and $A = 30°$. (You may recall from geometry that there was no congruence theorem SSA.) Figure 1 illustrates what went wrong here.

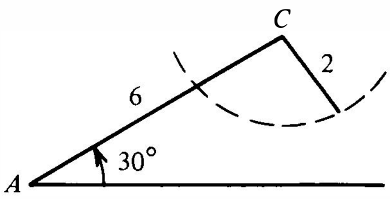


Figure 1

When we are given two sides and an angle opposite one of them (SSA), we have several possibilities for the triangle or triangles that result. As was the case in Example 1, one of the possibilities is that no triangle will fit the given information. If side a in Example 1 had been longer than the altitude drawn from vertex C but shorter than side b, we would have had two triangles that fit the given information, as shown in Figure 2. On the other hand, if side a in triangle ABC of Example 1 had been longer than side b, we would have had only one triangle that fit the given information, as illustrated in Figure 3.

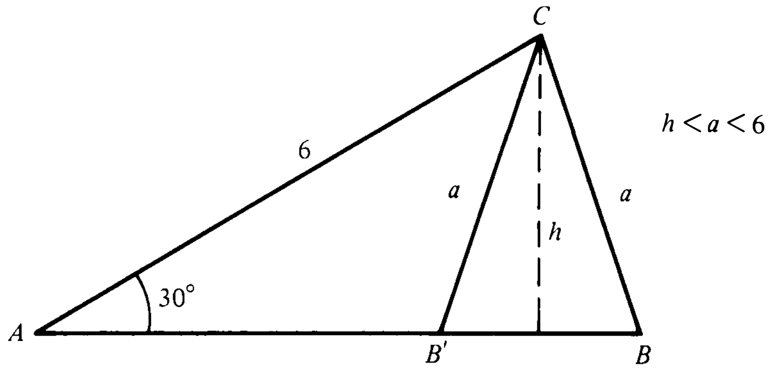


Figure 2

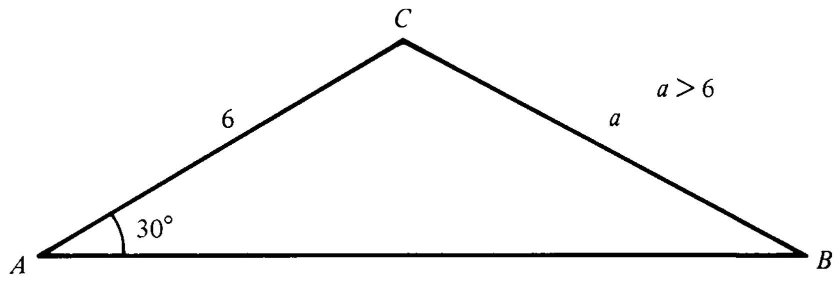


Figure 3

Because of the different possibilities that arise in solving a triangle in which we are given two sides and an angle opposite one of the given sides, we call this situation the *ambiguous case*.

▼ **Example 2** Find the missing parts in triangle ABC if $a = 54$ centimeters, $b = 62$ centimeters, and $A = 40°$.

Solution First we solve for sin B with the law of sines.

Angle B

$$\sin B = \frac{b \sin A}{a}$$

$$= \frac{62 \sin 40°}{54}$$

$$= 0.7380$$

Now, since sin B is positive for any angle in quadrants I or II, we have two possibilities. We will call one of them B and the other B'.

$$B = 48° \quad \text{or} \quad B' = 180° - 48° = 132°$$

We have two different triangles that can be found with $a = 54$ centimeters, $b = 62$ centimeters, and $A = 40°$. Figure 4 shows both of them. One is labeled ABC, while the other is labeled $AB'C$.

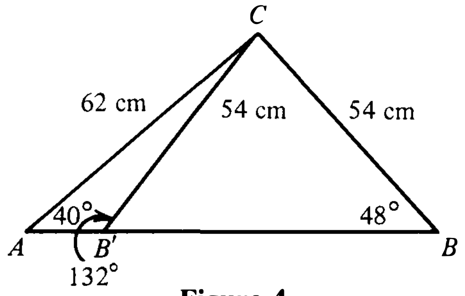


Figure 4

Angles C and C'

Since there are two values for B, we have two values for C.

$$\begin{aligned} C &= 180 - (A + B) \\ &= 180 - (40° + 48°) \\ &= 92° \end{aligned} \quad \text{and} \quad \begin{aligned} C' &= 180 - (A + B') \\ &= 180 - (40° + 132°) \\ &= 8° \end{aligned}$$

Sides c and c'

$$\begin{aligned} c &= \frac{a \sin C}{\sin A} \\ &= \frac{54 \sin 92°}{\sin 40°} \\ &= 84 \text{ centimeters} \end{aligned} \quad \text{and} \quad \begin{aligned} c' &= \frac{a \sin C'}{\sin A} \\ &= \frac{54 \sin 8°}{\sin 40°} \\ &= 12 \text{ centimeters} \end{aligned}$$

Figure 5 shows both triangles.

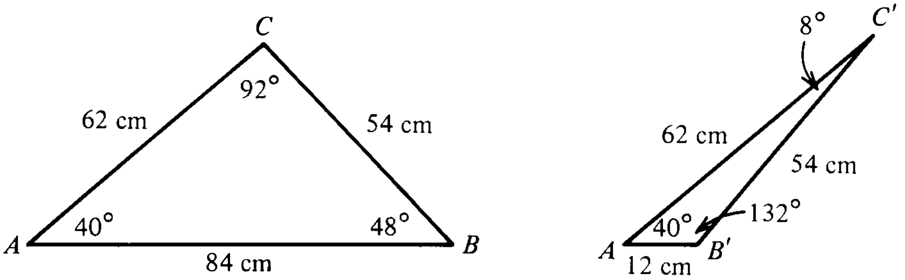


Figure 5

▲

▼ **Example 3** Find the missing parts of triangle ABC if $C = 35.4°$, $a = 205$ feet, and $c = 314$ feet.

Solution Applying the law of sines, we find sin A.

Angle A

$$\sin A = \frac{a \sin C}{c} = \frac{205 \sin 35.4°}{314} = 0.3782$$

Since sin A is positive in quadrants I and II, we have two possible values for A.

$$A = 22.2° \quad \text{and} \quad A' = 180° - 22.2° = 157.8°$$

The second possibility, $A' = 157.8°$, will not work however, since C is already 35.4° and therefore

$$C + A' = 35.4° + 157.8° = 193.2°$$

which is larger than 180°. This result indicates that there is exactly one triangle that fits the description given in Example 3. In that triangle

$$A = 22.2°$$

Angle B

$$B = 180° - (35.4° + 22.2°) = 122.4°$$

Side b

$$b = \frac{c \sin B}{\sin C} = \frac{314 \sin 122.4°}{\sin 35.4°} = 458 \text{ feet}$$

Figure 6 is a diagram of this triangle.

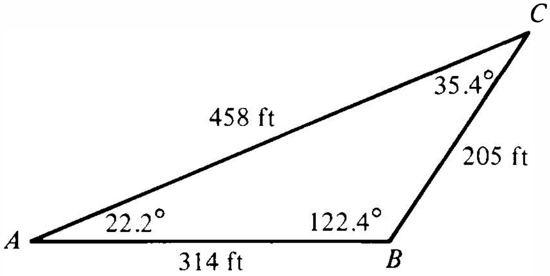


Figure 6

The different cases that can occur when we solve the kinds of triangles we have been given in Examples 1, 2, and 3 become apparent in the process of solving for the missing parts. Nevertheless, we can make a table that shows the set of conditions under which we will have 1, 2, or no triangles in the ambiguous case.

In Table 1, we are assuming that we are given angle A and sides a and b in triangle ABC, and that h is the altitude from vertex C.

Table 1

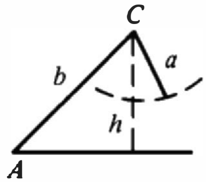
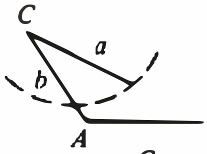
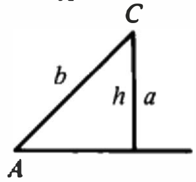
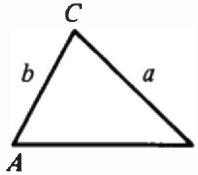
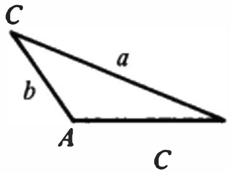
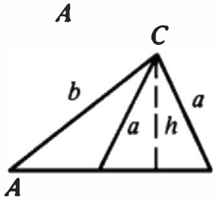
Conditions	Number of Triangles	Diagram
$A < 90°$ and $a < h$	0	
$A > 90°$ and $a < b$	0	
$A < 90°$ and $a = h$	1	
$A < 90°$ and $a \geq b$	1	

Table 1 (continued)

Conditions	Number of Triangles	Diagram
$A > 90°$ and $a > b$	1	
$A < 90°$ and $h < a < b$	2	

Problem Set 7.2

For each triangle below, solve for B and use the results to explain why the triangle has the given number of solutions.

1. $A = 30°$, $b = 40$ feet, $a = 10$ feet; no solution
2. $A = 150°$, $b = 30$ feet, $a = 10$ feet; no solution
3. $A = 120°$, $b = 20$ centimeters, $a = 30$ centimeters; one solution
4. $A = 30°$, $b = 12$ centimeters, $a = 6$ centimeters; one solution
5. $A = 60°$, $b = 18$ meters, $a = 16$ meters; two solutions
6. $A = 20°$, $b = 40$ meters, $a = 30$ meters; two solutions

Find all solutions to each of the following triangles:

7. $A = 38°$, $a = 41$ feet, $b = 54$ feet
8. $A = 43°$, $a = 31$ feet, $b = 37$ feet
9. $A = 112.2°$, $a = 43.8$ centimeters, $b = 22.3$ centimeters
10. $A = 124.3°$, $a = 27.3$ centimeters, $b = 50.2$ centimeters
11. $C = 27°50'$, $c = 347$ meters, $b = 425$ meters
12. $C = 51°30'$, $c = 707$ meters, $b = 821$ meters
13. $B = 45°10'$, $b = 1.79$ inches, $c = 1.12$ inches
14. $B = 62°40'$, $b = 6.78$ inches, $c = 3.48$ inches
15. $B = 118°$, $b = 0.68$ centimeters, $a = 0.92$ centimeters
16. $B = 30°$, $b = 4.2$ centimeters, $a = 8.4$ centimeters
17. $A = 142°$, $b = 2.9$ yards, $a = 1.4$ yards
18. $A = 65°$, $b = 7.6$ yards, $a = 7.1$ yards
19. $C = 26.8°$, $c = 36.8$ kilometers, $b = 36.8$ kilometers
20. $C = 73.4°$, $c = 51.1$ kilometers, $b = 92.4$ kilometers
21. A 50 foot wire running from the top of a tent pole to the ground makes an angle of 58° with the ground. If the length of the tent pole is 44 feet, how far is it from the bottom of the tent pole to the point where the wire is fastened to the ground?

22. A hot-air balloon is held at a constant altitude by two ropes that are anchored to the ground. One rope is 120 feet long and makes an angle of 65° with the ground. The other rope is 115 feet long. What is the distance between the points on the ground at which the two ropes are anchored?

Review Problems The problems that follow review material we covered in Sections 5.3 and 5.4.

Let A terminate in quadrant I with $\sin A = 4/5$ and find

23. $\sin 2A$

24. $\cos 2A$

25. $\sin \dfrac{A}{2}$

26. $\cos \dfrac{A}{2}$

27. $\tan 2A$

28. $\tan \dfrac{A}{2}$

29. Use a half-angle formula to find the exact value of sin 15°.

30. Use a half-angle formula to find the exact value of cos 15°.

7.3 The Law of Cosines

In this section, we will derive another relationship that exists between the sides and angles in any triangle. It is called *the law of cosines* and is stated like this

Law of Cosines

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

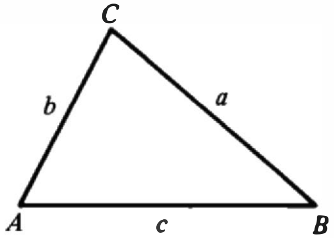


Figure 1

Derivation

To derive the formulas stated in the law of cosines, we apply the Pythagorean theorem and some of our basic trigonometric identities. Applying the Pythagorean theorem to right triangle BCD in Figure 2, we have

$$a^2 = (c - x)^2 + h^2$$
$$= c^2 - 2cx + x^2 + h^2$$

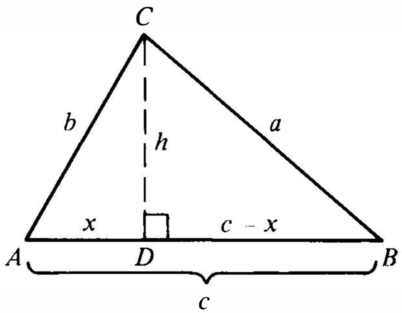


Figure 2

But, from right triangle ACD, we have $x^2 + h^2 = b^2$, so

$$\begin{aligned} a^2 &= c^2 - 2cx + b^2 \\ &= b^2 + c^2 - 2cx \end{aligned}$$

Now, since $\cos A = x/b$ we have $x = b \cos A$, or

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Applying the same sequence of substitutions and reasoning to the right triangles formed by the altitudes from vertices A and B will give us the other two formulas listed in the law of cosines.

We can use the law of cosines to solve triangles in which we are given two sides and the angle included between them (SAS) or to solve triangles in which we are given all three sides (SSS).

Two Sides and the Included Angle

▼ **Example 1** Find the missing parts of triangle ABC if $A = 60°$, $b = 20$ inches, and $c = 30$ inches.

Solution The solution process will include the use of both the law of cosines and the law of sines. We begin by using the law of cosines to find a.

Side a

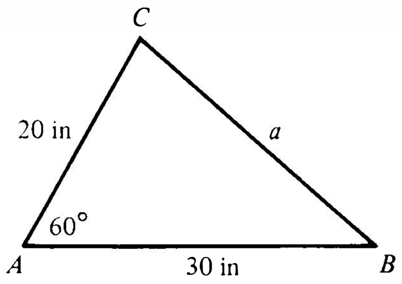


Figure 3

$a^2 = b^2 + c^2 - 2\,bc \cos A$	Law of cosines
$= 20^2 + 30^2 - 2(20)(30)\cos 60°$	Substitute in given values
$= 400 + 900 - 1200(0.5000)$	Table or calculator
$a^2 = 700$	
$a = 26$ inches	To the nearest integer

Now that we have a, we can use the law of sines to solve for either B or C. When we have a choice of angles to solve for, and we are using the law of sines to do so, it is usually best to solve for the smaller angle. Since side b is smaller than side c, angle B will be smaller than angle C.

Angle B

$$\sin B = \frac{b \sin A}{a}$$

$$= \frac{20 \sin 60°}{26}$$

$$\sin B = 0.6662$$

$$\text{so} \quad B = 42° \qquad \text{To the nearest degree}$$

Note that we don't have to check $B' = 180° - 42° = 138°$ because we know B is an acute angle since it is smaller than angles A and C.

Angle C

$$\begin{aligned} C &= 180° - (A + B) \\ &= 180° - (60° + 42°) \\ &= 78° \end{aligned}$$

▲

▼ **Example 2** The diagonals of a parallelogram are 24.2 centimeters and 35.4 centimeters and intersect at an angle of 65.5°. Find the length of the shorter side of the parallelogram.

Solution A diagram of the parallelogram is shown in Figure 4. The variable x represents the length of the shorter side. Note also that, since the diagonals bisect each other, we labeled the length of half of each.

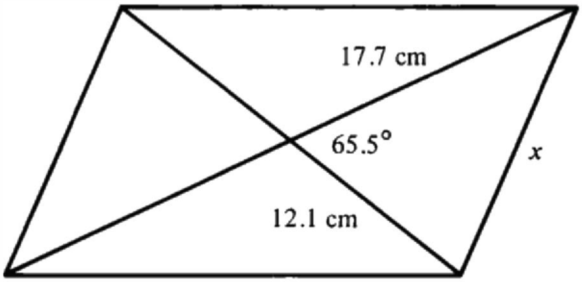


Figure 4

$$\begin{aligned} x^2 &= (12.1)^2 + (17.7)^2 - 2(12.1)(17.7)\cos 65.5° \\ x^2 &= 282.07 \\ x &= 16.8 \text{ centimeters} \qquad \text{To the nearest tenth} \end{aligned}$$

▲

Next we will see how the law of cosines can be used to find the missing parts of a triangle in which all three sides are given.

Three Sides

To use the law of cosines to solve a triangle in which we are given all three sides, it is convenient to rewrite the equations with the cosines isolated on one side. Here is an equivalent form of the law of cosines.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Here is how we arrived at the first of these formulas.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$b^2 + c^2 - 2bc \cos A = a^2$	Exchange sides
$-2bc \cos A = -b^2 - c^2 + a^2$	Add $-b^2$ and $-c^2$ to both sides
$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$	Divide both sides by $-2bc$

▼ **Example 3** Solve triangle ABC if $a = 34$ kilometers, $b = 20$ kilometers, and $c = 18$ kilometers.

Solution We will use the law of cosines to solve for one of the angles and then use the law of sines to find one of the remaining angles. Since there is never any confusion as to whether an angle is acute or obtuse if we have its cosine (the cosine of an obtuse angle is negative) it is best to solve for the largest angle first. Since the longest side is a, we solve for A first.

Angle A

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{20^2 + 18^2 - 34^2}{(2)(20)(18)}$$

$$\cos A = -0.6000$$

$$\text{so} \quad A = 127° \qquad \text{To the nearest degree}$$

Now we use the law of sines to find angle C.

Angle C

$$\sin C = \frac{c \sin A}{a}$$

$$= \frac{18 \sin 127°}{34}$$

$$\sin C = 0.4234$$

$$\text{so} \quad C = 25°$$

Angle B

$$\begin{aligned} B &= 180° - (A + C) \\ &= 180° - (127° + 25°) \\ &= 28° \end{aligned}$$

Example 4 A plane is flying with an air speed of 185 miles per hour with bearing 120°. The wind currents are running at a constant 28 miles per hour with a bearing of 165°. Find the true direction and speed of the plane with respect to the ground.

Solution Figure 5 is a diagram of the situation with the vector **V** representing the air speed and direction of the plane and **W** representing the speed and direction of the wind currents.

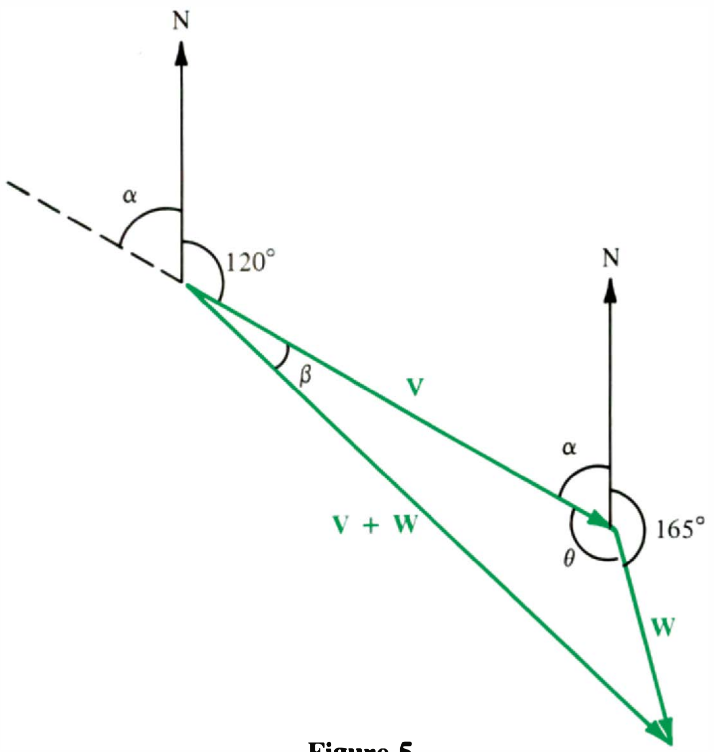


Figure 5

From Figure 5, $\alpha = 180° - 120° = 60°$ and $\theta = 360° - (\alpha + 165°) = 135°$.

The magnitude of $\mathbf{V} + \mathbf{W}$ can be found from the law of cosines

$$\begin{aligned} |\mathbf{V} + \mathbf{W}|^2 &= |\mathbf{V}|^2 + |\mathbf{W}|^2 - 2|\mathbf{V}||\mathbf{W}| \cos \theta \\ &= 185^2 + 28^2 - 2(185)(28)\cos 135° \\ &\cong 42{,}335 \\ \text{so } |\mathbf{V} + \mathbf{W}| &= 206 \text{ miles per hour} \end{aligned}$$

To find the direction of $\mathbf{V} + \mathbf{W}$, we must first find β using the law of sines.

$$\frac{\sin \beta}{28} = \frac{\sin \theta}{206}$$

$$\sin \beta = \frac{28 \sin 135°}{206}$$

$$= 0.0961$$

$$\text{so } \beta = 5.5°$$

The bearing of $\mathbf{V} + \mathbf{W}$ is $120° + \beta = 120° + 5.5° = 125.5°$. The speed of the plane with respect to the ground is 206 miles per hour and it is actually on a course with a bearing of 125.5°. ▲

Problem Set 7.3

Each problem below refers to triangle *ABC*.

1. If $a = 100$ inches, $b = 60$ inches, and $C = 60°$, find c.
2. If $a = 100$ inches, $b = 60$ inches, and $C = 120°$, find c.
3. If $a = 5$ yards, $b = 6$ yards, and $c = 8$ yards, find the largest angle.
4. If $a = 10$ yards, $b = 14$ yards, and $c = 8$ yards, find the largest angle.
5. If $b = 4.2$ meters, $c = 6.8$ meters, and $A = 116°$, find a.
6. If $a = 3.7$ meters, $c = 6.4$ meters, and $B = 23°$, find b.
7. If $a = 38$ centimeters, $b = 10$ centimeters, and $c = 31$ centimeters, find the largest angle.
8. If $a = 51$ centimeters, $b = 24$ centimeters, and $c = 31$ centimeters, find the largest angle.

Solve each triangle below.

9. $a = 50$ centimeters, $b = 70$ centimeters, $C = 60°$
10. $a = 10$ centimeters, $b = 12$ centimeters, $C = 120°$
11. $a = 4$ inches, $b = 6$ inches, $c = 8$ inches. (*Remember*: Solve for the largest angle first.)
12. $a = 5$ inches, $b = 10$ inches, $c = 12$ inches
13. $a = 410$ meters, $c = 340$ meters, $B = 151.5°$

14. $a = 76.3$ meters, $c = 42.8$ meters, $B = 16.3°$
15. $a = 0.048$ yards, $b = 0.063$ yards, $c = 0.075$ yards
16. $a = 48$ yards, $b = 75$ yards, $c = 63$ yards
17. $b = 0.923$ kilometers, $c = 0.387$ kilometers, $A = 43°20'$
18. $b = 63.4$ kilometers, $c = 75.2$ kilometers, $A = 124°40'$
19. $a = 4.38$ feet, $b = 3.79$ feet, $c = 5.22$ feet
20. $a = 832$ feet, $b = 623$ feet, $c = 345$ feet
21. Use the law of cosines to show that, if $C = 90°$, then $a^2 = b^2 + c^2$.
22. Use the law of cosines to show that, if $a^2 = b^2 + c^2$, then $C = 90°$.
23. The diagonals of a parallelogram are 56 inches and 34 inches and intersect at an angle of 120°. Find the length of the shorter side.
24. The diagonals of a parallelogram are 14 meters and 16 meters and intersect at an angle of 60°. Find the length of the longer side.
25. Two planes leave an airport at the same time. Their speeds are 130 miles per hour and 150 miles per hour, and the angle between their courses is 36°. How far apart are they after 1.5 hours?
26. Two ships leave the harbor at the same time. One ship is traveling at 14 miles per hour on a course with a bearing of S 13° W, while the other is traveling at 12 miles per hour on a course with a bearing of N 75° E. How far apart are they after three hours?
27. A plane is flying with an air speed of 160 miles per hour with a bearing of 150°. The wind currents are running at 35 miles per hour with a bearing of 165°. Use vectors to find the true direction and speed of the plane with respect to the ground.
28. Vector **U** forms an angle of 22.3° with the positive x-axis and has a magnitude of 4.82. Vector **V** forms an angle of 63.8° with the positive x-axis and has a magnitude of 2.41. Find the magnitude and direction of the resultant vector **V** + **W**.

Review Problems The problems below review material we covered in Sections 6.1, 6.2, and 6.3

Find all solutions if $0° \le \theta \le 360°$.

29. $2 \sin \theta = 1$
30. $2 \cos \theta = \sqrt{3}$
31. $2 \sin^2 \theta - \sin \theta - 1 = 0$
32. $4 \cos^2 \theta + 4 \cos \theta + 1 = 0$
33. $\sin 2\theta - \sin \theta = 0$
34. $\cos 2\theta + \sin \theta = 0$
35. $4 \sin \theta - 2 \csc \theta = 0$
36. $4 \cos \theta - 3 \sec \theta = 0$

7.4 The Area of a Triangle

In this section, we will derive three formulas for the area S of a triangle. We will start by deriving the formula used to find the area of a triangle in which two sides and the included angle are given.

Two Sides and the Included Angle

To derive our first formula we begin with the general formula for the area of a triangle

$$S = \frac{1}{2}\text{(base)(height)}$$

The base of triangle ABC in Figure 1 is c and the height is h. So the formula for S becomes, in this case,

$$S = \frac{1}{2}ch$$

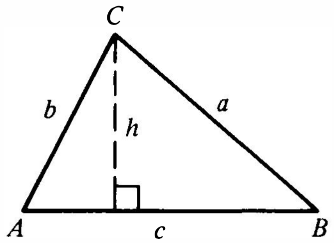


Figure 1

Suppose that, for triangle ABC, we are given the lengths of sides b and c and the measure of angle A. Then we can write sin A as

$$\sin A = \frac{h}{b}$$

or, by solving for h, $\qquad h = b \sin A$

Substituting this expression for h into the formula

$$S = \frac{1}{2}ch$$

we have

$$S = \frac{1}{2}bc \sin A$$

Applying the same kind of reasoning to the heights drawn from A and C, we also have

$$S = \frac{1}{2}ab \sin C$$

$$S = \frac{1}{2}ac \sin B$$

Each of these three formulas indicates that to find the area of a triangle in which we are given two sides and the angle included between them, we multiply half the product of the two sides times the sine of the angle included between them.

▼ **Example 1** Find the area of triangle ABC if $A = 35.1°$, $b = 2.43$ centimeters, and $c = 3.57$ centimeters.

Solution Applying the first formula we derived, we have

$$S = \frac{1}{2} bc \sin A$$

$$= \frac{1}{2}(2.43)(3.57)\sin 35.1°$$

$$= \frac{1}{2}(2.43)(3.57)(0.5750)$$

$$= 2.49 \text{ centimeters}^2 \qquad \text{To three significant digits}$$ ▲

The next area formula we will derive is used to find the area of triangles in which we are given two angles and one side.

Two Angles and One Side

Suppose we were given angles A and B and side a in triangle ABC in Figure 1. We could easily solve for C by subtracting the sum of A and B from 180°.

To find side b, we use the law of sines

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

solving this equation for b would give us

$$b = \frac{a \sin B}{\sin A}$$

Substituting this expression for b into the formula

$$S = \frac{1}{2} ab \sin C$$

$$S = \frac{1}{2} a\left(\frac{a \sin B}{\sin A}\right)\sin C$$

$$= \frac{a^2 \sin B \sin C}{2 \sin A}$$

A similar sequence of steps can be used to derive

$$S = \frac{b^2 \sin A \sin C}{2 \sin B}$$

and

$$S = \frac{c^2 \sin A \sin B}{2 \sin C}$$

The formula we use depends on the side we are given.

▼ **Example 2** Find the area of triangle ABC if $A = 24°10'$, $B = 120°40'$, and $a = 4.25$ feet.

Solution We begin by finding C.

$$\begin{aligned} C &= 180° - (24°10' + 120°40') \\ &= 35°10' \end{aligned}$$

Now, applying the formula

$$S = \frac{a^2 \sin B \sin C}{2 \sin A}$$

with $a = 4.25$, $A = 24°10'$, $B = 120°40'$, and $C = 35°10'$, we have

$$\begin{aligned} S &= \frac{(4.25)^2 (\sin 120°40')(\sin 35°10')}{2 \sin 24°10'} \\ &= \frac{(4.25)^2(0.8601)(0.5760)}{2(0.4094)} \\ &= 10.93 \text{ feet}^2 \end{aligned}$$

▲

Three Sides

The last area formula is called Heron's formula and it is used to find the area of a triangle in which all three sides are known.

Heron's Formula The area of a triangle with sides of length a, b, and c is given by

$$S = \sqrt{s(s-a)(s-b)(s-c)}$$

where s is half the perimeter of the triangle; that is,

$$s = \frac{1}{2}(a + b + c) \quad \text{or} \quad 2s = a + b + c$$

Proof We begin our proof by squaring both sides of the formula

$$S = \frac{1}{2} ab \sin C$$

to obtain

$$S^2 = \frac{1}{4} a^2b^2\sin^2 C$$

Next, we multiply both sides of the equation by $4/a^2 b^2$ to isolate $\sin^2 C$ on the right side.

$$\frac{4S^2}{a^2b^2} = \sin^2 C$$

Replacing $\sin^2 C$ with $1 - \cos^2 C$ and then factoring as the difference of two squares, we have

$$= 1 - \cos^2 C$$
$$= (1 + \cos C)(1 - \cos C)$$

From the law of cosines we know that $\cos C = (a^2 + b^2 - c^2)/2ab$.

$$= \left[1 + \frac{a^2 + b^2 - c^2}{2ab}\right]\left[1 - \frac{a^2 + b^2 - c^2}{2ab}\right]$$
$$= \left[\frac{2ab + a^2 + b^2 - c^2}{2ab}\right]\left[\frac{2ab - a^2 - b^2 + c^2}{2ab}\right]$$
$$= \left[\frac{(a^2 + 2ab + b^2) - c^2}{2ab}\right]\left[\frac{c^2 - (a^2 - 2ab + b^2)}{2ab}\right]$$
$$= \left[\frac{(a + b)^2 - c^2}{2ab}\right]\left[\frac{c^2 - (a - b)^2}{2ab}\right]$$

Now we factor each numerator as the difference of two squares and multiply the denominators.

$$= \frac{[(a + b + c)(a + b - c)][(c + a - b)(c - a + b)]}{4a^2b^2}$$

Now, since $a + b + c = 2s$, it is also true that

$$a + b - c = a + b + c - 2c = 2s - 2c$$
$$c + a - b = a + b + c - 2b = 2s - 2b$$
$$c - a + b = a + b + c - 2a = 2s - 2a$$

Substituting these expressions into our last equation, we have

$$= \frac{2s(2s - 2c)(2s - 2b)(2s - 2a)}{4a^2b^2}$$

Factoring out a 2 from each term in the numerator and showing the left side of our equation along with the right side, we have

$$\frac{4S^2}{a^2b^2} = \frac{16s(s-a)(s-b)(s-c)}{4a^2b^2}$$

Multiplying both sides by $a^2b^2/4$ we have

$$S^2 = s(s-a)(s-b)(s-c)$$

Taking the square root of both sides of the equation we have Heron's formula.

$$S = \sqrt{s(s-a)(s-b)(s-c)}$$

▼ **Example 3** Find the area of triangle ABC if $a = 12$ meters, $b = 14$ meters, and $c = 8$ meters.

Solution We begin by calculating the formula for s, half the perimeter of ABC.

$$\begin{aligned} s &= \frac{1}{2}(12 + 14 + 8) \\ &= 17 \end{aligned}$$

Substituting this value of s into Heron's formula along with the given values of a, b, and c, we have

$$\begin{aligned} S &= \sqrt{17(17-12)(17-14)(17-8)} \\ &= \sqrt{17(5)(3)(9)} \\ &= \sqrt{2{,}295} \\ &= 47.9 \text{ meters}^2 \qquad \text{To three significant digits} \end{aligned}$$

▲

Problem Set 7.4

Each problem below refers to triangle ABC. In each case find the area of the triangle.

1. $a = 50$ centimeters, $b = 70$ centimeters, $C = 60°$.
2. $a = 10$ centimeters, $b = 12$ centimeters, $C = 120°$.
3. $a = 41$ meters, $c = 34$ meters, $B = 151.5°$.
4. $a = 76.3$ meters, $c = 42.8$ meters, $B = 16.3°$.
5. $b = 0.923$ kilometers, $c = 0.387$ kilometers, $A = 43°20'$.
6. $b = 63.4$ kilometers, $c = 75.2$ kilometers, $A = 124°40'$.
7. $A = 46°$, $B = 95°$, $c = 6.8$ meters.
8. $B = 57°$, $C = 31°$, $a = 7.3$ meters.
9. $A = 42.5°$, $B = 71.4°$, $a = 210$ inches.

10. $A = 110.4°$, $C = 21.8°$, $c = 240$ inches.
11. $A = 43°30'$, $C = 120°30'$, $a = 3.48$ feet.
12. $B = 14°20'$, $C = 75°40'$, $b = 2.72$ feet.
13. $a = 4$ inches, $b = 6$ inches, $c = 8$ inches.
14. $a = 5$ inches, $b = 10$ inches, $c = 12$ inches.
15. $a = 4.8$ yards, $b = 6.3$ yards, $c = 7.5$ yards.
16. $a = 48$ yards, $b = 75$ yards, $c = 63$ yards.
17. $a = 4.38$ feet, $b = 3.79$ feet, $c = 5.22$ feet.
18. $a = 8.32$ feet, $b = 6.23$ feet, $c = 3.45$ feet.
19. Find the area of a parallelogram if the angle between two of the sides is 120° and the two sides are 15 inches and the longest side is 12 inches.
20. Find the area of a parallelogram if the two sides measure 24.1 inches and 32.4 inches and the longest diagonal is 31.4 inches.
21. The area of a triangle is 40 centimeters². Find the length of the side included between the angles $A = 30°$ and $B = 50°$.
22. The area of a triangle is 80 inches². Find the length of the side included between $A = 25°$ and $C = 110°$.

Review Problems The problems that follow review material we covered in Section 4.6.

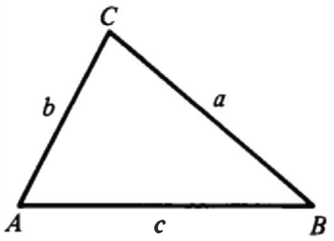
Evaluate each expression.

23. $\text{Cos}^{-1}(-1)$
24. $\text{Sin}^{-1}(-1)$
25. $\text{Arcsin}(-1/2)$
26. $\text{Arccos}(-1/2)$
27. $\text{Tan}^{-1}(\sqrt{3})$
28. $\text{Tan}^{-1}(-1)$
29. $\sin(\text{Sin}^{-1}(3/5))$
30. $\cos(\text{Sin}^{-1}(-3/5))$
31. $\cos(\text{Arcsin}(2/3))$
32. $\sin(\text{Arccos}(2/3))$

Chapter 7 Summary and Review

THE LAW OF SINES [7.1]

For any triangle ABC, the following relationships are always true:



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

or, equivalently,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Examples

1. If $A = 30°$, $B = 70°$, and $a = 8$ centimeters in triangle ABC, then, by the law of sines,

$$b = \frac{a \sin B}{\sin A} = \frac{8 \sin 70°}{\sin 30°} = 15 \text{ centimeters}$$

2. In triangle ABC, if $a = 54$ centimeters, $b = 62$ centimeters, and $A = 40°$, then

$$\sin B = \frac{b \sin A}{a} = \frac{62 \sin 40°}{54} = 0.7380$$

Since $\sin B$ is positive for any angle in quadrant I or II, we have two possibilities for B.

$$B = 48° \quad \text{or} \quad B' = 180° - 48° = 132°$$

This indicates that two triangles exist, both of which fit the given information.

THE AMBIGUOUS CASE [7.2]

When we are given two sides and an angle opposite one of them (SSA), we have several possibilities for the triangle or triangles that result. One of the possibilities is that no triangle will fit the given information. Another possibility is that two different triangles can be obtained from the given information and a third possibility is that exactly one triangle will fit the given information. Because of these different possibilities, we call the situation where we are solving a triangle in which we are given two sides and the angle opposite one of them the *ambiguous case*.

3. In triangle ABC, if $a = 34$ kilometers, $b = 20$ kilometers, and $c = 18$ kilometers, then we can find A using the law of cosines.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{20^2 + 18^2 - 34^2}{(2)(20)(18)}$$

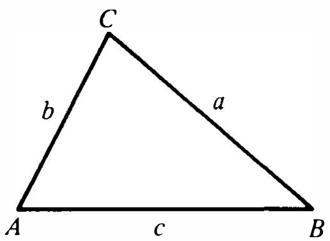
$$\cos A = -0.6000$$

$$\text{so} \quad A = 127°$$

THE LAW OF COSINES [7.3]

In any triangle ABC, the following relationships are always true:

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$



Another form of the law of cosines looks like this

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

THE AREA OF A TRIANGLE [7.4]

The area of a triangle in which we are given two sides and the included angle is given by

$$S = \frac{1}{2} ab \sin C$$

$$S = \frac{1}{2} ac \sin B$$

$$S = \frac{1}{2} bc \sin A$$

The area of a triangle in which we are given all three sides is given by the formula

$$S = \sqrt{s(s - a)(s - b)(s - c)}$$

$$\text{where } s = \frac{1}{2}(a + b + c)$$

The area of a triangle in which we are given two angles and the side included between them is given by

$$S = \frac{a^2 \sin B \sin C}{2 \sin A}$$

$$S = \frac{b^2 \sin C \sin A}{2 \sin B}$$

$$S = \frac{c^2 \sin A \sin B}{2 \sin C}$$

4. For triangle ABC,

a. If $a = 12$ centimeters, $b = 15$ centimeters, and $C = 20°$, then the area of ABC is

$$S = \frac{1}{2}(12)(15)\sin 20°$$

$$= 30.8 \text{ centimeters}^2 \text{ to the nearest tenth}$$

b. If $a = 24$ inches, $b = 14$ inches, and $c = 18$ inches, then the area of ABC is

$$S = \sqrt{28(28 - 24)(28 - 14)(28 - 18)}$$
$$= \sqrt{28(4)(14)(10)}$$
$$= \sqrt{15680}$$
$$= 125.2 \text{ inches}^2 \text{ to the nearest tenth}$$

c. If $A = 40°$, $B = 72°$, and $c = 45$ meters, then the area of ABC is

$$S = \frac{45^2 \sin 40° \sin 72°}{2 \sin 68°}$$

$$= \frac{2025(0.6428)(0.9511)}{2(0.9272)}$$

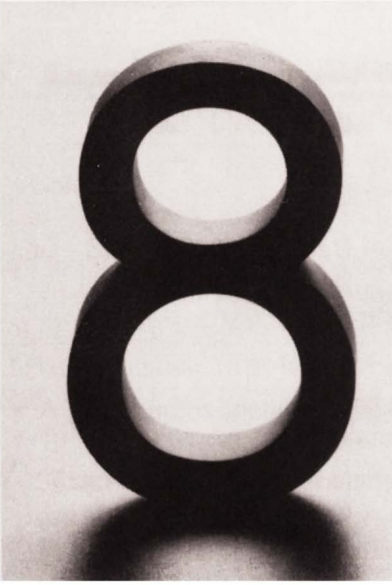
$$= 667.6 \text{ meters}^2 \text{ to the nearest tenth}$$

Chapter 7 Test

Problems 1 through 14 refer to triangle ABC which is not necessarily a right triangle.

1. If $A = 32°$, $B = 70°$, and $a = 3.8$ inches, use the law of sines to find b.
2. If $B = 118°$, $C = 37°$, and $c = 2.9$ inches, use the law of sines to find b.
3. If $A = 38.2°$, $B = 63.4°$, and $c = 42.0$ centimeters, find all the missing parts.
4. If $A = 24.7°$, $C = 106.1°$, and $b = 34.0$ centimeters, find all the missing parts.
5. Use the law of sines to show that no triangle exists for which $A = 60°$, $a = 12$ inches, and $b = 42$ inches.
6. Use the law of sines to show that exactly one triangle exists for which $A = 42°$, $a = 29$ inches, and $b = 21$ inches.

7. Find two triangles for which $A = 51°$, $a = 6.5$ feet, and $b = 7.9$ feet.
8. Find two triangles for which $A = 26°$, $a = 4.8$ feet, and $b = 9.4$ feet.
9. If $C = 60°$, $a = 10$ centimeters, and $b = 12$ centimeters, use the law of cosines to find c.
10. If $C = 120°$, $a = 10$ centimeters, and $b = 12$ centimeters, use the law of cosines to find c.
11. If $a = 5$ kilometers, $b = 7$ kilometers, and $c = 9$ kilometers, use the law of cosines to find C to the nearest tenth of a degree.
12. If $a = 10$ kilometers, $b = 12$ kilometers, and $c = 11$ kilometers, use the law of cosines to find B to the nearest tenth of a degree.
13. Find all the missing parts if $a = 6.4$ meters, $b = 2.8$ meters, and $C = 119°$.
14. Find all the missing parts if $b = 3.7$ meters, $c = 6.2$ meters, and $A = 35°$.
15. The two equal sides of an isosceles triangle are each 38 centimeters. If the base measures 48 centimeters, find the measure of the two equal angles to the nearest tenth of a degree.
16. A lamp pole casts a shadow 53 feet long when the angle of elevation of the sun is 48.1°. Find the height of the lamp pole, to the nearest foot.
17. A man standing near a building notices the angle of elevation to the top of the building is 64°. He then walks 240 feet farther away from the building and finds the angle of elevation to the top to be 43°. To the nearest foot, how tall is the building?
18. The diagonals of a parallelogram are 26.8 meters and 39.4 meters. If they meet at an angle of 134°, find the length of the shorter side of the parallelogram.
19. Find the area of the triangle in Problem 3.
20. Find the area of the triangle in Problem 4.
21. Find the area of the triangle in Problem 9.
22. Find the area of the triangle in Problem 10.
23. Find the area of the triangle in Problem 11.
24. Find the area of the triangle in Problem 12.



Complex Numbers and Polar Coordinates

To the student:

The first four sections of this chapter are a study of complex numbers. You may have already studied some of this material in algebra. If so, the first section of this chapter will be review for you. The material on complex numbers that may not be review for you is based on a new definition that makes it possible to look at complex numbers from a trigonometric point of view. In Section 8.2, we use this new definition to write complex numbers in what is called trigonometric form. Once we can write complex numbers in trigonometric form, many of the operations with complex numbers, such as multiplication, become much easier. In Section 8.4, we use trigonometric form for complex numbers to discover that every number has n distinct nth roots. For example, in that section, we will find that -1 has exactly 3 cube roots.

In the last two sections of this chapter we will study polar coordinates. Polar coordinates are used to name points in the plane and are an alternative to rectangular coordinates. The definition for the polar coordinates of a point in the plane is based on our original definition for sine and cosine. In Section 8.5, we introduce the definition for polar coordinates and then look at the relationship between polar and rectangular coordinates. In Section 8.6, we extend our work with polar coordinates to include graphing equations in which the variables are given in polar coordinates. As you will see, the graphs of the equations written in polar coordinates are interesting and pleasing to look at.

8.1 Complex Numbers

The equation $x^2 = -9$ has no real solutions because there is no real number whose square is -9. We have been unable to work with square roots of negative numbers, such as $\sqrt{-25}$ and $\sqrt{-16}$, for the same reason. In this section, we will introduce a set of numbers which will allow us to handle square roots of negative numbers. These new numbers are called *complex numbers*. Our work with complex numbers is based on the following definition:

DEFINITION The number i is such that $i^2 = -1$. (That is, i is the number whose square is -1.)

The number i is not a real number. We can use it to write square roots of negative numbers without a negative sign. To do so we reason that if $a > 0$, then $\sqrt{-a} = \sqrt{ai^2} = i\sqrt{a}$.

Example 1 Write each expression in terms of i.

a. $\sqrt{-9}$ **b.** $\sqrt{-12}$ **c.** $\sqrt{-17}$

Solution

a. $\sqrt{-9} = i\sqrt{9} = 3i$
b. $\sqrt{-12} = i\sqrt{12} = 2i\sqrt{3}$
c. $\sqrt{-17} = i\sqrt{17}$ ▲

Note In order to simplify expressions that contain square roots of negative numbers by using the properties of radicals developed in algebra, it is necessary to write each square root in terms of i before applying the properties of radicals. For example,

this is correct: $\sqrt{-4}\,\sqrt{-9} = (i\sqrt{4})\,(i\sqrt{9}) = (2i)(3i) = 6i^2 = -6$
this is incorrect: $\sqrt{-4}\,\sqrt{-9} = \sqrt{-4(-9)} = \sqrt{36} = 6$

Remember, the properties of radicals you developed in algebra hold only for expressions in which the numbers under the radical sign are positive. When the radicals contain negative numbers, you must first write each radical in terms of i and then simplify.

Next, we use i to write a definition for complex numbers.

DEFINITION A *complex number* is any number that can be written in the form

$$a + bi$$

where a and b are real numbers and $i^2 = -1$. The form $a + bi$ is called *standard form* for complex numbers. The number a is called the *real part* of the complex number. The number b is called the *imaginary part* of the complex number. If $b \neq 0$, we say $a + bi$ is also an *imaginary number*.

▼ Example 2

a. The number $3 + 2i$ is a complex number. (It is in standard form.) The number 3 is the real part and the number 2 (not $2i$) is the imaginary part. Since $b \neq 0$, $3 + 2i$ is also an imaginary number.
b. The number $-7i$ is a complex number since it can be written as $0 + (-7)i$. The real part is 0. The imaginary part is -7. $-7i$ is also an imaginary number since $b \neq 0$.
c. The number 4 is a complex number since it can be written as $4 + 0i$. The real part is 4 and the imaginary part is 0.
d. The number $3 + \sqrt{-25}$ is a complex number since it can be written as $3 + 5i$. The real part is 3 and the imaginary part is 5. ▲

From part c in Example 2, it is apparent that real numbers are also complex numbers. The real numbers are a subset of the complex numbers.

Equality for Complex Numbers

DEFINITION Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal. That is, for real numbers a, b, c, and d,

$$a + bi = c + di \text{ if and only if } a = c \text{ and } b = d$$

▼ Example 3 Find x and y if $(-3x - 9) + 4i = 6 + (3y - 2)i$.

Solution The real parts are $-3x - 9$ and 6. The imaginary parts are 4 and $3y - 2$.

$$\begin{aligned} -3x - 9 &= 6 \qquad\text{and}\qquad & 4 &= 3y - 2 \\ -3x &= 15 & 6 &= 3y \\ x &= -5 & y &= 2 \end{aligned}$$

▲

Addition and Subtraction of Complex Numbers

DEFINITION If $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ are complex numbers, then the sum and difference of z_1 and z_2 are defined as follows:

$$z_1 + z_2 = (a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i$$
$$z_1 - z_2 = (a_1 + b_1 i) - (a_2 + b_2 i) = (a_1 - a_2) + (b_1 - b_2)i$$

As you can see, we add and subtract complex numbers in the same way we would add and subtract polynomials: by combining similar terms.

▼ **Example 4** If $z_1 = 3 - 5i$ and $z_2 = -6 - 2i$, find $z_1 + z_2$ and $z_1 - z_2$.

Solution

$$z_1 + z_2 = (3 - 5i) + (-6 - 2i) = -3 - 7i$$
$$z_1 - z_2 = (3 - 5i) - (-6 - 2i) = 9 - 3i$$

▲

Powers of i

If we assume the properties of exponents hold when the base is i, we can write any integer power of i as either i, -1, $-i$, or 1. Using the fact that $i^2 = -1$, we have

$$i^1 = i$$
$$i^2 = -1$$
$$i^3 = i^2 \cdot i = -1(i) = -i$$
$$i^4 = i^2 \cdot i^2 = -1(-1) = 1$$

Since $i^4 = 1$, i^5 will simplify to i and we will begin repeating the sequence i, -1, $-i$, 1 as we increase our exponent by one each time.

$$i^5 = i^4 \cdot i = 1(i) = i$$
$$i^6 = i^4 \cdot i^2 = 1(-1) = -1$$
$$i^7 = i^4 \cdot i^3 = 1(-i) = -i$$
$$i^8 = i^4 \cdot i^4 = 1(1) = 1$$

We can simplify higher powers of i by writing them in terms of i^4 since i^4 is always 1.

▼ **Example 5** Simplify each power of i.

a. $i^{20} = (i^4)^5 = 1^5 = 1$
b. $i^{23} = (i^4)^5 \cdot i^3 = 1(-i) = -i$
c. $i^{30} = (i^4)^7 \cdot i^2 = 1(-1) = -1$

▲

Multiplication with Complex Numbers

DEFINITION If $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ are complex numbers, then their product is defined as follows:

$$\begin{aligned} z_1 z_2 &= (a_1 + b_1 i)(a_2 + b_2 i) \\ &= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i \end{aligned}$$

This formula comes from binomial multiplication but is actually more trouble than it is worth. Since complex numbers have the form of binomials with i as the variable, we can multiply two complex numbers using the same methods we use to multiply binomials and not have another formula to memorize.

▼ **Example 6** Multiply $(3 - 4i)(2 - 5i)$.

Solution Multiplying as if these were two binomials, we have

$$\begin{aligned} (3 - 4i)(2 - 5i) &= 3 \cdot 2 - 3 \cdot 5i - 2 \cdot 4i + 4i \cdot 5i \\ &= 6 - 15i - 8i + 20i^2 \\ &= 6 - 23i + 20i^2 \end{aligned}$$

Now, since $i^2 = -1$, we can simplify further.

$$\begin{aligned} &= 6 - 23i + 20(-1) \\ &= 6 - 23i - 20 \\ &= -14 - 23i \end{aligned}$$

▲

▼ **Example 7** Multiply $(4 - 5i)(4 + 5i)$.

Solution This product has the form $(a - b)(a + b)$ which we know results in the difference of two squares, $a^2 - b^2$.

$$\begin{aligned} (4 - 5i)(4 + 5i) &= 4^2 - (5i)^2 \\ &= 16 - 25i^2 \\ &= 16 - 25(-1) \\ &= 16 + 25 \\ &= 41 \end{aligned}$$

▲

The product of the two complex numbers $4 - 5i$ and $4 + 5i$ is the real number 41. This fact is very useful and leads to the following definition.

DEFINITION The complex numbers $a + bi$ and $a - bi$ are called *complex conjugates*. Their product is the real number $a^2 + b^2$. Here's why

$$\begin{aligned}(a + bi)(a - bi) &= a^2 - (bi)^2 \\ &= a^2 - b^2i^2 \\ &= a^2 - b^2(-1) \\ &= a^2 + b^2\end{aligned}$$

The fact that the product of two complex conjugates is a real number is the key to division with complex numbers.

▼ **Example 8** Divide $\dfrac{5i}{2 - 3i}$.

Solution We want a complex number in standard form that is equivalent to the quotient $5i/(2 - 3i)$. To do so, we need to replace the denominator with a real number. We can accomplish this by multiplying both the numerator and denominator by $2 + 3i$, which is the conjugate of $2 - 3i$.

$$\begin{aligned}\frac{5i}{2 - 3i} &= \frac{5i}{2 - 3i} \cdot \frac{\mathbf{(2 + 3i)}}{\mathbf{(2 + 3i)}} \\ &= \frac{5i(2 + 3i)}{(2 - 3i)(2 + 3i)} \\ &= \frac{10i + 15i^2}{4 - 9i^2} \\ &= \frac{10i + 15(-1)}{4 - 9(-1)} \\ &= \frac{-15 + 10i}{13} \\ &= -\frac{15}{13} + \frac{10}{13}i\end{aligned}$$

Notice that we have written our answer in standard form. The real part is $-15/13$ and the imaginary part is $10/13$. ▲

Problem Set 8.1

Write each expression in terms of i.

1. $\sqrt{-16}$
2. $\sqrt{-49}$
3. $\sqrt{-121}$
4. $\sqrt{-400}$
5. $\sqrt{-18}$
6. $\sqrt{-45}$
7. $\sqrt{-8}$
8. $\sqrt{-20}$
9. $\sqrt{-13}$
10. $\sqrt{-11}$

Find x and y so that each of the following equations is true:

11. $4 + 7i = 6x - 14yi$
12. $2 - 5i = -x + 10yi$
13. $(4x - 3) - 2i = 8 + yi$
14. $(2x - 4) - 3i = 10 - 6yi$
15. $(5x + 2) - 7i = 4 + (2y + 1)i$
16. $(7x - 1) + 4i = 2 + (5y + 2)i$
17. $(x^2 - 6) + 9i = x + y^2i$
18. $(x^2 - 2x) + y^2i = 8 + (2y - 1)i$

Find all x and y between 0 and 2π so that each of the following equations is true:

19. $\cos x + i \sin y = \sin x + i$
20. $\sin x + i \cos y = -\cos x - i$
21. $(\sin^2 x + 1) + i \tan y = 2 \sin x + i$
22. $(\cos^2 x + 1) + i \tan y = 2 \cos x - i$

Combine the following complex numbers:

23. $(7 + 2i) + (3 - 4i)$
24. $(3 - 5i) + (2 + 4i)$
25. $(6 + 7i) - (4 + i)$
26. $(5 + 2i) - (3 + 6i)$
27. $(7 - 3i) - (4 + 10i)$
28. $(11 - 6i) - (2 - 4i)$
29. $(3 \cos x + 4i \sin y) + (2 \cos x - 7i \sin y)$
30. $(2 \cos x - 3i \sin y) + (3 \cos x - 2i \sin y)$
31. $[(3 + 2i) - (6 + i)] + (5 + i)$
32. $[(4 - 5i) - (2 + i)] + (2 + 5i)$
33. $(7 - 4i) - [(-2 + i) - (3 + 7i)]$
34. $(10 - 2i) - [(2 + i) - (3 - i)]$

Simplify each power of i.

35. i^{12}
36. i^{13}
37. i^{14}
38. i^{15}
39. i^{32}
40. i^{34}
41. i^{33}
42. i^{35}

Find the following products:

43. $-6i(3 - 8i)$
44. $6i(3 + 8i)$
45. $(2 - 4i)(3 + i)$
46. $(2 + 4i)(3 - i)$
47. $(3 + 2i)^2$
48. $(3 - 2i)^2$
49. $(5 + 4i)(5 - 4i)$
50. $(4 + 5i)(4 - 5i)$
51. $(7 + 2i)(7 - 2i)$
52. $(2 + 7i)(2 - 7i)$
53. $2i(3 + i)(2 + 4i)$
54. $3i(1 + 2i)(3 + i)$

Find the following quotients. Write all answers in standard form for complex numbers.

55. $\dfrac{3}{4 - 5i}$
56. $\dfrac{4}{2 - 3i}$
57. $\dfrac{2i}{3 + i}$
58. $\dfrac{3i}{2 + i}$
59. $\dfrac{2 + 3i}{2 - 3i}$
60. $\dfrac{3 + 2i}{3 - 2i}$

61. $\dfrac{5-2i}{i}$

62. $\dfrac{5-2i}{-i}$

63. $\dfrac{2+i}{5-6i}$

64. $\dfrac{5+4i}{3+6i}$

Let $z_1 = 2 + 3i$, $z_2 = 2 - 3i$, and $z_3 = 4 + 5i$ and find

65. $z_1 z_2$

66. $z_2 z_1$

67. $z_1 z_3$

68. $z_3 z_1$

69. $2z_1 + 3z_2$

70. $3z_1 + 2z_2$

71. $z_3(z_1 + z_2)$

72. $z_3(z_1 - z_2)$

73. Assume x represents a real number and multiply $(x + 3i)(x - 3i)$.

74. Assume x represents a real number and multiply $(x - 4i)(x + 4i)$.

75. The opposite of i is $-i$. The reciprocal of i is $1/i$. Multiply the numerator and denominator of $1/i$ by i and simplify the result to see that the opposite of i and the reciprocal of i are the same number.

76. We know that the product of complex conjugates $a + bi$ and $a - bi$ is the real number $a^2 + b^2$. What is the sum of $a + bi$ and $a - bi$? Is it a real number?

77. Is addition of complex numbers a commutative operation? That is, if z_1 and z_2 are two complex numbers, is it always true that $z_1 + z_2 = z_2 + z_1$?

78. Is multiplication with complex numbers a commutative operation?

79. Is subtraction with complex numbers a commutative operation?

80. Show that $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ for the three complex numbers $z_1 = 3 + 2i$, $z_2 = 4 - i$, and $z_3 = 3 - 5i$.

Review Problems The problems that follow review material we covered in Sections 1.3 and 3.1. Reviewing these problems will help you with some of the material in the next section.

Find $\sin\theta$ and $\cos\theta$ if the given point lies on the terminal side of θ.

81. $(3, -4)$

82. $(-5, 12)$

83. (a, b)

84. $(1, -1)$

Find θ between 0° and 360° if

85. $\sin\theta = \dfrac{1}{\sqrt{2}}$ and $\cos\theta = -\dfrac{1}{\sqrt{2}}$

86. $\tan\theta = 1$ and θ terminates in QIII

87. $\sin\theta = \dfrac{1}{2}$ and θ terminates in QII

88. $\cos\theta = \dfrac{\sqrt{3}}{2}$ and θ terminates in QIV

8.2 Trigonometric Form for Complex Numbers

We will begin this section with a definition that will give us a way to represent complex numbers graphically. We will then develop a way to relate the work we did in Section 8.1 to some of the concepts in trigonometry.

DEFINITION The graph of the complex number $a + bi$ is a vector (arrow) that extends from the origin out to the point (a, b).

