HANDBOOK OF DIFFERENTIAL EQUATIONS

Evolutionary Equations

VOLUME 4

Edited by

C.M. Dafermos

M. Pokorny

N-H

HANDBOOK OF DIFFERENTIAL EQUATIONS

EVOLUTIONARY EQUATIONS

Volume IV

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Preface

The present, fourth volume in the series Evolutionary Equations of the Handbook of Differential Equations develops further the program initiated in the past three volumes, namely to provide a panorama of this amazingly rich field, whose roots and fruits are related to the physical world while its flowers belong to the world of mathematics. With an eye towards retaining the proper balance between basic theory and its applications, we are including here review articles by leading experts on the following topics.

Chapter 1, by D. Chae, deals with equations related to the Euler equations for incompressible fluids, and examines the development of singularities in finite time.

The recent development in the mathematical theory of the compressible Navier–Stokes equations is addressed in Chapter 2 by E. Feireisl.

In Chapter 3, A. Miranville and S. Zelik discuss the large time behavior of solutions of dissipative partial differential equations, in bounded or unbounded domains, and establish, in particular, the existence of global and exponential attractors.

The aim of Chapter 4, by A. Novick-Cohen, is to present recent results in the theory of the Cahn–Hilliard equation as well as related problems.

The problem of existence, regularity and stability of solutions to systems of evolutionary equations governing the flow of viscoelastic fluids is the focus of Chapter 5, by M. Renardy.

The following Chapter 6, by L. Simon, is devoted to the application of the theory of monotone operators to parabolic and functional-parabolic equations or systems thereof.

In Chapter 7, by A. Vasseur, the recent results in hydrodynamic limits, especially those corresponding to hyperbolic scaling, are addressed.

Chapter 8, by A. Visintin, gives a detailed introduction into the modeling of phenomena which can be described by the Stefan-type problems together with analysis of their weak formulation.

Finally, A. Wazwaz's Chapter 9 deals with the Korteweg–deVries equation and some of its modifications and describes various methods for constructing solutions.

We are indebted to the authors, for their valuable contributions, to the referees, for their helpful comments, and to the editors and staff of Elsevier, for their assistance.

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CHAPTER 1

Incompressible Euler Equations: The Blow-up Problem and Related Results

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HANDBOOK OF DIFFERENTIAL EQUATIONS
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Abstract

The question of spontaneous apparition of singularity in the 3D incompressible Euler equations is one of the most important and challenging open problems in mathematical fluid mechanics. In this survey article we review some of recent approaches to the problem. We first review Kato's classical local well-posedness result in the Sobolev space and derive the celebrated Beale–Kato–Majda criterion for finite time blow-up. Then, we discuss recent refinements of the criterion as well as geometric type of theorems on the sufficiency condition for the regularity of solutions. After that we review results excluding some of the scenarios leading to finite time singularities. We also survey studies of various simplified model problems. A dichotomy type of result between the finite time blow-up and the global in time regular dynamics is presented, and a spectral dynamics approach to study local in time behaviors of the enstrophy is also reviewed. Finally, progresses on the problem of optimal regularity for solutions to have conserved quantities are presented.

1. Introduction

The motion of homogeneous incompressible ideal fluid in a domain $\Omega \subset \mathbb{R}^n$ is described by the following system of the Euler equations.

$$\text{(E)} \quad \begin{cases} \dfrac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p, & (x,t) \in \Omega \times (0,\infty), \\ \operatorname{div} v = 0, \quad (x,t) \in \Omega \times (0,\infty), \\ v(x,0) = v_0(x), \quad x \in \Omega, \end{cases}$$

where $v = (v^1, v^2, \dots, v^n)$, $v^j = v^j(x,t)$, $j = 1,2,\dots,n$, is the velocity of the fluid flows, $p = p(x,t)$ is the scalar pressure, and $v_0(x)$ is a given initial velocity field satisfying $\operatorname{div} v_0 = 0$. Here we use the standard notion of vector calculus, denoting

$$\nabla p = \left(\frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}, \dots, \frac{\partial p}{\partial x_n}\right), \quad (v \cdot \nabla)v^j = \sum_{k=1}^n v^k \frac{\partial v^j}{\partial x_k}, \quad \operatorname{div} v = \sum_{k=1}^n \frac{\partial v^k}{\partial x_k}.$$

The first equation of (E) follows from the balance of momentum for each portion of fluid, while the second equation can be derived from the conservation of mass of fluid during its motion, combined with the homogeneity(constant density) assumption on the fluid. The system (E) is first derived by L. Euler in 1755 [77]. Unless otherwise stated, we are concerned on the Cauchy problem of the system (E) on $\Omega = \mathbb{R}^n$, but many of the results presented here are obviously valid also for $\Omega = \mathbb{R}^n/\mathbb{Z}^n$ (periodic domain), and even for the bounded domain with the smooth boundary with the boundary condition $v \cdot \nu = 0$, where ν is the outward unit normal vector. We also suppose $n = 2$ or 3 throughout this paper. *In this article our aim to survey recent results on the mathematical aspects the 3D Euler equations closely related to the problem of spontaneous apparition of singularity starting from a classical solutions having finite energy*. If we add the dissipation term $\mu \Delta v = \mu \sum_{j=1}^n \frac{\partial^2 v}{\partial x_j^2}$, where $\mu > 0$ is the viscosity coefficient, to the right-hand side of the first equation of (E), then we have the Navier–Stokes equations, the regularity/singularity question of which is one of the seven millennium problems in mathematics. In this article we do not treat the Navier–Stokes equations. For details of mathematical studies on the Navier–Stokes equations see e.g. [144,57,112,84,107,117,109]. We also omit other important topics such as existence and uniqueness questions of the weak solutions of the 2D Euler equations, and the related vortex patch problems, vortex sheet problems, and so on. These are well treated in the other papers and monographs [117,37,45,112,133,135,153,154,148, 139] and the references therein. For the survey related the stability question please see for example [79] and references therein. For the results on the regularity of the Euler equations with uniformly rotating external force we refer [2], while for the numerical studies on the blow-up problem of the Euler equations there are many articles including [101,102, 94,7,80,11,89–91,127]. For various mathematical and physical aspects of the Euler equations there are many excellent books, review articles including [1,8,45,47,49,52,79,81,82, 86,115,117,118,29,152]. Obviously, the references are not complete mainly due to author's ignorance.

1.1. *Basic properties*

In the study of the Euler equations the notion of vorticity, $\omega = \operatorname{curl} v$, plays important roles. We can reformulate the Euler system in terms of the vorticity fields only as follows. We first consider the 3D case. Let us first rewrite the first equation of (E) as

$$\frac{\partial v}{\partial t} - v \times \operatorname{curl} v = -\nabla\left(p + \frac{1}{2}|v|^2\right). \tag{1.1}$$

Then, taking curl of (1.1), and using elementary vector identities, we obtain the following vorticity formulation:

$$\frac{\partial \omega}{\partial t} + (v \cdot \nabla)\omega = \omega \cdot \nabla v, \tag{1.2}$$

$$\operatorname{div} v = 0, \qquad \operatorname{curl} v = \omega, \tag{1.3}$$

$$\omega(x, 0) = \omega_0(x). \tag{1.4}$$

The linear elliptic system (1.3) for v can be solved explicitly in terms of ω, assuming ω decays sufficiently fast near spatial infinity, to provides us with the Biot–Savart law,

$$v(x, t) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x - y) \times \omega(y, t)}{|x - y|^3} \, dy. \tag{1.5}$$

Substituting this v into (1.2), we obtain an integro-differential system for ω. The term in the right-hand side of (1.2) is called the vortex stretching term, and is regarded as the main source of difficulties in the mathematical theory of the 3D Euler equations. Let us introduce the deformation matrix $S(x, t) = (S_{ij}(x, t))_{i,j=1}^3$ defined as the symmetric part of the velocity gradient matrix,

$$S_{ij} = \frac{1}{2}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right).$$

From the Biot–Savart law in (1.5) we can explicitly compute

$$S(x, t) = \frac{3}{8\pi} \, p.v. \int_{\mathbb{R}^3} \frac{[(y \times \omega(x + y, t)) \otimes y + y \otimes (y \times \omega(x + y, t))]}{|y|^5} \, dy \tag{1.6}$$

(see e.g. [117] for the details on the computation). The kernel in the convolution integral of (1.6) defines a singular integral operator of the Calderon–Zygmund type (see e.g. [137, 138] for more details). Since the vortex stretching term can be written as $(\omega \cdot \nabla)v = S\omega$, we see that the singular integral operator and related harmonic analysis results could have important roles to study the Euler equations.

In the two-dimensional case we take the vorticity as the scalar, $\omega = \frac{\partial v^2}{\partial x_1} - \frac{\partial v^1}{\partial x_2}$, and the evolution equation of ω becomes

$$\frac{\partial \omega}{\partial t} + (v \cdot \nabla)\omega = 0, \tag{1.7}$$

where the velocity is represented in terms of the vorticity by the 2D Biot–Savart law,

$$v(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(-y_2 + x_2, y_1 - x_1)}{|x - y|^2} \omega(y, t) \, dy. \tag{1.8}$$

Observe that there is no vortex stretching term in (1.7), which makes the proof of global regularity in 2D Euler equations easily accessible. In many studies of the Euler equations it is convenient to introduce the notion of 'particle trajectory mapping', $X(\cdot,t)$ defined by

$$\frac{\partial X(a,t)}{\partial t} = v\bigl(X(a,t),t\bigr), \quad X(a,0)=a, \quad a\in\Omega. \tag{1.9}$$

The mapping $X(\cdot,t)$ transforms from the location of the initial fluid particle to the location at time t, and the parameter a is called the Lagrangian particle marker. If we denote the Jacobian of the transformation, $\det(\nabla_a X(a,t)) = J(a,t)$, then we can show easily (see e.g. [117] for the proof) that

$$\frac{\partial J}{\partial t} = (\operatorname{div} v)J,$$

which implies that the velocity field v satisfies the incompressibility, $\operatorname{div} v = 0$ if and only if the mapping $X(\cdot,t)$ is volume preserving. At this moment we note that, although the Euler equations are originally derived by applying the physical principles of mass conservation and the momentum balance, we could also derive them by applying the least action principle to the action defined by

$$\mathcal{I}(A) = \frac{1}{2}\int_{t_1}^{t_2}\int_\Omega \left|\frac{\partial X(x,t)}{\partial t}\right|^2 \mathrm{d}x\,\mathrm{d}t.$$

Here, $X(\cdot,t):\Omega\to\Omega\subset\mathbb{R}^n$ is a parameterized family of volume preserving diffeomorphism. This variational approach to the Euler equations implies that we can view solutions of the Euler equations as a geodesic curve in the $L^2(\Omega)$ metric on the infinite dimensional manifold of volume preserving diffeomorphisms (see e.g. [1,8,75] and references therein for more details on the geometric approaches to the Euler equations).

The 3D Euler equations have many conserved quantities. We list some important ones below.

(i) Energy,

$$E(t) = \frac{1}{2}\int_\Omega \bigl|v(x,t)\bigr|^2\,\mathrm{d}x.$$

(ii) Helicity,

$$H(t) = \int_\Omega v(x,t)\cdot\omega(x,t)\,\mathrm{d}x.$$

(iii) Circulation,

$$\Gamma_{\mathcal{C}(t)} = \oint_{\mathcal{C}(t)} v\cdot\mathrm{d}l,$$

where $\mathcal{C}(t) = \{X(a,t) \mid a\in\mathcal{C}\}$ is a curve moving along with the fluid.

(iv) Impulse,

$$I(t) = \frac{1}{2}\int_\Omega x\times\omega\,\mathrm{d}x.$$

(v) Moment of Impulse,

$$M(t) = \frac{1}{3}\int_{\Omega} x \times (x \times \omega)\,\mathrm{d}x.$$

The proof of conservations of the above quantities for the classical solutions can be done without difficulty using elementary vector calculus (for details see e.g. [117,118]). The helicity, in particular, represents the degree of knotedness of the vortex lines in the fluid, where the vortex lines are the integral curves of the vorticity fields. In [1] there are detailed discussions on this aspects and other topological implications of the helicity conservation. For the 2D Euler equations there is no analogue of helicity, while the circulation conservation is replaced by the vorticity flux integral,

$$\int_{D(t)} \omega(x,t)\,\mathrm{d}x,$$

where $D(t) = \{X(a,t) \mid a \in D \subset \Omega\}$ is a planar region moving along the fluid in Ω. The impulse and the moment of impulse integrals in the 2E Euler equations are replace by

$$\frac{1}{2}\int_{\Omega}(x_2, -x_1)\omega\,\mathrm{d}x \quad \text{and} \quad -\frac{1}{3}\int_{\Omega}|x|^2\omega\,\mathrm{d}x \quad \text{respectively.}$$

In the 2D Euler equations we have extra conserved quantities; namely for any continuous function f the integral

$$\int_{\Omega} f\big(\omega(x,t)\big)\,\mathrm{d}x$$

is conserved. There are also many known explicit solutions to the Euler equations, for which we just refer [108,117]. In the remained part of this subsection we introduce some notations to be used later for 3D Euler equations. Given velocity $v(x,t)$, and pressure $p(x,t)$, we set the 3×3 matrices,

$$V_{ij} = \frac{\partial v_j}{\partial x_i}, \quad S_{ij} = \frac{V_{ij} + V_{ji}}{2}, \quad A_{ij} = \frac{V_{ij} - V_{ji}}{2}, \quad P_{ij} = \frac{\partial^2 p}{\partial x_i \partial x_j},$$

with $i, j = 1, 2, 3$. We have the decomposition $V = (V_{ij}) = S + A$, where the symmetric part $S = (S_{ij})$ represents the deformation tensor of the fluid introduced above, while the antisymmetric part $A = (A_{ij})$ is related to the vorticity ω by the formula,

$$A_{ij} = \frac{1}{2}\sum_{k=1}^{3} \varepsilon_{ijk}\omega_k, \quad \omega_i = \sum_{j,k=1}^{3} \varepsilon_{ijk}A_{jk}, \tag{1.10}$$

where ε_{ijk} is the skewsymmetric tensor with the normalization $\varepsilon_{123} = 1$. Note that $P = (P_{ij})$ is the Hessian of the pressure. We also frequently use the notation for the vorticity direction field,

$$\xi(x,t) = \frac{\omega(x,t)}{|\omega(x,t)|},$$

defined whenever $\omega(x,t) \neq 0$. Computing partial derivatives $\partial/\partial x_k$ of the first equation of (E), we obtain the matrix equation

$$\frac{DV}{Dt} = -V^2 - P, \qquad \frac{D}{Dt} = \frac{\partial}{\partial t} + (v \cdot \nabla)v. \tag{1.11}$$

Taking symmetric part of this, we obtain

$$\frac{DS}{Dt} = -S^2 - A^2 - P,$$

from which, using the formula (1.10), we have

$$\frac{DS_{ij}}{Dt} = -\sum_{k=1}^{3} S_{ik}S_{kj} + \frac{1}{4}\big(|\omega|^2\delta_{ij} - \omega_i\omega_j\big) - P_{ij}, \tag{1.12}$$

where $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$. The antisymmetric part of (1.11), on the other hand, is

$$\frac{DA}{Dt} = -SA - AS,$$

which, using the formula (1.10) again, we obtain easily

$$\frac{D\omega}{Dt} = S\omega, \tag{1.13}$$

which is the vorticity evolution equation (1.2). Taking dot product (1.13) with ω, we immediately have

$$\frac{D|\omega|}{Dt} = \alpha|\omega|, \tag{1.14}$$

where we set

$$\alpha(x,t) = \begin{cases} \displaystyle\sum_{i,j=1}^{3} \xi_i(x,t)S_{ij}(x,t)\xi_j(x,t) & \text{if } \omega(x,t) \neq 0, \\ 0 & \text{if } \omega(x,t) = 0. \end{cases}$$

1.2. *Preliminaries*

Here we introduce some notations and function spaces to be used in the later sections. Given $p \in [1,\infty]$, the Lebesgue space $L^p(\mathbb{R}^n)$, $p \in [1,\infty]$, is the Banach space defined by the norm

$$\|f\|_{L^p} := \begin{cases} \left(\displaystyle\int_{\mathbb{R}^n} \big|f(x)\big|^p \,\mathrm{d}x\right)^{\frac{1}{p}}, & p \in [1,\infty), \\ \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \big|f(x)\big|, & p = \infty. \end{cases}$$

For $j = 1, \dots, n$ the Riesz transform R_j of f is given by

$$R_j(f)(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} p.v. \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\,\mathrm{d}y$$

whenever the right-hand side makes sense. The Hardy space $\mathcal{H}^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ is defined by

$$f \in \mathcal{H}^1(\mathbb{R}^n) \quad \text{if and only if} \quad \|f\|_{\mathcal{H}^1} := \|f\|_{L^1} + \sum_{j=1}^n \|R_j f\|_{L^1} < \infty.$$

The space $\mathrm{BMO}(\mathbb{R}^n)$ denotes the space of functions of bounded mean oscillations, defined by

$$\begin{aligned} &f \in \mathrm{BMO}(\mathbb{R}^n) \quad \text{if and only if} \\ &\|f\|_{\mathrm{BMO}} := \sup_{Q \subset \mathbb{R}^n} \frac{1}{\mathrm{Vol}(Q)} \int_Q |f - f_Q|\,\mathrm{d}x < \infty, \end{aligned}$$

where $f_Q = \frac{1}{\mathrm{Vol}(Q)} \int_Q f\,\mathrm{d}x$. For more details on the Hardy space and BMO we refer [137,138]. Let us set the multi-index $\alpha := (\alpha_1, \alpha_2, \dots, \alpha_n) \in (\mathbb{Z}_+ \cup \{0\})^n$ with $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Then, $D^\alpha := D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$, where $D_j = \partial/\partial x_j$, $j = 1, 2, \dots, n$. Given $k \in \mathbb{Z}$ and $p \in [1, \infty)$ the Sobolev space, $W^{k,p}(\mathbb{R}^n)$ is the Banach space of functions consisting of functions $f \in L^p(\mathbb{R}^n)$ such that

$$\|f\|_{W^{k,p}} := \left(\int_{\mathbb{R}^n} \left| D^\alpha f(x) \right|^p \mathrm{d}x \right)^{\frac{1}{p}} < \infty,$$

where the derivatives are in the sense of distributions. For $p = \infty$ we replace the $L^p(\mathbb{R}^n)$ norm by the $L^\infty(\mathbb{R}^n)$ norm. In particular, we denote $H^m(\mathbb{R}^n) = W^{m,2}(\mathbb{R}^n)$. In order to handle the functions having fractional derivatives of order $s \in \mathbb{R}$, we use the Bessel potential space $L^s_p(\mathbb{R}^n)$ defined by the Banach spaces norm,

$$\|f\|_{L^{s,p}} := \left\| (1-\Delta)^{\frac{s}{2}} f \right\|_{L^p},$$

where $(1-\Delta)^{s/2} f = \mathcal{F}^{-1}[(1+|\xi|^2)^{s/2} \mathcal{F}(f)(\xi)]$. Here $\mathcal{F}(\cdot)$ and $\mathcal{F}^{-1}(\cdot)$ denoting the Fourier transform and its inverse, defined by

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}x\cdot\xi} f(x)\,\mathrm{d}x,$$

and

$$\mathcal{F}^{-1}(f)(x) = \check{f}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi} f(\xi)\,\mathrm{d}\xi,$$

whenever the integrals make sense. Next we introduce the Besov spaces. We follow [145] (see also [141,109,45,130]). Let $\mathfrak{S}$ be the Schwartz class of rapidly decreasing functions. We consider $\varphi \in \mathfrak{S}$ satisfying $\mathrm{Supp}\,\hat{\varphi} \subset \{\xi \in \mathbb{R}^n \mid \frac{1}{2} \leqslant |\xi| \leqslant 2\}$, and $\hat{\varphi}(\xi) > 0$ if $\frac{1}{2} < |\xi| < 2$. Setting $\hat{\varphi_j} = \hat{\varphi}(2^{-j}\xi)$ (In other words, $\varphi_j(x) = 2^{jn}\varphi(2^j x)$.), we can adjust the normalization constant in front of $\hat{\varphi}$ so that

$$\sum_{j\in\mathbb{Z}} \hat{\varphi}_j(\xi) = 1 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

Let $s \in \mathbb{R}$, $p, q \in [0, \infty]$. Given $f \in \mathfrak{S}'$, we denote $\Delta_j f = \varphi_j * f$. Then the homogeneous Besov seminorm $\|f\|_{\dot{B}^s_{p,q}}$ is defined by

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} \left(\sum_{j\in\mathbb{Z}} 2^{jqs} \|\varphi_j * f\|^q_{L^p}\right)^{\frac{1}{q}} & \text{if } q \in [1, \infty), \\ \sup_{j\in\mathbb{Z}} \left(2^{js} \|\varphi_j * f\|_{L^p}\right) & \text{if } q = \infty. \end{cases}$$

For $(s, p, q) \in [0, \infty) \times [1, \infty] \times [1, \infty]$ the homogeneous Besov space $\dot{B}^s_{p,q}$ is a quasi-normed space with the quasi-norm given by $\|\cdot\|_{\dot{B}^s_{p,q}}$. For $s > 0$ we define the inhomogeneous Besov space norm $\|f\|_{B^s_{p,q}}$ of $f \in \mathfrak{S}'$ as $\|f\|_{B^s_{p,q}} = \|f\|_{L^p} + \|f\|_{\dot{B}^s_{p,q}}$. Similarly, for $(s, p, q) \in [0, \infty) \times [1, \infty) \times [1, \infty]$, the homogeneous Triebel–Lizorkin seminorm $\|f\|_{\dot{F}^s_{p,q}}$ is defined by

$$\|f\|_{\dot{F}^s_{p,q}} = \begin{cases} \left\| \left(\sum_{j\in\mathbb{Z}} 2^{jqs} \left|\varphi_j * f(\cdot)\right|^q\right)^{\frac{1}{q}} \right\|_{L^p} & \text{if } q \in [1, \infty), \\ \left\| \sup_{j\in\mathbb{Z}} \left(2^{js} \left|\varphi_j * f(\cdot)\right|\right) \right\|_{L^p} & \text{if } q = \infty. \end{cases}$$

The homogeneous Triebel–Lizorkin space $\dot{F}^s_{p,q}$ is a quasi-normed space with the quasi-norm given by $\|\cdot\|_{\dot{F}^s_{p,q}}$. For $s > 0$, $(p, q) \in [1, \infty) \times [1, \infty)$ we define the inhomogeneous Triebel–Lizorkin space norm by

$$\|f\|_{F^s_{p,q}} = \|f\|_{L^p} + \|f\|_{\dot{F}^s_{p,q}}.$$

The inhomogeneous Triebel–Lizorkin space is a Banach space equipped with the norm, $\|\cdot\|_{F^s_{p,q}}$. We observe that $B^s_{p,p}(\mathbb{R}^n) = F^s_{p,p}(\mathbb{R}^n)$. The Triebel–Lizorkin space is a generalization of many classical function spaces. Indeed, the followings are well established (see e.g. [145])

$$F^0_{p,2}(\mathbb{R}^n) = \dot{F}^0_{p,2}(\mathbb{R}^n) = L^p(\mathbb{R}^n) \quad (1 < p < \infty),$$
$$\dot{F}^0_{1,2}(\mathbb{R}^n) = \mathcal{H}^1(\mathbb{R}^n) \quad \text{and} \quad \dot{F}^0_{\infty,2} = \mathrm{BMO}(\mathbb{R}^n),$$
$$F^s_{p,2}(\mathbb{R}^n) = L^{s,p}(\mathbb{R}^n).$$

We also note sequence of continuous embeddings for the spaces close to $L^\infty(\mathbb{R}^n)$ [145,95].

$$\dot{B}^{n/p}_{p,1}(\mathbb{R}^n) \hookrightarrow \dot{B}^0_{\infty,1}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n) \hookrightarrow \mathrm{BMO}(\mathbb{R}^n) \hookrightarrow \dot{B}^0_{\infty,\infty}(\mathbb{R}^n). \tag{1.15}$$

Given $0 < s < 1$, $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, we introduce another function spaces $\dot{\mathcal{F}}^s_{p,q}$ defined by the seminorm,

$$\|f\|_{\dot{\mathcal{F}}^s_{p,q}} = \begin{cases} \left\| \left(\int_{\mathbb{R}^n} \frac{|f(x)-f(x-y)|^q}{|y|^{n+sq}} \, \mathrm{d}y \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n,\,\mathrm{d}x)} \\ \quad \text{if } 1 \leqslant p \leqslant \infty,\ 1 \leqslant q < \infty, \\ \left\| \operatorname*{ess\,sup}_{|y|>0} \frac{|f(x)-f(x-y)|}{|y|^s} \right\|_{L^p(\mathbb{R}^n,\,\mathrm{d}x)} \\ \quad \text{if } 1 \leqslant p \leqslant \infty,\ q = \infty. \end{cases}$$

On the other hand, the space $\dot{\mathcal{B}}^s_{p,q}$ is defined by the seminorm,

$$\|f\|_{\dot{\mathcal{B}}^s_{p,q}} = \begin{cases} \left(\int_{\mathbb{R}^n} \frac{\|f(\cdot)-f(\cdot-y)\|^q_{L^p}}{|y|^{n+sq}} \, \mathrm{d}y \right)^{\frac{1}{q}} & \text{if } 1 \leqslant p \leqslant \infty,\ 1 \leqslant q < \infty, \\ \operatorname*{ess\,sup}_{|y|>0} \frac{\|f(\cdot)-f(\cdot-y)\|_{L^p}}{|y|^s} & \text{if } 1 \leqslant p \leqslant \infty,\ q = \infty. \end{cases}$$

Observe that, in particular, $\dot{\mathcal{F}}^s_{\infty,\infty} = \dot{\mathcal{B}}^s_{\infty,\infty} = C^s$, which is the usual Hölder seminormed space for $s \in \mathbb{R}_+\mathbb{Z}$. We also note that if $q = \infty$, $\dot{\mathcal{B}}^s_{p,\infty} = \dot{\mathcal{N}}^s_p$, which is the Nikolskii space.

The inhomogeneous version of those spaces, $\mathcal{F}^s_{p,q}$ and $\mathcal{B}^s_{p,q}$ are defined by their norms,

$$\|f\|_{\mathcal{F}^s_{p,q}} = \|f\|_{L^p} + \|f\|_{\dot{\mathcal{F}}^s_{p,q}}, \qquad \|f\|_{\dot{\mathcal{B}}^s_{p,q}} = \|f\|_{L^p} + \|f\|_{\dot{\mathcal{B}}^s_{p,q}},$$

respectively. We note that for $0 < s < 1$, $2 \leqslant p < \infty$, $q = 2$, $\mathcal{F}^s_{p,2} \cong L^p_s(\mathbb{R}^n)$, introduced above (see [137, p. 163]). If $\frac{n}{\min\{p,q\}} < s < 1$, $n < p < \infty$ and $n < q \leqslant \infty$, then $\mathcal{F}^s_{p,q}$ coincides with the Triebel–Lizorkin space $F^s_{p,q}(\mathbb{R}^n)$ defined above (see [145, p. 101]). On the other hand, for wider range of parameters, $0 < s < 1$, $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, $\mathcal{B}^s_{p,q}$ coincides with the Besov space $B^s_{p,q}(\mathbb{R}^n)$ defined above.

2. Local well-posedness and blow-up criteria

2.1. *Kato's local existence and the BKM criterion*

We review briefly the key elements in the classical local existence proof of solutions in the Sobolev space $H^m(\mathbb{R}^n)$, $m > n/2 + 1$, essentially obtained by Kato in [97] (see also [117]). After that we derive the celebrated Beale, Kato and Majda's criterion on finite time blow-up of the local solution in $H^m(\mathbb{R}^n)$, $m > n/2 + 1$ in [4]. Taking derivatives D^α on the first equation of (E) and then taking L^2 inner product it with $D^\alpha v$, and summing over the multi-indices α with $|\alpha| \leqslant m$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{H^m} &= -\sum_{|\alpha|\leqslant m} \left(D^\alpha(v\cdot\nabla)v - (v\cdot\nabla)D^\alpha v,\, D^\alpha v \right)_{L^2} \\ &\quad - \sum_{|\alpha|\leqslant m} \left((v\cdot\nabla)D^\alpha v,\, D^\alpha v \right)_{L^2} - \sum_{|\alpha|\leqslant m} \left(D^\alpha \nabla p,\, D^\alpha v \right)_{L^2} \\ &= I + II + III. \end{aligned}$$

Integrating by part, we obtain

$$III = \sum_{|\alpha|\leqslant m} \big(D^\alpha p, D^\alpha \operatorname{div} v\big)_{L^2} = 0.$$

Integrating by part again, and using the fact $\operatorname{div} v = 0$, we have

$$II = -\frac{1}{2}\sum_{|\alpha|\leqslant m}\int_{\mathbb{R}^n}(v\cdot\nabla)\big|D^\alpha v\big|^2\,\mathrm{d}x = \frac{1}{2}\sum_{|\alpha|\leqslant m}\int_{\mathbb{R}^n}\operatorname{div} v\big|D^\alpha v\big|^2\,\mathrm{d}x = 0.$$

We now use the so-called *commutator type of estimate* [104],

$$\sum_{|\alpha|\leqslant m}\big\|D^\alpha(fg) - fD^\alpha g\big\|_{L^2} \leqslant C\big(\|\nabla f\|_{L^\infty}\|g\|_{H^{m-1}} + \|f\|_{H^m}\|g\|_{L^\infty}\big),$$

and obtain

$$I \leqslant \sum_{|\alpha|\leqslant m}\big\|D^\alpha(v\cdot\nabla)v - (v\cdot\nabla)D^\alpha v\big\|_{L^2}\|v\|_{H^m} \leqslant C\|\nabla v\|_{L^\infty}\|v\|^2_{H^m}.$$

Summarizing the above estimates, I, II, III, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{H^m} \leqslant C\|\nabla v\|_{L^\infty}\|v\|^2_{H^m}. \tag{2.1}$$

Further estimate, using the *Sobolev inequality*, $\|\nabla v\|_{L^\infty} \leqslant C\|v\|_{H^m}$ for $m > n/2+1$, gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{H^m} \leqslant C\|v\|^3_{H^m}.$$

Thanks to Gronwall's lemma we have the local in time uniform estimate

$$\big\|v(t)\big\|_{H^m} \leqslant \frac{\|v_0\|_{H^m}}{1-Ct\|v_0\|_{H^m}} \leqslant 2\|v_0\|_{H^m} \tag{2.2}$$

for all $t \in [0,T]$, where $T = \frac{1}{2C\|v_0\|_{H^m}}$. Using this estimate we can also deduce the estimate

$$\sup_{0\leqslant t\leqslant T}\left\|\frac{\partial v}{\partial t}\right\|_{H^{m-1}} \leqslant C\big(\|v_0\|_{H^m}\big) \tag{2.3}$$

directly from (E). The estimates (2.2) and (2.3) are the two key a priori estimates for the construction of the local solutions. For actual elaboration of the proof we approximate the Euler system by mollification, Galerkin projection, or iteration of successive linear systems, and construct a sequence of smooth approximate solutions to (E), say $\{v_k(\cdot,t)\}_{k\in\mathbb{N}}$ corresponding to the initial data $\{v_{0,k}\}_{k\in\mathbb{N}}$ respectively with $v_k \to v_0$ in $H^m(\mathbb{R}^n)$. The estimates for the approximate solution sequence provides us with the uniform estimates of $\{v_k\}$ in $L^\infty([0,T];H^m(\mathbb{R}^n)) \cap \mathrm{Lip}([0,T];H^{m-1}(\mathbb{R}^n))$. Then, applying the standard Aubin–Nitche compactness lemma, we can pass to the limit $k\to\infty$ in the equations for the approximate solutions, and can show that the limit $v = v_\infty$ is a solution of the (E) in $L^\infty([0,T]); H^m(\mathbb{R}^n)$. By further argument we can actually show that the limit v belongs to $C([0,T];H^m(\mathbb{R}^n)) \cap AC([0,T];H^{m-1}(\mathbb{R}^n))$, where $AC([0,T];X)$ denotes the space of X valued absolutely continuous functions on $[0, T]$. The general scheme of such existence proof is standard, and is described in detail in [114] in the general type of hyperbolic conservation laws. The approximation of the Euler system by mollification was done for the construction of local solution of the Euler (and the Navier–Stokes) system in [117].

Regarding the question of finite time blow-up of the local classical solution in $H^m(\mathbb{R}^n)$, $m > n/2 + 1$, constructed above, the celebrated Beale–Kato–Majda theorem (called the BKM criterion) states that

$$\limsup_{t \nearrow T_*} \|v(t)\|_{H^s} = \infty \quad \text{if and only if} \quad \int_0^{T_*} \|\omega(s)\|_{L^\infty} \, \mathrm{d}s = \infty. \tag{2.4}$$

We outline the proof of this theorem below (for more details see [4,117]). We first recall the Beale–Kato–Majda's version of the *logarithmic Sobolev inequality*,

$$\|\nabla v\|_{L^\infty} \leqslant C\|\omega\|_{L^\infty}\big(1 + \log\big(1 + \|v\|_{H^m}\big)\big) + C\|\omega\|_{L^2} \tag{2.5}$$

for $m > n/2+1$. Now suppose $\int_0^{T_*} \|\omega(t)\|_{L^\infty} \, \mathrm{d}t := M(T_*) < \infty$. Taking L^2 inner product the first equation of (E) with ω, then after integration by part we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^2}^2 = \big((\omega \cdot \nabla)v, \omega\big)_{L^2} \leqslant \|\omega\|_{L^\infty}\|\nabla v\|_{L^2}\|\omega\|_{L^2} = \|\omega\|_{L^\infty}\|\omega\|_{L^2}^2,$$

where we used the identity $\|\nabla v\|_{L^2} = \|\omega\|_{L^2}$. Applying the Gronwall lemma, we obtain

$$\|\omega(t)\|_{L^2} \leqslant \|\omega_0\|_{L^2} \exp\left(\int_0^{T_*} \|\omega(s)\|_{L^\infty} \, \mathrm{d}s\right) = \|\omega_0\|_{L^2} \exp\big[M(T_*)\big] \tag{2.6}$$

for all $t \in [0, T_*]$. Substituting (2.6) into (2.5), and combining this with (2.1), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{H^m}^2 \leqslant C\big[1 + \|\omega\|_{L^\infty}\big[1 + \log\big(1 + \|v\|_{H^m}\big)\big]\|v\|_{H^m}^2\big].$$

Applying the Gronwall lemma we deduce

$$\|v(t)\|_{H^m} \leqslant \|v_0\|_{H^m} \exp\left[C_1 \exp\left(C_2 \int_0^{T_*} \|\omega(\tau)\|_{L^\infty} \, \mathrm{d}\tau\right)\right] \tag{2.7}$$

for all $t \in [0, T_*]$ and for some constants C_1 and C_2 depending on $M(T_*)$. The inequality (2.7) provides the with the necessity part of (2.4). The sufficiency part is an easy consequence of the Sobolev inequality,

$$\int_0^{T_*} \|\omega(s)\|_{L^\infty} \, \mathrm{d}s \leqslant T_* \sup_{0 \leqslant t \leqslant T_*} \|\nabla v(t)\|_{L^\infty} \leqslant CT_* \sup_{0 \leqslant t \leqslant T_*} \|v(t)\|_{H^m}$$

for $m > n/2 + 1$. There are many other results of local well-posedness in various function spaces (see [14,15,17,20,44,45,96,98,99,111,142,143,147,148,153]). For the local existence proved in terms of a geometric formulation see [75]. For the BKM criterion for solutions in the Hölder space see [3]. Immediately after the BKM result appeared, Ponce derive similar criterion in terms of the deformation tensor [128]. Recently, Constantin proved local well-posedness and a blow-up criterion in terms of the active vector formulation [51].

2.2. *Refinements of the BKM criterion*

The first refinement of the BKM criterion was done by Kozono and Taniuchi in [105], where they proved

THEOREM 2.1. *Let $s > n/p + 1$. A solution v of the Euler equations belonging to $C([0, T_*); W^{s,p}(\mathbb{R}^n)) \cap C^1([0, T_*); W^{s-2,p}(\mathbb{R}^n))$ blows up at T_* in $W^{s,p}(\mathbb{R}^n)$, namely*

$$\limsup_{t \nearrow T_*} \|v(t)\|_{W^{s,p}} = \infty \quad \textit{if and only if} \quad \int_0^{T_*} \|\omega\|_{\mathrm{BMO}} = \infty.$$

The proof is based on the following version of the logarithmic Sobolev inequality for $f \in W^{s,p}(\mathbb{R}^n)$, $s > n/p$, $1 < p < \infty$,

$$\|f\|_{L^\infty} \leqslant C\big(1 + \|f\|_{\mathrm{BMO}}\big(1 + \log^+ \|f\|_{W^{s,p}}\big)\big)$$

(see [105] for details of the proof). We recall now the embedding relations (1.15). Further refinement of the above theorem is the following (see [14,20]).

THEOREM 2.2.

(i) (super-critical case) *Let $s > n/p + 1$, $p \in (1, \infty)$, $q \in [1, \infty]$. Then, the local in time solution $v \in C([0, T_*); B^s_{p,q}(\mathbb{R}^n))$ blows up at T_* in $B^s_{p,q}(\mathbb{R}^n)$, namely*

$$\limsup_{t \nearrow T_*} \|v(t)\|_{B^s_{p,q}} = \infty \quad \textit{if and only if} \quad \int_0^{T_*} \|\omega(t)\|_{\dot{B}^0_{\infty,\infty}} \,\mathrm{d}t = \infty.$$

(ii) (critical case) *Let $p \in (1, \infty)$. Then, the local in time solution $v \in C([0, T_*); B^{n/p+1}_{p,1}(\mathbb{R}^n))$ blows up at T_* in $B^{n/p+1}_{p,1}(\mathbb{R}^n)$, namely*

$$\limsup_{t \nearrow T_*} \|v(t)\|_{B^{n/p+1}_{p,1}} = \infty \quad \textit{if and only if} \quad \int_0^{T_*} \|\omega(t)\|_{\dot{B}^0_{\infty,1}} \,\mathrm{d}t = \infty.$$

The proof of (i) is based on the following version of the logarithmic Sobolev inequality for $f \in B^s_{p,q}(\mathbb{R}^n)$ with $s > n/p$ with $p \in (1, \infty)$, $q \in [1, \infty]$.

$$\|f\|_{L^\infty} \leqslant C\big(1 + \|f\|_{\dot{B}^0_{\infty,\infty}}\big(\log^+ \|f\|_{B^s_{p,q}} + 1\big)\big).$$

In [106] Kozono, Ogawa and Taniuchi obtained similar results to (i) above independently.

In all of the above criteria, including the BKM theorem, we need to control all of the three components of the vorticity vector to obtain regularity. The following theorem proved in [22] states that actually we only need to control two components of the vorticity in the slightly stronger norm than the L^∞ norm (recall again the embedding (1.15)).

THEOREM 2.3. *Let $m > 5/2$. Suppose $v \in C([0, T_*); H^m(\mathbb{R}^3))$ is the local classical solution of* (E) *for some $T_1 > 0$, corresponding to the initial data $v_0 \in H^m(\mathbb{R}^3)$, and $\omega = \operatorname{curl} v$ is its vorticity. We decompose $\omega = \tilde{\omega} + \omega^3 e_3$, where $\tilde{\omega} = \omega^1 e_1 + \omega^2 e_2$, and*

$\{e_1, e_2, e_3\}$ *is the canonical basis of* $\mathbb{R}^3$. *Then,*

$$\limsup_{t \nearrow T_*} \|v(t)\|_{H^m} = \infty \quad \textit{if and only if} \quad \int_0^{T_*} \|\tilde{\omega}(t)\|^2_{\dot{B}^0_{\infty,1}} \, \mathrm{d}t = \infty.$$

Note that $\tilde{\omega}$ could be the projected component of ω onto any plane in $\mathbb{R}^3$. For the solution $v = (v^1, v^2, 0)$ of the Euler equations on the $x_1 - x_2$ plane, the vorticity is $\omega = \omega^3 e_3$ with $\omega_3 = \partial_{x_1} v^2 - \partial_{x_2} v^1$, and $\tilde{\omega} \equiv 0$. Hence, as a trivial application of the above theorem we reproduce the well-known global in time regularity for the 2D Euler equations.

Next we present recent results on the blow up criterion in terms of Hessian of the pressure. As in the introduction we use $P = (P_{ij})$, $S = (S_{ij})$ and ξ to denote the Hessian of the pressure, the deformation tensor and the vorticity direction field respectively, introduced in Section 1. We also introduce the notations

$$\frac{S\xi}{|S\xi|} = \zeta, \quad \zeta \cdot P\xi = \mu.$$

The following is proved in [30].

THEOREM 2.4. *If the solution* $v(x,t)$ *of the* 3D *Euler system with* $v_0 \in H^m(\mathbb{R}^3)$, $m > \frac{5}{2}$, *blows up at* T_*, *namely* $\limsup_{t \nearrow T_*} \|v(t)\|_{H^m} = \infty$, *then necessarily,*

$$\int_0^{T_*} \exp\left(\int_0^{\tau} \|\mu(s)\|_{L^\infty} \, \mathrm{d}s \right) \mathrm{d}\tau = \infty.$$

Similar criterion in terms of the Hessian of pressure, but with different detailed geometric configuration from the above theorem is obtained by Gibbon, Holm, Kerr and Roulstone in [87]. Below we denote $\xi_p = \xi \times P\xi$.

THEOREM 2.5. *Let* $m \geqslant 3$ *and* $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ *be a periodic box. Then, there exists a global solution of the Euler equations* $v \in C([0,\infty); H^m(\mathbb{T}^3)) \cap C^1([0,\infty); H^{m-1}(\mathbb{T}^3))$ *if*

$$\int_0^T \|\xi_p(t)\|_{L^\infty} \, \mathrm{d}t < \infty, \quad \forall t \in (0, T)$$

excepting the case where ξ *becomes collinear with the eigenvalues of* P *at* T.

Next, we consider the axisymmetric solution of the Euler equations, which means velocity field $v(r, x_3, t)$, solving the Euler equations, and having the representation

$$v(r, x_3, t) = v^r(r, x_3, t) e_r + v^\theta(r, x_3, t) e_\theta + v^3(r, x_3, t) e_3$$

in the cylindrical coordinate system, where

$$e_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0 \right), \quad e_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0 \right), \quad e_3 = (0,0,1), \quad r = \sqrt{x_1^2 + x_2^2}.$$

In this case also the question of finite time blow-up of solution is wide open (see e.g. [89, 90, 11] for studies in such case). The vorticity $\omega = \operatorname{curl} v$ is computed as

$$\omega = \omega^r e_r + \omega^\theta e_\theta + \omega^3 e_3,$$

where

$$\omega^r = -\partial_{x_3} v^\theta, \quad \omega^\theta = \partial_{x_3} v^r - \partial_r v^3, \quad \omega^3 = \frac{1}{r}\partial_r\big(r v^\theta\big).$$

We denote

$$\tilde{v} = v^r e_r + v^3 e_3, \qquad \tilde{\omega} = \omega^r e_r + \omega^3 e_3.$$

Hence, $\omega = \tilde{\omega} + \vec{\omega}_\theta$, where $\vec{\omega}_\theta = \omega^\theta e_\theta$. The Euler equations for the axisymmetric solution are

$$\begin{cases} \dfrac{\partial v^r}{\partial t} + (\tilde{v} \cdot \tilde{\nabla}) v^r = -\dfrac{\partial p}{\partial r}, \\ \dfrac{\partial v^\theta}{\partial t} + (\tilde{v} \cdot \tilde{\nabla}) v^\theta = -\dfrac{v^r v^\theta}{r}, \\ \dfrac{\partial v^3}{\partial t} + (\tilde{v} \cdot \tilde{\nabla}) v^3 = -\dfrac{\partial p}{\partial x_3}, \\ \operatorname{div} \tilde{v} = 0, \\ v(r, x_3, 0) = v_0(r, x_3), \end{cases}$$

where $\tilde{\nabla} = e_r \frac{\partial}{\partial r} + e_3 \frac{\partial}{\partial x_3}$. In the axisymmetric Euler equations the vorticity formulation becomes

$$\begin{cases} \dfrac{\partial \omega^r}{\partial t} + (\tilde{v} \cdot \tilde{\nabla}) = \omega^r (\tilde{\omega} \cdot \tilde{\nabla}) v^r, \\ \dfrac{\partial \omega^3}{\partial t} + (\tilde{v} \cdot \tilde{\nabla}) = \omega^3 (\tilde{\omega} \cdot \tilde{\nabla}) v^3, \\ \left[\dfrac{\partial}{\partial t} + \tilde{v} \cdot \tilde{\nabla}\right]\left(\dfrac{\omega^\theta}{r}\right) = (\tilde{\omega} \cdot \tilde{\nabla})\left(\dfrac{v^\theta}{r}\right), \\ \operatorname{div} \tilde{v} = 0, \quad \operatorname{curl} \tilde{v} = \vec{\omega}^\theta. \end{cases}$$

In the case of axisymmetry we only need to control just one component of vorticity (the angular component) to get the regularity of solution. The following theorem is proved in [40].

THEOREM 2.6. *Let $v \in C([0, T_*); H^m(\mathbb{R}^3))$, $m > 5/2$, be the local classical axisymmetric solution of* (E), *corresponding to an axisymmetric initial data $v_0 \in H^m(\mathbb{R}^3)$. Then, the solution blows up in $H^m(\mathbb{R}^3)$ at T_* if and only if for all $(\gamma, p) \in (0, 1) \times [1, \infty]$ we have*

$$\begin{aligned} &\int_0^{T_*} \|\omega_\theta(t)\|_{L^\infty} \,\mathrm{d}t \\ &\quad + \int_0^{T_*} \exp\left[\int_0^t \left\{\|\omega_\theta(s)\|_{L^\infty}\big(1 + \log^+\big(\|\omega_\theta(s)\|_{C^\gamma}\|\omega_\theta(s)\|_{L^p}\big)\big)\right.\right. \\ &\quad \left.\left. + \|\omega_\theta(s)\log^+ r\|_{L^\infty}\right\} \mathrm{d}s\right] \mathrm{d}t = \infty. \end{aligned} \tag{2.8}$$

We observe that although we need to control only ω_θ to get the regularity, the its norm, which is in C^γ, is higher than the L^∞ norm used in the BKM criterion. If we use the 'critical' Besov space $\dot{B}^0_{\infty,1}(\mathbb{R}^3)$ we can derive slightly sharper criterion than Theorem 2.6 as follows (see [22] for the proof).

THEOREM 2.7. *Let $v \in C([0,T_*); H^m(\mathbb{R}^3))$ be the local classical axisymmetric solution of* (E), *corresponding to an axisymmetric initial data $v_0 \in H^m(\mathbb{R}^3)$. Then,*

$$\limsup_{t\nearrow T_*} \|v(t)\|_{H^m} = \infty \quad \textit{if and only if} \quad \int_0^{T_*} \|\vec{\omega}_\theta(t)\|_{\dot{B}^0_{\infty,1}}\, \mathrm{d}t = \infty. \tag{2.9}$$

We observe that contrary to (2.8) we do not need to control the high regularity norm, the C^γ norm of vorticity in (2.9). We can also have the regularity of the axisymmetric Euler equation by controlling only one component of the velocity, the swirl velocity v^θ as in the follows proved in [38].

THEOREM 2.8. *Let $v \in C([0,T_*); H^m(\mathbb{R}^3))$, $m > 5/2$, be the local classical axisymmetric solution of* (E), *corresponding to an axisymmetric initial data $v_0 \in H^m(\mathbb{R}^3)$. Then, the solution blows up in $H^m(\mathbb{R}^3)$ at T_* if and only if*

$$\int_0^{T_*} \left(\|\tilde{\nabla} v^\theta\|_{L^\infty} + \left\| \frac{\partial v^\theta}{\partial r} \right\|_{L^\infty} \left\| \frac{1}{r} \frac{\partial v^\theta}{\partial x_3} \right\|_{L^\infty} \right) \mathrm{d}t = \infty.$$

2.3. *Constantin–Fefferman–Majda's and other related results*

In order to study the regularity problem of the 3D Navier–Stokes equations Constantin and Fefferman investigated the geometric structure of the integral kernel in the vortex stretching term more carefully, and discovered the phenomena of 'depletion effect' hidden in the integration ([55], see also [48] for detailed exposition related to this fact). Later similar geometric structure of the vortex stretching term was studied extensively also in the blow-up problem of the 3D Euler equations by Constantin, Fefferman and Majda [56]. Here we first present their results in detail, and results in [25], where the BKM criterion and the Constantin–Fefferman–Majda's criterion are interpolated in some sense. Besides those results presented in this subsection we also mention that there are other interesting geometric approaches to the Euler equations such as the quaternion formulation by Gibbon and his collaborators [83,85–87]. We begin with a definition in [56]. Given a set $W \in \mathbb{R}^3$ and $r > 0$ we use the notation $B_r(W) = \{y \in B_r(x);\ x \in W\}$.

DEFINITION 2.1. A set $W_0 \subset \mathbb{R}^3$ is called smoothly directed if there exists $\rho > 0$ and r, $0 < r \leqslant \rho/2$ such that the following three conditions are satisfied.

(i) For every $a \in W_0^* = \{q \in W_0;\ |\omega_0(q)| \neq 0\}$, and all $t \in [0,T)$, the vorticity direction field $\xi(\cdot,t)$ has a Lipschitz extension (denoted by the same letter) to the Euclidean ball of radius 4ρ centered at $X(a,t)$ and

$$M = \lim_{t\to T} \sup_{a\in W_0^*} \int_0^t \|\nabla\xi(\cdot,t)\|_{L^\infty(B_{4\rho}(X(a,t)))}\, \mathrm{d}t < \infty.$$

(ii) The inequality

$$\sup_{B_{3r}(W_t)} \left|\omega(x,t)\right| \leqslant m \sup_{B_r(W_t)} \left|\omega(x,t)\right|$$

holds for all $t \in [0, T)$ with $m \geqslant 0$ constant.

(iii) The inequality

$$\sup_{B_{4\rho}(W_t)} \left|v(x,t)\right| \leqslant U$$

holds for all $t \in [0, T)$.

The assumption (i) means that the direction of vorticity is well behaved in a neighborhood of a bunch of trajectories. The assumption (ii) states that this neighborhood is large enough to capture the local intensification of ω. Under these assumptions the following theorem is proved in [56].

THEOREM 2.9. *Assume W_0 is smoothly directed. Then there exists $\tau > 0$ and Γ such that*

$$\sup_{B_r(W_t)} \left|\omega(x,t)\right| \leqslant \Gamma \sup_{B_\rho(W_{t_0})} \left|\omega(x,t_0)\right|$$

holds for any $0 \leqslant t_0 < T$ and $0 \leqslant t - t_0 \leqslant \tau$.

They also introduced the notion of regularly directed set, closely related to the geometric structure of the kernel defining vortex stretching term.

DEFINITION 2.2. We sat that a set W_0 is regularly directed if there exists $\rho > 0$ such that

$$\sup_{aW_0^*} \int_0^T K_\rho\big(X(a,t)\big)\,\mathrm{d}t < \infty,$$

where

$$K_\rho(x) = \int_{|y|\leqslant\rho} \left|D\big(\hat{y}, \xi(x+y), \xi(x)\big)\right|\left|\omega(x+y)\right| \frac{\mathrm{d}y}{|y|^3}$$

and

$$D\big(\hat{y}, \xi(x+y), \xi(x)\big) = \big(\hat{y}\cdot\xi(x)\big)\,\mathrm{Det}\big(\hat{y}, \xi(x+y), \xi(x)\big).$$

Under the above assumption on the regularly directed sets the following is proved also in [56].

THEOREM 2.10. *Assume W_0 is regularly directed. Then there exists a constant Γ such that*

$$\sup_{a\in W_0} \left|\omega\big(X(a,t),t\big)\right| \leqslant \Gamma \sup_{a\in W_0} \left|\omega_0(a)\right|$$

holds for all $t \in [0, T]$.

The original studies by Constantin and Fefferman in [55] about the Navier–Stokes equations, which motivated the above theorems, are concerned mainly about the regularity of solutions in terms of the vorticity direction fields ξ. We recall, on the other hand, that the BKM type of criterion controls the magnitude of vorticity to obtain regularity. Incorporation of both the direction and the magnitude of vorticity to obtain regularity for the 3D Navier–Stokes equations was first initiated by Beirão da Veiga and Berselli in [6], and developed further by Beirão da Veiga in [5], and finally refined in an 'optimal' form in [35] (see also [39] for a localized version). We now present the Euler equation version of the result in [35].

Below we use the notion of particle trajectory $X(a,t)$, which is defined by the classical solution $v(x,t)$ of (E). Let us denote

$$\Omega_0 = \left\{x \in \mathbb{R}^3 \mid \omega_0(x) \neq 0\right\}, \qquad \Omega_t = X(\Omega_0, t).$$

We note that the direction field of the vorticity, $\xi(x,t) = \omega(x,t)/|\omega(x,t)|$, is well-defined if $x \in \Omega_t$ for $v_0 \in C^1(\mathbb{R}^3)$ with $\Omega_0 \neq \emptyset$. The following is the main theorem proved in [25].

THEOREM 2.11. *Let $v(x,t)$ be the local classical solution to* (E) *with initial data $v_0 \in H^m(\mathbb{R}^3)$, $m > 5/2$, and $\omega(x,t) = \operatorname{curl} v(x,t)$. We assume $\Omega_0 \neq \emptyset$. Then, the solution can be continued up to $T + \delta$ as the classical solution, namely $v(t) \in C([0, T+\delta]; H^m(\mathbb{R}^3))$ for some $\delta > 0$, if there exists $p, p', q, q', s, r_1, r_2, r_3$ satisfying the following conditions,*

$$\frac{1}{p} + \frac{1}{p'} = 1, \qquad \frac{1}{q} + \frac{1}{q'} = 1, \tag{2.10}$$

and

$$\frac{1}{r_1} + \frac{p'}{r_2}\left(1 - \frac{sq'}{3}\right) + \frac{1}{r_3}\left\{1 - p'\left(1 - \frac{sq'}{3}\right)\right\} = 1 \tag{2.11}$$

with

$$0 < s < 1, \quad 1 \leqslant \frac{3}{sq'} < p \leqslant \infty, \quad 1 \leqslant q \leqslant \infty, \tag{2.12}$$

and

$$\begin{aligned} &r_1 \in [1, \infty], \quad r_2 \in \left[p'\left(1 - \frac{sq'}{3}\right), \infty\right], \\ &r_3 \in \left[1 - p'\left(1 - \frac{sq'}{3}\right), \infty\right] \end{aligned} \tag{2.13}$$

such that for direction field $\xi(x,t)$, and the magnitude of vorticity $|\omega(x,t)|$ the followings hold

$$\int_0^T \left\|\xi(t)\right\|^{r_1}_{\dot{\mathcal{F}}^s_{\infty,q}(\Omega_t)} \mathrm{d}t < \infty, \tag{2.14}$$

and

$$\int_0^T \left\|\omega(t)\right\|^{r_2}_{L^{pq'}(\Omega_t)} \mathrm{d}t + \int_0^T \left\|\omega(t)\right\|^{r_3}_{L^{q'}(\Omega_t)} \mathrm{d}t < \infty. \tag{2.15}$$

In order to get insight implied by the above theorem let us consider the special case of $p=\infty$, $q=1$. In this case the conditions (2.14)–(2.15) are satisfied if

$$\xi(x,t)\in L^{r_1}\big(0,T;C^s\big(\mathbb{R}^3\big)\big), \tag{2.16}$$

$$\omega(x,t)\in L^{r_2}\big(0,T;L^\infty\big(\mathbb{R}^3\big)\big)\cap L^{r_3}\big(0,T;L^\infty\big(\mathbb{R}^3\big)\big), \tag{2.17}$$

with

$$\frac{1}{r_1}+\frac{1}{r_2}\left(1-\frac{s}{3}\right)+\frac{s}{3r_3}=1. \tag{2.18}$$

Let us *formally* pass $s\to 0$ in (2.16) and (2.18), and choose $r_1=\infty$ and $r_2=r_3=1$, then we find that the conditions (2.16)–(2.17) reduce to the BKM condition, since the condition $\xi(x,t)\in L^\infty(0,T;C^0(\mathbb{R}^3))\cong L^\infty((0,T)\times\mathbb{R}^3)$ is obviously satisfied due to the fact that $|\xi(x,t)|\equiv 1$.

The other case of interest is $q'=3/s$, where (2.14)–(2.15) are satisfied if

$$\xi(x,t)\in L^{r_1}\big(0,T;\dot{\mathcal{F}}^s_{\infty,\frac{3}{3-s}}\big(\mathbb{R}^3\big)\big),\qquad \big|\omega(x,t)\big|\in L^{r_2}\big(0,T;L^{3/s}\big(\mathbb{R}^3\big)\big). \tag{2.19}$$

with $1/r_1+1/r_2=1$. The condition (2.19) shows explicitly the mutual compensation between the regularity of the direction field and the integrability of the vorticity magnitude in order to control regularity/singularity of solutions of the Euler equations.

Next we review the result of non-blow-up conditions due to Deng, Hou and Yu [71,72]. We consider a time t and a vortex line segment L_t such that the maximum of vorticity over the whole domain is comparable to the maximum of vorticity on over L_t, namely

$$\Omega(t):=\sup_{x\in\mathbb{R}^3}\big|\omega(x,t)\big|\sim\max_{x\in L_t}\big|\omega(x,t)\big|.$$

We denote $L(t):=$ arc length of L_t; ξ, $\mathbf{n}$ and κ are the unit tangential and the unit normal vectors to L_t and the curvature of L_t respectively. We also use the notations,

$$U_\xi(t):=\max_{x,y\in L_t}\big|(v\cdot\xi)(x,t)-(v\cdot\xi)(y,t)\big|,$$

$$U_n(t):=\max_{x\in L_t}\big|(v\cdot\mathbf{n})(x,t)\big|,$$

$$M(t):=\max_{x\in L_t}\big|(\nabla\cdot\xi)(x,t)\big|,$$

$$K(t):=\max_{x\in L_t}\kappa(x,t).$$

We denote by $X(A,s,t)$ the image by the trajectory map at time $t>s$ of fluid particles at A at time s. Then, the following is proved in [72].

THEOREM 2.12. *Assume that there is a family of vortex line segment L_t and $T_0\in[0,T^*)$, such that $X(L_{t_1},t_1,t_2)\supseteq L_{t_2}$ for all $T_0<t_1<t_2<T^*$. Also assume that $\Omega(t)$ is monotonically increasing and $\max_{x\in L_t}|\omega(x,t)|\geqslant c_0\Omega(t)$ for some c_0 when t is sufficiently close to T^*. Furthermore, we assume there are constants C_U, C_0, c_L such that*

1. $[U\xi(t)+U_n(t)K(t)L(t)]\leqslant C_U(T^*-t)^{-A}$ *for some constant* $A\in(0,1)$,
2. $M(t)L(t),\ K(t)L(t)\leqslant C_0$,

3. $L(t) \geqslant c_L(T^* - t)^B$ *for some constant* $B \in (0,1)$.
Then there will be no blow-up in the 3D *incompressible Euler flow up to time* T^*, *as long as* $B < 1 - A$.

In the endpoint case of $B = 1 - A$ they deduced the following theorem [71].

THEOREM 2.13. *Under the same assumption as in Theorem* 2.10, *there will be no blow-up in the Euler system up to time* T^* *in the case* $B = 1 - A$, *as long as the following condition is satisfied*:

$$R^3 K < y_1\big(R^{A-1}(1-A)^{1-A}/(2-A)^{2-A}\big),$$

where $R = \mathrm{e}^{C_0}/c_0$, $K := C_U c_0/(c_L(1-A))$, *and* $y_1(m)$ *denotes the smallest positive* y *such that* $m = y/(1+y)^{2-A}$.

We refer [71,72] for discussions on the various connections of Theorems 2.10 and 2.11 with numerical computations.

3. Blow-up scenarios

3.1. *Vortex tube collapse*

We recall that a vortex line is an integral curve of the vorticity, and a vortex tube is a tubular neighborhood in $\mathbb{R}^3$ foliated by vortex lines. Numerical simulations (see e.g. [46]) show that vortex tubes grow and thinner (stretching), and folds before singularity happens. We review here the result by Córdoba and Fefferman [66] excluding a type of vortex tube collapse.

Let $Q = I_1 \times I_2 \times I_3 \subset \mathbb{R}^3$ be a closed rectangular box, and let $T > 0$ be given. A regular tube is a relatively open set $\Omega_t \subset Q$ parameterized by time $t \in [0,T)$, having the form $\Omega_t = \{(x_1,x_2,x_3) \in Q\colon \theta(x_1,x_2,x_3,t) < 0\}$ with $\theta \in C^1(Q \times [0,T))$, and satisfying the following properties:

$$|\nabla_{x_1,x_2}\theta| \neq 0 \quad \text{for } (x_1,x_2,x_3,t) \in Q \times [0,T), \theta(x_1,x_2,x_3,t) = 0;$$
$$\Omega_t(x_3) := \big\{(x_1,x_2) \in I_1 \times I_2\colon\ (x_1,x_2,x_3) \in \Omega_t\big\} \text{ is non-empty},$$

for all $x_3 \in I_3$, $t \in [0,T)$;

$$\operatorname{closure}\big(\Omega_t(x_3)\big) \subset \operatorname{interior}(I_1 \times I_2)$$

for all $x_3 \in I_3$, $t \in [0,T)$.

Let $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ be a C^1 velocity field defined on $Q \times [0,T)$. We say that the regular tube Ω_t moves with the velocity field u, if we have

$$\left(\frac{\partial}{\partial t} + u \cdot \nabla_x\right)\theta = 0 \quad \text{whenever } (x,t) \in Q \times [0,T),\ \theta(x,t) = 0.$$

By the Helmholtz theorem we know that a vortex tube arising from a 3D Euler solution moves with the fluid velocity. The following theorem proved by Córdoba and Fefferman [66] says for the 3D Euler equations that a vortex tube cannot reach zero thickness in finite time, unless it bends and twists so violently that no part of it forms a regular tube.

THEOREM 3.1. *Let $\Omega_t \subset Q(t \in [0, T))$ be a regular tube that moves with C^1, divergence free velocity field $u(x, t)$.*

$$\text{If } \int_0^T \sup_{x \in Q} \left|u(x,t)\right| \mathrm{d}t < \infty, \quad \text{then } \lim \inf_{t \to T_-} \operatorname{Vol}(\Omega_t) > 0.$$

3.2. *Squirt singularity*

The theorem of excluding the regular vortex tube collapse was generalized by Córdoba, Fefferman and de la Lave [70], which we review here. We first recall their definition of squirt singularities. Let $\Omega \subset \mathbb{R}^n$ be an open set. We denote $X_t(a) = X(a, t)$, which is a particle trajectory generated by a C^1 vector field $u : \Omega \times [0, T) \to \mathbb{R}^n$ such that $\operatorname{div} u = 0$. We also set $X_{t,s}(a)$ as the position at time t of the trajectory which at time $t = s$ is a. We have obvious relations,

$$X_t(a) = X_{t,0}(a), \quad X_{t,s} = X_t \circ X_s^{-1}, \quad X_{t,s} \circ X_{s,s_1} = X_{t,s_1}.$$

For $\mathcal{S} \subset \Omega$, we denote by

$$X_{t,s}^{\Omega} \mathcal{S} = \left\{x \in \Omega \mid x = X_t(a), a \in \mathcal{S}, X_s(a) \in \Omega, 0 \leqslant s \leqslant t\right\}.$$

In other words, $X_{t,s}^{\Omega} \mathcal{S}$ is the evolution of the set $\mathcal{S}$, starting at time a, after we eliminate the trajectories which step out of Ω at some time. By the incompressibility condition on u, we have that $\operatorname{Vol}(X_{t,s}\mathcal{S})$ is independent of t, and the function $t \mapsto \operatorname{Vol}(X_{t,s}\mathcal{S})$ is non-increasing.

DEFINITION 3.1. Let Ω_-, Ω_+ be open and bounded sets. $\overline{\Omega_-} \subset \Omega_+$. Therefore,

$$\operatorname{dist}\left(\Omega_-, \mathbb{R}^n - \Omega_+\right) \geqslant r > 0.$$

We say that u experiences a squirt singularity in Ω_-, at time $T > 0$, when for every $0 \leqslant s < T$, we can find a set $\mathcal{S}_s \subset \Omega_+$ such that

(i) $\mathcal{S}_s \cap \Omega_-$ has positive measure, $0 \leqslant s < T$,
(ii) $\lim_{t \to T} \operatorname{Vol}(X_{t,s}^{\Omega^+} \mathcal{S}_s) = 0$.

The physical intuition behind the above definition is that there is a region of positive volume so that all the fluid occupying it gets ejected from a slightly bigger region in finite time. Besides the vortex tube collapse singularity introduced in the previous subsection the potato chip singularity and the saddle collapse singularity, which will be defined below, are also special examples of the squirt singularity, connected with real fluid mechanics phenomena.

DEFINITION 3.2 (potato chip singularity). We say that u experiences a potato chip singularity when we can find continuous functions

$$f_\pm : \mathbb{R}^{n-1} \times [0, T) \to \mathbb{R}$$

such that

$$\begin{aligned}
&f_+(x_1, \ldots, x_{n-1}, t) \geqslant f_-(x_1, \ldots, x_{n-1}, t),\\
&\quad t \in [0, T], x_1, \ldots, x_{n-1} \in B_{2r}(\Pi x^0),\\
&f_+(x_1, \ldots, x_{n-1}, 0) \geqslant f_-(x_1, \ldots, x_{n-1}, 0), \quad x_1, \ldots, x_{n-1} \in B_r(\Pi x^0),\\
&\lim_{t\to T_-} \big[f_+(x_1, \ldots, x_{n-1}, t) - f_-(x_1, \ldots, x_{n-1}, t)\big] = 0\\
&\quad \forall x_1, \ldots, x_{n-1} \in B_{2r}(\Pi x^0)
\end{aligned}$$

and such that the surfaces

$$\Sigma_{\pm,t} = \big\{x_n = f_\pm(x_1, \ldots, x_{n-1}, t)\big\} \subset \Omega$$

are transformed into each other by the flow

$$X(\Sigma_{\pm,0}, t) \supset \Sigma_{\pm,t}.$$

In the above Π is projection on the first $n-1$ coordinates.

Previously to [70] potato chip singularities were considered in the 2D and 3D flows by Córdoba and Fefferman [69,67] respectively in the name of 'sharp front'. In particular the exclusion of sharp front in the 2D quasi-geostrophic equation is proved in [69]. The following notion of saddle collapse singularity is relevant only for 2D flows.

DEFINITION 3.3 (saddle collapse singularity). We consider foliation of a neighborhood of the origin (with coordinates x_1, x_2) whose leaves are given by equations of the form

$$\rho := \big(y_1\beta(t) + y_2\big) \cdot \big(y_1\delta(t) + y_2\big) = const$$

and $(y_1, y_2) = F_t(x_1, x_2)$, where $\beta, \delta : [0, T) \to \mathbb{R}^+$ are C^1 foliations, F is a C^2 function of x, t, for a fixed t, and F_t is an orientation preserving diffeomorphism. We say that the foliation experiences a saddle collapse when

$$\lim\inf_{t\to T} \beta(t) + \delta(t) = 0.$$

If the leaves of the foliation are transported by a vector field u, we say that the vector field u experiences a saddle collapse.

The exclusion of saddle point singularity in the 2D quasi-geostrophic equation (see Section 4.3 below) was proved by Córdoba in [65]. The following 'unified' theorem is proved in [70].

THEOREM 3.2. *If u has a squirt singularity at T, then $\int_s^T \sup_x |u(x,t)|\,\mathrm{d}t = \infty$ for all $s \in (0,T)$. Moreover, if u has a potato chip singularity, then*

$$\int_s^T \sup_x \left|\Pi u(x,t)\right| \mathrm{d}t = \infty.$$

3.3. *Self-similar blow-up*

In this subsection we review the scenario of self-similar singularity studied in [32] (see also [50] for a related study). We first observe that the Euler system (E) has scaling property that if (v, p) is a solution of the system (E), then for any $\lambda > 0$ and $\alpha \in \mathbb{R}$ the functions

$$v^{\lambda,\alpha}(x,t) = \lambda^{\alpha} v(\lambda x, \lambda^{\alpha+1} t), \qquad p^{\lambda,\alpha}(x,t) = \lambda^{2\alpha} p(\lambda x, \lambda^{\alpha+1} t) \tag{3.1}$$

are also solutions of (E) with the initial data $v_0^{\lambda,\alpha}(x) = \lambda^{\alpha} v_0(\lambda x)$. In view of the scaling properties in (3.1), the self-similar blowing up solution $v(x,t)$ of (E), if it exists, should be of the form,

$$v(x,t) = \frac{1}{(T_* - t)^{\frac{\alpha}{\alpha+1}}} V\left(\frac{x}{(T_* - t)^{\frac{1}{\alpha+1}}}\right) \tag{3.2}$$

for $\alpha \neq -1$ and t sufficiently close to T_*. If we assume that initial vorticity ω_0 has compact support, then the non-existence of self-similar blow-up of the form given by (3.2) is rather immediate from the well-known formula, $\omega(X(a,t),t) = \nabla_a X(a,t)\omega_0(a)$. We want to generalize this to a non-trivial case. Substituting (3.2) into (E), we find that V should be a solution of the system

$$\text{(SE)} \qquad \begin{cases} \dfrac{\alpha}{\alpha+1} V + \dfrac{1}{\alpha+1}(x \cdot \nabla) V + (V \cdot \nabla) V = -\nabla P, \\ \operatorname{div} V = 0 \end{cases}$$

for some scalar function P, which could be regarded as the Euler version of the Leray equations introduced in [110]. The question of existence of non-trivial solution to (SE) is equivalent to the that of existence of non-trivial self-similar finite time blowing up solution to the Euler system of the form (3.2). Similar question for the 3D Navier–Stokes equations was raised by Leray in [110], and answered negatively by Necas, Ruzicka and Sverak [122], the result of which was refined later by Tsai in [146] (see also [119] for a generalization). Combining the energy conservation with a simple scaling argument, the author of this article showed that if there exists a non-trivial self-similar finite time blowing up solution, then its helicity should be zero [18]. Mainly due to lack of the Laplacian term in the right-hand side of the first equations of (SE), we cannot expect the maximum principle, which was crucial in the works in [122] and [146] for the 3D Navier–Stokes equations. Using a completely different argument from those previous ones, in [32] it is proved that there cannot be self-similar blowing up solution to (E) of the form (3.2), if the vorticity decays sufficiently fast near infinity. Given a smooth velocity field $v(x,t)$, the particle trajectory mapping $a \mapsto X(a,t)$. The inverse $A(x,t) := X^{-1}(x,t)$ is called the back to label map, which satisfies $A(X(a,t),t) = a$, and $X(A(x,t),t) = x$. The existence of the back-to-label map $A(\cdot, t)$ for our smooth velocity $v(x,t)$ for $t \in (0, T_*)$, is guaranteed if we assume a uniform decay of $v(x,t)$ near infinity, independent of the decay rate (see [51]). The following is proved in [32].

THEOREM 3.3. *There exists no finite time blowing up self-similar solution $v(x,t)$ to the* 3D *Euler equations of the form* (3.2) *for $t \in (0, T_*)$ with $\alpha \neq -1$, if v and V satisfy the following conditions*:

(i) *For all $t \in (0, T_*)$ the particle trajectory mapping $X(\cdot, t)$ generated by the classical solution $v \in C([0, T_*); C^1(\mathbb{R}^3; \mathbb{R}^3))$ is a C^1 diffeomorphism from $\mathbb{R}^3$ onto itself.*

(ii) *The vorticity satisfies $\Omega = \operatorname{curl} V \neq 0$, and there exists $p_1 > 0$ such that $\Omega \in L^p(\mathbb{R}^3)$ for all $p \in (0, p_1)$.*

We note that the condition (ii) is satisfied, for example, if $\Omega \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ and there exist constants R, K and $\varepsilon_1, \varepsilon_2 > 0$ such that $|\Omega(x)| \leqslant K \mathrm{e}^{-\varepsilon_1 |x|^{\varepsilon_2}}$ for $|x| > R$, then we have $\Omega \in L^p(\mathbb{R}^3; \mathbb{R}^3)$ for all $p \in (0,1)$. Indeed, for all $p \in (0,1)$, we have

$$\begin{aligned}\int_{\mathbb{R}^3} \left|\Omega(x)\right|^p \mathrm{d}x &= \int_{|x| \leqslant R} \left|\Omega(x)\right|^p \mathrm{d}x + \int_{|x|>R} \left|\Omega(x)\right|^p \mathrm{d}x \\ &\leqslant |B_R|^{1-p} \left(\int_{|x| \leqslant R} \left|\Omega(x)\right| \mathrm{d}x\right)^p + K^p \int_{\mathbb{R}^3} \mathrm{e}^{-p\varepsilon_1 |x|^{\varepsilon_2}} \mathrm{d}x < \infty,\end{aligned}$$

where $|B_R|$ is the volume of the ball B_R of radius R.

In the zero vorticity case $\Omega = 0$, from $\operatorname{div} V = 0$ and $\operatorname{curl} V = 0$, we have $V = \nabla h$, where $h(x)$ is a harmonic function in $\mathbb{R}^3$. Hence, we have an easy example of self-similar blow-up,

$$v(x,t) = \frac{1}{(T_* - t)^{\frac{\alpha}{\alpha+1}}} \nabla h\left(\frac{x}{(T_* - t)^{\frac{1}{\alpha+1}}}\right),$$

in $\mathbb{R}^3$, which is also the case for the 3D Navier–Stokes with $\alpha = 1$. We do not consider this case in the theorem.

The above theorem is actually a corollary of the following more general theorem.

THEOREM 3.4. *Let $v \in C([0,T); C^1(\mathbb{R}^3; \mathbb{R}^3))$ be a classical solution to the* 3D *Euler equations generating the particle trajectory mapping $X(\cdot, t)$ which is a C^1 diffeomorphism from $\mathbb{R}^3$ onto itself for all $t \in (0, T)$. Suppose we have representation of the vorticity of the solution, by*

$$\omega(x,t) = \Psi(t)\Omega\big(\Phi(t)x\big) \quad \forall t \in [0, T) \tag{3.3}$$

where $\Psi(\cdot) \in C([0,T); (0,\infty))$, $\Phi(\cdot) \in C([0,T); \mathbb{R}^{3\times 3})$ with $\det(\Phi(t)) \neq 0$ on $[0,T)$; $\Omega = \operatorname{curl} V$ for some V, and there exists $p_1 > 0$ such that $|\Omega|$ belongs to $L^p(\mathbb{R}^3)$ for all $p \in (0, p_1)$. Then, necessarily either $\det(\Phi(t)) \equiv \det(\Phi(0))$ on $[0,T)$, or $\Omega = 0$.

For the detailed proof of Theorems 3.3 and 3.4 we refer [32].

3.4. *Asymptotically self-similar blow-up*

In this subsection we consider the possibility of more refined scenario of self-similar singularity than in the previous subsection, called the asymptotically self-similar singularity. This means that the local in time smooth solution evolves into a self-similar profile as the possible singularity time is approached. The similar notion was considered previously by Giga and Kohn in their study of semilinear heat equation [88]. Their sense of convergence of solution to the self-similar profile is the pointwise sense with a time difference weight to make it scaling invariant, and cannot apply directly to the case of Euler system. It is found in [33] that if we make the sense of convergence strong enough, then we can apply the notion of asymptotically self-similar singularity to the Euler and the Navier–Stokes equations. The following theorem is proved in [33].

THEOREM 3.5. *Let $v \in C([0,T); B^{3/p+1}_{p,1}(\mathbb{R}^3))$ be a classical solution to the* 3D *Euler equations. Suppose there exist $p_1 > 0, \alpha > -1, \bar{V} \in C^1(\mathbb{R}^3)$ with $\lim_{R\to\infty} \sup_{|x|=R} |\bar{V}(x)| = 0$ such that $\bar{\Omega} = \operatorname{curl} \bar{V} \in L^q(\mathbb{R}^3)$ for all $q \in (0, p_1)$, and the following convergence holds true*:

$$\lim_{t\nearrow T} (T-t)^{\frac{\alpha-3}{\alpha+1}} \left\| v(\cdot,t) - \frac{1}{(T-t)^{\frac{\alpha}{\alpha+1}}} \bar{V}\left(\frac{\cdot}{(T-t)^{\frac{1}{\alpha+1}}}\right) \right\|_{L^1} = 0,$$

and

$$\sup_{t\in(0,T)} (T-t) \left\| \omega(\cdot,t) - \frac{1}{T-t} \bar{\Omega}\left(\frac{\cdot}{(T-t)^{\frac{1}{\alpha+1}}}\right) \right\|_{\dot{B}^0_{\infty,1}} < \infty.$$

Then, $\bar{\Omega} = 0$, and $v(x,t)$ can be extended to a solution of the 3D *Euler system in $[0, T+\delta] \times \mathbb{R}^3$, and belongs to $C([0, T+\delta]; B^{3/p+1}_{p,1}(\mathbb{R}^3))$ for some $\delta > 0$.*

We note that the above theorem still does not exclude the possibility that the sense of vorticity convergence to the asymptotically self-similar singularity is weaker than L^∞ sense. Namely, a self-similar vorticity profile could be approached from a local classical solution in the pointwise sense in space, or in the $L^p(\mathbb{R}^3)$ sense for some p with $1 \leqslant p < \infty$. In [33] we also proved non-existence of asymptotically self-similar solution to the 3D Navier–Stokes equations with appropriate change of functional setting (see also [93] for related results).

The proof of the above theorem follows without difficulty from the following blow-up rate estimate [33], which is interesting in itself.

THEOREM 3.6. *Let $p \in [1, \infty)$ and $v \in C([0,T); B^{3/p+1}_{p,1}(\mathbb{R}^3))$ be a classical solution to the* 3D *Euler equations. There exists an absolute constant $\eta > 0$ such that if*

$$\inf_{0\leqslant t<T} (T-t) \|\omega(t)\|_{\dot{B}^0_{\infty,1}} < \eta, \tag{3.4}$$

then $v(x,t)$ can be extended to a solution of the 3D *Euler system in $[0, T+\delta] \times \mathbb{R}^3$, and belongs to $C([0, T+\delta]; B^{3/p+1}_{p,1}(\mathbb{R}^3))$ for some $\delta > 0$.*

We note that the proof of the local existence for $v_0 \in B^{3/p+1}_{p,1}(\mathbb{R}^3)$ is done in [14,20] (see also [147]). The above theorem implies that if T_* is the first time of singularity, then we have the lower estimate of the blow-up rate,

$$\left\|\omega(t)\right\|_{\dot{B}^0_{\infty,1}} \geqslant \frac{C}{T_* - t} \quad \forall t \in [0, T_*) \tag{3.5}$$

for an absolute constant C. The estimate (3.5) was actually derived previously by a different argument in [18]. We observe that (3.5) is consistent with both of the BKM criterion [4] and the Kerr's numerical calculation in [101] respectively.

The above continuation principle for a local solutions in $B^{3/p+1}_{p,1}(\mathbb{R}^3)$ has obvious applications to the solutions belonging to more conventional function spaces, due to the embeddings,

$$H^m(\mathbb{R}^3) \hookrightarrow C^{1,\gamma}(\mathbb{R}^3) \hookrightarrow B^{3/p+1}_{p,1}(\mathbb{R}^3)$$

for $m > 5/2$ and $\gamma = m - 5/2$. For example the local solution $v \in C([0, T); H^m(\mathbb{R}^3))$ can be continued to be $v \in C([0, T + \delta]; H^m(\mathbb{R}^3))$ for some δ, if (5.4) is satisfied. Regarding other implication of the above theorem on the self-similar blowing up solution to the 3D Euler equations, we have the following corollary (see [33] for the proof).

Corollary 3.1. *Let $v \in C([0, T_*); B^{3/p+1}_{p,1}(\mathbb{R}^3))$ be a classical solution to the* 3D *Euler equations. There exists $\eta > 0$ such that if we have representation for the velocity by* (3.2), *and $\bar{\Omega} = \operatorname{curl} \bar{V}$ satisfies $\|\bar{\Omega}\|_{\dot{B}^0_{\infty,1}} < \eta$, then $\bar{\Omega} = 0$, and $v(x, t)$ can be extended to a solution of the* 3D *Euler system in $[0, T_* + \delta] \times \mathbb{R}^3$, and belongs to $C([0, T_* + \delta]; B^{3/p+1}_{p,1}(\mathbb{R}^3))$ for some $\delta > 0$.*

4. Model problems

Since the blow-up problem of the 3D Euler equations looks beyond the capability of current analysis, people proposed simplified model equations to get insight on the original problem. In this section we review some of them. Besides those results presented in the following subsections there are also studies on the other model problems. In [73] Dinaburg, Posvyanskii and Sinai analyzed a quasi-linear approximation of the infinite system of ODE arising when we write the Euler equation in a Fourier mode. Friedlander and Pavlović [80] considered a vector model, and Katz and Pavlović [100] studied dyadic model, both of which are resulted from the representation of the Euler equations in the wave number space. Okamoto and Ohkitani proposed model equations in [126], and a 'dual' system to the Euler equations was considered in [21].

4.1. *Distortions of the Euler equations*

Taking trace of the matrix equation (1.11) for V, we obtain $\Delta p = -\operatorname{tr} V^2$, and hence the Hessian of the pressure is given by

$$P_{ij} = -\partial_i \partial_j (\Delta)^{-1} \operatorname{tr} V^2 = -R_i R_j \operatorname{tr} V^2,$$

where R_j denotes the Riesz transform (see Section 1). Hence we can rewrite the Euler equations as

$$\frac{DV}{Dt} = -V^2 - R\big[\mathrm{tr}\, V^2\big], \quad R[\cdot] = \big(R_i R_j[\cdot]\big). \tag{4.1}$$

In [47] Constantin studied a distorted version of the above system,

$$\frac{\partial V}{\partial t} = -V^2 - R\big[\mathrm{tr}\, V^2\big], \quad R[\cdot] = \big(R_i R_j[\cdot]\big), \tag{4.2}$$

where the convection term of the original Euler equations is deleted, and showed that a solution indeed blows up in finite time. Note that the incompressibility condition, $\mathrm{tr}\, V = 0$, is respected in the system (4.2). Thus we find that the convection term should have significant role in the study of the blow-up problem of the original Euler equations.

On the other hand, in [113] Liu and Tadmor studied another distorted version of (4.1), called the restricted Euler equations,

$$\frac{DV}{Dt} = -V^2 + \frac{1}{n}\big(\mathrm{tr}\, V^2\big)I. \tag{4.3}$$

We observe that in (4.3) the convection term is kept, but the non-local operator $R_i R_j(\cdot)$ is changed into a local one $-1/n\delta_{ij}$, where the numerical factor $-1/n$ is to keep the incompressibility condition. Analyzing the dynamics of eigenvalues of the matrix V, they showed that the system (4.3) actually blows up in finite time [113].

4.2. *The Constantin–Lax–Majda equation*

In 1985 Constantin, Lax and Majda constructed a one-dimensional model of the vorticity formulation of the 3D Euler equations, which preserves the feature of non-locality in vortex stretching term. Remarkably enough this model equation has an explicit solution for general initial data [58]. In this subsection we briefly review their result. We first observe from Section 1 that vorticity formulation of the Euler equations is $\frac{D\omega}{Dt} = S\omega$, where $S = \mathcal{P}(\omega)$ defines a singular integral operator of the Calderon–Zygmund type on ω. Let us replace $\omega(x,t) \Rightarrow \theta(x,t)$, $\frac{D}{Dt} \Rightarrow \frac{\partial}{\partial t}$, $\mathcal{P}(\cdot) \Rightarrow H(\cdot)$, where $\theta(x,t)$ is a scalar function on $\mathbb{R} \times \mathbb{R}_+$, and $H(\cdot)$ is the *Hilbert transform* defined by

$$Hf(x) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{f(y)}{x-y}\, \mathrm{d}y.$$

Then we obtain, the following 1D scalar equation from the 3D Euler equation,

$$(\mathrm{CLM}): \quad \frac{\partial \theta}{\partial t} = \theta H\theta.$$

This model preserve the feature of non-locality of the Euler system (E), in contrast to the more traditional one-dimensional model, the inviscid Burgers equation. We recall the identities for the Hilbert transform:

$$H(Hf) = -f, \qquad H(fHg + gHf) = (Hf)(Hg) - fg, \tag{4.4}$$

which imply $H(\theta H\theta) = \frac{1}{2}[(H\theta)^2 - \theta^2]$. Applying H on both sides of the first equation of (CLM), and using the formula (4.4), we obtain

$$\text{(CLM)}^*: \quad (H\theta)_t + \frac{1}{2}\big((H\theta)^2 - (\theta)^2\big) = 0.$$

We introduce the complex valued function,

$$z(x,t) = H\theta(x,t) + \mathrm{i}\theta(x,t).$$

Then, (CLM) and (CLM)* are the imaginary and the real parts of the complex Riccati equation,

$$z_t(x,t) = \frac{1}{2}z^2(x,t).$$

The explicit solution to the complex equation is

$$z(x,t) = \frac{z_0}{1 - \frac{1}{2}tz_0(x)}.$$

Taking the imaginary part, we obtain

$$\theta(x,t) = \frac{4\theta_0(x)}{(2 - tH\theta_0(x))^2 + t^2\theta_0^2(x)}.$$

The finite time blow-up occurs if and only if

$$Z = \big\{x \mid \theta_0(x) = 0 \text{ and } H\theta_0(x) > 0\big\} \neq \varnothing.$$

In [134] Schochet find that even if we add viscosity term to (CLM) there is a finite time blow-up. See also [131,132] for the studies of other variations of (CLM).

4.3. *The 2D quasi-geostrophic equation and its 1D model*

The 2D quasi-geostrophic equation (QG) models the dynamics of the mixture of cold and hot air and the fronts between them.

$$\text{(QG)} \quad \begin{cases} \theta_t + (u\cdot\nabla)\theta = 0, \\ v = -\nabla^\perp(-\Delta)^{-\frac{1}{2}}\theta, \\ \theta(x,0) = \theta_0(x), \end{cases}$$

where $\nabla^\perp = (-\partial_2, \partial_1)$. Here, $\theta(x,t)$ represents the temperature of the air at $(x,t) \in \mathbb{R}^2 \times \mathbb{R}_+$. Besides its direct physical significance (QG) has another important feature that it has very similar structure to the 3D Euler equations. Indeed, taking $\nabla^\perp$ to (QG), we obtain

$$\left(\frac{\partial}{\partial t} + v\cdot\nabla\right)\nabla^\perp\theta = (\nabla^\perp\theta\cdot\nabla)v,$$

where

$$v(x,t) = \int_{\mathbb{R}^2} \frac{\nabla^\perp\theta(y,t)}{|x-y|}\,\mathrm{d}y.$$

This is exactly the vorticity formulation of 3D Euler equation if we identify

$$\nabla^{\perp}\theta \Longleftrightarrow \omega.$$

After first observation and pioneering analysis of these feature by Constantin, Majda and Tabak [59] there have been so many research papers devoted to the study of this equation (also the equation with the viscosity term, $-(-\Delta)^{\alpha}\theta$, $\alpha > 0$, added) [48,53,60,61,65,64, 62,68–70,63,149–151,16,25,26,42,124,73,103,10]. We briefly review some of them here concentrating on the inviscid equation (QG).

The local existence can be proved easily by standard method. The BKM type of blow-up criterion proved by Constantin, Majda and Tabak in [59] is

$$\limsup_{t \nearrow T_*} \|\theta(t)\|_{H^s} = \infty \quad \text{if and only if} \quad \int_0^{T_*} \|\nabla^{\perp}\theta(s)\|_{L^\infty}\,\mathrm{d}s = \infty. \tag{4.5}$$

This criterion has been refined, using the Triebel–Lizorkin spaces [16]. The question of finite time singularity/global regularity is still open. Similarly to the Euler equations case we also have the following geometric type of condition for the regularity. We define the direction field $\xi = \nabla^{\perp}\theta/|\nabla^{\perp}\theta|$ whenever $|\nabla^{\perp}\theta(x,t)| \neq 0$.

DEFINITION 4.1. We say that a set Ω_0 is smoothly directed if there exists $\rho > 0$ such that

$$\sup_{q\in\Omega_0} \int_0^T \left|v\big(X(q,t),t\big)\right|^2 \mathrm{d}t < \infty,$$

$$\sup_{q\in\Omega_0^*} \int_0^T \left\|\nabla\xi(\cdot,t)\right\|^2_{L^\infty(B_\rho(X(q,t)))} \mathrm{d}t < \infty,$$

where $B_\rho(X)$ is the ball of radius ρ centered at X and

$$\Omega_0^* = \big\{q \in \Omega_0;\ \left|\nabla\theta_0(q)\right| \neq 0\big\}.$$

We denote $\mathfrak{O}_T(\Omega_0) = \{(x,t) \mid x \in X(\Omega_0,t), 0 \leqslant t \leqslant T\}$. Then, the following theorem is proved [59].

THEOREM 4.1. *Assume that Ω_0 is smoothly directed. Then*

$$\sup_{(x,t)\in\mathfrak{O}_T(\Omega_0)} \left|\nabla\theta(x,t)\right| < \infty,$$

and no singularity occurs in $\mathfrak{O}_T(\Omega_0)$.

Next we present an 'interpolated' result between the criterion (4.5) and Theorem 4.1, obtained in [25]. Let us denote bellow,

$$D_0 = \big\{x \in \mathbb{R}^2 \mid \left|\nabla^{\perp}\theta_0(x)\right| \neq 0\big\}, \qquad D_t = X(D_0,t).$$

The following theorem [25] could be also considered as the (QG) version of Theorem 2.9.

THEOREM 4.2. *Let $\theta(x,t)$ be the local classical solution to* (QG) *with initial data $\theta_0 \in H^m(\mathbb{R}^2)$, $m > 3/2$, for which $D_0 \neq \emptyset$. Let $\xi(x,t) = \nabla^\perp\theta(x,t)/|\nabla^\perp\theta(x,t)|$ be the direction field defined for $x \in D_t$. Then, the solution can be continued up to $T < \infty$ as the classical solution, namely $\theta(t) \in C([0,T]; H^m(\mathbb{R}^2))$, if there exist parameters $p, p', q, q', s, r_1, r_2, r_3$ satisfying the following conditions*,

$$\frac{1}{p} + \frac{1}{p'} = 1, \qquad \frac{1}{q} + \frac{1}{q'} = 1, \tag{4.6}$$

and

$$\frac{1}{r_1} + \frac{p'}{r_2}\left(1 - \frac{sq'}{2}\right) + \frac{1}{r_3}\left\{1 - p'\left(1 - \frac{sq'}{2}\right)\right\} = 1 \tag{4.7}$$

with

$$0 < s < 1, \quad 1 \leqslant \frac{2}{sq'} < p \leqslant \infty, \quad 1 \leqslant q \leqslant \infty, \tag{4.8}$$

and

$$r_1 \in [1, \infty], \quad r_2 \in \left[p'\left(1 - \frac{sq'}{2}\right), \infty\right], \quad r_3 \in \left[1 - p'\left(1 - \frac{sq'}{2}\right), \infty\right] \tag{4.9}$$

such that the followings hold:

$$\int_0^T \|\xi(t)\|^{r_1}_{\dot{\mathcal{F}}^s_{\infty,q}(D_t)} \,\mathrm{d}t < \infty, \tag{4.10}$$

and

$$\int_0^T \|\nabla^\perp\theta(t)\|^{r_2}_{L^{pq'}(D_t)} \,\mathrm{d}t + \int_0^T \|\nabla^\perp\theta(t)\|^{r_3}_{L^{q'}(D_t)} \,\mathrm{d}t < \infty. \tag{4.11}$$

In order to compare this theorem with the Constantin–Majda–Tabak criterion (4.5), let us consider the case of $p = \infty$, $q = 1$. In this case the conditions (4.10)–(4.11) are satisfied if

$$\xi(x,t) \in L^{r_1}\big(0,T; C^s\big(\mathbb{R}^2\big)\big), \tag{4.12}$$

$$\big|\nabla^\perp\theta(x,t)\big| \in L^{r_2}\big(0,T; L^\infty\big(\mathbb{R}^2\big)\big) \cap L^{r_3}\big(0,T; L^\infty\big(\mathbb{R}^2\big)\big). \tag{4.13}$$

with

$$\frac{1}{r_1} + \frac{1}{r_2}\left(1 - \frac{s}{2}\right) + \frac{s}{2r_3} = 1.$$

If we *formally* passing $s \to 0$, and choosing $r_1 = \infty, r_2 = r_3 = 1$, we find that the conditions (4.12)–(4.13) are satisfied if the Constantin–Majda–Tabak condition in (4.5) holds, since the condition

$$\xi(x,t) \in L^\infty\big(0,T; C^0\big(\mathbb{R}^2\big)\big) \cong L^\infty\big((0,T) \times \mathbb{R}^2\big)$$

is automatically satisfied. The other is the case $q' = 2/s$, where (4.10)–(4.11) are satisfied if

$$\xi(x,t) \in L^{r_1}\big(0,T;\dot{\mathcal{F}}^s_{\infty,\frac{2}{2-s}}\big(\mathbb{R}^2\big)\big), \quad \big|\nabla^\perp\theta(x,t)\big| \in L^{r_2}\big(0,T;L^{\frac{2}{s}}\big(\mathbb{R}^2\big)\big) \tag{4.14}$$

with $1/r_1 + 1/r_2 = 1$, which shows mutual compensation of the regularity of the direction field $\xi(x,t)$ and the integrability of the magnitude of gradient $|\nabla^\perp\theta(x,t)|$ to obtain smoothness of $\theta(x,t)$.

There had been a conjectured scenario of singularity in (QG) in the form of hyperbolic saddle collapse of level curves of $\theta(x,t)$ (see Definition 3.3). This was excluded by Córdoba in 1998 ([65], see also Section 3.2 of this article). Another scenario of singularity, the sharp front singularity, which is a two-dimensional version of potato chip singularity (see Definition 3.2 with $n = 2$) was excluded by Córdoba and Fefferman in [69] under the assumption of suitable velocity control (see Section 3.2).

We can also consider the possibility of self-similar singularity for (QG). We first note that (QG) has the scaling property that if θ is a solution of the system, then for any $\lambda > 0$ and $\alpha \in \mathbb{R}$ the functions

$$\theta^{\lambda,\alpha}(x,t) = \lambda^\alpha \theta(\lambda x, \lambda^{\alpha+1} t) \tag{4.15}$$

are also solutions of (QG) with the initial data $\theta_0^{\lambda,\alpha}(x) = \lambda^\alpha \theta_0(\lambda x)$. Hence, the self-similar blowing up solution should be of the form,

$$\theta(x,t) = \frac{1}{(T_* - t)^{\frac{\alpha}{\alpha+1}}} \Theta\left(\frac{x}{(T_* - t)^{\frac{1}{\alpha+1}}}\right) \tag{4.16}$$

for t sufficiently close T_* and $\alpha \neq -1$. The following theorem is proved in [32].

THEOREM 4.3. *Let v generates a particle trajectory, which is a C^1 diffeomorphism from $\mathbb{R}^2$ onto itself for all $t \in (0, T_*)$. There exists no non-trivial solution θ to the system* (QG) *of the form* (4.16), *if there exists $p_1, p_2 \in (0, \infty]$, $p_1 < p_2$, such that $\Theta \in L^{p_1}(\mathbb{R}^2) \cap L^{p_2}(\mathbb{R}^2)$.*

We note that the integrability condition on the self-similar representation function Θ in the above theorem is 'milder' than the case of the exclusion of self-similar Euler equations in Theorem 3.3, in the sense that the decay condition is of Θ (not $\nabla^\perp\Theta$) near infinity is weaker than that of $\Omega = \operatorname{curl} V$.

In the remained part of this subsection we discuss a 1D model of the 2D quasi-geostrophic equation studied in [36] (see [121] for related results). The construction of the one-dimensional model can be done similarly to the Constantin–Lax–Majda equation introduced in Section 4.2. We first note that

$$v = -R^\perp\theta = (-R_2\theta, R_1\theta),$$

where R_j, $j = 1, 2$, is the two-dimensional Riesz transform (see Section 1). We can rewrite the dynamical equation of (QG) as

$$\theta_t + \operatorname{div}\big[(R^\perp\theta)\theta\big] = 0,$$

since $\operatorname{div}(R^{\perp}\theta) = 0$. To construct the one-dimensional model we replace:

$$R^{\perp}(\cdot) \Rightarrow H(\cdot), \qquad \operatorname{div}(\cdot) \Rightarrow \partial_x$$

to obtain

$$\theta_t + \big(H(\theta)\theta\big)_x = 0.$$

Defining the complex valued function $z(x,t) = H\theta(x,t) + \mathrm{i}\theta(x,t)$, and following Constantin–Lax–Majda [58], we find that our equation is the imaginary part of

$$z_t + zz_x = 0,$$

which is complex Burgers' equation. The characteristics method does not work here. Even in that case we can show that the finite time blow-up occurs for the generic initial data as follows.

THEOREM 4.4. *Given a periodic non-constant initial data $\theta_0 \in C^1([-\pi,\pi])$ such that $\int_{-\pi}^{\pi} \theta_0(x)\,\mathrm{d}x = 0$, there is no $C^1([-\pi,\pi] \times [0,\infty))$ periodic solution to the model equation.*

For the proof we refer [36]. Here we give a brief outline of the construction of an explicit blowing up solution. We begin with the complex Burgers equation:

$$z_t + zz_x = 0, \qquad z = u + \mathrm{i}\theta$$

with $u(x,t) \equiv H\theta(x,t)$. Expanding it to real and imaginary parts, we obtain the system:

$$\begin{cases} u_t + uu_x - \theta\theta_x = 0, \\ \theta_t + u\theta_x + \theta u_x = 0. \end{cases}$$

In order to perform the *hodograph transform* we consider $x(u,\theta)$ and $t(u,\theta)$ We have,

$$u_x = Jt_\theta, \qquad \theta_x = -Jt_u,$$
$$u_t = -Jx_\theta, \qquad \theta_t = Jx_u,$$

where $J = (x_u t_\theta - x_\theta t_u)^{-1}$. By direct substitution we obtain,

$$\begin{cases} -x_\theta + ut_\theta + \theta t_u = 0, \\ x_u - ut_u + \theta t_\theta = 0 \end{cases}$$

as far as $J^{-1} \neq 0$. This system can be written more compactly in the form:

$$-(x - tu)_\theta + (t\theta)_u = 0,$$
$$(x - tu)_u + (t\theta)_\theta = 0,$$

which leads to the following Cauchy–Riemann system,

$$\xi_u = \eta_\theta, \qquad \xi_\theta = -\eta_u,$$

where we set $\eta(u,\theta) := x(u,\theta) - t(u,\theta)u$, $\xi(u,\theta) := t(u,\theta)\theta$. Hence, $f(z) = \xi(u,\theta) + \mathrm{i}\eta(u,\theta)$ with $z = u + \mathrm{i}\theta$ is an analytic function. Choosing $f(z) = \log z$, we find,

$$t\theta = \log\sqrt{u^2+\theta^2}, \qquad x - tu = \arctan\frac{\theta}{u}, \tag{4.17}$$

which corresponds to the initial data, $z(x,0) = \cos x + \mathrm{i}\sin x$. The relation (4.17) defines implicitly the real and imaginary parts $(u(x,t), \theta(x,t))$ of the solution. Removing θ from the system, we obtain

$$tu\tan(x-tu) = \log\left|\frac{u}{\cos(x-tu)}\right|,$$

which defines $u(x,t)$ implicitly. By elementary computations we find both u_x and θ_x blow up at $t = \mathrm{e}^{-1}$.

4.4. *The 2D Boussinesq system and Moffatt's problem*

The 2D Boussinesq system for the incompressible fluid flows in $\mathbb{R}^2$ is

$$(\mathrm{B})_{\nu,\kappa} \qquad \begin{cases} \dfrac{\partial v}{\partial t} + (v\cdot\nabla)v = -\nabla p + \nu\Delta v + \theta e_2, \\ \dfrac{\partial \theta}{\partial t} + (v\cdot\nabla)\theta = \kappa\Delta\theta, \\ \operatorname{div} v = 0, \\ v(x,0) = v_0(x), \qquad \theta(x,0) = \theta_0(x), \end{cases}$$

where $v = (v_1, v_2)$, $v_j = v_j(x,t)$, $j = 1,2$, $(x,t) \in \mathbb{R}^2 \times (0,\infty)$, is the velocity vector field, $p = p(x,t)$ is the scalar pressure, $\theta(x,t)$ is the scalar temperature, $\nu \geqslant 0$ is the viscosity, and $\kappa \geqslant 0$ is the thermal diffusivity, and $e_2 = (0,1)$. The Boussinesq system has important roles in the atmospheric sciences (see e.g. [116]). The global in time regularity of $(B)_{\nu,\kappa}$ with $\nu > 0$ and $\kappa > 0$ is well-known (see e.g. [13]). On the other hand, the regularity/singularity questions of the fully inviscid case of $(B)_{0,0}$ is an outstanding open problem in the mathematical fluid mechanics. It is well-known that inviscid 2D Boussinesq system has exactly same structure to the axisymmetric 3D Euler system *off* the axis of symmetry (see e.g. [115] for this observation). This is why the inviscid 2D Boussinesq system can be considered as a model equation of the 3D Euler system. The problem of the finite time blow-up for the fully inviscid Boussinesq system is an outstanding open problem. The BKM type of blow-up criterion, however, can be obtained without difficulty (see [38,41,43,74,140] for various forms of blow-up criteria for the Boussinesq system). We first consider the partially viscous cases, i.e. either the zero diffusivity case, $\kappa = 0$ and $\nu > 0$, or the zero viscosity case, $\kappa > 0$ and $\nu = 0$. Even the regularity problem for partial viscosity cases has been open recently. Actually, in an article appeared in 2001, M.K. Moffatt raised a question of finite time singularity in the case $\kappa = 0$, $\nu > 0$ and its possible development in the limit $\kappa \to 0$ as one of the 21th century problems(see the Problem no. 3 in [120]). For this problem Córdoba, Fefferman and de la Llave [70] proved that special type of singularities, called 'squirt singularities', is absent. In [27] the author considered the both of two partial viscosity cases, and prove the global in time regularity for both of the cases. Furthermore it is proved that as diffusivity (viscosity) goes to zero the solutions of $(\mathrm{B})_{\nu.\kappa}$ converge strongly to those of zero diffusivity (viscosity) equations [27]. In particular the Problem no. 3 in [120] is solved. More precise statements of these results are stated in Theorems 4.5 and 4.6 below.

THEOREM 4.5. *Let $\nu > 0$ be fixed, and $\operatorname{div} v_0 = 0$. Let $m > 2$ be an integer, and $(v_0, \theta_0) \in H^m(\mathbb{R}^2)$. Then, there exists unique solution (v, θ) with $\theta \in C([0,\infty); H^m(\mathbb{R}^2))$ and $v \in C([0,\infty); H^m(\mathbb{R}^2)) \cap L^2(0,T; H^{m+1}(\mathbb{R}^2))$ of the system $(\mathrm{B})_{\nu,0}$. Moreover, for each $s < m$, the solutions (v, θ) of $(\mathrm{B})_{\nu,\kappa}$ converge to the corresponding solutions of $(\mathrm{B})_{\nu,0}$ in $C([0,T]; H^s(\mathbb{R}^2))$ as $\kappa \to 0$.*

We note that Hou and Li also obtained the existence part of the above theorem independently in [92]. The following theorem is concerned with zero viscosity problem with fixed positive diffusivity.

THEOREM 4.6. *Let $\kappa > 0$ be fixed, and $\operatorname{div} v_0 = 0$. Let $m > 2$ be an integer. Let $m > 2$ be an integer, and $(v_0, \theta_0) \in H^m(\mathbb{R}^2)$. Then, there exists unique solutions (v, θ) with $v \in C([0,\infty); H^m(\mathbb{R}^2))$ and $\theta \in C([0,\infty); H^m(\mathbb{R}^2)) \cap L^2(0,T; H^{m+1}(\mathbb{R}^2))$ of the system $(\mathrm{B})_{0,\kappa}$. Moreover, for each $s < m$, the solutions (v, θ) of $(\mathrm{B})_{\nu,\kappa}$ converge to the corresponding solutions of $(\mathrm{B})_{0,\kappa}$ in $C([0,T]; H^s(\mathbb{R}^2))$ as $\nu \to 0$.*

The proof of the above two theorems in [27] crucially uses the Brezis–Wainger inequality in [9,76]. Below we consider the fully inviscid Boussinesq system, and show that there is no self-similar singularities under milder decay condition near infinity than the case of the 3D Euler system. The inviscid Boussinesq system $(\mathrm{B}) = (\mathrm{B})_{0,0}$ has scaling property that if (v, θ, p) is a solution of the system (B), then for any $\lambda > 0$ and $\alpha \in \mathbb{R}$ the functions

$$v^{\lambda,\alpha}(x,t) = \lambda^\alpha v\big(\lambda x, \lambda^{\alpha+1} t\big), \qquad \theta^{\lambda,\alpha}(x,t) = \lambda^{2\alpha+1}\theta\big(\lambda x, \lambda^{\alpha+1} t\big), \tag{4.18}$$

$$p^{\lambda,\alpha}(x,t) = \lambda^{2\alpha} p\big(\lambda x, \lambda^{\alpha+1} t\big) \tag{4.19}$$

are also solutions of (B) with the initial data

$$v_0^{\lambda,\alpha}(x) = \lambda^\alpha v_0(\lambda x), \qquad \theta_0^{\lambda,\alpha}(x) = \lambda^{2\alpha+1}\theta_0(\lambda x).$$

In view of the scaling properties in (4.18), the self-similar blowing-up solution $(v(x,t), \theta(x,t))$ of (B) should of the form,

$$v(x,t) = \frac{1}{(T_* - t)^{\frac{\alpha}{\alpha+1}}} V\left(\frac{x}{(T_* - t)^{\frac{1}{\alpha+1}}}\right), \tag{4.20}$$

$$\theta(x,t) = \frac{1}{(T_* - t)^{2\alpha+1}} \Theta\left(\frac{x}{(T_* - t)^{\frac{1}{\alpha+1}}}\right), \tag{4.21}$$

where $\alpha \neq -1$. We have the following non-existence result of such type of solution (see [32]).

THEOREM 4.7. *Let v generates a particle trajectory, which is a C^1 diffeomorphism from $\mathbb{R}^2$ onto itself for all $t \in (0, T_*)$. There exists no non-trivial solution (v, θ) of the system* (B) *of the form (4.20)–(4.21), if there exists $p_1, p_2 \in (0, \infty]$, $p_1 < p_2$, such that $\Theta \in L^{p_1}(\mathbb{R}^2) \cap L^{p_2}(\mathbb{R}^2)$, and $V \in H^m(\mathbb{R}^2)$, $m > 2$.*

Recalling the fact that the system (B) has the similar form as the axisymmetric 3D Euler system, we can also deduce the non-existence of self-similar blowing up solution to the axisymmetric 3D Euler equations of the form (3.2), if $\Theta = rV^\theta$ satisfies the condition of Theorem 4.7, and $\operatorname{curl} V = \Omega \in H^m(\mathbb{R}^3)$, $m > 5/2$, where $r = \sqrt{x_1^2 + x_2^2}$, and V^θ is the angular component of V. Note that in this case we do not need to assume strong decay of Ω as in Theorem 3.3. See [32] for more details.

4.5. *Deformations of the Euler equations*

Let us consider the following system considered in [34].

$$(\mathrm{P}_1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla q + (1+\varepsilon) \|\nabla u(t)\|_{L^\infty} u, \\ \operatorname{div} u = 0, \\ u(x,0) = u_0(x), \end{cases}$$

where $u = (u_1, \dots, u_n)$, $u_j = u_j(x,t)$, $j = 1, \dots, n$, is the unknown vector field $q = q(x,t)$ is the scalar, and u_0 is the given initial vector field satisfying $\operatorname{div} u_0 = 0$. The constant $\varepsilon > 0$ is fixed. Below denote $\operatorname{curl} u = \omega$ for 'vorticity' associated the 'velocity' u. We first note that the system of (P_1) has the similar non-local structure to the Euler system (E), which is implicit in the pressure term combined with the divergence free condition. Moreover it has the same scaling properties as the original Euler system in (E). Namely, if $u(x,t)$, $q(x,t)$ is a pair of solutions to (P_1) with initial data $u_0(x)$, then for any $\alpha \in \mathbb{R}$

$$u^\lambda(x,t) = \lambda^\alpha u\big(\lambda x, \lambda^{\alpha+1} t\big), \qquad q^\lambda(x,t) = \lambda^{2\alpha} q\big(\lambda x, \lambda^{\alpha+1} t\big)$$

is also a pair of solutions to (P_1) with initial data $u_0^\lambda(x) = \lambda^\alpha u_0(x)$. As will be seen below, we can have the local well-posedness in the Sobolev space, $H^m(\mathbb{R}^n)$, $m > n/2 + 2$, as well as the BKM type of blow-up criterion for (P_1), similarly to the Euler system (E). Furthermore, we can prove actual finite time blow-up for smooth initial data if $\omega_0 \neq 0$. This is rather surprising in the viewpoint that people working on the Euler system often have speculation that the divergence free condition might have the role of 'desingularization', and might make the singularity disappear. Obviously this is not the case for the system (P_1). Furthermore, there is a canonical functional relation between the solution of (P_1) and that of the Euler system (E); hence the word 'deformation'. Using this relation we can translate the blow-up condition of the Euler system in terms of the solution of (P_1). The precise contents of the above results on (P_1) are stated in the following theorem.

THEOREM 4.8. *Given $u_0 \in H^m(\mathbb{R}^n)$ with* $\operatorname{div} u_0 = 0$, *where $m > \frac{n}{2} + 2$, the following statements hold true for* (P_1).

(i) *There exists a local in time unique solution* $u(t) \in C([0,T]:H^m(\mathbb{R}^n))$ *with* $T = T(\|u_0\|_{H^m})$.

(ii) *The solution* $u(x,t)$ *blows-up at* $t = t_*$, *namely*

$$\limsup_{t\to t_*} \|u(t)\|_{H^m} = \infty \quad \textit{if and only if} \quad \int_0^{t_*} \|\omega(t)\|_{L^\infty}\,\mathrm{d}t = \infty,$$

where $\omega = \operatorname{curl} u$. *Moreover, if the solution* $u(x,t)$ *blows up at* t_*, *then necessarily,*

$$\int_0^{t_*} \exp\left[(2+\varepsilon)\int_0^\tau \|\nabla u(s)\|_{L^\infty}\,\mathrm{d}s\right]\mathrm{d}\tau = \infty$$

for $n = 3$, *while*

$$\int_0^{t_*} \exp\left[(1+\varepsilon)\int_0^\tau \|\nabla u(s)\|_{L^\infty}\,\mathrm{d}s\right]\mathrm{d}\tau = \infty$$

for $n = 2$.

(iii) *If* $\|\omega_0\|_{L^\infty} \neq 0$, *then there exists time* $t_* \leqslant 1/(\varepsilon\|\omega_0\|_{L^\infty})$ *such that solution* $u(x,t)$ *of* (P$_1$) *actually blows up at* t_*. *Moreover, at such* t_* *we have*

$$\int_0^{t_*} \exp\left[(1+\varepsilon)\int_0^\tau \|\nabla u(s)\|_{L^\infty}\,\mathrm{d}s\right]\mathrm{d}\tau = \infty.$$

(iv) *The functional relation between the solution* $u(x,t)$ *of* (P$_1$) *and the solution* $v(x,t)$ *of the Euler system* (E) *is given by*

$$u(x,t) = \varphi'(t)v\big(x,\varphi(t)\big),$$

where

$$\varphi(t) = \lambda\int_0^t \exp\left[(1+\varepsilon)\int_0^\tau \|\nabla u(s)\|_{L^\infty}\,\mathrm{d}s\right]\mathrm{d}\tau.$$

(*The relation between the two initial datum is* $u_0(x) = \lambda v_0(x)$.)

(v) *The solution* $v(x,t)$ *of the Euler system* (E) *blows up at* $T_* < \infty$ *if and only if for* $t_* := \varphi^{-1}(T_*) < 1/(\varepsilon\|\omega_0\|_{L^\infty})$ *both of the followings hold true*

$$\int_0^{t_*} \exp\left[(1+\varepsilon)\int_0^\tau \|\nabla u(s)\|_{L^\infty}\,\mathrm{d}s\right]\mathrm{d}\tau < \infty,$$

and

$$\int_0^{t_*} \exp\left[(2+\varepsilon)\int_0^\tau \|\nabla u(s)\|_{L^\infty}\,\mathrm{d}s\right]\mathrm{d}\tau = \infty.$$

For the proof we refer [34]. In the above theorem the result (ii) combined with (v) shows indirectly that there is no finite time blow-up in 2D Euler equations, consistent with the well-known result. Following the argument on p. 542 of [18], the following fact can be verified without difficulty:

We set

$$a(t) = \exp\left(\int_0^t (1+\varepsilon)\|\nabla u(s)\|_{L^\infty}\,\mathrm{d}s\right). \tag{4.22}$$

Then, the solution (u,q) of (P_1) is given by

$$u(x,t) = a(t)U(x,t), \qquad q(x,t) = a(t)P(x,t),$$

where (U,P) is a solution of the following system,

$$(\mathrm{aE}) \qquad \begin{cases} \dfrac{\partial U}{\partial t} + a(t)(U\cdot\nabla)U = -\nabla P, \\ \operatorname{div} U = 0, \\ U(x,0) = U_0(x). \end{cases}$$

The system (aE) was studied in [18], when $a(t)$ is a prescribed function of t, in which case the proof of local existence of (aE) in [18] is exactly same as the case of (E). In the current case, however, we need an extra proof of local existence, as is done in the next section, since the function $a(t)$ defined by (4.22) depends on the solution $u(x,t)$ itself. As an application of Theorem 4.8 we can prove the following lower estimate of the possible blow-up time (see [34] for the detailed proof).

THEOREM 4.9. *Let $p\in(1,\infty)$ be fixed. Let $v(t)$ be the local classical solution of the* 3D *Euler equations with initial data $v_0\in H^m(\mathbb{R}^3)$, $m>7/2$. If T_* is the first blow-up time, then*

$$T_* - t \geqslant \frac{1}{C_0\|\omega(t)\|_{\dot{B}^{3/p}_{p,1}}}, \qquad \forall t\in(0,T_*) \tag{4.23}$$

where C_0 is the absolute constant in (Q_2).

In [18] the following form of lower estimate for the blow-up rate is derived.

$$T_* - t \geqslant \frac{1}{\tilde{C}_0\|\omega(t)\|_{\dot{B}^0_{\infty,1}}}, \tag{4.24}$$

where $\tilde{C}_0$ is another absolute constant (see also the remarks after Theorem 3.6). Although there is (continuous) embedding relation, $\dot{B}^{3/p}_{p,1}(\mathbb{R}^3) \hookrightarrow \dot{B}^0_{\infty,1}(\mathbb{R}^3)$ for $p\in[1,\infty]$ (see Section 1), it is difficult to compare the two estimates (4.23) and (4.24) and decide which one is sharper, since the precise evaluation of the optimal constants C_0, $\tilde{C}_0$ in those inequalities could be very difficult problem.

Next, given $\varepsilon \geqslant 0$, we consider the following problem.

$$(\mathrm{P}_2) \qquad \begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)u = -\nabla q - (1+\varepsilon)\big\|\nabla u(t)\big\|_{L^\infty} u, \\ \operatorname{div} u = 0, \\ u(x,0) = u_0(x). \end{cases}$$

Although the system of (P_2) has also the same non-local structure and the scaling properties as the Euler system and (P_1), we have the result of the global regularity stated in the following theorem (see [34] for the proof).

THEOREM 4.10. *Given $u_0 \in H^m(\mathbb{R}^n)$ with $\operatorname{div} u_0 = 0$, where $m > \frac{n}{2} + 2$, then the solution $u(x,t)$ of* (P$_2$) *belongs to $C([0,\infty): H^m(\mathbb{R}^n))$. Moreover, we have the following decay estimate for the vorticity,*

$$\left\|\omega(t)\right\|_{L^\infty} \leqslant \frac{\|\omega_0\|_{L^\infty}}{1+\varepsilon\|\omega_0\|_{L^\infty} t} \quad \forall t \in [0,\infty).$$

We also note that solution of the system (P$_2$) has also similar functional relation with that of the Euler system as given in (iv) of Theorem 4.8.

Next, given $\varepsilon > 0$, we consider the following perturbed systems of (E).

$$(\mathrm{E})_\pm^\varepsilon \qquad \begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)u = -\nabla q \pm \varepsilon\|\nabla u\|_{L^\infty}^{1+\varepsilon} u, \\ \operatorname{div} u = 0, \\ u(x,0) = u_0(x). \end{cases}$$

If we set $\varepsilon = 0$ in the above, then the system $(\mathrm{E})_\pm^0$ becomes (E). For $\varepsilon > 0$ we have finite time blow-up for the system $(\mathrm{E})_+^\varepsilon$ with certain initial data, while we have the global regularity for $(\mathrm{E})_-^\varepsilon$ with all solenoidal initial data in $H^m(\mathbb{R}^3)$, $m > 5/2$. More precisely we have the following theorem (see [31] for the proof).

THEOREM 4.11.

(i) *Given $\varepsilon > 0$, suppose $u_0 = u_0^\varepsilon \in H^m(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$ satisfies $\|\omega_0\|_{L^\infty} > (2/\varepsilon)^{1/\varepsilon}$, then there exists T_* such that the solution $u(x,t)$ to $(\mathrm{E})_\varepsilon^+$ blows up at T_*, namely*

$$\limsup_{t\nearrow T_*} \left\|u(t)\right\|_{H^m} = \infty.$$

(ii) *Given $\varepsilon > 0$ and $u_0 \in H^m(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$, there exists unique global in time classical solution $u(t) \in C([0,\infty); H^m(\mathbb{R}^3))$ to $(\mathrm{E})_-^\varepsilon$. Moreover, we have the global in time vorticity estimate for the solution of $(\mathrm{E})_-^\varepsilon$,*

$$\left\|\omega(t)\right\|_{L^\infty} \leqslant \max\left\{\|\omega_0\|_{L^\infty}, \left(\frac{1}{\varepsilon}\right)^{\frac{1}{\varepsilon}}\right\} \quad \forall t \geqslant 0.$$

The following theorem relates the finite time blow-up/global regularity of the Euler system with those of the system $(\mathrm{E})_\pm^\varepsilon$.

THEOREM 4.12. *Given $\varepsilon > 0$, let $u_\pm^\varepsilon$ denote the solutions of $(\mathrm{E})_\pm^\varepsilon$ respectively with the same initial data $u_0 \in H^m(\mathbb{R}^3)$, $m > 5/2$. We define*

$$\varphi_\pm^\varepsilon(t,u_0) := \int_0^t \exp\left[\pm\varepsilon \int_0^\tau \left\|\nabla u_\pm^\varepsilon(s)\right\|_{L^\infty}^{1+\varepsilon} \mathrm{d}s\right] \mathrm{d}\tau.$$

(i) *If $\varphi_-^\varepsilon(\infty,u_0) = \infty$, then the solution of the Euler system with initial data u_0 is regular globally in time.*

(ii) *Let t_* be the first blow-up time for a solution u_+^ε of* $(\mathrm{E})_+^\varepsilon$ *with initial data u_0 such that*

$$\int_0^{t_*} \left\|\omega_+^\varepsilon(t)\right\|_{L^\infty} \mathrm{d}t = \infty, \quad \textit{where } \omega_+^\varepsilon = \operatorname{curl} u_+^\varepsilon.$$

If $\varphi_+^\varepsilon(t_*, u_0) < \infty$, then the solution of the Euler system blows up at the finite time $T_* = \varphi_+^\varepsilon(t_*, u_0)$.

We refer [31] for the proof of the above theorem.

5. Dichotomy: singularity or global regular dynamics?

In this section we review results in [28]. Below S, P and $\xi(x,t)$ are the deformation tensor, the Hessian of the pressure and the vorticity direction field, associated with the flow, v, respectively as introduced in Section 1. Let $\{(\lambda_k, \eta^k)\}_{k=1}^3$ be the eigenvalue and the normalized eigenvectors of S. We set $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, and

$$|\lambda| = \left(\sum_{k=1}^3 \lambda_k^2\right)^{\frac{1}{2}}, \quad \rho_k = \eta^k \cdot P\eta^k \quad \text{for } k = 1, 2, 3.$$

We also denote

$$\begin{aligned} &\eta^k(x,0) = \eta_0^k(x), \quad \lambda_k(x,0) = \lambda_{k,0}(x), \quad \lambda(x,0) = \lambda_0(x), \\ &\rho_k(x,0) = \rho_{k,0}(x) \end{aligned}$$

for the quantities at $t = 0$. Let $\omega(x,t) \neq 0$. At such point (x,t) we define the scalar fields

$$\alpha = \xi \cdot S\xi, \qquad \rho = \xi \cdot P\xi.$$

At the points where $\omega(x,t) = 0$ we define $\alpha(x,t) = \rho(x,t) = 0$. We denote $\alpha_0(x) = \alpha(x,0)$, $\rho_0(x) = \rho(x,0)$. Below we denote $f(X(a,t),t)' = \frac{Df}{Dt}(X(a,t),t)$ for simplicity.

Now, suppose that there is no blow-up of the solution on $[0, T_*]$, and the inequality

$$\alpha\big(X(a,t),t\big)\big|\omega\big(X(a,t),t\big)\big| \geqslant \varepsilon\big|\omega\big(X(a,t),t\big)\big|^2 \tag{5.1}$$

persists on $[0, T_*]$. We will see that this leads to a contradiction. Combining (5.1) with (1.14), we have

$$|\omega|' \geqslant \varepsilon|\omega|^2.$$

Hence, by Gronwall's lemma, we obtain

$$\big|\omega\big(X(a,t),t\big)\big| \geqslant \frac{|\omega_0(a)|}{1 - \varepsilon|\omega_0(a)|t},$$

which implies that

$$\limsup_{t \nearrow T_*} \big|\omega\big(X(a,t),t\big)\big| = \infty.$$

Thus we are lead to the following lemma.

LEMMA 5.1. *Suppose* $\alpha_0(a) > 0$, *and there exists* $\varepsilon > 0$ *such that*

$$\alpha_0(a)\big|\omega_0(a)\big| \geqslant \varepsilon\big|\omega_0(a)\big|^2. \tag{5.2}$$

Let us set

$$T_* = \frac{1}{\varepsilon\alpha_0(a)}. \tag{5.3}$$

Then, either the vorticity blows up no later than T_*, *or there exists* $t \in (0, T_*)$ *such that*

$$\alpha\big(X(a,t),t\big)\big|\omega\big(X(a,t),t\big)\big| < \varepsilon\big|\omega\big(X(a,t),t\big)\big|^2. \tag{5.4}$$

From this lemma we can derive the following:

THEOREM 5.1 *(vortex dynamics). Let* $v_0 \in H^m(\Omega)$, $m > 5/2$, *be given. We define*

$$\Phi_1(a,t) = \frac{\alpha(X(a,t),t)}{|\omega(X(a,t),t)|}$$

and

$$\Sigma_1(t) = \big\{a \in \Omega \mid \alpha\big(X(a,t),t\big) > 0\big\}$$

associated with the classical solution $v(x,t)$. *Suppose* $a \in \Sigma_1(0)$ *and* $\omega_0(a) \neq 0$. *Then one of the following holds true.*

(i) (finite time singularity) *The solution of the Euler equations blows-up in finite time along the trajectory* $\{X(a,t)\}$.

(ii) (regular dynamics) *On of the following holds true*:

 (a) (finite time extinction of α) *There exists* $t_1 \in (0,\infty)$ *such that* $\alpha(X(a,t_1),t_1) = 0$.

 (b) (long time behavior of Φ_1) *There exists an infinite sequence* $\{t_j\}_{j=1}^{\infty}$ *with* $t_1 < t_2 < \cdots < t_j < t_{j+1} \to \infty$ *as* $j \to \infty$ *such that for all* $j = 1, 2, \ldots$ *we have* $\Phi_1(a,0) > \Phi_1(a,t_1) > \cdots > \Phi_1(a,t_j) > \Phi_1(a,t_{j+1}) > 0$ *and* $\Phi_1(a,t) \geqslant \Phi_1(a,t_j) > 0$ *for all* $t \in [0,t_j]$.

As an illustration of proofs for Theorems 5.2 and 5.3 below, we give outline of the proof of the above theorem. Let us first observe that the formula

$$\big|\omega\big(X(a,t),t\big)\big| = \exp\left[\int_0^t \alpha\big(X(a,s),s\big)\,\mathrm{d}s\right]\big|\omega_0(a)\big|,$$

which is obtained from (1.14) immediately shows that $\omega(X(a,t),t) \neq 0$ if and only if $\omega_0(a) \neq 0$ for the particle trajectory $\{X(a,t)\}$ of the classical solution $v(x,t)$ of the Euler equations. Choosing $\varepsilon = \alpha_0(a)/|\omega_0(a)|$ in Lemma 5.1, we see that either the vorticity blows up no later than $T_* = 1/\alpha_0(a)$, or there exists $t_1 \in (0,T_*)$ such that

$$\Phi_1(a,t_1) = \frac{\alpha(X(a,t_1),t_1)}{|\omega(X(a,t_1),t_1)|} < \frac{\alpha_0(a)}{|\omega_0(a)|} = \Phi_1(a,0).$$

Under the hypothesis that (i) and (ii)-(a) do not hold true, we may assume $a \in \Sigma_1(t_1)$ and repeat the above argument to find $t_2 > t_1$ such that $\Phi_1(a,t_2) < \Phi_1(a,t_1)$, and also

$a \in \Sigma_1(t_2)$. Iterating the argument, we find a monotone increasing sequence $\{t_j\}_{j=1}^{\infty}$ such that $\Phi_1(a, t_j) > \Phi_1(a, t_{j+1})$ for all $j = 1, 2, 3, \ldots$. In particular we can choose each t_j so that $\Phi_1(a, t) \geqslant \Phi_1(a, t_j)$ for all $t \in (t_{j-1}, t_j]$. If $t_j \to t_\infty < \infty$ as $j \to \infty$, then we can proceed further to have $t_* > t_\infty$ such that $\Phi_1(a, t_\infty) > \Phi_1(a, t_*)$. Hence, we may set $t_\infty = \infty$, which finishes the proof.

The above argument can be extended to prove the following theorems.