▼ **Example 1** Graph each complex number.

$$2 + 4i, \quad -2 - 4i, \quad 2 - 4i$$

Solution

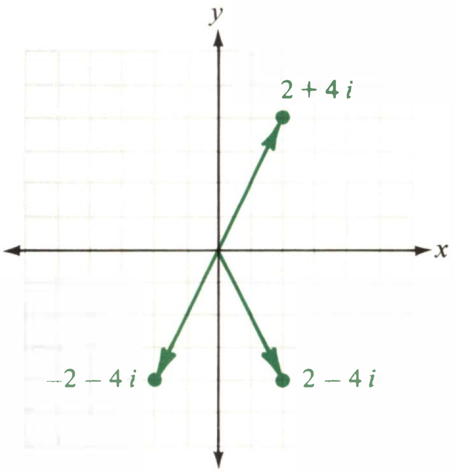


Figure 1

Notice how the graph of $2 + 4i$ and $2 - 4i$, which are conjugates, have symmetry about the x-axis. Note also that the graphs of $2 + 4i$ and $-2 - 4i$, which are opposites, have symmetry about the origin. ▲

▼ **Example 2** Graph the complex numbers 1, i, -1, and $-i$.

Solution Here are the four complex numbers written in standard form and the corresponding graphs of those numbers.

$$1 = 1 + 0i \qquad i = 0 + i$$
$$-1 = -1 + 0i \qquad -i = 0 - i$$

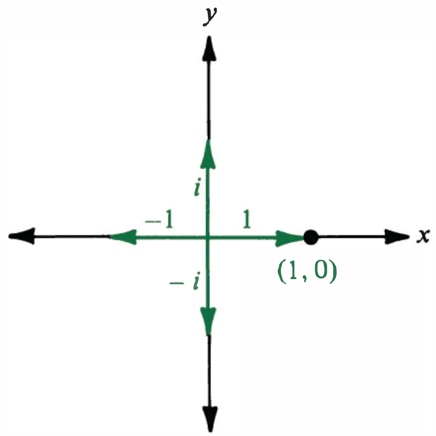


Figure 2 ▲

If we write a real number as a complex number in standard form, its graph will fall on the x-axis. Therefore, we call the x-axis the *real axis* when we are graphing complex numbers. Likewise, because the imaginary numbers i and $-i$ fall on the y-axis, we call the y-axis the *imaginary axis* when we are graphing complex numbers.

DEFINITION The *absolute value* or *modulus* of the complex number $z = a + bi$ is the distance from the origin to the point (a, b). If this distance is denoted by r, then

$$r = |z| = |a + bi| = \sqrt{a^2 + b^2}$$

▼ **Example 3** Find the modulus of each of the complex numbers $5i$, 7, and $3 + 4i$.

Solution Writing each number in standard form and then applying the definition of modulus, we have

$$\begin{aligned}
&\text{For } z = 5i = 0 + 5i, && r = |z| = |0 + 5i| = \sqrt{0^2 + 5^2} = 5 \\
&\text{For } z = 7 = 7 + 0i, && r = |z| = |7 + 0i| = \sqrt{7^2 + 0^2} = 7 \\
&\text{For } z = 3 + 4i, && r = |z| = |3 + 4i| = \sqrt{3^2 + 4^2} = 5
\end{aligned}$$ ▲

DEFINITION The *argument* of the complex number $z = x + yi$ is the smallest positive angle θ from the positive real axis to the graph of z.

Figure 3 illustrates the relationships between the complex number $z = x + yi$, its graph, and the modulus r and argument θ of z.

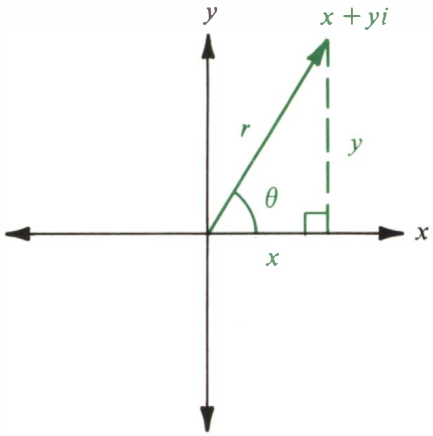


Figure 3

From Figure 3 we see that

$$\cos\theta = \frac{x}{r} \text{ or } x = r\cos\theta$$

and

$$\sin\theta = \frac{y}{r} \text{ or } y = r\sin\theta$$

We can use this information to write z in terms of r and θ.

$$\begin{aligned} z &= x + yi \\ &= r\cos\theta + (r\sin\theta)i \\ &= r\cos\theta + ri\sin\theta \\ &= r(\cos\theta + i\sin\theta) \end{aligned}$$

This last expression is called the *trigonometric form* for z. The formal definition follows.

Definition If $z = x + yi$ is a complex number in standard form, then the *trigonometric form* for z is given by

$$z = r(\cos\theta + i\sin\theta)$$

Where r is the modulus of z and θ is the argument of z.

We can convert back and forth between standard form and trigonometric form by using the relationships that follow.

For $z = x + yi = r(\cos\theta + i\sin\theta)$

$r = \sqrt{x^2 + y^2}$ and θ is such that

$$\cos\theta = \frac{x}{r}, \quad \sin\theta = \frac{y}{r}, \quad \text{and } \tan\theta = \frac{y}{x}$$

Example 4 Write $z = -1 + i$ in trigonometric form.

Solution We have $x = -1$ and $y = 1$, therefore

$$r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

Angle θ is the smallest positive angle for which $\cos\theta = x/r = -1/\sqrt{2}$ and $\sin\theta = y/r = 1/\sqrt{2}$, (or $\tan\theta = y/x = 1/-1 = -1$). Therefore, θ must be 135°.

Using these values of r and θ in the formula for trigonometric form we have

$$\begin{aligned} z &= r(\cos\theta + i\sin\theta) \\ &= \sqrt{2}(\cos 135^\circ + i\sin 135^\circ) \end{aligned}$$

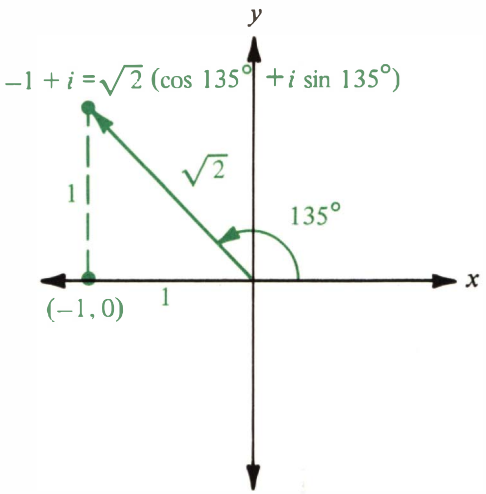


Figure 4

Example 5 Write $z = 2(\cos 60^\circ + i\sin 60^\circ)$ in standard form.

Solution Using exact values for cos 60° and sin 60° we have

$$z = 2(\cos 60° + i \sin 60°)$$
$$= 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$
$$= 1 + i\sqrt{3}$$

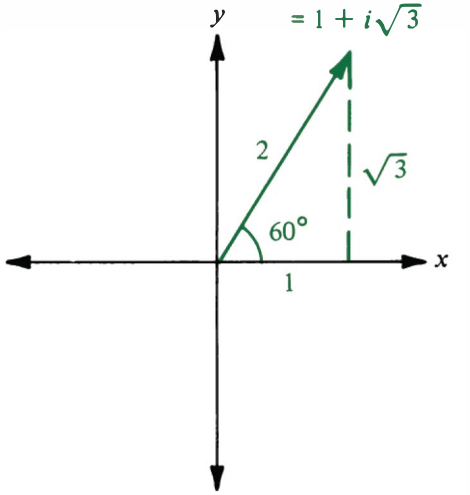


Figure 5

As you can see, converting from trigonometric form to standard form is usually more direct than converting from standard form to trigonometric form.

Problem Set 8.2

Graph each complex number. In each case give the absolute value of the number.

1. $3 + 4i$
2. $3 - 4i$
3. $1 + i$
4. $1 - i$
5. $-5i$
6. $4i$
7. 2
8. -4
9. $-4 - 3i$
10. $-3 - 4i$

Graph each complex number along with its opposite and conjugate.

11. $2 - i$
12. $2 + i$
13. $4i$
14. $-3i$
15. -3
16. 5
17. $-5 - 2i$
18. $-2 - 5i$

Write each complex number in standard form.

19. $2(\cos 30° + i \sin 30°)$
20. $4(\cos 30° + i \sin 30°)$
21. $4(\cos 120° + i \sin 120°)$
22. $8(\cos 120° + i \sin 120°)$
23. $\cos 210° + i \sin 210°$
24. $\cos 240° + i \sin 240°$
25. $\cos 315° + i \sin 315°$
26. $\sqrt{2}(\cos 315° + i \sin 315°)$

Use the tables at the back of the book to help write each complex number in standard form. Round the numbers in your answers to the nearest hundredth.

27. $10(\cos 12° + i \sin 12°)$
28. $100(\cos 70° + i \sin 70°)$
29. $100(\cos 143° + i \sin 143°)$
30. $100(\cos 171° + i \sin 171°)$
31. $\cos 205° + i \sin 205°$
32. $\cos 261° + i \sin 261°$
33. $10(\cos 342° + i \sin 342°)$
34. $10(\cos 318° + i \sin 318°)$

Write each complex number in trigonometric form. In each case begin by sketching the graph to help with finding the argument θ.

35. $-1 + i$
36. $1 + i$
37. $1 - i$
38. $-1 - i$
39. $3 + 3i$
40. $5 + 5i$
41. $8i$
42. $-8i$
43. -9
44. 2
45. $-2 + 2i\sqrt{3}$
46. $-2\sqrt{3} + 2i$
47. $3 - 3i\sqrt{3}$
48. $3\sqrt{3} - 3i$

49. We know that $2i \cdot 3i = 6i^2 = -6$. Change $2i$ and $3i$ to trigonometric form and then show that their product in trigonometric form is still -6.

50. Change $4i$ and 2 to trigonometric form and then multiply. Show that this product is $8i$.

51. Show that $2(\cos 30° + i \sin 30°)$ and $2[\cos(-30°) + i \sin(-30°)]$ are conjugates.

52. Show that $2(\cos 60° + i \sin 60°)$ and $2[\cos(-60°) + i \sin(-60°)]$ are conjugates.

53. Let $z_1 = 2 + 3i$ and $z_2 = z_1 i$ (the product of z_1 and i). Graph both z_1 and z_2. Then find $z_3 = z_2 i$ and graph z_3 on the same coordinate system used to graph z_1 and z_2. Multiplying a complex number by i should produce a complex number with a graph that is rotated 90° from the graph of the original number. Multiply z_3 by i to obtain z_4 and show that the graph of z_4 follows the same pattern.

54. Let $z_1 = 3 - i$ and find $z_2 = z_1 i$, $z_3 = z_2 i$, and $z_4 = z_3 i$. Graph z_1, z_2, z_3, and z_4 on the same coordinate system.

55. Show that if $z = \cos\theta + i \sin\theta$, then $|z| = 1$.

56. Show that if $z = \cos\theta - i \sin\theta$, then $|z| = 1$.

Review Problems The problems that follow review material we covered in Section 5.2. Reviewing these problems will help you understand the next section.

57. Use the formula for $\cos(A + B)$ to find the exact value of $\cos 75°$.

58. Use the formula for $\sin(A + B)$ to find the exact value of $\sin 75°$.

Let $\sin A = 3/5$ with A in QI and $\sin B = 5/13$ with B in QI and find

59. $\sin(A + B)$
60. $\cos(A + B)$

Simplify each expression to a single trigonometric function.

61. $\sin 30°\cos 90° + \cos 30°\sin 90°$

62. $\cos 30°\cos 90° - \sin 30°\sin 90°$

63. $\cos 18°\cos 32° - \sin 18°\sin 32°$

64. $\sin 18°\cos 32° + \cos 18°\sin 32°$

8.3 Products and Quotients in Trigonometric Form

Multiplication and division with complex numbers becomes a very simple process when the numbers are written in trigonometric form. Let's state the rule for finding the product of two complex numbers written in trigonometric form as a theorem and then prove the theorem.

THEOREM (MULTIPLICATION) If

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$$

and

$$z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$$

are two complex numbers in trigonometric form, then their product, z_1z_2 is

$$\begin{aligned} z_1z_2 &= [r_1(\cos \theta_1 + i \sin \theta_1)][r_2(\cos \theta_2 + i \sin \theta_2)] \\ &= r_1r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \end{aligned}$$

In words: To multiply two complex numbers in trigonometric form, multiply absolute values and add angles.

Proof We begin by multiplying algebraically. Then we simplify our product by using the sum formulas we introduced in Section 5.2.

$$\begin{aligned} z_1z_2 &= [r_1(\cos \theta_1 + i \sin \theta_1)][r_2(\cos \theta_2 + i \sin \theta_2)] \\ &= r_1r_2(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \\ &= r_1r_2(\cos \theta_1\cos \theta_2 + i\cos \theta_1\sin \theta_2 + i\sin \theta_1\cos \theta_2 + i^2\sin \theta_1\sin \theta_2) \\ &= r_1r_2[\cos \theta_1\cos \theta_2 + i(\cos \theta_1\sin \theta_2 + \sin \theta_1\cos \theta_2) - \sin \theta_1\sin \theta_2] \\ &= r_1r_2[(\cos \theta_1\cos \theta_2 - \sin \theta_1\sin \theta_2) + i(\sin \theta_1\cos \theta_2 + \cos \theta_1\sin \theta_2)] \\ &= r_1r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \end{aligned}$$

This completes our proof. As you can see, to multiply two complex numbers in trigonometric form, we multiply absolute values, r_1r_2, and add angles, $\theta_1 + \theta_2$.

▼ **Example 1** Multiply $3(\cos 40° + i \sin 40°)$ and $5(\cos 10° + i \sin 10°)$.

Solution Applying the formula from our theorem on products, we have

$$\begin{aligned}&[3(\cos 40° + i \sin 40°)][5(\cos 10° + i \sin 10°)]\\ &\quad = 3 \cdot 5[\cos(40° + 10°) + i \sin(40° + 10°)]\\ &\quad = 15(\cos 50° + i \sin 50°)\end{aligned}$$

▲

▼ **Example 2** Find the product of $z_1 = 1 + i\sqrt{3}$ and $z_2 = -\sqrt{3} + i$ in standard form, and then write z_1 and z_2 in trigonometric form and find their product again.

Solution Leaving each complex number in standard form and multiplying, we have

$$\begin{aligned}z_1 z_2 &= (1 + i\sqrt{3})(-\sqrt{3} + i)\\ &= -\sqrt{3} + i - 3i + i^2\sqrt{3}\\ &= -2\sqrt{3} - 2i\end{aligned}$$

Changing z_1 and z_2 to trigonometric form and multiplying looks like this

$$\begin{aligned}z_1 &= 1 + i\sqrt{3} = 2(\cos 60° + i \sin 60°)\\ z_2 &= -\sqrt{3} + i = 2(\cos 150° + i \sin 150°)\\ z_1 z_2 &= [2(\cos 60° + i \sin 60°)][2(\cos 150° + i \sin 150°)]\\ &= 4(\cos 210° + i \sin 210°)\end{aligned}$$

To compare our two products, we convert our product in trigonometric form to standard form.

$$\begin{aligned}4(\cos 210° + i \sin 210°) &= 4\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)\\ &= -2\sqrt{3} - 2i\end{aligned}$$

As you can see, both methods of multiplying complex numbers produce the same result. ▲

The next theorem is an extension of the work we have done so far with multiplication. We will not give a formal proof of the theorem.

DeMoivre's Theorem If $z = r(\cos\theta + i \sin\theta)$ is a complex number in trigonometric form and n is an integer, then

$$z^n = [r(\cos\theta + i \sin\theta)]^n = r^n(\cos n\theta + i \sin n\theta)$$

The theorem seems reasonable after the work we have done with multiplication. For example, if n is 2,

$$\begin{aligned}[r(\cos\theta + i \sin\theta)]^2 &= r(\cos\theta + i \sin\theta)r(\cos\theta + i \sin\theta)\\ &= r^2(\cos 2\theta + i \sin 2\theta)\end{aligned}$$

Example 3 Find $(1 + i)^{10}$.

Solution First we write $1 + i$ in trigonometric form

$$1 + i = \sqrt{2}(\cos 45° + i \sin 45°)$$

Then we use DeMoivre's theorem to raise this expression to the 10th power.

$$\begin{aligned}(1 + i)^{10} &= [\sqrt{2}(\cos 45° + i \sin 45°)]^{10} \\ &= (\sqrt{2})^{10}(\cos 10 \cdot 45° + i \sin 10 \cdot 45°) \\ &= 32(\cos 450° + i \sin 450°)\end{aligned}$$

Which we can simplify to

$$= 32(\cos 90° + i \sin 90°)$$

since 90° and 450° are coterminal. In standard form our result is

$$\begin{aligned}&= 32(0 + i) \\ &= 32i\end{aligned}$$

that is,

$$(1 + i)^{10} = 32i$$

Since multiplication with complex numbers in trigonometric form is accomplished by multiplying absolute values and adding angles, we should expect that division is accomplished by dividing absolute values and subtracting angles.

THEOREM (DIVISION) If

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$$

and

$$z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$$

are two complex numbers in trigonometric form, then the quotient, z_1/z_2, is

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)} \\ &= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]\end{aligned}$$

Proof As was the case with division of complex numbers in standard form, the major step in this proof is multiplying the numerator and denominator of our quotient by the conjugate of the denominator.

$$\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \cdot \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos^2\theta_2 + \sin^2\theta_2)}$$

$$= \frac{r_1}{r_2}(\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - i^2\sin\theta_1\sin\theta_2)$$

$$= \frac{r_1}{r_2}[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)] + i[(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)]$$

$$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

Example 4 Divide $20(\cos 75^\circ + i\sin 75^\circ)$ by $4(\cos 40^\circ + i\sin 40^\circ)$.

Solution We divide according to the formula given in our theorem on division.

$$\frac{20(\cos 75^\circ + i\sin 75^\circ)}{4(\cos 40^\circ + i\sin 40^\circ)} = \frac{20}{4}[\cos(75^\circ - 40^\circ) + i\sin(75^\circ - 40^\circ)]$$

$$= 5(\cos 35^\circ + i\sin 35^\circ)$$

Example 5 Divide $z_1 = 1 + i\sqrt{3}$ by $z_2 = \sqrt{3} + i$ and leave the answer in standard form. Then change each to trigonometric form and divide again.

Solution Dividing in standard form, we have

$$\frac{z_1}{z_2} = \frac{1 + i\sqrt{3}}{\sqrt{3} + i}$$

$$= \frac{1 + i\sqrt{3}}{\sqrt{3} + i} \cdot \frac{\sqrt{3} - i}{\sqrt{3} - i}$$

$$= \frac{\sqrt{3} - i + 3i - i^2\sqrt{3}}{3 + 1}$$

$$= \frac{2\sqrt{3} + 2i}{4}$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

Changing z_1 and z_2 to trigonometric form and multiplying again, we have

$$z_1 = 1 + i\sqrt{3} = 2(\cos 60° + i \sin 60°)$$

$$z_2 = \sqrt{3} + i = 2(\cos 30° + i \sin 30°)$$

$$\frac{z_1}{z_2} = \frac{2(\cos 60° + i \sin 60°)}{2(\cos 30° + i \sin 30°)}$$

$$= \frac{2}{2}[\cos(60° - 30°) + i \sin(60° - 30°)]$$

$$= \cos 30° + i \sin 30°$$

which, in standard form, is

$$\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

▲

Problem Set 8.3

Multiply. Leave all answers in trigonometric form.

1. $3(\cos 20° + i \sin 20°) \cdot 4(\cos 30° + i \sin 30°)$
2. $5(\cos 15° + i \sin 15°) \cdot 2(\cos 25° + i \sin 25°)$
3. $7(\cos 110° + i \sin 110°) \cdot 8(\cos 47° + i \sin 47°)$
4. $9(\cos 115° + i \sin 115°) \cdot 4(\cos 51° + i \sin 51°)$
5. $2(\cos 135° + i \sin 135°) \cdot 2(\cos 45° + i \sin 45°)$
6. $2(\cos 120° + i \sin 120°) \cdot 4(\cos 30° + i \sin 30°)$

Find the product z_1z_2 in standard form. Then write z_1 and z_2 in trigonometric form and find their product again. Finally, convert the answer that is in trigonometric form to standard form to show that the two products are equal.

7. $z_1 = 1 + i,\ z_2 = -1 + i$
8. $z_1 = 1 + i,\ z_2 = 2 + 2i$
9. $z_1 = 1 + i\sqrt{3},\ z_2 = -\sqrt{3} + i$
10. $z_1 = -1 + i\sqrt{3},\ z_2 = \sqrt{3} + i$
11. $z_1 = 3i,\ z_2 = -4i$
12. $z_1 = 2i,\ z_2 = -5i$
13. $z_1 = 1 + i,\ z_2 = 4i$
14. $z_1 = 1 + i,\ z_2 = 3i$
15. $z_1 = -5,\ z_2 = 1 + i\sqrt{3}$
16. $z_1 = -3,\ z_2 = \sqrt{3} + i$

Use DeMoivre's theorem to find each of the following. Write your answer in standard form.

17. $[2(\cos 10° + i \sin 10°)]^6$
18. $[4(\cos 15° + i \sin 15°)]^3$
19. $(\cos 12° + i \sin 12°)^{10}$
20. $(\cos 18° + i \sin 18°)^{10}$
21. $[3(\cos 60° + i \sin 60°)]^4$
22. $[3(\cos 30° + i \sin 30°)]^4$
23. $[\sqrt{2}(\cos 45° + i \sin 45°)]^{10}$
24. $[\sqrt{2}(\cos 70° + i \sin 70°)]^6$
25. $(1 + i)^4$
26. $(1 + i)^5$
27. $(-\sqrt{3} + i)^4$
28. $(\sqrt{3} + i)^4$
29. $(1 - i)^6$
30. $(-1 + i)^8$
31. $(-2 + 2i)^3$
32. $(-2 - 2i)^3$

Divide. Leave your answers in trigonometric form.

33. $\dfrac{20(\cos 75^\circ + i \sin 75^\circ)}{5(\cos 40^\circ + i \sin 40^\circ)}$

34. $\dfrac{30(\cos 80^\circ + i \sin 80^\circ)}{10(\cos 30^\circ + i \sin 30^\circ)}$

35. $\dfrac{18(\cos 51^\circ + i \sin 51^\circ)}{12(\cos 32^\circ + i \sin 32^\circ)}$

36. $\dfrac{21(\cos 63^\circ + i \sin 63^\circ)}{14(\cos 44^\circ + i \sin 44^\circ)}$

37. $\dfrac{4(\cos 90^\circ + i \sin 90^\circ)}{8(\cos 30^\circ + i \sin 30^\circ)}$

38. $\dfrac{6(\cos 120^\circ + i \sin 120^\circ)}{8(\cos 90^\circ + i \sin 90^\circ)}$

Find the quotient z_1/z_2 in standard form. Then write z_1 and z_2 in trigonometric form and find their quotient again. Finally, convert the answer that is in trigonometric form to standard form to show that the two quotients are equal.

39. $z_1 = 2 + 2i,\ z_2 = 1 + i$

40. $z_1 = 2 - 2i,\ z_2 = 1 - i$

41. $z_1 = \sqrt{3} + i,\ z_2 = 2i$

42. $z_1 = 1 + i\sqrt{3},\ z_2 = 2i$

43. $z_1 = 4 + 4i,\ z_2 = 2 - 2i$

44. $z_1 = 6 + 6i,\ z_2 = -3 - 3i$

45. $z_1 = 8,\ z_2 = -4$

46. $z_1 = -6,\ z_2 = 3$

Convert all complex numbers to trigonometric form and then simplify each expression. Write all answers in standard form.

47. $\dfrac{(1 + i)^4(2i)^2}{-2 + 2i}$

48. $\dfrac{(\sqrt{3} + i)^4(2i)^5}{(1 + i)^{10}}$

49. $\dfrac{(1 + i\sqrt{3})^4(\sqrt{3} - i)^2}{(1 - i\sqrt{3})^3}$

50. $\dfrac{(2 + 2i)^5(-3 + 3i)^3}{(\sqrt{3} + i)^{10}}$

51. Show that $x = 2(\cos 60^\circ + i \sin 60^\circ)$ is a solution to the quadratic equation $x^2 - 2x + 4 = 0$ by replacing x with $2(\cos 60^\circ + i \sin 60^\circ)$ and simplifying.

52. Show that $x = 2(\cos 300^\circ + i \sin 300^\circ)$ is a solution to the equation $x^2 - 2x + 4 = 0$.

53. Show that $w = 2(\cos 15^\circ + i \sin 15^\circ)$ is a fourth root of $z = 8 + 8i\sqrt{3}$ by raising w to the fourth power and simplifying to get z. (The number w is a fourth root of z, $w = z^{1/4}$, if the fourth power of w is z, $w^4 = z$.)

54. Show that $x = 1/2 + (\sqrt{3}/2)i$ is a cube root of -1.

DeMoivre's theorem can be used to find reciprocals of complex numbers. Recall from algebra that the reciprocal of x is $1/x$, which can be expressed as x^{-1}. Use this fact, along with DeMoivre's theorem, to find the reciprocal of each number below.

55. $1 + i$

56. $1 - i$

57. $\sqrt{3} - i$

58. $\sqrt{3} + i$

59. $1 + i\sqrt{3}$

60. $1 - i\sqrt{3}$

Review Problems The problems that follow review material we covered in Sections 2.1 and 3.1.

Use the tables at the back of the book to find the following:

61. sin 15°	**62.** cos 15°
63. cos 165°	**64.** sin 165°
65. cos 195°	**66.** sin 195°
67. sin 345°	**68.** cos 345°

8.4 Roots of a Complex Number

In this section, we want to develop a formula for finding roots of a complex number. The discussion that follows will lead to the theorem containing the formula for these roots.

Suppose that z and w are complex numbers such that w is the nth root of z. That is,

$$\sqrt[n]{z} = w$$

If we raise both sides of this last expression to the nth power, we have

$$w^n = z$$

Now suppose $z = r(\cos\theta + i\sin\theta)$ and $w = s(\cos\alpha + i\sin\alpha)$. Substituting these expressions into the equation $w^n = z$, we have

$$[s(\cos\alpha + i\sin\alpha)]^n = r(\cos\theta + i\sin\theta)$$

We can rewrite the left side of this last equation using DeMoivre's theorem.

$$s^n(\cos n\alpha + i\sin n\alpha) = r(\cos\theta + i\sin\theta)$$

The only way these two expressions can be equal is if their absolute values are equal and their angles are coterminal.

Absolute Values Equal	*Angles Coterminal*
$s^n = r$	$n\alpha = \theta + 360°k \quad k = \text{an integer}$

Solving for s and α, we have

$$s = r^{1/n} \qquad \alpha = \frac{\theta + 360°k}{n}$$

To summarize, we find the nth roots of a complex number by first finding the real nth root of the absolute value and then adding multiples of 360° to θ and dividing the result by n. The multiples of 360° that we add on will range from $360° \cdot 0$ up to $360° \cdot (n - 1)$. After that we start repeating angles.

THEOREM (ROOTS) The nth roots of the complex number

$$z = r(\cos \theta + i \sin \theta)$$

are given by

$$w_k = r^{1/n}\left[\cos \frac{\theta + 360°k}{n} + i \sin \frac{\theta + 360°k}{n}\right]$$

where $k = 0, 1, 2, \ldots, n - 1$. That is,

$$w_0 = r^{1/n}\left[\cos \frac{\theta}{n} + i \sin \frac{\theta}{n}\right]$$

$$w_1 = r^{1/n}\left[\cos \frac{\theta + 360°}{n} + i \sin \frac{\theta + 360°}{n}\right]$$

$$w_2 = r^{1/n}\left[\cos \frac{\theta + 720°}{n} + i \sin \frac{\theta + 720°}{n}\right]$$

$$\vdots$$

$$w_{n-1} = r^{1/n}\left[\cos \frac{\theta + 360°(n-1)}{n} + i \sin \frac{\theta + 360°(n-1)}{n}\right]$$

▼ **Example 1** Find the 4 fourth roots of $z = 16(\cos 60° + i \sin 60°)$.

Solution According to the formula given in our theorem on roots, the 4 fourth roots will be

$$w_k = 16^{1/4}\left[\cos \frac{60° + 360°k}{4} + i \sin \frac{60° + 360°k}{4}\right] \quad k = 0, 1, 2, 3$$

$$= 2[\cos (15° + 90°k) + i \sin (15° + 90°k)]$$

Replacing k with 0, 1, 2, and 3, we have

$$w_0 = 2(\cos 15° + i \sin 15°)$$
$$w_1 = 2(\cos 105° + i \sin 105°)$$
$$w_2 = 2(\cos 195° + i \sin 195°)$$
$$w_3 = 2(\cos 285° + i \sin 285°)$$

It is interesting to note the relationships among the graphs of these 4 roots. As Figure 1 indicates, the graphs of the four roots are evenly distributed around the coordinate plane.

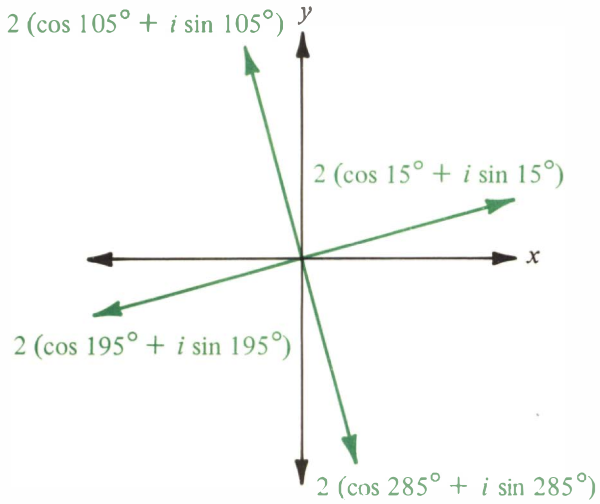


Figure 1

Example 2 Find the 3 cube roots of -1.

Solution We already know that one of the cube roots of -1 is -1. There are two other cube roots that are imaginary numbers. To find them, we write -1 in trigonometric form and then apply the formula from our theorem on roots. Writing -1 in trigonometric form, we have

$$-1 = 1(\cos 180° + i \sin 180°)$$

The 3 cube roots are given by

$$\begin{aligned} w_k &= 1^{1/3}\left[\cos \frac{180° + 360°k}{3} + i \sin \frac{180° + 360°k}{3}\right] \\ &= \cos (60° + 120°k) + i \sin (60° + 120°k) \end{aligned}$$

where $k = 0, 1$, and 2. Replacing k with 0, 1, and 2 and then simplifying each result, we have

$$w_0 = \cos 60° + i \sin 60° = \frac{1}{2} + \frac{\sqrt{3}}{2}\, i$$

$$w_1 = \cos 180° + i \sin 180° = -1$$

$$w_2 = \cos 300° + i \sin 300° = \frac{1}{2} - \frac{\sqrt{3}}{2}\, i$$

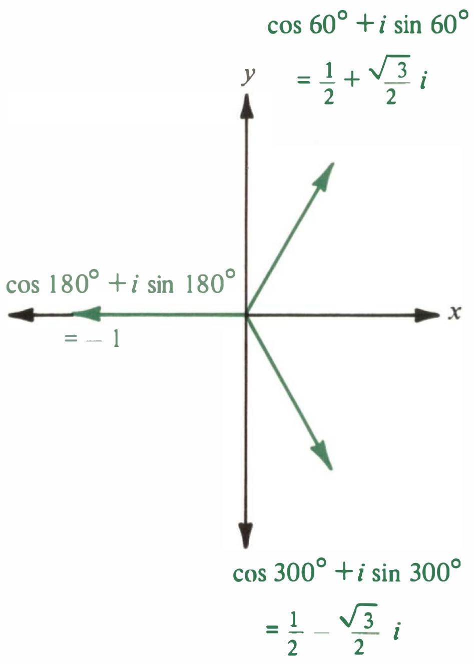


Figure 2

Note that the two complex roots are conjugates. Let's check root w_0 by cubing it.

$$\begin{aligned} w_0{}^3 &= (\cos 60° + i \sin 60°)^3 \\ &= \cos 3 \cdot 60° + i \sin 3 \cdot 60° \\ &= \cos 180° + i \sin 180° \\ &= -1 \end{aligned}$$

▲

▼ **Example 3** Solve the equation $x^4 - 2\sqrt{3}x^2 + 4 = 0$.

Solution The equation is quadratic in x^2. We can solve for x^2 by applying the quadratic formula.

$$\begin{aligned} x^2 &= \frac{2\sqrt{3} \pm \sqrt{12 - 4(1)(4)}}{2} \\ &= \frac{2\sqrt{3} \pm \sqrt{-4}}{2} \\ &= \frac{2\sqrt{3} \pm 2i}{2} \\ &= \sqrt{3} \pm i \end{aligned}$$

The two solutions for x^2 are $\sqrt{3} + i$ and $\sqrt{3} - i$, which we write in trigonometric form as follows:

$$x^2 = \sqrt{3} + i \qquad \text{or} \qquad x^2 = \sqrt{3} - i$$
$$= 2(\cos 30° + i \sin 30°) \qquad\qquad = 2(\cos 330° + i \sin 330°)$$

Now each of these expressions has two square roots, each of which is a solution to our original equation.

When $x^2 = 2(\cos 30° + i \sin 30°)$

$$x = 2^{1/2}\left[\cos \frac{30° + 360°k}{2} + i \sin \frac{30° + 360°k}{2}\right]$$

for $k = 0$ and 1

When $k = 0$, we have $x = 2^{1/2}(\cos 15° + i \sin 15°)$

When $k = 1$, we have $x = 2^{1/2}(\cos 195° + i \sin 195°)$

When $x^2 = 2(\cos 330° + i \sin 330°)$

$$x = 2^{1/2}\left[\cos \frac{330° + 360°k}{2} + i \sin \frac{330° + 360°k}{2}\right]$$

for $k = 0$ and 1

When $k = 0$, we have $x = 2^{1/2}(\cos 165° + i \sin 165°)$

When $k = 1$, we have $x = 2^{1/2}(\cos 345° + i \sin 345°)$

Using the tables in the back of the book or a calculator and rounding to the nearest hundredth, we can write decimal approximations to each of these four solutions.

Solutions

Trigonometric Form		*Decimal Approximation*
$2^{1/2}(\cos 15° + i \sin 15°)$	=	$1.37 + 0.37i$
$2^{1/2}(\cos 165° + i \sin 165°)$	=	$-1.37 + 0.37i$
$2^{1/2}(\cos 195° + i \sin 195°)$	=	$-1.37 - 0.37i$
$2^{1/2}(\cos 345° + i \sin 345°)$	=	$1.37 - 0.37i$

Problem Set 8.4

Find two square roots for each of the following complex numbers. Leave your answers in trigonometric form. In each case, graph the two roots.

1. $4(\cos 30° + i \sin 30°)$

2. $16(\cos 30° + i \sin 30°)$

3. $25(\cos 210° + i \sin 210°)$

4. $9(\cos 310° + i \sin 310°)$

Find two square roots for each of the following complex numbers. Write your answers in standard form.

5. $2 + 2i\sqrt{3}$
6. $-2 + 2i\sqrt{3}$
7. $4i$
8. $-4i$
9. -25
10. 25

Find three cube roots for each of the following complex numbers. Leave your answers in trigonometric form.

11. $8(\cos 210° + i \sin 210°)$
12. $27(\cos 303° + i \sin 303°)$
13. $4\sqrt{3} + 4i$
14. $-4\sqrt{3} + 4i$
15. -27
16. 8

17. Find 4 fourth roots of $z = 16(\cos 120° + i \sin 120°)$. Write each root in standard form.

18. Find 4 fourth roots of $z = \cos 240° + i \sin 240°$. Leave your answers in trigonometric form.

19. Find 5 fifth roots of $z = 10^5(\cos 15° + i \sin 15°)$. Write each root in trigonometric form and then give a decimal approximation, accurate to the nearest hundredth, for each one.

20. Find 5 fifth roots of $z = 10^{10}(\cos 75° + i \sin 75°)$. Write each root in trigonometric form and then give a decimal approximation, accurate to the nearest hundredth, for each one.

21. Find 6 sixth roots of $z = -1$. Leave your answers in trigonometric form. Graph all six roots on the same coordinate system.

22. Find 6 sixth roots of $z = 1$. Leave your answers in trigonometric form. Graph all six roots on the same coordinate system.

Solve each of the following equations. Leave your solutions in trigonometric form.

23. $x^4 + 2\sqrt{3}x^2 + 4 = 0$
24. $x^4 - 4\sqrt{3}x^2 + 16 = 0$
25. $x^4 - 2x^2 + 4 = 0$
26. $x^4 + 2x^2 + 4 = 0$
27. $x^4 + 2x^2 + 2 = 0$
28. $x^4 - 2x^2 + 2 = 0$

Review Problems The problems below review material we covered in Sections 4.2 and 4.3.

Graph each equation on the given interval.

29. $y = -2 \sin(-3x),\ 0 \le x \le 2\pi$
30. $y = -2 \cos(-3x),\ 0 \le x \le 2\pi$

Graph one complete cycle of each of the following:

31. $y = -\cos\left(2x + \frac{\pi}{2}\right)$
32. $y = -\cos\left(2x - \frac{\pi}{2}\right)$
33. $y = 3 \sin\left(\frac{\pi}{3}x - \frac{\pi}{3}\right)$
34. $y = 3 \cos\left(\frac{\pi}{3}x - \frac{\pi}{3}\right)$

In the rectangular coordinate system, we locate points in the plane by indicating their distance from two perpendicular lines (the x-axis and y-axis).

The *polar coordinate system* also allows us to locate points in the plane but by a slightly different method.

DEFINITION (POLAR COORDINATES) The ordered pair (r, θ) names the point that is r units from the origin along the number line (polar axis) that has been rotated through and angle θ from the positive x-axis. The coordinates r and θ are said to be the polar *coordinates* of the point they name. In polar coordinates, the origin is sometimes referred to as the pole.

Graphing in polar coordinates will be a little easier if we revise our coordinate system somewhat. It helps to have angles that are multiples of 15°, along with circles centered at the origin with radii of 1, 2, 3, 4, 5, and 6.

▼ **Example 1** Graph the points (3, 45°), (2, 120°), (−4, 60°), and (−5, 150°) on a polar coordinate system.

Solution To graph (3, 45°), we locate the point that is 3 units from the origin along the terminal side of 45°.

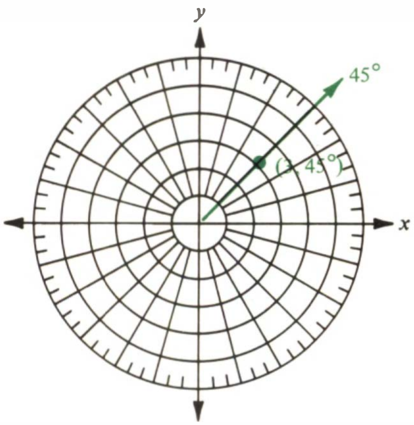


Figure 1

The point (2, 120°) is 2 units out on the terminal side of 120°.

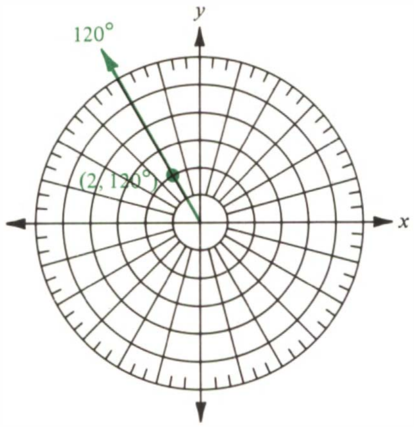


Figure 2

As you can see from Figures 1 and 2, if r is positive, we locate the point (r, θ) along the terminal side of θ. The next two points we will graph have negative values of r. To graph a point (r, θ) in which r is negative, we look for the point on the *projection* of the terminal side of θ through the origin.

To graph (−4, 60°), we locate the point that is 4 units from the origin on the projection of 60° through the origin.

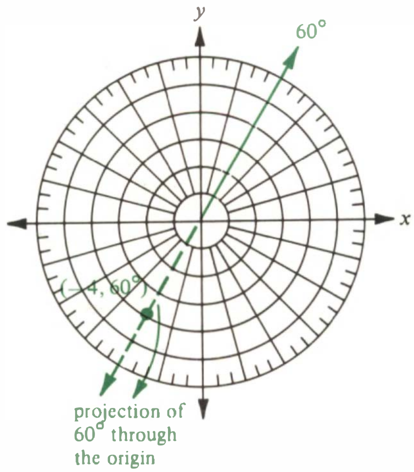


Figure 3

To graph $(-5, 150°)$, we look for the point that is 5 units from the origin along the projection of 150° through the origin.

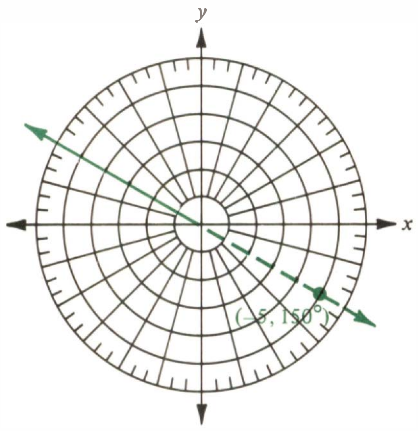


Figure 4

In rectangular coordinates, each point in the plane is named by a unique ordered pair (x, y). That is, no point can be named by two different ordered pairs. The same is not true of points named by polar coordinates, as Example 2 illustrates.

Example 2 Give three other ordered pairs that name the same point as $(3, 60°)$.

Solution As Figure 5 illustrates, the points $(-3, 240°)$, $(-3, -120°)$, and $(3, -300°)$ all name the point $(3, 60°)$.

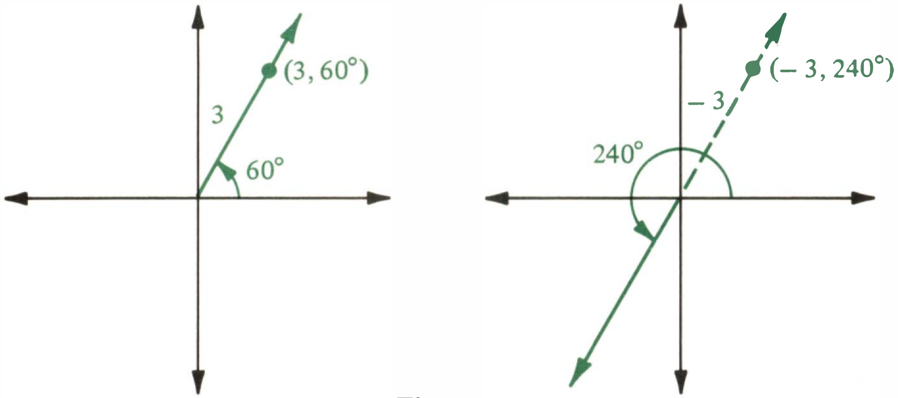


Figure 5

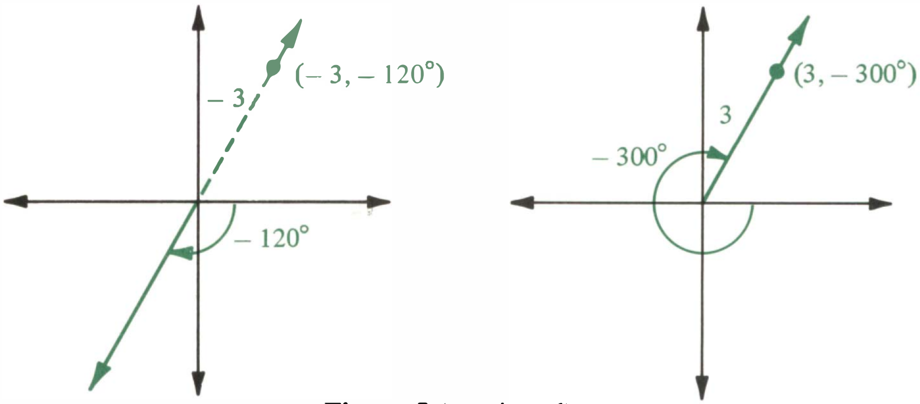


Figure 5 (continued)

There are actually an infinite number of ordered pairs that name the point (3, 60°). Any angle that is coterminal with 60° will have its terminal side pass through (3, 60°). Therefore, all points of the form

$$(3,\ 60^\circ + 360^\circ k) \qquad k = \text{an integer}$$

will name the point (3, 60°). ▲

Polar Coordinates and Rectangular Coordinates

To derive the relationship between polar coordinates and rectangular coordinates, we consider a point P with rectangular coordinates (x, y) and polar coordinates (r, θ).

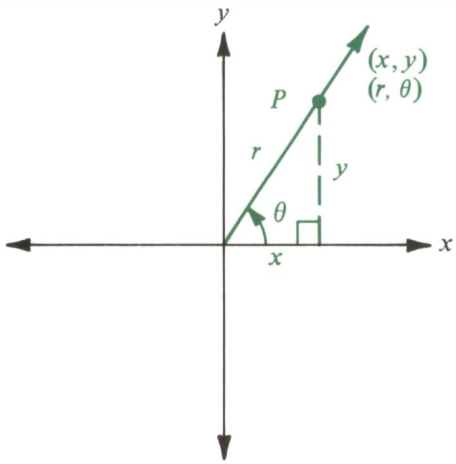


Figure 6

To convert back and forth between polar and rectangular coordinates, we simply use the relationships that exist among x, y, r, and θ in Figure 6.

To Convert from Rectangular Coordinates to Polar Coordinates

$$\text{Let } r = \pm\sqrt{x^2 + y^2} \text{ and } \tan\theta = \frac{y}{x}$$

Where the sign of r and the choice of θ place the point (r, θ) in the same quadrant as (x, y).

To Convert from Polar Coordinates to Rectangular Coordinates

$$\text{Let } x = r\cos\theta \text{ and } y = r\sin\theta$$

The process of converting to rectangular coordinates is simply a matter of substituting r and θ into the equations given above. To convert to polar coordinates we have to choose θ and the sign of r so that the point (r, θ) is in the same quadrant as the point (x, y).

Example 3 Convert to rectangular coordinates.

a. $(4, 30°)$ **b.** $(-\sqrt{2}, 135°)$ **c.** $(3, 270°)$

Solution To convert from polar coordinates to rectangular coordinates, we substitute the given values of r and θ into the equations

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

Here are the conversions for each point along with the graphs in both rectangular and polar coordinates.

a. $x = 4\cos 30° = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$

$y = 4\sin 30° = 4\left(\frac{1}{2}\right) = 2$

The point $(2\sqrt{3}, 2)$ in rectangular coordinates is equivalent to $(4, 30°)$ in polar coordinates.

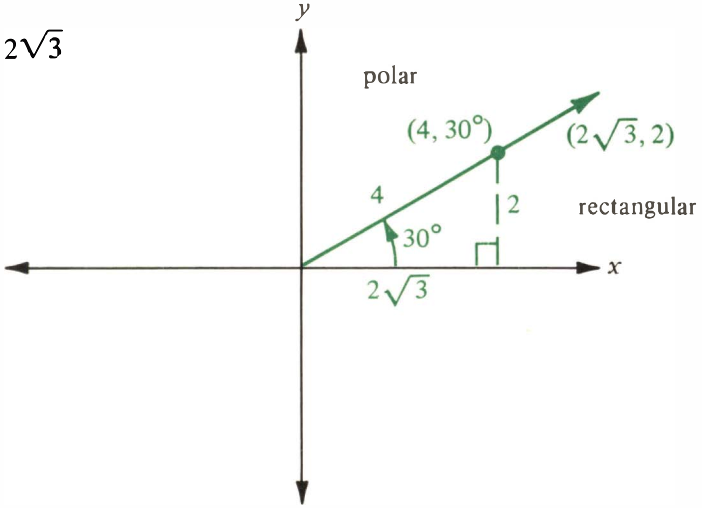


Figure 7

b. $x = -\sqrt{2}\cos 135° = -\sqrt{2}\left(-\frac{1}{\sqrt{2}}\right) = 1$

$y = -\sqrt{2}\sin 135° = -\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = -1$

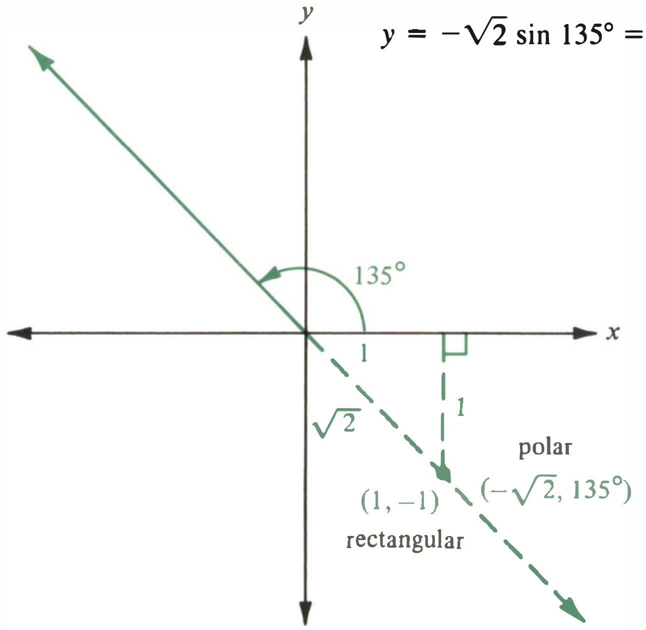


Figure 8

The point $(1, -1)$ in rectangular coordinates is equivalent to $(-\sqrt{2}, 135°)$ in polar coordinates.

c. $x = 3\cos 270° = 3(0) = 0$
$y = 3\sin 270° = 3(-1) = -3$

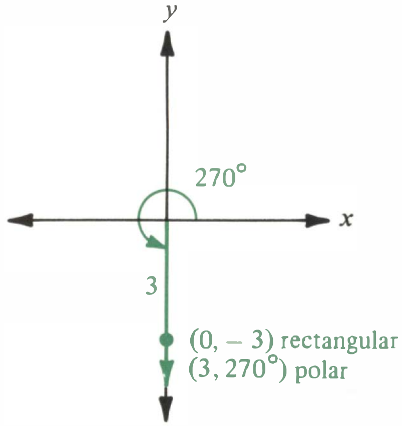


Figure 9

The point $(0, -3)$ in rectangular coordinates is equivalent to $(3, 270°)$ in polar coordinates. ▲

Example 4 **Convert to polar coordinates.**

a. $(3, 3)$ **b.** $(-2, 0)$ **c.** $(-1, \sqrt{3})$

Solution

a. Since x is 3 and y is 3, we have

$$r = \pm\sqrt{9 + 9} = \pm 3\sqrt{2} \text{ and } \tan\theta = \frac{3}{3} = 1$$

Since $(3, 3)$ is in quadrant I, we can choose $r = 3\sqrt{2}$ and $\theta = 45°$. Remember, there are an infinite number of ordered pairs in polar coordinates that name the point $(3, 3)$. The point $(3\sqrt{2}, 45°)$ is just one of them. Generally, we choose r and θ so that θ is between 0° and 360°, and r is positive.

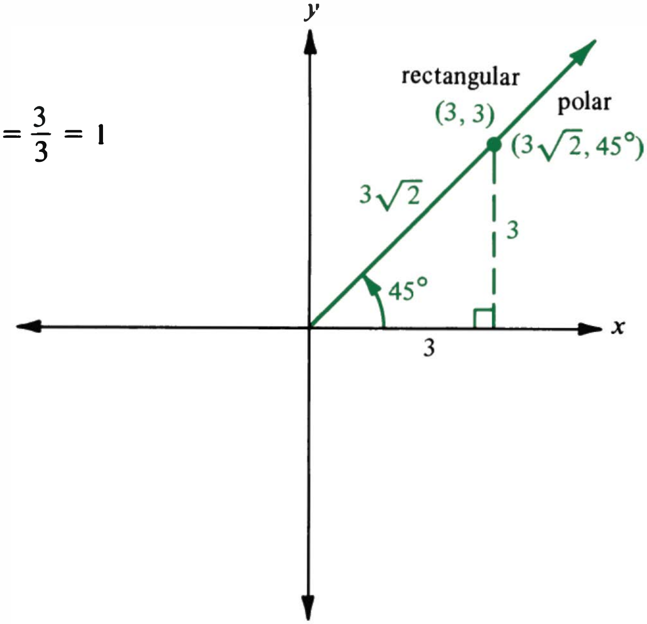


Figure 10

b. We have $x = -2$ and $y = 0$, so

$$r = \pm\sqrt{4 + 0} = \pm 2 \text{ and } \tan\theta = \frac{0}{-2} \quad 0$$

Since $(-2, 0)$ is on the negative x-axis, we can choose $r = 2$ and $\theta = 180°$ to get the point $(2, 180°)$.

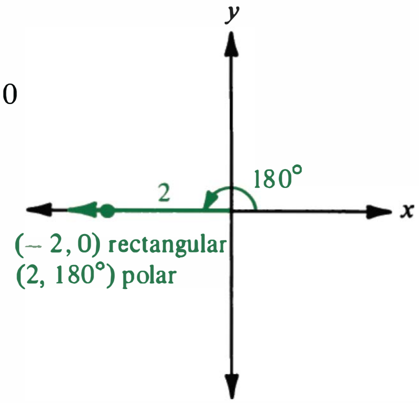


Figure 11

c. Since $x = -1$ and $y = \sqrt{3}$, we have

$$r = \pm\sqrt{1 + 3} = \pm 2 \text{ and } \tan\theta = \frac{\sqrt{3}}{-1}$$

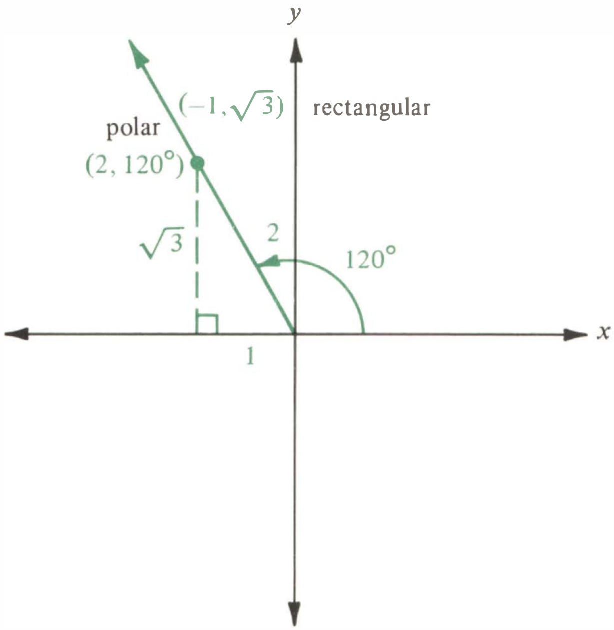


Figure 12

Since $(-1, \sqrt{3})$ is in quadrant II, we can let $r = 2$ and $\theta = 120°$. In polar coordinates the point is $(2, 120°)$. ▲

Equations in Polar Coordinates

Equations in polar coordinates have variables r and θ instead of x and y. The conversions we used to change ordered pairs from polar coordinates to rectangular coordinates and from rectangular coordinates to polar coordinates are the same ones we use to convert back and forth between equations given in polar coordinates and those in rectangular coordinates.

▼ **Example 5** Change $r^2 = 9 \sin 2\theta$ to rectangular coordinates.

Solution Before we substitute to clear the equation of r and θ, we must use a double-angle identity to write $\sin 2\theta$ in terms of $\sin\theta$ and $\cos\theta$.

$$r^2 = 9 \sin 2\theta$$

$$r^2 = 9 \cdot 2 \sin\theta \cos\theta \qquad \text{Double-angle identity}$$

$$r^2 = 18 \cdot \frac{y}{r} \cdot \frac{x}{r} \qquad \text{Substitute } y/r \text{ for } \sin\theta \text{ and } x/r \text{ for } \cos\theta$$

$$r^2 = \frac{18xy}{r^2} \qquad \text{Multiply}$$

$$r^4 = 18xy \qquad \text{Multiply both sides by } r^2$$

$$(x^2 + y^2)^2 = 18xy \qquad \text{Substitute } x^2 + y^2 \text{ for } r^2$$

▲

▼ **Example 6** Change $x + y = 4$ to polar coordinates.

Solution Since $x = r \cos \theta$ and $y = r \sin \theta$, we have

$$r \cos \theta + r \sin \theta = 4$$

$$r(\cos \theta + \sin \theta) = 4 \qquad \text{Factor out } r$$

$$r = \frac{4}{\cos \theta + \sin \theta} \qquad \text{Divide both sides by } \cos \theta + \sin \theta$$

The last equation gives us r in terms of θ. ▲

Problem Set 8.5

Graph each ordered pair on a polar coordinate system.

1. (2, 45°)
2. (3, 60°)
3. (3, 150°)
4. (4, 135°)
5. (1, −225°)
6. (2, −240°)
7. (−3, 45°)
8. (−4, 60°)
9. (−4, −210°)
10. (−5, −225°)
11. (−2, 0°)
12. (−2, 270°)

For each ordered pair, give three other ordered pairs with θ between −360° and 360° that name the same point.

13. (2, 60°)
14. (1, 30°)
15. (5, 135°)
16. (3, 120°)
17. (−3, 30°)
18. (−2, 45°)

Convert to rectangular coordinates. Use exact values.

19. (2, 60°)
20. (−2, 60°)
21. (3, 270°)
22. (1, 180°)
23. $(\sqrt{2}, -135°)$
24. $(\sqrt{2}, -225°)$
25. $(-4\sqrt{3}, 30°)$
26. $(4\sqrt{3}, -30°)$

Convert to polar coordinates with $r \geq 0$ and θ between 0° and 360°.

27. (−3, 3)
28. (−3, −3)
29. $(-2\sqrt{3}, 2)$
30. $(2, -2\sqrt{3})$
31. (2, 0)
32. (−2, 0)
33. $(-\sqrt{3}, -1)$
34. $(-1, -\sqrt{3})$

Convert to polar coordinates. Use a calculator or table to find θ to the nearest tenth of a degree. Keep r positive and θ between 0° and 360°.

35. (3, 4)
36. (4, 3)
37. (−1, 2)
38. (1, −2)
39. (−2, −3)
40. (−3, −2)

Write each equation with rectangular coordinates.

41. $r^2 = 9$
42. $r^2 = 4$
43. $r = 6 \sin \theta$
44. $r = 6 \cos \theta$
45. $r^2 = 4 \sin 2\theta$
46. $r^2 = 4 \cos 2\theta$
47. $r(\cos \theta + \sin \theta) = 3$
48. $r(\cos \theta - \sin \theta) = 2$

Write each equation in polar coordinates.

49. $x - y = 5$
50. $x + y = 5$
51. $x^2 + y^2 = 4$
52. $x^2 + y^2 = 9$
53. $x^2 + y^2 = 6x$
54. $x^2 + y^2 = 4x$
55. $y = x$
56. $y = -x$

Review Problems The problems that follow review material we covered in Sections 4.2 and 4.4. Reviewing these problems will help you with the next section.

Graph one complete cycle of each equation.

57. $y = 6 \sin x$
58. $y = 6 \cos x$
59. $y = 4 \sin 2x$
60. $y = 2 \sin 4x$
61. $y = 4 + 2 \sin x$
62. $y = 4 + 2 \cos x$

8.6 Equations in Polar Coordinates and Their Graphs

In this section we will consider the graphs of polar equations. The solutions to these equations are ordered pairs (r, θ), where r and θ are the polar coordinates we defined in Section 8.5

Example 1 Sketch the graph of $r = 6 \sin \theta$.

Solution We can find ordered pairs (r, θ) that satisfy the equation by making a table. The table is a little different than the ones we made for rectangular coordinates. With polar coordinates, we substitute in convenient values for θ and then use the equation to find corresponding values of r. Let's use multiples of 30° and 45° for θ.

Table 1

θ	$r = 6 \sin \theta$	r	(r, θ)
0°	$r = 6 \sin 0° = 0$	0	(0, 0°)
30°	$r = 6 \sin 30° = 3$	3	(3, 30°)
45°	$r = 6 \sin 45° = 4.2$	4.2	(4.2, 45°)
60°	$r = 6 \sin 60° = 5.2$	5.2	(5.2, 60°)
90°	$r = 6 \sin 90° = 6$	6	(6, 90°)
120°	$r = 6 \sin 120° = 5.2$	5.2	(5.2, 120°)
135°	$r = 6 \sin 135° = 4.2$	4.2	(4.2, 135°)
150°	$r = 6 \sin 150° = 3$	3	(3, 150°)
180°	$r = 6 \sin 180° = 0$	0	(0, 180°)
210°	$r = 6 \sin 210° = -3$	−3	(−3, 210°)
225°	$r = 6 \sin 225° = -4.2$	−4.2	(−4.2, 225°)
240°	$r = 6 \sin 240° = -5.2$	−5.2	(−5.2, 240°)
270°	$r = 6 \sin 270° = -6$	−6	(−6, 270°)
300°	$r = 6 \sin 300° = -5.2$	−5.2	(−5.2, 300°)
315°	$r = 6 \sin 315° = -4.2$	−4.2	(−4.2, 315°)
330°	$r = 6 \sin 330° = -3$	−3	(−3, 330°)
360°	$r = 6 \sin 360° = 0$	0	(0, 360°)

Note If we were to continue past 360° with values of θ, we would simply start to repeat the values of r we have already obtained, since $\sin \theta$ is periodic with period 360°.

Plotting each point on a polar coordinate system and then drawing a smooth curve through them, we have the graph in Figure 1.

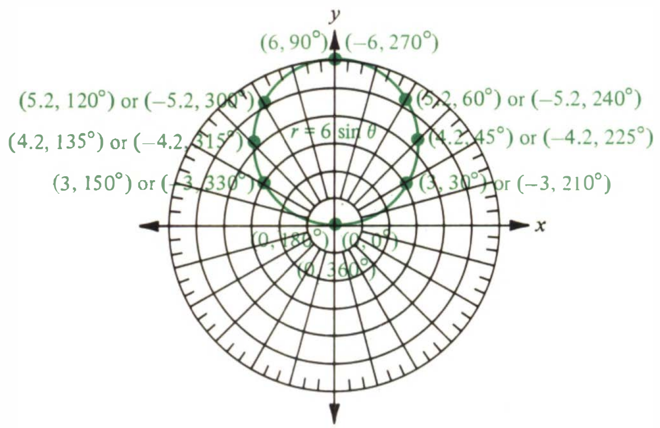


Figure 1

Could we have found the graph of $r = 6 \sin \theta$ in Example 1 without making a table? The answer is yes, there are a couple of other ways to do so. One way is to convert to rectangular coordinates and see if we recognize the graph from the rectangular equation. We begin by replacing $\sin \theta$ with y/r.

$$r = 6\frac{y}{r} \qquad \sin\theta = \frac{y}{r}$$

$$r^2 = 6y \qquad \text{Multiply both sides by } r$$

$$x^2 + y^2 = 6y \qquad r^2 = x^2 + y^2$$

The equation is now written in terms of rectangular coordinates. If we add $-6y$ to both sides and then complete the square on y, we will obtain the rectangular equation of a circle with center at (0, 3) and a radius of 3.

$$x^2 + y^2 - 6y = 0 \qquad \text{Add } -6y \text{ to both sides}$$

$$x^2 + y^2 - 6y + 9 = 9 \qquad \text{Complete the square on } y \text{ by adding 9 to both sides.}$$

$$x^2 + (y - 3)^2 = 3^2 \qquad \text{Standard form for the equation of a circle}$$

This method of graphing, by changing to rectangular coordinates, only works well in some cases. Many of the equations we will encounter in polar coordinates do not have graphs that are recognizable in rectangular form.

In Example 2 we will look at another method of graphing polar equations that does not depend on the use of a table.

▼ **Example 2** Sketch the graph of $r = 4 \sin 2\theta$.

Solution One way to visualize the relationship between r and θ as given by the equation $r = 4 \sin 2\theta$ is to sketch the graph of $y = 4 \sin 2x$ on a rectangular coordinate system. (Since we have been using degree measure for our angles in polar coordinates, we will label the x-axis for the graph of $y = 4 \sin 2x$ in degrees rather than radians as we usually do.) The graph of $y = 4 \sin 2x$ will have an amplitude of 4 and a period of $360°/2 = 180°$. Figure 2 shows the graph of $y = 4 \sin 2x$ between 0° and 360°.

As you can see in Figure 2, as x goes from 0° to 45°, y goes from 0 to 4. This means that, for the equation $r = 4 \sin 2\theta$, as θ goes from 0° to 45°, r will go from 0 up to 4. A diagram of this is shown in Figure 3.

As x continues from 45° to 90°, y decreases from 4 down to 0. Likewise, as θ rotates through 45° to 90°, r will decrease from 4 down to 0. A diagram of this is shown in Figure 4. The numbers 1 and 2 in Figure 4 indicate the order in which those sections of the graph are drawn.

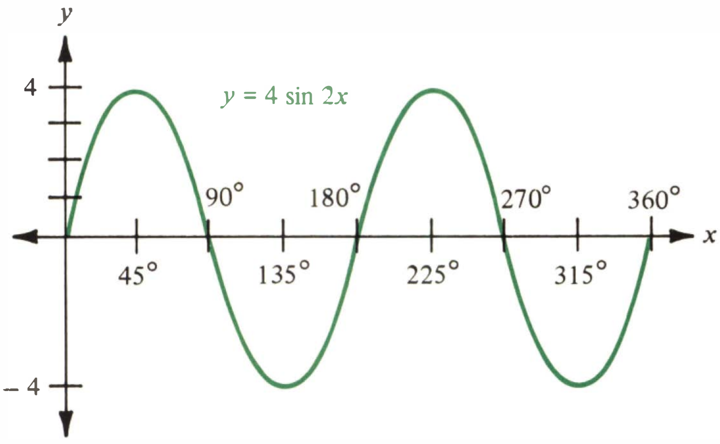


Figure 2

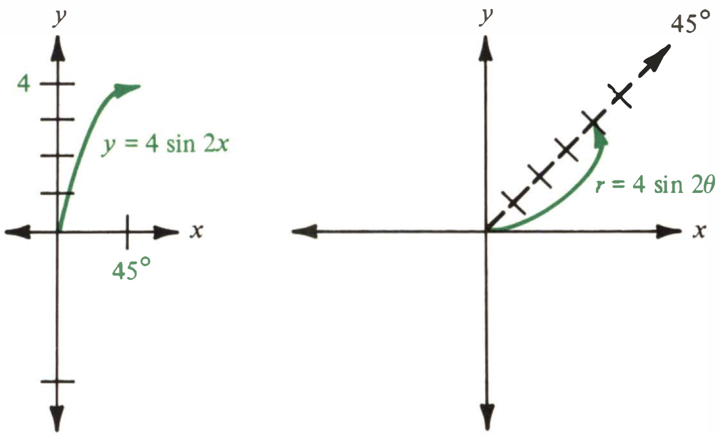


Figure 3

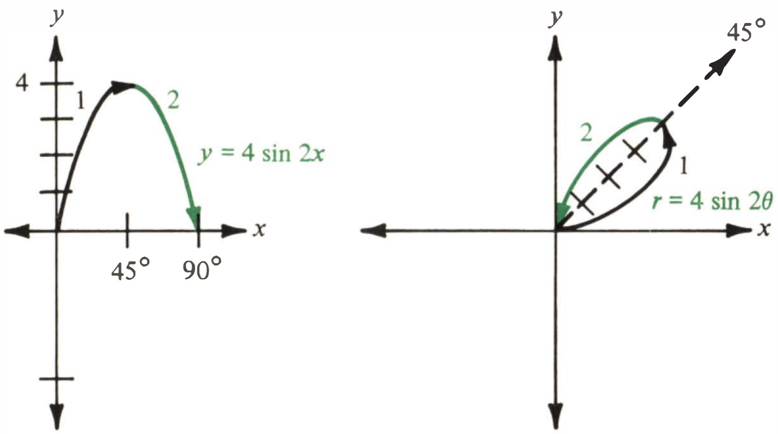


Figure 4

If we continue to reason in this manner, we will obtain a good sketch of the graph of $r = 4 \sin 2\theta$ by watching how y is affected by changes in x on the graph of $y = 4 \sin 2x$. Table 2 summarizes this information, and Figure 5 contains both the graph of $y = 4 \sin 2x$ and $r = 4 \sin 2\theta$.

Table 2

Reference Number on Graphs	Variations in x (or θ)	Corresponding Variations in y (or r)
1.	0° to 45°	0 to 4
2.	45° to 90°	4 to 0
3.	90° to 135°	0 to −4
4.	135° to 180°	−4 to 0
5.	180° to 225°	0 to 4
6.	225° to 270°	4 to 0
7.	270° to 315°	0 to −4
8.	315° to 360°	−4 to 0

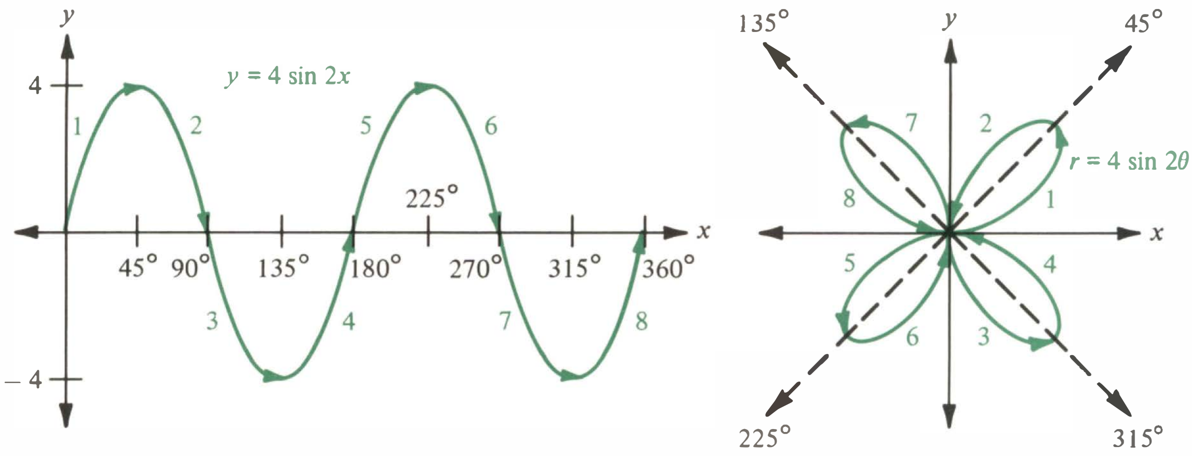


Figure 5

Example 3 Sketch the graph of $r = 4 + 2 \sin \theta$.

Solution The graph of $r = 4 + 2 \sin \theta$ (Figure 7) is obtained by first graphing $y = 4 + 2 \sin x$ (Figure 6) and then noticing the relationship between variations in x and the corresponding variations in y. These variations are equivalent to those that exist between θ and r.

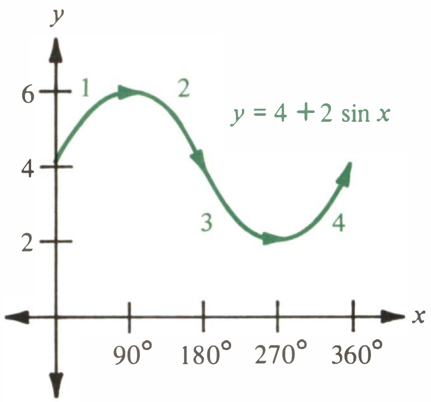


Figure 6

Table 3

Reference Number on Graphs	Variations in x (or θ)	Corresponding Variations in y (or r)
1.	0° to 90°	4 to 6
2.	90° to 180°	6 to 4
3.	180° to 270°	4 to 2
4.	270° to 360°	2 to 4

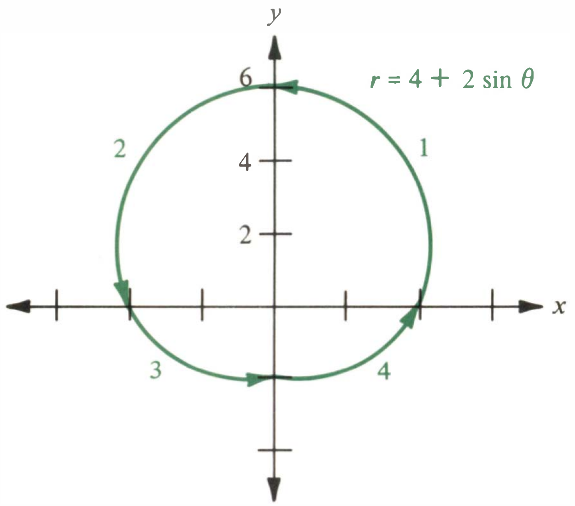


Figure 7

Although the method of graphing presented in Examples 2 and 3 is sometimes difficult to comprehend at first, with a little practice it becomes much easier. In any case, the usual alternative is to make a table and plot points until the shape of the curve can be recognized. Probably the best way to graph these equations is to use a combination of both methods.

Here are some other common graphs in polar coordinates along with the equations that produce them. When you start graphing some of the equations in Problem Set 8.6, you may want to keep these graphs handy for reference. It is sometimes easier to get started when you can anticipate the general shape of the curve.

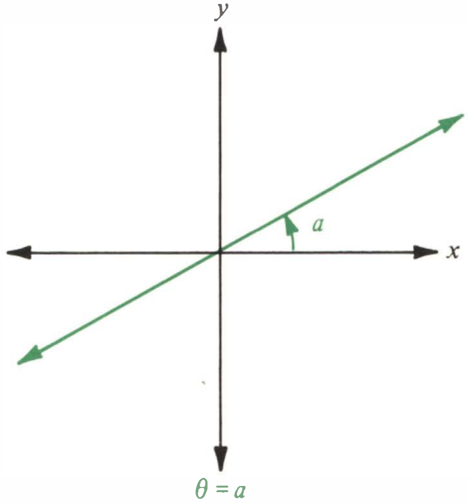


Figure 8

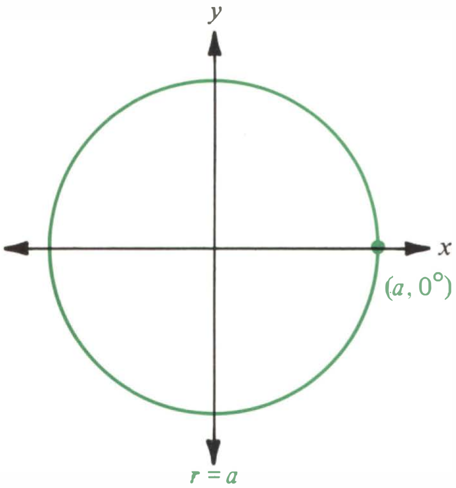


Figure 9

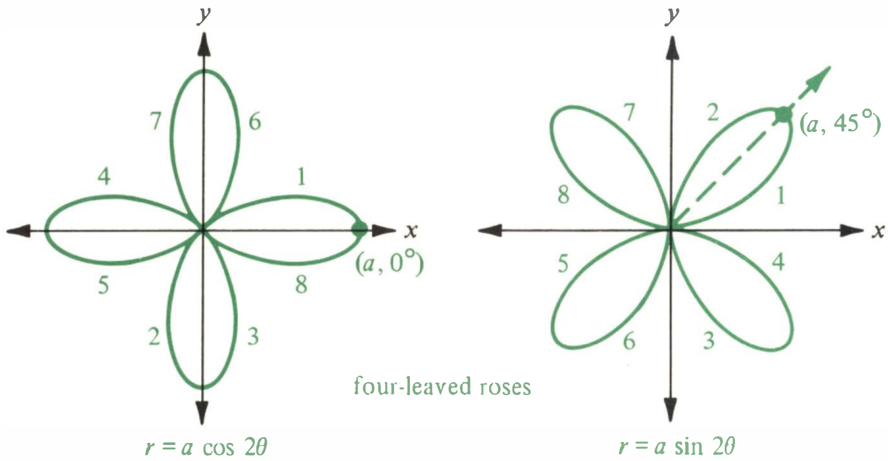


Figure 10

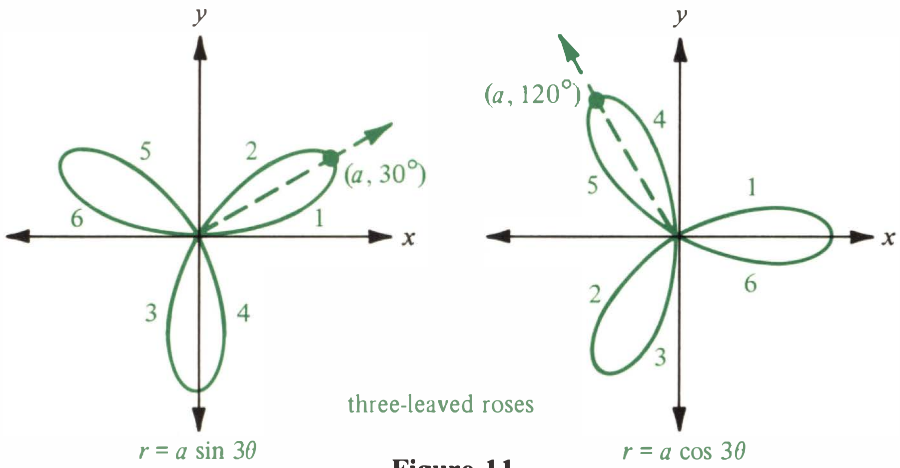


Figure 11

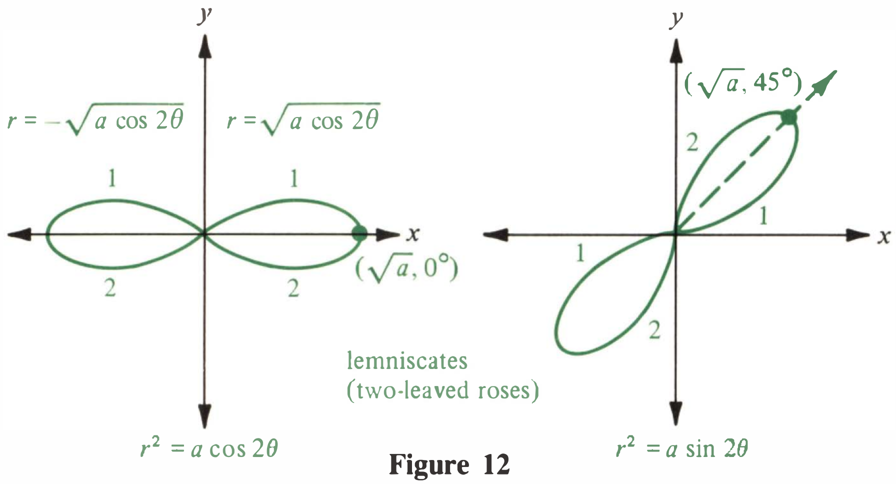


Figure 12

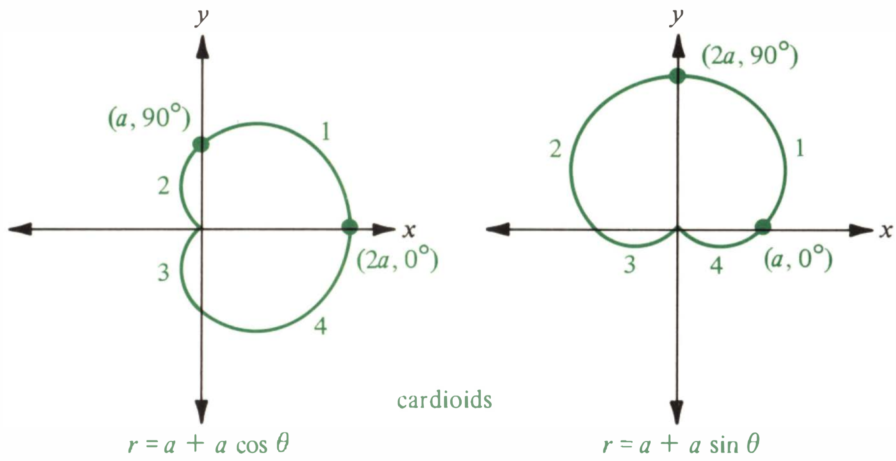


Figure 13

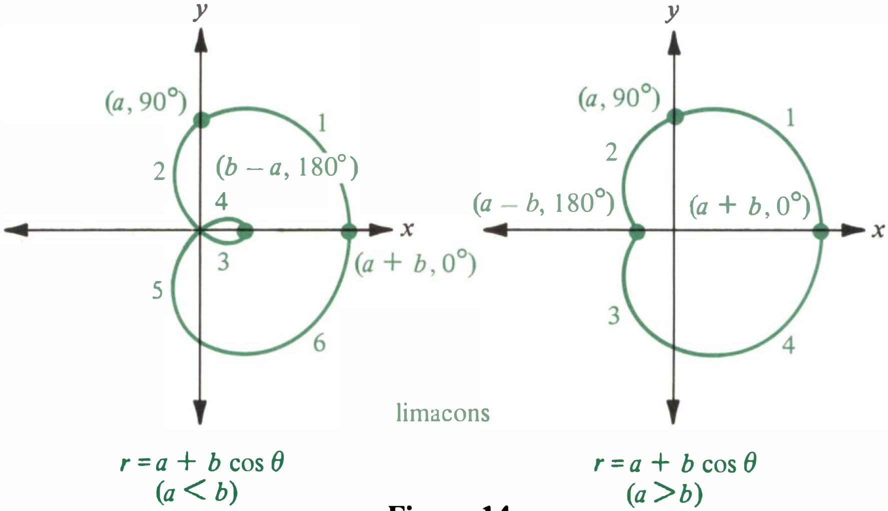


Figure 14

Problem Set 8.6

Sketch the graph of each equation by making a table using values of θ that are multiples of 45°.

1. $r = 6\cos\theta$
2. $r = 4\sin\theta$
3. $r = \sin 3\theta$
4. $r = \cos 3\theta$

Graph each equation.

5. $r = 3$
6. $r = 2$
7. $\theta = 45°$
8. $\theta = 135°$
9. $r = 3\sin\theta$
10. $r = 3\cos\theta$
11. $r = 4 + 2\sin\theta$
12. $r = 4 + 2\cos\theta$
13. $r = 2 + 4\cos\theta$
14. $r = 2 + 4\sin\theta$
15. $r = 2 + 2\sin\theta$
16. $r = 2 + 2\cos\theta$
17. $r^2 = 9\sin 2\theta$
18. $r^2 = 4\cos 2\theta$
19. $r = 2\sin 2\theta$
20. $r = 2\cos 2\theta$
21. $r = 4\cos 3\theta$
22. $r = 4\sin 3\theta$

Convert each equation to polar coordinates and then sketch the graph.

23. $x^2 + y^2 = 16$
24. $x^2 + y^2 = 25$
25. $x^2 + y^2 = 6x$
26. $x^2 + y^2 = 6y$
27. $(x^2 + y^2)^2 = 2xy$
28. $(x^2 + y^2)^2 = x^2 - y^2$

Change each equation to rectangular coordinates and then graph.

29. $r(2\cos\theta + 3\sin\theta) = 6$
30. $r(3\cos\theta - 2\sin\theta) = 6$
31. $r(1 - \cos\theta) = 1$
32. $r(1 - \sin\theta) = 1$
33. $r = 4\sin\theta$
34. $r = 6\cos\theta$

35. Graph $r = 2 \sin \theta$ and $r = 2 \cos \theta$ and then name two points they have in common.

36. Graph $r = 2 + 2 \cos \theta$ and $r = 2 - 2 \cos \theta$ and name three points they have in common.

Review Problems The problems that follow review material we covered in Section 4.4.

Graph each equation.

37. $y = \sin x - \cos x,\ 0 \le x \le 4\pi$

38. $y = \cos x - \sin x,\ 0 \le x \le 4\pi$

39. $y = x + \sin \pi x,\ 0 \le x \le 8$

40. $y = x + \cos \pi x,\ 0 \le x \le 8$

41. $y = 3 \sin x + \cos 2x,\ 0 \le x \le 4\pi$

42. $y = \sin x + \frac{1}{2} \cos 2x, 0 \le x \le 4\pi$

Chapter 8 Summary and Review

DEFINITIONS [8.1]

The number i is such that $i^2 = -1$. If $a > 0$, the expression $\sqrt{-a}$ can be written as $\sqrt{ai^2} = i\sqrt{a}$.

A *complex number* is any number that can be written in the form

$$a + bi$$

where a and b are real numbers and $i^2 = -1$. The number a is called the *real part* of the complex number and b is called the *imaginary part*. The form $a + bi$ is called *standard form*.

All real numbers are also complex numbers since they can be put in the form $a + bi$, where $b = 0$. If $b \neq 0$, then $a + bi$ is also called an *imaginary number*.

Examples

1. Each of the following is a complex number.

$$5 + 4i$$
$$-\sqrt{3} + i$$
$$7i$$
$$-8$$

the number $7i$ is complex because

$$7i = 0 + 7i$$

the number -8 is complex because

$$-8 = -8 + 0i$$

EQUALITY FOR COMPLEX NUMBERS [8.1]

Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal. That is,

$$a + bi = c + di \text{ if and only if } a = c \text{ and } b = d$$

2. If $3x + 2i = 12 - 4yi$, then

$3x = 12$ and $2 = -4y$

$x = 4$ $\quad$ $y = -1/2$

OPERATIONS ON COMPLEX NUMBERS IN STANDARD FORM [8.1]

If $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ are two complex numbers in standard form, then the following definitions and operations apply.

Addition

$z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$
Add real parts, add imaginary parts.

Subtraction

$z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2)i$
Subtract real parts, subtract imaginary parts.

Multiplication

$z_1 z_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i$
In actual practice, simply multiply as you would multiply two binomials.

Conjugates

The conjugate of $a + bi$ is $a - bi$. Their product is the real number $a^2 + b^2$.

Division

Multiply the numerator and denominator of the quotient by the conjugate of the denominator.

3. If $z_1 = 2 - i$ and $z_2 = 4 + 3i$, then

$$z_1 + z_2 = 6 + 2i$$

$$z_1 - z_2 = -2 - 4i$$

$$\begin{aligned} z_1 z_2 &= (2 - i)(4 + 3i) \\ &= 8 + 6i - 4i - 3i^2 \\ &= 11 + 2i \end{aligned}$$

the conjugate of $4 + 3i$ is $4 - 3i$ and $(4 + 3i)(4 - 3i) = 16 + 9 = 25$

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{2 - i}{4 + 3i} \cdot \frac{\mathbf{4 - 3i}}{\mathbf{4 - 3i}} \\ &= \frac{5 - 10i}{25} \\ &= \frac{1}{5} - \frac{2}{5}i \end{aligned}$$

POWERS OF *i* [8.1]

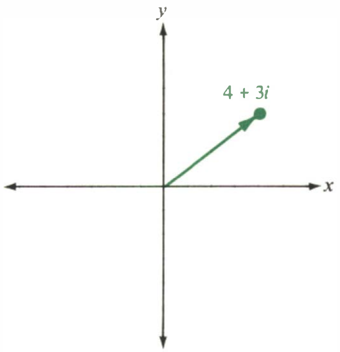
If n is an integer, then i^n can always be simplified to either i, -1, $-i$, or 1.

4.
$$\begin{aligned} i^{20} &= (i^4)^5 = 1 \\ i^{21} &= (i^4)^5 \cdot i = i \\ i^{22} &= (i^4)^5 \cdot i^2 = -1 \\ i^{23} &= (i^4)^5 \cdot i^3 = -i \end{aligned}$$

GRAPHING COMPLEX NUMBERS [8.2]

The graph of the complex number $z = a + bi$ is the arrow (vector) that extends from the origin to the point (a, b).

5. The graph of $4 + 3i$ is



ABSOLUTE VALUE OF A COMPLEX NUMBER [8.2]

The *absolute value* (or *modulus*) of the complex number $z = a + bi$ is the distance from the origin to the point (a, b). If this distance is denoted by r then

$$r = |z| = |a + bi| = \sqrt{a^2 + b^2}$$

6. If $z = \sqrt{3} + i$, then

$$|z| = |\sqrt{3} + i| = \sqrt{3 + 1} = 2$$

ARGUMENT OF A COMPLEX NUMBER [8.2]

The argument of the complex number $z = a + bi$ is the smallest positive angle from the positive x-axis to the graph of z. If the argument of z is denoted by θ then,

$$\sin \theta = \frac{b}{r}, \cos \theta = \frac{a}{r}, \text{ and } \tan \theta = \frac{b}{a}$$

7. For $z = \sqrt{3} + i$, θ is the smallest positive angle for which

$$\sin \theta = \frac{1}{2} \text{ and } \cos \theta = \frac{\sqrt{3}}{2}$$

which means $\theta = 30°$

TRIGONOMETRIC FORM OF A COMPLEX NUMBER [8.2]

The complex number $z = a + bi$ is written in trigonometric form when it is written as

$$z = r(\cos \theta + i \sin \theta)$$

where r is the absolute value of z and θ is the argument of z.

8. If $z = \sqrt{3} + i$, then in trigonometric form

$$z = 2(\cos 30° + i \sin 30°)$$

PRODUCTS AND QUOTIENTS IN TRIGONOMETRIC FORM [8.3]

If $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ are two complex numbers in trigonometric form, then

their product is

$$z_1 z_2 = r_1 r_2[\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)]$$

their quotient is

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2)]$$

9. If $z_1 = 8(\cos 40° + i \sin 40°)$ and $z_2 = 4(\cos 10° + i \sin 10°)$, then

$$z_1 z_2 = 32(\cos 50° + i \sin 50°)$$

$$\frac{z_1}{z_2} = 2(\cos 30° + i \sin 30°)$$

10. If $z = \sqrt{2}(\cos 30° + i \sin 30°)$, then

$$z^{10} = \sqrt{2}^{10}(\cos 10 \cdot 30° + i \sin 10 \cdot 30°)$$
$$= 32(\cos 300° + i \sin 300°)$$

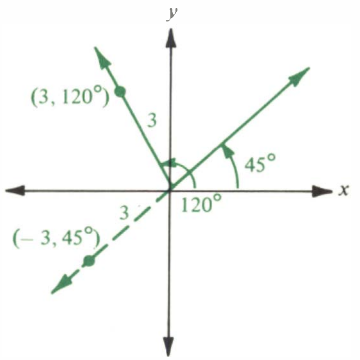
11. The 3 cube roots of $z = 8(\cos 60° + i \sin 60°)$ are given by

$$w_k = 8^{1/3}\left(\cos \frac{60° + 360°k}{3} + i \sin \frac{60° + 360°k}{3}\right)$$
$$= 2[\cos(20° + 120°k) + i \sin(20° + 120°k)]$$

when $k = 0, 1, 2$. That is,

$$w_0 = 2(\cos 20° + i \sin 20°)$$
$$w_1 = 2(\cos 140° + i \sin 140°)$$
$$w_2 = 2(\cos 260° + i \sin 260°)$$

12.



DEMOIVRE'S THEOREM [8.4]

If $z = r(\cos \theta + i \sin \theta)$ is a complex number in trigonometric form and n is an integer, then

$$z^n = r^n(\cos n\theta + i \sin n\theta)$$

ROOTS OF A COMPLEX NUMBER [8.4]

The n nth roots of the complex number $z = r(\cos \theta + i \sin \theta)$ are given by the formula

$$w_k = r^{1/n}\left(\cos \frac{\theta + 360°k}{n} + i \sin \frac{\theta + 360°k}{n}\right)$$

where $k = 0, 1, 2, \ldots, n - 1$. That is, the n nth roots are

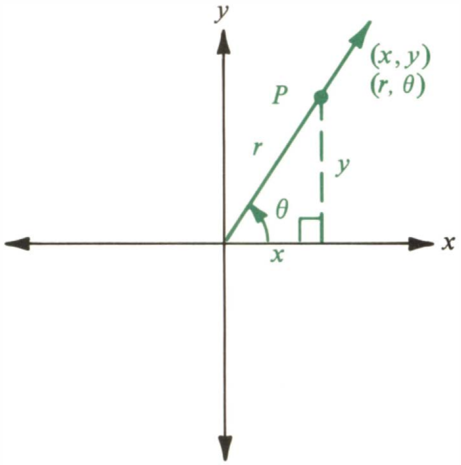
$$w_0 = r^{1/n}\left(\cos \frac{\theta}{n} + i \sin \frac{\theta}{n}\right)$$
$$w_1 = r^{1/n}\left(\cos \frac{\theta + 360°}{n} + i \sin \frac{\theta + 360°}{n}\right)$$
$$w_2 = r^{1/n}\left(\cos \frac{\theta + 720°}{n} + i \sin \frac{\theta + 720°}{n}\right)$$
$$\vdots$$
$$w_{n-1} = r^{1/n}\left(\cos \frac{\theta + 360°(n - 1)}{n} + i \sin \frac{\theta + 360°(n - 1)}{n}\right)$$

POLAR COORDINATES [8.5]

The ordered pair (r, θ) names the point that is r units from the origin along the axis rotated through θ degrees from the positive x-axis. The coordinates r and θ are said to be the *polar coordinates* of the point they name.

POLAR COORDINATES AND RECTANGULAR COORDINATES [8.5]

To derive the relationship between polar coordinates and rectangular coordinates, we consider a point P with rectangular coordinates (x, y) and polar coordinates (r, θ).



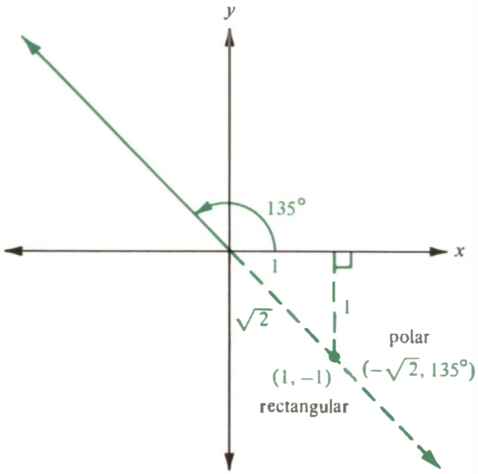
13. Convert $(-\sqrt{2}, 135°)$ to rectangular coordinates.

To convert from polar coordinates to rectangular coordinates, we substitute the given values of r and θ into the equations

$x = r \cos \theta$ and $y = r \sin \theta$

$$x = -\sqrt{2} \cos 135° = -\sqrt{2}\left(-\frac{1}{\sqrt{2}}\right) = 1$$

$$y = -\sqrt{2} \sin 135° = -\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = -1$$



EQUATIONS IN POLAR COORDINATES [8.5]

Equations in polar coordinates have variables r and θ instead of x and y. The conversions we use to change ordered pairs from polar coordinates to rectangular coordinates and from rectangular coordinates to polar coordinates are the same ones we use to convert back and forth between equations given in polar coordinates and those in rectangular coordinates.

14. Change $x + y = 4$ to polar coordinates.

Since $x = r \cos \theta$ and $y = r \sin \theta$, we have

$r \cos \theta + r \sin \theta = 4$	
$r(\cos \theta + \sin \theta) = 4$	Factor out r
$r = \dfrac{4}{\cos \theta + \sin \theta}$	Divide both sides by $\cos \theta + \sin \theta$

The last equation gives us r in terms of θ.

Chapter 8 Test

Write in terms of i.

1. $\sqrt{-25}$
2. $\sqrt{-12}$

Find x and y so that each of the following equations is true.

3. $7x - 6i = 14 - 3yi$
4. $(x^2 - 3x) + 16i = 10 + 8yi$

Combine the following complex numbers:

5. $(6 - 3i) + [(4 - 2i) - (3 + i)]$
6. $(7 + 3i) - [(2 + i) - (3 - 4i)]$

Simplify each power of i.

7. i^{16}
8. i^{17}

Multiply. Leave your answer in standard form.

9. $(8 + 5i)(8 - 5i)$
10. $(3 + 5i)^2$

Divide. Write all answers in standard form.

11. $\dfrac{5 - 4i}{2i}$
12. $\dfrac{6 + 5i}{6 - 5i}$

For each of the following complex numbers give: (a) the absolute value; (b) the opposite; and (c) the conjugate.

13. $3 + 4i$
14. $3 - 4i$
15. $8i$
16. -4

Write each complex number in standard form.

17. $8(\cos 330° + i \sin 330°)$
18. $2(\cos 135° + i \sin 135°)$

Write each complex number in trigonometric form.

19. $2 + 2i$
20. $-\sqrt{3} + i$
21. $5i$
22. -3

Multiply or divide as indicated. Leave your answer in trigonometric form.

23. $5(\cos 25° + i \sin 25°) \cdot 3(\cos 40° + i \sin 40°)$
24. $\dfrac{10(\cos 50° + i \sin 50°)}{2(\cos 20° + i \sin 20°)}$
25. $[2(\cos 10° + i \sin 10°)]^5$
26. $[3(\cos 20° + i \sin 20°)]^4$
27. Find two square roots of $z = 49(\cos 50° + i \sin 50°)$. Leave your answer in trigonometric form.
28. Find four fourth roots for $z = 2 + 2i\sqrt{3}$. Leave your answer in trigonometric form.

Solve each equation. Write your solutions in trigonometric form.

29. $x^4 - 2\sqrt{3}x^2 + 4 = 0$

30. $x^3 = -1$

For each of the following points, name two other ordered pairs in polar coordinates that name the same point and then convert each of the original ordered pairs to rectangular coordinates.

31. $(4, 225°)$

32. $(-6, 60°)$

Convert to polar coordinates with r positive and θ between 0° and 360°.

33. $(-3, 3)$

34. $(0, 5)$

Write each equation with rectangular coordinates.

35. $r = 6 \sin \theta$

36. $r = \sin 2\theta$

Write each equation in polar coordinates.

37. $x + y = 2$

38. $x^2 + y^2 = 8y$

Graph each equation.

39. $r = 4$

40. $\theta = 45°$

41. $r = 4 + 2 \cos \theta$

42. $r = \sin 2\theta$

Appendix Logarithms

To the student

Logarithms are exponents. The properties of logarithms are actually the properties of exponents. As it turns out, writing some expressions with logarithms instead of exponents allows us to solve some problems we would otherwise be unable to solve. There are many applications of logarithms to both sciences and higher mathematics. For example, the pH of a liquid is defined in terms of logarithms. (That's the same pH that is given on the label of many hair conditioners.) The Richter scale for measuring earthquake intensity is a logarithmic scale, as is the decibel scale used for measuring the intensity of sound.

A.1 Logarithms Are Exponents

As you know from your work in algebra, equations of the form

$$y = b^x \qquad (b > 0,\ b \neq 1)$$

are called exponential functions. The inverse of an exponential function is a logarithmic function. Since the equation of the inverse of a function can be obtained by exchanging x and y in the equation of the original function, the inverse of an exponential function must have the form

$$x = b^y \qquad (b > 0,\ b \neq 1)$$

Now, this last equation is actually the equation of a logarithmic function, as the following definition indicates:

Definition The expression $y = \log_b x$ is read "y is the logarithm to the base b of x" and is equivalent to the expression

$$x = b^y \qquad (b > 0,\ b \neq 1)$$

In words, we say "y is the number we raise b to in order to get x."

Notation When an expression is in the form $x = b^y$, it is said to be in exponential form. On the other hand, if an expression is in the form $y = \log_b x$, it is said to be in logarithmic form.

Here are some equivalent statements written in both forms.

Exponential form		Logarithmic form
$8 = 2^3$	$\Leftrightarrow$	$\log_2 8 = 3$
$25 = 5^2$	$\Leftrightarrow$	$\log_5 25 = 2$
$.1 = 10^{-1}$	$\Leftrightarrow$	$\log_{10} .1 = -1$
$\frac{1}{8} = 2^{-3}$	$\Leftrightarrow$	$\log_2 \frac{1}{8} = -3$
$r = z^s$	$\Leftrightarrow$	$\log_z r = s$

▼ **Example 1** Solve for x: $\log_3 x = -2$

Solution In exponential form the equation looks like this:

$$x = 3^{-2}$$

$$\text{or} \quad x = \frac{1}{9}$$

The solution is 1/9. ▲

▼ **Example 2** Solve $\log_x 4 = 3$.

Solution Again, we use the definition of logarithms to write the expression in exponential form:

$$4 = x^3$$

Taking the cube root of both sides, we have

$$\sqrt[3]{4} = \sqrt[3]{x^3}$$
$$x = \sqrt[3]{4}$$

▲

▼ **Example 3** Solve $\log_8 4 = x$.

Solution We write the expression again in exponential form:

$$4 = 8^x$$

Since both 4 and 8 can be written as powers of 2, we write them in terms of powers of 2:

$$2^2 = (2^3)^x$$
$$2^2 = 2^{3x}$$

The only way the left and right sides of this last line can be equal is if the exponents are equal—that is, if $2 = 3x$. Solving for x, we have $x = 2/3$, which is our solution. As with all equations, we can check our solution by substitution.

If $\log_8 4 = 2/3$, then
$$4 = 8^{2/3}$$
$$4 = (\sqrt[3]{8})^2$$
$$4 = 2^2$$
$$4 = 4 \quad \text{A true statement}$$

▲

Graphing Logarithmic Functions

As we indicated in Section 4.5, the graph of a logarithmic function can be drawn using the graph of its associated exponential function. Since the graph of a function and the graph of its inverse have symmetry about the line $y = x$, we reason as follows:

The equation $y = \log_2 x$ is, by definition, equivalent to the exponential equation

$$x = 2^y$$

which is the equation of the inverse of the function

$$y = 2^x$$

Therefore, to graph $y = \log_2 x$, we simply reflect the graph of $y = 2^x$ about the line $y = x$. Figure 1 shows both graphs.

One thing to notice from Figure 1 is the fact that the x in $y = \log_2 x$ is never negative. In general, the only quantity in the expression $y = \log_b x$ that can be negative is y. Both b and x are always positive. [Try to find $\log(-2)$ on your calculator.]

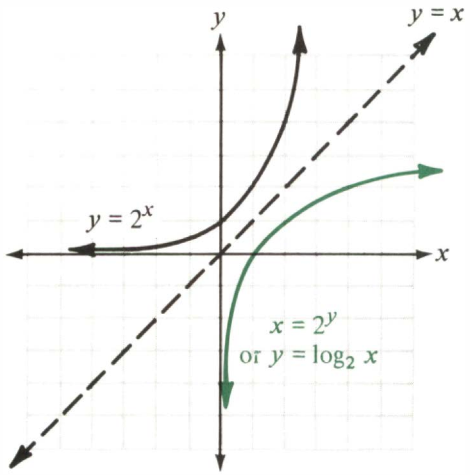


Figure 1

Two Special Identities

If b is a positive real number other than 1, then each of the following is a consequence of the definition of a logarithm:

$$b^{\log_b x} = x \qquad \log_b b^x = x$$

The justifications for these identities are similar. Let's consider only the first one. Consider the expression

$$y = \log_b x$$

By definition, it is equivalent to

$$x = b^y$$

Substituting $\log_b x$ for y in the last line gives us

$$x = b^{\log_b x}$$

▼ **Example 4** Simplify $\log_2 8$.

Solution Substitute 2^3 for 8:

$$\begin{aligned} \log_2 8 &= \log_2 2^3 \\ &= 3 \end{aligned}$$

▲

▼ **Example 5** Simplify $\log_b b$ $(b > 0,\ b \neq 1)$.

Solution Since $b^1 = b$, we have

$$\begin{aligned} \log_b b &= \log_b b^1 \\ &= 1 \end{aligned}$$

▲

Example 6 Simplify $\log_b 1$ $(b > 0,\ b \neq 1)$.

Solution Since $1 = b^0$, we have

$$\log_b 1 = \log_b b^0 = 0$$

Properties of Logarithms

For the following three properties, x, y, and b are all positive real numbers, $b \neq 1$, and r is any real number:

Property 1 $\log_b (xy) = \log_b x + \log_b y$

In words: The logarithm of a product is the sum of the logarithms.

Property 2 $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

In words: The logarithm of a quotient is the difference of the logarithms.

Property 3 $\log_b x^r = r \log_b x$

In words: The logarithm of a base raised to a power is the product of the power and the logarithm of the base.

Proof of Property 1 To prove property 1, we simply apply the first of our two special identities.

$$b^{\log_b xy} = xy = b^{\log_b x} b^{\log_b y} = b^{\log_b x + \log_b y}$$

Since the first and last expressions are equal and the bases are the same, the exponents $\log_b xy$ and $\log_b x + \log_b y$ must be equal. Therefore,

$$\log_b xy = \log_b x + \log_b y$$

The proofs of Properties 2 and 3 proceed in much the same manner, so we will omit them here.

Example 7 Expand the following using the properties of logarithms:

$$\log_5 \frac{3xy}{z}$$

Solution Applying Property 2, we can write the quotient of $3xy$ and z in terms of a difference:

$$\log_5 \frac{3xy}{z} = \log_5 3xy - \log_5 z$$

Applying Property 1 to the product $3xy$, we write it in terms of addition:

$$\log_5 \frac{3xy}{z} = \log_5 3 + \log_5 x + \log_5 y - \log_5 z$$

Example 8 Expand the following using the properties of logarithms:

$$\log_2 \frac{x^4}{\sqrt{y} \cdot z^3}$$

Solution We write $\sqrt{y}$ as $y^{1/2}$ and apply the properties:

$$\log_2 \frac{x^4}{\sqrt{y} \cdot z^3} = \log_2 \frac{x^4}{y^{1/2}z^3}$$

$$= \log_2 x^4 - \log_2 (y^{1/2} \cdot z^3) \quad \text{Property 2}$$

$$= \log_2 x^4 - (\log_2 y^{1/2} + \log_2 z^3) \quad \text{Property 1}$$

$$= \log_2 x^4 - \log_2 y^{1/2} - \log_2 z^3 \quad \text{Remove parentheses}$$

$$= 4 \log_2 x - \frac{1}{2} \log_2 y - 3 \log_2 z \quad \text{Property 3}$$

We can also use the three properties to write an expression in expanded form as just one logarithm.

Example 9 Write the following as a single logarithm:

$$2 \log_{10} a + 3 \log_{10} b - \frac{1}{3} \log_{10} c$$

Solution We begin by applying Property 3:

$$2 \log_{10} a + 3 \log_{10} b - \frac{1}{3} \log_{10} c$$

$$= \log_{10} a^2 + \log_{10} b^3 - \log_{10} c^{1/3} \quad \text{Property 3}$$

$$= \log_{10} (a^2 \cdot b^3) - \log_{10} c^{1/3} \quad \text{Property 1}$$

$$= \log_{10} \frac{a^2b^3}{c^{1/3}} \quad \text{Property 2}$$

$$= \log_{10} \frac{a^2b^3}{\sqrt[3]{c}} \quad c^{1/3} = \sqrt[3]{c}$$

Problem Set A.1

Write each of the following expressions in logarithmic form:

1. $125 = 5^3$
2. $16 = 4^2$
3. $.01 = 10^{-2}$
4. $.001 = 10^{-3}$
5. $2^{-5} = 1/32$
6. $4^{-2} = 1/16$

Write each of the following expressions in exponential form:

7. $\log_{10} 100 = 2$
8. $\log_2 8 = 3$
9. $\log_8 1 = 0$
10. $\log_9 9 = 1$
11. $\log_{10} .001 = -3$
12. $\log_{10} .0001 = -4$

Solve each of the following equations for x:

13. $\log_5 x = -3$
14. $\log_2 x = -4$
15. $\log_2 16 = x$
16. $\log_3 27 = x$
17. $\log_8 2 = x$
18. $\log_{25} 5 = x$
19. $\log_x 4 = 2$
20. $\log_x 16 = 4$

Simplify each of the following:

21. $\log_2 16$
22. $\log_3 9$
23. $\log_{25} 125$
24. $\log_9 27$
25. $\log_{10} 1000$
26. $\log_{10} 10{,}000$
27. $\log_3 3$
28. $\log_4 4$
29. $\log_5 1$
30. $\log_{10} 1$
31. $\log_3 (\log_6 6)$
32. $\log_5 (\log_3 3)$

Use the three properties of logarithms given in this section to expand each expression as much as possible.

33. $\log_3 4x$
34. $\log_2 5x$
35. $\log_6 \frac{5}{x}$
36. $\log_3 \frac{x}{5}$
37. $\log_2 y^5$
38. $\log_7 y^3$
39. $\log_9 \sqrt[3]{z}$
40. $\log_8 \sqrt{z}$
41. $\log_6 x^2y^3$
42. $\log_{10} x^2y^4$
43. $\log_5 \sqrt{x} \cdot y^4$
44. $\log_8 \sqrt[3]{xy^6}$
45. $\log_b \frac{xy}{z}$
46. $\log_b \frac{3x}{y}$
47. $\log_{10} \frac{4}{xy}$
48. $\log_{10} \frac{5}{4y}$
49. $\log_{10} \frac{x^2y}{\sqrt{z}}$
50. $\log_{10} \frac{\sqrt{x} \cdot y}{z^3}$

51. $\log_{10} \frac{x^3\sqrt{y}}{z^4}$

52. $\log_{10} \frac{x^4\sqrt[3]{y}}{\sqrt{z}}$

Write each expression as a single logarithm.

53. $\log_b x + \log_b z$

54. $\log_b x - \log_b z$

55. $2 \log_3 x - 3 \log_3 y$

56. $4 \log_2 x + 5 \log_2 y$

57. $\log_{10} x + \frac{1}{3} \log_{10} y$

58. $\frac{1}{3} \log_{10} x - \frac{1}{4} \log_{10} y$

59. $3 \log_2 x + \frac{1}{2} \log_2 y - \log_2 z$

60. $2 \log_3 x + 3 \log_3 y - \log_3 z$

61. $\frac{1}{2} \log_2 x - 3 \log_2 y - 4 \log_2 z$

62. $3 \log_{10} x - \log_{10} y - \log_{10} z$

63. The formula $M = 0.21(\log_{10} a - \log_{10} b)$ is used in the food processing industry to find the number of minutes M of heat processing a certain food should undergo at 250°F to reduce the probability of survival of *C. botulinum* spores. The letter a represents the number of spores per can before heating, and b represents the total number of spores per can after heating. Find M if $a = 1$ and $b = 10^{-12}$.

64. The formula $N = 10 \log_{10} \frac{P_1}{P_2}$ is used in radio electronics to find the ratio of the acoustic powers of two electric circuits in terms of their electric powers. Find N if P_1 is 50 and P_2 is 1/2.

A.2 Common Logarithms and Computations

In the past, logarithms have been very useful in simplifying calculations. With the widespread availability of hand-held calculators, the importance of logarithms in calculations has been decreased considerably. Working through arithmetic problems using logarithms is, however, good practice. The problems in this section all involve the properties of logarithms developed in the preceding section and indicate how some fairly difficult computation problems can be simplified using logarithms.

Definition A common logarithm is a logarithm with a base of 10. Since common logarithms are used so frequently, it is customary, in order to save time, to omit notating the base. That is,

$$\log_{10} x = \log x$$

When the base is not shown, it is assumed to be 10.

The reason logarithms base 10 are so common is that our number system is a base-10 number system.

Common logarithms of powers of 10 are very simple to evaluate. We need only recognize that $\log 10 = \log_{10} 10 = 1$ and apply the third property of logarithms: $\log_b x^r = r \log_b x$.

$$\begin{aligned}
\log 1000 &= \log 10^3 &&= 3 \log 10 &&= 3(1) &&= 3\\
\log 100 &= \log 10^2 &&= 2 \log 10 &&= 2(1) &&= 2\\
\log 10 &= \log 10^1 &&= 1 \log 10 &&= 1(1) &&= 1\\
\log 1 &= \log 10^0 &&= 0 \log 10 &&= 0(1) &&= 0\\
\log .1 &= \log 10^{-1} &&= -1 \log 10 &&= -1(1) &&= -1\\
\log .01 &= \log 10^{-2} &&= -2 \log 10 &&= -2(1) &&= -2\\
\log .001 &= \log 10^{-3} &&= -3 \log 10 &&= -3(1) &&= -3
\end{aligned}$$

For common logarithms of numbers that are not powers of 10, we have to resort to a table or a calculator. Table IV at the back of the book gives common logarithms of numbers between 1.00 and 9.99. To find the common logarithm of, say, 2.76, we read down the left-hand column until we get to 2.7, then across until we are below the 6 in the top row (or above the 6 in the bottom row):

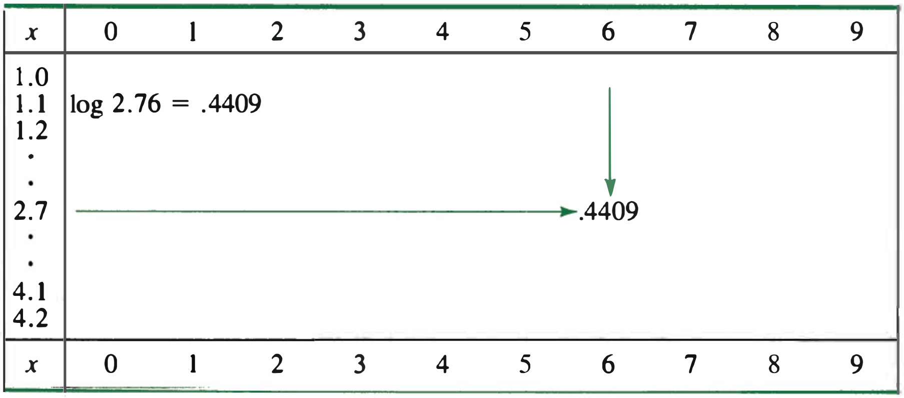


Table IV contains only logarithms of numbers between 1.00 and 9.99. Check the following logarithms in the table to be sure you know how to use the table:

$$\begin{aligned}
\log 7.02 &= .8463\\
\log 1.39 &= .1430\\
\log 6.00 &= .7782\\
\log 9.99 &= .9996
\end{aligned}$$

To find the common logarithm of a number that is not between 1.00 and 9.99, we simply write the number in scientific notation, apply Property 2 of logarithms, and use the table. The following examples illustrate the procedure.

Note A number is written in scientific notation when it is written as the product of a number between 1 and 10 and a power of 10. For example, $39{,}800 = 3.98 \times 10^4$ in scientific notation.

▼ **Example 1** Use Table IV to find log 2760.

Solution
$$\begin{aligned} \log 2760 &= \log(2.76 \times 10^3) \\ &= \log 2.76 + \log 10^3 \\ &= .4409 + 3 \\ &= 3.4409 \end{aligned}$$

The 3 in the answer is called the *characteristic*, and its main function is to keep track of the decimal point. The decimal part of this logarithm is called the *mantissa*. It is found from the table.

Following are more of the same type of problems: ▲

▼ **Example 2** Find log 843.

Solution
$$\begin{aligned} \log 843 &= \log(8.43 \times 10^2) \\ &= \log 8.43 + \log 10^2 \\ &= .9258 + 2 \\ &= 2.9258 \end{aligned}$$
▲

▼ **Example 3** Find log .0391.