THEOREM 5.2 *(dynamics of α). Let $v_0 \in H^m(\Omega)$, $m > 5/2$, be given. In case $\alpha(X(a,t),t) \neq 0$ we define*

$$\Phi_2(a, t) = \frac{|\xi \times S\xi|^2(X(a,t),t) - \rho(X(a,t),t)}{\alpha^2(X(a,t),t)},$$

and

$$\Sigma_2^+(t) = \left\{a \in \Omega \mid \alpha\big(X(a,t),t\big) > 0, \Phi_2\big(X(a,t),t\big) > 1\right\},$$
$$\Sigma_2^-(t) = \left\{a \in \Omega \mid \alpha\big(X(a,t),t\big) < 0, \Phi_2\big(X(a,t),t\big) < 1\right\},$$

associated with $v(x,t)$. Suppose $a \in \Sigma_2^+(0) \cup \Sigma_2^-(0)$. Then one of the following holds true.

(i) (finite time singularity) *The solution of the Euler equations blows-up in finite time along the trajectory $\{X(a,t)\}$.*

(ii) (regular dynamics) *One of the following holds true*:

   (a) (finite time extinction of α) *There exists $t_1 \in (0, \infty)$ such that $\alpha(X(a,t_1),t_1) = 0$.*

   (b) (long time behaviors of Φ_2) *Either there exists $T_1 \in (0, \infty)$ such that $\Phi_2(a, T_1) = 1$, or there exists an infinite sequence $\{t_j\}_{j=1}^{\infty}$ with $t_1 < t_2 < \cdots < t_j < t_{j+1} \to \infty$ as $j \to \infty$ such that one of the followings hold*:

      (b.1) *In the case $a \in \Sigma_2^+(0)$, for all $j = 1, 2, \ldots$ we have $\Phi_2(a,0) > \Phi_2(a,t_1) > \cdots > \Phi_2(a,t_j) > \Phi_2(a,t_{j+1}) > 1$ and $\Phi_2(a,t) \geqslant \Phi_2(a,t_j) > 1$ for all $t \in [0, t_j]$.*

      (b.2) *In the case $a \in \Sigma_2^-(0)$, for all $j = 1, 2, \ldots$ we have $\Phi_2(a,0) < \Phi_2(a,t_1) < \cdots < \Phi_2(a,t_j) < \Phi_2(a,t_{j+1}) < 1$ and $\Phi_2(a,t) \leqslant \Phi_2(a,t_j) < 1$ for all $t \in [0, t_j]$.*

THEOREM 5.3 *(spectral dynamics). Let $v_0 \in H^m(\Omega)$, $m > 5/2$, be given. In case $\lambda(X(a,t),t) \neq 0$ we define*

$$\Phi_3(a, t) = \frac{\sum_{k=1}^{3}[-\lambda_k^3 + \frac{1}{4}|\eta_k \times \omega|^2\lambda_k - \rho_k\lambda_k](X(a,t),t)}{|\lambda(X(a,t),t)|^3},$$

and

$$\Sigma_3(t) = \left\{a \in \Omega \mid \lambda\big(X(a,t),t\big) \neq 0, \Phi_3\big(X(a,t),t\big) > 0\right\}$$

associated with $v(x,t)$. Suppose $a \in \Sigma_3(0)$. Then one of the following holds true:

(i) (finite time singularity) *The solution of the Euler equations blows-up in finite time along the trajectory $\{X(a,t)\}$.*

(ii) (regular dynamics) *One of the followings hold true*:
   (a) (finite time extinction of λ) *There exists* $t_1 \in (0, \infty)$ *such that* $\lambda(X(a, t_1), t_1) = 0$.
   (b) (long time behavior of Φ_3) *Either there exists* $T_1 \in (0, \infty)$ *such that* $\Phi_3(a, T_1) = 0$, *or there exists an infinite sequence* $\{t_j\}_{j=1}^{\infty}$ *with* $t_1 < t_2 < \cdots < t_j < t_{j+1} \to \infty$ *as* $j \to \infty$ *such that for all* $j = 1, 2, \ldots$ *we have* $\Phi_2(a, 0) > \Phi_3(a, t_1) > \cdots > \Phi_3(a, t_j) > \Phi_3(a, t_{j+1}) > 0$ *and* $\Phi_3(a, t) \geqslant \Phi_3(a, t_j) > 0$ *for all* $t \in [0, t_j]$.

For the details of the proof of Theorems 5.2 and 5.3 we refer [28].
Here we present a refinement of Theorem 2.1 of [30], which is proved in [28].

THEOREM 5.4. *Let* $v_0 \in H^m(\Omega)$, $m > 5/2$, *be given. For such* v_0 *let us define a set* $\Sigma \subset \Omega$ *by*

$$\Sigma = \big\{a \in \Omega \mid \alpha_0(a) > 0, \omega_0(a) \neq 0, \exists\, \varepsilon \in (0, 1) \text{ such that } \rho_0(a) + 2\alpha_0^2(a) - |\xi_0 \times S_0\xi_0|^2(a) \leqslant (1 - \varepsilon)^2\alpha_0^2(a)\big\}.$$

Let us set

$$T_* = \frac{1}{\varepsilon\alpha_0(a)}. \tag{5.5}$$

Then, either the solution blows up no later than T_*, *or there exists* $t \in (0, T_*)$ *such that*

$$\rho\big(X(a, t), t\big) + 2\alpha^2\big(X(a, t), t\big) - |\xi \times S\xi|^2\big(X(a, t), t\big) > (1 - \varepsilon)^2\alpha^2\big(X(a, t), t\big). \tag{5.6}$$

We note that if we ignore the term $|\xi_0 \times S_0\xi_0|^2(a)$, then we have the condition,

$$\rho_0(a) + \alpha_0^2(a) \leqslant \big(-2\varepsilon + \varepsilon^2\big)\alpha_0^2(a) < 0,$$

since $\varepsilon \in (0, 1)$. Thus $\Sigma \subset \mathcal{S}$, where $\mathcal{S}$ is the set defined in Theorem 2.1 of [30]. One can verify without difficulty that $\Sigma = \emptyset$ for the 2D Euler flows. Regarding the question if $\Sigma \neq \emptyset$ or not for 3D Euler flows, we have the following proposition (see [30] for more details).

PROPOSITION 5.1. *Let us consider the system the domain* $\Omega = [0, 2\pi]^3$ *with the periodic boundary condition. In* Ω *we consider the Taylor–Green vortex flow defined by*

$$u(x_1, x_2, x_3) = (\sin x_1 \cos x_2 \cos x_3, -\cos x_1 \sin x_2 \cos x_3, 0). \tag{5.7}$$

Then, the set

$$\mathcal{S}_0 = \left\{\left(0, \frac{\pi}{4}, \frac{7\pi}{4}\right), \left(0, \frac{7\pi}{4}, \frac{\pi}{4}\right)\right\}$$

is included in Σ *of Theorem* 4.4. *Moreover, for* $x \in \mathcal{S}_0$ *we have the explicit values of* α *and* ρ,

$$\alpha(x) = \frac{1}{2}, \qquad \rho(x) = -\frac{1}{2}.$$

We recall that the Taylor–Green vortex has been the first candidate proposed for a finite time singularity for the 3D Euler equations, and there have been many numerical calculations of solution of (E) with the initial data given by it (see e.g. [7]).

6. Spectral dynamics approach

Spectral dynamics approach in the fluid mechanics was initiated by Liu and Tadmor [113]. They analyzed the restricted Euler system (4.3) in terms of (pointwise) dynamics of the eigenvalues of the velocity gradient matrix V. More specifically, multiplying left and right eigenvectors of V to (4.3), they derived

$$\frac{D\lambda_j}{Dt} = -\lambda_j^2 + \frac{1}{n}\sum_{k=1}^{n}\lambda_k^2, \quad j = 1, 2, \ldots, n,$$

where λ_j, $j = 1, 2, \ldots, n$, are eigenvalues V, which are not necessarily real values. In this model system they proved finite time blow-up for suitable initial data. In this section we review the results in [23], where the full Euler system is concerned. Moreover, the we are working on the dynamics of eigenvalues of the deformation tensor S (hence real valued), not the velocity gradient matrix. We note that there were also application of the spectral dynamics of the deformation tensor in the study of regularity problem of the Navier–Stokes equations by Neustupa and Penel [123]. In this section for simplicity we consider the 3D Euler system (E) in the periodic domain, $\Omega = \mathbb{T}^3$ $(= \mathbb{R}^3/\mathbb{Z}^3)$. Below we denote $\lambda_1, \lambda_2, \lambda_3$ for the eigenvalues of the deformation tensor $S = (S_{ij})$ for the velocity fields of the 3D Euler system. We will first establish the following formula,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right)\mathrm{d}x = -4\int_{\mathbb{T}^3}\lambda_1\lambda_2\lambda_3\,\mathrm{d}x, \tag{6.1}$$

which has important implications (Theorems 6.1–6.3 below). Indeed, using (1.12), we can compute

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3} S_{ij}S_{ij}\,\mathrm{d}x &= \int_{\mathbb{T}^3} S_{ij}\frac{DS_{ij}}{Dt}\,\mathrm{d}x \\
&= -\int_{\mathbb{T}^3} S_{ik}S_{kj}S_{ij}\,\mathrm{d}x - \frac{1}{4}\int_{\mathbb{T}^3}\omega_i S_{ij}\omega_j\,\mathrm{d}x \\
&\quad + \frac{1}{4}\int_{\mathbb{T}^3}|\omega|^2 S_{ii}\,\mathrm{d}x + \int_{\mathbb{T}^3} P_{ij}S_{ij}\,\mathrm{d}x \\
&= -\int_{\mathbb{T}^3} S_{ik}S_{kj}S_{ij}\,\mathrm{d}x - \frac{1}{8}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3}|\omega|^2\,\mathrm{d}x,
\end{aligned}$$

where we used the summation convention for the repeated indices, and used the L^2-version of the vorticity equation,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3}|\omega|^2\,\mathrm{d}x = \int_{\mathbb{T}^3}\omega_i S_{ij}\omega_j\,\mathrm{d}x \tag{6.2}$$

which is immediate from (1.13). We note

$$\int_{\mathbb{T}^3} |\omega|^2\,\mathrm{d}x = \int_{\mathbb{T}^3} |\nabla v|^2\,\mathrm{d}x = \int_{\mathbb{T}^3} V_{ij}V_{ij}\,\mathrm{d}x = \int_{\mathbb{T}^3} (S_{ij}+A_{ij})(S_{ij}+A_{ij})\,\mathrm{d}x$$
$$= \int_{\mathbb{T}^3} S_{ij}S_{ij}\,\mathrm{d}x + \int_{\mathbb{T}^3} A_{ij}A_{ij}\,\mathrm{d}x = \int_{\mathbb{T}^3} S_{ij}S_{ij}\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^3} |\omega|^2\,\mathrm{d}x.$$

Hence,

$$\int_{\mathbb{R}^n} S_{ij}S_{ij}\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{T}^3} |\omega|^2\,\mathrm{d}x.$$

Substituting this into (6.2), we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3} S_{ij}S_{ij}\,\mathrm{d}x = -\frac{4}{3}\int_{\mathbb{T}^3} S_{ik}S_{kj}S_{ij}\,\mathrm{d}x,$$

which, in terms of the spectrum of S, can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3} \left(\lambda_1^2+\lambda_2^2+\lambda_3^2\right)\mathrm{d}x = -\frac{4}{3}\int_{\mathbb{T}^3} \left(\lambda_1^3+\lambda_2^3+\lambda_3^3\right)\mathrm{d}x. \tag{6.3}$$

We observe from the divergence free condition, $0 = \operatorname{div} v = \operatorname{Tr} S = \lambda_1+\lambda_2+\lambda_3$,

$$\begin{aligned} 0 &= (\lambda_1+\lambda_2+\lambda_3)^3 \\ &= \lambda_1^3+\lambda_2^3+\lambda_3^3+3\lambda_1^2(\lambda_2+\lambda_3)+3\lambda_2^2(\lambda_1+\lambda_3)+3\lambda_3(\lambda_1+\lambda_2)+6\lambda_1\lambda_2\lambda_3 \\ &= \lambda_1^3+\lambda_2^3+\lambda_3^3-3\left(\lambda_1^3+\lambda_2^3+\lambda_3^3\right)+6\lambda_1\lambda_2\lambda_3. \end{aligned}$$

Hence, $\lambda_1^3+\lambda_2^3+\lambda_3^3 = 3\lambda_1\lambda_2\lambda_3$. Substituting this into (6.3), we completes the proof of (6.1).

Using the formula (6.1), we can first prove the following new a priori estimate for the L^2 norm of vorticity for the 3D incompressible Euler equations (see [23] for the proof). We denote

$$\mathbb{H}^m_\sigma = \left\{ v \in \left[H^m\left(\mathbb{T}^3\right)\right]^3 \,\middle|\, \operatorname{div} v = 0 \right\}.$$

THEOREM 6.1. *Let $v(t) \in C([0,T);\mathbb{H}^m_\sigma)$, $m > 5/2$ be the local classical solution of the* 3D *Euler equations with initial data $v_0 \in \mathbb{H}^m_\sigma$ with $\omega_0 \neq 0$. Let $\lambda_1(x,t) \geqslant \lambda_2(x,t) \geqslant \lambda_3(x,t)$ are the eigenvalues of the deformation tensor $S_{ij}(v) = \frac{1}{2}(\frac{\partial v_j}{\partial x_i}+\frac{\partial v_i}{\partial x_j})$. We denote $\lambda_2^+(x,t) = \max\{\lambda_2(x,t),0\}$, and $\lambda_2^-(x,t) = \min\{\lambda_2(x,t),0\}$. Then, the following estimates hold.*

$$\exp\left[\int_0^t \left(\frac{1}{2}\inf_{x\in\mathbb{T}^3}\lambda_2^+(x,t) - \sup_{x\in\mathbb{T}^3}\left|\lambda_2^-(x,t)\right|\right)\mathrm{d}t\right]$$
$$\leqslant \frac{\|\omega(t)\|_{L^2}}{\|\omega_0\|_{L^2}} \leqslant \exp\left[\int_0^t \left(\sup_{x\in\mathbb{T}^3}\lambda_2^+(x,t) - \frac{1}{2}\inf_{x\in\mathbb{T}^3}\left|\lambda_2^-(x,t)\right|\right)\mathrm{d}t\right]$$

for all $t \in (0,T)$.

The above estimate says, for example, that if we have the following compatibility conditions,

$$\sup_{x\in\mathbb{T}^3} \lambda_2^+(x,t) \simeq \inf_{x\in\Omega} \left|\lambda_2^-(x,t)\right| \simeq g(t)$$

for some time interval $[0, T]$, then

$$\left\|\omega(t)\right\|_{L^2} \lesssim \mathrm{O}\left(\exp\left[C\int_0^t g(s)\,\mathrm{d}s\right]\right) \quad \forall t\in[0,T]$$

for some constant C. On the other hand, we note the following connection of the above result to the previous one. From the equation

$$\frac{D|\omega|}{Dt} = \alpha|\omega|, \qquad \alpha(x,t) = \frac{\omega\cdot S\omega}{|\omega|^2}$$

we immediately have

$$\begin{aligned}
\left\|\omega(t)\right\|_{L^2} &\leqslant \|\omega_0\|_{L^2}\exp\left(\int_0^t \sup_{x\in\mathbb{T}^3}\alpha(x,s)\,\mathrm{d}s\right)\\
&\leqslant \|\omega_0\|_{L^2}\exp\left(\int_0^t \sup_{x\in\mathbb{T}^3}\lambda_1(x,s)\,\mathrm{d}\tau\right),
\end{aligned}$$

where we used the fact $\lambda_3 \leqslant \alpha \leqslant \lambda_1$, the well-known estimate for the Rayleigh quotient. We note that $\lambda_2^+(x,t) > 0$ implies we have stretching of infinitesimal fluid volume in two directions and compression in the other one direction (planar stretching) at (x,t), while $|\lambda_2^-(x,t)| > 0$ implies stretching in one direction and compressions in two directions (linear stretching). The above estimate says that the dominance competition between planar stretching and linear stretching is an important mechanism controlling the growth/decay in time of the L^2 norm of vorticity.

In order to state our next theorem we introduce some definitions. Given a differentiable vector field $f = (f_1, f_2, f_3)$ on $\mathbb{T}^3$, we denote by the scalar field $\lambda_i(f)$, $i = 1, 2, 3$, the eigenvalues of the deformation tensor associated with f. Below we always assume the ordering, $\lambda_1(f) \geqslant \lambda_2(f) \geqslant \lambda_3(f)$. We also fix $m > 5/2$ below. We recall that if $f \in \mathbb{H}^m_\sigma$, then $\lambda_1(f) + \lambda_2(f) + \lambda_3(f) = 0$, which is another representation of $\operatorname{div} f = 0$.

Let us begin with introduction of admissible classes $\mathcal{A}_\pm$ defined by

$$\mathcal{A}_+ = \left\{f\in\mathbb{H}^m_\sigma\left(\mathbb{T}^3\right) \mid \inf_{x\in\mathbb{T}^3}\lambda_2(f)(x) > 0\right\},$$

and

$$\mathcal{A}_- = \left\{f\in\mathbb{H}^m_\sigma\left(\mathbb{T}^3\right) \mid \sup_{x\in\mathbb{T}^3}\lambda_2(f)(x) < 0\right\}.$$

Physically $\mathcal{A}_+$ consists of solenoidal vector fields with planar stretching everywhere, while $\mathcal{A}_-$ consists of everywhere linear stretching vector fields. Although they do not represent real physical flows, they might be useful in the study of searching initial data leading to finite time singularity for the 3D Euler equations. Given $v_0 \in \mathbb{H}^m_\sigma$, let $T_*(v_0)$ be the maximal time of unique existence of solution in $\mathbb{H}^m_\sigma$ for the system (E). Let $S_t : \mathbb{H}^m_\sigma \to \mathbb{H}^m_\sigma$

be the solution operator, mapping from initial data to the solution $v(t)$. Given $f \in \mathcal{A}_+$, we define the first zero touching time of $\lambda_2(f)$ as

$$T(f) = \inf\bigl\{t \in \bigl(0, T_*(v_0)\bigr) \bigm| \exists x \in \mathbb{T}^3 \text{ such that } \lambda_2(S_t f)(x) < 0\bigr\}.$$

Similarly for $f \in \mathcal{A}_-$, we define

$$T(f) = \inf\bigl\{t \in \bigl(0, T_*(v_0)\bigr) \bigm| \exists x \in \mathbb{T}^3 \text{ such that } \lambda_2(S_t f)(x) > 0\bigr\}.$$

The following theorem is actually an immediate corollary of Theorem 6.1, combined with the above definition of $\mathcal{A}_\pm$ and $T(f)$. We just observe that for $v_0 \in \mathcal{A}_+$ (resp. $\mathcal{A}_-$) we have $\lambda_2^- = 0$, $\lambda_2^+ = \lambda_2$ (resp. $\lambda_2^+ = 0$, $\lambda_2^- = \lambda_2$) on $\Omega \times (0, T(v_0))$.

THEOREM 6.2. *Let $v_0 \in \mathcal{A}_\pm$ be given. We set $\lambda_1(x,t) \geqslant \lambda_2(x,t) \geqslant \lambda_3(x,t)$ as the eigenvalues of the deformation tensor associated with $v(x,t) = (S_t v_0)(x)$ defined $t \in (0, T(v_0))$. Then, for all $t \in (0, T(v_0))$ we have the following estimates*:

(i) *If $v_0 \in \mathcal{A}_+$, then*

$$\exp\biggl(\frac{1}{2}\int_0^t \inf_{x\in\mathbb{T}^3}\bigl|\lambda_2(x,s)\bigr|\,\mathrm{d}s\biggr) \leqslant \frac{\|\omega(t)\|_{L^2}}{\|\omega_0\|_{L^2}} \leqslant \exp\biggl(\int_0^t \sup_{x\in\mathbb{T}^3}\bigl|\lambda_2(x,s)\bigr|\,\mathrm{d}s\biggr).$$

(ii) *If $v_0 \in \mathcal{A}_-$, then*

$$\exp\biggl(-\int_0^t \sup_{x\in\mathbb{T}^3}\bigl|\lambda_2(x,s)\bigr|\,\mathrm{d}s\biggr) \leqslant \frac{\|\omega(t)\|_{L^2}}{\|\omega_0\|_{L^2}} \leqslant \exp\biggl(-\frac{1}{2}\int_0^t \inf_{x\in\mathbb{T}^3}\bigl|\lambda_2(x,s)\bigr|\,\mathrm{d}s\biggr).$$

(See [23] for the proof.) If we have the compatibility conditions,

$$\inf_{x\in\mathbb{T}^3}\bigl|\lambda_2(x,t)\bigr| \simeq \sup_{x\in\mathbb{T}^3}\bigl|\lambda_2(x,t)\bigr| \simeq g(t) \quad \forall t \in \bigl(0, T(v_0)\bigr),$$

which is the case for sufficiently small box $\mathbb{T}^3$, then we have

$$\frac{\|\omega(t)\|_{L^2}}{\|\omega_0\|_{L^2}} \simeq \begin{cases} \exp\Bigl(\int_0^t g(s)\,\mathrm{d}s\Bigr) & \text{if } v_0 \in \mathcal{A}_+, \\ \exp\Bigl(-\int_0^t g(s)\,\mathrm{d}s\Bigr) & \text{if } v_0 \in \mathcal{A}_- \end{cases}$$

for $t \in (0, T(v_0))$. In particular, if we could find $v_0 \in \mathcal{A}_+$ such that

$$\inf_{x\in\mathbb{T}^3}\bigl|\lambda_2(x,t)\bigr| \gtrsim \mathrm{O}\biggl(\frac{1}{t_* - t}\biggr)$$

for time interval near t_*, then such data would lead to singularity at t_*.

As another application of the formula (6.1) we have some decay in time estimates for some ratio of eigenvalues (see [23] for the proof).

THEOREM 6.3. *Let $v_0 \in \mathcal{A}_\pm$ be given, and we set $\lambda_1(x,t) \geqslant \lambda_2(x,t) \geqslant \lambda_3(x,t)$ as in Theorem 3.1. We define*

$$\varepsilon(x,t) = \frac{|\lambda_2(x,t)|}{\lambda(x,t)} \quad \forall (x,t) \in \mathbb{T}^3 \times \big(0, T(v_0)\big),$$

where we set

$$\lambda(x,t) = \begin{cases} \lambda_1(x,t) & \text{if } v_0 \in \mathcal{A}_+, \\ -\lambda_3(x,t) & \text{if } v_0 \in \mathcal{A}_-. \end{cases}$$

Then, there exists a constant $C = C(v_0)$ such that

$$\inf_{(x,s)\in\mathbb{T}^3\times(0,t)} \varepsilon(x,s) < \frac{C}{\sqrt{t}} \quad \forall t \in \big(0, T(v_0)\big).$$

Regarding the problem of searching finite time blowing up solution, the proof of the above theorem suggests the following:

Given $\delta > 0$, let us suppose we could find $v_0 \in \mathcal{A}_+$ such that for the associated solution $v(x,t) = (S_t v_0)(x)$ the estimate

$$\inf_{(x,s)\in\mathbb{T}^3\times(0,t)} \varepsilon(x,s) \gtrsim \mathrm{O}\left(\frac{1}{t^{1/2+\delta}}\right), \tag{6.4}$$

holds true, for sufficiently large time t. Then such v_0 will lead to the finite time singularity. In order to check the behavior (6.4) for a given solution we need a sharper and/or localized version of Eq. (6.1) for the dynamics of eigenvalues of the deformation tensor.

7. Conservation laws for singular solutions

For the smooth solutions of the Euler equations there are many conserved quantities as described in Section 1 of this article. One of the most important conserved quantities is the total kinetic energy. For non-smooth (weak) solutions it is not at all obvious that we still have energy conservation. Thus, there comes very interesting question of how much smoothness we need to assume for the solution to have energy conservation property. Regarding this question L. Onsager conjectured that a Hölder continuous weak solution with the Hölder exponent $1/3$ preserve the energy, and this is sharp [125]. Considering Kolmogorov's scaling argument on the energy correlation in the homogeneous turbulence the exponent $1/3$ is natural. A sufficiency part of this conjecture is proved in a positive direction by an ingenious argument due to Constantin–E–Titi [54] (see also [78]), using a special Besov type of space norm, $\dot{\mathcal{B}}^s_{3,\infty}$ with $s > 1/3$ (more precisely, the Nikolskii space norm) for the velocity. See also [12] for related results in the magnetohydrodynamics. Remarkably enough Shnirelman [136] later constructed an example of weak solution of 3D Euler equations, which does not preserve energy. The problem of finding optimal regularity condition for a weak solution to have conservation property can also be considered for the helicity. Since the helicity is closely related to the topological invariants, e.g. the knottedness of vortex tubes, the non-conservation of helicity is directly related to the spontaneous apparition of singularity from local smooth solutions, which is the main theme of this article. In [19] the author of this article obtained a sufficient regularity condition for the helicity conservation, using the function space $\dot{\mathcal{B}}^s_{\frac{9}{5},\infty}$, $s > 1/3$, for the vorticity. These results on the energy and the helicity are recently refined in [24], using the Triebel–Lizorkin type of spaces, $\dot{\mathcal{F}}^s_{p,q}$, and the Besov spaces $\dot{\mathcal{B}}^s_{p,q}$ (see Section 1 for the definitions) with similar values for s, p, but allowing full range of values for $q \in [1, \infty]$.

By a weak solution of (E) in $\mathbb{R}^n \times (0, T)$ with initial data v_0 we mean a vector field $v \in C([0, T); L^2_{\text{loc}}(\mathbb{R}^n))$ satisfying the integral identity:

$$\begin{aligned}
&-\int_0^T\int_{\mathbb{R}^n} v(x,t)\cdot\frac{\partial\phi(x,t)}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}^n} v_0(x)\cdot\phi(x,0)\,\mathrm{d}x\\
&\quad -\int_0^T\int_{\mathbb{R}^n} v(x,t)\otimes v(x,t)\colon \nabla\phi(x,t)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad -\int_0^T\int_{\mathbb{R}^n} \operatorname{div}\phi(x,t)p(x,t)\,\mathrm{d}x\,\mathrm{d}t = 0,
\end{aligned}\tag{7.1}$$

$$\int_0^T\int_{\mathbb{R}^n} v(x,t)\cdot\nabla\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{7.2}$$

for every vector test function $\phi = (\phi_1, \ldots, \phi_n) \in C_0^\infty(\mathbb{R}^n \times [0, T))$, and for every scalar test function $\psi \in C_0^\infty(\mathbb{R}^n \times [0, T))$. Here we used the notation $(u \otimes v)_{ij} = u_i v_j$, and $A : B = \sum_{i,j=1}^n A_{ij}B_{ij}$ for $n \times n$ matrices A and B. In the case when we discuss the helicity conservation of the weak solution we impose further regularity for the vorticity, $\omega(\cdot, t) \in L^{\frac{3}{2}}(\mathbb{R}^3)$ for almost every $t \in [0, T]$ in order to define the helicity for such weak solution. *Hereafter, we use the notation* $\dot{\mathcal{X}}^s_{p,q}$ *(resp.* $X^s_{p,q}$*) to represent* $\dot{\mathcal{F}}^s_{p,q}$ *(resp.* $\dot{\mathcal{F}}^s_{p,q}$*) or* $\dot{\mathcal{B}}^s_{p,q}$ *(resp.* $\dot{\mathcal{B}}^s_{p,q}$*)*. The following is proved in [24].

THEOREM 7.1. *Let $s > 1/3$ and $q \in [2, \infty]$ be given. Suppose v is a weak solution of the n-dimensional Euler equations with $v \in C([0, T]; L^2(\mathbb{R}^n)) \cap L^3(0, T; \dot{X}^s_{3,q}(\mathbb{R}^n))$. Then, the energy is preserved in time, namely*

$$\int_{\mathbb{R}^n} \left|v(x,t)\right|^2 \mathrm{d}x = \int_{\mathbb{R}^n} \left|v_0(x)\right|^2 \mathrm{d}x \tag{7.3}$$

for all $t \in [0, T)$.

When we restrict $q = \infty$, the above theorem reduce to the one in [54]. On the other hand, the results for Triebel–Lizorkin type of space are completely new.

THEOREM 7.2. *Let $s > 1/3$, $q \in [2, \infty]$, and $r_1 \in [2, \infty]$, $r_2 \in [1, \infty]$ be given, satisfying $2/r_1 + 1/r_2 = 1$. Suppose v is a weak solution of the 3D Euler equations with $v \in C([0, T]; L^2(\mathbb{R}^3)) \cap L^{r_1}(0, T; \dot{X}^s_{\frac{9}{2},q}(\mathbb{R}^3))$ and $\omega \in L^{r_2}(0, T; \dot{\mathcal{X}}^s_{\frac{9}{5},q}(\mathbb{R}^3))$, where the* curl *operation is in the sense of distribution. Then, the helicity is preserved in time, namely*

$$\int_{\mathbb{R}^3} v(x,t)\cdot\omega(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^3} v_0(x)\cdot\omega_0(x)\,\mathrm{d}x \tag{7.4}$$

for all $t \in [0, T)$.

Similarly to the case of Theorem 7.1, when we restrict $q = \infty$, the above theorem reduce to the one in [19]. The results for the case of the Triebel–Lizorkin type of space, however, is new in [24].

As an application of the above theorem we have the following estimate from below of the vorticity by a constant depending on the initial data for the weak solutions of the 3D Euler equations. We estimate the helicity,

$$\int_{\mathbb{R}^3} v(x,t)\cdot\omega(x,t)\,\mathrm{d}x \leqslant \|v(\cdot,t)\|_{L^3}\|\omega(\cdot,t)\|_{L^{\frac{3}{2}}} \leqslant C\|\nabla v(\cdot,t)\|_{L^{\frac{3}{2}}}\|\omega(\cdot,t)\|_{L^{\frac{3}{2}}} \leqslant C\|\omega(\cdot,t)\|^2_{L^{\frac{3}{2}}},$$

where we used the Sobolev inequality and the Calderon–Zygmund inequality. Combining this estimate with (7.4), we obtain the following:

COROLLARY 7.1. *Suppose v is a weak solution of the* 3D *Euler equations satisfying the conditions of Theorem* 7.2. *Then*, *we have the following estimate*:

$$\|\omega(\cdot,t)\|^2_{L^{\frac{3}{2}}} \geqslant CH_0, \quad \forall t \in [0,T)$$

where $H_0 = \int_{\mathbb{R}^3} v_0(x)\cdot\omega_0(x)\,\mathrm{d}x$ is the initial helicity, and C is an absolute constant.

Next we are concerned on the L^p-norm conservation for the weak solutions of (QG). Let $p \in [2,\infty)$. By a weak solution of (QG) in $D\times(0,T)$ with initial data v_0 we mean a scalar field $\theta \in C([0,T); L^p(\mathbb{R}^2)\cap L^{\frac{p}{p-1}}(\mathbb{R}^2))$ satisfying the integral identity:

$$-\int_0^T\int_{\mathbb{R}^2}\theta(x,t)\left[\frac{\partial}{\partial t}+v\cdot\nabla\right]\phi(x,t)\,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}^2}\theta_0(x)\phi(x,0)\,\mathrm{d}x = 0, \tag{7.5}$$

$$v(x,t) = -\nabla^\perp\int_{\mathbb{R}^2}\frac{\theta(y,t)}{|x-y|}\,\mathrm{d}y \tag{7.6}$$

for every test function $\phi \in C_0^\infty(\mathbb{R}^2\times[0,T))$, where $\nabla^\perp$ in (7.6) is in the sense of distribution. We note that contrary to the case of 3D Euler equations there is a global existence result for the weak solutions of (QG) for $p = 2$ due to Resnick [129]. The following is proved in [24].

THEOREM 7.3. *Let $s > 1/3$, $p \in [2,\infty)$, $q \in [1,\infty]$, and $r_1 \in [p,\infty]$, $r_2 \in [1,\infty]$ be given*, *satisfying $p/r_1 + 1/r_2 = 1$. Suppose θ is a weak solution of* (QG) *with $\theta \in C([0,T]; L^p(\mathbb{R}^2)\cap L^{\frac{p}{p-1}}(\mathbb{R}^2))\cap L^{r_1}(0,T; X^s_{p+1,q}(\mathbb{R}^2))$ and $v \in L^{r_2}(0,T; \dot{\mathcal{X}}^s_{p+1,q}(\mathbb{R}^2))$. Then*, *the L^p norm of $\theta(\cdot,t)$ is preserved*, *namely*

$$\|\theta(t)\|_{L^p} = \|\theta_0\|_{L^p} \tag{7.7}$$

for all $t \in [0,T]$.

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CHAPTER 2

Mathematical Methods in the Theory of Viscous Fluids*

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Contents

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1. Balance laws

Continuum mechanics describes a fluid in terms of observable and measurable macroscopic quantities: the density, the velocity, the absolute temperature, etc. The basic physical principles are expressed through balance laws that can be written in a general form:

$$\int_B r(t_2,\cdot)\,\mathrm{dx} - \int_B r(t_1,\cdot)\,\mathrm{dx} + \int_{t_1}^{t_2}\int_{\partial B} \mathbf{F}(t,\cdot)\cdot\mathbf{n}\,\mathrm{d}\sigma_x\,\mathrm{d}t = \int_{t_1}^{t_2}\int_B s(t,\cdot)\,\mathrm{dx}\,\mathrm{d}t, \tag{1.1}$$

where the symbol r denotes the volumic density, $\mathbf{F}$ is the flux, and s stands for the production rate of an observable quantity. Here, the time $t \in [0,T) \subset \mathbb{R}$ and the spatial position $x \in \Omega$, where $\Omega \subset \mathbb{R}^3$ is the physical domain occupied by the fluid, play a role of independent reference variables, while B is an arbitrary subset of Ω. The reference system attached to the physical space corresponds to the Eulerian description of motion.

It is easy to check that (1.1) gives rise to

$$\partial_t r + \mathrm{div}_x\,\mathbf{F} = s \quad \text{in } (0,T)\times\Omega \tag{1.2}$$

as soon as all quantities are continuously differentiable. However, the hypothesis of smoothness of the state variables is questionable, in particular in the case of the fluid density and other extensive quantities. Thus we should always keep in mind, that the "correct" formulation of a balance law is represented by the integral identity (1.1) rather than the partial differential equation (1.2).

On the other hand, given a vector field $[r,\mathbf{F}]$ satisfying (1.2), we can define its normal trace on a space–time cylinder $[t_1,t_2]\times B$ by means of the classical Gauss–Green theorem as

$$\begin{aligned}
&\int_B r(t_2,\cdot)\varphi(t_2,x)\,\mathrm{dx} - \int_B r(t_1,\cdot)\varphi(t_1,x)\,\mathrm{dx} + \int_{t_1}^{t_2}\int_{\partial B}\varphi(t,x)\mathbf{F}(t,x)\cdot\mathbf{n}\,\mathrm{d}\sigma_x\,\mathrm{d}t\\
&\quad = \int_{t_1}^{t_2}\int_B s(t,x)\varphi(t,x)\,\mathrm{dx}\,\mathrm{d}t\\
&\qquad - \int_{t_1}^{t_2}\int_B \big[r(t,x)\partial_t\varphi(t,x) + \mathbf{F}(t,x)\cdot\nabla_x\varphi(t,x)\big]\,\mathrm{dx}\,\mathrm{d}t
\end{aligned} \tag{1.3}$$

to be satisfied for all test functions $\varphi \in C^1([0,T)\times\overline{\Omega})$.

Motivated by the previous discussion, we introduce a *weak formulation* of the balance law (1.1) as a family of integral identities

$$\int_0^T\int_\Omega (r\partial_t\varphi + \mathbf{F}\cdot\nabla_x\varphi)\,\mathrm{dx}\,\mathrm{d}t = \langle s,\varphi\rangle, \tag{1.4}$$

for any $\varphi \in C^1([0,T)\times\overline{\Omega})$, where the production rate s can be a measure distributed on the set $\overline{(0,T)\times\Omega}$.

In accordance with formula (1.5), the measure s can capture the boundary behavior of the normal trace of the vector $[r,\mathbf{F}]$ on the space–time cylinder $[0,T)\times\Omega$, in particular, we recover the initial distribution $r_0 = r(0,\cdot)$, together with the boundary flux $F_b = \mathbf{F}\cdot\mathbf{n}$

taking

$$\langle s, \varphi \rangle = -\int_{\Omega} r_0 \varphi(0, \cdot)\, \mathrm{dx} - \int_0^T \int_{\partial\Omega} F_b \varphi \, \mathrm{d}\sigma_x \, \mathrm{d}t + \langle g, \varphi \rangle, \tag{1.5}$$

where g is a (bounded) Radon measure on $(0, T) \times \Omega$.

Thus relations (1.4), (1.5) can be *formally* interpreted as a partial differential equation

$$\partial_t r + \mathrm{div}_x \, \mathbf{F} = g \quad \text{in } (0, T) \times \Omega, \tag{1.6}$$

supplemented with the initial condition

$$r(0, \cdot) = r_0, \tag{1.7}$$

and the boundary condition

$$\mathbf{F} \cdot \mathbf{n}|_{\partial\Omega} = F_b. \tag{1.8}$$

Although the *classical* formulation (1.6)–(1.8) is widely used in the literature, the *weak* formulation expressed through (1.4), (1.5) seems to reflect better our understanding of macroscopic variables in continuum fluid mechanics as integral means rather than quantities that are well defined at each particular point of the underlying physical space. For further aspects of the weak formulation of conservation laws, the reader may consult the monograph by Dafermos [33], or a recent study by Chen and Torres [27].

2. Formulation of basic physical principles

Following the approach discussed in the previous section, we adopt the "weak" interpretation of the basic physical principles expressed through families of integral identities although they will be written in the classical way as a system of partial differential equations. Otherwise, the material presented below is classical and may be found in all standard texts devoted to continuum fluid mechanics: Batchelor [9], Chorin and Marsden [28], Gallavotti [61], Lamb [84], Lighthill [86], Truesdell [121,120,123], Truesdell and Rajagopal [122], among others.

2.1. *Conservation of mass*

The total mass m_B of the fluid contained in a set $B \subset \Omega$ at an instant t is given as

$$m_B = \int_B \varrho(t, \cdot)\, \mathrm{dx},$$

where ϱ stands for the *density*. Accordingly, the physical principle of mass conservation can be expressed in terms of the integral identity

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0, \tag{2.1}$$

where $\mathbf{u}$ denotes the velocity of the fluid. Equation (2.1) is supplemented with the initial condition

$$\varrho(0, \cdot) = \varrho_0, \tag{2.2}$$

and the boundary conditions

$$\varrho \mathbf{u} \cdot \mathbf{n}|_{\partial \Omega} = 0. \tag{2.3}$$

As already pointed out, relations (2.1)–(2.3) are to be understood in the weak sense specified in (1.4)–(1.5).

2.2. *Balance of momentum*

Following the same line of arguments as in the preceding sections we can write the balance of momentum in the form

$$\partial_t (\varrho \mathbf{u}) + \operatorname{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) = \operatorname{div}_x \mathbb{T} - \varrho \mathbf{f}, \tag{2.4}$$

where $\mathbb{T}$ denotes the Cauchy stress tensor, and $\mathbf{f}$ is a given external force. In addition, the fluids are characterized by Stokes' relation

$$\mathbb{T} = \mathbb{S} - p\mathbb{I}, \tag{2.5}$$

where $\mathbb{S}$ denotes the viscous stress tensor, and p is a scalar function termed pressure.

The initial distribution of momentum is given through

$$(\varrho \mathbf{u})(0, \cdot) = (\varrho \mathbf{u})_0. \tag{2.6}$$

A proper choice of the boundary conditions for the fluid velocity offers more possibilities. Taking (2.3) for granted, we can assume that $\mathbf{u}$ satisfies the complete slip boundary condition

$$(\mathbb{S}\mathbf{n}) \times \mathbf{n}|_{\partial \Omega} = 0 \tag{2.7}$$

or, alternatively, the no-slip boundary condition

$$\mathbf{u}|_{\partial \Omega} = 0. \tag{2.8}$$

Note that both (2.7) and (2.8) are conservative in the sense that the kinetic energy flux vanishes on the boundary of Ω.

Similarly to (1.4), (1.5), the weak formulation of (2.4)–(2.6) reads

$$\begin{aligned} &\int_0^T \int_\Omega (\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + p \operatorname{div}_x \varphi) \, \mathrm{d}x \, \mathrm{d}t \\ &\quad = \int_0^T \int_\Omega (\mathbb{S} : \nabla_x \varphi - \varrho \mathbf{f} \cdot \varphi) \, \mathrm{d}x - \int_\Omega (\varrho \mathbf{u})_0 \cdot \varphi(0, \cdot) \, \mathrm{d}x \end{aligned} \tag{2.9}$$

for any $\varphi \in C_0^1([0, T) \times \overline{\Omega}; \mathbb{R}^3)$ satisfying

$$\varphi \cdot \mathbf{n}|_{\partial \Omega} = 0.$$

If the no-slip boundary conditions (2.8) are imposed, we have to require, in addition, that

$$\varphi|_{\partial\Omega} = 0.$$

In contrast with the weak formulation of conservation laws introduced in the previous section, the satisfaction of the "vectorial" boundary conditions (2.3), (2.8) must be incorporated both in the choice of the functional space for $\mathbf{u}$ and the space of test functions. Furthermore, the no-slip boundary condition requires the existence of a *trace* of $\mathbf{u}$ on $\partial\Omega$.

2.3. *First law of thermodynamics, total energy balance*

Formally, we can take the scalar product of (2.4) with $\mathbf{u}$ in order to deduce the kinetic energy balance

$$\begin{aligned}&\partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2\right) + \operatorname{div}_x\left(\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + p\right)\mathbf{u}\right) - \operatorname{div}_x(\mathbb{S}\mathbf{u})\\ &\quad = -\mathbb{S} : \nabla_x\mathbf{u} + p\operatorname{div}_x\mathbf{u} + \varrho\mathbf{f}\cdot\mathbf{u}.\end{aligned} \tag{2.10}$$

The first law of thermodynamics asserts that the energy of the fluid is a conserved quantity provided $\mathbf{f} \equiv 0$ and there is no energy flux through the boundary. Accordingly, introducing the specific internal energy e we get the energy balance equation in the form

$$\begin{aligned}&\partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e\right) + \operatorname{div}_x\left(\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + e + p\right)\mathbf{u}\right) + \operatorname{div}_x\mathbf{q} - \operatorname{div}_x(\mathbb{S}\mathbf{u})\\ &\quad = \varrho\mathbf{f}\cdot\mathbf{u},\end{aligned} \tag{2.11}$$

where the symbol $\mathbf{q}$ denotes the internal energy flux. If the system is energetically isolated, in particular if the boundary conditions (2.3), (2.7), or, alternatively (2.8), are supplemented with

$$\mathbf{q}\cdot\mathbf{n}|_{\partial\Omega} = 0, \tag{2.12}$$

Eq. (2.11) integrated over Ω gives rise to the total energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e\right)\mathrm{dx} = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{dx}. \tag{2.13}$$

2.4. *Second law of thermodynamics, entropy*

Subtracting (2.10) from (2.11) we obtain

$$\partial_t(\varrho e) + \operatorname{div}_x(\varrho e\mathbf{u}) + \operatorname{div}_x\mathbf{q} = \mathbb{S} : \nabla_x\mathbf{u} - p\operatorname{div}_x\mathbf{u}, \tag{2.14}$$

which is an equation governing the time evolution of the internal energy.

In the absence of any dissipative mechanism in the system, meaning when $\mathbf{q} = 0$, $\mathbb{S} = 0$, Eq. (2.14) takes the form

$$\partial_t(\varrho e) + \operatorname{div}_x(\varrho e\mathbf{u}) + p\operatorname{div}_x\mathbf{u} = 0. \tag{2.15}$$

The basic idea leading to the concept of entropy asserts that (2.15) can be written as a conservation law for a new state variable s termed entropy:

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s\mathbf{u}) = 0. \tag{2.16}$$

Now, assume that both e and s depend on ϱ and another internal variable ϑ called absolute temperature. Consequently, by help of (2.1), Eq. (2.15) can be written as

$$De \cdot \partial_t \begin{bmatrix} \varrho \\ \vartheta \end{bmatrix} + \mathbf{u} \cdot De \cdot \nabla_x \begin{bmatrix} \varrho \\ \vartheta \end{bmatrix} - \frac{p}{\varrho^2}(\partial_t \varrho + \mathbf{u} \cdot \nabla_x \varrho) = 0;$$

whence, necessarily,

$$H(\varrho, \vartheta) Ds = De + pD\left(\frac{1}{\varrho}\right),$$

where the factor H can be adjusted by a suitable choice of the temperature scale. Adopting the standard relation $\vartheta \approx \partial e/\partial s$ we arrive at Gibbs' equation

$$\vartheta Ds(\varrho, \vartheta) = De(\varrho, \vartheta) + p(\varrho, \vartheta) D\left(\frac{1}{\varrho}\right). \tag{2.17}$$

Accordingly, the internal energy balance (2.14) divided on ϑ gives rise to the entropy balance equation

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma, \tag{2.18}$$

with the entropy production rate

$$\sigma = \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right). \tag{2.19}$$

In addition, the second law of thermodynamics asserts that

$$\mathbb{S} : \nabla_x \mathbf{u} \geqslant 0, \qquad -\mathbf{q} \cdot \nabla_x \vartheta \geqslant 0 \tag{2.20}$$

for any admissible fluid motion.

Note that equations (2.11), (2.14), and (2.18) are *equivalent*, meaning they provide the same information on the state of the system, as long as all state variables are sufficiently smooth. However, the situation may be rather different in the framework of the weak solutions considered in this paper.

2.5. *Kinetic energy dissipation*

As already pointed out above, the two terms appearing in the entropy production rate introduced in (2.19) are responsible for the irreversible transfer of the mechanical energy into heat. Accordingly, both (2.4) and (2.18) may be considered as "parabolic", while (2.1) represents a hyperbolic equation. We assume the simplest possible form of $\mathbb{S}$, $\mathbf{q}$, namely that these quantities are *linear* functions of the velocity gradient and the temperature gradient, respectively. It can be shown (see e.g. Chorin and Marsden [28]) that the only form of $\mathbb{S}$

conformable with the principle of material frame indifference reads

$$\mathbb{S} = \mu\left(\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^t - \frac{2}{3}\operatorname{div}_x \mathbf{u}\mathbb{I}\right) + \eta \operatorname{div}_x \mathbf{u}\mathbb{I}, \tag{2.21}$$

while

$$\mathbf{q} = -\kappa \nabla_x \vartheta, \tag{2.22}$$

where the transport coefficients μ, η, and κ are non-negative scalar functions of ϑ and ϱ, as the case may be. Relation (2.22) is usually called Fourier's law.

2.6. *Navier–Stokes–Fourier system*

The mathematical theory to be developed in this study is based on the Navier–Stokes–Fourier system of equations

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0, \tag{2.23}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \operatorname{div}_x \mathbb{S} + \varrho \mathbf{f}, \tag{2.24}$$

$$\partial_t(\varrho s) + \operatorname{div}_x(\varrho s \mathbf{u}) + \operatorname{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma, \tag{2.25}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e\right)\mathrm{dx} = \int_\Omega \varrho \mathbf{f}\cdot \mathbf{u}\,\mathrm{dx}, \tag{2.26}$$

supplemented with the boundary conditions

$$\mathbf{u}\cdot\mathbf{n} = (\mathbb{S}\mathbf{n}) \times \mathbf{n}|_{\partial\Omega} = 0, \quad \text{or} \quad \mathbf{u}|_{\partial\Omega} = 0, \tag{2.27}$$

$$\mathbf{q}\cdot\mathbf{n}|_{\partial\Omega} = 0, \tag{2.28}$$

where the thermodynamics functions p, e, and s are interrelated through Gibbs' equation

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right), \tag{2.29}$$

$\mathbb{S}$ and $\mathbf{q}$ obey

$$\mathbb{S} = \mu\left(\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^t - \frac{2}{3}\operatorname{div}_x \mathbf{u}\mathbb{I}\right) + \eta \operatorname{div}_x \mathbf{u}\mathbb{I}, \tag{2.30}$$

$$\mathbf{q} = -\kappa \nabla_x \vartheta, \tag{2.31}$$

and the entropy production rate is a non-negative measure on the set $[0, T] \times \overline{\Omega}$ satisfying

$$\sigma \geqslant \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right). \tag{2.32}$$

The inequality sign in (2.32) certainly deserves some comment. To begin with, it is well-known that solutions of the inviscid (Euler) system may produce entropy although, formally, the dissipative terms $\mathbb{S}$, $\mathbf{q}$ are not present in the equations. On the other hand, even if the dissipation effect due to viscosity and heat conductivity is taken into account, it is not known, at least in the framework of the weak solutions, that the kinetic energy balance is given by (2.10). Thus, in general, we are not able to exclude a hypothetical possibility of the production of entropy, which is not captured by the quantity on the right-hand side of (2.19), the latter being absolutely continuous with respect to the standard Lebesgue measure. Indeed this very interesting scenario has been studied by Duchon and Robert [40], Eyink [46], among others. Related results concerning the validity of the kinetic energy balance have been obtained by Nagasawa [101]. On the point of conclusion, let us remark that (2.23)–(2.32) imply (2.19) as soon as all quantities are smooth enough (see [50]).

2.7. *Bibliographical comments*

Since the truly pioneering work of Leray [85], extended in an essential way by Ladyzhenskaya [82], Temam [118], Caffarelli et al. [25], Antontsev et al. [5], P.-L. Lions [88,89], among many others, the theory of weak solutions based on the function spaces of Sobolev type has become an important part of modern mathematical physics. In particular, the theory of compressible fluid flow for general (large) data is more likely to rely on the concept of "genuinely weak" solutions incorporating various types of discontinuities and other irregular phenomena that are expected to come into play (see, for instance, Desjardins [36], Hoff [66,68], Hoff and Serre [69], Vaigant [124], among others). Pursuing further this direction some authors developed the theory of measure valued solutions in order to deal with rapid oscillations that solutions may develop in a finite time (see DiPerna [38], Málek et al. [93]).

The weak formulation of the full Navier–Stokes–Fourier system was introduced in [50], following the previous studies [42,49]. The idea of replacing the entropy equation by entropy inequality + total energy balance is reminiscent of the concept of weak solution with "defect measure" employed, for instance, by Alexandre and Villani [3] in the context of the Boltzmann equation.

An alternative way to avoid the mathematical difficulties inherent to the theory of Newtonian fluids consists in introducing more complex constitutive equations relating the viscous stress and the heat flux to the affinities $\nabla_x \mathbf{u}$, $\nabla_x \vartheta$. Thus, for example, the mathematical theory of viscous *multipolar* fluids, based on the general ideas of Green and Rivlin [62], was developed by Nečas and Šilhavý [102] in order to provide a general framework for studying viscous, compressible fluids and to present a suitable alternative to the boundary layer theory (see Bellout et al. [15]). The reader may consult the review paper by Málek and Rajagopal [94] for other constitutive theories as well as a comprehensive list of relevant literature.

3. Constitutive theory

In order to develop a rigorous mathematical theory of the Navier–Stokes–Fourier system introduced in Section 2.6, the material properties of the fluid in question must be specified. To this end, we discuss briefly the constitutive theory based on the principles of statistical mechanics applicable to *gases*. Note that for *liquids*, the constitutive theory is mostly based on heuristic and phenomenological arguments that lead in some cases to rather different constitutive equations, in particular for the pressure and the transport coefficients. The basic reference material for this part are the monographs by Becker [12], Bridgman [22], Chapman and Cowling [26], Eliezer et al. [44], Gallavotti [60], Müller and Ruggeri [99], among others.

3.1. *Equation of state*

The thermal equation of state is a relation determining the pressure p in terms of the state variables ϱ, ϑ. The caloric equation of state relates the specific internal energy e to the pressure p.

The simplest example of a caloric equation of state is that of a monoatomic gas, where the molecular pressure p_M is interrelated to the internal energy e_M as

$$p_M(\varrho, \vartheta) = \frac{2}{3}\varrho e_M(\varrho, \vartheta) \tag{3.1}$$

(see Eliezer et al. [44]). As p_M, e_M have to comply with Gibbs' equation (2.29), the only admissible form of p_M reads

$$p_M(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) = \varrho\vartheta \frac{P(Z)}{Z}, \qquad Z = \frac{\varrho}{\vartheta^{3/2}}. \tag{3.2}$$

We recover the perfect gas law provided $P(Z)/Z = \text{const}$.

It is easy to check that the corresponding entropy s_M satisfies

$$s_M(\varrho, \vartheta) = S\left(\frac{\varrho}{\vartheta^{3/2}}\right), \tag{3.3}$$

where

$$S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2}. \tag{3.4}$$

Note that s is uniquely determined by (3.4) up to an additive constant. The specific shape of the entropy function with regards to the so-called third law of thermodynamics is discussed by Belgiorno [13,14].

3.2. *Thermodynamics stability*

The hypothesis of thermodynamics stability asserts that

- the compressibility $\frac{\partial p}{\partial \varrho}$ is strictly positive,
- the specific heat at constant volume $\frac{\partial e}{\partial \vartheta}$ is strictly positive.

Consequently, in the case of a general monoatomic gas discussed above, we deduce from (3.1), (3.2) that

$$\left\{ \begin{array}{l} P'(Z) > 0, \\ \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} > 0 \end{array} \right\} \tag{3.5}$$

for all $Z \geqslant 0$. It can be shown that the hypothesis of thermodynamics stability implies linear stability of any static state of the system (see Bechtel et al. [11]).

Now, as a direct consequence of (3.5), we get

$$Z \mapsto \frac{P(Z)}{Z^{5/3}} \quad \text{is decreasing function in } Z;$$

whence we may denote

$$p_\infty = \lim_{Z \to \infty} \frac{P(Z)}{Z^{5/3}}. \tag{3.6}$$

The value of p_∞ characterizes the properties of the gas in the degenerate regime where $Z \gg 1$. Most of gases in the degenerate regime behave like a Fermi gas for which

$$p_\infty > 0 \tag{3.7}$$

(see Ruggeri and Trovato [112]). This follows from the fact that a gas under these circumstances is a mixture of monoatomic gases among which one component is formed by free electrons behaving as a Fermi gas (see Battaner [10], Eliezer et al. [44]). From the mathematical viewpoint, property (3.7) is crucial providing relatively strong *a priori* estimates on the density.

3.3. *Transport coefficients*

The transport coefficients μ, η, and κ appearing in Newton's and Fourier's law (2.21) and (2.22), respectively, are scalar functions of the absolute temperature ϑ and the density ϱ as the case may be. Here, we assume that the dependence on ϱ is negligible, which seems to be the case at least for gases under normal conditions (see Becker [12]).

Accordingly, we suppose that $\mu = \mu(\vartheta)$, $\eta = \eta(\vartheta)$, and $\kappa = \kappa(\vartheta)$ are continuously differentiable functions of ϑ satisfying, in accordance with the second law of thermodynamics,

$$\mu(\vartheta) \geqslant \mu_0 > 0, \quad \eta(\vartheta) \geqslant 0, \quad \kappa(\vartheta) \geqslant \kappa_0 > 0 \quad \text{for all } \vartheta > 0. \tag{3.8}$$

As a matter of fact, the existence theory discussed below requires certain coercivity properties of μ and κ, in particular,

$$\mu(\vartheta) \to \infty, \quad \kappa(\vartheta) \to \infty \quad \text{for } \vartheta \to \infty.$$

Note that for monoatomic gases $\mu(\vartheta) \approx \sqrt{\vartheta}$, while $\eta \equiv 0$. The interesting fact that the viscosity of a gas is independent of the density is called Maxwell's paradox (see Becker [12]).

3.4. *Effect of thermal radiation*

We report briefly on the effect of thermal radiation on the fluid motion, where a purely phenomenological, or macroscopic, description is adopted, the radiation being treated as a continuous field, and both the wave (classical) and photonic (quantum) aspects are taken into account.

In the quantum picture, the total pressure p in the fluid is augmented, due to the presence of the photon gas, by a radiation component p_R related to the absolute temperature ϑ through the Stefan–Boltzmann law:

$$p_R = \frac{a}{3}\vartheta^4, \quad \text{with a constant } a > 0. \tag{3.9}$$

Furthermore, in accordance with Gibbs' relation (2.29), the specific internal energy e of the fluid must by supplemented with a term

$$e_R = e_R(\varrho, \vartheta) = \frac{a}{\varrho}\vartheta^4, \quad \text{equivalently,} \quad p_R = \frac{1}{3}\varrho e_R, \tag{3.10}$$

while the related specific entropy reads

$$s_R = s_R(\varrho, \vartheta) = \frac{4a}{3}\frac{\vartheta^3}{\varrho}. \tag{3.11}$$

Similarly, the heat conductivity of the fluid is enhanced by a radiation component

$$\mathbf{q}_R = -\kappa_R \vartheta^3 \nabla_x \vartheta, \quad \text{with a constant } \kappa_R > 0. \tag{3.12}$$

It seems worth-noting that certain models take into account also a radiative component of the viscosity although its impact on the motion becomes relevant only under extreme temperature regimes occurring, for instance, in the interiors of big stars (see Oxenius [107]).

At the level of mathematical analysis, the presence of the radiation component in the state equation gives rise to a compactification effect on the temperature field. The fact that the radiation entropy is an *intensive* quantity prevents the fast time oscillations of the temperature regardless the hypothetical appearance of vacuum zones (see [41,42]).

3.5. *Real gas state equation*

Summing up the results discussed in Sections 3.1, 3.4, the state equation of a real gas takes the form

$$p(\varrho, \vartheta) = p_M(\varrho, \vartheta) + p_R(\vartheta). \tag{3.13}$$

In particular, for a (mixture of) monoatomic gas(es), we have

$$p(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4, \tag{3.14}$$

and, similarly,

$$e(\varrho, \vartheta) = \frac{3}{2}\vartheta \frac{\vartheta^{3/2}}{\varrho} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + a\frac{\vartheta^4}{\varrho}, \tag{3.15}$$

$$s(\varrho, \vartheta) = S\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{4a}{3}\frac{\vartheta^3}{\varrho}, \tag{3.16}$$

where the functions P, S were introduced in Section 3.1.

3.6. *Bibliographical comments*

It should be noted that the coupling of radiation effects with fluid dynamics is a complex physical problem, where the complete system of equations should include a new balance law for the intensity of radiation (see e.g. Buet and Després [24], Mihalas [97]), and, strictly speaking, the classical formulation of the equations of motion should be replaced by a relativistic one as the zero mass particles (photons) are involved.

However, there are simplified models based on asymptotic analysis and certain physical hypotheses (specifically, the matter together with the radiation are in local thermodynamical equilibrium), which give rise to the same system of field equations (2.23)–(2.26), where the constitutive equations for the pressure and the heat conductivity coefficient κ contain the extra "radiative" terms depending on the temperature discussed in Section 3.4 (see Oxenius [107]). As a matter of fact, these models were introduced in astrophysics in order to study the dynamics of radiative stars (see [41,42]). More material on this issue can be found in the monographs by Battaner [10] and Bose [17].

4. A priori estimates

A priori estimates represent a corner stone of any mathematical theory related to a non-linear problem. These are bounds imposed in a natural way on the solutions by the fact that they satisfy a system of differential equations endowed with a family of data (initial, boundary, driving forces, among others). The modern theory of non-linear partial differential equations is based on the abstract function spaces notably the Sobolev spaces, that have been identified by means of *a priori* bounds associated to certain classes of model equations. *A priori* estimates are of purely formal character, being derived under the hypothesis that the solution in question is smooth. However, as we shall see below, all *a priori* bounds that can be derived for the Navier–Stokes–Fourier system actually hold even within the much larger class of the weak solutions discussed in Section 1. This is mainly because all nowadays available *a priori* bounds arise as a direct consequence of the energy conservation or the entropy balance already included in the weak formulation of the problem. In this section, we review a complete list of known *a priori* estimates that can be deduced for the Navier–Stokes–Fourier system. The proofs of several estimates are mostly sketched whereas a more detailed analysis may be found in [50].

4.1. *Total mass conservation*

The total mass conservation follows as a direct consequence of (2.23). Indeed it is easy to see that

$$\int_\Omega \varrho(t,\cdot)\,\mathrm{dx} = \int_\Omega \varrho(0,\cdot)\,\mathrm{dx} \quad \text{for any } t \in [0,T]. \tag{4.1}$$

Since ϱ is a non-negative quantity, we deduce that

$$\operatorname*{ess\,sup}_{t\in(0,T)} \left\|\varrho(t)\right\|_{L^1(\Omega)} \leqslant c(\text{data}). \tag{4.2}$$

Such a bound is of particular interest on unbounded domains, where it provides a valuable piece of information concerning the asymptotic behavior of ϱ for $|x| \to \infty$.

4.2. *Energy estimates*

The energy estimates follow directly from the total energy balance (2.26). Indeed we can use a Gronwall type argument in order to see that the total energy of the fluid remains bounded in terms of the initial data on any compact time interval $[0, T]$ as soon as

$$\operatorname*{ess\,sup}_{(t,x)\in(0,T)\times\Omega} \left|\mathbf{f}(t,x)\right| \leqslant c. \tag{4.3}$$

Thus we infer that

$$\operatorname*{ess\,sup}_{t\in(0,T)} \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right)(t)\,\mathrm{dx} \leqslant c(T, \text{data}). \tag{4.4}$$

Hypothesis (4.3) can be relaxed to boundedness of the driving force in a certain Lebesgue norm. On the other hand, the bound (4.3) is satisfactory in most applications.

Under the hypotheses of Section 3.5, we can deduce from (4.4) that

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\sqrt{\varrho}\,\mathbf{u}\|_{L^2(\Omega;R^3)} \leqslant c, \tag{4.5}$$

and, on condition that $p_\infty > 0$ in (3.6),

$$\operatorname*{ess\,sup}_{t\in(0,T)} \left\|\varrho(t)\right\|_{L^{5/3}(\Omega)} \leqslant c. \tag{4.6}$$

Finally, taking the radiation effects into account (see Section 3.5), we obtain

$$\operatorname*{ess\,sup}_{t\in(0,T)} \left\|\vartheta(t)\right\|_{L^4(\Omega)} \leqslant c. \tag{4.7}$$

The energy estimates provide *a priori* bounds uniform in time. They are "conservative" in nature and as such completely reversible in time. In particular, the bounds imposed on the initial data are preserved by the flow, there is no smoothing effect.

4.3. *A priori estimates based on the energy dissipation*

Integrating the entropy balance (2.25) we obtain

$$\int_\Omega \varrho s(\varrho,\vartheta)(\tau)\,\mathrm{dx} + \sigma\big[[0,\tau]\times\overline{\Omega}\big] = \int_\Omega \varrho s(\varrho,\vartheta)(0)\,\mathrm{dx} \quad \text{for a.a. } \tau\in[0,T]. \tag{4.8}$$

Similarly to the preceding considerations, we assume that we can control the right-hand side of (4.8) in terms of the initial data. On the other hand, if Ω is a *bounded* domain, it can be shown that

$$\operatorname*{ess\,sup}_{t\in(0,T)} \int_\Omega \varrho s(\varrho,\vartheta)(t)\,\mathrm{dx} \leqslant c(\text{data}) \tag{4.9}$$

in terms of the energy estimates established above.

Consequently, the entropy production represented by the measure σ must be bounded, in particular,

$$\int_0^T \int_\Omega \frac{1}{\vartheta} \mathbb{S} : \nabla_x \mathbf{u} \, \mathrm{dx} + \int_0^T \int_\Omega \kappa(\vartheta) \frac{|\nabla_x \vartheta|^2}{\vartheta^2} \, \mathrm{dx} \leqslant c(\text{data}). \tag{4.10}$$

Since our basic hypothesis requires the transport coefficients μ and κ to be bounded below away from zero, we immediately deduce that

$$\int_0^T \int_\Omega \left| \nabla_x \log(\vartheta) \right|^2 \mathrm{dx} \, \mathrm{d}t \leqslant c(\text{data}), \tag{4.11}$$

together with

$$\int_0^T \int_\Omega \frac{\mu(\vartheta)}{\vartheta} \left| \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{I} \right|^2 \mathrm{dx} \, \mathrm{d}t \leqslant c. \tag{4.12}$$

If, in addition, the heat conductivity coefficient κ satisfies certain coercivity properties as, for instance, (3.12), we get

$$\int_0^T \int_\Omega |\nabla_x \vartheta|^2 \, \mathrm{dx} \, \mathrm{d}t \leqslant c(\text{data}). \tag{4.13}$$

The estimates on the velocity gradient are more delicate. The easy way, of course, is to assume that

$$\mu(\vartheta) \approx \vartheta \quad \text{for } \vartheta \gg 1.$$

Under these circumstances, a generalized version of Korn's inequality can be used in order to deduce from (4.5), (4.12) that

$$\int_0^T \int_\Omega |\nabla_x \mathbf{u}|^2 \, \mathrm{dx} \, \mathrm{d}t \leqslant c(\text{data}). \tag{4.14}$$

Unfortunately, in accordance with the physical background, a realistic behavior of μ is rather

$$\underline{\mu}(1 + \sqrt{\vartheta}\,) \leqslant \mu(\vartheta) \leqslant \overline{\mu}(1 + \sqrt{\vartheta}\,), \tag{4.15}$$

yielding only

$$\int_0^T \int_\Omega \frac{1}{\sqrt{\vartheta}} \left| \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{I} \right|^2 \mathrm{dx} \, \mathrm{d}t \leqslant c.$$

Thus the resulting estimate must be "interpolated" with (4.7), (4.13), in order to obtain

$$\left\| \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{I} \right\|_{L^2(0,T;L^q(\Omega;R^{3\times 3}))} \leqslant c(\text{data}), \tag{4.16}$$

where, in general, $q < 2$.

The concrete value of the Lebesgue exponent q in (4.16) depends on μ, specifically, it can be shown that

$$q = \frac{8}{5 - \alpha} \tag{4.17}$$

as soon as

$$\underline{\mu}\big(1+\vartheta^{\alpha}\big) \leqslant \mu(\vartheta) \leqslant \bar{\mu}\big(1+\vartheta^{\alpha}\big) \quad \text{for all } \vartheta \geqslant 0. \tag{4.18}$$

It turns out that the critical value of α equals $2/5$, more precisely,

$$\frac{2}{5} < \alpha \leqslant 1, \tag{4.19}$$

for which (4.16), (4.17), together with (4.5), (4.6), and the standard Sobolev embedding relations, guarantee that

$$\varrho\mathbf{u} \otimes \mathbf{u} \text{ is bounded in } L^{p}\big((0,T)\times\Omega;\mathbb{R}^{3\times 3}\big) \text{ for a certain } p > 1$$

in terms of the data (see [50] for details).

Unlike their counterparts derived in Section 4.2, the estimates based on dissipation are "irreversible" in time, yielding higher regularity of solutions than that assumed for the initial data. This regularizing effect is instantaneous, meaning available at any *positive* time but not *uniform* in time. These estimates are absolutely necessary in order to develop the existence theory as they prevent fast *oscillations* of $\mathbf{u}$ and ϑ with respect to the spatial variable. On the other hand, the technical condition (4.19) was needed in order to eliminate possible *concentrations* in the convective term. The presence of oscillations and concentrations in the families of solutions to non-linear problems represents one of the principal difficulties to be handled by the mathematical theory (cf. Evans [45]).

4.4. *Renormalized equation of continuity and refined density estimates*

A priori estimates related to a non-linear problem should be at least so strong that all quantities may be equi-integrable, meaning the set of solutions is pre-compact in the Lebesgue space L^1 endowed with the weak topology. The estimates obtained in the previous part were based on boundedness of the total energy of the system, in particular, they are not strong enough in order to guarantee equi-integrability of this quantity, and a similar difficulty occurs for the pressure. In this section, we derive refined density and pressure estimates "computing" directly the pressure term in (2.9).

We start with a renormalized formulation of the equation of continuity (2.1) introduced by DiPerna and P.-L. Lions [39]. Formally, multiplying (2.1) on $b'(\varrho)$, where b is an arbitrary function, we obtain

$$\partial_t b(\varrho) + \operatorname{div}_x\big(b(\varrho)\mathbf{u}\big) + \big(b'(\varrho)\varrho - b(\varrho)\big)\operatorname{div}_x \mathbf{u} = 0 \tag{4.20}$$

to be satisfied in $(0,T)\times\Omega$. Equation (4.20) is equivalent to (2.1) as soon as all quantities in question are smooth. At the level of the weak solutions introduced in Section 1, however, Eq. (4.20) represents an extra piece of information that must be incorporated in the concept of the weak solutions. Note that a slightly more convenient formulation of (4.20) reads

$$\partial_t\big(\varrho B(\varrho)\big) + \operatorname{div}_x\big(\varrho B(\varrho)\mathbf{u}\big) + b(\varrho)\operatorname{div}_x \mathbf{u} = 0 \tag{4.21}$$

to be satisfies in the sense specified in Section 1 for any b, B such that

$$b \in C[0,\infty) \cap L^\infty(0,\infty), \quad b(0) = 0, \quad B(\varrho) = \int_1^\varrho \frac{b(z)}{z^2}\,\mathrm{d}z. \tag{4.22}$$

In the following, we always tacitly suppose that the continuity equation is satisfied also in the sense of renormalized solutions.

In order to deduce more precise pressure and density estimates, we first assume that $\mathbf{u}$ satisfies the complete slip boundary condition (2.3), (2.7). In particular, the quantity

$$\varphi(t,x) = \psi(t)\nabla_x \Delta_N^{-1}\left[b(\varrho) - \frac{1}{|\Omega|}\int_\Omega b(\varrho)\,\mathrm{dx}\right], \quad \psi \in \mathcal{D}(0,T),$$

where Δ_N^{-1} denotes the inverse of the standard Laplacean endowed with the homogeneous Neumann boundary condition, can be taken as an admissible test function in (2.9). Notably the pressure term can be expressed in the form

$$\begin{aligned}
&\int_0^T\int_\Omega \psi p(\varrho,\vartheta)b(\varrho)\,\mathrm{dx} - \frac{1}{|\Omega|}\int_0^T \left(\psi\int_\Omega b(\varrho)\,\mathrm{dx}\right)\left(\int_\Omega p(\varrho,\vartheta)\,\mathrm{dx}\right)\mathrm{d}t \\
&\quad = \int_0^T\int_\Omega \psi\mathbb{S} : \nabla_x\nabla_x\Delta_N^{-1}\left[b(\varrho) - \frac{1}{|\Omega|}\int_\Omega b(\varrho)\,\mathrm{dx}\right]\mathrm{dx}\,\mathrm{d}t \\
&\qquad - \int_0^T\int_\Omega \psi\mathbf{f}\cdot\nabla_x\Delta_N^{-1}\left[b(\varrho) - \frac{1}{|\Omega|}\int_\Omega b(\varrho)\,\mathrm{dx}\right]\mathrm{dx}\,\mathrm{d}t \\
&\qquad - \int_0^T\int_\Omega \psi\varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\nabla_x\Delta_N^{-1}\left[b(\varrho) - \frac{1}{|\Omega|}\int_\Omega b(\varrho)\,\mathrm{dx}\right]\mathrm{dx}\,\mathrm{d}t \\
&\qquad + \int_0^T\int_\Omega \psi\varrho\mathbf{u}\cdot\nabla_x\Delta_N^{-1}\operatorname{div}_x\big(b(\varrho)\mathbf{u}\big)\,\mathrm{dx}\,\mathrm{d}t \\
&\qquad - \int_0^T\int_\Omega \partial_t\psi\varrho\mathbf{u}\cdot\nabla_x\Delta_N^{-1}\left[b(\varrho) - \frac{1}{|\Omega|}\int_\Omega b(\varrho)\,\mathrm{dx}\right]\mathrm{dx}\,\mathrm{d}t \\
&\qquad + \int_0^T\int_\Omega \psi\varrho\mathbf{u}\cdot\nabla_x\Delta_N^{-1}\Big[\big(b(\varrho) - b'(\varrho)\varrho\big)\operatorname{div}_x\mathbf{u} \\
&\qquad - \int_\Omega \big(b(\varrho) - b'(\varrho)\varrho\big)\operatorname{div}_x\mathbf{u}\,\mathrm{dx}\Big]\mathrm{dx}\,\mathrm{d}t.
\end{aligned} \tag{4.23}$$

At this stage, it is a bit tedious but entirely routine matter to combine the uniform estimates obtained in Sections 4.2, 4.3 with the standard elliptic estimates for Δ_N, in order to deduce that all integrals on the right-hand side of (4.23) are already bounded in terms of the data as soon as

$$b(\varrho) \approx \varrho^\beta, \quad \text{with } \beta > 0 \text{ small enough}$$

(see [50,55] for details). Consequently, we conclude that

$$\int_0^T\int_\Omega p(\varrho,\vartheta)\varrho^\beta\,\mathrm{dx}\,\mathrm{d}t \leqslant c(\text{data}). \tag{4.24}$$

Thus we get, by virtue of (3.6), (4.7), (4.13) that

$$\int_0^T \int_\Omega p_M(\varrho, \vartheta)^q \, \mathrm{dx} \, \mathrm{d}t \leqslant c(\text{data}) \tag{4.25}$$

for a certain $q > 1$, and the same estimate can be shown for the internal energy density $\varrho e_M(\varrho, \vartheta)$. Note that similar bounds on the radiation components follow from estimates (4.7), (4.13).

The situation becomes more delicate when $\mathbf{u}$ satisfies the no-slip boundary condition (2.8). In this case, the operator $\nabla_x \Delta^{-1}$ must be replaced by a generalized inverse $\operatorname{div}_x^{-1}$ satisfying the homogeneous Dirichlet boundary condition. More specifically, we can replace $\nabla_x \Delta^{-1}$ by $\mathcal{B}$, where the operator $\mathcal{B}$ enjoys the following properties:

- $\mathcal{B}$ is a bounded operator from $\tilde{L}^p(\Omega)$ to $W^{1,p}(\Omega; \mathbb{R}^3)$ for any $1 < p < \infty$, where $\tilde{L}^p$ is the subspace of $L^p(\Omega)$ of functions of zero mean,
- $$\operatorname{div}_x \mathcal{B}[v] = v, \quad \mathcal{B}[v]|_{\partial\Omega} = 0 \quad \text{for any } v \in \tilde{L}^p(\Omega),$$
- if, in addition, $v = \operatorname{div}_x \mathbf{h}$, where $\mathbf{h} \cdot \mathbf{n}|_{\partial\Omega} = 0$, we have

$$\big\| \mathcal{B}[\operatorname{div}_x \mathbf{h}] \big\|_{L^q(\Omega;\mathbb{R}^3)} \leqslant c(p, q) \|\mathbf{h}\|_{L^q(\Omega;\mathbb{R}^3)} \quad \text{for any } 1 < q < \infty.$$

An example of operator $\mathcal{B}$ was constructed by Bogovskii [16], a detailed analysis of its basic properties may be found in Galdi [59], or Novotný and Straškraba [105].

The pressure estimates derived in this part have "dispersive" character. They do not improve smoothness but assert better integrability of the density in the space–time cylinder.

4.5. *A priori estimates, summary*

In the preceding part we have derived practically all known *a priori* estimates available for the Navier–Stokes–Fourier system. These bounds are strong enough in order to guarantee weak compactness of all quantities in the Lebesgue space L^1 except for the entropy production rate σ, which is known to be bounded only as a non-negative measure on the set $[0, T] \times \Omega$.

The spatial gradients $\nabla_x \mathbf{u}$ and $\nabla_x \vartheta$ are bounded in some Lebesgue norm, in particular, $\mathbf{u}$ and ϑ do not exhibit uncontrollable spatial oscillations. On the other hand, there is no such a bound available for the density ϱ.

The time derivatives of the state variables are bounded only in a very weak sense, in particular, there is no control on the time oscillations of the extensive quantities on the "vacuum" region where $\varrho = 0$.

There is an interesting feature of the weak formulation of the Navier–Stokes–Fourier system used in the present study, namely, all *a priori* estimates are real estimates imposed by the data on *any* weak solution of the problem. This fact plays a crucial role in the asymptotic analysis discussed in Sections 6, 7 below.

4.6. *Bibliographical comments*

The fundamental importance of *a priori* estimates in the theory of non-linear partial differential equations was recognized in the classical monographs by Friedman [58], Ladyzhenskaya et al. [83], or J.-L. Lions [87], among many others. The reader should keep in mind that all *a priori* estimates discussed in the present study are related to the large data solutions without any restriction on the length of the existence interval.

The energy and entropy estimates based on the mechanical energy dissipation are straightforward. The renormalized solutions were introduced by DiPerna and Lions [39] in order to identify a physically relevant class of uniqueness for general transport equations. Their pioneering result have been generalized recently by Ambrosio et al. [4].

A local version of the pressure estimates was used by P.-L. Lions [89] in the context of the isentropic Navier–Stokes system. Their extension up to the boundary of the physical domain was obtained in [55] (see also P.-L. Lions [90]).

Quite recently, a new entropy-like identity conditioned by a very particular form of the viscosity coefficients depending on the *density* ϱ was discovered by Bresch and Desjardins [18,19], Bresch et al. [20] (see also Mellet and Vasseur [96]). This identity gives rise to very strong *a priori* estimates of $\nabla_x \varrho$ in a certain Lebesgue space on condition that the viscosity coefficients $\mu = \mu(\varrho)$, $\eta = \eta(\varrho)$ are interrelated in a specific way.

5. Weak sequential stability

The problem of weak sequential stability represents a central issue of the analysis of any non-linear problem. Having established all *a priori* bounds available we consider a family $\{\varrho_n, \mathbf{u}_n, \vartheta_n\}_{n=1}^{\infty}$ of solutions of the full Navier–Stokes–Fourier system (2.23)–(2.32) assuming, in accordance with (4.5)–(4.7) and (4.16), that

$$\varrho_n \to \varrho \quad \text{weakly-}(*) \text{ in } L^\infty\big(0,T;L^{5/3}(\Omega)\big), \tag{5.1}$$

$$\begin{aligned} \vartheta_n \to \vartheta \quad &\text{weakly-}(*) \text{ in } L^\infty\big(0,T;L^4(\Omega)\big) \\ &\text{and weakly in } L^2\big(0,T;W^{1,2}\big(\Omega;\mathbb{R}^3\big)\big), \end{aligned} \tag{5.2}$$

and

$$\mathbf{u}_n \to \mathbf{u} \quad \text{weakly in } L^2\big(0,T;W^{1,q}\big(\Omega;\mathbb{R}^3\big)\big), \tag{5.3}$$

with $q = 8/(5-\alpha)$, $2/5 < \alpha \leqslant 1$.

Our goal in this section is to show that the limit quantity $\{\varrho, \mathbf{u}, \vartheta\}$ represents another weak solution of the same problem. In particular, we must be able to perform the limit in all non-linear constitutive relations involving both ϱ_n and ϑ_n, which amounts to proving strong (pointwise) convergence of $\{\varrho_n\}_{n=1}^{\infty}$, $\{\vartheta_n\}_{n=1}^{\infty}$. Let us point out that the velocity fields $\mathbf{u}_n$ *do not*, or at least are not known to converge almost everywhere in the cylinder $(0,T)\times\Omega$ but only on the set where the limit density ϱ is strictly positive. The hypothetical existence of the vacuum zones, that means, zones where ϱ vanishes, seems to be one of the major stumbling blocks of the present theory.

5.1. *Preliminaries, Div–Curl lemma, and related results*

Div–Curl lemma developed in the framework of the theory of compensated compactness became one of the most efficient tools of the modern theory of partial differential equations (see Murat [100], Tartar [117], Yi [126]).

LEMMA 5.1 *(DIV–CURL lemma). Let* $\{\mathbf{U}_n\}_{n=1}^\infty$, $\{\mathbf{V}_n\}_{n=1}^\infty$ *be two sequences of vector fields such that*

$$\mathbf{U}_n \to \mathbf{U} \quad \textit{weakly in } L^p\big(\mathbb{R}^N;\mathbb{R}^N\big),$$
$$\mathbf{V}_n \to \mathbf{V} \quad \textit{weakly in } L^q\big(\mathbb{R}^N,\mathbb{R}^N\big),$$

where

$$1 < p, q < \infty, \qquad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1.$$

Assume, *in addition*, *that*

$$\{\operatorname{div}_x \mathbf{U}\}_{n=1}^\infty \quad \textit{is precompact in } W^{-1,s}\big(\mathbb{R}^N\big),$$

and

$$\{\operatorname{curl} \mathbf{V}_n\}_{n=1}^\infty \quad \textit{is precompact in } W^{-1,s}\big(\mathbb{R}^N;\mathbb{R}^{N\times N}\big)$$

for a certain $s > 1$.

Then

$$\mathbf{U}_n \cdot \mathbf{V}_n \to \mathbf{U} \cdot \mathbf{V} \quad \textit{weakly in } L^r\big(\mathbb{R}^N\big).$$

In order to realize the strength of this result, let us point out that any sequence of (weak) solutions of a conservation law

$$\partial_t r_n + \operatorname{div}_x \mathbf{F}_n = s_n$$

can be written in the form

$$\mathrm{DIV}_{t,x}[r_n, \mathbf{F}_n] = s_n$$

in a 4-dimensional space–time cylinder $(0, T) \times \Omega$. Note that the argument of Lemma 5.1 can be easily localized.

On the other hand, consider a family of functions $\{V_n\}_{n=1}^\infty$ such that

$$\|\nabla_x V_n\|_{L^q(0,T;L^q(\Omega;\mathbb{R}^3))} \leqslant c \quad \text{for a certain } q > 1,$$

in particular,

$$\big\|\mathrm{CURL}_{t,x}\big[H(V_n), 0, 0, 0\big]\big\|_{L^q((0,T)\times\Omega,\mathbb{R}^{4\times 4})} \leqslant c$$

for any $H \in W^{1,\infty}(R)$.

Thus a direct application of DIV–CURL lemma implies

$$r_n H(V_n) \to r\,\overline{H(V)} \quad \text{weakly in } L^r\big((0,T)\times\Omega\big) \text{ for any } H \in W^{1,\infty}(\mathbb{R}) \tag{5.4}$$

as soon as

$$r_n \to r, \quad \mathbf{F}_n \to \mathbf{F} \quad \text{weakly in } L^r\big((0,T)\times\Omega\big) \text{ for a certain } r>1,$$
$$H(V_n) \to \overline{H(V)} \quad \text{weakly-}(*) \text{ in } L^\infty\big((0,T)\times\Omega\big),$$

and

$$\{s_n\}_{n=1}^\infty \quad \text{is bounded in } \mathcal{M}\big([0,T]\times\overline{\Omega}\big).$$

Note that (5.4), being a kind of "biting limit" (see Brooks and Chacon [23]), yields

$$r_n V_n \to rV \quad \text{weakly in } L^1\big((0,T)\times\Omega\big) \tag{5.5}$$

as soon as both $\{r_n\}_{n=1}^\infty$ and $\{V_n\}_{n=1}^\infty$ are equi-integrable (weakly precompact) in $L^1((0,T)\times\Omega)$. Here V denotes the corresponding weak limit of $\{V_n\}_{n=1}^\infty$.

In such a way, we can easily identify the weak limit of all convective terms in (2.23)–(2.26). Accordingly, letting $n\to\infty$ we obtain

$$\partial_t\varrho + \operatorname{div}_x(\varrho\mathbf{u}) = 0, \tag{5.6}$$
$$\partial_t(\varrho\mathbf{u}) + \operatorname{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x\overline{p(\varrho,\vartheta)} = \operatorname{div}_x\overline{\mathbb{S}} + \varrho\mathbf{f}, \tag{5.7}$$
$$\partial_t\big(\overline{\varrho s(\varrho,\vartheta)}\big) + \operatorname{div}_x\big(\overline{\varrho s(\varrho,\vartheta)}\mathbf{u}\big) + \operatorname{div}_x\overline{\left(\frac{\mathbf{q}}{\vartheta}\right)} = \overline{\sigma}, \tag{5.8}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \overline{\varrho e(\varrho,\vartheta)}\right)\mathrm{d}x = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x, \tag{5.9}$$

supplemented with the boundary conditions

$$\mathbf{u}\cdot\mathbf{n} = (\mathbb{S}\mathbf{n})\times\mathbf{n}|_{\partial\Omega} = 0, \quad \text{or} \quad \mathbf{u}|_{\partial\Omega} = 0, \tag{5.10}$$
$$\overline{\left(\frac{\mathbf{q}}{\vartheta}\right)}\cdot\mathbf{n}|_{\partial\Omega} = 0. \tag{5.11}$$

We have used the standard notation, where bar denotes a weak L^1-limit of a composed function.

Similarly, we deduce from (4.20)

$$\partial_t\big(\overline{\varrho B(\varrho)}\big) + \operatorname{div}_x\big(\overline{\varrho B(\varrho)}\mathbf{u}\big) + \overline{b(\varrho)\operatorname{div}_x\mathbf{u}} = 0. \tag{5.12}$$

Here, we should note that the same results could be obtained by means of a "more classical" tool, namely the Lions–Aubin lemma (see Lions [87]). However, the arguments via DIV–CURL lemma seem more straightforward and apply also to the case when the production term is represented by a measure.

5.2. *Strong convergence of the temperature*

Our goal is to show that the sequence $\{\vartheta_n\}_{n=1}^\infty$ converges a.a. in $(0,T)\times\Omega$. By virtue of (4.7), (4.13),

$$\vartheta_n \to \vartheta \quad \text{weakly in } L^2\big(0,T;W^{1,2}(\Omega)\big),$$

therefore we have to exclude possible time oscillations. At this stage, the presence of the radiation component of the entropy plays a crucial role.

Let us start with a preliminary result that can be considered as a fundamental theorem of the theory of Young measures (see Ball [7], Pedregal [108]).

THEOREM 5.1. *Let* $\{\mathbf{U}_n\}_{n=1}^{\infty}$ *be an equi-integrable* (*weakly precompact*) *sequence of functions in* $L^1(Q;\mathbb{R}^M)$, $Q \subset \mathbb{R}^N$.

Then $\{\mathbf{U}_n\}_{n=1}^{\infty}$ *possesses a subsequence* (*not relabeled*) *such that there exists a parametrized family of probability measures* $\{\nu_y\}_{y\in Q}$ *on* R^M *enjoying the following property*:

$$\overline{F(\cdot,\mathbf{U})}(y) = \langle \nu_y, F(y,\cdot)\rangle \quad \textit{for a.a. } y \in Q,$$

whenever $F = F(y,\mathbf{U})$ *is a Caratheodory function on* $Q \times \mathbb{R}^M$ *and*

$$F(\cdot,\mathbf{U}_n) \to \overline{F(\cdot,\mathbf{U})} \quad \textit{weakly in } L^1(Q).$$

In accordance with the hypothesis of thermodynamics stability, the entropy is a strictly increasing function of ϑ, more specifically,

$$\int_0^T\int_\Omega \big(\varrho_n s(\varrho_n,\vartheta_n) - \varrho_n s(\varrho_n,\vartheta)\big)(\vartheta_n - \vartheta)\,\mathrm{d}x\,\mathrm{d}t \geqslant \frac{4a}{3}\int_0^T\int_\Omega |\vartheta_n - \vartheta|^4\,\mathrm{d}x\,\mathrm{d}t. \tag{5.13}$$

Here again, we have taken advantage of the radiation component ϱs_R. Consequently, it is enough to observe that the left-hand side of (5.13) tends to zero.

To this end, we first repeat the arguments of Section 5.1 in order to show that

$$\overline{\varrho s(\varrho,\vartheta)\vartheta} = \overline{\varrho s(\varrho,\vartheta)}\vartheta. \tag{5.14}$$

On the other hand, by the same token, we can use the renormalized equation (4.21) to deduce that

$$\overline{b(\varrho)h(\vartheta)} = \overline{b(\varrho)}\,\overline{h(\vartheta)} \quad \text{for all bounded continuous functions } b, h.$$

Such a relation can be expressed in terms of the Young measures as

$$\nu_{t,x}^{(\varrho,\vartheta)} = \nu_{t,x}^{\varrho} \otimes \nu_{t,x}^{\vartheta} \quad \text{for a.a. } (t,x) \in (0,T)\times\Omega, \tag{5.15}$$

where $\nu^{(\varrho,\vartheta)}$, ν^{ϱ}, and ν^{ϑ} denote the Young measure associated to $\{(\varrho_n,\vartheta_n)\}_{n=1}^{\infty}$, $\{\varrho_n\}_{n=1}^{\infty}$, and $\{\vartheta_n\}_{n=1}^{\infty}$, respectively.

Thus, as a direct consequence of Theorem 5.1, we get

$$\varrho_n s(\varrho_n,\vartheta)(\vartheta_n - \vartheta) \to 0 \quad \text{weakly in } L^1\big((0,T)\times\Omega\big),$$

which, together with (5.14) yields the desired conclusion. We infer that

$$\vartheta_n \to \vartheta \quad \text{in } L^4\big((0,T)\times\Omega\big). \tag{5.16}$$

Without the radiation component of s, we would only get pointwise convergence of the temperature on the set where the limit density ϱ is strictly positive. This is the main reason why the effect of radiation is taken into account in the present theory.

5.3. *Strong convergence of the density*

The pointwise convergence of the densities, necessary in order to pass to the limit in the non-linear constitutive relations, represents one of the most delicate tasks of the theory. The main idea is to use the renormalized form of the equation of continuity in order to describe the time evolution of oscillations.

To begin with, let us introduce the cut-off functions

$$T_k(z) = kT\left(\frac{z}{k}\right), \quad z \geqslant 0,\ k \geqslant 1, \tag{5.17}$$

where $T \in C^\infty[0, \infty)$ satisfies

$$T(z) = \begin{cases} z & \text{for } 0 \leqslant z \leqslant 1, \\ \text{concave} & \text{for } 1 \leqslant z \leqslant 3, \\ 2 & \text{for } z \geqslant 3. \end{cases}$$

By virtue of (5.12), we have

$$\partial_t\left(\overline{\varrho L_k(\varrho)}\right) + \operatorname{div}_x\left(\overline{\varrho L_k(\varrho)}\mathbf{u}\right) + \overline{T_k(\varrho)\operatorname{div}_x \mathbf{u}} = 0, \tag{5.18}$$

where

$$L_k(\varrho) = \int_1^\varrho \frac{T_k(z)}{z^2}\,\mathrm{d}z.$$

The next natural step is to write the renormalized equation for the limit ϱ, $\mathbf{u}$, namely

$$\partial_t\left(\varrho L_k(\varrho)\right) + \operatorname{div}_x\left(\varrho L_k(\varrho)\mathbf{u}\right) + T_k(\varrho)\operatorname{div}_x \mathbf{u} = 0. \tag{5.19}$$

However, at this stage, it is not obvious that (5.19) holds. It is worth noting that the regularizing technique introduced by DiPerna and P.-L. Lions [39] does not apply here because of the low degree of integrability of ϱ (and also $\mathbf{u}$). Consequently, in order to proceed, we evoke the method developed in [47] introducing the oscillations defect measure associated to the sequence $\{\varrho_n\}_{n=1}^\infty$ as

$$\mathbf{osc}_p[\varrho_n \to \varrho](Q) = \sup_{k\geqslant 1}\left(\limsup_{n\to\infty}\int_Q \left|T_k(\varrho_n) - T_k(\varrho)\right|^p\,\mathrm{d}x\,\mathrm{d}t\right). \tag{5.20}$$

We report the following result (see [53,50]):

PROPOSITION 5.1. *Let $Q \subset (0,T)\times\Omega$ be a domain. Suppose that*

$$\varrho_n \to \varrho \quad \textit{weakly in } L^1(Q),$$
$$\mathbf{u}_n \to \mathbf{u} \quad \textit{weakly in } L^r\left(Q;\mathbb{R}^3\right),$$
$$\nabla_x \mathbf{u}_n \to \nabla_x \mathbf{u} \quad \textit{weakly in } L^r\left(Q;\mathbb{R}^{3\times 3}\right),\ r > 1,$$

and

$$\mathbf{osc}_p[\varrho_n \to \varrho](Q) < \infty \quad \textit{for a certain } p \textit{ such that } \frac{1}{p} + \frac{1}{r} < 1.$$

Let, in addition, $\{\varrho_n, \mathbf{u}_n\}_{n=1}^\infty$ satisfy the renormalized equation (4.21) in $\mathcal{D}'(Q)$. Then ϱ, $\mathbf{u}$ satisfy (4.21) in $\mathcal{D}'(Q)$.

In view of Proposition 5.1, we have to find a suitable oscillations defect measure in order to justify (5.19). To this end, we make use of the quantity termed *effective viscous flux*.

Take

$$\varphi(t,x)=\psi(t)\phi(x)\nabla_x\Delta^{-1}\big[1_\Omega T_k(\varrho)\big],\quad \psi\in\mathcal{D}(0,T),\ \phi\in\mathcal{D}(\Omega)$$

as a test function in the momentum balance (2.9). The symbol Δ^{-1} denotes the inverse of the Laplace operator defined on the whole space $\mathbb{R}^3$ by means of convolution with the Poisson kernel. After a bit lengthy but entirely straightforward manipulation, we obtain

$$\begin{aligned}&\int_0^T\int_\Omega \psi\phi\big(p(\varrho_n,\vartheta_n)T_k(\varrho_n)-\mathbb{S}_n:\mathcal{R}\big[1_\Omega T_k(\varrho_n)\big]\big)\,\mathrm{dx}\,\mathrm{d}t\\&\quad=\int_0^T\int_\Omega \psi\phi\big(\varrho_n\mathbf{u}_n\cdot\mathcal{R}\big[1_\Omega T_k(\varrho_n)\mathbf{u}_n\big]-(\varrho_n\mathbf{u}_n\otimes\mathbf{u}_n):\mathcal{R}\big[1_\Omega T_k(\varrho_n)\big]\big)\,\mathrm{dx}\,\mathrm{d}t\\&\qquad+\sum_{j=1}^6 I_{j,n},\end{aligned}\tag{5.21}$$

where

$$\begin{aligned}I_{1,n}&=-\int_0^T\int_\Omega \psi\phi\varrho_n\mathbf{u}_n\cdot\nabla_x\Delta^{-1}\big[1_\Omega\big(T_k(\varrho_n)-T_k'(\varrho_n)\varrho_n\big)\operatorname{div}_x\mathbf{u}_n)\big]\,\mathrm{dx}\,\mathrm{d}t,\\I_{2,n}&=-\int_0^t\int_\Omega \psi\phi\varrho_n\mathbf{f}\cdot\nabla_x\Delta^{-1}\big[1_\Omega T_k(\varrho_n)\big]\,\mathrm{dx}\,\mathrm{d}t,\\I_{3,n}&=-\int_0^T\int_\Omega \psi p(\varrho_n,\vartheta_n)\nabla_x\phi\cdot\nabla_x\Delta^{-1}\big[1_\Omega T_k(\varrho_n)\big]\,\mathrm{dx}\,\mathrm{d}t,\\I_{4,n}&=\int_0^T\int_\Omega \psi\mathbb{S}_n:\nabla_x\phi\otimes\nabla_x\Delta^{-1}\big[1_\Omega T_k(\varrho_n)\big]\,\mathrm{dx}\,\mathrm{d}t,\\I_{5,n}&=-\int_0^T\int_\Omega \psi(\varrho_n\mathbf{u}_n\otimes\mathbf{u}_n):\nabla_x\phi\otimes\nabla_x\Delta^{-1}\big[1_\Omega T_k(\varrho_n)\big]\,\mathrm{dx}\,\mathrm{d}t,\end{aligned}$$

and

$$I_{6,n}=-\int_0^T\int_\Omega \partial_t\psi\phi\varrho_n\mathbf{u}_n\cdot\nabla_x\Delta^{-1}\big[1_\Omega T_k(\varrho_n)\big]\,\mathrm{dx}\,\mathrm{d}t.$$

The symbol $\mathcal{R}=\mathcal{R}_{i,j}$ denotes a pseudodifferential operator of zero order $\mathcal{R}_{i,j}=\partial_{x_i}\Delta^{-1}\partial_{x_j}$, or, in terms of the Fourier symbol

$$\mathcal{R}_{i,j}[v]=\mathcal{F}^{-1}_{\xi\to x}\left[\frac{\xi_i\xi_j}{|\xi|^2}\mathcal{F}_{x\to\xi}[v]\right],$$

where $\mathcal{F}$ denotes the standard Fourier transform.

Similarly, using

$$\varphi(t,x)=\psi(t)\phi(x)\nabla_x\Delta^{-1}\big[1_\Omega\overline{T_k(\varrho)}\big],\quad \psi\in\mathcal{D}(0,T),\ \phi\in\mathcal{D}(\Omega),$$

as a test function in the weak formulation of the limit equation (5.7) we obtain

$$\int_0^T \int_\Omega \psi\phi\big(\overline{p(\varrho,\vartheta)}\,\overline{T_k(\varrho)} - \mathbb{S} : \mathcal{R}\big[1_\Omega \overline{T_k(\varrho)}\,\big]\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T \int_\Omega \psi\phi\big(\varrho\mathbf{u}\cdot\mathcal{R}\big[1_\Omega \overline{T_k(\varrho)}\mathbf{u}\big] - (\varrho\mathbf{u}\otimes\mathbf{u}) : \mathcal{R}\big[1_\Omega \overline{T_k(\varrho)}\,\big]\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \sum_{j=1}^{6} I_j, \tag{5.22}$$

where

$$I_1 = -\int_0^T \int_\Omega \psi\phi\varrho\mathbf{u}\cdot\nabla_x\Delta^{-1}\big[1_\Omega\overline{\big(T_k(\varrho_n) - T_k'(\varrho_n)\varrho_n\big)\operatorname{div}_x\mathbf{u}_n\big)}\big]\,\mathrm{d}x\,\mathrm{d}t,$$
$$I_2 = -\int_0^t \int_\Omega \psi\phi\varrho\mathbf{f}\cdot\nabla_x\Delta^{-1}\big[1_\Omega \overline{T_k(\varrho)}\,\big]\,\mathrm{d}x\,\mathrm{d}t,$$
$$I_3 = -\int_0^T \int_\Omega \psi\,\overline{p(\varrho,\vartheta)}\nabla_x\phi\cdot\nabla_x\Delta^{-1}\big[1_\Omega \overline{T_k(\varrho)}\,\big]\,\mathrm{d}x\,\mathrm{d}t,$$
$$I_4 = \int_0^T \int_\Omega \psi\mathbb{S} : \nabla_x\phi\otimes\nabla_x\Delta^{-1}\big[1_\Omega \overline{T_k(\varrho)}\,\big]\,\mathrm{d}x\,\mathrm{d}t,$$
$$I_5 = -\int_0^T \int_\Omega \psi(\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla_x\phi\otimes\nabla_x\Delta^{-1}\big[1_\Omega \overline{T_k(\varrho)}\,\big]\,\mathrm{d}x\,\mathrm{d}t,$$

and

$$I_6 = -\int_0^T \int_\Omega \partial_t\psi\phi\varrho\mathbf{u}\cdot\nabla_x\Delta^{-1}\big[1_\Omega \overline{T_k(\varrho)}\,\big]\,\mathrm{d}x\,\mathrm{d}t.$$

Now, we claim that all quantities on the right-hand side of (5.21) tend to their counterparts in (5.22), in particular,

$$\lim_{n\to\infty}\int_0^T \int_\Omega \psi\phi\big(p(\varrho_n,\vartheta_n)T_k(\varrho_n) - \mathbb{S}_n : \mathcal{R}\big[1_\Omega T_k(\varrho_n)\big]\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T \int_\Omega \psi\phi\big(\overline{p(\varrho,\vartheta)}\,\overline{T_k(\varrho)} - \mathbb{S} : \mathcal{R}\big[1_\Omega \overline{T_k(\varrho)}\,\big]\big)\,\mathrm{d}x\,\mathrm{d}t. \tag{5.23}$$

The easy part of the proof of (5.23) is to observe that, by virtue of the regularizing effect of the operator $\nabla_x\Delta^{-1}$, $I_{j,n} \to I_j$ as $n \to \infty$ for any $j = 1, \dots, 6$ (see [50,53] for details).

In order to handle the remaining term, we report the following result that can be viewed as a direct consequence of DIV–CURL lemma.

LEMMA 5.2. *Let*

$$\mathbf{U}_n \to \mathbf{U} \quad \textit{weakly in } L^p\big(\mathbb{R}^3;\mathbb{R}^3\big),$$
$$\mathbf{V}_m \to \mathbf{V} \quad \textit{weakly in } L^q\big(\mathbb{R}^3;\mathbb{R}^3\big),$$

where $p, q \geqslant 1$

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1.$$

Then

$$\mathbf{U}_n \cdot \mathcal{R}[\mathbf{V}_n] - \mathbf{V}_n \cdot \mathcal{R}[\mathbf{U}_n] \to \mathbf{U} \cdot \mathcal{R}[\mathbf{V}] - \mathbf{V} \cdot \mathcal{R}[\mathbf{U}] \quad \textit{weakly in } L^r\big(\mathbb{R}^3\big).$$

In order to see the conclusion of Lemma 5.2, it is enough to rewrite

$$\mathbf{U}_n \cdot \mathcal{R}[\mathbf{V}_n] - \mathbf{V}_n \cdot \mathcal{R}[\mathbf{U}_n] = \big(\mathbf{U}_n - \mathcal{R}[\mathbf{U}_n]\big) \cdot \mathcal{R}[\mathbf{V}_n] - \big(\mathbf{V}_n - \mathcal{R}[\mathbf{V}_n]\big) \cdot \mathcal{R}[\mathbf{U}_n],$$

and to apply Lemma 5.1 as

$$\begin{aligned} &\operatorname{div}_x\big(\mathbf{U}_n - \mathcal{R}[\mathbf{U}_n]\big) = \operatorname{div}_x\big(\mathbf{V}_n - \mathcal{R}[\mathbf{V}_n]\big) = 0, \\ &\quad \mathbf{curl}\, \mathcal{R}[\mathbf{U}_n] = \mathbf{curl}\, \mathcal{R}[\mathbf{V}_n] = 0. \end{aligned}$$

Since

$$\begin{aligned} &\int_0^T \int_\Omega \psi\phi\big(\varrho_n \mathbf{u}_n \cdot \mathcal{R}\big[1_\Omega T_k(\varrho_n)\mathbf{u}_n\big] - (\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n) : \mathcal{R}\big[1_\Omega T_k(\varrho_n)\big]\big)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_0^T \int_{\mathbb{R}^3} \psi \mathbf{u}_n \cdot \big(\mathcal{R}[\phi\varrho_n\mathbf{u}_n]1_\Omega T_k(\varrho_n) - \mathcal{R}\big[1_\Omega T_k(\varrho_n)\big]\phi\varrho_n\mathbf{u}_n\big)\,\mathrm{d}x\,\mathrm{d}t, \end{aligned}$$

we deduce, by means of Lemma 5.2, that

$$\begin{aligned} &\big(\mathcal{R}[\phi\varrho_n\mathbf{u}_n]1_\Omega T_k(\varrho_n) - \mathcal{R}\big[1_\Omega T_k(\varrho_n)\big]\phi\varrho_n\mathbf{u}_n\big) \\ &\quad \to \big(\mathcal{R}[\phi\varrho\mathbf{u}]1_\Omega \overline{T_k(\varrho)} - \mathcal{R}\big[1_\Omega \overline{T_k(\varrho)}\,\big]\phi\varrho\mathbf{u}\big) \quad \text{in } L^2\big(0,T;W^{-1,q'}\big(\mathbb{R}^3;\mathbb{R}^3\big)\big), \\ &\mathbf{u}_n \to \mathbf{u} \quad \text{weakly in } L^2\big(0,T;W^{1,q}\big(\mathbb{R}^3;\mathbb{R}^3\big)\big), \end{aligned}$$

for certain conjugate exponents q, q' provided $\mathbf{u}_n$ was extended as a function belonging to $W^{1,q}(\mathbb{R}^3;\mathbb{R}^3)$ outside Ω. Accordingly, relation (5.23) follows.

At this stage, the crucial observation is that relation (5.23), rewritten in the form

$$\begin{aligned} &\lim_{n\to\infty} \int_0^T \int_\Omega \psi\big(\phi p(\varrho_n,\vartheta_n)T_k(\varrho_n) - T_k(\varrho_n)\mathcal{R} : [\eta\mathbb{S}_n]\big)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_0^T \int_\Omega \psi\big(\phi\,\overline{p(\varrho,\vartheta)}\,\overline{T_k(\varrho)} - \overline{T_k(\varrho)}\mathcal{R} : [\phi\mathbb{S}]\big)\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{5.24}$$

gives rise to

$$\begin{aligned} &\lim_{n\to\infty} \int_0^T \int_\Omega \psi\phi\bigg(p(\varrho_n,\vartheta_n)T_k(\varrho_n) - T_k(\varrho_n)\bigg(\frac{4}{3}\mu(\vartheta_n) + \eta(\vartheta_n)\bigg)\operatorname{div}_x \mathbf{u}_n\bigg)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_0^T \int_\Omega \psi\phi\bigg(\overline{p(\varrho,\vartheta)}\;\overline{T_k(\varrho)} - \overline{T_k(\varrho)}\bigg(\frac{4}{3}\mu(\vartheta) + \eta(\vartheta)\bigg)\operatorname{div}_x \mathbf{u}\bigg)\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{5.25}$$

where the quantity $p - ((4/3)\mu + \eta)\operatorname{div}_x \mathbf{u}$ is usually termed the effective viscous flux.

Note that quantities appearing (5.24) and (5.25) differ by a *commutator* of $\mathcal{R}$ with the operator of multiplication on a scalar function μ. Consequently, in order to see how (5.24) yields (5.25), we need the following result that may be viewed as a particular application of the general theory developed by Coifman and Meyer [30] (see Coifman et al. [29,49]).

LEMMA 5.3. *Let $\mu \in W^{1,2}(\mathbb{R}^3)$ be a scalar function and $\mathbf{V} \in L^r \cap L^1(\mathbb{R}^3;\mathbb{R}^3)$ a vector field, $r > 6/5$.*

Then

$$\begin{aligned}&\big\|\big(\nabla_x\Delta^{-1}\operatorname{div}_x\big)[\mu\mathbf{V}] - \mu\big(\nabla_x\Delta^{-1}\operatorname{div}_x\big)[\mathbf{V}]\big\|_{W^{\omega,p}(\mathbb{R}^3)}\\&\quad\leqslant c\|\mu\|_{W^{1,2}(\mathbb{R}^3)}\|\mathbf{V}\|_{L^r\cap L^1(\mathbb{R}^3;\mathbb{R}^3)}\end{aligned}$$

for a certain $\omega > 0$, $p > 1$.

Since

$$\overline{T_k(\varrho_n)} \to \overline{T_k(\varrho)} \quad \text{in } C_{\mathrm{weak}}\big([0,T];L^q(\Omega)\big) \text{ for any } 1 < q < \infty,$$

Lemma 5.1, together with (5.24), imply (5.25).

On the other hand, relation (5.25) yields immediately

$$\begin{aligned}&\overline{p_M(\varrho,\vartheta)T_k(\varrho)} - \left(\frac{4}{3}\mu(\vartheta)+\eta(\vartheta)\right)\overline{T_k(\varrho)\operatorname{div}_x\mathbf{u}}\\&\quad= \overline{p_M(\varrho,\vartheta)}\,\overline{T_k(\varrho)} - \left(\frac{4}{3}\mu(\vartheta)+\eta(\vartheta)\right)\overline{T_k(\varrho)}\operatorname{div}_x\mathbf{u}.\end{aligned}\tag{5.26}$$

Now, it can be shown that

$$\overline{p_M(\varrho,\vartheta)T_k(\varrho)} - \overline{p_M(\varrho,\vartheta)}\,\overline{T_k(\varrho)} \geqslant c\,\mathbf{osc}_{8/3}[\varrho_n \to \varrho]\big((0,T)\times\Omega\big);\tag{5.27}$$

whence, after a certain manipulation (see [50]), we deduce that

$$\mathbf{osc}_p[\varrho_n \to \varrho]\big((0,T)\times\Omega\big) < \infty \quad \text{for a certain } p > \frac{8}{3+\alpha}.\tag{5.28}$$

Indeed the pressure p_M can be written in the form $p_M = p_{\mathrm{mon}}(\varrho,\vartheta) + p_{\mathrm{conv}}(\varrho)$, where p_{mon} is non-decreasing in ϱ, while $p_{\mathrm{conv}}(\varrho)$ is a convex function $p_{\mathrm{conv}} \approx \varrho^{5/3}$. Moreover, it can be checked by direct inspection that

$$\overline{\varrho^{5/3}T_k(\varrho)} - \overline{\varrho^{5/3}}\,\overline{T_k(\varrho)} \geqslant \lim_{n\to\infty}\int_0^T\!\!\int_\Omega \big|T_k(\varrho_n) - T_k(\varrho)\big|^{8/3}\,\mathrm{d}x\,\mathrm{d}t$$

(for details see [50]).

By virtue of (4.16), (4.17), and Proposition 5.1, the limit functions ϱ, $\mathbf{u}$ satisfy equation (5.19), and, consequently,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\big(\overline{\varrho\log(\varrho)} - \varrho\log(\varrho)\big)\,\mathrm{d}x + \int_\Omega\big(\overline{T_k(\varrho)\operatorname{div}_x\mathbf{u}} - \overline{T_k(\varrho)}\operatorname{div}_x\mathbf{u}\big)\,\mathrm{d}x\\&\quad= \int_\Omega\big(T_k(\varrho)\operatorname{div}_x\mathbf{u} - \overline{T_k(\varrho)}\operatorname{div}_x\mathbf{u}\big)\,\mathrm{d}x.\end{aligned}$$

Letting $k \to \infty$ we conclude that

$$\int_\Omega \big(\overline{\varrho \log(\varrho)} - \varrho \log(\varrho)\big)(\tau)\,\mathrm{dx} \leqslant \int_\Omega \big(\overline{\varrho \log(\varrho)} - \varrho \log(\varrho)\big)(0)\,\mathrm{dx}, \tag{5.29}$$

in other words

$$\varrho_n \to \varrho \quad \text{a.a. on } (0,T) \times \Omega$$

as soon as we choose the initial distribution of the densities precompact in $L^1(\Omega)$.

5.4. *Existence theory*

The *a priori* estimates derived in Section 4, together with the stability property established in Section 5, form a suitable platform for the existence theory based on the concept of the weak solutions in the sense specified in Section 1. We report the following result.

THEOREM 5.2. *Let $\Omega \subset \mathbb{R}^3$ be a domain of class $C^{2+\nu}$. Suppose the viscous stress is given by Newton's rheological law (2.21), while the heat flux satisfies Fourier's law (2.22), where the transport coefficients $\mu = \mu(\vartheta)$, $\eta = \eta(\vartheta)$, $\kappa = \kappa(\vartheta)$ are continuously differentiable functions of the absolute temperature ϑ such that*

$$\big|\mu'(\vartheta)\big| \leqslant c, \qquad \underline{\mu}\big(1+\vartheta^\alpha\big) \leqslant \mu(\vartheta) \leqslant \bar{\mu}(1+\vartheta), \tag{5.30}$$

$$0 \leqslant \eta(\vartheta) \leqslant \overline{\eta}\big(1+\vartheta^\alpha\big), \tag{5.31}$$

for a certain $2/5 < \alpha \leqslant 1$,

$$\underline{\kappa}\big(1+\vartheta^3\big) \leqslant \kappa(\vartheta) \leqslant \bar{\kappa}\big(1+\vartheta^3\big) \tag{5.32}$$

for all $\vartheta \geqslant 0$. Let the pressure be given through formula

$$p(\varrho,\vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4, \quad a > 0, \tag{5.33}$$

where the function $P \in C^1[0,\infty)$ satisfies (3.5), (3.6), and e, s are given by (3.15), (3.16). Finally, assume the initial data take the form

$$\begin{aligned} &\varrho(0,\cdot) = \varrho_0, \quad (\varrho\mathbf{u})(0,\cdot) = \varrho_0\mathbf{u}_0, \quad \textit{and} \\ &\varrho s(\varrho,\vartheta)(0,\cdot) = \varrho_0 s(\varrho_0,\vartheta_0), \end{aligned} \tag{5.34}$$

where

$$\begin{aligned} &0 < \underline{\varrho} \leqslant \varrho_0(x) \leqslant \bar{\varrho}, \quad \mathbf{u}_0 \in L^2\big(\Omega;\mathbb{R}^3\big), \\ &\quad 0 < \underline{\vartheta} \leqslant \vartheta_0(x) \leqslant \overline{\vartheta} \quad \textit{for a.a. } x \in \Omega. \end{aligned} \tag{5.35}$$

Then problem (2.23)–(2.32), supplemented with the initial conditions (5.34), admits a weak solution on the set $(0,T) \times \Omega$ for any $T > 0$.

Since for any weak solution we have $t \mapsto \varrho(t,\cdot)$, $t \mapsto (\varrho\mathbf{u})(t,\cdot)$ weakly continuous in L^1, the initial conditions for ϱ and $\varrho\mathbf{u}$ make sense. On the other hand, it can be shown that

$$\varrho s(\varrho,\vartheta)(t,\cdot) \to \varrho_0 s(\varrho_0,\vartheta_0) \quad \text{weakly in } L^1(\Omega)$$

as soon as $\vartheta_0 \in W^{1,\infty}(\Omega)$.

In the remaining part of this study, we take advantage of the existence theory in order to discuss two fundamental issues concerning the global in time solutions: the behavior of solutions for large times, and the problems involving singular limits.

5.5. *Bibliographical comments*

The problem of *existence* of solutions to various systems studied in the framework of the mathematical fluid dynamics has a long history and very active present. To begin with, a complete list of relevant results lies definitely beyond the scope of the present study. Here, we focus only on the problems related to viscous fluid flows with large data defined on an arbitrary time interval – global-in-time solutions.

After the seminal work of Leray [85], Hopf [73] established the existence of global in time weak solutions of the incompressible Navier–Stokes system on a bounded domain in $\mathbb{R}^3$. Later on, Ladyzhenskya [80,81] showed uniqueness and regularity of the weak solutions in the two-dimensional case.

Another significant step in the theory is achieved by the proof of global existence for the barotropic compressible fluid flow by P.-L. Lions [89]. An extension of this result to the case of more realistic values of the adiabatic coefficient has been obtained in [47,53]. In this context, it is worth-noting that Vaigant and Kazhikhov [125] succeeded in showing global existence of *regular* solutions in the barotropic case in the *two-dimensional* physical space under rather artificial conditions imposed on the bulk viscosity coefficient depending on the density.

The crucial quantity of the existence theory for compressible fluids is the effective viscous flux introduced in Section 5.3. Although the central role of this quantity has been already recognized by Hoff [65] and Serre [116], the real breakthrough was accomplished by P.-L. Lions [89], who first observed the “weak continuity” property stated in (5.25). This result was later generalized to the case of variable viscosity coefficients in [49].

The existence theory for the full Navier–Stokes–Fourier system, based on the entropy inequality and the total energy balance equation, was developed in [50]. Recently, Bresch and Desjardins [19] proposed an alternative approach based on strong *a priori* estimates of the density gradient under the main hypothesis that the shear and bulk viscosity coefficient are explicit functions of the density interrelated in a very specific way. Results for the system subjected by certain symmetry condition were obtained by Jiang and Zhang [75].

Quite recently, a new approach to the existence problem was proposed by Hoff [72], based on the concept of irregular but still unique solutions emanating from small initial data.

The corresponding stationary problem has been studied by many authors, the most recent results can be found in Frehse et al. [57], Mucha and Pokorný [98], or Plotnikov and Sokolowski [109].

In order to conclude, let us point out that almost all the above-mentioned results concern the large data – large time problems in the natural 3-dimensional physical space. On the other hand, the theory is much more complete in the one-dimensional geometry (see the monograph Antontsev et al. [5], Hoff [64,66], Jiang [74], Serre [115], Zlotnik [130], among many others), while the 2-D case differs from the 3-D case only by technical details.

6. Long-time behavior

The theory and applications of infinite dimensional dynamical systems have attracted the attention of scientists for a long time. The long-time behavior as well as other dynamical issues arise in many equations modeling phenomena that change in time. A typical example is provided by the *Navier–Stokes–Fourier system* governing the motion of a general viscous, compressible and heat conducting fluid introduced in Section 2.6. Because of the nonlinearities occurring in these equations, the long-term dynamics has always been expected to be quite complicated and possibly intimately related to the phenomena of turbulence. Most of the recent results is related to reduced system: either the isentropic or isothermal models or, even more restrictedly, the incompressible Navier–Stokes equations. The common feature of these systems is that the mechanical energy is converted into heat while the effect of the resulting temperature changes on the dynamics is completely neglected. Such an assumption may be quite satisfactory on medium time scales but it is definitely not suitable for describing the long-time behavior, in particular, for energetically isolated systems. The main objective of this section is to examine the full system of equations taking into account the second law of thermodynamics and, in particular, the entropy production corresponding to the irreversible transfer of the mechanical energy into heat. Our results may be thought of, in a certain sense, as a mathematical verification of the celebrated minimum entropy production principle (see Onsager [106], Rajagopal and Srinivasa [111]).

Given a bounded driving force $\mathbf{f} = \mathbf{f}(t, x)$, the main issue to be discussed below is the long time behavior of solutions to the complete Navier–Stokes–Fourier system. The central idea is based on the fact that in entropy producing processes like those we have to deal with, the long time dynamics is surprisingly simple: either the system is truly conservative, that means, $\mathbf{f}(t, x) \approx \nabla_x F(x)$, and then all solutions tend to a unique equilibrium (static) state, or all mechanical energy is converted (dissipated) into heat, and, accordingly, the total energy becomes infinite:

$$E(t) = \int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) \mathrm{dx} \to \infty \quad \text{as } t \to \infty. \tag{6.1}$$

As a matter of fact, some *simplified* models provide a more complicated picture. Consider, for instance, an isentropic flow governed by the Navier–Stokes system:

$$\left\{ \begin{array}{l} \partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0, \\ \partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + a \nabla_x \varrho^\gamma = \operatorname{div}_x \mathbb{S} + \varrho \mathbf{f}, \\ \mathbf{u}|_{\partial\Omega} = 0. \end{array} \right\} \tag{6.2}$$

In this model, the entropy production is rather artificially set to be zero therefore the "reversible" part of the internal energy is not converted into heat, which, strangely enough, leads to a more complicated dynamics in the long run. It can be shown that all (weak) solutions of system (6.2) enter, after a certain time depending on the size of the initial data, a bounded absorbing set provided $\gamma > 5/3$, and $\mathbf{f}$ is a bounded measurable function of x and t (see [55]). Moreover, all trajectories are asymptotically compact with respect to a natural "energy" norm, and the system admits a global attractor (see [48]). If, in addition, the function $\mathbf{f}$ is time-periodic, there exists a time periodic solution (see [51]).

An interesting situation occurs when we take $\mathbf{f}(t, x) = \nabla_x F(x)$. In this case, it is known that

$$\left.\begin{array}{l} \varrho\mathbf{u}(t) \to 0 \quad \text{in } L^1\big(\Omega; \mathbb{R}^3\big), \\ \varrho(t) \to \{\text{static states}\} \equiv \{\varrho_s \mid a\nabla_x \varrho_s^\gamma = \varrho_s \nabla_x F\} \quad \text{in } L^\gamma(\Omega) \end{array}\right\} \quad \text{as } t \to \infty$$

provided $\gamma > 3/2$. If, in addition, the level sets $\{x \in \Omega \mid F(x) > k\}$ are connected for any $k \in \mathbb{R}$, then $\varrho(t) \to \varrho_s$ as $t \to \infty$ for a suitable static solution ϱ_s (see [54]). To the best of our knowledge, the problem of convergence to a *single* equilibrium remains open for a general potential F. This problem is closely related to inevitable occurrence of the *vacuum zones* in the static density distribution if the total mass of the fluid is sufficiently small. Such a problem, although very interesting mathematically, seems to be of purely theoretical character as in practical situations the *isentropic*, meaning constant entropy process, is definitely not a good approximation as far as the long-time behavior of the underlying physical system is concerned.

Pursuing further the path of simplification, one can consider the *incompressible* Navier–Stokes system

$$\left\{\begin{array}{l} \operatorname{div}_x \mathbf{u} = 0, \\ \partial_t \mathbf{u} + \operatorname{div}_x(\mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mu\Delta\mathbf{u} + \mathbf{f}, \\ \mathbf{u}|_{\partial\Omega} = 0. \end{array}\right\} \tag{6.3}$$

Here, of course, the case $\mathbf{f} = \nabla_x F$ is not very interesting since $\mathbf{f}$ can be "absorbed" by the pressure, and $\mathbf{u} \to 0$ in $L^2(\Omega; \mathbb{R}^3)$ for $t \to \infty$. On the other hand, for a general $\mathbf{f}$, the dynamics can be quite complex and still not well-understood.

6.1. *Stationary driving force*

Let us examine the complete Navier–Stokes–Fourier system introduced in Section 2.6 driven by a time-independent force $\mathbf{f} = \mathbf{f}(x)$. In addition, in many physically realistic cases, we have $\mathbf{f} = \nabla_x F$ for a Lipschitz potential F. Accordingly, it is easy to deduce from (2.23), (2.26) that the system admits a Lyapunov function, specifically,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F\right](t)\,\mathrm{d}x = 0. \tag{6.4}$$

Moreover, it follows from (2.25) that the total entropy production is finite,

$$\sigma\big[[0, \infty) \times \overline{\Omega}\big] < \infty, \tag{6.5}$$

and

$$\int_{\Omega} \varrho s(\varrho, \vartheta)(t)\,\mathrm{dx} \nearrow S_{\infty} \quad \text{as } t \to \infty. \tag{6.6}$$

Consequently, it is easy to deduce from (6.5) that

$$\varrho\mathbf{u}(t) \to 0 \quad \text{as } t \to \infty,$$

while

$$(\varrho, \vartheta)(t) \to \big\{(\varrho_s, \vartheta_s) \mid \varrho_s = \varrho_s(x),\ \vartheta_s = \text{const} > 0,\ p(\varrho_s, \vartheta_s) = \varrho_s \nabla_x F\big\}.$$

In other words, the ω-limit set of each trajectory $\{\varrho, \varrho\mathbf{u}, \vartheta\}_{t\geqslant 0}$ is formed by *static states* – solutions of the system with zero velocity. In order to determine the limit state, we use the three Lyapunov functionals: (i) the total mass

$$\int_{\Omega} \varrho_s\,\mathrm{dx} = \int_{\Omega} \varrho(t, \cdot)\,\mathrm{dx} \quad \text{for all } t > 0;$$

(ii) the energy

$$\int_{\Omega} \big(\varrho_s e(\varrho_s, \vartheta_s) - \varrho_s F\big)\,\mathrm{dx} = \lim_{t\to\infty} \int_{\Omega} \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F\right](t)\,\mathrm{dx};$$

and (iii) the entropy

$$\int_{\Omega} \varrho_s s(\varrho_s, \vartheta_s)\,\mathrm{dx} = \lim_{t\to\infty} \int_{\Omega} \varrho s(\varrho, \vartheta)(t)\,\mathrm{dx}.$$

It turns out that these three quantities fully determine the limit state ϱ_s, ϑ_s.

If $\mathbf{f} \neq \nabla_x F$, the total energy of the system is no longer a conserved quantity. The continuous supply of the mechanical energy provided by $\mathbf{f}$ is converted into heat by the dissipative effects of viscosity and heat conductivity of the fluid. The resulting "degenerate" internal energy tends to infinity, more precisely,

$$\lim_{t\to\infty} \int_{\Omega} \frac{1}{2}\big(\varrho|\mathbf{u}|^2 + \varrho e(\varrho, \vartheta)\big)(t)\,\mathrm{dx} = \infty.$$

The previous considerations can be summarized in the following assertion (see [56]):

THEOREM 6.1. *Under the hypotheses of Theorem 5.2, assume that the driving force* $\mathbf{f} = \mathbf{f}(x)$ *is a bounded measurable function independent of the time* t.

Then either $\mathbf{f} \neq \nabla_x F$, *and*

$$\int_{\Omega} \frac{1}{2}\big(\varrho|\mathbf{u}|^2 + \varrho e(\varrho, \vartheta)\big)(t)\,\mathrm{dx} \to \infty \quad \textit{as } t \to \infty,$$

or $\mathbf{f} = \nabla_x F$, *and there exist a positive bounded function* ϱ_s *and positive constant* ϑ_s *such that*

$$\varrho\mathbf{u}(t) \to 0 \quad \textit{in } L^1\big(\Omega; \mathbb{R}^3\big),$$
$$\varrho(t) \to \varrho_s \quad \textit{in } L^1(\Omega),$$

and

$$\int_{\Omega} \varrho s(\varrho, \vartheta)(t)\,\mathrm{dx} \to \int_{\Omega} \varrho_s s(\varrho_s, \vartheta_s)\,\mathrm{dx}$$

as $t \to \infty$.

6.2. *Time-dependent driving force*

The general case when $\mathbf{f} = \mathbf{f}(t, x)$ can be handled by means of the following result (see [56]).

THEOREM 6.2. *Under the hypotheses of Theorem 5.2, assume that the driving force* $\mathbf{f}$ *belongs to* $L^\infty(0, \infty; L^\infty(\Omega; \mathbb{R}^3))$.

Then either

$$\int_{\Omega} \frac{1}{2}\big(\varrho|\mathbf{u}|^2 + \varrho e(\varrho, \vartheta)\big)(t)\,\mathrm{dx} \to \infty \quad \text{as } t \to \infty,$$

or

$$\operatorname{ess}\sup_{t\in(0,T)} \int_{\Omega} \frac{1}{2}\big(\varrho|\mathbf{u}|^2 + \varrho e(\varrho, \vartheta)\big)(t)\,\mathrm{dx} = E_\infty < \infty.$$

Furthermore, in the latter case, any sequence of times $\tau_n \to \infty$ *contains a subsequence such that*

$$\mathbf{f}_n \to \nabla_x F \quad \text{weakly-}(*) \text{ in } L^\infty\big(0, T; L^\infty\big(\Omega; \mathbb{R}^3\big)\big)$$

for any $T > 0$, *where we have set*

$$\mathbf{f}_n(t, x) = \mathbf{f}(t + \tau_n, x),$$

and where $F = F(x)$, $F \in W^{1,\infty}(\Omega)$. *The potential* F *may be different for different choices of* $\tau_n \to \infty$.

As a direct consequence of the previous theorem we obtain, in contrast with the isentropic model, that the complete Navier–Stokes–Fourier system under the conservative boundary conditions does not admit time-periodic solutions for time periodic driving forces unless the latter is a gradient of a time independent potential F.

6.3. *Bibliographical comments*

There is a vast literature concerning the long-time behavior of solutions to the incompressible Navier–Stokes system. A good reference material are the monographs by Babin and Vishik [6], Constantin et al. [31,32], Ladyzhenskaya [82], or Temam [119,118], among others. The most recent results are available in the survey of Bardos and Nicolaenko [8].

On the other hand, the results on the long-time behavior for compressible and/or heat conductive fluids emanating form large data are in relatively short supply. Rigorous results in the 1-D case were obtained by Hoff and Ziane [70,71], the 3-D case was studied by Novotný and Straškraba [104,103].