Solution
$$\begin{aligned} \log .0391 &= \log(3.91 \times 10^{-2}) \\ &= \log 3.91 + \log 10^{-2} \\ &= .5922 + (-2) \end{aligned}$$

Now there are two ways to proceed from here. We could add .5922 and −2 to get −1.4078. (If you were using a calculator to find log .0391, this is the answer you would see.) The problem with −1.4078 is that the mantissa is negative. Our table contains only positive numbers. If we have to use log .0391 again, and it is written as −1.4078, we will not be able to associate it with an entry in the table. It is most common, when using a table, to write the characteristic −2 as 8 + (−10) and proceed as follows:

$$\begin{aligned} \log .0391 &= .5922 + 8 + (-10) \\ &= 8.5922 - 10 \end{aligned}$$
▲

Here is another example:

▼ **Example 4** Find log .00523.

Solution

$$\begin{aligned}\log .00523 &= \log(5.23 \times 10^{-3})\\ &= \log 5.23 + \log 10^{-3}\\ &= .7185 + (-3)\\ &= .7185 + 7 + (-10)\\ &= 7.7185 - 10\end{aligned}$$

▲

It is not necessary to show any of the steps we have shown in the examples above. As a matter of fact, it is better if we don't. With a little practice the steps can be eliminated. Once we have become familiar with using the table to find logarithms, we can go in the reverse direction and use the table to solve equations like $\log x = 3.8774$.

▼ **Example 5** Find x if $\log x = 3.8774$.

Solution We are looking for the number whose logarithm is 3.8774. The mantissa is .8774, which appears in the table across from 7.5 and under (or above) 4.

x	0	1	2	3	4	5	6	7	8	9
4.3					↑					
4.4										
·										
·										
7.5 ←					.8774					
·										
·										
7.8										
7.9										
x	0	1	2	3	4	5	6	7	8	9

The characteristic is 3 and it came from the exponent on 10.

$\log x = 3.8774$	
$\log x = .8774 + 3$	Separate characteristic and mantissa
$x = 7.54 \times 10^3$	Table IV for 7.54
$x = 7540$	

The number 7540 is called the antilogarithm or just antilog of 3.8774. That is, 7540 is the number whose logarithm is 3.8774. ▲

Calculator Note To do this same problem on a calculator you would use the button labeled 10^x.

3.8774 [10^x]

Example 6 Find x if $\log x = 7.5821 - 10$.

Solution The mantissa is .5821, which comes from 3.82 in the table. The characteristic is $7 + (-10)$ or -3, which comes from a power of 10:

$$\begin{aligned}\log x &= 7.5821 - 10\\ \log x &= .5821 + 7 + (-10)\\ \log x &= .5821 + (-3)\\ x &= 3.82 \times 10^{-3}\\ x &= .00382\end{aligned}$$

The antilog of $7.5821 - 10$ is .00382. That is, the logarithm of .00382 is $7.5821 - 10$. ▲

The following examples illustrate how logarithms are used as an aid in computations:

Example 7 Use logarithms to find (3780)(45,200).

Solution We will let n represent the answer to this problem:

$$n = (3780)(45{,}200)$$

Since these two numbers are equal, so are their logarithms. We therefore take the common logarithm of both sides:

$$\begin{aligned}\log n &= \log(3780)(45{,}200)\\ &= \log 3780 + \log 45{,}200\\ &= 3.5775 + 4.6551\\ \log n &= 8.2326\end{aligned}$$

The number 8.2326 is the logarithm of the answer. The mantissa is .2326, which is not in the table. Since .2326 is closest to .2330, and .2330 is the logarithm of 1.71, we write

$$n = 1.71 \times 10^8$$ ▲

Note The answer 1.71×10^8 is not exactly equal to (3780)(45,200). It is an approximation to it. If we want to be more accurate and have an answer with more significant digits, we would have to use a table with more significant digits.

Example 8 Use logarithms to find $\sqrt[3]{875}$.

Solution

$$\text{If} \quad n = \sqrt[3]{875},$$

$$\text{then} \quad n = (875)^{1/3}$$

$$\text{and} \quad \log n = \log 875^{1/3}$$
$$= \frac{1}{3} \log 875$$
$$= \frac{1}{3}(2.9420)$$
$$\log n = .9807$$
$$n = 9.56$$

A good approximation to the cube root of 875 is 9.56. ▲

▼ **Example 9** Use logarithms to find $35^{2.7}$.

Solution

$$\begin{aligned} \text{Let } n &= 35^{2.7} \\ \text{Then } \log n &= \log 35^{2.7} \\ &= 2.7 \log 35 \\ &= 2.7(1.5441) \\ \log n &= 4.1691 \\ n &= 1.48 \times 10^4 \\ &= 14{,}800 \end{aligned}$$

▲

▼ **Example 10** Use logarithms to find $\frac{(34.5)^2\sqrt{1080}}{(2.76)^3}$.

Solution

$$\text{If} \quad n = \frac{(3.45)^2\sqrt{1080}}{(2.76)^3}$$

$$\text{then} \quad \log n = \log\left[\frac{(34.5)^2\sqrt{1080}}{(2.76)^3}\right]$$
$$= 2 \log 34.5 + \frac{1}{2} \log 1080 - 3 \log 2.76$$
$$= 2(1.5378) + \frac{1}{2}(3.0334) - 3(.4409)$$
$$= 3.0756 + 1.5167 - 1.3227$$
$$\log n = 3.2696$$
$$n = 1.86 \times 10^3$$
$$= 1860$$

▲

Problem Set A.2

Use Table IV to find the following:

1. log 378
2. log 426
3. log 37.8
4. log 42,600
5. log 3780
6. log .4260
7. log .0378
8. log .0426
9. log 37,800
10. log 4900
11. log 600
12. log 900
13. log 2010
14. log 10,200
15. log .00971
16. log .0312
17. log .0314
18. log .00052
19. log .399
20. log .111

Find x in the following equations:

21. $\log x = 2.8802$
22. $\log x = 4.8802$
23. $\log x = 7.8802 - 10$
24. $\log x = 6.8802 - 10$
25. $\log x = 3.1553$
26. $\log x = 5.5911$
27. $\log x = 4.6503 - 10$
28. $\log x = 8.4330 - 10$
29. $\log x = 2.9628 - 10$
30. $\log x = 5.8000 - 10$

Use logarithms to evaluate the following:

31. $(378)(24.5)$
32. $(921)(2630)$
33. $\dfrac{496}{391}$
34. $\dfrac{512}{216}$
35. $\sqrt{401}$
36. $\sqrt[3]{92.3}$
37. $\sqrt[3]{1030}$
38. $\sqrt{.641}$
39. $\dfrac{(2390)(28.4)}{176}$
40. $\dfrac{(32.9)(5760)}{11.1}$
41. $(296)^2(459)$
42. $(3250)(24.2)^3$
43. $\sqrt{526(1080)}$
44. $(32.7)\sqrt{.580}$
45. $45^{-3.1}$
46. $72^{1.8}$
47. $\dfrac{(895)^3\sqrt{41.1}}{17.8}$
48. $\dfrac{(925)^2\sqrt{1.99}}{243}$

49. The weight W of a sphere with diameter d can be found by using the formula

$$W = \frac{\pi}{6} d^3 w$$

where w is the weight of a cubic unit of material. Use logarithms to find W if $d = 2.98$ inches, $w = 3.03$ pounds and $\pi = 3.14$.

50. The following formula is used in hydraulics:

$$d = 2.57\sqrt[5]{\frac{flQ^2}{h}}$$

Use logarithms to find d, when $f = 0.022$, $l = 2{,}820$, $h = 133$, and $Q = 12.5$.

51. The area A of a triangle in which all three sides are of equal length l (equilateral) is given by the formula

$$A = \frac{l^2}{4}\sqrt{3}$$

Find A when $l = 276$. (Use logarithms.)

52. The formula $H = \dfrac{PLAN}{33{,}000}$ is used to compute the horsepower of a steam or gas engine. In this formula, H is the horsepower; P, the average effective pressure on the piston, in pounds per square inch; L, the distance the piston travels per stroke, in feet; A, the area of cross-section of the cylinder, in square inches; and N, the number of working strokes per minute. Find H, when $P = 62.8$ pounds, $L = 2.63$ feet, $A = 18.4$ square inches, and $N = 92.4$.

A.3 Exponential Equations, Logarithmic Equations, and Change of Base

Logarithms are very important in solving equations in which the variable appears as an exponent. For example, consider the equation

$$5^x = 12$$

Since the quantities 5^x and 12 are equal, so are their common logarithms. To solve this equation we begin by taking the logarithm of both sides:

$$\log 5^x = \log 12$$

We now apply Property 3 for logarithms, $\log x^r = r \log x$, to turn x from an exponent into a coefficient:

$$x \log 5 = \log 12$$

Dividing both sides by log 5 gives us

$$x = \frac{\log 12}{\log 5}$$

If we want a decimal approximation to the solution, we can find log 12 and log 5 in Table IV or on a calculator and divide:

$$x = \frac{1.0792}{.6990}$$

$$= 1.5439$$

The complete problem looks like this:

$$5^x = 12$$
$$\log 5^x = \log 12$$
$$x \log 5 = \log 12$$
$$x = \frac{\log 12}{\log 5}$$
$$= \frac{1.0792}{.6990}$$
$$= 1.5439$$

Note A very common mistake can occur in the third-from-the-last step. Many times the expression

$$\frac{\log 12}{\log 5}$$

is mistakenly simplified as $\log 12 - \log 5$. There is no property of logarithms that allows us to do this. The only property of logarithms that deals with division is Property 2, which is this

$$\log_b \frac{x}{y} = \log_b x - \log_b y \qquad \text{(Right)}$$

not this

$$\frac{\log_b x}{\log_b y} = \log_b x - \log_b y \qquad \text{(Wrong)}$$

The second statement is simply not a property of logarithms, although it is sometimes mistaken for one.

Here is another example of solving an exponential equation using logarithms.

▼ **Example 1** Solve for x: $25^{2x+1} = 15$

Solution Taking the logarithms of both sides and then writing the exponent $(2x + 1)$ as a coefficient, we proceed as follows:

$$25^{2x+1} = 15$$
$$\log 25^{2x+1} = \log 15$$
$$(2x + 1)\log 25 = \log 15$$

$$2x + 1 = \frac{\log 15}{\log 25}$$

$$2x = \frac{\log 15}{\log 25} - 1$$

$$x = \frac{1}{2}\left(\frac{\log 15}{\log 25} - 1\right)$$

Using Table IV or a calculator, we can write a decimal approximation to the answer:

$$x = \frac{1}{2}\left(\frac{1.1761}{1.3979} - 1\right)$$

$$= \frac{1}{2}(.8413 - 1)$$

$$= \frac{1}{2}(-.1587)$$

$$= -.0793$$

▲

The properties of logarithms along with the definition of logarithms are useful in solving equations that involve logarithms to begin with.

▼ **Example 2** Solve for x: $\log_2 (x + 2) + \log_2 x = 3$

Solution Applying Property 1 to the left side of the equation allows us to write it as a single logarithm:

$$\log_2 (x + 2) + \log_2 x = 3$$
$$\log_2 [(x + 2)(x)] = 3$$

The last line can be written in exponential form using the definition of logarithms:

$$(x + 2)(x) = 2^3$$

which simplifies to

$$x^2 + 2x = 8$$
$$x^2 + 2x - 8 = 0$$
$$(x + 4)(x - 2) = 0$$
$$x + 4 = 0 \quad \text{or} \quad x - 2 = 0$$
$$x = -4 \quad \text{or} \quad x = 2$$

Earlier we noted the fact that x in the expression $y = \log_b x$ cannot be a negative number. Since substitution of $x = -4$ into the original equation gives

$$\log_2(-2) + \log_2(-4) = 3$$

which contains logarithms of negative numbers, we cannot use -4 as a solution. The only solution is $x = 2$. ▲

There is a fourth property of logarithms we have not yet considered. This last property is called the change-of-base property.

Property 4 (Change of Base) If a and b are both positive numbers other than 1, and if $x > 0$, then

$$\underset{\substack{\uparrow \\ \text{Base } a}}{\log_a x} = \underset{\substack{\uparrow \\ \text{Base } b}}{\frac{\log_b x}{\log_b a}}$$

The logarithm on the left side has a base of a, while both logarithms on the right side have a base of b. This allows us to change from base a to another base b that is a positive number other than 1. Here is a proof of Property 4 for logarithms.

Proof We begin by writing the identity

$$a^{\log_a x} = x$$

Taking the logarithm base b of both sides and setting the exponent $\log_a x$ as a coefficient, we have

$$\log_b a^{\log_a x} = \log_b x$$
$$\log_a x \log_b a = \log_b x$$

Dividing both sides by $\log_b a$, we have the desired result:

$$\frac{\log_a x \log_b a}{\log_b a} = \frac{\log_b x}{\log_b a}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

We can use this property to find logarithms for which we do not have a table. The next two examples illustrate the use of this property.

Example 3 Find $\log_8 24$.

Solution Since we do not have a table for base-8 logarithms, we change this expression to an equivalent expression that only contains base-10 logarithms:

$$\log_8 24 = \frac{\log 24}{\log 8}$$

Don't be confused. We did not just drop the base, we changed to base 10. We could have written the last line like this:

$$\log_8 24 = \frac{\log_{10} 24}{\log_{10} 8}$$

Looking up log 24 and log 8 in Table IV, or using a calculator, we write

$$\log_8 24 = \frac{1.3802}{.9031}$$

$$= 1.5283$$

Problem Set A.3

Solve each exponential equation. Use the Table IV to write the answer in decimal form.

1. $3^x = 5$
2. $4^x = 3$
3. $5^x = 3$
4. $3^x = 4$
5. $5^{-x} = 12$
6. $7^{-x} = 8$
7. $12^{-x} = 5$
8. $8^{-x} = 7$
9. $8^{x+1} = 4$
10. $9^{x+1} = 3$
11. $4^{x-1} = 4$
12. $3^{x-1} = 9$
13. $3^{2x+1} = 2$
14. $2^{2x+1} = 3$
15. $3^{1-2x} = 2$
16. $2^{1-2x} = 3$
17. $15^{3x-4} = 10$
18. $10^{3x-4} = 15$
19. $6^{5-2x} = 4$
20. $9^{7-3x} = 5$

Solve each of the following equations:

21. $\log_2 x + \log_2 3 = 1$
22. $\log_2 x - \log_2 3 = 1$
23. $\log_3 x - \log_3 2 = 2$
24. $\log_3 x + \log_3 2 = 2$
25. $\log_3 x + \log_3 (x - 2) = 1$
26. $\log_6 x + \log_6 (x - 1) = 1$
27. $\log_3 (x + 3) - \log_3 (x - 1) = 1$
28. $\log_4 (x - 2) - \log_4 (x + 1) = 1$
29. $\log_2 x + \log_2 (x - 2) = 3$
30. $\log_4 x + \log_4 (x + 6) = 2$
31. $\log_8 x + \log_8 (x - 3) = 2/3$
32. $\log_{27} x + \log_{27} (x + 8) = 2/3$

Use the change-of-base property and Table IV to find a decimal approximation to each of the following logarithms:

33. $\log_8 16$
34. $\log_9 27$
35. $\log_{16} 8$
36. $\log_{27} 9$
37. $\log_7 15$
38. $\log_3 12$
39. $\log_{15} 7$
40. $\log_{12} 3$
41. $\log_{12} 11$
42. $\log_{14} 15$
43. $\log_{11} 12$
44. $\log_{15} 14$
45. $\log_8 240$
46. $\log_6 180$
47. $\log_4 321$
48. $\log_5 462$

49. The formula $A = P(1 + r)^n$ shows how to find the amount of money A that will be in an account if P dollars is invested at interest rate r for n years if the interest is compounded yearly. Use logarithms to solve this formula for n.

50. If P dollars is invested in an account that pays an annual rate of interest r that is compounded semiannually, the amount of money in the account at the end of n years is given by the formula

$$A = P\left(1 + \frac{r}{2}\right)^{2n}$$

Use logarithms to solve this formula for n.

A.4 Word Problems

There are many practical applications of exponential equations and logarithms. Many times an exponential equation will describe a situation arising in nature. Logarithms are then used to solve the exponential equation. Many of the problems in this section deal with expressions and equations that have been derived or defined in some other discipline.

In chemistry the pH of a solution is defined in terms of logarithms as

$$\text{pH} = -\log(H_3O+)$$

where (H_3O+) is the concentration of the hydronium ion, H_3O+, in solution. An acid solution has a pH lower than 7, and a basic solution has a pH above 7.

▼ **Example 1** Find the pH of a solution in which $(H_3O+) = 5.1 \times 10^{-3}$.

Solution Using the definition given above, we write

$$\begin{aligned} pH &= -\log(H_3O+) \\ &= -\log(5.1 \times 10^{-3}) \\ &= -[.7076 + (-3)] \\ &= -.7076 + 3 \\ &= 2.29 \end{aligned}$$

▲

According to our previous discussion, this solution would be considered an acid solution, since its pH is less than 7.

▼ **Example 2** What is (H_3O+) for a solution with a pH of 9.3?

Solution We substitute 9.3 for pH in the definition of pH and proceed as follows:

$$\begin{aligned} 9.3 &= -\log(H_3O+) \\ \log(H_3O+) &= -9.3 \end{aligned}$$

Because it is negative, it is impossible to look up -9.3 in Table IV (but a calculator would work just fine). We can get around this problem by adding and subtracting 10 to the -9.3:

$$\begin{aligned} \log(H_3O+) &= -9.3 + 10 - 10 \qquad \text{(Actually adding 0)} \\ &= .7 - 10 \end{aligned}$$

The closest thing to .7 in the table is .6998, which is the logarithm of 5.01:

$$(H_3O+) = 5.01 \times 10^{-10}$$

The concentration of H_3O+ is 5.01×10^{-10}. (This quantity is usually given in moles per liter. We have left off the units to make things simpler.) ▲

Carbon-14 dating is used extensively in science to find the age of fossils. If at one time a nonliving substance contains an amount A_0 of carbon 14, then t years later it will contain an amount A of carbon 14, where

$$A = A_0 \cdot 2^{-t/5600}$$

▼ **Example 3** If a nonliving substance has 3 grams of carbon 14, how much carbon 14 will be present 500 years later?

Solution The original amount of carbon 14, A_0, is 3 grams. The number of years, t, is 500. Substituting these quantities into the equation given above, we have the expression

$$A = (3)2^{-500/5600}$$
$$= (3)2^{-5/56}$$

In order to evaluate this expression we must use logarithms. We begin by taking the logarithm of both sides:

$$\log A = \log[(3)2^{-5/56}]$$
$$= \log 3 - \frac{5}{56}\log 2$$
$$= .4771 - \frac{5}{56}(.3010)$$
$$= .4771 - .0269$$
$$\log A = .4502$$
$$A = 2.82$$

The amount remaining after 500 years is 2.82 grams. ▲

If an amount of money P (P for principal) is invested in an account that pays a rate r of interest compounded annually, then the total amount of money A (the original amount P plus all the interest) in the account after t years is given by the equation

$$A = P(1 + r)^t$$

▼ **Example 4** How much money will accumulate over 20 years if a person invests \$5,000 in an account that pays 6% interest per year?

Solution The original amount P is \$5,000, the rate of interest r is .06 (6% = .06), and the length of time t is 20 years. We substitute these values into the equation and solve for A:

$$A = 5000(1 + .06)^{20}$$
$$A = 5000(1.06)^{20}$$
$$\log A = \log[5000(1.06)^{20}]$$
$$\log A = \log 5000 + 20 \log 1.06$$
$$= 3.6990 + 20(.0253)$$
$$= 3.6990 + .5060$$
$$\log A = 4.2050$$
$$A = 1.60 \times 10^4$$
$$= 16{,}000$$

The original \$5,000 will more than triple to become \$16,000 in 20 years at 6% annual interest. ▲

Example 5 How long does it take for $5,000 to double if it is deposited in an account that yields 5% per year?

Solution The original amount P is $5,000, the total amount after t years is $A = \$10,000$, and the interest rate r is .05. Substituting into

$$A = P(1 + r)^t$$

we have

$$\begin{aligned} 10{,}000 &= 5000(1 + .05)^t \\ &= 5000(1.05)^t \end{aligned}$$

This is an exponential equation. We solve it by taking the logarithms of both sides:

$$\begin{aligned} \log 10{,}000 &= \log[(5000)(1.05)^t] \\ \log 10{,}000 &= \log 5000 + t \log 1.05 \\ 4 &= 3.6990 + t(.0212) \end{aligned}$$

Subtract 3.6990 from both sides:

$$.3010 = .0212t$$

Dividing both sides by .0212, we have

$$t = 14.20$$

It takes a little over 14 years for $5,000 to double if it earns 5% per year.

Problem Set A.4

For problems 1–8 find the pH of the solution using the formula $pH = -\log(H_3O+)$.

1. $(H_3O+) = 4 \times 10^{-3}$
2. $(H_3O+) = 3 \times 10^{-4}$
3. $(H_3O+) = 5 \times 10^{-6}$
4. $(H_3O+) = 6 \times 10^{-5}$
5. $(H_3O+) = 4.2 \times 10^{-5}$
6. $(H_3O+) = 2.7 \times 10^{-7}$
7. $(H_3O+) = 8.6 \times 10^{-2}$
8. $(H_3O+) = 5.3 \times 10^{-4}$

For problems 9–12 find (H_3O+) for solutions with the given pH.

9. pH = 3.4
10. pH = 5.7
11. pH = 6.5
12. pH = 2.1

13. A nonliving substance contains 3 micrograms of carbon 14. How much carbon 14 will be left at the end of

 a. 5000 years?
 b. 10,000 years?
 c. 56,000 years?
 d. 112,000 years?

14. A nonliving substance contains 5 micrograms of carbon 14. How much carbon 14 will be left at the end of

a. 500 years?
b. 5000 years?
c. 56,000 years?
d. 112,000 years?

15. At one time a certain nonliving substance contained 10 micrograms of carbon 14. How many years later did the same substance contain only 5 micrograms of carbon 14?

16. At one time a certain nonliving substance contained 20 micrograms of carbon 14. How many years later did the same substance contain only 5 micrograms of carbon 14?

17. How much money is in an account after 10 years if $5,000 was deposited originally at 6% per year?

18. If you put $2,000 in an account that earns 5% interest annually, how much will you have in the account at the end of 10 years?

19. If you deposit $2,000 in an account that earns 7.5% annually, how much will you have in the account after 10 years?

20. Suppose $15,000 is deposited in an account that yields 10% per year. How much is in the account 5 years later?

21. If $4,000 is in an account that earns 5% per year, how long does it take before the account has $8,000 in it?

22. If $4,000 is deposited in an account that yields 7% annually, how long does it take the account to reach $8,000?

23. How long does it take to double $10,000 if it is in an account that earns 6% per year?

24. How long does it take to double $10,000 if it is in an account that earns 10% per year?

Table I Powers, Roots, and Prime Factors

n	n^2	$\sqrt{n}$	n^3	$\sqrt[3]{n}$	Prime factors
1	1	1.000	1	1.000	—
2	4	1.414	8	1.260	prime
3	9	1.732	27	1.442	prime
4	16	2.000	64	1.587	2 · 2
5	25	2.236	125	1.710	prime
6	36	2.449	216	1.817	2 · 3
7	49	2.646	343	1.913	prime
8	64	2.828	512	2.000	2 · 2 · 2
9	81	3.000	729	2.080	3 · 3
10	100	3.162	1,000	2.154	2 · 5
11	121	3.317	1,331	2.224	prime
12	144	3.464	1,728	2.289	2 · 2 · 3
13	169	3.606	2,197	2.351	prime
14	196	3.742	2,744	2.410	2 · 7
15	225	3.873	3,375	2.466	3 · 5
16	256	4.000	4,096	2.520	2 · 2 · 2 · 2
17	289	4.123	4,913	2.571	prime
18	324	4.243	5,832	2.621	2 · 3 · 3
19	361	4.359	6,859	2.668	prime
20	400	4.472	8,000	2.714	2 · 2 · 5
21	441	4.583	9,261	2.759	3 · 7
22	484	4.690	10,648	2.802	2 · 11
23	529	4.796	12,167	2.844	prime
24	576	4.899	13,824	2.884	2 · 2 · 2 · 3
25	625	5.000	15,625	2.924	5 · 5
26	676	5.099	17,576	2.962	2 · 13
27	729	5.196	19,683	3.000	3 · 3 · 3
28	784	5.292	21,952	3.037	2 · 2 · 7
29	841	5.385	24,389	3.072	prime
30	900	5.477	27,000	3.107	2 · 3 · 5
31	961	5.568	29,791	3.141	prime
32	1,024	5.657	32,768	3.175	2 · 2 · 2 · 2 · 2
33	1,089	5.745	35,937	3.208	3 · 11
34	1,156	5.831	39,304	3.240	2 · 17
35	1,225	5.916	42,875	3.271	5 · 7
36	1,296	6.000	46,656	3.302	2 · 2 · 3 · 3
37	1,369	6.083	50,653	3.332	prime
38	1,444	6.164	54,872	3.362	2 · 19
39	1,521	6.245	59,319	3.391	3 · 13
40	1,600	6.325	64,000	3.420	2 · 2 · 2 · 5
41	1,681	6.403	68,921	3.448	prime
42	1,764	6.481	74,088	3.476	2 · 3 · 7
43	1,849	6.557	79,507	3.503	prime
44	1,936	6.633	85,184	3.530	2 · 2 · 11
45	2,025	6.708	91,125	3.557	3 · 3 · 5
46	2,116	6.782	97,336	3.583	2 · 23
47	2,209	6.856	103,823	3.609	prime
48	2,304	6.928	110,592	3.634	2 · 2 · 2 · 2 · 3
49	2,401	7.000	117,649	3.659	7 · 7

Table I (Continued)

n	n^2	$\sqrt{n}$	n^3	$\sqrt[3]{n}$	Prime factors
50	2,500	7.071	125,000	3.684	2 · 5 · 5
51	2,601	7.141	132,651	3.708	3 · 17
52	2,704	7.211	140,608	3.733	2 · 2 · 13
53	2,809	7.280	148,877	3.756	prime
54	2,916	7.348	157,464	3.780	2 · 3 · 3 · 3
55	3,025	7.416	166,375	3.803	5 · 11
56	3,136	7.483	175,616	3.826	2 · 2 · 2 · 7
57	3,249	7.550	185,193	3.849	3 · 19
58	3,364	7.616	195,112	3.871	2 · 29
59	3,481	7.681	205,379	3.893	prime
60	3,600	7.746	216,000	3.915	2 · 2 · 3 · 5
61	3,721	7.810	226,981	3.936	prime
62	3,844	7.874	238,328	3.958	2 · 31
63	3,969	7.937	250,047	3.979	3 · 3 · 7
64	4,096	8.000	262,144	4.000	2 · 2 · 2 · 2 · 2 · 2
65	4,225	8.062	274,625	4.021	5 · 13
66	4,356	8.124	287,496	4.041	2 · 3 · 11
67	4,489	8.185	300,763	4.062	prime
68	4,624	8.246	314,432	4.082	2 · 2 · 17
69	4,761	8.307	328,509	4.102	3 · 23
70	4,900	8.367	343,000	4.121	2 · 5 · 7
71	5,041	8.426	357,911	4.141	prime
72	5,184	8.485	373,248	4.160	2 · 2 · 2 · 3 · 3
73	5,329	8.544	389,017	4.179	prime
74	5,476	8.602	405,224	4.198	2 · 37
75	5,625	8.660	421,875	4.217	3 · 5 · 5
76	5,776	8.718	438,976	4.236	2 · 2 · 19
77	5,929	8.775	456,533	4.254	7 · 11
78	6,084	8.832	474,552	4.273	2 · 3 · 13
79	6,241	8.888	493,039	4.291	prime
80	6,400	8.944	512,000	4.309	2 · 2 · 2 · 2 · 5
81	6,561	9.000	531,441	4.327	3 · 3 · 3 · 3
82	6,724	9.055	551,368	4.344	2 · 41
83	6,889	9.110	571,787	4.362	prime
84	7,056	9.165	592,704	4.380	2 · 2 · 3 · 7
85	7,225	9.220	614,125	4.397	5 · 17
86	7,396	9.274	636,056	4.414	2 · 43
87	7,569	9.327	658,503	4.431	3 · 29
88	7,744	9.381	681,472	4.448	2 · 2 · 2 · 11
89	7,921	9.434	704,969	4.465	prime
90	8,100	9.487	729,000	4.481	2 · 3 · 3 · 5
91	8,281	9.539	753,571	4.498	7 · 13
92	8,464	9.592	778,688	4.514	2 · 2 · 23
93	8,649	9.644	804,357	4.531	3 · 31
94	8,836	9.695	830,584	4.547	2 · 47
95	9,025	9.747	857,375	4.563	5 · 19
96	9,216	9.798	884,736	4.579	2 · 2 · 2 · 2 · 2 · 3
97	9,409	9.849	912,673	4.595	prime
98	9,604	9.899	941,192	4.610	2 · 7 · 7
99	9,801	9.950	970,299	4.626	3 · 3 · 11
100	10,000	10.000	1,000,000	4.642	2 · 2 · 5 · 5

Table II Trigonometric Functions—Decimal Degrees

θ	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ	
0.0	0.0000	1.000	0.0000	—	1.000	—	90.0
.1	0.0017	1.000	0.0017	573.0	1.000	573.0	.9
.2	0.0035	1.000	0.0035	286.5	1.000	286.5	.8
.3	0.0052	1.000	0.0052	191.0	1.000	191.0	.7
.4	0.0070	1.000	0.0070	143.2	1.000	143.2	.6
.5	0.0087	1.000	0.0087	114.6	1.000	114.6	.5
.6	0.0105	0.9999	0.0105	95.49	1.000	95.49	.4
.7	0.0122	0.9999	0.0122	81.85	1.000	81.85	.3
.8	0.0140	0.9999	0.0140	71.62	1.000	71.62	.2
.9	0.0157	0.9999	0.0157	63.66	1.000	63.66	.1
1.0	0.0175	0.9998	0.0175	57.29	1.000	57.30	89.0
.1	0.0192	0.9998	0.0192	52.08	1.000	52.09	.9
.2	0.0209	0.9998	0.0209	47.74	1.000	47.75	.8
.3	0.0227	0.9997	0.0227	44.07	1.000	44.08	.7
.4	0.0244	0.9997	0.0244	40.92	1.000	40.93	.6
.5	0.0262	0.9997	0.0262	38.19	1.000	38.20	.5
.6	0.0279	0.9996	0.0279	35.80	1.000	35.81	.4
.7	0.0297	0.9996	0.0297	33.69	1.000	33.71	.3
.8	0.0314	0.9995	0.0314	31.82	1.000	31.84	.2
.9	0.0332	0.9995	0.0332	30.14	1.001	30.16	.1
2.0	0.0349	0.9994	0.0349	28.64	1.001	28.65	88.0
.1	0.0366	0.9993	0.0367	27.27	1.001	27.29	.9
.2	0.0384	0.9993	0.0384	26.03	1.001	26.05	.8
.3	0.0401	0.9992	0.0402	24.90	1.001	24.92	.7
.4	0.0419	0.9991	0.0419	23.86	1.001	23.88	.6
.5	0.0436	0.9990	0.0437	22.90	1.001	22.93	.5
.6	0.0454	0.9990	0.0454	22.02	1.001	22.04	.4
.7	0.0471	0.9989	0.0472	21.20	1.001	21.23	.3
.8	0.0488	0.9988	0.0489	20.45	1.001	20.47	.2
.9	0.0506	0.9987	0.0507	19.74	1.001	19.77	.1
3.0	0.0523	0.9986	0.0524	19.08	1.001	19.11	87.0
.1	0.0541	0.9985	0.0542	18.46	1.001	18.49	.9
.2	0.0558	0.9984	0.0559	17.89	1.002	17.91	.8
.3	0.0576	0.9983	0.0577	17.34	1.002	17.37	.7
.4	0.0593	0.9982	0.0594	16.83	1.002	16.86	.6
.5	0.0610	0.9981	0.0612	16.35	1.002	16.38	.5
.6	0.0628	0.9980	0.0629	15.89	1.002	15.93	.4
.7	0.0645	0.9979	0.0647	15.46	1.002	15.50	.3
.8	0.0663	0.9978	0.0664	15.06	1.002	15.09	.2
.9	0.0680	0.9977	0.0682	14.67	1.002	14.70	.1
4.0	0.0698	0.9976	0.0699	14.30	1.002	14.34	86.0
	cos θ	sin θ	cot θ	tan θ	csc θ	sec θ	θ

Table II (Continued)

θ	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ	
4.0	0.0698	0.9976	0.0699	14.30	1.002	14.34	86.0
.1	0.0715	0.9974	0.0717	13.95	1.003	13.99	.9
.2	0.0732	0.9973	0.0734	13.62	1.003	13.65	.8
.3	0.0750	0.9972	0.0752	13.30	1.003	13.34	.7
.4	0.0767	0.9971	0.0769	13.00	1.003	13.03	.6
.5	0.0785	0.9969	0.0787	12.71	1.003	12.75	.5
.6	0.0802	0.9968	0.0805	12.43	1.003	12.47	.4
.7	0.0819	0.9966	0.0822	12.16	1.003	12.20	.3
.8	0.0837	0.9965	0.0840	11.91	1.004	11.95	.2
.9	0.0854	0.9963	0.0857	11.66	1.004	11.71	.1
5.0	0.0872	0.9962	0.0875	11.43	1.004	11.47	85.0
.1	0.0889	0.9960	0.0892	11.20	1.004	11.25	.9
.2	0.0906	0.9959	0.0910	10.99	1.004	11.03	.8
.3	0.0924	0.9957	0.0928	10.78	1.004	10.83	.7
.4	0.0941	0.9956	0.0945	10.58	1.004	10.63	.6
.5	0.0958	0.9954	0.0963	10.39	1.005	10.43	.5
.6	0.0976	0.9952	0.0981	10.20	1.005	10.25	.4
.7	0.0993	0.9951	0.0998	10.02	1.005	10.07	.3
.8	0.1011	0.9949	0.1016	9.845	1.005	9.895	.2
.9	0.1028	0.9947	0.1033	9.677	1.005	9.728	.1
6.0	0.1045	0.9945	0.1051	9.514	1.006	9.567	84.0
.1	0.1063	0.9943	0.1069	9.357	1.006	9.411	.9
.2	0.1080	0.9942	0.1086	9.205	1.006	9.259	.8
.3	0.1097	0.9940	0.1104	9.058	1.006	9.113	.7
.4	0.1115	0.9938	0.1122	8.915	1.006	8.971	.6
.5	0.1132	0.9936	0.1139	8.777	1.006	8.834	.5
.6	0.1149	0.9934	0.1157	8.643	1.007	8.700	.4
.7	0.1167	0.9932	0.1175	8.513	1.007	8.571	.3
.8	0.1184	0.9930	0.1192	8.386	1.007	8.446	.2
.9	0.1201	0.9928	0.1210	8.264	1.007	8.324	.1
7.0	0.1219	0.9925	0.1228	8.144	1.008	8.206	83.0
.1	0.1236	0.9923	0.1246	8.028	1.008	8.091	.9
.2	0.1253	0.9921	0.1263	7.916	1.008	7.979	.8
.3	0.1271	0.9919	0.1281	7.806	1.008	7.870	.7
.4	0.1288	0.9917	0.1299	7.700	1.008	7.764	.6
.5	0.1305	0.9914	0.1317	7.596	1.009	7.661	.5
.6	0.1323	0.9912	0.1334	7.495	1.009	7.561	.4
.7	0.1340	0.9910	0.1352	7.396	1.009	7.463	.3
.8	0.1357	0.9907	0.1370	7.300	1.009	7.368	.2
.9	0.1374	0.9905	0.1388	7.207	1.010	7.276	.1
8.0	0.1392	0.9903	0.1405	7.115	1.010	7.185	82.0
	cos θ	sin θ	cot θ	tan θ	csc θ	sec θ	θ

Table II (Continued)

θ	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ	
8.0	0.1392	0.9903	0.1405	7.115	1.010	7.185	82.0
.1	0.1409	0.9900	0.1423	7.026	1.010	7.097	.9
.2	0.1426	0.9898	0.1441	6.940	1.010	7.011	.8
.3	0.1444	0.9895	0.1459	6.855	1.011	6.927	.7
.4	0.1461	0.9893	0.1477	6.772	1.011	6.845	.6
.5	0.1478	0.9890	0.1495	6.691	1.011	6.765	.5
.6	0.1495	0.9888	0.1512	6.612	1.011	6.687	.4
.7	0.1513	0.9885	0.1530	6.535	1.012	6.611	.3
.8	0.1530	0.9882	0.1548	6.460	1.012	6.537	.2
.9	0.1547	0.9880	0.1566	6.386	1.012	6.464	.1
9.0	0.1564	0.9877	0.1584	6.314	1.012	6.392	81.0
.1	0.1582	0.9874	0.1602	6.243	1.013	6.323	.9
.2	0.1599	0.9871	0.1620	6.174	1.013	6.255	.8
.3	0.1616	0.9869	0.1638	6.107	1.013	6.188	.7
.4	0.1633	0.9866	0.1655	6.041	1.014	6.123	.6
.5	0.1650	0.9863	0.1673	5.976	1.014	6.059	.5
.6	0.1668	0.9860	0.1691	5.912	1.014	5.996	.4
.7	0.1685	0.9857	0.1709	5.850	1.015	5.935	.3
.8	0.1702	0.9854	0.1727	5.789	1.015	5.875	.2
.9	0.1719	0.9851	0.1745	5.730	1.015	5.816	.1
10.0	0.1736	0.9848	0.1763	5.671	1.015	5.759	80.0
.1	0.1754	0.9845	0.1781	5.614	1.016	5.702	.9
.2	0.1771	0.9842	0.1799	5.558	1.016	5.647	.8
.3	0.1788	0.9839	0.1817	5.503	1.016	5.593	.7
.4	0.1805	0.9836	0.1835	5.449	1.017	5.540	.6
.5	0.1822	0.9833	0.1853	5.396	1.107	5.487	.5
.6	0.1840	0.9829	0.1871	5.343	1.107	5.436	.4
.7	0.1857	0.9826	0.1890	5.292	1.018	5.386	.3
.8	0.1874	0.9823	0.1908	5.242	1.018	5.337	.2
.9	0.1891	0.9820	0.1926	5.193	1.018	5.288	.1
11.0	0.1908	0.9816	0.1944	5.145	1.019	5.241	79.0
.1	0.1925	0.9813	0.1962	5.097	1.019	5.194	.9
.2	0.1942	0.9810	0.1980	5.050	1.019	5.148	.8
.3	0.1959	0.9806	0.1998	5.005	1.020	5.103	.7
.4	0.1977	0.9803	0.2016	4.959	1.020	5.059	.6
.5	0.1994	0.9799	0.2035	4.915	1.020	5.016	.5
.6	0.2011	0.9796	0.2053	4.872	1.021	4.973	.4
.7	0.2028	0.9792	0.2071	4.829	1.021	4.931	.3
.8	0.2045	0.9789	0.2089	4.787	1.022	4.890	.2
.9	0.2062	0.9785	0.2107	4.745	1.022	4.850	.1
12.0	0.2079	0.9781	0.2126	4.705	1.022	4.810	78.0
	cos θ	sin θ	cot θ	tan θ	csc θ	sec θ	θ

Table II (Continued)

θ	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ	
12.0	0.2079	0.9781	0.2126	4.705	1.022	4.810	78.0
.1	0.2096	0.9778	0.2144	4.665	1.023	4.771	.9
.2	0.2113	0.9774	0.2162	4.625	1.023	4.732	.8
.3	0.2130	0.9770	0.2180	4.586	1.023	4.694	.7
.4	0.2147	0.9767	0.2199	4.548	1.024	4.657	.6
.5	0.2164	0.9763	0.2217	4.511	1.024	4.620	.5
.6	0.2181	0.9759	0.2235	4.474	1.025	4.584	.4
.7	0.2198	0.9755	0.2254	4.437	1.025	4.549	.3
.8	0.2215	0.9751	0.2272	4.402	1.025	4.514	.2
.9	0.2233	0.9748	0.2290	4.366	1.026	4.479	.1
13.0	0.2250	0.9744	0.2309	4.331	1.026	4.445	77.0
.1	0.2267	0.9740	0.2327	4.297	1.027	4.412	.9
.2	0.2284	0.9736	0.2345	4.264	1.027	4.379	.8
.3	0.2300	0.9732	0.2364	4.230	1.028	4.347	.7
.4	0.2317	0.9728	0.2382	4.198	1.028	4.315	.6
.5	0.2334	0.9724	0.2401	4.165	1.028	4.284	.5
.6	0.2351	0.9720	0.2419	4.134	1.029	4.253	.4
.7	0.2368	0.9715	0.2438	4.102	1.029	4.222	.3
.8	0.2385	0.9711	0.2456	4.071	1.030	4.192	.2
.9	0.2402	0.9707	0.2475	4.041	1.030	4.163	.1
14.0	0.2419	0.9703	0.2493	4.011	1.031	4.134	76.0
.1	0.2436	0.9699	0.2512	3.981	1.031	4.105	.9
.2	0.2453	0.9694	0.2530	3.952	1.032	4.077	.8
.3	0.2470	0.9690	0.2549	3.923	1.032	4.049	.7
.4	0.2487	0.9686	0.2568	3.895	1.032	4.021	.6
.5	0.2504	0.9681	0.2586	3.867	1.033	3.994	.5
.6	0.2521	0.9677	0.2605	3.839	1.033	3.967	.4
.7	0.2538	0.9673	0.2623	3.812	1.034	3.941	.3
.8	0.2554	0.9668	0.2642	3.785	1.034	3.915	.2
.9	0.2571	0.9664	0.2661	3.758	1.035	3.889	.1
15.0	0.2588	0.9659	0.2679	3.732	1.035	3.864	75.0
.1	0.2605	0.9655	0.2698	3.706	1.036	3.839	.9
.2	0.2622	0.9650	0.2717	3.681	1.036	3.814	.8
.3	0.2639	0.9646	0.2736	3.655	1.037	3.790	.7
.4	0.2656	0.9641	0.2754	3.630	1.037	3.766	.6
.5	0.2672	0.9636	0.2773	3.606	1.038	3.742	.5
.6	0.2689	0.9632	0.2792	3.582	1.038	3.719	.4
.7	0.2706	0.9627	0.2811	3.558	1.039	3.695	.3
.8	0.2723	0.9622	0.2830	3.534	1.039	3.673	.2
.9	0.2740	0.9617	0.2849	3.511	1.040	3.650	.1
16.0	0.2756	0.9613	0.2867	3.487	1.040	3.628	74.0
	cos θ	sin θ	cot θ	tan θ	csc θ	sec θ	θ

Table II (Continued)

θ	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ	
16.0	0.2756	0.9613	0.2867	3.487	1.040	3.628	74.0
.1	0.2773	0.9608	0.2886	3.465	1.041	3.606	.9
.2	0.2790	0.9603	0.2905	3.442	1.041	3.584	.8
.3	0.2807	0.9598	0.2924	3.420	1.042	3.563	.7
.4	0.2823	0.9593	0.2943	3.398	1.042	3.542	.6
.5	0.2840	0.9588	0.2962	3.376	1.043	3.521	.5
.6	0.2857	0.9583	0.2981	3.354	1.043	3.500	.4
.7	0.2874	0.9578	0.3000	3.333	1.044	3.480	.3
.8	0.2890	0.9573	0.3019	3.312	1.045	3.460	.2
.9	0.2907	0.9568	0.3038	3.291	1.045	3.440	.1
17.0	0.2924	0.9563	0.3057	3.271	1.046	3.420	73.0
.1	0.2940	0.9558	0.3076	3.251	1.046	3.401	.9
.2	0.2957	0.9553	0.3096	3.230	1.047	3.382	.8
.3	0.2974	0.9548	0.3115	3.211	1.047	3.363	.7
.4	0.2990	0.9542	0.3134	3.191	1.048	3.344	.6
.5	0.3007	0.9537	0.3153	3.172	1.049	3.326	.5
.6	0.3024	0.9532	0.3172	3.152	1.049	3.307	.4
.7	0.3040	0.9527	0.3191	3.133	1.050	3.289	.3
.8	0.3057	0.9521	0.3211	3.115	1.050	3.271	.2
.9	0.3074	0.9516	0.3230	3.096	1.051	3.254	.1
18.0	0.3090	0.9511	0.3249	3.078	1.051	3.236	72.0
.1	0.3107	0.9505	0.3269	3.060	1.052	3.219	.9
.2	0.3123	0.9500	0.3288	3.042	1.053	3.202	.8
.3	0.3140	0.9494	0.3307	3.024	1.053	3.185	.7
.4	0.3156	0.9489	0.3327	3.006	1.054	3.168	.6
.5	0.3173	0.9483	0.3346	2.989	1.054	3.152	.5
.6	0.3190	0.9478	0.3365	2.971	1.055	3.135	.4
.7	0.3206	0.9472	0.3385	2.954	1.056	3.119	.3
.8	0.3223	0.9466	0.3404	2.937	1.056	3.103	.2
.9	0.3239	0.9461	0.3424	2.921	1.057	3.087	.1
19.0	0.3256	0.9455	0.3443	2.904	1.058	3.072	71.0
.1	0.3272	0.9449	0.3463	2.888	1.058	3.056	.9
.2	0.3289	0.9444	0.3482	2.872	1.059	3.041	.8
.3	0.3305	0.9438	0.3502	2.856	1.060	3.026	.7
.4	0.3322	0.9432	0.3522	2.840	1.060	3.011	.6
.5	0.3338	0.9426	0.3541	2.824	1.061	2.996	.5
.6	0.3355	0.9421	0.3561	2.808	1.062	2.981	.4
.7	0.3371	0.9415	0.3581	2.793	1.062	2.967	.3
.8	0.3387	0.9409	0.3600	2.778	1.063	2.952	.2
.9	0.3404	0.9403	0.3620	2.762	1.064	2.938	.1
20.0	0.3420	0.9397	0.3640	2.747	1.064	2.924	70.0
	cos θ	sin θ	cot θ	tan θ	csc θ	sec θ	θ

Table II (Continued)

θ	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ	
20.0	0.3420	0.9397	0.3640	2.747	1.064	2.924	70.0
.1	0.3437	0.9391	0.3659	2.733	1.065	2.910	.9
.2	0.3453	0.9385	0.3679	2.718	1.066	2.896	.8
.3	0.3469	0.9379	0.3699	2.703	1.066	2.882	.7
.4	0.3486	0.9373	0.3719	2.689	1.067	2.869	.6
.5	0.3502	0.9367	0.3739	2.675	1.068	2.855	.5
.6	0.3518	0.9361	0.3759	2.660	1.068	2.842	.4
.7	0.3535	0.9354	0.3779	2.646	1.069	2.829	.3
.8	0.3551	0.9348	0.3799	2.633	1.070	2.816	.2
.9	0.3567	0.9432	0.3819	2.619	1.070	2.803	.1
21.0	0.3584	0.9336	0.3839	2.605	1.071	2.790	69.0
.1	0.3600	0.9330	0.3859	2.592	1.072	2.778	.9
.2	0.3616	0.9323	0.3879	2.578	1.073	2.765	.8
.3	0.3633	0.9317	0.3899	2.565	1.073	2.753	.7
.4	0.3649	0.9311	0.3919	2.552	1.074	2.741	.6
.5	0.3665	0.9304	0.3939	2.539	1.075	2.729	.5
.6	0.3681	0.9298	0.3959	2.526	1.076	2.716	.4
.7	0.3697	0.9291	0.3979	2.513	1.076	2.705	.3
.8	0.3714	0.9285	0.4000	2.500	1.077	2.693	.2
.9	0.3730	0.9278	0.4020	2.488	1.078	2.681	.1
22.0	0.3746	0.9272	0.4040	2.475	1.079	2.669	68.0
.1	0.3762	0.9265	0.4061	2.463	1.079	2.658	.9
.2	0.3778	0.9259	0.4081	2.450	1.080	2.647	.8
.3	0.3795	0.9252	0.4101	2.438	1.081	2.635	.7
.4	0.3811	0.9245	0.4122	2.426	1.082	2.624	.6
.5	0.3827	0.9239	0.4142	2.414	1.082	2.613	.5
.6	0.3843	0.9232	0.4163	2.402	1.083	2.602	.4
.7	0.3859	0.9225	0.4183	2.391	1.084	2.591	.3
.8	0.3875	0.9219	0.4204	2.379	1.085	2.581	.2
.9	0.3891	0.9212	0.4224	2.367	1.086	2.570	.1
23.0	0.3907	0.9205	0.4245	2.356	1.086	2.559	67.0
.1	0.3923	0.9198	0.4265	2.344	1.087	2.549	.9
.2	0.3939	0.9191	0.4286	2.333	1.088	2.538	.8
.3	0.3955	0.9184	0.4307	2.322	1.089	2.528	.7
.4	0.3971	0.9178	0.4327	2.311	1.090	2.518	.6
.5	0.3987	0.9171	0.4348	2.300	1.090	2.508	.5
.6	0.4003	0.9164	0.4369	2.289	1.091	2.498	.4
.7	0.4019	0.9157	0.4390	2.278	1.092	2.488	.3
.8	0.4035	0.9150	0.4411	2.267	1.093	2.478	.2
.9	0.4051	0.9143	0.4431	2.257	1.094	2.468	.1
24.0	0.4067	0.9135	0.4452	2.246	1.095	2.459	66.0
	cos θ	sin θ	cot θ	tan θ	csc θ	sec θ	θ

Table II (Continued)

θ	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ	
24.0	0.4067	0.9135	0.4452	2.246	1.095	2.459	66.0
.1	0.4083	0.9128	0.4473	2.236	1.095	2.449	.9
.2	0.4099	0.9121	0.4494	2.225	1.096	2.439	.8
.3	0.4115	0.9114	0.4515	2.215	1.097	2.430	.7
.4	0.4131	0.9107	0.4536	2.204	1.098	2.421	.6
.5	0.4147	0.9100	0.4557	2.194	1.099	2.411	.5
.6	0.4163	0.9092	0.4578	2.184	1.100	2.402	.4
.7	0.4179	0.9085	0.4599	2.174	1.101	2.393	.3
.8	0.4195	0.9078	0.4621	2.164	1.102	2.384	.2
.9	0.4210	0.9070	0.4642	2.154	1.102	2.375	.1
25.0	0.4226	0.9063	0.4663	2.145	1.103	2.366	65.0
.1	0.4242	0.9056	0.4684	2.135	1.104	2.357	.9
.2	0.4258	0.9048	0.4706	2.125	1.105	2.349	.8
.3	0.4274	0.9041	0.4727	2.116	1.106	2.340	.7
.4	0.4289	0.9033	0.4748	2.106	1.107	2.331	.6
.5	0.4305	0.9026	0.4770	2.097	1.108	2.323	.5
.6	0.4321	0.9018	0.4791	2.087	1.109	2.314	.4
.7	0.4337	0.9011	0.4813	2.078	1.110	2.306	.3
.8	0.4352	0.9003	0.4834	2.069	1.111	2.298	.2
.9	0.4368	0.8996	0.4856	2.059	1.112	2.289	.1
26.0	0.4384	0.8988	0.4877	2.050	1.113	2.281	64.0
.1	0.4399	0.8980	0.4899	2.041	1.114	2.273	.9
.2	0.4415	0.8973	0.4921	2.032	1.115	2.265	.8
.3	0.4431	0.8965	0.4942	2.023	1.115	2.257	.7
.4	0.4446	0.8957	0.4964	2.014	1.116	2.249	.6
.5	0.4462	0.8949	0.4986	2.006	1.117	2.241	.5
.6	0.4478	0.8942	0.5008	1.997	1.118	2.233	.4
.7	0.4493	0.8934	0.5029	1.988	1.119	2.226	.3
.8	0.4509	0.8926	0.5051	1.980	1.120	2.218	.2
.9	0.4524	0.8918	0.5073	1.971	1.121	2.210	.1
27.0	0.4540	0.8910	0.5095	1.963	1.122	2.203	63.0
.1	0.4555	0.8902	0.5117	1.954	1.123	2.195	.9
.2	0.4571	0.8894	0.5139	1.946	1.124	2.188	.8
.3	0.4586	0.8886	0.5161	1.937	1.125	2.180	.7
.4	0.4602	0.8878	0.5184	1.929	1.126	2.173	.6
.5	0.4617	0.8870	0.5206	1.921	1.127	2.166	.5
.6	0.4633	0.8862	0.5228	1.913	1.128	2.158	.4
.7	0.4648	0.8854	0.5250	1.905	1.129	2.151	.3
.8	0.4664	0.8846	0.5272	1.897	1.130	2.144	.2
.9	0.4679	0.8838	0.5295	1.889	1.132	2.137	.1
28.0	0.4695	0.8829	0.5317	1.881	1.133	2.130	62.0
	cos θ	sin θ	cot θ	tan θ	csc θ	sec θ	θ

Table II (Continued)

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$	
28.0	0.4695	0.8829	0.5317	1.881	1.133	2.130	62.0
.1	0.4710	0.8821	0.5340	1.873	1.134	2.123	.9
.2	0.4726	0.8813	0.5362	1.865	1.135	2.116	.8
.3	0.4741	0.8805	0.5384	1.857	1.136	2.109	.7
.4	0.4756	0.8796	0.5407	1.849	1.137	2.103	.6
.5	0.4772	0.8788	0.5430	1.842	1.138	2.096	.5
.6	0.4787	0.8780	0.5452	1.834	1.139	2.089	.4
.7	0.4802	0.8771	0.5475	1.827	1.140	2.082	.3
.8	0.4818	0.8763	0.5498	1.819	1.141	2.076	.2
.9	0.4833	0.8755	0.5520	1.811	1.142	2.069	.1
29.0	0.4848	0.8746	0.5543	1.804	1.143	2.063	61.0
.1	0.4863	0.8738	0.5566	1.797	1.144	2.056	.9
.2	0.4879	0.8729	0.5589	1.789	1.146	2.060	.8
.3	0.4894	0.8721	0.5612	1.782	1.147	2.043	.7
.4	0.4909	0.8712	0.5635	1.775	1.148	2.037	.6
.5	0.4924	0.8704	0.5658	1.767	1.149	2.031	.5
.6	0.4939	0.8695	0.5681	1.760	1.150	2.025	.4
.7	0.4955	0.8686	0.5704	1.753	1.151	2.018	.3
.8	0.4970	0.8678	0.5727	1.746	1.152	2.012	.2
.9	0.4985	0.8669	0.5750	1.739	1.154	2.006	.1
30.0	0.5000	0.8660	0.5774	1.732	1.155	2.000	60.0
.1	0.5015	0.8652	0.5797	1.725	1.156	1.994	.9
.2	0.5030	0.8643	0.5820	1.718	1.157	1.988	.8
.3	0.5045	0.8634	0.5844	1.711	1.158	1.982	.7
.4	0.5060	0.8625	0.5867	1.704	1.159	1.976	.6
.5	0.5075	0.8616	0.5890	1.698	1.161	1.970	.5
.6	0.5090	0.8607	0.5914	1.691	1.162	1.964	.4
.7	0.5105	0.8599	0.5938	1.684	1.163	1.959	.3
.8	0.5120	0.8590	0.5961	1.678	1.164	1.953	.2
.9	0.5135	0.8581	0.5985	1.671	1.165	1.947	.1
31.0	0.5150	0.8572	0.6009	1.664	1.167	1.942	59.0
.1	0.5165	0.8563	0.6032	1.658	1.168	1.936	.9
.2	0.5180	0.8554	0.6056	1.651	1.169	1.930	.8
.3	0.5195	0.8545	0.6080	1.645	1.170	1.925	.7
.4	0.5210	0.8536	0.6104	1.638	1.172	1.919	.6
.5	0.5225	0.8526	0.6128	1.632	1.173	1.914	.5
.6	0.5240	0.8517	0.6152	1.625	1.174	1.908	.4
.7	0.5255	0.8508	0.6176	1.619	1.175	1.903	.3
.8	0.5270	0.8499	0.6200	1.613	1.177	1.898	.2
.9	0.5284	0.8490	0.6224	1.607	1.178	1.892	.1
32.0	0.5299	0.8480	0.6249	1.600	1.179	1.887	58.0
	$\cos\theta$	$\sin\theta$	$\cot\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	θ

Table II (Continued)