7. Singular limits

Many recent papers and research monographs explain the role of scaling arguments in the rigorous analysis of complex models arising in mathematical fluid dynamics. Such a procedure leads often to simplified systems of equations that capture the essential piece of information on a concrete fluid flow suppressing irrelevant phenomena. These systems typically arise because of a singularity in the governing equations related to the flow regime in question. This approach has become of particular relevance in meteorology, where the huge scale differences in atmospheric flows give rise to a large variety of qualitatively different models (see the survey papers by Klein et al. [79], Klein [77], or the lecture notes of Majda [92]).

Many interesting applications of fluid dynamics involve the asymptotic behavior of solutions as certain parameters vanish or become infinite. As a model case, consider the full Navier–Stokes–Fourier system introduced in Section 2.6:

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0, \tag{7.1}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{Ma^2}\nabla_x p = \operatorname{div}_x \mathbb{S} + \frac{1}{Fr^2}\varrho \nabla_x F, \tag{7.2}$$

$$\partial_t(\varrho s) + \operatorname{div}_x(\varrho s \mathbf{u}) + \operatorname{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma, \tag{7.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{Ma^2}{2}\varrho|\mathbf{u}|^2 + \varrho e - \frac{Ma^2}{Fr^2}\varrho F\right)\mathrm{dx} = 0, \tag{7.4}$$

supplemented with the complete slip boundary conditions

$$\mathbf{u}\cdot\mathbf{n} = (\mathbb{S}\mathbf{n}) \times \mathbf{n}|_{\partial\Omega} = 0, \tag{7.5}$$

and the no-flux boundary conditions

$$\mathbf{q}\cdot\mathbf{n}|_{\partial\Omega} = 0, \tag{7.6}$$

where the thermodynamics functions p, e, and s are interrelated through Gibbs' equation

$$\vartheta D s = De + pD\left(\frac{1}{\varrho}\right), \tag{7.7}$$

$\mathbb{S}$ and $\mathbf{q}$ obey

$$\mathbb{S} = \mu\left(\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^t - \frac{2}{3}\operatorname{div}_x \mathbf{u}\mathbb{I}\right) + \eta \operatorname{div}_x \mathbf{u}\mathbb{I}, \tag{7.8}$$

$$\mathbf{q} = -\kappa \nabla_x \vartheta, \tag{7.9}$$

and the entropy production rate is a non-negative measure on the set $[0,T] \times \overline{\Omega}$ satisfying

$$\sigma \geqslant \frac{1}{\vartheta}\left(Ma^2\,\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right). \tag{7.10}$$

The dimensionless parameters Ma and Fr are called the Mach number and the Froude number, respectively. Our aim is to examine the singular limit

$$Ma = \varepsilon, \quad Fr = \sqrt{\varepsilon}, \quad \varepsilon \to 0. \tag{7.11}$$

The low Mach number flows play a dominant role in many important areas of fluid mechanics including incompressible viscous non-steady aerodynamics, non-linear acoustics, and non-adiabatic atmospheric flows. When *Ma* approaches zero, the pressure becomes almost constant, while the speed of sound tends to infinity. Accordingly, the fluid flow in this asymptotic regime becomes incompressible (isochoric). If simultaneously the Froude number is small, a formal asymptotic expansion produces a very useful model – the Oberbeck–Boussinesq approximation (see Rajagopal et al. [110], Zeytounian [128]) :

$$\mathrm{div}_x\, \mathbf{U} = 0, \tag{7.12}$$

$$\bar{\varrho}\big(\partial_t \mathbf{U} + \mathrm{div}_x(\mathbf{U}\times\mathbf{U})\big) + \nabla_x \Pi = \mathrm{div}_x\big(\mu(\bar{\vartheta})\big(\nabla_x\mathbf{U} + \nabla_x^t\mathbf{U}\big)\big) + r\nabla_x F, \tag{7.13}$$

$$\bar{\varrho}c_p(\bar{\varrho},\bar{\vartheta})\big(\partial_t\Theta + \mathrm{div}_x(\Theta\mathbf{U})\big) - \mathrm{div}_x(G\mathbf{U}) - \mathrm{div}_x\big(\kappa(\bar{\vartheta})\nabla_x\Theta\big) = 0, \tag{7.14}$$

$$r + \bar{\varrho}\alpha(\bar{\varrho},\bar{\vartheta})\Theta = 0, \tag{7.15}$$

where $\bar{\varrho}$ and $\bar{\vartheta}$ represent constant reference values of the density and the temperature, respectively, and

$$G = \bar{\varrho}\bar{\vartheta}\alpha(\bar{\varrho},\bar{\vartheta})F, \tag{7.16}$$

$$c_p(\varrho,\vartheta) = \frac{\partial e(\varrho,\vartheta)}{\partial\vartheta} + \alpha(\varrho,\vartheta)\frac{\vartheta}{\varrho}\frac{\partial p(\varrho,\vartheta)}{\partial\vartheta},$$

$$\alpha(\varrho,\vartheta) = \frac{1}{\varrho}\frac{\partial_\varrho p}{\partial_\vartheta p}(\varrho,\vartheta). \tag{7.17}$$

System (7.12)–(7.15) is supplemented with the boundary conditions

$$\mathbf{U}\cdot\mathbf{n} = \Sigma\mathbf{n}\times\mathbf{n}|_{\partial\Omega} = 0, \quad \text{where } \Sigma = \nabla_x\mathbf{U} + \nabla_x^t\mathbf{U}, \tag{7.18}$$

$$\nabla_x\Theta\cdot\mathbf{n}|_{\partial\Omega} = 0. \tag{7.19}$$

The remaining part of this chapter is devoted to a sketch of a rigorous justification of the limit passage from (7.1)–(7.10) to (7.12)–(7.19). The basic reference material to be consulted for all details is [52].

7.1. *Uniform estimates*

Let us start with a simple heuristic argument. In order to recover the limit system, we need uniform bounds independent of ε that are at least as strong as the standard energy estimates available for the incompressible Navier–Stokes system. In particular, the velocity gradients must be uniformly square integrable. Such estimates are typically provided by (7.10) as soon as we can show that the entropy production rate is of order ε^2. In particular, we get

$$\nabla_x\vartheta \approx \varepsilon,$$

meaning the temperature perturbations of order ε must be bounded, specifically,

$$\frac{\vartheta - \bar{\vartheta}}{\varepsilon} \approx 1 \quad \text{for a certain } \bar{\vartheta}. \tag{7.20}$$

Let $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ be a family of weak solutions to the Navier–Stokes–Fourier system (7.1)–(7.10) defined on a fixed time interval $(0, T)$. To begin with, it is easy to see that the total mass of the fluid is a conserved quantity, specifically,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho_\varepsilon(t,\cdot)\,\mathrm{dx} = 0.$$

Accordingly, we suppose

$$\int_\Omega \big(\varrho_\varepsilon(t,\cdot) - \bar\varrho\big)\,\mathrm{dx} = 0 \quad \text{for all } t \in [0, T], \tag{7.21}$$

where $\bar\varrho > 0$ is a constant independent of ε.

In order to obtain more estimates, we introduce an auxiliary function

$$H_{\bar\vartheta}(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \overline{\vartheta}\varrho s(\varrho, \vartheta), \tag{7.22}$$

which is reminiscent of the Helmholtz free energy.

As a direct consequence of the hypothesis of thermodynamics stability discussed in Section 3.2, the function $H_{\bar\vartheta}$ enjoys two remarkable properties of coercivity:

- $\varrho \mapsto H_{\bar\vartheta}(\varrho, \bar\vartheta)$ is a strictly convex function on $[0, \infty)$, (7.23)
- $\vartheta \mapsto H_{\bar\vartheta}(\varrho, \vartheta)$ admits a strong, non-degenerate local minimum at $\vartheta = \bar\vartheta$. (7.24)

Now the total energy balance (7.4), together with the entropy production equation (7.3), give rise to the total dissipation balance

$$\begin{aligned}
&\int_\Omega \bigg(\frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\bigg[H_{\bar\vartheta}(\varrho_\varepsilon, \vartheta_\varepsilon) - \frac{\partial H_{\bar\vartheta}(\bar\varrho, \bar\vartheta)}{\partial\varrho}(\varrho_\varepsilon - \bar\varrho) - H_{\bar\vartheta}(\bar\varrho, \bar\vartheta)\bigg] \\
&\qquad - \frac{1}{\varepsilon}\varrho_\varepsilon F\bigg)(\tau)\,\mathrm{dx} + \frac{\bar\vartheta}{\varepsilon^2}\sigma_\varepsilon\big[[0,\tau]\times\overline{\Omega}\big] \\
&\quad = \int_\Omega \bigg(\frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\bigg[H_{\bar\vartheta}(\varrho_\varepsilon, \vartheta_\varepsilon) - \frac{\partial H_{\bar\vartheta}(\bar\varrho, \bar\vartheta)}{\partial\varrho}(\varrho_\varepsilon - \bar\varrho) - H_{\bar\vartheta}(\bar\varrho, \bar\vartheta)\bigg] \\
&\qquad - \frac{1}{\varepsilon}\varrho_\varepsilon F\bigg)(0)\,\mathrm{dx}
\end{aligned} \tag{7.25}$$

for a.a. $\tau \in [0, T]$, where we have used (7.21).

The quantity on the right-hand side of (7.25) can be controlled in terms of the initial data. In particular, taking

$$\left\|\frac{\varrho_\varepsilon(0,\cdot) - \bar\varrho}{\varepsilon}\right\|_{L^\infty(\Omega)}, \left\|\frac{\vartheta_\varepsilon(0,\cdot) - \bar\vartheta}{\varepsilon}\right\|_{L^\infty(\Omega)}, \|\mathbf{u}_\varepsilon\|_{L^2(\Omega;\mathbb{R}^3)} \leqslant c \tag{7.26}$$

we easily observe that the right-hand side of (7.25) remains bounded uniformly for $\varepsilon \to 0$.

In order to facilitate future considerations, we introduce the "essential" and "residual" sets

$$\mathcal{M}_{\mathrm{ess}} = \big\{(\varrho, \vartheta) \in \mathbb{R}^2 \mid \bar\varrho/2 < \varrho < 2\bar\varrho,\ \bar\vartheta/2 < \vartheta < 2\bar\vartheta\big\},$$

$$\mathcal{M}_{\rm res} = [0,\infty)^2 \setminus \mathcal{M}_{\rm ess},$$

together with the associated "essential" and "residual" parts of a measurable quantity h_ε:

$$h_\varepsilon = [h_\varepsilon]_{\rm ess} + [h_\varepsilon]_{\rm res},$$
$$[h_\varepsilon]_{\rm ess} = h_\varepsilon 1_{\{(t,x) | [\varrho_\varepsilon(t,x),\vartheta_\varepsilon(t,x)] \in \mathcal{M}_{\rm ess}\}},$$
$$[h]_{\rm res} = h_\varepsilon 1_{\{(t,x) | [\varrho_\varepsilon(t,x),\vartheta_\varepsilon(t,x)] \in \mathcal{M}_{\rm res}\}}.$$

In view of the coercivity properties established in (7.23), (7.24), it is not difficult to show that

$$\left[H_{\bar\vartheta}(\varrho_\varepsilon,\vartheta_\varepsilon) - \frac{\partial H_{\bar\vartheta}(\bar\varrho,\bar\vartheta)}{\partial\varrho}(\varrho_\varepsilon - \bar\varrho) - H_{\bar\vartheta}(\bar\varrho,\bar\vartheta)\right]$$
$$\geqslant \begin{cases} c\big(|\varrho-\bar\varrho|^2 + |\vartheta-\bar\vartheta|^2\big) & \text{if } [\varrho,\vartheta]\in\mathcal{M}_{\rm ess}, \\ c\big(\varrho e(\varrho,\vartheta) + \bar\vartheta\varrho\big|s(\varrho,\vartheta)\big|\big) & \text{if } [\varrho,\vartheta]\in\mathcal{M}_{\rm res}. \end{cases} \tag{7.27}$$

In addition

$$\inf_{[\varrho,\vartheta]\in\mathcal{M}_{\rm res}} H_{\bar\vartheta}(\varrho,\vartheta) \geqslant \inf_{[\varrho,\vartheta]\in\partial\mathcal{M}_{\rm ess}} H_{\bar\vartheta}(\varrho,\vartheta) > 0 \tag{7.28}$$

(see [52]).

Accordingly, as a direct consequence of the total dissipation balance (7.25), we obtain the following uniform estimates

$$\operatorname*{ess\,sup}_{t\in[0,T]} \left\| \left[\frac{\varrho_\varepsilon - \bar\varrho}{\varepsilon}\right]_{\rm ess}\right\|_{L^2(\Omega)} \leqslant c, \qquad \operatorname*{ess\,sup}_{t\in[0,T]} \left\| \left[\frac{\vartheta_\varepsilon - \bar\vartheta}{\varepsilon}\right]_{\rm ess}\right\|_{L^2(\Omega)} \leqslant c, \tag{7.29}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\sqrt{\varrho_\varepsilon}\,\mathbf{u}_\varepsilon\|_{L^2(\Omega;\mathbb{R}^3)} \leqslant c, \tag{7.30}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \big\| \big[\varrho_\varepsilon e(\varrho_\varepsilon,\vartheta_\varepsilon) + \bar\vartheta\varrho_\varepsilon\big|s(\varrho_\varepsilon,\vartheta_\varepsilon)\big|\big]_{\rm res}\big\|_{L^1(\Omega)} \leqslant c, \tag{7.31}$$

and

$$\operatorname*{ess\,sup}_{t\in(0,T)} \big|\big\{x\in\Omega \;\big|\; \big[\varrho_\varepsilon(t,x),\vartheta_\varepsilon(t,x)\big]\in\mathcal{M}_{\rm res}\big\}\big| \leqslant \varepsilon^2 c. \tag{7.32}$$

Moreover,

$$\|\sigma_\varepsilon\|_{\mathcal{M}^+([0,T]\times\overline{\Omega})} \leqslant \varepsilon^2 c, \tag{7.33}$$

in particular, by virtue of (7.10),

$$\left\|\nabla_x \frac{\vartheta_\varepsilon - \bar\vartheta}{\varepsilon}\right\|_{L^2((0,T)\times\Omega;\mathbb{R}^3)}, \left\|\nabla_x \frac{\log(\vartheta_\varepsilon) - \log(\bar\vartheta)}{\varepsilon}\right\|_{L^2((0,T)\times\Omega;\mathbb{R}^3)} \leqslant c, \tag{7.34}$$

and, by means of the standard Korn inequality,

$$\|\nabla_x \mathbf{u}_\varepsilon\|_{L^2((0,T)\times\Omega;\mathbb{R}^{3\times 3})} \leqslant c. \tag{7.35}$$

The uniform estimates obtained above are sufficient in order to pass to a limit in the family $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$, in particular, we obtain

$$\frac{\varrho_\varepsilon - \bar{\varrho}}{\varepsilon} \to \varrho^{(1)} \quad \text{weakly-}(*) \in L^\infty\big(0, T; L^q(\Omega)\big), \tag{7.36}$$

$$\frac{\vartheta_\varepsilon - \bar{\vartheta}}{\varepsilon} \to \vartheta^{(1)} \quad \text{weakly in } L^2\big(0, T; W^{1,2}(\Omega)\big), \tag{7.37}$$

and

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \bar{\varrho}\mathbf{U} \quad \text{weakly-}(*) \text{ in } L^\infty\big(0, T; L^2\big(\Omega; \mathbb{R}^3\big)\big), \tag{7.38}$$

$$\mathbf{u}_\varepsilon \to \mathbf{U} \quad \text{weakly in } L^2\big(0, T; W^{1,2}\big(\Omega; \mathbb{R}^3\big)\big). \tag{7.39}$$

The value of the Lebesgue exponent q in (7.36) is determined by the coercivity properties of the function $H_{\bar{\vartheta}}$ established in (7.23). In particular, if the constant p_∞ in (3.6) is strictly positive, we have $q = 5/3$.

7.2. *Asymptotic limit*

With the uniform estimates established in the preceding part, it is a routine matter to identify the limit system of equations satisfied by the quantities $\varrho^{(1)}$, $\mathbf{U}$, $\vartheta^{(1)}$, with the only problematic issue represented by the convective term in the momentum equation (7.2). Note that

$$\nabla_x p(\varrho_\varepsilon, \vartheta_\varepsilon) = \nabla_x\big(p(\varrho_\varepsilon, \vartheta_\varepsilon) - p(\bar{\varrho}, \bar{\vartheta})\big),$$
$$\frac{1}{\varepsilon}\varrho_\varepsilon \nabla_x F = \frac{\varrho_\varepsilon - \bar{\varrho}}{\varepsilon}\nabla_x F + \frac{\bar{\varrho}}{\varepsilon}\nabla_x F,$$

therefore

$$\nabla_x\left(\frac{p(\varrho_\varepsilon, \vartheta_\varepsilon) - p(\bar{\varrho}, \bar{\vartheta})}{\varepsilon} - \bar{\varrho}F\right)$$
$$\approx \nabla_x\left(\frac{\partial p(\bar{\varrho}, \bar{\vartheta})}{\partial \varrho}\frac{\varrho_\varepsilon - \bar{\varrho}}{\varepsilon} + \frac{\partial p(\bar{\varrho}, \bar{\vartheta})}{\partial \vartheta}\frac{\vartheta_\varepsilon - \bar{\vartheta}}{\varepsilon} - \bar{\varrho}F\right) \to 0,$$

while, in accordance with (7.36), (7.37),

$$\left(\frac{\partial p(\bar{\varrho}, \bar{\vartheta})}{\partial \varrho}\frac{\varrho_\varepsilon - \bar{\varrho}}{\varepsilon} + \frac{\partial p(\bar{\varrho}, \bar{\vartheta})}{\partial \vartheta}\frac{\vartheta_\varepsilon - \bar{\vartheta}}{\varepsilon} - \bar{\varrho}F\right)$$
$$\to \left(\frac{\partial p(\bar{\varrho}, \bar{\vartheta})}{\partial \varrho}\varrho^{(1)} + \frac{\partial p(\bar{\varrho}, \bar{\vartheta})}{\partial \vartheta}\vartheta^{(1)} - \bar{\varrho}F\right). \tag{7.40}$$

Fixing F so that $\int_\Omega F \, \mathrm{d}x = 0$ we conclude that

$$\frac{\partial p(\bar{\varrho}, \bar{\vartheta})}{\partial \varrho}\varrho^{(1)} + \frac{\partial p(\bar{\varrho}, \bar{\vartheta})}{\partial \vartheta}\vartheta^{(1)} - \bar{\varrho}F = 0, \tag{7.41}$$

which yields a desired relation between $\varrho^{(1)}$ and $\vartheta^{(1)}$ facilitating the limit process in the entropy equation (7.3). Indeed setting $\Theta = \vartheta^{(1)}$ we recover (7.14), (7.15) performing the asymptotic limit $\varepsilon \to 0$ in (7.3) (see [52]).

Thus the major unsolved problem is to identify the asymptotic limit of the convective term $\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon$ in (7.2). To this end, let us introduce the Helmholtz decomposition

$$\mathbf{v} = \mathbf{H}[\mathbf{v}] + \mathbf{H}^\perp[\mathbf{v}], \tag{7.42}$$

where

$$\begin{aligned} &\mathbf{H}^\perp[\mathbf{v}] = \nabla_x \Psi, \quad \Delta\Psi = \operatorname{div}_x \mathbf{v} \quad \text{in } \Omega, \\ &\nabla_x \Psi \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \int_\Omega \Psi \,\mathrm{dx} = 0. \end{aligned} \tag{7.43}$$

Applying $\mathbf{H}$ to (7.2), meaning taking $\mathbf{H}[\varphi]$ as a test function in the weak formulation, we deduce that

$$\begin{aligned} &t \mapsto \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathbf{w} \,\mathrm{dx} \text{ is a precompact family in } C[0,T] \\ &\quad \text{for any fixed } \mathbf{w} \in \mathcal{D}\big(\Omega;\mathbb{R}^3\big),\ \operatorname{div}_x \mathbf{w} = 0 \end{aligned}$$

as both singular terms in (7.2) vanish. Such a piece of information, in combination with (7.36), (7.39), is sufficient in order to conclude that

$$\mathbf{H}[\mathbf{u}_\varepsilon] \to \mathbf{H}[\mathbf{U}] = \mathbf{U} \quad \text{in } L^2\big((0,T)\times\Omega;\mathbb{R}^3\big). \tag{7.44}$$

Consequently, in order to identify the asymptotic limit of (7.2), we have to handle the term $\mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}^\perp[\mathbf{u}_\varepsilon]$. More specifically, it is enough to show that

$$\int_0^T \int_\Omega \mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}^\perp[\mathbf{u}_\varepsilon] : \nabla_x \varphi \,\mathrm{dx}\,\mathrm{d}t \to 0 \tag{7.45}$$

for any $\varphi \in C^1((0,T)\times\overline{\Omega};\mathbb{R}^3)$, $\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$, $\operatorname{div}_x \varphi = 0$. It is important to note that we need (7.45) only for *solenoidal* test functions φ. As a matter of fact, strong convergence of the gradient components $\mathbf{H}^\perp[\mathbf{u}_\varepsilon]$ is not expected. As we shall see below, the quantity $\operatorname{div}_x \mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}^\perp[\mathbf{u}_\varepsilon]$ can be written as a sum of a vanishing part and a gradient which is sufficient for (7.45) to hold.

7.3. *Acoustic equation*

In order to describe the possible time oscillations of the gradient part of the velocity, we use the acoustic equation:

$$\varepsilon\partial_t r_\varepsilon + \operatorname{div}_x \mathbf{V}_\varepsilon = \varepsilon \operatorname{div}_x \mathbf{h}^1_\varepsilon + \sigma_\varepsilon, \tag{7.46}$$

$$\varepsilon\partial_t \mathbf{V}_\varepsilon + \omega\nabla_x r_\varepsilon = \varepsilon\big(\operatorname{div}_x \mathbb{H}^2_\varepsilon + \mathbf{h}^3_\varepsilon\big), \tag{7.47}$$

where we have set

$$r_\varepsilon = \frac{1}{\omega}\left(\omega\frac{\varrho_\varepsilon - \bar\varrho}{\varepsilon} + \Lambda\frac{\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon) - \varrho_\varepsilon s(\bar\varrho,\bar\vartheta)}{\varepsilon} - \bar\varrho F\right), \qquad \mathbf{V}_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon,$$

$$\omega = \partial_\varrho p(\bar\varrho, \bar\vartheta) + \frac{|\partial_\vartheta p(\bar\varrho, \bar\vartheta)|^2}{\bar\varrho^2 \partial_\vartheta s(\bar\varrho, \bar\vartheta)|}, \qquad \Lambda = \frac{\partial_\vartheta p(\bar\varrho, \bar\vartheta)}{\bar\varrho \partial_\vartheta s(\bar\varrho, \bar\vartheta)}.$$

Equations (7.46), (7.47) can be deduced directly from (7.1)–(7.3). Moreover, it can be shown, by means of the uniform estimates (7.29)–(7.35), that

$$\|\mathbf{h}_\varepsilon^1\|_{L^1((0,T)\times\Omega;\mathbb{R}^3)}, \|\mathbb{H}_\varepsilon^2\|_{L^1((0,T)\times\Omega;\mathbb{R}^{3\times 3})}, \|\mathbf{h}_\varepsilon^3\|_{L^1((0,T)\times\Omega;\mathbb{R}^3)} \leqslant c$$

uniformly for $\varepsilon \to 0$.

As already pointed out, in order to show (7.45), we have to observe that

$$\operatorname{div}_x\big(\mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}^\perp[\mathbf{u}_\varepsilon]\big) \approx \nabla_x \chi_\varepsilon + \text{“small terms”}. \tag{7.48}$$

We give a formal argument assuming all quantities are sufficiently smooth. However, this formal argument can be used in the rigorous proof as the problem can be reduced to a finite number of modes in the spectral decomposition associated to the wave operator in (7.46), (7.47) (see [52]).

To begin with, we write

$$\mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}^\perp[\mathbf{u}_\varepsilon] \approx \frac{1}{\bar\varrho}\mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon],$$

where, in accordance with (7.43),

$$\mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] = \nabla_x \Psi_\varepsilon, \qquad \Delta\Psi_\varepsilon = \operatorname{div}_x[\varrho_\varepsilon \mathbf{u}_\varepsilon].$$

Consequently,

$$\begin{aligned} &\operatorname{div}_x\big(\mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon]\big) \\ &\quad = \operatorname{div}_x(\nabla_x \Psi_\varepsilon \otimes \nabla_x \Psi_\varepsilon) = \frac{1}{2}\nabla_x|\nabla_x \Psi_\varepsilon|^2 + \Delta\Psi_\varepsilon \nabla_x \Psi_\varepsilon, \end{aligned} \tag{7.49}$$

where, by means of the acoustic system (7.46), (7.47),

$$\begin{aligned} \Delta\Psi_\varepsilon \nabla_x \Psi_\varepsilon = \omega \frac{1}{2}\nabla_x r_\varepsilon^2 + \varepsilon\bigg[&\bigg(\operatorname{div}_x \mathbf{h}_\varepsilon^1 - \frac{\sigma_\varepsilon}{\varepsilon}\bigg)\nabla_x \Psi_\varepsilon - \partial_t(r_\varepsilon \nabla_x \Psi_\varepsilon) \\ &+ \mathbf{H}^\perp\big(\operatorname{div}_x \mathbb{H}_\varepsilon^2 + \mathbf{h}_\varepsilon^3\big)\bigg]. \end{aligned} \tag{7.50}$$

Relations (7.49), (7.49) give rise, at least formally, to (7.48).

7.4. *Bibliographical comments*

The approach pursued in this section leans on the concept of weak solutions to the complete Navier–Stokes–Fourier system developed in [50]. Similarly to the results by Bresch et al. [21], Desjardins et al. [37], Lions and Masmoudi [91] (for more references see the survey paper by Masmoudi [95]) devoted to the barotropic Navier–Stokes system, our theory is based on the uniform bounds available in the framework of weak solutions defined on an arbitrarily large time interval $(0, T)$.

Note that there is an alternative approach proposed in the pioneering paper by Klainerman and Majda [76] (see also Ebin [43]) followed by Danchin [34,35], Hoff [67], Schochet [114,113], among others, which is based on uniform estimates in Sobolev spaces of higher order confined to a possibly very short existence time interval $(0, T)$. The most relevant results for the complete Navier–Stokes–Fourier system in this direction were obtained quite recently by Alazard [1,2].

Formal results oriented towards applications and numerical analysis can be found in Klein et al. [79], and Klein [78]. A nice survey and many open problems are provided by the monographs of Majda [92], Zeytounian [129,127].

Another application of the singular limit approach was given by Hagstrom and Lorentz [63], where they show global-in-time existence for the Navier–Stokes system provided the Mach number is low and the solutions are close to a regular solution of the incompressible system.

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CHAPTER 3

Attractors for Dissipative Partial Differential Equations in Bounded and Unbounded Domains

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1. Introduction

The study of the asymptotic behavior of dynamical systems arising from mechanics and physics is a capital issue, as it is essential, for practical applications, to be able to understand, and even predict, the long time behavior of the solutions of such systems.

A dynamical system is a (deterministic) system which evolves with respect to the time. Such a time evolution can be continuous or discrete (i.e., one only measures the state of the system at given times, e.g., every hour or every day). We will essentially consider continuous dynamical systems in this survey.

In many situations, the evolution of the system can be described by a system of ordinary differential equations (ODEs) of the form

$$y' = f(t, y), \quad y = (y_1, \ldots, y_N), \tag{1.1}$$

together with the initial condition

$$y(\tau) = y_\tau, \quad \tau \in \mathbb{R}. \tag{1.2}$$

Assuming that the above Cauchy problem is well-posed, we can define a family of solving operators $U(t, \tau)$, $t \geqslant \tau$, $\tau \in \mathbb{R}$, acting on some subset Φ of $\mathbb{R}^N$ (called the phase space), i.e.,

$$\begin{aligned} U(t, \tau) : \Phi &\to \Phi, \\ y_\tau &\mapsto y(t), \end{aligned}$$

where $y(t)$ is the solution of (1.1)–(1.2) at time t. It is easy to see that this family of operators satisfies

$$U(\tau, \tau) = \mathrm{Id}, \quad U(t, s) \circ U(s, \tau) = U(t, \tau), \quad t \geqslant s \geqslant \tau,\ \tau \in \mathbb{R},$$

where Id denotes the identity operator. We say that this family of operators forms a process. When the function f does not depend explicitly on the time (in that case, we say that the system is autonomous), we can write

$$U(t, \tau) = S(t - \tau),$$

where the family of operators $S(t)$, $t \geqslant 0$, satisfies

$$S(0) = \mathrm{Id}, \quad S(t) \circ S(s) = S(t + s), \quad t, s \geqslant 0.$$

We say that this family of solving operators $S(t)$, $t \geqslant 0$, which maps the initial datum at $t = 0$ onto the solution at time t, forms a semigroup. Furthermore, we say that the pair $(S(t), \Phi)$ (or $(U(t, \tau), \Phi)$ for a nonautonomous system) is the dynamical system associated with our problem.

The qualitative study of such finite dimensional dynamical systems goes back to the pioneering works of Poincaré on the N-body problem in the beginning of the 20th century (see, e.g., [25]; see also [64] and the references therein for the study of discrete dynamical systems in finite dimensions). In particular, it was discovered, at the very beginning of the theory, that even relatively simple systems of ODEs can generate very complicated (chaotic) behaviors. Furthermore, these systems are extremely sensitive to perturbations, in the sense that trajectories with close, but different, initial data may diverge exponentially. As a consequence, in spite of the deterministic nature of the system, its temporal evolution is unpredictable on time scales larger than some critical value which depends on the error of approximation and on the rate of divergence of close trajectories, and can show typical stochastic behaviors.

Such behaviors have first been observed and established for the pendulum equation perturbed by time periodic external forces, namely,

$$y''(t) + \sin\bigl(y(t)\bigr)\bigl(1 + \epsilon \sin(\omega t)\bigr) = 0,$$

$\epsilon, \omega > 0$. Another, important, example is the Lorenz system, obtained by truncation of the Navier–Stokes equations (more precisely, one considers here a three-mode Galerkin approximation (one in velocity and two in temperature) of the Boussinesq equations),

$$\begin{aligned} x' &= \sigma(y - x), \\ y' &= -xy + rx - y, \\ z' &= xy - bz, \end{aligned}$$

where the positive constants σ, r, and b correspond to the Prandtl number, the Rayleigh number, and the aspect ratio, respectively; in the original work of Lorenz (see [146]), these numbers take the values 10, 28, and $\frac{8}{3}$, respectively. This system gives an approximate description of a two-dimensional layer of fluid heated from below: the warmer fluid formed at the bottom tends to rise, creating convection currents, which is similar to what is observed in the atmosphere. For a sufficiently intense heating, the time evolution has a sensitive dependence on the initial conditions, thus representing a very irregular (chaotic) convection. This fact was used by Lorenz to justify the so-called “butterfly effect”, a metaphor for the imprecision of weather forecast. Other well-known relatively simple systems which exhibit chaotic behaviors are the Minea system [170] and the Rössler system [202].

Very often, the trajectories of such chaotic systems are localized, up to some transient process, in some subset of the phase space having a very complicated geometric structure, e.g., locally homeomorphic to the Cartesian product of $\mathbb{R}^m$ and some Cantor set, which thus accumulates the nontrivial dynamics of the system, the so-called strange attractor (see, e.g., [27]). One noteworthy feature of a strange attractor is its dimension. First, in order for the sensitivity to initial conditions to be possible on the strange attractor, this dimension has to be strictly greater than 2, so that the dimension of the phase space has to be greater than 3; let us assume, for simplicity, that this dimension is equal to 3, as in the Lorenz system. Then the volume of the strange attractor must be equal to 0; indeed, in systems having a strange attractor, one observes a contraction of volumes in the phase space. Thus, the dimension of a strange attractor is noninteger, strictly between 2 and 3, and we need to use other dimensions than the Euclidean dimension to measure it. Several dimensions, which are not equivalent and yield different values of the dimension in concrete applications, can be used (roughly speaking, some notions of dimensions are related to the connectedness of the sets that one measures, others are related to the way that these sets are embedded into the ambient space, for instance). We will mainly consider in this article the box-counting (or entropy) dimension (see below; see also [84]), which we will call the fractal dimension. Other possible notions of dimensions are the Hausdorff dimension or the Lyapunov dimension (see [84]). Thus, the main features of a strange attractor are

- the trajectories (at least those starting from a neighborhood) are attracted to it;
- close, but different, trajectories may diverge;
- it has a noninteger (fractal) dimension (for instance, for the Lorenz system, numerical investigations show that this dimension is close to, but greater than, 2, namely, $2.05\ldots$, which means that there is a "strong" contraction of volumes).

Now, for a distributed system whose initial state is described by functions depending on the spatial variable, the time evolution is usually governed by a system of partial differential equations (PDEs). In that case, the phase space Φ is (a subset of) an infinite dimensional function space; typically, $\Phi = L^2(\Omega)$ or $L^\infty(\Omega)$, where Ω is some domain of $\mathbb{R}^N$. We will thus speak of infinite dimensional dynamical systems.

A first, important, difference, when compared with ODEs, is that the analytical structure of a PDE is much more complicated. In particular, we do not have a unique solvability result in general, or such a result can be very difficult to obtain. We can, for instance, mention the three-dimensional Navier–Stokes equations, for which a proper global well-posedness result is not known yet (see, e.g., [218]). Nevertheless, the global existence and uniqueness of solutions has been proven for a large class of PDEs arising from mechanics and physics, and it is therefore natural to investigate whether the features mentioned above for dynamical systems generated by systems of ODEs, and, in particular, the strange attractor, generalize to systems of PDEs.

Such behaviors can be observed in a large class of PDEs which exhibit some energy dissipation and are called dissipative PDEs. Roughly speaking, the highly complicated behaviors observed in such systems usually arise from the interaction of the following mechanisms:

- energy dissipation in the higher part of the Fourier spectrum;
- external energy income in its lower part (in order to have nontrivial dynamics, the system has to also account for the energy income);
- energy flux from the lower to the higher Fourier modes, due to the nonlinear terms of the equations.

As already mentioned, this class of PDEs contains a large number of equations from mechanics and physics; we can mention for instance reaction–diffusion equations, the incompressible Navier–Stokes equations, pattern formation equations (e.g., the Cahn–Hilliard equation in materials science and the Kuramoto–Sivashinsky equation in combustion), and damped wave equations.

It is worth emphasizing once more that the phase space is an infinite dimensional function space. However, experiments showed that, as in the case of finite dimensional dynamical systems, the trajectories are localized, up to some transient process, in a "thin" invariant subset of the phase space having a very complicated geometric structure, which thus accumulates all the essential dynamics of the system.

From a mathematical point of view, this led to the notion of a global attractor (see [22, 49,51,119,136,137,197,211], and [217]; see also [15] and [195] for some historical comments). Assuming that the problem is well-posed and that the system is autonomous (i.e., that the time does not appear explicitly in the equations), we have, as in the finite dimensional case, the semigroup $S(t)$, $t \geqslant 0$, acting on the phase space Φ, which maps the initial condition onto the solution at time t. Then we say that $\mathcal{A} \subset \Phi$ is the global attractor for $S(t)$ if

(i) it is compact in Φ;

(ii) it is invariant, i.e., $S(t)\mathcal{A} = \mathcal{A}$, $\forall t \geqslant 0$;

(iii) $\forall B \subset \Phi$ bounded,

$$\lim_{t\to+\infty} \operatorname{dist}\big(S(t)B, \mathcal{A}\big) = 0,$$

where dist denotes the Hausdorff semi-distance between sets (we assume that Φ is a metric space with distance d) defined by

$$\operatorname{dist}(A, B) := \sup_{a\in A} \inf_{b\in B} d(a, b).$$

This is equivalent to the following: $\forall B \subset \Phi$ bounded, $\forall \epsilon > 0$, $\exists t_0 = t_0(B, \epsilon)$ such that $t \geqslant t_0$ implies $S(t)B \subset \mathcal{U}_\epsilon$, where $\mathcal{U}_\epsilon$ is the ϵ-neighborhood of $\mathcal{A}$.

We note that it follows from (ii) and (iii) that the global attractor, if it exists, is unique. Furthermore, it follows from (i) that it is essentially thinner than the initial phase space Φ; indeed, in infinite dimensions, a compact set cannot contain a ball and is nowhere dense. It is also not difficult to prove that the global attractor is the smallest (for the inclusion) closed set enjoying the attraction property (iii); it thus appears as a suitable object in view of the study of the long time behavior of the system. It is also the maximal bounded invariant set. We finally note that $\mathcal{A}$ attracts all the trajectories (uniformly with respect to bounded sets of initial data), and not just those starting from a neighborhood. The global attractor is sometimes called the maximal or the universal attractor (which is reasonable in view of the above considerations), although these denominations are less used nowadays.

It has also been early conjectured that the invariant attracting sets mentioned above, and, in particular, the global attractor, should be, in a proper sense, finite dimensional and that the dynamics, restricted to these sets, should be effectively described by a finite number of parameters. The notions of dimensions mentioned above, and, in particular, the fractal dimension, should again be appropriate to measure the dimension of these sets. So, when this conjecture is true, the effective dynamics, restricted to the global attractor, is finite dimensional, even though the initial phase space is infinite dimensional. This also suggests that such systems cannot produce any new dynamics which are not observed in finite dimensions, the infinite dimensionality only bringing (possibly essential) technical difficulties.

Starting from the pioneering works of Ladyzhenskaya (see, e.g., [135,136], and the references therein), this finite dimensional reduction, based on the global attractor, has been given solid mathematical grounds in the past decades for dissipative systems in *bounded domains*. In particular, the existence of the finite dimensional global attractor has been proven for many classes of dissipative PDEs, including the examples mentioned above. We refer the reader to [22,49,119,136,137,197,211], and [217] for extensive reviews on this subject.

Now, the global attractor may present several defaults. Indeed, it may attract the trajectories at a slow rate. Furthermore, in general, it is very difficult, if not impossible, to express the convergence rate in terms of the physical parameters of the problem. This can be seen on the following real Ginzburg–Landau equation in one space dimension:

$$\partial_t u - \nu\partial_x^2 u + u^3 - u = 0, \quad x \in [0, 1],\ \nu > 0,$$
$$u(0, t) = u(1, t) = -1, \quad t \geqslant 0,$$

see Remark 2.25. A second drawback, which can also be seen as a consequence of the first one, is that the global attractor may be sensitive to perturbations; a given system is only an approximation of reality and it is thus essential that the objects that we study are robust under small perturbations. Actually, in general, the global attractor is upper semicontinuous with respect to perturbations, i.e.,

$$\mathrm{dist}(\mathcal{A}_\epsilon, \mathcal{A}_0) \to 0 \quad \text{as } \epsilon \to 0^+,$$

where $\mathcal{A}_0$ is the global attractor associated with the nonperturbed system and $\mathcal{A}_\epsilon$ that associated with the perturbed one, $\epsilon > 0$ being the perturbation parameter. Very roughly speaking, this property means that the global attractor cannot explode under small perturbations. Now, the lower semicontinuity, i.e.,

$$\mathrm{dist}(\mathcal{A}_0, \mathcal{A}_\epsilon) \to 0 \quad \text{as } \epsilon \to 0^+,$$

which, roughly speaking, means that the global attractor cannot implode also, is much more difficult to prove (see, e.g., [195]). Furthermore, this property may not hold. This can already be seen in finite dimensions by considering the following ODE (see [195]):

$$x' = \left(1 - x^2\right)\left(1 - \lambda^2\right), \quad \lambda \in [-1, 1].$$

Then, when $\lambda = 0$, $\mathcal{A}_\lambda = [0, 1]$, whereas $\mathcal{A}_\lambda = \{1\}$ for $\lambda < 0$ and $\mathcal{A}_\lambda = [-\sqrt{\lambda}, 1]$ for $\lambda > 0$. Thus, there is a bifurcation phenomenon at $\lambda = 0$ and the global attractor is not lower semicontinuous at $\lambda = 0$. It thus follows that the global attractor may change drastically under small perturbations. Furthermore, in many situations, the global attractor may not be observable in experiments or in numerical simulations. This can be due to the fact that it has a very complicated geometric structure, but not necessarily. Indeed, we can again consider the above Ginzburg–Landau equation. Then, due to the boundary conditions, $\mathcal{A} = \{-1\}$. Now, this problem possesses many metastable "almost stationary" equilibria which live up to a time $t_\star \sim \mathrm{e}^{\nu^{-1/2}}$. Thus, for ν small, one will not see the global attractor in numerical simulations. Finally, in some situations, the global attractor may fail to capture important transient behaviors. This can be observed, e.g., on some models of one-dimensional Burgers equations with a weak dissipation term (see [28]). In that case, the global attractor is trivial, it is reduced to one exponentially attracting point, but the system presents very rich and important transient behaviors which resemble some modified version of the Kolmogorov law. We can also mention models of pattern formation equations in chemotaxis for which one observes important transient behaviors, i.e., patterns, which are not contained in the global attractor (see [215]).

It is thus also important to construct and study larger objects which contain the global attractor, are more robust under perturbations, attract the trajectories at a fast (typically, exponential) rate, and are still finite dimensional. Two such objects have been proposed, namely, an inertial manifold (see [95]) and an exponential attractor (see [65]). We will discuss these objects in more details in the next sections, with an emphasis on exponential attractors (which are as general as global attractors).

An interesting question is whether one has a similar reduction principle for nonautonomous dissipative PDEs (in bounded domains). A first difference, compared with autonomous systems, is that both the initial and final times play an important role; assuming that the problem is well-posed, it defines a process $U(t,\tau)$, $t \geqslant \tau$, $\tau \in \mathbb{R}$, which maps the initial condition at time τ onto the solution at time t. For such systems, the notion of a global attractor is no longer adequate (in particular, we will not be able to construct proper time independent invariant sets), and one needs to consider other notions of attractors.

A first approach, initiated by Haraux (see [121]) and further studied and developed by Chepyzhov and Vishik (see, e.g., [45] and [49]), is based on the notion of a uniform attractor. Actually, in order to construct the uniform attractor, one considers, together with the initial equations, a whole family of equations. Then one proves the existence of the global attractor for a proper semigroup on an extended phase space, and, finally, projecting this global attractor onto the first component, one obtains the uniform attractor. The major drawback of this approach is that the extended dynamical system is essentially more complicated than the initial one, which leads, for general (translation compact, see Section 3; see also [45] and [49]) time dependences, to an artificial infinite dimensionality of the uniform attractor. This can already be seen on the following simple linear equation:

$$\partial_t u - \Delta_x u = h(t), \quad u|_{\partial\Omega} = 0,$$

in a bounded smooth domain Ω, whose dynamics is simple, namely, one has one exponentially attracting trajectory. However, for more or less general external forces h, the associated uniform attractor has infinite dimension and infinite topological entropy (see [49]).

Nevertheless, for periodic and quasiperiodic time dependences, one has in general finite dimensional uniform attractors (i.e., if the same is true for the global attractor of the corresponding autonomous system). Furthermore, one can derive sharp upper and lower bounds on the dimension of the uniform attractor, so that this approach is appropriate and relevant in those cases.

A second approach, which resembles the so-called kernel sections proposed by Chepyzhov and Vishik (see [44] and [49]), but was studied and developed independently, is based on the notion of a pullback attractor (see, e.g., [62,129], and [207]). In that case, one has a time dependent attractor $\{\mathcal{A}(t),\ t \in \mathbb{R}\}$, contrary to the uniform attractor which is time independent. More precisely, a family $\{\mathcal{A}(t),\ t \in \mathbb{R}\}$ is a pullback attractor for the process $U(t,\tau)$ if

(i) the set $\mathcal{A}(t)$ is compact in Φ, $\forall t \in \mathbb{R}$;
(ii) it is invariant, i.e., $U(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(t)$, $\forall t \geqslant \tau$, $\tau \in \mathbb{R}$;
(iii) it satisfies the following pullback attraction property:

$$\forall B \subset \Phi \text{ bounded}, \ \forall t \in \mathbb{R}, \quad \lim_{s\to+\infty} \operatorname{dist}\bigl(U(t,t-s)B, \mathcal{A}(t)\bigr) = 0.$$

One can prove that, in general, $\mathcal{A}(t)$ has finite dimension, $\forall t \in \mathbb{R}$, see, e.g., [38] and [139]. Now, the attraction property essentially means that, at time t, the attractor $\mathcal{A}(t)$ attracts the bounded sets of initial data coming from the past (i.e., from $-\infty$). However, in (iii), the rate of attraction is not uniform in t, so that the forward convergence does not hold in general (see nevertheless [35,40], and [138] for cases where the forward convergence can be proven). We can illustrate this on the following nonautonomous ODE:

$$y' = f(t, y),$$

where $f(t, y) := -y$ if $t \leqslant 0$, $(-1 + 2t)y - ty^2$ if $t \in [0, 1]$, and $y - y^2$ if $t \geqslant 1$. Then one has the existence of a pullback attractor $\{\mathcal{A}(t),\ t \in \mathbb{R}\}$, namely, $\mathcal{A}(t) = \{0\}$, $\forall t \in \mathbb{R}$. However, for $t \geqslant 1$, every trajectory, different from $\{0\}$, starting from a small neighborhood of 0, will leave this neighborhood, never to enter it again. This clearly contradicts our intuitive understanding of attractors.

So, these two theories of attractors for nonautonomous systems do not yield an entirely satisfactory finite dimensional reduction principle, contrary to the autonomous case, since we have either an artificial infinite dimensionality or no forward attraction in general. We will see below that the construction of exponential attractors allows to overcome the main drawback of pullback attractors, namely, the problem of the forward attraction, as proven in [73]; indeed, one has an exponential uniform control on the rate of attraction. This yields a satisfactory reduction principle for nonautonomous dynamical systems associated with dissipative PDEs in bounded domains.

Now, while the theory of attractors for dissipative dynamical systems in bounded domains is rather well understood, the situation is different for systems in unbounded domains and such a theory has only recently been addressed (and is still progressing), starting from the pioneering works of Abergel [1] and Babin and Vishik [21]. The main difficulty in this theory is the fact that, in contrast to the case of bounded domains discussed above, the dynamics generated by dissipative PDEs in unbounded domains is (as a rule) *purely* infinite dimensional and does not possess any finite dimensional reduction principle. Furthermore, the additional spatial “unbounded” directions lead to the so-called *spatial* chaos and the interactions between spatial and temporal chaotic modes generate a *space–time* chaos which also has no analogue in finite dimensions.

As a consequence, most of the ideas and methods of the classical (finite dimensional) theory of dynamical systems do not work here (as such systems have infinite Lyapunov dimension, infinite topological entropy, ...). Thus, we are faced with dynamical phenomena with new levels of complexity which do not have analogues in the finite dimensional theory and we need to develop a new theory in order to describe such phenomena in an accurate way.

It is also interesting to note that, in the case of bounded domains, the dimension of the global attractor grows at least linearly with respect to the volume of the spatial domain and, thus, for sufficiently large domains, the reduced dynamical system may be too large for reasonable investigations. Furthermore, as shown in [16], the spatial complexity of the system (e.g., the number of topologically different equilibria) grows *exponentially* with respect to the volume of the spatial domain. Therefore, even in the case of relatively small dimensions, the reduced system can be out of reach of reasonable investigations, due to its extremely complicated structure. As a consequence, it seems more natural, at least from a physical point of view, to replace large bounded domains by their limit unbounded ones (e.g., the whole space or cylindrical domains), which, of course, requires a systematic study of dissipative dynamical systems associated with PDEs in unbounded domains.

We will discuss such (for most of them new) developments in Section 5 of this survey.

In a last section, we will briefly discuss extensions of the theory of attractors to ill-posed problems, with an emphasis on the so-called trajectory attractor, see, e.g., [46,47,49], and [210]. Indeed, for many interesting problems, including the three-dimensional Navier–Stokes equations, various types of damped hyperbolic equations (e.g., damped wave equations with supercritical nonlinearities), ..., the well-posedness of the solution operator $S(t)$ has not been proven yet or/and the proper choice of the phase space is not known. Furthermore, e.g., for dissipative systems with non-Lipschitz nonlinearities or for systems arising from the dynamical approach of elliptic boundary value problems in unbounded domains, nonuniqueness results and the ill-posedness of the associated solution operator are known.

2. The global attractor

2.1. *Main definitions*

Let E be a Banach space with norm $\|\cdot\|_E$ (actually, in most results, E can more generally be a complete metric space; furthermore, in some cases, e.g., for the so-called trajectory attractors, see Section 6 (see also Theorem 2.20), even metric spaces may be inadequate). We consider a semigroup $S(t)$, $t \geqslant 0$, acting on E, i.e., we assume that the phase space Φ is the whole space E (it is not difficult to adapt the definitions when Φ is a subset of E),

$$S(t): E \to E, \quad \forall t \geqslant 0, \tag{2.1}$$

$$S(0) = \mathrm{Id}, \tag{2.2}$$

$$S(t+s) = S(t) \circ S(s), \quad \forall t, s \geqslant 0, \tag{2.3}$$

where Id denotes the identity operator. We will also need some continuity property on $S(t)$, and we assume from now on that

$$S(t) \text{ is continuous from } E \text{ into itself, } \forall t \geqslant 0. \tag{2.4}$$

REMARK 2.1.

(a) It was recently proven in [186] that condition (2.4) can be relaxed and that one can prove the existence of global attractors under the following, much weaker, condition:

$$\text{if } x_k \to x \text{ and } S(t)x_k \to y, \text{ then } y = S(t)x. \tag{2.5}$$

A semigroup satisfying (2.5) is called a closed semigroup (see also [244] for another type of condition, contained in (2.5)). Condition (2.5) is also important for concrete applications; indeed, there are situations in which $x_k \to x$ only implies that $S(t)x_k \to S(t)x$ for the weak topology (this is the case, e.g., for the damped wave equation with a nonlinear damping, see [186]). However, in contrast to the usual continuous case, the global attractor may not be connected (see the next subsection) for closed semigroups (even if the initial absorbing set is connected) and some additional assumptions are necessary to guarantee this property.

(b) In general, the operators $S(t)$, $t \geqslant 0$, are not one-to-one (this property is equivalent to the backward uniqueness property, see, e.g., [217]). When $S(t)$, $t > 0$, is one-to-one, we can define its inverse, which we denote by $S(-t)$. It is then easy to see that the family $S(t)$, $t \in \mathbb{R}$, enjoys properties (2.1)–(2.3), and we say that it forms a group acting on E. One new feature of the infinite dimensional theory, compared with the finite dimensional one, is that, in general, as already mentioned, the operators $S(t)$, $t < 0$, are not defined everywhere.

DEFINITION 2.2. A set $X \subset E$ is invariant for $S(t)$ if

$$S(t)X = X, \quad \forall t \geqslant 0.$$

If $S(t)X \subset X$, $\forall t \geqslant 0$, we say that X is positively invariant and, if $X \subset S(t)X$, $\forall t \geqslant 0$, we say that X is negatively invariant.

A first, simple, example of invariant sets is given by fixed points (also called stationary trajectories or solutions) or by sets of fixed points ($a \in E$ is a fixed point if $S(t)a = a$, $\forall t \geqslant 0$). A second example is given by complete trajectories or by sets of complete trajectories. Let u_0 belong to E. Then the forward, or positive, trajectory starting at u_0 is the set

$$\left\{S(t)u_0,\ t \geqslant 0\right\}.$$

A backward, or negative, trajectory ending at u_0, if it exists, is a set of points of the form

$$\bigcup_{t \leqslant 0} u(t), \quad u(t) \in S(-t)^{-1}u_0,\ \forall t \geqslant 0$$

(we can note that a negative trajectory, if it exists, is not necessarily unique). Finally, a complete trajectory through u_0, if it exists, is the union of the positive and a negative trajectories. It is not difficult to show that the positive trajectory is positively invariant, a negative trajectory is negatively invariant, and a complete trajectory is invariant.

Another, more complicated, example of invariant sets is given by ω-limit sets; these sets are also essential in view of the construction of global attractors.

DEFINITION 2.3. Let u_0 belong to E. The ω-limit set of u_0 is the set

$$\omega(u_0) := \overline{\bigcap_{s \geqslant 0} \bigcup_{t \geqslant s} S(t)u_0},$$

where the closure is taken in E. Similarly, for $B \subset E$, the ω-limit set of B is the set

$$\omega(B) := \overline{\bigcap_{s \geqslant 0} \bigcup_{t \geqslant s} S(t)B}.$$

We have the following important characterization of ω-limit sets: $x \in \omega(B)$ if and only if there exist sequences $\{x_k,\ k \in \mathbb{N}\}$ and $\{t_k,\ k \in \mathbb{N}\}$, with $x_k \in B$, $\forall k \in \mathbb{N}$, and $t_k \to +\infty$ as $k \to +\infty$, such that $S(t_k)x_k \to x$ as $k \to +\infty$.

REMARK 2.4. Similarly, we define the α-limit set of B, if it exists, by

$$\alpha(B) := \overline{\bigcap_{s\leqslant 0}\bigcup_{t\leqslant s} S(-t)^{-1}B}.$$

We then have the

PROPOSITION 2.5. *We assume that $B \subset E$, $B \neq \emptyset$, and that there exists $t_0 \geqslant 0$ such that $\bigcup_{t\geqslant t_0} S(t)B$ is relatively compact in E. Then $\omega(B)$ is nonempty, compact, and invariant.*

We are now ready to formulate some *mathematical* concepts of dissipativity. To this end, we need to recall the notions of absorbing and attracting sets for the semigroup $S(t)$.

DEFINITION 2.6.

(i) A bounded set $\mathcal{B}_0 \subset E$ is a bounded absorbing set for $S(t)$ if, $\forall B \subset E$ bounded, $\exists t_0 = t_0(B)$ such that $t \geqslant t_0$ implies $S(t)B \subset \mathcal{B}_0$.

(ii) A set $K \subset E$ is attracting if, $\forall B \subset E$ bounded,

$$\lim_{t\to+\infty} \operatorname{dist}\big(S(t)B, K\big) = 0,$$

where dist (or dist_E if it is necessary to precise the topology) is the Hausdorff semi-distance between sets in E, defined by

$$\operatorname{dist}(A, B) := \sup_{a\in A}\inf_{b\in B} \|a - b\|_E$$

(note that $\operatorname{dist}(A, B) = 0$ only implies $A \subset \bar{B}$).

The existence of an absorbing set is often used as a mathematical definition of a dissipative system. Following this tradition, we give the following definition.

DEFINITION 2.7. The semigroup $S(t)$ is *dissipative* in E if it possesses a bounded absorbing set $B \subset E$.

In applications, this property is usually verified by proving a so-called *dissipative estimate* of the form

$$\big\|S(t)u_0\big\|_E \leqslant Q\big(\|u_0\|_E\big)e^{-\alpha t} + C_*, \quad t \geqslant 0, \tag{2.6}$$

where the monotonic function Q and the positive constants α and C_* are independent of $u_0 \in E$.

REMARK 2.8. A different notion of an absorbing set is considered in [119]: the semigroup $S(t)$ is called point dissipative if there exists a bounded set $\mathcal{B}_0 \subset E$ such that, $\forall u_0 \in E$, $\exists t_0 = t_0(u_0)$ such that $t \geqslant t_0$ implies $S(t)u_0 \in \mathcal{B}_0$.

We however have to note that the above mathematical definition of dissipativity is not sufficient to capture the typical physical properties of dissipative systems (see the introduction). Indeed, let us consider the semigroup generated by the following equation in an infinite dimensional Hilbert space E:

$$y'(t) = y(t)\bigl(1 - \|y(t)\|_E^2\bigr), \quad y(0) = y_0 \in E.$$

Then this semigroup obviously satisfies a dissipative estimate as above and is dissipative according to the above mathematical definition. However, we do not have here an energy dissipation in the higher Fourier modes (in fact, the energy increases or decreases *simultaneously* in all modes depending on whether or not $\|y(t)\|_E \geqslant 1$). Thus, it is difficult to consider this semigroup as a dissipative system from a physical point of view.

In order to avoid such a situation, some kind of *asymptotic compactness* of the semigroup (e.g., the existence of a *compact* absorbing/attracting set; this naturally gives a decay in the higher part of the Fourier spectrum) should be postulated. This asymptotic compactness can naturally be expressed in terms of the so-called Kuratowski measure of noncompactness.

DEFINITION 2.9. Let B be a bounded subset of E. The Kuratowski measure of noncompactness of B is the quantity

$$\kappa(B) := \inf\{d,\ B \text{ has a finite covering with balls of } E \text{ with diameter less than } d\}.$$

The Kuratowski measure of noncompactness enjoys the following properties (see [119]):
- $\kappa(B) = 0$ if and only if B is relatively compact in E;
- $\kappa(B) = \kappa(\bar{B})$;
- $B_1 \subset B_2$ implies $\kappa(B_1) \leqslant \kappa(B_2)$;
- $\kappa(B_1 + B_2) \leqslant \kappa(B_1) + \kappa(B_2)$.

DEFINITION 2.10. We say that the semigroup $S(t)$ is *asymptotically* compact if, for every bounded set $B \subset E$, the Kuratowski measure of noncompactness of the image $S(t)B$ tends to zero as $t \to +\infty$,

$$\lim_{t\to+\infty} \kappa\bigl(S(t)B\bigr) = 0, \quad \forall B \text{ bounded in } E.$$

We are now ready to define the main object of this survey, namely, a global attractor.

DEFINITION 2.11. A set $\mathcal{A} \subset E$ is a global attractor of the semigroup $S(t)$ on E if the following properties are satisfied:
(i) it is a compact subset of E;
(ii) it is invariant, $S(t)\mathcal{A} = \mathcal{A}$, $\forall t \geqslant 0$;
(iii) it is an attracting set for $S(t)$ on E.

It follows from this definition that the dissipativity and asymptotic compactness of the associated semigroup are *necessary* for the existence of a global attractor. As we will see below, these conditions are also *sufficient*.

As already mentioned in the introduction, the global attractor, if it exists, is unique. Furthermore, it is the smallest closed set which attracts the bounded subsets of E and the maximal bounded invariant set. We also note that it attracts the trajectories starting from the whole phase space (uniformly with respect to bounded sets of initial data), and not just those starting from a neighborhood.

We now formulate a simple, but very useful, result on the structure of the global attractor. To do so, we first give the following definition.

DEFINITION 2.12. The kernel (in the terminology of Chepyzhov and Vishik) $\mathcal{K} \subset L^\infty(\mathbb{R}, E)$ of the semigroup $S(t)$ is the set of all bounded complete trajectories of the semigroup $S(t)$, i.e., the functions $u : \mathbb{R} \to E$ such that

$$S(t)u(s) = u(t+s) \quad \text{and} \quad \|u(s)\|_E \leqslant C_u < +\infty, \quad \forall s \in \mathbb{R},\ t \in \mathbb{R}_+.$$

Then we have the following result, which follows from the invariance of $\mathcal{A}$, see, e.g., [22].

THEOREM 2.13. *The global attractor* $\mathcal{A}$ (*if it exists*) *is generated by the set* $\mathcal{K}$ *of all bounded complete trajectories of* $S(t)$,

$$\mathcal{A} = \mathcal{K}(0) := \{u(0),\ u \in \mathcal{K}\}. \tag{2.7}$$

In other words, $u_0 \in \mathcal{A}$ *if and only if there exists a bounded complete trajectory* u *such that* $u(0) = u_0$. *Furthermore*, $\mathcal{A} = \mathcal{K}(s)$, *for every* $s \in \mathbb{R}$.

REMARK 2.14. Together with the concept of a *global* attractor given above, the so-called *local* attractors are widely used in the theory of dynamical systems. Such an attractor only attracts the images of all bounded subsets of some *neighborhood* $\mathcal{U}$ ($\mathcal{A} \subset \mathcal{U}$). The largest neighborhood which satisfies this property is then called the *basin* of attraction of the attractor $\mathcal{A}$. Another weaker concept of an attractor can be obtained by relaxing the attraction property. To be more precise, instead of requiring that all trajectories starting from a bounded subset of the phase space have a *uniform* rate of attraction to the attractor (see Definition 2.6), one may allow every trajectory to have its own (nonuniform) rate of attraction. This leads to the so-called pointwise attractor which has been used, e.g., in the original works of Ladyzhenskaya, see [137] and the references therein.

In some situations, e.g., for equations in unbounded domains, the attraction holds in a weaker topology, defined by some topological space E_1, $E \subset E_1$. To describe such a situation, Babin and Vishik proposed the terminology (E, E_1)-attractor, see [22]. Roughly speaking, an (E, E_1)-attractor attracts the bounded subsets of E in the topology of the space E_1 (thus, the space E is used here only to determine the class of bounded sets). In particular, if E_1 corresponds to E endowed with the weak topology, then one speaks of weak attractors. Furthermore, it is sometimes more convenient (especially, in the theory of the so-called trajectory attractors, see Section 6 below) to use more general classes of "bounded" sets which are not generated by any Banach space E and can be fixed almost arbitrarily. The only property of "bounded" sets which seems to be important for the theory of attractors is the following one.

DEFINITION 2.15. A class $\mathcal{B}$ of subsets of E is called a class of "bounded" sets if

$$B \in \mathcal{B} \text{ and } B_1 \subset B \text{ imply } B_1 \in \mathcal{B}. \tag{2.8}$$

Then, naturally, a set $B \in \mathcal{B}$ is an absorbing set for the semigroup $S(t)$ if it absorbs the images of all "bounded" sets (i.e., all sets belonging to $\mathcal{B}$), see [207].

2.2. *Existence of the global attractor*

As mentioned in the previous subsection, ω-limit sets play an important role in the construction of global attractors. Indeed, one has the following result, based on Proposition 2.5 (see, e.g., [22] and [217]).

THEOREM 2.16. *We assume that the semigroup $S(t)$ is continuous and has a compact absorbing set $\mathcal{B}_0$. Then it possesses the global attractor $\mathcal{A}$ such that $\mathcal{A} = \omega(\mathcal{B}_0)$. Furthermore*, *$\mathcal{A}$ is connected.*

We note that, owing to Proposition 2.5, one only needs to prove the attraction property to have the existence of the global attractor; this property follows from the fact that $\mathcal{B}_0$ is an absorbing set. In concrete situations, the above result will apply to (most) parabolic systems in bounded domains, since one has some compact regularizing effect in finite time. For damped hyperbolic equations and for parabolic equations in unbounded domains, we need a more general result, since such a regularizing effect is not available. However, noting that one has, in some sense, some compact regularizing effect at infinity, the following existence result, due to Babin and Vishik (see, e.g., [22]), can be used in most situations.

THEOREM 2.17. *We assume that the semigroup $S(t)$ is continuous and possesses a compact attracting set. Then it possesses the connected global attractor $\mathcal{A}$. Furthermore*, *if K is a compact attracting set*, *then $\mathcal{A} = \omega(K)$.*

REMARK 2.18. In order to prove that the attractor $\mathcal{A}$ is connected, one only needs the existence of a connected bounded absorbing set. Since the balls in a Banach space are always connected, this property holds automatically if the phase space E is the whole Banach space. In a more general setting, i.e., when E is a metric, or even a topological, space, this assumption should be added in order to ensure the connectedness.

We give another attractor's existence result which exploits the Kuratowski measure of noncompactness (see [119]). Although it is formally equivalent to Theorem 2.17, in practice, it can be used in a more general setting, namely, when the existence of a compact attracting set is difficult to verify directly (however, the existence of such a set a posteriori follows from that of the global attractor), see, e.g., [185] and [198].

THEOREM 2.19. *We assume that the semigroup $S(t)$ is continuous*, *dissipative* (*i.e.*, *it possesses a bounded absorbing set $\mathcal{B}_0$*), *and asymptotically compact* (*in the sense of Definition* 2.10). *Then it possesses the connected global attractor $\mathcal{A}$ such that $\mathcal{A} = \omega(\mathcal{B}_0)$.*

We now discuss a general strategy to verify the conditions of the above attractor's existence theorems in applications.

The existence of a compact *absorbing* set (for Theorem 2.16) is typical of *parabolic* problems in bounded domains for which the semigroup $S(t)$ usually consists of *compact* operators for $t > 0$. In that case, one usually has a *smoothing* property of the form

$$\|S(t)\|_{E_1} \leqslant t^{-\beta} Q(\|u_0\|_E), \quad t \in (0,1],$$

where E_1 is some stronger space (i.e., it is compactly embedded into E) and where the monotonic function Q and the positive constant β are independent of u_0, see [22,119, 217], and the references therein. Then, together with the dissipative estimate (2.6), this smoothing property guarantees that a ball in E_1 with a sufficiently large radius R is a compact absorbing set for $S(t)$. According to Theorem 2.16, this yields the existence of the global attractor $\mathcal{A} \subset E_1$ and its boundedness in E_1.

However, for more general classes of dissipative systems (e.g., damped hyperbolic equations), the smoothing property in *finite* time does not hold and should be replaced by an *asymptotically* smoothing property,

$$S(t) = S_1(t) + S_2(t), \quad S_i(t): E \to E, \quad i = 1, 2, \tag{2.9}$$

where the operators $S_2(t)$ are *compact* for every fixed $t \geqslant 0$ (i.e., $S_2(t)B$ is precompact in E for every bounded subset B of E and every $t \geqslant 0$) and the operators $S_1(t)$ tend to zero as $t \to +\infty$,

$$\lim_{t\to+\infty} \|S_1(t)B\|_E = 0, \quad \text{for every } B \subset E \text{ bounded},$$

where $\|B\|_E := \sup_{x\in B} \|x\|_E$, $B \subset E$ (we emphasize here that only the maps $S(t)$ should be continuous in E, and no additional *continuity* assumption on $S_1(t)$ and $S_2(t)$ is required).

It is not difficult to see that decomposition (2.9) is formally equivalent to the asymptotic compactness (in the sense of the Kuratowski measure of noncompactness, see Definition 2.10) and, consequently, together with the dissipative estimate (2.6), this gives the existence of the global attractor, due to Theorem 2.19 (when the space E is a uniformly convex Banach space, this decomposition can be artificially reduced to that of *continuous* operators $S_1(t)$ and $S_2(t)$, see [114] and [217]).

Very often, in applications, $S_2(t)$ maps E into some stronger space E_1 (which is compactly embedded into E). If, in addition, the operators $S_2(t)$ are uniformly bounded in E_1 as $t \to +\infty$,

$$\sup_{t\in\mathbb{R}_+} \|S_2(t)B\|_{E_1} < +\infty, \quad \text{for every } B \subset E \text{ bounded}, \tag{2.10}$$

decomposition (2.9), together with the dissipative estimate (2.6), guarantee that a ball in E_1 with a sufficiently large radius is a compact *attracting* set for $S(t)$ and one can apply Theorem 2.17 to prove the existence of the global attractor $\mathcal{A}$ and to verify, in addition, that $\mathcal{A}$ is *bounded* in E_1. This is the usual way to verify the further regularity of global attractors, see [22] and [217] for details.

However, it is sometimes very difficult, if not impossible, to verify the additional boundedness (2.10) and the operators $S_2(t)$ may a priori grow as $t \to +\infty$, see, e.g., [54]. In that case, decomposition (2.9) is not strong enough to construct a compact attracting set (at least in a direct way) and Theorem 2.17 is not applicable. Nevertheless, as already mentioned, the boundedness property (2.10) is not necessary to verify the asymptotic compactness and Theorem 2.19 gives the existence of the global attractor $\mathcal{A}$. The main drawback of this scheme is that we now only have the compactness of $\mathcal{A}$ in E and cannot conclude that $\mathcal{A}$ belongs to a more regular space E_1. So, Theorem 2.19 cannot be used to prove further regularity results on the attractors.

We also recall another equivalent definition of asymptotic compactness, namely, in the terminology of Ladyzhenskaya (see, e.g., [137]), $S(t)$ is asymptotically compact if

$$\begin{aligned}&\text{for every } \{x_k,\ k \in \mathbb{N}\} \text{ bounded and } \{t_k,\ k \in \mathbb{N}\} \text{ such that } t_k \to +\infty,\\ &\big\{S(t_k)x_k,\ k \in \mathbb{N}\big\} \text{ is relatively compact in } E.\end{aligned} \tag{2.11}$$

Ball proposed in [24] a general method in order to verify (2.11), based on energy functionals. Roughly speaking, this method is based on the simple observation that a *weakly* convergent sequence in a Hilbert (and, more generally, a reflexive Banach) space converges *strongly* if the corresponding sequence of norms converges to the norm of the limit function. Then, in order to verify (2.11), one first extracts a weakly convergent subsequence from $\{S(t_k)x_k\}$ by using the dissipativity and the fact that bounded subsets are precompact in the weak topology and then verifies the convergence of the norms by passing to the limit in the associated energy equality.

This method was applied with success to many equations, both in bounded and unbounded domains, see [24,34,63,110,113–115,125,126,148,180–182,198,199,217,226–228], and [229].

To conclude, we give a result related to global attractors for abstract classes of "bounded" sets (see [207]) which generalizes the concept of (E, E_1)-attractors (in the terminology of Babin and Vishik) and is very useful, e.g., in the theory of attractors in unbounded domains, for ill-posed dissipative systems, and for attractors in weak topologies.

THEOREM 2.20. *Let E be a topological space and $S(t)$ be a semigroup acting on E. Assume also that a class of "bounded" subsets $\mathcal{B}$ of E satisfying* (2.8) *is given. Let finally $S(t)$ possess a "bounded", compact* (*in E*), *and metrizable absorbing set $B_0 \in \mathcal{B}$ and be continuous on B_0, for every fixed $t \geqslant 0$. Then there exists a compact and "bounded" global attractor $\mathcal{A} \subset B_0$ which is generated by all "bounded" complete trajectories of $S(t)$ in E.*

REMARK 2.21. We refer the reader to [119,135,150,186], and [244] for other existence results for the global attractor.

2.3. *Attractors for semigroups having a global Lyapunov function*

DEFINITION 2.22. Let X be a subset of E and $L : X \to \mathbb{R}$ be a continuous function. The function L is a global Lyapunov function for $S(t)$ on X if

(i) $\forall u_0 \in X$, the function $t \mapsto L(S(t)u_0)$ is decreasing (i.e., L is decreasing along the trajectories);

(ii) if $L(S(t)u_0) = L(u_0)$ for some $t > 0$, then u_0 is a fixed point of $S(t)$ (i.e., L is strictly decreasing along the trajectories which are not reduced to fixed points).

Let $\mathcal{N}$ be the set of all fixed points of $S(t)$,

$$\mathcal{N} := \{z \in E,\ S(t)z = z,\ \forall t \geqslant 0\}.$$

Let $z \in \mathcal{N}$. The unstable set $\mathcal{M}^{\mathrm{un}}(z)$ of z is the set of all points $u \in E$ such that $S(t)u$ is defined for all $t \leqslant 0$ and $\lim_{t\to-\infty} S(t)u = z$. Similarly, the stable set $\mathcal{M}^{\mathrm{s}}(z)$ of z is the set of all points $u \in E$ such that $\lim_{t\to+\infty} S(t)u = z$. More generally, let X be an invariant subset of E. Then the unstable set of X is the (possibly empty) set

$$\mathcal{M}^{\mathrm{un}}(X) := \Big\{u_\star \in E,\ u_\star \text{ belongs to a complete trajectory } u(t),\ t \in \mathbb{R}, \\ \text{and } \lim_{t\to-\infty} \operatorname{dist}\big(u(t), X\big) = 0\Big\}.$$

Similarly, the stable set of X is the (possibly empty) set

$$\mathcal{M}^{\mathrm{s}}(X) := \Big\{u_\star \in E,\ u_\star \text{ belongs to a complete trajectory } u(t),\ t \in \mathbb{R}, \\ \text{and } \lim_{t\to+\infty} \operatorname{dist}\big(u(t), X\big) = 0\Big\}.$$

REMARK 2.23. We assume that $S(t)$ possesses the global attractor $\mathcal{A}$. We can note that $\mathcal{N} \subset \mathcal{A}$. Furthermore, it is not difficult to show that $\mathcal{M}^{\mathrm{un}}(z) \subset \mathcal{A}$, $\forall z \in \mathcal{N}$; we also note that $\mathcal{M}^{\mathrm{un}}(z)$ and $\mathcal{M}^{\mathrm{s}}(z)$ are invariant by $S(t)$. Finally, if X is an invariant set, then $\mathcal{M}^{\mathrm{un}}(X) \subset \mathcal{A}$, and $\mathcal{M}^{\mathrm{un}}(\mathcal{A}) = \mathcal{A}$.

We have the following result on the structure of the global attractor for a semigroup having a global Lyapunov function.

THEOREM 2.24. *We assume that the semigroup $S(t)$ possesses a continuous global Lyapunov function. Then*

$$\mathcal{A} = \mathcal{M}^{\mathrm{un}}(\mathcal{N}).$$

If, furthermore, $\mathcal{N}$ is finite, $\mathcal{N} = \{z_1, \dots, z_m\}$, and $t \mapsto S(t)x$ is continuous, $\forall x \in E$, then

$$\mathcal{A} = \bigcup_{i=1}^{m} \mathcal{M}^{\mathrm{un}}(z_i)$$

and every trajectory $u(t)$, $t \in \mathbb{R}$, lying on $\mathcal{A}$ satisfies

$$\lim_{t\to-\infty} u(t) = z_i, \quad \lim_{t\to+\infty} u(t) = z_j, \quad z_i \neq z_j,\ z_i, z_j \in \mathcal{N}.$$

REMARK 2.25. We further assume that $S(t)$ is differentiable in E (to be more precise, $S(t) \in \mathcal{C}^{1+\delta}(E, E)$, $\delta > 0$), $\forall t \in \mathbb{R}_+$. A fixed point z is hyperbolic if the spectrum of $S'(t)z$ does not intersect the unit circle, $t > 0$. In that case, the unstable set of z, $\mathcal{M}^{\mathrm{un}}(z)$, is a k-dimensional submanifold of E, where k is the stability index of z (see [22] for more details). Therefore, if $\mathcal{N}$ is finite and all the fixed points are hyperbolic, the global attractor $\mathcal{A}$ of a semigroup having a continuous global Lyapunov function is a finite union of smooth finite dimensional submanifolds of the phase space. Such global attractors are called regular attractors by Babin and Vishik (see, e.g., [22]). They also possess several additional "good" properties and, to the best of our knowledge, it is the only general class of attractors for which a more or less complete description of their structure is available. In particular, regular attractors are automatically *exponential*, i.e., for every bounded subset $B \subset E$, the following estimate holds:

$$\operatorname{dist}\bigl(S(t)B, \mathcal{A}\bigr) \leqslant Q\bigl(\|B\|_E\bigr)\mathrm{e}^{-\alpha t}, \quad t \geqslant 0, \tag{2.12}$$

where the positive constant α and the monotonic function Q are independent of B. Furthermore, regular attractors are preserved under general sufficiently regular perturbations (the perturbed system may not have a Lyapunov function and may even be nonautonomous, see [22,77,111,225], and the references therein). Finally, for one-dimensional scalar parabolic equations, it is even possible to find *explicitly* the so-called permutation matrix of the attractor (which shows whether or not two equilibria are connected by a heteroclinic trajectory) and, on some occasions, to describe the topological structure of the attractor in terms of the physical parameters of the problem, see [31,91], and [92] for details. We however note that, although the finiteness of the set of fixed points and the hyperbolicity of these fixed points are, in some proper sense, generic properties, see [22], they are very difficult, if not impossible, to prove for concrete examples and given values of the physical parameters of the problem, except for scalar parabolic equations in one space dimension. Furthermore, even if the regularity of the attractor can be proven, one usually *cannot* compute explicitly the constant α and the function Q in the exponential attraction property (2.12) and these quantities can be extremely bad. Indeed, in the example

$$\begin{aligned} &\partial_t u - \nu\partial_x^2 u + u^3 - u = 0, \quad x \in [0,1],\ \nu > 0, \\ &u(0,t) = u(1,t) = -1, \quad t \geqslant 0, \end{aligned}$$

mentioned in the introduction, the global attractor $\mathcal{A} = \{-1\}$ is obviously regular and one can take $\alpha = 2$ in formula (2.12) (this is determined by the hyperbolicity constant of the equilibrium $u_0 = 1$). However, the function Q satisfies

$$Q(r) \geqslant \mathrm{e}^{2\mathrm{e}^{C\nu^{-1/2}}}, \quad r \geqslant 0.$$

Thus, even for a reasonably small ν, one will never "see" this regular attractor in numerical simulations. This phenomenon is related to the existence of *metastable* almost-equilibria with an extremely large lifetime in the phase space of this equation (it is also worth mentioning that they are situated far from the global attractor and have "nothing in common" with the properties of the global attractor). As we will see in the next section, this confusing drawback can be overcome by using the general concept of an *exponential* attractor, for which the constant α and the function Q can reasonably be found in terms of the physical parameters of the problem.

We conclude this subsection by the following result on the existence of the global attractor for a semigroup having a global Lyapunov function (see [59]; see also [119] and [135]) which can be useful in applications (see, e.g., [59] and [174]).

THEOREM 2.26. *We make the following assumptions*:

(i) $t \mapsto S(t)x$ *is continuous*, $\forall x \in E$;

(ii) $S(t)$ *possesses a continuous global Lyapunov function* L *such that* $L(x) \to +\infty$ *if and only if* $\|x\|_E \to +\infty$;

(iii) *the set of fixed points of* $S(t)$, $\mathcal{N}$, *is bounded in* E;

(iv) $S(t)$ *is asymptotically compact, i.e.*, $\forall B \subset E$ *bounded*,

$$\lim_{t\to+\infty} \kappa\big(S(t)B\big) = 0,$$

where κ *is the Kuratowski measure of noncompactness*.

Then $S(t)$ *possesses the connected global attractor* $\mathcal{A}$ *such that* $\mathcal{A} = \mathcal{M}^{\mathrm{un}}(\mathcal{N})$.

REMARK 2.27. Theorem 2.26 can be useful, e.g., when the dissipative estimate (2.6) and the existence of a bounded absorbing set can be difficult to establish (see, e.g., [59] and [174]), although the existence of the global attractor implies the existence of a bounded absorbing set (it suffices to take any ϵ-neighborhood of the global attractor). Thus, the dissipativity can be obtained in an implicit way by using the Lyapunov function and the fact that the set of equilibria is bounded. Roughly speaking, the dissipativity is related to the fact that every trajectory converges to the set of equilibria (due to the Lyapunov function and the asymptotic compactness) and, since the set of equilibria is bounded, the energy of a "large" solution must decay (due to property (ii) of a Lyapunov function).

2.4. *Dimension of the global attractor*

As mentioned in the introduction, we will essentially consider the fractal (or box-counting) dimension here.

DEFINITION 2.28. Let $X \subset E$ be a (relatively) compact set. For $\epsilon > 0$, let $N_\epsilon(X)$ be the minimal number of balls of radius ϵ which are necessary to cover X. Then the fractal dimension of X is the quantity

$$\dim_F X := \limsup_{\epsilon\to0^+} \frac{\log_2 N_\epsilon(X)}{\log_2 (1/\epsilon)} \quad \left(= \limsup_{\epsilon\to0^+} \frac{\ln N_\epsilon(X)}{\ln (1/\epsilon)}\right) \tag{2.13}$$

(note that $\dim_F X \in [0, +\infty]$). Furthermore, the quantity $\mathcal{H}_\epsilon(X) := \log_2 N_\epsilon(X)$ is called the Kolmogorov ϵ-entropy of X.

The fractal dimension satisfies the following properties (see [84]):

- $\dim_F(X_1 \times X_2) \leqslant \dim_F X_1 + \dim_F X_2$;
- if $f : X_1 \to X_2$ is Lipschitz, then $\dim_F X_2 \leqslant \dim_F X_1$;
- if X is a smooth m-dimensional manifold, then $\dim_F X = m$.

It is important to note that, for sets which are not manifolds, the fractal dimension can be noninteger; for instance, if X is the ternary Cantor set in $\mathbb{R}$, then

$$\dim_F X = \frac{\ln 2}{\ln 3} < 1$$

(see [84]). Furthermore, it follows from the definition that, if the minimal number of balls of radius ϵ which are necessary to cover X satisfies

$$N_\epsilon(X) \leqslant c\left(\frac{1}{\epsilon}\right)^d,$$

where c and d are independent of ϵ, then

$$\dim_F X \leqslant d.$$

A strong interest, for considering the fractal dimension over other dimensions, is given by the (modified) Hölder-Mañé theorem (see [65,94], and [122]). We start with the following definition.

DEFINITION 2.29. (See [123].) A Borel subset X of a Banach space E is prevalent if there exists a compactly supported probability measure μ such that $\mu(X + x) = 1$, $\forall x \in E$. A non-Borel set which contains a prevalent set is also prevalent.

REMARK 2.30. Prevalence extends the notion of "Lebesgue almost every" from Euclidean spaces to infinite dimensional spaces (see [123] for a discussion on this subject).

THEOREM 2.31 *(Modified Hölder–Mañé theorem, [122]). Let $X \subset E$ be compact and such that $\dim_F X = d$ and $N > 2d$ be an integer. Then almost every* (*in the sense of prevalence*) *bounded linear projector $P : E \to \mathbb{R}^N$ is one-to-one on X and has a Hölder continuous inverse.*

It follows from Theorem 2.31 that, if the global attractor has finite fractal dimension, then, fixing a projector P satisfying the assumptions of the theorem, the reduced dynamical system $(\bar{S}(t), \bar{\mathcal{A}})$, where $\bar{S}(t) := P \circ S(t) \circ P^{-1}$ and $\bar{\mathcal{A}} := P(\mathcal{A})$, is a finite dimensional dynamical system (i.e., in $\mathbb{R}^N$) which is Hölder continuous with respect to the initial data. This result, and the fractal dimension, thus play an important role in the finite dimensional reduction theory of infinite dimensional dynamical systems.

REMARK 2.32. The Hausdorff dimension (see [84]) is also frequently used to measure the dimension of the global attractor (see, e.g., [22,49], and [217]). However, Theorem 2.31 does not hold for the Hausdorff dimension.

The next result (see [234]; see also [136]) gives a general method to prove the finite fractal dimensionality of a compact set.

THEOREM 2.33. *Let X be a compact subset of E. We assume that there exist a Banach space E_1 such that E_1 is compactly embedded into E and a mapping $L : X \to X$ such that $L(X) = X$ and*

$$\|Lx_1 - Lx_2\|_{E_1} \leqslant c\|x_1 - x_2\|_E, \quad \forall x_1,\ x_2 \in X. \tag{2.14}$$

Then the fractal dimension of X is finite and satisfies

$$\dim_F X \leqslant \mathcal{H}_{\frac{1}{4c}}\big(B_{E_1}(0, 1)\big),$$

where c is the constant in (2.14) *and* $B_{E_1}(0, 1)$ *is the unit ball in* E_1 (*note that it is relatively compact in* E).

In applications to parabolic systems in bounded domains, one usually proves that, for instance, (2.14) is satisfied for $L = S(1)$. Then, owing to the invariance property, one deduces from Theorem 2.33 that the global attractor has finite fractal dimension. We will come back to the "smoothing" property (2.14), and its generalizations (in particular, to damped hyperbolic systems), in the next section when discussing the construction of exponential attractors.

It is essential, in view of the finite dimensional reduction principle given by Theorem 2.31, to find sharp estimates on the dimension of the global attractor in terms of the physical parameters of the problem. In general, the best upper bounds are obtained by the so-called volume contraction method, which is based on the study of the evolution of infinitesimal k-dimensional volumes in the neighborhood of the attractor (see [22,49,197], and [217]); see however [67] for a sharp upper bound based on (2.14). One then proves that, if the dynamical system contracts the k-dimensional volumes, then the fractal dimension of $\mathcal{A}$ is less than k. This method requires some differentiability property of the semigroup $S(t)$.

DEFINITION 2.34. A map $L : X \to X$, $X \subset E$, is uniformly quasidifferentiable on X if, for every $x \in X$, there exists a linear operator $L'(x)$ (called quasidifferential) such that

$$\left\| L(x+v) - L(x) - L'(x)v \right\|_E = \mathrm{o}\big(\|v\|_E\big)$$

holds uniformly with respect to $x \in X$, $v \in X$, $x + v \in E$.

We now assume that E is a Hilbert space. We have the following result (see [41]; see also [22,49], and [217]).

THEOREM 2.35. *We assume that X is an invariant subset of E and that $S(t)$ is uniformly quasidifferentiable on X, with $x \mapsto S'(t)x$ continuous*, $\forall t \geqslant 0$, *and that, for some $t_\star > 0$*,

$$\bar{\omega}_d(X) := \sup_{x \in X} \omega_d\big(S'(t_\star)x\big) < 1,$$

where, for a bounded linear operator $L : E \to E$,

$$\omega_d(L) := \sup_{B_d} \frac{\mathrm{Vol}_d(L(B_d))}{\mathrm{Vol}_d(B_d)},$$

Vol_d *being the d-dimensional volume and the supremum being taken over all d-dimensional ellipsoids. Then*

$$\dim_F X \leqslant d.$$

We can note that, when E is a Hilbert space, then, if E_d is a vector subspace of E of dimension d, a bounded linear operator L maps a d-dimensional ellipsoid $B_d \subset E_d$ onto the d-dimensional ellipsoid $L(B_d) \subset L(E_d)$. Furthermore, $\mathrm{Vol}_d(B_d)$ is well-defined. The quantity $\omega_d(L)$ measures the changes of d-dimensional volumes under the action of L.

REMARK 2.36. Another powerful and useful method to prove the finite dimensionality of the global attractor is based on the so-called l-trajectories: one needs minimal regularity on the solutions in order to apply this method, see [32,53,54,152–154,173,190], and [192]. In particular, this method allows to prove the finite dimensionality of the global attractor associated with generalized Navier–Stokes equations (see [134]) for which the smoothing property (2.14) and the quasidifferentiability are not known (see [152,153], and [154]); the quasidifferentiability was however recently proven in [124] for some of these models in two space dimensions.

It is also essential to derive lower bounds on the dimension of the global attractor and to compare them with the known upper bounds. The derivation of lower bounds is based on the following observation: the global attractor always contains the unstable sets of equilibria. Thus, the stability index of a properly constructed (hyperbolic) equilibrium yields a lower bound on the dimension of the global attractor (see [22] for more details; see also [144,145,171,179], and [217] for examples).

2.5. *Robustness of the global attractor*

Very often, one needs to consider regular or singular perturbations of the system under study; indeed, as mentioned in the introduction, a given system is only an approximation of reality. A natural question is how these perturbations will affect the asymptotic behavior of the system. One natural idea is to "compare" the global attractors of the perturbed and nonperturbed systems; such results were first established in [118] for systems having a global Lyapunov function and then in [20] for general systems.

We thus consider a family of semigroups $\{S_\lambda(t),\ \lambda \in I\}$, $I \subset \mathbb{R}$ interval (more generally, I can be some topological space), acting on E such that $S_\lambda(t)$ possesses the global attractor $\mathcal{A}_\lambda$, $\forall \lambda \in I$.

DEFINITION 2.37.

(i) The attractors $\mathcal{A}_\lambda$ are upper semicontinuous at $\lambda_0 \in I$ if

$$\lim_{\lambda \in I \to \lambda_0} \operatorname{dist}(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0}) = 0.$$

(ii) The attractors $\mathcal{A}_\lambda$ are lower semicontinuous at $\lambda_0 \in I$ if

$$\lim_{\lambda \in I \to \lambda_0} \operatorname{dist}(\mathcal{A}_{\lambda_0}, \mathcal{A}_\lambda) = 0.$$

(iii) The attractors $\mathcal{A}_\lambda$ are continuous at $\lambda_0 \in I$ if they are both upper and lower semicontinuous at λ_0.

In general, global attractors are upper semicontinuous, i.e., we can prove the upper semicontinuity property under natural, and relatively easy to check in applications, conditions. We have, for instance, the following results (see [119]).

THEOREM 2.38. *Let λ_0 belong to I. We assume that there exist $\delta > 0$, $t_0 > 0$, and a compact subset K of E such that*

(i) $\bigcup_{\lambda\in(\lambda_0-\delta,\lambda_0+\delta)\cap I}\mathcal{A}_\lambda \subset K$;
(ii) *for every sequences* $\{\lambda_k,\ k\in\mathbb{N}\}$ *and* $\{x_k,\ k\in\mathbb{N}\}$, $\lambda_k\in I$, $x_k\in\mathcal{A}_{\lambda_k}$, *such that* $\lambda_k\to\lambda_0$ *and* $x_k\to x_0$ *as* $k\to+\infty$, *then*

$$S_{\lambda_k}(t_0)x_k \to S_{\lambda_0}(t_0)x_0 \quad \text{as } k\to+\infty.$$

Then the attractors $\mathcal{A}_\lambda$ *are upper semicontinuous at* λ_0.

THEOREM 2.39. *Let* λ_0 *belong to* I. *We make the following assumptions*:
(i) *there exist* $\delta>0$, $t_0>0$, *and a bounded subset* B_0 *of* E *such that*

$$\bigcup_{\lambda\in(\lambda_0-\delta,\lambda_0+\delta)\cap I}\mathcal{A}_\lambda \subset B_0;$$

(ii) $\forall\epsilon>0, \forall t\geqslant t_0$, *there exists* $\theta=\theta(\epsilon,t)$, $0<\theta<\delta$, *such that*

$$\left\|S_\lambda(t)x_\lambda - S_{\lambda_0}(t)x_\lambda\right\|_E \leqslant \epsilon, \quad \forall x_\lambda\in\mathcal{A}_\lambda,\ \forall\lambda\in(\lambda_0-\theta,\lambda_0+\theta)\cap I.$$

Then the attractors $\mathcal{A}_\lambda$ *are upper semicontinuous at* λ_0.

Thus, roughly speaking, if the perturbation $S_\lambda(t)$ is continuous with respect to λ and the associated absorbing sets are uniformly bounded, the attractors $\mathcal{A}_\lambda$ are upper semicontinuous.

We also mention a simple theorem which follows in a straightforward way from the definition of upper semicontinuity and which, however, is especially useful for singular perturbations and, as a rule, gives the "simplest" way to establish the upper semicontinuity of attractors (see [22]).

THEOREM 2.40. *Let the attractors* $\mathcal{A}_\lambda$ *possess the following property*: *for every sequences* $\{\lambda_k,\ k\in\mathbb{N}\}$ *and* $\{x_k,\ k\in\mathbb{N}\}$, $\lambda_k\in I$, $x_k\in\mathcal{A}_{\lambda_k}$, *such that* $\lambda_k\to\lambda_0\in I$, *there exists a subsequence* x_{k_n} *which converges to some* $x_0\in\mathcal{A}_{\lambda_0}$. *Then the attractors* $\mathcal{A}_\lambda$ *are upper semicontinuous at* λ_0.

In applications, the assumption of Theorem 2.40 is verified based on the fact that the global attractor is generated by bounded complete trajectories (see Theorem 2.13). Thus, there only remains to extract, from a sequence of complete trajectories $u_{\lambda_k}\in\mathcal{K}_{\lambda_k}$, a subsequence converging to some complete trajectory $u_{\lambda_0}\in\mathcal{K}_{\lambda_0}$ of the limit system. The advantage of this approach is that the semigroups $S_\lambda(t)$ (which, for singular perturbations, may have bad properties such as boundary layers, lack of regularity in finite time, ...) are not involved in the process and the result can be obtained by directly passing to the limit in the associated equations for u_{λ_k}, see [22] for details.

REMARK 2.41. Although everything seems satisfactory as far as the upper semicontinuity of global attractors is concerned, the situation changes drastically if one is interested in *estimating* the distance between the perturbed and nonperturbed attractors in terms of the physical parameters of the problem. Indeed, this distance is naturally related to the rate of attraction to the limit attractor and, as already mentioned, this rate of attraction is, in general, impossible to find in terms of the physical parameters of the problem.

Now, the lower semicontinuity property is much more difficult to prove; actually, as mentioned in the introduction, it may even not hold. We need, in order to prove this property, much more restrictive assumptions. For instance, we have the following result (see [22]).

THEOREM 2.42. *Let $\lambda_0 \in I$, the attractors $\mathcal{A}_\lambda$ be uniformly bounded in E, i.e., $\mathcal{A}_\lambda \subset B_0$ for every $\lambda \in I$, for some bounded subset B_0 of E, and the following uniform* (*with respect to λ*) *attraction property hold*:

$$\operatorname{dist}\big(S_\lambda(t)B_0, \mathcal{A}_\lambda\big) \leqslant \beta(t), \quad t \geqslant t_0,\ \lambda \in I, \tag{2.15}$$

where $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ is some monotonic function which tends to zero as $t \to +\infty$. Assume also that S_λ is continuous at λ_0 in the following sense: *for every $T \in \mathbb{R}_+$,*

$$\sup_{t\in[0,T]} \sup_{x\in B_0} \big\| S_\lambda(t)x - S_{\lambda_0}(t)x \big\|_E \to 0 \quad \text{as } \lambda \to \lambda_0. \tag{2.16}$$

Then the attractors $\mathcal{A}_\lambda$ are lower semicontinuous at λ_0.

REMARK 2.43. Under some natural additional assumptions, condition (2.15) of a uniform rate of attraction is *necessary and sufficient* to have the lower semicontinuity. However, it is completely unclear how to verify such a condition in applications (to the best of our knowledge, no general method to prove this uniform rate of attraction has been developed). An exception is again the case where the limit attractor $\mathcal{A}_{\lambda_0}$ possesses a global Lyapunov function and is *regular*, see Remark 2.25. Indeed, as already mentioned, regular attractors attract bounded subsets *exponentially*, see (2.12), and persist under sufficiently regular perturbations. Furthermore, the rate of attraction to the perturbed regular attractor $\mathcal{A}_\lambda$ remains exponential and uniform with respect to λ, for λ close to λ_0, i.e., (2.15) holds with $\beta(t) := C\mathrm{e}^{-\alpha t}$, $\alpha > 0$. This, in turn, gives the upper and lower semicontinuity, together with the estimate

$$\operatorname{dist}_{\mathrm{sym}}(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0}) \leqslant C|\lambda - \lambda_0|^\gamma, \tag{2.17}$$

where $\operatorname{dist}_{\mathrm{sym}}$ denotes the symmetric Hausdorff distance between sets defined by

$$\operatorname{dist}_{\mathrm{sym}}(A, B) := \max\big(\operatorname{dist}(A, B), \operatorname{dist}(B, A)\big)$$

and for some positive constants C and $0 < \gamma < 1$, see, e.g., [22] for details. In some cases, it is also possible to prove that the dynamical system considered is Morse–Smale, which means that the dynamics, restricted to the regular attractor, is also preserved, up to homeomorphisms, under perturbations (see [31,119], and [195] for more details). Finally, for some one-dimensional scalar parabolic equations, the uniform rate of attraction is possible to establish even when the equilibria are nonhyperbolic (due to relatively simple and completely understood structures of degenerate equilibria, see [133]). However, as mentioned in Remark 2.25, even though regular attractors are, in some proper sense, generic, it is in general very difficult, if not impossible, to prove that the global attractor is regular for given values of the physical parameters of the problem. Furthermore, even when the regularity can be proven, it is also impossible, in general, to obtain *explicit* estimates on the rate of exponential attraction. So, the constants C and γ in (2.17) are also implicit.

REMARK 2.44. We refer the reader to [120] for different approaches for the comparison of attractors under perturbations.

REMARK 2.45. It follows from the above considerations that the existing perturbation theory of global attractors has a purely qualitative nature and no *quantitative* result (e.g., explicit estimates in terms of the physical parameters of the problem) is available in general. As we will see in the next section, this drawback can be overcome by using the so-called *exponential* attractors for which the analogues of estimates (2.12) and (2.17) hold without any assumption on the hyperbolicity of the equilibria and the existence of a Lyapunov function and all the constants can be computed explicitly.

3. Exponential attractors

3.1. *Inertial manifolds*

We established in Subsection 2.4 a finite dimensional reduction principle for infinite dimensional dynamical systems based on the finite fractal dimensionality of the global attractor, via the Hölder–Mañé theorem. However, even though it is very important, this finite dimensional reduction principle has essential drawbacks. Indeed, the reduced dynamical system $(\bar{S}(t), \bar{\mathcal{A}})$ given by the Hölder–Mañé theorem is only Hölder continuous and cannot thus be realized in a satisfactory way as a dynamical system generated by a system of ODEs, i.e., a system of ODEs which is well-posed. Furthermore, reasonable conditions on the global attractor which would guarantee that the Mañé projectors are Lipschitz are not known. A second drawback is that the complicated geometric structure of the attractors $\mathcal{A}$ and $\bar{\mathcal{A}}$ make the use of this finite dimensional reduction principle in computations hazardous: essentially, one only has a heuristic estimate on the number of unknowns which are necessary to capture all the dynamical effects in approximations.

It thus appears reasonable to embed the global attractor into a proper smooth finite dimensional manifold. The dynamics, reduced to this manifold, would then be realized as a (at least Lipschitz) system of ODEs which could be used in numerical simulations and would be a good approximation of the dynamics of the original system. This led Foias, Sell, and Temam to propose the notion of an inertial manifold in [95].

DEFINITION 3.1. A Lipschitz finite dimensional manifold $\mathcal{M} \subset E$ is an inertial manifold for the semigroup $S(t)$ if

(i) it is positively invariant, i.e., $S(t)\mathcal{M} \subset \mathcal{M}$, $\forall t \geqslant 0$;

(ii) it satisfies the following asymptotic completeness property:

$$\forall u_0 \in E,\ \exists v_0 \in \mathcal{M} \text{ such that}$$
$$\big\| S(t)u_0 - S(t)v_0 \big\|_E \leqslant Q\big(\|u_0\|_E\big)\mathrm{e}^{-\alpha t}, \quad t \geqslant 0, \tag{3.1}$$

where the positive constant α and the monotonic function Q are independent of u_0.

It follows from this definition that an inertial manifold, if it exists, contains the global attractor and attracts the trajectories exponentially fast (and uniformly with respect to bounded sets of initial data).

Furthermore, the existence of such a set would confirm, in a perfect way, the heuristic conjecture on a finite dimensional reduction principle of infinite dimensional dissipative dynamical systems. Indeed, the dynamics, restricted to an inertial manifold, can be described by a system of ODEs which is Lipschitz continuous (and thus well-posed), called the inertial form of the system. Furthermore, the asymptotic completeness property gives, in a particularly strong form, the equivalence of the initial dynamical system $(S(t), E)$ with its inertial form $(S(t), \mathcal{M})$.

REMARK 3.2. In turbulence, i.e., for the incompressible Navier–Stokes equations, the existence of an inertial manifold would also yield an exact interaction law between the small and the large structures of the flow (see, e.g., [93]).

Several methods have been proposed to construct inertial manifolds (by the Lyapunov–Perron method, by constructing converging sequences of approximate inertial manifolds, by the so-called graph-transform method, ...); we refer the interested reader to [58,95,197, 211,217], and the references therein for more details.

However, all the known constructions of inertial manifolds make use of a restrictive condition, namely, the so-called spectral gap condition (see [95]), which requires arbitrarily large gaps in the spectrum of the linearization of the initial system (see [95] for more details). In general, this property can only be verified in one space dimension. Nevertheless, the existence of inertial manifolds has been proven for a large number of equations, essentially in one and two space dimensions; we refer the reader to [58,95,197,211,217], and the numerous references therein. However, the existence of an inertial manifold is still an open problem for several physically important equations, such as the two-dimensional incompressible Navier–Stokes equations. Furthermore, nonexistence results have been proven for damped Sine–Gordon equations by Mora and Solà-Morales [183].

REMARK 3.3. Notions of approximate inertial manifolds have been proposed when the existence of an (exact) inertial manifold is not known and, in particular, for the incompressible Navier–Stokes equations. We refer the reader to, e.g., [95,96,99], and [217] for more details.

3.2. *Construction of exponential attractors*

It follows from the previous subsection that it is not always possible to embed the global attractor into a proper smooth finite dimensional manifold. Nevertheless, and also in view of the possible defaults of the global attractor as discussed in the introduction, it can be useful to construct larger (not necessarily smooth) sets which contain the global attractor, are still finite dimensional, and attract the trajectories exponentially fast. This led Eden, Foias, Nicolaenko, and Temam to propose the notion of an exponential attractor (also sometimes called an inertial set) in [65].

DEFINITION 3.4. A compact set $\mathcal{M} \subset E$ is an exponential attractor for $S(t)$ if

(i) it has finite fractal dimension, $\dim_F \mathcal{M} < +\infty$;

(ii) it is positively invariant, $S(t)\mathcal{M} \subset \mathcal{M}$, $\forall t \geqslant 0$;

(iii) it attracts exponentially the bounded subsets of E in the following sense:

$$\forall B \subset E \text{ bounded}, \quad \operatorname{dist}\bigl(S(t)B, \mathcal{M}\bigr) \leqslant Q\bigl(\|B\|_E\bigr)\mathrm{e}^{-\alpha t}, \quad t \geqslant 0,$$

where the positive constant α and the monotonic function Q are independent of B.

It follows from this definition that an exponential attractor, if it exists, contains the global attractor (actually, the existence of an exponential attractor $\mathcal{M}$ yields the existence of the global attractor $\mathcal{A} \subset \mathcal{M}$, since it is a compact attracting set, see Theorem 2.17; note that $S(t)$ is still assumed to satisfy the continuity assumption (2.4)).

Thus, an exponential attractor is still finite dimensional, like the global attractor (and one still has the finite dimensional reduction principle given by the Hölder–Mañé theorem); actually, proving the existence of an exponential attractor is also one way of proving that the global attractor has finite fractal dimension. Compared with an inertial manifold, an exponential attractor is not smooth in general, but one still has a uniform exponential control on the rate of attraction of the trajectories.

Now, the main drawback of exponential attractors is that the relaxation to positive invariance makes these objects nonunique; actually, once we have the existence of an exponential attractor, we have the existence of a whole family of exponential attractors (see [65]). Therefore, the question of the best choice of an exponential attractor, if this makes sense, is a crucial one. One possibility, to overcome this drawback, is to find a "simple" algorithm which maps a semigroup $S(t)$ onto an exponential attractor $\mathcal{M}(S)$; by simple, we have in particular in mind the numerical realization of such an algorithm.

The first construction of exponential attractors, due to Eden, Foias, Nicolaenko, and Temam [65], was not constructible; indeed, Zorn's lemma had to be used in order to construct exponential attractors. This construction consists in a way in constructing a "fractal expansion" of the global attractor $\mathcal{A}$. Very roughly speaking, one considers an iterative process in which one adds, at each step, a "cloud" of points around the global attractor. The difficulty is that, at each step, one needs to control the dimension of this new cloud of points around the global attractor, and also ensure that the new set remains positively invariant, without increasing its dimension. The key idea which allows to control the number of points added at each step is the so-called squeezing property which says, roughly speaking, that either the higher modes are dominated by the lower ones or that the flow is contracted exponentially: a mapping $S : X \to X$, where X is a compact subset of a Hilbert space E, enjoys the squeezing property on X if, for some $\delta \in (0, \frac{1}{4})$, there exists an orthogonal projector $P = P(\delta)$ with finite rank such that, for every $u,\ v \in X$, either

$$\bigl\|(I - P)(Su - Sv)\bigr\|_E \leqslant \bigl\|P(Su - Sv)\bigr\|_E$$

or

$$\|Su - Sv\|_E \leqslant \delta\|u - v\|_E.$$

We can note that this property makes an essential use of orthogonal projectors with finite rank, so that the corresponding construction is valid in Hilbert spaces only.

The construction of [65] essentially applies to semigroups which possess a compact absorbing set (although a construction valid for damped wave equations is also given in [65]).

It was then improved by Babin and Nicolaenko (in the sense that one could also consider semigroups which possess a compact attracting set) in [18] (see also [66]). We have, based on the construction of [18], the following result (see [80,81], and [172]).

THEOREM 3.5. *Let E and E_1 be two Hilbert spaces such that E_1 is compactly embedded into E and $S(t): X \to X$ be a semigroup acting on a closed subset X of E. We assume that*

(i) *there exist orthogonal projectors $P_k : E \to E$, $k \in \mathbb{N}$, with finite rank such that*

$$\|(I - P_k)y\|_E \leqslant c(k)\|y\|_{E_1}, \quad \forall y \in E_1,\ c(k) \to 0 \text{ as } k \to +\infty;$$

(ii) $\forall x_1, x_2 \in X, \forall t > 0,$

$$\|S(t)x_1 - S(t)x_2\|_{E_1} \leqslant h(t)\|x_1 - x_2\|_E,$$

where the function h is continuous;

(iii) $(t, x) \mapsto S(t)x$ *is Lipschitz on* $[0, T] \times B, \forall T > 0, \forall B \subset X$ *bounded.*

Then $S(t)$ possesses an exponential attractor $\mathcal{M}$ on X (i.e., $\mathcal{M}$ satisfies all the assertions of Definition 3.4 with E replaced by X).

REMARK 3.6.

(a) Actually, (i) follows from the compact embedding $E_1 \subset E$. Furthermore, it follows from (i) and (ii) that the squeezing property is satisfied for some $t_\star > 0$.

(b) Condition (ii) can be replaced by the more general condition

$$\forall x_1, x_2 \in X,\ \forall t \geqslant 0, \quad S(t)x_1 - S(t)x_2 = S_1(t, x_1, x_2) + S_2(t, x_1, x_2),$$

where

$$\|S_1(t, x_1, x_2)\|_E \leqslant d(t)\|x_1 - x_2\|_E,$$
$$d \text{ continuous},\ t \geqslant 0,\ d(t) \to 0 \text{ as } t \to +\infty,$$

and

$$\|S_2(t, x_1, x_2)\|_{E_1} \leqslant h(t)\|x_1 - x_2\|_E, \quad t > 0,\ h \text{ continuous}.$$

This more general condition allows to construct exponential attractors for damped hyperbolic equations (see [82] and [98]).

(c) One essential difficulty, when constructing exponential attractors for damped hyperbolic equations, is to prove that the exponential attractors attract the bounded subsets of the whole phase space, and not those starting from a subspace of the phase space only (typically, consisting of more regular functions), see [65]. This difficulty was overcome in [83] by proving the following transitivity property of the exponential attraction: let (E, d) be a metric space and $S(t)$ be a semigroup acting on E such that

$$d\big(S(t)x_1, S(t)x_2\big) \leqslant c_1 e^{\alpha_1 t} d(x_1, x_2), \quad t \geqslant 0,\ x_1, x_2 \in E,$$

for some positive constants c_1 and α_1. We further assume that there exist three subsets M_1, M_2, and M_3 of E such that

$$\operatorname{dist}\big(S(t)M_1, M_2\big) \leqslant c_2 e^{-\alpha_2 t}, \quad t \geqslant 0,\ \alpha_2 > 0,$$

and

$$\operatorname{dist}\big(S(t)M_2, M_3\big) \leqslant c_3 e^{-\alpha_3 t}, \quad t \geqslant 0,\ \alpha_3 > 0.$$

Then

$$\operatorname{dist}\big(S(t)M_1, M_3\big) \leqslant c_4 e^{-\alpha_4 t}, \quad t \geqslant 0,$$

where $c_4 := c_1 c_2 + c_3$ and $\alpha_4 := \alpha_2\alpha_3/(\alpha_1 + \alpha_2 + \alpha_3)$.

We note that condition (ii) in Theorem 3.5 resembles the smoothing property (2.14); actually, in order to prove Theorem 3.5, one only needs to prove that $S(t_\star)$ satisfies (2.14) for a proper $t_\star$. Now, this smoothing property is sufficient in order to construct exponential attractors and one does not need the squeezing property (and, thus, one does not need orthogonal projectors with finite rank); therefore, exponential attractors can also be constructed in Banach spaces.

Let thus E and E_1 be two Banach spaces such that E_1 is compactly embedded into E and let X be a bounded subset of E. Let finally $S : X \to X$ be a (nonlinear) mapping. We then consider the discrete dynamical system (or semigroup) generated by S, i.e., we set

$$S(0) := \mathrm{Id}, \quad S(n) := S \circ \cdots \circ S \ (n \text{ times}), \quad n \in \mathbb{N}.$$

It is easy to see that this family of operators satisfies (2.2)–(2.3), but for $t, s \in \mathbb{N}$. Then we say that $\mathcal{M} \subset X$ is an exponential attractor for this discrete semigroup on X if

(i) it is compact in E and has finite fractal dimension;

(ii) it is positively invariant, i.e., $S\mathcal{M} \subset \mathcal{M}$;

(iii) $\operatorname{dist}(S(n)X, \mathcal{M}) \leqslant c e^{-\alpha n}$, $n \in \mathbb{N}$, where c and $\alpha > 0$ only depend on X.

We then have the

THEOREM 3.7. *(See [68].) We assume that the mapping S enjoys the smoothing property (2.14) on X, i.e.,*

$$\|Sx_1 - Sx_2\|_{E_1} \leqslant c\|x_1 - x_2\|_E, \quad \forall x_1, x_2 \in E.$$

Then the discrete dynamical system generated by the iterations of S possesses an exponential attractor $\mathcal{M} \subset X$.

Let us now consider a continuous semigroup $S(t)$ acting on X, i.e.,

$$S(t) : X \to X, \quad t \geqslant 0.$$

In order to construct an exponential attractor for $S(t)$ on X, we usually proceed as follows. We assume that $S(t_\star)$ satisfies the smoothing property (2.14) for some $t_\star > 0$. We then have, owing to Theorem 3.7, an exponential attractor $\mathcal{M}_\star$ for the discrete dynamical system generated by the mapping $S_\star := S(t_\star)$ and we set

$$\mathcal{M} := \bigcup_{t\in[0,t_\star]} S(t)\mathcal{M}_\star.$$

Finally, if $(t,x) \mapsto S(t)x$ is Lipschitz (or even Hölder) on $[0,t_\star]\times X$, we can prove that $\mathcal{M}$ is an exponential attractor for $S(t)$ on X (see [65]).

REMARK 3.8.

(a) In applications to PDEs, it is in general not restrictive at all to consider a bounded invariant subset $X \subset E$ instead of the whole space E. Indeed, X usually is a positively invariant bounded absorbing set; we note that, if $\mathcal{B}_0$ is a bounded absorbing set for $S(t)$, then $\mathcal{B}_1 := \overline{\bigcup_{t\geqslant t_0} S(t)\mathcal{B}_0}$, where t_0 is such that $t \geqslant t_0$ implies $S(t)\mathcal{B}_0 \subset \mathcal{B}_0$ and the closure is taken in E, is a positively invariant bounded absorbing set. Therefore, the exponential attractors still attract all the bounded subsets of E.

(b) For applications to damped hyperbolic equations, we will need a weaker form of a smoothing property, and, more precisely, some asymptotically smoothing property (see Remark 3.6(b)). More precisely, the existence of an exponential attractor still holds if (2.14) is replaced by one of the following weaker conditions (see [68]):

$$\begin{aligned}
&S = S_1 + S_2, \quad \text{where}\\
&\|S_1x_1 - S_1x_2\|_E \leqslant \alpha\|x_1 - x_2\|_E, \quad \forall x_1,x_2\in X,\ \alpha < \frac{1}{2},\\
&\|S_2x_1 - S_2x_2\|_{E_1} \leqslant c\|x_1 - x_2\|_E, \quad \forall x_1,x_2\in X,
\end{aligned}\tag{3.2}$$

or

$$\begin{aligned}
&Sx_1 - Sx_2 = S_1(x_1,x_2) + S_2(x_1,x_2), \quad \forall x_1,x_2\in X, \quad \text{where}\\
&\big\|S_1(x_1,x_2)\big\|_E \leqslant \alpha\|x_1 - x_2\|_E, \quad \alpha < \frac{1}{2},\\
&\big\|S_2(x_1,x_2)\big\|_{E_1} \leqslant c\|x_1 - x_2\|_E.
\end{aligned}\tag{3.3}$$

(c) If E_1 and E_2 are Hilbert spaces, then we can prove that, if $\alpha < \frac{1}{8}$, (3.2) and (3.3) imply the squeezing property (see [67]; see also Remark 3.6(a)).

(d) Based on the above results, one has been able to prove the existence of exponential attractors in many situations, see [6,7,60,61,68,70,71,74,85,100–103,105–109,116, 117,157,175–177,184], and [216]. Actually, exponential attractors are as general as global attractors: to the best of our knowledge, exponential attractors exist indeed for all equations of mathematical physics for which we can prove the existence of the finite dimensional global attractor.

(e) Another construction of exponential attractors in Banach spaces was proposed by Le Dung and Nicolaenko in [140]. This construction consists in adapting the original construction of [65] to a Banach setting. We can note that it is based on conditions which are contained in (and are more restrictive than) those given above. Furthermore, it is worth noting that the construction given in [68] is very simple, in particular, when compared to those of [65] and [140].

(f) The method of l-trajectories is also very efficient to construct exponential attractors. In particular, this method allows to prove the smoothing property in a simple way. Furthermore, as already mentioned, it requires minimal regularity on the solutions. We refer the reader to [32,78,153,173,178,191,192,203], and [206] for more details; a necessary and sufficient condition on the existence of an exponential attractor is also given in [191].

3.3. *Robust families of exponential attractors*

As already mentioned in the introduction and Subsection 2.5, global attractors can be sensitive to perturbations; more precisely, the lower semicontinuity property may not hold. Furthermore, even though this property is, in some proper sense, generic (see, e.g., [195]), it is in general very difficult, if not impossible, to prove it for given values of the physical parameters in applications. Similarly, regular attractors (see Remark 2.25) are robust (see [22]), and, in particular, lower semicontinuous, but, again, it is in general very difficult, if not impossible, to prove the existence of such sets for given values of the physical parameters.

It is also worth noting that inertial manifolds are robust under perturbations; indeed, they are hyperbolic manifolds, see [200]. However, as mentioned in Subsection 3.1, the existence of such sets is not known for several important equations and may even not hold.

Now, since exponential attractors attract exponentially fast the trajectories, with a uniform control on the rate of attraction, it is reasonable to expect that these sets are robust under perturbations and that one should be able to construct robust families of exponential attractors, of course, up to the "best choice", since they are not unique.

It is possible, based on the initial construction of [65], to construct families of exponential attractors which are upper and lower semicontinuous (see, e.g., [65,82], and [98]). However, this continuity only holds up to some time shift, i.e., one has a result of the form

$$\lim_{\epsilon\to 0^+}\limsup_{t\to+\infty}\big[\operatorname{dist}\big(S_\epsilon(t)\mathcal{M}_\epsilon,\mathcal{M}_0\big)+\operatorname{dist}\big(S_0(t)\mathcal{M}_0,\mathcal{M}_\epsilon\big)\big]=0,$$

where $(S_\epsilon(t),\mathcal{M}_\epsilon)$ and $(S_0(t),\mathcal{M}_0)$ are the perturbed and nonperturbed dynamical systems, respectively, $\epsilon>0$ being the perturbation parameter. Consequently, we essentially have, as far as the lower semicontinuity is concerned,

$$\lim_{\epsilon\to 0^+}\operatorname{dist}(\mathcal{A}_0,\mathcal{M}_\epsilon)=0,$$

where $\mathcal{A}_0$ is the global attractor associated with the nonperturbed system, which is not satisfactory.

This result was improved in [70] (see also [74]) and one has the

THEOREM 3.9. *(See [70].) Let E and E_1 be two Banach spaces such that E_1 is compactly embedded into E and let X be a bounded subset of E. We assume that the family of operators $S_\epsilon : X\mapsto X$, $\epsilon\in[0,\epsilon_0]$, $\epsilon_0>0$, satisfies the following assumptions*:

(i) (*Uniform, with respect to ϵ, smoothing property*) $\forall\epsilon\in[0,\epsilon_0]$, $\forall x_1,\ x_2\in X$,

$$\|S_\epsilon x_1-S_\epsilon x_2\|_{E_1}\leqslant c_1\|x_1-x_2\|_E,$$

where c_1 is independent of ϵ.

(ii) (*The trajectories of the perturbed system approach those of the nonperturbed one, uniformly with respect to* ϵ, *as* ϵ *tends to* 0) $\forall \epsilon \in [0, \epsilon_0]$, $\forall i \in \mathbb{N}$, $\forall x \in X$,

$$\left\| S_\epsilon^i x - S_0^i x \right\|_E \leqslant c_2^i \epsilon,$$

where c_2 *is independent of* ϵ *and, for a mapping* L, $L^i := L \circ \cdots \circ L$ (i *times*).

Then, $\forall \epsilon \in [0, \epsilon_0]$, *the discrete dynamical system generated by the iterations of* S_ϵ *possesses an exponential attractor* $\mathcal{M}_\epsilon$ *on* X *such that*

1. *the fractal dimension of* $\mathcal{M}_\epsilon$ *is bounded, uniformly with respect to* ϵ,

$$\dim_F \mathcal{M}_\epsilon \leqslant c_3,$$

where c_3 *is independent of* ϵ;

2. $\mathcal{M}_\epsilon$ *attracts* X, *uniformly with respect to* ϵ,

$$\operatorname{dist}\bigl(S_\epsilon^i X, \mathcal{M}_\epsilon\bigr) \leqslant c_4 e^{-c_5 i}, \quad c_5 > 0,\ i \in \mathbb{N},$$

where c_4 *and* c_5 *are independent of* ϵ;

3. *the family* $\{\mathcal{M}_\epsilon,\ \epsilon \in [0, \epsilon_0]\}$ *is continuous at* 0,

$$\operatorname{dist}_{\mathrm{sym}}(\mathcal{M}_\epsilon, \mathcal{M}_0) \leqslant c_6 \epsilon^{c_7},$$

where c_6 *and* $c_7 \in (0, 1)$ *are independent of* ϵ *and* $\operatorname{dist}_{\mathrm{sym}}$ *denotes the symmetric Hausdorff distance between sets defined by*

$$\operatorname{dist}_{\mathrm{sym}}(A, B) := \max\bigl(\operatorname{dist}(A, B), \operatorname{dist}(B, A)\bigr).$$

REMARK 3.10.

(a) The constants c_i, $i = 3, \ldots, 7$, can be computed explicitly in terms of the physical parameters of the problem in concrete situations. It is worth noting that this is not the case in general for the constants c_6 and c_7 in the estimate of the symmetric distance when such a result can be proven for global attractors, e.g., for regular attractors.

(b) In [70], in order to construct this family of exponential attractors, one first constructs $\mathcal{M}_0$ and one then constructs $\mathcal{M}_\epsilon$, $\epsilon > 0$, based on $\mathcal{M}_0$. Therefore, $\mathcal{M}_\epsilon$ depends on S_ϵ, but also on S_0, and the continuity only holds at $\epsilon = 0$.

(c) We also mention [7] for robustness results with respect to numerical approximations.

In applications to PDEs, Theorem 3.9 applies to parabolic systems (in bounded domains). In order to construct a robust family of exponential attractors $\mathcal{M}_\epsilon$ for the continuous semigroups $S_\epsilon(t)$, $\epsilon \in [0, \epsilon_0]$, associated with such systems, we usually first prove the existence of a uniform (with respect to ϵ) bounded absorbing set, i.e., a bounded subset $\mathcal{B}_0$ of E, independent of ϵ, such that, $\forall B \subset E$ bounded, $\exists T_0$ independent of ϵ such that

$$t \geqslant T_0 \text{ implies } S_\epsilon(t) B \subset \mathcal{B}_0, \quad \forall \epsilon \in [0, \epsilon_0].$$

We then consider the discrete mappings $S_\epsilon^{T_0} := S_\epsilon(T_0)$, $\forall \epsilon \in [0, \epsilon_0]$ (possibly for a larger, but still independent of ϵ, T_0). We thus have $S_\epsilon^{T_0} : \mathcal{B}_0 \to \mathcal{B}_0$, $\forall \epsilon \in [0, \epsilon_0]$, and we then prove that the $S_\epsilon^{T_0}$, $\epsilon \in [0, \epsilon_0]$, satisfy the assumptions of Theorem 3.9, which yields the existence of a robust family of discrete exponential attractors $\mathcal{M}_\epsilon^{T_0}$, $\epsilon \in [0, \epsilon_0]$. Finally, we set

$$\mathcal{M}_\epsilon := \bigcup_{t\in[0,T_0]} S_\epsilon(t)\mathcal{M}_\epsilon^{T_0}.$$

Then, if $(t,x)\mapsto S_\epsilon(t)x$ is Lipschitz, or even Hölder, on $[0,T_0]\times\mathcal{B}_0$, the exponential attractors $\mathcal{M}_\epsilon$, $\epsilon\in[0,\epsilon_0]$, satisfy

- $\dim_F\mathcal{M}_\epsilon\leqslant c_1'$, $\epsilon\in[0,\epsilon_0]$;
- $\forall B\subset E$ bounded,

$$\operatorname{dist}\big(S_\epsilon(t)B,\mathcal{M}_\epsilon\big)\leqslant c_2'\mathrm{e}^{-c_3't},\quad t\geqslant 0,\ \epsilon\in[0,\epsilon_0],\ c_3'>0;$$

- $\operatorname{dist}_{\mathrm{sym}}(\mathcal{M}_\epsilon,\mathcal{M}_0)\leqslant c_4'\epsilon^{c_5'}$, $\epsilon\in[0,\epsilon_0]$, $c_5'\in(0,1)$;

where the constants c_i', $i=1,\dots,5$, are independent of ϵ and can be computed explicitly in terms of the physical parameters of the problem.

For damped hyperbolic equations, we should replace (2.14) by some asymptotically smoothing property (see Remark 3.8(b)). More generally, we have, for singularly perturbed problems, the following result, proven in [83] (see also [104] for a reformulation of this result).

THEOREM 3.11. *We consider two families of Banach spaces $E(\epsilon)$ and $E_1(\epsilon)$, $\epsilon\in[0,\epsilon_0]$ (which are embedded into a larger topological space V), such that, $\forall\epsilon\in[0,\epsilon_0]$, $E_1(\epsilon)$ is compactly embedded into $E(\epsilon)$. We further assume that these compact embeddings are uniform with respect to ϵ in the sense that*

$$\mathcal{H}_\delta\big(B_{E_1(\epsilon)}(0,1),E(\epsilon)\big)\leqslant c_1(\delta),\quad\forall\delta>0,$$

where $\mathcal{H}_\delta(\cdot,E(\epsilon))$ denotes the Kolmogorov δ-entropy in the topology of $E(\epsilon)$ and c_1 is a monotonic function which is independent of ϵ. We then consider a family of closed sets $B_\epsilon\subset E(\epsilon)$, with B_0 bounded in $E(0)$, and a family of maps $S_\epsilon:B_\epsilon\to B_\epsilon$, $\epsilon\in[0,\epsilon_0]$, such that

(i) *$\forall\epsilon\in[0,\epsilon_0]$, $B_0\subset E(\epsilon)$ and*

$$\|b_0\|_{E(\epsilon)}\leqslant c_2\|b_0\|_{E(0)}+c_3\epsilon,\quad\forall b_0\in B_0,$$

where c_2 and c_3 are independent of ϵ;

(ii) *$\forall\epsilon\in[0,\epsilon_0]$, $S_\epsilon=\mathcal{C}_\epsilon+\mathcal{K}_\epsilon$, where $\mathcal{C}_\epsilon$ and $\mathcal{K}_\epsilon'$ map B_ϵ into $E(\epsilon)$ and, $\forall b_\epsilon^1,b_\epsilon^2\in B_\epsilon$,*

$$\big\|\mathcal{C}_\epsilon b_\epsilon^1-\mathcal{C}_\epsilon b_\epsilon^2\big\|_{E(\epsilon)}\leqslant c_4\big\|b_\epsilon^1-b_\epsilon^2\big\|_{E(\epsilon)},$$
$$\big\|\mathcal{K}_\epsilon b_\epsilon^1-\mathcal{K}_\epsilon b_\epsilon^2\big\|_{E_1(\epsilon)}\leqslant c_5\big\|b_\epsilon^1-b_\epsilon^2\big\|_{E(\epsilon)},$$

where $c_4<\frac{1}{2}$ and c_5 are independent of ϵ;

(iii) *there exist nonlinear "projectors" $\Pi_\epsilon:B_\epsilon\to B_0$, $\epsilon\in[0,\epsilon_0]$, such that $\Pi_\epsilon B_\epsilon=B_0$ and*

$$\big\|S_\epsilon b_\epsilon-S_0^k\Pi_\epsilon b_\epsilon\big\|_{E(\epsilon)}\leqslant c_6c_7^k\epsilon,\quad\epsilon\in[0,\epsilon_0],$$

where c_6 and c_7 are independent of ϵ.

Then there exists a family of exponential attractors $\mathcal{M}_\epsilon \subset B_\epsilon$ for the dynamical systems generated by the maps S_ϵ, $\epsilon \in [0, \epsilon_0]$, such that

1. $\dim_F \mathcal{M}_\epsilon \leqslant c_8$, $\epsilon \in [0, \epsilon_0]$;
2. $\operatorname{dist}_{E(\epsilon)}(S_\epsilon^k B_\epsilon, \mathcal{M}_\epsilon) \leqslant c_9 \mathrm{e}^{-c_{10}k}$, $\epsilon \in [0, \epsilon_0]$, $k \in \mathbb{N}$, $c_{10} > 0$;
3. $\operatorname{dist}_{\mathrm{sym}}(\mathcal{M}_\epsilon, \mathcal{M}_0) \leqslant c_{11}\epsilon^{c_{12}}$, $\epsilon \in [0, \epsilon_0]$, $c_{12} \in (0, 1)$; *where the constants c_i, $i = 8, \dots, 12$, are independent of ϵ and can be computed explicitly.*

REMARK 3.12.

(a) In order to construct a robust family of exponential attractors for continuous semigroups $S_\epsilon(t)$, $\epsilon \in [0, \epsilon_0]$, we essentially proceed as indicated above (see, e.g., [83]).

(b) Condition (ii) in Theorem 3.11 can be replaced by the more general condition

$$S_\epsilon b_\epsilon^1 - S_\epsilon b_\epsilon^2 = \mathcal{C}_\epsilon\big(b_\epsilon^1, b_\epsilon^2\big) + \mathcal{K}_\epsilon\big(b_\epsilon^1, b_\epsilon^2\big),$$

where

$$\big\|\mathcal{C}_\epsilon\big(b_\epsilon^1, b_\epsilon^2\big)\big\|_{E(\epsilon)} \leqslant c_4\big\|b_\epsilon^1 - b_\epsilon^2\big\|_{E(\epsilon)}, \quad c_4 < \frac{1}{2},$$
$$\big\|\mathcal{K}_\epsilon\big(b_\epsilon^1, b_\epsilon^2\big)\big\|_{E_1(\epsilon)} \leqslant c_5\big\|b_\epsilon^1 - b_\epsilon^2\big\|_{E(\epsilon)},$$

$\forall \epsilon \in [0, \epsilon_0]$, $\forall b_\epsilon^1, b_\epsilon^2 \in B_\epsilon$.

(c) We refer the reader to [60,61,70,74,83,101–105,108,109,116,117,175,176], and [177] for applications of Theorems 3.9 and 3.11 (or generalizations).