θ	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ	
32.0	0.5299	0.8480	0.6249	1.600	1.179	1.887	58.0
.1	0.5314	0.8471	0.6273	1.594	1.180	1.882	.9
.2	0.5329	0.8462	0.6297	1.588	1.182	1.877	.8
.3	0.5344	0.8453	0.6322	1.582	1.183	1.871	.7
.4	0.5358	0.8443	0.6346	1.576	1.184	1.866	.6
.5	0.5373	0.8434	0.6371	1.570	1.186	1.861	.5
.6	0.5388	0.8425	0.6395	1.564	1.187	1.856	.4
.7	0.5402	0.8415	0.6420	1.558	1.188	1.851	.3
.8	0.5417	0.8406	0.6445	1.552	1.190	1.846	.2
.9	0.5432	0.8396	0.6469	1.546	1.191	1.841	.1
33.0	0.5446	0.8387	0.6494	1.540	1.192	1.836	57.0
.1	0.5461	0.8377	0.6519	1.534	1.194	1.831	.9
.2	0.5476	0.8368	0.6544	1.528	1.195	1.826	.8
.3	0.5490	0.8358	0.6569	1.522	1.196	1.821	.7
.4	0.5505	0.8348	0.6594	1.517	1.198	1.817	.6
.5	0.5519	0.8339	0.6619	1.511	1.199	1.812	.5
.6	0.5534	0.8329	0.6644	1.505	1.201	1.807	.4
.7	0.5548	0.8320	0.6669	1.499	1.202	1.802	.3
.8	0.5563	0.8310	0.6694	1.494	1.203	1.798	.2
.9	0.5577	0.8300	0.6720	1.488	1.205	1.793	.1
34.0	0.5592	0.8290	0.6745	1.483	1.206	1.788	56.0
.1	0.5606	0.8281	0.6771	1.477	1.208	1.784	.9
.2	0.5621	0.8271	0.6796	1.471	1.209	1.779	.8
.3	0.5635	0.8261	0.6822	1.466	1.211	1.775	.7
.4	0.5650	0.8251	0.6847	1.460	1.212	1.770	.6
.5	0.5664	0.8241	0.6873	1.455	1.213	1.766	.5
.6	0.5678	0.8231	0.6899	1.450	1.215	1.761	.4
.7	0.5693	0.8221	0.6924	1.444	1.216	1.757	.3
.8	0.5707	0.8211	0.6950	1.439	1.218	1.752	.2
.9	0.5721	0.8202	0.6976	1.433	1.219	1.748	.1
35.0	0.5736	0.8192	0.7002	1.428	1.221	1.743	55.0
.1	0.5750	0.8181	0.7028	1.423	1.222	1.739	.9
.2	0.5764	0.8171	0.7054	1.418	1.224	1.735	.8
.3	0.5779	0.8161	0.7080	1.412	1.225	1.731	.7
.4	0.5793	0.8151	0.7107	1.407	1.227	1.726	.6
.5	0.5807	0.8141	0.7133	1.402	1.228	1.722	.5
.6	0.5821	0.8131	0.7159	1.397	1.230	1.718	.4
.7	0.5835	0.8121	0.7186	1.392	1.231	1.714	.3
.8	0.5850	0.8111	0.7212	1.387	1.233	1.710	.2
.9	0.5864	0.8100	0.7239	1.381	1.235	1.705	.1
36.0	0.5878	0.8090	0.7265	1.376	1.236	1.701	54.0
	cos θ	sin θ	cot θ	tan θ	csc θ	sec θ	θ

Table II (Continued)

θ	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ	
36.0	0.5878	0.8090	0.7265	1.376	1.236	1.701	54.0
.1	0.5892	0.8080	0.7292	1.371	1.238	1.697	.9
.2	0.5906	0.8070	0.7319	1.366	1.239	1.693	.8
.3	0.5920	0.8059	0.7346	1.361	1.241	1.689	.7
.4	0.5934	0.8049	0.7373	1.356	1.242	1.685	.6
.5	0.5948	0.8039	0.7400	1.351	1.244	1.681	.5
.6	0.5962	0.8028	0.7427	1.347	1.246	1.677	.4
.7	0.5976	0.8018	0.7454	1.342	1.247	1.673	.3
.8	0.5990	0.8007	0.7481	1.337	1.249	1.669	.2
.9	0.6004	0.7997	0.7508	1.332	1.250	1.666	.1
37.0	0.6018	0.7986	0.7536	1.327	1.252	1.662	53.0
.1	0.6032	0.7976	0.7563	1.322	1.254	1.658	.9
.2	0.6046	0.7965	0.7590	1.317	1.255	1.654	.8
.3	0.6060	0.7955	0.7618	1.313	1.257	1.650	.7
.4	0.6074	0.7944	0.7646	1.308	1.259	1.646	.6
.5	0.6088	0.7934	0.7673	1.303	1.260	1.643	.5
.6	0.6101	0.7923	0.7701	1.299	1.262	1.639	.4
.7	0.6115	0.7912	0.7729	1.294	1.264	1.635	.3
.8	0.6129	0.7902	0.7757	1.289	1.266	1.632	.2
.9	0.6143	0.7891	0.7785	1.285	1.267	1.628	.1
38.0	0.6157	0.7880	0.7813	1.280	1.269	1.624	52.0
.1	0.6170	0.7869	0.7841	1.275	1.271	1.621	.9
.2	0.6184	0.7859	0.7869	1.271	1.272	1.617	.8
.3	0.6198	0.7848	0.7898	1.266	1.274	1.613	.7
.4	0.6211	0.7837	0.7926	1.262	1.276	1.610	.6
.5	0.6225	0.7826	0.7954	1.257	1.278	1.606	.5
.6	0.6239	0.7815	0.7983	1.253	1.280	1.603	.4
.7	0.6252	0.7804	0.8012	1.248	1.281	1.599	.3
.8	0.6266	0.7793	0.8040	1.244	1.283	1.596	.2
.9	0.6280	0.7782	0.8069	1.239	1.285	1.592	.1
39.0	0.6293	0.7771	0.8098	1.235	1.287	1.589	51.0
.1	0.6307	0.7760	0.8127	1.230	1.289	1.586	.9
.2	0.6320	0.7749	0.8156	1.226	1.290	1.582	.8
.3	0.6334	0.7738	0.8185	1.222	1.292	1.579	.7
.4	0.6347	0.7727	0.8214	1.217	1.294	1.575	.6
.5	0.6361	0.7716	0.8243	1.213	1.296	1.572	.5
.6	0.6374	0.7705	0.8273	1.209	1.298	1.569	.4
.7	0.6388	0.7694	0.8302	1.205	1.300	1.566	.3
.8	0.6401	0.7683	0.8332	1.200	1.302	1.562	.2
.9	0.6414	0.7672	0.8361	1.196	1.304	1.559	.1
40.0	0.6428	0.7660	0.8391	1.192	1.305	1.556	50.0
	cos θ	sin θ	cot θ	tan θ	csc θ	sec θ	θ

Table II (Continued)

θ	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ	
40.0	0.6428	0.7660	0.8391	1.192	1.305	1.556	50.0
.1	0.6441	0.7649	0.8421	1.188	1.307	1.552	.9
.2	0.6455	0.7638	0.8451	1.183	1.309	1.549	.8
.3	0.6468	0.7627	0.8481	1.179	1.311	1.546	.7
.4	0.6481	0.7615	0.8511	1.175	1.313	1.543	.6
.5	0.6494	0.7604	0.8541	1.171	1.315	1.540	.5
.6	0.6508	0.7593	0.8571	1.167	1.317	1.537	.4
.7	0.6521	0.7581	0.8601	1.163	1.319	1.534	.3
.8	0.6534	0.7570	0.8632	1.159	1.321	1.530	.2
.9	0.6547	0.7559	0.8662	1.154	1.323	1.527	.1
41.0	0.6561	0.7547	0.8693	1.150	1.325	1.524	49.0
.1	0.6574	0.7536	0.8724	1.146	1.327	1.521	.9
.2	0.6587	0.7524	0.8754	1.142	1.329	1.518	.8
.3	0.6600	0.7513	0.8785	1.138	1.331	1.515	.7
.4	0.6613	0.7501	0.8816	1.134	1.333	1.512	.6
.5	0.6626	0.7490	0.8847	1.130	1.335	1.509	.5
.6	0.6639	0.7478	0.8878	1.126	1.337	1.506	.4
.7	0.6652	0.7466	0.8910	1.122	1.339	1.503	.3
.8	0.6665	0.7455	0.8941	1.118	1.341	1.500	.2
.9	0.6678	0.7443	0.8972	1.115	1.344	1.497	.1
42.0	0.6691	0.7431	0.9004	1.111	1.346	1.494	48.0
.1	0.6704	0.7420	0.9036	1.107	1.348	1.492	.9
.2	0.6717	0.7408	0.9067	1.103	1.350	1.489	.8
.3	0.6730	0.7396	0.9099	1.099	1.352	1.486	.7
.4	0.6743	0.7385	0.9131	1.095	1.354	1.483	.6
.5	0.6756	0.7373	0.9163	1.091	1.356	1.480	.5
.6	0.6769	0.7361	0.9195	1.087	1.359	1.477	.4
.7	0.6782	0.7349	0.9228	1.084	1.361	1.475	.3
.8	0.6794	0.7337	0.9260	1.080	1.363	1.472	.2
.9	0.6807	0.7325	0.9293	1.076	1.365	1.469	.1
43.0	0.6820	0.7314	0.9325	1.072	1.367	1.466	47.0
.1	0.6833	0.7302	0.9358	1.069	1.370	1.464	.9
.2	0.6845	0.7290	0.9391	1.065	1.372	1.461	.8
.3	0.6858	0.7278	0.9424	1.061	1.374	1.458	.7
.4	0.6871	0.7266	0.9457	1.057	1.376	1.455	.6
.5	0.6884	0.7254	0.9490	1.054	1.379	1.453	.5
.6	0.6896	0.7242	0.9523	1.050	1.381	1.450	.4
.7	0.6909	0.7230	0.9556	1.046	1.383	1.447	.3
.8	0.6921	0.7218	0.9590	1.043	1.386	1.445	.2
.9	0.6934	0.7206	0.9623	1.039	1.388	1.442	.1
44.0	0.6947	0.7193	0.9657	1.036	1.390	1.440	46.0
	cos θ	sin θ	cot θ	tan θ	csc θ	sec θ	θ

Table II (Continued)

θ	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ	
44.0	0.6947	0.7193	0.9657	1.036	1.390	1.440	46.0
.1	0.6959	0.7181	0.9691	1.032	1.393	1.437	.9
.2	0.6972	0.7169	0.9725	1.028	1.395	1.434	.8
.3	0.6984	0.7157	0.9759	1.025	1.397	1.432	.7
.4	0.6997	0.7145	0.9793	1.021	1.400	1.429	.6
.5	0.7009	0.7133	0.9827	1.018	1.402	1.427	.5
.6	0.7022	0.7120	0.9861	1.014	1.404	1.424	.4
.7	0.7034	0.7108	0.9896	1.011	1.407	1.422	.3
.8	0.7046	0.7096	0.9930	1.007	1.409	1.419	.2
.9	0.7059	0.7083	0.9965	1.003	1.412	1.417	.1
45.0	0.7071	0.7071	1.000	1.000	1.414	1.414	45.0
	cos θ	sin θ	cot θ	tan θ	csc θ	sec θ	θ

Table III Trigonometric Functions—Degrees and Minutes

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$	
0°00′	0.0000	1.0000	0.0000	—	1.0000	—	90°00′
10	0.0029	1.0000	0.0029	343.8	1.000	343.8	50
20	0.0058	1.0000	0.0058	171.9	1.000	171.9	40
30	0.0087	1.0000	0.0087	114.6	1.000	114.6	30
40	0.0116	0.9999	0.0116	85.94	1.000	85.95	20
50	0.0145	0.9999	0.0145	68.75	1.000	68.76	10
1°00′	0.0175	0.9998	0.0175	57.29	1.000	57.30	89°00′
10	0.0204	0.9998	0.0204	49.10	1.000	49.11	50
20	0.0233	0.9997	0.0233	42.96	1.000	42.98	40
30	0.0262	0.9997	0.0262	38.19	1.000	38.20	30
40	0.0291	0.9996	0.0291	34.37	1.000	34.38	20
50	0.0320	0.9995	0.0320	31.24	1.001	31.26	10
2°00′	0.0349	0.9994	0.0349	28.64	1.001	28.65	88°00′
10	0.0378	0.9993	0.0378	26.43	1.001	26.45	50
20	0.0407	0.9992	0.0407	24.54	1.001	24.56	40
30	0.0436	0.9990	0.0437	22.90	1.001	22.93	30
40	0.0465	0.9989	0.0466	21.47	1.001	21.49	20
50	0.0494	0.9988	0.0495	20.21	1.001	20.23	10
3°00′	0.0523	0.9986	0.0524	19.08	1.001	19.11	87°00′
10	0.0552	0.9985	0.0553	18.07	1.002	18.10	50
20	0.0581	0.9983	0.0582	17.17	1.002	17.20	40
30	0.0610	0.9981	0.0612	16.35	1.002	16.38	30
40	0.0640	0.9980	0.0641	15.60	1.002	15.64	20
50	0.0669	0.9978	0.0670	14.92	1.002	14.96	10
4°00′	0.0698	0.9976	0.0699	14.30	1.002	14.34	86°00′
10	0.0727	0.9974	0.0729	13.73	1.003	13.76	50
20	0.0756	0.9971	0.0758	13.20	1.003	13.23	40
30	0.0785	0.9969	0.0787	12.71	1.003	12.75	30
40	0.0814	0.9967	0.0816	12.25	1.003	12.29	20
50	0.0843	0.9964	0.0846	11.83	1.004	11.87	10
5°00′	0.0872	0.9962	0.0875	11.43	1.004	11.47	85°00′
10	0.0901	0.9959	0.0904	11.06	1.004	11.10	50
20	0.0929	0.9957	0.0934	10.71	1.004	10.76	40
30	0.0958	0.9954	0.0963	10.39	1.005	10.43	30
40	0.0987	0.9951	0.0992	10.08	1.005	10.13	20
50	0.1016	0.9948	0.1022	9.788	1.005	9.839	10
6°00′	0.1045	0.9945	0.1051	9.514	1.006	9.567	84°00′
	$\cos\theta$	$\sin\theta$	$\cot\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	θ

Table III (Continued)

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$	
6°00′	0.1045	0.9945	0.1051	9.514	1.006	9.567	84°00′
10	0.1074	0.9942	0.1080	9.255	1.006	9.309	50
20	0.1103	0.9939	0.1110	9.010	1.006	9.065	40
30	0.1132	0.9936	0.1139	8.777	1.006	8.834	30
40	0.1161	0.9932	0.1169	8.556	1.007	8.614	20
50	0.1190	0.9929	0.1198	8.345	1.007	8.405	10
7°00′	0.1219	0.9925	0.1228	8.144	1.008	8.206	83°00′
10	0.1248	0.9922	0.1257	7.953	1.008	8.016	50
20	0.1276	0.9918	0.1287	7.770	1.008	7.834	40
30	0.1305	0.9914	0.1317	7.596	1.009	7.661	30
40	0.1334	0.9911	0.1346	7.429	1.009	7.496	20
50	0.1363	0.9907	0.1376	7.269	1.009	7.337	10
8°00′	0.1392	0.9903	0.1405	7.115	1.010	7.185	82°00′
10	0.1421	0.9899	0.1435	6.968	1.010	7.040	50
20	0.1449	0.9894	0.1465	6.827	1.011	6.900	40
30	0.1478	0.9890	0.1495	6.691	1.011	6.765	30
40	0.1507	0.9886	0.1524	6.561	1.012	6.636	20
50	0.1536	0.9881	0.1554	6.435	1.012	6.512	10
9°00′	0.1564	0.9877	0.1584	6.314	1.012	6.392	81°00′
10	0.1593	0.9872	0.1614	6.197	1.013	6.277	50
20	0.1622	0.9868	0.1644	6.084	1.013	6.166	40
30	0.1650	0.9863	0.1673	5.976	1.014	6.059	30
40	0.1679	0.9858	0.1703	5.871	1.014	5.955	20
50	0.1708	0.9853	0.1733	5.769	1.015	5.855	10
10°00′	0.1736	0.9848	0.1763	5.671	1.015	5.759	80°00′
10	0.1765	0.9843	0.1793	5.576	1.016	5.665	50
20	0.1794	0.9838	0.1823	5.485	1.016	5.575	40
30	0.1822	0.9833	0.1853	5.396	1.017	5.487	30
40	0.1851	0.9827	0.1883	5.309	1.018	5.403	20
50	0.1880	0.9822	0.1914	5.226	1.018	5.320	10
11°00′	0.1908	0.9816	0.1944	5.145	1.019	5.241	79°00′
10	0.1937	0.9811	0.1974	5.066	1.019	5.164	50
20	0.1965	0.9805	0.2004	4.989	1.020	5.089	40
30	0.1994	0.9799	0.2035	4.915	1.020	5.016	30
40	0.2022	0.9793	0.2065	4.843	1.021	4.945	20
50	0.2051	0.9787	0.2095	4.773	1.022	4.876	10
12°00′	0.2079	0.9781	0.2126	4.705	1.022	4.810	78°00′
	$\cos\theta$	$\sin\theta$	$\cot\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	θ

Table III (Continued)

θ	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ	
12°00′	0.2079	0.9781	0.2126	4.705	1.022	4.810	78°00′
10	0.2108	0.9775	0.2156	4.638	1.023	4.745	50
20	0.2136	0.9769	0.2186	4.574	1.024	4.682	40
30	0.2164	0.9763	0.2217	4.511	1.024	4.620	30
40	0.2193	0.9757	0.2247	4.449	1.025	4.560	20
50	0.2221	0.9750	0.2278	4.390	1.026	4.502	10
13°00′	0.2250	0.9744	0.2309	4.331	1.026	4.445	77°00′
10	0.2278	0.9737	0.2339	4.275	1.027	4.390	50
20	0.2306	0.9730	0.2370	4.219	1.028	4.336	40
30	0.2334	0.9724	0.2401	4.165	1.028	4.284	30
40	0.2363	0.9717	0.2432	4.113	1.029	4.232	20
50	0.2391	0.9710	0.2642	4.061	1.030	4.182	10
14°00′	0.2419	0.9703	0.2493	4.011	1.031	4.134	76°00′
10	0.2447	0.9696	0.2524	3.962	1.031	4.086	50
20	0.2476	0.9689	0.2555	3.914	1.032	4.039	40
30	0.2504	0.9681	0.2586	3.867	1.033	3.994	30
40	0.2532	0.9674	0.2617	3.821	1.034	3.950	20
50	0.2560	0.9667	0.2648	3.776	1.034	3.906	10
15°00′	0.2588	0.9659	0.2679	3.732	1.035	3.864	75°00′
10	0.2616	0.9652	0.2711	3.689	1.036	3.822	50
20	0.2644	0.9644	0.2742	3.647	1.037	3.782	40
30	0.2672	0.9636	0.2773	3.606	1.038	3.742	30
40	0.2700	0.9628	0.2805	3.566	1.039	3.703	20
50	0.2728	0.9621	0.2836	3.526	1.039	3.665	10
16°00′	0.2756	0.9613	0.2867	3.487	1.040	3.628	74°00′
10	0.2784	0.9605	0.2899	3.450	1.041	3.592	50
20	0.2812	0.9596	0.2931	3.412	1.042	3.556	40
30	0.2840	0.9588	0.2962	3.376	1.043	3.521	30
40	0.2868	0.9580	0.2994	3.340	1.044	3.487	20
50	0.2896	0.9572	0.3026	3.305	1.045	3.453	10
17°00′	0.2924	0.9563	0.3057	3.271	1.046	3.420	73°00′
10	0.2952	0.9555	0.3089	3.237	1.047	3.388	50
20	0.2979	0.9546	0.3121	3.204	1.048	3.356	40
30	0.3007	0.9537	0.3153	3.172	1.049	3.326	30
40	0.3035	0.9528	0.3185	3.140	1.049	3.295	20
50	0.3062	0.9520	0.3217	3.108	1.050	3.265	10
18°00′	0.3090	0.9511	0.3249	3.078	1.051	3.236	72°00′
	cos θ	sin θ	cot θ	tan θ	csc θ	sec θ	θ

Table III (Continued)

θ	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ	
18°00′	0.3090	0.9511	0.3249	3.078	1.051	3.236	72°00′
10	0.3118	0.9502	0.3281	3.047	1.052	3.207	50
20	0.3145	0.9492	0.3314	3.018	1.053	3.179	40
30	0.3173	0.9483	0.3346	2.989	1.054	3.152	30
40	0.3201	0.9474	0.3378	2.960	1.056	3.124	20
50	0.3228	0.9465	0.3411	2.932	1.057	3.098	10
19°00′	0.3256	0.9455	0.3443	2.904	1.058	3.072	71°00′
10	0.3283	0.9446	0.3476	2.877	1.059	3.046	50
20	0.3311	0.9436	0.3508	2.850	1.060	3.021	40
30	0.3338	0.9426	0.3541	2.824	1.061	2.996	30
40	0.3365	0.9417	0.3574	2.798	1.062	2.971	20
50	0.3393	0.9407	0.3607	2.773	1.063	2.947	10
20°00′	0.3420	0.9397	0.3640	2.747	1.064	2.924	70°00′
10	0.3448	0.9387	0.3673	2.723	1.065	2.901	50
20	0.3475	0.9377	0.3706	2.699	1.066	2.878	40
30	0.3502	0.9367	0.3739	2.675	1.068	2.855	30
40	0.3529	0.9356	0.3772	2.651	1.069	2.833	20
50	0.3557	0.9346	0.3805	2.628	1.070	2.812	10
21°00′	0.3584	0.9336	0.3839	2.605	1.071	2.790	69°00′
10	0.3611	0.9325	0.3872	2.583	1.072	2.769	50
20	0.3638	0.9315	0.3906	2.560	1.074	2.749	40
30	0.3665	0.9304	0.3939	2.539	1.075	2.729	30
40	0.3692	0.9293	0.3973	2.517	1.076	2.709	20
50	0.3719	0.9283	0.4006	2.496	1.077	2.689	10
22°00′	0.3746	0.9272	0.4040	2.475	1.079	2.669	68°00′
10	0.3773	0.9261	0.4074	2.455	1.080	2.650	50
20	0.3800	0.9250	0.4108	2.434	1.081	2.632	40
30	0.3827	0.9239	0.4142	2.414	1.082	2.613	30
40	0.3854	0.9228	0.4176	2.394	1.084	2.596	20
50	0.3881	0.9216	0.4210	2.375	1.085	2.577	10
23°00′	0.3907	0.9205	0.4245	2.356	1.086	2.559	67°00′
10	0.3934	0.9194	0.4279	2.337	1.088	2.542	50
20	0.3961	0.9182	0.4314	2.318	1.089	2.525	40
30	0.3987	0.9171	0.4348	2.300	1.090	2.508	30
40	0.4014	0.9159	0.4383	2.282	1.092	2.491	20
50	0.4041	0.9147	0.4417	2.264	1.093	2.475	10
24°00′	0.4067	0.9135	0.4452	2.246	1.095	2.459	66°00′
	cos θ	sin θ	cot θ	tan θ	csc θ	sec θ	θ

Table III (Continued)

θ	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ	
24°00′	0.4067	0.9135	0.4452	2.246	1.095	2.459	66°00′
10	0.4094	0.9124	0.4487	2.229	1.096	2.443	50
20	0.4120	0.9112	0.4522	2.211	1.097	2.427	40
30	0.4147	0.9100	0.4557	2.194	1.099	2.411	30
40	0.4173	0.9088	0.4592	2.177	1.100	2.396	20
50	0.4200	0.9075	0.4628	2.161	1.102	2.381	10
25°00′	0.4226	0.9063	0.4663	2.145	1.103	2.366	65°00′
10	0.4253	0.9051	0.4699	2.128	1.105	2.352	50
20	0.4279	0.9038	0.4734	2.112	1.106	2.337	40
30	0.4305	0.9026	0.4770	2.097	1.108	2.323	30
40	0.4331	0.9013	0.4806	2.081	1.109	2.309	20
50	0.4358	0.9001	0.4841	2.066	1.111	2.295	10
26°00′	0.4384	0.8988	0.4877	2.050	1.113	2.281	64°00′
10	0.4410	0.8975	0.4913	2.035	1.114	2.268	50
20	0.4436	0.8962	0.4950	2.020	1.116	2.254	40
30	0.4462	0.8949	0.4986	2.006	1.117	2.241	30
40	0.4488	0.8936	0.5022	1.991	1.119	2.228	20
50	0.4514	0.8923	0.5059	1.977	1.121	2.215	10
27°00′	0.4540	0.8910	0.5095	1.963	1.122	2.203	63°00′
10	0.4566	0.8897	0.5132	1.949	1.124	2.190	50
20	0.4592	0.8884	0.5169	1.935	1.126	2.178	40
30	0.4617	0.8870	0.5206	1.921	1.127	2.166	30
40	0.4643	0.8857	0.5243	1.907	1.129	2.154	20
50	0.4669	0.8843	0.5280	1.894	1.131	2.142	10
28°00′	0.4695	0.8829	0.5317	1.881	1.133	2.130	62°00′
10	0.4720	0.8816	0.5354	1.868	1.134	2.118	50
20	0.4746	0.8802	0.5392	1.855	1.136	2.107	40
30	0.4772	0.8788	0.5430	1.842	1.138	2.096	30
40	0.4797	0.8774	0.5467	1.829	1.140	2.085	20
50	0.4823	0.8760	0.5505	1.816	1.142	2.074	10
29°00′	0.4848	0.8746	0.5543	1.804	1.143	2.063	61°00′
10	0.4874	0.8732	0.5581	1.792	1.145	2.052	50
20	0.4899	0.8718	0.5619	1.780	1.147	2.041	40
30	0.4924	0.8704	0.5658	1.767	1.149	2.031	30
40	0.4950	0.8689	0.5696	1.756	1.151	2.020	20
50	0.4975	0.8675	0.5735	1.744	1.153	2.010	10
30°00′	0.5000	0.8660	0.5774	1.732	1.155	2.000	60°00′
	cos θ	sin θ	cot θ	tan θ	csc θ	sec θ	θ

Table III (Continued)

θ	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ	
30°00′	0.5000	0.8660	0.5774	1.732	1.155	2.000	60°00′
10	0.5025	0.8646	0.5812	1.720	1.157	1.990	50
20	0.5050	0.8631	0.5851	1.709	1.159	1.980	40
30	0.5075	0.8616	0.5890	1.698	1.161	1.970	30
40	0.5100	0.8601	0.5930	1.686	1.163	1.961	20
50	0.5125	0.8587	0.5969	1.675	1.165	1.951	10
31°00′	0.5150	0.8572	0.6009	1.664	1.167	1.942	59°00′
10	0.5175	0.8557	0.6048	1.653	1.169	1.932	50
20	0.5200	0.8542	0.6088	1.643	1.171	1.923	40
30	0.5225	0.8526	0.6128	1.632	1.173	1.914	30
40	0.5250	0.8511	0.6168	1.621	1.175	1.905	20
50	0.5275	0.8496	0.6208	1.611	1.177	1.896	10
32°00′	0.5299	0.8480	0.6249	1.600	1.179	1.887	58°00′
10	0.5324	0.8465	0.6289	1.590	1.181	1.878	50
20	0.5348	0.8450	0.6330	1.580	1.184	1.870	40
30	0.5373	0.8434	0.6371	1.570	1.186	1.861	30
40	0.5398	0.8418	0.6412	1.560	1.188	1.853	20
50	0.5422	0.8403	0.6453	1.550	1.190	1.844	10
33°00′	0.5446	0.8387	0.6494	1.540	1.192	1.836	57°00′
10	0.5471	0.8371	0.6536	1.530	1.195	1.828	50
20	0.5495	0.8355	0.6577	1.520	1.197	1.820	40
30	0.5519	0.8339	0.6619	1.511	1.199	1.812	30
40	0.5544	0.8323	0.6661	1.501	1.202	1.804	20
50	0.5568	0.8307	0.6703	1.492	1.204	1.796	10
34°00′	0.5592	0.8290	0.6745	1.483	1.206	1.788	56°00′
10	0.5616	0.8274	0.6787	1.473	1.209	1.781	50
20	0.5640	0.8258	0.6830	1.464	1.211	1.773	40
30	0.5664	0.8241	0.6873	1.455	1.213	1.766	30
40	0.5688	0.8225	0.6916	1.446	1.216	1.758	20
50	0.5712	0.8208	0.6959	1.437	1.218	1.751	10
35°00′	0.5736	0.8192	0.7002	1.428	1.221	1.743	55°00′
10	0.5760	0.8175	0.7046	1.419	1.223	1.736	50
20	0.5783	0.8158	0.7089	1.411	1.226	1.729	40
30	0.5807	0.8141	0.7133	1.402	1.228	1.722	30
40	0.5831	0.8124	0.7177	1.393	1.231	1.715	20
50	0.5854	0.8107	0.7221	1.385	1.233	1.708	10
36°00′	0.5878	0.8090	0.7265	1.376	1.236	1.701	54°00′
	cos θ	sin θ	cot θ	tan θ	csc θ	sec θ	θ

Table III (Continued)

θ	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ	
36°00′	0.5878	0.8090	0.7265	1.376	1.236	1.701	54°00′
10	0.5901	0.8073	0.7310	1.368	1.239	1.695	50
20	0.5925	0.8056	0.7355	1.360	1.241	1.688	40
30	0.5948	0.8039	0.7400	1.351	1.244	1.681	30
40	0.5972	0.8021	0.7445	1.343	1.247	1.675	20
50	0.5995	0.8004	0.7490	1.335	1.249	1.668	10
37°00′	0.6018	0.7986	0.7536	1.327	1.252	1.662	53°00′
10	0.6041	0.7969	0.7581	1.319	1.255	1.655	50
20	0.6065	0.7951	0.7627	1.311	1.258	1.649	40
30	0.6088	0.7934	0.7673	1.303	1.260	1.643	30
40	0.6111	0.7916	0.7720	1.295	1.263	1.636	20
50	0.6134	0.7898	0.7766	1.288	1.266	1.630	10
38°00′	0.6157	0.7880	0.7813	1.280	1.269	1.624	52°00′
10	0.6180	0.7862	0.7860	1.272	1.272	1.618	50
20	0.6202	0.7844	0.7907	1.265	1.275	1.612	40
30	0.6225	0.7826	0.7954	1.257	1.278	1.606	30
40	0.6248	0.7808	0.8002	1.250	1.281	1.601	20
50	0.6271	0.7790	0.8050	1.242	1.284	1.595	10
39°00′	0.6293	0.7771	0.8098	1.235	1.287	1.589	51°00′
10	0.6316	0.7753	0.8146	1.228	1.290	1.583	50
20	0.6338	0.7735	0.8195	1.220	1.293	1.578	40
30	0.6361	0.7716	0.8243	1.213	1.296	1.572	30
40	0.6383	0.7698	0.8292	1.206	1.299	1.567	20
50	0.6406	0.7679	0.8342	1.199	1.302	1.561	10
40°00′	0.6428	0.7660	0.8391	1.192	1.305	1.556	50°00′
10	0.6450	0.7642	0.8441	1.185	1.309	1.550	50
20	0.6472	0.7623	0.8491	1.178	1.312	1.545	40
30	0.6494	0.7604	0.8541	1.171	1.315	1.540	30
40	0.6517	0.7585	0.8591	1.164	1.318	1.535	20
50	0.6539	0.7566	0.8642	1.157	1.322	1.529	10
41°00′	0.6561	0.7547	0.8693	1.150	1.325	1.524	49°00′
10	0.6583	0.7528	0.8744	1.144	1.328	1.519	50
20	0.6604	0.7509	0.8796	1.137	1.332	1.514	40
30	0.6626	0.7490	0.8847	1.130	1.335	1.509	30
40	0.6648	0.7470	0.8899	1.124	1.339	1.504	20
50	0.6670	0.7451	0.8952	1.117	1.342	1.499	10
42°00′	0.6691	0.7431	0.9004	1.111	1.346	1.494	48°00′
	cos θ	sin θ	cot θ	tan θ	csc θ	sec θ	θ

Table III (Continued)

θ	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ	
42°00′	0.6691	0.7431	0.9004	1.111	1.346	1.494	48°00′
10	0.6713	0.7412	0.9057	1.104	1.349	1.490	50
20	0.6734	0.7392	0.9110	1.098	1.353	1.485	40
30	0.6756	0.7373	0.9163	1.091	1.356	1.480	30
40	0.6777	0.7353	0.9217	1.085	1.360	1.476	20
50	0.6799	0.7333	0.9271	1.079	1.364	1.471	10
43°00′	0.6820	0.7314	0.9325	1.072	1.367	1.466	47°00′
10	0.6841	0.7294	0.9380	1.066	1.371	1.462	50
20	0.6862	0.7274	0.9435	1.060	1.375	1.457	40
30	0.6884	0.7254	0.9490	1.054	1.379	1.453	30
40	0.6905	0.7234	0.9545	1.048	1.382	1.448	20
50	0.6926	0.7214	0.9601	1.042	1.386	1.444	10
44°00′	0.6947	0.7193	0.9657	1.036	1.390	1.440	46°00′
10	0.6967	0.7173	0.9713	1.030	1.394	1.435	50
20	0.6988	0.7153	0.9770	1.024	1.398	1.431	40
30	0.7009	0.7133	0.9827	1.018	1.402	1.427	30
40	0.7030	0.7112	0.9884	1.012	1.406	1.423	20
50	0.7050	0.7092	0.9942	1.006	1.410	1.418	10
45°00′	0.7071	0.7071	1.0000	1.0000	1.414	1.414	45°00′
	cos θ	sin θ	cot θ	tan θ	csc θ	sec θ	θ

Table IV Common Logarithms

x	0	1	2	3	4	5	6	7	8	9
1.0	.0000	.0043	.0086	.0128	.0170	.0212	.0253	.0294	.0334	.0374
1.1	.0414	.0453	.0492	.0531	.0569	.0607	.0645	.0682	.0719	.0755
1.2	.0792	.0828	.0864	.0899	.0934	.0969	.1004	.1038	.1072	.1106
1.3	.1139	.1173	.1206	.1239	.1271	.1303	.1335	.1367	.1399	.1430
1.4	.1461	.1492	.1523	.1553	.1584	.1614	.1644	.1673	.1703	.1732
1.5	.1761	.1790	.1818	.1847	.1875	.1903	.1931	.1959	.1987	.2014
1.6	.2041	.2068	.2095	.2122	.2148	.2175	.2201	.2227	.2253	.2279
1.7	.2304	.2330	.2355	.2380	.2405	.2430	.2455	.2480	.2504	.2529
1.8	.2553	.2577	.2601	.2625	.2648	.2672	.2695	.2718	.2742	.2765
1.9	.2788	.2810	.2833	.2856	.2878	.2900	.2923	.2945	.2967	.2989
2.0	.3010	.3032	.3054	.3075	.3096	.3118	.3139	.3160	.3181	.3201
2.1	.3222	.3243	.3263	.3284	.3304	.3324	.3345	.3365	.3385	.3404
2.2	.3424	.3444	.3464	.3483	.3502	.3522	.3541	.3560	.3579	.3598
2.3	.3617	.3636	.3655	.3674	.3692	.3711	.3729	.3747	.3766	.3784
2.4	.3802	.3820	.3838	.3856	.3874	.3892	.3909	.3927	.3945	.3962
2.5	.3979	.3997	.4014	.4031	.4048	.4065	.4082	.4099	.4116	.4133
2.6	.4150	.4166	.4183	.4200	.4216	.4232	.4249	.4265	.4281	.4298
2.7	.4314	.4330	.4346	.4362	.4378	.4393	.4409	.4425	.4440	.4456
2.8	.4472	.4487	.4502	.4518	.4533	.4548	.4564	.4579	.4594	.4609
2.9	.4624	.4639	.4654	.4669	.4683	.4698	.4713	.4728	.4742	.4757
3.0	.4771	.4786	.4800	.4814	.4829	.4843	.4857	.4871	.4886	.4900
3.1	.4914	.4928	.4942	.4955	.4969	.4983	.4997	.5011	.5024	.5038
3.2	.5051	.5065	.5079	.5092	.5105	.5119	.5132	.5145	.5159	.5172
3.3	.5185	.5198	.5211	.5224	.5237	.5250	.5263	.5276	.5289	.5302
3.4	.5315	.5328	.5340	.5353	.5366	.5378	.5391	.5403	.5416	.5428
3.5	.5441	.5453	.5465	.5478	.5490	.5502	.5514	.5527	.5539	.5551
3.6	.5563	.5575	.5587	.5599	.5611	.5623	.5635	.5647	.5658	.5670
3.7	.5682	.5694	.5705	.5717	.5729	.5740	.5752	.5763	.5775	.5786
3.8	.5798	.5809	.5821	.5832	.5843	.5855	.5866	.5877	.5888	.5899
3.9	.5911	.5922	.5933	.5944	.5955	.5966	.5977	.5988	.5999	.6010
4.0	.6021	.6031	.6042	.6053	.6064	.6075	.6085	.6096	.6107	.6117
4.1	.6128	.6138	.6149	.6160	.6170	.6180	.6191	.6201	.6212	.6222
4.2	.6232	.6243	.6253	.6263	.6274	.6284	.6294	.6304	.6314	.6325
4.3	.6335	.6345	.6355	.6365	.6375	.6385	.6395	.6405	.6415	.6425
4.4	.6435	.6444	.6454	.6464	.6474	.6484	.6493	.6503	.6513	.6522
4.5	.6532	.6542	.6551	.6561	.6571	.6580	.6590	.6599	.6609	.6618
4.6	.6628	.6637	.6646	.6656	.6665	.6675	.6684	.6693	.6702	.6712
4.7	.6721	.6730	.6739	.6749	.6758	.6767	.6776	.6785	.6794	.6803
4.8	.6812	.6821	.6830	.6839	.6848	.6857	.6866	.6875	.6884	.6893
4.9	.6902	.6911	.6920	.6928	.6937	.6946	.6955	.6964	.6972	.6981
5.0	.6990	.6998	.7007	.7016	.7024	.7033	.7042	.7050	.7059	.7067
5.1	.7076	.7084	.7093	.7101	.7110	.7118	.7126	.7135	.7143	.7152
5.2	.7160	.7168	.7177	.7185	.7193	.7202	.7210	.7218	.7226	.7235
5.3	.7243	.7251	.7259	.7267	.7275	.7284	.7292	.7300	.7308	.7316
5.4	.7324	.7332	.7340	.7348	.7356	.7364	.7372	.7380	.7388	.7396
x	0	1	2	3	4	5	6	7	8	9

Table IV (Continued)

x	0	1	2	3	4	5	6	7	8	9
5.5	.7404	.7412	.7419	.7427	.7435	.7443	.7451	.7459	.7466	.7474
5.6	.7482	.7490	.7497	.7505	.7513	.7520	.7528	.7536	.7543	.7551
5.7	.7559	.7566	.7574	.7582	.7589	.7597	.7604	.7612	.7619	.7627
5.8	.7634	.7642	.7649	.7657	.7664	.7672	.7679	.7686	.7694	.7701
5.9	.7709	.7716	.7723	.7731	.7738	.7745	.7752	.7760	.7767	.7774
6.0	.7782	.7789	.7796	.7803	.7810	.7818	.7825	.7832	.7839	.7846
6.1	.7853	.7860	.7868	.7875	.7882	.7889	.7896	.7903	.7910	.7917
6.2	.7924	.7931	.7938	.7945	.7952	.7959	.7966	.7973	.7980	.7987
6.3	.7993	.8000	.8007	.8014	.8021	.8028	.8035	.8041	.8048	.8055
6.4	.8062	.8069	.8075	.8082	.8089	.8096	.8102	.8109	.8116	.8122
6.5	.8129	.8136	.8142	.8149	.8156	.8162	.8169	.8176	.8182	.8189
6.6	.8195	.8202	.8209	.8215	.8222	.8228	.8235	.8241	.8248	.8254
6.7	.8261	.8267	.8274	.8280	.8287	.8293	.8299	.8306	.8312	.8319
6.8	.8325	.8331	.8338	.8344	.8351	.8357	.8363	.8370	.8376	.8382
6.9	.8388	.8395	.8401	.8407	.8414	.8420	.8426	.8432	.8439	.8445
7.0	.8451	.8457	.8463	.8470	.8476	.8482	.8488	.8494	.8500	.8506
7.1	.8513	.8519	.8525	.8531	.8537	.8543	.8549	.8555	.8561	.8567
7.2	.8573	.8579	.8585	.8591	.8597	.8603	.8609	.8615	.8621	.8627
7.3	.8633	.8639	.8645	.8651	.8657	.8663	.8669	.8675	.8681	.8686
7.4	.8692	.8698	.8704	.8710	.8716	.8722	.8727	.8733	.8739	.8745
7.5	.8751	.8756	.8762	.8768	.8774	.8779	.8785	.8791	.8797	.8802
7.6	.8808	.8814	.8820	.8825	.8831	.8837	.8842	.8848	.8854	.8859
7.7	.8865	.8871	.8876	.8882	.8887	.8893	.8899	.8904	.8910	.8915
7.8	.8921	.8927	.8932	.8938	.8943	.8949	.8954	.8960	.8965	.8971
7.9	.8976	.8982	.8987	.8993	.8998	.9004	.9009	.9015	.9020	.9025
8.0	.9031	.9036	.9042	.9047	.9053	.9058	.9063	.9069	.9074	.9079
8.1	.9085	.9090	.9096	.9101	.9106	.9112	.9117	.9122	.9128	.9133
8.2	.9138	.9143	.9149	.9154	.9159	.9165	.9170	.9175	.9180	.9186
8.3	.9191	.9196	.9201	.9206	.9212	.9217	.9222	.9227	.9232	.9238
8.4	.9243	.9248	.9253	.9258	.9263	.9269	.9274	.9279	.9284	.9289
8.5	.9294	.9299	.9304	.9309	.9315	.9320	.9325	.9330	.9335	.9340
8.6	.9345	.9350	.9355	.9360	.9365	.9370	.9375	.9380	.9385	.9390
8.7	.9395	.9400	.9405	.9410	.9415	.9420	.9425	.9430	.9435	.9440
8.8	.9445	.9450	.9455	.9460	.9465	.9469	.9474	.9479	.9484	.9489
8.9	.9494	.9499	.9504	.9509	.9513	.9518	.9523	.9528	.9533	.9538
9.0	.9542	.9547	.9552	.9557	.9562	.9566	.9571	.9576	.9581	.9586
9.1	.9590	.9595	.9600	.9605	.9609	.9614	.9619	.9624	.9628	.9633
9.2	.9638	.9643	.9647	.9652	.9657	.9661	.9666	.9671	.9675	.9680
9.3	.9685	.9689	.9694	.9699	.9703	.9708	.9713	.9717	.9722	.9727
9.4	.9731	.9736	.9741	.9745	.9750	.9754	.9759	.9763	.9768	.9773
9.5	.9777	.9782	.9786	.9791	.9795	.9800	.9805	.9809	.9814	.9818
9.6	.9823	.9827	.9832	.9836	.9841	.9845	.9850	.9854	.9859	.9863
9.7	.9868	.9872	.9877	.9881	.9886	.9890	.9894	.9899	.9903	.9908
9.8	.9912	.9917	.9921	.9926	.9930	.9934	.9939	.9943	.9948	.9952
9.9	.9956	.9961	.9965	.9969	.9974	.9978	.9983	.9987	.9991	.9996
x	0	1	2	3	4	5	6	7	8	9

Answers to Odd-Numbered Exercises and Chapter Tests

CHAPTER 1

Problem Set 1.1

1. Acute, complement is 80°, supplement is 170°

3. Acute, complement is 45°, supplement is 135°

5. Obtuse, complement is $-30°$ [because $120° + (-30°) = 90°$], supplement is 60°

7. We can't tell if x is acute or obtuse (or neither), complement is $90° - x$, supplement is $180° - x$.

9. 60°

11. 45°

13. 50° (Look at it in terms of the big triangle *ABC*.)

15. Use the proportion $\frac{x}{360°} = \frac{4 \text{ hrs}}{12 \text{ hrs}}$, then $x = 120°$.

17. 5 (This triangle is called a 3–4–5 right triangle. You will see it again.)

19. 15

21. 5

Note: Whenever the three sides in a right triangle are whole numbers, those three numbers are called a Pythagorean triple.

23. 2 (Note that this must be a 30°–60°–90° triangle.)

25. $3\sqrt{2}$ (Note that this must be a 45°–45°–90° triangle.)

27. 4 (This is another 30°–60°–90° triangle.)

29. Find x by solving the equation $(\sqrt{10})^2 = (x + 2)^2 + x^2$ to get $x = 1$.

31. 25 feet

33. Longest side 2, third side $\sqrt{3}$

35. Shortest side 4, third side $4\sqrt{3}$

37. Shortest side 3, longest side 6

39. Shortest side $6/\sqrt{3} = 2\sqrt{3}$, longest side $4\sqrt{3}$

41. 40 feet

43. $4\sqrt{2}/5$

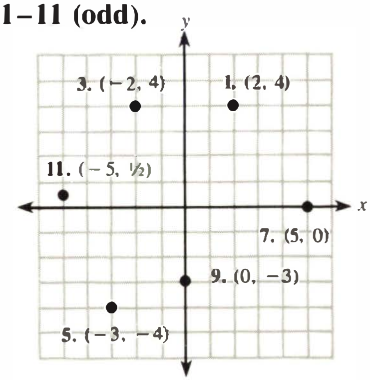
45. 8

47. $4/\sqrt{2} = 2\sqrt{2}$

49. 1414 feet

Problem Set 1.2

1–11 (odd).



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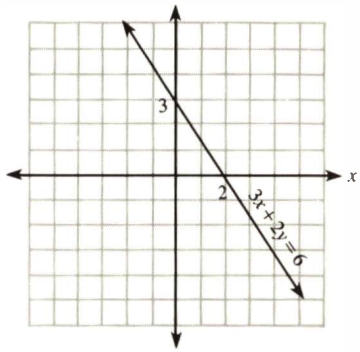

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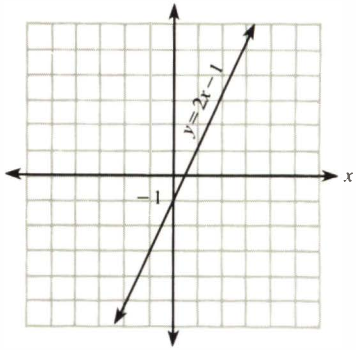

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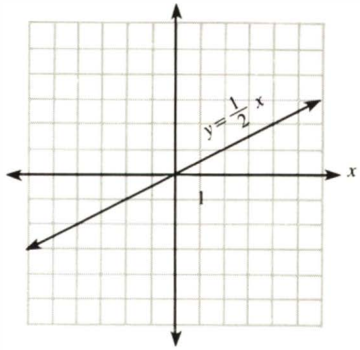

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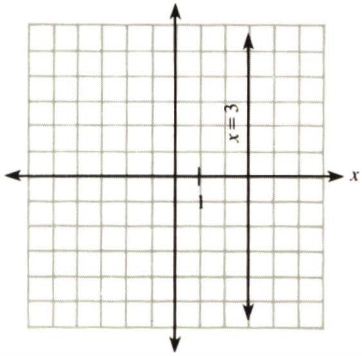

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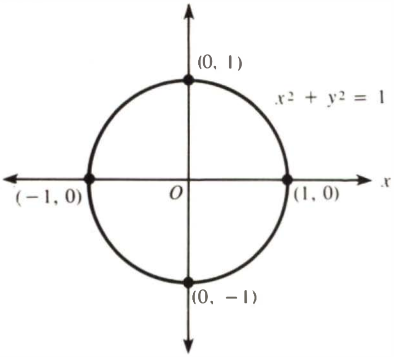


23. 5

25. 13

27. $\sqrt{61}$

29. $\sqrt{130}$

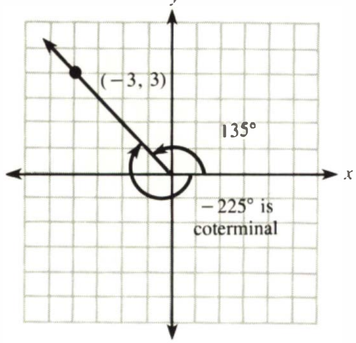
31. 5

33. −1, 3

35. 1.3 miles

37. homeplate: (0,0); first base: (60,0); second base: (60, 60); third base: (0, 60)

39.



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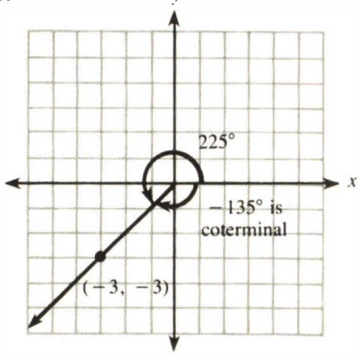

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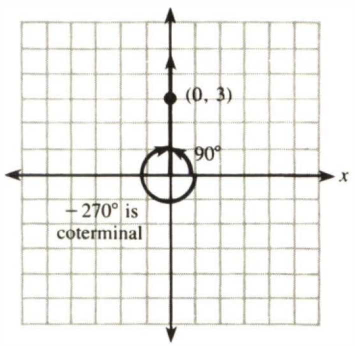

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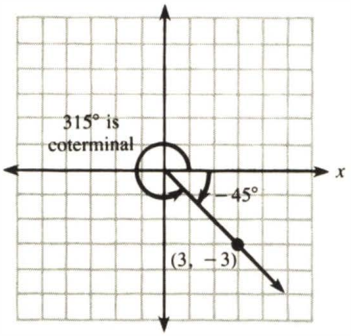

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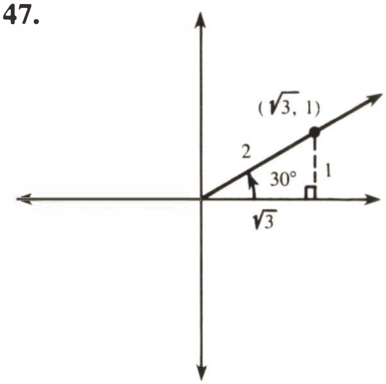

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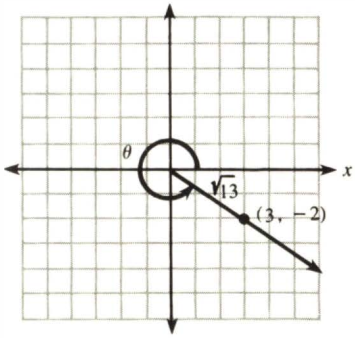

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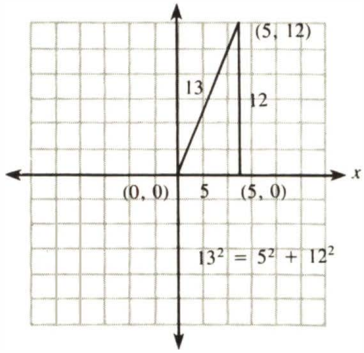
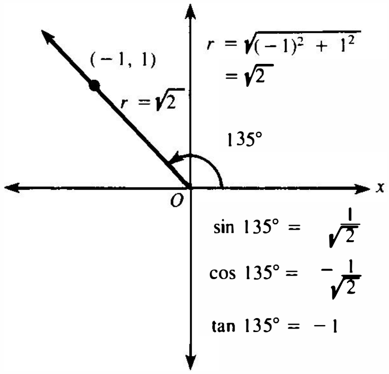


Problem Set 1.3

1.



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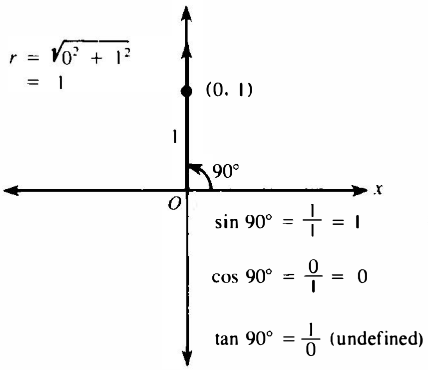

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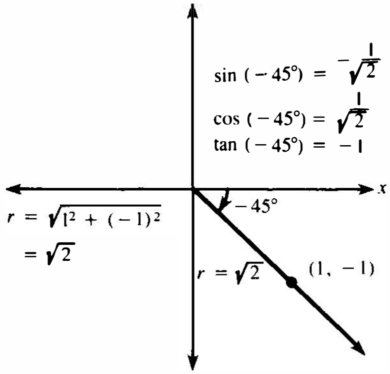


7. QI, QIV because cos $\theta = x/r$ and x is positive in QI and QIV. (Remember, r is always positive.)

9. QIII, QIV

11. QI, QIII

13. QIII

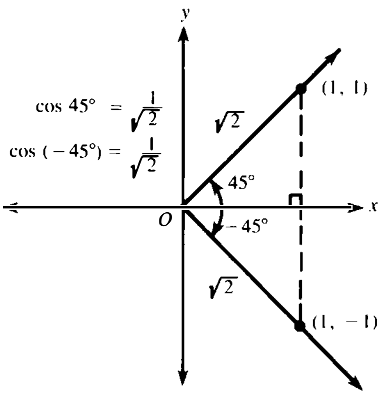
15. QI (both positive), QIV (both negative)

	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
17.	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{4}{3}$	$\frac{3}{4}$	$\frac{5}{3}$	$\frac{5}{4}$
19.	$\frac{4}{5}$	$-\frac{3}{5}$	$-\frac{4}{3}$	$-\frac{3}{4}$	$-\frac{5}{3}$	$\frac{5}{4}$
21.	$\frac{12}{13}$	$-\frac{5}{13}$	$-\frac{12}{5}$	$-\frac{5}{12}$	$-\frac{13}{5}$	$\frac{13}{12}$
23.	$-\frac{2}{\sqrt{5}}$	$-\frac{1}{\sqrt{5}}$	2	$\frac{1}{2}$	$-\sqrt{5}$	$-\frac{\sqrt{5}}{2}$
25.	$\frac{b}{\sqrt{a^2+b^2}}$	$\frac{a}{\sqrt{a^2+b^2}}$	$\frac{b}{a}$	$\frac{a}{b}$	$\frac{\sqrt{a^2+b^2}}{a}$	$\frac{\sqrt{a^2+b^2}}{b}$
27.	$\frac{12}{13}$	$\frac{5}{13}$	$\frac{12}{5}$	$\frac{5}{12}$	$\frac{13}{5}$	$\frac{13}{12}$
29.	$-\frac{4}{5}$	$-\frac{3}{5}$	$\frac{4}{3}$	$\frac{3}{4}$	$-\frac{5}{3}$	$-\frac{5}{4}$
31.	$-\frac{12}{13}$	$\frac{5}{13}$	$-\frac{12}{5}$	$-\frac{5}{12}$	$\frac{13}{5}$	$-\frac{13}{12}$
33.	$\frac{2}{\sqrt{5}}$	$\frac{1}{\sqrt{5}}$	2	$\frac{1}{2}$	$\sqrt{5}$	$\frac{\sqrt{5}}{2}$

35. A point on the terminal side of θ is $(1, 2)$, because it satisfies the equation $y = 2x$; $r = \sqrt{1^2 + 2^2} = \sqrt{5}$; $\sin\theta = 2/\sqrt{5}$; $\cos\theta = 1/\sqrt{5}$.

37. Find a point in QII that is on the terminal side of $y = -3x$ to obtain $\sin\theta = 3/\sqrt{10}$ and $\tan\theta = -3$.

39.



41. If $\sin\theta = -3/5$, let $y = -6$ and $r = 10$. Then $x = \pm\sqrt{10^2 - (-6)^2} = \pm\sqrt{64} = \pm 8$.

43. $\sin\theta = 0$ when $y = 0$, so θ is 0°, 180°, or 360°

45. 90°, 270°

47. $\tan\theta$ is undefined when $x = 0$, so θ is 90° and 270°

Problem Set 1.4

1.	$1/7$	**3.**	$5/3$	**5.**	$-3/2$	**7.**	$-\sqrt{2}$
9.	$1/x\ (x \neq 0)$	**11.**	$a + b$	**13.**	$5/4$	**15.**	$-1/2$
17.	$1/a$	**19.**	$1/2$	**21.**	21	**23.**	-0.5
25.	$12/5$	**27.**	$-13/5$	**29.**	$144/169$	**31.**	1
33.	$-3/5$	**35.**	$1/2$	**37.**	$1/2\sqrt{2}$		

	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
39.	$\frac{5}{13}$	$\frac{12}{13}$	$\frac{5}{12}$	$\frac{12}{5}$	$\frac{13}{12}$	$\frac{13}{5}$
41.	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$\frac{2}{\sqrt{3}}$	-2
43.	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$
45.	$\frac{1}{a}$	$\frac{\sqrt{a^2-1}}{a}$	$\frac{1}{\sqrt{a^2-1}}$	$\sqrt{a^2-1}$	$\frac{a}{\sqrt{a^2-1}}$	a

47. $-1/2$

49. 3

51. m

Note: As Problems 49 through 52 indicate, the slope of a line through the origin is the same as the tangent of the angle the line makes with the positive x-axis.