(d) As in [70], the exponential attractors $\mathcal{M}_\epsilon$, $\epsilon > 0$, constructed in [83] depend both on S_ϵ and S_0. These constructions were improved in [73], where the following result was proven (we will come back to this construction, and its generalizations, in the next section when discussing nonautonomous systems). Let E and E_1 be two Banach spaces such that E_1 is compactly embedded into E. We then consider a mapping S which satisfies the following conditions:
   - it maps the δ-neighborhood (for the topology of E) $\mathcal{O}_\delta(B)$ of a bounded subset B of E into B, for a proper constant $\delta > 0$;
   - $\forall x_1, x_2 \in \mathcal{O}_\delta(B)$, one has the smoothing property (2.14),

$$\|Sx_1 - Sx_2\|_{E_1} \leqslant K\|x_1 - x_2\|_E.$$

Then the discrete dynamical system generated by the iterations of S possesses an exponential attractor $\mathcal{M}(S) \subset B$ such that

- it is compact in E_1 and

$$\dim_F \mathcal{M}(S) \leqslant c_1;$$

- $\operatorname{dist}_{E_1}(S^k B, \mathcal{M}(S)) \leqslant c_2 \mathrm{e}^{-c_3 k}$, $k \in \mathbb{N}$, $c_3 > 0$;
- the map $S \mapsto \mathcal{M}(S)$ is Hölder continuous in the following sense: $\forall S_1, S_2$ satisfying the above conditions (for the same constants δ and K),

$$\operatorname{dist}_{\mathrm{sym}, E_1}\big(\mathcal{M}(S_1), \mathcal{M}(S_2)\big) \leqslant c_4\|S_1 - S_2\|^{c_5}, \quad c_5 > 0,$$

where

$$\|S\| := \sup_{h \in \mathcal{O}_\delta(B)} \|Sh\|_{E_1}.$$

Furthermore, all the constants c_i, $i = 1, \dots, 5$, only depend on B, E, E_1, δ, and K (in particular, they are independent of the concrete choice of S) and can be computed explicitly. We thus now have a mapping $S \mapsto \mathcal{M}(S)$ and, owing to the Hölder continuity of this mapping, we can now construct robust families of exponential attractors which are continuous at every point, and not just at $\epsilon = 0$ as in the previous constructions.

4. Nonautonomous systems

We now consider a system of the form

$$\frac{\partial u}{\partial t} = F(t, u), \quad u|_{t=\tau} = u_\tau, \quad \tau \in \mathbb{R},$$

in a Banach space E, i.e., we now assume that the time appears explicitly in the equations (e.g., in the forcing terms). Assuming that the problem is well-posed, we have the process $U(t, \tau)$, $t \geqslant \tau$, $\tau \in \mathbb{R}$, acting on E,

$$\begin{aligned} U(t, \tau) : E &\to E, \\ u_\tau &\mapsto u(t), \end{aligned}$$

which maps the initial datum at time τ onto the solution at time t.

For such a system, both the initial and final times are important, i.e., the trajectories are no longer (positively) invariant by time translations. Thus, the notion of a global attractor is no longer adequate and has to be adapted.

4.1. *Uniform attractors*

We consider in this subsection an approach initiated by Haraux [121] and further developed by Chepyzhov and Vishik [45] and [49].

We rewrite the equations in the form

$$\frac{\partial u}{\partial t} = F_{\sigma_0(t)}(u), \tag{4.1}$$

where $\sigma_0(t)$ consists of all the time dependent terms of the equations and is called the symbol of the system. For instance, if $F(t, u) = \tilde{F}(u) + f(t)$, then $\sigma_0(t) := f(t)$.

The idea in the approach described here is to actually consider, together with (4.1), a whole family of equations. To do so, we assume that σ_0 belongs to some complete metric space Θ (e.g., $\Theta := \mathcal{C}_b(\mathbb{R}, M)$, where M is a complete metric space and $\mathcal{C}_b$ denotes the bounded continuous functions). We then consider the translations group $T(h)$, $h \in \mathbb{R}$, defined by

$$T(h) f(s) := f(s + h), \quad s, h \in \mathbb{R},$$

and we assume that $T(h)\Theta \subset \Theta$ and $T(h)$ is continuous on Θ, $\forall h \in \mathbb{R}$. We finally define the hull of σ_0 as the set

$$\mathcal{H}(\sigma_0) := \overline{\{T(h)\sigma_0,\ h \in \mathbb{R}\}},$$

where the closure is taken in Θ. We say that $\mathcal{H}(\sigma_0)$ is the symbol space and, for simplicity, we denote it as Σ (see also Remark 4.1 below). It is not difficult to see that Σ is invariant by the translations group, i.e.,

$$T(h)\Sigma = \Sigma, \quad \forall h \in \mathbb{R}. \tag{4.2}$$

REMARK 4.1. More generally, we can take any subset of Θ which is invariant by the translations group as symbol space Σ; we will however restrict ourselves to the above symbol space in this subsection.

Now, together with Eq. (4.1), we consider the whole family of equations

$$\frac{\partial u}{\partial t} = F_{\sigma(t)}(u), \quad \sigma \in \Sigma. \tag{4.3}$$

Assuming that (4.3) is well-posed, $\forall \sigma \in \Sigma$, we have the family of processes $\{U_\sigma(t, \tau),\ t \geqslant \tau,\ \tau \in \mathbb{R},\ \sigma \in \Sigma\}$ acting on E.

DEFINITION 4.2. A set $\mathcal{A}_\Sigma \subset E$ is a uniform (with respect to σ) attractor for the family of processes $\{U_\sigma(t, \tau), t \geqslant \tau,\ \tau \in \mathbb{R},\ \sigma \in \Sigma\}$ if

(i) it is compact in E;

(ii) it attracts the bounded subsets of E, uniformly with respect to σ, i.e.,

$$\forall B \subset E \text{ bounded}, \qquad \lim_{t\to+\infty} \sup_{\sigma\in\Sigma} \operatorname{dist}\big(U_\sigma(t, \tau)B, \mathcal{A}_\Sigma\big) = 0;$$

(iii) it is minimal among the closed sets which enjoy the attraction property (ii).

REMARK 4.3. In general, the uniform attractor is not invariant (we say that $X \subset E$ is invariant if $U_\sigma(t, \tau)X = X$, $\forall t \geqslant \tau$, $\tau \in \mathbb{R}$, $\forall \sigma \in \Sigma$) and, in some sense, the invariance is replaced by the minimality property (iii); in particular, it follows from (ii) and (iii) that the uniform attractor, if it exists, is unique.

We then have the following result which is the analogue of Theorem 2.17, see [45] and [49].

THEOREM 4.4. *We assume that the family of processes* $\{U_\sigma(t, \tau),\ t \geqslant \tau,\ \tau \in \mathbb{R},\ \sigma \in \Sigma\}$ *possesses a compact uniformly (with respect to* σ*) attracting set, i.e., a compact subset* K *of* E *such that*

$$\forall B \subset E \text{ bounded}, \qquad \lim_{t\to+\infty} \sup_{\sigma\in\Sigma} \operatorname{dist}\big(U_\sigma(t, \tau)B, K\big) = 0.$$

Then it possesses the uniform attractor $\mathcal{A}_\Sigma$.

REMARK 4.5.

(a) It is easy to extend the other notions and definitions given for semigroups, e.g., bounded absorbing sets, to families of processes (see [45] and [49]).

(b) Theorem 4.4 does not require any continuity assumption on the processes, contrary to Theorem 2.17. This is due to the fact that the invariance property is replaced by the minimality property.

In applications, we need further assumptions on the symbol space in order to prove the existence of the uniform attractor, and we assume from now on that the initial symbol σ_0 is translation compact, i.e., that Σ is compact in Θ (see however [147,149], and [151] in which the translation compactness is relaxed; more precisely, one considers classes of time dependences which are translation bounded (i.e., Σ is bounded), but not translation compact).

A first example of translation compact symbols is given by quasiperiodic symbols. More precisely, σ_0 is quasiperiodic (with values in a metric space M) if it can be written in the form

$$\sigma_0(s) = \varphi(\alpha s), \quad \alpha = (\alpha_1, \dots, \alpha_k),\ k \in \mathbb{N},$$

where φ is 2π-periodic in each argument and $\alpha_1, \dots, \alpha_k$ are rationally independent (for $k = 1$, the symbol is periodic). We further assume that $\varphi \in \mathcal{C}(\mathbb{T}^k, M)$, where $\mathbb{T}^k$ is the k-dimensional torus. Then the hull of σ_0 in $\mathcal{C}_b(\mathbb{R}, M)$ coincides with $\{\varphi(\alpha s + \omega),\ \omega \in \mathbb{T}^k\}$. Actually, in that case, we take the torus $\mathbb{T}^k$ as symbol space; we can note that the mapping $\omega \mapsto \varphi(\alpha s+\omega)$ is continuous, but not necessarily one-to-one. Furthermore, the translations group $T(h)$, $h \in \mathbb{R}$, acts on $\mathbb{T}^k$ by the relation

$$T(h)\omega = h(1, \dots, 1) + \omega \left(\operatorname{mod} \mathbb{T}^k\right), \quad \omega \in \mathbb{T}^k,\ h \in \mathbb{R}.$$

Other examples of translation compact symbols are given by almost periodic (in Bochner–Amerio sense) symbols in $\mathcal{C}_b(\mathbb{R}, M)$ (see [45] and [49] for more details and other examples of translation compact symbols).

One interesting feature of nonautonomous systems with translation compact symbols is that we can reduce the construction of the uniform attractor to that of the global attractor for a semigroup acting on a proper extended phase space; this also yields further properties on the uniform attractor.

Noting that, owing to the well-posedness,

$$U_{T(h)\sigma}(t, \tau) = U_\sigma(t + h, \tau + h), \quad \forall t \geqslant \tau,\ \tau \in \mathbb{R},\ \forall \sigma \in \Sigma,\ \forall h \in \mathbb{R},$$

it is not difficult to show that the family of operators

$$\begin{aligned} S(t) : E \times \Sigma &\to E \times \Sigma, \\ (u, \sigma) &\mapsto \left(U_\sigma(t, 0)u, T(t)\sigma\right), \end{aligned} \tag{4.4}$$

$t \geqslant 0$, forms a semigroup on $E \times \Sigma$.

We further assume that, $\forall t \geqslant \tau$, $\tau \in \mathbb{R}$,

$$(u, \sigma) \mapsto U_\sigma(t, \tau)u \text{ is continuous from } E \times \Sigma \text{ into } E$$

(we say that the family of processes $\{U_\sigma(t, \tau),\ t \geqslant \tau,\ \tau \in \mathbb{R},\ \sigma \in \Sigma\}$ is $(E \times \Sigma, E)$-continuous). Then the semigroup $S(t)$ satisfies the continuity property (2.4) on $E \times \Sigma$.

We can now use the results of Subsection 2.2 to construct the global attractor $\mathcal{A}$ for $S(t)$ on the extended phase space $E \times \Sigma$. In particular, if the family of processes $\{U_\sigma(t,\tau),\ t \geqslant \tau,\ \tau \in \mathbb{R},\ \sigma \in \Sigma\}$ possesses a compact uniformly attracting set, then $S(t)$ possesses a compact attracting set (note that Σ is compact) and we have the following result.

THEOREM 4.6. *We assume that the family of processes $\{U_\sigma(t,\tau),\ t \geqslant \tau,\ \tau \in \mathbb{R},\ \sigma \in \Sigma\}$ is $(E \times \Sigma)$-continuous and possesses a compact uniformly attracting set. Then the semigroup $S(t)$ defined in (4.4) possesses the connected global attractor $\mathcal{A}$. Furthermore, if Π_1 (resp., Π_2) denotes the projector onto E (resp., Σ), then*

$$\mathcal{A}_\Sigma = \Pi_1 \mathcal{A}$$

is the uniform attractor for the family of processes $\{U_\sigma(t,\tau),\ t \geqslant \tau,\ \tau \in \mathbb{R},\ \sigma \in \Sigma\}$ and

$$\Pi_2 \mathcal{A} = \Sigma.$$

REMARK 4.7. It follows from Theorem 4.6 that, under the assumptions of this theorem, the uniform attractor $\mathcal{A}_\Sigma$ is connected.

We say that $u(s)$, $s \in \mathbb{R}$, is a complete trajectory for the process $U(t,\tau)$, $t \geqslant \tau$, $\tau \in \mathbb{R}$, acting on E if

$$U(t,\tau)u(\tau) = u(t), \quad \forall t \geqslant \tau,\ \tau \in \mathbb{R}$$

(as in Subsection 2.1, we can also define the forward and backward trajectories) and we define the kernel of this process as the set

$$\mathcal{K} := \Big\{ u : \mathbb{R} \to E,\ u \text{ is a complete trajectory of the process } U(t,\tau), \\ \sup_{t \in \mathbb{R}} \big\| u(t) \big\|_E < +\infty \Big\}.$$

We then have the

THEOREM 4.8. *Under the assumptions of Theorem 4.6, the global attractor $\mathcal{A}$ associated with the semigroup $S(t)$ defined by (4.4) satisfies*

$$\mathcal{A} = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma(0) \times \{\sigma\},$$

where $\mathcal{K}_\sigma$ is the kernel of the process $U_\sigma(t,\tau)$. Furthermore, the uniform attractor $\mathcal{A}_\Sigma = \Pi_1 \mathcal{A}$ satisfies

$$\mathcal{A}_\Sigma = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma(0).$$

REMARK 4.9.

(a) It follows from the invariance of $\mathcal{A}$ that

$$\mathcal{A}_\Sigma = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma(s), \quad \forall s \in \mathbb{R}.$$

(b) It follows from the above results that, under the assumptions of Theorem 4.6, the process $U_\sigma(t,\tau)$ possesses at least one bounded complete trajectory, $\forall\sigma\in\Sigma$.

REMARK 4.10. It is also possible to construct, for the initial process $U_{\sigma_0}(t,\tau)$, the uniform, now with respect to $\tau\in\mathbb{R}$, attractor. We refer the reader to [49] for more details and conditions which ensure that this attractor coincides with $\mathcal{A}_\Sigma$.

An important issue is whether the uniform attractor $\mathcal{A}_\Sigma$ has finite (fractal) dimension. A natural way to prove such a result would be to prove that the global attractor $\mathcal{A}$ for the semigroup $S(t)$ defined by (4.4) has finite (fractal) dimension. Then, since the projector Π_1 is Lipschitz, we would infer that $\mathcal{A}_\Sigma$ has also finite dimension. Unfortunately, as mentioned in the introduction, the dynamics of $S(t)$ is much more complicated than that of the initial system in general and $\mathcal{A}$ has infinite dimension in general; we also saw that the uniform attractor can already be infinite dimensional for simple linear equations.

Thus, in general, the uniform attractor does not yield a finite dimensional reduction principle. Essentially, we are only able to prove the finite dimensionality of the uniform attractor for quasiperiodic processes (see [49]; see however [212] for a finite dimensional result for asymptotically periodic processes).

REMARK 4.11. A direct way to study the dimension of $\mathcal{A}_\Sigma$ consists in computing its Kolmogorov ϵ-entropy (see Definition 2.28; see also [49] for details). In particular, if the Kolmogorov ϵ-entropy of $\mathcal{A}_\Sigma$ satisfies

$$\mathcal{H}_\epsilon(\mathcal{A}_\Sigma)\leqslant d\log_2\frac{1}{\epsilon}+c,$$

where c and d are independent of ϵ, then

$$\dim_F\mathcal{A}_\Sigma\leqslant d.$$

The use of the Kolmogorov entropy allows in particular to obtain sharp bounds on the dimension of $\mathcal{A}_\Sigma$ for quasiperiodic processes, see [49].

4.2. *Pullback attractors*

We saw in the previous subsection that the uniform attractor does not yield a satisfactory finite dimensional reduction principle in general, i.e., for a general translation compact symbol. Furthermore, even though the time appears explicitly in the equations, the uniform attractor is time independent.

In this subsection, we introduce a second notion of a nonautonomous attractor, now time dependent.

We consider a process $U(t,\tau)$, $t\geqslant\tau$, $\tau\in\mathbb{R}$, acting on a Banach space E,

$$U(t,\tau):E\to E,\quad t\geqslant\tau,\ \tau\in\mathbb{R},$$

and we assume that

$$U(t,\tau)\text{ is continuous on }E,\quad\forall t\geqslant\tau,\ \tau\in\mathbb{R}.$$

DEFINITION 4.12. A family $\{\mathcal{A}(t),\ t \in \mathbb{R}\}$ is a pullback attractor for the process $U(t,\tau)$ if

(i) $\mathcal{A}(t)$ is compact in E, $\forall t \in \mathbb{R}$;

(ii) it is invariant in the following sense:

$$U(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(t), \quad \forall t \geqslant \tau,\ \forall \tau \in \mathbb{R};$$

(iii) it satisfies the following attraction property, called pullback attraction:

$$\begin{aligned} &\forall B \subset E \text{ bounded},\ \forall t \in \mathbb{R}, \\ &\lim_{s\to+\infty} \operatorname{dist}\big(U(t,t-s)B, \mathcal{A}(t)\big) = 0. \end{aligned} \tag{4.5}$$

REMARK 4.13. The pullback attraction (4.5) essentially means that, at time t, the set $\mathcal{A}(t)$ attracts the bounded sets of initial data coming from $-\infty$.

REMARK 4.14.

(a) We can note that Definition 4.12 is too general to have the uniqueness of a pullback attractor, if it exists. Indeed, let us consider the following simple dissipative *autonomous* ODE:

$$y' + y = 0.$$

Then it possesses the global attractor $\mathcal{A} = \{0\}$. However, *any* trajectory $y(t) = C\mathrm{e}^{-t}$, $t \in \mathbb{R}$, generates a pullback attractor (e.g., $\mathcal{A}(t) = \{0, Ce^{-t}\}$, $t \in \mathbb{R}$)! Thus, the uniqueness of a pullback attractor fails and additional conditions must be added in order to restore such a property (see [36] and [40]). For instance, the uniqueness holds if one has some "backward boundedness", e.g.,

$$\sup_{s\in\mathbb{R}_+} \|\mathcal{A}(t-s)\|_E \quad \Big(:= \sup_{s\in\mathbb{R}_+}\ \sup_{x\in\mathcal{A}(t-s)} \|x\|_E\Big) \leqslant C_t, \quad t \in \mathbb{R}. \tag{4.6}$$

(b) If the system is autonomous and we further assume that (4.6) holds, then we recover the global attractor. Indeed, in that case, we can write $U(t,\tau) = S(t-\tau)$, where $S(t)$ is a semigroup, and we have, in the pullback attraction property, $U(t,t-s) = S(s)$.

REMARK 4.15. The above definition of a pullback attractor resembles that of the so-called kernel sections introduced by Chepyzhov and Vishik, see [44,45], and [49]. Actually, in order to prove that these two objects are equivalent, i.e.,

$$\mathcal{A}(t) = \mathcal{K}(t) := \big\{u(t),\ u \in \mathcal{K}\big\}, \quad t \in \mathbb{R}, \tag{4.7}$$

where $\mathcal{K}$ is the kernel (i.e., the set of all bounded complete trajectories of the associated process), we need to make further assumptions. In particular, this equivalence holds if one has the backward boundedness (4.6), together with some forward dissipativity (e.g., the existence of a (forward) bounded uniformly absorbing set for the process). Furthermore, as proven, e.g., in [45] (see also [49]), the kernel sections (i.e., the pullback attractor here) have finite fractal dimension in E,

$$\dim_F \mathcal{A}(t) < +\infty, \quad t \in \mathbb{R},$$

under assumptions which are very close to those given in Subsection 2.4 for autonomous systems, see [44,45], and [49] for more details. However, pullback attractors have been introduced and further studied independently; it is also worth noting that they have been extended to cocycles in the context of random dynamical systems as well, see, e.g., [62].

DEFINITION 4.16. The family $\{K(t),\ t \in \mathbb{R}\}$ is pullback attracting for the process $U(t,\tau)$ if, $\forall t \in \mathbb{R}$, $\forall B \subset E$ bounded,

$$\lim_{s\to+\infty} \operatorname{dist}\bigl(U(t,t-s)B, K(t)\bigr) = 0.$$

The following result is the analogue of Theorem 2.17 for pullback attractors (see, e.g., [35]).

THEOREM 4.17. *We assume that the process $U(t,\tau)$ possesses a compact pullback attracting family $\{K(t),\ t \in \mathbb{R}\}$ (i.e., $K(t)$ is compact, $\forall t \in \mathbb{R}$). Then it possesses a pullback attractor $\{\mathcal{A}(t),\ t \in \mathbb{R}\}$. Furthermore, if the compact pullback attracting family $\{K(t),\ t \in \mathbb{R}\}$ satisfies (4.6), then a pullback attractor $\{\mathcal{A}(t) \subset K(t),\ t \in \mathbb{R}\}$ also satisfies (4.6) and is unique (in this class).*

REMARK 4.18.

(a) As in the case of semigroups, one usually proves the existence of a compact pullback attracting family $\{K(t),\ t \in \mathbb{R}\}$ by introducing a proper decomposition $U(t,\tau) = U_1(t,\tau) + U_2(t,\tau)$, see [35].
(b) Actually, all notions, definitions, and properties introduced for global attractors have a "pullback counterpart", see, e.g., [39,214], and [230]. For instance, the pullback version of Theorem 2.19 is given in [214] (see also [39]).

An interesting feature of pullback attractors is that, in general, $\mathcal{A}(t)$ has finite fractal dimension, $\forall t \in \mathbb{R}$ (see, e.g., [38] and [139]; see also Remark 4.15), so that the finite dimensional reduction principle given by the Hölder–Mañé theorem holds. Unfortunately, as mentioned in the introduction, the forward convergence, i.e.,

$$\lim_{t\to+\infty} \operatorname{dist}\bigl(U(t,\tau)B, \mathcal{A}(t)\bigr) = 0, \quad \forall B \subset E \text{ bounded},\ \forall \tau \in \mathbb{R},$$

does not hold in general, due to the fact that the pullback attraction property (4.5) is not uniform with respect to $t \in \mathbb{R}$ (see however [35] and [40] for examples for which the forward attraction holds); we also gave an example of an equation for which the pullback attractor satisfying (4.6) (we recall that we have the uniqueness in this class) does not reflect the asymptotic behavior of the solutions of the system. Thus, again, this notion of a nonautonomous attractor does not yield a satisfactory finite dimensional reduction principle in general.

REMARK 4.19. Nonautonomous inertial manifolds (also called integral manifolds) were studied, e.g., in [42] (see also [12,130,131], and [162]). In that case, under natural assumptions, the forward exponential convergence, and even the asymptotic completeness (i.e., a property similar to (3.1)) hold. However, we also have here the drawbacks mentioned in Subsection 3.1 and, in particular, very restrictive spectral gap conditions are necessary to prove the existence of such objects. We also mention that, when the process $U(t,\tau)$ is, in some proper sense, close to an autonomous semigroup $S(t)$ which possesses a global Lyapunov function and has a *regular* attractor, the associated pullback attractor $\mathcal{A}(t)$, $t \in \mathbb{R}$, is also regular (i.e., it is a finite union of finite dimensional submanifolds of E, of course, now depending on t) and uniformly (forward and pullback) exponentially attracting, see [42,43, 77,111], and [225] for details.

4.3. *Finite dimensional reduction of nonautonomous systems*

We saw in the two previous subsections that neither the uniform attractor nor a pullback attractor yield a satisfactory finite dimensional reduction principle in general. We noted however that the problem of the forward attraction for pullback attractors is due to the fact that the pullback attraction (4.5) may not be uniform with respect to t. Therefore, if we are able to construct (possibly) larger sets for which the pullback attraction is uniform with respect to t, then we will also obtain the forward attraction: the concept of an exponential attractor appears as a natural one to reach this goal and it is thus important to extend it to processes.

We first consider a discrete process $U(l,m)$, $l,m \in \mathbb{Z}$, $l \geqslant m$, acting on E, i.e.,

$$U(l,l) = \mathrm{Id}, \quad \forall l \in \mathbb{Z},$$
$$U(l,m) \circ U(m,n) = U(l,n), \quad \forall l \geqslant m \geqslant n,\ l,m,n \in \mathbb{Z}.$$

We set $\mathcal{U}(n) := U(n+1,n)$, $n \in \mathbb{Z}$. It is then easy to see that the process $U(l,m)$ is uniquely determined by the family $\{\mathcal{U}(l),\ l \in \mathbb{Z}\}$; indeed,

$$U(n+k,n) = \mathcal{U}(n+k-1) \circ \mathcal{U}(n+k-2) \circ \cdots \circ \mathcal{U}(n), \quad n \in \mathbb{Z},\ k \in \mathbb{N}.$$

We have the following result, which extends to (discrete) processes that given in Remark 3.12(d).

THEOREM 4.20. *(See [73].) We consider a second Banach space E_1 such that E_1 is compactly embedded into E and a bounded subset B of E_1. We make the following assumptions*:

(i) $\forall l \in \mathbb{Z}, \mathcal{U}(l)$ *maps the δ-neighborhood (for the topology of E_1) $\mathcal{O}_\delta(B)$ of B onto B, where δ is independent of l*;

(ii) $\forall l \in \mathbb{Z}, \mathcal{U}(l)$ *satisfies the smoothing property* (2.14) *on* $\mathcal{O}_\delta(B)$,

$$\left\|\mathcal{U}(l)x_1 - \mathcal{U}(l)x_2\right\|_{E_1} \leqslant K\|x_1 - x_2\|_E, \quad \forall x_1, x_2 \in \mathcal{O}_\delta(B),$$

where K is independent of l, x_1, and x_2.

Then the discrete process $U(l,m)$ possesses a nonautonomous exponential attractor

$$\left\{\mathcal{M}_U(n),\ n \in \mathbb{Z}\right\}$$

such that

1. $\forall n \in \mathbb{Z}$, $\mathcal{M}_U(n) \subset B$ *and is compact in* E_1;
2. $\forall n \in \mathbb{Z}$, $\mathcal{M}_U(n)$ *has finite fractal dimension* (*in the topology of* E_1),

$$\dim_F \mathcal{M}_U(n) \leqslant c_1,$$

where c_1 *is independent of* n;
3. *it is positively invariant in the following sense*:

$$U(l,m)\mathcal{M}_U(m) \subset \mathcal{M}_U(l), \quad l \geqslant m,\ l,m \in \mathbb{Z};$$

4. *it satisfies the following exponential attraction property*:

$$\operatorname{dist}_{E_1}\big(U(l+m,l)B, \mathcal{M}_U(l+m)\big) \leqslant c_2 \mathrm{e}^{-c_3 m}, \quad l \in \mathbb{Z},\ m \in \mathbb{N}, \tag{4.8}$$

where c_2 *and* c_3 *are independent of* l *and* m;
5. *the map* $U \mapsto \{\mathcal{M}_U(n),\ n \in \mathbb{Z}\}$ *is uniformly Hölder continuous in the following sense*: *for every processes* $U_1(l,m)$ *and* $U_2(l,m)$ *such that* $\mathcal{U}_i(l)$, $i = 1, 2$, *satisfy* (i) *and* (ii), $\forall l \in \mathbb{Z}$ (*for the same constants* δ *and* K), *there holds*

$$\begin{aligned}&\operatorname{dist}_{\mathrm{sym}, E_1}\big(\mathcal{M}_{U_1}(l), \mathcal{M}_{U_2}(l)\big)\\ &\quad\leqslant c_4 \sup_{m \in (-\infty, l) \cap \mathbb{Z}} \big\{\mathrm{e}^{-c_5(l-m)} \big\|\mathcal{U}_1(m) - \mathcal{U}_2(m)\big\|^{c_6}\big\},\end{aligned} \tag{4.9}$$

where $c_4, c_5 > 0$, *and* $c_6 > 0$ *are independent of* l, U_1, *and* U_2 *and*

$$\|S\| := \sup_{h \in \mathcal{O}_\delta(B)} \|Sh\|_{E_1}.$$

Furthermore, *all the constants only depend on* B, E, E_1, δ, *and* K *and can be computed explicitly in terms of the physical parameters of the problem.*

REMARK 4.21.

(a) It follows from (4.8) that the pullback attraction holds, but one now has the forward attraction (and, even better, one has a uniform forward attraction). Since, $\forall l \in \mathbb{Z}$, $\mathcal{M}_U(l)$ has finite fractal dimension, this shows that the asymptotic behavior of (discrete) nonautonomous systems is also, in some proper sense, finite dimensional in general, as in the case of autonomous systems.

(b) It also follows from (4.9) that the influence of the past decays exponentially, in agreement with our physical intuition.

(c) We can also construct the exponential attractor $\{\mathcal{M}_U(n),\ n \in \mathbb{Z}\}$ such that the following cocycle identity holds:

$$\mathcal{M}_U(l+m) = \mathcal{M}_{T_m U}(l), \quad l, m \in \mathbb{Z},$$

where $T_k U(l,m) := U(l+k, m+k)$, $k, l, m \in \mathbb{Z}$, $l \geqslant m$.

(d) If $\mathcal{U}(l) \equiv S$, $\forall l \in \mathbb{Z}$, i.e., if the system is autonomous, we recover the exponential attractor constructed in Remark 3.12(d).

(e) If the dependence of $\mathcal{U}(l)$ on l is periodic or quasiperiodic, then the same holds for the dependence of $\mathcal{M}_U(l)$ on l.

REMARK 4.22. As mentioned several times, the smoothing property (2.14) is typical of parabolic systems and, e.g., for damped hyperbolic systems, it has to be generalized. In particular, if the second assumption of Theorem 4.20 is replaced by the following: $\forall l \in \mathbb{Z}$, $\forall x_1, x_2 \in \mathcal{O}_\delta(B)$, B being a proper closed subset of E_1,

$$\left\|\mathcal{U}(l)x_1 - \mathcal{U}(l)x_2\right\|_{E_1} \leqslant (1-\epsilon)\|x_1 - x_2\|_{E_1} + K\|x_1 - x_2\|_E,$$

where $\epsilon \in (0,1)$ and K are independent of l, x_1, and x_2, then, assuming that B can be covered by a finite number of balls of radius δ (in the topology of E_1) with centers belonging to B, Theorem 4.20 also holds. We can obtain a similar result under the following more general (asymptotically) smoothing property: $\forall l \in \mathbb{Z}$, $\forall x_1, x_2 \in \mathcal{O}_\delta(B)$, $\mathcal{U}(l)x_1 - \mathcal{U}(l)x_2 = v_1 + v_2$, where

$$\|v_1\|_E \leqslant (1-\epsilon)\|x_1 - x_2\|_E,$$
$$\|v_2\|_{E_1} \leqslant K\|x_1 - x_2\|_E,$$

where $\epsilon \in (0,1)$ and K are independent of l, x_1, and x_2. However, in that case, all properties are obtained for the topology of E instead of that of E_1, see [73] for more details.

The next step is to extend such constructions to continuous processes $U(t,\tau)$, $t \geqslant \tau$, $\tau \in \mathbb{R}$.

For instance, for a parabolic system in a bounded domain, we usually proceed as follows. We first consider a uniform (with respect to $\tau \in \mathbb{R}$) bounded absorbing set B in E_1 (i.e., $\forall B_0 \subset E_1$ bounded, $\exists t_0 = t_0(B_0)$ such that $t \geqslant t_0$ implies $U(t+\tau,\tau)B_0 \subset B$, $\forall \tau \in \mathbb{R}$). We further assume that the map $U(T+\tau,\tau)$ satisfies the assumptions of Theorem 4.20, $\forall \tau \in \mathbb{R}$, for B as above and for some $T > 0$, $\delta > 0$, and K which are independent of τ (typically, in applications, we can take $\delta = 1$). Then, for every $\tau \in \mathbb{R}$, we consider the discrete process

$$U^\tau(l,m) := U(\tau + lT, \tau + mT), \quad l, m \in \mathbb{Z},\ l \geqslant m.$$

Thus, owing to Theorem 4.20, we can construct, for every $\tau \in \mathbb{R}$, a discrete exponential attractor $\{\mathcal{M}_U(l,\tau),\ l \in \mathbb{Z}\}$ which satisfies all the assertions of this theorem. In addition, it satisfies the following properties:

$$\mathcal{M}_U(l,\tau) = \mathcal{M}_U(0, lT + \tau), \quad l \in \mathbb{Z},\ \tau \in \mathbb{R};$$
$$\mathcal{M}_{T_s U}(l,\tau) = \mathcal{M}_U(l, \tau + s),\ l \in \mathbb{Z},\ s,\ \tau \in \mathbb{R},$$

where $T_s U(t,\tau) := U(t+s, \tau+s)$, $t \geqslant \tau$, $s, \tau \in \mathbb{R}$. We finally set

$$\mathcal{M}_U(t) := \bigcup_{s \in [0,T]} U(t, t-T-s)\mathcal{M}_U(0, t-T-s), \quad t \in \mathbb{R}.$$

Then, assuming that $L : (t,\tau,x) \mapsto U(t,\tau)x$ is Lipschitz with respect to the x-variable and satisfies proper Hölder type properties with respect to t and τ, typically,

$$\left\|U(\tau + s + t, \tau)x - U(\tau + t, \tau)x\right\|_E \leqslant c|s|^{1/2},$$

where c is independent of $t \geqslant 0$, $\tau \in \mathbb{R}$, $s \geqslant 0$, and $x \in B$, and

$$\left\| U(t+\tau+s, \tau+s)x - U(t+\tau, \tau)x \right\|_E \leqslant c\mathrm{e}^{c't}|s|^\gamma,$$
$$t \geqslant T,\ s \in \left[0, \frac{T}{2}\right],\ \gamma > 0,$$

where c is independent of t, τ, s, and $x \in B$ (see [73] for more details), we can prove the following result.

THEOREM 4.23. *(See [73].) The family* $\{\mathcal{M}_U(t),\ t \in \mathbb{R}\}$ *is a nonautonomous exponential attractor for the process* $U(t, \tau)$ *in* E_1 *which satisfies the following properties*:

1. $\forall t \in \mathbb{R}$, $\mathcal{M}_U(t)$ *is compact in* E_1 *and has finite fractal dimension*,

$$\dim_F \mathcal{M}_U(t) \leqslant c'_1, \quad \forall t \in \mathbb{R},$$

where c'_1 *is independent of* t;

2. *it is positively invariant*,

$$U(t, \tau)\mathcal{M}_U(\tau) \subset \mathcal{M}_U(t), \quad t \geqslant \tau,\ \tau \in \mathbb{R};$$

3. *it satisfies the following exponential attraction property*:

$$\operatorname{dist}_{E_1}\big(U(t+\tau, \tau)B, \mathcal{M}_U(t+\tau)\big) \leqslant c'_2\mathrm{e}^{-c'_3 t}, \quad \tau \in \mathbb{R},\ t \geqslant 0,$$

where c'_2 *and* $c'_3 > 0$ *are independent of* t *and* τ *and where* B *is the bounded absorbing set introduced above*;

4. *it satisfies the following Hölder continuity property*: *for every processes* $U_1(t, \tau)$ *and* $U_2(t, \tau)$ *such that* $U_i(t+T, t)$, $i = 1, 2$, *satisfy the assumptions of Theorem* 4.20 (*for the constants* δ *and* K *introduced above*), $\forall t \in \mathbb{R}$, *then*

$$\operatorname{dist}_{\mathrm{sym}, E_1}\big(\mathcal{M}_{U_1}(t), \mathcal{M}_{U_2}(t)\big)$$
$$\leqslant c'_4 \sup_{s \geqslant 0}\left\{\mathrm{e}^{-c'_5 s}\left\| U_1(t, t-s) - U_2(t, t-s)\right\|^{c'_6}\right\},$$

where c'_4, $c'_5 > 0$, *and* $c'_6 > 0$ *are independent of* $t \in \mathbb{R}$.

Furthermore, *all the constants can be computed explicitly.*

REMARK 4.24.

(a) We can give a more precise Hölder continuity result in concrete applications, see [73].

(b) We also have the following Hölder continuity with respect to the time:

$$\operatorname{dist}_{\mathrm{sym}, E_1}\big(\mathcal{M}_U(t+s), \mathcal{M}_U(t)\big) \leqslant c'_7|s|^{c'_8}, \quad t \in \mathbb{R},\ s \geqslant 0,$$

where c'_7 and $c'_8 > 0$ are independent of t and s.

REMARK 4.25.
(a) We again have properties which are similar to those listed in Remark 4.21. In particular, we have the (uniform) forward attraction and, since $\mathcal{M}_U(t)$ has finite fractal dimension, $\forall t \in \mathbb{R}$, we obtain a satisfactory finite dimensional reduction principle for nonautonomous systems in bounded domains.
(b) Such exponential attractors were constructed for nonautonomous reaction–diffusion equations (in bounded domains) in [73]. However, this construction has a universal nature and should be applicable to most equations (in bounded domains) for which the finite dimensionality of pullback attractors can be proven (e.g., the two-dimensional incompressible Navier–Stokes equations, the Cahn–Hilliard equation, damped hyperbolic equations, ...).

REMARK 4.26. It follows from the Hölder continuity that we can construct robust families of nonautonomous exponential attractors which are continuous at every point, as in the autonomous case.

5. Dissipative PDEs in unbounded domains

As mentioned in the introduction, the study of the dynamics of dissipative systems in large and unbounded domains necessitates to develop new ideas and methods, when compared with the above sections, devoted to systems in bounded domains. Indeed, we are faced here with new phenomena which do not have analogues in the finite dimensional theory.

Our aim in this section is to give a short survey of the recent progress in this direction, including the so-called entropy theory, the description of the space–time chaos via Bernoulli schemes with an infinite number of symbols and its relations with the Kotelnikov formula, the Sinai–Bunimovich space–time chaos for continuous media, ...

We start by introducing and discussing the appropriate class of weighted and uniformly local Sobolev spaces, which is one of the main technical tools in the theory.

5.1. *Weighted and uniformly local phase spaces: basic dissipative estimates*

We first note that, in contrast to the case of bounded domains, many physically relevant and interesting solutions of PDEs in unbounded domains (such as spatially periodic patterns, traveling waves, wave trains, spiral waves, ...) are not spatially localized and, thus, usually have *infinite* energy. Therefore, the typical, for bounded domains, choice of the phase space as $\Phi = L^2(\Omega)$ or $W^{l,p}(\Omega)$ does not seem to be reasonable here. On the other hand, all the above mentioned structures are bounded as $|x| \to +\infty$ and, therefore, belong to the phase space $\Phi = L^\infty(\Omega)$. However, the analytical properties of PDEs in L^∞-spaces are very bad (there is no maximal regularity, no analytic semigroups, ...), so that this choice of a phase space only works for relatively simple equations (for which the maximum principle holds).

Instead, it was suggested in [1,21], and [165], to use *weighted* and so-called *uniformly local* Sobolev spaces which, on the one hand, contain all the sufficiently regular spatially bounded solutions and, on the other hand, enjoy regularity, embedding, and interpolation properties which are very similar to those of usual Sobolev spaces in bounded domains.

In order to introduce these spaces, we first need to define the appropriate class of admissible weight functions (see [75] and [237] for details).

DEFINITION 5.1. Let $\mu > 0$ be arbitrary. A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is a weight function with an exponential growth μ if there holds

$$\phi(x + y) \leqslant C_\phi e^{\mu|y|}\phi(x), \quad \phi(x) > 0, \tag{5.1}$$

for every $x,\ y \in \mathbb{R}^n$.

The most important examples of such weight functions are the following ones:

$$\phi_{\varepsilon,x_0}(x) := e^{-\varepsilon|x-x_0|}, \quad \varepsilon \in \mathbb{R},\ x_0 \in \mathbb{R}^n, \tag{5.2}$$

or their smooth analogues,

$$\varphi_{\varepsilon,x_0}(x) := e^{-\varepsilon\sqrt{1+|x-x_0|^2}}, \quad \varepsilon \in \mathbb{R},\ x_0 \in \mathbb{R}^n. \tag{5.3}$$

Obviously, these weight functions are weight functions with an exponential growth $|\varepsilon|$ and the constant C_ϕ is independent of x_0. Another important class of weight functions consists of the so-called polynomial weights,

$$\theta_{N,x_0}(x) := \bigl(1 + |x - x_0|^2\bigr)^{-N/2}, \quad N \in \mathbb{R},\ x_0 \in \mathbb{R}^n, \tag{5.4}$$

which are also sometimes useful. Obviously, these weight functions have an exponential growth μ, for every $\mu > 0$.

We are now ready to introduce the proper classes of Sobolev spaces.

DEFINITION 5.2. Let Ω be a sufficiently regular unbounded domain, ϕ be a weight function with an exponential growth, and $1 \leqslant p \leqslant +\infty$. Then the associated weighted spaces $L^p_\phi(\Omega)$ and weighted uniformly local spaces $L^p_{b,\phi}(\Omega)$ are defined by the following norms:

$$\begin{aligned} \|u\|^p_{L^p_\phi} &:= \int_\Omega \phi^p(x)\bigl|u(x)\bigr|^p\,dx, \\ \|u\|_{L^p_{b,\phi}} &:= \sup_{x_0\in\Omega}\bigl\{\phi(x_0)\|u\|_{L^p(\Omega\cap B^1_{x_0})}\bigr\}, \end{aligned} \tag{5.5}$$

where $B^R_{x_0}$ denotes the R-ball in the space $\mathbb{R}^n$ centered at x_0. For simplicity, we will write $L^p_b(\Omega)$ instead of $L^p_{b,1}(\Omega)$ and we naturally define the Sobolev spaces $W^{l,p}_\phi(\Omega)$ (resp., $W^{l,p}_{b,\phi}(\Omega)$) as the spaces of distributions whose derivatives up to the order l belong to $L^p_\phi(\Omega)$ (resp., $L^p_{b,\phi}(\Omega)$).

We note that $L^\infty(\Omega) \subset L^2_b(\Omega)$ and, consequently, all the dissipative structures mentioned above indeed belong to the uniformly local phase space $\Phi := L^2_b(\Omega)$. Furthermore, we also have the embedding $L^\infty(\Omega) \subset L^2_\phi(\Omega)$ if the weight function ϕ is integrable (i.e., $\phi \in L^1(\Omega)$). The important relations between weighted and uniformly local spaces are collected in the following proposition (see [237]).

PROPOSITION 5.3. *Let Ω be a sufficiently regular unbounded domain and let ϕ be a weight function with an exponential growth ε. Then, for every $\mu > \varepsilon$, the following norms are equivalent*:

$$\|u\|^p_{L^p_\phi} \sim \int_\Omega \phi^p(x_0) \int_\Omega \mathrm{e}^{-\mu|x-x_0|} \big|u(x)\big|^p \,\mathrm{d}x\,\mathrm{d}x_0,$$
$$\|u\|^p_{L^p_{b,\phi}} \sim \sup_{x_0\in\Omega} \left\{ \phi^p(x_0) \int_\Omega \mathrm{e}^{-\mu|x-x_0|} \big|u(x)\big|^p \,\mathrm{d}x \right\}, \tag{5.6}$$

with constants which depend on ε, μ, and C_ϕ, but are independent of the concrete form of ϕ.

REMARK 5.4. Relations (5.6) allow to reduce the calculation and estimation of any weighted norm (for a weight function with an exponential growth) to those of the special exponential weight functions $\phi_{\mu,x_0}(x)$. In particular, relations of this type allow to reduce most results concerning embeddings and interpolation estimates for the weighted and uniformly local spaces, together with the associated regularity results for linear elliptic and parabolic operators, to the corresponding ones for the weight functions $\phi_{\varepsilon,x_0}(x)$ or $\varphi_{\varepsilon,x_0}(x)$ (and, thanks to the natural change of function $\tilde{u} = u\varphi_{-\varepsilon,x_0}$, to the classical spaces without weight). Furthermore, all the constants in such estimates only depend on the weight exponent and C_ϕ (and on some regularity constants of the boundary) and are independent of the concrete choice of ϕ and the shape of Ω, see [75,165,237,242], and the references therein. This also explains why the linear theory of PDEs in uniformly local spaces is very similar to that in the unweighted spaces.

We are now ready to return to the main issues of this subsection, namely, the definition of the proper phase spaces Φ for dissipative PDEs in unbounded domains and the derivation of the basic dissipative estimate

$$\big\|S(t)u_0\big\|_\Phi \leqslant Q\big(\|u_0\|_\Phi\big)\mathrm{e}^{-\alpha t} + C, \quad u_0 \in \Phi,\ t \geqslant 0,\ \alpha > 0. \tag{5.7}$$

A "general" answer to these questions can be formulated as follows:

(1) use the *uniformly local* Sobolev spaces $W^{l,p}_b(\Omega)$ or $L^p_b(\Omega)$ as phase spaces, e.g., in a Hilbert setting, i.e., $p = 2$;

(2) use the so-called weighted energy estimates and weighted regularity theory to obtain a dissipative estimate in the spaces $W^{l,p}_{\phi_{\varepsilon,x_0}}(\Omega)$;

(3) pass from the weighted to the uniformly local spaces by using the second estimate of (5.6) with $\phi = 1$.

This machinery has been successfully applied to many physically relevant PDEs in unbounded domains, including various types of reaction–diffusion equations (see [9,75,76,165], and [237]), damped wave equations (see [87] and [235]), elliptic equations in unbounded domains (see [166] and [224]), and even the Navier–Stokes equations in a strip (see [242]).

For the reader's convenience, we illustrate below such a scheme on the relatively simple example of a reaction–diffusion system in $\Omega = \mathbb{R}^3$ (see [237] for more details):

$$\partial_t u = a\Delta_x u - \lambda u - f(u) + g, \quad u|_{t=0} = u_0. \tag{5.8}$$

Here, $u = (u^1, \dots, u^k)$ is an unknown vector-valued function, a is a constant diffusion matrix satisfying the standard assumption $a + a^* > 0$, $\lambda > 0$ is a fixed constant, g corresponds to the external forces and belongs to $L^2_b(\mathbb{R}^3)^k$, and f is a given nonlinear interaction function satisfying the following standard dissipativity assumptions:

$$\begin{cases} 1.\ f \in C^1\big(\mathbb{R}^k, \mathbb{R}^k\big); \\ 2.\ f(u) \cdot u \geqslant -C; \\ 3.\ f'(u) \geqslant -K; \\ 4.\ \big|f(u)\big| \leqslant C\big(1 + |u|^p\big); \end{cases} \tag{5.9}$$

$u \in \mathbb{R}^k$, $C,\ K \geqslant 0$, where $u.v$ denotes the usual inner product in $\mathbb{R}^k$ and $p \geqslant 0$ is arbitrary.

THEOREM 5.5. *Let the above assumptions hold. Then, for every $u_0 \in \Phi_b := L^2_b(\mathbb{R}^3)^k$, problem (5.8) possesses a unique solution $u(t) \in \Phi_b$ and the following dissipative estimate holds*:

$$\big\|u(t)\big\|_{L^2_b} \leqslant C\|u_0\|_{L^2_b} e^{-\alpha t} + C\big(1 + \|g\|_{L^2_b}\big), \quad t \geqslant 0, \tag{5.10}$$

where the positive constants α and C are independent of u_0.

PROOF. We only give a formal derivation of the dissipative and uniqueness estimates. The remaining details can be found in [237]. We multiply Eq. (5.8) by $u\phi^2$, where $\phi(x) = \phi_{\varepsilon,x_0}(x) := e^{-\varepsilon|x-x_0|}$, for a sufficiently small ε which will be fixed below, and integrate with respect to $x \in \mathbb{R}^n$. Then we have

$$\begin{aligned} &1/2\partial_t \big\|u(t)\big\|^2_{L^2_\phi} + \big(a\nabla_x u(t), \nabla_x\big[\phi^2 u(t)\big]\big) + \lambda\big\|u(t)\big\|^2_{L^2_\phi} \\ &\quad = -\big(f\big(u(t)\big) \cdot u(t), \phi^2\big) + \big(\phi^2 u(t), g\big) \end{aligned} \tag{5.11}$$

(here and below, $(\cdot,\cdot)$ denotes the scalar products in $L^2(\mathbb{R}^3)$, $L^2(\mathbb{R}^3)^k$, and $L^2(\mathbb{R}^3)^{3k}$). According to the dissipativity assumption (5.9)(2), we see that

$$-\big(\phi^2, f(u) \cdot u\big) \leqslant C\|\phi\|^2_{L^2} = C\varepsilon^{-3} \tag{5.12}$$

and, thus, the nonlinear term can be controlled. Furthermore, thanks to the obvious inequality

$$\big|\nabla_x \phi_{\varepsilon,x_0}(x)\big| \leqslant C\varepsilon\phi_{\varepsilon,x_0}(x), \tag{5.13}$$

together with the positivity of a, we conclude that, if $\varepsilon > 0$ is small enough, the following estimate holds:

$$\begin{aligned} \big(a\nabla_x u, \nabla_x\big[\phi^2 u\big]\big) + \lambda\|u\|^2_{L^2_\phi} &\geqslant 2\alpha\big(\|\nabla_x u\|^2_{L^2_\phi} + \|u\|^2_{L^2_\phi}\big) - C\varepsilon\big(|u|, |\nabla_x u|\phi^2\big) \\ &\geqslant \alpha\big(\|\nabla_x u\|^2_{L^2_\phi} + \|u\|^2_{L^2_\phi}\big), \end{aligned} \tag{5.14}$$

for some positive constant α which is independent of x_0. Inserting these estimates into (5.11), we deduce that

$$\partial_t \big\|u(t)\big\|^2_{L^2_\phi} + 2\alpha\big\|u(t)\big\|^2_{W^{1,2}_\phi} \leqslant C\big(1 + \|g\|^2_{L^2_\phi}\big)$$

and the Gronwall inequality gives

$$\begin{aligned}\|u(T)\|^2_{L^2_{\phi_{\varepsilon,x_0}}} &+ \int_T^{T+1} \|u(t)\|^2_{W^{1,2}_{\phi_{\varepsilon,x_0}}}\,dt \\ &\leqslant C\|u(0)\|^2_{L^2_{\phi_{\varepsilon,x_0}}} \mathrm{e}^{-2\alpha T} + C\big(1 + \|g\|^2_{L^2_{\phi_{\varepsilon,x_0}}}\big).\end{aligned} \tag{5.15}$$

It is crucial here that the constants C and α in this inequality are independent of x_0. Therefore, taking the supremum over $x_0 \in \mathbb{R}^3$ and using the second relation of (5.6) with $\phi = 1$, we deduce the required dissipative estimate (5.10) in the uniformly local phase space $\Phi_b = L^2_b(\mathbb{R}^3)^k$.

We now verify the uniqueness. Let $u_1(t)$ and $u_2(t)$ be two solutions of (5.8) and set $v(t) := u_1(t) - u_2(t)$. Then this function solves the linear equation

$$\partial_t v = a\Delta_x v - \lambda v - l(t)v, \quad l(t) := \int_0^1 f'\big(su_1(t) + (1-s)u_2(t)\big)\,\mathrm{d}s. \tag{5.16}$$

We note that, due to the third (quasimonotonicity) assumption of (5.9), we have $l(t) \geqslant -K$. Multiplying now equation (5.16) by $v\phi^2_{\varepsilon,x_0}$, using the last inequality, and arguing exactly as in the derivation of the dissipative estimate, we obtain

$$\|v(t)\|^2_{L^2_{\phi_{\varepsilon,x_0}}} \leqslant C\mathrm{e}^{Kt}\|v(0)\|^2_{L^2_{\phi_{\varepsilon,x_0}}}, \tag{5.17}$$

for some positive constant C which is independent of x_0. This estimate gives the uniqueness and finishes the proof of the theorem. □

REMARK 5.6. We see that the growth restriction (5.9)(4) has not been used in the proof of uniqueness and of the derivation of the dissipative estimate. However, this assumption is necessary in order to show that the associated solution satisfies Eq. (5.8) in the sense of distributions. Furthermore, as shown in [237], $f(u(t))$ and $\Delta_x u(t)$ belong to $L^2_b(\mathbb{R}^3)^k$, for every $t > 0$, so that the equation can be understood as an equality in $L^2_b(\mathbb{R}^3)^k$.

REMARK 5.7. Estimates (5.15) and (5.17) show that the reaction–diffusion problem (5.8) is well-defined not only in the uniformly local phase space $\Phi_b = L^2_b(\mathbb{R}^3)^k$, but also in the larger phase space $\Phi_\varepsilon := L^2_{\mathrm{e}^{-\varepsilon|x|}}(\mathbb{R}^3)^k$, provided that $\varepsilon > 0$ is small enough. Roughly speaking, this space contains not only all functions which are bounded as $|x| \to +\infty$, but also functions which grow at most like $\mathrm{e}^{\varepsilon|x|}$ at infinity. Thus, the alternative choice of the *weighted* phase space Φ_ε (or the choice of weighted spaces with polynomial weights as in the first articles on this subject, see [21]) is also possible here, see also [18,75], and [227]. However, such a choice has essential drawbacks related to the addition of the above spatially unbounded solutions. Indeed, on the one hand, all the dissipative structures mentioned above are *bounded* as $|x| \to +\infty$, so that the class of bounded (uniformly local) solutions seems physically natural and sufficient. On the other hand, the analytical properties of the equations in spaces of spatially unbounded functions are essentially more complicated (in particular, even in the case that we consider, there is no differentiability with respect to the initial data in Φ_ε). Furthermore, even the uniqueness in such classes is *strongly* related to the restrictive quasimonotonicity assumption (5.9)(3) and can be violated if it is not satisfied, see [75]. Thus, the choice of the *uniformly local* phase spaces seems more general and preferable.

REMARK 5.8. We note that, in contrast to bounded domains, the space $\mathcal{C}^\infty(\Omega)$ *is not dense* in the uniformly local space $L^2_b(\Omega)$. As a consequence, even the linear equation (5.8) with $f = g = 0$ does not generate an analytic semigroup in $L^2_b(\mathbb{R}^3)^k$ and, in particular, the solution $u(t)$ is not continuous at $t = 0$ for generic u_0 (i.e., $u \notin \mathcal{C}([0,T], L^2_b(\mathbb{R}^3)^k)$). However,

$$u \in \mathcal{C}\big([0,T], L^2_\phi\big(\mathbb{R}^3\big)^k\big),$$

for every $\phi \in L^1(\mathbb{R}^3)$, see, e.g., [236]. This inconvenience can be overcome by introducing a more restrictive uniformly local space $\tilde{L}^2_b(\Omega)$ as follows:

$$\tilde{L}^2_b(\Omega) := \big[\mathcal{C}^\infty(\Omega)\big]_{L^2_b(\Omega)},$$

where $[\cdot]_V$ denotes the closure in the space V. Roughly speaking, $u \in \tilde{L}^2_b(\Omega)$ means the boundedness of the L^2_b-norm, plus some kind of "translation compactness". Indeed, as proven in [236], at least for $\Omega = \mathbb{R}^n$, the space $\tilde{L}^2_b(\Omega)$ coincides with the space of the so-called translation compact functions introduced by Chepyzhov and Vishik for the theory of nonautonomous attractors, see [49]. We recall that the function $u \in L^2_b(\mathbb{R}^n)$ is translation compact if its hull,

$$\mathcal{H}(u) := \big[T_s u, s \in \mathbb{R}^n\big]_{L^2_{\mathrm{loc}}}, \quad T_s u(x) := u(x+s), \quad s, x \in \mathbb{R}^n,$$

is compact in the local space $L^2_{\mathrm{loc}}(\mathbb{R}^n)$. Under such a more restrictive choice of the phase space, the analytic semigroups theory works and the continuity of $u(t)$ also holds, see [9] and the references therein. We note however that, although it is crucial for the general analytic semigroups approach, the above density problem does not seem to be essential for the weighted energy methods considered here, since every $u \in L^2_b(\Omega)$ can obviously be approximated by smooth functions in the *local* topology of $L^2_{\mathrm{loc}}(\Omega)$, and this is enough in order to establish the existence of a solution, see, e.g., [75] and [237]. Furthermore, verifying the artificial translation compactness requirement is an extremely difficult (unsolvable?) problem for more complicated equations (such as the two-dimensional and, especially, the three-dimensional Navier–Stokes equations in cylindrical domains, see [242] and [243]). Thus, we will no longer consider the space $\tilde{L}^2_b(\Omega)$ in this survey.

5.2. *Attractors and locally compact attractors*

We now consider the theory of attractors in the uniformly local phase spaces. The first essential difference here is that, contrary to bounded domains, the embedding

$$W^{1,2}_b(\Omega) \subset L^2_b(\Omega) \tag{5.18}$$

is not compact. Thus, the usual smoothing (or asymptotically smoothing) properties are *not sufficient* to establish the existence of a compact attractor in the uniformly local phase spaces. As a consequence, the global attractor only exists in some exceptional cases (which will be considered in the next subsection) in the initial phase space and, in order to construct a general theory, the compactness assumption must be weakened. In particular, as shown in [237], already for the simple real Ginzburg–Landau equation in $\mathbb{R}$,

$$\partial_t u = \partial_x^2 u + u - u^3,$$

the associated set of equilibria is not compact in $L^2_b(\mathbb{R})$. Thus, this equation cannot have a compact global attractor in the phase space $L^2_b(\mathbb{R})$.

This difficulty is overcome by a systematic use of the local topology of $L^2_{\mathrm{loc}}(\Omega)$ and the related locally compact global attractors. To be more precise, the set $\mathcal{A}$ is the locally compact global attractor for the semigroup $S(t)$ acting on the uniformly local phase space $\Phi_b := W^{l,p}_b(\Omega)$ if:

(i) it is bounded in Φ_b and compact in the local topology of $\Phi_{\mathrm{loc}} := W^{l,p}_{\mathrm{loc}}(\bar{\Omega})$;
(ii) it is invariant, $S(t)\mathcal{A} = \mathcal{A}$, $\forall t \geqslant 0$;
(iii) it attracts the bounded subsets of the phase space Φ_b in the local topology of Φ_{loc}.

This means that, for every bounded subset B of Φ_b and every bounded subdomain Ω_1 of Ω,

$$\lim_{t\to+\infty} \mathrm{dist}_{W^{l,p}(\Omega_1)}\big(S(t)B|_{\Omega_1}, \mathcal{A}|_{\Omega_1}\big) = 0,$$

where $u|_{\Omega_1}$ denotes the restriction of the function u (defined in Ω) to the subdomain Ω_1.

REMARK 5.9. It is not difficult to see that the attractor defined above is a $(\Phi_b, \Phi_{\mathrm{loc}})$-attractor in the terminology of Babin and Vishik, and, consequently, its existence can be verified, e.g., by using the general attractor's existence Theorem 2.20. However, in contrast to the case of usual global attractors, the compactness assumption on the absorbing/attracting sets should now be verified in the *local* topology of Φ_{loc} only. Since the embedding

$$W^{l+\alpha,p}_b(\Omega) \subset W^{l,p}_{\mathrm{loc}}(\bar{\Omega})$$

is compact for $\alpha > 0$, verifying such a compactness assumption can be reduced (exactly as in the case of bounded domains) to the derivation of an appropriate smoothing property for the equations under study.

For the reader's convenience, we illustrate the above theory on the reaction–diffusion system (5.8) (see [68,75,76,161], and [165] for more general classes of reaction–diffusion equations, [52,87], and [235] for damped wave equations, and [242] and [243] for the Navier–Stokes equations in unbounded domains).

THEOREM 5.10. *Let the assumptions of Theorem 5.5 hold. Then the semigroup $S(t)$ possesses the locally compact global attractor $\mathcal{A}$ in the uniformly local phase space $\Phi_b = L^2_b(\mathbb{R}^3)^k$.*

SKETCH OF THE PROOF. According to the abstract attractor's existence theorem mentioned above, we need to verify that the semigroup $S(t)$ is continuous in the local topology of $\Phi_{\mathrm{loc}} := L^2_{\mathrm{loc}}(\mathbb{R}^3)^k$ on every bounded subset of Φ_b and that there exists a compact (in the topology of Φ_{loc}) absorbing set (for bounded subsets of Φ_b).

The continuity follows in a standard way from estimate (5.17). Thus, we only need to construct a compact absorbing set.

As usual, the basic dissipative estimate (5.10) guarantees that the ball of radius R, $B_R := \{u,\ \|u\|_{L^2_b} \leqslant R\}$, in the phase space Φ_b is an absorbing set if R is large enough. However, this ball is obviously not compact in Φ_{loc}. For this reason, we construct a new absorbing set in the following standard way:

$$\mathcal{B} := \big[S(1)B_R\big]_{\Phi_{\mathrm{loc}}}.$$

Since the embedding $W^{1,2}_b(\mathbb{R}^3)^k \subset \Phi_{\mathrm{loc}}$ is compact, it is sufficient, in order to prove that the set $\mathcal{B}$ is compact in Φ_{loc} (and, thus, finish the proof of the theorem), to prove a *smoothing* property on the solutions of problem (5.8) of the following form:

$$\big\|u(1)\big\|^2_{W^{1,2}_b} \leqslant C\big(1 + \big\|u(0)\big\|^2_{L^2_b} + \|g\|^2_{L^2_b}\big), \tag{5.19}$$

where the constant C is independent of u.

In order to prove (5.19), we multiply Eq. (5.8) by the following expression:

$$t\sum_{i=1}^{3} \partial_{x_i}\big(\phi^2 \partial_{x_i} u(t)\big) =: t.T_\phi u(t), \tag{5.20}$$

where $\phi(x) = \phi_{\varepsilon,x_0}(x) := \mathrm{e}^{-\varepsilon|x-x_0|}$ and $\varepsilon > 0$ is small enough. Then, integrating with respect to x, we have

$$\begin{aligned}&\frac{1}{2}\partial_t\big(t\big\|\nabla_x u(t)\big\|^2_{L^2_\phi}\big) + \lambda t\big\|\nabla_x u(t)\big\|^2_{L^2_\phi} + t\big(a\Delta_x u(t), T_\phi u(t)\big)\\ &\quad = \big\|\nabla_x u(t)\big\|^2_{L^2_\phi} - t\big(\phi^2 f'\big(u(t)\big)\nabla_x u(t), \nabla_x u(t)\big) + t\big(g, T_\phi u(t)\big).\end{aligned} \tag{5.21}$$

Using now the positivity of a and estimate (5.13), we note that

$$\begin{aligned}(a\Delta_x u, T_\phi u) &\geqslant \big(a\Delta_x u, \phi^2\Delta_x u\big) - C\varepsilon\big(\phi^2|\Delta_x u|, |\nabla_x u|\big)\\ &\geqslant \alpha\|\Delta_x u\|^2_{L^2_\phi} - C\varepsilon^2\|\nabla_x u\|^2_{L^2_\phi},\end{aligned} \tag{5.22}$$

for some positive constant α. Using this estimate, together with the quasimonotonicity assumption $f'(u) \geqslant -K$, we deduce from (5.21) that

$$\begin{aligned}&\partial_t\big(t\big\|\nabla_x u(t)\big\|^2_{L^2_\phi}\big) + \lambda t\big\|\nabla_x u(t)\big\|^2_{L^2_\phi} + t\big\|\Delta_x u(t)\big\|^2_{L^2_\phi}\\ &\quad \leqslant C(t+1)\big(\|g\|^2_{L^2_\phi} + \big\|\nabla_x u(t)\big\|^2_{L^2_\phi}\big).\end{aligned}$$

Integrating this estimate with respect to $t \in [0,1]$ and using (5.15) to estimate the time integral of $\nabla_x u$, we find

$$\big\|u(1)\big\|^2_{W^{1,2}_{\phi_{\varepsilon,x_0}}} \leqslant C\big(1 + \|g\|^2_{L^2_{\phi_{\varepsilon,x_0}}} + \big\|u(0)\big\|^2_{L^2_{\phi_{\varepsilon,x_0}}}\big), \tag{5.23}$$

where the constant C is independent of x_0. Taking the supremum over $x_0 \in \mathbb{R}^3$ in both sides of this inequality and using (5.6), we obtain the required smoothing property (5.19), which finishes the proof of Theorem 5.10. $\square$

REMARK 5.11. The trick consisting in multiplying equation (5.8) by the expression $T_\phi u$ (suggested in [237]) allows to estimate the nonlinear term $f(u)$ in an optimal way by only using the quasimonotonicity assumption (exactly as in bounded domains). In contrast to this, the straightforward multiplication of the equation by $\phi^2 \Delta_x u$ (as performed in [18] and [21]) gives, when integrating by parts in the nonlinear term, the additional "bad" term $(\nabla_x \phi^2 f(u)^T, \nabla_x u)$ and the extremely restrictive growth assumption $p \leqslant 1 + \min\{4/n, 2/(n-2)\}$ in (5.9)(4) is necessary in order to handle it. Thus, in three space dimensions, this yields that $p < 7/3$, and even cubic nonlinearities cannot be treated. The above mentioned simple trick allows to avoid to impose a growth restriction to prove the existence of attractors.

5.3. *The finite dimensional case*

Before discussing the general infinite dimensional case in the next sections, we consider some rather exceptional cases in which the global attractor remains finite dimensional. As we will see below, in such cases, in spite of the fact that the underlying domain is unbounded, the attractor is localized (up to exponentially decaying terms) in some *bounded* domain (due to some special structural assumptions on the nonlinearity and the external forces). Thus, the corresponding theory is very similar to that in bounded domains and seems to be well-understood now (see [1,18,21,68,71,75,90,160,235], and the references therein).

As above, we consider, for simplicity, the reaction–diffusion system (5.8), although the approach described below has a general nature, see, e.g., [235] for nonlinear damped wave equations, [88] for degenerate parabolic equations, and [10] for the Navier–Stokes equations.

The most commonly used structural assumption on the nonlinearity f (suggested in [21]) is the following one:

$$f(u) \cdot u \geqslant 0, \quad \forall u \in \mathbb{R}^k \tag{5.24}$$

(compare with (5.9)(2)). In addition, some *decay* assumptions on the external forces g as $|x| \to +\infty$ are necessary. In order to formulate them, we need to introduce some more specific classes of uniformly local spaces.

DEFINITION 5.12. Let Ω be a sufficiently smooth unbounded domain. The space $\dot{W}_b^{l,p}(\Omega)$ consists of all functions $u \in W_b^{l,p}(\Omega)$ which satisfy

$$\lim_{|x_0| \to +\infty} \|u\|_{W^{l,p}(\Omega \cap B^1_{x_0})} = 0. \tag{5.25}$$

Roughly speaking, the space $\dot{W}_b^{l,p}(\Omega)$ consists of all functions $u \in W_b^{l,p}(\Omega)$ which decay as $|x| \to +\infty$. In particular, obviously, $W^{l,p}(\Omega) \subset \dot{W}_b^{l,p}(\Omega)$.

Finally, following [68], we impose a decay assumption on the external forces g of the form

$$g \in \dot{L}_b^2\big(\mathbb{R}^3\big)^k. \tag{5.26}$$

The following simple lemma (see [71]) is a key technical tool in the theory.

LEMMA 5.13. *Let $g \in \dot{L}_b^2(\Omega)^k$ and set*

$$R_g(x_0) := \|g\|^2_{L^2_{\phi_{\varepsilon,x_0}}},$$

for some positive ε. Then

$$\lim_{|x_0|\to+\infty} R_g(x_0) = 0. \tag{5.27}$$

Returning to the reaction–diffusion system (5.8) and to the weighted dissipative estimate (5.15), we see that, owing to the structural assumption (5.24) (instead of $f(u)\cdot u \geqslant -C$), the constant 1 disappears in the right-hand side of (5.15) and we have an homogeneous estimate,

$$\|u(t)\|^2_{L^2(B^1_{x_0})^k} \leqslant C_1\|u(t)\|^2_{L^2_{\phi_{\varepsilon,x_0}}} \leqslant C_2 e^{-\alpha t}\|u(0)\|^2_{L^2_{\phi_{\varepsilon,x_0}}} + C_2\|g\|^2_{L^2_{\phi_{\varepsilon,x_0}}}, \tag{5.28}$$

where the positive constants C_2 and α are independent of x_0 and u. In particular, the first term in the right-hand side vanishes on the attractor and we have

$$\|u\|^2_{L^2(B^1_{x_0})^k} \leqslant C_2 R_g(x_0), \quad \forall u \in \mathcal{A}. \tag{5.29}$$

Thus, owing to Lemma 5.13, $\mathcal{A} \subset \dot{L}_b^2(\mathbb{R}^3)^k$ and, besides, (5.29) gives a uniform "tail estimate" as $|x| \to +\infty$ with respect to all functions on the attractor. This tail estimate, together with the embedding $\mathcal{A} \subset W_b^{1,2}(\mathbb{R}^3)^k$ which follows from the smoothing property (5.19), guarantee the *compactness* of the (locally compact) attractor $\mathcal{A}$ on the initial topology of the phase space as well. Finally, a slightly more accurate analysis of estimate (5.28) allows to check the asymptotic compactness of the associated semigroup $S(t)$ in Φ_b. Thus, we have the following result (see [68] and [71] for a detailed proof).

THEOREM 5.14. *Let the assumptions of Theorem 5.5 hold and let, in addition, the structural assumptions (5.24) and (5.26) be satisfied. Then the semigroup $S(t)$ associated with the reaction–diffusion system (5.8) possesses the compact global attractor $\mathcal{A}$ on the initial uniformly local phase space Φ_b (exactly as in bounded domains).*

Furthermore, exactly as in bounded domains, we have the finite dimensionality of the above global attractor in the phase space.

THEOREM 5.15. *Let the assumptions of the previous theorem hold. Then the global attractor $\mathcal{A}$ has finite fractal dimension. Furthermore, the associated semigroup possesses a finite dimensional exponential attractor $\mathcal{M}$ in the phase space Φ_b.*

The proof of this theorem is also based on the uniform tail estimate (5.29) and can be found in [68] and [71].

REMARK 5.16. In the original article [21], the uniform tail estimate on the global attractor was proven in an alternative and more complicated way. To be more precise, the equations were considered in the phase space $\Phi_\phi := L^2_\phi(\Omega)^k$, with *growing* weight functions of the form $\phi(x) := (1+|x|^2)^N$, $N > 0$ (thus, Φ_ϕ consists of functions which *decay* sufficiently fast at infinity). Then the compactness of the global attractor in Φ_ϕ was deduced by proving the embedding

$$\mathcal{A} \subset L^2_{\phi^\alpha}(\Omega)^k \cap W^{1,2}_\phi(\Omega)^k,$$

for some $\alpha > 1$. This however requires the artificial restriction $g \in L^2_\phi(\Omega)^k$ and some additional assumptions on f. In particular, the Hilbert case $\phi = 1$ was not covered by this approach. This drawback was overcome in [227], in which a more accurate method to estimate the tails in the Hilbert case $\Phi = L^2(\Omega)^k$ was suggested and the compactness of the attractor for $\phi = 1$ was proven. An alternative very simple and effective way to handle the Hilbert case $\phi = 1$ is based on the so-called *energy* method, see [24,198], and [217]. This approach is based on the elementary fact that a weakly convergent sequence in a Hilbert (reflexive) space converges strongly if the associated sequence of norms converges to the norm of the limit function. The convergence of the norms is then verified by passing to the limit in the energy equality. Thus, the asymptotic compactness of the semigroup can be verified *without* requiring to work on weighted spaces. This approach is especially helpful for complicated equations (such as the Navier–Stokes equations) for which estimates in weighted spaces are rather difficult to obtain, see [217]. A drawback of this approach is that it does not give any qualitative nor quantitative information on the spatial structure of the global attractor, which are available when using weighted spaces, and only works in the Hilbert case. However, it is worth noting that, as usual, the global attractor (if it exists) is independent of the choice of the admissible phase space, see [75], so that all cases mentioned above are actually contained in the general Theorems 5.14 and 5.15.

REMARK 5.17. We finally mention that the constant λ in (5.8) can be replaced by x-dependent functions $\lambda(x)$ which are not necessarily positive everywhere, see [9] and [160]; actually, it is sufficient to require that

$$(a\nabla_x v, \nabla_x v) + (\lambda v, v) \geqslant \lambda_0 \|v\|^2_{W^{1,2}}, \quad \forall v \in W^{1,2}\big(\mathbb{R}^3\big)^k,\ \lambda_0 > 0.$$

Indeed, all the estimates given above can be obtained by repeating word by word the corresponding proofs. Another slight generalization consists in considering functions f which depend on x and requiring that, instead of (5.24),

$$f(x,u)\cdot u \geqslant -C(x), \quad x \in \mathbb{R}^3,\ u \in \mathbb{R}^k,$$

where C belongs to $\dot{L}^1_b(\mathbb{R}^3)$.

We now formulate, following essentially [235] (see also [71]), some natural generalizations of the structural assumption (5.24) and discuss the spatial asymptotics of the global attractor.

ASSUMPTION A. Let the nonlinearity f and the external forces $g \in L_b^2(\mathbb{R}^3)^k$ be such that there exists a solution $Z_0(x)$ of the associated elliptic equilibrium problem

$$a\Delta_x Z_0 - \lambda Z_0 - f(Z_0) + g = 0, \quad x \in \mathbb{R}^3 \backslash B_0^R,\ Z_0 \in W_b^{2,2}\big(\mathbb{R}^3 \backslash B_0^R\big)^k, \tag{5.30}$$

outside a large ball B_0^R of $\mathbb{R}^3$ which satisfies the following property:

$$\big[f\big(v + Z_0(x)\big) - f\big(Z_0(x)\big)\big] \cdot v \geqslant 0, \quad v \in \mathbb{R}^k,\ x \in \mathbb{R}^3 \backslash B_0^R. \tag{5.31}$$

The following generalization of Theorem 5.14 gives the spatial asymptotics of the global attractor up to exponentially small terms.

THEOREM 5.18. *Let the assumptions of Theorem 5.5 hold and let Assumption A be satisfied. Then the associated semigroup $S(t)$ possesses the global attractor $\mathcal{A}$ in the phase space Φ_b which satisfies the following estimate*:

$$\|u_0 - Z_0\|_{L^2(B_{x_0}^1)^k} \leqslant C\mathrm{e}^{-\alpha|x_0|}, \quad |x_0| > R + 1,\ u_0 \in \mathcal{A}, \tag{5.32}$$

where the positive constants C and α are independent of $u_0 \in \mathcal{A}$ and x_0.

SKETCH OF THE PROOF. Let $\tilde{Z}_0(x)$, $\tilde{Z}_0 \in W_b^{2,2}(\mathbb{R}^3)^k$, be some extension of $Z_0(x)$ inside the ball B_0^R. Then this function satisfies

$$a\Delta_x \tilde{Z}_0 - \lambda \tilde{Z}_0 - f(\tilde{Z}_0) + g = \tilde{g}, \tag{5.33}$$

where $\tilde{g} \in L_b^2(\mathbb{R}^3)^k$ and $\operatorname{supp} \tilde{g} \subset B_0^R$.

We now set $v(t) := u(t) - \tilde{Z}_0$. Then this function solves

$$\partial_t v = a\Delta_x v - \lambda v - \big[f(v + \tilde{Z}_0) - f(\tilde{Z}_0)\big] + \tilde{g}. \tag{5.34}$$

We recall that, owing to Assumption A, $[f(v + \tilde{Z}_0) - f(\tilde{Z}_0)] \cdot v \geqslant 0$, for $x \notin B_0^R$. Using the quasimonotonicity assumption $f'(v) \geqslant -K$ to estimate this term inside the ball B_0^R, we infer

$$\begin{aligned} &\big[f\big(v(x) + \tilde{Z}_0(x)\big) - f\big(Z_0(x)\big)\big] \cdot v(x) \\ &\quad \geqslant -K\big|v(x)\big|^2 \chi_R(x) \\ &\quad \geqslant -K\big(\big|u(x)\big|^2 + \big|\tilde{Z}_0(x)\big|^2\big)\chi_R(x), \quad x \in \mathbb{R}^3, \end{aligned} \tag{5.35}$$

where $\chi_R(x)$ is the characteristic function of the ball B_0^R.

Multiplying now equation (5.34) by $\phi_{\varepsilon,x_0}^2 v(t)$ and arguing exactly as in the derivation of (5.15) and (5.28), we conclude that

$$\|v\|_{L^2_{\phi_{\varepsilon,x_0}}}^2 \leqslant C\big(\|u\chi_R\|_{L^2_{\phi_{\varepsilon,x_0}}}^2 + \|\tilde{Z}_0\chi_R\|_{L^2_{\phi_{\varepsilon,x_0}}}^2 + \|\tilde{g}\chi_R\|_{L^2_{\phi_{\varepsilon,x_0}}}^2\big), \tag{5.36}$$

where the constant C is independent of x_0 and $v \in \mathcal{A} - \tilde{Z}_0$. Multiplying this inequality by the weight function

$$\phi(x_0) := \inf_{z \in B_0^R} \mathrm{e}^{\alpha|x_0 - z|},$$

with $\alpha < \varepsilon$ (which, obviously, is a weight function with an exponential growth α), taking the supremum over $x_0 \in \mathbb{R}^3$, and using the second equivalence in (5.6), we finally find

$$\|v\|^2_{L^2_{b,\phi}} \leqslant C\big(\|u\chi_R\|^2_{L^2_b} + \|\tilde{Z}_0\chi_R\|^2_{L^2_b} + \|\tilde{g}\|^2_{L^2_b}\big) \leqslant C_1, \tag{5.37}$$

where we have implicitly used the fact that the L^2_b-norm of the attractor is bounded. There remains to note that (5.37) is equivalent to (5.32) to finish the proof of Theorem 5.18. □

REMARK 5.19. If, in addition, the attractor $\mathcal{A}$ is bounded in $\mathcal{C}_b(\mathbb{R}^3)^k$ by some constant L, it is, obviously, sufficient to verify estimate (5.31) from Assumption A only for $|v| \leqslant 2L$. We also note that Theorem 5.18 shows, in particular, that the spatial asymptotics (5.32) holds with Z_0 replaced by any true equilibrium of the problem.

We conclude the section by giving two examples to illustrate the above theorem.

EXAMPLE 5.20. Let the assumptions of Theorem 5.14 hold. We claim that Assumption A is automatically satisfied here and, therefore, the global attractor $\mathcal{A}$ possesses the spatial asymptotics (5.32). Indeed, as proven in [237], $\mathcal{A}$ is globally bounded in $W^{2,2}_b(\mathbb{R}^3)^k$. This fact, together with a proper interpolation inequality and the tail estimate (5.29), yield

$$\|u\|_{\mathcal{C}(B^1_{x_0})^k} \leqslant C\big[R_g(x_0)\big]^{1/4}. \tag{5.38}$$

Therefore, the global attractor also belongs to $\dot{\mathcal{C}}_b(\mathbb{R}^3)^k$ and is bounded in this space. In particular, any equilibrium $z_0(x)$ of this problem satisfies $\lim_{|x|\to+\infty} z_0(x) = 0$. Thus, in order to verify Assumption A, with $Z_0 = z_0$, it is sufficient to check that there exists $\varepsilon > 0$ such that

$$\big[f(v+z) - f(z)\big]\cdot v \geqslant -\lambda/2|v|^2, \tag{5.39}$$

for every $v,\ z \in \mathbb{R}^k$, $|v| \leqslant 2L$ (L is the $\mathcal{C}$-diameter of the attractor) and $|z| \leqslant \varepsilon$. Indeed, Assumption A then holds with f replaced by $f(u) + \lambda/2u$, for a sufficiently large $R = R(\varepsilon)$. In order to verify inequality (5.39), we consider two cases, namely, $|v| \leqslant \delta$ and $|v| > \delta$, where $\delta > 0$ is a sufficiently small number to be fixed. In the first case, both v and z are small, so that inequality (5.39) follows from the continuity of f' and the fact that, owing to assumption (5.24), $f(0) = 0$ and $f'(0) \geqslant 0$. We now consider the second case ($\delta > 0$ has been fixed at this stage). It is sufficient, in view of inequality (5.24) and the assumption $|v| > \delta$, to find $\varepsilon > 0$ such that

$$f(v+z)\cdot z + f(z)\cdot v \leqslant \lambda\delta/2,$$

for every $|z| \leqslant \varepsilon$ and $|v| \leqslant 2L$. Since $f(0) = 0$, the existence of such an $\varepsilon = \varepsilon(\delta, L)$ is straightforward, see [235] for more details. Thus, Assumption A is verified and Theorem 5.18 (together with the $W^{2,2}_b$-bound on the attractor) now gives

$$\big|u(x) - z_0(x)\big| \leqslant C\mathrm{e}^{-\alpha|x|}, \quad \forall u \in \mathcal{A},\ x \in \mathbb{R}^3. \tag{5.40}$$

REMARK 5.21. In particular, we see that, although the rate of convergence to zero of the external forces g determines that of any function belonging to the global attractor, the "thickness" of the attractor decays *exponentially*, no matter how slow this rate is. Thus, the attractor is, in fact, concentrated (up to exponentially small terms) in a bounded domain. This property clarifies the nature of the finite dimensionality of the attractor in that case. Furthermore, to the best of our knowledge, such an exponential localization holds for all examples for which the finite dimensionality is known.

EXAMPLE 5.22. We consider the real Ginzburg–Landau equation in $\mathbb{R}^3$,

$$\partial_t u = \Delta_x u + u - u^3 + g. \tag{5.41}$$

We claim that Assumption A is satisfied if $g \in L^2_b(\mathbb{R}^3)$ and

$$\liminf_{|x|\to+\infty} g(x) > \frac{2}{3\sqrt{3}}. \tag{5.42}$$

Indeed, an elementary analysis shows that

$$\big[f(v+z) - f(z)\big] \cdot v \geqslant 0, \quad \forall v \in \mathbb{R}, \quad f(u) := u^3 - u,$$

if and only if $|z| > \frac{2}{\sqrt{3}}$. On the other hand, assumption (5.42) guarantees that the function $W_0(x) := \frac{2}{\sqrt{3}} + \varepsilon$ is a subsolution of (5.41) if ε is small enough and $|x|$ is large enough. Therefore, by the comparison principle, there exists a solution Z_0 of the equilibrium equation (5.30) outside a large ball which satisfies $Z_0(x) > \frac{2}{\sqrt{3}} + \varepsilon$ and Assumption A is verified. Thus, we see that, under assumption (5.42), the global attractor is spatially localized (in the sense of estimate (5.40)) and, for this reason, it is compact in $L^2_b(\mathbb{R}^3)$ and finite dimensional. As already mentioned in the previous section, when $g = 0$, the associated global attractor is not compact in $L^2_b(\mathbb{R}^3)$ (and is infinite dimensional).

5.4. *The infinite dimensional case: entropy estimates*

Starting from this section, we consider the general case in which the dimension of the global attractor is infinite. Indeed, the simplest way to understand why this dimension must be infinite in general is to consider the real one-dimensional Ginzburg–Landau equation (5.41) with zero external forces; we also consider the space periodic solutions with period $2L$. Then the associated dynamical system acting on the space $L^2_{\mathrm{per}}([-L, L])$ of $2L$-periodic functions is dissipative and possesses the (finite dimensional) global attractor $\mathcal{A}_L$. Furthermore, we see that, by computing the dimension of the unstable set at $u = 0$,

$$\dim_F \mathcal{A}_L \geqslant \dim \mathcal{M}^{\mathrm{un}}(0) \geqslant \frac{2L}{\pi}.$$

On the other hand, since the phase space $L^2_{\mathrm{per}}([-L, L])$ is contained in the phase space $\Phi_b := L^2_b(\mathbb{R})$, we also have the embedding

$$\mathcal{A}_L \subset \mathcal{A},$$

where $\mathcal{A}$ is the (locally compact) global attractor of the equation in the whole space. Thus, since the dimension of $\mathcal{A}_L$ grows as $L \to +\infty$, the dimension of $\mathcal{A}$ cannot be finite.

This simple example shows that, in contrast to bounded domains, we cannot now expect any finite dimensional reduction in general and the dynamics reduced to the global attractor remains infinite dimensional. However, it is intuitively clear that the attractor $\mathcal{A}$ is essentially "thinner" than the initial phase space and, in some proper sense, the reduced dynamics can be described by less degrees of freedom here as well. Now, in order to make this observation rigorous, we need to be able to compare the "thickness" of infinite dimensional sets.

One possible approach to this problem (which is widespread in the approximation theory, see, e.g., [219]) consists in using the Kolmogorov ε-entropy, see Definition 2.28. Indeed, owing to the Hausdorff criterium, the entropy $\mathcal{H}_\varepsilon(X, M)$ is finite for every $\varepsilon > 0$ and every compact subset X of the metric space M. Then, according to formula (2.13), the set X is finite dimensional if and only if

$$\mathcal{H}_\varepsilon(X, M) \leqslant d \log_2 \frac{1}{\varepsilon} + C,$$

for some constants C and d which are independent of $\varepsilon \to 0^+$. So, under this approach, the infinite dimensionality of X just means that the quantity $\mathcal{H}_\varepsilon(X)$ has another, more complicated, asymptotics as $\varepsilon \to 0^+$, which is to be found or estimated.

To the best of our knowledge, the idea of using the Kolmogorov ε-entropy in the theory of attractors was suggested by Chepyzhov and Vishik in [221] in order to study the infinite dimensional *uniform* attractors of nonautonomous dynamical systems in *bounded* domains. However, such an approach appears as especially adapted to the study of equations in unbounded domains and, starting from [56] and [233], the ε-entropy has become one of the most powerful technical tools in view of the study of the locally compact attractors in large and unbounded domains.

We start our considerations by giving several examples of asymptotics of the ε-entropy for some typical infinite dimensional function spaces.

EXAMPLE 5.23. Let Ω be a regular bounded domain, $M := W^{l_1,p_1}(\Omega)$, and X be the unit ball of the space $W^{l_2,p_2}(\Omega)$, with

$$\frac{1}{p_1} - \frac{l_1}{n} > \frac{1}{p_2} - \frac{l_2}{n}.$$

Then it is well-known that X is (pre)compact in M, so that $\mathcal{H}_\varepsilon(X, M)$ is well-defined and satisfies

$$C_1\left(\frac{1}{\varepsilon}\right)^{n/(l_2-l_1)} \leqslant \mathcal{H}_\varepsilon(X, M) \leqslant C_2\left(\frac{1}{\varepsilon}\right)^{n/(l_2-l_1)}, \tag{5.43}$$

where the constants C_1 and C_2 are independent of ε, see, e.g., [219].

Thus, the typical asymptotics of the entropy of Sobolev spaces embeddings are *polynomial* with respect to ε^{-1}. The next example shows the typical behavior of the entropy for classes of *analytic* functions embeddings.

EXAMPLE 5.24. Let K be the set of all analytic functions f in a ball B_R of radius R in $\mathbb{C}^n$ such that $\|f\|_{\mathcal{C}(B_R)} \leqslant 1$ and let M be the space $\mathcal{C}(B^{\mathrm{Re}})$, where $B^{\mathrm{Re}} := \{z = (z_1, \dots, z_n) \in \mathbb{C}^n,\ \operatorname{Im} z_i = 0,\ i = 1, \dots, n,\ |z| \leqslant 1\}$. Thus, K consists of all functions of $\mathcal{C}(B^{\mathrm{Re}})$ which can be holomorphically extended to the ball B_R and for which the $\mathcal{C}$-norm of this extension is less than one. Then

$$C_1\left(\log_2 \frac{1}{\varepsilon}\right)^{n+1} \leqslant \mathcal{H}_\varepsilon(K, M) \leqslant C_2\left(\log_2 \frac{1}{\varepsilon}\right)^{n+1}, \tag{5.44}$$

see [132].

In particular, the above asymptotics show, in a mathematically rigorous way, that the set of real analytic functions is indeed essentially smaller than that of functions with finite smoothness $\mathcal{C}^k$.

We now recall that, here, the global attractor is not compact, but only *locally compact*, in the phase space. In order to compare such types of sets, we need to introduce, following [132], the so-called entropy per unit volume or *mean* ε-entropy.

DEFINITION 5.25. Let K be a locally compact set in some uniformly local space $\Phi_b := W_b^{l,p}(\mathbb{R}^n)$. Then, for every hypercube $[-R, R]^n$, the entropy $\mathcal{H}_\varepsilon(K|_{[-R,R]^n})$ of the restriction of K to this hypercube is well-defined. By definition, the mean ε-entropy of $\mathcal{K}$ is the following (finite or infinite) quantity:

$$\bar{\mathcal{H}}_\varepsilon(K, \Phi_b) := \limsup_{R\to+\infty} \frac{1}{(2R)^n} \mathcal{H}_\varepsilon(K|_{[-R,R]^n}). \tag{5.45}$$

As we will see below, the next example is crucial for the theory of attractors in unbounded domains.

EXAMPLE 5.26. Let $\mathbb{B}_\sigma(\mathbb{R}^n)$, $\sigma \in \mathbb{R}_+$, be the subspace of $L^\infty(\mathbb{R}^n)$ consisting of all functions u whose Fourier transform $\hat{u}$ has a compact support,

$$\operatorname{supp} \hat{u} \subset B_0^\sigma := \left\{\xi \in \mathbb{R}^n,\ \|\xi\| \leqslant \sigma\right\}.$$

It is well known that the space $\mathbb{B}_\sigma(\mathbb{R}^n)$ consists of entire functions (i.e., functions which are analytic on the whole space $\mathbb{R}^n$) with an exponential growth. Furthermore, if $\mathcal{B}(\sigma)$ is the unit ball in this space (endowed with the usual L^∞-metric), then

$$C_1 \log_2 \frac{1}{\varepsilon} \leqslant \bar{\mathcal{H}}_\varepsilon\big(\mathcal{B}(\sigma)\big) \leqslant C_2 \log_2 \frac{1}{\varepsilon}, \tag{5.46}$$

where C_1 and C_2 depend on σ, but are independent of ε, see [132]. Moreover, we have, concerning the entropy of the restrictions $\mathcal{B}(\sigma)|_{[-R,R]^n}$,

$$C_1 R^n \log_2 \frac{1}{\varepsilon} \leqslant \mathcal{H}_\varepsilon\big(\mathcal{B}(\sigma)|_{[-R,R]^n}\big) \leqslant C_2\left(R + \log_2 \frac{1}{\varepsilon}\right)^n \log_2 \frac{1}{\varepsilon}, \tag{5.47}$$

where C_1 and C_2 are independent of ε and R. We can note that these estimates are sharp for $R \gg \log_2 \frac{1}{\varepsilon}$ and for $R \sim \log_2 \frac{1}{\varepsilon}$, but, for $R \ll \log_2 \frac{1}{\varepsilon}$, the lower bound is far from being optimal and can be corrected as follows:

$$\mathcal{H}_\varepsilon\big(\mathcal{B}(\sigma)|_{[-R,R]^n}\big) \geqslant C_R\left(\frac{\log_2(1/\varepsilon)}{(\log_2\log_2 1/\varepsilon)^n}\right)^{n+1}, \tag{5.48}$$

where C_R depends on R, but is independent of ε. The proof of estimates (5.47) and (5.48) can be found in [236].

Finally, we also mention the analogue of Example 5.23 for the uniformly local case.

EXAMPLE 5.27. Let the exponents l_i and p_i, $i = 1, 2$, be the same as in Example 5.23. Let also K be the unit ball in the space $W_b^{l_2,p_2}(\mathbb{R}^n)$ and set $M := W_b^{l_1,p_1}(\mathbb{R}^n)$. Then

$$C_1\left(\frac{1}{\varepsilon}\right)^{n/(l_2-l_1)} \leqslant \bar{\mathcal{H}}_\varepsilon(K, M) \leqslant C_2\left(\frac{1}{\varepsilon}\right)^{n/(l_2-l_1)}, \tag{5.49}$$

where the constants C_1 and C_2 are independent of ε. Actually, these estimates immediately follow from (5.43).

We are now ready to formulate the *universal* entropy estimates for the *uniformly local* attractors of dissipative systems in unbounded domains which, as we will see below, are natural generalizations of the fractal dimension estimates to systems in unbounded domains. These estimates have the following form:

$$\mathcal{H}_\varepsilon\big(\mathcal{A}|_{\Omega\cap B^R_{x_0}}, \Phi_b\big(\Omega\cap B^R_{x_0}\big)\big) \leqslant C\,\mathrm{vol}\big(\Omega\cap B_{x_0}^{R+L\log_2 1/\varepsilon}\big)\log_2\frac{1}{\varepsilon}, \tag{5.50}$$

where $\mathrm{vol}(\cdot)$ denotes the usual Lebesgue measure in $\mathbb{R}^n$ and the constants C and L are independent of R, x_0, and ε. Thus, (5.50) gives upper bounds on the entropy of the restrictions of the attractor $\mathcal{A}$ to all bounded subdomains $\Omega \cap B^R_{x_0}$ which depend on the three parameters R, x_0, and ε.

The above formula has a general nature, independent of the concrete class of dissipative systems considered, and has been verified for various classes of reaction–diffusion systems (see [76,233,236], and [237]), for damped wave equations (see [235]), and even for elliptic boundary value problems in unbounded domains (see [166]). Indeed, roughly speaking, it is sufficient, in order to prove such estimates, to verify a *weighted* analogue of the "parabolic" smoothing property (2.14),

$$\big\|S(1)u_1 - S(1)u_2\big\|_{W^{1,2}_{\phi_\mu,x_0}} \leqslant L\|u_1-u_2\|_{L^2_{\phi_\mu,x_0}}, \quad u_1, u_2 \in \mathcal{A}, \tag{5.51}$$

for some fixed μ and every x_0 in Ω (or its "hyperbolic" analogues (3.2) and (3.3)), see [235] and [236]. Thus, these entropy estimates are also based on rather simple and general (weighted) energy estimates and do not use any specific property of the dissipative system under study. This somehow clarifies the nature of their universality. We also mention that the upper entropy estimates are sharp with respect to the three parameters R, x_0, and R (appropriate examples of lower bounds will be given in the next subsections).

In order to further clarify these universal entropy estimates, we conclude this subsection by considering the most interesting particular cases and by comparing them with the typical asymptotics given above.

EXAMPLE 5.28. Let Ω be a bounded domain. Then $\mathrm{vol}(\Omega \cap B^R_{x_0}) = \mathrm{vol}(\Omega)$ if R is large enough. Therefore, (5.50) gives

$$\mathcal{H}_\varepsilon(\mathcal{A}) \leqslant C\,\mathrm{vol}(\Omega)\log_2 \frac{1}{\varepsilon}.$$

Thus, in the case of bounded domains, the entropy formula allows to recover the standard result on the finite dimensionality of the global attractor and reflects in a correct way the typical dependence of the dimension on the size of the domain ($\dim_F \mathcal{A} \sim \mathrm{vol}(\Omega)$, see [22]). However, even in that case, the entropy estimate gives some additional information which may be important, especially for large bounded domains, namely, it allows to estimate the entropy of the restrictions $\mathcal{A}|_{B^1_{x_0}}$ and, thus, to study the "thickness" of the attractor with respect to the position inside the domain.

EXAMPLE 5.29. We now assume that $\Omega = \mathbb{R}^n$. Then $\mathrm{vol}(\Omega \cap B^R_{x_0}) = cR^n$ and (5.50) reads

$$\mathcal{H}_\varepsilon\big(\mathcal{A}|_{B^R_{x_0}}\big) \leqslant C\left(R + L\log_2 \frac{1}{\varepsilon}\right)^n \log_2 \frac{1}{\varepsilon}. \tag{5.52}$$

We see that this estimate coincides with the upper bound (5.47) for the space $\mathbb{B}_\sigma(\mathbb{R}^n)$ of entire functions and, in particular, dividing (5.52) by R^n and passing to the limit $R \to +\infty$, we also obtain the analogue of (5.46),

$$\bar{\mathcal{H}}_\varepsilon(\mathcal{A}) \leqslant C\log_2 \frac{1}{\varepsilon} \tag{5.53}$$

(for the one-dimensional real Ginzburg–Landau and damped wave equations, this estimate was obtained independently in [55] and [57]). Thus, we see that the "thickness" of the attractor $\mathcal{A}$ is of the order of that of the class $\mathbb{B}_\sigma(\mathbb{R}^n)$ of entire functions and is essentially less than that of the class of finite smoothness, see Examples 5.23 and 5.27 (and, in particular, it is essentially less than the thickness of any absorbing set). However, even when all the terms in the equations are entire, the attractor is usually not entire (the simplest example is the real Ginzburg–Landau equation) and only the analyticity in a strip $\mathbb{R}_\mu := \mathrm{i}[-\mu, \mu]^n \times \mathbb{R}^n$ takes place. The mean entropy for such classes of functions has an asymptotics of the form $(\log_2 \frac{1}{\varepsilon})^{1+p}$, for some $p > 0$, and is *worse* than (5.53). Therefore, even in the real analytic case, the nature of the universal entropy estimates cannot be explained by regularity arguments and reflects the *dynamical* reduction of the number of degrees of freedom by the dissipative dynamics. Furthermore, we emphasize here that the analyticity is *not necessary* for the validity of the entropy estimates. In particular, these estimates hold for the reaction–diffusion system (5.8) under the assumptions of Theorem 5.5, see [237]. In that case, the regularity of f and g only yields that $\mathcal{A} \subset W^{2,2}_b(\Omega)^k$, so that the best entropy estimates which can be extracted from this regularity is *polynomial* with respect to ε^{-1} ($\varepsilon^{-3/2}$ to be more precise).

REMARK 5.30. Estimates (5.50) can be rewritten in the more compact equivalent form

$$\mathcal{H}_\varepsilon(\mathcal{A}, \Phi_{\mathrm{e}^{-|x-x_0|}}) \leqslant C\left(\log_2 \frac{1}{\varepsilon}\right)^{n+1}, \tag{5.54}$$

i.e., the entropy of the attractor can be equivalently computed in *weighted* phase spaces with the *exponential* weight functions $e^{-|x-x_0|}$. In particular, in the spatially homogeneous case, the sole space $\Phi_{e^{-|x|}}$ with $x_0 = 0$ is sufficient. Indeed, using the simple "summation" properties of the Kolmogorov entropy, one can easily show that (5.54) implies (5.50). Actually, estimate (5.50) has first been obtained precisely in this form, see [233]. However, we prefer to use the more complicated formulation (5.50) in order to avoid artificial weight functions in the formulation and to prevent from the confusing feeling that $\log_2$ terms in the entropy estimates are related to the artificial choice of exponential weight functions and are, thus, also artificial.

5.5. *Infinite dimensional exponential attractors*

In this subsection, we discuss the theory of *exponential* attractors for systems in unbounded domains, following essentially [72]. Since even the global attractor (which is always contained in an exponential attractor) is now infinite dimensional, one cannot expect an exponential attractor to be finite dimensional. Thus, this assumption must be relaxed in Definition 3.4. On the other hand, this assumption cannot be simply omitted, since, otherwise, any compact absorbing set would be an exponential attractor, which does not make sense. In any case, one wants to make an exponential attractor as small as possible (i.e., to add a "minimal number" of new artificial points to the global attractor) and, therefore, it is natural to use the Kolmogorov entropy to control its "thickness"; in particular, it is natural to look for an exponential attractor which satisfies the *universal* entropy estimates (5.50) known for global attractors (an analogous idea was also used in [69] for infinite dimensional exponential attractors for nonautonomous problems in bounded domains).

Another difference, when compared with bounded domains, is the fact that the locally compact global attractor only attracts the bounded sets in the *local* topology (counterexamples for the attraction in the uniform topology of the initial phase space can be easily constructed, see [236]). Thus, one would expect the same type of attraction for exponential attractors as well. However, as shown in [72], this drawback of the theory of global attractors can be overcome by constructing proper exponential attractors and one can obtain the (exponential) attraction in the topology of the initial phase space.

Thus, based on the above considerations, the following modifications of the concept of an exponential attractor are natural.

DEFINITION 5.31. Let $S(t)$ be a dissipative semigroup in the uniformly local Sobolev space $\Phi_b := W_b^{l,p}(\Omega)$, for a regular unbounded domain Ω. A set $\mathcal{M}$ is an (infinite dimensional) exponential attractor for the semigroup $S(t)$ if the following conditions are satisfied:

(i) it is bounded in Φ_b and compact in Φ_{loc};
(ii) it is positively invariant, $S(t)\mathcal{M} \subset \mathcal{M}$, $t \geqslant 0$;
(iii) it attracts exponentially the bounded subsets of Φ_b in the *uniform* topology of Φ_b, i.e., there exist a monotonic function Q and a positive constant α such that, for every bounded subset $B \subset \Phi_b$, the following estimate:

$$\mathrm{dist}_{\Phi_b}\big(S(t)B, \mathcal{M}\big) \leqslant Q\big(\|B\|_{\Phi_b}\big)e^{-\alpha t} \tag{5.55}$$

holds, for every $t \geqslant 0$;

(iv) it satisfies the universal entropy estimates (5.50), for some positive constants C and L which are independent of R, x_0, and ε.

The following theorem, proven in [72], gives the existence of such an object for the reaction–diffusion system (5.8).

THEOREM 5.32. *Let the assumptions of Theorem 5.5 be satisfied. Then the associated semigroup $S(t)$ possesses an infinite dimensional exponential attractor $\mathcal{M}$ in the phase space $\Phi_b = L^2_b(\mathbb{R}^3)^k$ in the sense of the above definition.*

This result is, to the best of our knowledge, the only one on the existence of infinite dimensional exponential attractors in unbounded domains. However, its construction mainly exploits the smoothing estimate (5.51) on the difference of two solutions, but does not involve the specific properties of the reaction–diffusion system (5.8). Thus, we expect that the existence of such an exponential attractor is a general fact which can be established for all dissipative systems in unbounded domains for which the validity of the universal entropy estimates is satisfied (for the global attractor).

We conclude this subsection by considering the problem of the approximation of equations in an unbounded domain by appropriate equations in large bounded domains. It is well-known that the global attractor is not robust with respect to this singular limit and can change drastically. To illustrate this, we consider the one-dimensional real Ginzburg–Landau equation with a transport term,

$$\partial_t u = \partial_x^2 u - L\partial_x u + u - u^3, \quad L > 2, \tag{5.56}$$

and approximate it by analogous equations in the bounded domains $\Omega_R := [-R, R]$, endowed with Dirichlet boundary conditions. Then, as shown in [72], the global attractor $\mathcal{A}_R$ for the approximate problem is trivial for every (finite) R, $\mathcal{A}_R = \{0\}$. However, the limit attractor for $R = +\infty$ is completely nontrivial and has infinite dimension and infinite topological entropy. Thus, this approximation problem seems to be very difficult as far as global attractors are concerned and, probably, cannot be solved in a reasonable way.

In contrast to this, as the following theorem (proven in [72]) shows, this approximation problem has a natural and adequate solution in terms of exponential attractors.

THEOREM 5.33. *Let the reaction–diffusion system (5.8) in the unbounded domain $\Omega = \mathbb{R}^3$ satisfy the assumptions of Theorem 5.5 and let $S_\infty(t)$ be the associated dissipative semigroup acting on $\Phi_b = L^2_b(\mathbb{R}^3)^k$. We also consider the same problem in the large, but bounded, ball $\Omega_R = B^R_0$ in $\mathbb{R}^3$ with Dirichlet boundary conditions and we let $S_R(t)$ be the dissipative semigroup associated with this problem on $\Phi_b(R) := L^2_b(B^R_0)^k$. Then there exists a family of closed bounded sets $\mathcal{M}_R$, $R \in [R_0, +\infty]$, of $\Phi_b(R)$ such that, for every finite R, $\mathcal{M}_R$ is an exponential attractor for $S_R(t)$ in the usual sense and, for $R = +\infty$, the corresponding set is an infinite dimensional exponential attractor for $S_\infty(t)$. Furthermore, the following additional properties are satisfied:*

(1) *the sets $\mathcal{M}_R$ are uniformly (with respect to R) bounded in $\Phi_b(R)$;*

(2) *there exist a positive constant α and a monotonic function Q such that, for every R and every bounded subset B of $\Phi_b(R)$,*

$$\operatorname{dist}_{\Phi_b(R)}\big(S_R(t)B, \mathcal{M}_R\big) \leqslant Q\big(\|B\|_{\Phi_b(R)}\big)\mathrm{e}^{-\alpha t}$$

(*uniform exponential attraction*);

(3) *uniform entropy estimates*:

$$\mathcal{H}_\varepsilon\big(\mathcal{M}_R|_{\Omega_R\cap B^r_{x_0}}\big) \leqslant C \operatorname{vol}\big(\Omega_R \cap B^{r+L\log_2 1/\varepsilon}_{x_0}\big)\log_2\frac{1}{\varepsilon},$$

where the constants C and L are independent of R, $r \leqslant R$, x_0, and ε;

(4) *the attractors $\mathcal{M}_R$ tend to $\mathcal{M}_\infty$ in the following sense*:

$$\operatorname{dist}_{\mathrm{sym},\Phi_b(r)}(\mathcal{M}_R|_{\Omega_r}, \mathcal{M}_\infty|_{\Omega_r}) \leqslant C\mathrm{e}^{-\gamma(R-r)}, \tag{5.57}$$

where the positive constants C and γ are independent of R and $r \leqslant R$.

In particular, estimate (5.57) shows that, if we want to approximate the attractor $\mathcal{M}_\infty$ with an accuracy ε inside the ball Ω_r, it is sufficient to construct the usual finite dimensional exponential attractor $\mathcal{M}_{R(\varepsilon)}$ for the reaction–diffusion problem in a ball of radius $R(\varepsilon) = r + L\log_2\frac{1}{\varepsilon}$. We also note that one cannot expect that $\mathcal{M}_R$ approximates $\mathcal{M}_\infty$ in the whole ball Ω_R, since the additional boundary conditions on $\partial\Omega_R$ for the approximate problems should be satisfied. Nevertheless, estimate (5.57) also shows that the influence of the boundary and the boundary conditions decays exponentially with respect to the distance to the boundary (in agreement with our physical intuition).

5.6. *Complexity of space–time dynamics: entropy theory*

In the previous subsections, we gave sharp upper bounds on the Kolmogorov ε-entropy which characterize the "size" or "thickness" of the attractors. Starting from this subsection, we describe some general *dynamical* properties of a dissipative system in a large or an unbounded domain, restricted to its global attractor.

As already mentioned, contrary to bounded domains, the reduced dynamics now remains infinite dimensional and dynamical effects of essentially new higher levels of complexity (which are not observable in the classical finite dimensional theory of dynamical systems) may appear. In particular, the Lyapunov and topological entropy dimensions for such dynamics are usually infinite, see [239]. For this reason, most ideas and methods from the classical theory fail (at least in a straightforward way) to describe these new types of dynamics. Thus, a new theory, which is only developing now, is required.

Another essential difference from the classical theory is the fact that, in addition to complicated temporal dynamics, the solutions may have very irregular (chaotic) spatial structures, i.e., the so-called *spatial* chaos may appear. Furthermore, as a result of the chaotic temporal evolution of spatially chaotic structures, the so-called *space–time* chaos may appear.

The most studied case is that of spatial chaos which is already observable on the set of temporal equilibria of the dynamical system. Indeed, the equilibria satisfy some *elliptic* equation of the form

$$a\Delta_x u - f(u) + g = 0, \tag{5.58}$$

so that the number of independent variables is reduced by one, which is an essential simplification. So, in the particular case of one space variable, (5.58) becomes an ODE and the (spatially) chaotic behaviors of its solutions can be successfully studied by classical theories (homoclinic bifurcation analysis, variational methods for constructing complex solutions, ..., see [4,127,187], and the references therein). Furthermore, many interesting multi-dimensional problems in *cylindrical* domains can be reduced to this one-dimensional one by using the so-called spatial center manifold reduction, see [3,30,128,163,164], and the references therein. Also, direct generalizations of the techniques from ODEs to multi-dimensional elliptic PDEs of the form (5.58) (e.g., the shadowing lemma, variational methods, ...) are available, see, e.g., [8] and [194]. We finally mention a rather simple and very effective method to construct spatially chaotic patterns which are, in addition, stable with respect to the time developed in [2,13,14], and [16]. This method is based on the study of homotopy properties of the level sets of the nonlinear term f and related energy functionals and is somehow close to the variational methods, see the recent survey [15] for more details.

We however note that all the above mentioned methods give examples of spatial chaotic behaviors with *finite* topological entropy (usually related to the Bernoulli scheme $\mathbb{M} := \{0, 1\}^{\mathbb{Z}}$ or $\mathbb{M}_n := \{0, 1\}^{\mathbb{Z}^n}$ in the multi-dimensional case), which is typical of ODEs, but *does not* capture the "whole" complexity of the spatial dynamics, since its topological entropy is usually infinite, see [55,166,237], and [239]. In order to overcome this problem, an alternative method, related to the so-called infinite dimensional essentially unstable manifolds and the Kotelnikov formula, which gives a description of the spatial chaos in terms of the Bernoulli scheme $\mathbb{M}_n := [0, 1]^{\mathbb{Z}^n}$ with a continuous number of symbols and an infinite topological entropy, has been suggested in [237]. This method will be discussed in more details in the next subsection.

Now, the case of full space–time dynamics is essentially less understood. However, even here, some reasonable progress related to the so-called Sinai–Bunimovich space–time for continuous media has recently been obtained. This topic will be discussed in a subsequent subsection.

In the remaining of this subsection, we discuss (following essentially [239] and [241]) topological and smooth invariants for the space–time dynamics which are strongly based on the universal entropy estimates on the global attractor and give useful "upper bounds" on the possible complexity of the dynamics. For simplicity, we restrict ourselves to $\Omega = \mathbb{R}^n$ and to spatially homogeneous dissipative systems (i.e., the coefficients and external forces do not depend explicitly on x; this constitutes a natural analogue of "autonomous" systems for space–time dynamics). In that case, the attractor $\mathcal{A}$ possesses a very important additional structure, namely, the group $\{T_h,\ h \in \mathbb{R}^n\}$ of spatial shifts acts on it,

$$T_h\mathcal{A} = \mathcal{A}, \quad h \in \mathbb{R}^n, \quad T_h u(x) := u(x+h), \quad h, x \in \mathbb{R}^n. \tag{5.59}$$

Thus, in addition to the temporal evolution semigroup $S(t)$, we also have the action of the spatial shifts group T_h on the attractor which, obviously, commutes with $S(t)$. As a result, the extended $(n+1)$-parametric spatio-temporal semigroup $\mathbb{S}(t,h)$,

$$\mathbb{S}(t,h)\mathcal{A}=\mathcal{A},\quad \mathbb{S}(t,h):=S(t)\circ T_h,\quad t\geqslant 0,\ h\in\mathbb{R}^n, \tag{5.60}$$

acts on the attractor.

Following [237] and [239], we will treat this multi-parametric semigroup as a dynamical system with multi-dimensional "time" $(t,h)\in\mathbb{R}_+\times\mathbb{R}^n$, which describes the space–time behavior of the dissipative system under study, and we will describe the space–time chaos by finding appropriate dynamical invariants of this action. In particular, under this approach, the spatial x and temporal t directions are treated in a unified way. Some justifications for such a unification will be given at the end of the next subsection when giving examples for which these directions are indeed equivalent (in spite of the fact that they seem essentially different from an intuitive point of view).

In order to introduce these invariants, we need to make some reasonable assumptions on the attractor $\mathcal{A}$, namely,

(i) it is locally compact on some uniformly local Sobolev phase space $\Phi_b=\Phi_b(\mathbb{R}^n)$ which is embedded into $L^\infty(\mathbb{R}^n)$;

(ii) the dissipative system is spatially homogeneous, i.e., the extended semigroup (5.60) acts on the attractor;

(iii) the universal entropy estimates (5.52) hold;

(iv) the evolution semigroup $S(t)$ is Lipschitz continuous in a weighted space $\Phi_{e^{-\varepsilon|x|}}(\mathbb{R}^n)$ on the attractor,

$$\begin{aligned}&\bigl\|S(t)u_0-S(t)u_1\bigr\|_{\Phi_{e^{-\varepsilon|x|}}}\leqslant C e^{kt}\|u_0-u_1\|_{\Phi_{e^{-\varepsilon|x|}}},\\ &u_0,u_1\in\mathcal{A},\ t\geqslant 0,\end{aligned} \tag{5.61}$$

for some fixed $\varepsilon>0$ and positive constants C and L which are independent of t, u_0, and u_1.

We note that the assumption $\Phi_b\subset L^\infty(\mathbb{R}^n)$ is not essential and was introduced in [239] just to avoid additional technicalities.

The first, and most natural, dynamical invariant of the action of (5.60) is its topological entropy, see [127] for details.

DEFINITION 5.34. We endow the attractor $\mathcal{A}$ with the topology of $L^\infty_{e^{-|x|}}(\mathbb{R}^n)$ and define, for every $R\in\mathbb{R}_+$, an equivalent metric d_R on $\mathcal{A}$ by

$$\begin{aligned}&d_R(u_0,u_1):=\sup_{(t,h)\in R\cdot[0,1]^{n+1}}\bigl\|\mathbb{S}(t,h)u_0-\mathbb{S}(t,h)u_1\bigr\|_{L^\infty_{e^{-|x|}}},\\ &\quad u_0,u_1\in L^\infty_{e^{-|x|}}\bigl(\mathbb{R}^n\bigr).\end{aligned} \tag{5.62}$$

Since $\mathcal{A}$ is bounded in Φ_b and compact in Φ_{loc}, it is compact in $L^\infty_{e^{-|x|}}(\mathbb{R}^n)$ (thanks to the embedding $\Phi_b\subset L^\infty(\mathbb{R}^n)$) and, therefore, it is also compact in the metric of d_R and the Kolmogorov ε-entropy $\mathcal{H}_\varepsilon(\mathcal{A},d_R)$ is well-defined. Then the topological entropy of the action of $\mathbb{S}(t,h)$ on $\mathcal{A}$ is the following quantity:

$$h_{\mathrm{top}}\bigl(\mathbb{S}(t,h),\mathcal{A}\bigr):=\lim_{\varepsilon\to 0^+}\limsup_{R\to+\infty}\frac{1}{R^{n+1}}\mathcal{H}_\varepsilon(\mathcal{A},d_R). \tag{5.63}$$

REMARK 5.35. Although this definition depends on the specific metric, it is well known (see, e.g., [127]) that the topological entropy only depends on the *topology* and is independent of the choice of the equivalent metric on $\mathcal{A}$. Furthermore, it is also not difficult to show that the space $L^\infty(\mathbb{R}^n)$ in the definition of d_R can be replaced by $\Phi_{e^{-|x|}}(\mathbb{R}^n)$, see [239]. However, we define d_R by special exponentially weighted metrics keeping in mind other invariants which will depend on this choice.

We now recall that the topological entropy for one-parametric evolution semigroups $S(t)$ is usually finite in the classical theory of dynamical systems. The following theorem, proven in [239], can be considered as a generalization of this principle to spatially extended systems.

THEOREM 5.36. *Let the attractor $\mathcal{A}$ satisfy the above conditions. Then the topological entropy of the action of the extended space–time semigroup $\mathbb{S}(t,h)$ is finite*,

$$h_{\mathrm{top}}\big(\mathbb{S}(t,h),\mathcal{A}\big) < +\infty.$$

Furthermore, it coincides with the so-called topological entropy per unit volume (introduced by Collet and Eckmann, see [55] and [57]) and can be computed by the following simplified formula:

$$\begin{aligned} h_{\mathrm{top}}&(\mathbb{S}(t,h),\mathcal{A})\\ &= \lim_{\varepsilon\to 0^+}\bar{\mathcal{H}}_\varepsilon\big(\mathcal{K},L^\infty\big(\mathbb{R}^{n+1}\big)\big)\\ &= \lim_{\varepsilon\to 0^+}\lim_{R\to+\infty}\frac{1}{R^{n+1}}\mathcal{H}_\varepsilon\big(\mathcal{K}|_{R\cdot[0,1]^{n+1}},L^\infty\big(R\cdot[0,1]^{n+1}\big)\big), \end{aligned} \tag{5.64}$$

where $\mathcal{K}\subset L^\infty(\mathbb{R},\Phi_b)\subset L^\infty(\mathbb{R}^{n+1})$ is the set of all bounded trajectories of the dissipative system (the so-called kernel in the terminology of Chepyzhov and Vishik, see [49]) and $\bar{\mathcal{H}}_\varepsilon(\mathcal{K})$ denotes its mean ε-entropy, see Definition 5.25.

REMARK 5.37. It can also be shown that any sufficiently regular bounded subdomain $V\subset\mathbb{R}^n$ can be chosen as a "window" instead of $[0,1]^{n+1}$ in (5.64), namely,

$$\bar{\mathcal{H}}_\varepsilon(\mathcal{K},L^\infty\big(\mathbb{R}^{n+1}\big) = \lim_{R\to+\infty}\frac{1}{\mathrm{vol}(R\cdot V)}\mathcal{H}_\varepsilon\big(\mathcal{K}|_{R\cdot V},L^\infty(R\cdot V)\big).$$

We now note that the complexity of the dynamical behaviors of the extended system (5.60) may essentially differ in different directions. In particular, for the so-called extended gradient systems, see [97,213,238], and [239], the space–time topological entropy $h_{\mathrm{top}}(\mathbb{S}(t,h),\mathcal{A})$ vanishes, due to the simpler *temporal* dynamics induced by the gradient structure, which however does not reduce the complexity of the *spatial* dynamics. In order to capture these directional dynamical effects, it seems natural to consider the k-parametric subsemigroups $\mathbb{S}^{V_k}(t,h)$ of the extended space–time dynamical system $\mathbb{S}(t,h)$ generated by the restrictions of the argument (t,h) to k-dimensional linear subspaces of the space–time $\mathbb{R}^{n+1}$,

$$\mathbb{S}^{V_k}(t,h) := \big\{\mathbb{S}(t,h),\ (t,h)\in V_k,\ t\geqslant 0,\ h\in\mathbb{R}^n\big\}, \tag{5.65}$$

and to study their invariants with respect to the linear space V_k and its dimension k. For instance, the choice $V_1 = \mathbb{R}_t$ gives the purely temporal dynamics, $\mathbb{S}^{V_1}(t,h) = S(t)$, the choice $V_n = \mathbb{R}^n_x$ gives the spatial dynamics and spatial chaos, $\mathbb{S}^{V_n}(t,h) = T_h$, and the intermediate choices of planes V_k describe the interactions between the temporal and the spatial chaotic modes, e.g., the complexity of profiles of traveling waves.

In particular, it seems natural to study the topological entropies $h_{\text{top}}(\mathbb{S}^{V_k}(t,h),\mathcal{A})$ of the action of these semigroups on the attractor (i.e., the directional topological entropies introduced by Milnor for cellular automata, see [169]). Now, in contrast to the cellular automata, these entropies are typically infinite for dissipative dynamics if $k < n+1$. In order to overcome this difficulty, it was suggested in [239] to modify the definition of the topological entropy by taking into account the typical rate of divergence of the mean entropy as $\varepsilon \to 0^+$.

DEFINITION 5.38. Let V_k be a k-dimensional plane in $\mathbb{R}^{n+1}$ and let $[0,1]^k_{V_k}$ be its unit hypercube generated by some orthonormal basis in V_k. Analogously to (5.62), for every $R > 0$, we introduce a new metric $d^{V_k}_R$ by

$$d^{V_k}_R(u_0,u_1) := \sup_{(t,h)\in R\cdot[0,1]^k_{V_k}} \big\|\mathbb{S}(t,h)u_0 - \mathbb{S}(t,h)u_1\big\|_{L^\infty_{e^{-|x|}}}, \qquad u_0,u_1 \in L^\infty_{e^{-|x|}}\big(\mathbb{R}^n\big). \tag{5.66}$$

Then a modified topological entropy $\hat{h}_{\text{top}}(\mathbb{S}^{V_k})$ of the action of the directional dynamical system $\mathbb{S}^{V_k}(t,h)$ on the attractor is given by the following quantity:

$$\hat{h}_{\text{top}}\big(\mathbb{S}^{V_k}(t,h),\mathcal{A}\big) := \limsup_{\varepsilon\to 0^+}\left(\log_2\frac{1}{\varepsilon}\right)^{k-n-1}\limsup_{R\to+\infty}\frac{1}{R^k}\mathcal{H}_\varepsilon\big(\mathcal{A},d^{V_k}_R\big), \tag{5.67}$$

see [239] for details.

We see that the above definition differs from the classical one by the presence of a normalizing factor $(\log_2 1/\varepsilon)^{k-n-1}$ which guarantees that this quantity is finite. In particular, if the modified entropy is strictly positive (examples of such cases will be given in the next subsection), then the corresponding classical topological entropy must be infinite.

The next theorem from [239] establishes the finiteness of these modified quantities and gives some of their basic relations.

THEOREM 5.39. *Let the assumptions of Theorem 5.36 hold. Then, for every k and every k-dimensional plane V_k, the associated modified entropy $\hat{h}_{\text{top}}(\mathbb{S}^{V_k})$ is finite,*

$$\hat{h}_{\text{top}}\big(\mathbb{S}^{V_k}(t,h),\mathcal{A}\big) < +\infty.$$

Furthermore, if $V_{k_1} \subset V_{k_2}$, then

$$\hat{h}_{\text{top}}\big(\mathbb{S}^{V_{k_2}}(t,h),\mathcal{A}\big) \leqslant L^{k_2-k_1}\hat{h}_{\text{top}}\big(\mathbb{S}^{V_{k_1}}(t,h),\mathcal{A}\big), \tag{5.68}$$

where L is some constant which is independent of k_i and V_{k_i}, $i = 1,2$.

REMARK 5.40. Inequalities (5.68) can be considered as a natural generalization of the classical inequality relating the fractal dimension to the topological entropy to the spatially extended case. Indeed, in the case of an ODE (without spatial directions), we have $n = 0$ and, as it can easily be shown, $\hat{h}_{\text{top}}(\mathbb{S}^{V_0})$ coincides with the fractal dimension of $\mathcal{A}$, $\hat{h}_{\text{top}}(\mathbb{S}^{V_1})$ gives the classical topological entropy, and (5.68) reads

$$h_{\text{top}}\big(S(t), \mathcal{A}\big) \leqslant L \dim_F \mathcal{A},$$

which coincides with a classical inequality, see [127]. Roughly speaking, the invariant $\hat{h}_{\text{top}}(\mathbb{S}^{V_k})$ has the structure of a topological entropy in the directions of V_k and of a (generalized) fractal dimension in the orthogonal directions, see [239] and [241] for details.

REMARK 5.41. Inequalities (5.68) are particularly useful to verify whether or not some modified directional entropy is strictly positive. In particular, the positivity of the full space–time entropy $h_{\text{top}}(\mathbb{S}(t, h), \mathcal{A})$ (which, e.g., corresponds to the presence of the so-called Sinai–Bunimovich space–time chaos in the system, see the next subsections) implies that *all* the above modified entropies are strictly positive. On the contrary, if

$$\lim_{\varepsilon \to 0^+} \left(\log_2 \frac{1}{\varepsilon}\right)^{-n-1} \mathcal{H}_\varepsilon\big(\mathcal{A}, L^\infty_{\mathrm{e}^{-|x|}}\big(\mathbb{R}^n\big)\big) = 0,$$

then all the above modified entropies automatically vanish.

REMARK 5.42. Analogues of the simplified formulas (5.64) to compute the directional entropies are also deduced in [239]. In particular, for the spatial dynamics $V_n = \mathbb{R}^n_x$, we have a particularly simple formula,

$$\hat{h}_{\text{sp}}(\mathcal{A}) := \hat{h}_{\text{top}}(T_h, \mathcal{A}) = \limsup_{\varepsilon \to 0^+} \left(\log_2 \frac{1}{\varepsilon}\right)^{-1} \bar{\mathcal{H}}_\varepsilon(\mathcal{A}). \tag{5.69}$$

Thus, contrary to the usual Kolmogorov ε-entropy which measures the "thickness" of a set (of the attractor here), the mean ε-entropy is more related to the complexity of its spatial structure.

REMARK 5.43. To conclude this subsection, it is worth noting that, in contrast to the full space–time topological entropy, the directional entropies introduced above are not *topological* invariants, but only Lipschitz continuous invariants (like the fractal dimension), due to the presence of the term $\log_2 \frac{1}{\varepsilon}$ in the definition. Furthermore, it is possible to show that there is no topological invariant which is typically finite and strictly positive when $k < n$. When $k = n$ (e.g., for spatial dynamics and spatial chaos), such an invariant exists, namely, the so-called mean topological dimension introduced in [142] (for the Bernoulli scheme with a continuous number of symbols) which can be obtained as in Definition 5.38, but by taking the additional infimum with respect to all metrics which induce the local topology on $\mathcal{A}$, see [239] for details.

5.7. *Lower bounds on the entropy, the Kotelnikov formula, and spatial chaos*

In this subsection, we discuss, following essentially [237] and [239], the derivation of lower bounds on the Kolmogorov ε-entropy and related lower bounds on the complexity of the dynamics. We start by recalling that, in bounded domains, one usually estimates the dimension of the global attractor from below by finding a proper equilibrium with a large instability index and by constructing the associated unstable set. Since an unstable set always belongs to the global attractor, the instability index of this equilibrium then gives a lower bound on its dimension, see [22,217], and the references therein.

Thus, it seems natural to try to extend this theory to unbounded domains and to obtain lower bounds on the ε-entropy from the existence of large (infinite dimensional) unstable sets for appropriate equilibria. However, the main difficulty here is that, contrary to bounded domains, the spectrum of an equilibrium usually consists of continuous curves (or continuous sets) and does not have reasonable spectral gaps in order to use the usual theory of unstable manifolds. As a consequence, the unstable set of an equilibrium is usually *not* a manifold and a straightforward extension fails.

This obstacle can be overcome by using (following [75,237], and [239]) the so-called essentially unstable manifolds which consist of the initial data of the solutions which tend to an equilibrium as $t \to -\infty$ with a sufficiently fast exponential rate. As the following theorem (proven in [239]) shows, no spectral gap condition is required for the existence of such manifolds.

THEOREM 5.44. *Let X be a Banach space and let $S : X \to X$ be a nonlinear map satisfying*

$$S(u) = S_0 u + K(u), \quad K \in \mathcal{C}^{1+\alpha}(X, X), \quad K(0) = K'(0) = 0, \tag{5.70}$$

for some $0 < \alpha \leqslant 1$ and some linear operator $S_0 \in \mathcal{L}(X, X)$. Let then the linearization S_0 of the operator S at zero be exponentially unstable, i.e.,

$$r(S_0) := \sup\bigl|\sigma(S_0, X)\bigr| > 1,$$

where $\sigma(L, V)$ denotes the spectrum of the operator L in the space V. We finally assume that there exists a closed invariant subspace X_+ of S_0 such that

$$\inf\bigl|\sigma(S_0|_{X_+}, X_+)\bigr| > \theta_0 > 1, \quad \theta_0^{1+\alpha} > r(S_0). \tag{5.71}$$

Then there exists a ball $\mathcal{B} := B_{X_+}(0, \rho)$ and a $\mathcal{C}^{1,\alpha}$ map $\mathbb{V} : \mathcal{B} \to X$ such that

$$\bigl\|\mathbb{V}(x_+) - x_+\bigr\|_X \leqslant C\|x_+\|_X^{1+\alpha}, \quad x_+ \in \mathcal{B}.$$

Furthermore, for every $u_0 \in \mathbb{V}(\mathcal{B})$, there exists a backward trajectory $\{u(n)\}_{n\in\mathbb{Z}_-}$ such that

$$u(n+1) = S\bigl(u(n)\bigr), \quad u(0) = u_0, \quad \bigl\|u(n)\bigr\|_X \leqslant C\theta_0^n, \quad n \in \mathbb{Z}_-,$$

and, consequently, $\mathbb{V}(\mathcal{B})$ is an essentially unstable manifold of the equilibrium $u = 0$ of the map S.

We see that, in contrast to the usual theory of unstable manifolds, see, e.g., [22], neither the finite dimensionality of X_+ nor any spectral gap assumption (and nor even the existence of a complement to X_+ in X) are required.

We illustrate the application of this theorem to dissipative dynamical systems on the simple example of the real Ginzburg–Landau equation in $\mathbb{R}^n$,

$$\partial_t u = \Delta_x u + u - u^3, \tag{5.72}$$

and we consider the equilibrium $u = 0$. In that case, the first variation equation reads

$$\partial_t v = \Delta_x v + v. \tag{5.73}$$

Let $S(t)$ and $S_0(t)$ be the solution operators associated with equations (5.72) and (5.73) in $X := L^\infty(\mathbb{R}^n)$, respectively. Then condition (5.70) is obviously satisfied for $S = S(1)$, $S_0 = S_0(1)$, and $\alpha = 1$. In order to find the space X_+, it is sufficient to write the Fourier transform of S_0,

$$\widehat{S_0(t)u_0}(\xi) = \mathrm{e}^{(1-|\xi|^2)t}\hat{u}_0(\xi).$$

This shows that $r(S_0) = e$ and that the unstable part of the spectrum is related to the functions $\mathbb{B}_1(\mathbb{R}^n)$, the support of the Fourier transform of which belongs to the unit ball, see Example 5.26. Furthermore, condition (5.71) is satisfied if we take $X_+ := \mathbb{B}_\sigma(\mathbb{R}^n)$, with $\sigma < \frac{1}{\sqrt{2}}$.

Thus, thanks to Theorem 5.44 and to the fact that an unstable manifold always belongs to the global attractor, we have verified that the attractor $\mathcal{A}$ contains a smooth image of a ball $\mathcal{B}$ of the space $\mathbb{B}_\sigma(\mathbb{R}^n)$ (of entire functions with an exponential growth). Combining this embedding with the lower bounds on the ε-entropy of the spaces $\mathbb{B}_\sigma(\mathbb{R}^n)$ collected in Example 5.26, we obtain the following result.

THEOREM 5.45. *The Kolmogorov ε-entropy of the global attractor $\mathcal{A}$ of the real Ginzburg–Landau equation has lower bounds which are analogous to estimates* (5.47) *and* (5.48) *and, consequently, the universal entropy estimates* (5.52) *are sharp.*

Of course, the approach based on infinite dimensional essentially unstable manifolds described above is not related to any specific property of the Ginzburg–Landau equation, but has a universal nature. Actually, only the existence of at least one spatially homogeneous exponentially unstable equilibrium is necessary to apply this method (and, as a consequence, to obtain sharp lower bounds on the entropy), see [75,76,237], and [239] for applications of this method to various types of reaction–diffusion systems and [235] for damped hyperbolic equations.

As a next step, we mention that the embedding $\mathbb{V} : \mathcal{B}(\sigma) \to \mathcal{A}$ of the unit ball $\mathcal{B}(\sigma) = B_{\mathbb{B}_\sigma}(0, 1)$ in the space of entire functions into the attractor $\mathcal{A}$ gives much more than just estimates on the ε-entropy. Indeed, since the dissipative system and the equilibrium are spatially homogeneous, the unstable manifold map $\mathbb{V}$ commutes with the spatial shifts T_h,

$$T_h \circ \mathbb{V} = \mathbb{V} \circ T_h, \quad h \in \mathbb{R}^n,$$

and, consequently, we have obtained a smooth embedding of the spatial dynamics on the space $\mathbb{B}_\sigma(\mathbb{R}^n)$ of entire functions into that on the attractor $\mathcal{A}$,

$$\mathbb{V}:\big(\mathcal{B}(\sigma), T_h\big) \to (\mathcal{A}, T_h), \tag{5.74}$$

see [237] for details. Thus, the shifts dynamics on the unit ball $\mathcal{B}(\sigma)$ gives a universal model for the spatial dynamics on the attractor.

In order to clarify the complexity of this model dynamics, we need to introduce a special type of Bernoulli shift dynamics.

DEFINITION 5.46. Let $\mathbb{M}_n := [0,1]^{\mathbb{Z}^n}$ be endowed with the Tikhonov topology. We recall that $\mathbb{M}_n$ consists of all functions $v:\mathbb{Z}^n \to [0,1]$ and the Tikhonov topology can be generated by the following metric:

$$\|v_1 - v_2\|_\phi := \sup_{m\in\mathbb{Z}^n}\big\{\phi(m)\big|v_1(m) - v_2(m)\big|\big\}, \quad v_1, v_2 \in \mathbb{M}_n, \tag{5.75}$$

where ϕ is an arbitrary weight function such that $\lim_{|m|\to+\infty}\phi(m) = 0$. We define the action of the group $\mathbb{Z}^n$ on $\mathbb{M}_n$ in the following standard way:

$$\mathcal{T}_l v(m) := v(l+m), \quad v \in \mathbb{M}_n,\ l, m \in \mathbb{Z}^n,$$

and interpret the group $(\mathbb{M}_n, \mathcal{T}_l)$ as a multi-dimensional Bernoulli scheme with a continuum of symbols $\omega \in [0,1]$.

Our approach to the study of the dynamics generated by the shifts group $(\mathcal{B}(\sigma), T_h)$ is based on the following elementary observation: according to the classical Kotelnikov formula (see [29] and [132]), every function $w \in \mathbb{B}_\sigma(\mathbb{R}) \cap L^2(\mathbb{R})$ can be uniquely recovered from its values on the lattice $\rho\mathbb{Z}$, $\rho = \frac{\pi}{\sigma}$,

$$w(x) = \sum_{l=-\infty}^{+\infty} w(\rho l)\frac{\sin(\sigma x - \pi l)}{\sigma x - \pi l} \tag{5.76}$$

(see also the Whittaker–Shennon–Kotelnikov formula, e.g., in [29], which allows to recover an arbitrary function $w \in \mathbb{B}_\sigma(\mathbb{R}^n)$ from its values on a lattice). Given an arbitrary sequence $v = \{v_l\}_{l\in\mathbb{Z}} \in l^2$, formula (5.76) allows to construct a function $w \in \mathbb{B}_\sigma(\mathbb{R}) \cap L^2(\mathbb{R})$ such that $w(\rho l) = v_l$, for every $l \in \mathbb{Z}$. Furthermore, the spatial shifts $T_{\rho l}w$ of this function obviously correspond to the shifts $\mathcal{T}_l v$ of the sequence v. This leads to a description of the spatial dynamics on $\mathbb{B}_\sigma(\mathbb{R}) \cap L^2(\mathbb{R})$ in terms of the Bernoulli scheme introduced above (with the additional restriction $v \in l^2$). The extension of representation (5.76) in the spirit of the Whittaker–Shennon–Kotelnikov formula leads to the following result, see [239].

LEMMA 5.47. *For every $\sigma > 0$, there exist $\rho = \rho(\sigma)$ and a map*

$$\mathbb{U}:\mathbb{M}_n \to \mathcal{B}(\sigma) \quad \text{such that} \quad T_{\rho l}\circ\mathbb{U} = \mathbb{U}\circ\mathcal{T}_l,\ l \in \mathbb{Z}^n. \tag{5.77}$$

Furthermore, for every polynomial weight $\theta = \theta_{N,x_0}$ (see (5.4) with $N > 0$), there holds

$$C_1^{-1}\|v_1 - v_2\|_\theta \leqslant \big\|\mathbb{U}(v_1) - \mathbb{U}(v_2)\big\|_{L^\infty_\theta} \leqslant C_1\|v_1 - v_2\|_\theta,$$

where C_1 depends on N, but is independent of $v_i \in \mathbb{M}_n$, $i = 1, 2$.

Combining this lemma with (5.74), we obtain the following result, see [237] for details.

THEOREM 5.48. *Let $\mathcal{A}$ be the global attractor of the real Ginzburg–Landau equation* (5.72). *Then there exist a positive constant ρ and a map*

$$\mathcal{U}:\mathbb{M}_n \to \mathcal{A} \quad \textit{such that} \quad \mathcal{U}\circ\mathcal{T}_l = T_{\rho l}\circ\mathcal{U}, \quad l\in\mathbb{Z}^n. \tag{5.78}$$

Furthermore, $\mathcal{U}$ is continuous in the local topology (*and even Lipschitz continuous in appropriate weighted spaces*).

Thus, we see that the Bernoulli scheme $(\mathbb{M}_n, \mathcal{T}_l)$ can be considered as a universal model for the spatial dynamics on the attractor $\mathcal{A}$. Indeed, on the one hand, this model has infinite topological entropy and strictly positive modified entropy $\hat{h}_{\mathrm{sp}}(\mathbb{M}_n, \mathcal{T}_l)$ (see (5.69)),

$$1 = \hat{h}_{\mathrm{sp}}(\mathbb{M}_n, \mathcal{T}_l) = \rho^n \hat{h}_{\mathrm{sp}}\big(\mathcal{U}(\mathbb{M}_n), T_h\big) \leqslant \hat{h}_{\mathrm{sp}}(\mathcal{A}, T_h) < +\infty, \tag{5.79}$$

and, therefore, this gives an example of spatial dynamics of "maximal" complexity (in the sense of the entropy theory). On the other hand, (5.78) holds under very weak assumptions on the dissipative system under study (namely, the existence of at least one spatially homogeneous exponentially unstable equilibrium, see [237] and [239]) and thus has a universal nature.

To conclude this section, we briefly discuss the possibility of extending such a complexity description from the spatial dynamics $V_n = \mathbb{R}^n_x$ to the dynamics of $\mathbb{S}^{V_n}(t,h)$, where V_n contains the *temporal* direction, e.g., $V_n = \operatorname{span}\{e_t, e_{x_2}, \dots, e_{x_n}\}$. As above, we restrict ourselves to the real Ginzburg–Landau equation, but now with a transport term along the x_1-axis,

$$\partial_t u = \Delta_x u - L\partial_{x_1} u + u - u^3, \tag{5.80}$$

although the result also holds for the general reaction–diffusion system (5.8) under the assumptions of Theorem 5.5, plus the spatial homogeneity and the exponential instability of the zero equilibrium, see [241].

The main idea here is to "change" the temporal t and spatial x_1 directions by considering x_1 as a new "time" and t as one of the "spatial" variables. Then, describing the spatial chaos in this new dissipative system by the scheme introduced above, we would automatically obtain the description of the n-directional space–time chaos in the plane V_n. In order to realize this strategy, we consider equation (5.80) in the half-space $x_1 > 0$, endow it with the following unusual "initial" condition:

$$\begin{cases} \partial_t u = \Delta_x u - L\partial_{x_1} u + u - u^3, \quad t\in\mathbb{R},\ (x_2,\dots,x_n)\in\mathbb{R}^{n-1},\ x_1>0, \\ u|_{x_1=0} = u^0 \in \Psi_b := L^\infty\big(\mathbb{R}^n\big), \end{cases} \tag{5.81}$$

and treat it as an "evolutionary" equation with respect to the time variable x_1 and the spatial variables $t, x_2, \dots, x_n$. Clearly, this problem is ill-posed if $L = 0$ (as well as for small L). However, as proven in [239] and [241], it indeed generates a well-posed and smooth dissipative system in Ψ_b if L is large enough ($L > 2$ for the real Ginzburg–Landau equation). Furthermore, the zero equilibrium remains exponentially unstable for this new system, so that the theory of essentially unstable manifolds is applicable and gives the following result, see [239].

THEOREM 5.49. *Let $\mathcal{A}$ be the global attractor of the real Ginzburg–Landau equation (5.80) with a sufficiently large transport term* ($L > 2$). *Then there exist a positive constant σ and a map $\mathcal{W} : \mathcal{B}(\sigma) \to \mathcal{A}$, which is continuous in the local topology, such that*

$$\begin{aligned} &S(t) \circ \mathcal{W} = \mathcal{W} \circ T_{te_{x_1}}, \quad T_{he_{x_i}} \circ \mathcal{W} = \mathcal{W} \circ T_{he_{x_i}}, \\ &\quad i = 2, \dots, n, \ h \in \mathbb{R}, \ t \geqslant 0, \end{aligned} \tag{5.82}$$

where $T_{h\vec{e}}v(x) := v(x + h\vec{e})$, $h \in \mathbb{R}$, $x \in \mathbb{R}^n$.

A combination of this result and Lemma 5.47 gives the desired embedding of the Bernoulli scheme $(\mathbb{M}_n, \mathcal{T}_l)$ into the n-directional space–time dynamics of $\mathbb{S}^{V_n}(t, h)$. In particular, this embedding shows that the modified topological entropy of this dynamics is strictly positive,

$$\hat{h}_{\text{top}}\big(\mathbb{S}^{V_n}(t, h), \mathcal{A}\big) > 0,$$

and, owing to inequalities (5.68), the modified entropy of the temporal evolution group $S(t)$ ($V_1 = \mathbb{R}_t$) is also strictly positive,

$$\hat{h}_{\text{top}}\big(S(t), \mathcal{A}\big) > 0.$$

We also recall that the modified entropy for $S(t)$ differs from the classical topological entropy by the presence of a factor $(\log_2 \frac{1}{\varepsilon})^{-n}$ in the definition and, consequently, its positivity implies that the classical topological entropy is infinite,

$$h_{\text{top}}\big(S(t), \mathcal{A}\big) = +\infty. \tag{5.83}$$

To the best of our knowledge, this is the first example of a reasonable dissipative system with an infinite topological entropy.

REMARK 5.50. To conclude, we note that, although the above method gives an adequate description of the n-directional complexity and the n-directional space–time chaos for an arbitrary plane V_n under weak assumptions on the system, it does not give reasonable information on the full $(n+1)$-directional space–time complexity, since one direction should be interpreted as the time and we should have exponential divergence in this direction. Furthermore, the $(n + 1)$-dimensional Bernoulli scheme $\mathbb{M}_{n+1}$ cannot be embedded into the global attractor, since its space–time entropy $h_{\text{top}}(\mathbb{S}(t, h), \mathcal{A})$ is finite, see Theorem 5.36.

5.8. *Sinai–Bunimovich space–time chaos in PDEs*

In this concluding subsection, we discuss very recent results concerning the full $(n + 1)$-directional space–time chaos and, in particular, we give examples of dissipative systems in unbounded domains with a strictly positive space–time topological entropy,

$$h_{\text{top}}\big(\mathbb{S}(t, h), \mathcal{A}\big) > 0, \tag{5.84}$$

which shows that space–time dynamics with a maximum level of complexity (from the point of view of the entropy theory) can indeed appear in dissipative systems generated by PDEs.

We first recall that, in spite of a huge amount of numerical and experimental data on various types of space–time irregular and turbulent behaviors in various physical systems, see, e.g., [112,155,156,193], and the references therein, there are very few rigorous mathematical results on this topic and mathematically relevant models which describe such phenomena.

The simplest and most natural known model which exhibits such phenomena is the so-called Sinai–Bunimovich space–time chaos which was initially defined and found for discrete lattice dynamics, see [5,33,188], and [189]. We also recall that this model consists of a $\mathbb{Z}^n$-grid of temporally chaotic oscillators coupled by a weak interaction. Then, if a single chaotic oscillator of this grid is described by the Bernoulli scheme $\mathcal{M}^1 := \{0,1\}^{\mathbb{Z}}$ (now with only two symbols $\omega \in \{0,1\}$, in contrast to the previous subsection!), the uncoupled system naturally has an infinite dimensional hyperbolic set which is homeomorphic to the multi-dimensional Bernoulli scheme $\mathcal{M}^{n+1} := \{0,1\}^{\mathbb{Z}^{n+1}} = (\mathcal{M}^1)^{\mathbb{Z}^n}$. The temporal evolution operator is then conjugated to the shift in $\mathcal{M}^{n+1}$ along the first coordinate vector and the other n coordinate shifts are associated with the spatial shifts on the grid. Finally, owing to the stability of hyperbolic sets, the above structure survives under a sufficiently small coupling. Thus, according to this model, the space–time chaos can naturally be described in terms of the multi-dimensional Bernoulli scheme $\mathcal{M}^{n+1}$.

It is worth noting that, although this model is clearly not relevant to describe the space–time chaos in the so-called fully developed turbulence (since it does not reproduce the typical properties, such as energy cascades and the Kolmogorov laws, which are believed to be crucial for the understanding of this phenomenon), it can be useful and relevant to describe weak space–time chaos and weak turbulence (close to the threshold), where the generation and long-time survival of such global spatial patterns are still possible. Furthermore, to the best of our knowledge, it is the only mathematically rigorous model which gives positive space–time topological entropy and an associated space–time dynamics with maximal complexity.

Thus, the possibility of having $h_{\mathrm{top}}(\mathbb{S}(t,h))$ positive is clear for space-discrete lattice dissipative systems. However, verifying the existence of such space–time dynamics in continuous media described by PDEs is an extremely complicated problem. Furthermore, even the existence of a single PDE which possesses such an infinite dimensional Bernoulli scheme has been a long-standing open problem.

The first examples of reaction–diffusion systems in $\mathbb{R}^n$ with Sinai–Bunimovich space–time chaos were recently constructed in [167]. We describe below this construction in more details.

We consider the following special *space–time periodic* reaction–diffusion equation:

$$\partial_t u = \gamma \Delta_x u - f_\lambda(t,x,u) \quad \text{in } \mathbb{R}^n,\ \gamma > 0, \tag{5.85}$$

where the nonlinearity f_λ has the following structure: there exists a smooth bounded domain $\Omega_0 \Subset (0,1)^n$ such that, for every $x \in [0,1]^n$, there holds

$$f_\lambda(t,x,u) := \begin{cases} f(t,u) & \text{for } x \in \Omega_0, \\ \lambda u & \text{for } x \in [0,1]^n \backslash \Omega_0, \end{cases} \tag{5.86}$$

where $f(t,u)$ is a given function (which is assumed to be 1-periodic with respect to t) and $\lambda \gg 1$ is a large parameter. Then we extend (5.86) by space-periodicity from $[0,1]^n$ to the whole space $\mathbb{R}^n$. Thus, we have a periodic grid of "islands" $\Omega_l := l + \Omega_0$, $l \in \mathbb{Z}^n$, on which the nonlinearity f_λ coincides with $f(t,u)$ and can generate nontrivial dynamics. These islands are separated from each other by the "ocean" $\Omega_- := \mathbb{R}^n \setminus (\bigcup_{l \in \mathbb{Z}^n} \Omega_l)$, where we have the strong absorption provided by the nonlinearity $f_\lambda(t,x,u) \equiv \lambda u$.

It is intuitively clear that, for a sufficiently large absorption coefficient λ, the solutions of Eq. (5.85) should be small in the absorption domain Ω_- and, consequently, the interactions between the islands are also expected to be small, and the dynamics inside the islands are "almost-independent". Thus, if the reaction–diffusion system in Ω_0,

$$\partial_t v = \gamma \Delta_x v - f(t,v) \quad \text{in } \Omega_0, \qquad v = 0 \quad \text{on } \partial\Omega_0, \tag{5.87}$$

which describes the limit dynamics inside one "island" as $\lambda = +\infty$, possesses a hyperbolic set Γ_0, then, according to the structural stability principle, the whole system (5.85) should have a hyperbolic set which is homeomorphic to $(\Gamma_0)^{\mathbb{Z}^n}$ if the absorption parameter λ is large enough. Furthermore, if, in addition, the initial hyperbolic set Γ_0 is homeomorphic to the Bernoulli scheme $\{0,1\}^{\mathbb{Z}}$, then (5.85) contains an $(n+1)$-dimensional Bernoulli scheme $\{0,1\}^{\mathbb{Z}^{n+1}} \sim (\{0,1\}^{\mathbb{Z}})^{\mathbb{Z}^n}$, in a complete analogy with the Sinai–Bunimovich lattice model.

These intuitive arguments were rigorously justified in [167], where the following result was obtained.

THEOREM 5.51. *Let the limit equation (5.87) possess a hyperbolic set which is homeomorphic to the usual Bernoulli scheme $\mathcal{M}^1 = \{0,1\}^{\mathbb{Z}}$ and let some natural assumptions on f be satisfied. Then there exists $\lambda_0 = \lambda_0(f, \mathcal{M}^1)$ such that, for every $\lambda > \lambda_0$, problem (5.85) possesses an infinite dimensional hyperbolic set which is homeomorphic to $\mathcal{M}^{n+1} = \{0,1\}^{\mathbb{Z}^{n+1}}$. Furthermore, the action of the space–time dynamics on this set (restricted to $(t,h) \in \mathbb{Z}^{n+1}$) is conjugated to the Bernoulli shift on $\mathcal{M}^{n+1}$.*

Since the existence of a hyperbolic set which is homeomorphic to $\mathcal{M}^1$ for the reaction–diffusion system (5.87) in a *bounded* domain is well-known (the existence of a single transversal homoclinic trajectory is sufficient in order to have such a result, see [127]; see also [167] for an explicit construction), the above theorem indeed provides examples for Sinai–Bunimovich space–time chaos in reaction–diffusion systems and, in particular, examples of reaction–diffusion systems with a strictly positive space–time topological entropy. Furthermore, owing to the stability of hyperbolic sets, the space discontinuous nonlinearity f_λ can then be replaced by close $\mathcal{C}^\infty$ ones and, finally, by embedding the space–time periodic system that we obtain into a larger autonomous one (one creates the space–time periodic modes by using the additional equations), examples of space–time autonomous reaction–diffusion systems of the form (5.8) were also constructed in [167].

REMARK 5.52. We note that the spatial grid $\mathbb{Z}^n$ (which is crucial for the Sinai–Bunimovich model) is directly modulated by the special spatial structure of the nonlinearity f_λ in the continuous model (5.85) (see (5.86)) and, therefore, the above approach does not allow to find such phenomena in many physically relevant equations for which the structure of the nonlinearity is a priori given (such as the Navier–Stokes equations, the real and complex Ginzburg–Landau equations, ...). In order to overcome this drawback, an alternative, potentially more promising, approach was suggested in [168], where the spatial grid is obtained by using the so-called spatially-localized solutions (pulses, standing solitons, ...) initially situated in the nodes of the grid. Then, due to the "tail"-interaction between solitons, a weak temporal dynamics appears and this dynamics allows a center manifold reduction to a *lattice* system of ODEs (roughly speaking, this system describes the temporal evolution of the soliton centers, see [79,168], and [204] for details). Finding then the Sinai–Bunimovich space–time chaos in these reduced lattice equations, one can lift it to the initial PDE. The advantage of this method is that the spatial grid is now modulated in an implicit way by the positions of localized solutions in space and the center manifold reduction, and the underlying dissipative system may be autonomous and spatially homogeneous. In particular, this approach was realized in [168] for the one-dimensional space–time periodically perturbed Swift–Hohenberg equation,

$$\partial_t u + \left(\partial_x^2 + 1\right)^2 u + \beta^2 u + u^3 + \kappa u^2 = h(t, x), \tag{5.88}$$

for values of β and κ for which the existence of a spatially localized soliton is known. To be more precise, for these values of β and κ, the existence of a hyperbolic set $\mathcal{M}^2$ for (5.88) is proven for special (rather artificial) space–time periodic external forces h with *arbitrary small* amplitudes. The presence of these external forces are unavoidable for the Swift–Hohenberg model, since it belongs to the class of the so-called extended gradient systems and, when $h = 0$, its space–time topological entropy vanishes, see [239], and the Sinai–Bunimovich space–time chaos is then impossible. Finally, we also mention a very recent result [220] in which the above method allowed to prove the existence of a Sinai–Bunimovich space–time chaos for the one-dimensional complex Ginzburg–Landau equation,

$$\partial_t u = (1 + \mathrm{i}\beta)\partial_x^2 u + \gamma u - \delta u|u|^3 + \varepsilon,$$

where $\beta \in \mathbb{R}$, $\gamma, \delta \in \mathbb{C}$, and ε is an arbitrary small real parameter. Contrary to the above examples, this equation is already space–time homogeneous and does not contain any artificial nonlinearity or external forces. This confirms that the Sinai–Bunimovich space–time chaos may appear in *natural* PDEs arising from mathematical physics.

6. Ill-posed dissipative systems and trajectory attractors

In this concluding subsection, we briefly discuss possible extensions of the theory of attractors to ill-posed problems. Indeed, in all the above results, we required the solution operator

$$S(t) : u_0 \mapsto u(t) \tag{6.1}$$

to be well-defined and continuous (in a proper phase space). However, as mentioned in the introduction, in several cases, such a result is not known or does not hold.

There exist two approaches to handle dissipative systems without uniqueness.

The first one allows the solution operator (6.1) to be *multi-valued* (set-valued) and then extends the theory of attractors to semigroups of multi-valued maps. Actually, all the results on the existence of the global attractor given in Subsection 2.2 have their natural analogues in the multi-valued setting, see [11,19,23,24,50,158,159,201,208,209], and the references therein; see also [37,141], and [231] for nonautonomous systems.

An alternative, more geometric, approach consists in changing the phase space of the problem and in passing to the so-called *trajectory* phase space and the associated trajectory dynamical system, which is single-valued, and, thus, the usual theory of attractors can be applied, see [46–49,89,210,211,222,223,232], and the references therein. We illustrate this approach on the simple example of an ODE in $E = \mathbb{R}^n$, see [49] for details,

$$u' + f(u) = 0, \quad u(0) = u_0, \tag{6.2}$$

for some, at least, continuous nonlinearity f. We also assume that the system is *dissipative*, so that it is globally solvable for every $u_0 \in E$ and the following estimate holds:

$$\|u(t)\|_E \leqslant Q\big(\|u(0)\|_E\big)\mathrm{e}^{-\alpha t} + C_F, \quad t \geqslant 0, \tag{6.3}$$

for some positive constants α and C_F and monotonic function Q and for every solution u of (6.2). This holds, e.g., if f satisfies a dissipativity assumption of the form

$$f(u)\cdot u \geqslant -C + \beta|u|^2, \quad u \in \mathbb{R}^n,\ C, \beta > 0.$$

Let us assume for a while that f is Lipschitz continuous. Then we have the uniqueness of solutions and, for every two solutions $u_1(t)$ and $u_2(t)$, the following estimate holds:

$$\|u_1(t) - u_2(t)\|_E \leqslant C\mathrm{e}^{Kt}\|u_1(0) - u_2(0)\|_E, \quad t \geqslant 0, \tag{6.4}$$

where the constants C and K depend on $\|u_i(0)\|_E$, $i = 1,\ 2$.

In this classical case, the dissipative estimate (6.3) guarantees the existence of a compact absorbing set for the semigroup $S(t)$ associated with problem (6.2) via (6.1) (we recall that $\dim E < +\infty$). Thus, this semigroup possesses the global attractor $\mathcal{A}_{\mathrm{gl}}$ on E which has the usual structure,

$$\mathcal{A}_{\mathrm{gl}} = \mathcal{K}|_{t=0}, \tag{6.5}$$

where $\mathcal{K} \subset \mathcal{C}_b(\mathbb{R}, E)$ is the kernel (i.e., the set of all bounded complete trajectories of problem (6.2), see Subsection 2.1).

We now define a *trajectory* phase space for problem (6.2) as follows:

$$K_{\mathrm{tr}} := \big\{u \in \mathcal{C}_b(\mathbb{R}_+, E),\ u(t) = S(t)u_0,\ u_0 \in E,\ t \geqslant 0\big\}. \tag{6.6}$$

In other words, K_{tr} consists of all positive trajectories of (6.2) starting from all points $u_0 \in E$. Then, owing to the uniqueness, K_{tr} is isomorphic to E by the solution operator $\mathcal{S}u_0 := u(\cdot) = S(\cdot)u_0$,

$$\begin{aligned} &\mathcal{S} : E \to \Phi_b := \mathcal{C}_b(\mathbb{R}_+, E), \quad \mathcal{S}(E) = K_{\mathrm{tr}}, \\ &\mathcal{S}^{-1}u = u(0), \quad u \in K_{\mathrm{tr}}, \quad \mathcal{S}^{-1}K_{\mathrm{tr}} = E. \end{aligned} \tag{6.7}$$

Furthermore, as it is not difficult to see, the semigroup $S(t)$ is conjugated to the time translations on K_{tr} under this isomorphism,

$$\begin{aligned} &T_t : K_{\mathrm{tr}} \to K_{\mathrm{tr}}, \quad T_t = \mathcal{S} \circ S(t) \circ \mathcal{S}^{-1}, \\ &T_t u(s) := u(t+s), \quad t \geqslant 0,\ s \in \mathbb{R}. \end{aligned} \tag{6.8}$$

We call the shifts semigroup $\{T_t,\ t \geqslant 0\}$ acting on the trajectory phase space the *trajectory* dynamical system associated with problem (6.2).

We now fix a class of bounded sets and a topology on K_{tr} via this isomorphism. Indeed, obviously, the set $B \subset E$ is bounded if and only if $\mathcal{S}(B)$ is bounded in Φ_b (see the dissipative estimate (6.3)) and $\mathcal{S}$ is an homeomorphism if we endow the phase space K_{tr} with the topology of $\Phi_{\mathrm{loc}} := \mathcal{C}_{\mathrm{loc}}(\mathbb{R}_+, E)$ (due to the Lipschitz continuity (6.4)).

Thus, owing to the homeomorphism $\mathcal{S}$, the existence of the global attractor $\mathcal{A}_{\mathrm{gl}}$ for the semigroup $S(t)$ on E is *equivalent* to that of the $(\Phi_b, \Phi_{\mathrm{loc}})$-global attractor $\mathcal{A}_{\mathrm{tr}}$ of the trajectory dynamical system (T_t, K_{tr}) (we recall that a $(\Phi_b, \Phi_{\mathrm{loc}})$-attractor attracts the bounded subsets of Φ_b in the topology of Φ_{loc}, see [22]). We refer the attractor $\mathcal{A}_{\mathrm{tr}}$ as the *trajectory* attractor associated with problem (6.2). As usual, this trajectory attractor is also generated by the set $\mathcal{K}$ of all bounded complete trajectories of the problem,

$$\mathcal{A}_{\mathrm{tr}} = \mathcal{K}|_{t \geqslant 0}, \quad \mathcal{A}_{\mathrm{gl}} = \mathcal{A}_{\mathrm{tr}}|_{t=0}. \tag{6.9}$$

A key observation here is that, although *crucial* for the usual global attractor $\mathcal{A}_{\mathrm{gl}}$, the uniqueness and continuity (6.4) are *not necessary* for the existence of the global attractor $\mathcal{A}_{\mathrm{tr}}$ for the *trajectory* dynamical system (T_t, K_{tr}) and can be relaxed. Indeed, the phase space K_{tr} is well-defined and the shifts semigroup T_t acts continuously on it, no matter whether or not the uniqueness holds (only the *dissipative* estimate (6.3) is necessary to ensure that $K_{\mathrm{tr}} \subset \Phi_b$; of course, we also need the continuity of f to ensure that the solutions exist and K_{tr} is not empty). Furthermore, the dissipative estimate (6.3) also guarantees that the set

$$B_{\mathrm{tr}} := \left\{u \in K_{\mathrm{tr}},\ \|u\|_{\Phi_b} \leqslant R\right\}$$

is a Φ_b-absorbing set for T_t. Finally, this absorbing set is *compact* in the Φ_{loc}-topology (since E is finite dimensional and we have a uniform control on the norm of $\mathrm{d}u/\mathrm{d}t$ for every $u \in B_{\mathrm{tr}}$ from Eq. (6.2)). Thus, the existence of the trajectory attractor $\mathcal{A}_{\mathrm{tr}}$ is verified when f is only continuous and we have the following theorem.

THEOREM 6.1. *Let the nonlinearity f in* (6.2) *be continuous and let the dissipative estimate* (6.3) *be satisfied for all solutions. Then the trajectory dynamical system* (T_t, K_{tr}) *possesses the* $(\Phi_b, \Phi_{\mathrm{loc}})$*-global attractor* $\mathcal{A}_{\mathrm{tr}}$ (*which is the trajectory attractor associated with problem* (6.2)) *which is generated by all bounded complete trajectories of the system*,

$$\mathcal{A}_{\mathrm{tr}} = \mathcal{K}|_{t \geqslant 0}. \tag{6.10}$$

It is also worth noting that, projecting this trajectory attractor $\mathcal{A}_{\mathrm{tr}}$, we obtain the global attractor $\mathcal{A}_{\mathrm{gl}}^{\mathrm{m-v}}$ for the multi-valued semigroup $S(t)$ associated with problem (6.2) in a standard way,

$$\mathcal{A}_{\mathrm{gl}}^{\mathrm{m-v}} = \mathcal{A}_{\mathrm{tr}}|_{t=0}, \tag{6.11}$$

see [49] for details.

Thus, we see that, although the trajectory approach usually essentially gives the same object as the multi-valued semigroup (see (6.11)), it allows, on the one hand, to avoid the use of "unfriendly" multi-valued maps and, on the other hand, to study the long time behavior for ill-posed problems by using the classical theory of attractors for single-valued semigroups. We also note that the trick consisting in passing from the usual to the trajectory dynamical system may be useful even when the uniqueness holds. In particular, the aforementioned l-trajectories method for estimating the dimension of global attractors and constructing exponential attractors is essentially based on this trick, see [153] and the references therein. Furthermore, this trick was also used in [167] to prove the persistence of hyperbolic trajectories when the perturbation is not small in the initial phase space, but only in some averaged time-integral norms.

However, it is worth mentioning that the above trajectory approach has not been applied to artificial problems like (6.2) (for which the nonuniqueness appears due to the lack of regularity on f), but to extremely complicated equations such as the three-dimensional Navier–Stokes equations and even to equations in compressible fluid mechanics for which only minimal information on the associated weak solutions is available. This leads to several unusual "common" delicate points in the theory which we would like to outline before passing to more relevant examples.

REMARK 6.2. (a) Very often, the dissipative estimate (6.3) can be verified not for every solution belonging to some function space, but *only* for some special weak solutions (e.g., obtained by Galerkin approximations, as for the three-dimensional Navier–Stokes equations). So, one should somehow exclude the "pathological", possibly non-dissipative, trajectories from the trajectory phase space K_{tr}. By doing this, one should, however, take a special care to preserve the action of the shifts semigroup on K_{tr}. In particular, the direct way which consists in incorporating the dissipative estimate into the phase space K_{tr}, i.e., in defining K_{tr} as the set of all trajectories satisfying a dissipative estimate of the form (6.3), may fail for this very reason. Indeed, typically, for ill-posed problems, we can construct a solution which satisfies the energy inequality between $t = 0$ and any $t = T$ (which gives the dissipative estimate), but not between $t = \tau$ and $t = T$ for $\tau > 0$ (see the example of a damped wave equation below). So, in that case, we cannot verify a dissipative estimate of the form (6.3) starting from $t = \tau$ and, for this reason, we lose the invariance $T_t K_{\mathrm{tr}} \subset K_{\mathrm{tr}}$ which is crucial in the theory! This problem can be overcome (following [47]) by using, instead of (6.3), a weaker dissipativity assumption of the form

$$\left\|u(t)\right\|_E \leqslant C_u \mathrm{e}^{-\alpha t} + C_F \quad \text{or} \quad \|T_t u\|_{\Phi_b} \leqslant C_u \mathrm{e}^{-\alpha t} + C_F, \quad t \geqslant 0, \tag{6.12}$$

where the positive constants α and C_F are the same as in (6.3), except that C_u is now some constant depending on u (without specifying any relation with $u(0)$). Such dissipative inequalities are, obviously, invariant with respect to time shifts and the action of T_t on K_{tr} is recovered.

(b) In order to prove the existence of the global attractor, one usually uses a *compact* absorbing/attracting set $B_{\mathrm{tr}} \subset K_{\mathrm{tr}}$. The semi-compactness is usually not a problem, since the weak and weak-$*$ topologies are used, and immediately follows from energy estimates. The fact that the limit points of B_{tr} solve the equations is also not an essential problem, since, with a proper choice of the topology of Φ_{loc}, it can usually be done as in the proof of existence of a weak solution (which should be done before proving the existence of attractors!). However, since K_{tr} does not contain all the solutions of the problem, these limit points may not belong to K_{tr} and the existence of a compact absorbing set may be lost in such a procedure. For instance, without a special care, the limits of solutions which are all obtained by Galerkin approximations may not satisfy this property. Analogously, concerning the dissipative inequalities (6.12), if one defines an absorbing set in the natural way, namely, $B_{\mathrm{tr}} := \{u \in K_{\mathrm{tr}},\ \|u\|_{\Phi_b} \leqslant R\}$, then it may very well be *not closed*, since an estimate of the form (6.12) may be lost under the limit procedure. Thus, the closure of the absorbing/attracting set indeed requires an additional attention. These considerations show that the use of the space Φ_b to define the class of bounded sets is *not sufficient* and more general abstract definitions of "bounded" sets should be used instead, see Definition 2.15. In particular, for dissipative inequalities of the form (6.12), it is sufficient to define the class of bounded sets in the following natural way:

$$B \subset K_{\mathrm{tr}} \text{ is "bounded" if and only if } C_u \leqslant C_B < +\infty,\ u \in B. \tag{6.13}$$

In other words, B is "bounded" if there exists a uniform constant C_B such that (6.12) holds with C_u replaced by C_B, for every $u \in B$. Then the existence of a "bounded" absorbing set is an immediate consequence of (6.12) and such estimates are preserved under the limit procedure, see [49] for details. This problem, when K_{tr} only consists of solutions obtained by some (e.g., Galerkin) approximation scheme can also be solved in a similar way, see [240] and the examples below.

EXAMPLE 6.3. Here, we briefly consider the application of the trajectory approach to the three-dimensional Navier–Stokes equations in a bounded domain Ω (see [47,49], and [211] for more detailed expositions),

$$\begin{cases} \partial_t u + (u, \nabla_x)u = \nu\Delta_x u - \nabla_x p + g, \quad \nu > 0, \\ \operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0. \end{cases} \tag{6.14}$$

Let, as usual, H and H_1 be the closures of the smooth divergent free vector fields in Ω which vanish on the boundary in the metrics of $L^2(\Omega)^3$ and $W^{1,2}(\Omega)^3$, respectively. Then, as is well-known (see, e.g., [49,143], and [217]), for every $u_0 \in H$, the Navier–Stokes problem possesses at least one global weak energy solution

$$u \in \Phi_b := L^\infty(\mathbb{R}_+, H) \cap L^2_b(\mathbb{R}_+, H_1)$$

which satisfies, in addition, an *energy inequality* in the following differential form:

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_H^2 + \nu\|\nabla_x u(t)\|_H^2 \leqslant (u, g)_H. \tag{6.15}$$

To be more precise, this inequality should be understood in the sense of distributions, i.e.,

$$\begin{aligned} &-\frac{1}{2}\int_0^{+\infty} \|u(t)\|_H^2 \cdot \phi'(t)\,\mathrm{d}t + \nu\int_0^{+\infty} \|\nabla_x u(t)\|^2 \cdot \phi(t)\,\mathrm{d}t \\ &\quad\leqslant \int_0^{+\infty} \phi(t) \cdot \big(g, u(t)\big)_H\,\mathrm{d}t \end{aligned} \tag{6.16}$$

holds for every $\phi \in C_0^\infty(\mathbb{R}_+)$ such that $\phi(t) \geqslant 0$. In particular, this energy inequality implies that, for almost every $t, \tau \in \mathbb{R}_+$, $t \geqslant \tau$, the following dissipative estimate holds:

$$\begin{aligned}
&\|u(t)\|_H^2 + \nu \int_\tau^t \mathrm{e}^{-\alpha(s-\tau)} \|\nabla_x u(s)\|_H^2 \,\mathrm{d}s \\
&\quad \leqslant \|u(\tau)\|_H^2 \mathrm{e}^{-\alpha(t-\tau)} + C\|g\|_{L^2}^2,
\end{aligned} \tag{6.17}$$

for some positive constants C and α which only depend on ν and Ω. Since the existence or the nonexistence of other weak solutions $u \in \Phi_b$ (which do not satisfy the energy inequality and, thus, are nondissipative) is not known yet, it is natural to define the trajectory phase space K_{tr} as the set of all weak solutions satisfying this energy inequality,

$$K_{\mathrm{tr}} := \{u \in \Phi_b,\ u \text{ solves (6.14) and satisfies (6.15)}\}. \tag{6.18}$$

Indeed, since the energy inequality is shift-invariant, the phase space K_{tr} thus defined is also invariant with respect to the shifts semigroup T_t, $T_t : K_{\mathrm{tr}} \to K_{\mathrm{tr}}$, and, therefore, the trajectory dynamical system (T_t, K_{tr}) is well-defined. Furthermore, the dissipative estimate (6.17) implies that

$$\|T_t u\|_{\Phi_b}^2 \leqslant C\|u\|_{L^\infty(\mathbb{R}_+,H)}^2 \mathrm{e}^{-\alpha t} + C\|g\|_{L^2}^2, \quad t \geqslant 0, \tag{6.19}$$

for every $u \in K_{\mathrm{tr}}$ and for positive constants C and α which are independent of u and t. Therefore, the R-ball in Φ_b, intersected with K_{tr},

$$B_R := \{u \in K_{\mathrm{tr}},\ \|u\|_{\Phi_b} \leqslant R\},$$

is a Φ_b-absorbing set for the trajectory semigroup T_t on K_{tr} if R is large enough. Thus, there only remains to fix the topology of Φ_{loc} on K_{tr} in such a way that this ball is compact. To be more precise, we set

$$\Phi_{\mathrm{loc}} := L_{\mathrm{loc}}^{\infty,w^*}(\mathbb{R}_+, H) \cap L_{\mathrm{loc}}^{2,w}(\mathbb{R}_+, H_1),$$

where w and w^* denote the weak and weak-$*$ topologies, respectively. We recall that a sequence u_n converges to u in the space Φ_{loc} if and only if, for every $T > 0$, the sequence $u_n|_{[0,T]}$ converges to $u|_{[0,T]}$ weakly in $L^2([0,T], H_1)$ and weakly-$*$ in $L^\infty([0,T], H)$, see [49] for details. Then every bounded subset of Φ_b is precompact and metrizable in Φ_{loc}, see, e.g., [205], and we only need to verify that B is closed in K_{tr} in the Φ_{loc}-topology. As already mentioned in Remark 6.2, this can be done as in the justification of the passage to the limit $N \to +\infty$ in Galerkin approximations for weak energy solutions u, see [49] for details. Thus, the assumptions of the $(\Phi_b, \Phi_{\mathrm{loc}})$-attractor's existence theorem (see Theorem 2.20 and [22]) are verified and, consequently, the trajectory dynamical system (T_t, K_{tr}) possesses the global attractor $\mathcal{A}_{\mathrm{tr}}$ which attracts the bounded subsets of Φ_b in the topology of Φ_{loc}. As usual, the trajectory attractor $\mathcal{A}_{\mathrm{tr}}$ is generated by all bounded complete solutions of the Navier–Stokes system (of course, satisfying the energy inequality) via (6.10) and its restriction at $t = 0$ gives the global attractor of the associated semigroup of multi-valued maps (i.e., (6.11) holds), see [47] and [49] for details. To conclude with the Navier–Stokes equations, we mention that, although the above trajectory attractor attracts the bounded subsets of Φ_b in the *weak* topology of Φ_{loc} only, this weak convergence implies the *strong* convergence in slightly larger spaces (due to compactness arguments). In particular, for every bounded subset $B \subset K_{\mathrm{tr}}$, every $T \in \mathbb{R}_+$, and every $\delta > 0$, we have

$$\lim_{t \to +\infty} \mathrm{dist}_{\mathcal{C}([0,T],H_{-\delta}) \cap L^2([0,T],H_{1-\delta})}(T_t B|_{[0,T]}, \mathcal{A}_{\mathrm{tr}}|_{[0,T]}) = 0, \tag{6.20}$$

where H_s is a scale of Hilbert spaces associated with the Stokes operator in Ω, see [49]. We also recall that, although nothing is known, in general, concerning the additional regularity and/or the structure of the attractor $\mathcal{A}_{\mathrm{tr}}$ of the three-dimensional Navier–Stokes problem, there are several special cases for which such results can be proven. In particular, for thin domains $\Omega = [-h, h] \times \Omega_0$, where Ω_0 is a bounded two-dimensional domain and h is a small parameter (depending on ν), endowed with Dirichlet boundary conditions on $\partial\Omega_0$ and Neumann boundary conditions on $\{-h, h\} \times \Omega$, the smoothness $\mathcal{A}_{\mathrm{tr}} \subset \mathcal{C}_b(\mathbb{R}_+, H_1)$, which is enough for the uniqueness on the attractor, follows from [196]. Another nontrivial example is a three-dimensional Navier–Stokes system with an additional rotation term,

$$\partial_t u + (u, \nabla_x)u + \omega \times u = \nu \Delta_x u - \nabla_x p + g$$

in the domain $\Omega = [0, T_1] \times [0, T_2] \times [0, T_3]$ with periodic boundary conditions. As proven in [17], if ω is large enough and the periods T_i, $i = 1, 2, 3$, satisfy some nonresonance conditions, the attractor $\mathcal{A}_{\mathrm{tr}}$ is also smooth and the uniqueness holds on the attractor.

EXAMPLE 6.4. As a next example, we consider "the second" (after the three-dimensional Navier–Stokes equations) classical ill-posed problem, namely, a damped wave equation with a supercritical nonlinearity,

$$\begin{cases} \varepsilon \partial_t^2 u + \gamma \partial_t u - \Delta_x u + f(u) = g, \\ u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_0', \end{cases} \tag{6.21}$$

in a bounded smooth domain Ω of $\mathbb{R}^3$. Here, ε and γ are positive parameters, $g \in L^2(\Omega)$ corresponds to given external forces and the nonlinearity $f \in \mathcal{C}^2(\mathbb{R})$ is assumed to satisfy the following dissipative and growth assumptions:

$$1. \quad f'(u) \geqslant -C + C_1|u|^{p-1}, \qquad 2. \quad \bigl|f''(u)\bigr| \leqslant C\bigl(1 + |u|^{p-2}\bigr), \tag{6.22}$$

$C, C_1 > 0, u \in \mathbb{R}, p \geqslant 0$. It is well-known, see, e.g., [22], that, in the subcritical $p < 3$ and critical $p = 3$ cases, problem (6.21) is well-posed in the energy phase space $W_0^{1,2}(\Omega) \times L^2(\Omega)$ and possesses the global attractor $\mathcal{A}$, see also [119,136,217], and the references therein. In contrast to this, in the supercritical case $p > 3$, the well-posedness of (6.21) in a proper phase space is still an open problem (the limit exponent $p = 3$ can be shifted till $p = 5$ when $\Omega = \mathbb{R}^3$, see [86], but, to the best of our knowledge, this result is not known for bounded domains). On the other hand, it is well-known (see, e.g., [49] and [143]) that, for every $(u_0, u_0') \in E := W_0^{1,2}(\Omega) \cap (L^{p+1}(\Omega) \times L^2(\Omega))$, equation (6.21) possesses at least one global weak energy solution

$$u \in \Phi_b := L^\infty\bigl(\mathbb{R}_+, W_0^{1,2}(\Omega) \cap L^{p+1}(\Omega)\bigr) \times W^{1,\infty}\bigl(\mathbb{R}_+, L^2(\Omega)\bigr)$$

which satisfies the dissipative estimate

$$\bigl\|\bigl(u(t), \partial_t u(t)\bigr)\bigr\|_E \leqslant Q\bigl(\bigl\|\bigl(u(0), \partial_t u(0)\bigr)\bigr\|_E\bigr)e^{-\alpha t} + Q\bigl(\|g\|_{L^2}\bigr), \quad t \geqslant 0, \tag{6.23}$$

where the positive constant α and monotonic function Q are independent of u. As shown in [47] and [49], this is sufficient to verify the existence of the trajectory attractor. However, we are now exactly in the situation mentioned in Remark 6.2. Indeed, in contrast to the three-dimensional Navier–Stokes equations, here, the dissipativity estimate cannot be formulated as a differential inequality similar to (6.15) and the dissipative estimate (6.23) is the best known one. As already mentioned, this estimate is *not* shift-invariant and, therefore, *cannot* be directly used to define the space K_{tr} (this is related to the fact that, when constructing the solution $u(t)$, e.g., by Galerkin approximations, we can easily guarantee that $u_n(0)$ converges *strongly* to $u(0)$ in E, but, for $u_n(\tau)$ with $\tau > 0$, we only have a weak convergence, which does not yield the convergence of the norms and, consequently, $t = 0$ cannot be replaced by $t = \tau$ in (6.23)). Thus, following the general scheme described in Remark 6.2, we consider a dissipative estimate in a weaker, but shift-invariant form,

$$\|T_t u\|_{\Phi_b} \leqslant C_u \mathrm{e}^{-\alpha t} + C_*, \quad C_* = Q\big(\|g\|_{L^2}\big),\ \alpha > 0,\ t \geqslant 0, \tag{6.24}$$

where C_u is a constant which depends on u, and define the trajectory phase space K_{tr} by using this dissipative estimate,

$$K_{\mathrm{tr}} := \big\{u \in \Phi_b,\ u \text{ solves (6.21) and satisfies (6.24)}\big\}.$$

Thus, the trajectory dynamical system (T_t, K_{tr}) is well-defined. Furthermore, following the general scheme, we also define a class of "bounded" sets via (6.13). Then the existence of a "bounded" absorbing set, e.g., of the form $B := \{u \in K_{\mathrm{tr}},\ C_u \leqslant 1\}$, immediately follows from the dissipative estimate (6.24) and the definition of "bounded" sets. So, there only remains to fix a topology on K_{tr} in such a way that the absorbing set B is compact. This can be done by using the local weak-$*$ topology on Φ_b, exactly as in the case of the three-dimensional Navier–Stokes equations, namely,

$$\Phi_{\mathrm{loc}} := L^{\infty,w^*}_{\mathrm{loc}}\big(\mathbb{R}_+, W^{1,2}_0(\Omega) \cap L^{p+1}(\Omega)\big) \times L^{\infty,w^*}_{\mathrm{loc}}\big(\mathbb{R}_+, L^2(\Omega)\big). \tag{6.25}$$

Then B is precompact and metrizable in Φ_{loc}, since it is bounded in Φ_b, see [205], and the fact that it is closed in K_{tr} can be verified in a standard way, see [47] and [49] for details. Thus, the assumptions of the attractor's existence theorem (see Theorem 2.20) are verified and the semigroup (T_t, K_{tr}) possesses the global attractor $\mathcal{A}_{\mathrm{tr}}$ (i.e., the trajectory attractor) which attracts the "bounded" (in the sense of (6.13)) subsets of K_{tr} in the topology of Φ_{loc}. Again, the trajectory attractor $\mathcal{A}_{\mathrm{tr}}$ is generated by all bounded complete solutions (satisfying $\|u\|_{\Phi_b} \leqslant C_*$) via (6.10) and its restriction at $t = 0$ gives the global attractor of the associated semigroup of multi-valued maps constructed in [19] (i.e., (6.11) holds), see [47] and [49] for details.

EXAMPLE 6.5. In this example, we consider, following [240], an alternative way to construct a trajectory attractor for the damped wave equation (6.21) which a priori contains a "smaller number" of possible pathological solutions and, as a consequence, some reasonable results concerning its structure are available. To this end, we first recall a construction of Galerkin approximations for (6.21). Let $\{e_k\}_{k=1}^{+\infty}$ be an orthonormal basis in $L^2(\Omega)$ (say, generated by the eigenvectors of the Laplacian with Dirichlet boundary conditions) and denote by P_N the orthoprojector onto the first N vectors of this basis. Then the N-th Galerkin approximation for (6.21) reads

$$\epsilon\partial_t^2 u_N + \gamma\partial_t u_N - \Delta_x u_N + P_N f(u_N) = P_N g, \quad u_N \in P_N L^2(\Omega). \tag{6.26}$$

Actually, the weak energy solutions mentioned in the previous example are usually constructed by solving the Galerkin ODEs (6.26) and by then passing to the limit $N \to +\infty$ in a proper sense, namely,

$$u := \Phi_{\mathrm{loc}} - \lim_{k\to+\infty} u_{N_k}, \tag{6.27}$$

where the space Φ_{loc} is the same as in the previous example, see (6.25). The main idea is now to restrict ourselves to the solutions which can be obtained via (6.27) only and to define the trajectory phase space K_{tr} as follows:

$$K_{\mathrm{tr}} := \big\{u \in \Phi_b,\ u \text{ solves (6.21) and is obtained via (6.27)}\big\}. \tag{6.28}$$

The main problem here is that the weak limit of solutions which can all be obtained by the above Galerkin approximations may a priori not satisfy this property, so that the usual bounded subsets of Φ_b may not be closed in K_{tr}. In order to overcome this difficulty and to define the proper class of "bounded" sets, we need to introduce the following functional on K_{tr}:

$$M(u) := \inf\Big\{\liminf_{k\to+\infty}\big\|\big(u_{N_k}(0), \partial_t u_{N_k}(0)\big)\big\|_E,\ u = \Phi_{\mathrm{loc}} - \lim_{k\to+\infty} u_{N_k}\Big\}, \tag{6.29}$$

where the infimum is taken over all sequences u_{N_k} of Galerkin solutions which converge weakly to a given solution u. We now define the class of M-bounded sets of K_{tr} as the sets on which the functional M is uniformly bounded. Then, as shown in [240], the trajectory dynamical system (T_t, K_{tr}) possesses an M-bounded absorbing set and the weak limit of a sequence u_n belonging to any M-bounded set belongs to K_{tr} (i.e., it can be obtained by the above Galerkin approximations). Thus, according to the abstract attractor's existence theorem, the trajectory dynamical system (T_t, K_{tr}) possesses the global attractor $\mathcal{A}_{\mathrm{tr}}^{\mathrm{Gal}}$ which attracts all M-bounded sets in the topology of Φ_{loc}. It is worth noting once more that, in contrast to the trajectory attractor $\mathcal{A}_{\mathrm{tr}}$ constructed above, this new attractor $\mathcal{A}_{\mathrm{tr}}^{\mathrm{Gal}} \subset \mathcal{A}_{\mathrm{tr}}$ possesses several good properties which are not available for $\mathcal{A}_{\mathrm{tr}}$ and, in particular,

(1) it is connected in Φ_{loc} (since the simplest M-bounded sets $B_R := \{u \in K_{\mathrm{tr}}, M(u) \leqslant R\}$ are connected; this follows from the fact that they can be approximated, in Φ_{loc}, by analogous sets for the Galerkin approximations which are clearly connected);

(2) every complete trajectory belonging to this attractor tends in a proper sense to the set of equilibria as time goes to plus or minus infinity;

(3) every complete trajectory $u(t)$ on $\mathcal{A}_{\mathrm{tr}}^{\mathrm{Gal}}$ is *smooth* for sufficiently small t, i.e., there exists $T = T_u$ such that $u(t) \in W^{2,2}(\Omega) \subset \mathcal{C}(\Omega)$ for $t \leqslant T_u$, and every solution is unique (in the above class) as long as it is smooth. So, the only way for a singular solution to appear on the attractor $\mathcal{A}_{\mathrm{tr}}^{\mathrm{Gal}}$ is by a blow up of a strong solution, see [240] for details;

(4) as proven in [240], the whole attractor $\mathcal{A}_{\mathrm{tr}}^{\mathrm{Gal}}$ is *smooth* if the coefficient $\varepsilon > 0$ is small enough, $\mathcal{A}_{\mathrm{tr}}^{\mathrm{Gal}} \subset \mathcal{C}_b(\mathbb{R}_+, W^{2,2}(\Omega))$.

A drawback of such a construction is that $\mathcal{A}_{\mathrm{tr}}^{\mathrm{Gal}}$ depends on the concrete approximation scheme (e.g., different Galerkin bases may lead to different attractors). However, the attractor $\mathcal{A}_{\mathrm{tr}}$ constructed in the previous example also depends a priori on the artificial constant C_* in the dissipative estimate (6.24).

REMARK 6.6. To conclude, we note that the above trajectory approach has been successfully applied not only to ill-posed *evolutionary* problems, but also to elliptic boundary value problems in unbounded domains (for which the nonuniqueness does not appear as a consequence of poorly understood analytical properties of the equations under study, but is related to the classical ill-posedness of the Cauchy problem for elliptic equations), see [166,222], and [223] for trajectory attractors for elliptic problems in cylindrical domains and [26] and [232] for more general classes of unbounded domains. Finally, we also note that most of the results considered in this subsection can naturally be extended to nonautonomous ill-posed problems as well, see [49] for details.

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CHAPTER 4

The Cahn–Hilliard Equation

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Contents

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1. Introduction

The present chapter is devoted to the Cahn–Hilliard equation [16,15]:

$$u_t = \nabla \cdot M(u)\nabla\big[f(u) - \epsilon^2 \Delta u\big], \quad (x,t) \in \Omega \times \mathbb{R}^+, \tag{1}$$

$$n \cdot \nabla u = n \cdot M(u)\nabla\big[f(u) - \epsilon^2 \Delta u\big] = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}^+, \tag{2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{3}$$

Here $0 < \epsilon^2 \ll 1$ is a "coefficient of gradient energy", $M = M(u)$ is a "mobility" coefficient, and $f = f(u)$ is a "homogeneous free energy". The equation was initially developed to describe phase separation is a two component system, with $u = u(x,t)$ representing the *concentration* of one of the two components. Typically, the domain Ω is assumed to be a bounded domain with a "sufficiently smooth" boundary, $\partial\Omega$, with n in (2) representing the unit exterior normal to $\partial\Omega$. It is reasonable to consider evolution for times $t > 0$, or on some finite time interval, $0 < t < T < \infty$.

Concentration should be understood as referring either to *volume fraction* or to *mass fraction*, depending on the physical system under investigation. By *volume fraction* we mean the volume fraction per unit volume of say component "A", in a system containing two components which we shall denote by "A" and "B". The meaning of *mass fraction* is analogous. Thus the Cahn–Hilliard equation constitutes a continuous, as opposed to a discrete or lattice description, of the material undergoing phase separation. Such a description is appropriate under many but not all circumstances. Note that the definition of $u(x,t)$ implies that $u(x,t)$ should satisfy $0 \leqslant u(x,t) \leqslant 1$. Moreover, if $u(x,t)$, the concentration of component A, is known, then the concentration of the second component is given by $1-u$ and is hence also known; thus the evolution of the composition of the two component system is being predicted by a single scalar Cahn–Hilliard equation.

In the context of the Cahn–Hilliard equation, the two components could refer, for example, to a system with two metallic components, or two polymer components, or say, two glassy components. Frequently in materials science literature, concentration is given in terms of *mole fraction* or equivalently *number fraction*, rather than in terms of volume fraction or mass fraction. A mole refers to 6.02252×10^{23} molecules (Avogadro's number of molecules), and the mole fraction of component A refers to the number of A molecules per mole of the two component system, locally evaluated. Mole fraction of number fraction are equivalent to volume fraction if the molar volume (the volume occupied by one mole) is independent of composition, which is rarely strictly correct [15]. For example, in a two component polymer systems when many of the polymers are long, the configuration of the polymers, i.e. whether they are "stretched out" or "rolled up", typically depends on composition, which in turn influences the molar volume. Notice also that often temperature does not appear explicitly in the Cahn–Hilliard equation, since the model is based on the assumption that the temperature is constant; such an assumption requires careful temperature control and is also rarely strictly fulfilled in reality. The model also assumes *isotropy* of the system, which can also only be approximately correct for metallic systems [5,77,87], for which the equations were designed, which have an inherent crystalline structure unless they are in a liquid phase. Nevertheless, the Cahn–Hilliard equation has been seen to contain many of the dominant *paradigms* for phase separation dynamics, and as such, has played, and continues to play, an important role in understanding the evolution of phase separation.

Why does the Cahn–Hilliard equation appear in so many different contexts, and what behavior is predicted by the Cahn–Hilliard equation which is common to all these systems? Off-hand, what is being modeled with the Cahn–Hilliard equation is phase separation, in other words, the segregation of the system into spatial domains predominated by one of the components, in the presence of a mass constraint, and what one wishes to accomplish here is to model the dynamics in a sufficiently accurate fashion so that many of the various features of the resultant pattern formation evolution that one sees in nature during phase separation can be explained and predicted. In materials science this pattern formation is referred to as the *microstructure* of the material, and the microstructure is highly influential in determining many of the properties of the material, such as strength, hardness, and conductivity. The Cahn–Hilliard model is rather broad ranged in its evolutionary scope; it can serve as a good model for many systems during early times, it can give a reasonable qualitative description for these systems during intermediary times, and it can serve as a good model for even more systems at late times. Often, the late time evolution is so slow that the pattern formation or microstructure becomes effectively frozen into the system over time scales of interest, and hence it is the long time behavior of the system which is seen in practice.

The Cahn–Hilliard equation also appears in modeling many other phenomena. These include the evolution of two components of intergalactic material [80], the dynamics of two populations [19], the biomathematical modeling of a bacterial film [46], and certain thin film problems [69,79]. We apologies to the reader that most of the details pertaining to the modeling of these phenomena are outside the scope of the present survey. Nevertheless, we invite the interested reader to have a look at the forthcoming book by the author of this survey, entitled *From Backwards Diffusion to Surface Diffusion: the Cahn–Hilliard Equation* [65], where these and other details will be treated in greater depth.

We hope that this survey will clarify for the reader the notions of backwards diffusion and surface diffusion and their connection with the Cahn–Hilliard equation, and will convey something of the nature of the physical phenomena which accompany phase separation and how the Cahn–Hilliard equation manages to capture these features.

2. Backwards diffusion and regularization

Let us consider a simple variant of the Cahn–Hilliard equation in which $f(u) = -u + u^3$ and $M(u) = M_0$, where $M_0 > 0$ is constant. Let $t \in (0, T)$, $0 < T < \infty$, and $\Omega = (0, L)$. In most applications, $\Omega \in \mathbb{R}^n$ with $n = 2$ or $n = 3$ is most physically relevant. However, let us focus temporarily on the $n = 1$ case for simplicity. Thus,

$$\begin{cases} u_t = M_0\big[-u + u^3 - \epsilon^2 u_{xx}\big]_{xx}, & (x,t) \in \Omega_T, \\ u_x = M_0\big[-u + u^3 - \epsilon^2 u_{xx}\big]_x = 0, & (x,t) \in \partial\Omega_T, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{4}$$

where $\Omega_T = (0,T)\times\Omega$ and $\partial\Omega_T = \{0,L\}\times(0,T)$. Note that $u(x,t)=\bar{u}$ constitutes a steady state of (4), where $\bar{u}$ is an arbitrary constant; however if $u(x,t)$ is to represent concentration, clearly one must assume that $0\leqslant\bar{u}\leqslant 1$.

Let us now suppose that $u_0(x)=\bar{u}+\tilde{u}_0(x)$, where $\tilde{u}_0(x)$ represents a small perturbation from spatial uniformity. Setting $u(x,t)=\bar{u}+\tilde{u}(x,t)$, (4) yields that

$$\begin{cases} \tilde{u}_t = M_0\big[-\tilde{u}+[\bar{u}+\tilde{u}]^3-\epsilon^2\tilde{u}_{xx}\big]_{xx}, & (x,t)\in\Omega_T,\\ \tilde{u}_x = M_0\big[-\tilde{u}+[\bar{u}+\tilde{u}]^3-\epsilon^2\tilde{u}_{xx}\big]_{x}=0, & (x,t)\in\partial\Omega_T,\\ \tilde{u}(x,0)=\tilde{u}_0(x):=u_0(x)-\bar{u}, & x\in\Omega. \end{cases} \tag{5}$$

Assuming (5) to be well-posed and $\tilde{u}(x,t)$ to be small, we neglect terms which are nonlinear in $\tilde{u}(x,t)$ and obtain to leading order the linearized problem

$$\begin{cases} \tilde{u}_t = M_0\big[-\big(1-3\bar{u}^2\big)\tilde{u}-\epsilon^2\tilde{u}_{xx}\big]_{xx}, & (x,t)\in\Omega_T,\\ \tilde{u}_x = M_0\big[-\big(1-3\bar{u}^2\big)\tilde{u}-\epsilon^2\tilde{u}_{xx}\big]_{x}=0, & (x,t)\in\partial\Omega_T,\\ \tilde{u}(x,0)=\tilde{u}_0(x), & x\in\Omega. \end{cases} \tag{6}$$

We recall that we have assumed earlier that $0<\epsilon^2\ll 1$. Suppose that we optimistically neglect terms in the system (6) which contain a factor of ϵ^2. This yields

$$\begin{cases} \tilde{u}_t = -M_0(1-3\bar{u}^2)\tilde{u}_{xx}, & (x,t)\in\Omega_T,\\ \tilde{u}_x = -M_0(1-3\bar{u}^2)\tilde{u}_x=0, & (x,t)\in\partial\Omega_T,\\ \tilde{u}(x,0)=\tilde{u}_0(x), & x\in\Omega. \end{cases} \tag{7}$$

If we stop and consider for a moment (7), we can see that for $3\bar{u}^2-1>0$, it is equivalent to the classical diffusion equation with Neumann boundary conditions

$$\begin{cases} \tilde{u}_t = D\tilde{u}_{xx}, & (x,t)\in\Omega_T,\\ \tilde{u}_x = 0, & (x,t)\in\partial\Omega_T,\\ \tilde{u}(x,0)=\tilde{u}_0(x), & x\in\Omega, \end{cases} \tag{8}$$

whose solutions decay to $\frac{1}{L}\int_0^L\tilde{u}_0(x)\,\mathrm{d}x$. For $3\bar{u}^2-1>0$, it is equivalent to

$$\begin{cases} \tilde{u}_t = -D\tilde{u}_{xx}, & (x,t)\in\Omega_T,\\ \tilde{u}_x = 0, & (x,t)\in\partial\Omega_T,\\ \tilde{u}(x,0)=\tilde{u}_0(x), & x\in\Omega. \end{cases} \tag{9}$$

Now (9) is precisely the *backwards diffusion equation*, which can be obtained from the classical diffusion equation by redefining time $t\to -t$ so that time will "run backwards." The problem (9) is notoriously ill-posed as can be verified by noting that for $\tilde{u}_0\in L^2(\Omega)$, it possesses the formal separation of variables solution

$$\tilde{u}(x,t)=\frac{A_0}{2}+\sum_{n=1}^{\infty}A_n\mathrm{e}^{\frac{n^2\pi^2}{L^2}t}\cos(n\pi x/L), \tag{10}$$

where the coefficients A_i, $i=0,1,2,\dots$, correspond to the Fourier coefficients of the initial conditions,

$$\tilde{u}(x,0)=\tilde{u}_0(x)=\frac{A_0}{2}+\sum_{n=1}^{\infty}A_n\cos(n\pi x/L). \tag{11}$$

Its amplitude grows without bound

$$\left\|\tilde{u}(x,t)\right\|^2_{L^2[0,L]} = \frac{A_0^2}{2} + \sum_{n=1}^{\infty} A_n^2 \mathrm{e}^{\frac{2n^2\pi^2}{L^2}t}; \tag{12}$$

even for initial data based on a single mode, $\tilde{u}_0(x) = A_k \cos(k\pi x/L)$,

$$\left\|\tilde{u}(x,t)\right\|^2_{L^2[0,L]} = A_k^2 \mathrm{e}^{\frac{2k^2\pi^2}{L^2}t}. \tag{13}$$

This clearly makes little physical sense in terms of a model for phase separation, although in other contexts, such as image processing [17], it has been successfully implemented. In particular, we see that the solution, $u(x,t) = \bar{u} + \tilde{u}(x,t)$ does not remain bounded within the interval [0, 1] over time.

Thus both problems, (8) and (9), make little physical sense as models for phase separation. Hence, the higher order terms proportional to ϵ^2 are truly necessary in the physical model, and cannot be made light of easily. Seemingly this would provide a compelling reason to include such regularizing terms, but in fact regularizing terms were already added much before the dynamics for phase separation came under consideration, when equilibrium considerations lead to the search for a free energy with "phase separated" steady states possessing certain regularity and uniqueness properties. This reflects the independent scientific contribution of Gibbs (1893) [35] and van der Waals (1973) [81].

The reader should have no difficulty in ascertaining that (6), where the regularizing terms have been included, can be formulated as a well-posed problem, and it is fairly straightforward to verify that (5) and (4) can be carefully formulated as well-posed problems as well. However, before discussing existence, uniqueness, and well-posedness, we first briefly consider what are the physical phenomena one should like to model with the Cahn–Hilliard equation, and which are the most important variants of the Cahn–Hilliard equation which one should like to consider.

3. The Cahn–Hilliard equation and phase separation

We now outline what are the physical features and phenomena which one should like to be described by the Cahn–Hilliard equation. The process of phase separation in two component systems is accompanied by *pattern formation* and evolution. A typical scenario we should like to model is that of *quick quenching*. Let $\Omega \subset \mathbb{R}^3$ initially contain two components which are roughly uniformly distributed, so that $u(x,0) \approx u_0(x) \equiv \bar{u}$. We should suppose that $\bar{u} \in [0,1]$ if $u(x,t)$ is to represent concentration. If there is no flux of material into or out of Ω, then the total amount of each component should be conserved,

$$\frac{1}{|\Omega|}\int_{\Omega} u(x,t)\,\mathrm{d}x = \bar{u}, \quad 0 \leqslant t \leqslant T. \tag{14}$$

Let the initial temperature be given by Θ_0, and let the temperature of the system be now rapidly lowered (*quick quenched*) to some new temperature, $\Theta_1 \ll \Theta_0$. In two component metallic alloy systems, the average thermal conductivity is high, and the temperature of the system will equilibrate rapidly to the new temperature. With this in mind, the assumption

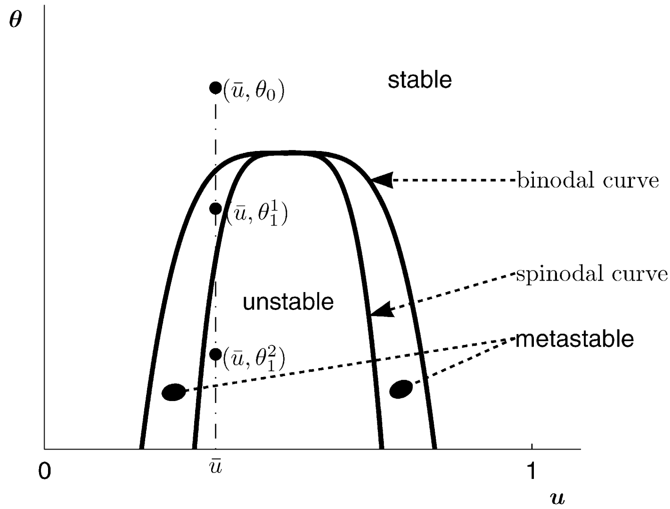

Fig. 1. A typical phase diagram. Here $(\bar{u}, \Theta_0)$ lies above the binodal curve in the stable region, $(\bar{u}, \Theta_1)$ lies in the "metastable region" which lies between the spinodal curve and the binodal curve, and $(\bar{u}, \Theta_2)$ lies below the spinodal curve.

is made that the temperature equilibrates immediately to lower temperature, Θ_1. The equilibration process can be modeled by coupling the Cahn–Hilliard equation with an energy balance equation. This augmented system is known as a *conserved phase field model* [12].

The dynamics which appears in the system Ω in the wake of quick quenching can be roughly explained with the help of *phase diagrams* as developed by Gibbs [35] within the framework of classical thermodynamics. In the present context, this implies that whether or not phase separation is predicted, as well as the nature of the phase separation which can be expected, are determined by the location of $(\bar{u}, \Theta_0)$ and $(\bar{u}, \Theta_1)$ within the phase diagram. While phase diagrams of varying levels of complexity can occur, a simplest nontrivial level of phase diagram which can describe phase separation is portrayed in Figure 1.

In the phase diagram, there are two curves which should be noted. One is an upper curve, known as the *binodal* or the *coexistence curve*, and other is a lower curve, known as the *spinodal*. The two curves intersect at point, $(\bar{u}_{\text{crit}}, \Theta_{\text{crit}})$, known as the *critical point*. If both $(\bar{u}, \Theta_0)$ and $(\bar{u}, \Theta_1)$ lie above the binodal, no phase separation is expected to occur and the system is expected to persist in its initially uniform state, $u(x, t) \equiv \bar{u}$. Hence the region above the binodal is known as the *stable* or *one-phase region*. For phase separation to occur, the initial state $(\bar{u}, \Theta_0)$ should lie above both the binodal and spinodal, and the final state $(\bar{u}, \Theta_1)$ should lie somewhere below the binodal, either above or below the spinodal.

If $(\bar{u}, \Theta_1)$ lies below the spinodal curve and $\bar{u} \neq \bar{u}_{\text{crit}}$, then phase separation is predicted to onset via *spinodal decomposition*. During the onset of spinodal decomposition, the system is distinguished by a certain "fogginess" reflecting the simultaneous growth of perturbations with many different wavelengths. Spinodal decomposition is fairly well described by the Cahn–Hilliard equation. If $(\bar{u}, \Theta_1)$ lies below the binodal but above the spinodal, phase separation can be expected to occur by *nucleation and growth*. During this process, phase separation occurs via the appearance or *nucleation* of localized perturbations in the uniform state $\bar{u}$ which persist and grow if they are sufficiently large. Though we mention the nucleation and growth process, it is not well modeled by the Cahn–Hilliard equation, and alternative approaches have been developed for purpose such as the Lifshitz–Slyozov theory of *Oswald ripening* [49] and its extensions [42,4].

We caution the reader that if $(\bar{u}, \Theta_0)$ or $(\bar{u}, \Theta_1)$ are too close to $(\bar{u}_{\text{crit}}, \Theta_{\text{crit}})$, then the above descriptions are inappropriate, since *critical phenomena* [23], such as critical slowing down, will accompany the phase separation. Such effects are characteristic of *second order* phase transitions, as opposed to our earlier description, which was appropriate for *first order* phase transitions. Arguably Θ_1 should be taken not too far from Θ_{crit}, otherwise inertial and higher order effects may become important; these effects would render the Cahn–Hilliard model inaccurate, and make it difficult to control the phase separation process and the resultant microstructure. What distinguishes a first order phase transition from a second or higher order phase transition is the degree of continuity or regularity of the system as the system crosses from the stable regime above the binodal into the *unstable* regime which lies below it, see e.g. [50].

Whether phase separation occurs via spinodal decomposition or via nucleation and growth, eventually the system saturates into well-defined spatial domains in which one of the two components dominates, so that $u \approx u_A$ or by $u \approx u_B$, where u_A and u_B denote the binodal or limiting miscibility gap concentrations when $\Theta = \Theta_1$. See Figure 1. The average size of these spatial domains increases over time, as larger domains grow at the expense of smaller domains. This process is called *coarsening*, and the dynamics of the system may now be characterized by the motion of the boundaries or *interfaces* between these various domains. Because of mass balance, (14), the relative volume or area of the domains where $u \approx u_A$ and $u \approx u_B$ remains unchanged, but the overall amount of “domain interface” decreases as some limiting configuration is seemingly approached. While nucleation and growth is somewhat of a weak spot for the Cahn–Hilliard theory, the Cahn–Hilliard equation can give some reasonable description of the coarsening process, even if the initial stages of the phase separations were dominated by nucleation and growth.

Let us now consider two important cases of the Cahn–Hilliard equation formulation given in (1)–(3), to which we shall refer to later repeatedly.

4. Two prototype formulations

Perhaps the easiest formulation to consider is that given in (4) which was discussed in Section 2. We shall refer to this case as the *constant mobility-quartic polynomial* case, or more briefly, the *constant mobility Cahn–Hilliard equation*, and it is summarized below.

4.1. *The constant mobility – quartic polynomial case*

Let

$$M(u) = M_0 > 0, \quad \text{where } M_0 \text{ is a constant, and} \quad f(u) = -u + u^3. \tag{15}$$

It follows from (15) that

$$f(u) = F'(u), \qquad F(u) = \frac{1}{4}\left(u^2 - 1\right)^2. \tag{16}$$

Within this framework, the Cahn–Hilliard equation is given by

$$\begin{cases} u_t = M_0 \Delta(-u + u^3 - \epsilon^2 \Delta u), & (x,t) \in \Omega \times (0,T), \\ n \cdot \nabla u = n \cdot \nabla \Delta u = 0, & (x,t) \in \partial\Omega \times (0,T), \end{cases} \tag{17}$$

in conjunction with appropriate initial conditions. The value of M_0 may be set to unity by rescaling time, but we maintain M_0 in the formulation since it is frequently maintained in the literature, [62]. Note that (17) is invariant under the transformation $u \to -u$, because u in (17) represents the difference between the two concentrations, $u \equiv u_A - u_B = 2u_A - 1$. Thus, in terms of the physical interpretation, $u(x,t)$ should assume values in the interval $[-1, 1]$.

The analysis and treatment of this case is relatively easy since (17) constitutes a fourth order semilinear parabolic equation, whose treatment is similar to that of second semilinear parabolic equations such as the reaction diffusion equation,

$$u_t = \epsilon^2 \Delta u - f(u), \tag{18}$$

which arises in a wide variety of applications, from populations genetics to tiger spots, [56,57]. Nevertheless, one of the mainstays in the treatment of second order equations, the maximum principle, does not carry over easily into the fourth order setting [53]. An existence theory can be given, for example, in terms of Galerkin approximations [78] which can also be used to construct finite element approximations that can be implemented numerically. From numerical calculations and analytical consideration, it can be seen that for a sensible choice of initial conditions, (17) gives a reasonable description of spinodal decomposition and of coarsening.

An unfortunate feature of the constant mobility Cahn–Hilliard variant (17) is that its solutions need not remain bounded between -1 and 1, even if the initial data lies in this interval. This drawback can be avoided by employing a formulation, written in terms of one of the concentrations, in which the mobility is taken to degenerate when $u = 0$ and $u = 1$, and the free energy is taken to be well behaved, as was demonstrated in one space dimension by Jingxue [44]. Such a formulation does not occur so naturally in the context of phase separation, but is does occur naturally in other contexts, such as in structure formation in biofilms [46]. In the context of phase separation, it is natural in including a degenerate mobility to also include logarithmic terms in the free energy. This seemingly less natural formulation is in fact well-based in terms of the physics; the logarithmic terms reflect entropy contributions and the vanishing of the mobilities reflects jump probability considerations [72]. We shall refer to this formulation as the *degenerate mobility-logarithmic free energy case*, or for short, the *degenerate Cahn–Hilliard equation*, and it is explained below.

4.2. *The degenerate mobility – logarithmic free energy case*

Here we assume that

$$M(u) = u(1-u) \quad \text{and} \quad f(u) = F'(u), \tag{19}$$

where

$$F(u) = \frac{\Theta}{2}\big\{u \ln u + (1-u)\ln(1-u)\big\} + \alpha u(1-u), \tag{20}$$

with $\Theta > 0$, $\alpha > 0$. In (20), Θ denotes temperature, or more accurately a scaled temperature. The resultant Cahn–Hilliard formulation is now:

$$\begin{cases} u_t = \nabla \cdot M(u)\nabla\left\{\frac{\Theta}{2}\ln\left[\frac{u}{1-u}\right] + \alpha(1-2u) - \epsilon^2\Delta u\right\}, & (x,t) \in \Omega_T, \\ n \cdot \nabla u = 0, & (x,t) \in \partial\Omega_T, \\ n \cdot M(u)\left\{\frac{\Theta}{2u(1-u)}\nabla u - 2\alpha\nabla u - \epsilon^2\nabla\Delta u\right\} = 0, & (x,t) \in \partial\Omega_T, \end{cases} \tag{21}$$

where $\Omega_T = \Omega \times (0,T)$ and $\partial\Omega_T = \partial\Omega \times (0,T)$, and the equation and boundary conditions are to be solved in conjunction with appropriate initial data, $u_0(x)$. Since $u(x,t)$ represents here the concentration of one of the two components, $u_0(x)$ and $u(x,t)$ should satisfy $0 \leqslant u_0(x), u(x,t) \leqslant 1$.

Formally, referring to (19), (21) can be written more simply as

$$\begin{cases} u_t = \frac{\Theta}{2}\Delta u - \nabla \cdot M(u)\nabla\big\{2\alpha u + \epsilon^2\Delta u\big\}, & (x,t) \in \Omega_T, \\ n \cdot \nabla u = n \cdot M(u)\nabla\Delta u = 0, & (x,t) \in \partial\Omega_T. \end{cases} \tag{22}$$

Note though that (22) is in fact only meaningful for $u \in [0,1]$, and $M(u)$ has only been defined on that interval.

The mobility in (19) is referred to as a *degenerate* mobility, since it is not strictly positive. A concentration dependent mobility was already considered by Cahn in 1961 [15], and a *degenerate mobility* similar to (19) appeared in the work by Hillert in 1956 [40,39] on a one-dimensional discretely defined precursor of the Cahn–Hilliard equation. The use of logarithmic terms in the free energy, which arises naturally due to thermodynamic entropy consideration, also appeared in the papers [40,39] as well as in the 1958 paper of Cahn and Hilliard [16]. See also the discussions in [26,41]. The problem formulated in (21) constitutes a degenerate fourth order semilinear parabolic problem. Galerkin approximations can be used to prove existence and to construct finite element schemes by first regularizing the free energy. The payoff for working with the more complicated formulation is that it yields more physical results; namely, for (21) and for $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, if $u_0 \in [0,1]$, then $u(x,t) \in [0,1]$ for $t \geqslant 0$. Details follow in the next section.

However, early on the degenerate and concentration dependent mobilities were replaced by constant mobilities and logarithmic terms in the free energy were expanded into polynomials, to simplify the analysis and to enable some qualitative understanding of the equation. In fact, very early analyzes were totally linear. Surprisingly this was not such a bad path to take since the dominant unstable modes are typically sustained longer that a straight forward linear analysis would suggest, see Section 6. It seems that nonlinear effects were first included by de Fontaine in 1967 [22], who did so in the context of early numerical studies of the Cahn–Hilliard equation.

For detailed derivations of both variants, see [65,66,34]. Physically speaking, it is more natural to first justify the degenerate Cahn–Hilliard equation with logarithmic free energy terms and then to obtain the constant mobility Cahn–Hilliard equation with a polynomial free energy by making suitable approximations.

5. Existence, uniqueness, and regularity

For the constant mobility Cahn–Hilliard equation with a polynomial free energy, a proof of existence and uniqueness was given in 1986 by Elliott and Songmu [27], which also contains a finite element Galerkin approximation scheme. To be more precise, setting

$$H_E^2(\Omega) = \{v \in H^2(\Omega) \mid n \cdot \nabla v = 0 \text{ on } \partial\Omega\},$$

where n denotes the unit exterior normal to $\partial\Omega$, and $\Omega_T = \Omega \times (0, T)$, it follows from [27] that

THEOREM 5.1. *If Ω is a bounded domain in $\mathbb{R}^n$, $n \leqslant 2$, with a smooth boundary, then for any initial data $u_0 \in H_E^2(\Omega)$ and $T > 0$, there exists a unique global solution in $H^{4,1}(\Omega_T)$.*

The proof relies on Picard iteration and on a priori estimates obtained by multiplying (17) by u, $f(u)-\epsilon^2\Delta u$, and $\Delta^2 u$. By taking more regular initial data, classical solutions may also be obtained. Of some physical interest is the estimate obtained by multiplying (17) by $f(u) - \epsilon^2\Delta u$, namely

$$\mathcal{F}(t) - \mathcal{F}(0) = -\int_{\Omega_T} \left|\nabla\{f(u) - \epsilon\Delta u\}\right|^2 \mathrm{d}x\,\mathrm{d}t, \tag{23}$$

where

$$\mathcal{F}(t) = \int_\Omega \left\{F(u) + \frac{\epsilon^2}{2}|\nabla u|^2\right\} \mathrm{d}x. \tag{24}$$

The quantity $f(u) - \epsilon^2\Delta u$ is frequently identified as the chemical potential, $\mu = \mu(x, t)$.

Of interest also is the estimate obtained by multiplying (17) by $\phi \equiv 1$, namely

$$\int_\Omega u(x, t)\,\mathrm{d}x = \int_\Omega u_0(x)\,\mathrm{d}x, \tag{25}$$

which can be understood as a statement of conservation of mass or conservation of the mean.

From (23), (25), it also follows that

$$\begin{aligned}\mathcal{F}(t) - \mathcal{F}(0) &= \int_{\Omega_T} \langle f(u) - \epsilon^2\Delta u, u_t\rangle_{H^1(\Omega),(H^1(\Omega))'}\,\mathrm{d}t \\ &= -\|u_t\|^2_{L^2(0,T;H^{-1}(\Omega))},\end{aligned} \tag{26}$$

and hence the Cahn–Hilliard equation is frequently referred to as H^{-1} gradient flow.

In [58] (see [78] for an extended explanation), using essentially the same estimates and a Galerkin approximation based on the eigenfunctions of $\mathcal{A}$, where $\mathcal{A}$ is the Laplacian with Neumann boundary conditions, it is proven that

THEOREM 5.2. *For $u_0(x) \in L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$, $n \leqslant 3$, there exists a unique solution, $u(x,t)$, to the constant mobility Cahn–Hilliard equation, and $u(x,t)$ satisfies*

$$u \in \mathcal{C}\big([0,T]; L^2(\Omega)\big) \cap L^2\big(0,T; H^1(\Omega)\big) \cap L^4\big(0,T; L^4(\Omega)\big), \quad \forall T > 0, \tag{27}$$

and $\mathcal{F}(t)$ decays along orbits. If, moreover, $u_0(x) \in H^2_E(\Omega)$, then

$$u \in \mathcal{C}\big([0,T]; H^2_E(\Omega)\big) \cap L^2\big(0,T; \mathcal{D}\big(\mathcal{A}^2\big)\big), \quad \forall T > 0. \tag{28}$$

To get (27), (17) needs only to be tested by u. The result (28) follows by testing (17) by $\Delta^2 u$. Uniqueness may be demonstrated by testing with the inverse of $\mathcal{A}$, suitably defined, acting on the difference of two solutions, see the discussion in [11].

Proofs of similar existence results for (17) can also be given within the framework of the theory of semilinear operators [61]. More specifically, taking $L^2(\Omega)$ to be the underlying space, and defining the $\mathcal{A}_1 = \epsilon^2\Delta$ with domain $\mathcal{D}(\mathcal{A}_1)$, the operator $\mathcal{A}_1$ can be shown to be a sectorial operator and existence may be proved by using a variation of constant formulation and results of Henry [38] and Miklavčič [54]. Within the framework of dynamical systems [61,78], it is easy to prove using (23) that

THEOREM 5.3. *As $t \to \infty$, $u(x,t)$ converges to its ω-limit cycle which is compact, connected, and invariant. If the steady states are isolated, then solutions converge to a steady state.*

In a sense, Theorem 5.3 has served as the starting point for many rich studies with regard to the identification of steady states [63,64,36,28,82–86], the existence and properties of attractors [58,59], the behavior of solutions in the neighborhood of attractors [3], the stability of steady states [61], and the list given here is admittedly very far from being complete.

As to existence theories for the degenerate Cahn–Hilliard equation, apparently the first result in this direction was given in 1992 by Jingxue [44]. The existence theory given there is for $\Omega = [0,1]$, and it is for the Cahn–Hilliard equation with a degenerate mobility but with a nonsingular free energy.

THEOREM 5.4. *Let $M(s)$ be a Hölder continuous function and $f'(s)$ be a continuous function,*

$$M(0) = M(1) = 0, \qquad M(s) \geqslant 0 \quad \textit{for } s \in (0,1).$$

Let $u_0 \in H^3_0(I)$, $0 \leqslant u_0(x) \leqslant 1$. Then problem (1)–(3) has a generalized solution u satisfying $0 \leqslant u(t,x) \leqslant 1$.

Here $u \in \mathcal{C}^\alpha(\bar{\Omega}_T)$, $\alpha \in (0,1)$ is said to be a *generalized solution* if

(1) $D^3 u \in L^2_{\mathrm{loc}}(G_u)$ and $\int_{G_u} M(u)(D^3u)^2 < \infty$, where

$$G_u = \big\{(x,t) \in \bar{\Omega}_T \mid M\big(u(x,t)\big) > 0\big\}.$$

(2) $u \in L^\infty(0,T;H^1(0,1))$, Du is locally Hölder continuous in G_u and $Du|_{\Gamma\cap G_u} = 0$ holds in the classical sense, where $\Gamma = \{\{(0,t),(1,t)\} \mid t \in [0,T]\}$.

(3) For any $\phi \in \mathcal{C}^1(\bar{\Omega}_T)$, the following integral equality holds*:*

$$-\int_0^1 u(x,T)\phi(x,T)\,\mathrm{d}x + \int_0^1 u_0(x)\phi(x,0)\,\mathrm{d}x + \int_{\Omega_T} u\phi_t + \int_{G_u} M(u)\big(\epsilon^2 D^3 u - Df(u)\big)D\phi = 0.$$

The definition of generalized solution given here and the method of proof are in the spirit of the analysis by Bernis and Friedman [9] of the thin film equation.

For the degenerate Cahn–Hilliard equation with logarithmic free energy, one has the following results due primarily to Elliott and Garcke [24,47,65],

THEOREM 5.5. *Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, where $\partial\Omega \in \mathcal{C}^{1,1}$ or Ω is convex. Suppose that $u_0 \in H^1(\Omega)$ and $0 \leqslant u_0 \leqslant 1$. Then there exists a pair of functions (u, J) such that*

$$\begin{aligned}
&\text{(a)} \quad u \in L^2\big(0,T;H^2(\Omega)\big) \cap L^\infty\big(0,T;H^1(\Omega)\big) \cap \mathcal{C}\big([0,T];L^2(\Omega)\big),\\
&\text{(b)} \quad u_t \in L^2\big(0,T;\big(H^1(\Omega)\big)'\big),\\
&\text{(c)} \quad u(0) = u_0 \text{ and } \nabla u\cdot n = 0 \text{ on } \partial\Omega\times(0,T),\\
&\text{(d)} \quad 0 \leqslant u \leqslant 1 \text{ a.e. in } \Omega_T := \Omega\times(0,T),\\
&\text{(e)} \quad J \in L^2\big(\Omega_T,\mathbb{R}^n\big)
\end{aligned} \tag{29}$$

which satisfies $u_t = -\nabla\cdot J$ in $L^2(0,T;(H^1(\Omega))')$, i.e.,

$$\int_0^T \big\langle \zeta(t), u_t(t)\big\rangle_{H^1,(H^1)'} = \int_{\Omega_T} J\cdot\nabla\zeta$$

for all $\zeta \in L^2(0,T;H^1(\Omega))$ and

$$J = -M(u)\nabla\cdot\big(-\epsilon^2\Delta u + f(u)\big)$$

in the following weak sense:

$$\int_{\Omega_T} J\cdot\eta = -\int_{\Omega_T}\big[\epsilon^2\Delta u\nabla\cdot\big(M(u)\eta\big) + (Mf')(u)\nabla u\cdot\eta\big]$$

for all $\eta \in L^2(0,T;H^1(\Omega,\mathbb{R}^n)) \cap L^\infty(\Omega_T,\mathbb{R}^n)$ which fulfill $\eta\cdot n = 0$ on $\partial\Omega\times(0,T)$.

(f) *Moreover, letting $\mathcal{F}(t)$ be as defined in (24), then for a.e. $t_1 < t_2$, $t_1, t_2 \in [0,T]$,*

$$\mathcal{F}(t_2) - \mathcal{F}(t_1) \geqslant -\int_{t_1}^{t_2}\int_\Omega \frac{1}{M(u)}|J|^2\,\mathrm{d}x.$$

The proof here is based on existence results for a regularized equation, where the mobility is given by $M_\epsilon(u)$ and the free energy is given by $f_\epsilon(u)$, and implementation of an additional estimate obtained by testing the equation with $\Phi_\epsilon{}'(u)$, where $\Phi_\epsilon{}''(u) = \frac{1}{M_\epsilon}$, which yields an entropy like estimate [9], which enables the bounds $0 \leqslant u(x,t) \leqslant 1$ to be demonstrated. We note that the "entropy" Φ, such that $\Phi''(u) = \frac{1}{M}$, had been employed earlier in the Cahn–Hilliard context in stability studies [60]. For a discussion of uniqueness and numerical schemes, see [8].

6. Linear stability and spinodal decomposition

In Section 2, linear stability of the spatially uniform state $u(x,t) = \bar{u}$ was considered in one spatial dimension for the constant mobility Cahn–Hilliard equation. Setting $\Omega = [0, L]$ and $u(x,t) = \bar{u} + \tilde{u}(x,t)$, the following linear stability problem was obtained

$$\begin{cases} \tilde{u}_t = M_0\big[-\big(1 - 3\bar{u}^2\big)\tilde{u} - \epsilon^2 \tilde{u}_{xx}\big]_{xx}, & (x,t) \in \Omega_T, \\ \tilde{u}_x = M_0\big[-\big(1 - 3\bar{u}^2\big)\tilde{u} - \epsilon^2 \tilde{u}_{xx}\big]_x = 0, & (x,t) \in \partial\Omega_T, \\ \tilde{u}(x,0) = \tilde{u}_0(x), & x \in \Omega. \end{cases} \tag{30}$$

It was shown in Section 2 that when ϵ is set to zero and the regularizing terms are dropped from the analysis, then (30) is equivalent to the backwards diffusion equation for $\bar{u}^2 < 1/3$, and it is equivalent to the (forward) diffusion equation for $\bar{u}^2 > 1/3$. We have already seen that when the regularizing terms in ϵ are included, then (17) is well-posed, so no problems with ill-posedness are expected here.

It is easy to verify that in the multi-dimensional case, linearization of the constant mobility Cahn–Hilliard equation about the spatially homogeneous steady state, $u(x,t) = \bar{u}$, yields the *linear stability problem*,

$$\begin{cases} \tilde{u}_t = M_0\big(\big(1 - 3\bar{u}^2\big)\Delta\tilde{u} - \epsilon\Delta^2\tilde{u}\big), & (x,t) \in \Omega_T, \\ n \cdot \nabla\tilde{u} = n \cdot \nabla\Delta\tilde{u} = 0, & (x,t) \in \partial\Omega_T, \\ \tilde{u}_0(x,0) = \tilde{u}_0(x), & x \in \Omega. \end{cases} \tag{31}$$

If we wish, we may proceed as in the analysis in [58,78,24] and construct a solution of (31) based on the eigenfunctions of $\mathcal{A}$, the Laplacian with Neumann boundary conditions. This yields

$$\tilde{u}(x,t) = \frac{A_0(0)}{2} + \sum_{k=1}^{\infty} A_k(0) \mathrm{e}^{\sigma(\lambda_k)t} \Phi_k(x),$$

where λ_k and Φ_k are the eigenvalues and the eigenfunctions of $\mathcal{A}$, $A_k(0)$ are the coefficients in the eigenfunction expansion for $\tilde{u}_0(x)$, and

$$\sigma(\lambda_k) = \big(\big(1 - 3\bar{u}^2\big) - \epsilon^2\lambda_k\big)\lambda_k. \tag{32}$$

One question of physical interest is number of unstable (or "growing") modes, in other words, the number of $k \in \mathcal{Z}^+$ such that $\sigma(\lambda_k) > 0$. Another question of physical interest is the identification of the dominant (or "fastest growing") mode, in other words, identifying λ_k such that $\sigma(\lambda_k)$ is maximal.

In one dimension with $\Omega = [0, L]$, $\lambda_k = (k\pi/L)^2$ and (32) yields the "dispersion relation"

$$\bar{\sigma}(k) := \sigma(\lambda_k) = \frac{k^2\pi^2}{L^2}\left[\frac{1}{4} - \frac{\epsilon^2 k^2 \pi^2}{L^2}\right], \tag{33}$$

for $k \in \mathcal{Z}^+$. Examining $\bar{\sigma}(k)$ it is easily seen that $\bar{\sigma}(k)$ vanishes at $k_1 = 0$ and $k_2 = L/(2\epsilon\pi)$, it is positive for $k \in (k_1, k_2)$, it has a unique critical point (a maximum) at $k_3 = L/(2\sqrt{2}\,\epsilon\pi)$, and it is negative elsewhere. Even if $k_3 \notin \mathcal{Z}^+$, the mode k_3 is known as the *fastest growing mode*. From (33), it follows that

$$\#\text{ growing modes} = \begin{cases} \left[\dfrac{L\sqrt{1-3\bar{u}^2}}{\epsilon\pi}\right], & |\bar{u}| < \dfrac{1}{\sqrt{3}}, \\ 0, & \text{otherwise}, \end{cases} \tag{34}$$

where $[s]$ refers to the *integer value* of s. From (34), it follows that as L increases or as ϵ decreases, the number of growing modes increases. Note that if L is sufficiently small or ϵ is sufficiently large, then there are no growing modes at all. Thus the parameter range for *linear instability* depends on L and ϵ, as well as on $\bar{u}$. While ϵ reflects a material property of the system, L, which reflects the size of the system, can be varied with relative ease. Since in most systems, the size of the system is very large relative to the size of the (micro-)structures under consideration, the limit of the parameter range of instability as $\epsilon/L \to 0$ is of physical relevance. And in this limit, the parameter range for instability is given by

$$\frac{-1}{\sqrt{3}} \leqslant \bar{u} \leqslant \frac{1}{\sqrt{3}}. \tag{35}$$

The limiting compositions $\lim_{\epsilon/L\to 0} \bar{u}_\pm = \pm\frac{1}{\sqrt{3}}$ are known as the *spinodal* compositions.

What does this have to do with the way the terminology *spinodal* was used in Section 3? We note first that the one dimensional analysis may be readily generalized to higher dimensions by recalling that also in higher dimensions one has that $\lambda_k \sim k^2$. Moreover, the analysis may also be readily generalized to treat the degenerate Cahn–Hilliard equation, (21), if $\bar{u}$ is taken to lie strictly in the interval (0, 1) and perturbations are taken sufficiently small. (For the special cases, $\bar{u} = 0$ or 1, there are no perturbations which conserve the original mass constraint, and it make some physical sense to impose such a constraint.) For (21), the spinodal compositions can be easily verified to depend also on temperature, and hence the parameter range for linear stability can be prescribed in terms of $(\bar{u}, \Theta)$, as was done in Section 3.

As time goes on, the importance of the nonlinear terms becomes more and more pronounced. It is the nonlinear effects which keep the amplitude of the solution from becoming unbounded and which cause the system to saturate near the binodal values, u_A and u_B. After the initial stages of saturation, certain regions, in which u_A or u_B dominate, grow at the expense of other regions and *coarsening* begins. As the nonlinear effects set in, the differences between the two Cahn–Hilliard variants become more pronounced, as we shall see shortly. One would expect, however, that the patterning in the phase separation would be dominated by the fastest growing mode over a period of time roughly proportional to the inverse of the growth rate of the fastest growing mode. Actually, often it remains dominant over a considerably longer time interval. This rather surprising result has been demonstrated for the constant mobility Cahn–Hilliard equation, see [75,51,52].

7. Comparison with experiment

What can be said with regard to is experimental verification of the Cahn–Hilliard theory? While qualitative comparison between numerical calculation and experimental data has been known for years to be reasonable [15,43], more quantitative indicators are clearly desirable. At the onset on spinodal decomposition, linear theory predicts a *dominant growing mode* (see Section 6), and as the system evolves into phase separated domains which coarsen, the dominant length scale in the system gets larger. Two approaches have been developed to quantitatively compare the evolution of length scales.

One approach is based on the *structure function*

$$S(k,t) \equiv \left|\widehat{\{u-\bar{u}\}}(k,t)\right|^2,$$

where $\bar{u} = \bar{u}(t) := \frac{1}{|\Omega|}\int_\Omega u(x,t)\,\mathrm{d}x$, and "$\widehat{\ }$" denotes the Fourier transform. If the length scale characterizing the patterns of the phase separation are much smaller than the length scales of Ω, edge effects should become negligible. In this case if $\Omega \subset \mathbb{R}^2$, then

$$S(k,t) \approx \frac{1}{4\pi^2}\left|\int_{\mathbb{R}^2\times\mathbb{R}^2} f(\bar{x},t) f(\bar{y},t)\,\mathrm{e}^{-k\cdot(\bar{x}-\bar{y})}\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{y}\right|^2, \quad \forall k \in \mathbb{R}^2, \tag{36}$$

where $f(s,t) = u(s,t) - \bar{u}(t)$. Structure function analysis can be implemented from the earliest stages of phase separation and throughout the coarsening regime. Various conjectures and predictions have been made with regard to possible self-similar behavior and scaling laws for growth of the characteristic length, based in part on analysis of the evolution of the structure factor, see e.g. [30]. Although there has been no rigorously verification of these prediction, some rigorous upper bounds on coarsening rates can be given [47,67].

Another approach which has been developed more recently is computational evaluation of Betti numbers to study the topological changes occurring during phase separation [32]. Betti numbers, β_k, $k = 0, 1, \ldots$, are topological invariants which reflect the topological properties of the structure [45]. The first Betti number, β_0 counts of the number of connected components, and the second Betti number, β_1 counts of the number of loops (in two dimensions) or the number of tunnels (in three dimensions). Reasonable qualitative agreement between theory and experiment [43] has been reported.

8. Long time behavior and limiting motions

It is constructive to be able to describe coarsening, and to obtain an accurate description of the motion of the interfaces. It turns out that to leading order, the Mullins–Sekerka problem and motion by surface diffusion give such a description. They both constitute *free boundary problems* where in the present context, the *free boundaries* refer to the interfaces between the phases. The constant mobility and the degenerate mobility Cahn–Hilliard equations differ in their behavior during coarsening stages. More specifically, the behavior of the constant mobility Cahn–Hilliard equation during coarsening can be described by the *Mullins–Sekerka* problem, and the behavior for the degenerate mobility Cahn–Hilliard equation is approximated by *surface diffusion* if $\Theta = \mathcal{O}(\epsilon^{1/2})$. It is of interest to note that the Mullins–Sekerka problem and motion by surface diffusion appeared in various other problems, especially in materials science [55,6], long before their connection with the Cahn–Hilliard equation became known.

How does one pass from the Cahn–Hilliard equation which describes the evolution of the concentration at all points in the system, to a description of the evolution which focuses on the motion of the interfaces? One such approach is to derive limiting motions by utilizing certain formal asymptotic expansions. Such an approach was developed to describe limiting motions for the Allen–Cahn equation [74] and for the phase field equations [13], and could be generalized to the Cahn–Hilliard context by Pego [71] for the case of constant mobility and by Cahn, Elliott and Novick-Cohen [14] in the case of degenerate mobility. As to the justification of the formal asymptotic analysis, under appropriate assumptions the passage from the Cahn–Hilliard equation to the Mullins–Sekerka problem can be made rigorous [1,2,18]. The passage from the degenerate Cahn–Hilliard equation to motion by surface diffusion has yet to be rigorously justified, however numerical computations indicate that the limiting motion has been correctly identified [8].

Since during coarsening the system has already saturated into domains dominated by one of the two binodal concentrations, we can envision the domain Ω during coarsening as being partitioned by N interfaces, Γ_i, $i = 1, \dots, N$, and the description of the evolution of the system can be given in terms of these N partitions.

8.1. *The Mullins–Sekerka problem*

In the Mullins–Sekerka problem [55], the following laws govern the evolution of the interfaces for $t \in (0, T)$, $0 < T < \infty$. See Figure 2. Away from the interfaces

$$\Delta\mu = 0, \quad x \in \Omega \backslash \Gamma, \tag{37}$$

and along the interfaces

$$V = -[n \cdot \nabla\mu]_-^+, \quad x \in \Gamma, \tag{38}$$

and

$$\mu = -\kappa. \tag{39}$$

Along $\partial\Omega$, the boundary of Ω,

$$n \cdot \nabla\mu = 0, \quad x \in \Gamma \cap \partial\Omega, \tag{40}$$

and

$$\Gamma \perp \partial\Omega, \quad x \in \Gamma \cap \partial\Omega. \tag{41}$$

In (37)–(39), $\mu = \mu(x, t)$ denotes the chemical potential which in the context of the formulation of the Cahn–Hilliard equation can be identified as $\mu = f(u) - \epsilon^2 \Delta u$. Note that here, in the limiting problem, the concentration $u = u(x, t)$ no longer appears explicitly, but only via the chemical potential, μ. In (38), $V = V(x, t)$ denotes the normal velocity at the point $x \in \Gamma$, and $n = n(x, t)$ denotes an unit exterior normal to one of the components

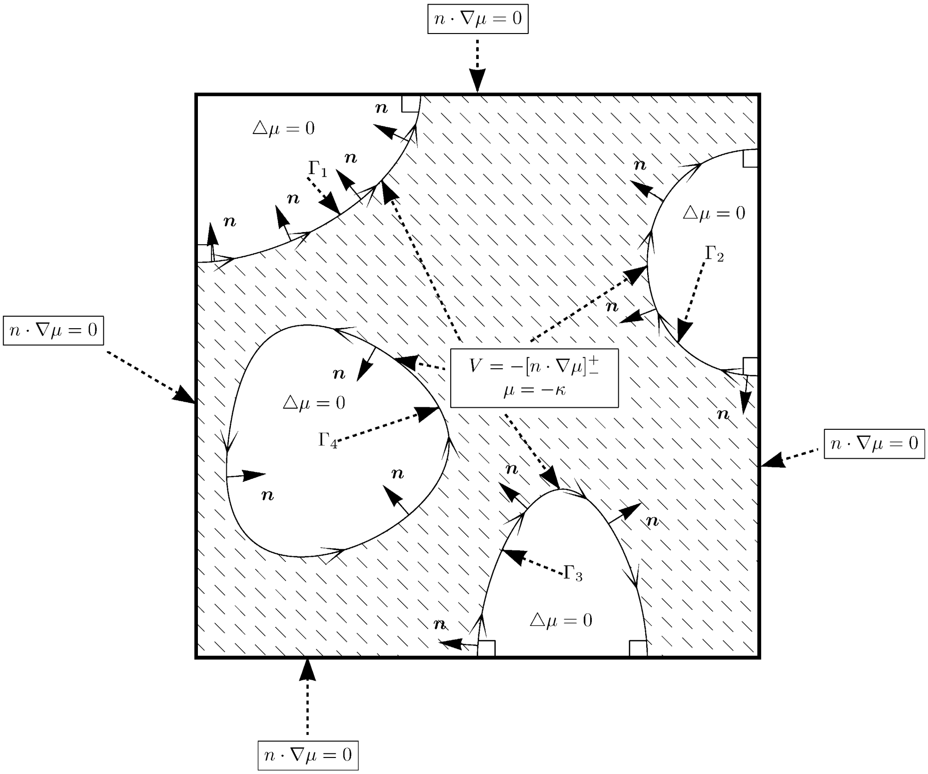

Fig. 2. Limiting motion as $t \to \infty$ for Case I: the Mullins–Sekerka problem.

Γ_i which comprises Γ. The orientations can be chosen arbitrarily for the parameterizations of the curves Γ_i, $i = 1, \ldots, N$. The normal velocity V can be defined by $V = n \cdot \vec{V}$ where $\vec{V} = \vec{V}(x,t)$ is the velocity of the interface at $x \in \Gamma$. See e.g. Gurtin [37] for background. One should note that Γ is time dependent in this formulation. In (38), $[n \cdot \nabla\mu]_-^+$ denotes the jump in the normal derivative of μ across the interface at $x \in \Gamma$. In (39), κ denotes the mean curvature. For curves in the plane,

$$\kappa = \frac{1}{R},$$

where R is the *signed radius* of the inscribed circle which is tangent to Γ at $x \in \Gamma$, and the sign of the radius is taken here to be positive if the inscribed circle lies on the "exterior" or "left" side of the curve whose orientation has been fixed. In $\mathbb{R}^3$,

$$\kappa = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right),$$

where R_1, R_2 are the *principle radii of curvature*. See Gurtin [37] or Finn [29].

Clearly the Mullins–Sekerka problem is a *nonlocal problem* in that the motion of the interfaces cannot be ascertained without taking into account what is happening within the domains bounded by the interfaces. For existence results for the Mullins–Sekerka problem, and a discussion of some of its qualitative properties, see for example, [2,18].

8.2. *Surface diffusion*

For the degenerate Cahn–Hilliard equation, if the scaled temperature Θ is sufficiently small and if logarithmic terms are included in the free energy, then the long time coarsening behavior can be formally shown to be governed by *surface diffusion*. By this we mean that the evolution of the interfaces $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_N$ is given by

$$V = -\frac{\pi^2}{16}\Delta_s \kappa, \quad x \in \Gamma, \tag{42}$$

$$n \cdot \nabla_s \kappa = 0, \quad x \in \Gamma \cap \partial\Omega, \tag{43}$$

$$\Gamma_i \perp \partial\Omega, \quad i = 1, \ldots, N,\ x \in \Gamma \cap \partial\Omega. \tag{44}$$

The boundary condition (43) is an analogue of the no-flux boundary condition, and the boundary condition (44) is a geometric analogue of the Neumann boundary condition.

In (42)–(44), V, κ and Γ have the same connotation as in our earlier discussion of the Mullins–Sekerka problem, and Δ_s denotes the *surface Laplacian* or *Laplace–Beltrami operator*, see [31]. Here the motion is *geometric* in that the motion of the interfaces is determined by the local geometry of the interfaces themselves. A formal asymptotic derivation of (42)–(44) is given in [14]. The system (42)–(44) can also be shown to describe the long time coarsening behavior for the *deep quench limit* [68].

To gain some intuition into the predicted motion, note that in the plane (see Figure 3) the system (42)–(44) can be written as

$$\begin{cases} V = -\dfrac{\pi^2}{16}\kappa_{ss}, & x \in \Gamma, \\ \kappa_s = 0, & x \in \Gamma \cap \partial\Omega, \\ \Gamma_i \perp \partial\Omega, & i = 1, \ldots, N,\ x \in \Gamma \cap \partial\Omega. \end{cases} \tag{45}$$

Here s is an arc-length parameterization of the components; i.e., along Γ_i, $i \in \{1, \ldots, N\}$,

$$s(p) = \int_{p_0}^{p} \sqrt{\dot{x}^2 + \dot{y}^2}\,\mathrm{d}\tau,$$

where $\{(x(\tau), y(\tau)) \mid p_0 \leqslant \tau \leqslant p\}$ is an arbitrary parameterization of Γ_i and p_0 refers to an arbitrary point on Γ_i. For (45), local existence can be demonstrated for smooth initial data, and perturbation of circles can be shown to evolve towards circles while preserving area [25].

9. Upper bounds for coarsening

In this section we present some rigorous results on upper bounds for coarsening. The first results given in this direction are by Kohn and Otto [47] in the context of the Cahn–Hilliard

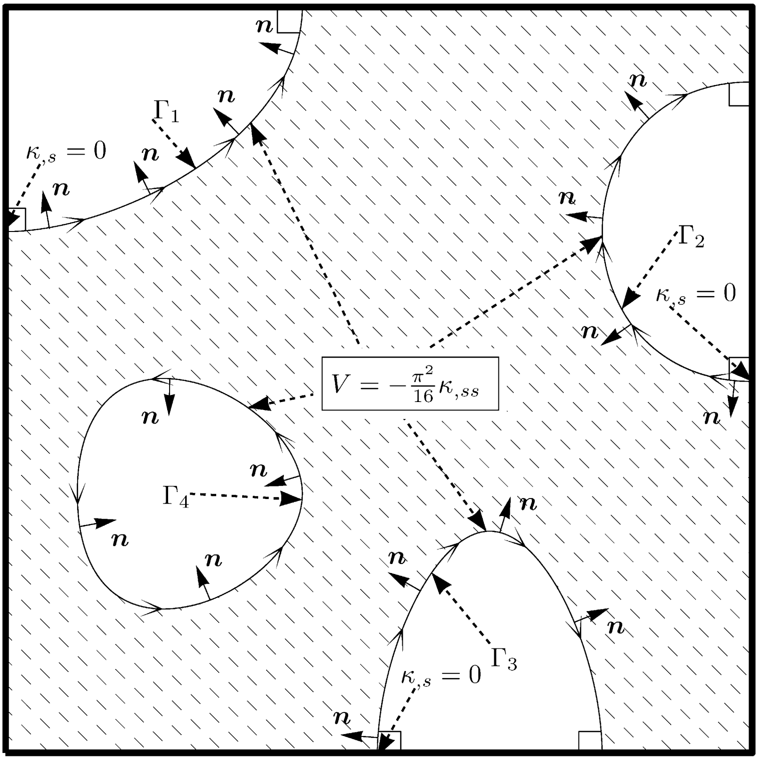

Fig. 3. Limiting motion as $t \to \infty$ for Case II: motion by surface diffusion.

equation. Their results are for (a) the Cahn–Hilliard equation with constant mobility, (17), and for (b) the degenerate Cahn–Hilliard equation, (21), where the mobility is taken as (19) and the temperature, Θ, is set to zero. The $\Theta = 0$ limit problem described in (b) in fact constitutes a free boundary obstacle problem [10], though solutions for it may be obtained via limits of solutions of (21) with $\Theta > 0$, for which the existence and regularity results of Section 5 apply. For simplicity, in [47] periodic boundary conditions are assumed and the mean mass, $\bar{u}$, is taken to be equal to $1/2$. They demonstrate upper bounds for the dominant length scale during coarsening, of the form $\propto t^{1/3}$ for (a), and of the form $\propto t^{1/4}$ for (b). Stated more precisely, they proved that there exist constants C_α such that if $L^{3+\alpha}(0) \gg 1 \gg E(0)$ and $T \gg L^{3+\alpha}(0)$, where E denotes a scaled free energy and L is a $(W^{1,\infty})^*$ norm of u, then

$$\frac{1}{T}\int_0^T E^{\theta r} L^{-(1-\theta)r}\,\mathrm{d}t \geqslant C_\alpha T^{-r/(3+\alpha)},$$

for all r, θ such that

$$0 \leqslant \theta \leqslant 1, \quad r < 3+\alpha, \quad \theta r > 1+\alpha, \quad (1-\theta)r < 2,$$

where $\alpha = 0$ for (a) and $\alpha = 1$ for (b). Their analysis is based on three lemmas which should hold at long times when the system has sufficiently coarsened. The first of these lemmas gives a bound of the form $1 \leqslant dEL$ where d is an O(1) constant, the second lemma gives a differential inequality involving E, L, and their time derivatives, and the third lemma uses the results of the first two lemmas to obtain upper bounds. Similar analyses have appeared more recently in various related settings [48,21,70].

While the predictions of Kohn and Otto are quite elegant, various deviations from the results in [47] have been seen [33,76,7], in particular strong mean mass dependence and slower than predicted rates. Moreover, the validity of their results requires that sufficiently large systems must be considered at sufficiently large times, which hinders ready numerical verification. As a partial remedy, the results of Kohn and Otto have been generalized in [67], and upper bounds for coarsening have now been given for all temperatures $\Theta \in (0, \Theta_{\text{crit}})$, where Θ_{crit} denotes the "critical temperature", and for arbitrary mean masses, $\bar{u} \in (u_A, u_B)$, where u_A and u_B denote the binodal concentrations. In [67], the domain $\Omega \subset \mathbb{R}^N$, $N = 1, 2, 3$, is taken to be bounded and convex, and the analysis applies either to the Neumann and no flux boundary conditions given in (22) or to periodic boundary conditions. Moreover, the upper bounds for the length scale are valid for all times $t > 0$, even before coarsening has truly commenced. By giving the upper bounds in terms of explicit temperature and mean mass dependent coefficients, it becomes clear that transitional and cross-over behavior may be occur, as has been reported in [33,73]. The remainder of this section is devoted to explaining some of the assumptions, analysis, and results of [47,67] in greater depth.

The starting point for the analysis in both [47,67] is the following scaled variant of the degenerate Cahn–Hilliard equation,

$$\begin{cases} u_t = \nabla\cdot\big(1-u^2\big)\nabla\left[\dfrac{\theta}{2}\ln\left[\dfrac{1+u}{1-u}\right] - u - \Delta u\right], & (x,t)\in\Omega_T,\\ n\cdot\nabla u = 0, & (x,t)\in\partial\Omega_T,\\ n\cdot\big(1-u^2\big)\nabla\left[\dfrac{\theta}{2}\ln\left[\dfrac{1+u}{1-u}\right] - u - \Delta u\right] = 0, & (x,t)\in\partial\Omega_T,\\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{46}$$

which may be obtained by writing (21) in terms of the variables

$$u' = 2u - 1, \quad x' = \big(\alpha^{1/2}/\epsilon\big)x, \quad t' = \big(\alpha^2 M_0/\epsilon^2\big)t, \quad \theta = \Theta/\alpha, \tag{47}$$

then dropping the primes. In the context of (46), $\theta = 1$ corresponds to the critical temperature. By setting $\theta = 1 - \delta$, $x' = (\delta/2)^{1/2}$, $t' = (\delta^2/4)t$, and $u' = (3\delta)^{-1/2}u$ in (46), and letting $\delta \to 0$ and dropping the primes, the constant mobility Cahn–Hilliard equation, (17), with $M_0 = 1$ is obtained. For this reason, case (a) treated in [47] is referred to there as the "shallow quench" limit. Letting $\theta \to 0$ in (46), case (b), which is referred to in [47] as the "deep quench" limit, is obtained.

Why consider $E^{-\lambda}(t)L^{1-\lambda}(t)$, $0 \leqslant \lambda \leqslant 1$, as a reasonable measure for the dominant length scale in the system? Since the mean mass, $\bar{u} = \frac{1}{|\Omega|}\int_\Omega u(x,t)\,\mathrm{d}x$, is time invariant for (46), it is convenient to define a first length scale, $L(t)$ as

$$L(t) := \sup_{\xi\in A}\frac{1}{|\Omega|}\int_\Omega u(x,t)\,\xi(x)\,\mathrm{d}x,$$

where

$$A := \left\{ \xi \in W^{1,\infty} \;\Big|\; \int_\Omega \xi \,\mathrm{d}x = 0 \text{ and } \sup_\Omega |\nabla \xi| = 1 \right\}.$$

A second length scale, $E^{-1}(t)$, can be defined based on the free energy, $\mathcal{F}(t)$, which was introduced in (24). In terms of the rescalings (47), we obtain that

$$E(t) := \frac{1}{|\Omega|}\mathcal{F}(t) = \frac{1}{2|\Omega|}\int_\Omega \left\{ |\nabla u|^2 + \left[\frac{\partial W}{\partial u}\right]^2 \right\}_{|u=u(x,t)} \mathrm{d}x, \tag{48}$$

where

$$\frac{\partial W}{\partial u} = \left[\left(1-u^2\right) + \theta\left\{(1+u)\ln(1+u) + (1-u)\ln(1-u)\right\} + e(\theta)\right]^{1/2}. \tag{49}$$

In (49), $e(\theta)$ is determined by requiring that $\frac{\partial W}{\partial u} = 0$ at $u = u_\pm$, where $u_\pm$ denote here the two unique minima of $\frac{\partial W}{\partial u}$, such that $u_+ = -u_- > 0$. A straightforward calculation yields that

$$\theta = \frac{2u_\pm}{\ln(1+u_\pm) - \ln(1-u_\pm)} = \left[\sum_{k=0}^\infty \frac{1}{2k+1} u_\pm^{2k}\right]^{-1}, \tag{50}$$

and hence, in particular, $u_\pm = u_\pm(\theta)$, as one would expect. That $E^{-1}(t)$ acts as a length scale measuring the amount of perimeter during coarsening can be seen by noting that (48) implies that

$$E(t) \geqslant \frac{1}{|\Omega|}\int_\Omega \left|\nabla W(u)\right| \mathrm{d}x. \tag{51}$$

During the later stages of coarsening when the system is approximately partitioned into regions in which $u = u_+$ and in which $u = u_-$, the inequality in (51) can be expected to be closely approximated by equality. The expression on the right-hand side of (51) scales as length^{-1} and gives, for such partitioned systems, a measure of the amount of interfacial surface area per unit volume times the "surface energy", $\sigma = W(u_+) - W(u_-)$. Note that for well partitioned systems, $(u_+ - u_-)\|u\|^{-1}_{W^{1,\infty}}$ gives a rough lower bound on interfacial widths, hence $|\Omega|(u_+ - u_-)^{-1}\|u\|_{W^{1,\infty}}$ gives an upper bound on the amount of interfacial area within the volume $|\Omega|$, and therefore, in some sense, $L(t)$ and $E^{-1}(t)$ are measuring similar quantities. If $L(t)$ and E^{-1} both act as reasonable measures of "length" during coarsening, clearly $E^{-\lambda}L^{(1-\lambda)}(t)$, $0 \leqslant \lambda \leqslant 1$ also constitutes a reasonable measure.

In treating temperatures $\theta \in [0, 1]$ and mean masses $u_- < \bar{u} < u_+$, the following technical results are useful:

CLAIM 9.1. *Let* $0 < \theta < 1$, $u_- < \bar{u} < u_+$, *and let* $u(x,t)$ *denote a solution to* (46). *Then*

$$\frac{\partial W}{\partial u}(u) \geqslant \Psi(\theta)\left|u^2 - u_\pm^2\right|,$$

where

$$\Psi(\theta) := \frac{1}{u_\pm^2}\left[-1 + \frac{2}{u_\pm}\left\{\frac{\ln(1-u_\pm) + \ln(1+u_\pm)}{\ln(1-u_\pm) - \ln(1+u_\pm)}\right\}\right],$$

and

$$\frac{1}{|\Omega|}\int_\Omega \bigl(u_\pm^2 - u^2\bigr)\,\mathrm{d}x \leqslant 2[E+\theta\ln 2].$$

The following lemmas [67], which make use of the estimates in Claim 9.1, are extensions and generalization Lemmas 1, 2, and 3 from [47].

LEMMA 9.1. *Let* $0<\theta<1$ *and* $u_- < \bar{u} < u_+$. *Then*

$$\bigl(u_\pm^2 - \bar{u}^2\bigr) \leqslant \left[32L(t)\left(\frac{5E(t)}{u_+[\Psi(\theta)]^{1/2}} + \frac{3|\partial\Omega|}{|\Omega|}\right)\right]^{1/2} + F(E;\theta), \quad 0<t, \tag{52}$$

where

$$F(E;\theta) = \min\left\{\left[\frac{2E}{\Psi(\theta)}\right]^{1/2}, 2[\theta\ln 2 + E]\right\}. \tag{53}$$

LEMMA 9.2. *Let* $0<\theta<1$ *and* $u_- < \bar{u} < u_+$. *Then*

$$|\dot{L}|^2 \leqslant -\bigl(1-u_\pm^2\bigr)\dot{E} - F(E;\theta)\dot{E}, \quad 0<t, \tag{54}$$

where $F(E;\theta)$ *is as defined in* (53).

LEMMA 9.3. *Suppose that*

$$|\dot{L}|^2 \leqslant -AE^\alpha\dot{E}, \quad 0\leqslant t\leqslant T, \tag{55}$$

and

$$0\leqslant\alpha\leqslant 1, \quad 0\leqslant\varphi\leqslant 1, \quad r<3+\alpha, \quad \varphi r>1+\alpha, \quad (1-\varphi)r<2. \tag{56}$$

If, in addition to (55), (56),

$$LE\geqslant B, \quad 0\leqslant t\leqslant T, \tag{57}$$

then

$$\frac{1}{T}\left[\int_0^T E^{r\varphi}L^{-(1-\varphi)r}\,\mathrm{d}t + L(0)^{(3+\alpha)-r}\right] \geqslant \vartheta_1 T^{-r/(3+\alpha)}, \tag{58}$$

where $\vartheta_1 = \vartheta_1(A,B,\alpha,r,\varphi)$.

If, in addition to (55), (56),

$$E\geqslant C, \quad 0\leqslant t\leqslant T, \tag{59}$$

then

$$\frac{1}{T}\left[\int_0^T E^{\varphi r}L^{-(1-\varphi)r}\,\mathrm{d}t + L(0)^{2-(1-\varphi)r}\right] \geqslant \vartheta_2 T^{-(1-\varphi)r/2}, \tag{60}$$

where $\vartheta_2 = \vartheta_2(A,C,\alpha,r,\varphi)$.

REMARK 9.1. The inequalities in (56) imply that $\frac{2}{1-\varphi} > 3+\alpha$, hence the upper bound predicted by (60) is slower than in (58).

We shall now see how Lemmas 9.1, 9.2, and 9.3 imply upper bounds for coarsening. Let us first consider the expression for $F(E;\theta)$. If $0<\theta<1$ and E is sufficiently small, then $F(E;\theta) = [2E/\Psi(\theta)]^{1/2}$. If the term $\frac{3|\partial\Omega|}{|\Omega|}$ in (52), which represents boundary effects, is sufficiently small, Lemma 9.1 can be used to imply either a bound of the form (57) or a bound of the form (59). In particular, if E is sufficiently small, then a bound of the form (57) is implied. It now follows from Lemma 9.2, depending on the relative size of the terms $(1-u_{\pm}^2)$ and $[2E/\Psi(\theta)]^{1/2}$, that Lemma 9.3 holds with either $\alpha = 0$ or $\alpha = 1/2$. In particular, if E is sufficiently small, then Lemma 9.3 holds with $\alpha = 0$. This yields the shallow quench result of [47].

Suppose that $\theta = 0$. If E is sufficiently small, then $F(E;\theta) = 2E$. Again, Lemma 9.1 can be seen to imply either a bound of the form (57) or a bound of the form (59), with a bound of the form (57) being implied if E is sufficiently small. When $\theta = 0$, then referring to (50), $u_{\pm} = \pm 1$. Hence if $F(E;\theta) = 2E$, Lemma 9.2 implies that (55) holds with $\alpha = 1$. This yields the deep quench result of [47].

More generally, Lemmas 9.1 and 9.2 can be used to demonstrate that if $\bar{u} \in (u_-, u_+)$ and $\theta \in [0,1)$, then for any $t > 0$, there exists times $0 \leqslant T_1 < T_2$ such that for all $t \in (T_1, T_2)$, (55) holds for some $\alpha \in \{0, \frac{1}{2}, 1\}$ and either (57) or (59) holds. Noting the autonomy of the differential inequality, (55), it is possible to conclude

THEOREM 9.1. *Let $u(x,t)$ be a solution to (46) in the sense of Theorem 5.5 such that $u_- < \bar{u} < u_+$ and $0 < \theta < 1$, then at any given time $t \geqslant 0$, if boundary effects are negligible then upper bounds of the form*

$$\frac{1}{t-T_1}\left[\int_{T_1}^{t} E^{r\varphi} L^{-(1-\varphi)r}\, \mathrm{d}t + L(T_1)^{(3+\alpha)-r}\right] \geqslant \vartheta_1 (t-T_1)^{-r/(3+\alpha)},$$

or

$$\frac{1}{t-T_2}\left[\int_{T_2}^{t} E^{\varphi r} L^{-(1-\varphi)r}\, \mathrm{d}t + L(T_2)^{2-(1-\varphi)r}\right] \geqslant \vartheta_2 (t-T_2)^{-(1-\varphi)r/2},$$

may be prescribed, for appropriate values of the parameters.