Problem Set 1.5

1. $\cos\theta = \pm\sqrt{1 - \sin^2\theta}$

3. $\cot\theta = \pm\dfrac{\sqrt{1-\sin^2\theta}}{\sin\theta}$

5. $\sec\theta = 1/\cos\theta$

7. $\tan\theta = \pm\dfrac{\sqrt{1-\cos^2\theta}}{\cos\theta}$

9. $\csc\theta\cot\theta = \dfrac{1}{\sin\theta}\cdot\dfrac{\cos\theta}{\sin\theta} = \dfrac{\cos\theta}{\sin^2\theta}$

11. $1/\cos\theta$

13. $\dfrac{\sec\theta}{\csc\theta} = \dfrac{1/\cos\theta}{1/\sin\theta} = \dfrac{\sin\theta}{\cos\theta}$

15. $\dfrac{\sec\theta}{\tan\theta} = \dfrac{1/\cos\theta}{\sin\theta/\cos\theta} = \dfrac{1}{\sin\theta}$

17. $\dfrac{\tan\theta}{\cot\theta} = \dfrac{\sin\theta/\cos\theta}{\cos\theta/\sin\theta} = \dfrac{\sin^2\theta}{\cos^2\theta}$

19. $\sin^2\theta$

21. $\dfrac{\sin\theta + 1}{\cos\theta}$

23. $2\cos\theta$

25. $\cos\theta$

27. $\dfrac{\sin\theta}{\cos\theta} + \dfrac{1}{\sin\theta} = \dfrac{\sin\theta}{\cos\theta}\cdot\dfrac{\boldsymbol{\sin\theta}}{\boldsymbol{\sin\theta}} + \dfrac{1}{\sin\theta}\cdot\dfrac{\boldsymbol{\cos\theta}}{\boldsymbol{\cos\theta}}$

$$= \frac{\sin^2\theta + \cos\theta}{\sin\theta\cdot\cos\theta}$$

29. $\dfrac{\sin\theta\cos\theta + 1}{\cos\theta}$

31. $\dfrac{1 - \sin^2\theta}{\sin\theta} = \dfrac{\cos^2\theta}{\sin\theta}$

33. $\sin^2\theta + 7\sin\theta + 12$

35. $1 - \sin^2\theta = \cos^2\theta$

37. $\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta$
$= 1 - 2\sin\theta\cos\theta$

39. $\cos\theta\tan\theta = \cos\theta \cdot \dfrac{\sin\theta}{\cos\theta} = \sin\theta$

43. $\dfrac{\csc\theta}{\cot\theta} = \dfrac{1/\sin\theta}{\cos\theta/\sin\theta} = \dfrac{1}{\cos\theta} = \sec\theta$

47. $\sin\theta\tan\theta + \cos\theta = \sin\theta \cdot \dfrac{\sin\theta}{\cos\theta} + \cos\theta$

$$= \frac{\sin^2\theta}{\cos\theta} + \cos\theta$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta}$$

$$= \sec\theta$$

51. $\csc\theta - \sin\theta = \dfrac{1}{\sin\theta} - \sin\theta = \dfrac{1}{\sin\theta} - \dfrac{\sin^2\theta}{\sin\theta}$

$$= \frac{1 - \sin^2\theta}{\sin\theta}$$

$$= \frac{\cos^2\theta}{\sin\theta}$$

57. $(\sin\theta - \cos\theta)^2 - 1 = \sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta - 1$
$= 1 - 2\sin\theta\cos\theta - 1$
$= -2\sin\theta\cos\theta$

Chapter 1 Test

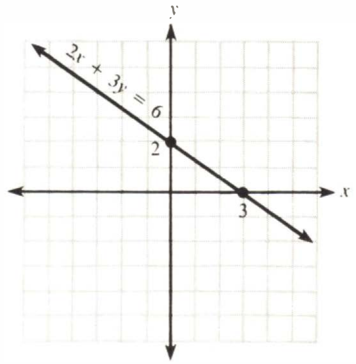
1. $4\sqrt{2}$

2. 8

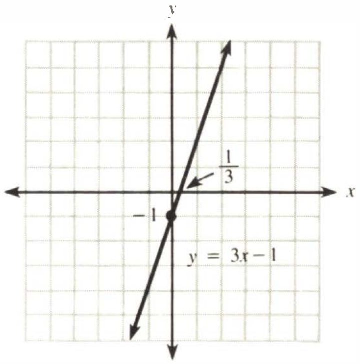
3. $3\sqrt{3}$

4. 3

5.



6.

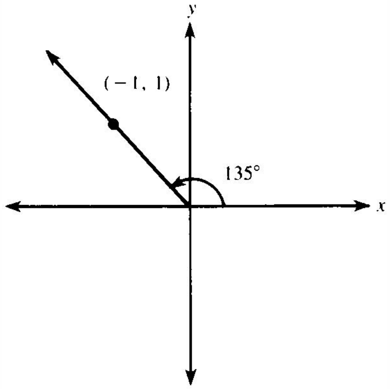


7. 13

8. $\sqrt{a^2 + b^2}$

9. $-5, 1$

10.



11. $\sin 90° = 1$, $\cos 90° = 0$, $\tan 90°$ is undefined
12. $\sin(-45°) = -1/\sqrt{2}$, $\cos(-45°) = 1/\sqrt{2}$, $\tan(-45°) = -1$
13. QIV
14. QII

	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
15.	$\frac{4}{5}$	$-\frac{3}{5}$	$-\frac{4}{3}$	$-\frac{3}{4}$	$-\frac{5}{3}$	$\frac{5}{4}$
16.	$-\frac{1}{\sqrt{10}}$	$-\frac{3}{\sqrt{10}}$	$\frac{1}{3}$	3	$-\frac{\sqrt{10}}{3}$	$-\sqrt{10}$
17.	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$-\frac{2}{\sqrt{3}}$	2
18.	$-\frac{12}{13}$	$-\frac{5}{13}$	$\frac{12}{5}$	$\frac{5}{12}$	$-\frac{13}{5}$	$-\frac{13}{12}$

19. $\sin\theta = -2/\sqrt{5}$ $\cos\theta = 1/\sqrt{5}$
20. $\theta = 0°, 180°, 360°$
21. $-4/3$
22. $-1/2$
23. $2\sqrt{2}/3$ and $3/2\sqrt{2}$
24. $1/2\sqrt{2}$ and $2\sqrt{2}$
25. 9
26. 9/8
27. 2
28. $-25/27$
29. $\pm\dfrac{\sqrt{1-\cos^2\theta}}{\cos^2\theta}$
30. $\sin^2\theta - 4\sin\theta - 21$
31. $1 - 2\sin\theta\cos\theta$
32. $\cos^2\theta/\sin\theta$

CHAPTER 2

Problem Set 2.1

	$\sin A$	$\cos A$	$\tan A$	$\cot A$	$\sec A$	$\csc A$
1.	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{4}{3}$	$\frac{3}{4}$	$\frac{5}{3}$	$\frac{5}{4}$
3.	$\frac{2}{\sqrt{5}}$	$\frac{1}{\sqrt{5}}$	2	$\frac{1}{2}$	$\sqrt{5}$	$\frac{\sqrt{5}}{2}$
5.	$\frac{2}{3}$	$\frac{\sqrt{5}}{3}$	$\frac{2}{\sqrt{5}}$	$\frac{\sqrt{5}}{2}$	$\frac{3}{\sqrt{5}}$	$\frac{3}{2}$

	sin A	cos A	tan A	sin B	cos B	tan B
7.	$\frac{5}{6}$	$\frac{\sqrt{11}}{6}$	$\frac{5}{\sqrt{11}}$	$\frac{\sqrt{11}}{6}$	$\frac{5}{6}$	$\frac{\sqrt{11}}{5}$
9.	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
11.	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{4}{3}$
13.	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$

15. $90° - 10° = 80°$ **17.** $90° - 8° = 82°$ **19.** $17°$ **21.** $90° - x$

23. x **25.** $4\left(\frac{1}{2}\right) = 2$ **27.** 3 **29.** $1/2\sqrt{2}$ or $\sqrt{2}/4$

31. $\left(\frac{1 + \sqrt{3}}{2}\right)^2 = \frac{1 + 2\sqrt{3} + 3}{4} = \frac{4 + 2\sqrt{3}}{4} = \frac{2 + \sqrt{3}}{2}$ **33.** 0

35. $4 + 2\sqrt{3}$ **37.** 1 **39.** $2\sqrt{3}$ **41.** $-3\sqrt{3}/2$

43. $\sqrt{2}$ **45.** $\sec 30° = \frac{1}{\cos 30°} = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}$

47. $2/\sqrt{3}$

49. 1 **51.** $\sqrt{2}$ **53.** 0.866025403 **55.** 0.866025403

57. 0.707106781 **59.** 0.707106781 **61.** 1 **63.** $\sqrt{9 + 16} = 5$

65. $-1, 3$ **67.** See Example 5, Section 1.2

Problem Set 2.2

1. $64°\ 9'$ **3.** $89°\ 40'$ **5.** $106°\ 49'$

7. $\begin{array}{r} 89°\ 60' \\ -34°\ 12' \\ \hline 55°\ 48' \end{array}$ **9.** $59°\ 43'$ **11.** $\begin{array}{r} 76°\ 24' \\ -22°\ 34' \\ \hline \end{array} = \begin{array}{r} 75°\ 84' \\ -22°\ 34' \\ \hline 53°\ 50' \end{array}$

13. $39°\ 50'$ **15.** $35°\ 24'$ (*Calculator*: .4 [×] 60 [=])

17. $16°\ 15'$ **19.** $92°\ 33'$ **21.** $19°\ 54'$ **23.** $28°\ 21'$

25. $45.2°$ (*Calculator*: 12 [÷] 60 [+] 45 [=]) **27.** $62.6°$

29. $17.3\bar{3}°$ (the $\bar{3}$ means the 3's repeat indefinitely) **31.** $48.45°$

33. 0.4446 **35.** 0.9494 **37.** 0.9598 **39.** 19.74

41. 0.7373 (*Calculator*: 53.6 [tan] [1/x]) **43.** 0.9100

45. 0.9057 **47.** 0.8355 **49.** 0.3365 **51.** 0.0204

53. 34.0° **55.** 48.0° **57.** 20.8° **59.** 45.3°
61. 37.0° **63.** 8° **65.** 85° 10′
67. 85° 20′ (*Calculator*: 12.30 [1/x] [cos⁻¹]) **69.** 38° 30′
71. 45° 40′ **73.** 1 **75.** $\csc\theta \geq 1$
77. $\sin 90° = 1$, $\cos 90° = 0$, $\tan 90°$ is undefined
79. $\sin\theta = -12/13$, $\tan\theta = 12/5$, $\cot\theta = 5/12$, $\sec\theta = -13/5$, $\csc\theta = -13/12$
81. QII

Problem Set 2.3

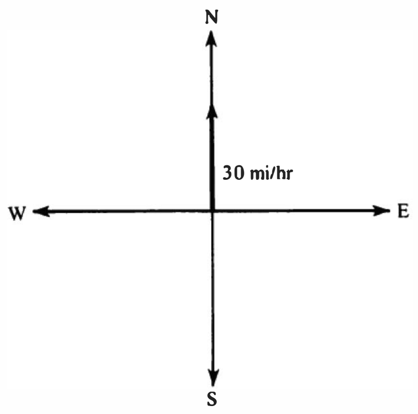
1. 3 **3.** 2 **5.** 5 **7.** 4
9. 2 **11.** 2 **13.** 3 **15.** 4
17. 4 **19.** 9.6 **21.** 4.1 **23.** 31°
25. 53° **27.** $B = 65°$, $a = 10$, $b = 22$
29. $B = 57.4°$, $b = 67.9$, $c = 80.6$ **31.** $B = 79°\,20'$, $a = 1.12$, $c = 6.03$
33. $A = 14°$, $a = 1.4$, $b = 5.6$ **35.** $A = 63°\,40'$, $a = 654$, $c = 730$
37. $A = 23°$, $B = 67°$, $c = 95$ **39.** $A = 42.8°$, $B = 47.2°$, $b = 2.97$
41. $A = 53.9°$, $B = 36.1°$, $a = 208$ **43.** 49°
45. 43° **47.** $-12/5$ **49.** $13/5$ **51.** $2\sqrt{2}/3$

Problem Set 2.4

1. Height: 39 centimeters; base angles: 69° **3.** 39 feet **5.** 36.6°
7. 55.1° **9.** 5 tan 11° + 5 tan 12° = 2 feet **11.** 8.5 miles
13. 39 miles **15.** 63.3 miles north, 48.0 miles west **17.** $1 - 2\sin\theta\cos\theta$

Problem Set 2.5

1.



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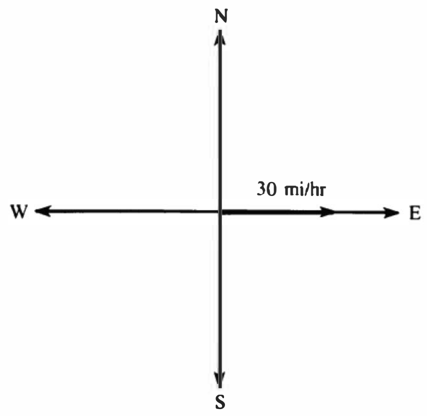
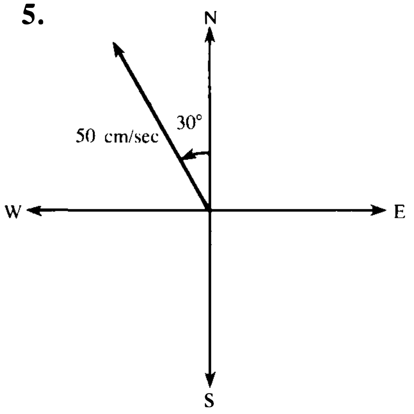
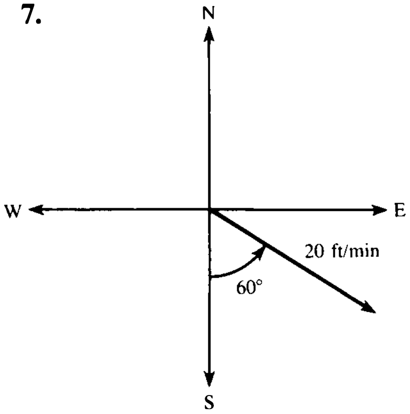


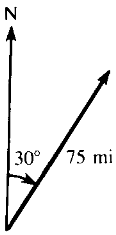
5.



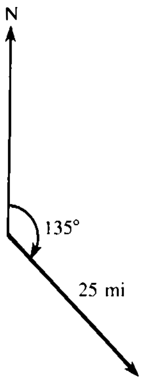
7.



9.



11.



13. 12.4 miles per hour at N 76.0° E

15. 202 miles per hour N 21.5° E

17. 2.6 miles per hour

19. $|\mathbf{V}_x| = 35 \quad |\mathbf{V}_y| = 20$

21. $|\mathbf{V}_x| = 12.6 \quad |\mathbf{V}_y| = 5.66$

23. $|\mathbf{V}_x| = 339 \quad |\mathbf{V}_y| = 248$

25. $|\mathbf{V}_x| = 55 \quad |\mathbf{V}_y| = 32$

27. $|\mathbf{V}| = 50$

29. $|\mathbf{V}| = 43.6$

31. $|\mathbf{V}| = 5.9$

33. 850 feet per second

35. 2550 feet

37. 38.1 feet per second at an inclination of 23.2°

39. 100 miles south, 90.3 miles east

41. 245 miles north, 145 miles east

43. $\sin 135° = 1/\sqrt{2}$, $\cos 135° = -1/\sqrt{2}$, $\tan 135° = -1$

45. $\sin \theta = 2/\sqrt{5}$, $\cos \theta = 1/\sqrt{5}$

47. $x = \pm 6$

Chapter 2 Test

	sin *A*	cos *A*	tan *A*	sin *B*	cos *B*	tan *B*
1.	$\frac{1}{\sqrt{5}}$	$\frac{2}{\sqrt{5}}$	$\frac{1}{2}$	$\frac{2}{\sqrt{5}}$	$\frac{1}{\sqrt{5}}$	2
2.	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
3.	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{4}{3}$
4.	$\frac{5}{13}$	$\frac{12}{13}$	$\frac{5}{12}$	$\frac{12}{13}$	$\frac{5}{13}$	$\frac{12}{5}$

5. 76°

6. 17°

7. 5/4

8. 2

9. 0

10. $\sqrt{3}/2$

11. 73° 10′

12. 9° 43′

13. 73° 12′

14. 16° 27′

15. 2.8°

16. 79.5°

17. 0.4120 **18.** 0.7902 **19.** 2.035 **20.** 0.3378
21. 4.7° or 4° 40′ **22.** 58.7° or 58° 40′ **23.** 71.2° or 71° 10′ **24.** 13.3° or 13° 20′
25. 4 **26.** 2
27. $A = 33.2°$, $B = 56.8°$, $c = 124$ **28.** $A = 30.3°$, $B = 59.7°$, $b = 41.5$
29. $A = 65.1°$, $a = 657$, $c = 724$ **30.** $B = 54.5°$, $a = 0.268$, $b = 0.376$
31. 86 centimeters **32.** 5.8 feet
33. 112 feet **34.** 96.4 miles west, 84.3 miles south
35. $|\mathbf{V}_x| = 4.3$ $|\mathbf{V}_y| = 2.5$ **36.** 72°
37. $|\mathbf{V}_x| = 400$ feet per second $|\mathbf{V}_y| = 690$ feet per second **38.** 60 miles south, 104 miles east

CHAPTER 3

Problem Set 3.1

1. 30° **3.** 36.6° **5.** 48.3° **7.** 15° 10′
9. 28° 40′ **11.** 60° **13.** 60° **15.** −0.9744
17. −0.2045 **19.** −0.7427
21. 1.503 (*Calculator*: 311.7 [cos] [1/x]) **23.** 1.732
25. −0.8151 **27.** −0.5000 **29.** 0.7071 **31.** 0.2711
33. 1.423 **35.** 1.302 **37.** −0.8660 **39.** 0.5851
41. 198° **43.** 140° **45.** 210.5° or 210° 30′
47. 74.7° or 74° 40′ **49.** 105.2° or 105° 10′ **51.** $\sqrt{3}/2$
53. −1 **55.** −1/2 **57.** −2 **59.** 2
61. 1/2 **63.** $-1/\sqrt{3}$ **65.** 240° **67.** 135°
69. 300° **71.** 240° **73.** 65.3 (2.3931 [1/x] [$\cos^{-1}$])

75.

θ	85°	87°	89°	89.9°	89.99°
$\tan\theta$	11.4	19.1	57.3	573.0	5730

77. Complement 20°, supplement 110° **79.** Complement $90° - x$, supplement $180° - x$
81. Side opposite 30° is 5, side opposite 60° is $5\sqrt{3}$ **83.** 1/4 **85.** 1

Problem Set 3.2

1. 3 **3.** 1/2 **5.** 3π **7.** 2
9. $\theta = s/r = 450/4000 = 0.1125$ radians **11.** $\pi/6 \cong 0.52$ **13.** $\pi/2 \cong 1.57$
15. $13\pi/9 \cong 4.54$ **17.** $-5\pi/6 \cong -2.62$ **19.** $7\pi/3 \cong 7.33$ **21.** $-3\pi/4 \cong -2.36$

23. 2.11

25. 0.000291

27. Use the answer to Problem 27 to write $\theta = 1'$ $= 0.000291$ radians, then substitute this value of θ into $\theta = s/r$ along with $r = 4000$ and solve for S. 1.16 miles

29. $60°$	**31.** $120°$	**33.** $-210°$	**35.** $300°$
37. $720°$	**39.** $15°$	**41.** $-70°$	**43.** $74.5°$
45. $43.0°$	**47.** $286.5°$	**49.** $-\sqrt{3}/2$	**51.** $1/\sqrt{3}$
53. -2	**55.** 2	**57.** $-2\sqrt{2}$	**59.** $-1/\sqrt{2}$
61. $\sqrt{3}$	**63.** $\sqrt{3}/2$	**65.** 0	**67.** $\sqrt{3}/2$
69. -2			

	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
71.	$-\frac{3}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	-3	$-\frac{1}{3}$	$\sqrt{10}$	$-\frac{\sqrt{10}}{3}$
73.	$\frac{n}{r}$	$\frac{m}{r}$	$\frac{n}{m}$	$\frac{m}{n}$	$\frac{r}{m}$	$\frac{r}{n}$ where $r = \sqrt{m^2 + n^2}$
75.	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$-\frac{2}{\sqrt{3}}$	2
77.	$\frac{2}{\sqrt{5}}$	$\frac{1}{\sqrt{5}}$	2	$\frac{1}{2}$	$\sqrt{5}$	$\frac{\sqrt{5}}{2}$

Problem Set 3.3

		$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
1.	$\theta = 150°$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$-\frac{2}{\sqrt{3}}$	2
3.	$\theta = \frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$\frac{2}{\sqrt{3}}$	-2
5.	$\theta = 180°$	0	-1	0	undefined	-1	undefined
7.	$\theta = \frac{3\pi}{4}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$

9. $\cos(-60°) = \cos 60°$ cosine is an even function
$= 1/2$ from the unit circle

11. $\cos(-5\pi/6) = \cos 5\pi/6$
$= -\sqrt{3}/2$

13. $\sin(-30°) = -\sin 30°$ sine is an odd function
$= -1/2$ from the unit circle

15. $\sin(-3\pi/4) = \sin 3\pi/4$
$= -1/\sqrt{2}$

17. On the unit circle, we locate all points with a y-coordinate of $1/2$. The angles associated with these points are $\pi/6$ and $5\pi/6$.

19. $5\pi/6$, $7\pi/6$

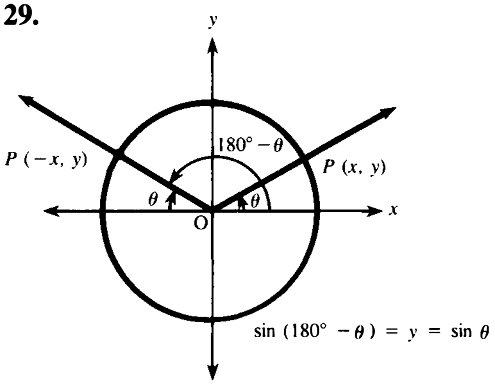
21. Look for points for which $y/x = -\sqrt{3}$. The angles associated with these points are $2\pi/3$ and $5\pi/3$.

23. $\sin\theta = -2/\sqrt{5}$, $\cos\theta = 1/\sqrt{5}$, $\tan\theta = -2$

25. $\tan\theta = y/x$, $\cot\theta = x/y$, $\sec\theta = 1/x$, $\csc\theta = 1/y$

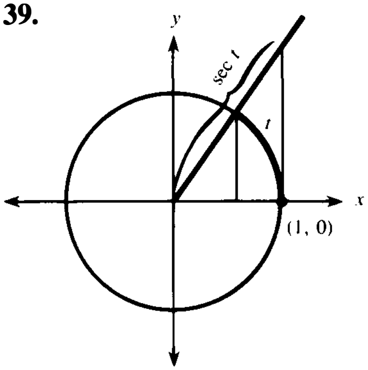
27. $\sin(-\theta) = -\sin\theta = -(-1/3) = 1/3$

29.



31. $\tan(-\theta) = \dfrac{\sin(-\theta)}{\cos(-\theta)} = \dfrac{-\sin\theta}{\cos\theta} = -\dfrac{\sin\theta}{\cos\theta} = -\tan\theta$

39.



41. 0.5807

43. 0.7314

45. −1.510

47. −1.211

Problem Set 3.4

1. 6 inches

3. 2.25 feet

5. 2π centimeters $\cong$ 6.28 centimeters

7. $4\pi/3$ millimeters $\cong$ 4.19 millimeters

9. $40\pi/3$ inches $\cong$ 41.9 inches

11. 5.03 centimeters

13. 1400π miles $\cong$ 4,400 miles

15. $4\pi/9$ feet $\cong$ 1.40 feet

17. 2100 miles

19. 0.5 feet

21. 3 inches

23. 4 centimeters

25. 1 meter

27. 31,000 feet

29. 9 centimeters^2

31. 19.2 inches^2

33. $9\pi/10$ meters² $\cong 2.83$ meters²
35. $25\pi/24$ meters² $\cong 3.27$ meters²
37. 4 inches²
39. 2 centimeters
41. $4/\sqrt{3}$ inches $\cong 2.31$ inches
43. 900π feet² $\cong 2{,}830$ feet²
45. $B = 50°$, $a = 23$, $b = 28$
47. $A = 33.1°$, $B = 56.9°$, $c = 37.5$

Problem Set 3.5

1. 1.5 feet per minute
3. 3 centimeters per second
5. 15 miles per hour
7. 80 feet
9. 22.5 miles
11. 7 miles (first change 20 minutes to 1/3 hour)
13. $2\pi/15$ radians per second $\cong 0.419$ radians per second
15. 4 radians per minute
17. 8/3 radians per second $\cong 2.67$ radians per second
19. 37.5π radians per hour = 118 radians per hour
21. $d = 100 \tan 1/2\pi t$; when $t = 1/2$, $d = 100$ feet; when $t = 3/2$, $d = -100$ feet; when $t = 1$, d is undefined because the light rays are parallel to the wall.
23. 40 inches
25. 180π meters $\cong 565$ meters
27. 4500 feet
29. 20π radians per minute $\cong 62.8$ radians per minute
31. $200\pi/3$ radians per minute $\cong 209$ radians per minute
33. 11.6π radians per minute $\cong 36.4$ radians per minute
35. 10 inches per second
37. 0.5 radians per second
39. 80π feet per minute $\cong 251$ feet per minute
41. $\pi/12$ radians per hour $\cong 0.262$ radians per hour
43. 10π feet $\cong 31.4$ feet
45. $33{,}000/\pi$ rpm $\cong 10{,}500$ rpm
47. 576π centimeters per day $\cong 1810$ centimeters per day
49. 889 radians per minute (53,300 radians per hour)
51. 60°
53. 2.2 feet

Chapter 3 Test

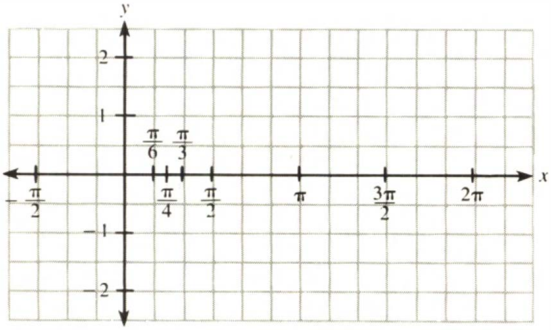
1. 55°
2. 62.2°
3. 50° 20′
4. 45°
5. -0.8391
6. -0.4663
7. -0.5490
8. -0.6115
9. -1.299
10. -6.512
11. 174°
12. 241° 30° or 241.5°
13. 226°
14. 310° 20′ or 310.3°
15. $-1/\sqrt{2}$
16. $-1/\sqrt{2}$
17. $-1/\sqrt{3}$
18. $2/\sqrt{3}$
19. $25\pi/18$
20. $-13\pi/6$
21. 240°
22. 105°
23. $\sqrt{3}/2$
24. $-1/2$
25. $-2\sqrt{2}$
26. 1
27. $-2/\sqrt{3}$
28. 2
29. 0
30. $2\sqrt{2}$
31. $\cot(-\theta) = \dfrac{\cos(-\theta)}{\sin(-\theta)} = \dfrac{\cos\theta}{-\sin\theta} = -\cot\theta$
32. First use odd and even functions to write everything in terms of θ instead of $-\theta$.

33. 2π meters $\cong$ 6.28 meters
34. 2π feet $\cong$ 6.28 feet
35. 4 centimeters^2
36. 3/8 centimeters^2 $\cong$ 0.375 centimeters^2
37. 4π inches^2 $\cong$ 12.6 inches^2
38. 10.8 centimeters^2
39. 2π centimeters
40. 8 inches^2
41. 90 feet
42. 3960 feet
43. 72 inches
44. 120π feet $\cong$ 377 feet
45. 12π radians per minute $\cong$ 37.7 radians per minute
46. 4π radians per minute $\cong$ 12.6 radians per minute
47. 0.5 radians per second
48. 5/3 radians per second
49. 80π feet per minute $\cong$ 251 feet per minute
50. 20π feet per minute $\cong$ 62.8 feet per minute
51. 4 radians per second for the 6 cm pulley and 3 radians per second for the 8 cm pulley
52. 2700π feet per minute $\cong$ 8,480 feet per minute

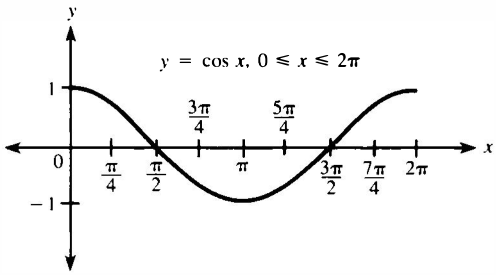
CHAPTER 4

Problem Set 4.1

If you want your graphs to match the ones here, use graph paper on which each square is 1/4 inch on each side. Then let two squares equal one unit. This way you can let the number π be approximately 6 units. Here is an example:



1.



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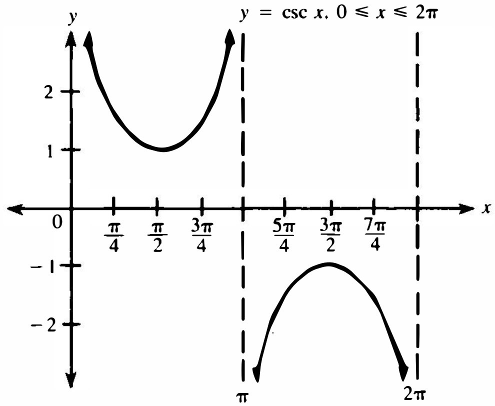

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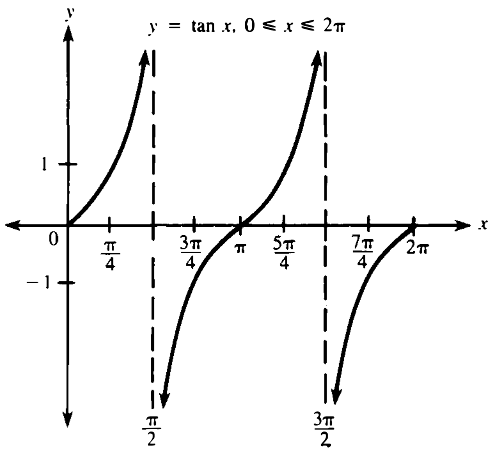


7. $\pi/3$, $5\pi/3$

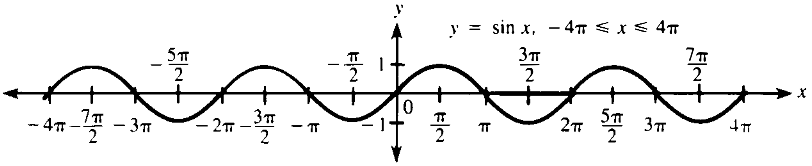
9. $3\pi/2$

11. $3\pi/4$, $5\pi/4$

13. $\pi/4$, $5\pi/4$

15. $\pi/6$, $11\pi/6$

17.



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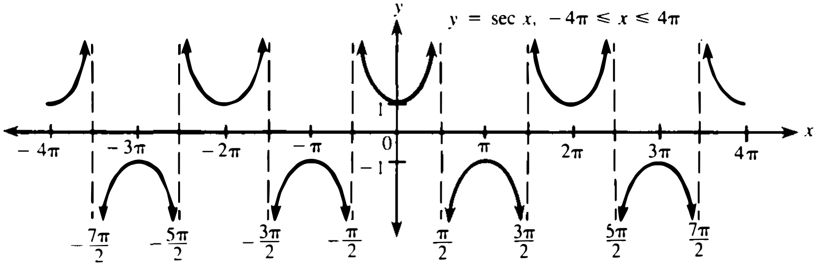

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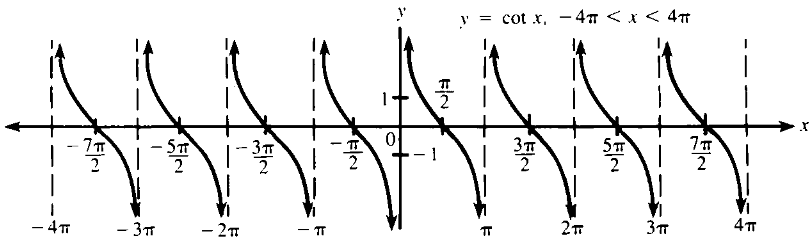


23. $\pi/3$, $2\pi/3$, $7\pi/3$, $8\pi/3$

25. π, 3π

27. $7\pi/6$, $11\pi/6$, $19\pi/6$, $23\pi/6$

29. $\pi/4$, $5\pi/4$, $9\pi/4$, $13\pi/4$

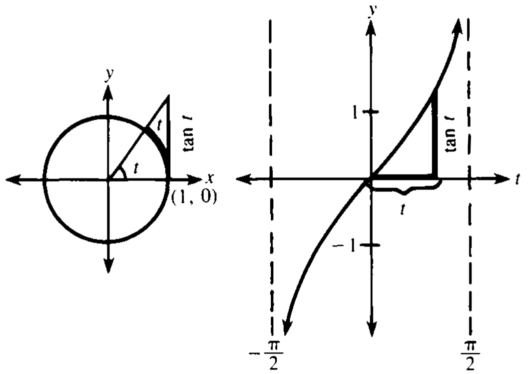
31. $\pi/4$, $3\pi/4$, $9\pi/4$, $11\pi/4$

33. $\pi/2 + k\pi$

35. $\pi/2 + k\pi$

37. $k\pi$

39.



41. Amplitude = 3, period = π

43. Amplitude = 2, period = 2

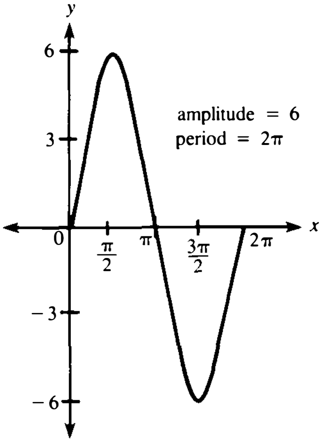
45. Amplitude = 3, period = π

47. After you have tried the problem yourself, look at Example 1 in Section 4.2.

49. After you have tried the problem yourself, look at Example 4 in Section 4.2.

Problem Set 4.2

1.



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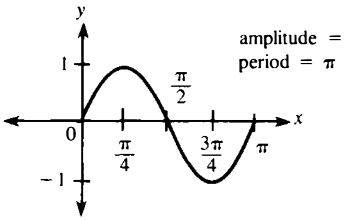

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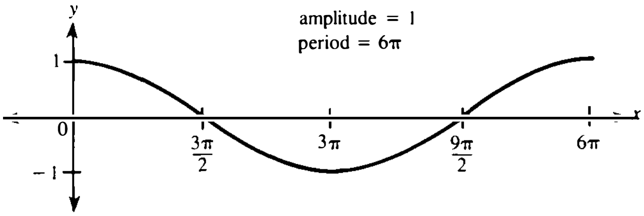

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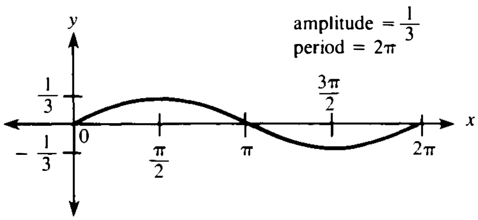

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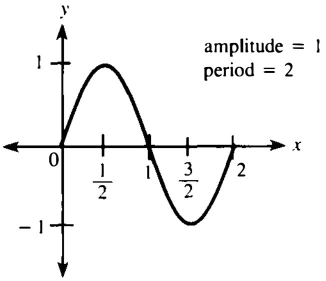

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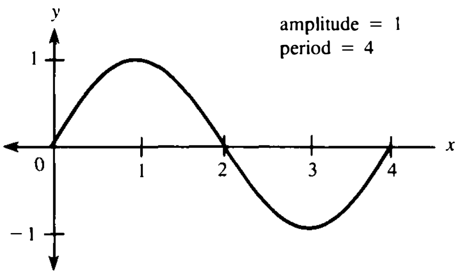

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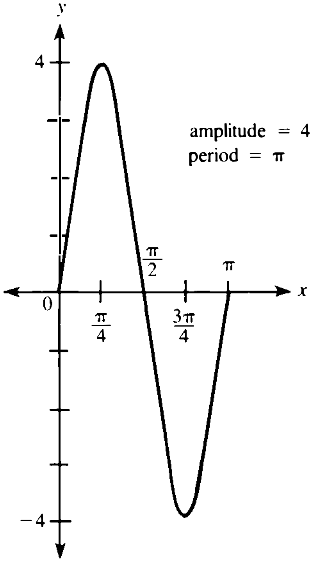

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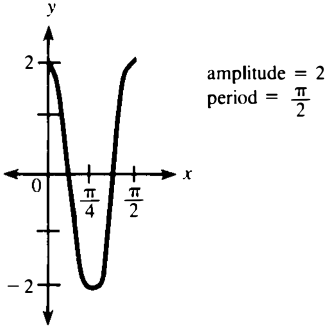

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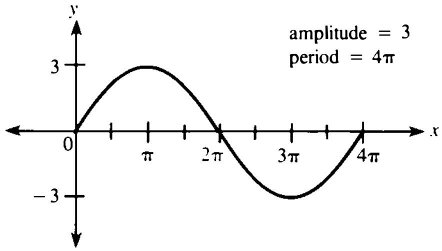

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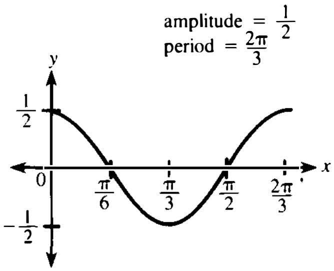

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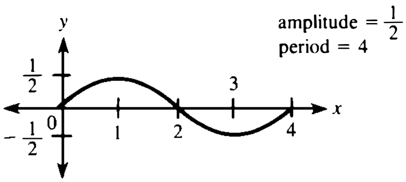

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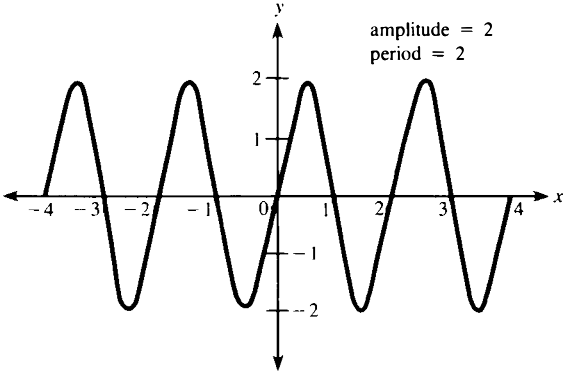

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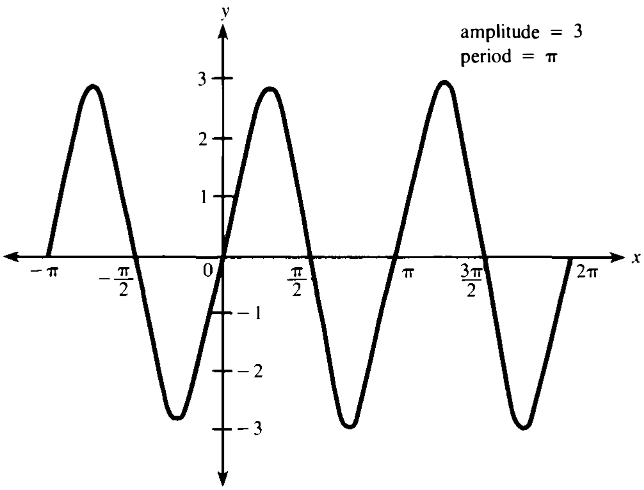

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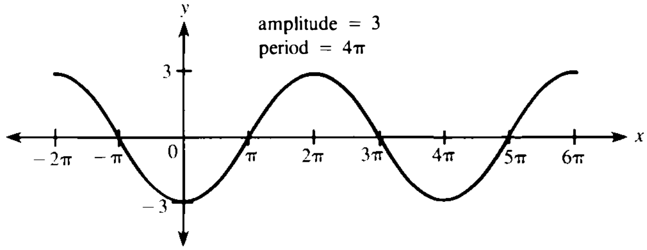

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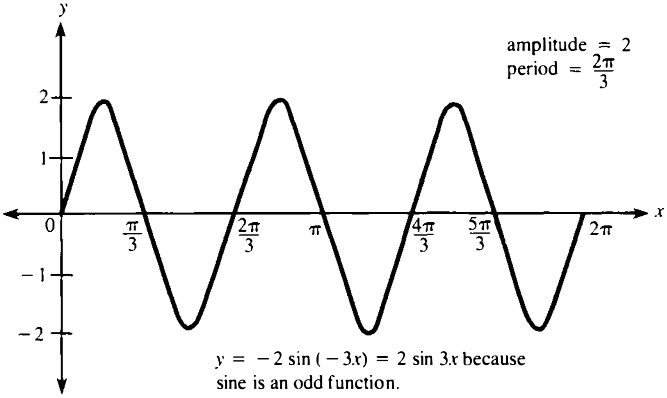
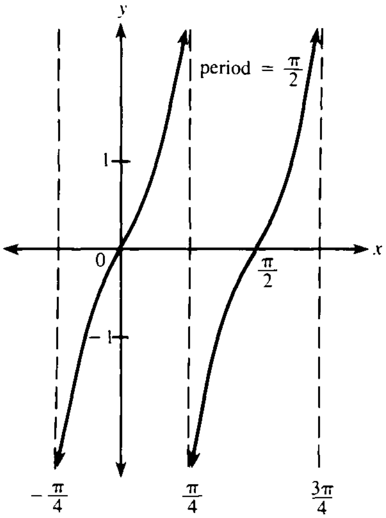


31. Maximum value of I is 20 amperes, one complete cycle takes $2\pi/120\pi = 1/60$ seconds.

33.



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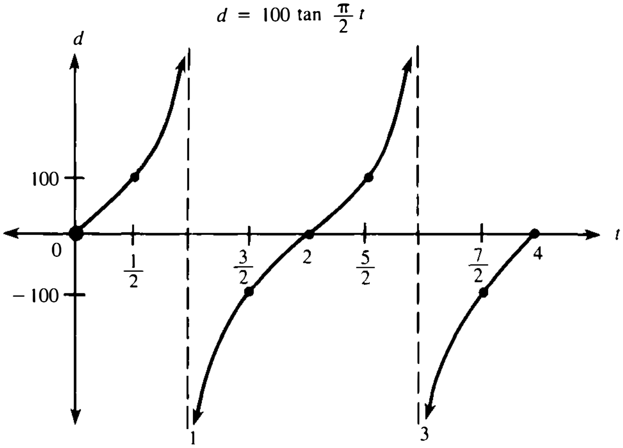

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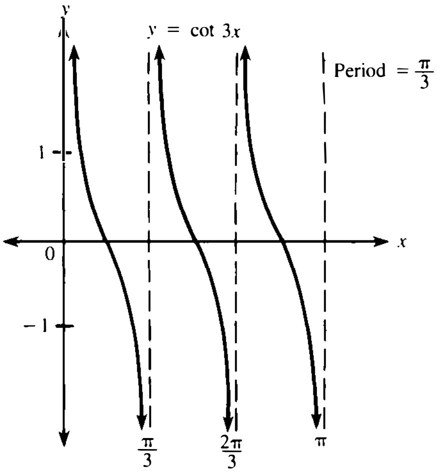

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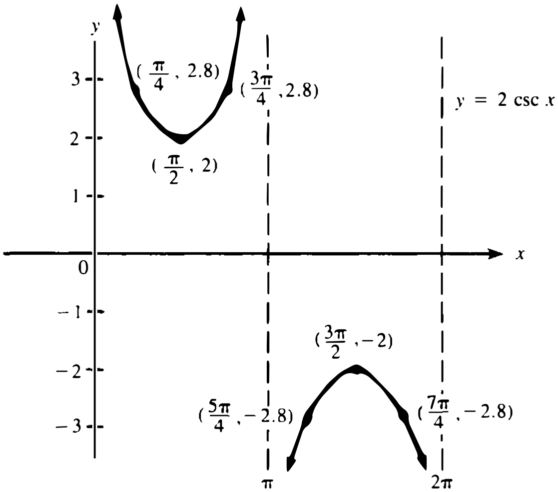

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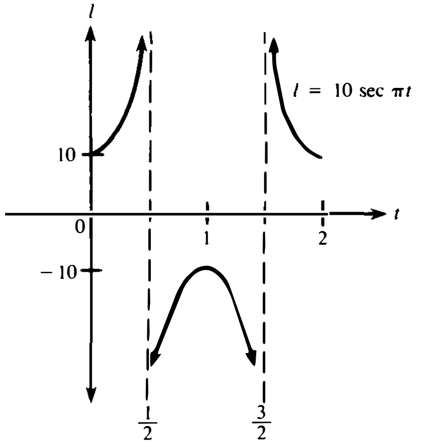

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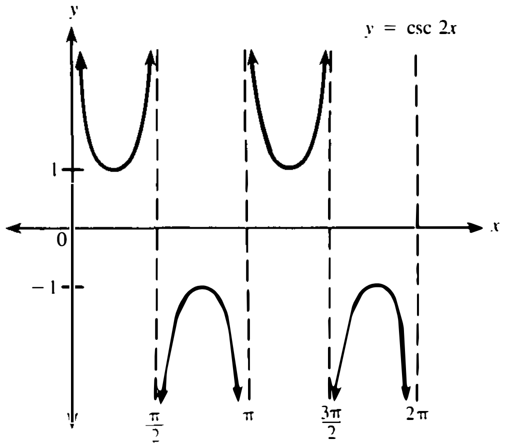

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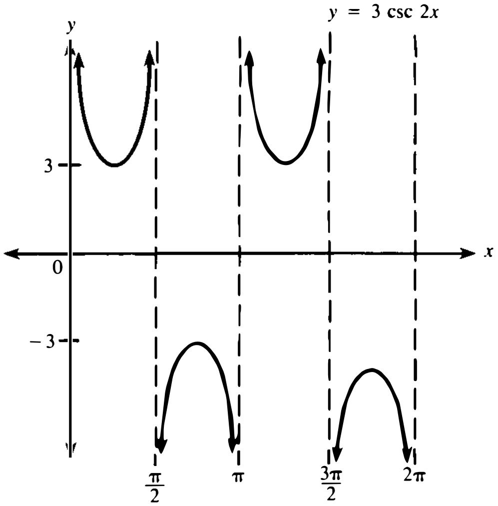
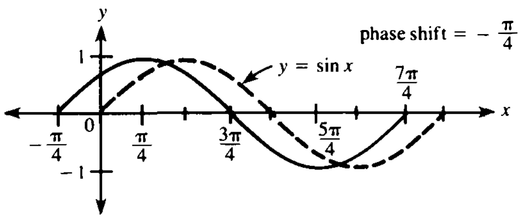


47. 0
49. 1
51. $\sqrt{3}/2$
53. $3/2$

Problem Set 4.3

1.



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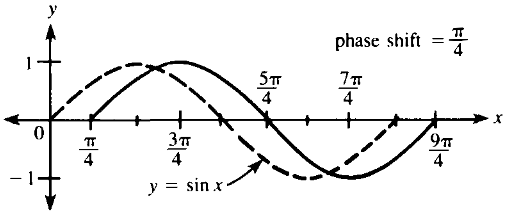

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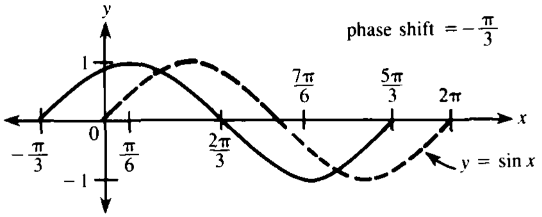

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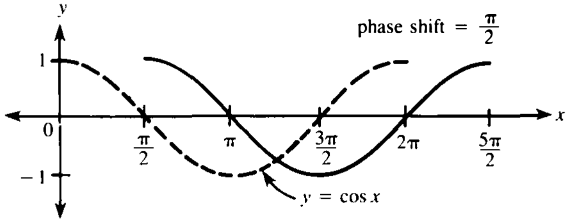

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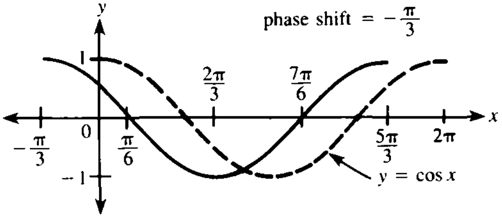

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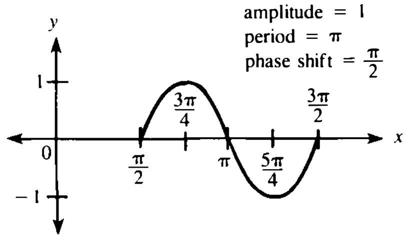

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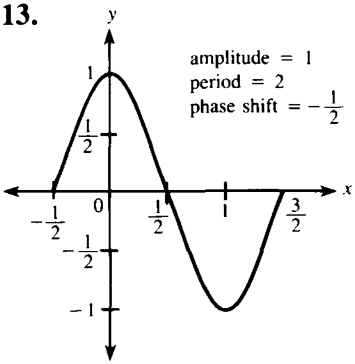
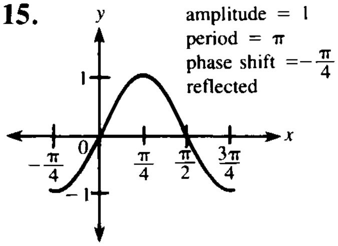
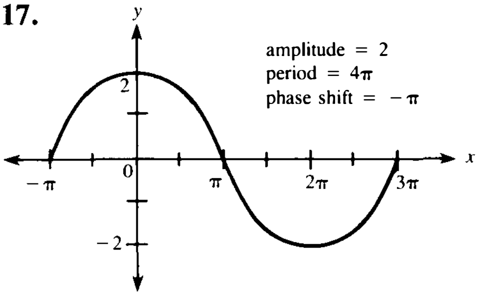


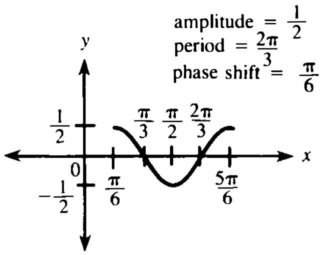
15.



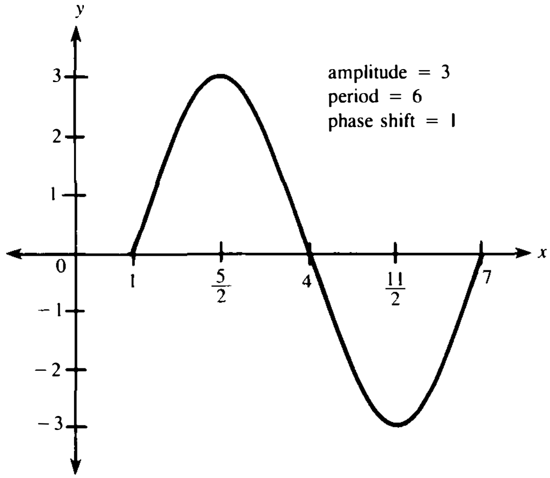
17.



19.



21.



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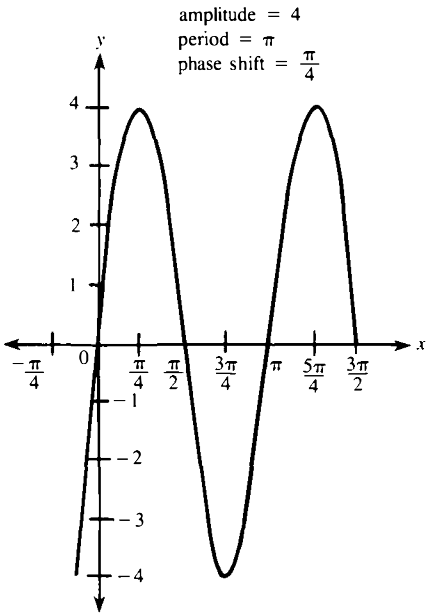
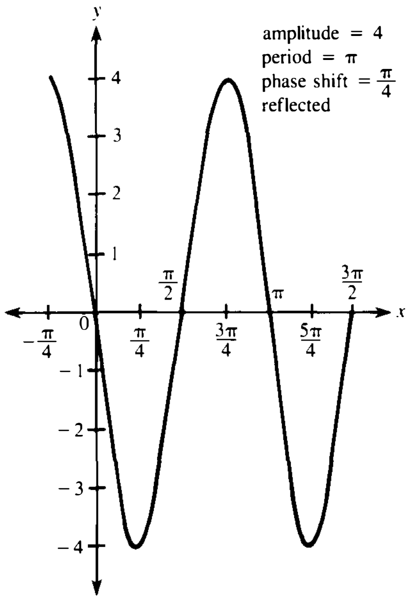
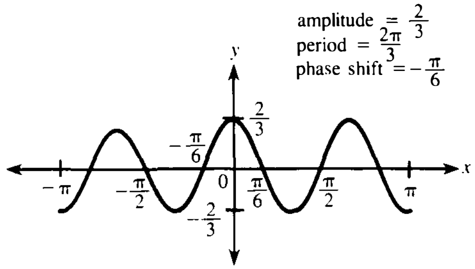


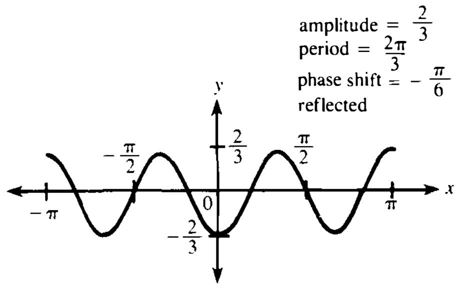
25.



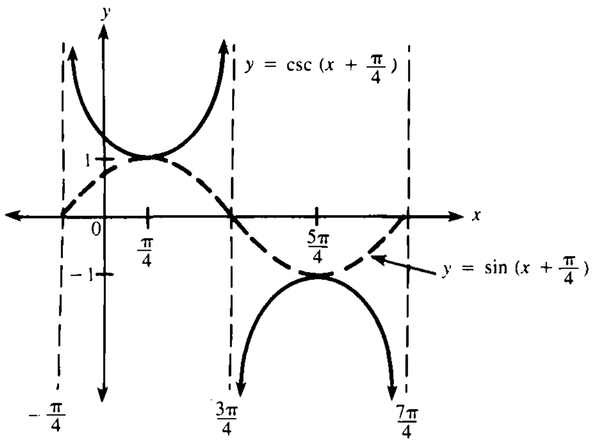
27.



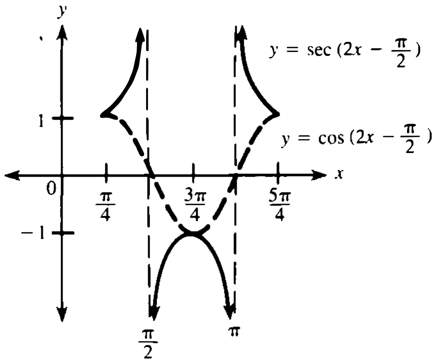
29.



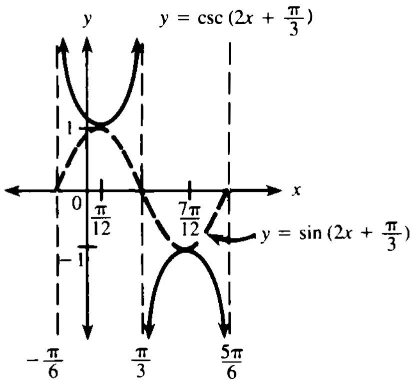
31.



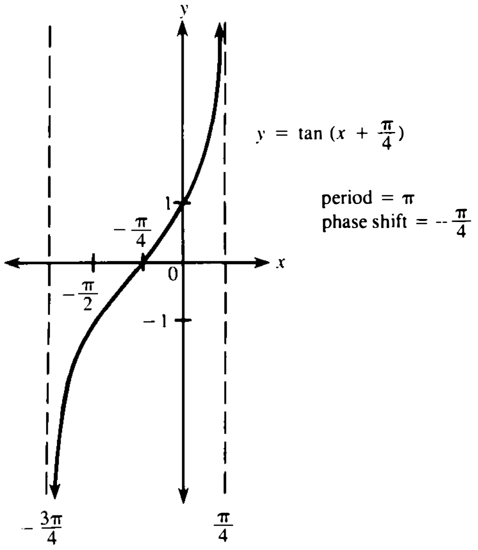
33.



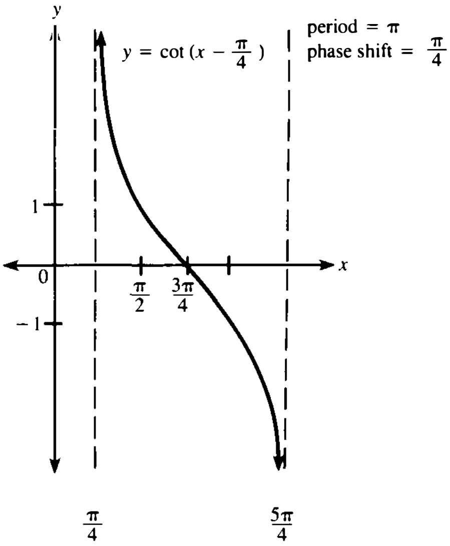
35.



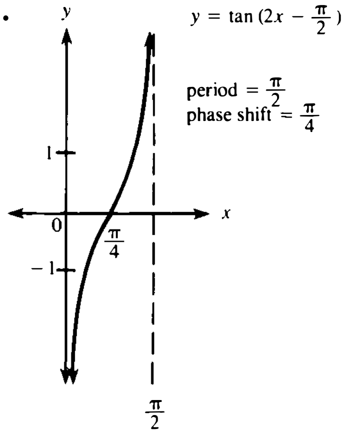
37.



39.



41.

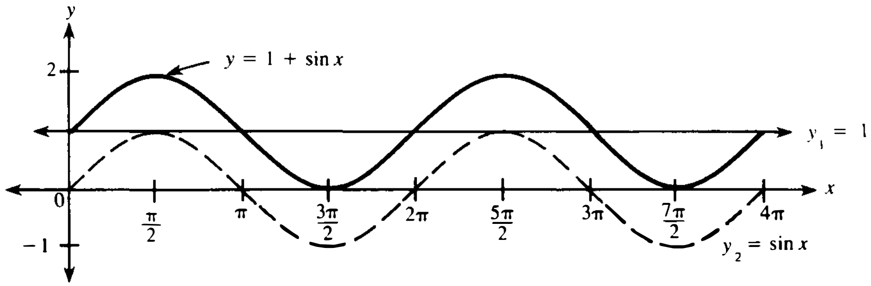


43. $5\pi/3$ centimeters

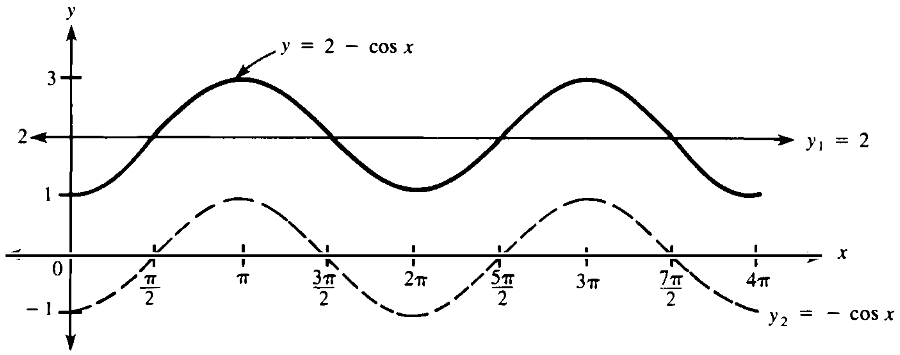
45. 2/3 feet

Problem Set 4.4

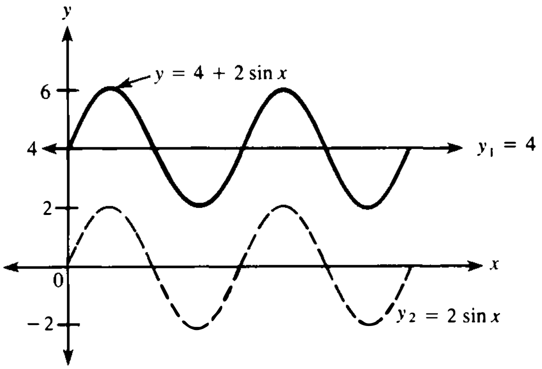
1.