The boundary terms, which are neglected in Theorem 9.1, may be incorporated by suitably redefining E. Over time, E decreases, and the relative size of the terms on the right-hand side of (52), (54) changes in accordance also with the size of $\bar{u}$ and θ. In this manner, a variety of time depend predictions for upper bounds on coarsening follow from Theorem 9.1, with transitions which may clearly depend on both $\bar{u}$ and θ, [67,76,33]. A complete discussion of these results is quite involved [67], and a complete understanding of the actual coarsening rates requires refinement of the bounds [20] and considerable further work.

A CLOSING REMARK. Roughly fifty years have passed since the Cahn–Hilliard equation was proposed as a model for phase separation [16,15]. While many aspects of its dynamics have been studied, many aspects remain to be analyzed. The author of this chapter apologies that the list of references which follow cannot claim to be complete. Clearly it is a tribute to the robustness of the equation, that the details that have been forthcoming from the analysis all seem to contribute to the overall picture and not to lead to the dismissal of the model. The Cahn–Hilliard equation continues to be proposed as a relevant model in a variety of new contexts, and it continues to be generalized in a variety of new directions, [62,65].

Illustrations: Courtesy of Niv Aharonov.

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CHAPTER 5

Mathematical Analysis of Viscoelastic Fluids

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Contents

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1. The equations describing viscoelastic flows

In Section 9 of Book II of his Principia, Newton [123] states the hypothesis that "The resistance arising from the want of lubricity in the parts of a fluid is, other things being equal, proportional to the velocity with which the parts of the fluid are separated from one another." He then proceeds to apply this hypothesis to analyze the velocity field surrounding a cylinder in uniform rotation. The generalization of Newton's hypothesis to more general flows had to await the work of Navier [122] and Stokes [203], who formulated the equations that now bear their name.

The motion of an incompressible continuous medium is described by the equation of momentum balance,

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = \operatorname{div}\mathbf{T} - \nabla p, \tag{1}$$

the incompressibility condition,

$$\operatorname{div}\mathbf{v} = 0, \tag{2}$$

and a constitutive law. Here ρ is the density, $\mathbf{v}$ is the velocity, p is the pressure, and $\mathbf{T}$ is the "extra" stress (extra meaning in addition to the pressure). The constitutive law describes how this extra stress is related to the motion of the fluid. Newton's hypothesis, as generalized by Navier and Stokes, states that this extra stress is proportional to the symmetric part of the velocity gradient,

$$\mathbf{T} = \eta\big(\nabla\mathbf{v} + (\nabla\mathbf{v})^T\big). \tag{3}$$

The proportionality constant η is called the viscosity.

The stress postulated by Newton's law is basically a friction force. The hypothesis is adequate for many fluids, such as water, air, gasoline, liquid metals etc. Other liquids, however, have a microstructure, which can be altered by a flow, and the stresses in such fluids depend on this flow-microstructure interaction. Examples include polymers, liquid crystals, suspensions, and foams. The first statement on a connection between microstructure and rheological properties appears to be due to Lucretius [109]:

"We see how quickly through a colander
The wines will flow; how, on the other hand,
The sluggish olive-oil delays: no doubt,
Because 'tis wrought of elements more large,
Or else more crook'd and intertangled. Thus
It comes that the primordials cannot be
So suddenly sundered one from other, and seep,
One through each several hole of anything."

The first generalization of the Newtonian fluid that comes to mind is to replace the linear dependence on the velocity gradient by a nonlinear one. Such models, known as the generalized Newtonian fluid, are useful and have inspired a substantial mathematical literature. They cannot, however, account for the physics of the fluid-microstructure interaction just alluded to.

This article will focus on the analysis of models which were developed primarily for polymeric liquids. The basic physics governing the flow of such liquids is quite simple to describe. In their rest state, the long chain molecules of the polymers assume a randomly coiled shape. The drag exerted in a flow can stretch them out. Since a stretched molecule is a state of lower entropy than a coiled molecule, this increases the free energy, leading to a restoring force, which produces an elastic stress.

Putting this insight into mathematical equations is far from easy. It is clear that the deformation of molecules in response to a flow is not instantaneous, and therefore the stress will depend not only on the instantaneous velocity gradient, but on the history of the motion. Attempts to formulate specific models have been of two kinds: "Phenomenological" models postulate a relationship chosen of a certain form and then attempt to fit such a relation to available data. "Molecular" models attempt to capture essential aspects of the physics governing the interaction between molecular structure and flow in a simplified system that might be viewed as a "caricature" of the real polymer molecules.

1.1. *General considerations*

In general, the motion of a fluid is described in terms of the relative deformation gradient. If $\mathbf{y}(\mathbf{x}, t, s)$ is the position at time s of the fluid particle which occupies position $\mathbf{x}$ at time t, we define the relative deformation gradient $\mathbf{F}(\mathbf{x}, t, s)$ by

$$F_{ij}(\mathbf{x}, t, s) = \frac{\partial y_i(\mathbf{x}, t, s)}{\partial x_j}. \tag{4}$$

In a viscoelastic fluid, the stress tensor depends on the history of this relative deformation gradient:

$$\mathbf{T}(\mathbf{x}, t) = \mathcal{F}\big(\mathbf{F}(\mathbf{x}, t, s)\big)_{s=-\infty}^{t}. \tag{5}$$

One general restriction on the form of this dependence is the principle of frame indifference [130,208], which expresses the condition that stresses result only from deformations and are not affected by merely rotating the medium. The result of this principle is that $\mathbf{T}$ depends only on the combination

$$\mathbf{C}(\mathbf{x}, t, s) = \mathbf{F}^T(\mathbf{x}, t, s)\mathbf{F}(\mathbf{x}, t, s), \tag{6}$$

known as the relative Cauchy strain, and that this functional is isotropic:

$$\mathbf{T}(\mathbf{x}, t) = \mathcal{G}\big(\mathbf{C}(\mathbf{x}, t, s)\big)_{s=-\infty}^{t}, \tag{7}$$

and

$$\mathcal{G}\big(\mathbf{Q}\mathbf{C}(\mathbf{x}, t, s)\mathbf{Q}^T\big)_{s=-\infty}^{t} = \mathbf{Q}\big[\mathcal{G}\big(\mathbf{C}(\mathbf{x}, t, s)\big)_{s=-\infty}^{t}\big]\mathbf{Q}^T \tag{8}$$

for every orthogonal matrix $\mathbf{Q}$. With the notable exception of the Newtonian fluid, the principle of material frame indifference specifically precludes linear constitutive relations.

It is not possible to formulate useful models without some a priori assumption about the nature of the functional dependence on the history. At the "phenomenological" level, three approaches have been used:

(1) Taylor expansion around $s = t$.
(2) A history dependence resulting from solving a differential system.
(3) A history dependence expressed by integrals.

Taylor expansion around $s = t$ leads to the Rivlin–Ericksen fluids [195]. Such models do not lead to mathematical equations with desirable well-posedness and stability properties [16] and should be used only within the context of perturbation expansions. Differential and integral models will be discussed below.

1.2. *Differential models*

Differential models attempt to formulate a system of differential equations for the stress. In general, the form of a differential model is

$$\frac{\partial \mathbf{T}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{T} = \mathbf{G}(\nabla \mathbf{v}, \mathbf{T}), \tag{9}$$

or, more generally, we may allow several parts of the stress, each governed by an equation of this form,

$$\mathbf{T} = \sum_{i=1}^{N} \mathbf{T}_i,$$
$$\frac{\partial \mathbf{T}_i}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{T}_i = \mathbf{G}_i(\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_N, \nabla \mathbf{v}). \tag{10}$$

Differential constitutive theories are nonlinear generalizations of Maxwell's theory of linear viscoelasticity [119]. It is possible to classify the possibilities allowed by material frame indifference if, for instance, $\mathbf{G}$ is required to be quadratic [131], and, not surprisingly, many popular constitutive models are of such a form. As pointed out by Oldroyd [130], there are two natural candidates for a frame indifferent version of Maxwell's theory, the upper convected Maxwell model,

$$\frac{\partial \mathbf{T}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{T} - (\nabla \mathbf{v})\mathbf{T} - \mathbf{T}(\nabla \mathbf{v})^T + \lambda \mathbf{T} = \mu\big(\nabla \mathbf{v} + (\nabla \mathbf{v})^T\big), \tag{11}$$

and the lower convected Maxwell model,

$$\frac{\partial \mathbf{T}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{T} + (\nabla \mathbf{v})^T\mathbf{T} + \mathbf{T}(\nabla \mathbf{v}) + \lambda \mathbf{T} = \mu\big(\nabla \mathbf{v} + (\nabla \mathbf{v})^T\big). \tag{12}$$

(When using the gradient of a vector in matrix products, we follow the convention that the first index refers to the vector component and the second index to the direction of differentiation.) The experimental facts on polymeric liquids as well as molecular theories (see below) heavily favor the upper convected model. A number of popular models modify the upper convected Maxwell model by adding additional quadratic terms to the equation, for instance, the Giesekus model [52],

$$\cdots + \kappa \mathbf{T}^2 = \cdots, \tag{13}$$

the Phan–Thien–Tanner (PTT) model [134],

$$\cdots + \kappa \mathbf{T}(\operatorname{tr} \mathbf{T}) = \cdots, \tag{14}$$

and the Johnson–Segalman model [84],

$$\cdots + \frac{1-a}{2}\big[\mathbf{T}\big(\nabla \mathbf{v} + (\nabla \mathbf{v})^T\big) + \big(\nabla \mathbf{v} + (\nabla \mathbf{v})^T\big)\mathbf{T}\big] = \cdots, \tag{15}$$

where, in each instance, the dots indicate the terms already there in the upper convected Maxwell model. For any of these models, an additional Newtonian term may be added to the stress. The upper convected Maxwell model, modified by addition of a Newtonian viscous contribution, is known in the literature as the Oldroyd B model.

1.3. *Integral models*

Boltzmann's theory of linear viscoelasticity [6] postulates a relationship of the form

$$\begin{aligned}\mathbf{T}(\mathbf{x}, t) = {} & \eta\big(\nabla \mathbf{v}(\mathbf{x}, t) + \big(\nabla \mathbf{v}(\mathbf{x}, t)\big)^T\big) \\ & + \int_0^\infty G(t-s)\big(\nabla \mathbf{v}(\mathbf{x}, s) + \big(\nabla \mathbf{v}(\mathbf{x}, s)\big)^T\big)\,\mathrm{d}s.\end{aligned} \tag{16}$$

Here the function G is usually assumed to be positive, monotone decreasing and convex, in fact, all molecular models lead to completely monotone functions as long as molecular inertia is neglected.

It is natural to attempt a nonlinear frame-invariant generalization. One such model is the K-BKZ model introduced by Kaye [89] and Bernstein, Kearsley and Zapas [4]. With

$$I_1 = \operatorname{tr} \mathbf{C}^{-1}(\mathbf{x}, t, s), \qquad I_2 = \operatorname{tr} \mathbf{C}(\mathbf{x}, t, s), \tag{17}$$

this model can be put in the form

$$\begin{aligned}\mathbf{T}(\mathbf{x}, t) = \int_{-\infty}^t & \frac{\partial W(I_1, I_2, t-s)}{\partial I_1}\big(\mathbf{C}^{-1}(\mathbf{x}, t, s) - \mathbf{I}\big) \\ & - \frac{\partial W(I_1, I_2, t-s)}{\partial I_2}\big(\mathbf{C}(\mathbf{x}, t, s) - \mathbf{I}\big)\,\mathrm{d}s.\end{aligned} \tag{18}$$

The model has a formal analogy with elasticity, if we replace the equilibrium position of a particle in an elastic body by the position $\mathbf{y}(\mathbf{x}, t, s)$ of a particle at a prior time. It contains the arbitrary assumption that, although the dependence for fixed s is nonlinear, the contributions from different times s superpose in an additive fashion. This assumption, of course, serves only to restrict the possibilities to a manageable class and has no compelling physical basis. Refinements of the model which relax this assumption have been considered.

1.4. *Molecular models*

The essential feature of polymeric liquids is the presence of long chain molecules. In modeling efforts, such chain molecules are represented by chains of beads and rods or beads and springs, which move under the influence of the forces in the springs, forces exerted by the surrounding fluid, and stochastic forces. The trickier part of the modeling, however, is how to represent the environment with which a polymer molecule interacts. There are three basic approaches:

(1) Dilute solution theories visualize the polymer molecule as surrounded by a Newtonian fluid (the "solvent"), and the interaction is by hydrodynamic drag forces depending on the difference between the velocity of a "bead" in a polymer molecule and the surrounding liquid.
(2) Network theories are motivated by theories of rubber elasticity. In rubber elasticity, a segment of a polymer molecule is thought of as being constrained at certain "end points," while being able to move in between. Thus molecules are linked together in a network. In a liquid, such networks are temporary, and hypotheses are needed to describe the formation and decay of network junctions and how they move with the fluid.
(3) Reptation theories visualize the molecule as slithering inside a tube formed by the other polymer molecules.

I refer to [206] for an account of the history of molecular modeling efforts and references to the original literature.

Only in the simplest cases and under additional ad hoc "approximations" do molecular models lead to constitutive models of differential or integral type as discussed above. We shall discuss this in the case of dilute solution theories, which are the conceptually simplest type of model and the one most amenable to "rigorous" development. The following description is somewhat sketchy, for a more detailed discussion see e.g. Volume 2 of [5].

In the simplest case, the polymer molecule is thought of as a "dumbbell" consisting of two beads connected by a spring. The spring connecting the beads exerts a spring force $\mathbf{F}(\mathbf{R})$. On each of the beads, we have a balance between this spring force, a friction force exerted by the surrounding fluid, modeled as a Stokes drag, and a stochastic force due to Brownian motion.

The result of this balance is an equation of the form

$$\dot{\mathbf{R}} = \nabla \mathbf{v} \cdot \mathbf{R} - \frac{2}{\zeta}\mathbf{F}(\mathbf{R}) + \frac{1}{\zeta}\mathbf{S} \tag{19}$$

for the vector $\mathbf{R}$ connecting the beads. Hence $\mathbf{F}(\mathbf{R})$ is the spring force, $\mathbf{S}$ is a stochastic term, and ζ is a constant related to the friction coefficient of the beads.

Under reasonable assumptions on the stochastic forces, the methods of stochastic differential equations can be used to convert the stochastic differential equation (19) to a Fokker–Planck equation. We assume hat each macroscopic volume element in space contains a large number of polymer molecules and that the distribution of their connector vectors $\mathbf{R}$ can be described by a probability density $\psi(\mathbf{R}, \mathbf{x}, t)$. For each $\mathbf{x}$ and t, the total probability is equal to 1:

$$\int_{\mathbb{R}^3} \psi(\mathbf{R}, \mathbf{x}, t)\, \mathrm{d}\mathbf{R} = 1. \tag{20}$$

If the stochastic forces on each bead are described by a Wiener process, and their magnitude is proportional to kT, then the Fokker–Planck equation is

$$\frac{\partial \psi}{\partial t} + (\mathbf{v} \cdot \nabla_{\mathbf{x}})\psi = \frac{2kT}{\zeta} \Delta_{\mathbf{R}} \psi + \mathrm{div}_{\mathbf{R}} \left[-\nabla \mathbf{v}(\mathbf{x}, t) \cdot \mathbf{R} \psi + \frac{2}{\zeta} \mathbf{F}(\mathbf{R}) \psi \right]. \tag{21}$$

Here $\Delta_{\mathbf{R}}$ and $\mathrm{div}_{\mathbf{R}}$ indicate differential operators with respect to the variable $\mathbf{R}$.

The contribution to the stress tensor resulting from the tension in the springs is given by

$$\mathbf{T}_p(\mathbf{x}, t) = n \int_{\mathbb{R}^3} \mathbf{R}\mathbf{F}(\mathbf{R}) \psi(\mathbf{R}, \mathbf{x}, t)\, \mathrm{d}\mathbf{R}. \tag{22}$$

The product $\mathbf{RF}$ which appears here is the dyadic product, and n is the number density of polymer molecules. The total stress in the polymer solution is then modeled as the sum of this "polymer contribution" and a viscous "solvent stress."

The Fokker–Planck equation is actually a PDE involving an additional variable $\mathbf{R}$ which needs to be solved in order to determine the constitutive behavior. Thus, in a three-dimensional time-dependent flow, we have a total of seven independent variables, one for time, three for $\mathbf{x}$ and another three for $\mathbf{R}$. If one tries to go from dumbbells to more "realistic" chains, the computational effort quickly becomes impractical. Only if the spring force is linear it is possible to derive a closed set of differential equation for the quadratic moments

$$\mathbf{C} = \int_{\mathbb{R}^3} \mathbf{R}\mathbf{R} \psi(\mathbf{R}, \mathbf{x}, t)\, \mathrm{d}\mathbf{R}. \tag{23}$$

The tensor $\mathbf{C}$ is called the configuration tensor. The resulting constitutive model is the upper convected Maxwell model. For nonlinear dumbbells, this is not possible unless we cheat. In a nonlinear dumbbell, the spring force is given by $\gamma(|\mathbf{R}|^2)\mathbf{R}$, where γ is the spring constant. If we replace the spring constant of the actual spring by the spring constant of an average spring, i.e. we change $\gamma(|\mathbf{R}|^2)$ by $\gamma(\mathrm{tr}\, \mathbf{C})$, then it is again possible to derive a closed system for the tensor $\mathbf{C}$ which does not require the solution of the Fokker–Planck equation. This is referred to as the Peterlin "approximation," but it is not an approximation in any rigorous asymptotic sense. Rather, it is an attempt to obtain a more tractable problem which retains the qualitative physics of the original problem. In this sense, most "molecular" models in use by rheologists are perhaps more aptly described as "molecularly inspired."

Comparisons between Peterlin and "real" dumbbells have recently been an active topic of numerical simulation. Usually, such simulations are based not on the Fokker–Planck equation, but on the original stochastic differential equation.

2. Existence results for initial value problems

2.1. *Local existence*

Viscoelastic materials are intermediate between the viscous and elastic case, and it is natural to approach existence by viewing them as perturbations of either case. We shall illustrate this in the example of one dimensional shear flow of a K-BKZ fluid. The equation of motion is of the form

$$\rho u_{tt} = \eta u_{xxt} + \int_0^\infty a\big(t-s, u_x(t) - u_x(s)\big)_x \, \mathrm{d}s. \tag{24}$$

Here u represents the displacement, and η is a possible Newtonian contribution to the viscosity. If η is positive, it is natural to construct solutions via the iteration

$$\rho u_{tt}^{n+1} = \eta u_{xxt}^{n+1} + \int_0^\infty a\big(t-s, u_x^n(t) - u_x^n(s)\big)_x \, \mathrm{d}s. \tag{25}$$

If, on the other hand, η is zero and a is smooth, solutions can be constructed using the iteration

$$\rho u_{tt}^{n+1} = \int_0^\infty a\big(t-s, u_x^{n+1}(t) - u_x^n(s)\big)_x \, \mathrm{d}s. \tag{26}$$

In the first case, each step of the iteration simply involves solving the heat equation; in the second case, a nonlinear wave equation needs to be solved at each step of the iterative process. Existence is based on proving that the mapping defined by the iteration (with the imposition of suitable initial and boundary conditions) is a contraction in an appropriately chosen function space. We note that the case of creeping flow is also important, since inertia is often negligible in viscoelastic flows. In that case, (26) simplifies to

$$\int_0^\infty a\big(t-s, u_x^{n+1}(t) - u_x^n(s)\big)_x \, \mathrm{d}s = 0, \tag{27}$$

i.e. the problem to be solved at each step of the iteration is elliptic.

There is no essential difference between viscoelastic fluids and solids when it comes to local existence results, and many results in the literature are stated for viscoelastic solids. For earlier reviews of the literature, I refer, for instance, to [147,152]. Creeping flow problems seem to have been the first to be solved, see e.g. [1] for an early reference. Early existence results for "parabolic" models of viscoelasticity appear, for instance in [73,143]. One-dimensional hyperbolic problems are discussed in [21,22,74,77,111]. In the three-dimensional case, the incompressibility constraint complicates matters. Even the elastic case was tackled only in the mid 1980s [32,81]. Kim [90] seems to have been the first to prove a three-dimensional existence result for a viscoelastic fluid, the techniques can be extended to general K-BKZ fluids [147]. A different approach for integral models can be found in [144]. Existence of solutions for differential models is proved in [155].

An interesting possibility arises when $\eta = 0$, but the kernel a in (24) has a singularity as $s \to t$. Such models are not perturbations of the hyperbolic case as models with smooth kernels, and an iteration along the lines of (26) would not work. An existence theory for models with singular kernels was developed in [79,80,148]. Regularity of solutions is interesting for such models, because the linear problem is known to have smoothing properties [142,78]. I refer to [26,27,39,60,61,136] for research on this issue.

The first existence result for molecular dumbbell models was established in [158]. For more recent additional work on this subject, see [2,87,88,104,215].

2.2. *Stability of the rest state*

The first rigorous proof for the linear stability of the rest state of viscoelastic fluids is due to Slemrod [199,200]. Since viscoelastic fluids are dissipative, we expect that solutions with small initial data will exist globally and decay to the rest state at a rate determined by the linearized problem. For parabolic equations, such a result can be established in a fashion which treats the nonlinearities as a perturbation to the linearization; for equations of hyperbolic type the situation is more delicate. Nishida [124] and Matsumura [117] were the first to establish global existence and asymptotic decay for quasilinear hyperbolic equations with damping under the hypothesis of small initial data.

Since then, an extensive literature has developed on using the energy method to prove analogous results for systems with viscoelastic damping. Much of it focuses on viscoelastic solids. There is a difference to the case of fluids, since in a fluid only velocities, but not displacements, can be expected to go to zero. The first global existence result for viscoelastic fluids is due to Kim [90], he considers a special case of the K-BKZ fluid, his results are extended to more general K-BKZ fluids in [147]. While Kim's result is for a fluid filling all of space, the result in [147] is for periodic boundary conditions. Brandon and Hrusa [8] consider one-dimensional shearing motions and specifically address the issues associated with the spatially unbounded case, which is more difficult since decay is not exponential.

Results on asymptotic decay for differential models of viscoelastic flow were pioneered by Guillopé and Saut [63,64]. For subsequent extensions and refinements, see e.g. [11,70, 100,105,121]. A result on asymptotic decay for molecular dumbbell models was recently obtained in [87].

2.3. *Global existence*

For the Navier–Stokes equations, it is well known that a global smooth solution for the Dirichlet initial-boundary value problem exists in two space dimensions. In three dimensions, global existence, but not uniqueness, of a weak solution is known (see for instance [207]). Whether a global smooth solution exists in three dimensions is one of the Millenium Problems of the Clay Mathematics Institute.

For viscoelastic fluids, our knowledge is much more fragmentary. Global existence of smooth solutions is known for some shear flow problems. All these results have in common that the constitutive model has a Newtonian part and a shear thinning viscoelastic part. For the Newtonian case, the problem is simply the heat equation, and the essence of the argument is that the viscoelastic terms can be kept under control. Results along those lines were established by Engler [37] for shear flows of certain K-BKZ fluids, by Guillopé and

Saut [63] for the Johnson–Segalman model, and by Renardy [191] for a class of differential models.

Lions and Masmoudi [106] prove the global existence of weak solutions for the corotational Oldroyd model. The estimates rely crucially on the corotational structure and the result is unlikely to carry over to more general models. In [2] global existence is proved for a regularized system of equations for dumbbell models. A number of papers exploit the regularizing effect of singular kernels to obtain global weak solutions of various model equations arising from viscoelasticity [3,38,60,61,108]. These results are not immediately applicable to multidimensional incompressible fluids.

Nonlinear hyperbolic systems in one dimension do not allow global smooth solutions and generally form shocks in finite time (see the next section below). It is of interest to show existence of weak solutions beyond that point. Few results along these lines are known which allow for the dissipative terms associated with viscoelasticity, and they are limited to rather special models, see [13,127].

3. Development of singularities

3.1. *Hyperbolic shocks*

The explicit solution of scalar hyperbolic conservation laws in terms of characteristics shows that, in general, characteristics will intersect in finite time, precluding the global existence of smooth solutions to initial value problems. For pairs of conservation laws, Lax [99] gave a proof of development of singularities in finite time. The proof is based on analyzing the evolution of Riemann invariants.

The first indication of what might happen if hyperbolic equations are augmented by viscoelastic damping is in the work of Coleman, Gurtin and Herrera [17,18] on acceleration waves propagating into a medium at rest. For concreteness, consider one-dimensional motions of a viscoelastic medium described by an integral model of the form

$$u_{tt}(x,t) = \int_{-\infty}^{t} m(t-s)h\big(u_x(x,t), u_x(x,s)\big)_x \, \mathrm{d}s. \tag{28}$$

We assume that $u = 0$ ahead of a wave front $x > ct$ and that there is a jump in the second derivatives of u across this front. The analysis of [17,18] shows that the wave speed is given by

$$c^2 = h_{,1}(0,0) \int_0^\infty m(s)\, \mathrm{d}s, \tag{29}$$

and that the amplitude A of the jump satisfies the equation

$$\dot{A} = \alpha A^2 - \beta A, \tag{30}$$

where

$$\alpha = -\frac{1}{2c^2} h_{,11}(0,0) \int_0^\infty m(s)\, \mathrm{d}s,$$

$$\beta = -\frac{1}{2c^2} h_{,2}(0,0) m(0). \tag{31}$$

Here the notation $h_{,i}$ indicates the derivative of h with respect to the ith argument. If $\alpha \neq 0$, then solutions to (30) will become infinite in finite time if the initial value of A has the same sign as α and is sufficiently large. On the other hand, if the initial value of A has absolute value less than $|\beta/\alpha|$, then A will tend to zero regardless of sign. Thus acceleration waves of sufficiently large amplitude will evolve into a stronger singularity. The analysis also makes suggestions for singular kernels. If m has a strong enough singularity that it fails to be integrable, then the wave speed is infinite. On the other hand, if m is integrable, but $m(0) = \infty$, then waves propagate with finite speed, but the amplitude goes to zero instantly.

The first result on development of singularities from smooth data in a model of one-dimensional viscoelasticity is due to Slemrod [201]. Subsequently, a number of results along these lines appeared in [59,72,112,139]. A fairly general approach was given by Nohel and Renardy [126] for differential models and by Dafermos [20] for integral models. The idea of the proof is an extension of Lax's argument. With viscoelastic damping, there are no Riemann invariants, but there are "approximate" Riemann invariants, and development of singularities can be established by mimicking the Lax proof and controlling the error terms.

Nothing is known on a rigorous basis on what happens beyond the development of singularities. As reviewed in the previous section, results on existence of weak solutions are very limited and even these results do not elucidate the structure of singularities beyond their formation. It is of course to be expected that singularities take the form of shock fronts with discontinuities in u_t and u_x, and numerical evidence supports this [116,198].

The case of "weakly" singular kernels were m is integrable, but $m(0)$ is infinite raises intriguing questions. As pointed out above, acceleration waves are not possible in this case. On the other hand, traveling wave solutions with discontinuities in first derivatives of u are possible [58,147]. It is not known whether such discontinuities can evolve from initially smooth data.

3.2. *Breakup of liquid jets*

The problem of global existence for the Navier–Stokes equations is open only in the case of flows bounded by walls. In free surface flows global existence does not hold. One example of development of a singularity in finite time is the breakup of liquid jets into droplets. This is a flow which involves strong elongation and is profoundly affected by the presence of polymers which have a strong resistance to elongation.

Linear stability of an inviscid cylindrical jet was first studied by Rayleigh [140,141], and the extension of the results to the viscous case was completed by Chandrasekhar [10]. While experiments on jets of viscoelastic fluids show a strong stabilizing effect of viscoelasticity [53,54], linear stability analysis actually shows a destabilizing effect [53,94, 120]. This observation led to the conclusion that the stabilizing effect of polymers arises in the later stages of deformation where the high elongational resistance of the polymer becomes important [40,53]. Numerical simulations confirm this [7,41].

Formal asymptotic solutions for the approach to breakup in Newtonian jets were found in the mid-1990s by Papageorgiou [132] for the case of Stokes flow and by Eggers [33–35] (see also [9]) for the case with inertia. Underlying these solutions is a slender body approximation which assumes that axial velocity and stresses are approximately constant in the cross-section of the jet, and a similarity ansatz. The inertialess and inertial solution are profoundly different, while the case without inertia leads to a solution that is symmetric about the pinch point, the inertial solution is highly asymmetric.

The breakup of jets is most easily analyzed in a Lagrangian formulation. In this formulation, we consider the deformation of the jet from a reference configuration, in which the jet has a uniform thickness δ. Let X be the position of a fluid particle in this reference configuration and let $x(X,t)$ be the actual position of a fluid particle in space. The stretch is defined by $s = \partial x/\partial X$. In the slender body approximation, the evolution of the jet is governed by the equations (see e.g. [187])

$$\begin{aligned} s_t &= u_X, \\ \rho u_t &= \frac{\partial}{\partial X}\left(\frac{T_{xx}-T_{rr}}{s} + \frac{\sigma}{\delta\sqrt{s}}\right). \end{aligned} \tag{32}$$

Here u is the axial velocity, T_{xx} and T_{rr} are the axial and radial components of stress, ρ is the density and σ is the coefficient of surface tension. In the Newtonian case, the stresses are given by $T_{xx} = 2\eta s_t/s$, $T_{rr} = -\eta s_t/s$. In the case without inertia, this leads to the equation

$$3\eta s_t = \lambda(t)s^2 - \frac{\sigma}{\delta}s^{3/2}, \tag{33}$$

where $\lambda(t)$ is an unknown integration constant representing the force in the jet. Equation (33) would not be difficult to analyze if $\lambda(t)$ were given, but $\lambda(t)$ must be determined as part of the solution. For the similarity solutions below, we should require that u does not blow up outside the self-similar region; integration of the first equation of (32) then yields the condition

$$\int_{-\infty}^{\infty} s_t(X,t)\,\mathrm{d}X = 0. \tag{34}$$

With the breakup time fixed at $t = 0$ and the pinchoff point at $X = 0$, self-similar solutions for breakup are of the form

$$s(X,t) = (-t)^{-\alpha}\phi\left(\frac{X}{(-t)^{\beta}}\right), \qquad u(X,t) = (-t)^{\beta-\alpha-1}\psi\left(\frac{X}{(-t)^{\beta}}\right). \tag{35}$$

Similarity solutions have been found for Stokes flow as well as the case with inertia. In the Newtonian case, we have $\alpha = 2$, while $\beta = 5/2$ in the inertial case, and $\beta \sim 2.17487$ is the solution of a transcendental equation in the Stokes case. Numerical evidence supports the belief that generic initial data will evolve towards self-similar breakup, but no proof for this exists at this point (see [178,190] for partial results).

Viscoelastic fluids have a strong resistance to elongational flow, which makes them resist breakup. Indeed, in [166], a global existence result for (32) without inertia is shown if the constitutive model is the Oldroyd B fluid. At very high elongation rates, however, the elongational viscosity of polymers decreases again, and many models reflect such a decrease. In contrast to the Oldroyd B fluid, these models allow for self-similar breakup in a similar fashion to the Newtonian case. In the inertialess case, this has been analyzed for a number of viscoelastic models, for specifics, I refer to the review article [187] and the original papers [47,179–181]. Two qualitative features arise in viscoelastic fluids which are new compared to the Newtonian case:

(1) Some models allow for a breakup mechanism driven by elastic forces in which surface tension is not involved. Basically, this requires a model which is sufficiently elongation thinning at large deformations. Elastic forces are then sufficient to pull the fluid out of the neck without a need for surface tension. This was first observed in the numerical simulations of Hassager and Kolte [71] and then confirmed by analysis for a number of models [47,180,181].

(2) While a Newtonian jet pinches at a point in space, some models of viscoelastic flow predict breakup over a finite length (i.e. at the moment of breakup the fragments are already a finite distance apart). This occurs if $\alpha = \beta$ in the similarity solutions discussed above. See [179–181] for details.

Viscoelastic jets with the inclusion of inertia are less well studied. Some results for the Giesekus model appear in [182]. The behavior is quite similar to the Newtonian case. On the other hand, the behavior of power law fluids [28,29,186] can be quite different from the Newtonian situation.

4. Steady flows

4.1. *Existence theory*

The first result on existence of steady flows of viscoelastic fluids is due to Renardy [145], who considered multimode Maxwell models. For the upper convected Maxwell fluid, we have the momentum equation

$$\rho(\mathbf{v}\cdot\nabla)\mathbf{v} = \operatorname{div}\mathbf{T} - \nabla p + \mathbf{f}, \tag{36}$$

the incompressibility constraint, and the constitutive law,

$$(\mathbf{v}\cdot\nabla)\mathbf{T} - (\nabla\mathbf{v})\mathbf{T} - \mathbf{T}(\nabla\mathbf{v})^T + \lambda\mathbf{T} = \eta\lambda\big(\nabla\mathbf{v} + (\nabla\mathbf{v})^T\big). \tag{37}$$

Here $\mathbf{f}$ is a given body force assumed sufficiently small, and we consider flows in a bounded domain with homogeneous Dirichlet boundary conditions for the velocity. By taking the divergence of the constitutive law and inserting the result in the momentum equation, we obtain the equation

$$\begin{aligned}\mathbf{T} : \partial^2\mathbf{v} &+ \eta\lambda\Delta\mathbf{v} - \rho(\mathbf{v}\cdot\nabla)(\mathbf{v}\cdot\nabla)\mathbf{v} - \nabla q \\ &= -\big(\nabla\mathbf{v} + (\nabla\mathbf{v})^T\big)\nabla p - \rho(\nabla\mathbf{v})(v\cdot\nabla)\mathbf{v} + \lambda\rho(\mathbf{v}\cdot\nabla)\mathbf{v} \\ &\quad - (\mathbf{v}\cdot\nabla)\mathbf{f} - \lambda\mathbf{f} + (\nabla\mathbf{v})\mathbf{f}.\end{aligned} \tag{38}$$

Here q stands for the combination $(\mathbf{v}\cdot\nabla)p + \lambda p$. For slow flows, equation (38) is a perturbation of the Stokes equation, and an existence result for steady flows was obtained by an iteration which alternates between solving (38) and integrating the constitutive law along streamlines. The reformulation involving equation (38) was later used in numerical simulations [15], where the equation was named the explicitly elliptic momentum equation (EEME). The result in [145] is for fluids with differential constitutive models. An adaptation of the method for integral models was given in [149].

Guillopé and Saut [65] assume that the constitutive law is a small perturbation of the Newtonian fluid. By doing so, they are able to prove existence of steady flows which are not necessarily perturbations of the rest state, but perturbations of a given Newtonian flow. They need to assume that the Newtonian flow is stable.

A number of papers have extended the method of [145] to consider flows in unbounded domains, such as exterior flows, see e.g. [129,128,135]. The method has also been extended to compressible flows [118,205,66–68].

For the Navier–Stokes equations, a continuation argument based on degree theory and a priori estimates can be used to show existence of steady flows even for large data (see e.g. Chapter II of [207]). For viscoelastic flows, no sufficiently good a priori estimates are known to prove such a result. A regularized problem which allows a global existence proof for steady flows is considered in [36].

4.2. *Inflow boundaries*

Many problems considered in the applications of fluid dynamics have inflow and outflow boundaries. For instance, in modeling a manufacturing process, it is usually necessary to focus on a specific part, where the fluid enters and leaves from and to other parts of the process. Boundary conditions imposed at such boundaries are a mathematical artifact, since in reality there is no boundary. For the Navier–Stokes equations, from a mathematical point of view, it makes little difference whether homogeneous or inhomogeneous conditions are imposed on the velocity. Viscoelastic fluids, however, have stresses which depend on the deformation history experienced before the fluid entered the flow domain. It is therefore not sufficient to prescribe velocities at inflow boundaries. Naturally, it is necessary to restrict the type of constitutive model to formulate meaningful boundary value problems here, and the work in the literature so far has focused on differential models of Maxwell or Jeffreys type.

To show the nature of the problem, we focus on the simplest case. We consider the linearization of a Maxwell model at uniform flow. The flow domain is the strip $0 < x < 1$ in the plane, with periodic boundary conditions in the y-direction. The unperturbed flow has uniform velocity $U > 0$ in the x-direction and zero stresses. The linearized equations are the momentum equation

$$\rho U \frac{\partial \mathbf{v}}{\partial x} = \operatorname{div} \mathbf{T} - \nabla p, \tag{39}$$

incompressibility,

$$\operatorname{div} \mathbf{v} = 0, \tag{40}$$

and the constitutive law

$$U\frac{\partial \mathbf{T}}{\partial x} + \lambda \mathbf{T} = \eta\lambda\big(\nabla \mathbf{v} + (\nabla \mathbf{v})^T\big). \tag{41}$$

We can derive an "EEME equation" by applying the operator $U\partial/\partial x + \lambda$ to the momentum equation and using the constitutive law. This yields the equation

$$\rho U^2 \frac{\partial^2 \mathbf{v}}{\partial x^2} + \rho\lambda\frac{\partial \mathbf{v}}{\partial x} = \eta\lambda\Delta\mathbf{v} - \nabla q, \tag{42}$$

where

$$q = U\frac{\partial p}{\partial x} + \lambda p. \tag{43}$$

If $\rho U^2 < \eta\lambda$, then (42), together with the incompressibility condition, is of elliptic type and allows for a unique (up to a constant in q) solution for $\mathbf{v}$ and q if Dirichlet conditions for the velocity (subject to balance of total flux) are imposed. Once we know $\mathbf{v}$ and q, we can determine $\mathbf{T}$ and p from (41) and (43). To do so, we need data for $\mathbf{T}$ and p on the inflow boundary $x = 0$.

This, however, overlooks the fact that (42) is in fact not equivalent to the original momentum equation. Since we applied the operator $U\partial/\partial x + \lambda$ to the momentum equation in order to get (42), we need to impose the original momentum equation at the inflow boundary to get back. Thus, at $x = 0$, we require that

$$\begin{aligned}
&\frac{\partial T_{11}}{\partial x} + \frac{\partial T_{12}}{\partial y} - \frac{\partial p}{\partial x} = \rho U\frac{\partial u}{\partial x},\\
&\frac{\partial T_{12}}{\partial x} + \frac{\partial T_{22}}{\partial y} - \frac{\partial p}{\partial y} = \rho U\frac{\partial v}{\partial x}.
\end{aligned} \tag{44}$$

We can use (41) and (43) to eliminate the x-derivatives on the left hand side of these equations:

$$\begin{aligned}
&\frac{1}{U}\left(2\eta\lambda\frac{\partial u}{\partial x} - q - \lambda T_{11} + \lambda p\right) + \frac{\partial T_{12}}{\partial y} = \rho U\frac{\partial u}{\partial x},\\
&\frac{1}{U}\left(\eta\lambda\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \lambda T_{12}\right) + \frac{\partial T_{22}}{\partial y} - \frac{\partial p}{\partial y} = \rho U\frac{\partial v}{\partial x}.
\end{aligned} \tag{45}$$

On the inflow boundary $x = 0$, these two equations give $\partial p/\partial y + \lambda T_{12}/U$ and $\partial T_{12}/\partial y + \lambda p/U$ in terms of the other variables. We can thus find T_{12} and p on the inflow boundary by solving these two ODEs, and only data for T_{11} and T_{22} need to be prescribed.

If nonlinear terms are included in the equations, solutions can be constructed by an iteration which alternates between solving the EEME equation, the ODEs on the inflow boundary as discussed above, and the constitutive law [150]. In three dimensions, the prescription of inflow boundary conditions is less straightforward. Basically, four conditions are needed in addition to velocities, but it is not possible to identify four components of the stress tensor. In [150], this is dealt with by expanding $\mathbf{T}$ on the inflow boundary in a Fourier series and choosing different components to be prescribed, depending on the Fourier component. Alternatively, certain combinations of derivatives of $\mathbf{T}$ can be prescribed on the inflow boundary [154]. For Jeffreys type models, which include a Newtonian viscous term in addition to a Maxwell stress, the situation is more straightforward; a well-posed problem results if velocities are prescribed on both boundaries, and viscoelastic stresses are prescribed on the inflow boundary, no conditions restricting these stresses arise [151]. Of course, even in the Newtonian case, boundary conditions on inflow boundaries need not be Dirichlet conditions; for viscoelastic fluids, problems combining traction boundary conditions with stress conditions on the inflow boundary are discussed in [153,167]. For Jeffreys fluids, an appropriate choice of boundary conditions is the prescription of traction and all components of viscoelastic stress. The same is true for Maxwell models in the case of two space dimensions [167]. Inflow boundaries may join walls at corners, as for instance in a Poiseuille flow. An existence result addressing the behavior at such corners is shown in [159].

If $\rho U^2 > \eta\lambda$ in (42), the equation is no longer elliptic. This change of type is reflected in the need for prescribing one more inflow boundary condition and one less outflow boundary condition. Well-posedness of such problems is addressed in [157]. In the two-dimensional case for a Maxwell fluid, for instance, a possible set of boundary conditions is the prescription of the stream function on both boundaries, and vorticity, its normal derivative, and the diagonal components of the stress tensor at the inflow boundary.

Little is known about time dependent flows with inflow boundaries. An existence result appears in [170], but it is limited to the upper convected Maxwell model and a choice of inflow boundary conditions specifically "concocted" to make the estimates work.

4.3. *The high Weissenberg number limit*

Newtonian flows at high Reynolds number become extremely complex. If we simply set the viscosity to zero, we obtain the Euler equations. However, solving the Euler equations does not actually tell us what happens at high Reynolds number for a number of reasons:

(1) The Euler equations allow for a high degree of nonuniqueness. For instance, every parallel velocity field in a pipe is a solution of the Euler equations. So this tells us nothing about the flow rate we should expect at a high Reynolds number.
(2) Solutions of the Euler equations cannot satisfy all the boundary conditions. This leads to boundary layers near walls.
(3) Instabilities play an important role, and the dynamics becomes very complex.

In viscoelastic flows, the Weissenberg or Deborah number is a dimensionless measure of the importance of elasticity. If the Weissenberg number is high, similar issues to those listed above arise. There is actually a connection between high Weissenberg number flows and the Euler equations, which was pointed out in [171] for the upper convected Maxwell model, but is actually more general. Recall that the source of elasticity in polymeric fluids is the stretching of polymer molecules by the flow. At high Weissenberg number, we can expect these stretched molecules to align in whatever direction the flow stretches them in, leading to a predominant component of the stress which is one-dimensional. Let us write this component in the form

$$\mathbf{T} = \gamma \mathbf{u}\mathbf{u}^T, \tag{46}$$

where $\mathbf{u}$ is a vector, and γ is a scalar. For given $\mathbf{T}$ of this form, we can always choose γ and $\mathbf{u}$ in such a way that

$$\operatorname{div}(\gamma\mathbf{u}) = 0. \tag{47}$$

If we now assume creeping flow and insert (46) into the equation of motion, we find

$$\operatorname{div}\mathbf{T} - \nabla p = \gamma(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p = \mathbf{0}. \tag{48}$$

Equations (47) and (48) are the compressible Euler equations. There are some crucial differences to compressible Newtonian flow, however. First of all, $\mathbf{u}$ is not the velocity and γ is not the density, although p is the pressure. Second, the sign of the pressure term is different, reflecting the fact that elastic stresses in polymers are tensile while Reynolds stresses are compressive. Finally, there is no equation of state linking p to γ.

We note that so far we have neither invoked a specific constitutive law nor even mentioned the velocity of the fluid. All we have used is that the stress is one-dimensional. For the upper convected Maxwell model, in the high Weissenberg number limit, we have the following approximation to the constitutive equation:

$$(\mathbf{v}\cdot\nabla)\mathbf{T} - (\nabla\mathbf{v})\mathbf{T} - \mathbf{T}(\nabla\mathbf{v})^T = \mathbf{0}. \tag{49}$$

It is shown in [171] that, in a two-dimensional flow, there are the following relationships between $\mathbf{v}$ and $\mathbf{u}$ and γ introduced above:

$$\mathbf{v}\cdot\nabla\gamma = 0, \qquad \mathbf{v}\times(\gamma\mathbf{u}) = K\mathbf{e}_3. \tag{50}$$

Here $\mathbf{e}_3$ is the out-of-plane unit vector and K is an arbitrary constant. The condition that $\mathbf{v}$ be divergence-free leads to the "equation of state"

$$\operatorname{div}\mathbf{u} = \phi(\gamma). \tag{51}$$

In this equation, ϕ is not a specific function, but an arbitrary function. In particular, ϕ can be zero, and γ can be constant; in this case we recover the incompressible Euler equations.

4.4. *Stress boundary layers*

In this subsection, we are concerned not with the impact of viscoelasticity on high Reynolds number boundary layers, but with a completely different type of boundary layer that is of purely elastic origin. Sharp stress gradients near walls or separating streamlines are ubiquitous in numerical simulations of flows at moderately high Weissenberg numbers. Unlike traditional boundary layers, these boundary layers have nothing to do with satisfying additional boundary conditions.

The classical "toy problem" for a boundary layer is an equation like

$$\epsilon u'(x) + u(x) = f(x), \qquad u(0) = 0, \tag{52}$$

where ϵ is a small parameter. If we just set $\epsilon = 0$, we get $u(x) = f(x)$, but unless $f(0)$ happens to be zero, this does not satisfy $u(0) = 0$. Near $x = 0$, we need to modify the solution. In fact, near $x = 0$, we can set $y = x/\epsilon$, $v(y) = u(\epsilon y)$, and we get the approximation

$$v'(y) + v(y) = f(0), \tag{53}$$

with the solution $v(y) = f(0)(1 - \exp(-y))$. The two approximations "match" in the sense that

$$\lim_{y\to\infty} f(0)\big(1 - \exp(-y)\big) = \lim_{x\to 0} f(x). \tag{54}$$

In contrast, let us consider the problem

$$Wy\frac{\partial u}{\partial x} + u = f(x, y) \tag{55}$$

for $y > 0$, where, for the sake of concreteness, we assume $f(x, y)$ is 2π-periodic in x. We assume that W is large. Note that no boundary condition is involved. If we formally set W to infinity, we find the solution

$$u(x, y) = \frac{1}{2\pi}\int_0^{2\pi} f(x, y)\,\mathrm{d}x. \tag{56}$$

However, setting W to infinity becomes an invalid approximation if y is small of order $1/W$. If we set $y = z/W$, we obtain the approximation

$$z\frac{\partial u}{\partial x} + u = f(x, 0). \tag{57}$$

Again, the two approximations match; the limit of the inner solution for $z \to \infty$ equals the limit of the outer solution for $y \to 0$.

An analogous type of boundary layer arises in viscoelastic flows near a wall. Basically, viscoelastic fluids have memory, and high Weissenberg number means long range memory. Near the wall, however, fluid particles move very slowly, and at the wall itself, the stresses are determined purely by the local velocity gradient. As an example, we consider two-dimensional flow of the upper convected Maxwell fluid in the half-plane $y > 0$. The governing equations in the case of creeping flow are, in dimensionless form,

$$\begin{aligned}
&W\left[u\frac{\partial T_{11}}{\partial x} + v\frac{\partial T_{11}}{\partial y} - 2\frac{\partial u}{\partial x}T_{11} - 2\frac{\partial u}{\partial y}T_{12}\right] + T_{11} = 2\frac{\partial u}{\partial x},\\
&W\left[u\frac{\partial T_{12}}{\partial x} + v\frac{\partial T_{12}}{\partial y} - \frac{\partial u}{\partial y}T_{22} - \frac{\partial v}{\partial x}T_{11}\right] + T_{12} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\\
&W\left[u\frac{\partial T_{22}}{\partial x} + v\frac{\partial T_{22}}{\partial y} - 2\frac{\partial v}{\partial x}T_{12} - 2\frac{\partial v}{\partial y}T_{22}\right] + T_{22} = 2\frac{\partial v}{\partial y},\\
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,\\
&\frac{\partial^2}{\partial x\partial y}(T_{11} - T_{22}) + \left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}\right)T_{12} = 0.
\end{aligned} \tag{58}$$

At the wall, $y = 0$, where the velocities are zero, we find that

$$T_{22} = 0, \qquad T_{12} = \frac{\partial u}{\partial y}, \qquad T_{11} = 2W\left(\frac{\partial u}{\partial y}\right)^2. \tag{59}$$

Let us assume that the wall shear rate $\partial u/\partial y$ is of order 1, and that variations in x occur over a length scale of order 1. Then T_{11} is of order W, while T_{12} is of order 1. To balance $\partial^2 T_{11}/\partial x\partial y$ with $\partial^2 T_{12}/\partial y^2$, we should scale y with a factor $1/W$. A fully consistent scaling for the boundary layer is obtained if we set

$$\begin{aligned} &y = z/W, \quad u = \tilde{u}(x,z)/W, \quad v = \tilde{v}(x,z)/W^2, \\ &T_{11} = W\tilde{T}_{11}(x,z), \quad T_{12} = \tilde{T}_{12}(x,z), \quad T_{22} = \tilde{T}_{22}(x,z)/W. \end{aligned} \tag{60}$$

Retaining only the leading order terms in W, we obtain the following set of equations (the tildes have been suppressed)

$$\begin{aligned} &u\frac{\partial T_{11}}{\partial x} + v\frac{\partial T_{11}}{\partial z} - 2\frac{\partial u}{\partial x}T_{11} - 2\frac{\partial u}{\partial z}T_{12} + T_{11} = 0, \\ &u\frac{\partial T_{12}}{\partial x} + v\frac{\partial T_{12}}{\partial z} - \frac{\partial u}{\partial z}T_{22} - \frac{\partial v}{\partial x}T_{11} + T_{12} = \frac{\partial u}{\partial z}, \\ &u\frac{\partial T_{22}}{\partial x} + v\frac{\partial T_{22}}{\partial z} - 2\frac{\partial v}{\partial x}T_{12} - 2\frac{\partial v}{\partial z}T_{22} + T_{22} = 2\frac{\partial v}{\partial z}, \\ &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} = 0, \\ &\frac{\partial^2 T_{11}}{\partial x\partial z} + \frac{\partial^2 T_{12}}{\partial z^2} = 0. \end{aligned} \tag{61}$$

These boundary layer equations for the upper convected Maxwell fluid were first derived in [172]. They can be put into a simpler form by transforming the stresses to a basis aligned with the velocity field and making the streamfunction an independent variable in place of z [172]. For other models, different scalings apply, due to different behavior of the stresses at the wall. For instance, for the Giesekus model the boundary layer thickness is of order $W^{-1/2}$, while for the PTT model it is of order $W^{-1/3}$ [69]. In [172] similarity solutions of the boundary layer equations are discussed. These similarity solutions play a role in the reentrant corner singularity, which we shall discuss in the next subsection. In [176], an existence result is proved for solutions of the boundary layer equations which match a given periodic shear rate in the outer flow, provided that this given shear rate is sufficiently close to a constant.

Singular features in the high Weissenberg number limit also appear along streamlines coming off a wall, e.g. the wake of an obstacle or the lid in a driven cavity. The structure of such layers remains to be understood. At this point, analysis has been limited to the simpler problem of stress integration in a given velocity field. See [174,212] for high Weissenberg number flow past a cylinder and [184] for flow in a driven cavity.

4.5. *The reentrant corner singularity*

The flow through a sudden contraction is one of the most studied problems in viscoelastic fluid mechanics, experimentally as well as numerically. From a mathematical point of view, one of the major interests in this problem is the presence of corners.

The regularity of steady flow solutions is restricted if the boundary of the flow domain is not smooth. For Newtonian fluids, the behavior near corners is dominated by Stokes flow, i.e. by a linear elliptic equation. Dean and Montagnon [23] were the first to obtain formal approximations for two-dimensional Stokes flow near a corner using separation of variables in polar coordinates for the biharmonic equation. Kondratiev [93] developed a rigorous theory of existence and regularity for solutions of the Navier–Stokes equations in domains with corners; see also the monograph of Grisvard [62] for an overview of this subject.

For a corner between two walls, the velocity gradient at the corner is zero if the angle is less than 180 degrees, while it is infinite if the angle is greater. For viscoelastic flows, this makes a profound difference. Formally, any fluid can be approximated by the Newtonian fluid in the limit of zero velocity gradient. We can therefore expect the behavior near a convex corner to be dominated by Stokes flow even in non-Newtonian fluids and hope to analyze the corner behavior as a perturbation. A rigorous existence result along such lines is obtained for instance in [48]. On the other hand, if the corner is concave, the velocity gradient and stress are expected to be infinite, and the corner behavior is governed by the high Weissenberg number asymptotics, i.e. it is intrinsically nonlinear and dependent on the constitutive law.

The problem of integrating stresses in a fixed velocity field (assumed Newtonian) is studied in [163,173,49]. The full problem is very difficult even for formal asymptotics. At the formal level, a solution has been obtained for the upper convected Maxwell model. This solution is constructed by matched asymptotics; in the core region away from walls a potential solution of the Euler equations (see the previous section) applies, while near the walls the leading asymptotics is given by a similarity solution of the boundary layer equations.

We shall focus our discussion on the case of a 270 degree corner which is relevant for the contraction flow, but similar arguments apply for any angle. Potential flow in a 270 degree corner is given by the stream function

$$\tilde{\psi} = r^{2/3} \sin\left(\frac{2\theta}{3}\right), \tag{62}$$

where r and θ are the usual polar coordinates. In potential flow of a UCM fluid at high Weissenberg number, we must have the same streamlines, so the actual streamfunction ψ is a function of $\tilde{\psi}$: $\psi = f(\tilde{\psi})$. For the corner asymptotics, only the behavior of f as $\psi \to 0$ is relevant. By matching the order of magnitude of stresses in the potential flow with viscometric stresses which must apply near the wall, it can be determined (see [76]) that

$$f(\tilde{\psi}) \sim \tilde{\psi}^{7/3}. \tag{63}$$

Moreover, the potential flow solution loses its validity if θ is of order $r^{1/3}$.

The boundary layer equations for the UCM fluid have similarity solutions which satisfy a system of nonlinear ODEs with the independent variable $\xi = r^{-1/3}\theta$. To complete the asymptotic solution, such a solution must be found which matches to the potential flow solution as $\xi \to \infty$. This can be done only numerically. For the upstream boundary, solutions were constructed by Renardy [168], and the more difficult downstream problem was recently solved by Rallison and Hinch [138].

The corner asymptotics of the Oldroyd B fluid is identical to that of the UCM fluid. If the Newtonian term is made to dominate over the viscoelastic term, however, the region where viscoelastic stresses dominate shrinks, and a transition to Stokes flow occurs a short distance from the corner. This limit is investigated in [45].

In the solution discussed above, it is assumed that there is an upstream boundary with flow towards the corner. Some experiments and numerical results show a lip vortex, which makes the flow along both walls away from the corner. A recent paper by Evans [46] aims to explain such flows in terms of a hypothesized double layer structure [44], but the results are not conclusive at this point.

Numerical simulations encounter significant difficulties near reentrant corners. A downstream instability in the integration along streamlines, pointed out in [165] is likely to be a significant contributor to such difficulties.

5. Instabilities and change of type

5.1. *Flow instabilities*

The numerous instabilities which occur in Newtonian flows have always been a topic of much interest for mathematicians and have inspired a great body of work in dynamical systems, asymptotics and other field of mathematics. I refer to [31] for an overview of Newtonian flow instabilities. Viscoelastic effects can significantly alter the characteristics of these instabilities or lead to entirely new instabilities. Over the last thirty years, a significant literature has studied instabilities in viscoelastic flows, using analysis, numerics and experiments. The following discussion will limit itself to highlighting a few points, and I shall not attempt a more thorough review. The review articles of Larson [96] and Shaqfeh [197] are an excellent starting point for further reading.

(1) Parallel shear flows: In the 1970s it was widely believed that flow instabilities in parallel shear flow might explain extrudate instabilities in polymer processing known as melt fracture (we shall discuss melt fracture in the next subsection). A number of claims of instability emerged, based on faulty approximations or spurious numerics. As numerical capabilities became more developed, it became clear that, although viscoelastic effects lower the threshold for the Newtonian instability in plane Poiseuille flow, the search for purely elastic instabilities turned out negative, at least for the upper convected Maxwell and Oldroyd B models [146,204]. On the other hand, Wilson and Rallison [213] have found a flow instability in the Poiseuille flow of a strongly shear thinning White–Metzner fluid. Instabilities associated with non-monotone constitutive laws will be discussed in the next subsection.

(2) Shear flows with curved streamlines: The viscoelastic Taylor problem has been studied extensively. Purely elastic instabilities without inertia were first discovered by Larson, Shaqfeh and Muller [97]. Analogous instabilities occur in other shear flows with curved streamlines, such as Dean flow, flow between rotating discs, cone and plate flow and more complex flows such as the driven cavity problem.

(3) The Bénard problem: Since the Bénard instability is an instability of the rest state, the onset for viscoelastic fluids is identical to that for Newtonian fluids if it is through

a zero eigenvalue. However, for sufficiently elastic fluids, an oscillatory onset is possible [57,211].

(4) Interfacial instabilities: In parallel shear flow of two layers, the shear stress is continuous at the interface, but first and second normal stresses can have jump discontinuities. These discontinuities can drive instabilities and lead to interfacial waves. The first studies demonstrating such instabilities are due to Renardy [193] and Chen [12].

(5) Elongational flows: The effect of viscoelasticity on elongational flow instabilities, such as draw resonance in fiber spinning, has been found to be stabilizing. The effect of elasticity on jet breakup has already been discussed in an earlier section.

From the point of view of rigorous mathematical analysis, flow instabilities raise a number of questions, which are well understood in Newtonian flows, but largely open in the viscoelastic case.

(1) Is linear stability actually determined by the spectrum as is commonly assumed?
(2) How can we characterize continuous parts of the spectrum?
(3) Does linear stability imply nonlinear stability for small perturbations?
(4) Is there a center manifold reduction which gives a rigorous underpinning to the study of bifurcation?

The abstract techniques used to resolve these issues in the Newtonian case do not work for non-Newtonian flows or, more generally, for problems involving nonlinear hyperbolic PDEs; in particular, results which allow the determination of linear stability from the location of spectra do not apply to hyperbolic PDEs, see [164] for a counterexample. On the other hand, linear stability can be inferred if a resolvent estimate is known in addition to the location of the spectrum [51,75,137,82]. Rigorous stability results in viscoelastic flows are at this point rather fragmentary.

The first rigorous linear stability proof is for creeping plane Couette flow of the upper convected Maxwell fluid [160]. It is based on solution of the linearized equations in closed form, building on earlier work of Gorodtsov and Leonov [55], who solved the eigenvalue problem in closed form. Other positive results concern parallel shear flows of fluids of Jeffreys type [162,164], i.e. fluids with differential constitutive laws that include a Newtonian contribution. The technique in these papers exploits the separation of variables which is possible in parallel flow, which allows the reduction of the problem to ordinary differential equations. For the latter, it is then possible to relate resolvent bounds to the location of spectra by using a reduction to finite dimensions which exploits the correspondence between solutions and initial data.

Guillope and Saut [64,65] consider viscoelastic flows which are a perturbation of Newtonian flows and prove that they inherit the Newtonian stability characteristics. Renardy [169] allows somewhat larger perturbations. The result of [169] does not guarantee that the viscoelastic flow inherits stability from the Newtonian case, but it does guarantee resolvent estimates for large imaginary part which allow the determination of stability from spectra. The results of these papers also include a nonlinear stability result for small disturbances.

In [183], the theory of evolution semigroups [14] is applied to show that creeping flow of an upper convected Maxwell fluid has spectrally determined linear stability. Unfortunately, the proof exploits a cancellation which occurs only for this specific constitutive law.

Some results characterizing the spectra for flows of the upper convected Maxwell and Oldroyd B fluids were obtained in [55,214,175].

The only rigorous result on bifurcations in viscoelastic flows that I am aware of is [161], where a version of the center manifold theorem is proved that is applicable to the viscoelastic Bénard problem.

5.2. *Constitutive instabilities*

Many popular models of viscoelastic flows, for instance, the Giesekus, Johnson–Segalman and Doi–Edwards models, allow for a nonmonotone dependence of shear stress on shear rate in steady shear flow. If this occurs, the flow is unstable in the range where shear stress decreases with shear rate and the flow domain separates into regions of increasing shear rate, separated by a jump in shear rate. In contrast to the analogous situation in elasticity or generalized Newtonian fluids, the instability need not be associated with ill-posedness of the equations, see e.g. the analysis of the Johnson–Segalman model in [85].

Consider, for instance, a plane Poiseuille flow of the Johnson–Segalman model. The flow domain is $[-L, L]$, and we assume that the shear stress is given by a Newtonian part ηv_x and a polymeric contribution τ. If $\bar{f}$ is the pressure gradient, we have, in creeping flow,

$$\tau + \eta v_x = -\bar{f} x. \tag{64}$$

Moreover, the constitutive law of the Johnson–Segalman fluid, with

$$\mathbf{T} = \begin{pmatrix} \sigma & \tau \\ \tau & \gamma \end{pmatrix}, \tag{65}$$

leads to

$$\begin{aligned} &\sigma_t - (1+a)\tau v_x + \lambda\sigma = 0, \\ &\tau_t - \left[\frac{1+a}{2}\gamma - \frac{1-a}{2}\sigma + \mu\right] v_x + \lambda\tau = 0, \\ &\gamma_t + (1-a)\tau v_x + \lambda\gamma = 0. \end{aligned} \tag{66}$$

In a steady flow, this leads to

$$\tau = \frac{\lambda\mu v_x}{\lambda^2 + (1-a^2)v_x^2}, \tag{67}$$

i.e. the total shear stress is

$$\frac{\lambda\mu v_x}{\lambda^2 + (1-a^2)v_x^2} + \eta v_x. \tag{68}$$

This is a nonmonotone function of v_x if $-1 < a < 1$ and $\eta < \mu/(8\lambda)$.

The system (66) can be reduced to a system of two equations by introducing the combination

$$Z = \frac{1+a}{2}\gamma - \frac{1-a}{2}\sigma. \tag{69}$$

In a Poiseuille flow, we can also eliminate v_x from (64):

$$v_x = \frac{-\bar{f}x - \tau}{\eta}. \tag{70}$$

For any fixed x, we can then analyze the dynamics of (66) using phase plane methods.

The stability of solutions with discontinuous shear rates and the dynamics of stresses when this situation occurs has been analyzed in a number of papers in the literature, see for instance [43,92,91,113–115,125]. Both the Johnson–Segalman model and the Giesekus model, which leads to a rather analogous situation, have been studied.

While the shear stress is continuous across a discontinuity in shear rate, the normal stresses are usually not continuous. This can lead to an instability of the interface driven by a normal stress jump, just like the instability at the interface between two different fluids [194].

Much of the work on constitutive instabilities was originally driven by an attempt to understand the phenomenon of melt fracture. Melt fracture is an instability which occurs when certain polymer melts are extruded at sufficient high flow rates. Associated phenomena include a sudden increase in flow rate ("spurt"), and surface distortions on the extrudate, which can take the form of small scale irregularities ("sharkskin"), a periodic oscillation between sharkskin and smooth regions, or a gross snakelike distortion ("gross melt fracture"). I refer to the review article of Denn [25] for a more detailed description of melt fracture and attempts to explain it. The idea of explaining it by constitutive instabilities is that the spurt occurs when the wall shear stress reaches the relative maximum on the shear stress vs. shear rate curve and that a Hopf bifurcation leading to periodic dynamics can explain the stick-slip oscillations. Such a Hopf bifurcation has been found [115] when the flow rate rather than the pressure gradient is held fixed. The explanation of melt fracture by constitutive instabilities is at this point a minority view among rheologists, primarily due to the lack of more direct evidence for constitutive instabilities in polymer melts. On the other hand, constitutive instabilities associated with nonmonotone shear stress appear to occur in wormlike micelles [202]. Wormlike micelles are surfactant solutions in which surfactant particles align to form larger structures, which in many ways influence the rheology in the same way as chain molecules in a polymer.

An alternative explanation that has been offered for the spurt phenomenon is slip at the wall. If the slip velocity is a nonmonotone function of the wall shear stress, slip will also result in instability [133] and there is a possibility of stick-slip oscillations. A more subtle instability mechanism arises if a memory dependence is assumed, i.e. the slip velocity depends not just on the current wall shear stress, but on the history of wall shear stress, see [156,56].

5.3. *Characteristics and change of type*

The equations governing a Maxwell-like viscoelastic fluid form a first order quasilinear system. The analysis of characteristics yields necessary conditions for well-posedness in the usual fashion. A change of type in the time dependent equations can lead to Hadamard instability, while a change of type in the steady flow equations leads to the possibility of shock fronts.

The first papers analyzing such issues in viscoelastic flows are due to Rutkevich [196] and Ultman and Denn [209]. In the mid-1980s, more comprehensive studies appeared, see Joseph, Renardy and Saut [85], Luskin [110] and Hulsen [83]. Further work on characteristics and change of type can be found in the monograph of Joseph [86] and, for instance, in [19,50,95].

We shall give a brief discussion of characteristics for the Johnson–Segalman model, along the lines of [85].

The equations are

$$\begin{aligned}
&\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = \operatorname{div}\mathbf{T} - \nabla p,\\
&\operatorname{div}\mathbf{v} = 0,\\
&\frac{\partial \mathbf{T}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{T} - \frac{1+a}{2}\big(\nabla\mathbf{v}\mathbf{T} + \mathbf{T}(\nabla\mathbf{v})^T\big)\\
&\quad + \frac{1-a}{2}\big(\mathbf{T}\nabla\mathbf{v} + (\nabla\mathbf{v})^T\mathbf{T}\big) + \lambda\mathbf{T} = \mu\big(\nabla\mathbf{v} + (\nabla\mathbf{v})^T\big).
\end{aligned} \tag{71}$$

This is a quasilinear system of 10 equations (six in 2-D flows), which we can put in the form

$$\mathbf{A}_0(\mathbf{q})\mathbf{q}_t + \mathbf{A}_1(\mathbf{q})\mathbf{q}_x + \mathbf{A}_2(\mathbf{q})\mathbf{q}_y + \mathbf{A}_3(\mathbf{q})\mathbf{q}_z = \mathbf{F}(\mathbf{q}). \tag{72}$$

To investigate characteristic surfaces (see e.g. [185] for the basic definitions), we need to consider the equation

$$\det\left(\omega\mathbf{A}_0 + \sum_{l=1}^{3}\xi_l\mathbf{A}_l\right) = 0. \tag{73}$$

This equation can be shown to reduce to the following:

$$\begin{aligned}
&|\xi|^2\beta^4\left(\rho\beta^2 - |\xi|^2\left(\mu + \frac{1+a}{2}T_{aa} - \frac{1-a}{2}\Lambda_1\right)\right)\\
&\quad\times\left(\rho\beta^2 - |\xi|^2\left(\mu + \frac{1+a}{2}T_{aa} - \frac{1-a}{2}\Lambda_2\right)\right) = 0.
\end{aligned} \tag{74}$$

Here,

$$\beta = \omega + v_1\xi_1 + v_2\xi_2 + v_3\xi_3, \qquad T_{aa} = \mathbf{n}\cdot\mathbf{T}\mathbf{n}, \tag{75}$$

where $\mathbf{n}$ is a unit vector in the direction of (ξ_1, ξ_2, ξ_3). Moreover, with $\mathbf{P}$ denoting the orthogonal projection along $\mathbf{n}$, Λ_1 and Λ_2 are the eigenvalues of $\mathbf{PTP}$. In two dimensions, we get β^2 instead of β^4, and there is only one of the last two factors.

The following consequences can be deduced:

(1) Well-posedness: If ω as determined by (74) is not real, localized short wave disturbances can be expected to grow in a catastrophic manner, and ill-posedness of the initial value problem occurs. It turns out that solutions of (74) with ω not real exist

if one of the terms

$$\mu + \frac{1+a}{2} T_{aa} - \frac{1-a}{2} \Lambda_i \tag{76}$$

is negative. This is the case for an appropriate choice of ξ if

$$\frac{1-a}{2} \Lambda_{\max} - \frac{1+a}{2} \Lambda_{\min} > \mu, \tag{77}$$

where $\Lambda_{\max}$ and $\Lambda_{\min}$ are the largest and smallest eigenvalues of $\mathbf{T}$. Eigenvalues of $\mathbf{T}$ are greater than $-\mu/a$ if a is positive, and less than $-\mu/a$ if a is negative. Hence no ill-posedness occurs if $a = \pm 1$.

(2) Change of type in steady flow: This occurs if the fluid speed exceeds the speed of propagation of shear waves. For $\mathbf{T} = \mathbf{0}$, the condition becomes

$$\rho |v|^2 > \mu. \tag{78}$$

For nonzero $\mathbf{T}$, wave speeds become stress-dependent and anisotropic. This change of type is analogous to transonic flow in gas dynamics. The phenomenon of delayed die swell, in particular, can be explained as analogous to a gas dynamic shock. Joseph [86] was the first to advance the hypothesis of such a connection, which is further established by the numerical simulations of Delvaux and Crochet [24] and the analysis of Entov [42].

(3) Boundary conditions: For subcritical flows (i.e. the speed of the fluid is less than the wave speed), we need four conditions at an inflow boundary, in addition to the usual Newtonian boundary conditions. If the flow becomes supercritical, we have to drop a velocity boundary condition at the outflow boundary, and we need an extra condition at the inflow boundary. See the discussion of inflow boundary conditions in the section on steady flows.

6. Controllability of viscoelastic flows

Controllability is the possibility of steering a system from a given initial state to a desired final state using a control input from a given class. In the context of continuum mechanics, the control is usually a body force given in a part of the spatial domain or a boundary condition. Questions of controllability have been widely studied in elasticity and Newtonian fluid mechanics, but viscoelastic flows raise new issues.

A number of papers on controllability of linear viscoelastic media [98,101–103,107] extend results from the elastic case to viscoelastic media. In those works, the variables which are controlled are the displacement and velocity. This, however, is in a sense not the right problem. Unlike the elastic case, displacement and velocity do not constitute a "state" which determines the further evolution of the system. For a fluid with a differential constitutive law of Maxwell type, for instance, a state of the system appropriately is determined by velocity and stresses. The question whether stresses, in addition to velocities, can be controlled turns out to be far from straightforward.

Doubova et al. [30] pose this question, unfortunately the controllability results they claim are not true as stated. For a linear Maxwell fluid, the stress obeys the equation

$$\mathbf{T}_t + \lambda \mathbf{T} = \mu\big(\nabla \mathbf{v} + (\nabla \mathbf{v})^T\big). \tag{79}$$

It is obvious from this equation that the stress is the symmetric part of a gradient of a divergence-free vector if this is initially the case. It is therefore not possible to steer to a state of stress not satisfying this constraint, whatever we may do with a body force in the equation of motion or with boundary conditions. It is likely that the results of [30] hold if stresses are restricted accordingly (the paper is a brief announcement and no full version seems to have appeared). This raises an interesting question of what happens in nonlinear situation where symmetric parts of gradients are no longer an invariant subspaces for the evolution of stresses.

A few recent articles by Renardy [188,189,192] investigate parallel shear flows of viscoelastic fluids. For a linear Maxwell model, the governing equations are

$$\rho v_t = \tau_x, \qquad \tau_t + \lambda \tau = \mu v_x + f, \tag{80}$$

where v is the velocity, τ the shear stress, and f is a given body force. For simplicity, we consider homogeneous Dirichlet boundary conditions: $v(0,t) = v(L,t) = 0$. For given initial conditions for v and τ, we want to pick the control f on $(a,b) \subset (0,L)$ in such a way that v and τ assume given values at a final time $t = T$. Equations (80) can be reformulated as a lower order perturbation of the wave equation. Existing results in the literature [210] then guarantee exact controllability under the usual assumption that T is large enough for waves to propagate back and forth across the uncontrolled region. In [188], multimode Maxwell models with several relaxation times are also considered. In this case, unless $(a,b) = (0,L)$, only approximate controllability holds.

For linear viscoelastic shear flow, there is only one nonzero component of stress, the shear stress. In the nonlinear case, there are first and second normal stresses. Thus the linear problem makes no suggestion on whether we can control the normal stresses. In [189], homogeneous shear flows are considered, with the shear rate in the role of the control. For the upper convected Maxwell model, the second normal stress difference is identically zero, and for the remaining two stress components the set of states reachable from a given initial condition is described by an inequality. Similar results are derived for a number of other constitutive models. In [192], the control of spatially inhomogeneous shear flows is analyzed. It is shown that the inequality between shear and normal stress derived in [189] ensures reachability of a final state only if a body force is allowed on the entire interval $(0,L)$. If (a,b) is a proper subset of $(0,L)$, additional restrictions of a nonlocal nature apply.

These fragmentary results show that the question of controllability of nonlinear viscoelastic flows is far from straightforward. In general, the characterization of final states which can be reached from a given initial state is likely to be quite difficult.

7. Concluding remarks

Like all articles of its kind, this article reflects the interests and activities of its author. In selecting topics, a number of choices were made. First of all, I have focused on equations arising from the modeling of viscoelastic polymeric liquids. This means that a sizable mathematical literature on other kinds of non-Newtonian fluids, e.g. generalized Newtonian fluids, micropolar fluids and liquid crystals has been ignored. On the mathematical side, I have focused on topics generally of interest to analysts working in partial differential equations, such as questions of existence, stability, and development of singularities. I have not attempted to review the extensive literature on a number of other aspects of viscoelastic flow, for instance numerical simulation.

While the analysis of viscoelastic flows has made major progress over the past thirty years, the article also points to a number of major open questions in the field, which have so far yielded only fragmentary results and which pose interesting challenges for the future. Among those, I mention in particular the following:

(1) Questions of global existence (or possibly the lack of it) for large data, both for initial value problems and steady flows.
(2) A rigorous proof for the asymptotics of jet breakup.
(3) The understanding of flows in the high Weissenberg number limit.
(4) The rigorous analysis of stability and bifurcation.
(5) The characterization of final states (in terms of stress as well as motion) to which a flow can be controlled.

A more comprehensive discussion of many of the topics touched upon in this article can be found in the author's monograph [177].

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CHAPTER 6

Application of Monotone Type Operators to Parabolic and Functional Parabolic PDE's

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1. Introduction

The aim of this chapter is to formulate main notions and results in the theory of monotone type operators and apply them to (possibly nonlinear) parabolic differential equations and parabolic functional differential equations.

It is well known the importance of linear and nonlinear parabolic partial differential equations in physical, chemical, biological etc. applications (e.g. in the classical models of heat conduction, diffusion). The classical results on linear and quasilinear second order parabolic equations can be found in the monographs [24,35,40] and also in books [6,50, 52,72].

Partial functional differential equations arise in biology, chemistry, physics, climatology (see, e.g., [4,5,11,16,18–20,30,31,42,54,74] and its references). The systematic study of such equations from the dynamical systems and semigroups point of view began in the 70s. Several results in this direction can be found in monographs [50,72,74]. This approach is mostly based on arguments used in the theory of ordinary differential equations and functional differential equations (see [2,3,21,27–29,48,49]).

In classical work [41] of J.L. Lions one can find the fundamental results on monotone type operators and their applications to nonlinear partial differential equations. Further important monographs are written by E. Zeidler [75] and H. Gajewski, K. Gröger, K. Zacharias [26], S. Fučik, A. Kufner in [25]. A good summary of further results on monotone type operators, based on degree theory (see, e.g., [17]) and its applications to nonlinear evolution equations is in the works [7] and [47] of V. Mustonen and J. Berkovits. By using the theory of monotone type operators one obtains directly global existence of solutions, also for higher order nonlinear parabolic equations, satisfying certain conditions which are more restrictive (in some sense) than in the case of the previous approach.

It turned out that one can apply the theory of monotone type operators (e.g. pseudomonotone operators) to nonlinear parabolic functional differential equations and systems to get existence, uniqueness theorems on weak solutions and results on qualitative properties of solutions, including "non-uniformly parabolic" equations.

The above mentioned works [26,41,75] contain also applications of monotone type operators to nonlinear elliptic equations. In works [23,36–39,45,46,73] one can find extensions of the applications to strongly nonlinear elliptic equations and boundary value problems on unbounded domains. Further, it was possible to apply the theory to functional elliptic equations and elliptic variational inequalities with usual and "non-local" boundary conditions (see, e.g. [55–57,71]).

Now we give the structure of this chapter. In Section 2 the abstract Cauchy problem is considered for first order evolution equations in a finite interval. These general results will be applied in Sections 3–7. In Section 3 the main results on existence, uniqueness and continuous dependence of the weak solutions of higher order nonlinear parabolic differential equations are shown. In Section 4 higher order nonlinear functional parabolic equations are considered, where only the lower order terms contain functional dependence. In Section 5 second order nonlinear parabolic functional differential equations are studied, where also the main part contains functional dependence. Section 6 is devoted to existence and qualitative properties of solutions of parabolic functional differential equations in $(0, \infty)$. Finally, in Section 7 we study further applications of monotone type operators, e.g. to systems of functional parabolic equations. In Sections 3–7 several examples are given for the "general" results.

2. Abstract Cauchy problem for first order evolution equations

Here we summarize the main definitions and theorems on first order (possibly nonlinear) evolution equations, based on the theory of monotone type operators. The detailed exposition of the results and proofs see, e.g., in the monographs [25,26,41,75] and in the works [7,47].

2.1. *Basic definitions*

DEFINITION 2.1. Let V be a Banach space, $0 < T < \infty$, $1 \leqslant p < \infty$. Denote by $L^p(0, T; V)$ the set of measurable functions $f : (0, T) \to V$ such that $\|f\|_V^p$ is integrable and define the norm by

$$\|f\|^p_{L^p(0,T;V)} = \int_0^T \|f(t)\|_V^p \, \mathrm{d}t.$$

$L^p(0, T; V)$ is a Banach space over $\mathbb{R}$ and $\mathbb{C}$, respectively (identifying functions that are equal almost everywhere on $(0, T)$). If V is separable then $L^p(0, T; V)$ is separable, too. If V is uniformly convex and $1 < p < \infty$ then $L^p(0, T; V)$ is uniformly convex.

Denoting by $V^\star$ the dual space of V and by $\langle \cdot, \cdot \rangle$ the dualities in spaces $V^\star$, V, we have for all $f \in L^p(0, T; V)$, $g \in L^q(0, T; V^\star)$ with $1 < p < \infty$, $1/p + 1/q = 1$ the Hölder inequality

$$\left| \int_0^T \langle g(t), f(t) \rangle \, \mathrm{d}t \right| \leqslant \left[\int_0^T \|g(t)\|^q_{V^\star} \, \mathrm{d}t \right]^{1/q} \left[\int_0^T \|f(t)\|_V^p \, \mathrm{d}t \right]^{1/p}.$$

Further, for $1 < p < \infty$ the dual space of $L^p(0, T; V)$ is isomorphic and isometric with $L^q(0, T; V^\star)$. Thus we may identify the dual space of $L^p(0, T; V)$ with $L^q(0, T; V^\star)$. Consequently, if V is reflexive then $L^p(0, T; V)$ is reflexive for $1 < p < \infty$.

DEFINITION 2.2. Let V be a real separable and reflexive Banach space and H a real separable Hilbert space with the scalar product $(\cdot, \cdot)$ such that the embedding $V \subset H$ is continuous and V is dense in H. Then the formula

$$\langle \tilde{v}, u \rangle = (v, u), \quad u \in V, \ v \in H$$

defines a linear continuous functional $\tilde{v}$ over V and it generates a bijection between H and a subset of $V^\star$, i.e. we may write

$$V \subset H \subset V^\star$$

which will be called an evolution triple.

A typical example for an evolution triple is the following. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, m a nonnegative integer and $1 \leqslant p < \infty$. Denote by $W^{m,p}(\Omega)$ the Sobolev space of (real valued) measurable functions $u : \Omega \to R$ with the norm

$$\|u\|_{W^{m,p}(\Omega)} = \left[\sum_{|\alpha| \leqslant m} \int_\Omega \left| D^\alpha u \right|^p \mathrm{d}x \right]^{1/p}$$

where $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $D_j = \partial/\partial x_j$. (The detailed investigation of Sobolev spaces see, e.g., in [1].) Let V be a closed linear subspace of $W^{m,p}(\Omega)$ with $2 \leqslant p < \infty$, $m \geqslant 1$ and $H = L^2(\Omega)$. Then $V \subset H \subset V^\star$ is an evolution triple.

Now we define generalized derivatives of functions $u \in L^p(0, T; V)$.

DEFINITION 2.3. Let $V \subset H \subset V^\star$ be an evolution triple, $u \in L^p(0, T; V)$. If there exits $w \in L^q(0, T; V^\star)$ such that

$$\int_0^T \varphi'(t) u(t)\, \mathrm{d}t = -\int_0^T \varphi(t) w(t)\, \mathrm{d}t$$

for all $\varphi \in C_0^\infty(0, T)$ (i.e. for all infinitely times differentiable functions on $(0, T)$ with compact support) then w is called generalized derivative of u.

In the above equality $u(t) \in V$ is considered as an element of $V^\star$. In this case we shall write shortly $u' \in L^q(0, T; V^\star)$; the generalized derivative is unique.

THEOREM 2.1. *Let $V \subset H \subset V^\star$ be an evolution triple*, $1 < p < \infty$, $1/p + 1/q = 1$, $0 < T < \infty$. *Then*

$$W_p^1(0, T; V, H) = \left\{ u \in L^p(0, T; V) \colon u' \in L^q(0, T; V^\star) \right\}$$

with the norm

$$\|u\| = \|u\|_{L^p(0,T;V)} + \|u'\|_{L^q(0,T;V^\star)}$$

is a Banach space. $W_p^1(0, T; V, H)$ is continuously embedded into $C([0, T]; H)$ (the space of continuous functions $v : [0, T] \to H$ with the supremum norm) in the following sense: to $u \in W_p^1(0, T; V, H)$ there is a uniquely defined $\tilde{u} \in C([0, T]; H)$ such that $u(t) = \tilde{u}(t)$ for a.e. $t \in [0, T]$. Further, the following integration by parts formula holds for arbitrary $u, v \in W_p^1(0, T; V, H)$ functions and $0 \leqslant s < t \leqslant T$:

$$\big(u(t), v(t)\big) - \big(u(s), v(s)\big) = \int_s^t \left[\langle u'(\tau), v(\tau) \rangle + \langle v'(\tau), u(\tau) \rangle \right] \mathrm{d}\tau. \tag{2.1}$$

(In the last formula $u(t), u(s)$ mean the values of the above $\tilde{u} \in C([0, T]; H)$ in t, s, respectively.) In the case $v = u$ we obtain from (2.1)

$$\left\| u(t) \right\|_H^2 - \left\| u(s) \right\|_H^2 = 2 \int_s^t \langle u'(\tau), u(\tau) \rangle\, \mathrm{d}\tau$$

for any $u \in W_p^1(0, T; V, H)$.

In [41] it is proved

THEOREM 2.2. *Let $V \subset H \subset V^\star$ be an evolution triple and V_1 a Banach space such that $V \subset V_1 \subset V^\star$ where the embedding $V_1 \subset V^\star$ is continuous and the embedding $V \subset V_1$ is compact.*

Then (with $1 < p < \infty$) the embedding $W_p^1(0, T; V, H) \subset L^p(0, T; V_1)$ is compact.

Now we can formulate the "abstract" Cauchy problem. Let $V \subset H \subset V^\star$ be an evolution triple, $1 < p < \infty$, $1/p + 1/q = 1$, $0 < T < \infty$ and $A : L^p(0, T; V) \to L^q(0, T; V^\star)$ a given (nonlinear) operator; $u_0 \in H$, $f \in L^q(0, T; V^\star)$. We want to find $u \in W_p^1(0, T; V, H)$ satisfying

$$u' + A(u) = f, \qquad u(0) = u_0. \tag{2.2}$$

By Theorem 2.1 the initial condition $u(0) = u_0$ makes sense. Sometimes problem (2.2) is considered in the case when the domain of the operator A is not the whole space $L^p(0, T; V)$.

2.2. *Evolution equations with monotone operators*

We shall formulate existence and uniqueness theorems on problem (2.2). Let X be a Banach space, $X^\star$ its dual space and denote by $[\cdot,\cdot]$ the dualities in $X^\star$, X.

DEFINITION 2.4. Operator $A : X \to X^\star$ is called monotone if

$$\big[A(x_1) - A(x_2), x_1 - x_2\big] \geqslant 0 \quad \text{for all } x_1, x_2 \in X.$$

A is called bounded if it maps bounded sets of X into bounded sets of $X^\star$.

A is called hemicontinuous if for arbitrary fixed $u, v, w \in X$ the function

$$\lambda \mapsto [A(u + \lambda v), w], \qquad \lambda \in \mathbb{R},$$

is continuous.

Finally, A is called demicontinuous if it is continuous with respect to the strong topology in X and the weak topology in $X^\star$.

DEFINITION 2.5. Operator $A : X \to X^\star$ is called pseudomonotone if

$$(u_n) \to u \quad \text{weakly in } X \quad \text{and} \quad \limsup_{n\to\infty}\big[A(u_n), u_n - u\big] \leqslant 0 \tag{2.3}$$

imply

$$\lim_{n\to\infty}\big[A(u_n), u_n - u\big] = 0 \quad \text{and} \quad \big(A(u_n)\big) \to A(u) \quad \text{weakly in } X^\star. \tag{2.4}$$

Operator $A : X \to X^\star$ is called of class (S_+) if (2.3) implies

$$(u_n) \to u \quad \text{strongly in } X.$$

It is not difficult to show (see, e.g., [41,75]) that if $A : X \to X^\star$ is monotone, bounded and hemicontinuous then it is pseudomonotone. Further, if A is bounded and pseudomonotone then it is demicontinuous. Finally, if A is demicontinuous and of class (S_+) then it is pseudomonotone.

DEFINITION 2.6. Operator $A : X \to X^\star$ is called coercive if

$$\lim_{\|x\|\to\infty} \frac{[A(x), x]}{\|x\|} = +\infty.$$

THEOREM 2.3. *Let $V \subset H \subset V^\star$ be an evolution triple, $1 < p < \infty$, $0 < T < \infty$. Assume that for all fixed $t \in [0, T]$, $\tilde{A}(t) : V \to V^\star$ is monotone, hemicontinuous and bounded in the sense*

$$\big\|\tilde{A}(t)(v)\big\|_{V^\star} \leqslant c_1\|v\|_V^{p-1} + k_1(t) \tag{2.5}$$

for all $v \in V$, $t \in [0, T]$ with suitable constant c_1 and function $k_1 \in L^q(0, T)$. Further, $\tilde{A}(t)$ is coercive in the sense: there are constant $c_2 > 0$, $k_2 \in L^1(0, T)$ such that

$$\big\langle\tilde{A}(t)(v), v\big\rangle \geqslant c_2\|v\|_V^p - k_2(t) \tag{2.6}$$

for all $v \in V$, $t \in [0, T]$. Finally, for arbitrary $u, v \in V$, the function

$$t \mapsto \big\langle\tilde{A}(t)(u), v\big\rangle, \quad t \in [0, T] \text{ is measurable.}$$

Then for arbitrary $f \in L^q(0, T; V^\star)$ and $u_0 \in H$ there exists a unique solution of problem (2.2) with the operator A defined by $[A(u)](t) = [\tilde{A}(t)](u(t))$.

PROOF. According to [75] the proof is based on Galerkin's approximation. Since V is separable, there exists a countable set of linearly independent elements $w_1, \dots, w_k, \dots$ such that their finite linear combinations are dense in V. We shall find the m-th approximation of a solution u in the form

$$u_m(t) = \sum_{k=1}^{m} a_{km}(t)w_k \quad \text{with some } a_{km} \in W^{1,q}(0, T)$$

such that for a.e. $t \in [0, T]$

$$\begin{aligned} &\big\langle u_m'(t), w_j\big\rangle + \big\langle\tilde{A}(t)\big[u_m(t)\big], w_j\big\rangle = \big\langle f(t), w_j\big\rangle, \\ &u_m(0) = u_{m0} \in \operatorname{span}(w_1, \dots, w_m), \quad j = 1, \dots, m \end{aligned} \tag{2.7}$$

where $(u_{m0}) \to u_0$ in H. (2.7) is a system of ordinary differential equations for a_{km}:

$$\begin{aligned} &\sum_{k=1}^{m} a_{km}'(t)(w_k, w_j) + \left\langle \tilde{A}(t)\left[\sum_{k=1}^{m} a_{km}(t)w_k\right], w_j\right\rangle = \big\langle f(t), w_j\big\rangle, \\ &a_{jm}(0) = \alpha_{j0}, \quad j = 1, \dots, m \end{aligned} \tag{2.8}$$

which can be transformed to explicit form since $\det(w_k, w_j) \neq 0$.

By the last assumption of Theorem 2.3

$$a_j(t,z) = a_j(t, z_1, \dots, z_m) = \left\langle \tilde{A}(t)\left[\sum_{k=1}^{m} z_k w_k\right], w_j \right\rangle$$

is measurable in t (with fixed z) and continuous in $z = (z_1, \dots, z_m)$, because

$$A : L^p(0,T;V) \to L^q(0,T;V^\star)$$

is monotone, hemicontinuous, bounded by the assumptions of the theorem, thus it is pseudomonotone and so it is demicontinuous. From (2.5) follows that $|a_j(t,z)|$ locally can be estimated by an integrable $M(t)$. Consequently, by theorem of Carathéodory, there exists a solution of (2.8) in a neighbourhood of 0. The coercivity condition (2.6) implies that the solution u_m can be extended to the whole $[0,T]$.

Further, by using the notations $X = L^p(0,T;V)$, $X^\star = L^q(0,T;V^\star)$, we obtain that

$$\|u_m\|_X, \sup_{t\in[0,t]} \|u_m(t)\|_H, \quad m = 1,2,\dots, \text{ are bounded,} \tag{2.9}$$

hence $\|A(u_m)\|_{X^\star}$ is bounded, too. Since X, $X^\star$ and H are reflexive, there exists a subsequence of (u_m), again denoted by (u_m), such that

$$\begin{aligned} &(u_m) \to u \quad \text{weakly in } X, \qquad \big(A(u_m)\big) \to w \quad \text{weakly in } X^\star, \\ &\big(u_m(T)\big) \to z \quad \text{weakly in } H. \end{aligned} \tag{2.10}$$

By using (2.10) and the relation $(u_m(0)) \to u_0$ in H, one can derive from (2.7), as in [75]

$$u' \in W_p^1(0,T;V,H), \quad u' + w = f, \quad u(0) = 0, \quad u(T) = z, \tag{2.11}$$

$$\limsup_{m\to\infty}\big[A(u_m), u_m - u\big] \leqslant 0. \tag{2.12}$$

Since A is pseudomonotone, (2.10), (2.12) imply $w = A(u)$ which means that u is a solution of (2.2). □

Uniqueness of the solution follows from monotone property of $A : X \to X^\star$.

REMARK 2.1. According to the above proof, a subsequence of the Galerkin solutions (u_m) converges weakly to a solution u of (2.2). Since u is unique, the total sequence (u_m) is also weakly converging to u. By Theorem 2.2, (u_m) converges to u in the norm of $L^p(0,T;V_1)$ if V is compactly embedded in V_1 and V_1 is continuously impeded in $V^\star$. Further, one can show (see, e.g. [75]) that $(u_m) \to u$ in the norm of $C([0,T];H)$, too.

If the operator $A : X \to X^\star$ is of class (S_+), then

$$\begin{aligned} &(u_m) \to u \quad \text{strongly in } X \text{ since} \\ &(u_m) \to u \quad \text{weakly in } X \quad \text{and} \quad \limsup_{m\to\infty}\big[A(u_m), u_m - u\big] \leqslant 0. \end{aligned}$$

REMARK 2.2. Assume that the conditions of Theorem 2.3 are satisfied such that $\tilde{A}(t)$ is uniformly monotone in the sense

$$\langle [\tilde{A}(t)](v_1) - [\tilde{A}(t)](v_2) \rangle \geqslant c\|v_1 - v_2\|_V^p, \quad v_1, v_2 \in V \tag{2.13}$$

with some constant $c > 0$, for all $t \in [0, T]$. Then the solution of (2.2) continuously depends on f and u_0: if u_j is a solution of (2.2) with $f = f_j$, $u_0 = u_{0j}$ $(j = 1, 2)$ then for all $t \in [0, T]$

$$\begin{aligned} &\|u_1(t) - u_2(t)\|_H^2 + c\|u_1 - u_2\|_{L^p(0,T;V)}^p \\ &\quad \leqslant \tilde{c}\|f_1 - f_2\|_{L^q(0,T;V^\star)}^q + \|u_{10} - u_{20}\|_H^2 \end{aligned} \tag{2.14}$$

with some constant $\tilde{c}$.

One gets inequality (2.14) by Young's inequality, applying (2.2) to $u_1 - u_2$.

2.3. *Evolution equations with pseudomonotone operators*

When proving existence of solutions of (2.2), we used that A is pseudomonotone, and not directly monotonicity of A. However, in applications to evolution equations, we can generally prove pseudomonotonicity of $A : L^p(0, T; V) \to L^q(0, T; V^\star)$ only if A is monotone. In order to get existence for more general operators A, it is convenient to introduce a generalization of pseudomonotone operators.

DEFINITION 2.7. Denote by L the operator defined by

$$D(L) = \{u \in W_p^1(0, T; V, H)\colon u(0) = 0\}, \quad Lu = u'.$$

Then L is a closed linear densely defined monotone map from $D(L) \subset X$ to $X^\star$. Further, L is maximal monotone, which means that its graph is not a proper subset of any monotone set in $X \times X^\star$.

DEFINITION 2.8. Let X be a reflexive Banach space and M a linear densely defined maximal monotone map from $D(M) \subset X$ to $X^\star$. A bounded, demicontinuous operator $A : X \to X^\star$ is called pseudomonotone with respect to $D(M)$ if for any sequence (u_n) in $D(M)$ with

$$(u_n) \to u \quad \text{weakly in } X, \quad (Mu_n) \to Mu \quad \text{weakly in } X^\star \quad \text{and}$$
$$\limsup_{m\to\infty} [A(u_n), u_n - u] \leqslant 0,$$

we have

$$\lim_{n\to\infty} [A(u_n), u_n - u] = 0 \quad \text{and} \quad (A(u_n)) \to A(u) \quad \text{weakly in } X^\star.$$

By using degree theory (see, e.g. [17]), in [7] it was proved

THEOREM 2.4. *Let X be a reflexive and uniformly convex Banach space. Assume that M is a closed linear densely defined maximal monotone map from $D(M) \subset X$ to $X^\star$ and $A : X \to X^\star$ is bounded, demicontinuous, pseudomonotone with respect to $D(M)$ and it is coercive.*

Then for arbitrary $f \in X^\star$ there exists $u \in D(M)$ such that

$$Mu + A(u) = f. \tag{2.15}$$

We shall apply the following particular case of Theorem 2.4

THEOREM 2.5. *Let V be a reflexive, separable and uniformly convex Banach space, $1 < p < \infty$, $V \subset H \subset V^\star$ an evolution triple, $X = L^p(0, T; V)$. Assume that $A : X \to X^\star$ is bounded, demicontinuous, pseudomonotone with respect to $D(L)$, given in Definition 2.7 and it is coercive in the sense*

$$\frac{\int_0^t \langle [A(u)](\tau), u(\tau)\rangle \, \mathrm{d}\tau}{\|u\|_{L^p(0,t;V)}} \to +\infty \quad \text{uniformly in } t \in [0, T]$$
$$\text{as } \|u\|_{L^p(0,t;V)} \to \infty. \tag{2.16}$$

Finally, assume that $A : L^p(0, T; V) \to L^q(0, T; V^\star)$ is of Volterra type: the restriction of $A(u)$ to $[0, t]$ depends only on the restriction of u to $[0, t]$ for all $u \in L^p(0, T; V)$, $t \in [0, T]$.

Then for arbitrary $f \in X^\star$ there exists $u \in D(L)$ such that

$$\begin{aligned} &Lu + A(u) = f, \quad \text{i.e.} \\ &u \in L^p(0, T; V), \quad u' \in L^q\bigl(0, T; V^\star\bigr), \\ &u'(t) + \bigl[A(u)\bigr](t) = f(t) \quad \text{for a.e. } t \in (0, T) \quad \text{and} \quad u(0) = 0. \end{aligned} \tag{2.17}$$

Theorem 2.5 can be proved also by Galerkin's method, similarly to the proof of Theorem 2.3. In the last case one obtains (instead of (2.8)) the problem

$$\begin{aligned} &\sum_{k=1}^m a'_{km}(t)(w_k, w_j) + \left\langle \left[A\left(\sum_{k=1}^m a_{km}(t) w_k \right) \right](t), w_j \right\rangle = \langle f(t), w_j\rangle, \\ &a_{jm}(0) = 0, \quad j = 1, \dots, m \end{aligned} \tag{2.18}$$

which is a system of functional differential equations for $a_{1m}, \dots, a_{mm}$ with the above homogeneous initial condition. By using the assumptions of Theorem 2.5, one can show that Carathéodory conditions for the system of functional differential equations are fulfilled and therefore (see [27]) there exist local solutions for all m.

By (2.16) the solutions can be extended to $[0, T]$ and (2.9) holds. Further, one shows in the same way that (2.10)–(2.12) are valid. Finally, it is not difficult to derive from

$$\begin{aligned} &\bigl\langle u'_m(t), w_j\bigr\rangle + \bigl\langle \bigl[A(u_m)\bigr](t), w_j\bigr\rangle = \bigl\langle f(t), w_j\bigr\rangle, \\ &u_m(0) = 0, \quad j = 1, \dots, m \end{aligned}$$

that (u'_m) is bounded in $L^q(0, T; V^\star)$. Thus there exists a subsequence of (u_m) (again denoted by (u_m)) for which

$$u_m \in D(L), \quad (Lu_m) \to Lu \quad \text{weakly in } X^\star = L^q\big(0, T; V^\star\big).$$

Since A is pseudomonotone with respect to $D(L)$, we obtain $w = A(u)$ and so u is a solution of (2.17).

REMARK 2.3. If operator $A : L^p(0, T; V) \to L^q(0, T; V^\star)$ is monotone then solution of (2.17) is unique (as in the case of Theorem 2.3).

Assume that A is strictly monotone in the sense:

$$\int_0^t \big\langle \big[A(u_1)\big](\tau) - \big[A(u_2)\big](\tau)\big\rangle \, \mathrm{d}\tau \geqslant c \int_0^t \big\|u_1(\tau) - u_2(\tau)\big\|_H^2 \, \mathrm{d}\tau \tag{2.19}$$

for all t with some constant $c > 0$. Then the solution of (2.17) depends continuously on $f \in L^2(Q_T)$ and $u_0 \in H$: if u_j is a solution of (2.17) with $f = f_j \in L^2(Q_T)$, $u_0 = u_{0j} \in H$ $(j = 1, 2)$ then for all $t \in [0, T]$

$$\begin{aligned} &\big\|u_1(t) - u_2(t)\big\|_H^2 + c\|u_1 - u_2\|_{L^2(0,T;H)}^2 \\ &\quad \leqslant \tilde{c}\|f_1 - f_2\|_{L^2(0,T;H)}^2 + \|u_{10} - u_{20}\|_H^2 \end{aligned} \tag{2.20}$$

with some constant $\tilde{c}$.

(See Remark 2.2.)

REMARK 2.4. Uniqueness and continuous dependence of the solution of (2.17) can be obtained also in the following way. Let $\tilde{u}(t) = \exp(-dt)u(t)$ be a new unknown function with some constant $d > 0$. Problem (2.17) is equivalent with

$$\begin{aligned} &\tilde{u}'(t) + \exp(-dt)\big[A\big(\exp(dt)\tilde{u}\big)\big](t) + d\tilde{u}(t) = \exp(-dt)f(t), \\ &\tilde{u}(0) = 0. \end{aligned} \tag{2.21}$$

If for some $d > 0$ operator B, defined by

$$\big[B(\tilde{u})\big](t) = \exp(-dt)\big[A\big(\exp(dt)\tilde{u}\big)\big](t) + d\tilde{u}(t) \tag{2.22}$$

is monotone then the solution of (2.17) is unique.

If the operator B defined by (2.22) is strictly monotone in the sense (2.19) then one obtains an analogous inequality to (2.20).

REMARK 2.5. According to the sketched proof of Theorem 2.5, a solution to (2.17) can be obtained as weak limit in $L^p(0, T; V)$ of a subsequence of (u_m) of Galerkin's solutions. If the solution is unique (see Remark 2.4), then the total sequence (u_m) converges weakly in $L^p(0, T; V)$ and (by Theorem 2.2) strongly in $L^p(0, T; V_1)$ to the solution u.

We shall use Theorem 2.5 to prove existence of weak solutions of nonlinear parabolic differential equations and functional parabolic equations. In the last case operator $A : L^p(0, T; V) \to L^q(0, T; V^\star)$ is such that $[A(u)](t)$ depends not only on $u(t)$.

Clearly, Theorem 2.3 follows from Theorem 2.5 in the case $u_0 = 0$, because the assumptions of Theorem 2.3 imply that $A : L^p(0, T; V) \to L^q(0, T; V^\star)$, defined by $[A(u)](t) = \tilde{A}(u(t))$, is bounded, hemicontinuous, monotone, thus pseudomonotone and it is coercive.

REMARK 2.6. Problem

$$u' + A(u) = f, \qquad u(0) = u_0 \quad \text{with some } u_0 \in V$$

has also a solution with the same assumptions on A, because this problem is equivalent to the problem for $\tilde{u} = u - u_0$

$$\tilde{u}' + A(\tilde{u} + u_0) = f, \qquad \tilde{u}(0) = 0$$

and $B(\tilde{u}) = A(\tilde{u} + u_0)$ satisfies the assumptions of Theorems 2.4, 2.5, respectively.

REMARK 2.7. Let $a < T$ be a fixed positive number and $B : [0, T] \times L^p(-a, 0; V) \to L^q(0, T; V^\star)$ a given (nonlinear) operator. Consider the following problem on functional differential equations: to find $v \in L^p(-a, T; V)$ such that $v' \in L^q(-a, T; V^\star)$,

$$\begin{aligned} &v'(t) + \big[B(t, v_t)\big](t) = f(t) \quad \text{for a.a. } t \in [0, T] \\ &\text{and } v(t) = \psi(t), \quad t \in [-a, 0] \end{aligned} \tag{2.23}$$

where $f \in L^q(0, T; V^\star)$ and $\psi \in L^p(-a, 0; V)$ are given functions with the property $\psi' \in L^q(-a, 0; V^\star)$ and v_t is defined by

$$v_t(\tau) = v(t + \tau), \quad \tau \in [-a, 0].$$

Problem (2.23) can be reduced to a problem of type (2.17) as follows.

Assume that we have a function $v_0 \in L^p(-a, T; V)$ such that $v_0' \in L^q(-a, T; V^\star)$ and the restriction of v_0 to $[-a, 0]$ is ψ (v_0 is an extension of ψ). Then v is a solution of (2.23) if and only if the restriction of $v - v_0$ to $[0, T]$ is a solution of

$$u'(t) + \big[A(u)\big](t) = f(t) + v_0'(t) \quad \text{for a.e. } t \in [0, T], \qquad u(0) = 0 \tag{2.24}$$

with $u \in L^p(0, T; V)$, $u' \in L^q(0, T; V^\star)$, where operator A is defined by

$$\begin{aligned} &\big[A(u)\big](t) = \big[B\big(t, (Nu + v_0)_t\big)\big](t), \quad t \in [0, T], \\ &[Nu](\tau) = u(\tau) \quad \text{for } \tau \in [0, T], \qquad [Nu](\tau) = 0 \quad \text{for } \tau \in [-a, 0). \end{aligned} \tag{2.25}$$

3. Second order and higher order nonlinear parabolic differential equations

In this section we shall apply results of Section 2 to initial-boundary value problems of certain nonlinear parabolic partial differential equations and prove existence (in some cases uniqueness and continuous dependence) of weak solutions of these problems in a finite time interval $[0, T]$.

3.1. *Definition of the weak solution*

In order to define weak solutions, at first for simplicity consider the following initial-boundary value problem:

$$
\begin{aligned}
&D_t u - \sum_{j=1}^{n} D_j\big[f_j(t,x,u,Du)\big] + f_0(t,x,u,Du) = g \\
&\quad \text{in } Q_T = (0,T)\times\Omega, && (3.1)\\
&u = 0 \quad \text{on } [0,T]\times\partial\Omega, && (3.2)\\
&u(0,x) = u_0(x), \quad x\in\Omega. && (3.3)
\end{aligned}
$$

where $\Omega\subset\mathbb{R}^n$ is a bounded domain with "sufficiently smooth" boundary, $D = (D_1,\dots,D_n)$, $D_j = \partial/\partial x_j$. Assume that $u\in C^{1,2}(\overline{Q_T})$ is a (classical) solution of (3.1) where f_0 is continuous and for $j = 1,\dots,n$ f_j is continuously differentiable (except of the variable t) in $\overline{Q_T}$. $C^{1,2}(\overline{Q_T})$ denotes the set of functions which are once continuously differentiable with respect to t and twice continuously differentiable with respect to x in $\overline{Q_T}$. Multiplying (3.1) by a test function $v\in C_0^1(\Omega)$ (i.e. with a continuously differentiable function with compact support), we obtain by Gauss theorem

$$
\begin{aligned}
&\int_\Omega (D_t u)v\,\mathrm{d}x + \sum_{j=1}^{n}\int_\Omega f_j(t,x,u,Du)D_j v\,\mathrm{d}x + \int_\Omega f_0(t,x,u,Du)v\,\mathrm{d}x \\
&\quad = \int_\Omega g v\,\mathrm{d}x. && (3.4)
\end{aligned}
$$

Clearly, a function $u\in C^{1,2}(\overline{Q_T})$ satisfies (3.1) if and only if (3.4) holds for all $v\in C_0^1(\Omega)$.

Assume that functions f_j satisfy the growth condition for all $(t,x)\in Q_T$, $\xi_0\in\mathbb{R}$, $\xi\in\mathbb{R}^n$

$$
\big|f_j(t,x,\xi_0,\xi)\big| \leqslant c_1\big(|\xi_0|^{p-1} + |\xi|^{p-1}\big) + k_1(t,x) \tag{3.5}
$$

with some constants $p>1$, $c_1>0$ and a function $k_1\in L^q(Q_T)$ (where q is defined by $1/p+1/q=1$). Let $V = W_0^{1,p}(\Omega)$, i.e the closure of $C_0^1(\Omega)$ in $W^{1,p}(\Omega)$. Then by Hölder's inequality, for each $t\in[0,T]$ the formula

$$
\begin{aligned}
&\big\langle\big[\tilde A(t)\big]u(t), v\big\rangle = \sum_{j=1}^{n}\int_\Omega f_j(t,x,u,Du)D_j v\,\mathrm{d}x + \int_\Omega f_0(t,x,u,Du)v\,\mathrm{d}x, \\
&\quad u\in L^p(0,T;V),\ v\in V && (3.6)
\end{aligned}
$$

defines a linear continuous functional $[\tilde A(t)]u(t)$ on V, i.e. $\tilde A(t)$ maps V into $V^\star$ and the operator A defined by

$$
\big[A(u)\big](t) = \big[\tilde A(t)\big]u(t), \quad t\in[0,T] \tag{3.7}
$$

maps $L^p(0,T;V)$ into $L^q(0,T;V^\star)$. (For a function $u \in L^p(0,T;W_0^{1,p}(\Omega))$ we denote also by u the function $(t,x) \mapsto [u(t)](x)$.) Further, formula

$$[G,v] = \int_0^T \langle G(t), v(t)\rangle \,\mathrm{d}t = \int_0^T\int_\Omega gv \,\mathrm{d}x\,\mathrm{d}t, \qquad v \in L^p(0,T;V) \tag{3.8}$$

defines a linear continuous functional G on $L^p(0,T;V)$ if $g \in L^q(Q_T)$.

Thus we may write equalities (3.3), (3.4) in the form

$$D_t u + A(u) = G, \qquad u(0) = u_0 \tag{3.9}$$

where

$$G \in L^q(0,T;V^\star), \qquad A : L^p(0,T;V) \to L^q(0,T;V^\star),$$

consequently, $D_t u \in L^q(0,T;V^\star)$. As before, in Eq. (3.9) we identify $u(t,\cdot)$ with $u(t)$ and $D_t u$ with u'.

DEFINITION 3.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $V = W_0^{1,p}(\Omega)$, $p \geqslant 2$ and define operator $A : L^p(0,T;V) \to L^q(0,T;V^\star)$ by (3.6), (3.7), $G \in L^q(0,T;V^\star)$ by (3.8) and let $H = L^2(\Omega)$. A function $u \in L^p(0,T;V)$, satisfying $D_t u = u' \in L^q(0,T;V^\star)$ and Eq. (3.9), is called a weak solution of (3.1)–(3.3).

Since $V = W_0^{1,p}(\Omega)$ and $H = L^2(\Omega)$ define an evolution triple $V \subset H \subset V^\star$, $u \in L^p(0,T;V)$, $D_t u = u' \in L^q(0,T;V^\star)$ imply $u \in C([0,T];L^2(\Omega))$ by Theorem 2.1 and so the initial condition $u(0) = u_0$ makes sense. Clearly, a sufficient smooth function $u \in L^p(0,T;V)$ satisfies the problem (3.1)–(3.3) if and only if it satisfies (3.9).

Similarly, in the case $V = W^{1,p}(\Omega)$ (with sufficiently smooth $\partial\Omega$) a sufficiently smooth function $u \in L^p(0,T;V)$ satisfies (3.9) if and only if it is a classical solution of (3.1), (3.3) and the (Neumann) boundary condition

$$\sum_{j=1}^n f_j(t,x,u,Du)\nu_j = 0 \quad \text{on } [0,T) \times \partial\Omega \tag{3.10}$$

where $\nu = (\nu_1, \dots, \nu_n)$ denotes the outer normal unit vector on $\partial\Omega$. Therefore, we define the weak solution of (3.1), (3.3), (3.10) as follows.

DEFINITION 3.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $V = W^{1,p}(\Omega)$, $p \geqslant 2$ and define operator $A : L^p(0,T;V) \to L^q(0,T;V^\star)$ by (3.6), (3.7), $G \in L^q(0,T;V^\star)$ by (3.8) and let $H = L^2(\Omega)$. A function $u \in L^p(0,T;V)$, satisfying $D_t u = u' \in L^q(0,T;V^\star)$ and equation (3.9), is called a weak solution of (3.1), (3.3), (3.10).

Weak solutions of initial-boundary value problems with nonhomogeneous Dirichlet boundary conditions can be defined in the following way.

DEFINITION 3.3. Let $u^\star \in W^{1,p}(Q_T)$ be a given function, $V = W_0^{1,p}(\Omega)$. If $\tilde{u} \in W_p^1(0,T;V,H)$ satisfies (similarly to (3.9))

$$D_t\tilde{u} + A(\tilde{u} + u^\star) = G - D_t u^\star, \qquad \tilde{u}(0) = u_0 - u^\star(0)$$

(where A is defined by (3.6), (3.7) for $v \in W^{1,p}(\Omega)$) then $u = \tilde{u} + u^\star$ is called a weak solution of (3.1), (3.3) and the boundary condition

$$u|_{\Gamma_T} = u^\star|_{\Gamma_T} \tag{3.11}$$

where $u^\star|_{\Gamma_T}$ is the trace of $u^\star \in W^{1,p}(Q_T)$ on $\Gamma_T = [0, T] \times \partial\Omega$.

The definition of the trace see, e.g., in [1]. Now we define the weak solution of (3.1), (3.3) with nonhomogeneous Neumann boundary condition.

DEFINITION 3.4. Weak solution of (3.1), (3.3) and

$$\sum_{j=1}^{n} f_j(t, x, u, Du)\nu_j = h(t, x, u) \quad \text{on } \Gamma_T \tag{3.12}$$

is a function $u \in W_p^1(0, T; V, H)$ with $V = W^{1,p}(\Omega)$ if u satisfies

$$D_t u + A(u) + B(u) = G, \qquad u(0) = u_0$$

where A is defined by (3.6), (3.7) and B is defined by

$$\big[B(u), v\big] = -\int_0^T \left[\int_{\Gamma_T} h\big(t, x, u(t)|_{\partial\Omega}\big)v(t)|_{\partial\Omega}\, \mathrm{d}\sigma\right] \mathrm{d}t.$$

Here $h : Q_T \times R \to R$ is a Carathéodory function, satisfying

$$\big|h(t, x, \xi_0)\big| \leqslant c_1|\xi_0|^{p-1} + \tilde{k}_1(t, x),$$

$\tilde{k}_1 \in L^q(\Gamma_T)$, $u(t)|_{\partial\Omega}$ denotes the trace of $u(t) \in W^{1,p}(\Omega)$ on $\partial\Omega$.

By Gauss theorem, a sufficiently smooth $u \in L^p(0, T; V)$ is a weak solution of (3.1), (3.3), (3.12) iff it is a classical solution.

3.2. *Application of monotone operators*

Now we shall apply the results in Section 2 to higher order parabolic differential equations. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $m \geqslant 1$ an integer, $p \geqslant 2$ and V be a closed linear subspace of $W^{m,p}(\Omega)$, $H = L^2(\Omega)$.

In order to define operator $\tilde{A}(t)$, introduce the following notation. Let M and N be the number of multiindices β and γ satisfying $|\beta| \leqslant m$, $|\gamma| \leqslant m-1$, respectively. The vectors $\xi \in \mathbb{R}^M$ will also be written in the form $\xi = (\eta, \zeta)$ where $\eta \in \mathbb{R}^N$ consists of coordinates ξ_γ for which $|\gamma| \leqslant m-1$ and $\zeta \in \mathbb{R}^{M-N}$ consists of coordinates ξ_β with $|\beta| = m$.

Assume that

(A_1) The functions $f_\alpha : Q_T \times \mathbb{R}^M \to R$ satisfy the Carathéodory conditions, i.e. they are measurable in (t, x) for each fixed $\xi \in \mathbb{R}^M$ and continuous in ξ for a.e. fixed $(t, x) \in Q_T$.

(A_2) There exist constant $c_1 > 0$ and $k_1 \in L^q(Q_T)$ such that

$$\left| f_\alpha(t,x,\xi) \right| \leqslant c_1 |\xi|^{p-1} + k_1(t,x) \quad (|\alpha| \leqslant m)$$

for a.e. $(t,x) \in Q_T$, all $\xi \in \mathbb{R}^M$.

(A_3) $\sum_{|\alpha|\leqslant m} [f_\alpha(t,x,\xi) - f_\alpha(t,x,\xi^\star)](\xi_\alpha - \xi_\alpha^\star) \geqslant 0$ for a.e. $(t,x) \in Q_T$, all $\xi, \xi^\star \in \mathbb{R}^M$.

(A_4) There exist constant $c_2 > 0$, $k_2 \in L^1(Q_T)$ such that

$$\sum_{|\alpha|\leqslant m} f_\alpha(t,x,\xi)\xi_\alpha \geqslant c_2|\xi|^p - k_2(t,x)$$

for a.e. $(t,x) \in Q_T$, each $\xi \in \mathbb{R}^M$.

THEOREM 3.1. *Assume* (A_1)–(A_4). *Then operator* $\tilde{A}(t)$, *defined by*

$$\left\langle \left[\tilde{A}(t)\right](\tilde{u}), \tilde{v} \right\rangle = \sum_{|\alpha|\leqslant m} \int_\Omega f_\alpha(t,x,\tilde{u},\dots,D^\beta \tilde{u},\dots) D^\alpha \tilde{v}\, dx, \tag{3.13}$$

where $|\beta| \leqslant m$, $\tilde{u}, \tilde{v} \in V$, *satisfies the conditions of Theorem* 2.3. *Consequently, the operator* A, *defined by* $[A(u)](t) = [\tilde{A}(t)](u(t))$, *i.e.*

$$\begin{aligned}\left[A(u), v\right] &= \int_0^T \left\langle \left[\tilde{A}(t)\right]\left(u(t)\right), v(t) \right\rangle dt \\ &= \sum_{|\alpha|\leqslant m} \int_{Q_T} f_\alpha\left(t,x,u,\dots,D_x^\beta u,\dots\right) D_x^\alpha v\, dx\, dt,\\ &u, v \in L^p(0,T;V)\end{aligned} \tag{3.14}$$

maps $L^p(0,T;V)$ *into* $L^q(0,T;V^\star)$, *it is bounded, demicontinuous, monotone* (*thus pseudomonotone*) *and coercive. Therefore, for arbitrary* $f \in L^q(0,T;V^\star)$ *and* $u_0 \in H$ *problem* (2.2) (*with operator* (3.14)) *has a unique solution.*

It is easy to see that by (A_1), (A_2), Hölder's inequality and Lebesgue's dominated convergence theorem $A : L^p(0,T;V) \to L^q(0,T;V^\star)$ is bounded and demicontinuous. Clearly, (A_3) implies that A is monotone and (A_4) implies that A is coercive.

A simple sufficient condition for (A_3) is (see [22]):

THEOREM 3.2. *Assume that the functions* f_α *are continuously differentiable with respect to* ξ *and the matrix*

$$\left(\frac{\partial f_\alpha}{\partial \xi_\beta} \right)_{|\alpha|,|\beta|\leqslant m}$$

is positive semidefinite for a.e. $(t,x) \in Q_T$, *all* $\xi \in \mathbb{R}^M$. *Then* (A_3) *holds.*

This theorem follows from the formula

$$\sum_{|\alpha|\leqslant m}\big[f_\alpha(t,x,\xi)-f_\alpha(t,x,\xi^\star)\big](\xi_\alpha-\xi_\alpha^\star)$$
$$=\sum_{|\alpha|\leqslant m}\left[\int_0^1\sum_{|\beta|\leqslant m}\frac{\partial f_\alpha}{\partial \xi_\beta}\big(t,x,\xi^\star+\tau(\xi-\xi^\star)\big)(\xi_\beta-\xi_\beta^\star)\,\mathrm{d}\tau\right](\xi_\alpha-\xi_\alpha^\star). \quad (3.15)$$

Clearly, the following assumption implies that operator $\tilde{A}(t)$ (given by (3.13)) is uniformly monotone in the sense (2.13):

(A_3') $\sum_{|\alpha|\leqslant m}[f_\alpha(t,x,\xi)-f_\alpha(t,x,\xi^\star)](\xi_\alpha-\xi_\alpha^\star)\geqslant c|\xi-\xi^\star|^p$ with some constant $c>0$, for a.e. $(t,x)\in Q_T$, all $\xi,\xi^\star\in\mathbb{R}^M$.

By using (3.15), in [22] it is shown that a sufficient condition for (A_3') is

$$\sum_{|\alpha|,|\beta|\leqslant m}\frac{\partial f_\alpha}{\partial \xi_\beta}(t,x,\xi)y_\alpha y_\beta\geqslant\tilde{c}\sum_{|\alpha|\leqslant m}|\xi_\alpha|^{p-2}|y_\alpha|^2 \quad (3.16)$$

with some constant $\tilde{c}>0$, for all $\xi,y\in\mathbb{R}^M$. Therefore, if (A_1), (A_2), (A_3'), (A_4) (or instead of (A_3'), (3.16)) hold then by Remark 2.2 the unique solution of problem (2.2) continuously depends on f and u_0 in the sense of (2.14). If (A_1), (A_2), (A_4) and (instead of (A_3'))

$$\sum_{|\alpha|\leqslant m}\big[f_\alpha(t,x,\xi)-f_\alpha(t,x,\xi^\star)\big]\big(\xi_\alpha-\xi_\alpha^\star\big)\geqslant c\big|\xi_0-\xi_0^\star\big|^2 \quad (3.17)$$

hold with some constant $c>0$ then the solution of problem (2.2) continuously depends on f and u_0 in the sense (2.20).

A simple example satisfying (A_1), (A_2), (3.16), (A_4) is in the case $m=1$ (considered in the first part of this section)

$$f_j(t,x,\xi)=\xi_j|\zeta|^{p-2},\quad j=1,\dots,n,\quad f_0(t,x,\xi)=k\eta|\eta|^{p-2} \quad (3.18)$$

where $\xi=(\eta,\zeta)$, $\eta=\xi_0\in\mathbb{R}$, $\zeta=(\xi_1,\dots,\xi_n)\in\mathbb{R}^n$, k is a positive constant and instead of multiindices α satisfying $|\alpha|\leqslant 1$, we use indices $j=1,\dots,n$. In this case the original differential operator (corresponding to A) is

$$-\Delta_p u+ku|u|^{p-2}=-\sum_{j=1}^n D_j\big[(D_ju)|\operatorname{grad}u|^{p-2}\big]+ku|u|^{p-2}. \quad (3.19)$$

(Operator Δ_p is called p-Laplacian.)

A simple case when assumptions (A_1)–(A_4) are fulfilled:

$$f_\alpha(t,x,\xi)=a_\alpha(t,x)\tilde{f}_\alpha(\xi_\alpha)+b_\alpha(t,x) \quad (3.20)$$

where a_α is measurable satisfying

$$0<c_3\leqslant a_\alpha(t,x)\leqslant c_4,\quad b_\alpha\in L^q(Q_T)$$

with constants c_3,c_4; $\tilde{f}_\alpha$ are monotone nondecreasing functions satisfying

$$c_5|\xi_\alpha|^{p-1}\leqslant|\tilde{f}_\alpha(\xi_\alpha|\leqslant c_6|\xi_\alpha|^{p-1},\quad \xi_\alpha\in\mathbb{R}$$

with positive constants c_5,c_6. ((A_4) can be shown by Young's inequality.)

REMARK 3.1. According to Definitions 3.1, 3.2, a solution of problem (2.2) with the operator (3.14) is a weak solution of (3.1)–(3.3) in the case $m = 1$, $V = W_0^{1,p}(\Omega)$ and it is a weak solution of (3.1), (3.3), (3.10) in the case $m = 1$, $V = W^{1,p}(\Omega)$.

Similarly, for $m > 1$, a solution of (2.2) with operator (3.14), $V = W_0^{m,p}(\Omega)$ and $V = W^{m,p}(\Omega)$ is considered as a weak solution of a classical initial-boundary value problem for the equation

$$D_t u + \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} D_x^\alpha \big[f_\alpha(t, x, u, \ldots, D_x^\beta u, \ldots) \big] = f$$

(where $|\beta| \leqslant m$) with homogeneous Dirichlet and Neumann boundary condition, respectively.

REMARK 3.2. Instead of (A_4) assume

(A_4') $\sum_{|\alpha| \leqslant m} f_\alpha(t, x, \xi)\xi_\alpha \geqslant c_2|\zeta|^p - k_2(t, x)$ with a constant $c_2 > 0$ and $k_2 \in L^1(Q_T)$.

Since in $W_0^{m,p}(\Omega)$ (with bounded $\Omega \subset \mathbb{R}^n$) the norm

$$\|\tilde{u}\|' = \left[\sum_{|\alpha| = m} \int_\Omega |D^\alpha \tilde{u}|^p \right]^{1/p}$$

is equivalent with the original norm

$$\|\tilde{u}\| = \left[\sum_{|\alpha| \leqslant m} \int_\Omega \left| D^\alpha \tilde{u} \right|^p \right]^{1/p},$$

we obtain that in the case $V = W_0^{m,p}(\Omega)$ the assertions of Theorem 3.1 hold if we assume (A_1)–(A_3), (A_4').

Thus we obtain e.g. existence of a weak solution of the problem for equation

$$D_t u - \Delta_p u = f$$

with homogeneous Dirichlet boundary condition.

3.3. *Application of pseudomonotone operators*

Now we formulate a theorem which is an application of Theorem 2.5. Instead of (A_3) assume

(A_3'') $\sum_{|\alpha| = m} [f_\alpha(t, x, \eta, \zeta) - f_\alpha(t, x, \eta, \zeta^\star)](\xi_\alpha - \xi_\alpha^\star) > 0$ for a.e. $(t, x) \in Q_T$, all $\eta \in \mathbb{R}^N$, and $\zeta, \zeta^\star \in \mathbb{R}^{M-N}$ if $\zeta \neq \zeta^\star$.

THEOREM 3.3. *Let $V \subset W^{m,p}(\Omega)$ a closed linear subspace and assume* (A_1), (A_2), (A_3''), (A_4). *Then operator $A: L^p(0, T; V) \to L^q(0, T; V^\star)$, defined by* (3.14) *is bounded, demicontinuous, pseudomonotone with respect to $D(L)$ and it is coercive* (*in the sense* (2.16)). *Consequently, for arbitrary $f \in L^q(0, T; V^\star)$ there exists a solution $u \in D(L)$ of* (2.17). *Finally, operator A is of class* (S_+).

By (A_1), (A_2), Hölder's inequality and Vitali's theorem we obtain that A is bounded and demicontinuous, further, by (A_4) A is coercive. In [44] it is proved that (by (A_1), (A_2), (A_3'')) A is pseudomonotone with respect to $D(L)$. (One can prove this fact also similarly to the elliptic case, considered in [12].) In [47] is proved: (A_1), (A_2), (A_3''), (A_4) imply that A is of class (S_+).

REMARK 3.3. In the case $V = W_0^{m,p}(\Omega)$ (A_1), (A_2), (A_3''), (A_4') imply the existence of a solution to (2.17) because according to Remark 3.2 A is coercive in this case.

REMARK 3.4. According to Remark 2.1, a subsequence of the Galerkin solutions converges strongly (not only weakly) in $L^p(0,T;V)$ and in $C([0,T],L^2(\Omega))$ to solution u. If the solution is unique, the total sequence is converging to u strongly in $L^p(0,T;V)$ and in $C([0,T],L^2(\Omega))$. By Remark 2.4, the solution is unique if

$$\sum_{|\alpha|\leqslant m}\big[f_\alpha(t,x,\xi)-f_\alpha(t,x,\xi^\star)\big](\xi_\alpha-\xi_\alpha^\star)\geqslant -c\big(\xi_0-\xi_0^\star\big)^2 \tag{3.21}$$

for some (sufficiently large) constant $c>0$. (By ξ_0 is denoted the coordinate of ξ, belonging to the multiindex $(0,\dots,0)$.) Further, it is not difficult to show that the solution continuously depends on $f\in L^2(Q_T)$, $u_0\in L^2(\Omega)$ in the sense (2.20).

REMARK 3.5. In Theorems 3.1, 3.3 it was assumed that $\Omega\subset\mathbb{R}^n$ is bounded and $p\geqslant 2$. If Ω is unbounded or $1<p<2$ then generally we do not have the continuous embedding $W^{m,p}(\Omega)\subset L^2(\Omega)$. In this case we obtain existence of solutions to problems (2.2), (2.17), respectively, if instead of $X=L^p(0,T;V)$, $X^\star=L^q(0,T;V^\star)$ we consider

$$X=L^p(0,T;V)\cap L^2(Q_T),\quad\text{and thus}\quad X^\star=L^q(0,T;V^\star)+L^2(Q_T).$$

(See [41,43,44,75].)