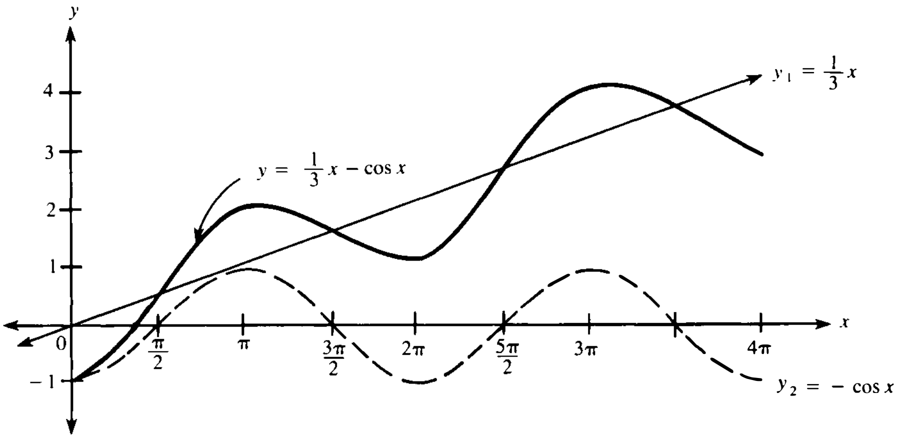
3.


y
$y = 2 - \cos x$
3
2
1
0
−1
$y_1 = 2$
x
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
$\frac{5\pi}{2}$
3π
$\frac{7\pi}{2}$
4π
$y_2 = -\cos x$

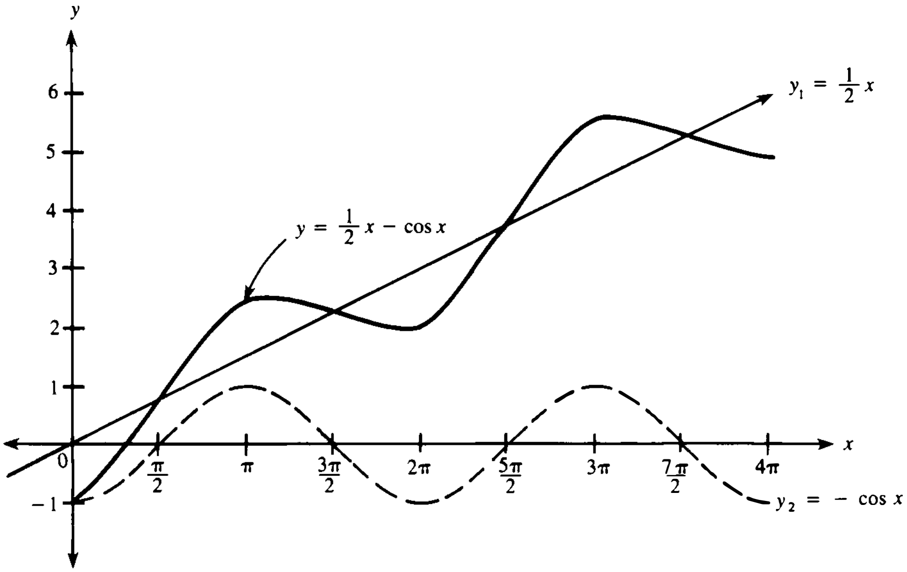
5.


y
$y = 4 + 2 \sin x$
6
4
2
0
−2
$y_1 = 4$
x
$y_2 = 2 \sin x$

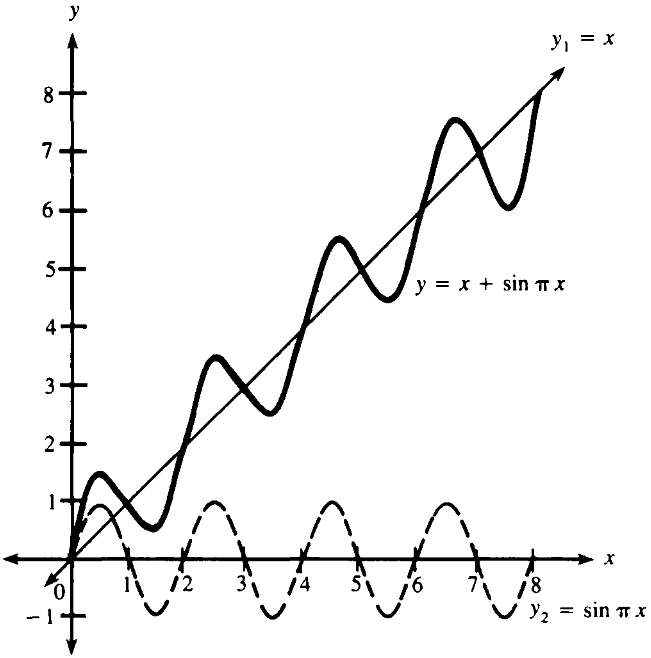
7.


y
4
3
2
1
0
−1
$y_1 = \frac{1}{3}x$
$y = \frac{1}{3}x - \cos x$
x
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
$\frac{5\pi}{2}$
3π
4π
$y_2 = -\cos x$

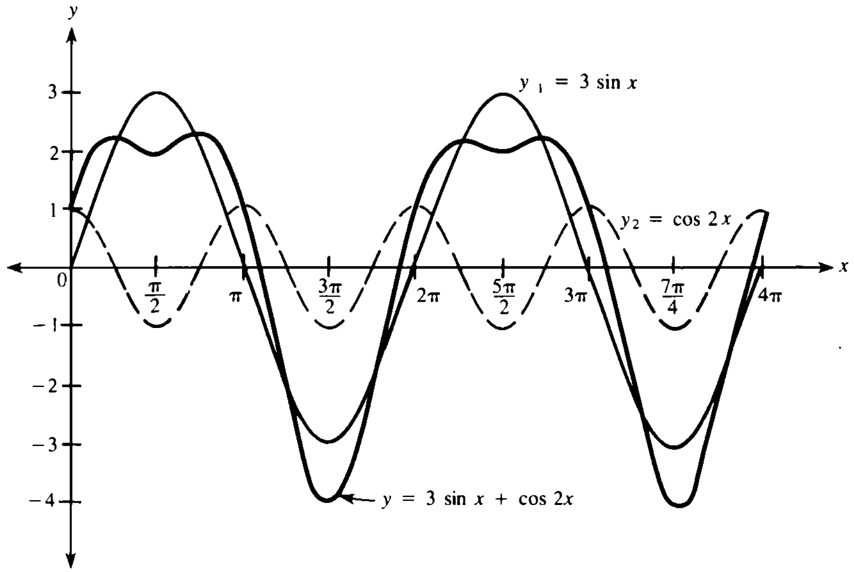
9.


y
6
5
4
3
2
1
0
−1
$y_1 = \frac{1}{2}x$
$y = \frac{1}{2}x - \cos x$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
$\frac{5\pi}{2}$
3π
$\frac{7\pi}{2}$
4π
x
$y_2 = -\cos x$

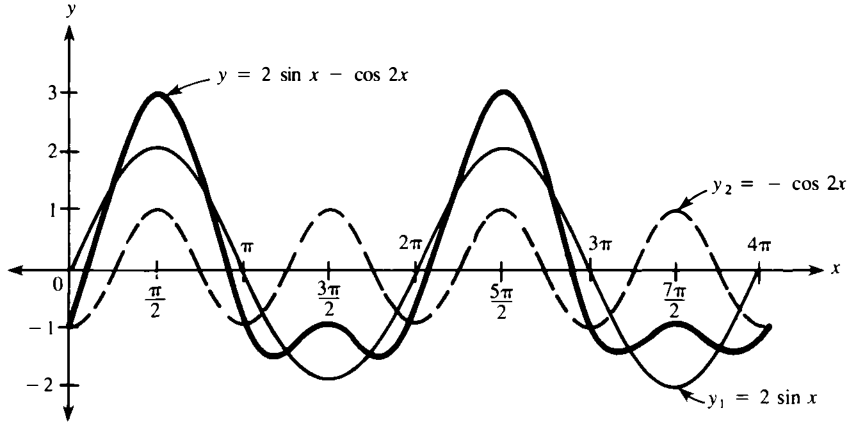
11.


y
$y_1 = x$
8
7
6
5
4
3
2
1
0
−1
$y = x + \sin \pi x$
1
2
3
4
5
6
7
8
x
$y_2 = \sin \pi x$

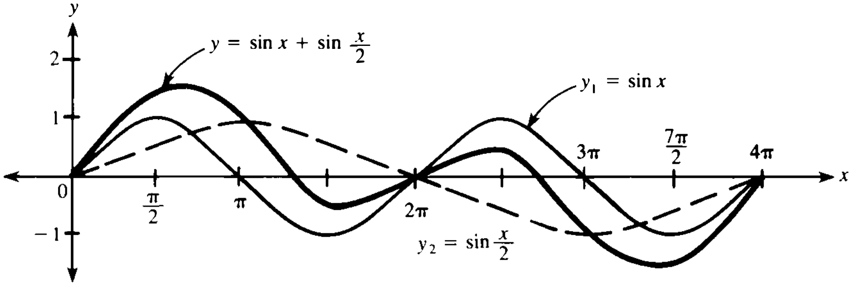
13.


y
x
3
2
1
0
−1
−2
−3
−4
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
$\frac{5\pi}{2}$
3π
$\frac{7\pi}{4}$
4π
$y_1 = 3 \sin x$
$y_2 = \cos 2x$
$y = 3 \sin x + \cos 2x$

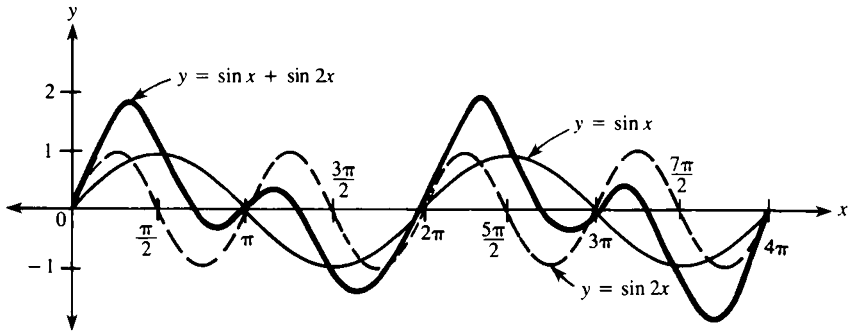
15.


y
x
3
2
1
0
−1
−2
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
$\frac{5\pi}{2}$
3π
$\frac{7\pi}{2}$
4π
$y = 2 \sin x - \cos 2x$
$y_2 = - \cos 2x$
$y_1 = 2 \sin x$

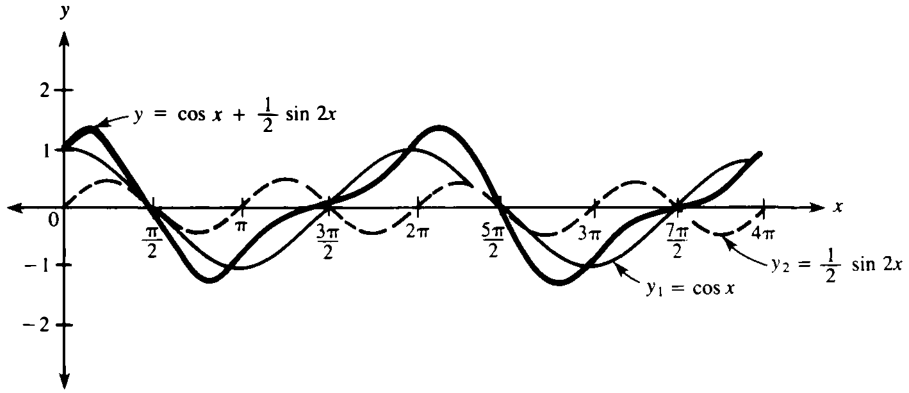
17.


y
x
2
1
0
−1
$\frac{\pi}{2}$
π
2π
3π
$\frac{7\pi}{2}$
4π
$y = \sin x + \sin \frac{x}{2}$
$y_1 = \sin x$
$y_2 = \sin \frac{x}{2}$

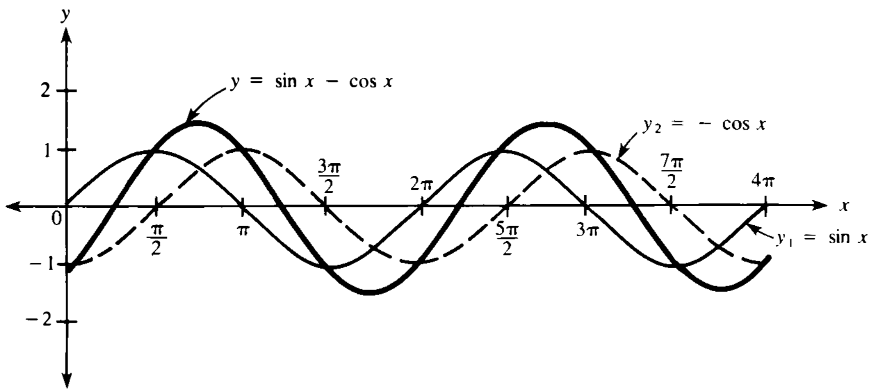
19.


y
y = sin x + sin 2x
y = sin x
y = sin 2x
2
1
0
−1
π/2
π
3π/2
2π
5π/2
3π
7π/2
4π
x

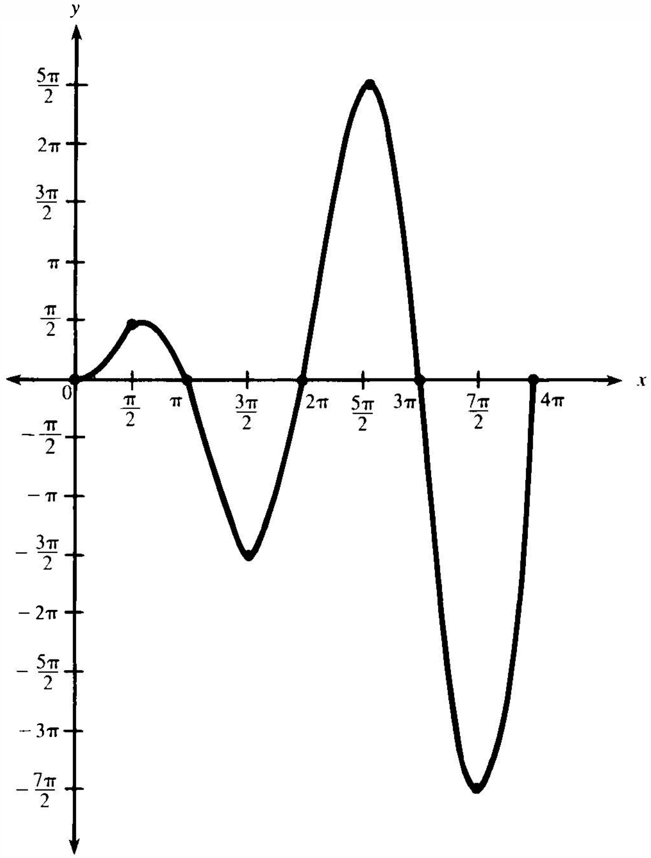
21.


y
y = cos x + 1/2 sin 2x
y_1 = cos x
y_2 = 1/2 sin 2x
2
1
0
−1
−2
π/2
π
3π/2
2π
5π/2
3π
7π/2
4π
x

23.


y
y = sin x − cos x
y_2 = − cos x
y_1 = sin x
2
1
0
−1
−2
π/2
π
3π/2
2π
5π/2
3π
7π/2
4π
x

25.

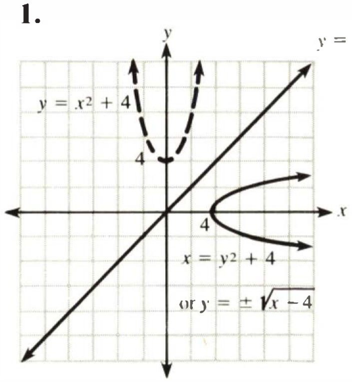


27. 1/4 feet per second

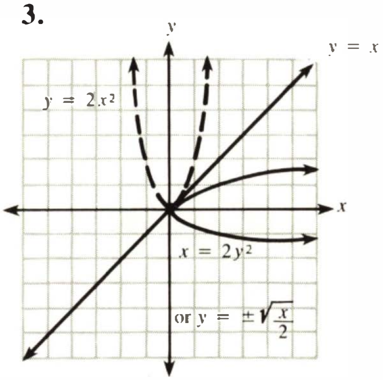
29. 180 meters

Problem Set 4.5

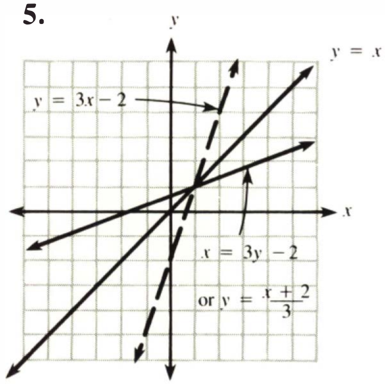
1.



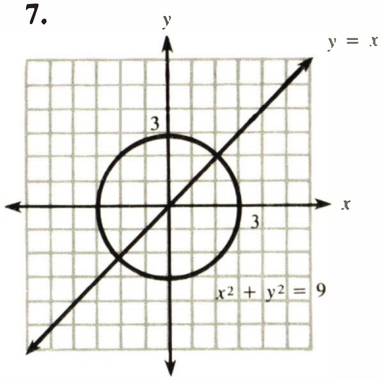
3.



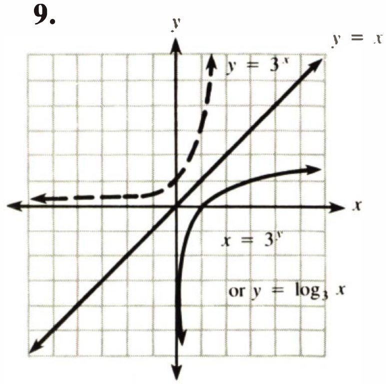
5.



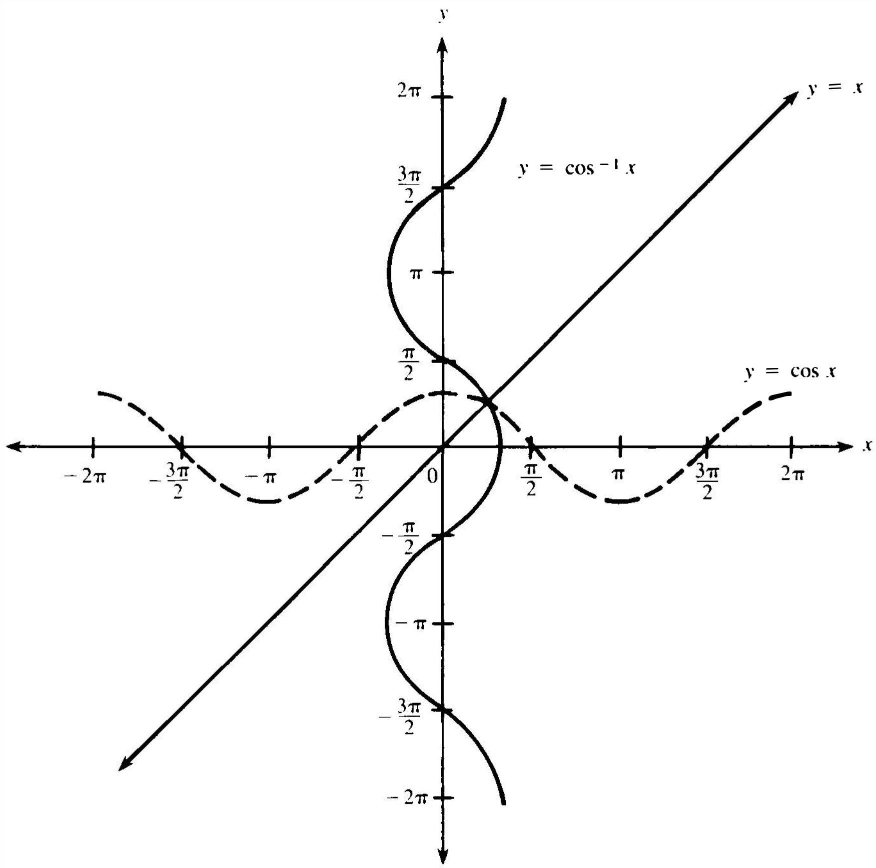
7.


y
y = x
3
x
3
x² + y² = 9

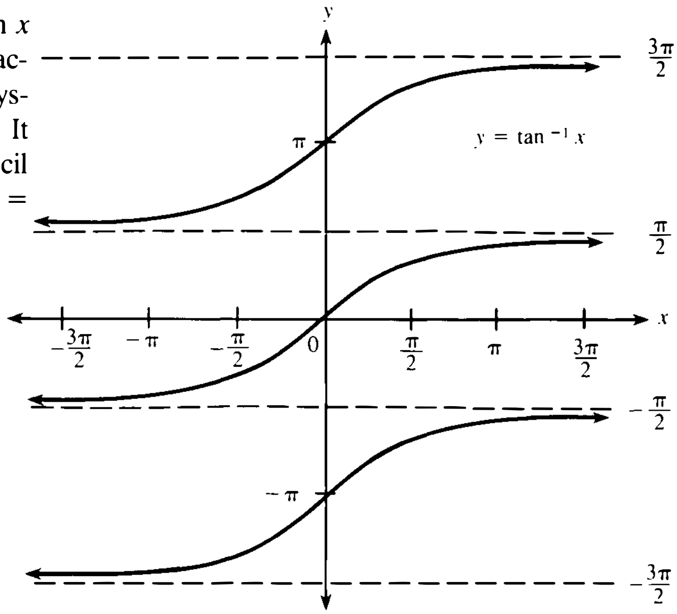
9.


y
y = x
y = 3ˣ
x
x = 3ʸ
or y = log₃ x

11.


y
2π
y = x
3π/2
y = cos⁻¹ x
π
π/2
y = cos x
x
−2π
−3π/2
−π
−π/2
0
π/2
π
3π/2
2π
−π/2
−π
−3π/2
−2π

13. We have not included the graph of $y = \tan x$ with the graph of $y = \tan^{-1}x$ because placing them both on the same coordinate system makes the diagram too complicated. It is best to graph $y = \tan x$ lightly in pencil and then reflect that graph about the line $y = x$ to get the graph of $y = \tan^{-1}x$.



15. $-1 \le x \le 1$ **17.** all real numbers **19.** $60° + 360°k$, $120° + 360°k$

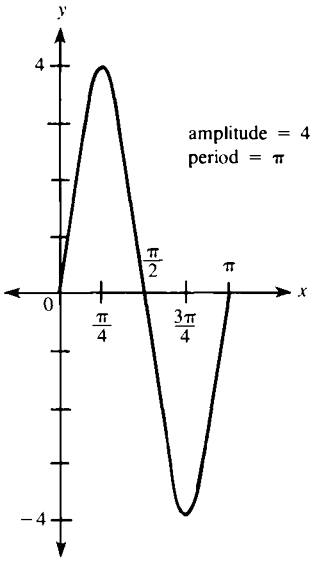
21. $60° + 360°k$, $300° + 360°k$ **23.** $210° + 360°k$, $330° + 360°k$ **25.** $90° + 180°k$

27. $270° + 360°k$ **29.** $45° + 180°k$ **31.** $60° + 360°k$, $300° + 360°k$

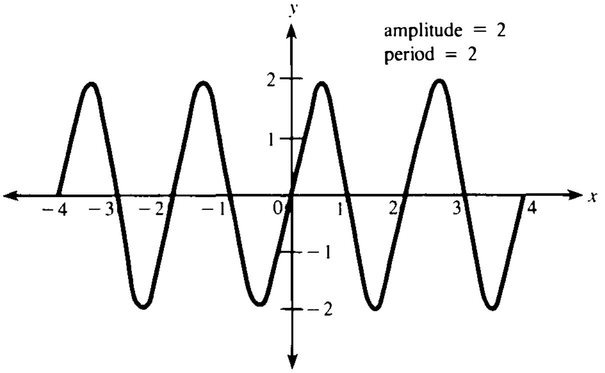
33. $\frac{5\pi}{4} + 2k\pi, \frac{7\pi}{4} + 2k\pi$ **35.** $\frac{\pi}{3} + k\pi$ **37.** $\frac{\pi}{6} + 2k\pi, \frac{11\pi}{6} + 2k\pi$

39. $\frac{\pi}{4} + 2k\pi, \frac{7\pi}{4} + 2k\pi$ **41.** $\frac{\pi}{2} + 2k\pi$ **43.** $\frac{\pi}{2} + k\pi$ **45.** $2k\pi$

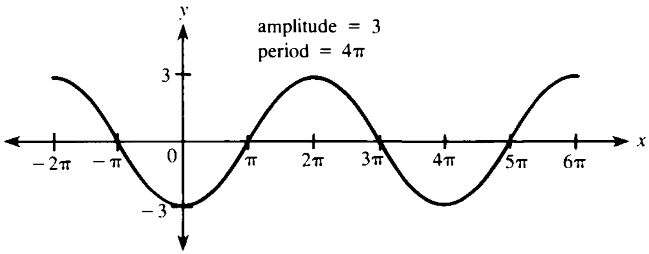
47.



49.



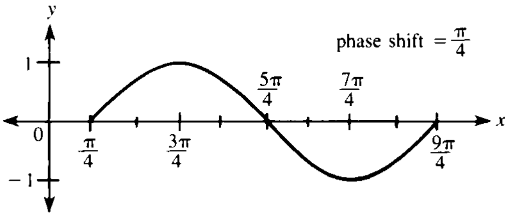
51.



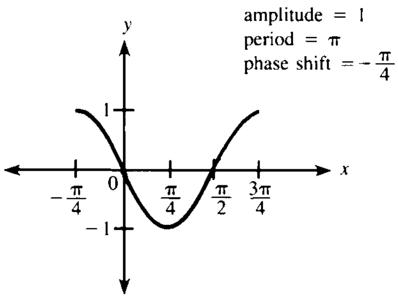
Problem Set 4.6

1.	$\pi/3$	**3.**	π	**5.**	$\pi/4$	**7.**	$3\pi/4$
9.	$-\pi/6$	**11.**	$\pi/3$	**13.**	0	**15.**	$-\pi/6$
17.	$2\pi/3$	**19.**	$\pi/6$	**21.**	9.8°	**23.**	147.4°
25.	20.8°	**27.**	74.3°	**29.**	117.8°	**31.**	−70°
33.	−50°	**35.**	4/5	**37.**	3/4	**39.**	$\sqrt{5}$
41.	$\sqrt{3}/2$	**43.**	2	**45.**	3/5	**47.**	1/2
49.	1/2	**51.**	x	**53.**	$\sqrt{1-x^2}$	**55.**	$\dfrac{x}{\sqrt{x^2+1}}$
57.	$\dfrac{\sqrt{x^2-1}}{x}$	**59.**	x				

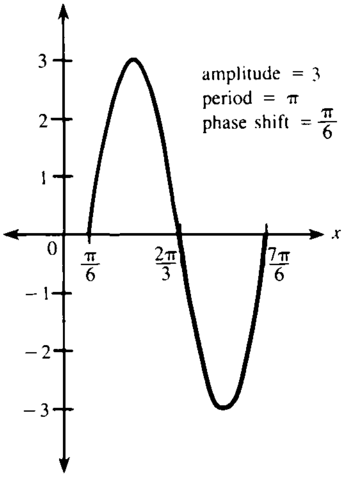
61.



63.

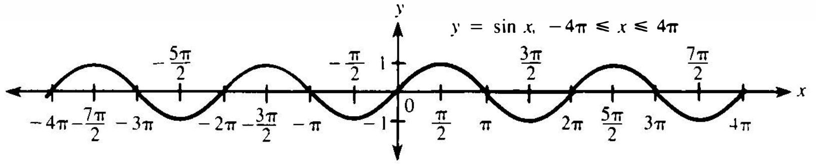


65.

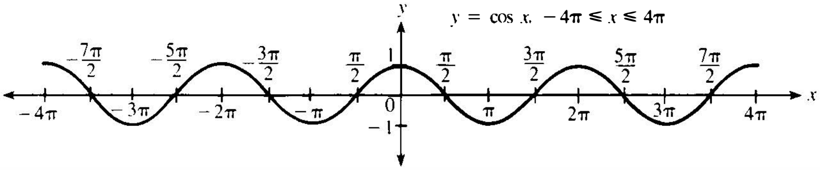


Chapter 4 Test

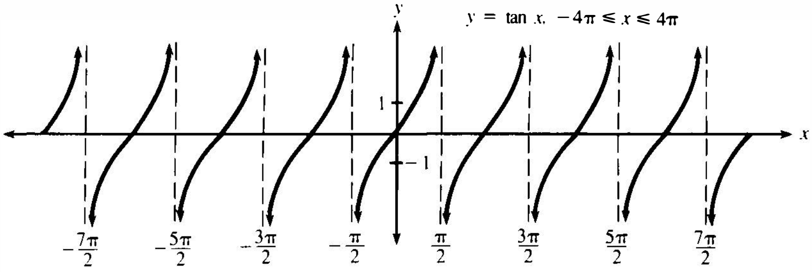
1.



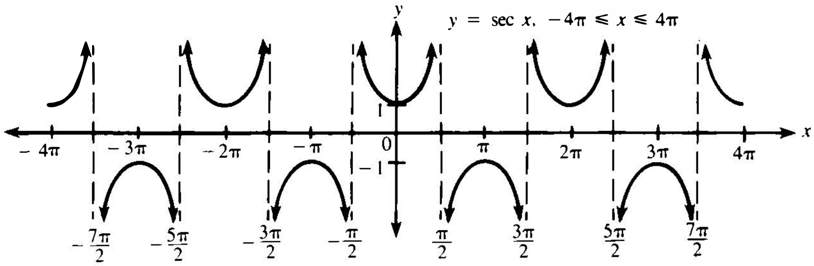
2.



3.



4.



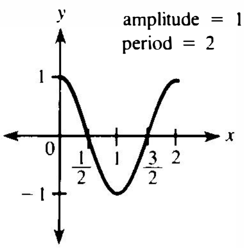
5. 4

6. 8

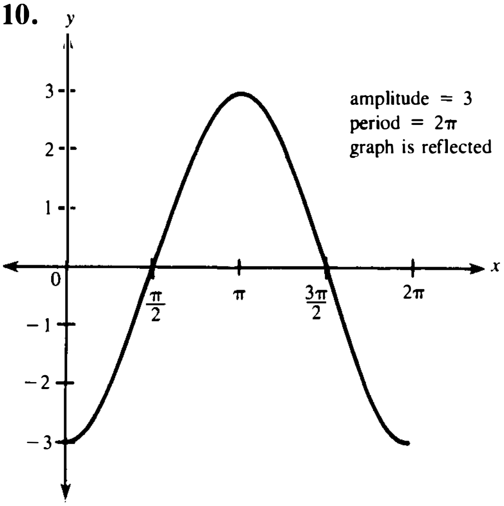
7. $-3\pi,\ -\pi,\ \pi,\ 3\pi$

8. $-11\pi/3,\ -7\pi/3,\ -5\pi/3,\ -\pi/3,\ \pi 3,\ 5\pi/3,\ 7\pi/3,\ 11\pi/3$

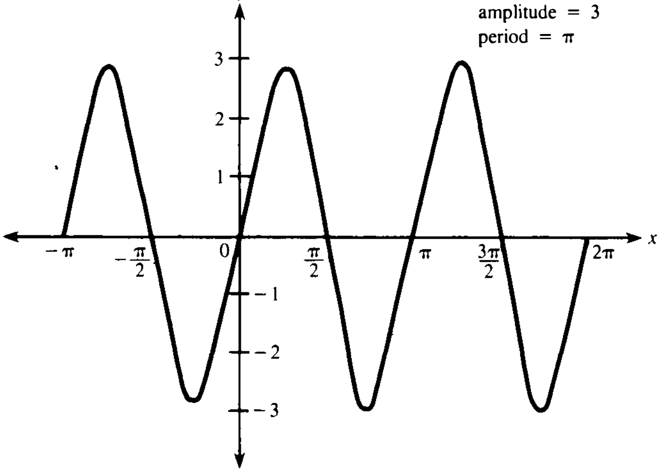
9.



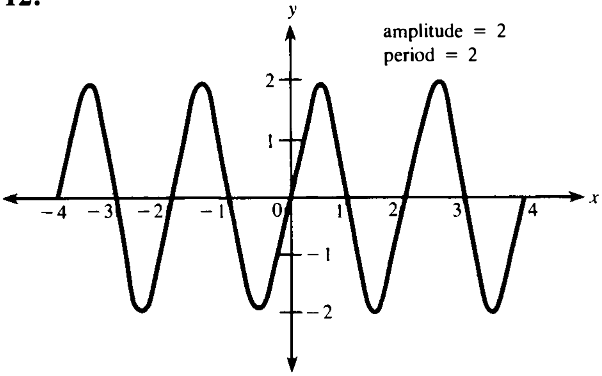
10.



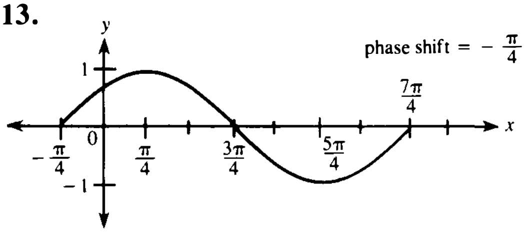
11.



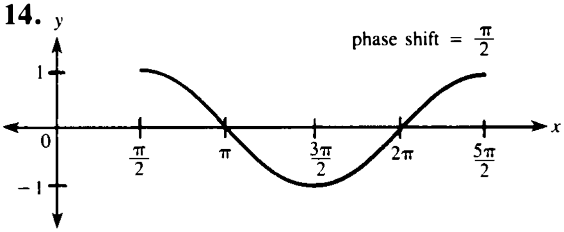
12.



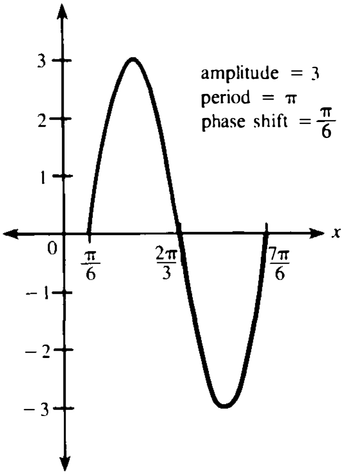
13.



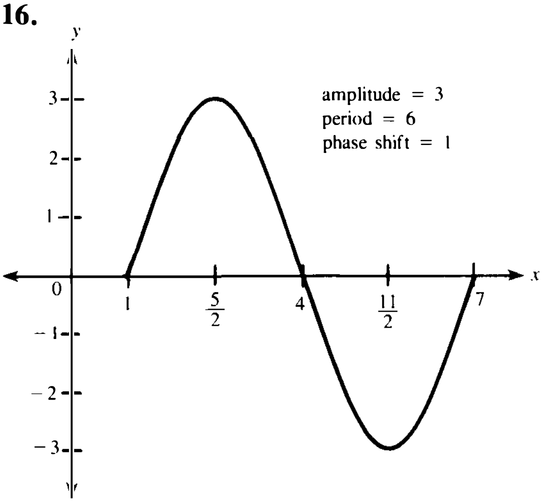
14.



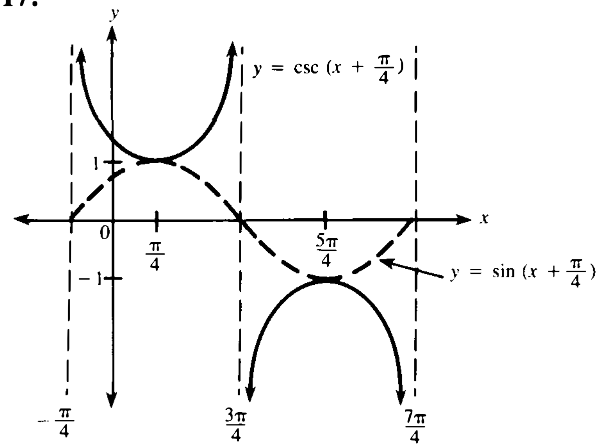
15.



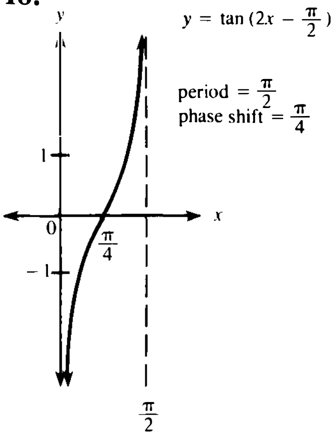
16.


amplitude = 3
period = 6
phase shift = 1

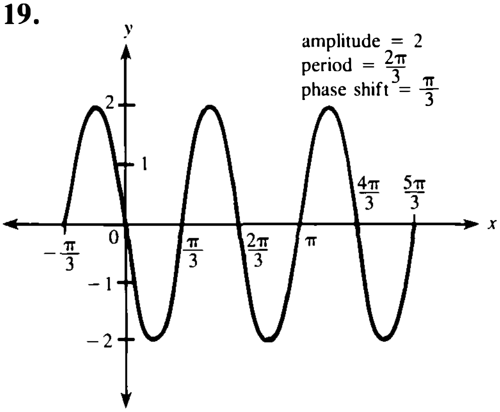
17.


y = csc (x + π/4)
y = sin (x + π/4)

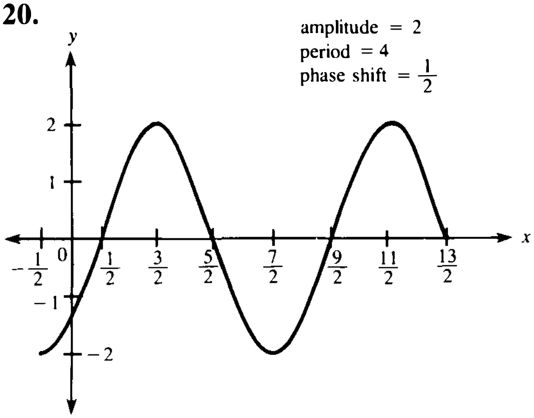
18.


y = tan (2x − π/2)
period = π/2
phase shift = π/4

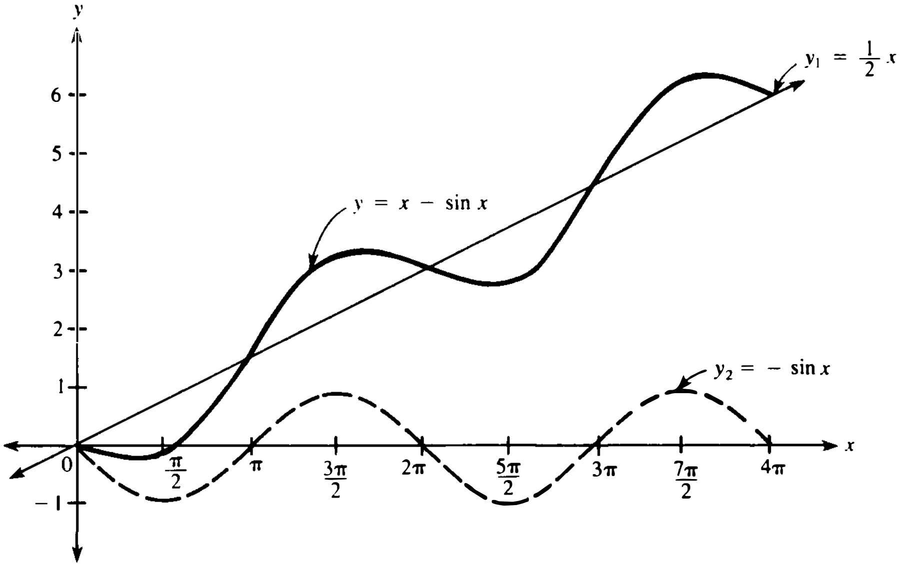
19.


amplitude = 2
period = 2π/3
phase shift = π/3

20.


amplitude = 2
period = 4
phase shift = 1/2

21.



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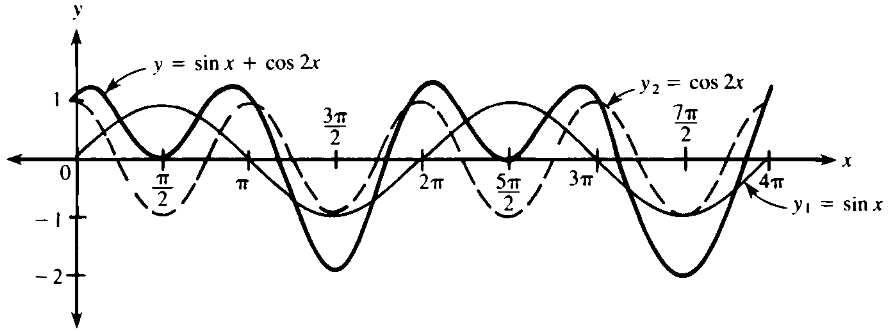
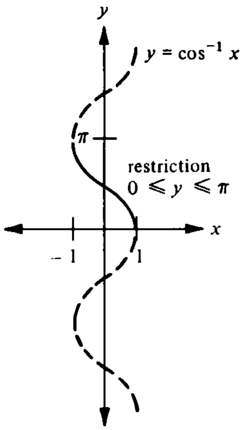


23. 180°k **24.** 180° + 360°k **25.**



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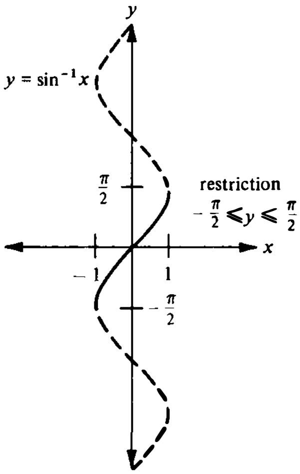


27. $\pi/6$ **28.** $5\pi/6$ **29.** $-\pi/4$ **30.** $\pi/2$
31. $36.4°$ **32.** $-39.7°$ **33.** $134.3°$ **34.** $-12.5°$
35. $\sqrt{5}/2$ **36.** $3/\sqrt{13}$ **37.** $\sqrt{1-x^2}$ **38.** $\dfrac{x}{\sqrt{1-x^2}}$

CHAPTER 5

Problem Set 5.1

For some of the problems in the beginning of this problem set we will give the complete proof. Remember, however, that there is often more than one way to prove an identity. You may have a correct proof even if it doesn't match the one you find here. As the problem set progresses, we will give hints on how to begin the proof instead of the complete proof.

1. $\cos\theta\tan\theta \stackrel{?}{=} \sin\theta$

$$\cos\theta\cdot\frac{\sin\theta}{\cos\theta}$$

$$\sin\theta = \sin\theta$$

9. $\cos x(\csc x + \tan x) \stackrel{?}{=} \cot x + \sin x$

$$\cos x\csc x + \cos x\tan x$$

$$\cos x\cdot\frac{1}{\sin x} + \cos x\cdot\frac{\sin x}{\cos x}$$

$$\frac{\cos x}{\sin x} + \sin x$$

$$\cot x + \sin x = \cot x + \sin x$$

17. $\dfrac{\cos^4 t - \sin^4 t}{\sin^2 t} \stackrel{?}{=} \cot^2 t - 1$

$$\frac{(\cos^2 t + \sin^2 t)(\cos^2 t - \sin^2 t)}{\sin^2 t}$$

$$\frac{\cos^2 t - \sin^2 t}{\sin^2 t}$$

$$\frac{\cos^2 t}{\sin^2 t} - \frac{\sin^2 t}{\sin^2 t}$$

$$\cot^2 t - 1 = \cot^2 t - 1$$

19. Write the numerator on the right side as $1 - \sin^2\theta$ and then factor it.

27. Change the left side to sines and cosines and then add the resulting fractions.

37. Rewrite the left side in terms of cosine and then simplify.

25. Factor the left side and then write in terms of sines and cosines.

33. See Example 6.

45. $\sin(30° + 60°) = \sin 90° = 1$

$$\sin 30° + \sin 60° = \frac{1}{2} + \frac{\sqrt{3}}{2} \neq 1$$

47. $\cos A = 4/5$, $\tan A = 3/4$

49. $\sqrt{3}/2$

51. $\sqrt{3}/2$

53. $15°$

Problem Set 5.2

1. $\dfrac{\sqrt{6}-\sqrt{2}}{4}$ **3.** $\dfrac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}$ **5.** $\dfrac{\sqrt{6}+\sqrt{2}}{4}$ **7.** $\dfrac{-\sqrt{6}-\sqrt{2}}{4}$

9. $\dfrac{-\sqrt{6}-\sqrt{2}}{4}$

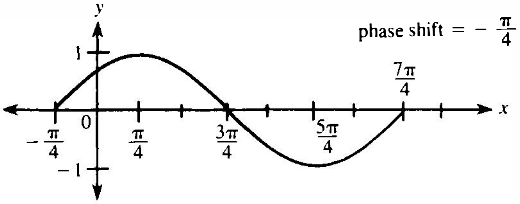
11. $\sin(x+2\pi) = \sin x \cos 2\pi + \cos x \sin 2\pi$
$= \sin x\,(1) + \cos x\,(0)$
$= \sin x$

For problems 13–19, proceed as in problem 11. Expand the left side and simplify.

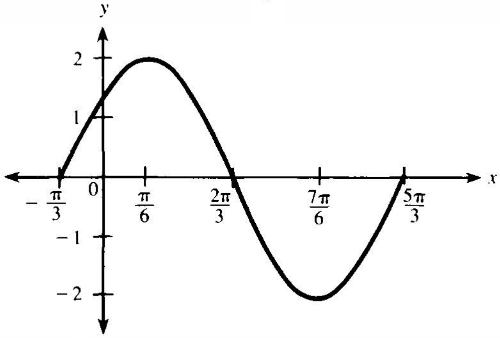
21. $\sin 5x$ **23.** $\cos 6x$ **25.** $\sin(45° + \theta)$ **27.** $\sin(30° + \theta)$

29. $\cos 90° = 0$

31. $y = \sin(x + \pi/4)$



33. $y = 2\sin(x + \frac{\pi}{3})$



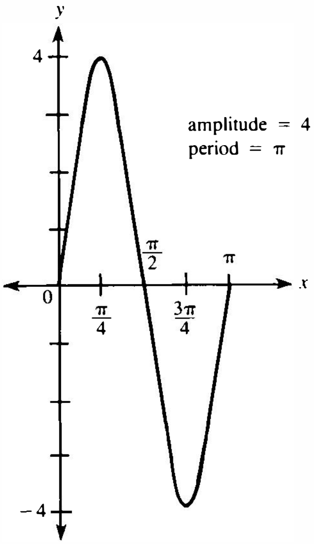
35. $-16/65$, $63/65$, $-16/63$, QIV

37. 2, 1/2, QI

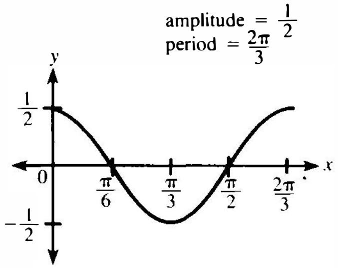
39. 1

41. $\sin 2x = 2 \sin x \cos x$

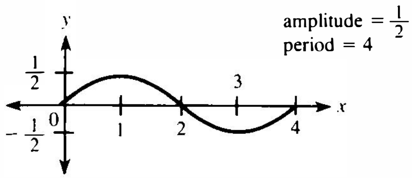
43.



45.



47.



Problem Set 5.3

1. 24/25 **3.** 24/7 **5.** −4/5 **7.** 4/3
9. 120/169 **11.** 169/120 **13.** 3/5 **15.** 5/3

17. $\cos 100° = \cos 2 \cdot 50° = 1 - 2\sin^2 50°$

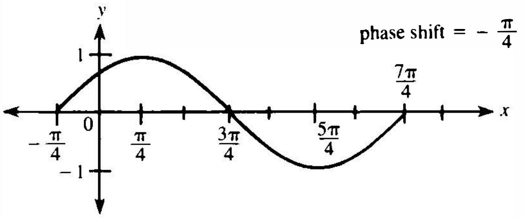
19. $\sin \frac{\pi}{5} = \sin 2 \cdot \frac{\pi}{10} = 2 \sin \frac{\pi}{10} \cos \frac{\pi}{10}$

29. Solve the given equation for sin 15° to get $\dfrac{\sqrt{2 - \sqrt{3}}}{2}$

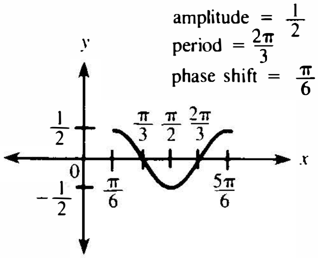
39. Write $\sin 3\theta$ as $\sin(2\theta + \theta)$ and expand according to the formula for $\sin(A + B)$, then write everything in terms of $\sin \theta$.

47. 24/7

49.



51.



Problem Set 5.4

1. 1/2 **3.** 2 **5.** $-1/\sqrt{10}$ **7.** $-\sqrt{10}$
9. $3/\sqrt{13}$ **11.** $-2/\sqrt{13}$ **13.** −3/2 **15.** $2/\sqrt{5}$
17. −7/25 **19.** −25/7 **21.** $3/\sqrt{10}$ **23.** 7/25

25. 0

27. $\cos \frac{\theta}{4} = \cos \frac{\theta/2}{2} = \sqrt{\frac{1 + \cos \theta/2}{2}} = 5/\sqrt{26}$

29. $\sin \theta = \sin 2 \cdot \frac{\theta}{2} = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = \frac{120}{169}$

37. Change $\sin^2 \theta/2$ to $(1 - \cos \theta)/2$ and then multiply numerator and denominator by $\csc \theta$.

41. Write $\tan x/2$ as $(\sin x/2)/(\cos x/2)$ and then apply the half-angle formulas to these.

47. 3/5 **49.** $1/\sqrt{5}$

51. $x/\sqrt{x^2 + 1}$ **53.** $x/\sqrt{1 - x^2}$

Problem Set 5.5

1. $-1/\sqrt{5}$ **3.** $(2\sqrt{3} - 1)/2\sqrt{5}$ **5.** 4/5 **7.** $x/\sqrt{1 - x^2}$
9. $2x\sqrt{1 - x^2}$ **11.** $2x^2 - 1$ **15.** $5(\sin 8x + \sin 2x)$

17. $\frac{1}{2}(\cos 10x + \cos 6x)$

19. $\frac{1}{2}(\sin 90° + \sin 30°) = \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4}$

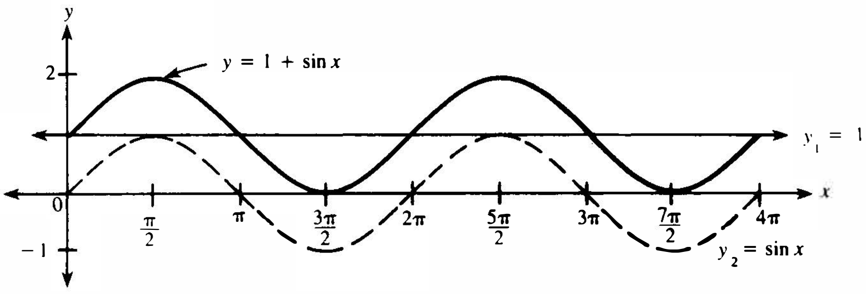
21. $\frac{1}{2}(\cos 6\pi - \cos 2\pi) = \frac{1}{2}(1 - 1) = 0$

25. $2 \sin 5x \cos 2x$

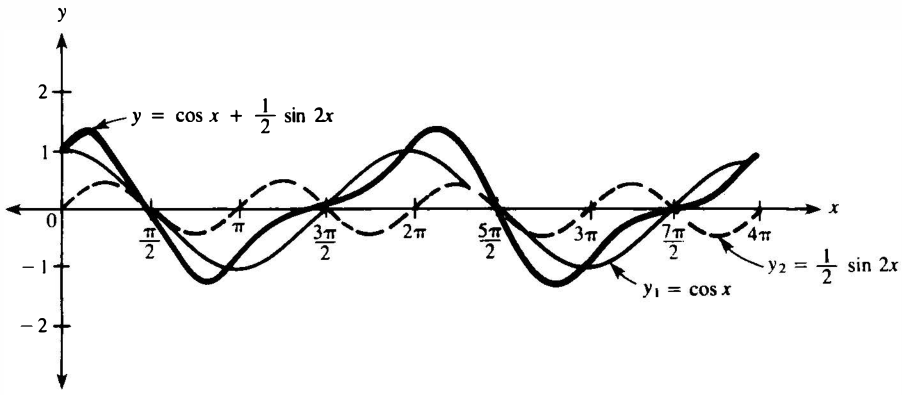
27. $2 \cos 30° \cos 15°$

29. $2 \cos \frac{\pi}{3} \sin \frac{\pi}{4} = 1/\sqrt{2}$

37.



39.



Chapter 5 Test

13. $63/65$ **14.** $-56/65$ **15.** $-119/169$ **16.** $-120/169$

17. $1/\sqrt{10}$ **18.** $-3/\sqrt{10}$ **19.** $\dfrac{\sqrt{6}+\sqrt{2}}{4}$ **20.** $\dfrac{\sqrt{6}+\sqrt{2}}{4}$

21. $\dfrac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}$ **22.** $\dfrac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}}$ **23.** $\cos 9x$ **24.** $\sin 90° = 1$

25. $3/5, -\sqrt{\dfrac{5-2\sqrt{5}}{10}}$ **26.** $3/5, \sqrt{\dfrac{10-\sqrt{10}}{20}}$ **27.** 1

28. $\pm\sqrt{3}/2$ **29.** $11/5\sqrt{5}$ **30.** $11/5\sqrt{5}$ **31.** $1 - 2x^2$

32. $2x\sqrt{1-x^2}$ **33.** $\dfrac{1}{2}(\cos 10x - \cos 2x)$ **34.** $2 \cos 45° \cos(-30°) = \sqrt{6}/2$

CHAPTER 6

Problem Set 6.1

1. 30°, 150° **3.** 30°, 330° **5.** 135°, 315° **7.** $\pi/3, 2\pi/3$

9. $\pi/6, 11\pi/6$ **11.** $3\pi/2$ **13.** 48.6°, 131.4° **15.** $\varnothing$

17. $228.6°, 311.4°$
19. $\pi/2, \pi/6, 5\pi/6$
21. $0, \pi/4, \pi, 5\pi/4, 2\pi$
23. $0, 2\pi/3, \pi, 4\pi/3, 2\pi$
25. $\pi/2, 7\pi/6, 11\pi/6$
27. $120°, 150°, 210°, 240°$
29. $0°, 60°, 180°, 120°, 360°$
31. $120°, 240°$
33. $201.5°, 338.5°$
35. $51.8°, 308.2°$
37. $30°, 150°$
39. $30° + 360°k, 150° + 360°k$
41. $\pi/3 + 2k\pi, 2\pi/3 + 2k\pi$
43. $3\pi/2 + 2k\pi$
45. $48.6° + 360°k, 131.4° + 360°k$
47. $h = -16t^2 + 750t$
49. $h = -16(2)^2 + 750(2) = 1{,}436$ feet
51. $15.7°$
53. $\sin 2A = 2 \sin A \cos A$
55. $\cos 2A = 2\cos^2 A - 1$
57. $\sin\theta\cos 45° + \cos\theta\sin 45° = \dfrac{1}{\sqrt{2}}\sin\theta + \dfrac{1}{\sqrt{2}}\cos\theta$

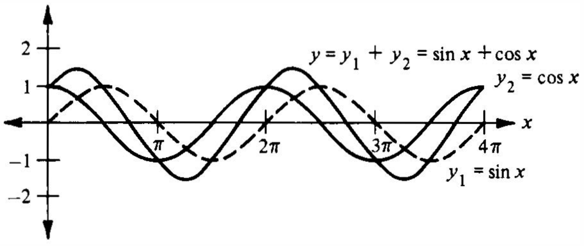
Problem Set 6.2

1. $30°, 330°$
3. $5\pi/4, 7\pi/4$
5. $\pi/4, 3\pi/4, 5\pi/4, 7\pi/4$
7. $30°, 150°$
9. $30°, 90°, 150°, 270°$
11. $60°, 180°, 300°$
13. $7\pi/6, 3\pi/2, 11\pi/6$
15. $0°, 120°, 240°, 360°$
17. $\pi/2, 7\pi/6, 11\pi/6$
19. $\pi/3, 5\pi/3$
21. $0, 2\pi/3, 4\pi/3, 2\pi$
23. $45°$
25. $30°, 90°$
27. $60°, 180°$
29. $\pi/3, 5\pi/3$
31. $2\pi/3, \pi$
33. $5\pi/4 + 2k\pi, 7\pi/4 + 2k\pi$
35. $45° + 360°k$
37. $2\pi/3 + 2k\pi, \pi + 2k\pi = \pi(2k + 1)$
41. $30°, 330°, 390°, 690°$
43. $\pi/4, 5\pi/4, 9\pi/4, 13\pi/4$

Problem Set 6.3

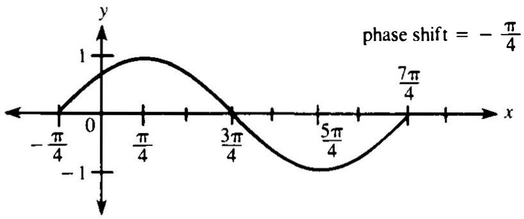
1. $30°, 60°, 210°, 240°$
3. $67.5°, 157.5°, 247.5°, 337.5°$
5. $60°, 180°, 300°$
7. $\pi/8, 3\pi/8, 9\pi/8, 11\pi/8$
9. $\pi/3, \pi, 5\pi/3$
11. $\pi/6, 2\pi/3, 7\pi/6, 5\pi/3$
13. $15° + 180°k, 75° + 180°k$
15. $30° + 120°k, 90° + 120°k$
17. $6° + 36°k, 12° + 36°k$
19. $\pi/18, 5\pi/18, 13\pi/18, 17\pi/18, 25\pi/18, 29\pi/18$
21. $7\pi/18, 11\pi/18, 19\pi/18, 23\pi/18, 31\pi/18, 35\pi/18$
23. $\dfrac{\pi}{10} + \dfrac{2k\pi}{5}$
25. $\dfrac{\pi}{8} + \dfrac{k\pi}{2}, \dfrac{3\pi}{8} + \dfrac{k\pi}{2}$
27. $\dfrac{\pi}{5} + \dfrac{2k\pi}{5}$
29. $10° + 120°k, 50° + 120°k, 90° + 120°k$
31. $60° + 180°k, 90° + 180°k, 120° + 180°k$
33. $20° + 60°k, 40° + 60°k$
35. $0°, 270°, 360°$
37. $180°, 270°$
39. 6
41. 1/4 of second (and every second after that)
43. 1/12

45.