Due to Theorem 3.3, we obtain the existence of solutions to (2.17) with operator (3.14) without monotonicity assumption on lower order terms f_α ($|\alpha|\leqslant m-1$) and without monotonicity assumption on f_α with respect to η if $|\alpha|=m$. However, for all α, $f_\alpha(t,x,\xi)$ are assumed to satisfy $(p-1)$-th power growth condition in ξ.

For $m=1$, we obtain existence of weak solutions to equations e.g. of the following type:

$$D_tu-\Delta_pu+f_0(t,x,u,Du)=f$$

where f_0 satisfies the Carathéodory condition and

$$\big|f_0(t,x,\xi)\big|\leqslant c_1|\xi|^{p-1}+k_1(t,x)$$

with some constant $c_1>0$, $k_1\in L^q(Q_T)$,

$$f_0(t,x,\eta,\zeta)\eta\geqslant c_2|\eta|^p-k_2(t,x) \tag{3.22}$$

with some constant $c_2>0$, $k_2\in L^1(Q_T)$. If $V=W_0^{1,p}(\Omega)$ then by Remark 3.3 instead of (3.22) it is sufficient to assume e.g. the sign condition

$$f_0(t,x,\eta,\zeta)\eta\geqslant 0.$$

3.4. *Strongly nonlinear equations*

In [10,13] F.E. Browder and H. Brezis considered "strongly nonlinear" parabolic equations containing a term $g(t, x, u)$ which could be quickly increasing in u. By using main ideas of these works, one can prove the following more general result on "strongly nonlinear" parabolic equations (see [58,59]).

Assume that

(S_1) For $|\alpha| \leqslant m-1$ functions $g_\alpha : Q_T \times \mathbb{R}^N \to \mathbb{R}$ satisfy the Carathéodory conditions.

(S_2) $g_0(t, x, \eta)\xi_0 \geqslant 0$ for a.e. $(t, x) \in Q_T$, all $\eta \in \mathbb{R}^N$, $\xi_0 \in \mathbb{R}$, further, there exists a continuous nondecreasing function $\psi : \mathbb{R} \to \mathbb{R}$ with $\psi(0) = 0$ and a constant c_3 such that

$$\bigl|g_0(t, x, \eta)\bigr| \leqslant \bigl|\psi(\xi_0)\bigr| \leqslant c_3\bigl[\bigl|g_0(t, x, \eta)\bigr| + |\eta|^{p-1} + 1\bigr].$$

(g_0 and ξ_0 denote g_α, ξ_α, respectively, for $|\alpha| = 0$, $p \geqslant 2$.)

(S_3) $|g_\alpha(t, x, \eta)|^q \leqslant k_3(\xi_0)g_0(t, x, \eta)\xi_0$ for

$$1 \leqslant |\alpha| \leqslant m - 1, \quad \text{a.e. } (t, x) \in Q_T, \text{ all } \eta \in \mathbb{R}^N$$

where k_3 is a continuous function satisfying $\lim_\infty k_3 = 0$, $1/p + 1/q = 1$.

THEOREM 3.4. *Let* $V = W_0^{m,p}(\Omega)$ *(with bounded* $\Omega \subset \mathbb{R}^n$*), assume* ($A_1$), ($A_2$), ($A_3''$), ($A_4$) *and* ($S_1$)–($S_3$). *Then for each* $f \in L^q(0, T; V^\star)$, $u_0 \in V \cap L^\infty(\Omega)$ *there exists* $u \in L^p(0, T; V) \cap C([0, T], L^2(\Omega))$ *such that*

$$g_0(t, x, \dots, D_x^\gamma u, \dots) \text{ and } ug_0(t, x, \dots, D_x^\gamma u, \dots) \in L^1(Q_T) \quad \bigl(|\gamma| \leqslant m-1\bigr),$$
$$g_\alpha(t, x, \dots, D_x^\gamma u, \dots) \in L^q(Q_T) \quad \text{for } 1 \leqslant |\alpha| \leqslant m-1 \quad \bigl(|\gamma| \leqslant m-1\bigr),$$

u is a distribution solution in Q_T of the equation

$$D_t u + \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} D_x^\alpha\bigl[f_\alpha(t, x, \dots, D_x^\beta u, \dots)\bigr] + \sum_{|\alpha| \leqslant m-1} (-1)^{|\alpha|} D_x^\alpha\bigl[g_\alpha(t, x, \dots, D_x^\gamma u, \dots)\bigr] = f \quad (|\beta| \leqslant m, |\gamma| \leqslant m-1)$$

and $u(0) = u_0$.

Further, u satisfies the following system of energy inequalities: for each $\tilde{t} \in [0, T]$ *and for each* $v \in L^p(0, T; V) \cap C^1([0, T], L^2(\Omega))$ *with* $v(0) = u_0$ *and* $v \in L^\infty(Q_T)$ *we have*

$$\begin{aligned}
&\int_0^{\tilde{t}} \bigl\langle D_t v(t), u(t) - v(t)\bigr\rangle \,\mathrm{d}t + \frac{1}{2}\bigl\|u(\tilde{t}) - v(\tilde{t})\bigr\|^2_{L^2(\Omega)} \\
&\quad + \sum_{|\alpha| \leqslant m} \int_{Q_{\tilde{t}}} f_\alpha(t, x, \dots, D_x^\beta u, \dots)\bigl(D_x^\alpha u - D_x^\alpha v\bigr) \,\mathrm{d}t \,\mathrm{d}x \\
&\quad + \sum_{|\alpha| \leqslant m-1} \int_{Q_{\tilde{t}}} g_\alpha\bigl(t, x, \dots, D_x^\gamma u, \dots\bigr)\bigl(D_x^\alpha u - D_x^\alpha v\bigr) \,\mathrm{d}t \,\mathrm{d}x \\
&= \int_0^{\tilde{t}} \bigl\langle f(t), u(t) - v(t)\bigr\rangle \,\mathrm{d}t.
\end{aligned} \tag{3.23}$$

If $g_0 = \psi$, $g_\alpha = 0$ *for* $|\alpha| \geqslant 1$ *and* (*the monotonicity condition*) (A_3) *is satisfied then the solution of* (3.23) *is unique.*

The proof of Theorem 3.4 is based on truncation of the functions g_α to obtain bounded functions g_α^k, a compact embedding theorem (see [13]) and also on an approximation theorem formulated in [10] and [13]. The proof of the last theorem combines the techniques of Hedberg's approximation theorem in Sobolev spaces with convolutions in time.

REMARK 3.6. According to assumptions (S_1)–(S_3) of Theorem 3.4, g_α may be "quickly increasing" in u.

4. Parabolic functional differential equations containing functional dependence in lower order terms

In this section we shall apply Theorem 2.5 to get existence of weak solutions (in a finite interval $[0, T]$) of initial-boundary value problems for functional parabolic equations which are perturbations of parabolic partial differential equations (with $2m$ order elliptic part) by lower order functional terms. Such equations arise e.g. in models considered in [11,18,19, 30,31,51,54,74].

4.1. *Existence theorems*

We shall consider equations of the form

$$D_t u + \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} D_x^\alpha \big[f_\alpha(t, x, \dots, D_x^\beta u, \dots) \big] + \sum_{|\alpha| \leqslant m-1} (-1)^{|\alpha|} D_x^\alpha \big[H_\alpha(u) \big] = f \quad \text{in } Q_T = (0, T) \times \Omega \tag{4.1}$$

where $|\beta| \leqslant m$, $\Omega \subset \mathbb{R}^n$ is a bounded domain (with sufficiently smooth boundary) and, denoting by V a closed linear subspace of the Sobolev space $W^{m,p}(\Omega)$ ($m \geqslant 1$, $p \geqslant 2$, $1/p + 1/q = 1$),

$$H_\alpha : L^p(0, T; V) \to L^q(Q_T)$$

is a bounded (possibly nonlinear) operator. We shall consider weak solutions of the above equations with some (possibly nonlinear) boundary conditions on $\Gamma_T = [0, T] \times \partial\Omega$ which may contain functional dependence.

Assume that

(A_5) $H_\alpha : L^p(0, T; V) \to L^q(Q_T)$, $G_\alpha : L^p(0, T; V) \to L^q(\Gamma_T)$ are bounded (possibly nonlinear) operators of Volterra type which are demicontinuous from $L^p(0, T; W^{m-\delta,p}(\Omega))$ to $L^q(Q_T)$ and $L^q(\Gamma_T)$, respectively, for some $\delta > 0$, $\delta < 1 - 1/p$. Further,

$$\lim_{\|u\| \to \infty} \frac{\|H_\alpha(u)\|^q_{L^q(Q_T)} + \|G_\alpha(u)\|^q_{L^q(\Gamma_T)}}{\|u\|^p_{L^p(0,T;V)}} = 0. \tag{4.2}$$

Then we may define operator $B_1 : L^p(0,T;V) \to L^q(0,T;V^\star)$ by

$$\begin{aligned}\big[B_1(u), v\big] = &\sum_{|\alpha|\leqslant m-1} \int_{Q_T} H_\alpha(u) D_x^\alpha v \,\mathrm{d}t\,\mathrm{d}x \\ &+ \sum_{|\alpha|\leqslant m-1} \int_{\Gamma_T} G_\alpha(u) D_x^\alpha v \,\mathrm{d}t\,\mathrm{d}\sigma_x, \quad u, v \in L^p(0,T;V). \end{aligned} \tag{4.3}$$

(In the last formula we consider the trace of $D_x^\alpha v(t,\cdot)$ on $\partial\Omega$.)

THEOREM 4.1. *Assume* (A_1), (A_2), (A_3''), (A_4), (A_5) *and consider the operator* $A = A_1$ *defined by* (3.14). *Then* $(A_1 + B_1) : L^p(0,T;V) \to L^q(0,T;V^\star)$ *is bounded, demicontinuous, pseudomonotone with respect to* $D(L)$, *it is coercive and of Volterra type. Consequently, for any* $f \in L^q(0,T;V^\star)$ *there exists* $u \in D(L)$ *satisfying*

$$D_t u + (A_1 + B_1)(u) = f, \qquad u(0) = 0. \tag{4.4}$$

Now we sketch the proof of Theorem 4.1 which can be found in [60]. By Theorem 3.3, $A_1 : L^p(0,T;V) \to L^q(0,T;V^\star)$ is bounded, demicontinuous, pseudomonotone with respect to $D(L)$ and it is coercive. By (A_5) and Hölder's inequality $B_1 : L^p(0,T;V) \to L^q(0,T;V^\star)$ is bounded and demicontinuous.

Further, assuming

$$\begin{aligned} &(u_j) \to u \quad \text{weakly in } L^p(0,T;V), \quad (Lu_j) \to Lu \quad \text{weakly in } L^q(0,T;V^\star) \\ &\quad \text{and} \quad \limsup_{j\to\infty}\big[(A_1 + B_1)(u_j), u_j - u\big] \leqslant 0, \end{aligned} \tag{4.5}$$

by Theorem 2.2 (on compact embedding) there is a subsequence of (u_j) (again denoted by (u_j)) for which

$$(u_j) \to u \text{ in } L^p\big(0,T;W^{m-\delta,p}(\Omega)\big) \tag{4.6}$$

where $\delta > 0$ is chosen according to (A_5) ($\delta < 1 - 1/p$). Since the trace operator $W^{m-\delta,p}(\Omega) \to W^{m-1,p}(\partial\Omega)$ is continuous (because $\delta + 1/p < 1$, see [1]),

$$\begin{aligned} &\lim_{j\to\infty}\big[B_1(u_j), u_j - u\big] = 0 \quad \text{and} \quad \big(B_1(u_j)\big) \to B_1(u) \\ &\quad \text{weakly in } L^q\big(0,T;V^\star\big) \end{aligned} \tag{4.7}$$

by (A_5). Consequently, (4.5) yields

$$\limsup_{j\to\infty}\big[A_1(u_j), u_j - u\big] \leqslant 0. \tag{4.8}$$

Since A_1 is pseudomonotone with respect to $D(L)$, we obtain from (4.5), (4.8)

$$\lim_{j\to\infty}\big[A_1(u_j), u_j - u\big] = 0, \qquad \big(A_1(u_j)\big) \to A_1(u) \quad \text{weakly in } L^q(0,T;V^\star)$$

thus by (4.7) $A_1 + B_1$ is pseudomonotone with respect to $D(L)$.

Finally, (A_4), (A_5) imply

$$\frac{[A_1(u)+B_1(u),u]}{\|u\|} \geqslant \frac{c_2\|u\|^p - c_2^\star}{\|u\|} - \frac{[B_1(u),u]}{\|u\|}$$
$$\geqslant \|u\|^{p-1}\left[c_2 - \frac{\|B_1(u)\|_{L^q(0,T;V^\star)}}{\|u\|^{p-1}}\right] - \frac{c_2^\star}{\|u\|} \to +\infty \tag{4.9}$$

if $\|u\| \to \infty$ because

$$\frac{\|B_1(u)\|_{L^q(0,T;V^\star)}}{\|u\|^{p-1}} = \left[\frac{\|B_1(u)\|^q}{\|u\|^p}\right]^{1/q} \to 0.$$

(4.9) means that $A_1 + B_1$ is coercive. Clearly, $A_1 + B_1$ is of Volterra type.

REMARK 4.1. In the case $V = W_0^{m,p}(\Omega)$, $G_\alpha = 0$ (for all α), the solution of (4.4) is called a weak solution of (4.1) with 0 initial and Dirichlet boundary condition. In the case $m = 1$, $V = W^{1,p}(\Omega)$ the solution of (4.4) can be considered as a weak solution of (4.1) with 0 initial condition and the following Neumann boundary condition of functional type (see Definition 3.4):

$$\sum_{|\alpha|=1} f_\alpha(t,x,u,Du)\nu_\alpha = -G_0(u) \text{ on } \Gamma_T.$$

REMARK 4.2. Assume that functions f_α satisfy (A_3) or (3.21), $G_\alpha = 0$ for all α and H_α are extended to $L^2(Q_T)$ such that they satisfy the Lipschitz condition with some constant c_3

$$\left\|\exp(-d\tau)\left[H_\alpha\left(\exp(d\tau)u_1\right)\right] - \exp(-d\tau)\left[H_\alpha\left(\exp(d\tau)u_2\right)\right]\right\|_{L^2(Q_t)}$$
$$\leqslant c_3\|u_1 - u_2\|_{L^2(Q_t)} \tag{4.10}$$

for all $u_1, u_2 \in L^p(0,T;V)$, $t \in [0,T]$ and $d > 0$. Then, choosing sufficiently large positive number d, we obtain uniqueness of the solution u and continuous dependence of u on f, u_0 in the sense (2.20).

REMARK 4.3. If the conditions of Remark 4.2 hold, $m = 1$, $f_\alpha(t,x,\xi)$ is linear in ξ with sufficiently smooth coefficients (depending on (t,x)), then, by using results on interior regularity of solutions of linear parabolic partial differential equations (see, e.g., [40] Theorem 6.6), we obtain for the solution u of (4.4) that $D_{ij}u, D_t u \in L^2_{\text{loc}}(Q_T)$ if $f \in L^2_{\text{loc}}(Q_T)$.

In a similar way, combining Remark 4.2 with regularity results on solutions of quasilinear parabolic differential equations, it is possible to obtain results on smoothness of solutions to functional parabolic problems.

4.2. *Examples*

Now we formulate several examples for H_α, G_α satisfying (A_5).

EXAMPLE 4.1. Let $G_\alpha = 0$ for all α and

$$[H_\alpha(u)](t,x) = \int_0^t h_\alpha\big(t,\tau,x,\dots,D_x^\gamma u(\tau,x),\dots\big)\,\mathrm{d}\tau, \quad (t,x)\in Q_T \quad \text{or}$$

$$[H_\alpha(u)](t,x) = \int_\Omega \left[\int_0^t h_\alpha(t,\tau,x,y,\dots,D_x^\gamma u(\tau,y),\dots)\,\mathrm{d}\tau\right]\mathrm{d}y, \quad (t,x)\in Q_T$$

where $|\gamma| \leqslant m-1$, functions h_α satisfy the Carathéodory conditions and

$$\begin{aligned} &\big|h_\alpha(t,\tau,x,\eta)\big| \leqslant c_3\big(|\eta|\big)|\eta|^{p-1} + k_3(t,x),\\ &\big|h_\alpha(t,\tau,x,y,\eta)\big| \leqslant c_3\big(|\eta|\big)|\eta|^{p-1} + k_3(t,x), \end{aligned} \tag{4.11}$$

respectively, with a continuous function c_3, $\lim_\infty c_3 = 0$, $k_3 \in L^q(Q_T)$.

First we show the first part of (A_5). By (4.11) and Hölder's inequality

$$H_\alpha : L^p(0,T;V) \to L^q(Q_T)$$

is bounded. If $(u_j) \to u$ in the norm of $L^p(0,T;W^{m-1,p}(\Omega))$ then $D_x^\gamma u_j(t,x) \to D_x^\gamma u(t,x)$ for a.e. $(t,x)\in Q_T$, for a subsequence ($|\gamma| \leqslant m-1$). By (4.11) the sequences of functions

$$\begin{aligned} &\tau \mapsto h_\alpha\big(t,\tau,x,\dots,D_x^\gamma u(\tau,x),\dots\big),\\ &(\tau,y) \mapsto h_\alpha\big(t,\tau,x,y,\dots,D_x^\gamma u(\tau,y),\dots\big), \end{aligned}$$

are equiintegrable, thus Vitali's theorem implies that

$$\big(H_\alpha(u_j)\big) \to H_\alpha(u) \quad \text{a.e. in } Q_T.$$

Similarly, by (4.11) functions $|H_\alpha(u_j) - H_\alpha(u)|^q$ are equiintegrable in Q_T, thus

$$\big(H_\alpha(u_j)\big) \to H_\alpha(u) \quad \text{in } L^q(Q_T)$$

for a subsequence. It is easy to show that the statement holds for the total sequence, too. (Assuming the converse, one gets a contradiction.)

Now we show (4.2) in the case $G_\alpha = 0$. By (4.11) for arbitrary $\varepsilon > 0$ number $a > 0$ can be chosen such that

$$\big|h_\alpha(t,\tau,x,\eta)\big|^q \leqslant \mathrm{const}\big(\varepsilon^q|\eta|^p + \big|k_3(t,x)\big|^q\big) \quad \text{if } |\eta| > a.$$

Consequently,

$$\begin{aligned} &\int_{Q_T} \big|h_\alpha\big(t,\tau,x,\dots,D_x^\gamma u(\tau,x),\dots\big)\big|^q\,\mathrm{d}\tau\,\mathrm{d}x\\ &\quad= \int_{Q_T^a} \big|h_\alpha\big(t,\tau,x,\dots,D_x^\gamma u(\tau,x),\dots\big)\big|^q\,\mathrm{d}\tau\,\mathrm{d}x\\ &\qquad+ \int_{Q_T\setminus Q_T^a} \big|h_\alpha\big(t,\tau,x,\dots,D_x^\gamma u(\tau,x),\dots\big)\big|^q\,\mathrm{d}\tau\,\mathrm{d}x\\ &\quad\leqslant \mathrm{const}\,\varepsilon^q\|u\|^p_{L^p(0,T;V)} + c(\varepsilon) \end{aligned}$$

where

$$Q_T^a = \{(\tau, x) \in Q_T\colon\ |(\ldots, D_x^\gamma u(\tau, x), \ldots)| \leqslant a\}$$

which implies (4.2) (with $G_\alpha = 0$) because

$$\int_{Q_T} |H_\alpha(u)|^q \, \mathrm{d}x \, \mathrm{d}t$$
$$\leqslant \text{const} \int_0^T \left[\int_{Q_T} |h_\alpha(t, \tau, x, \ldots, D_x^\gamma u(\tau, x), \ldots)|^q \, \mathrm{d}\tau \, \mathrm{d}x \right] \mathrm{d}t.$$

Similarly can be considered the other form of $H_\alpha(u)$.

If assumptions of Remark 4.2 hold such that $m = 1$ and $h_\alpha = h_0$ satisfies global Lipschitz condition in η then (4.10) is valid and we have uniqueness and continuous dependence of the solution.

REMARK. By a simple modification of (4.4) and this example we obtain a problem of type (2.23). Define operator

$$B : [0, T] \times L^p(-a, 0; V) \to L^q(0, T; V^\star)$$

of Remark 2.7 by

$$\begin{aligned}
\langle [B(t, w)](t), v \rangle &= \sum_{|\alpha| \leqslant m} \int_\Omega f_\alpha(t, x, \ldots, D_x^\beta w(0, x), \ldots) D^\alpha v(x) \, \mathrm{d}x \\
&\quad + \sum_{|\alpha| \leqslant m-1} \int_\Omega \left[\int_{-a}^0 h_\alpha(t, s, x, \ldots, D_x^\gamma w(s, x), \ldots) \right] \mathrm{d}s \, D^\alpha v(x) \, \mathrm{d}x, \\
&\quad t \in [0, T], \quad w \in L^p(-a, 0; V), \quad v \in V.
\end{aligned}$$

Then according to (2.25) for $u = w - v_0$

$$\begin{aligned}
\langle [A(u)](t), v \rangle &= \langle [B(t, (Nu + v_0)_t)](t), v \rangle \\
&= \sum_{|\alpha| \leqslant m} \int_\Omega f_\alpha(t, x, \ldots, D_x^\beta u(t, x) + D_x^\beta v_0(t, x), \ldots) D^\alpha v(x) \, \mathrm{d}x \\
&\quad + \sum_{|\alpha| \leqslant m-1} \int_\Omega \left[\int_0^t h_\alpha(t, \tau, x, \ldots, D_x^\gamma u(\tau, x) \right. \\
&\quad \left. + D_x^\gamma v_0(\tau, x), \ldots) \right] \mathrm{d}\tau \, D^\alpha v(x) \, \mathrm{d}x \\
&\quad + \sum_{|\alpha| \leqslant m-1} \int_\Omega \left[\int_{t-a}^0 h_\alpha(t, \tau, x, \ldots, D_x^\gamma \psi(\tau, x), \ldots) \right] \mathrm{d}\tau \, D^\alpha v(x) \, \mathrm{d}x,
\end{aligned}$$

for $t < a$, for $t \geqslant a$ instead of the last term we have 0. By Remark 2.7 problem (2.23) is equivalent to (2.24) which (in this case) is such a problem which was considered in Theorem 4.1 and in Example 4.1.

EXAMPLE 4.2. One proves similarly that the following operators H_α satisfy (A_5) (with $G_\alpha = 0$):

$$\big[H_\alpha(u)\big](t,x) = h_\alpha\big(t,x,\dots,F_\gamma(D_x^\gamma u),\dots\big), \quad (t,x) \in Q_T,\ |\alpha|,|\gamma| \leqslant m-1$$

where $F_\gamma : L^p(Q_T) \to L^p(Q_T)$ are linear continuous operators of Volterra type, functions h_α satisfy the Carathéodory conditions and (similarly to (4.11))

$$\big|h_\alpha(t,x,\eta)\big| \leqslant c_3\big(|\eta|\big)|\eta|^{p-1} + k_3(t,x) \tag{4.12}$$

with a continuous function c_3, $\lim_\infty c_3 = 0$, $k_3 \in L^q(Q_T)$.

The operators F_γ may have e.g. one of the forms

$$\big[F_\gamma(v)\big](t,x) = \int_0^t g_\gamma(t,\tau,x)v(\tau,x)\,\mathrm{d}\tau,$$

$$\big[F_\gamma(v)\big](t,x) = \int_0^t \int_\Omega g_\gamma(t,\tau,\xi,x)v(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi$$

(g_γ may be e.g. L^∞ functions),

$$\big[F_\gamma(v)\big](t,x) = v\big(\chi_\gamma(t),\psi_\gamma(x)\big)$$

where $0 \leqslant \chi_\gamma(t) \leqslant t$, $\chi_\gamma'(t) > 0$; the functions $\psi_\gamma : \bar\Omega \to \bar\Omega$ are C^1 diffeomorphisms.

From Example 4.2 one gets a problem of type (2.23) similarly to Example 4.1.

Analogous examples are for G_α satisfying (A_5):

EXAMPLE 4.3.

$$\big[G_\alpha(u)\big](t,x) = \int_0^t h_\alpha\big(t,\tau,x,\dots,D_x^\gamma u(\tau,\cdot)|_{\partial\Omega}(x),\dots\big)\,\mathrm{d}\tau, \quad (t,x) \in \Gamma_T,$$

$$\big[G_\alpha(u)\big](t,x) = \int_{\partial\Omega}\left[\int_0^t h_\alpha\big(t,\tau,x,y,\dots,D_x^\gamma u(\tau,\cdot)|_{\partial\Omega}(y),\dots\big)\,\mathrm{d}\tau\right]\mathrm{d}\sigma_y,$$
$$(t,x) \in \Gamma_T$$

where $D_x^\gamma u(\tau,\cdot)|_{\partial\Omega}$ denotes the trace of $D_x^\gamma u(\tau,\cdot)$ on $\partial\Omega$ ($|\gamma| \leqslant m-1$) and functions h_α satisfy conditions which are analogous to (4.11), (4.12).

EXAMPLE 4.4.

$$\big[G_\alpha(u)\big](t,x) = h_\alpha\big(t,x,\dots,F_\gamma(D_x^\gamma u|_{\Gamma_T}),\dots\big),$$
$$(t,x) \in \Gamma_T,\ |\alpha|,|\gamma| \leqslant m-1$$

where $F_\gamma : L^p(\Gamma_T) \to L^p(\Gamma_T)$ are linear continuous operators of Volterra type, functions h_α satisfy Carathéodory conditions and analogous conditions to (4.12).

Similar examples for F_γ see in Example 4.2. One can prove that Examples 4.3, 4.4 satisfy (A_5) as it was shown for Example 4.1.

4.3. *Strongly nonlinear equations*

Now we formulate an existence result on strongly nonlinear parabolic functional differential equations which is a generalization of Theorem 3.4 on strongly nonlinear parabolic differential equations. We assume that functions $g_\alpha : Q_T \times \mathbb{R}^N \to \mathbb{R}$ satisfy assumptions (S_1)–(S_3), further, assume

(S_4) $h^1_\alpha, h^2_\alpha : (0,T) \times Q_T \times \mathbb{R}^N \to \mathbb{R}$ satisfy the Carathéodory conditions for $|\alpha| \leqslant m-1$,

$$h_\alpha = h^1_\alpha + h^2_\alpha.$$

(S_5) The following estimates hold for a.e. (t,τ,x) and all $\eta \in \mathbb{R}^N$ ($|\alpha| \leqslant m-1$):

$$\begin{aligned}&\bigl|h^1_\alpha(t,\tau,x,\eta)\bigr| \leqslant c_5\bigl(|\eta|\bigr)|\eta|^{p-1} + k_5(t,x),\\ &\bigl|h^2_\alpha(t,\tau,x,\eta)\bigr|^q \leqslant c_6\bigl(|\xi_0|\bigr)g_0(\tau,x,\eta)\xi_0\end{aligned}$$

where c_5, c_6 are continuous functions, $\lim_\infty c_5 = 0$, $\lim_\infty c_6 = 0$, $k_5 \in L^q(Q_T)$. (The second estimation is analogous to (S_3).)

THEOREM 4.2. *Let $V = W^{m,p}_0(\Omega)$ with a bounded domain $\Omega \in \mathbb{R}^n$. Assume* (A_1), (A_2), (A''_3), (A_4), (S_1)–(S_5). (*I.e. we assume conditions of Theorem* 3.4 *and* (S_4), (S_5).)

Then for each $f \in L^q(0,T;V^\star)$, $u_0 \in V \cap L^\infty(\Omega)$ there exists $u \in L^p(0,T;V) \cap C([0,T],L^2(\Omega))$ such that

$$\begin{aligned}&g_0\bigl(t,x,u,\dots,D^\gamma_x u,\dots\bigr), ug_0\bigl(t,x,u,\dots,D^\gamma_x u,\dots\bigr) \in L^1(Q_T)\\ &\quad\bigl(|\gamma| \leqslant m-1\bigr),\\ &g_\alpha\bigl(t,x,u,\dots,D^\gamma_x u,\dots\bigr) \in L^q(Q_T)\\ &\quad\bigl(|\gamma| \leqslant m-1\bigr) \textit{ for } 1 \leqslant |\alpha| \leqslant m-1,\\ &(t,x) \mapsto \int_0^t h^2_\alpha\bigl(t,\tau,x,u(\tau,x),\dots,D^\gamma_x u(\tau,x),\dots\bigr)\,\mathrm{d}\tau \in L^q(Q_T)\\ &\quad\bigl(|\gamma| \leqslant m-1\bigr),\end{aligned}$$

u is a distribution solution in Q_T of

$$\begin{aligned}D_t u &+ \sum_{|\alpha|\leqslant m}(-1)^{|\alpha|}D^\alpha_x\bigl[f_\alpha(t,x,\dots,D^\beta_x u,\dots)\bigr]\\ &+ \sum_{|\alpha|\leqslant m-1}(-1)^{|\alpha|}D^\alpha_x\bigl[g_\alpha(t,x,\dots,D^\gamma_x u,\dots)\bigr]\\ &+ \sum_{|\alpha|\leqslant m-1}(-1)^{|\alpha|}D^\alpha_x\int_0^t\bigl[h_\alpha\bigl(t,\tau,x,\dots,D^\gamma_x u(\tau,x),\dots\bigr)\bigr]\,\mathrm{d}\tau = f\end{aligned}$$

and $u(0) = 0$.

Further, u satisfies the following system of energy inequalities: for each $\tilde{t} \in [0,T]$ and

$$v \in L^p(0,T;V) \cap C^1\bigl([0,T],L^2(\Omega)\bigr) \quad \textit{with } v(0) = u_0 \textit{ and } v \in L^\infty(Q_T)$$

we have

$$\begin{aligned}
&\int_0^{\tilde{t}} \langle D_t v(t), u(t) - v(t)\rangle \, dt + \frac{1}{2}\|u(\tilde{t}) - v(\tilde{t})\|^2_{L^2(\Omega)} \\
&\quad + \sum_{|\alpha|\leqslant m} \int_{Q_{\tilde{t}}} f_\alpha(t, x, \ldots, D_x^\beta u, \ldots)(D_x^\alpha u - D_x^\alpha v) \, dt \, dx \\
&\quad + \sum_{|\alpha|\leqslant m-1} \int_{Q_{\tilde{t}}} g_\alpha(t, x, \ldots, D_x^\gamma u, \ldots)(D_x^\alpha u - D_x^\alpha v) \, dt \, dx \\
&\quad + \sum_{|\alpha|\leqslant m-1} \int_{Q_{\tilde{t}}} \left[\int_0^t h_\alpha(t, \tau, x, \ldots, D_x^\gamma u, \ldots) \, d\tau\right](D_x^\alpha u - D_x^\alpha v) \, dt \, dx \\
&= \int_0^{\tilde{t}} \langle f(t), u(t) - v(t)\rangle \, dt.
\end{aligned}$$

The proof of Theorem 4.2 can be found in [58,59].

REMARK 4.4. According to assumptions of Theorem 4.2, g_α and h_α^2 may be "quickly increasing" in u.

5. Parabolic equations containing functional dependence in the main part

In this section we shall consider second order quasilinear parabolic functional differential equations where also the main part of the equation contains functional dependence on the unknown function. In [14,15] M. Chipot, L. Molinet and B. Lovat considered equation

$$D_t u - \sum_{i,j=1}^n D_i\big[a_{ij}\big(l\big(u(t,\cdot)\big)\big)D_j u\big] + a_0\big(l\big(u(t,\cdot)\big)\big)u = f \quad \text{in } \mathbb{R}^+ \times \Omega \tag{5.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary,

$$\sum_{i,j=1}^n a_{i,j}(\zeta)\xi_i\xi_j \geqslant \lambda|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n,\ \zeta \in \mathbb{R} \tag{5.2}$$

with some constant $\lambda > 0$,

$$l\big(u(t,\cdot)\big) = \int_\Omega g(x)u(t,x) \, dx$$

with a given $g \in L^2(\Omega)$. The existence and asymptotic properties (as $t \to \infty$) of solutions to initial-boundary value problems for (5.1) were proved. Equation (5.1) was motivated by the diffusion process (for heat or population) where the diffusion coefficient depends on a nonlocal quantity.

5.1. *Existence theorems*

By using the theory of monotone type operators, now we shall consider for simplicity second order quasilinear parabolic functional differential equations which are generalizations of (5.1). (It would be possible to consider higher order parabolic equations, too, by using analogous argument.) We shall apply the results of Section 2 to equations of the form

$$D_t u - \sum_{i=1}^{n} D_i\big[a_i\big(t,x,u(t,x),Du(t,x);u\big)\big] + a_0\big(t,x,u(t,x),Du(t,x);u\big) = f \tag{5.3}$$

where the functions

$$a_i : Q_T \times \mathbb{R}^{n+1} \times L^p(0,T;V) \to \mathbb{R}$$

with a closed linear subspace V of $W^{1,p}(\Omega)$, $2 \leqslant p < \infty$, satisfy conditions which are generalizations of assumptions (A_1), (A_2), (A_3''), (A_4) such that it will be possible to apply Theorem 2.5 and (also modifying the proof of Theorem 2.5) to obtain existence results for (5.3).

On functions a_i assume, by using the notation $X = X_T = L^p(0,T;V)$

(B_1) The functions $a_i : Q_T \times \mathbb{R}^{n+1} \times X \to \mathbb{R}$ satisfy the Carathéodory conditions for arbitrary fixed $u \in X$ and they have the Volterra property: $a_i(t,x,\zeta_0,\zeta;u)$ depends only on the restriction of u to $[0,t]$ $(i = 0,1,\dots,n)$.

(B_2) There exist bounded (nonlinear) operators $g_1 : X \to \mathbb{R}^+$ and $k_1 : X \to L^q(Q_T)$ such that

$$\big|a_i(t,x,\zeta_0,\zeta;u)\big| \leqslant g_1(u)\big[|\zeta_0|^{p-1} + |\zeta|^{p-1}\big] + \big[k_1(u)\big](t,x), \quad i = 0,1,\dots,n$$

for a.e. $(t,x) \in Q_T$, each $(\zeta_0,\zeta) \in \mathbb{R}^{n+1}$ and $u \in X$.

(B_3) $\sum_{i=1}^{n}[a_i(t,x,\zeta_0,\zeta;u) - a_i(t,x,\zeta_0,\zeta^\star;u)](\zeta_i - \zeta_i^\star) > 0$ if $\zeta \neq \zeta^\star$.

(B_4) There exist bounded operators $g_2 : X \to \mathbb{R}^+$, $k_2 : X \to L^1(Q_T)$ such that

$$\sum_{i=0}^{n} a_i(t,x,\zeta_0,\zeta;u)\zeta_i \geqslant \big[g_2(u)\big]\big[|\zeta_0|^p + |\zeta|^p\big] - \big[k_2(u)\big](t,x) \tag{5.4}$$

for a.e. $(t,x) \in Q_T$, all $(\zeta_0,\zeta) \in \mathbb{R}^{n+1}$, $u \in X$ and

$$\lim_{\|u\|\to\infty}\left[g_2(u)\|u\|_X^{p-1} - \frac{\|k_2(u)\|_{L^1(Q_T)}}{\|u\|_X}\right] = +\infty. \tag{5.5}$$

(B_5) There exists $\delta > 0$ such that if $(u_k) \to u$ weakly in $X = L^p(0,T;V)$, strongly in $L^p(0,T;W^{1-\delta,p}(\Omega))$ then for $i = 0,1,\dots,n$

$$a_i\big(t,x,u_k(t,x),Du_k(t,x);u_k\big) - a_i\big(t,x,u_k(t,x),Du_k(t,x);u\big) \to 0$$

in $L^q(Q_T)$.

DEFINITION 5.1. Define operator $A : X \to X^\star$ by

$$\big[A(u), v\big] = \int_{Q_T} \left[\sum_{i=1}^n a_i\big(t, x, u(t,x), Du(t,x); u\big) D_i v + a_0\big(t, x, u(t,x), Du(t,x); u\big) v \right] \mathrm{d}t\, \mathrm{d}x,$$

$$u, v \in X = L^p(0, T; V).$$

THEOREM 5.1. *Assume* (B_1)–(B_5). *Then* $A : X \to X^\star$ *is bounded, demicontinuous, pseudomonotone with respect to* $D(L)$, *coercive and of Volterra type. Consequently, by* Theorem 2.5 *for any* $f \in X^\star$ *there exists a solution* $u \in D(L)$ *of*

$$D_t u + A(u) = f, \quad u(0) = 0. \tag{5.6}$$

PROOF. Boundedness of A follows from (B_1), (B_2) and by (B_1) A is of Volterra type. Further, if $u(_k) \to u$ strongly in X then

$$\begin{aligned} &a_i(t, x, u_k, Du_k; u_k) - a_i(t, x, u, Du; u) \\ &\quad = \big[a_i(t, x, u_k, Du_k; u_k) - a_i(t, x, u_k, Du_k; u)\big] \\ &\qquad + \big[a_i(t, x, u_k, Du_k; u) - a_i(t, x, u, Du; u)\big] \end{aligned}$$

where the first term in the right hand side tends to 0 in $L^q(Q_T)$ by (B_5) and (for a subsequence) the second term also tends to 0 in $L^q(Q_T)$ by Vitali's theorem and (B_1), (B_2). Thus $A : X \to X^\star$ is continuous (and so it is demicontinuous). Clearly, (B_4) implies that A is coercive.

Finally, we show that A is pseudomonotone with respect to $D(L)$. For a fixed $\hat{u} \in X$ define operator $\tilde{A}_{\hat{u}}$ by

$$\big[\tilde{A}_{\hat{u}}(u), v\big] = \int_{Q_T} \left[\sum_{i=1}^n a_i\big(t, x, u(t,x), Du(t,x); \hat{u}\big) D_i v + a_0\big(t, x, u(t,x), Du(t,x); \hat{u}\big) v \right] \mathrm{d}t\, \mathrm{d}x,$$

where $u, v \in X$. Then (B_1)–(B_4) and Theorem 3.3 imply that $\tilde{A}_{\hat{u}}$ is pseudomonotone with respect to $D(L)$.

Assume that

$$\begin{aligned} &u_k \in D(L), \quad (u_k) \to u \quad \text{weakly in } X, \\ &(u_k') \to u' \quad \text{weakly in } X^\star \quad \text{and} \quad \limsup_{k\to\infty} \big[A(u_k), u_k - u\big] \leqslant 0. \end{aligned} \tag{5.7}$$

Then by Theorem 2.2

$$(u_k) \to u \quad \text{strongly in } L^p\big(0, T; W^{1-\delta, p}(\Omega)\big) \tag{5.8}$$

for a subsequence (again denoted by (u_k)). (5.7), (5.8) and (B$_5$) imply

$$\lim_{k\to\infty}\big[A(u_k)-\tilde{A}_u(u_k),u_k-u\big]=0, \tag{5.9}$$

hence

$$\limsup_{k\to\infty}\big[\tilde{A}_u(u_k),u_k-u\big]\leqslant 0. \tag{5.10}$$

Since $\tilde{A}_u$ is pseudomonotone with respect to $D(L)$ we obtain from (5.9), (5.10)

$$\lim_{k\to\infty}\big[A(u_k),u_k-u\big]=0 \quad\text{and}\quad \big(A(u_k)\big)\to A(u) \quad\text{weakly in } X^\star$$

because $(\tilde{A}_u(u_k)-A(u_k))\to 0$ in $X^\star$ by (5.7), (5.8) and (B$_5$). So we have shown that A is pseudomonotone with respect to $D(L)$.

Now instead of (B$_4$) we assume

(B$_4'$) There exist bounded operators

$$g_2: X\to C\big([0,T]\big),\qquad k_2: X\to L^1(Q_T)$$

such that $[g_2(u)](t)>0$ for $t\in[0,T]$ and for all $u\in X$

$$\sum_{i=0}^{n}a_i(t,x,\zeta_0,\zeta;u)\zeta_i\geqslant\big[g_2(u)\big](t)\big[|\zeta_0|^p+|\zeta|^p\big]-\big[k_2(u)\big](t,x) \tag{5.11}$$

for a.e. $(t,x)\in Q_T$, all $(\zeta_0,\zeta)\in\mathbb{R}^{n+1}$ and with some positive constants,

$$0\leqslant\sigma^\star<p-1,\qquad 0\leqslant\sigma<p-\sigma^\star$$

$$\big[g_2(u)\big](t)\geqslant\text{const}\,\|u\|_{X_t}^{-\sigma^\star}\quad\text{if } \|u\|_{X_t}\geqslant 1, \tag{5.12}$$

$$\big\|k_2(u)\big\|_{L^1(Q_t)}\leqslant\text{const}\,\|u\|_{X_t}^{\sigma}\quad\text{if } \|u\|_{X_t}\geqslant 1, \tag{5.13}$$

for any $u\in X_T=L^p(0,T;V)$, $t\in[0,T]$.

Then from Theorem 5.1 directly follows □

THEOREM 5.2. *Assume* (B$_1$)–(B$_3$), (B$_4'$), (B$_5$). *Then for arbitrary $f\in X^\star$ there exists a solution $u\in D(L)$ of* (5.6).

REMARK 5.1. According to (5.12), we have an existence theorem on (5.6) if the equation is not uniformly parabolic in the sense, analogous to the condition (5.2) in the linear case. (See Example 5.1.)

5.2. *Examples*

EXAMPLE 5.1. Let a_i have the form

$$a_i(t,x,\zeta_0,\zeta;u)=\big[B_1(u)\big](t,x)\alpha_i^1(t,x,\zeta_0,\zeta)+\big[B_2(u)\big](t,x)\alpha_i^2(t,x,\zeta_0,\zeta),$$
$$i=1,\dots,n,$$

$$a_0(t,x,\zeta_0,\zeta;u) = \big[B_1(u)\big](t,x)\alpha_0^1(t,x,\zeta_0,\zeta) + \big[B_3(u)\big](t,x)\alpha_0^2(t,x,\zeta_0,\zeta)$$

where α_i^l satisfy the Carathéodory conditions ($i = 0,1,\dots,n$, $l = 1,2$),

$$\big|\alpha_i^1(t,x,\zeta_0,\zeta)\big| \leqslant c_1\big(|\zeta_0|^{p-1} + |\zeta|^{p-1}\big) + k_1(x), \quad i = 0,1,\dots,n$$

with some constant c_1, $k_1 \in L^q(\Omega)$,

$$\sum_{i=1}^n \big[\alpha_i^1(t,x,\zeta_0,\zeta) - \alpha_i^1(t,x,\zeta_0,\zeta^\star)\big](\zeta_i - \zeta_i^\star) > 0 \quad \text{if } \zeta \neq \zeta^\star,$$

$$\sum_{i=0}^n \alpha_i^1(t,x,\zeta_0,\zeta)\zeta_i \geqslant c_2\big(|\zeta_0|^p + |\zeta|^p\big) - k_2(x)$$

with some constant $c_2 > 0$, $k_2 \in L^1(\Omega)$ (thus α_i^1 satisfy (A_1), (A_2), (A_3''), (A_4)),

$$\big|\alpha_i^2(t,x,\zeta_0,\zeta)\big| \leqslant c_1\big(|\zeta_0|^\rho + |\zeta|^\rho\big), \quad 0 \leqslant \rho,\ \sigma^\star + \rho < p-1,\ i = 0,1,\dots,n,$$

$$\sum_{i=1}^n \big[\alpha_i^2(t,x,\zeta_0,\zeta) - \alpha_i^2(t,x,\zeta_0,\zeta^\star)\big](\zeta_i - \zeta_i^\star) \geqslant 0,$$

$$\sum_{i=1}^n \alpha_i^2(t,x,\zeta_0,\zeta)\zeta_i \geqslant 0.$$

Further,

$$\begin{aligned}
&B_1 : L^p\big(0,T;W^{1-\delta,p}(\Omega)\big) \to L^\infty(Q_T),\\
&B_2 : L^p\big(0,T;W^{1-\delta,p}(\Omega)\big) \to L^{p/(p-1-\rho)}(Q_T),\\
&B_3 : L^p\big(0,T;W^{1-\delta,p}(\Omega)\big) \to L^{(p-\sigma^\star)/(p-\sigma^\star-1-\rho)}(Q_T)
\end{aligned}$$

are bounded and continuous (possibly nonlinear) operators of Volterra type satisfying the conditions

$$\big[B_1(u)\big](t,x) \geqslant \text{const}\,\|u\|_{X_t}^{-\sigma^\star} \quad \text{if } \|u\|_{X_t} \geqslant 1$$

with some positive constant and with $0 \leqslant \sigma < p - \sigma^\star$

$$\big[B_2(u)\big](t,x) \geqslant 0, \qquad \int_{Q_T} \big|B_3(u)\big|^{\frac{p-\sigma^\star}{p-1-\sigma^\star-\rho}} \leqslant \text{const}\,\|u\|^\sigma_{L^p(0,T;W^{1-\delta,p}(\Omega))}.$$

By using Young's and Hölder's inequalities, one can show that assumptions of Theorem 5.2 are fulfilled.

Operators B_1, B_2, B_3, satisfying the above conditions, may have e.g. the forms

$$\begin{aligned}
&\big[B_1(u)\big](t,x) = b_1\big(\big[H(u)\big](t,x)\big), \qquad \big[B_2(u)\big](t,x) = b_2\big(\big[G(u)\big](t,x)\big),\\
&\big[B_3(u)\big](t,x) = b_3\big(\big[G_0(u)\big](t,x)\big)
\end{aligned} \tag{5.14}$$

where functions b_j ($j = 1,2,3$) are continuous and satisfy (with some positive constants)

$$b_1(\theta) \geqslant \frac{\text{const}}{1 + |\theta|^{\sigma^\star}},$$

$$0 \leqslant b_2(\theta) \leqslant \text{const}\,|\theta|^{p-1-\hat{\rho}}, \qquad \bigl|b_3(\theta)\bigr| \leqslant \text{const}\,|\theta|^{p-1-\rho^\star}$$

with $\rho \leqslant \hat{\rho} \leqslant p-1$, $\sigma^\star + \rho < \rho^\star \leqslant p-1$. Further,

$$H : L^p\bigl(0,T;W^{1-\delta,p}(\Omega)\bigr) \to C(\overline{Q_T}),$$
$$G, G_0 : L^p\bigl(0,T;W^{1-\delta,p}(\Omega)\bigr) \to L^p(Q_T)$$

are linear continuous operators of Volterra type.

The operator H may have e.g. one of the forms

$$\bigl[H(u)\bigr](t,x) = \int_{Q_t} d(t,x,\tau,\xi)u(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi \text{ where } d \text{ is continuous in } (t,x),$$
$$\sup_{(t,x)\in Q_T} \int_{Q_T} \bigl|d(t,x,\tau,\xi)\bigr|^q\,\mathrm{d}\tau\,\mathrm{d}\xi < \infty,$$
$$\bigl[H(u)\bigr](t,x) = \int_{\Gamma_t} d(t,x,\tau,\xi)u(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\sigma_\xi \quad \text{where } d \text{ is continuous in } (t,x),$$
$$\sup_{(t,x)\in Q_T} \int_{\Gamma_T} \bigl|d(t,x,\tau,\xi)\bigr|^q\,\mathrm{d}\tau\,\mathrm{d}\sigma_\xi < \infty,$$
$$\Gamma_t = (0,t)\times\partial\Omega, \text{ assuming } \delta < 1-1/p.$$

The operators G, G_0 may have also the above forms with the following modified conditions:

$$\int_{Q_T}\biggl[\int_{Q_T}\bigl|d(t,x,\tau,\xi)\bigr|^q\,\mathrm{d}\tau\,\mathrm{d}\xi\biggr]^{p/q}\mathrm{d}t\,\mathrm{d}x < \infty,$$
$$\int_{Q_T}\biggl[\int_{\Gamma_T}\bigl|d(t,x,\tau,\xi)\bigr|^q\,\mathrm{d}\tau\,\mathrm{d}\sigma_\xi\biggr]^{p/q}\mathrm{d}t\,\mathrm{d}x < \infty, \quad \text{assuming } \delta < 1-1/p,$$

respectively, or they may have one of the forms

$$\int_0^t d(t,x,\tau)u(\tau,x)\,\mathrm{d}\tau, \qquad \int_\Omega d(t,x,\xi)u(t,\xi)\,\mathrm{d}\xi$$

where

$$\int_0^T \sup_{x\in\Omega}\biggl[\int_0^T\bigl|d(t,x,\tau)\bigr|^q\,\mathrm{d}\tau\biggr]^{p/q}\mathrm{d}t < \infty,$$
$$\int_\Omega \sup_{t\in[0,T]}\biggl[\int_\Omega\bigl|d(t,x,\xi)\bigr|^q\,\mathrm{d}\xi\biggr]^{p/q}\mathrm{d}x < \infty,$$

respectively.

The operators B_2, B_3, satisfying the above conditions, may have also the forms in the point (t,x)

$$\int_0^t h\bigl(t,\tau,x,u(\tau,x)\bigr)\,\mathrm{d}\tau \quad \text{or} \quad h\bigl(t,x,u\bigl(\chi(t),x\bigr)\bigr)$$

where

$$\left|h(t,\tau,x,\theta)\right| \leqslant \text{const}\,|\theta|^{p-1-\rho_0}, \qquad \left|h(t,x,\theta)\right| \leqslant \text{const}\,|\theta|^{p-1-\rho_0}$$

with $\rho < \rho_0 \leqslant p-1$ for B_2 and $\sigma^\star + \rho < \rho_0 \leqslant p-1$ for B_3, $0 \leqslant \chi(t) \leqslant t$, $\chi \in C^1$, $\chi' > 0$ and $h \geqslant 0$ for B_2.

5.3. *Non-uniformly parabolic equations*

Now we formulate an existence theorem when the condition of uniform parabolicity is not satisfied also for "small" u. Instead of (B_4') assume

(B_4'') The inequality (5.11) is satisfied such that (instead of (5.12), (5.13)) we have with some positive constants, for all $t \in [0,T]$

$$\begin{aligned}
&\big[g_2(u)\big](t) \geqslant \text{const}\,\|u\|_{L^{p_1}(Q_t)} \quad \text{if } \|u\|_{L^{p_1}(Q_t)} < 1,\\
&\big[g_2(u)\big](t) \geqslant \text{const}\,\|u\|^{-\sigma^\star}_{L^{p_1}(Q_t)} \quad \text{if } \|u\|_{L^{p_1}(Q_t)} \geqslant 1,\\
&\big\|k_2(u)\big\|_{L^1(Q_t)} \leqslant \text{const}\,\|u\|_{L^{p_1}(Q_t)} \quad \text{if } \|u\|_{L^{p_1}(Q_t)} < 1,\\
&\big\|k_2(u)\big\|_{L^1(Q_t)} \leqslant \text{const}\,\|u\|^{\sigma}_{L^{p_1}(Q_t)} \quad \text{if } \|u\|_{L^{p_1}(Q_t)} \geqslant 1,
\end{aligned}$$

where $1 \leqslant p_1 \leqslant p$, $0 \leqslant \sigma^\star < p-1$, $0 \leqslant \sigma < p - \sigma^\star$. Further,

$$\begin{aligned}
&a_i(\tau,x,\zeta_0,\zeta;u) = 0 \quad \text{for } \tau \in [0,t] \text{ if } u(\tau) = 0 \text{ for } \tau \in [0,t]\\
&\quad (i = 0,1,\ldots,n).
\end{aligned}$$

THEOREM 5.3. *Assume* (B_1)–(B_3), (B_4''), (B_5). *Then for any* $f \in L^{q_1}(Q_T)$ (*where* $1/p_1 + 1/q_1 = 1$) *there exists a solution of* (5.6).

The proof is a modification of the proof of Theorem 2.5 (based on Galerkin's method). Assumption (B_4'') does not imply the coerciveness of A (given in 2.16), but it is not difficult to show that by (B_4'') the sequence of Galerkin's approximations is bounded in $L^p(0,T;V)$. The detailed proof can be found in [69].

EXAMPLE 5.2. Let a_i have the form as in Example 5.1 and

$$\big[B_1(u)\big](t,x) = b_1\big(\big[H(u)\big](t,x)\big)$$

where the function b_1 satisfies

$$b_1(\theta) \geqslant \text{const}\,|\theta| \quad \text{if } |\theta| < 1, \qquad b_1(\theta) \geqslant \frac{\text{const}}{|\theta|^{\sigma^\star}} \quad \text{if } |\theta| > 1, \quad b_1(0) = 0$$

with some positive constants, $0 \leqslant \sigma^\star < p-1$ and

$$\big[H(u)\big](t,x) = \left[\int_{Q_t} d(t,x,\tau,\xi)\big|u(\tau,\xi)\big|^{p_1}\,\mathrm{d}\tau\,\mathrm{d}\xi\right]^{1/p_1}, \quad 1 \leqslant p_1 \leqslant p,$$

$d(t, x, \tau, \xi)$ is between two positive constants. Further,

$$B_2(u) = 0, \quad \big[B_3(u)\big](t,x) = b_3\big(\big[G_0(u)\big](t,x)\big), \quad |b_3(\theta)| \leqslant \text{const}\,|\theta|^{p-1-\rho^\star}$$

where $\sigma^\star + \rho < \rho^\star \leqslant p-1$, $G_0 : L^p(Q_T) \to L^p(Q_T)$ is a bounded linear operator of Volterra type,

$$(p-\sigma^\star)\frac{p-1-\rho^\star}{p-1-\sigma^\star-\rho} \geqslant 1.$$

Then assumptions of Theorem 5.3 hold.

5.4. *Modified conditions on nonlocal terms*

Now we formulate a modification of assumption (B_5) which makes possible to investigate equations considered in Example 5.1 where the operator H in B_1 may have more general form. At the same time, instead of (B_4) we have to assume a stronger condition:

$(\mathrm{B}_4^\star)$ $\sum_{i=0}^n a_i(t,x,\zeta_0,\zeta)\zeta_i \geqslant c_2[|\zeta_0|^p + |\zeta|^p] - [k_2(u)](t,x)$ with some constant $c_2 > 0$ for a.e. $(t,x) \in Q_T$, all $(\zeta_0,\zeta) \in \mathbb{R}^{n+1}$, $u \in L^p(0,T;V)$ where

$$\lim_{\|u\|_X \to \infty} \frac{\|k_2(u)\|_{L^1(Q_T)}}{\|u\|_X^p} = 0$$

and k_2 is continuous as a map from $L^p(0,T;W^{1-\delta,p}(\Omega))$ to $L^1(Q_T)$.

Instead of (B_5) assume

$(\mathrm{B}_5^\star)$ If $(u_k) \to u$ weakly in $X = L^p(0,T;V)$, strongly in $L^p(0,T;W^{1-\delta,p}(\Omega))$ and $(\zeta_{0k}) \to \zeta_0$, $(\zeta_k) \to \zeta$ then for a.e. $(t,x) \in Q_T$, $i = 0,1,\dots,n$

$$a_i(t,x,\zeta_{0k},\zeta_k;u_k) \to a_i(t,x,\zeta_0,\zeta;u) \quad \text{as } k \to \infty.$$

THEOREM 5.4. *Assume* (B_1)–(B_3), $(\mathrm{B}_4^\star)$, $(\mathrm{B}_5^\star)$. *Then operator* $A : X \to X^\star$ (*given in Definition* 5.1) *is bounded, demicontinuous, pseudomonotone with respect to* $D(L)$, *it is coercive and of Volterra type. Consequently, for any* $f \in X^\star$ *there exists a solution* $u \in D(L)$ *of* (5.6).

The proof of this theorem is based on arguments in the proof of existence theorem on nonlinear elliptic equations in [12] (the detailed proof see in [34]).

EXAMPLE 5.3. It is not difficult to show that assumptions of Theorem 5.4 are fulfilled for the equation, considered in Example 5.1 with operators B_j, defined in (5.14) if

$$c_3 \leqslant b_1(\theta) \leqslant c_4, \quad 0 \leqslant b_2(\theta) \leqslant c_5|\theta|^{p-1-\hat\rho}, \quad \big|b_3(\theta)\big| \leqslant c_5|\theta|^{p-1-\rho^\star}$$

with some positive constants c_3 - c_5, $\rho < \rho^\star \leqslant p-1$, $\rho \leqslant \hat\rho \leqslant p-1$. Now operator H may be more general, it may have the same forms as operator G and G_0 in Example 5.1.

6. Parabolic functional differential equations in $(0, \infty)$

In this section we shall study solutions of parabolic differential equations and functional parabolic equations in $(0, \infty)$. It will be proved a general existence theorem and certain qualitative properties of the solutions in $(0, \infty)$. The general results will be applied to problems considered in Sections 4 and 5.

6.1. *Existence of solutions in* $(0, \infty)$

First we formulate some basic definitions.

DEFINITION 6.1. Let V be a Banach space, $1 \leqslant p < \infty$. The set $L^p_{\mathrm{loc}}(0, \infty; V)$ consists of all functions $f : (0, \infty) \to V$ for which the restriction $f\,|_{(0,T)}$ of f to $(0, T)$ belongs to $L^p(0, T; V)$ for each finite $T > 0$.

Further, by using the notations $Q_\infty = (0, \infty) \times \Omega$, $\Gamma_\infty = (0, \infty) \times \Omega$, denote by $L^q_{\mathrm{loc}}(Q_\infty)$ and $L^q_{\mathrm{loc}}(\Gamma_\infty)$ the set of functions $f : Q_\infty \to R$ and $g : \Gamma_\infty \to R$, respectively, for which $f|_{Q_T} \in L^q(Q_T)$, $g|_{\Gamma_T} \in L^q(\Gamma_T)$ for arbitrary finite $T > 0$.

DEFINITION 6.2. Let $A : L^p_{\mathrm{loc}}(0, \infty; V) \to L^q_{\mathrm{loc}}(0, \infty; V^\star)$ be of Volterra type, i.e. for each $u \in L^p_{\mathrm{loc}}(0, \infty; V)$, $t > 0$, $[A(u)](t)$ depends only on $u|_{[0,t]}$. Then the "restriction of A to $[0, T]$", denoted by A_T, is the operator $A_T : L^p(0, T; V) \to L^q(0, T; V^\star)$, defined by

$$A_T(u) = A(u_T), \quad u \in L^p(0, T; V) \text{ where}$$
$$u_T(t) = u(t) \quad \text{for } t \in [0, T] \quad \text{and} \quad u_T(t) = 0 \quad \text{for } t > T.$$

THEOREM 6.1. *Let V be a reflexive, separable and uniformly convex Banach space, $1 < p < \infty$ and $V \subset H \subset V^\star$ an evolution triple (H is a Hilbert space), $A : L^p_{\mathrm{loc}}(0, \infty; V) \to L^q_{\mathrm{loc}}(0, \infty; V^\star)$ an operator of Volterra type such that for each finite $T > 0$, the restriction of A to $[0, T]$, $A_T : L^p(0, T; V) \to L^q(0, T; V^\star)$ satisfies the assumptions of Theorem 2.5, i.e. it is bounded, demicontinuous, pseudomonotone with respect to $D(L)$ and it is coercive in the sense (2.16).*

Then for arbitrary $f \in L^q_{\mathrm{loc}}(0, \infty; V^\star)$ there exists $u \in L^p_{\mathrm{loc}}(0, \infty; V)$ such that $u' \in L^q_{\mathrm{loc}}(0, \infty; V^\star)$ and

$$u'(t) + \big[A(u)\big](t) = f(t) \quad \textit{for a.e. } t \in (0, \infty), \qquad u(0) = 0. \tag{6.1}$$

By using Theorem 2.5, one can prove Theorem 6.1 as follows. Let (T_j) be an increasing sequence of positive numbers with $\lim(T_j) = +\infty$. Due to Theorem 2.5 there exist functions $u_j \in L^p(0, T_j; V)$ such that $u_j' \in L^q(0, T_j; V^\star)$ and

$$u_j'(t) + \big[A_{T_j}(u_j)\big](t) = f(t) \quad \text{for a.e. } t \in [0, T_j], \qquad u_j(0) = 0.$$

Volterra property implies that $u = u_j|_{[0,T_k]}$ satisfies

$$u'(t) + \big[A_{T_k}(u)\big](t) = f(t) \quad \text{for a.e. } t \in [0, T_k]$$

if $T_k < T_j$. Coercivity of A_T implies that for all fixed finite $T > 0$ (and sufficiently large j), $u_j|_{[0,T]}$ is bounded in $L^p(0, T; V)$. The boundedness of A_T implies that $A_T(u_j|_{[0,T]})$ is bounded in $L^q(0, T; V^\star)$.

Therefore, by a "diagonal process", one can select a subsequence of (u_j) (again denoted by (u_j)) such that for each fixed T_k, $(u_j|_{[0,T_k]})$ is weakly converging in $L^p(0, T_k; V)$ and $(u_j'|_{[0,T_k]})$ is weakly converging in $L^q(0, T_k; V^\star)$ as $j \to \infty$. Thus we obtain a function $u \in L^p_{\mathrm{loc}}(0, \infty; V)$ such that $u' \in L^q_{\mathrm{loc}}(0, \infty; V^\star)$, $u(0) = 0$, further,

$$
\begin{aligned}
&(u_j|_{[0,T_k]}) \to u|_{[0,T_k]} \quad \text{weakly in } L^p(0, T_k; V) \text{ and}\\
&(u_j'|_{[0,T_k]}) \to u'|_{[0,T_k]} \quad \text{weakly in } L^p\big(0, T_k; V^\star\big) \text{ as } j \to \infty.
\end{aligned}
$$

Since

$$
\begin{aligned}
&u_j'(t) + \big[A_{T_k}(u_j)\big](t) = f(t) \quad \text{for a.e. } t \in [0, T_k], \qquad u_j(0) = 0,\\
&\int_0^{T_k} \big\langle \big[A_{T_k}(u_j)\big](t), u_j(t) - u(t)\big\rangle \,\mathrm{d}t\\
&\quad = \big[A_{T_k}(u_j), u_j - u\big]_{T_k}\\
&\quad = [f, u_j - u]_{T_k} - [u_j', u_j - u]_{T_k}\\
&\quad = [f, u_j - u]_{T_k} - \frac{1}{2}\big\|u_j(T_k) - u(T_k)\big\|_H^2 + [u', u_j - u]_{T_k},
\end{aligned}
$$

hence

$$
\limsup_{j\to\infty} \big[A_{T_k}(u_j), u_j - u\big]_{T_k} \leqslant 0
$$

which implies (6.1) for $t \in [0, T_k]$.

Combining Theorem 6.1 with previous existence theorems in $[0, T]$, one obtains existence theorems in $[0, \infty)$. E.g., from Theorem 4.1 one gets

THEOREM 6.2. *Assume that for functions $f_\alpha : Q_\infty \times \mathbb{R}^M \to \mathbb{R}$ assumptions* (A_1), (A_2), (A_3''), (A_4) *are satisfied for any finite T and the restrictions $(H_\alpha)_T$, $(G_\alpha)_T$ of operators of Volterra type*

$$
H_\alpha : L^p_{\mathrm{loc}}(0, \infty; V) \to L^q_{\mathrm{loc}}(Q_\infty), \qquad G_\alpha : L^p_{\mathrm{loc}}(0, \infty; V) \to L^q_{\mathrm{loc}}(\Gamma_\infty)
$$

to $[0, T]$ satisfy (A_5) for any finite $T > 0$.

Then for arbitrary $f \in L^q_{\mathrm{loc}}(0, \infty; V^\star)$ there exists $u \in L^p_{\mathrm{loc}}(0, \infty; V)$ such that $u' \in L^q_{\mathrm{loc}}(0, \infty; V^\star)$ and (6.1) *holds.*

Similarly, from Theorem 5.2 one obtains

THEOREM 6.3. *Assume that the restrictions to*

$$
Q_T \times \mathbb{R}^{n+1} \times L^p(0, T; V) \quad \text{of } a_i : Q_\infty \times \mathbb{R}^{n+1} \times L^p_{\mathrm{loc}}(0, \infty; V) \to \mathbb{R}
$$

satisfy (B_1)–(B_3), (B_4'), (B_5) *for any finite $T > 0$. Then for arbitrary $f \in L^q_{\mathrm{loc}}(0, \infty; V^\star)$ there exists $u \in L^p_{\mathrm{loc}}(0, \infty; V)$ such that $u' \in L^q_{\mathrm{loc}}(0, \infty; V^\star)$ and* (6.1) *holds.*

REMARK 6.1. Combining the arguments in the proof of Theorems 4.2 and 6.1, one can formulate and prove existence theorems on strongly nonlinear equations (considered in Theorem 4.2) in $(0, \infty)$ (see also [66]).

6.2. *Boundedness of solutions*

Now we formulate a theorem on the boundedness of $\|u(t)\|_H$, $t \in (0, \infty)$ for the solutions of (6.1).

THEOREM 6.4. *Assume that the assumptions of Theorem 6.1 are fulfilled and for arbitrary $v \in L^p_{\mathrm{loc}}(0, \infty; V)$ with $v' \in L^q_{\mathrm{loc}}(0, \infty; V^\star)$ the inequality*

$$\begin{aligned} &\langle [A(v)](t), v(t) \rangle \\ &\quad \geqslant c_1 \|v(t)\|_V^p - c_2 \Big[\sup_{\tau \in [0,t]} \|v(\tau)\|_H^{p_1} + \varphi(t) \sup_{\tau \in [0,t]} \|v(\tau)\|_H^p + 1 \Big] \end{aligned} \tag{6.2}$$

holds where $c_1, c_2 > 0$ are constants, $0 < p_1 < p$, $\varphi \geqslant 0$ is a function with the property $\lim_\infty \varphi = 0$. *Further, $\|f(t)\|_{V^\star}$ is bounded for $t \in (0, \infty)$.*

Then for a solution u of (6.1) (with arbitrary initial condition) $\|u\|_H$ is bounded for $t \in (0, \infty)$ and

$$\int_{T_1}^{T_2} \|u(t)\|_V^p \, \mathrm{d}t \leqslant c_3 (T_2 - T_1), \quad 0 \leqslant T_1 < T_2 \tag{6.3}$$

with some constant c_3 not depending on T_1, T_2.

THE MAIN STEPS OF THE PROOF. By using the notation $y(t) = \|u(t)\|_H^2$ we obtain from (6.1), (6.2) the inequality for a.e. $t \in (0, \infty)$ (with some positive constants c_1, c_2)

$$\frac{1}{2} y'(t) + c_1 \|u(t)\|_V^p \leqslant \|f(t)\|_{V^\star} \|u(t)\|_V + c_2 \Big[\sup_{[0,t]} y^{p_1/2} + \varphi(t) \sup_{[0,t]} y^{p/2} + 1 \Big],$$

hence by Young's inequality (with some positive constants c_3, c_4)

$$\frac{1}{2} y'(t) + c_3 \|u(t)\|_V^p \leqslant c_4 \|f(t)\|_{V^\star}^q + c_2 \Big[\sup_{[0,t]} y^{p_1/2} + \varphi(t) \sup_{[0,t]} y^{p/2} + 1 \Big]. \tag{6.4}$$

Since the embedding $V \subset H$ is continuous and $\|f(t)\|_{V^\star}$ is bounded, we find the inequality

$$y'(t) + c^\star [y(t)]^{p/2} \leqslant c_5 \Big[\sup_{[0,t]} y^{p_1/2} + \varphi(t) \sup_{[0,t]} y^{p/2} + 1 \Big] \tag{6.5}$$

with some positive constants $c^\star$, c_5.

Assuming that $y(t)$ is not bounded, for any $M > 0$ there are $t_0 > 0$ and $t_1 \in [0, t_0]$ such that

$$M + 1 \geqslant y(t_1) = \sup_{[0,t_0]} y > M.$$

Since y is continuous, there is a $\delta > 0$ such that

$$y(t) > M \quad \text{if } t_1 - \delta \leqslant t \leqslant t_1.$$

Hence by (6.5)

$$\begin{aligned} &y(t_1) - y(t_1 - \delta) + c^\star \delta M^{p/2} \\ &\quad \leqslant c_5\left[(M+1)^{p_1/2} + (M+1)^{p/2}\int_{t_1-\delta}^{t_1} \varphi(t)\,\mathrm{d}t + 1\right] \end{aligned}$$

which is impossible for all $M > 0$ because $y(t_1) - y(t_1 - \delta) \geqslant 0$, $p_1 < p$ and $\lim_\infty \varphi = 0$. Finally, from (6.4) and boundedness of $y(t)$ we obtain (6.3). □

Theorem 6.4 directly implies

THEOREM 6.5. *Assume that conditions of Theorem 6.2 are fulfilled such that for arbitrary $v \in L^p_{\mathrm{loc}}(0, \infty; V)$ with $v' \in L^q_{\mathrm{loc}}(0, \infty; V^\star)$ the inequality*

$$\begin{aligned} &\int_\Omega \left|H_\alpha(v)\right|^q \mathrm{d}x + \int_{\partial\Omega} \left|G_\alpha(v)\right|^q \mathrm{d}\sigma_x \\ &\quad \leqslant c_2\left[\sup_{\tau\in[0,t]} \left\|v(\tau)\right\|^{p_1}_{L^2(\Omega)} + \varphi(t) \sup_{\tau\in[0,t]} \left\|v(\tau)\right\|^{p}_{L^2(\Omega)} + 1\right] \end{aligned} \tag{6.6}$$

holds for all $t \in (0, \infty)$ with constants $c_2 > 0$, $0 < p_1 < p$ and a function $\varphi \geqslant 0$ with the property $\lim_\infty \varphi = 0$. Further, $\|f(t)\|_{V^\star}$ is bounded for $t \in (0, \infty)$.

Then for a solution u of (6.1) (with arbitrary initial condition) $\|u\|_H$ is bounded for $t \in (0, \infty)$ and (6.3) holds.

Similarly to the proof of Theorem 6.4 one can prove (see [69])

THEOREM 6.6. *Assume that assumptions of Theorem 6.3 are satisfied such that for all $v \in L^p_{\mathrm{loc}}(0, \infty; V)$ with $v' \in L^q_{\mathrm{loc}}(0, \infty; V^\star)$ the inequalities*

$$\left[g_2(v)\right](t) \geqslant \mathrm{const}\left[\sup_{\tau\in[0,t]} \left\|v(\tau)\right\|^{-\sigma^\star}_{L^2(\Omega)} + 1\right], \tag{6.7}$$

$$\begin{aligned} &\int_\Omega \left[k_2(v)\right](t, x)\,\mathrm{d}x \\ &\quad \leqslant \mathrm{const}\left[\sup_{\tau\in[0,t]} \left\|v(\tau)\right\|^{\sigma}_{L^2(\Omega)} + \varphi(t) \sup_{\tau\in[0,t]} \left\|v(\tau)\right\|^{p-\sigma^\star}_{L^2(\Omega)} + 1\right] \end{aligned} \tag{6.8}$$

hold with some positive constants, $0 < \sigma^\star < p - 1$, $1 \leqslant \sigma < p - \sigma^\star$, $\lim_\infty \varphi = 0$ and $\|f(t)\|_{V^\star}$ is bounded for $t \in (0, \infty)$.

Then for a solution u of (6.1) (with arbitrary initial condition) $\|u\|_H$ is bounded for $t \in (0, \infty)$ and (6.3) holds.

6.3. *Attractivity*

Now we formulate conditions which imply that $\lim_{t\to\infty} \|u(t)\|_H = 0$.

THEOREM 6.7. *Assume that the conditions of Theorem 6.1 are satisfied such that for all $v \in L^p_{\text{loc}}(0,\infty; V)$ with $v' \in L^q_{\text{loc}}(0,\infty; V^\star)$ we have*

$$\big\langle \big[A(v)\big](t), v(t)\big\rangle \geqslant c_1 \big\|v(t)\big\|_V^p - \varphi(t)\Big[\sup_{[0,t]}\big\|v(t)\big\|_H^p + 1\Big] \tag{6.9}$$

with a constant $c_1 > 0$, $\lim_\infty \varphi = 0$ and $\lim_{t\to\infty} \|f(t)\|_{V^\star} = 0$.

Then for a solution u of (6.1) (with arbitrary initial condition)

$$\lim_{t\to\infty}\big\|u(t)\big\|_H = 0.$$

One can derive this theorem from Theorem 6.4 as follows. Since $y(t) = \|u(t)\|_H^2$ is bounded, we obtain similarly to (6.5) inequality

$$y'(t) + c^\star\big[y(t)\big]^{p/2} \leqslant \psi(t) \tag{6.10}$$

where $\lim_\infty \psi = 0$, $c^\star$ is a positive constant. Assuming that $y(t)$ does not converge to 0 as $t \to \infty$, it is not difficult to get a contradiction.

Now we formulate two corollaries (particular cases) of Theorem 6.7 (see [60,62,69]).

THEOREM 6.8. *Assume that the conditions of Theorem 6.2 are satisfied such that (for the function in* (A$_4$))

$$\lim_{t\to\infty}\int_\Omega k_2(t,x)\,\mathrm{d}x = 0$$

and for all $v \in L^p_{\text{loc}}(0,\infty; V)$ with $v' \in L^q_{\text{loc}}(0,\infty; V^\star)$

$$\int_\Omega \big|H_\alpha(v)\big|^q\,\mathrm{d}x + \int_{\partial\Omega}\big|G_\alpha(v)\big|^q\,\mathrm{d}\sigma_x \leqslant \varphi(t)\Big[\sup_{\tau\in[0,t]}\big\|v(\tau)\big\|_H^p + 1\Big]$$

where $\lim_\infty \varphi = 0$ and $\lim_{t\to\infty} \|f(t)\|_{V^\star} = 0$.

Then for a solution of equation (6.1)

$$\lim_{t\to\infty}\big\|u(t)\big\|_{L^2(\Omega)} = 0.$$

THEOREM 6.9. *Assume that the conditions of Theorem 6.6 are fulfilled such that for all $v \in L^p_{\text{loc}}(0,\infty; V)$ with $v' \in L^q_{\text{loc}}(0,\infty; V^\star)$*

$$\int_\Omega \big[k_2(v)\big](t,x)\,\mathrm{d}x \leqslant \varphi(t)\Big[\sup_{\tau\in[0,t]}\big\|v(\tau)\big\|_H^{p-\sigma^\star} + 1\Big]$$

where $0 < \sigma^\star < p-1$, $\lim_\infty \varphi = 0$ and $\lim_{t\to\infty} \|f(t)\|_{V^\star} = 0$.

Then for a solution of (6.1)

$$\lim_{t\to\infty}\big\|u(t)\big\|_{L^2(\Omega)} = 0.$$

6.4. *Stabilization of solutions*

In a similar way, one gets a result on the stabilization of the solution to (6.1) as $t \to \infty$.

THEOREM 6.10. *Assume that the conditions of Theorem 6.1 are satisfied. Further, there exist operators*

$$A_\infty : V \to V^\star \quad \text{and} \quad B : L^p_{\text{loc}}(0,\infty;V) \times V \to L^q_{\text{loc}}(0,\infty;V^\star)$$

such that for all $u \in L^p_{\text{loc}}(0,\infty;V)$, $v \in V$, $t \in (0,\infty)$

$$\langle [A(u)](t) - [B(u,v)](t), u(t) - v \rangle \geqslant c \| u(t) - v \|_V^p \tag{6.11}$$

with some constant $c > 0$ and

$$\lim_{t\to\infty} \| [B(u,v)](t) - A_\infty(v) \|_{V^\star} = 0. \tag{6.12}$$

Further, assume that there is $f_\infty \in V^\star$ such that

$$\lim_{t\to\infty} \| f(t) - f_\infty \|_{V^\star} = 0 \tag{6.13}$$

and $u_\infty \in V$ satisfies

$$A_\infty(u_\infty) = f_\infty. \tag{6.14}$$

Then for a solution u of (6.1) (with arbitrary initial condition)

$$\lim_{t\to\infty} \| u(t) - u_\infty \|_H = 0, \qquad \lim_{T\to\infty} \int_{T-a}^{T+a} \| u(t) - u_\infty \|_V^p \, \mathrm{d}t = 0 \tag{6.15}$$

for arbitrary fixed $a > 0$.

PROOF. From (6.1), (6.14) one obtains

$$\begin{aligned} &\langle (u(t) - u_\infty)', u(t) - u_\infty \rangle + \langle [A(u)](t) - A_\infty(u_\infty), u(t) - u_\infty \rangle \\ &\quad = \langle f(t) - f_\infty, u(t) - u_\infty \rangle. \end{aligned} \tag{6.16}$$

By (6.11) and Young's inequality for arbitrary $\varepsilon > 0$

$$\begin{aligned} &\langle [A(u)](t) - A_\infty(u_\infty), u(t) - u_\infty \rangle \\ &\quad = \langle [A(u)](t) - [B(u,u_\infty)](t), u(t) - u_\infty \rangle \\ &\qquad + \langle [B(u,u_\infty)](t) - A_\infty(u_\infty), u(t) - u_\infty \rangle \\ &\quad \geqslant c \| u(t) - u_\infty \|_V^p - \frac{\varepsilon^p}{p} \| u(t) - u_\infty \|_V^p \\ &\qquad - \frac{1}{q\varepsilon^q} \| [B(u,u_\infty)](t) - A_\infty(u_\infty) \|_{V^\star}^q \end{aligned} \tag{6.17}$$

and

$$|\langle f(t) - f_\infty, u(t) - u_\infty \rangle| \leqslant \frac{\varepsilon^p}{p} \| u(t) - u_\infty \|_V^p + \frac{1}{q\varepsilon^q} \| f(t) - f_\infty \|_{V^\star}^q. \tag{6.18}$$

Choosing sufficiently small $\varepsilon > 0$, one obtains from (6.16)–(6.18) (similarly to (6.10), (6.4), (6.5)) for $y(t) = \|u(t) - u_\infty\|_H^2$

$$y'(t) + \tilde{c}\|u(t) - u_\infty\|_V^p \leqslant \psi(t), \qquad y'(t) + c^\star[y(t)]^{p/2} \leqslant \psi(t) \tag{6.19}$$

where $c^\star > 0$, $\tilde{c} > 0$ and by (6.12), (6.13) $\lim_\infty \psi = 0$ which imply (6.15). □

Now we show how Theorem 6.10 can be applied to the problems considered in Sections 4, 5 and (in that cases) why exists a (unique) solution of (6.14).

THEOREM 6.11. *Assume that conditions of Theorem 6.2 and* (A_3') (*the condition of uniform monotonicity*) *are satisfied such that for all $u \in L^p_{\mathrm{loc}}(0,\infty;V)$ with $u' \in L^q_{\mathrm{loc}}(0,\infty;V^\star)$*

$$\lim_{t\to\infty} \|[H_\alpha(u)](t)\|_{L^q(\Omega)} = 0, \qquad \lim_{t\to\infty} \|[G_\alpha(u)](t)\|_{L^q(\partial\Omega)} = 0, \tag{6.20}$$

further, for a.e. $x \in \Omega$, all $\xi \in \mathbb{R}^M$

$$\lim_{t\to\infty} f_\alpha(t,x,\xi) = f_\alpha^\infty(x,\xi). \tag{6.21}$$

Then defining A_∞ by

$$\langle A_\infty(v), w\rangle = \sum_{|\alpha|\leqslant m} \int_\Omega f_\alpha^\infty(x, v, \ldots, D^\beta v, \ldots) D^\alpha w \,\mathrm{d}x, \quad v, w \in V, \tag{6.22}$$

for arbitrary $f_\infty \in V^\star$ there exists a solution u_∞ of (6.14) and (6.15) *holds*.

In this case conditions of Theorem 6.10 are satisfied with $A(u) = A_1(u) + B_1(u)$, $B(u,v) = B_1(u) + A_1(v)$ where A_1, B_1 are defined by (3.14) and (4.3), respectively. Because, (A_3') implies (6.11) and (6.20), (6.21), (A_2) imply (6.12). Further, by (A_1), (A_2), (A_3'), (A_4) and (6.21) $A_\infty : V \to V^\star$ is bounded, demicontinuous, monotone and coercive, thus there is a solution $u_\infty \in V$ of (6.14). (See, e.g., [41,75].)

THEOREM 6.12. *Assume that the conditions of Theorem 6.3 are fulfilled such that in* (B_2) *we have operators*

$$g_1 : L^p_{\mathrm{loc}}(0,\infty;V) \to R_+, \qquad k_1 : L^p_{\mathrm{loc}}(0,\infty;V) \to L^q(\Omega);$$

and for arbitrary $u \in L^p_{\mathrm{loc}}(0,\infty;V)$ with bounded $\|u\|_{L^2(\Omega)}$, for every ζ_0, ζ and a.e. x

$$\lim_{t\to\infty} a_i(t,x,\zeta_0,\zeta;u) = a_{i,\infty}(x,\zeta_0,\zeta), \quad i = 0,1,\ldots,n,$$

exist and is finite; *$a_{i,\infty}$ satisfy the Carathéodory conditions. Further, let $f_\infty \in V^\star$ be such that*

$$\lim_{t\to\infty} \|f(t) - f_\infty\|_{V^\star} = 0.$$

Finally, for every fixed $u \in L^p_{\mathrm{loc}}(0,\infty;V)$

$$\sum_{i=0}^{n}\big[a_i(t,x,\zeta_0,\zeta;u)-a_i(t,x,\zeta_0^\star,\zeta^\star;u)\big](\zeta_i-\zeta_i^\star)$$
$$\geqslant c\big[(\zeta_0-\zeta_0^\star)^p+|\zeta-\zeta^\star|^p\big]-\big[k_3(u)\big](t,x)$$

with some constant $c>0$,

$$\lim_{t\to\infty}\int_\Omega\big[k_3(u)\big](t,x)\,\mathrm{d}x=0\quad \textit{if}\ \int_\Omega\big|u(t,x)\big|^2\,\mathrm{d}x\ \textit{is bounded.}$$

Then there exists a unique $u_\infty \in V$ *satisfying* $A_\infty(u_\infty)=f_\infty$ *where* $A_\infty : V\to V^\star$ *is defined by*

$$\big\langle A_\infty(w),v\big\rangle=\sum_{i=1}^{n}\int_\Omega a_{i,\infty}\big(x,w(x),Dw(x)\big)D_iv\,\mathrm{d}x$$
$$+\int_\Omega a_{0,\infty}\big(x,w(x),Dw(x)\big)v\,\mathrm{d}x,\qquad w,v\in V.$$

Further, for the solution of (6.1) *we have* (6.15).

(The detailed proof see in [67].)

REMARK 6.2. It is not difficult to formulate conditions for the examples considered in Sections 4 and 5 when the theorems of this section can be applied. (See [60,62,66,67,69].

REMARK 6.3. One can formulate and prove analogous theorems on the boundedness of the solutions and on $\lim_{t\to\infty}\|u\|_{L^2(\Omega)}=0$ for the solutions of strongly nonlinear equations (considered in Theorem 4.2), see Remark 6.1, [62,66]. In this case one applies the system of energy equalities (in Theorem 4.2) to $v=0$ to get differential inequalities for $y(t)=\|u\|^2_{L^2(\Omega)}$ which are analogous to (6.5), (6.10), respectively.

Now we show another type of stabilization result on certain equations of particular type. Here we formulate it for Example 4.2. More general example see in [61].

THEOREM 6.13. *Let the functions* f_j *be defined by*

$$f_j(t,x,\zeta_0,\zeta)=f_j(x,\zeta_0,\zeta)=a_j(x)\zeta_j|\zeta_j|^{p-2},\quad j=1,\dots,n,$$
$$f_0(t,x,\zeta_0,\zeta)=f_0(x,\zeta_0,\zeta)=a_0(x)\zeta_0|\zeta_0|^{p-2}+g(x,\zeta_0)$$

where the measurable functions a_j *satisfy* $0<c_0\leqslant a_j(x)\leqslant c_0'$ *with some constants* c_0, c_0' *and* g *is a Carathéodory function satisfying*

$$\big|g(x,\zeta_0)\big|\leqslant c_1|\zeta_0|+k_1(x)\quad \textit{with a constant } c_1 \textit{ and } k_1\in L^q(\Omega).$$

Assume that h *is a Carathéodory function satisfying*

$$\big|h(t,x,\theta)\big|\leqslant \chi(t)\big[|\theta|+k_2(x)\big]$$

with some functions $\chi \in L^2(0,\infty)$, $k_2 \in L^2(\Omega)$ *and*

$$F_0 : L^p_{\mathrm{loc}}(Q_\infty) \to L^p_{\mathrm{loc}}(Q_\infty)$$

is a linear operator of Volterra type such that for any $u \in L^p_{\mathrm{loc}}(Q_\infty)$

$$\int_\Omega \big|F_0(u)\big|^2(t,x)\,\mathrm{d}x \leqslant c_2 \sup_{\tau\in[0,t]} \int_\Omega u(\tau,x)^2\,\mathrm{d}x$$

with some constant c_2. *Finally,*

$$f \in L^\infty\big(0,\infty; L^2(\Omega)\big), \qquad D_t f \in L^2_{\mathrm{loc}}\big(0,\infty; L^2(\Omega)\big),$$
$$\int_0^\infty \left[\int_\Omega \big|D_t f(t,x)\big|^2\,\mathrm{d}x\right]^{1/2} \mathrm{d}t < \infty.$$

Then for a weak solution $u \in L^p_{\mathrm{loc}}(0,\infty;V)$ (*with a closed linear subspace* V *of* $W^{1,p}(\Omega)$) *of the initial-boundary value problem for*

$$D_t u - \sum_{j=1}^n D_j\big[f_j(t,x,u,Du)\big] + f_0(t,x,u,Du) + g(x,u)$$
$$+ h\big(t,x,\big[F_0(u)\big](t,x)\big) = f, \quad (t,x)\in Q_\infty, \quad u(0)=u_0 \tag{6.23}$$

(*defined by* (6.1)) *we have*

$$D_t u \in L^2\big(0,\infty; L^2(\Omega)\big) \quad \text{and} \quad u \in L^\infty(0,\infty;V). \tag{6.24}$$

THE MAIN STEPS OF THE PROOF. Define the functional $\Phi : V \subset L^2(\Omega) \to \mathbb{R}$ by

$$\Phi(w) = \int_\Omega \left[\sum_{j=1}^n a_j(x)|D_j w|^p + a_0(x)|w|^p\right]\mathrm{d}x, \quad w \in V.$$

Then Φ is a convex nonnegative lower semicontinuous functional (see, e.g., [9,18]) and let $\partial\Phi$ be the subdifferential of Φ. One can show that the weak solution of the problem for (6.23) is the (unique) strong solution of $u(0)=u_0$,

$$\frac{\mathrm{d}u}{\mathrm{d}t} + (\partial\Phi)u(t) \ni b(t) = f - g(x,u) - h\big(t,x,F_0(u)\big) \in L^2\big(0,T;L^2(\Omega)\big)$$

for any finite T; for a.e. t

$$\left\|\frac{\mathrm{d}u}{\mathrm{d}t}\right\|^2_{L^2(\Omega)} + \frac{\mathrm{d}}{\mathrm{d}t}\big[\Phi\big(u(t)\big)\big] = \left(b(t), \frac{\mathrm{d}u}{\mathrm{d}t}(t)\right)_{L^2(\Omega)}$$

(see [9]). Integrating the last equality over (σ,τ), by the assumptions of the theorem we find

$$\frac{1}{2}\int_\sigma^\tau \left[\int_\Omega |D_t u|^2\,\mathrm{d}x\right]\mathrm{d}t + \Phi\big(u(\tau)\big) - \Phi\big(u(\sigma)\big) \leqslant \text{const}.$$

Since $\Phi \geqslant 0$, we obtain (6.24). (The detailed proof see in [61].) $\square$

Consider a sequence $(t_k) \to \infty$ and define functions U_k by

$$U_k(s, x) = u(t_k + s, x), \quad s \in (-a, b),\ x \in \Omega$$

with some fixed $a > 0$, $b > 0$.

REMARK 6.4. By Theorem 6.5 (U_k) is a bounded sequence in $L^p(-a, b; V)$ and in $L^\infty(-a, b; L^2(\Omega))$ for a weak solution u of (6.23).

Define the ω limit set associated to u by

$$\omega(u) = \bigl\{u_\infty \in V\colon \exists (t_k) \to \infty \text{ such that } u(t_k, \cdot) \to u_\infty \text{ in } L^p(\Omega)\bigr\}.$$

By using arguments of [18], one can prove (see [61])

THEOREM 6.14. *Let the assumptions of Theorem 6.13 be satisfied. On operator F_0 assume that there exists a finite $\rho > 0$ such that $[F_0(u)](t, x)$ depends only on the restriction of u to $(t - \rho, t) \times \Omega$ for arbitrary t. Further, there exists $f_\infty \in L^2(\Omega)$ such that*

$$\lim_{T \to \infty} \int_{T-1}^{T+1} \bigl\| f(t) - f_\infty \bigr\|_{L^2(\Omega)} \,\mathrm{d}t = 0.$$

Then for any weak solution of the problem for (6.23), $\omega(u) \neq \emptyset$. If $u_\infty \in \omega(u)$ then there is a sequence $(t_k) \to +\infty$ such that

$$U_k \to u_\infty \quad \textit{in } L^p\bigl((-1, 1) \times \Omega\bigr) \textit{ and weakly in } L^p(-1, 1; V).$$

Further, u_∞ is a solution of the stationary problem

$$\sum_{j=1}^{n} \int_\Omega f_j(x, u_\infty, Du_\infty) D_j w \,\mathrm{d}x + \int_\Omega f_0(x, u_\infty, Du_\infty) w \,\mathrm{d}x = \langle f_\infty, w\rangle,$$
$$w \in V.$$

7. Further applications

In this section we shall consider applications of Section 2 to systems of parabolic differential equations and functional parabolic equations, further, equations with contact conditions (transmission problems).

7.1. *Systems of parabolic equations and functional parabolic equations*

It is not difficult to extend the results of Sections 3–6 to systems of parabolic differential equations and functional parabolic equations. Set $V = V_1 \times \cdots \times V_r$ where V_l is a closed linear subspace of $W^{m,p}(\Omega)$. E.g. instead of (A_1), (A_2), (A_3''), (A_4) we assume

(A_1^r) The functions

$$f_\alpha^{(l)}: Q_T \times \mathbb{R}^{Mr} \to \mathbb{R}$$

are measurable in $(t,x) \in Q_T$ and continuous in $\xi = (\xi_1, \dots, \xi_r) \in \mathbb{R}^{Mr}$.

(A_2^r) $|f_\alpha^{(l)}(t,x,\xi)| \leqslant |\xi|^{p-1} + k_1(t,x)$ where $\xi = (\xi^{(1)}, \dots, \xi^{(r)})$, $|\alpha| \leqslant m$, $l = 1, \dots, r$, $k_1 \in L^q(Q_T)$.

(A_3^r) $\sum_{l=1}^r \sum_{|\alpha|=m} [f_\alpha^{(l)}(t,x,\eta^{(1)},\zeta^{(1)},\dots,\eta^{(r)},\zeta^{(r)}) - f_\alpha^{(l)}(t,x,\eta^{(1)},\zeta_\star^{(1)},\dots,\eta^{(r)}, \zeta_\star^{(r)})](\xi_\alpha^{(l)} - \xi_{\alpha,\star}^{(l)}) > 0$ if $\zeta \neq \zeta_\star$ ($\xi^{(l)} = (\eta^{(l)}, \zeta^{(l)})$, $\zeta = (\zeta^{(1)}, \dots, \zeta^{(r)})$).

(A_4^r) $\sum_{l=1}^r \sum_{|\alpha|\leqslant m} [f_\alpha^{(l)}(t,x,\xi)\xi_\alpha^{(l)} \geqslant c_2|\xi|^p - k_2(t,x)$ with $k_2 \in L^1(Q_T)$ and $c_2 > 0$.

Then we may define the operator A by

$$\begin{aligned}
\bigl[A(u), v\bigr] = {} & \sum_{l=1}^r \sum_{|\alpha|\leqslant m} f_\alpha^{(l)}\bigl(t,x,\dots,D_x^\beta u^{(1)}(t,x),\dots,D_x^\beta u^{(r)}(t,x),\dots\bigr) \\
& \times D_x^\alpha v^{(l)}(t,x)\,\mathrm{d}t\,\mathrm{d}x, \\
u = \bigl(u^{(1)},\dots,u^{(r)}\bigr) \in L^p(0,T;V), & \qquad v = \bigl(v^{(1)},\dots,v^{(r)}\bigr) \in L^p(0,T;V).
\end{aligned} \tag{7.1}$$

Further, define

$$D(L) = \bigl\{u \in L^p(0,T;V)\colon\ u' \in L^q(0,T;V^\star),\ u(0) = 0\bigr\}.$$

So we obtain the following extension of Theorem 3.3 to systems.

THEOREM 7.1. *Assume* (A_1^r)–(A_4^r). *Then the operator* $A: L^p(0,T;V) \to L^q(0,T;V^\star)$ *is bounded, demicontinuous, pseudomonotone with respect to* $D(L)$ *and coercive. Consequently, for arbitrary* $f \in L^q(0,T;V^\star)$ *there exists a solution* $u = (u^{(1)},\dots,u^{(r)}) \in D(L)$ *of the system*

$$\begin{aligned}
&\bigl[D_t u^{(l)}\bigr](t) + \bigl[A\bigl(u^{(l)}\bigr)\bigr](t) = f^{(l)}(t), \quad t \in (0,T), \\
&u^{(l)}(0) = 0, \quad l = 1,\dots,r.
\end{aligned}$$

This theorem can be proved by using arguments of [12] by F.E. Browder, similarly to the case of a single parabolic equation.

Similarly, one gets the following extension of Theorem 4.1 on parabolic functional differential equations to systems. Assume

(A_5^r) $H_\alpha^{(l)}: L^p(0,T;V) \to L^q(Q_T)$, $G_\alpha^{(l)}: L^p(0,T;V) \to L^q(Q_T)$ ($l = 1,\dots,r$) are bounded (possibly nonlinear) operators of Volterra type which are demicontinuous from $L^p(0,T;(W^{m-\delta,p}(\Omega))^r)$ to $L^q(Q_T)$ and $L^q(\Gamma_T)$, respectively, for some $0 < \delta < 1 - 1/p$ and have the property

$$\lim_{\|u\|\to\infty} \frac{\|H_\alpha^{(l)}(u)\|_{L^q(Q_T)}^q + \|G_\alpha^{(l)}(u)\|_{L^q(\Gamma_T)}^q}{\|u\|_{L^p(0,T;V)}^p} = 0, \quad l = 1,\dots,r.$$

Then we may define operator $B_1 : L^p(0,T;V) \to L^q(0,T;V^\star)$ by

$$[B_1(u), v] = \sum_{l=1}^{r} \sum_{|\alpha| \leqslant m-1} \int_{Q_T} H_\alpha^{(l)}(u) D_x^\alpha v^{(l)}\,\mathrm{d}t\,\mathrm{d}x \\ + \sum_{l=1}^{r} \sum_{|\alpha| \leqslant m-1} \int_{\Gamma_T} G_\alpha^{(l)}(u) D_x^\alpha v^{(l)}\,\mathrm{d}t\,\mathrm{d}\sigma_x, \quad u, v \in L^p(0,T;V). \tag{7.2}$$

THEOREM 7.2. *Assume* (A_1^r)–(A_5^r) *and consider* $A = A_1$ *defined by* (7.1). *Then* $(A_1 + B_1) : L^p(0,T;V) \to L^q(0,T;V^\star)$ *is bounded, demicontinuous, pseudomonotone with respect to* $D(L)$, *it is coercive and of Volterra type. Thus for any* $f \in L^q(0,T;V^\star)$ *there exists a solution* $u = (u^{(1)}, \ldots, u^{(r)}) \in D(L)$ *of*

$$D_t u + (A_1 + B_1)(u) = f, \qquad u(0) = 0. \tag{7.3}$$

REMARK 7.1. Similarly to the case of a single equation, assume

$$\sum_{l=1}^{r} \sum_{|\alpha| \leqslant m} \left[f_\alpha^{(l)}(t,x,\xi^{(1)},\ldots,\xi^{(r)}) - f_\alpha^{(l)}\big(t,x,\xi_\star^{(1)},\ldots,\xi_\star^{(r)}\big)\right]\big(\xi_\alpha^{(l)} - \xi_{\alpha,\star}^{(l)}\big) \\ \geqslant -c \sum_{l=1}^{r} \big(\xi_0^{(l)} - \xi_{0,\star}^{(l)}\big)^2$$

for (sufficiently large) constant $c > 0$, $G_\alpha^{(l)} = 0$ for all α, l and $H_\alpha^{(l)}$ satisfy the Lipschitz condition

$$\big\|\exp(-d\tau)\big[H_\alpha^{(l)}\big(\exp(d\tau u)\big) - H_\alpha^{(l)}\big(\exp(d\tau u^\star)\big)\big]\big\|_{L^2(Q_t)} \leqslant c_3 \|u - u^\star\|_{[L^2(Q_t)]^r}$$

with some constant $c_3 > 0$, not depending on $u, u^\star, d > 0$. Then (7.3) may have at most one solution. (See Remark 4.2.)

REMARK 7.2. One can apply "abstract" Theorems 6.1, 6.4, 6.7, 6.10 to systems considered in Theorem 7.2 and obtain results on systems which are analogous to Theorems 6.2, 6.5, 6.8, 6.11, respectively.

7.2. *Contact problems*

Now we formulate problems for nonlinear functional parabolic equations with "nonlocal" contact conditions (see [64,65]). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain having the uniform C^1 regularity property (see [1]) which is divided into two subdomains Ω_1, Ω_2 by means of a smooth surface S which has no intersection point with $\partial\Omega$; the boundary of Ω_1 is S and the boundary of Ω_2 is $S \cup \partial\Omega$ (such that Ω_1 and Ω_2 have the C_1 regularity property). We shall consider equations for $u^{(l)} = u|_{Q_T^l}$, $l = 1, 2$

$$D_t u^{(l)} - \sum_{j=1}^{n} D_j\big[f_j^{(l)}(t,x,u^{(l)},Du^{(l)})\big] + f_0^{(l)}(t,x,u^{(l)},Du^{(l)})$$

$$+ H^{(l)}(u^{(1)}, u^{(2)}) = f^{(l)}, \quad (t,x) \in Q_T^l = (0,T) \times \Omega_l,\ l = 1,2, \tag{7.4}$$

with (for simplicity) homogeneous initial and boundary conditions

$$u(0,x) = 0, \quad x \in \Omega_1 \cup \Omega_2, \tag{7.5}$$

$$u^{(2)} = 0 \quad \text{on } \Gamma_T = [0,T] \times \partial\Omega \tag{7.6}$$

where $H^{(l)} : L^p(Q_T^1) \times L^p(Q_T^2) \to L^q(Q_T^l)$ are bounded (possibly nonlinear) operators, $p \geqslant 2$. On the common part $S_T = [0,T] \times S$ of boundaries of Q_T^1 and Q_T^2 we shall formulate nonlocal "transmission conditions". Similar problems were considered by W. Jäger and N. Kutev in [32] for quasilinear elliptic equations with nonlinear contact condition of "Dirichlet type", in [33] similar problems were considered for parabolic equations (by using the theory of monotone type operators). Such problems are motivated e.g. by reaction–diffusion phenomena in porous medium.

First we consider problem (7.4)–(7.6) with the following (possibly nonlinear) contact condition:

$$\sum_{j=1}^n f_j^{(l)}(t,x,u^{(l)}, Du^{(l)})|_{S_T} \nu_j^l = G^{(l)}(u^{(1)}, u^{(2)}), \quad l = 1,2 \tag{7.7}$$

where $\nu^l = (\nu_1^l, \ldots, \nu_n^l)$ are the normal unit vectors on S ($\nu^1 = -\nu^2$), $G^{(l)} : L^p(0,T;V) \to L^q(S_T)$ are bounded (possibly nonlinear) operators, $V = V_1 \times V_2$ where $V_1 = W^{1,p}(\Omega_1)$, $V_2 = \{w \in W^{1,p}(\Omega_2):\ w|_{\partial\Omega} = 0\}$.

Assume that functions $f_j^{(l)}$ satisfy conditions (A$_1$), (A$_2$), (A$_3''$), (A$_4$) (for $m = 1$), then we may define operators

$$A = \left(A^{(1)}, A^{(2)}\right) : L^p(0,T;V) \to L^q(0,T;V^\star),$$
$$A^{(l)} : L^p(0,T;V_l) \to L^q(0,T;V_l^\star), \quad l = 1,2$$

(according to (3.14) or (7.1)) by

$$\left[A^{(l)}(u^{(l)}), v^{(l)}\right] = \int_{Q_T^l} \left[\sum_{j=1}^n f_j^{(l)}(t,x,u^{(l)}, Du^{(l)}) D_j v^{(l)} + f_0^{(l)}(t,x,u^{(l)}, Du^{(l)}) v^{(l)}\right] \mathrm{d}t\,\mathrm{d}x. \tag{7.8}$$

Further, assume

(A$_5^c$) $H^{(l)} : L^p(Q_T^1) \times L^p(Q_T^2) \to L^q(Q_T^l)$ are bounded (possibly nonlinear) and demicontinuous operators of Volterra type ($p \geqslant 2$); $G^{(l)} : L^p(0,T;V) \to L^q(S_T)$ are bounded (nonlinear) operators of Volterra type which are demicontinuous from $L^p(0,T; W^{1-\delta,p}(\Omega_1) \times W^{1-\delta,p}(\Omega_2))$ into $L^q(Q_T^l)$ and $L^q(S_T)$, respectively with some positive $\delta < 1 - 1/p$;

$$\lim_{\|u\| \to \infty} \frac{\|H^{(l)}(u)\|^q_{L^q(Q_T^l)} + \|G^{(l)}(u)\|^q_{L^q(S_T)}}{\|u\|^p_{L^p(0,T;V)}} = 0, \quad l = 1,2.$$

(Consequently, (A_5^r) is satisfied.)

Then we may define operators $B = (B^{(1)}, B^{(2)})$, $B^{(l)}: L^p(0,T;V) \to L^q(0,T;V_l^\star)$ by $[B(u), v] = [B^{(1)}(u), v^{(1)}] + [B^{(2)}(u), v^{(2)}]$,

$$\big[B^{(l)}(u), v^{(l)}\big] = \int_{Q_T^l} H^{(l)}(u) v^{(l)}\,\mathrm{d}t\,\mathrm{d}x - \int_{S_T} G^{(l)}(u) v^{(l)}\,\mathrm{d}t\,\mathrm{d}\sigma_x,$$

$$u = (u^{(1)}, u^{(2)}) \in L^p(0,T;V), \qquad (v^{(1)}, v^{(2)}) \in L^p(0,T;V).$$

Theorem 7.2 implies

THEOREM 7.3. *Assume that functions $f_j^{(l)}$ satisfy* (A$_1$), (A$_2$), (A$_3''$), (A$_4$) (*for $m = 1$*) *and operators $H^{(l)}, G^{(l)}$ satisfy* (A$_5^c$). *Then operator $(A + B): L^p(0,T;V) \to L^q(0,T;V^\star)$ is bounded, demicontinuous, pseudomonotone with respect to $D(L)$, it is coercive and of Volterra type. Consequently, for any $f = (f^{(1)}, f^{(2)}) \in L^q(0,T;V^\star)$ there exists $u = (u^{(1)}, u^{(2)}) \in L^p(0,T;V)$ such that $D_t u^{(l)} \in L^q(0,T;V_l^\star)$,*

$$D_t u^{(l)} + A^{(l)}(u^{(l)}) + B^{(l)}(u^{(1)}, u^{(2)}) = f^{(l)}, \quad u^{(l)}(0) = 0, \quad l = 1,2. \tag{7.9}$$

Similarly to Definition 3.4, a solution of (7.9) is called a weak solution of problem (7.4)–(7.7), because a "sufficiently smooth" function u satisfies (7.4)–(7.7) iff it is a solution of (7.9).

REMARK 7.3. Similarly to Remark 7.1, assume that

$$\sum_{j=0}^{n} \big[f_j^{(l)}(t,x,\xi) - f_j^{(l)}(t,x,\xi^\star)\big](\xi_j - \xi_j^\star) \geqslant -c(\xi_0 - \xi_0^\star)^2, \quad l = 1,2$$

with some constant $c > 0$,

$$\sum_{l=1}^{2} \int_{S_T} \big[G^{(l)}(u) - G^{(l)}(v)\big](u^{(l)} - v^{(l)})\,\mathrm{d}t\,\mathrm{d}\sigma_x \geqslant 0, \quad u, v \in L^p(0,T;V),$$

and for all $u, v \in L^p(0,T;V)$

$$\big\|\exp(-d\tau)\big[H^{(l)}\big(\exp(d\tau)u\big) - H^{(l)}\big(\exp(d\tau)v\big)\big]\big\|_{L^2(Q_t)} \leqslant c_3 \|u - v\|_{[L^2(Q_t)]^2}$$

with some constant c_3, not depending on u, v and the number $d > 0$. Then (7.9) may have at most one solution.

REMARK 7.4. Applying "abstract" Theorems 6.1, 6.4, 6.7, 6.10, we obtain results on solutions of (7.9) which are analogous to Theorems 6.2, 6.5, 6.8, 6.11, respectively (see Remark 7.2).

EXAMPLE 7.1. The assumptions of Theorem 7.3 are fulfilled on $H^{(l)}$ if

$$\big[H^{(l)}(u)\big](t,x) = \gamma^l\bigg(t, x, u^{(l)}\big(\chi_l(t), x\big), \int_{\Omega_{\hat{l}}} d^{\hat{l}}(y) u^{(\hat{l})}\big(\chi_{\hat{l}}(t), y\big)\,\mathrm{d}y\bigg), \quad (t,x) \in Q_T^l$$

where $\hat{l} = 1$ if $l = 2$ and $\hat{l} = 2$ if $l = 1$; $\chi_l \in C^1$, $\chi_l' > 0$, $0 \leqslant \chi_l(t) \leqslant t$, d^l are L^∞ functions; γ^l are Carathéodory functions satisfying

$$\left|\gamma^l(t, x, \theta_1, \theta_2)\right| \leqslant c^l(\theta_1, \theta_2)|\theta|^{p-1} + k_1^l(x) \tag{7.10}$$

with continuous functions c^l, $\lim_\infty c^l = 0$, $k_1^l \in L^q(\Omega_l)$.

Operators $H^{(l)}$ may have also the form

$$\begin{aligned}&\left[H^{(l)}(u)\right](t, x)\\&\quad = \int_0^t \gamma^l\left(t, \tau, x, u^{(l)}(\tau, x), \int_{\Omega_{\hat{l}}} d^{\hat{l}}(y) u^{(\hat{l})}(\tau, y)\,\mathrm{d}y\right)\mathrm{d}\tau, \quad (t, x) \in Q_T^l\end{aligned}$$

$l = 1, 2$ where γ^l satisfy analogous condition to (7.10).

Operators $G^{(l)}$ satisfy the assumptions of Theorem 7.3 if they have one of the forms

$$\begin{aligned}&\left[G^{(l)}(u)\right](t, x)\\&\quad = g^l\left(t, x, \int_S u^{(1)}\big(\chi(t), y\big)\,\mathrm{d}\sigma_y, \int_S u^{(2)}\big(\chi(t), y\big)\,\mathrm{d}\sigma_y\right), \quad (t, x) \in S_T,\\&\left[G^{(l)}(u)\right](t, x) = g^l\big(t, x, u^{(1)}\big(\chi(t), x\big), u^{(2)}\big(\chi(t), x\big)\big), \quad (t, x) \in S_T,\ l = 1, 2\end{aligned}$$

where functions g^l satisfy analogous conditions to (7.10)

The conditions of Remark 7.3 are satisfied for $H^{(l)}$, $G^{(l)}$ if $\gamma^l(t, x, \theta_1, \theta_2)$ satisfy global Lipschitz condition in (θ_1, θ_2),

$$\left[G^{(l)}(u)\right](t, x) = g^l\big(t, x, u^{(1)}(t, x), u^{(2)}(t, x)\big)$$

and the monotonicity condition

$$\sum_{l=1}^{2}\left[g^l(t, x, \theta_1, \theta_2) - g^l(t, x, \theta_1^\star, \theta_2^\star)\right](\theta_l - \theta_l^\star) \geqslant 0$$

holds.

Now we formulate (instead of (7.7)) another contact condition on S_T for the solutions of (7.4)–(7.6). Let $\psi : [0, T] \to [0, T]$ be a C^1 function satisfying

$$\psi' > 0, \quad 0 \leqslant \psi(t) \leqslant t, \quad \psi(0) = 0.$$

Further, let $L_S : L^p(S) \to L^p(S)$ be a linear and continuous operator. One of the contact boundary conditions on S_T is given by the equality on the traces

$$u_\psi^{(1)}(t, \cdot)|_S = L_S\big(u^{(2)}(t, \cdot)\ |_S\big) \quad \text{for a.e. } t \in [0, T], \tag{7.11}$$

where function $u_\psi^{(1)}$ is defined by

$$u_\psi^{(1)}(\tau, x) = u^{(1)}\big(\psi(\tau), x\big), \quad \tau \in [0, T],\ x \in \Omega_1.$$

EXAMPLE 7.2. One may consider e.g. the following linear and continuous operators L_S:

$$\big[L_S(w^{(2)})\big](x) = \int_S a(x,z)w^{(2)}(z)\,\mathrm{d}\sigma_z, \quad x \in S$$

where a is a given L^∞ function or

$$\big[L_S(w^{(2)})\big](x) = a(x)w^{(2)}\big(\varphi(x)\big), \quad x \in S$$

where $a \in L^\infty$, $\varphi : S \to S$ is a sufficiently smooth bijection.

Denote by V_0 the following closed linear subspace of V:

$$V_0 = \big\{(w^{(1)}, w^{(2)}) \in V\colon\ w^{(1)}|_S = L_S(w^{(2)}|_S)\big\}.$$

So the contact boundary condition (7.11) means that

$$\big(u_\psi^{(1)}, u^{(2)}\big) \in L^p(0,T;V_0).$$

Further, $(u^{(1)}, u^{(2)})$ is a (classical) solution of (7.4)–(7.6) iff $\tilde{u} = (\tilde{u}^{(1)}, \tilde{u}^{(2)}) = (u_\psi^{(1)}, u^{(2)})$ satisfies (by using the transformation $t = \psi(\tau)$)

$$\begin{aligned}
&D_\tau \tilde{u}^{(l)} - \sum_{j=1}^n D_j\big[\tilde{f}_j^{(l)}(\tau,x,\tilde{u}^{(l)}, D\tilde{u}^{(l)})\big] + \tilde{f}_0^{(l)}(\tau,x,\tilde{u}^{(l)}, D\tilde{u}^{(l)}) \\
&\quad + \tilde{H}^{(l)}(\tilde{u}^{(l)}, \tilde{u}^{(2)}) = \tilde{f}^{(l)}(\tau,x), \quad (\tau,x) \in (0,T)\times\Omega_l,\ l=1,2, && (7.12)\\
&\tilde{u}(0,x) = 0, \quad x \in \Omega_1 \cup \Omega_2, && (7.13)\\
&\tilde{u} = 0 \quad \text{on } \Gamma_T && (7.14)
\end{aligned}$$

where

$$\begin{aligned}
&\tilde{f}_j^{(1)}(\tau,x,\zeta_0,\zeta) = \psi'(\tau)f_j^{(1)}\big(\psi(\tau),x,\zeta_0,\zeta\big), \quad \tilde{f}_j^{(2)} = f_j^2, \quad j = 0,1,\dots,n,\\
&\big[\tilde{H}^{(1)}(\tilde{u}^{(l)}, \tilde{u}^{(2)})\big](\tau,x) = \psi'(\tau)\big[H^{(1)}(u^{(l)}, u^{(2)})\big]\big(\psi(\tau),x\big), \quad \tilde{H}^{(2)} = H^{(2)},\\
&\tilde{f}^{(1)}(\tau,x) = \psi'(\tau)f^{(1)}\big(\psi(\tau),x\big), \quad \tilde{f}^{(2)} = f^{(2)}.
\end{aligned}$$

Define operator $A = (A^{(l)}, A^{(2)}) : L^p(0,T;V_0) \to L^q(0,T;V_0^\star)$ by (7.8) such that functions $f_j^{(l)}$ are substituted by $\tilde{f}_j^{(l)}$ and define operator $B = (B^{(1)}, B^{(2)}) : L^p(0,T;V_0) \to L^q(0,T;V_0^\star)$ by

$$\begin{aligned}
&\big[B^{(l)}(\tilde{u}), \tilde{v}^{(l)}\big] = \int_{Q_T^l} \tilde{H}^{(l)}(\tilde{u})\tilde{v}^{(l)}\,\mathrm{d}t\,\mathrm{d}x, \quad \tilde{u} = (\tilde{u}^{(1)}, \tilde{u}^{(2)}) \in L^p(0,T;V_0),\\
&\tilde{v} = (\tilde{v}^{(1)}, \tilde{v}^{(2)}) \in L^p(0,T;V_0).
\end{aligned}$$

It is not difficult to show that if $f_j^{(l)}, H^{(l)}$ satisfy (A_1), (A_2), (A_3''), (A_4) and (A_5^c) then $\tilde{f}_j^{(l)}, \tilde{H}^{(l)}$ satisfy the same conditions, thus by Theorem 7.3 operator

$$(A+B) = \big(A^{(1)} + B^{(1)}, A^{(2)} + B^{(2)}\big) : L^p(0,T;V_0) \to L^q(0,T;V_0^\star)$$

is bounded, demicontinuous, pseudomonotone with respect to $D(L)$, coercive and it is of Volterra type. Consequently, for arbitrary $\tilde{f} \in L^q(0, T; V_0^\star)$ there exists $\tilde{u} \in L^p(0, T; V_0)$ such that $D_\tau \tilde{u} \in L^q(0, T; V_0^\star)$,

$$[D_\tau \tilde{u}^{(l)}, v^{(l)}] + \left[A^{(l)}(\tilde{u}^{(l)}), v^{(l)}\right] + \left[B^{(l)}(\tilde{u}^{(1)}, \tilde{u}^{(2)}), v^{(l)}\right] = [\tilde{f}^{(l)}, \tilde{v}^{(l)}], \quad (7.15)$$

for $l = 1, 2$, all $\tilde{v} = (\tilde{v}^{(1)}, \tilde{v}^{(2)}) \in L^p(0, T; V_0)$,

$$\tilde{u}^{(l)}(0) = 0, \quad l = 1, 2. \quad (7.16)$$

According to the above argument, if $u = (u^{(1)}, u^{(2)})$ satisfies (7.4)–(7.6), (7.11) then $\tilde{u} = (u_\psi^{(1)}, u^{(2)}) \in L^p(0, T; V_0)$ and (7.15) holds for all $\tilde{v} \in L^p(0, T; W_0^{1,p}(\Omega_1) \times W_0^{1,p}(\Omega_2))$. By using Gauss theorem, one obtains that for a "sufficiently smooth" function $\tilde{u}$ (7.15) holds for all $\tilde{v} \in L^p(0, T; V_0)$ iff u satisfies (7.4)–(7.6), (7.11) and the following contact (orthogonality) condition:

$$\sum_{j=1}^{n} \left[\int_{S_T} \tilde{f}_j^{(2)}(\tau, x, \tilde{u}^{(2)}, D\tilde{u}^{(2)}) \nu_j \tilde{v}^{(2)} \, \mathrm{d}\sigma \right.$$
$$\left. - \int_{S_T} \tilde{f}_j^{(1)}(\tau, x, \tilde{u}^{(1)}, D\tilde{u}^{(1)}) \nu_j \tilde{v}^{(1)} \, r d\sigma \right] = 0 \quad (7.17)$$

for each $(\tilde{v}^{(1)}, \tilde{v}^{(2)}) \in V_0$. ($\nu = (\nu_1, \ldots, \nu_n)$ denotes the normal unit vector on S_T.) Therefore, it is natural

DEFINITION 7.1. If $\tilde{u} = (u_\psi^{(1)}, u^{(2)}) \in L^p(0, T; V_0)$ is a solution of (7.15) (for each $\tilde{v} \in L^p(0, T; V_0)$) then u is called a weak solution of the contact problem (7.4)–(7.6), (7.11), (7.17).

Due to the above argument, Theorem 7.3 implies

THEOREM 7.4. *Assume that* $f_j^{(l)}$ *satisfy* (A_1), (A_2), (A_3''), (A_4) (*for* $m = 1$) *and* $H^{(l)}$ *satisfy* (A_5^c). *Then for arbitrary* $f \in L^q(0, T; V^\star)$ *there exists a weak solution of* (7.4)–(7.6), (7.11), (7.17).

REMARK 7.5. According to Remark 7.3 one can formulate conditions which imply the uniqueness of the weak solution.

At the end we mention several other applications of monotone type operators to nonlinear (functional) evolution equations.

By using arguments of the work [53] by J. Rauch, one can handle functional parabolic equations with discontinuous dependence on the unknown function (see [68]). Further, modifying the arguments of the limit process in Galerkin's approximation, it is possible to prove some results on approximation of the solution by solutions of perturbed problems (e.g. instead of unbounded domain Ω considering problems in "large" bounded domains. (See, e.g., [63].)

Finally, in applications arise problems for systems, consisting of a parabolic equation and another type of equations, e.g. of a one variable functional differential equation and (possibly) an elliptic partial differential equation (see [16,42]). Such problems are considered (by means of monotone type operators) in [8,70].

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CHAPTER 7

Recent Results on Hydrodynamic Limits

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Contents

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Abstract

Hydrodynamic limits study the relation between the fluid and kinetic descriptions of a transport phenomenon. In different area of physics and engineering, kinetic models have been developed and studied to shed some light on phenomenon far from thermodynamical equilibrium. Those models are far more accurate. However, they are still numerically too much costly, and the fluid descriptions cannot be completely avoided. It is then crucial to describe precisely the relation between these two kind of models, especially close to discontinuities, where the difference between the models are expected to be the deepest.

We present in this chapter a general theory of hydrodynamic limits based on the so-called relative entropy method. We give general conditions ensuring that a given conservation law (or balance law) can be obtained as hydrodynamic limit from a given kinetic equation. Several examples in gas dynamics are provided.

Relative entropy methods are known to provide asymptotic limits until the first time of appearance of singularities. We give first hints how to apply it beyond certain kind of singularities.

Keywords: Hydrodynamic limit, asymptotic limit, fluid equation, conservation law, balance law, dissipative solution, relative entropy, isentropic gas dynamic, isotherm gas dynamic, kinetic equation, nonlinear Fokker–Planck equation

1. Introduction

Hydrodynamic limits are the study of the links between fluid description of a transport phenomenon, like Euler equation, and its kinetic description, like Boltzmann equation. Different kinetic models have been introduced to describe motion of rarefied gases, far from thermodynamical equilibrium. The study of such models goes back to the work of Maxwell and Boltzmann. Since then, models have been specially designed for plasmas [64], quantum gases [72], dispersed particles [48,94] with application in nuclear physics, micro-electronic, medical science, chemical engineering.

Those models are very accurate. Compared to the fluid model, they involved a new variable, most of the time a velocity variable (or in some cases, energy variable), which increases a lot the complexity of the numerical tests. Today, they are still too much costly in computation time, and fluid description cannot be avoided. It is then very important to quantify the relation between the two descriptions, especially near singularities where the difference is expected to be the widest.

The relations between the fluid-dynamic description and the kinetic picture can be investigated from various points of view. A possible strategy is to study dissipative waves in the time-asymptotic sense (for the Boltzmann equation, see for instance [44] and [70]).

Another one is to study the so called small Knudsen number limit of the kinetic equation. This corresponds to enhance the kinetic collision operator, responsible for the trend to the thermodynamical equilibrium, at a factor $1/\varepsilon$.

Traditionally, we distinguish two kind of such limits (see [50]). One, called the diffusive scaling, considers a time scaling of the same order than the scaling of the collision operator. This is still a long time asymptotic, but with a strong collision operator. This kind of asymptotic leads to parabolic equations (see for instance [58,57,59,10,9,80]). The hydrodynamic limit of the Boltzmann equation in this regime leads to the incompressible Navier–Stokes equations. The complete, rigorous, global in time derivation of this limit has been performed in a series of stunning works [55,83,73,82,66], following the program initiated by Bardos, Golse and Levermore [7,8].

The second kind, called sometimes hyperbolic scaling, is the one studied in this chapter. It leads to conservation laws (or balance laws if source terms are in play). In the context of Boltzmann equation, it leads to the compressible temperature dependent Euler equations. Convergence on small time, depending on the regularity of the initial data, have been obtained by Caflisch [34]. Results after discontinuities are far more difficult. In the scalar case, and for the one-dimensional isentropic gas dynamics, general results in the large have been obtained thanks to the large family of entropies [68,67,13]. For Boltzmann equation in the one-dimensional case, stunning results have been recently obtained on situation involving shocks, following the fine study of micro-macro decomposition of Boltzmann equation by Tai-Ping Liu and Shih-Hsien Yu [69], see [70,63,96]. In the multi-dimensional setting, general global in time results are, at this time, completely out of reach since the existence of the limit problem is not even known. However, the hydrodynamic limits to special discontinuous solutions of the limit conservation law (or balance law) is already very interesting since it sheds some light on the intimate consistence between the kinetic and the related fluid model.

This chapter is dedicated to recent results obtained in the multi-dimensional case. It is based on the so-called relative entropy method. This method has been first used in the context of asymptotic limits by Horng-Tzer Yau [95]. It is based on the weak/strong uniqueness result on conservation laws with strictly convex entropy. This result can be find in [42]. It combines ideas of DiPerna [45] and Dafermos [41]. This method has been widely used to obtained asymptotic limits to different conservation laws (see for instance [27,30,29,81]). In the context of Hydrodynamic limit to hyperbolic conservation laws, it has been used in [15,56] and [75]. For incompressible limits from Boltzmann equation, it has been used in [24,51], and [66]. A elegant connection between relative entropy and relaxation has been established by Tzavaras [89]. If we consider a regular initial value for the limit equation, the method gives the convergence up to the first time of appearance of discontinuities. On a special one dimensional problem, the method was already used after shocks by Chen and Frid [40]. We present in this article some particular examples where the method provides convergence beyond discontinuities. For this matter, very weak notion of global in time solutions, the dissipative solutions, are considered. This notion is a direct extension of the dissipative solution introduced by Lions in [65] for incompressible Euler equations. This gives a framework where hydrodynamic limits can be performed for large time. However, the result is meaningful, only if we can have some uniqueness result on the dissipative solutions, to ensure that we get the expected solution at the limit. This can be performed on some special examples. In particular, we consider the convergence to an axisymmetric solution involving vacuum at the origin. The strength of the method is that there is no need of compactness tool. It requires only some abstract structure on the models, and some compatibility conditions between the kinetic and the fluid descriptions. The scope of application of the method will depend a lot on the validity of the dissipative solutions. It can be expected that those kind of solutions are meaningful only for some kind of discontinuities.

2. Fluid equations, relative entropy, and dissipative solutions

The modeling of the motion of a fluid involved conservation laws (or balance laws if external forces are in play). Generally, the state of the fluid, at any time $t \geqslant 0$ and any space position $x \in \Omega$, can be described by a finite number of conserved quantities: $U(t,x)$. This can be, for instance, the density, the momentum (product of the density by the velocity) and possibly the energy. To avoid boundary problem, we consider only $\Omega = \mathbb{R}^N$ or $\Omega = \mathbb{T}^N$. To fix the notation, let us denote $\mathcal{V} \subset \mathbb{R}^p$, convex open set, the interior of the domain of values of the conserved quantities.

2.1. *Conservation laws and balance laws*

The evolution in time of the conserved quantities is described by a system of equations of the form:

$$\partial_t U + \operatorname{div}_x A(U) = Q(U, x), \tag{1}$$

where $t \in \mathbb{R}^+$, $x \in \Omega$, $A : \mathcal{V} \to \mathbb{R}^{Np}$ and $Q : \mathcal{V} \times \Omega \to \mathbb{R}^p$. Note that this system of equations is completely characterized by the matrix valued flux function A and the vector valued source term function Q. When $Q = 0$ we say that (1) is a conservation law. In the general case, we talk of balance law. For a general presentation of conservation laws and balance laws, we refer to the books of Dafermos [42], Serre [85], and Bressan [31] for the theory of solutions with small bounded variations (see also [71]).

REMARK. In fluid dynamics, most of the time the flux function is continuously defined only on the open set $\mathcal{V}$, but the solutions can reach some values on the boundary of $\mathcal{V}$. For a model like the isentropic gas dynamics system, involving as conserved quantity the density ρ and the momentum ρu, we would have: $\mathcal{V} = (0, \infty) \times \mathbb{R}^N$. But the solutions can reach the vacuum $(\rho, P) = (0, 0)$. Unfortunately, the flux of momentum, $(1/\rho)(P \otimes P) + I_N \rho^\gamma$, where I_N is the $N \times N$ identity matrix, is not continuous at $(0, 0)$. The problem will be dealt later using the notion of entropy.

2.2. *Convex entropy*

A function $\eta : \mathcal{V} \to \mathbb{R}$ of class $C^2(\mathcal{V})$ is called an entropy of system (1) if there exists a vector valued function $G : \mathcal{V} \to \mathbb{R}^N$, $G = (G_1, \ldots, G_N)$, satisfying for every $j = 1, \ldots, p$, $k = 1, \ldots, n$, and $W \in \mathcal{V}$:

$$\partial_j G_k(W) = \sum_{i=1}^{p} \partial_i \eta(W) \partial_j A_{ki}(W). \tag{2}$$

The concept is important due to the following easy result.

LEMMA 1. *Consider $U \in W^{1,\infty}([0, T) \times \Omega)$ with values in $\mathcal{V}$, solution to (1). Then U verifies also the entropy equality*

$$\partial_t \eta(U) + \operatorname{div}_x G(U) = \eta'(U) \cdot Q(U, x). \tag{3}$$

PROOF. Taking the dot product of (1) with $\eta'(U)$ we find:

$$\sum_{i=1}^{p} \partial_i \eta(U) \partial_t U_i + \sum_{i,j,k} \partial_i \eta(U) \partial_j A_{ki}(U) \partial_{x_k} U_j = \eta'(U) \cdot Q(U).$$

Using (2), this gives:

$$\sum_{i} \partial_i \eta(U) \partial_t U_i + \sum_{j} \partial_j G_k(U) \partial_{x_k} U_j = \eta'(U) \cdot Q(U).$$

The chain rule formula gives the desired result. □

Even if we consider a regular initial value $U^0 \in W^{1,\infty}(\Omega)$ with values in $\mathcal{V}$, there exists a maximal time $T > 0$, which could be finite (see Sideris [86] for the Euler system), such that there exists a unique solution $U \in W^{1,\infty}([0, T) \times \Omega)$ with values in $\mathcal{V}$ to (1). We will consider only system for which the total entropy $\int \eta(U(t,x))\,\mathrm{d}x$ is uniformly bounded in time. At the blow-up time T, we can have a blow-up of the Lipschitz norm (appearance of shocks), and we can also have the values of U reaching some points of the boundary of $\mathcal{V}$ (typically, appearance of vacuum). Thanks to the conservation of total entropy, the meaningful such values W which can be reached are such that they are limits of sequence $U_k \in \mathcal{V}$ with $\limsup \eta(U_k) < \infty$.