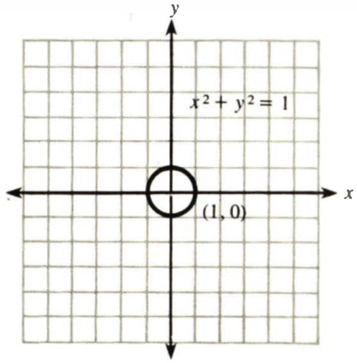
47. $\sin(x + 45°)$

49.

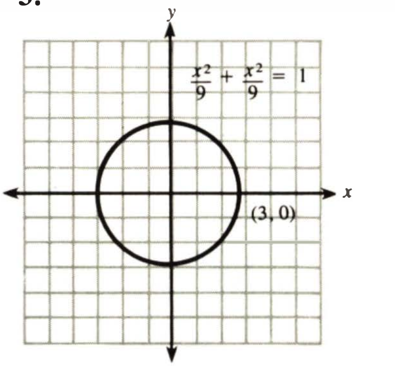


Problem Set 6.4

1.



3.



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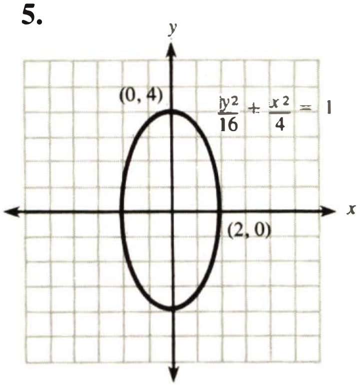

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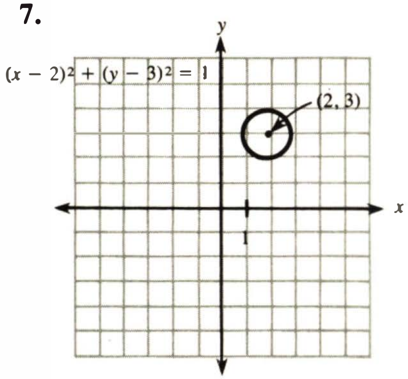
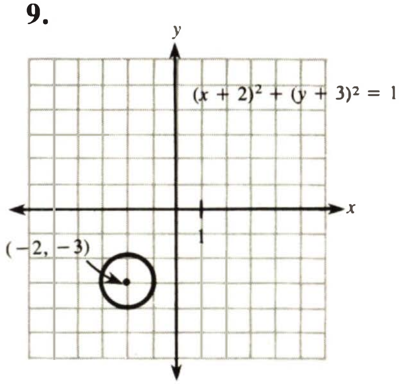
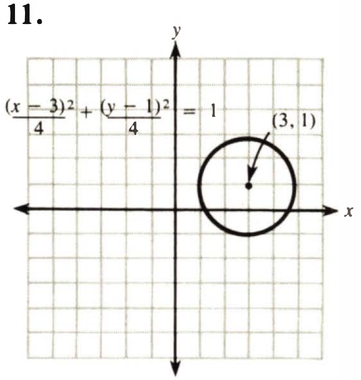


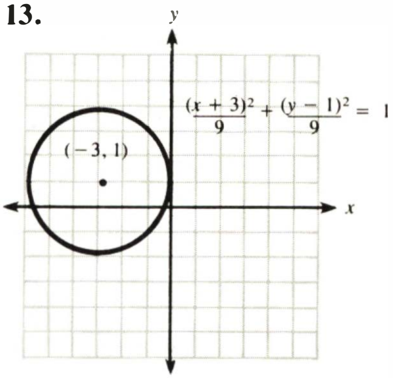
9.



11.



13.



15. $x^2 - y^2 = 1$

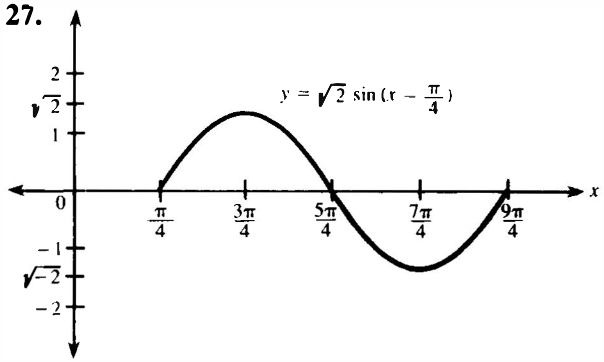
17. $\dfrac{x^2}{9} - \dfrac{y^2}{9} = 1$

19. $\dfrac{(y-4)^2}{9} - \dfrac{(x-2)^2}{9} = 1$

For problems 21 through 26, x and y are restricted to values between -1 and 1 inclusive.

21. $x = 1 - 2y^2$ **23.** $y = x$ **25.** $2x = 3y$

27.



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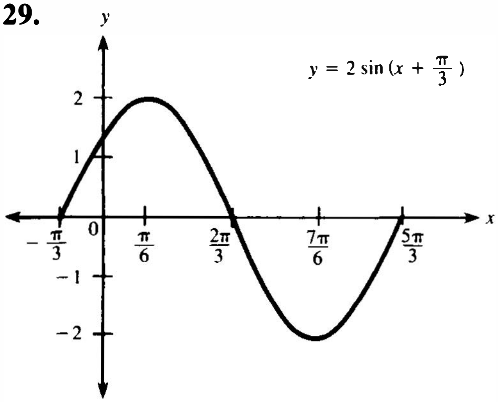

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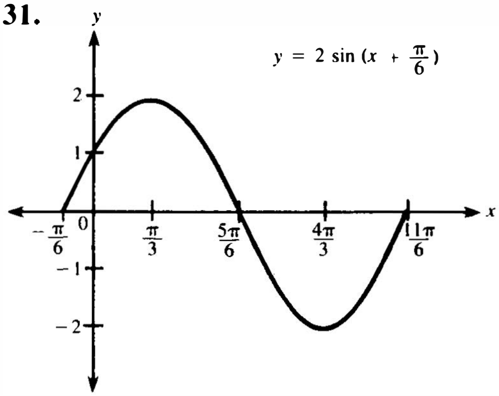


Chapter 6 Test

1. 30°, 150°

2. 150°, 330°

3. 30°, 90°, 150°, 270°

4. 0°, 60°, 180°, 300°, 360°

5. 45°, 135°, 225°, 315°

6. 90°, 210°, 330°

7. 180°

8. 0°, 240°

9. 0°, 30°, 150°, 180°, 210°, 330°, 360°

10. 15°, 75°, 105°, 165°, 195°, 255°, 285°, 345°

11. 0°, 90°, 360°

12. 90°, 180°

13. $2k\pi, \dfrac{\pi}{3} + 2k\pi, \dfrac{5\pi}{3} + 2k\pi$

14. $\dfrac{\pi}{6} + 2k\pi, \dfrac{7\pi}{6} + 2k\pi$

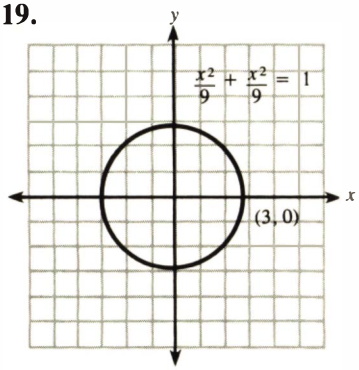
15. $\dfrac{\pi}{2} + \dfrac{2k\pi}{3}$

16. $\dfrac{\pi}{8} + \dfrac{k\pi}{4}$

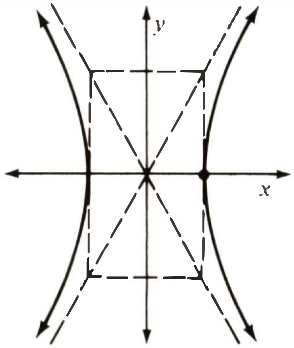
17. 90°, 203.6°, 336.4°

18. 111.5°, 248.5°

19.



20.



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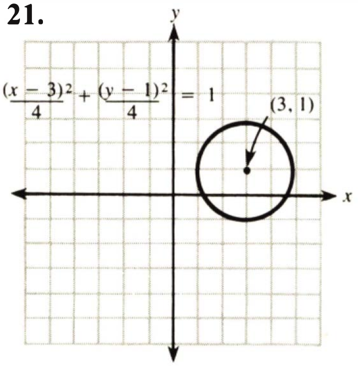

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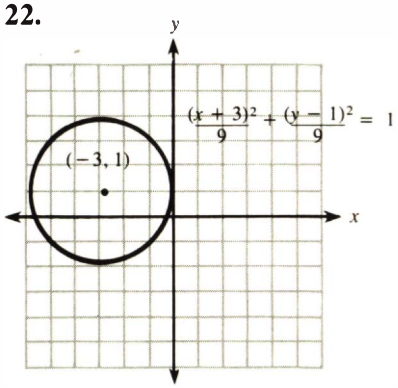
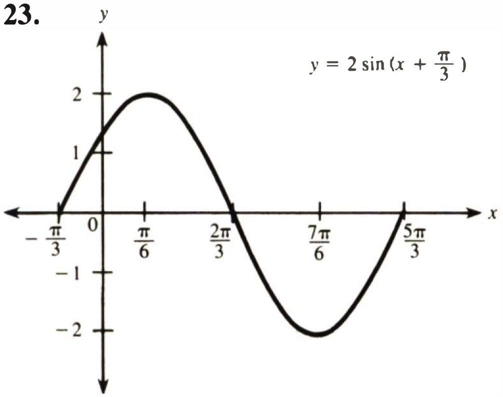
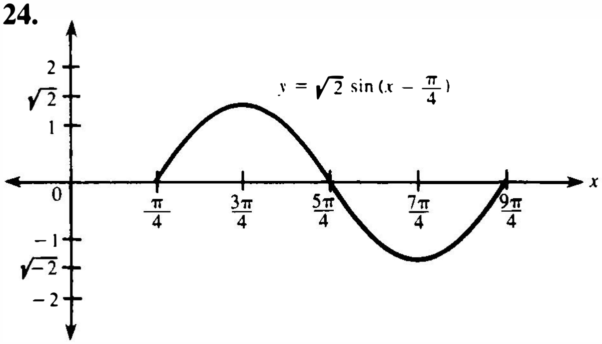


23.



24.



CHAPTER 7

Problem Set 7.1

1. 16 centimeters
3. 71 inches
5. 136 yards
7. $C = 70°$, $c = 44$ kilometers
9. $C = 80°$, $a = 11$ centimeters
11. $C = 66.1°$, $b = 295$ inches, $c = 284$ inches
13. $C = 39°$, $a = 7.8$ meters, $b = 11$ meters
15. $B = 16°$, $b = 1.39$ feet, $c = 4.36$ feet
17. $A = 141.8°$, $b = 118$ centimeters, $c = 214$ centimeters
19. $B = 121°$, $a = 0.445$ kilometers, $c = 0.499$ kilometers
21. $\sin B = 5$, which is impossible
23. 209 feet
25. 273 feet
27. 47.6° and 132.4°
29. 42.4° and 317.6°
31. 75.2° and 104.8°

Problem Set 7.2

1. $\sin B = 2$ is impossible
3. $B = 35.3$ is the only possibility for B
5. $B = 77°$ or $B' = 103°$
7. $B = 54°$, $C = 88°$, $c = 67$ feet or $B' = 126°$, $C' = 16°$, $c' = 18$ feet
9. $B = 28.1°$, $C = 39.7°$, $c = 30.2$ centimeters
11. $B = 34°50'$, $A = 117°20'$, $a = 660$ centimeters or $B' = 145°10'$, $A' = 7°$, $a' = 90.6$ centimeters
13. $C = 26°20'$, $A = 108°30'$, $a = 2.39$ inches
15. no solution
17. no solution
19. $B = 26.8°$, $A = 126.4°$, $a = 65.7$ kilometers
21. 15 feet or 38 feet
23. 24/25
25. $1/\sqrt{5}$
27. $-24/7$
29. $\dfrac{\sqrt{2-\sqrt{3}}}{2}$

Problem Set 7.3

1. 87 inches
3. $C = 93°$
5. 9.4 meters
7. $A = 128°$
9. $A = 44°$, $B = 76°$, $c = 62$ centimeters
11. $A = 29°$, $B = 47°$, $C = 104°$
13. $A = 15.6°$, $C = 12.9°$, $b = 727$ meters
15. $A = 39°$, $B = 57°$, $C = 84°$
17. $B = 114°10'$, $C = 22°30'$, $a = 0.694$
19. $A = 55.4°$, $B = 45.5°$, $C = 79.1°$
21. $a^2 = b^2 + c^2 - 2bc \cos 90° = b^2 + c^2 - 2bc(0) = b^2 + c^2$
23. 24 inches
25. 133 miles
27. 194 miles per hour with bearing 153°
29. 30°, 150°
31. 90°, 210°, 330°
33. 0°, 60°, 180°, 300°, 360°
35. 45°, 135°, 225°, 315°

Problem Set 7.4

The answers to problems 1 through 21 have been rounded to three significant digits.

1. 1,520 centimeters^2 **3.** 333 meters^2 **5.** 0.123 kilometers^2
7. 26.3 meters^2 **9.** 28,300 inches^2 **11.** 2.09 feet^2
13. $\sqrt{135} \cong 11.6$ square inches **15.** 15.0 square yards
17. 8.15 square feet **19.** 156 square inches **21.** 14.3 centimeters **23.** 180° or π
25. $-30°$ or $-\pi/6$ **27.** 60° or $\pi/3$ **29.** 3/5 **31.** $\sqrt{5}/3$

Chapter 7 Test

1. 6.7 inches **2.** 4.3 inches
3. $C = 78.4°$, $a = 26.5$ centimeters, $b = 38.3$ centimeters **4.** $B = 49.2°$, $a = 18.8$ centimeters, $c = 43.2$ centimeters
5. $\sin B = 3.0311$, which is impossible **6.** $B = 29°$ is the only possibility for B
7. $B = 71°$, $C = 58°$, $c = 7.1$ feet or $B' = 109°$, $C' = 20°$, $c' = 2.9$ feet **8.** $B = 59°$, $C = 95°$, $c = 11$ feet or $B' = 121°$, $C' = 33°$, $c' = 6.0$ feet
9. 11 centimeters **10.** 19 centimeters **11.** 95.7° **12.** 69.5°
13. $A = 43°$, $B = 18°$, $c = 8.1$ centimeters **14.** $B = 34°$, $C = 111°$, $a = 3.8$ meters
15. 50.8° **16.** 59 feet **17.** 411 feet **18.** 14.2 meters
19. 498 centimeters^2 **20.** 307 centimeters^2 **21.** 52 centimeters^2 **22.** 52 centimeters^2
23. 17 kilometers^2 **24.** 52 kilometers^2

CHAPTER 8

Problem Set 8.1

1. $4i$ **3.** $11i$ **5.** $3i\sqrt{2}$ **7.** $2i\sqrt{2}$
9. $i\sqrt{13}$ **11.** $x = 2/3$, $y = -1/2$ **13.** $x = 11/4$, $y = -2$
15. $x = 2/5$, $y = -4$ **17.** $x = -2$ or 3, $y = \pm 3$ **19.** $x = \pi/4$ or $5\pi/4$, $y = \pi/2$
21. $x = \pi/2$, $y = \pi/4$ or $5\pi/4$ **23.** $10 - 2i$ **25.** $2 + 6i$
27. $3 - 13i$ **29.** $5 \cos x - 3i \sin y$ **31.** $2 + 2i$
33. $12 + 2i$ **35.** 1 **37.** -1 **39.** 1
41. i **43.** $-48 - 18i$ **45.** $10 - 10i$ **47.** $5 + 12i$
49. 41 **51.** 53 **53.** $-28 + 4i$ **55.** $\frac{12}{41} + \frac{15}{41}i$
57. $\frac{1}{5} + \frac{3}{5}i$ **59.** $-\frac{5}{13} + \frac{12}{13}i$ **61.** $-2 - 5i$ **63.** $\frac{4}{61} + \frac{17}{61}i$
65. 13 **67.** $-7 + 22i$ **69.** $10 - 3i$ **71.** $16 + 20i$
73. $x^2 + 9$ **75.** $\frac{1}{i} = \frac{1}{i} \cdot \frac{i}{i} = \frac{i}{-1} = -i$

77. Yes, addition with complex numbers is commutative.

79. No. For example, $(3 + 2i) - (5 + 4i) = -2 - 2i$ while $(5 + 4i) - (3 + 2i) = 2 + 2i$

81. $\sin \theta = -4/5$, $\cos \theta = 3/5$

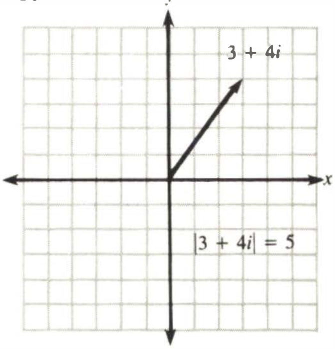
83. $\sin \theta = \dfrac{b}{\sqrt{a^2 + b^2}}$, $\cos \theta = \dfrac{a}{\sqrt{a^2 + b^2}}$

85. 135°

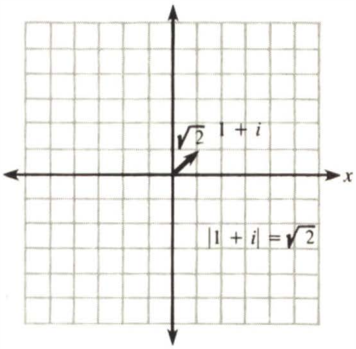
87. 150°

Problem Set 8.2

1.



3.



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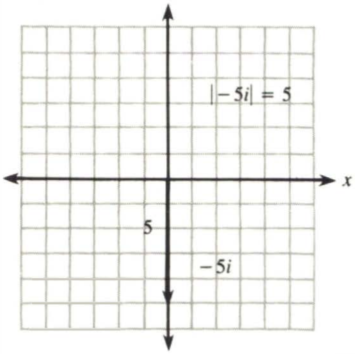

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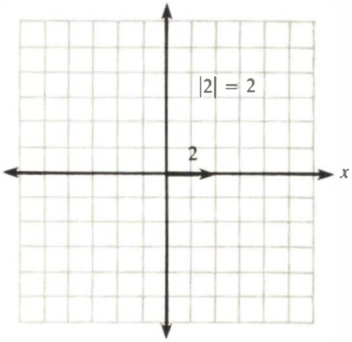
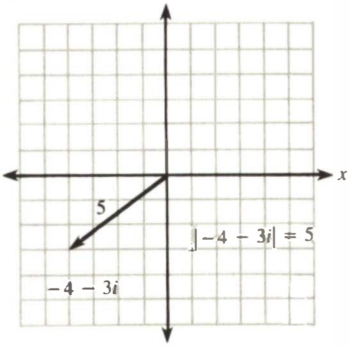


9.



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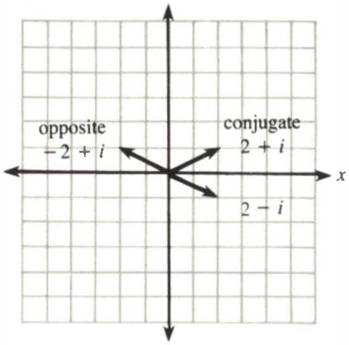

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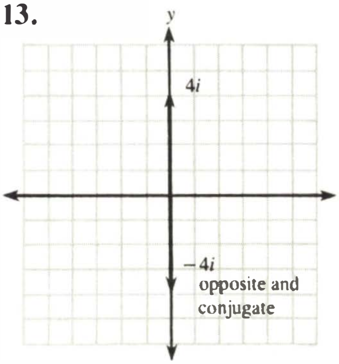

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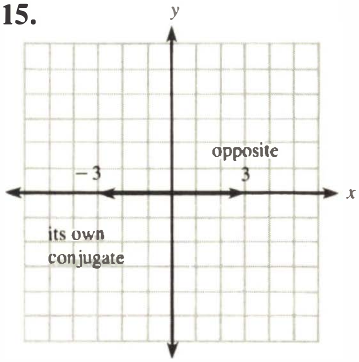
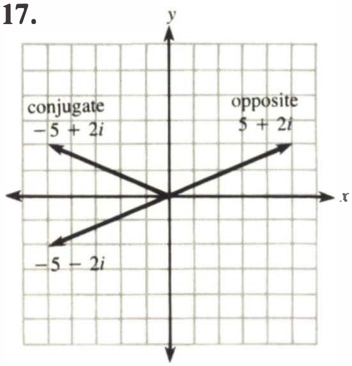


17.



19. $\sqrt{3} + i$ **21.** $-2 + 2i\sqrt{3}$ **23.** $-\frac{\sqrt{3}}{2} - \frac{1}{2}i$ **25** $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

27. $9.78 + 2.08i$ **29.** $-79.86 + 60.18i$ **31.** $-0.91 - 0.42i$

33. $9.51 - 3.09i$ **35.** $\sqrt{2}(\cos 135° + i \sin 135°)$ **37.** $\sqrt{2}(\cos 315° + i \sin 315°)$

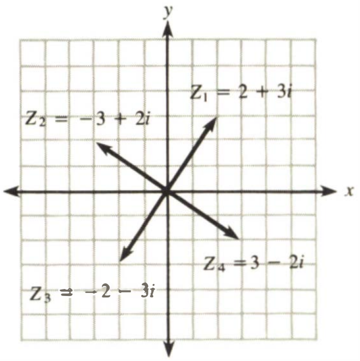
39. $3\sqrt{2}(\cos 45° + i \sin 45°)$ **41.** $8(\cos 90° + i \sin 90°)$ **43.** $9(\cos 180° + i \sin 180°)$

45. $4(\cos 120° + i \sin 120°)$ **47.** $6(\cos 300° + i \sin 300°)$

49.
$$\begin{aligned}(2i)(3i) &= [2(\cos 90° + i \sin 90°)][3(\cos 90° + i \sin 90°)]\\ &= 2 \cdot 3(\cos 90° \cos 90° + i \cos 90° \sin 90° + i \sin 90° \cos 90° + i^2 \sin 90° \sin 90°)\\ &= 6[0 \cdot 0 + i \cdot 0 \cdot 1 + i \cdot 1 \cdot 0 + (-1)(1)(1)]\\ &= 6(-1)\\ &= -6\end{aligned}$$

51.
$$2(\cos 30° + i \sin 30°) = 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \sqrt{3} + i$$
$$2[\cos(-30°) + i \sin(-30°)] = 2\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = \sqrt{3} - i$$

53.
$$\begin{aligned}z_1 &= 2 + 3i\\ z_2 &= z_1 \cdot i = -3 + 2i\\ z_3 &= z_2 \cdot i = -2 - 3i\\ z_4 &= z_3 \cdot i = 3 - 2i\end{aligned}$$



55. If $z = \cos\theta + i \sin\theta$, then $|z| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$

57.
$$\begin{aligned}\cos 75° &= \cos(45° + 30°)\\ &= \cos 45° \cos 30° - \sin 45° \sin 30°\\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

59.
$$\begin{aligned}\sin(A + B) &= \sin A \cos B + \cos A \sin B\\ &= \frac{3}{5} \cdot \frac{12}{13} + \frac{4}{5} \cdot \frac{5}{12} = \frac{56}{65}\end{aligned}$$

61. $\sin(30° + 90°) = \sin 120° = \sqrt{3}/2$

63. $\cos(18° + 32°) = \cos 50°$

Problem Set 8.3

1. $12(\cos 50° + i \sin 50°)$ **3.** $56(\cos 157° + i \sin 157°)$

5. $4(\cos 180° + i \sin 180°)$ **7.** $2(\cos 180° + i \sin 180°) = -2$

9. $4(\cos 210° + i \sin 210°) = -2\sqrt{3} - 2i$ **11.** $12(\cos 360° + i \sin 360°) = 12$

13. $4\sqrt{2}(\cos 135° + i \sin 135°) = -4 + 4i$ **15.** $10(\cos 240° + i \sin 240°) = -5 - 5i\sqrt{3}$

17. $64(\cos 60° + i \sin 60°) = 32 + 32i\sqrt{3}$ **19.** $\cos 120° + i \sin 120° = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

21. $81(\cos 240° + i \sin 240°) = -\frac{81}{2} - \frac{81\sqrt{3}}{2}i$ **23.** $32(\cos 450° + i \sin 450°) = 32i$

25. $(1+i)^4 = [\sqrt{2}(\cos 45° + i \sin 45°)]^4$
$= 4(\cos 180° + i \sin 180°) = -4$

27. $-8 - 8i\sqrt{3}$

29. $8i$

31. $16 + 16i$

33. $4(\cos 35° + i \sin 35°)$

35. $1.5(\cos 19° + i \sin 19°)$

37. $0.5(\cos 60° + i \sin 60°)$

39. $2(\cos 0° + i \sin 0°) = 2$

41. $\cos(-60°) + i \sin(-60°) = \frac{1}{2} - \frac{\sqrt{3}}{2} i$

43. $2[\cos(-270°) + i \sin(-270°)] = 2i$

45. $2[\cos(-180°) + i \sin(-180°)] = -2$

47. $-4 - 4i$

49. 8

51.
$$[2(\cos 60° + i \sin 60°)]^2 - 2[2(\cos 60° + i \sin 60°)] + 4 = 0$$
$$4(\cos 120° + i \sin 120°) - 4(\cos 60° + i \sin 60°) + 4 = 0$$
$$-2 + 2i\sqrt{3} - 2 - 2i\sqrt{3} + 4 = 0$$
$$0 = 0$$

53. $w^4 = [2(\cos 15° + i \sin 15°)]^4 = 16(\cos 60° + i \sin 60°)$
$= 8 + 8i\sqrt{3}$

55. $(1+i)^{-1} = [\sqrt{2}(\cos 45° + i \sin 45°)]^{-1}$
$= \sqrt{2}^{-1}[\cos(-45°) + i \sin(-45°)]$
$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i\right) = \frac{1}{2} - \frac{1}{2} i$$

57. $\frac{\sqrt{3}}{4} + \frac{1}{4} i$

59. $\frac{1}{4} - \frac{\sqrt{3}}{4} i$

61. 0.2588

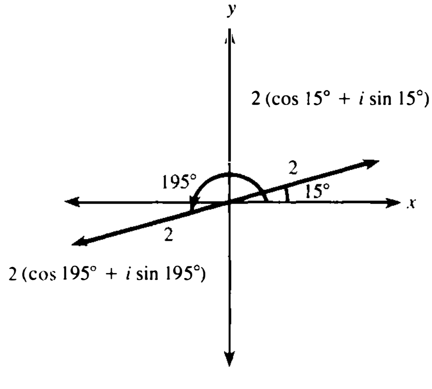
63. -0.9659

65. -0.9659

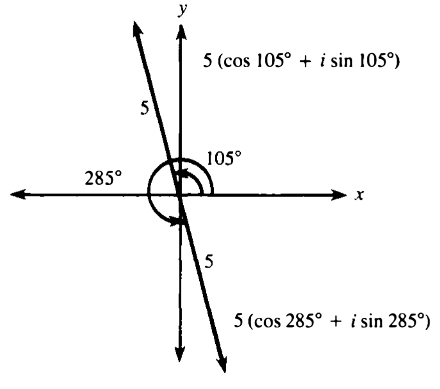
67. -0.2588

Problem Set 8.4

1.



3.



5. $\sqrt{3} + i, -\sqrt{3} - i$

7. $\sqrt{2} + i\sqrt{2}, -\sqrt{2} - i\sqrt{2}$

9. $5i, -5i$

11. $2(\cos 70° + i \sin 70°), 2(\cos 190° + i \sin 190°), 2(\cos 310° + i \sin 310°)$

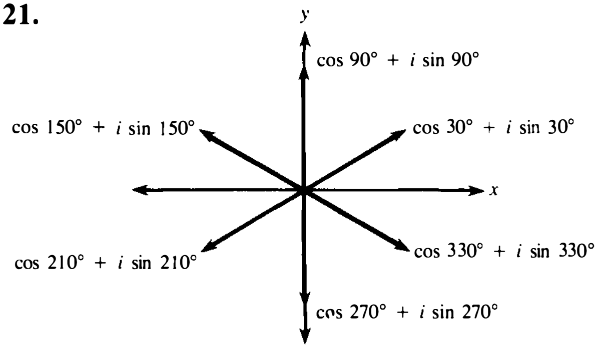
13. $2(\cos 10° + i \sin 10°), 2(\cos 130° + i \sin 130°), 2(\cos 250° + i \sin 250°)$

15. $3(\cos 60° + i \sin 60°), 3(\cos 180° + i \sin 180°), 3(\cos 300° + i \sin 300°)$

17. $\sqrt{3} + i, -1 + i\sqrt{3}, -\sqrt{3} - i, 1 - i\sqrt{3}$

19. $10(\cos 3° + i \sin 3°) \cong 9.99 + 0.52i$
$10(\cos 75° + i \sin 75°) \cong 2.59 + 9.66i$
$10(\cos 147° + i \sin 147°) \cong -8.39 + 5.45i$
$10(\cos 219° + i \sin 219°) \cong -7.77 - 6.29i$
$10(\cos 291° + i \sin 291°) \cong 3.58 - 9.34i$

21.

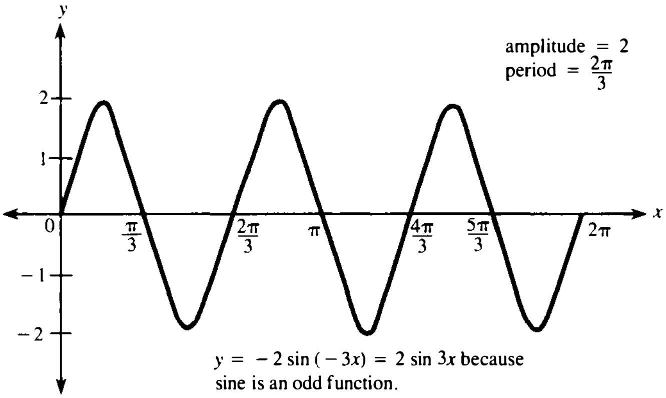


23. $\sqrt{2}(\cos \theta + i \sin \theta)$ where $\theta = 75°, 105°, 255°, 285°$

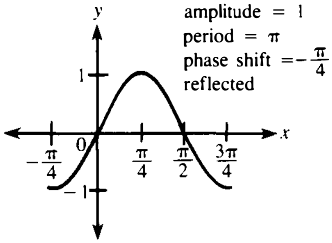
25. $\sqrt{2}(\cos \theta + i \sin \theta)$ where $\theta = 30°, 150°, 210°, 330°$

27. $\sqrt[4]{2}(\cos \theta + i \sin \theta)$ where $\theta = 67.5°, 112.5°, 247.5°, 292.5°$

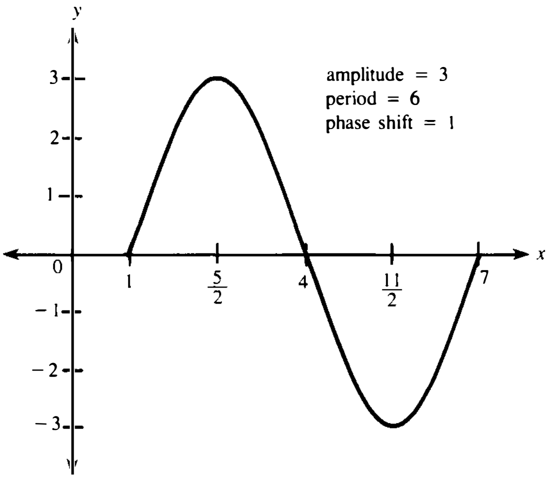
29.



31.

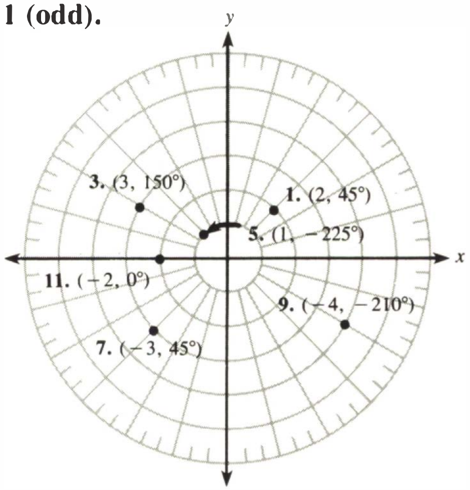


33.



Problem Set 8.5

1–11 (odd).



13. $(2, -300°), (-2, 240°), (-2, -120°)$

15. $(5, -225°), (-5, 315°), (-5, -45°)$

17. $(-3, -330°), (3, 210°), (3, -150°)$

19. $(1, \sqrt{3})$

21. $(0, -3)$

23. $(-1, -1)$

25. $(-6, -2\sqrt{3})$

27. $(3\sqrt{2}, 135°)$

29. $(4, 150°)$

31. $(2, 0°)$

33. $(2, 210°)$

35. $(5, 53.1°)$

37. $(\sqrt{5}, 116.6°)$

39. $(\sqrt{13}, 236.3°)$

41. $x^2 + y^2 = 9$

43. $x^2 + y^2 = 6y$

45. $(x^2 + y^2)^2 = 8xy$

47. $x + y = 3$

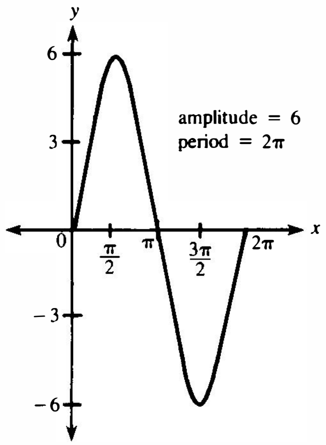
49. $r(\cos\theta - \sin\theta) = 5$

51. $r^2 = 4$

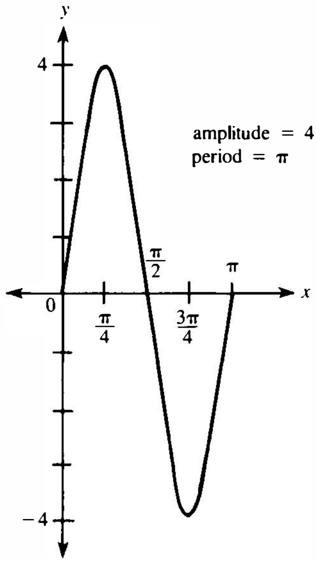
53. $r = 6\cos\theta$

55. $\theta = 45°$ or $\cos\theta = \sin\theta$

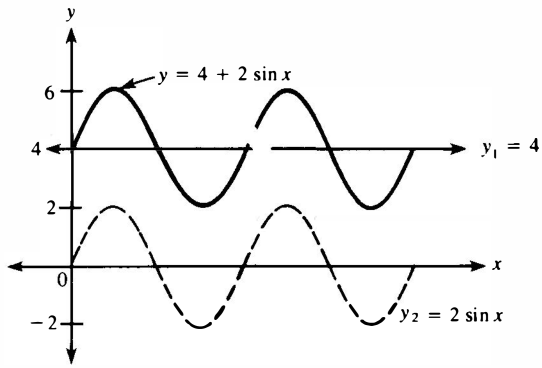
57.



59.

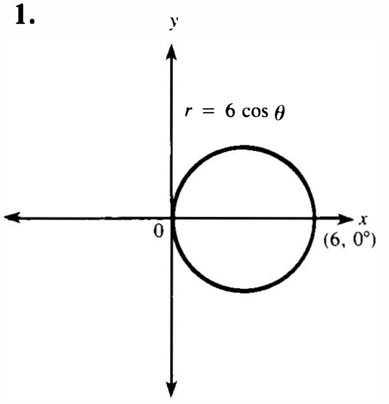


61.



Problem Set 8.6

1.



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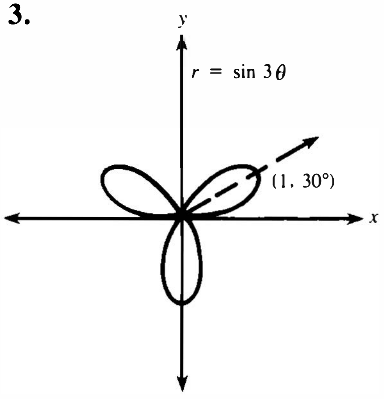
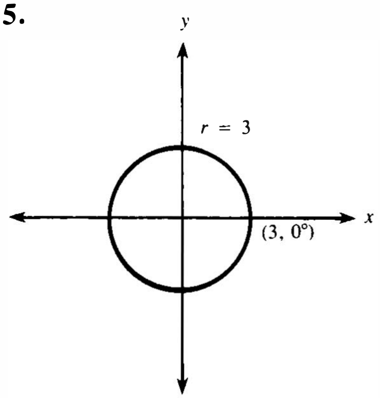


5.



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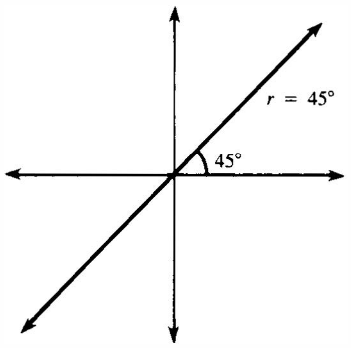
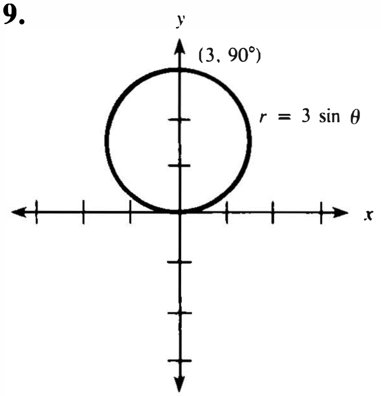


9.



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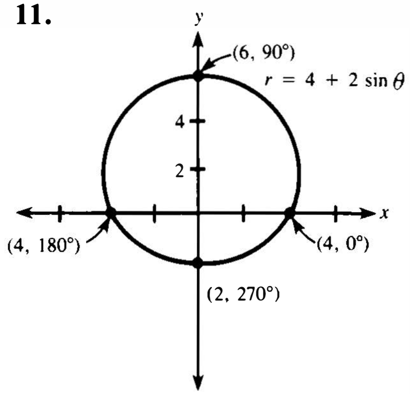

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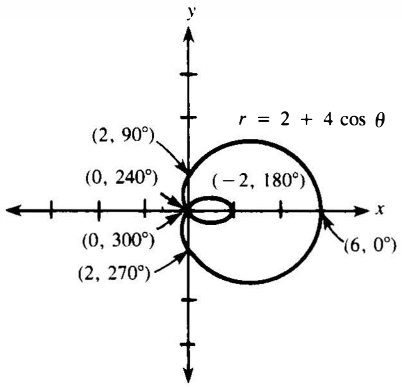

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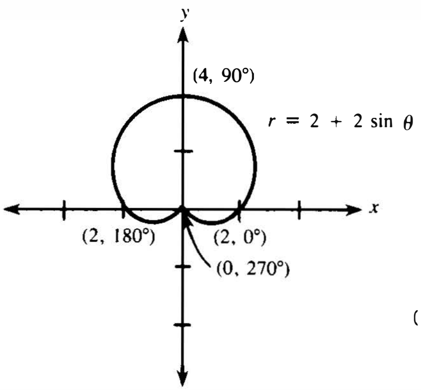

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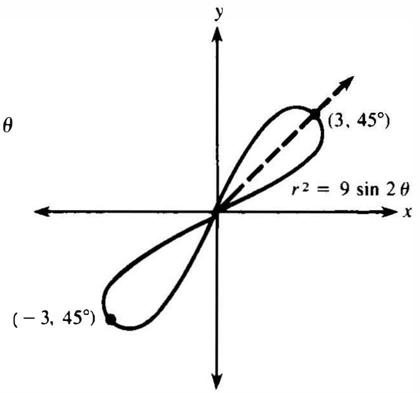

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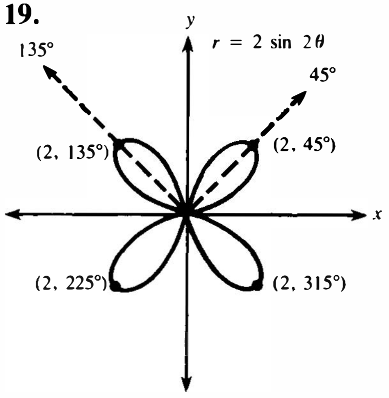

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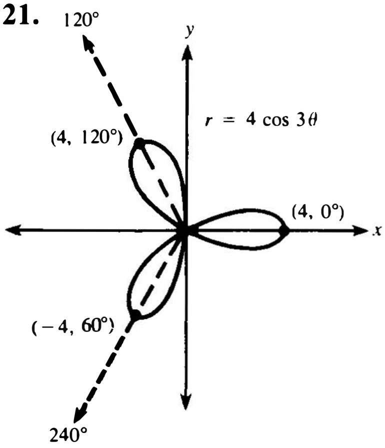

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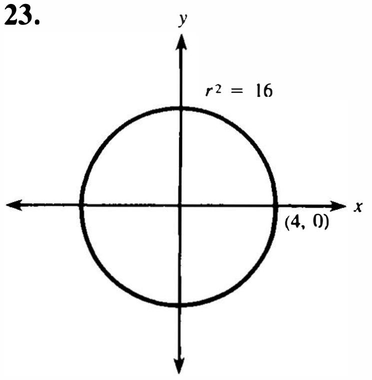
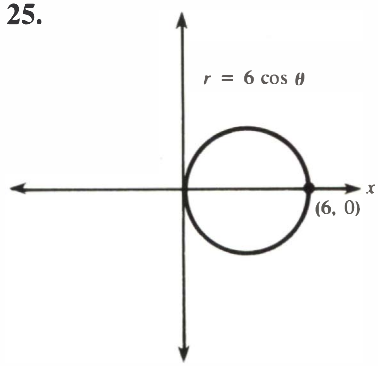
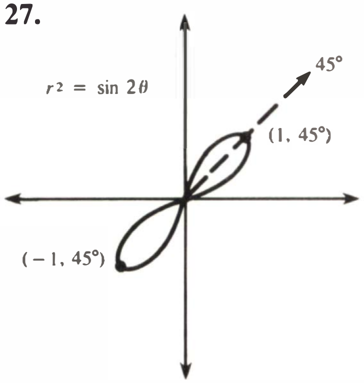


25.



27.



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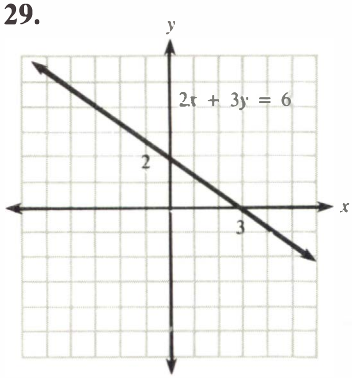

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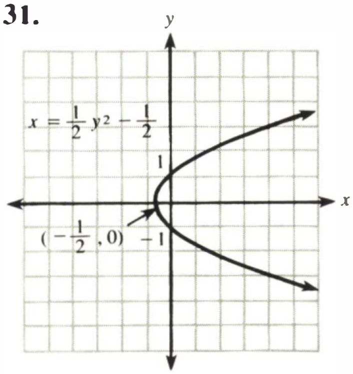

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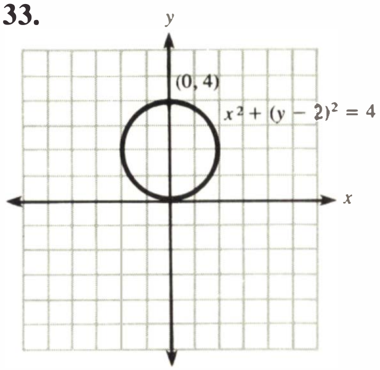

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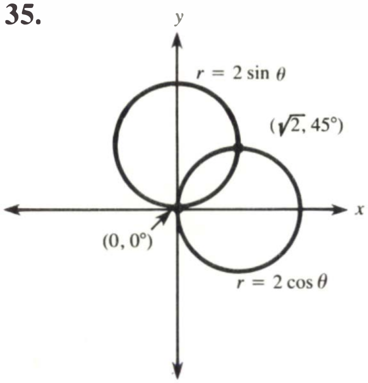

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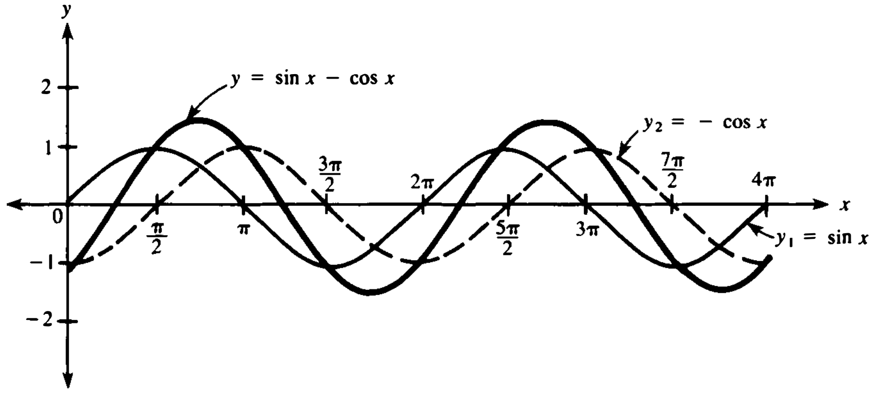
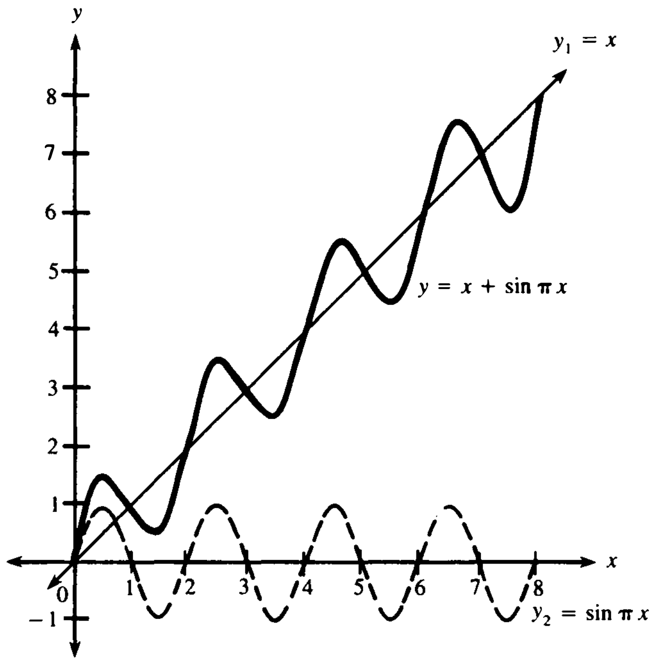
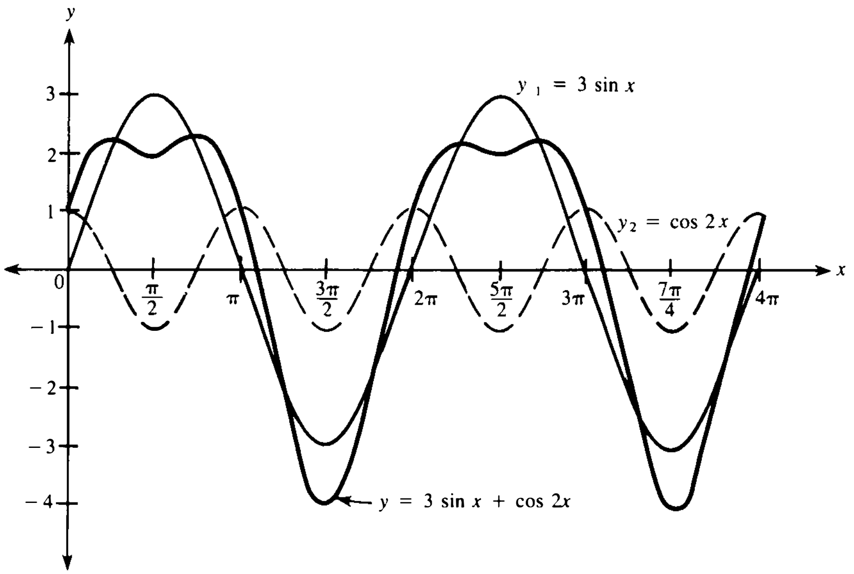

y
2
1
0
−1
−2
$y = \sin x - \cos x$
$y_2 = -\cos x$
$\frac{3\pi}{2}$
2π
$\frac{7\pi}{2}$
4π
x
$\frac{\pi}{2}$
π
$\frac{5\pi}{2}$
3π
$y_1 = \sin x$

39.


y
$y_1 = x$
8
7
6
5
4
3
2
1
0
−1
$y = x + \sin \pi x$
x
1 2 3 4 5 6 7 8
$y_2 = \sin \pi x$

41.



Chapter 8 Test

1. $5i$

2. $2i\sqrt{3}$

3. $x = 2,\ y = 2$

4. $x = -2$ or $5,\ y = 2$

5. $7 - 6i$

6. $8 - 2i$

7. 1

8. i

9. 89

10. $-16 + 30i$

11. $-2 - \frac{5}{2}i$

12. $\frac{11}{61} + \frac{60}{61}i$

13. a. 5 b. $-3 - 4i$ c. $3 - 4i$

14. a. 5 b. $-3 + 4i$ c. $3 + 4i$

15. a. 8 b. $-8i$ c. $-8i$

16. a. 4 b. 4 c. -4

17. $4\sqrt{3} - 4i$

18. $-\sqrt{2} + i\sqrt{2}$

19. $2\sqrt{2}(\cos 45° + i \sin 45°)$

20. $2(\cos 150° + i \sin 150°)$

21. $5(\cos 90° + i \sin 90°)$

22. $3(\cos 180° + i \sin 180°)$

23. $15(\cos 65° + i \sin 65°)$

24. $5(\cos 30° + i \sin 30°)$

25. $32(\cos 50° + i \sin 50°)$

26. $81(\cos 80° + i \sin 80°)$

27. $7(\cos 25° + i \sin 25°)$, $7(\cos 205° + i \sin 205°)$

28. $\sqrt{2}(\cos \theta + i \sin \theta)$ where $\theta = 15°, 105°, 195°, 285°$

29. $x = \sqrt{2}(\cos \theta + i \sin \theta)$ where $\theta = 15°, 165°, 195°, 345°$

30. $x = \cos \theta + i \sin \theta$ where $\theta = 60°, 180°, 300°$

31. $(-4, 45°), (4, -135°); (-2\sqrt{2}, -2\sqrt{2})$

32. $(6, 240°), (6, -120°); (-3, -3\sqrt{3})$

33. $(3\sqrt{2}, 135°)$

34. $(5, 90°)$

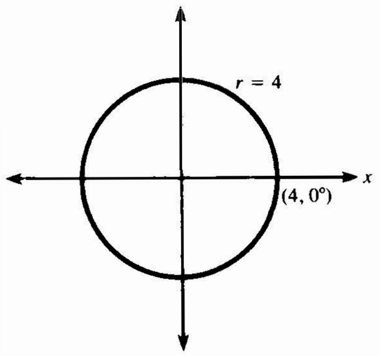
35. $x^2 + y^2 = 6y$

36. $(x^2 + y^2)^{3/2} = 2xy$

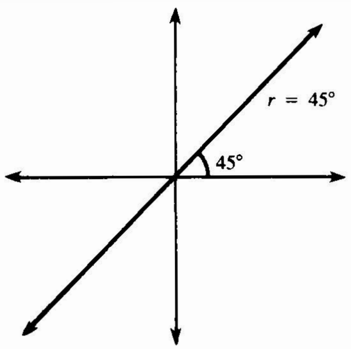
37. $r(\cos \theta + \sin \theta) = 2$

38. $r = 8 \sin \theta$

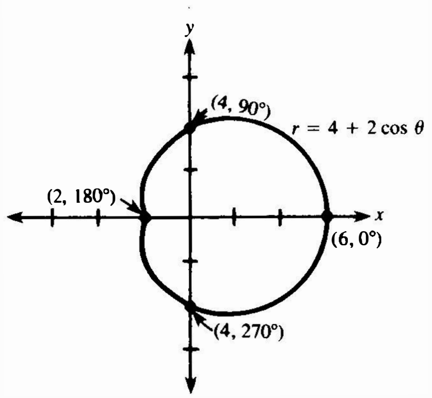
39.



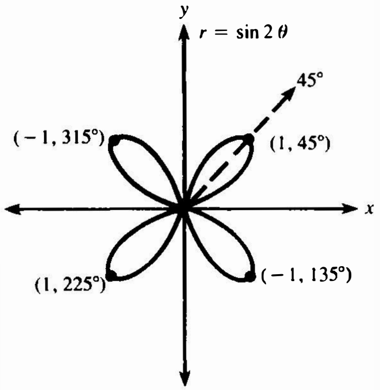
40.



41.



42.



APPENDIX

Appendix A.1

1.	$\log_5 125 = 3$	**3.**	$\log_{10} .01 = -2$	**5.**	$\log_2 \frac{1}{32} = -5$	**7.**	$10^2 = 100$
9.	$8^0 = 1$	**11.**	$10^{-3} = .001$	**13.**	1/125	**15.**	4
17.	1/3	**19.**	2	**21.**	4	**23.**	3/2
25.	3	**27.**	1	**29.**	0	**31.**	0
33.	$\log_3 4 + \log_3 x$	**35.**	$\log_6 5 - \log_6 x$	**37.**	$5 \log_2 y$	**39.**	$\frac{1}{3} \log_9 z$

41. $2 \log_6 x + 3 \log_6 y$

43. $\frac{1}{2} \log_5 x + 4 \log_5 y$

45. $\log_b x + \log_b y - \log_b z$

47. $\log_{10} 4 - \log_{10} x - \log_{10} y$

49. $2 \log_{10} x + \log_{10} y - \frac{1}{2} \log_{10} z$

51. $3 \log_{10} x + \frac{1}{2} \log_{10} y - 4 \log_{10} z$

53. $\log_b xz$
55. $\log_3 \frac{x^2}{y^3}$
57. $\log_{10} x\sqrt[3]{y}$
59. $\log_2 \frac{x^3\sqrt{y}}{z}$
61. $\log_2 \frac{\sqrt{x}}{y^3 z^4}$
63. 2.52 minutes

Appendix A.2

1. 2.5775
3. 1.5775
5. 3.5775
7. 8.5775 − 10
9. 4.5775
11. 2.7782
13. 3.3032
15. 7.9872 − 10
17. 8.4969 − 10
19. 9.6010 − 10
21. 759
23. .00759
25. 1430
27. .00000447
29. .0000000918
31. 9260
33. 1.27
35. 20
37. 10.1
39. 386
41. 40,200,000
43. 24,800
45. .0000075
47. 258,000,000
49. 42 pounds
51. 3.3×10^4

Appendix A.3

1. 1.4651
3. .6825
5. −1.5439
7. −.6477
9. −.3333
11. 2.0000
13. −.1846
15. .1846
17. 1.6168
19. 2.1132
21. 2/3
23. 18
25. Possible solutions −1 and 3; only 3 checks
27. 3
29. Possible solutions −2 and 4; only 4 checks
31. Possible solutions −1 and 4; only 4 checks
33. 1.3333
35. .7500
37. 1.3917
39. .7186
41. .9650
43. 1.0363
45. 2.6356
47. 4.1629
49. $n = \frac{\log A - \log P}{\log(1 + r)}$

Appendix A.4

1. 2.40
3. 5.30
5. 4.38
7. 1.07
9. 3.98×10^{-4}
11. 3.16×10^{-7}
13. a. 1.62 b. .87 c. .00293 d. 2.86×10^{-6}
15. 5600
17. $8,950
19. $4,120
21. 14.2 years
23. 11.9 years

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