Hence, to study global problem in time, we introduce:

$$\mathcal{U} = \Big\{ W \in \mathbb{R}^p \setminus \exists (W_k)_{k\in\mathbb{N}} \in (\mathcal{V})^{\mathbb{N}},\ \lim_{k\to\infty} W_k = W,\ \limsup_{k\to\infty} \eta(W_k) < \infty \Big\}. \quad (4)$$

The set $\mathcal{U}$ is the natural set of values of solution of (1) with finite entropy η.

We then extend the values of the entropy η onto $\mathcal{U}$ in the following way:

$$\eta(W) = \liminf_{\mathcal{V}\ni \underline{W}\to W} \eta(\underline{W}), \qquad W \in \mathcal{U}. \quad (5)$$

An entropy η is called a convex entropy if for any $W \in \mathcal{V}$, the matrix $(\partial_i\partial_j\eta(W))_{ij}$ is nonnegative. We have the following result:

LEMMA 2. *Let η be a convex entropy, then $\mathcal{U}$ is a convex subset of $\mathbb{R}^p$ and η is a convex function on $\mathcal{U}$, that is*:

$$\eta\big(sW_1 + (1-s)W_2\big) \leqslant s\eta(W_1) + (1-s)\eta(W_2),$$

for every $W_1, W_2 \in \mathcal{U}$, and every $0 \leqslant s \leqslant 1$.

PROOF. Consider $W_1, W_2 \in \mathcal{U}$. Then there exists $W_{1,k}, W_{2,k} \in \mathcal{V}$ converging respectively to W_1, W_2 with $\limsup \eta(W_{1,k}) < \infty$ and $\limsup \eta(W_{2,k}) < \infty$. Since $\mathcal{V}$ is convex, for every $0 \leqslant s \leqslant 1$, $sW_{1,k} + (1-s)W_{2,k} \in \mathcal{V}$ converges to $sW_1 + (1-s)W_2$. The function η is convex on $\mathcal{V}$, so we get:

$$\begin{aligned}&\limsup \eta\big(sW_{1,k} + (1-s)W_{2,k}\big)\\ &\quad \leqslant s\limsup \eta(W_{1,k}) + (1-s)\limsup \eta(W_{2,k}) < \infty.\end{aligned}$$

Hence $sW_1 + (1-s)W_2 \in \mathcal{U}$ and $\mathcal{U}$ is convex. Now, for any sequences $W_{1,k}, W_{2,k}$ valued in $\mathcal{V}$ converging respectively to W_1, W_2, from the convexity of η in $\mathcal{V}$ and the definition of the extension of η on $\mathcal{U}$, we have:

$$\begin{aligned}\eta\big(sW_1 + (1-s)W_2\big) &\leqslant \liminf \eta\big(sW_{1,k} + (1-s)W_{2,k}\big)\\ &\leqslant \liminf\big(s\eta(W_{1,k}) + (1-s)\eta(W_{2,k})\big).\end{aligned}$$

For every $\varepsilon > 0$, considering sequences such that:

$$\lim_{k\to\infty} \eta(W_{1,k}) \leqslant \eta(W_1) + \varepsilon,$$
$$\lim_{k\to\infty} \eta(W_{2,k}) \leqslant \eta(W_2) + \varepsilon,$$

we get

$$\eta\big(sW_1 + (1-s)W_2\big) \leqslant s\eta(W_1) + (1-s)\eta(W_2) + \varepsilon.$$

Passing in the limit $\varepsilon \to 0$ gives the desired result. □

We say that η is a strictly convex entropy if it is a convex entropy and it verifies that for any bounded subset $\mathcal{D}$ of $\mathcal{V}$, there exists a constant $C_0 > 0$ such that

$$\nabla^2 \eta(W) \geqslant C_0 I_N \quad \text{for any } W \in \mathcal{D},$$

where I_N is the $N \times N$ identity matrix.

REMARK 1. Most of the physically relevant systems carry a strictly convex entropy. This condition is known also to ensure the hyperbolicity of the system, which is necessary to get the stability of linear waves (see [85]).

REMARK 2. However, in most of the cases involved in compressible mechanics, the entropy functional is not continuously defined in the whole set $\mathcal{U}$. Indeed a degeneracy occurs at vanishing density $\rho = 0$. This motivates to define the entropy only on the interior of $\mathcal{U}$. This is why we first define η on $\mathcal{V}$ and then extend it on $\mathcal{U}$.

2.3. *Relative entropy*

Consider a function $Z \in C^1(\mathcal{V})$ with values in $\mathbb{R}^q$, where q is a positive integer. We introduce a so-called relative quantity associated to Z, $Z(.|.) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}^q$ defined by

$$Z(U|V) = Z(U) - Z(V) - Z'(V) \cdot (U - V). \tag{6}$$

For example, the relative flux is defined, for $q = np$, by

$$A(U|V) = A(U) - A(V) - A'(V) \cdot (U - V), \tag{7}$$

and the relative entropy, for $q = 1$, by

$$\eta(U|V) = \eta(U) - \eta(V) - \eta'(V) \cdot (U - V), \quad U, V \in \mathcal{V}. \tag{8}$$

The relative entropy (8) is extended to $\mathcal{U} \times \mathcal{V}$ using (5). If η is strictly convex we have the following result.

LEMMA 3. *Let η be a strictly convex entropy, then the relative entropy $\eta(\cdot|\cdot)$ is nonnegative on $\mathcal{U} \times \mathcal{V}$ and, for $(U, V) \in \mathcal{U} \times \mathcal{V}$, $\eta(U|V) = 0$ if and only if $U = V$.*

PROOF. Since $\eta \in C^2(\mathcal{V})$, for $U, V \in \mathcal{V}$ we have:

$$\eta(U|V) = \int_0^1 \int_0^1 \eta''\big(V + st(U - V)\big) : \big[(U - V) \otimes (U - V)\big] t \, \mathrm{d}s \, \mathrm{d}t,$$

which gives the result in this case.

Consider now $U \in \mathcal{U} \setminus \mathcal{V}$ and $V \in \mathcal{V}$. Let us denote $r = |U - V|$. Take $0 < \delta < r/2$ such that $\overline{B(V, 2\delta)} \subset \mathcal{V}$. Since η is strictly convex, $\eta''(W) \geqslant C I_N$ on the ball $B(V, \delta)$. For every ε we consider $U_k \in B(U, r/2)$ converging to U such that $\lim \eta(U_k) \leqslant \eta(U) + \varepsilon$. For every k, and for $1/2 \leqslant t \leqslant 1$ and $s \leqslant \delta/(2r)$ we have $V + st(U_k - V) \in B(V, \delta)$, since

$$st|U_k - V| \leqslant st\frac{3r}{2} \leqslant \frac{3}{4}\delta.$$

So

$$\eta''\big(V + st(U_k - V)\big) : \big[(U_k - V) \otimes (U_k - V)\big] \geqslant Cr^2/4.$$

Hence:

$$\eta(U_k|V) \geqslant \frac{Cr\delta}{16}.$$

Passing to the limit gives

$$\eta(U|V) \geqslant \frac{Cr\delta}{16} - \varepsilon.$$

Passing to the limit in $\varepsilon \to 0$ gives that $\eta(U|V) > 0$. □

REMARK. The relative entropy associated to a strictly convex entropy can be then used to measure the gap between two states U and V, whenever U lies in $\mathcal{U}$ and V in $\mathcal{V}$.

2.4. *Admissible balance laws*

We restrict our study to system whose nonlinearity are completely controlled by a relative entropy. Especially, it will be shown that regular solutions, and some special class of discontinuous solutions, of an admissible conservation law (or balance law) on $\mathcal{V}$ are stable in the class of dissipative solutions, without additional bound requirement. The notion of admissible balance laws was introduced by Berthelin and Vasseur in [15].

We say that the system (1) is an admissible balance law on $\mathcal{V}$ if it has a strictly convex entropy η on $\mathcal{V}$ such that for any $U, V \in \mathcal{V}$:

$$\big|A(U|V)\big| \leqslant C\eta(U|V), \tag{9}$$

and

$$\big|Q(U)\eta'(U|V) + \eta''(V)\big[Q(U) - Q(V)\big](U - V)\big| \leqslant C\eta(U|V), \tag{10}$$

and if for any $i = 1, \ldots, p$, there exists an increasing nonnegative function $\Phi_i \in C^0(\mathbb{R}^+)$ such that

$$\liminf_{y\to\infty} \frac{\Phi_i(y)}{|y|} = \infty,$$
$$\Phi_i\big(|W_i|\big) \leqslant |W| + \eta(W), \quad \text{for every } W \in \mathcal{V}. \tag{11}$$

We will show later that the systems of isentropic gas dynamics, isothermal gas dynamics, the shallow water system with barometry, and a bi-fluid model are admissible conservation laws or balance laws on the natural space of conserved quantities.

We say that the system (1) is locally admissible on $\mathcal{V}$, if it has a strictly convex entropy η on $\mathcal{V}$ such that for any $i = 1, \ldots, p$, there exists an increasing nonnegative function

$\Phi_i \in C^0(\mathbb{R}^+)$ verifying (11), and such that for every bounded subset $\mathcal{D} \subset \mathcal{V}$, there exists $C(\mathcal{D})$ such that for every $V \in \mathcal{D}$, $U \in \mathcal{V}$:

$$\left|A(U|V)\right| \leqslant C(\mathcal{D})\eta(U|V), \tag{12}$$

and

$$\left|Q(U)\eta'(U|V) + \eta''(V)\left[Q(U) - Q(V)\right](U - V)\right| \leqslant C(\mathcal{D})\eta(U|V). \tag{13}$$

REMARK. Note that, at least when U and V are small, $\eta(U|V)$ has a quadratic form. This justifies the form of (10) and (13).

2.5. *Weak entropy solutions*

Solutions to balance laws can exhibit some discontinuities in finite time. Therefore, when considering global solutions in time, it is necessary to consider a weaker notion of solution. A natural notion in the context of conservation laws and balance laws is the so-called Weak entropy solution.

Consider a balance law (1) bearing a strictly convex entropy. We say that U is a weak entropy solution of (1) with initial value U^0 if for every test function $V \in C_c^\infty([0,\infty) \times \Omega)$ with values in $\mathbb{R}^N$, $\phi \in C_c^\infty([0,\infty) \times \Omega)$ with values in $\mathbb{R}$, we have:

$$\begin{aligned}
&\int_\Omega V(0,x) \cdot U^0(x)\,\mathrm{d}x + \int_0^\infty \int_\Omega U(t,x) \cdot \partial_t V\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^\infty \int_\Omega A(U)(t,x) : \nabla V\,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty \int_\Omega Q(U)(t,x) \cdot V\,\mathrm{d}x\,\mathrm{d}t = 0, \\
&\int_\Omega \phi(0,x)\eta(U^0)(x)\,\mathrm{d}x + \int_0^\infty \int_\Omega \eta(U)(t,x)\partial_t\phi\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^\infty \int_\Omega G(U)(t,x) \cdot \nabla\phi\,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty \int_\Omega \eta'(U) \cdot Q(U)\phi\,\mathrm{d}x\,\mathrm{d}t \geqslant 0.
\end{aligned}$$

Hence, a weak entropy solution verifies (1) in the sense of distribution and the following inequality, also in the sense of distribution:

$$\partial_t \eta(U) + \operatorname{div}_x G(U) \leqslant \eta'(U) \cdot Q(U,x). \tag{14}$$

Lemma 1 ensures that Lipschitz solutions to (1) verify (14) with equality. However this is not anymore the case when solutions develop singularities. Moreover those solutions are not unique if only (1) is considered. In one dimension ($N = 1$), solutions of (1) (14) are unique, provided that the initial value is small in BV (see [31,71]). However, no result of this kind is known in the multi-dimensional case. Up to now, weak entropy solutions are not known to exist for $N \geqslant 2$.

2.6. *Dissipative solutions*

Dissipative solutions are a very weak class of solutions. They are not even known to be solution in the sense of distribution. This notion of solution has been first introduced for incompressible Euler equations by Lions [65]. Its justification is that it verifies the so-called strong/weak principle, that is, if there exists a classical solution, then it is the unique possible dissipative solution. Portilheiro introduced in [79] a related notion of dissipative solution for scalar conservation laws, reminiscent to the perturbed test function method developed by Evans in [46] for viscosity solutions. Portilheiro showed that his notion of dissipative solution is equivalent with the classical notion of entropy solution. Note, however, that our notion of dissipative solution is weaker than the one of Portilheiro. It would be of great interest to know if its result can be extended to this weaker class.

First we define the dissipative test functions which will be used to "test" the dissipative solution. We call dissipative test functions, any $V \in W^{1,\infty}([0,T]\times\Omega)\cap L^\infty(0,T;L^1(\Omega))$ with values in $\mathcal{V}$ and satisfying

$$\begin{aligned}
&X(V) = \Psi_A(V)\nabla_x\big[\eta'(V)\big] \in L^\infty\big([0,T]\times\Omega\big)\\
&\quad\text{where } \Psi_A(V) = \mathbf{1}_{\{V\in\mathcal{V}\mid \sup(\nabla_x[\eta'(V)]:A(U|V)|U\in\mathcal{U})<0\}},\\
&\big|E(V)\big|\big|\eta''(V)\big| \in L^\infty\big([0,T]\times\Omega\big),\\
&\quad\text{where } E(V) = \partial_t V + \operatorname{div}_x A(V) - Q(V),\\
&V(0,\cdot) = V^0 \in L^1(\Omega), \qquad \eta(V^0)\in L^1(\Omega), \qquad \eta'(V^0)\in L^\infty(\Omega).
\end{aligned}\tag{15}$$

REMARK. If $\Omega = \mathbb{T}^N$, then the two first requirements in (15) are consequences of $V \in W^{1,\infty}([0,T]\times\mathbb{T}^N)$. Indeed, $[0,T]\times\mathbb{T}^N$ is compact, so $V([0,T]\times\mathbb{T}^N)\in\mathcal{V}$ is compact. But $\eta\in C^2(\mathcal{V})$ so there exists C_T such that $|E(V)| + |\partial_{ij}\eta(V)| \leqslant C_T$ on $[0,T]\times\mathbb{T}^N$.

Consider an initial value $U^0 \in L^1(\Omega)$ with values in $\mathcal{U}$ and with finite entropy $\eta(U^0)\in L^1(\Omega)$. We call dissipative solution of (1) on $[0,T]$, with initial value U^0, any function $U \in L^\infty(0,T;L^1(\Omega))$ with values in $\mathcal{U}$ verifying $\eta(U)\in L^\infty(0,T;L^1(\Omega))$, such that for any dissipative test function $V\in W^{1,\infty}([0,T]\times\Omega)$ verifying (15)

$$\begin{aligned}
\int_\Omega \eta(U|V)(t,x)\,\mathrm{d}x &\leqslant \left(\int_\Omega \eta(U^0|V^0)\,\mathrm{d}x\right)\exp\left(\int_0^t \sigma_V(\tau)\,\mathrm{d}\tau\right)\\
&\quad+\int_0^t \exp\left(\int_\tau^t \sigma_V(s)\,\mathrm{d}s\right)\int_\Omega \eta''(V):\big(E(V)\otimes(V-U)\big)\,\mathrm{d}x\,\mathrm{d}\tau,
\end{aligned}\tag{16}$$

for all $t\in[0,T]$, where:

$$\sigma_V(s) = C_V\big(1+\big\|X(V)(s)\big\|_{L^\infty(\Omega)}\big).$$

If (1) is admissible on $\mathcal{V}$, $C_V = C$, the constant defined in (9) and (10). If (1) is locally admissible on $\mathcal{V}$ and $\Omega=\mathbb{T}^N$, $C_V = C(\mathcal{D}_V)$ with $\mathcal{D}_V = V([0,T]\times\mathbb{T}^N)$, where $C(\mathcal{D})$ is defined by (12) and (13).

We say that U is a dissipative solution of (1) on $[0,T)$ if it is a dissipative solution on $[0,\overline{T}]$ for every $\overline{T} < T$.

2.6.1. *A structure lemma* To clarify the link between classical solution and dissipative solution, we first present the following lemma, first due to Dafermos [41] and DiPerna [45]:

LEMMA 4. *For the entropy $\eta \in C^2(\mathcal{V})$ verifying (2) on $\mathcal{V}$, and for any*

$$V, U \in W^{1,\infty}\big([0,T]\times\Omega\big)$$

with values in $\mathcal{V}$, we have

$$\begin{aligned}\partial_t\eta(U|V) = {} & \big[\partial_t\eta(U) + \mathrm{div}_x\, G(U) - \eta'(U)Q(U)\big]\\ & - \eta''(V)\cdot\big[\partial_t V + \mathrm{div}_x\, A(V) - Q(V)\big]\cdot(U-V)\\ & - \eta'(V)\cdot\big[\partial_t U + \mathrm{div}_x\, A(U) - Q(U)\big]\\ & - \mathrm{div}_x\big[G(U|V)\big] + \eta'(V)\cdot\mathrm{div}_x\big[A(U|V)\big]\\ & + Q(V)\cdot\eta'(U|V) + \big[Q(U)-Q(V)\big]\cdot\big(\eta'(U)-\eta'(V)\big).\end{aligned}$$

REMARK. Notice that if V and U are regular solutions to (1), the 3 first lines vanish. The fourth line has a divergence form, hence its integral is vanishing. Finally the two last terms are quadratic with respect to $U-V$ (at least when $|U-V|\leqslant R$) as η is. Hence, from this proposition, we can expect to have a good structure to use Gronwall lemma on $\int \eta(U|V)\,\mathrm{d}x$.

REMARK. This lemma is true for any Lipschitz functions U and V. The equality depends only on the structure of system (1) endowed with a strictly convex entropy (2).

PROOF. From the definition of relative quantity (6), we have

$$\begin{aligned}\partial_t\eta(U|V) &= \partial_t\eta(U) - \partial_t\eta(V) - \partial_t\big[\eta'(V)\big]\cdot(U-V) - \eta'(V)\cdot\partial_t(U-V)\\ &= \big[\partial_t\eta(U) + \mathrm{div}_x\, G(U) - \eta'(U)Q(U)\big]\\ &\quad - \big[\partial_t\eta(V) + \mathrm{div}_x\, G(V) - \eta'(V)Q(V)\big]\\ &\quad - \eta''(V)\cdot\big[\partial_t V + \mathrm{div}_x\, A(V) - Q(V)\big]\cdot(U-V)\\ &\quad - \eta'(V)\cdot\big[\partial_t U + \mathrm{div}_x\, A(U) - Q(U)\big]\\ &\quad + \eta'(V)\cdot\big[\partial_t V + \mathrm{div}_x\, A(V) - Q(V)\big] + R_1 + R_2,\end{aligned}\tag{17}$$

where

$$\begin{aligned}R_1 &= \eta'(U)\cdot Q(U) - \eta'(V)\cdot Q(V) - \eta''(V)\cdot Q(V)\cdot(U-V)\\ &\quad - \eta'(V)\cdot Q(U) + \eta'(V)\cdot Q(V)\\ &= Q(V)\eta'(U|V) + \big[\eta'(V)-\eta'(U)\big]\cdot\big[Q(V)-Q(U)\big],\end{aligned}\tag{18}$$

and

$$\begin{aligned}R_2 &= \mathrm{div}_x\big[G(V)-G(U)\big] + \eta''(V)\cdot\mathrm{div}_x\, A(V)\cdot(U-V)\\ &\quad + \eta'(V)\cdot\mathrm{div}_x\big[A(U)-A(V)\big].\end{aligned}$$

The existence of the associated entropy flux G gives the relation (see (2)).

$$\partial_i G_k(W) = \sum_j \partial_j \eta(W)\, \partial_i A_{kj}(W), \quad \forall k, i,\ \forall W \in \mathcal{V}.$$

A derivation of this relation with respect to W_l gives

$$\sum_j \partial_{lj}\eta(W)\partial_i A_{kj}(W) = \partial_{il} G_k(W) - \sum_j \partial_j \eta(W)\partial_{il} A_{kj}(W).$$

We use this relation with $W = V$ and get

$$\begin{aligned}
&\eta''(V)\cdot \operatorname{div}_x A(V)\cdot (U-V)\\
&\quad = \sum \partial_{lj}\eta(V)\partial_{x_k}\big[A_{kj}(V)\big](U_l - V_l)\\
&\quad = \sum \partial_{lj}\eta(V)\partial_i A_{kj}(V)\partial_{x_k} V_i (U_l - V_l)\\
&\quad = \sum \partial_{il} G_k(V)\partial_{x_k} V_i (U_l - V_l) - \sum \partial_j \eta(V)\partial_{il} A_{kj}(V)\partial_{x_k} V_i (U_l - V_l),
\end{aligned}$$

now

$$\begin{aligned}
&-\partial_j \eta(V)\partial_{il} A_{kj}(V)\partial_{x_k} V_i (U_l - V_l)\\
&\quad = \partial_j \eta(V)\big[-\partial_{x_k}\big[\partial_l A_{kj}(V)\big](U_l - V_l)\big]\\
&\quad = \partial_j \eta(V)\big[-\partial_{x_k}\big[\partial_l A_{kj}(V)(U_l - V_l)\big] + \partial_l A_{kj}(V)\partial_{x_k}(U_l - V_l)\big],
\end{aligned}$$

therefore, we obtain

$$\begin{aligned}
R_2 &= \operatorname{div}_x\big[G(V) - G(U)\big] + \sum \partial_{il} G_k(V)\partial_{x_k} V_i (U_l - V_l)\\
&\quad + \sum \partial_j \eta(V)\big[-\partial_{x_k}\big[\partial_l A_{kj}(V)(U_l - V_l)\big] + \partial_l A_{kj}(V)\partial_{x_k}(U_l - V_l)\big]\\
&\quad + \eta'(V)\cdot \operatorname{div}_x\big[A(U) - A(V)\big]\\
&= \operatorname{div}_x\big[G(V) - G(U)\big] + \sum \partial_{x_k}\big[\partial_l G_k(V)\big](U_l - V_l)\\
&\quad - \sum \partial_j \eta(V)\partial_{x_k}\big[\partial_l A_{kj}(V)(U_l - V_l)\big]\\
&\quad + \sum \partial_j \eta(V)\partial_l A_{kj}(V)\partial_{x_k}(U_l - V_l)\\
&\quad + \sum \partial_j \eta(V)\partial_{x_k}\big[A_{kj}(U) - A_{kj}(V)\big].
\end{aligned}$$

We can rewrite (2) in the following way

$$\sum_j \partial_j \eta(V)\partial_l A_{kj}(V) = \partial_l G_k(V).$$

Thus we find

$$\begin{aligned}
R_2 &= \operatorname{div}_x\big[G(V) - G(U)\big] + \sum \partial_{x_k}\big[\partial_i G_k(V)(U_i - V_i)\big]\\
&\quad + \sum \partial_j \eta(V)\partial_{x_k}\big[A_{kj}(U|V)\big]\\
&= -\operatorname{div}_x G(U|V) + \eta'(V)\cdot \operatorname{div}_x A(U|V).
\end{aligned} \tag{19}$$

Equation (17) with (18) and (19) gives the desired relation. □

REMARK. We notice that in particular, the term R_2 of the proof satisfies

$$\int_\Omega R_2 \,\mathrm{d}x = \int_\Omega \sum_{jk} \partial_j \eta(V) \partial_{x_k} \big[A_{kj}(U|V) \big] \,\mathrm{d}x$$
$$= - \int_\Omega \sum_{jk} \partial_{x_k} \big[\partial_j \eta(V) \big] A_{kj}(U|V) \,\mathrm{d}x.$$

2.6.2. *Weak/strong principle* We consider a system (1) which is either admissible on $\mathcal{V}$, or locally admissible on $\mathcal{V}$ with $\Omega = \mathbb{T}^N$. The following result is due to Dafermos [41] and DiPerna [45].

PROPOSITION 1 *(Weak/strong principle). Let $T > 0$, and $U \in W^{1,\infty}([0, \overline{T}] \times \Omega) \cap L^\infty(0, \overline{T}; L^1(\Omega))$ satisfy (15) for every $\overline{T} < T$ with values in $\mathcal{V}$ and be solution to (1). If (1) is admissible on $\mathcal{V}$ then U is the unique dissipative solution on $[0, T)$ of (1) with initial value U^0. If (1) is locally admissible on $\mathcal{V}$ and $\Omega = \mathbb{T}^N$ then the result is still true.*

PROOF. Let us begin with the uniqueness result. Consider $\overline{U}$ a dissipative solution. The function U is a dissipative test function verifying $U(0, \cdot) = U^0$ and $E(U) = 0$. Hence we get for every $t \geqslant 0$:

$$\int_\Omega \eta(\overline{U}|U)(t, x) \,\mathrm{d}x \leqslant 0.$$

Thanks to Lemma 3, $\eta(\overline{U}|U) \geqslant 0$, so $\eta(\overline{U}|U) = 0$ almost everywhere. Hence, thanks to Lemma 3, $U = \overline{U}$ almost everywhere.

We show now that U is a dissipative solution. Consider a test function $V \in W^{1,\infty}([0, T] \times \Omega)$ with values in $\mathcal{V}$ and verifying (15). We use Lemma 4 with V and U. Since U is a strong solution of (1), the third line vanishes. But thanks to Lemma 1, U verifies also the entropy equality (3), hence the first line vanishes too. Integrating in x, the fourth line vanishes, and integrating by part the fifth line gives:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \eta(U|V) \,\mathrm{d}x = &- \int_\Omega \eta''(V) \cdot \big[\partial_t V + \mathrm{div}_x A(V) - Q(V) \big] \cdot (U - V) \,\mathrm{d}x \\
&- \int_\Omega \sum_{jk} \partial_{x_k} \big[\partial_j \eta(V) \big] A_{kj}(U|V) \,\mathrm{d}x \\
&+ \int_\Omega \big[Q(V) \cdot \eta'(U|V) + \big[Q(U) - Q(V) \big] \\
&\times \big(\eta'(U) - \eta'(V) \big) \big] \,\mathrm{d}x.
\end{aligned}$$

If (1) is admissible on $\mathcal{V}$, we get:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \eta(U|V) \,\mathrm{d}x \leqslant &\; C\big(1 + \|X(V)\|_{L^\infty(\Omega)}\big) \int_\Omega \eta(U|V) \,\mathrm{d}x \\
&- \int_\Omega \eta''(V) \cdot \big[\partial_t V + \mathrm{div}_x A(V) - Q(V) \big] \cdot (U - V) \,\mathrm{d}x.
\end{aligned}$$

Gronwall's lemma gives the desired result. We consider now the case where (1) is locally admissible on $\mathcal{V}$ and $\Omega = \mathbb{T}^N$. We have $V \in W^{1,\infty}([0,T] \times \mathbb{T}^N)$ with values in $\mathcal{V}$ open. The function V is continuous, $[0,T] \times \mathbb{T}^N$ is compact so $V([0,T] \times \mathbb{T}^N) \subset \mathcal{V}$ is compact. So the result holds with

$$C_V = C\big(V\big([0,T] \times \mathbb{T}^N\big)\big),$$

where $C(\mathcal{D})$ is defined in (12) and (13). □

2.6.3. *Stability* Another feature of the dissipative solution is that they are stable on strong perturbation on the initial value. This is part of the strength of the concept, since it makes easier to construct such solution. We still assume that (1) is (at least) locally admissible on $\mathcal{V}$.

PROPOSITION 2 *(Stability). Let U_k be a sequence of dissipative solutions to (1) on $[0,T)$, with $T \leqslant \infty$, with values in $\mathcal{U}$, such that U_k and $\eta(U_k)$ are uniformly bounded in $L^\infty(0,\overline{T}; L^1(\Omega))$ for every $\overline{T} < T$, and such that $U_k(0,\cdot)$ and $\eta(U_k(0,\cdot))$ converge strongly to U^0 and $\eta(U^0)$ in $L^1(\Omega)$. Then, up to a subsequence, U_k converges weakly in $L^p_{\mathrm{loc}}(0,T; L^1_{\mathrm{loc}}(\Omega))$ to a dissipative solution to (1) with initial value U^0, for $1 \leqslant p < \infty$. If $\Phi_i(y) \geqslant |y|^{\gamma_i}$ for $y > 0$, then the component $U_{i,k}$ converges weakly to U_i in $L^p_{\mathrm{loc}}(0,T; L^q(\Omega))$ for $1 \leqslant p < \infty$ and $1 \leqslant q < \gamma_i$.*

PROOF. Consider an increasing sequence $\overline{T}_n < T$, $T_n \to T$. The functions U_k and $\eta(U_k)$ are uniformly bounded in $L^\infty(0,\overline{T}_n; L^1(\Omega))$, so, since (1) is (at least) locally admissible on $\mathcal{V}$, $\Phi_i(U_{i,k})$ is uniformly bounded in $L^\infty(0,\overline{T}_n; L^1(\Omega))$. By a diagonal extraction, up to a subsequence, $U_{i,k}$ converges weakly to a limit U_i in $L^p(0,\overline{T}_n; L^1_{\mathrm{loc}}(\Omega))$ for $1 \leqslant p < \infty$, for any $n \geqslant 1$. If $U_{i,k}$ is uniformly bounded in $L^\infty(0,\overline{T}_n; L^\gamma_i(\Omega))$, then, up to a subsequence, $U_{i,k}$ converges weakly in $L^p_{\mathrm{loc}}(0,T; L^q(\Omega))$ for $1 \leqslant p < \infty$ and $1 \leqslant q < \gamma_i$. Passing to the limit, this gives that for any test function $V \in W^{1,\infty}([0,T] \times \Omega)$ with values in $\mathcal{V}$ (and decreasing fast enough for large $|x|$ if $\Omega = \mathbb{R}^N$), and verifying (15), we have for any $t < T$:

$$\begin{aligned}\lim_{k\to\infty} \int_\Omega \eta(U_k|V)(t,x)\,\mathrm{d}x &\leqslant \left(\int_\Omega \eta(U^0|V^0)\,\mathrm{d}x\right)\exp\left(\int_0^t \sigma_V(\tau)\,\mathrm{d}\tau\right)\\ &\quad - \int_0^t \exp\left(\int_\tau^t \sigma_V(s)\,\mathrm{d}s\right)\int \eta''(V) : \big(E(V)\otimes(V-U)\big)\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned}$$

Since, thanks to Lemma 3, $\eta(\cdot|\cdot)$ is convex with respect to the first variable on $\mathcal{U}$, we get:

$$\int_\Omega \eta(U|V)(t,x)\,\mathrm{d}x \leqslant \lim_{k\to\infty}\int_\Omega \eta(U_k|V)(t,x)\,\mathrm{d}x.$$

This gives the desired result for those test function V. Notice that we can then relax the condition of fast decrease in $|x|$, in the case of $\Omega = \mathbb{R}^N$, on the test function V. Hence U is a dissipative solution on $[0,T)$. □

2.6.4. *Approximated solutions* Dissipative solution is a useful concept for asymptotic limits. We give here a general framework. We fix $T \leqslant \infty$. Consider a sequence of functions U_ε, A_ε, Q_ε, η_ε, G_ε, and R_ε, in $L^1(0,\overline{T};L^1(\Omega))$ for any $\overline{T} < T$ such that $A(U_\varepsilon)$, $Q(U_\varepsilon)$, $\eta'(U_\varepsilon)\cdot Q(U_\varepsilon)$ are in $L^1(0,\overline{T};L^1(\Omega))$ for $\overline{T} < T$ and:

$$\partial_t U_\varepsilon + \operatorname{div}_x A_\varepsilon = Q_\varepsilon, \tag{20}$$

$$\partial_t \eta_\varepsilon + \operatorname{div}_x G_\varepsilon \leqslant R_\varepsilon, \tag{21}$$

$$U_\varepsilon(0,\cdot) = U_\varepsilon^0, \qquad \eta_\varepsilon(0,\cdot) = \eta_\varepsilon^0, \tag{22}$$

in the sense of distribution. We assume in addition that for every t, x, ε:

$$\eta_\varepsilon(t,x) \geqslant \eta\big(U_\varepsilon(t,x)\big). \tag{23}$$

Then we have the following result:

PROPOSITION 3. *For every ε, every $\overline{T} < T$, and every dissipative test function*

$$V \in W^{1,\infty}\big([0,\overline{T}]\times\Omega\big) \cap L^\infty\big(0,\overline{T};L^1(\Omega)\big)$$

verifying (15), we have for every $t < \overline{T}$:

$$\begin{aligned}
&\int_\Omega \big\{\big(\eta_\varepsilon - \eta(U_\varepsilon)\big)(t) + \eta(U_\varepsilon|V)(t)\big\}\,\mathrm{d}x \\
&\quad\leqslant \left(], \int_\Omega \big\{\eta\big(U_\varepsilon^0|V^0\big) + \eta_\varepsilon^0 - \eta\big(U_\varepsilon^0\big)\big\}\,\mathrm{d}x\right)\exp\left(\int_0^t \sigma_V(\tau)\,\mathrm{d}\tau\right) \\
&\qquad + \int_0^t \exp\left(\int_\tau^t \sigma_V(s)\,\mathrm{d}s\right)\int\big[\eta''(V):\big(E(V)\otimes(V-U_\varepsilon)\big) + \mathcal{R}_\varepsilon\big]\,\mathrm{d}x\,\mathrm{d}\tau,
\end{aligned}$$

where

$$\mathcal{R}_\varepsilon = \big[R_\varepsilon - \eta'(U_\varepsilon)Q(U_\varepsilon)\big] + \nabla_x\eta'(V)\big[A(U_\varepsilon) - A_\varepsilon\big] + \eta'(V)\big[Q_\varepsilon - Q(U_\varepsilon)\big].$$

Hence, if the quantities

$$A(U_\varepsilon) - A_\varepsilon, \qquad Q(U_\varepsilon) - Q_\varepsilon, \qquad R_\varepsilon - \eta'(U_\varepsilon)Q(U_\varepsilon), \tag{24}$$

converges to 0 weakly in $L^1((0,\overline{T})\times\Omega)$, for every $\overline{T} < T$, U_ε^0 converges strongly to U^0 in $L^1(\Omega)$, and both $\eta(U_\varepsilon^0)$ and η_ε^0 converge strongly to $\eta(U^0)$ in $L^1(\Omega)$, then up to a subsequence, U_ε converges weakly in $L^p_{\mathrm{loc}}(L^1_{\mathrm{loc}})$ for $1 \leqslant p < \infty$, to a dissipative solution U of (1) on $[0,T)$ with initial value U^0. If $\Phi_i(y) \geqslant |y|^{\gamma_i}$ for $y > 0$, then the component $U_{\varepsilon,i}$ converges weakly to U_i in $L^p_{\mathrm{loc}}(0,T;L^q(\Omega))$ for $1 \leqslant p < \infty$ and $1 \leqslant q < \gamma_i$.

PROOF. By density, Lemma 4 is still valid for V a dissipative test function and $U = U_\varepsilon$ such that U_ε, $\eta(U_\varepsilon)$, $A(U_\varepsilon)$, $Q(U_\varepsilon)$, and $\eta'(U_\varepsilon)\cdot Q(U_\varepsilon)$ are in $L^1([0,\overline{T}]\times\Omega)$. It gives:

$$\begin{aligned}
\partial_t\big\{\eta_\varepsilon - \eta(U_\varepsilon) + \eta(U_\varepsilon|V)\big\} &= \operatorname{div}_x\big(G(U_\varepsilon) - G_\varepsilon\big) - \big(\eta'(U_\varepsilon)\cdot Q(U_\varepsilon) - R_\varepsilon\big) \\
&\quad - \eta''(V):\big[E(V)\otimes(U_\varepsilon - V)\big] \\
&\quad + \eta'(V)\big[\operatorname{div}_x\big(A_\varepsilon - A(U_\varepsilon)\big) - \big(Q_\varepsilon - Q(U_\varepsilon)\big)\big]
\end{aligned}$$

$$
-\operatorname{div}_x\bigl(G(U_\varepsilon|V)\bigr)+\sum_{j,k}\partial_j\eta(V)\partial_{x_k}\bigl[A_{kj}(U_\varepsilon|V)\bigr]
$$
$$
+Q(V)\cdot\eta'(U_\varepsilon|V)+\bigl[Q(U_\varepsilon)-Q(V)\bigr]\cdot\bigl[\eta'(U_\varepsilon)-\eta'(V)\bigr].
$$

Integrating with respect to x, we find:

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\bigl\{\eta_\varepsilon-\eta(U_\varepsilon)+\eta(U_\varepsilon|V)\bigr\}\,\mathrm{d}x\leqslant\bigl(1+\|X(V)\|_{L^\infty(\Omega)}\bigr)C_V\int_\Omega\eta(U_\varepsilon|V)\,\mathrm{d}x
$$
$$
+\int_\Omega\bigl[R_\varepsilon-\eta'(U_\varepsilon)Q(U_\varepsilon)\bigr]\,\mathrm{d}x+\int_\Omega\eta''(V):\bigl[E(V)\otimes(V-U_\varepsilon)\bigr]\,\mathrm{d}x
$$
$$
+\int_\Omega\nabla_x\eta'(V):\bigl[A(U_\varepsilon)-A_\varepsilon\bigr]\,\mathrm{d}x+\int_\Omega\eta'(V)\bigl[Q_\varepsilon-Q(U_\varepsilon)\bigr]\,\mathrm{d}x.
$$

Integrating in time gives the desired result. □

REMARK. Indeed, it is enough to have the quantities (24) which converge to 0 in the sense of distribution. Then we can pass to the limit for regular test function $V\in\mathcal{D}([0,T]\times\Omega)$. Then, by density, we check that the inequality of dissipative solution holds, indeed, for any dissipative test functions.

2.7. *Examples of admissible balance laws*

We give some examples of admissible systems for which the previous study is valid. We will also show why the system of full Euler system of heat conducting flow is not included. Those two first results can be found in [15]. The bi-fluid model was studied by Mellet and Vasseur in [75] with viscosity and boundary conditions.

2.7.1. *Isentropic gas dynamics* The multidimensional system of isentropic gas dynamics reads:

$$
\begin{cases}\partial_t\rho+\operatorname{div}_x(\rho u)=0, & t\in\mathbb{R}^+,\ x\in\mathbb{R}^N,\\ \partial_t(\rho u)+\operatorname{div}_x(\rho u\otimes u+I\rho^\gamma)=\rho F, & t\in\mathbb{R}^+,\ x\in\mathbb{R}^N.\end{cases}\tag{25}
$$

for $1<\gamma\leqslant\frac{N+2}{N}$, and a given extern force field F. The associated entropy is

$$
\eta(\rho,\rho u)=\rho\frac{u^2}{2}+h(\rho),\tag{26}
$$

where $h(\rho)=\frac{1}{\gamma-1}\rho^\gamma$.

The conservative variables are $U=(\rho,\rho u)=(\rho,P)$ and the set $\mathcal{V}=(0,\infty)\times\mathbb{R}^N$. The entropy written in the conservative variables is:

$$
\eta(\rho,P)=\frac{|P|^2}{2\rho}+h(\rho).
$$

This entropy is regular in $\mathcal{V}$ and strictly convex. Indeed,

$$
\eta''(\rho,P)=\frac{1}{\rho}\begin{pmatrix}|P|^2/\rho^2+h''(\rho) & -P^T/\rho\\ -P/\rho & I_N\end{pmatrix},
$$

where P^T is the transpose of the vector P and I is the $N \times N$ identity matrix. The eigenvalue 1 has multiplicity $(n-1)$. The sum of the two other eigenvalues is given from the trace $S = |P|^2/\rho^2 + 1 + h''(\rho)$ and the product is the determinant $Pr = h''(\rho)$, which are nonnegative since h is convex. The smallest λ of those two eigenvalues is such that:

$$\lambda = \frac{Pr}{S} + \lambda^2 \geqslant \frac{Pr}{S} \geqslant C > 0,$$

for $(\rho, P) \in (\alpha, \beta) \times B(0, R)$, $0 < \alpha < \beta < \infty$, $R > 0$.

Following (4), we find that:

$$\mathcal{U} = \mathcal{V} \cup (0, 0).$$

The relative entropy is

$$\eta(U|U^*) = \frac{\rho}{2}|u - u^*|^2 + h(\rho|\rho^*).$$

The relative flux of the system is

$$A(U|U^*) = \big(0, \rho(u - u^*) \otimes (u - u^*) + h(\rho|\rho^*)I\big).$$

We clearly have the existence of a constant C such that

$$\big|A(U|U^*)\big| \leqslant C\eta(U|U^*), \tag{27}$$

for every $U, U^* \in \mathcal{V}$. For the system (25), the source terms reads

$$Q(\rho, P, x) = \big(0, \rho F(x)\big).$$

This gives

$$Q(U^*)\eta'(U|U^*) = -(u^* - u)(\rho^* - \rho)F,$$

and

$$\big[Q(U) - Q(U^*)\big]\big(\eta'(U) - \eta'(U^*)\big) = (u^* - u)(\rho^* - \rho)F,$$

and finally

$$Q(U^*)\eta'(U|U^*) + \big[Q(U) - Q(U^*)\big]\big(\eta'(U) - \eta'(U^*)\big) = 0. \tag{28}$$

Moreover, since $\gamma > 1$:

$$\frac{\rho^\gamma}{\gamma - 1} \leqslant \eta(\rho, P),$$

$$|P|^{2\gamma/(\gamma+1)} \leqslant (\rho^\gamma)^{1/(\gamma+1)}\left(\frac{|P|^2}{\rho}\right)^{\gamma/(\gamma+1)} \leqslant \eta(\rho, P).$$

This, together with (27) and (28), ensures that (25) is admissible in $\mathcal{V} = (0, \infty) \times \mathbb{R}^N$ with $\gamma_1 = \gamma$ and $\gamma_i = 2\gamma/(\gamma + 1)$ for $2 \leqslant i \leqslant N$.

2.7.2. *Isothermal gas dynamic* The multidimensional system of isothermal gas dynamics in $\mathbb{T}^N$ reads:

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho u) = 0, & t \in \mathbb{R}^+,\ x \in \mathbb{T}^N, \\ \partial_t(\rho u) + \mathrm{div}_x(\rho u \otimes u + I\rho) = \rho F, & t \in \mathbb{R}^+,\ x \in \mathbb{T}^N, \end{cases} \tag{29}$$

A nonnegative associated entropy is

$$\eta(\rho, \rho u) = \rho \frac{u^2}{2} + \rho \ln \rho + \frac{1}{e}. \tag{30}$$

The conservative variables are $U = (\rho, \rho u) = (\rho, P)$ and the set $\mathcal{V} = (0, \infty) \times \mathbb{R}^N$. The entropy written in the conservative variables is:

$$\eta(\rho, P) = \frac{|P|^2}{2\rho} + \rho \ln \rho + \frac{1}{e}.$$

This entropy is regular in $\mathcal{V}$ and strictly convex. The proof is the same than for the isentropic case since $h(\rho) = \rho \ln \rho + 1/e$ is convex. Following (4), we find that:

$$\mathcal{U} = \mathcal{V} \cup (0, 0).$$

The relative entropy is

$$\eta(U|U^*) = \frac{\rho}{2}|u - u^*|^2 + \rho \ln(\rho/\rho^*) - (\rho - \rho^*).$$

The relative flux of the system is

$$A(U|U^*) = \big(0, \rho(u - u^*) \otimes (u - u^*) + \big(\rho \ln(\rho/\rho^*) - (\rho - \rho^*)\big)I\big).$$

We clearly have the existence of a constant C such that

$$\big|A(U|U^*)\big| \leqslant C\eta(U|U^*), \tag{31}$$

for every $U, U^* \in \mathcal{V}$. For the system (29), the source term is dealt in the same way:

$$Q(U^*)\eta'(U|U^*) + \big[Q(U) - Q(U^*)\big]\big(\eta'(U) - \eta'(U^*)\big) = 0. \tag{32}$$

In the isotherm case, (11) is also verified:

$$\big[\rho \ln(\rho)\big]_+ \leqslant \eta(\rho, P),$$
$$|P|\sqrt{\ln\big(|P| + 1\big)} \leqslant C\big(\big|(\rho, P)\big| + \eta(\rho, P)\big).$$

For the last inequality we have to separate the cases $|P| > \rho^2$ and $|P| < \rho^2$. In the second case, it is smaller than

$$\frac{|P|}{\sqrt{\rho}}\sqrt{\rho}\sqrt{2\big(\ln(\rho + 1)\big)} \leqslant \frac{|P|^2}{2\rho} + \rho \ln(1 + \rho).$$

In the first case, $\eta(\rho, P) \geqslant \rho^3/2$ and we can use the proof for the isentropic system with $\gamma = 3$. This, with estimates (31) and (32) ensures that (29) is admissible in $\mathcal{V} = (0, \infty) \times \mathbb{R}^N$.

REMARK. We study this system in $\mathbb{T}^N$ to deal with the non integrability of the entropy in $\mathbb{R}^N$. This problem can be address in $\mathbb{R}^N$ considering the entropy

$$|x|^2\rho + \rho \ln \rho + \frac{|P|^2}{2\rho}.$$

But this is not strictly speaking included in the previous theory since the entropy depends also on x. We have chosen to restrict the theory to entropy depending only on U for the sake of clarity.

2.7.3. *A bi-fluid model* We consider the following system of hydrodynamic equations in the torus $x \in \mathbb{T}^N$:

$$\begin{cases} \partial_t n + \mathrm{div}_x(nu) = 0, \quad t > 0, x \in \mathbb{T}^N, \\ \partial_t \rho + \mathrm{div}_x(\rho u) = 0, \quad t > 0, x \in \mathbb{T}^N, \\ \partial_t\big((\rho+n)u\big) + \mathrm{div}_x\big((\rho+n)u \otimes u\big) + \nabla_x(n + \rho^\gamma) = 0, \quad t > 0, x \in \mathbb{T}^N. \end{cases} \tag{33}$$

This kind of multi-fluid system is widely used in the modeling of particle/fluid interaction. The conservative variables are $U = (n, \rho, P) = (n, \rho, (\rho+n)u)$. The set $\mathcal{V} = (0, \infty) \times (0, \infty) \times \mathbb{R}^N$. The following function is a nonnegative entropy for this system

$$\begin{aligned} \eta(n, \rho, P) &= (n+\rho)\frac{u^2}{2} + \frac{1}{\gamma - 1}\rho^\gamma + n \log n + \frac{1}{e} \\ &= \frac{P^2}{2(n+\rho)} + \frac{1}{\gamma-1}\rho^\gamma + n\log n + \frac{1}{e}, \end{aligned}$$

with entropy flux function

$$G(U) = \frac{P}{n+\rho}\left[\frac{P^2}{2(n+\rho)} + \frac{\gamma}{\gamma-1}\rho^\gamma + n + n\log n\right].$$

From definition (4), we have

$$\mathcal{U} = \big([0,\infty) \times (0,\infty) \cup (0,\infty) \times [0,\infty)\big) \times \mathbb{R}^N \cup (0,0,0).$$

We can show, as for the isentropic and isotherm cases, that η is strictly convex. A simple computation shows that the relative entropy associated with (33) is

$$\eta(U|U^*) = (n+\rho)\frac{|u-u^*|^2}{2} + \frac{1}{\gamma-1}p_1(\rho|\rho^*) + p_2(n|n^*)$$

with

$$\begin{aligned} p_1(\rho|\rho^*) &= \rho^\gamma - \rho^{*\gamma} - \gamma\rho^{*\gamma-1}(\rho - \rho^*), \\ p_2(n|n^*) &= n\log n - n^*\log n^* - (\log n^* + 1)(n - n^*) \\ &= n \log\frac{n}{n^*} + (n^* - n). \end{aligned}$$

(Note that the $p_i(\cdot|\cdot)$ are the relative quantities associated to $p_1(\rho) = \rho^\gamma$ and $p_2(n) = n\log n$.) The system (33) can be written in the form (1), with $U = (n, \rho, P)$ (we recall that

$P = (n+\rho)u)$ and

$$A(U) = \frac{1}{n+\rho}\begin{pmatrix} nP_1 & nP_2 & nP_3 \\ \rho P_1 & \rho P_2 & \rho P_3 \\ C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix},$$

where

$$C_{ij} = P_i P_j + (n+\rho^\gamma)(n+\rho)\delta_{ij}.$$

LEMMA 5. *Assume* $1<\gamma<2$, *then the system* (33) *is locally admissible in* $\mathcal{V} = (0,\infty)\times(0,\infty)\times\mathbb{R}^N$ *with*:

$$[n\ln n]_+ \leqslant \eta(n,\rho,P),$$
$$\rho^\gamma/(\gamma-1) \leqslant \eta(n,\rho,P),$$
$$|P|\sqrt{\ln\bigl(1+|P|\bigr)} \leqslant C\bigl(\eta(n,\rho,P) + \bigl|(n,\rho,P)\bigr|\bigr).$$

PROOF. First we can show, following the proofs for the isentropic gas and isotherm gases:

$$n\ln(n+1) \leqslant 2\eta(n,\rho,P) + n\ln 2,$$
$$\rho^\gamma \leqslant \eta(n,\rho,P),$$
$$|P|\sqrt{\ln\bigl(1+|P|\bigr)} \leqslant C\bigl(\eta(n,\rho,P) + \bigl|(n,\rho,P)\bigr|\bigr).$$

Let $\mathcal{D}$ be a compact set of $\mathcal{V}$. For any such compact set, there exists $0<\lambda<\Lambda$ verifying for any $(n^*,\rho^*,P^*)\in\mathcal{D}$

$$\lambda \leqslant n^* + \rho^* \leqslant \Lambda.$$

We want to show that there exists $C(\lambda^{-1},\Lambda)$ such that

$$\bigl|A(U|U^*)\bigr| \leqslant C\eta(U|U^*)$$

for any $U=(n,\rho,(n+\rho)u)$, $U^*=(n^*,\rho^*,(n^*+\rho^*)u^*)$ satisfying

$$\lambda \leqslant \rho^* + n^* \leqslant \Lambda.$$

First, we check that the relative flux is given by

$$A(U|U^*) = \begin{pmatrix} (\alpha-\alpha^*)(\rho+n)(u_i-u_i^*) \\ (\beta-\beta^*)(\rho+n)(u_i-u_i^*) \\ (\rho+n)(u_i-u_i^*)(u_j-u_j^*) + p_1(\rho|\rho^*)\delta_{ij} \end{pmatrix}$$

with $\alpha = \frac{n}{n+\rho}$, $\alpha^* = \frac{n^*}{n^*+\rho^*}$, $\beta = \frac{\rho}{n+\rho}$ and $\beta^* = \frac{\rho^*}{n^*+\rho^*}$. and we recall that the relative entropy satisfies

$$\eta(U|U^*) = (n+\rho)\frac{|u-u^*|^2}{2} + \frac{1}{\gamma-1}p_1(\rho|\rho^*) + p_2(n|n^*),$$

with

$$p_1(\rho|\rho^*) = \rho^\gamma - \rho^{*\gamma} - \gamma\rho^{*\gamma-1}(\rho-\rho^*) = \frac{\gamma}{2}\xi_1^{\gamma-2}(\rho-\rho^*)^2,$$
$$p_2(n|n^*) = n\log n - n^*\log n^* - (\log n^* + 1)(n-n^*) = \frac{1}{2}\frac{1}{\xi_2}(n-n^*)^2$$

for some ξ_1 between ρ and ρ^* and ξ_2 between n and n^*.

The relative flux $A(U|U^*)$ involves the following terms:

$$\left|(\alpha-\alpha^*)(\rho+n)(u_i-u_i^*)\right|, \qquad \left|(\beta-\beta^*)(\rho+n)(u_i-u_i^*)\right| \tag{34}$$

and

$$(\rho+n)(u_i-u_i^*)(u_j-u_j^*), \qquad p_1(\rho|\rho^*) \tag{35}$$

and it is readily seen that the last two terms (35) are bounded above by $2\eta(U|U^*)$. Moreover, the terms in (34) are equal since $\alpha+\beta=1$, $\alpha^*+\beta^*=1$. We note that

$$\left|(\alpha-\alpha^*)(\rho+n)(u_i-u_i^*)\right| \leqslant (\alpha-\alpha^*)^2(\rho+n) + (\rho+n)\left|(u_i-u_i^*)\right|^2,$$

where the second term is bounded above by $2\eta(U|U^*)$. So we are left with the task of showing that the quantity

$$I = (\alpha-\alpha^*)^2(n+\rho)$$

is bounded above by $\eta(U|U^*)$.

To that purpose, we need to distinguish the case where $n+\rho$ is larger than Λ and the case where $n+\rho$ is smaller than Λ. In each case, we will use one of the following expression for $\alpha-\alpha^*$:

$$\alpha-\alpha^* = \frac{\rho(n-n^*) + n(\rho^*-\rho)}{(n+\rho)(n^*+\rho^*)} \tag{36}$$

or

$$\alpha-\alpha^* = \frac{\rho^*(n-n^*) + n^*(\rho^*-\rho)}{(n+\rho)(n^*+\rho^*)}. \tag{37}$$

- When $n+\rho<\Lambda$, using the fact that $\rho<\Lambda$ and $\rho^*\leqslant\Lambda$, we get $\xi_1<\Lambda$. Since $\gamma-2<0$ we deduce

  $$p_1(\rho|\rho^*) \geqslant C(\Lambda)(\rho-\rho^*)^2.$$

  Similarly, using the fact that $n<\Lambda$ and $n^*\leqslant\Lambda$, we have $\xi_2<\Lambda$ which yields

  $$p_2(n|n^*) \geqslant C(\Lambda)(n-n^*)^2.$$

  Finally, using (36) together with the fact that $n/(n+\rho)\leqslant 1$ and $\rho/(n+\rho)\leqslant 1$, we get

  $$\begin{aligned} I &\leqslant (n+\rho)\left(\frac{|n-n^*|}{(n^*+\rho^*)} + \frac{|\rho^*-\rho|}{(n^*+\rho^*)}\right)^2 \\ &\leqslant \frac{\Lambda}{\lambda}\big((n-n^*)^2 + (\rho^*-\rho)^2\big) \end{aligned}$$

and therefore

$$I \leqslant C(\Lambda, \lambda)\big[p_1(\rho|\rho^*) + p_2(n|n^*)\big].$$

- When $n + \rho > \Lambda$, we first note that using (37) and the fact that $n^*/(n^* + \rho^*) \leqslant 1$, we have

$$I \leqslant \frac{(|\rho^* - \rho| + |n - n^*|)^2}{(n + \rho)} \leqslant \frac{(n - n^*)^2}{n + \rho} + \frac{(\rho - \rho^*)^2}{n + \rho}.$$

  In order to control the first term, we again distinguish two situations:
  - When $n \geqslant \Lambda$, then $n^* < n$, and so $\xi_2 < n$. Therefore

$$p_2(n|n^*) > \frac{1}{n}(n - n^*)^2 > \frac{(n - n^*)^2}{n + \rho}.$$

  - When $n < \Lambda$, then $1/\xi_2 > 1/\max(n, n^*) > 1/\Lambda$, and since $\frac{(n-n^*)^2}{\rho+n} < \frac{1}{\Lambda}(n - n^*)^2$, we get

$$p_2(n|n^*) > C(\Lambda)(n - n^*)^2 > C(\Lambda)\Lambda\frac{(n - n^*)^2}{\rho + n},$$

    where we used the fact that $n + \rho \geqslant \Lambda$.

  In either case, we have

$$\frac{(n - n^*)^2}{n + \rho} \leqslant C(\Lambda)p_2(n|n^*).$$

  Finally, we proceed similarly to show that the term $\frac{(\rho-\rho^*)^2}{n+\rho}$ is controlled by $p_1(\rho|\rho^*)$:
  - When $\rho > \Lambda$, then $\xi_1 \leqslant \rho$ and so $\xi_1^{\gamma-2} > \rho^{\gamma-2} > C(\Lambda)/\rho$ (using the fact that $\gamma > 1$). Since

$$\frac{(\rho^* - \rho)^2}{n + \rho} \leqslant \frac{(\rho - \rho^*)^2}{\rho},$$

    we deduce

$$\frac{(\rho^* - \rho)^2}{n + \rho} < C\xi_1^{\gamma-2}(\rho^* - \rho)^2 \leqslant p_1(\rho|\rho^*).$$

  - When $\rho < \Lambda$, then $\xi_1^{\gamma-2} > (\max(n, n^*))^{\gamma-2} > \Lambda^{\gamma-2}$, and since

$$\frac{(\rho - \rho^*)^2}{\rho + n} < \frac{1}{\Lambda}(\rho - \rho^*)^2$$

    (we recall that we still have $n + \rho > \Lambda$), we get

$$p_1(n|n^*) > C(\Lambda)(\rho - \rho^*)^2 > C(\Lambda)\frac{(\rho - \rho^*)^2}{\rho + n}.$$

The proof of Lemma 5 is now complete. $\square$

2.7.4. *A counterexample: The Euler system with temperature* We show that the Euler system with temperature is not admissible on $\mathcal{V} = (0,\infty) \times \mathbb{R}^N \times (0,\infty)$. The full gas dynamics of Euler with temperature reads

$$\begin{aligned}
&\partial_t \rho + \operatorname{div}_x(\rho u) = 0, \quad t \in \mathbb{R}^+,\ x \in \mathbb{R}^N,\\
&\partial_t(\rho u) + \operatorname{div}_x(\rho u \otimes u + I_N \rho T) = 0, \quad t \in \mathbb{R}^+,\ x \in \mathbb{R}^N,\\
&\partial_t\left(\rho\frac{|u|^2}{2} + \frac{N}{2}\rho T\right) + \operatorname{div}_x\left(\rho u \frac{|u|^2}{2} + \frac{N+2}{2}\rho T u\right) = 0, \quad t \in \mathbb{R}^+,\ x \in \mathbb{R}^N.
\end{aligned}$$

The conservative variables are

$$U = (\rho, P, E) = \left(\rho, \rho u, \rho\frac{|u|^2}{2} + \frac{N}{2}\rho T\right)$$

and the flux is

$$A(U) = \left(\rho u, \rho u \otimes u + \rho T I_N, \rho u \frac{|u|^2}{2} + \frac{N+2}{2}\rho T u\right).$$

The entropy is

$$\eta(U) = \rho \ln\left(\frac{\rho}{(2\pi T)^{N/2}}\right) - \frac{N}{2}\rho$$

and the associated flux is $G(U) = \eta(U)u$. The expression of the flux A in conservative variables is

$$A(U) = \left(P, \frac{1}{\rho}P \otimes P + \frac{2E}{N}I_N - \frac{1}{N\rho}|P|^2 I_N, \frac{N+2}{N}\frac{P}{\rho}E - \frac{1}{N}\frac{P}{\rho^2}|P|^2\right).$$

Then we get

$$\begin{aligned}
&\partial_\rho A_P(U) = -u \otimes u + \frac{1}{N}|u|^2 I_N,\\
&\partial_{P_i}(A_P)_{jk}(U) = \delta_{ij}u_k + \delta_{ik}u_j - \delta_{jk}\frac{2u_i}{N},\\
&\partial_E A_P(U) = \frac{2}{N}I_N,\\
&\partial_\rho A_E(U) = -\frac{N-2}{2}u\frac{|u|^2}{2} - \frac{N+2}{2}uT,\\
&\partial_{P_i}(A_E)_j(U) = \delta_{ij}\left(\frac{|u|^2}{2} + \frac{N+2}{2}T\right) - \frac{2}{N}u_i u_j,\\
&\partial_E A_E(U) = \frac{N+2}{N}u,
\end{aligned}$$

and the relative flux is

$$A_\rho(U|U^*) = 0, \tag{38}$$

$$A_P(U|U^*) = \rho(u-u^*) \otimes (u-u^*) - \frac{1}{N}\rho|u-u^*|^2 I_N, \tag{39}$$

$$A_E(U|U^*) = \frac{1}{2}\rho\big(|u|^2 - |u^*|^2\big)(u-u^*) + \frac{N+2}{2}\rho(u-u^*)(T-T^*) \\ - \frac{1}{N}\rho u^*|u-u^*|^2. \tag{40}$$

We compute now the relative entropy. Since linear part in a function disappears in any relative quantity, we have to compute the flux of

$$\tilde\eta(U) = \left(1+\frac{N}{2}\right)\rho\ln\rho - \frac{N}{2}\rho\ln\left(\frac{2E}{N} - \frac{|P|^2}{N\rho}\right),$$

which satisfies

$$\partial_\rho\tilde\eta(U) = 1 + \ln\rho + \frac{N}{2} - \frac{N}{2}\ln T - \frac{|u|^2}{2T}, \quad \partial_P\tilde\eta = \frac{u}{T}, \quad \partial_E\tilde\eta = -\frac{1}{T},$$

and thus we get

$$\eta(U|U^*) = h(\rho|\rho^*) + \frac{N\rho}{2T^*}h(T^*|T) + \frac{\rho}{2T^*}|u-u^*|^2, \tag{41}$$

where $h(x) = x\ln x$.

We see that (9) or (10) are not verified in this case because of the cubic power in velocity in $A_E(U|U^*)$ since such a term do not appear in $\eta(U|U^*)$.

2.8. *Examples of uniqueness results for dissipative solutions*

The notion of dissipative solutions is very efficient to show asymptotic limits. However, the limit makes sense only if this notion is relevant. Global weak entropy solutions for general multi-dimensional conservation laws is not known. The validity of this notion can be tested only on meaningful examples where reasonable solutions are known to exist. This can be done through uniqueness result for this kind of solutions.

The weak/strong principle gives the first of this kind of uniqueness result. We consider now two other particular examples of uniqueness results for solutions of isentropic or isotherm gas dynamics, involving singularities.

We denote $h(\rho) = \rho^\gamma/(\gamma-1)$ for the isentropic gas and $h(\rho) = \rho\ln\rho$ for the isotherm gas. We recall that for those systems

$$\eta'(U) = \big(-|u|^2 + h'(\rho), u\big).$$

2.8.1. *Rarefaction waves* In this subsection, we consider a one-dimensional problem, $\Omega = \mathbb{R}$ or $\Omega = \mathbb{T}$.

PROPOSITION 4. *Fix $1 \leqslant \gamma < \infty$. Consider an initial value $U^0 = (\rho^0, \rho^0 u^0) \in L^1(\Omega) \cap L^\infty(\Omega)$ with $\rho^0 \geqslant C > 0$, piecewise smooth with a finite number of discontinuities, each of them corresponding to rarefaction waves. Then there exists $T > 0$, and $U^* \in C^0(0,T; L^1(\Omega)) \cap W^{1,1}((\varepsilon,T)\times\Omega)$, for every $0 < \varepsilon < T$, such that U^* is a weak entropy solution of (25) with $N = 1$. In addition U^* is the only dissipative solution on $[0,T)$.*

In this case the singularities exist only at $t = 0$.

PROOF. We have

$$\nabla_x \eta'(U^*) : A(U|U^*) = \partial_x u^* \big(\rho|u - u^*|^2 + h(\rho|\rho^*)\big).$$

For $t > 0$, in every rarefaction area, we have

$$\partial_x u^*(t, x) \geqslant 0,$$

and so $X(U^*(t, x)) = 0$, since h is convex and so $h(\rho|\rho^*) \geqslant 0$. Hence σ_{U^*} depends only on the smooth part of the initial datum, lies in $L^\infty([0, T])$.

For every $\varepsilon > 0$, $V_\varepsilon(t, x) = U^*(\varepsilon + t, x)$ is a dissipative test function, so, from the weak/strong principle, it is the only dissipative solution to (25) on $[0, T - \varepsilon)$ with initial value $U^*(\varepsilon, \cdot)$. This function converges in $L^1(\Omega)$ to U^*, so from the stability property of dissipative solutions, U^* is a dissipative solution on $[0, T)$.

Consider an other dissipative solution U. Since V_ε is a dissipative test function with $E(V_\varepsilon) = 0$, we have for $t < T$:

$$\int_0^1 \eta(U|V_\varepsilon)(t, x)\,\mathrm{d}x \leqslant \left(\int_0^1 \eta\big(U^0(x)|U^*(\varepsilon, x)\big)\,\mathrm{d}x\right) \exp\big(T\|\sigma_{U^*}\|_{L^\infty(0,T)}\big).$$

Passing to the limit $\varepsilon \to 0$ and using the fact that $U^* \in L^\infty((0, T) \times \Omega) \cap C^0(0, T; L^1(\Omega))$, we find that for every $t > 0$:

$$\int_0^1 \eta(U|U^*)(t, x)\,\mathrm{d}x \leqslant 0.$$

Lemma 3 ensures that $U = U^*$. □

2.8.2. *Axisymmetric solution with vacuum at the origin* Let $\psi \in C^1(0, \infty)$, be such that $\psi(r) = 0$ for $r > 1/3$ and $\psi(y) \leqslant y$ everywhere. We consider the following steady isentropic solution $V_\psi = (\rho, u)(r = \sqrt{x_1^2 + x_2^2}, \theta, x_3)$ in $\mathbb{T}^3$ of the system (25):

$$\rho(r) = \left(\frac{\gamma - 1}{\gamma}\int_0^r \frac{\psi^2(y)}{y}\,\mathrm{d}y\right)^{1/(\gamma-1)}, \quad 0 < r < 1/3,\ x_3 \in \mathbb{T},$$
$$u = \psi(r)\vec{e}_\theta(\theta), \quad 0 < r < 1/3,\ 0 \leqslant \theta < 2\pi,\ x_3 \in \mathbb{T},$$

where

$$\vec{e}_\theta(\theta) = (-\sin\theta, \cos\theta) = \frac{1}{r}(-x_2, x_1), \qquad \vec{e}_r(\theta) = (\cos\theta, \sin\theta) = \frac{1}{r}(x_1, x_2).$$

We complete (ρ, u) in $\mathbb{T}^3$ such that the density ρ is continuous and constant outside $B(0, 1/3)$ and $u = 0$ outside $B(1, 1/2)$. We have

$$\mathrm{div}_x(\rho u) = \frac{\rho(r)}{r}\,\mathrm{div}(-x_2, x_1) + \nabla\left(\frac{\rho(r)}{r}\right) \cdot r\vec{e}_\theta = 0,$$

$$\begin{aligned}\rho(u\cdot\nabla)u &= -\rho\frac{\psi(r)^2}{r}\vec{e}_r(\theta)\\ &= -\frac{\gamma\rho}{\gamma-1}\partial_r\big(\rho(r)^{\gamma-1}\big)\vec{e}_r(\theta)\\ &= -\partial_r(\rho^\gamma)\vec{e}_r(\theta) = -\nabla\rho^\gamma.\end{aligned}$$

Hence, V_ψ is solution to the isentropic system. In the neighborhood of $r = 0$, we have $\rho(r) = C_\gamma r^{2/(\gamma-1)}$. Hence, there is vacuum at $r = 0$, and ρ is not Lipschitz continuous at this point for $\gamma > 3$. We have the following uniqueness result.

PROPOSITION 5. *The axisymmetric flow V_ψ is the unique dissipative solution with same initial value.*

PROOF. In this context, the singularity comes from the vacuum. the family of solutions V_ε defined by

$$u_\varepsilon(r,\theta) = \psi(r)\vec{e}_\theta(\theta), \qquad \rho_\varepsilon(r) = \left(\varepsilon + \frac{\gamma-1}{\gamma}\int_0^r \frac{\psi^2(y)}{y}\,\mathrm{d}y\right)^{1/(\gamma-1)},$$

are dissipative test functions. Note that

$$\nabla_x u_\varepsilon = \psi'(r)\vec{e}_r\otimes\vec{e}_\theta - \frac{\psi(r)}{r}\vec{e}_\theta\otimes\vec{e}_r.$$

So $\|\nabla_x u_\varepsilon\|_{L^\infty} \leqslant 1 + \|\psi'\|_{L^\infty}$ is uniformly bounded. Hence u_ε is uniformly Lipschitz but not for sure C^1. From the stability of dissipative solutions, V_ψ is a dissipative solution. Consider now an other dissipative solution U with same initial value. Since $\nabla_x\eta'(V_\varepsilon) = \nabla_x u_e ps$, and (25) is admissible on $\mathcal{V}$, we get for $t < T$

$$\int_{\mathbb{T}^3}\eta(U|V_\varepsilon)(t,x)\,\mathrm{d}x \leqslant \left(\int_{T_3}\eta\big(V_\psi(x)|V_\varepsilon(x)\big)\,\mathrm{d}x\right)\exp\big(CT\big(1+\|\psi'\|_{L^\infty}\big)\big).$$

Passing to the limit gives that $\eta(U|V_\psi) = 0$ almost everywhere and so $U = V_\psi$ thanks to Lemma 3. □

3. Kinetic equations

We introduce a new variable $v \in \mathbb{R}^N$ called velocity variable, and we call kinetic function any nonnegative function $f : \mathbb{R}^+ \times \Omega \times \mathbb{R}^N \to \mathbb{R}$. In a physical setting, like the nonlinear Fokker–Planck equation (see Section 3.1), $f(t,x,v)$ represents the microscopic density of particles at time t, with velocity in the cube centered at v with radius $\mathrm{d}v$, and position in the cube centered in x with radius $\mathrm{d}x$. Most of the kinetic equations can be written in the following form.

$$\begin{aligned}&\partial_t f + v\cdot\nabla_x f + F(x)\cdot\nabla_v f = \mathcal{Q}(f,v),\\ &f(0,\cdot,\cdot) = f^0,\end{aligned}\tag{42}$$

where $F \in C^\infty(\Omega)$ is a given function, and $\mathcal{Q} : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a nonlinear operator called collision term. In the Fokker–Planck case, this collision operator models the collision of the particles on an inert gas (Brownian motion), see [94] or [35]. The typical assumption to derive kinetic models, is to neglect the size of the particle, which leads to an operator on f in v only. We assume the existence of $a : \mathbb{R}^N \to \mathbb{R}^p$ such that for every $f \in \mathbb{R}$

$$\int_{\mathbb{R}^N} a(v)\mathcal{Q}(f, v)\,\mathrm{d}v = 0. \tag{43}$$

The quantity $U(t, x) = \int_{\mathbb{R}^N} a(v) f(t, x, v)\,\mathrm{d}v$ are then called conserved quantities, or macroscopic quantities. This are derived from microscopic quantities conserved during the collisions in play. In the non-linear Fokker–Planck equation, they are the mass and the momentum. We assume also the existence of kinetic entropy functional $H(f, v)$ and a non-negative dissipation of entropy functional $D(f) \geqslant 0$, such that H is convex in f and such that for every nonnegative function $f \in C_c^\infty(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} \frac{\mathrm{d}H}{\mathrm{d}f}(f, v)\mathcal{Q}(f, v)\,\mathrm{d}v = -D(f) \leqslant 0. \tag{44}$$

Historically, the concept of entropy has been developed together with the kinetic equations, by the founders Boltzmann, and Maxwell. For a good review on kinetic equation, especially Boltzamnn equation, we refer to [38] (see also [39]). See also [72] for an introduction of models coming from the micro-electronics.

We introduce now some technical assumptions ensuring a good enough control (at least for this theory), of global in time solutions of those kinetic equations. We say that (42) is admissible if there exists constants $\alpha, C > 0$ such that for every nonnegative function $f \in C_c^\infty(\mathbb{R}^N)$, $f > 0$

$$\begin{aligned} &\left| \int_{\mathbb{R}^N} \big(|a(v)| + |a(v) \otimes v| + |H(f, v)| + |\mathrm{div}_v\, a(v)| \big) f + \left| \frac{\partial H}{\partial v}(f, v) \right| \mathrm{d}v \right| \\ &\quad \leqslant C \int_{\mathbb{R}^N} \big(\alpha f + H(f, v) \big)\,\mathrm{d}v. \end{aligned} \tag{45}$$

We say that (42) is admissible up to a constant if there exists constants $\alpha, C > 0$ such that for every nonnegative function $f \in C_c^\infty(\mathbb{R}^N)$, $f > 0$,

$$\begin{aligned} &\left| \int_{\mathbb{R}^N} \big(|a(v)| + |H(f, v)| + |a(v) \otimes v| + |\nabla_v a(v)| \big) f + \left| \frac{\partial H}{\partial v}(f, v) \right| \mathrm{d}v \right| \\ &\quad \leqslant C \bigg(\int_{\mathbb{R}^N} \big(\alpha f + H(f, v) \big)\,\mathrm{d}v + 1 \bigg). \end{aligned} \tag{46}$$

In this case we restrict ourselves to the case $\Omega = \mathbb{T}^N$.

For Eqs. (42) which are either admissible or admissible up to a constant, we call entropy solutions on $(0, \infty) \times \Omega \times \mathbb{R}^N$ with initial value $f^0 \in L^1(\Omega \times \mathbb{R}^N)$, $f^0 \geqslant 0$, $H(f^0, v) \in$

$L^1(\Omega \times \mathbb{R}^N)$, any function $f \in L^\infty_{\mathrm{loc}}(0, \infty; L^1(\Omega \times \mathbb{R}^N))$ such that (42) is verified in the sense of distribution and the following inequality is also verified in the sense of distribution

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \int_{\mathbb{R}^N} H(f, v)\, \mathrm{d}v\, \mathrm{d}x \leqslant -\int_\Omega D(f)\, \mathrm{d}x + \int_\Omega \int_{\mathbb{R}^N} F \cdot \frac{\partial H}{\partial v}(f, v)\, \mathrm{d}v\, \mathrm{d}x, \\
&\int_\Omega \int_{\mathbb{R}^N} H\big(f(0, x, v), v\big)\, \mathrm{d}v\ \mathrm{d}x = \int_\Omega \int_{\mathbb{R}^N} H\big(f^0(x, v), v\big)\, \mathrm{d}v\, \mathrm{d}x.
\end{aligned} \tag{47}$$

We introduce three different examples of such kinetic equations.

REMARK. We call admissible kinetic equations, the kinetic equations verifying (45), by analogy with admissible conservation laws verifying (9) and (10). But we should keep in mind that in (9) the relative entropy is involved when in (45), only the entropy is involved.

3.1. *Nonlinear Fokker–Planck equation*

We consider a gas subject to Brownian motion and an external force $F \in C^\infty(\mathbb{T}^N)$. The microscopic density functional is then solution to the following Fokker–Planck equation, see [35]

$$\begin{aligned}
&\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f + \mathrm{div}_v\big((u - v) f - \nabla_v f\big) = 0, \\
&\quad t > 0,\ x \in \mathbb{T}^N,\ v \in \mathbb{R}^N,
\end{aligned} \tag{48}$$

where u is the mean velocity of the gas defined for t, x such that $\int f(t, x, v)\, \mathrm{d}v > 0$ by

$$u(t, x) \int_{\mathbb{R}^N} f(t, x, v)\, \mathrm{d}v = \int_{\mathbb{R}^N} v f(t, x, v)\, \mathrm{d}v.$$

This models has been studied by several authors, even considering coupling with Poisson equation and boundary conditions [36,21,19]. We have $a(v) = (1, v)$ which means that the conserved quantities are the macroscopic density $\rho = \int f\, \mathrm{d}v$ and the macroscopic momentum $\rho u = \int v f\, \mathrm{d}v$. The kinetic entropy functional is given by

$$H(f, v) = \left(\frac{|v|^2}{2} + \ln(f)\right) f.$$

Note that H is not nonnegative. We have the following result.

LEMMA 6. *We denote*

$$\mathcal{Q}(f, v) = \mathrm{div}_v\Big((v - u) f + \nabla_v f\Big).$$

Then for every function $f \in C_c^\infty(\mathbb{R}^N)$, $f > 0$, (43), (44) *and* (46) *are fulfilled*, *with*:

$$D(f) = \int_{\mathbb{R}^N} \frac{|(v - u) f + \nabla_v f|^2}{f}\, \mathrm{d}v \geqslant 0.$$

Hence (48) *is admissible up to a constant*.

PROOF. Estimate (43) is fulfilled thanks to the divergence form of $\mathcal{Q}$ and the definition of u. A simple calculation gives

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{\mathrm{d}H}{\mathrm{d}f}(f,v)\mathcal{Q}(f,v)\,\mathrm{d}v &= \int_{\mathbb{R}^N}\left(\frac{|v|^2}{2}+1+\ln f\right)\mathrm{div}_v\big((v-u)f+\nabla_v f\big)\,\mathrm{d}v\\ &= -\int_{\mathbb{R}^N}(v+\nabla_v f/f)\big((v-u)f+\nabla_v f\big)\,\mathrm{d}v\\ &= -\int_{\mathbb{R}^N}\frac{|(v-u)f+\nabla_v f|^2}{f}\,\mathrm{d}v,\end{aligned}$$

where we have used the fact that

$$\int_{\mathbb{R}^N} u\cdot\big((v-u)f+\nabla_v f\big)\,\mathrm{d}v = 0.$$

This gives (44) with the desired form on D. To verify (46), it is enough to control $\int((1+|v|^2+|\ln f|)f\,\mathrm{d}v$. The difficulty comes from the fact that H is not nonnegative. This can be solved in a very classical way. The result follows the fact that there exists $C>0$ such that

$$\int_{f\leqslant 1} f|\ln f|\,\mathrm{d}v \leqslant \int_{\mathbb{R}^N}\left(1+\frac{|v|^2}{8}\right)f\,\mathrm{d}v + C. \tag{49}$$

Indeed this implies that:

$$\begin{aligned}3\int f\,\mathrm{d}v + \int H(f,v)\,\mathrm{d}v &= \int\left(3+\frac{|v|^2}{2}\right)f\,\mathrm{d}v + \int f\ln f\,\mathrm{d}v\\ &= \int\left(3+\frac{|v|^2}{2}\right)f\,\mathrm{d}v + \int f|\ln f|\,\mathrm{d}v - 2\int_{f\leqslant 1} f|\ln f|\,\mathrm{d}v\\ &\geqslant \int\frac{|v|^2}{4}f\,\mathrm{d}v + \int f\,\mathrm{d}v - 2C + \frac{1}{2}\int f|\ln f|\,\mathrm{d}v.\end{aligned}$$

So

$$\int\big(|\ln f|+1+|v|^2\big)f\,\mathrm{d}v \leqslant 4\left(3\int f\,\mathrm{d}v + \int H(f,v)\,\mathrm{d}v\right) + 8C,$$

which leads to (46). To show (49), we write:

$$\begin{aligned}\int_{f\leqslant 1} f|\ln f|\,\mathrm{d}v &\leqslant \int_{\mathrm{e}^{-|v|^2/8-1}\leqslant f\leqslant 1} f|\ln f|\,\mathrm{d}v + \int_{f\leqslant \mathrm{e}^{-|v|^2/8-1}} f|\ln f|\,\mathrm{d}v\\ &\leqslant \int_{\mathbb{R}^N}\big(1+|v|^2/8\big)f\,\mathrm{d}v + \int_{\mathbb{R}^N}\big(|v|^2/8+1\big)\,\mathrm{e}^{-|v|^2/8-1}\,\mathrm{d}v\\ &\leqslant C + \int_{\mathbb{R}^N}\big(1+|v|^2/8\big)f\,\mathrm{d}v.\end{aligned}$$

In the second line we have used that $f|\ln f|$ is decreasing for $f<1/e$. □

We have now the following result of global existence in time of solution of (48).

THEOREM 1. *Let $f^0 \in L^1(\mathbb{T}^N \times \mathbb{R}^N)$ be a nonnegative initial value with finite entropy*:

$$\int_{\mathbb{T}^N}\int_{\mathbb{R}^N} H(f^0, v)\,\mathrm{d}v\,\mathrm{d}x < \infty.$$

Then there exists an entropy solution $f \in L^\infty_{\mathrm{loc}}(0,\infty; L^1(\mathbb{T}^N \times \mathbb{R}^N))$ to (48).

The proof of this result can be found in [36]. It makes use of the so-called averaging lemmas giving some compactness on moments of f in v. See [1,53,52,78].

3.2. *Kinetic formulation of balance laws*

In this section, we introduce a nice kinetic theory which is not associated to a microscopic physical phenomena. The theory of kinetic formulation of conservation laws and balance laws has been introduced, mainly, for numerical purposes. Kaniel, in his pioneering work [62], introduced a nonphysical kinetic model to model the isentropic gas dynamic. The key idea was to develop equations involving local equilibrium function (or "Maxwellian functions") which are compactly supported in v (compared to the usual Gaussian for the Boltzmann equation). Brenier [25], and independently Giga and Myakawa [49], developed the numerical "transport-collapse" method to compute scalar conservation laws, based on this principle. (See also Brenier [26], for applications on systems.) This theory was later extensively developed, both for theoretical purpose (see Lions Perthame and Tadmor [68,67] but also [28,12,14,43,91,92]), and to construct efficient numerical schemes [24,23]. Note that the aim is not to compute the kinetic equation itself which is far more costly to compute, but rather, to take advantage of the kinetic structure to design effective macroscopic numerical schemes. The method has been extended also for balance laws involving extern sources [20,77,5,4,33,32]. In this context, the hydrodynamic limit is linked to the study of the validation of the numerical scheme. The study of the behavior near singularities can describe the consistence of those schemes in singular situations. For a review on the kinetic theory of conservation laws, see Perthame [76].

In the following we present a model for the multi-variable isentropic balance laws which can be found in Bouchut [22]. For $\gamma > 1$ we consider the following BGK kinetic equation associated to the system of balance laws of the isentropic gas dynamic (25) with the same γ.

$$\partial_t f + v \cdot \nabla_x f + F(x) \cdot \nabla_v f = Mf - f, \tag{50}$$

where the unknown is $f = f(t,x,v) \in \mathbb{R}$ with $t \in \mathbb{R}^+$, $x, v \in \mathbb{R}^N$. The force term $F : \mathbb{R}^N \to \mathbb{R}^N$ is given. The equilibrium function Mf is defined in the following way

$$Mf(t,x,v) = M\big(\rho(t,x), \rho u(t,x), v\big), \tag{51}$$

with

$$\rho(t,x) = \int_{\mathbb{R}^N} f(t,x,v)\,\mathrm{d}v,$$

$$\rho u(t,x) = \int_{\mathbb{R}^N} v f(t,x,v)\,\mathrm{d}v,$$

where the Maxwellian $M : \mathbb{R}^p \times \mathbb{R}^N \to \mathbb{R}$ is given by

$$M(\rho, \rho u, v) = \mathbf{1}_{|u-v|^N \leqslant c_n \rho} \quad \text{for } \gamma = \frac{N+2}{N}, \tag{52}$$

$$M(\rho, \rho u, v) = c\left(\frac{2\gamma}{\gamma-1}\rho^{\gamma-1} - |v-u|^2\right)_+^{d/2} \quad \text{else.} \tag{53}$$

The constants are given by

$$c_n = n/|\mathbb{S}_n|,$$
$$d = \frac{2}{\gamma-1} - n,$$
$$c = \left(\frac{2\gamma}{\gamma-1}\right)^{-1/(\gamma-1)} \frac{\Gamma(\gamma/(\gamma-1))}{\pi^{N/2}\Gamma(d/2+1)}.$$

We set $a(v) = (1, v)$. We have the following result (see Perthame [76]).

LEMMA 7. *The kinetic entropy is the following*:

$$H(f, v) = \frac{|v|^2}{2} f, \quad \textit{for } \gamma = \frac{N+2}{N},$$
$$H(f, v) = \frac{|v|^2}{2} f + \frac{1}{2c^{2/d}} \frac{f^{1+2/d}}{1+2/d} \quad \textit{else}.$$

Then for every function $f \in C_c^\infty(\mathbb{R}^N)$, $f > 0$, *we have*

$$\int_{\mathbb{R}^N} H\big(M(U, v), v\big)\, \mathrm{d}v \leqslant \int_{\mathbb{R}^N} H(f, v)\, \mathrm{d}v. \tag{54}$$

Moreover, (43), (44), *and* (45) *are fulfilled*, *with*:

$$D(f) = \int_{\mathbb{R}^N} H(Mf|f, v) + H(f, v) - H(Mf, v)\, \mathrm{d}v \geqslant 0,$$

where $H(Mf|f, v)$ *is the relative entropy with* v *fixed*, *namely*

$$H(Mf|f, v) = H(Mf, v) - H(f, v) - \frac{\partial H}{\partial f}(f, v)(Mf - f).$$

Hence (50) *is admissible*.

PROOF. Let us denote

$$U = (\rho, \rho u) = \int_{\mathbb{R}^N} a(v) f\, \mathrm{d}v.$$

Estimate (43) is fulfilled by construction of Mf since:

$$\int_{\mathbb{R}^N} a(v) M(U, v)\, \mathrm{d}v = U.$$

To get (44) it is enough to check that, from the definition of relative quantity:

$$\frac{\partial H}{\partial f}(Mf - f) = -\big(H(Mf|f, v) + H(f) - H(Mf)\big).$$

Since H is convex with respect to the variable f, $H(Mf|f, v) \geqslant 0$. Estimate (54) can be found in [76].

The estimate (45) is clearly fulfilled since all the term can be controlled by:

$$\int_{\mathbb{R}^N} \big[\big(1 + |v|^2\big) f + H(f, v)\big]\, \mathrm{d}v,$$

which is smaller than

$$\int_{\mathbb{R}^N} f + 2H(f, v)\, \mathrm{d}v. \qquad \square$$

We have now the following result of global existence in time of solution of (50) which can be found in [22].

THEOREM 2. *Let $f^0 \in L^1(\Omega \times \mathbb{R}^N)$ be a nonnegative initial value with finite entropy*:

$$\int_{\mathbb{T}^N}\int_{\mathbb{R}^N} H(f^0, v)\, \mathrm{d}v\, \mathrm{d}x < \infty.$$

Then there exists an entropy solution $f \in L^\infty_{\mathrm{loc}}(0, \infty; L^1(\Omega \times \mathbb{R}^N))$ to (50).

3.3. *Coupled system of Navier–Stokes Fokker–Planck equations*

We consider the following system of equations in $\Omega = \mathbb{T}^N$:

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v\big((u - v) f - \nabla_v f\big) = 0, \tag{55}$$

$$\partial_t \rho + \mathrm{div}_x(\rho u) = 0, \tag{56}$$

$$\partial_t(\rho u) + \mathrm{div}_x(\rho u \otimes u) + \nabla_x \rho^\gamma - \nu \Delta u = (J - nu), \tag{57}$$

where $n = \int f(x, v, t)\, \mathrm{d}v$ and $J = \int v f(x, v, t)\, \mathrm{d}v$.

This system of equations models the evolution of dispersed particles in a compressible fluid. This kind of system arises in a lot of industrial applications. One example is the analysis of sedimentation phenomenon, with applications in medicine, chemical engineering or waste water treatment (see [11,48,84,88]). Such systems are also used in the modeling of aerosols and sprays with applications, for instance, in the study of Diesel engines (see Williams [94,93]).

At the microscopic scale, the cloud of particles is described by its distribution function $f(x, v, t)$, solution to a Vlasov–Fokker–Planck equation. The fluid, on the other hand, is modeled by macroscopic quantities, namely its density $\rho(x, t) \geqslant 0$ and its velocity field $u(x, t) \in \mathbb{R}^N$. We assume that the fluid is compressible and isentropic, so that (ρ, u) solves the compressible Euler or Navier–Stokes system of equations. The fluid-particles interactions are modeled by a friction (or drag) force exerted by the fluid onto the particles. This force is assumed to be proportional to the relative velocity of the fluid and the particles:

$$F_d = \big(u(x,t) - v\big).$$

The right hand-side in the Euler equation takes into account the action of the cloud of particles on the fluid:

$$F_f = -\int F_d f \,\mathrm{d}v = \int \big(v - u(x,t)\big) f(x,v,t)\,\mathrm{d}v.$$

For the sake of simplicity, we assume that the pressure term is given by

$$p = \rho^\gamma,$$

though more general pressure terms could be taken into consideration.

This particular Vlasov–Navier–Stokes system of equations is used, for instance, in the modeling of reaction flows of sprays (see Williams [94,93]) and is at the basis of the code KIVA-II of the Los Alamos National Laboratory (see O'Rourke et al. [2] and Amsden [3]). We refer to the nice paper of Carrillo and Goudon [37] for a discussion on various modeling issues and stability properties of this system of equations. Mathematical studies on related model can be found in [60,61,16].

Although this system is not exactly of the form (42), the study of its hydrodynamic limit is very similar. For this reason we introduce it here.

We have the following global existence result which can be found in Mellet and Vasseur [74] in a more general setting dealing with boundary conditions. See also [36].

THEOREM 3. *Let* $f_0(x,v) \geqslant 0$, $\rho_0(x) \geqslant 0$ *and* $u_0(x)$ *be integrable and such that*:

$$\int_{\mathbb{T}^N} \left(\int_{\mathbb{R}^N} \left(\frac{|v|^2}{2} + \ln f^0 \right) f^0 \,\mathrm{d}v + \frac{\rho^0 |u^0|^2}{2} + \frac{\rho^{0\gamma}}{\gamma - 1} \right) \mathrm{d}x < \infty.$$

Assume moreover that $f_0 \in L^\infty(\mathbb{T}^N \times \mathbb{R}^N)$ *and that*

$$\nu > 0 \quad \textit{and} \quad \gamma > 3/2.$$

Then, there exists a weak solution (f, ρ, u) *of* (55)–(57) *defined globally in time. Moreover, this solution satisfies the usual entropy inequality*:

$$\begin{aligned} &\int_{\mathbb{T}^N} \mathcal{H}\big(f(t), \rho(t), u(t)\big)\,\mathrm{d}x + \int_0^t \int_{\mathbb{T}^N} D(f,u)\,\mathrm{d}x\,\mathrm{d}s + \nu \int_0^t \int_{\mathbb{T}^N} |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}s \\ &\quad \leqslant \int_{\mathbb{T}^N} \mathcal{H}(f_0, \rho_0, u_0)\,\mathrm{d}x, \end{aligned} \tag{58}$$

where:

$$\mathcal{H}(f,\rho,u) = \int_{\mathbb{R}^N} \left[\frac{|v|^2}{2} f + f \ln f \right] \mathrm{d}v + \rho \frac{|u|^2}{2} + \frac{1}{\gamma - 1}\rho^\gamma + \frac{1}{e},$$

$$D(f,u) = \int_{\mathbb{R}^N} \big|(u - v) f - \nabla_v f\big|^2 \frac{1}{f}\,\mathrm{d}v.$$

4. Hydrodynamic limits

Kinetic theory provides a more refined models, valid even when thermodynamical local equilibrium is not fulfilled. However, it is more complicated and far more costly to compute than the related macroscopic model. We investigate in this section the link between those two kinds of models.

4.1. *Compatibility conditions*

We consider a kinetic equation (42) with a convex entropy H. We say that this kinetic equation is consistent with the balance law (1) if there exists a constant $C_0 \geqslant 0$ such that for any $f \in C_c^\infty(\mathbb{R}^N)$, $f \geqslant 0$

$$\eta(U) = \inf\left(\mathcal{H}(f)\,\Big|\, \int_{\mathbb{R}^N} a(v) f \,\mathrm{d}v = U\right), \tag{59}$$

with

$$\mathcal{H}(f) = \int_{\mathbb{R}^N} H\big(f(v), v\big)\,\mathrm{d}v + C_0, \tag{60}$$

and there exists a constant C such that for any $f \in C_c^\infty(\mathbb{R}^N)$, $f \geqslant 0$

$$\left| A(U) - \int_{\mathbb{R}^N} v \otimes a(v) f \,\mathrm{d}v \right| \leqslant C\big(\sqrt{\mathcal{H}}\sqrt{D} + D\big), \tag{61}$$

$$\left| Q(U) - \int_{\mathbb{R}^N} \big(F(x)\cdot\nabla_v\big) a(v) f \,\mathrm{d}v \right| \leqslant C\big(\sqrt{\mathcal{H}}\sqrt{D} + D\big), \tag{62}$$

$$\left| \eta'(U)\cdot Q(U) - \int_{\mathbb{R}^N} (F\cdot\nabla_v)\left[\frac{\partial H}{\partial f}(f,v)\right] f \,\mathrm{d}v \right| \leqslant C\big(\sqrt{\mathcal{H}}\sqrt{D} + D\big), \tag{63}$$

where

$$U = \int_{\mathbb{R}^N} a(v) f(v)\,\mathrm{d}v. \tag{64}$$

This is a consistency property. It shows that, when the dissipation of kinetic entropy D is small, then the macroscopic flux quantities $\int v \otimes a(v) f \,\mathrm{d}v$ correspond to the fluid flux function taken at the conservative quantities $A(U)$. The condition (59) is a consistency property between the kinetic entropy functional and the fluid entropy functional. It requires that any kinetic functional f with conserved quantities a given U, $\int a(v) f \,\mathrm{d}v = U$, has an macroscopic entropy bigger than the entropy functional given at this U.

4.2. *Scaling*

We consider Eq. (42) at a scale where the collision operator $\mathcal{Q}$ is predominant. For this matter, we introduce a parameter $\varepsilon > 0$ and the related family of equation:

$$\partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon + F\cdot\nabla_v f_\varepsilon = \frac{\mathcal{Q}(f_\varepsilon, v)}{\varepsilon}, \tag{65}$$

$$f_\varepsilon(0,x,v) = f_\varepsilon^0(x,v). \tag{66}$$

PROPOSITION 6. *We consider a kinetic equation* (42). *We assume that it is either admissible, or admissible up to a constant with* $\Omega = \mathbb{T}^N$. *Let* $f_\varepsilon \in L^\infty_{\mathrm{loc}}(0,\infty; L^1(\Omega))$ *be a family of entropy solution to the rescaled equation* (65), (66) *with* $f_\varepsilon^0 \geqslant 0$ *such that* $\int f_\varepsilon^0 \,\mathrm{d}v\,\mathrm{d}x$ *and* $\iint H(f_\varepsilon^0, v)\,\mathrm{d}v\,\mathrm{d}x$ *are uniformly bounded. Then, for every* $T > 0$, *there exists* $C_T > 0$ *such that for every* $\varepsilon > 0$

$$\sup_{0\leqslant t\leqslant T} \int_\Omega \int_{\mathbb{R}^N} \big(f_\varepsilon + \big|H(f_\varepsilon, v)\big|\big)\,\mathrm{d}v\,\mathrm{d}x \leqslant C_T, \tag{67}$$

$$\int_0^T \int_\Omega D(f_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \leqslant C_T\varepsilon. \tag{68}$$

PROOF. Using (46) and (47) we find

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \int_{\mathbb{R}^N} \big(H(f_\varepsilon, v) + \alpha f_\varepsilon\big)\,\mathrm{d}v\,\mathrm{d}x + \int \frac{D(f_\varepsilon)}{\varepsilon}\,\mathrm{d}x \\
&\quad\leqslant C\bigg(\int_\Omega \int_{\mathbb{R}^N} \big(H(f_\varepsilon, v) + \alpha f_\varepsilon\big)\,\mathrm{d}v\,\mathrm{d}x + 1\bigg).
\end{aligned}$$

If (42) is admissible, this is verified even without the term $+1$. If it is admissible up to a constant, this is still true with $\Omega = \mathbb{T}^N$. Using Gronwall's argument gives that for every $0 \leqslant t \leqslant T$

$$\begin{aligned}
\iint \big(H(f_\varepsilon, v) + \alpha f_\varepsilon\big)\,\mathrm{d}v\,\mathrm{d}x &\leqslant \bigg(\int_\Omega \int_{\mathbb{R}^N} \big(H(f_\varepsilon^0, v) + \alpha f_\varepsilon^0\big)\,\mathrm{d}v\,\mathrm{d}x\bigg)\mathrm{e}^{Ct} + \frac{\mathrm{e}^{Ct}}{C} \\
&\leqslant C_T.
\end{aligned}$$

Then

$$\begin{aligned}
\int_0^t \int_\Omega \frac{D(f_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t}{\varepsilon} &\leqslant \bigg[\int_\Omega \int_{\mathbb{R}^N} \big(H(f_\varepsilon, v) + \alpha f_\varepsilon\big)\,\mathrm{d}v\,\mathrm{d}x\bigg]_t^0 \\
&\quad + C \int_0^t \int_\Omega \int_{\mathbb{R}^N} \big(H(f_\varepsilon, v) + \alpha f_\varepsilon\big)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s + Ct \leqslant C_T.
\end{aligned}$$

Finally, from (46), we have

$$\iint \big|H(f, v)\big|\,\mathrm{d}v\,\mathrm{d}x \leqslant C\bigg(\iint \big(H(f_\varepsilon, v) + \alpha f_\varepsilon\big)\,\mathrm{d}v\,\mathrm{d}x + 1\bigg) \leqslant C_T. \qquad \square$$

4.3. *Asymptotic limit*

We show in this section, how the previous formalism can be used to obtain asymptotic limits. The following theorem is mainly due to Berthelin and al [15].

THEOREM 4. *Let* (42) *be a kinetic equation, either admissible or admissible up to a constant, with convex kinetic entropy* H, *and consistent with a* (*at least*) *locally admissible* *balance law* (1) *on* $\mathcal{V}$. *If either* (42) *is only admissible only up to a constant, or* (1) *is only locally admissible on* $\mathcal{V}$, *then we consider only the case* $\Omega = \mathbb{T}^N$. *We consider a family of entropy solutions* f_ε *to the rescaled kinetic equations* (65). *We set*

$$U_\varepsilon(t,x) = \int_{\mathbb{R}^N} a(v) f_\varepsilon(t,x,v)\,\mathrm{d}v,$$

and assume that for every t, x

$$U_\varepsilon(t,x) \in \mathcal{U}.$$

We assume in addition the convergence of initial data, that is there exists $U^0 \in L^1(\Omega)$ *such that*

$$U_\varepsilon^0 \xrightarrow{L^1(\Omega)} U^0, \tag{69}$$

and

$$\int_\Omega \left| \int_{\mathbb{R}^N} H\big(f_\varepsilon^0, v\big)\,\mathrm{d}v - \eta\big(U^0\big) \right| \mathrm{d}x \to 0, \tag{70}$$

when ε *tends to* 0. *Then, there exists* U, *dissipative solution to* (1) *on* $(0,\infty)\times\Omega$ *with initial value* U^0 *such that, up to a subsequence,* U_ε *converges weakly to* U *in* $L^p_{\mathrm{loc}}(0,\infty; L^1_{\mathrm{loc}}(\Omega))$ *for any* $1 \leqslant p < \infty$. *If* $\Phi_i(y) \geqslant y^{\gamma_i}$, *then the* $U_{\varepsilon,i}$ *converges weakly to* U_i *in* $L^p_{\mathrm{loc}}(0,\infty; L^q(\Omega))$ *for* $1 \leqslant p < \infty$ *and* $1 \leqslant q < \gamma_i$. *In particular, we have the following special cases.*

- *If* U *is a dissipative test function on* $[0,T)$, *and there exists* $C_0 > 0$ *such that*

$$\int_\Omega \eta\big(U_\varepsilon^0|U^0\big)\,\mathrm{d}x + \int_\Omega \left| \int_{\mathbb{R}^N} H\big(f_\varepsilon^0, v\big)\,\mathrm{d}v - \eta\big(U_\varepsilon^0\big) \right| \mathrm{d}x \leqslant C_0\sqrt{\varepsilon}, \tag{71}$$

  then, for every $t < T$, *there exists a constant* C_t *such that*

$$\int_{\mathbb{R}^N} \eta(U_\varepsilon|U)(s,x)\,\mathrm{d}x \leqslant C_t\sqrt{\varepsilon} \quad \textit{for any } s \in [0,t]. \tag{72}$$

  Moreover, the whole family U_ε *converges strongly to* U *in* $C^0(0,\overline{T}; L^1_{\mathrm{loc}})$, *for* $\overline{T} < T$. *If* $\Phi_i(y) \geqslant |y|^{\gamma_i}$ *for* $y > 0$, *then the component* $U_{\varepsilon,i}$ *converges strongly to* U_i *in* $C^0(0,\overline{T}; L^q_{\mathrm{loc}}(\Omega))$ *for* $1 \leqslant p < \infty$, $1 \leqslant q < \gamma_i$ *and* $\overline{T} < T$.
- *If* U^0 *is a initial value verifying the hypothesis of Proposition* 4 (*discontinuities leading to rarefaction waves*) *then the whole family* U_ε *converges to* U^* *on* $(0,T)$.

PROOF OF THEOREM 4. Thanks to Proposition 6 and (59), $\eta(U_\varepsilon)$ is uniformly bounded in $L^\infty(0,T; L^1(\Omega))$ for every $T > 0$. Since (1) is admissible on $\mathcal{V}$, (11) implies that, up to a subsequence, U_ε converges weakly to a U in $L^p_{\mathrm{loc}}(L^1_{\mathrm{loc}})$, where $\eta(U) + |U|$ is in $L^\infty_{\mathrm{loc}}(L^1)$. Multiplying the kinetic equation by $a(v)$ and integrating with respect to v, we find that U_ε is solution to (20) with

$$A_\varepsilon = \int_{\mathbb{R}^N} v \times a(v) f_\varepsilon\,\mathrm{d}v,$$

$$Q_\varepsilon = \int_{\mathbb{R}^N} (F \cdot \nabla_v) a(v) f_\varepsilon \, \mathrm{d}v.$$

Multiplying the kinetic equation by $\frac{\partial H}{\partial f}(f_\varepsilon, v)$ and integrating with respect to v, we find that U_ε verifies (21) with

$$\eta_\varepsilon = \int_{\mathbb{R}^N} H(f_\varepsilon, v) \, \mathrm{d}v,$$

$$G_\varepsilon = \int_{\mathbb{R}^N} v H(f_\varepsilon, v) \, \mathrm{d}v,$$

$$R_\varepsilon = \int_{\mathbb{R}^N} (F \cdot \nabla_v) \left[\frac{\partial H}{\partial f}(f_\varepsilon, v) \right] f_\varepsilon \, \mathrm{d}v.$$

Estimate (23) follows (59). Proposition 6 gives that

$$\int_0^{\overline{T}} \int_\Omega D_\varepsilon \, \mathrm{d}x \, \mathrm{d}t \leqslant C_{\overline{T}} \varepsilon$$

converges to 0. Since (42) is consistent with (1), we have

$$\begin{aligned} &\int_0^{\overline{T}} \int_\Omega \mathcal{R}_\varepsilon \, \mathrm{d}x \, \mathrm{d}t \\ &\quad \leqslant C \left(\sqrt{\int_0^{\overline{T}} \int_\Omega \mathcal{H}(f_\varepsilon) \, \mathrm{d}x \, \mathrm{d}t} \sqrt{\int_0^{\overline{T}} \int_\Omega D_\varepsilon \, \mathrm{d}x \, \mathrm{d}t} + \int_0^{\overline{T}} \int_\Omega D_\varepsilon \, \mathrm{d}x \, \mathrm{d}t \right) \\ &\quad \leqslant C_{\overline{T}} \sqrt{\varepsilon}. \end{aligned}$$

Hence, Proposition 3 gives the result.

If U is an dissipative test function on $[0, T)$, using Proposition 3 with $V = U$, we have, thanks to (61), (62):

$$\begin{aligned} &\int_\Omega \big(\eta_\varepsilon - \eta(U_\varepsilon) \big)(t) \, \mathrm{d}x + \int_\Omega \eta(U_\varepsilon | U)(t) \, \mathrm{d}x \\ &\quad \leqslant C_0 \sqrt{\varepsilon} \exp\left(\int_0^t \sigma_U(\tau) \, \mathrm{d}\tau \right) \\ &\qquad + C \int_0^t \exp\left(\int_\tau^t \sigma_U(s) \, \mathrm{d}s \right) \left\{ \sqrt{\mathcal{H}_\varepsilon} \sqrt{D_\varepsilon} + D_\varepsilon \right\} \mathrm{d}x \, \mathrm{d}t \\ &\quad \leqslant C_T \sqrt{\varepsilon}. \end{aligned}$$

Hence, $\eta(U_\varepsilon | U)$ converges to 0 in $C^0(0, \overline{T}; L^1(\Omega))$, for $\overline{T} < T$. Thanks to (11), U_ε converges strongly to U in $C^0(0, T; L^1_{\mathrm{loc}}(\Omega))$. By uniqueness of the limit, the whole family is converging.

If U^0 verifies the hypothesis of Proposition 4, then the uniqueness result of this proposition gives the result. □

4.4. *Hydrodynamic limit from Fokker–Planck equation to isothermal system of gas*

In this section, we study the convergence from the rescaled Fokker–Planck kinetic equation to the isothermal system (29). We study the equation for $x \in \Omega = \mathbb{T}^N$. The result can be found in [15]. The rescaled kinetic equation reads

$$\begin{aligned} &\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + F \cdot \nabla_v f_\varepsilon + \frac{\operatorname{div}_v((u_\varepsilon - v) f_\varepsilon - \nabla_v f_\varepsilon)}{\varepsilon} = 0, \\ &\quad x \in \mathbb{T}^N, v \in \mathbb{R}^N, \\ &f_\varepsilon(0, x, v) = f_\varepsilon^0(x, v), \quad x \in \mathbb{T}^N,\ v \in \mathbb{R}^N, \end{aligned} \tag{73}$$

where u_ε is defined for t, x such that $\int f_\varepsilon(t, x, v)\,\mathrm{d}v > 0$ by

$$u_\varepsilon(t, x) \int_{\mathbb{R}^N} f_\varepsilon(t, x, v)\,\mathrm{d}v = \int_{\mathbb{R}^N} v f_\varepsilon(t, x, v)\,\mathrm{d}v.$$

We have the following results.

THEOREM 5. *Let F be in $C^2(\mathbb{T}^N) \cap L^\infty(\mathbb{T}^N)$. Let f_ε be a family of entropy solutions to (73). We set*

$$U_\varepsilon(t, x) = \int_{\mathbb{R}^N} (1, v) f_\varepsilon(t, x, v)\,\mathrm{d}v = (\rho_\varepsilon, \rho_\varepsilon u_\varepsilon).$$

We assume the convergence of initial data, that is

$$\left(\int_{\mathbb{R}^N} \big(f_\varepsilon^0, v f_\varepsilon^0, H(f_\varepsilon^0)\big)\,\mathrm{d}v \right) \xrightarrow{L^1(\mathbb{T}^N)} \big(\rho^0, \rho^0 u^0, \rho^0 {u^0}^2/2 + \rho^0 \ln\big(\rho^0/(2\pi^{N/2})\big)\big),$$

when ε tends to 0. Then, there exists (ρ, u), dissipative solution on $(0, \infty) \times \Omega$ to (29) with initial value $(\rho^0, \rho^0 u^0)$ such that ρ_ε converges weakly to ρ in $L^p_{\mathrm{loc}}(0, \infty; L^1_{\mathrm{loc}}(\mathbb{T}^N))$ for any $1 \leqslant p < \infty$ and $\rho_\varepsilon u_\varepsilon$ converges weakly to ρu in $L^p_{\mathrm{loc}}(0, \infty; L^1_{\mathrm{loc}}(\mathbb{T}^N))$ for any $1 \leqslant p < \infty$. In particular, we have the following special cases.

- *If there is $T > 0$ such that $U = (\rho, \rho u) \in W^{1,\infty}([0, \overline{T}] \times \mathbb{T}^N) \cap L^\infty(0, \overline{T}; L^1)$, with $\rho > 0$ and $u \in W^{1,\infty}([0, \overline{T}] \times \mathbb{T}^N)$, for every $\overline{T} < T$, then the whole family $(\rho_\varepsilon, \rho_\varepsilon u_\varepsilon)$ converges strongly in $C^0(0, \overline{T}; L^1_{\mathrm{loc}}(\mathbb{T}^N))$ to $(\rho, \rho u)$ for every $\overline{T} < T$.*
- *If U^0 is a discontinuous initial value verifying the hypothesis of Proposition 4 (rarefaction wave), or V_ψ defined in Section 2.8.2, then the whole sequence U_ε converges to U^* on $[0, T)$.*

The only technical point in the rest of the proof, is to show that (48) is consistent with (29). We can then use Theorem 4 since (48) and (29) are admissible. We have $a(v) = (1, v)$ and

$$\mathcal{Q}(f, v) = \operatorname{div}_v\big((v - u) f + \nabla_v f\big), \qquad (\rho, \rho u) = \int_{\mathbb{R}^N} a(v) f\,\mathrm{d}v.$$

The functional $\eta(U) = \rho|u|^2/2 + h(\rho) + 1/e$ is a strictly convex entropy, since it is the sum of the strictly convex entropy (30) and a linear function of U. Estimate (59) is verified since

$$\eta(U) = \int_{\mathbb{R}^N} H\left(\frac{\rho}{(2\pi)^{N/2}} e^{|v-u|^2/2}, v\right) \mathrm{d}v \leqslant \int_{\mathbb{R}^N} H(f, v)\, \mathrm{d}v.$$

The estimates (62) and (63) are straightforward since

$$\partial_f H(f, v) = \frac{|v|^2}{2} + 1 + \ln f.$$

Indeed

$$\begin{aligned}\int_{\mathbb{R}^N} \partial_f H(f, v) F(x) \cdot \nabla_v f \,\mathrm{d}v &= -\rho u F(x)\\ &= -\eta'(\rho, \rho u) Q(\rho, \rho u).\end{aligned}$$

To show (61), we have to estimate

$$\left| A(U) - \int_{\mathbb{R}^N} v \otimes a(v) f \,\mathrm{d}v \right|.$$

In this case, it is quite easy. The first component is zero. The second one is

$$E_2 = \left| \rho u \otimes u + \rho I - \int_{\mathbb{R}^N} v \otimes v f \,\mathrm{d}v \right|,$$

which can be rewritten as

$$E_2 = \left| \int_{\mathbb{R}^N} v \otimes \left[(u - v) f - \nabla_v f \right] \mathrm{d}v \right|,$$

since

$$\int_{\mathbb{R}^N} v \otimes \nabla_v f \,\mathrm{d}v = - \int_{\mathbb{R}^N} f \,\mathrm{d}v.$$

Thus we get

$$\begin{aligned}&\left| A_2(U) - \int_{\mathbb{R}^N} v \otimes a_2(v) f \,\mathrm{d}v \right|\\ &\quad \leqslant \left(\int_{\mathbb{R}^N} |v|^2 f \,\mathrm{d}v \right)^{1/2} \left(\int_{\mathbb{R}^N} \frac{((v - u) f + \nabla_v f)^2}{f} \,\mathrm{d}v \right)^{1/2},\end{aligned}$$

which concludes the proof.

4.5. *Hydrodynamic limits to isentropic gas dynamics*

As a consequence of the Theorem 4, we show the asymptotic limit of the rescaled BGK models to the system of isentropic gas dynamics. The results presented here are due to Berthelin and al [15]. The rescaled kinetic equation from (50) for $1 < \gamma \leqslant N/(N + 2)$ is

$$\begin{aligned}&\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + F(x) \cdot \nabla_v f_\varepsilon = \frac{M f_\varepsilon - f_\varepsilon}{\varepsilon}, \quad t > 0,\ x \in \mathbb{R}^N,\ v \in \mathbb{R}^N,\\ &f_\varepsilon(0, \cdot, \cdot) = f_\varepsilon^0, \quad x \in \mathbb{R}^N,\ v \in \mathbb{R}^N.\end{aligned} \tag{74}$$

THEOREM 6. *Let F be in $C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Let f_ε be a family of entropy solutions to (74). We set*

$$U_\varepsilon(t,x) = \int_{\mathbb{R}^N} (1,v) f_\varepsilon(t,x,v)\,\mathrm{d}v = (\rho_\varepsilon, \rho_\varepsilon u_\varepsilon).$$

We assume the convergence of initial data, that is

$$\left(\int_{\mathbb{R}^N} \bigl(f_\varepsilon^0, v f_\varepsilon^0, H(f_\varepsilon^0)\bigr)\,\mathrm{d}v \right) \stackrel{L^1(\mathbb{R}^N)}{\longrightarrow} \bigl(\rho^0, \rho^0 u^0, \rho^0 {u^0}^2 + {\rho^0}^\gamma/(\gamma-1)\bigr),$$

when ε tends to 0. Then, there exists (ρ, u), dissipative solution on $(0,\infty)\times\mathbb{R}^N$ to (25) with initial value $(\rho^0, \rho^0 u^0)$ such that ρ_ε converges weakly to ρ in $L^p_{\mathrm{loc}}(0,\infty; L^q_{\mathrm{loc}}(\mathbb{R}^N))$ for any $1 \leqslant p < \infty$ and $1 \leqslant q < \gamma$, and $\rho_\varepsilon u_\varepsilon$ converges weakly to ρu in $L^p_{\mathrm{loc}}(0,\infty; L^q_{\mathrm{loc}}(\mathbb{R}^N))$ for any $1 \leqslant p < \infty$ and $1 \leqslant q < 2\gamma/(\gamma+1)$. In particular we have the special following cases:

- *If there is $T > 0$ such that $U = (\rho, \rho u) \in W^{1,\infty}([0,\overline{T}] \times \mathbb{R}^N) \cap L^\infty(0,\overline{T}; L^1)$, with $\rho > 0$ and $u \in W^{1,\infty}([0,\overline{T}] \times \mathbb{R}^N)$, for every $\overline{T} < T$, then the whole family ρ_ε converges strongly in $C^0(0,\overline{T}; L^p_{\mathrm{loc}}(\mathbb{R}^N))$ to ρ for every $1 \leqslant p < \gamma$ and the whole family $\rho_\varepsilon u_\varepsilon$ converges strongly to ρu in $C^0(0,\overline{T}; L^q_{\mathrm{loc}}(\mathbb{R}^N))$ for every $1 \leqslant q < 2\gamma/(\gamma+1)$ and $\overline{T} < T$.*
- *If U^0 is a discontinuous initial value verifying the hypothesis of Proposition 4 (rarefaction wave), or the steady state V_ψ defined in Section 2.8.2, then the whole sequence U_ε converges to U^* on $[0,T)$.*

The only difficulty to apply Theorem 4 is to show that (50) is consistent with (25), namely:

PROPOSITION 7. *For every nonnegative $f \in C_c^\infty(\mathbb{R}^N)$ we have*

$$\left| \int_{\mathbb{R}^N} \bigl(v \otimes a(v)\bigr)(f - Mf)\,\mathrm{d}v \right| \leqslant C(\sqrt{\mathcal{H}}\sqrt{\mathcal{D}} + \mathcal{D}). \tag{75}$$

In particular (50) is consistent to (25).

We postpone the proof of this proposition to prove Theorem 6.

PROOF OF THEOREM 6. The system (25) is admissible on $\mathcal{V} = (0,\infty)\times\mathbb{R}^N$, $\mathcal{U} = \mathcal{V} \cup (0,0)$. Thanks to Lemma 7, (50) is admissible. Thanks to Proposition 7, (50) is consistent with (25). Hence, Theorem 6 is a direct consequence of Theorem 4, since

$$\Phi_\rho(y) = y^\gamma, \qquad \Phi_{\rho u}(y) = y^{2\gamma/(\gamma+1)}.$$

Indeed

$$|\rho u|^{\frac{2\gamma}{\gamma+1}} \leqslant (\sqrt{\rho})^{2\gamma}\bigl(\sqrt{\rho}\,|u|\bigr)^2. \qquad \square$$

Let us first show that (75) implies that (50) is consistent with (25). Estimate (59) comes from (54). Since

$$A(U) = \int_{\mathbb{R}^N} v \otimes a(v) Mf \, \mathrm{d}v,$$

and

$$Q(U) = \int_{\mathbb{R}^N} \big(F(x) \cdot \nabla_v\big) a(v) Mf \, \mathrm{d}v,$$

it gives

$$\left| A(U) - \int_{\mathbb{R}^N} v \otimes a(v) f \, \mathrm{d}v \right| = \left| \int_{\mathbb{R}^N} v \otimes a(v)(Mf - f) \, \mathrm{d}v \right|, \tag{76}$$

$$\left| Q(U, x) - \int_{\mathbb{R}^N} \big(F(x) \cdot \nabla_v\big) a(v) f \, \mathrm{d}v \right| = \left| \int_{\mathbb{R}^N} F(x) \nabla_v a(v)(f - Mf) \, \mathrm{d}v \right|, \tag{77}$$

and

$$\left| \eta'(U) Q(U) - \int F(x) \cdot \nabla_v \left[\frac{\partial H}{\partial f}(f, v) \right] f \, \mathrm{d}v \right| = 0.$$

We have

$$\int_{\mathbb{R}^N} \partial_{v_i} a_j(v)(f - Mf) \, \mathrm{d}v = \delta_{i+1,j} \int_{\mathbb{R}^N} (f - Mf) \, \mathrm{d}v = 0,$$

thus (62) and (63) are satisfied. Finally (75) and (76) give (61). It remains to show (75). We will consider only the simpler case $\gamma = (N+2)/N$. For the full range we refer to [15].

Case $\gamma = (N+2)/N$. We have

$$D(f) = \int_{\mathbb{R}^N} |v|^2 (f - Mf) \, \mathrm{d}v.$$

But we need to control

$$\left| \int_{\mathbb{R}^N} v \otimes a(v)(Mf - f) \, \mathrm{d}v \right|,$$

which is more delicate. We set $a_1(v) = 1$ and $a_2(v) = v$ such that $a = (a_1, a_2)$. Similarly, we define $A_1(U) = \rho u$ and $A_2(U) = \rho u \otimes u + I\rho^\gamma$. Since

$$A(U) = \int_{\mathbb{R}^N} v \otimes a(v) Mf \, \mathrm{d}v,$$

the first component of $|\int_{\mathbb{R}^N} v \otimes a(v)(Mf - f) \, \mathrm{d}v|$ is still zero here. Now we have

$$\begin{aligned} \left| A_2(U) - \int_{\mathbb{R}^N} v \otimes v f \, \mathrm{d}v \right| &= \left| \int_{\mathbb{R}^N} (v - u) \otimes (v - u)(Mf - f) \, \mathrm{d}v \right| \\ &\leqslant \int_{\mathbb{R}^N} |v - u|^2 |Mf - f| \, \mathrm{d}v. \end{aligned}$$

The first equality uses

$$\int a(v)[Mf - f]\,\mathrm{d}v = 0. \tag{78}$$

Thus to control the second component, we want to show that

$$\int_{\mathbb{R}^N} |v-u|^2 |Mf - f|\,\mathrm{d}v$$

can be controlled (at least for bounded entropy) by the dissipation of entropy

$$\int_{\mathbb{R}^N} |v|^2 (f - Mf)\,\mathrm{d}v.$$

For $\gamma = (N+2)/N$, Proposition 7 is then a consequence of the following result.

PROPOSITION 8. *For every $f \in L^1(\mathbb{R}^N)$ verifying $0 \leqslant f \leqslant 1$, and every $u \in \mathbb{R}^N$ we denote*

$$\begin{aligned}
\rho &= \int_{\mathbb{R}^N} f(v)\,\mathrm{d}v,\\
F &= \int_{\mathbb{R}^N} |v-u|^2 \big| f(v) - M(\rho,u,v)\big|\,\mathrm{d}v,\\
D &= \int_{\mathbb{R}^N} |v|^2 \big(f(v) - M(\rho,u,v)\big)\,\mathrm{d}v.
\end{aligned}$$

Then there exists a constant C_N such that, for every $f \in L^1(\mathbb{R}^N)$ verifying $0 \leqslant f \leqslant 1$,

$$\begin{aligned}
F &\leqslant C_N\big(\rho^{N+2/(2N)}\sqrt{D} + D\big)\\
&\leqslant C_N\big(\sqrt{\mathcal{H}}\sqrt{D} + D\big).
\end{aligned}$$

To prove this result, we first introduce some notations and prove preliminary results. Notice that, thanks to (78),

$$D = \int_{\mathbb{R}^N} |v-u|^2 \big(f(v) - M(\rho,u,v)\big)\,\mathrm{d}v.$$

Then changing v by $v+u$ if necessary, we see that we can restrict ourself to the case $u=0$. We first reduce the problem to a one dimensional problem. We introduce the following quantities:

$$\begin{aligned}
\bar{f}(r) &= \frac{1}{|\mathbb{S}_N|}\int_{\mathbb{S}_N} f(r\sigma)\,\mathrm{d}\sigma,\\
\overline{M}(r) &= \frac{1}{|\mathbb{S}_N|}\int_{\mathbb{S}_N} M(\rho,0,r\sigma)\,\mathrm{d}\sigma = \mathbb{1}_{\{r^N \leqslant c_N\rho\}}(r).
\end{aligned}$$

Since the integral of f is equal to the integral of $M(\rho,0,\cdot)$, we have

$$\int_0^\infty r^{N-1}\bar{f}(r)\,\mathrm{d}r = \int_0^\infty r^{N-1}\overline{M}(r)\,\mathrm{d}r. \tag{79}$$

We denote $r_1 = (c_N \rho)^{\frac{1}{N}}$, we have

$$
\begin{aligned}
F &= |\mathbb{S}_N| \int_0^\infty r^{N+1} \big| \bar{f}(r) - \overline{M}(r) \big| \, \mathrm{d}r \\
&= |\mathbb{S}_N| \bigg(\int_0^{r_1} r^{N+1} \big(1 - \bar{f}(r)\big) \, \mathrm{d}r + \int_{r_1}^\infty r^{N+1} \bar{f}(r) \, \mathrm{d}r \bigg), \\
D &= |\mathbb{S}_N| \int_0^\infty r^{N+1} \big(\bar{f}(r) - \overline{M}(r) \big) \, \mathrm{d}r \\
&= |\mathbb{S}_N| \bigg(- \int_0^{r_1} r^{N+1} \big(1 - \bar{f}(r)\big) \, \mathrm{d}r + \int_{r_1}^\infty r^{N+1} \bar{f}(r) \, \mathrm{d}r \bigg).
\end{aligned}
$$

We define in addition

$$
M = \int_0^{r_1} r^{N-1} \big(1 - \bar{f}(r)\big) \, \mathrm{d}r = \int_{r_1}^\infty r^{N-1} \bar{f}(r) \, \mathrm{d}r,
$$

the last equality comes from (79) and $\overline{M}(r) = \mathbb{1}_{\{r \leqslant r_1\}}(r)$. We have to do a different treatment for values close to r_1 and far from this value. For this purpose we consider $r_2 > r_1$ a new number which will be fixed later on. Then we denote

$$
\begin{aligned}
M_1 &= \int_{r_1}^{r_2} r^{N-1} \bar{f}(r) \, \mathrm{d}r, \\
M_2 &= \int_{r_2}^\infty r^{N-1} \bar{f}(r) \, \mathrm{d}r.
\end{aligned}
$$

We have $M = M_1 + M_2$. Then we define $0 < r_0 < r_1$ (in a unique way when r_2 is chosen) in the following way

$$
M_1 = \int_{r_0}^{r_1} r^{N-1} \big(1 - \bar{f}(r)\big) \, \mathrm{d}r.
$$

Then, from the definition to M and since M is the sum of M_1 and M_2, we have

$$
M_2 = \int_0^{r_0} r^{N-1} \big(1 - \bar{f}(r)\big) \, \mathrm{d}r.
$$

In the same way we define F_1, F_2, D_1, D_2 in the following way

$$
\begin{aligned}
F_1 &= \int_{r_0}^{r_2} r^{N+1} \big| \bar{f}(r) - \overline{M}(r) \big| \, \mathrm{d}r \\
&= \int_{r_0}^{r_1} r^{N+1} \big(1 - \bar{f}(r)\big) \, \mathrm{d}r + \int_{r_1}^{r_2} r^{N+1} \bar{f}(r) \, \mathrm{d}r, \\
F_2 &= \int_0^{r_0} r^{N+1} \big| \bar{f}(r) - \overline{M}(r) \big| \, \mathrm{d}r + \int_{r_2}^\infty r^{N+1} \big| \bar{f}(r) - \overline{M}(r) \big| \, \mathrm{d}r \\
&= \int_0^{r_0} r^{N+1} \big(1 - \bar{f}(r)\big) \, \mathrm{d}r + \int_{r_2}^\infty r^{N+1} \bar{f}(r) \, \mathrm{d}r, \\
D_1 &= \int_{r_0}^{r_2} r^{N+1} \big(\bar{f}(r) - \overline{M}(r) \big) \, \mathrm{d}r
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_{r_0}^{r_1} r^{N+1}\big(1-\bar f(r)\big)\,\mathrm{d}r + \int_{r_1}^{r_2} r^{N+1}\bar f(r)\,\mathrm{d}r,\\
D_2 &= \int_0^{r_0} r^{N+1}\big(\bar f(r)-\overline{M}(r)\big)\,\mathrm{d}r + \int_{r_2}^{\infty} r^{N+1}\big(\bar f(r)-\overline{M}(r)\big)\,\mathrm{d}r\\
&= -\int_0^{r_0} r^{N+1}\big(1-\bar f(r)\big)\,\mathrm{d}r + \int_{r_2}^{\infty} r^{N+1}\bar f(r)\,\mathrm{d}r.
\end{aligned}
$$

Notice that F_1, F_2, M_1, M_2 are nonnegative (as integrals of nonnegative functions) and verify

$$
\begin{aligned}
&M = M_1 + M_2,\\
&F = F_1 + F_2,\\
&D = D_1 + D_2.
\end{aligned}
$$

We can show, in addition, that D_1 and D_2 are nonnegative too.

LEMMA 8. *We have*

$$
D_1 \geqslant 0, \qquad D_2 \geqslant 0.
$$

PROOF. We show the result for D_1 (the proof is similar for D_2). We have

$$
\begin{aligned}
&\int_{r_1}^{r_2} r^{N+1}\bar f(r)\,\mathrm{d}r = \int_{r_1}^{r_2} r^2\big(r^{N-1}\bar f(r)\big)\,\mathrm{d}r \geqslant r_1^2 M_1,\\
&\int_{r_0}^{r_1} r^{N+1}\bar f(r)\,\mathrm{d}r = \int_{r_0}^{r_1} r^2\big(r^{N-1}\bar f(r)\big)\,\mathrm{d}r \leqslant r_1^2 M_1.
\end{aligned}
$$

Since D_1 is the difference of those two terms we find that D_1 is nonnegative. □

We first consider the values far from r_1.

LEMMA 9. *We can dominate F_2 by D_2 in the following way*:

$$
F_2 \leqslant D_2\left(\frac{r_1^2+r_2^2}{r_2^2-r_1^2}\right).
$$

PROOF. We have

$$
\begin{aligned}
\int_{r_2}^{\infty} r^{N+1}\bar f(r)\,\mathrm{d}r &\geqslant r_2^2 M_2\\
&\geqslant r_2^2\frac{1}{r_0^2}\int_0^{r_0} r^{N+1}\big(1-\bar f(r)\big)\,\mathrm{d}r\\
&\geqslant \frac{r_2^2}{r_1^2}\int_0^{r_0} r^{N+1}\big(1-\bar f(r)\big)\,\mathrm{d}r.
\end{aligned}
$$

Hence we have

$$D_2 \geqslant \left(\frac{r_2^2}{r_1^2} - 1\right) \int_0^{r_0} r^{N+1}\bigl(1 - \bar{f}(r)\bigr)\,\mathrm{d}r.$$

But F_2 can be expressed in the following way

$$F_2 = D_2 + 2\int_0^{r_0} r^{N+1}\bigl(1 - \bar{f}(r)\bigr)\,\mathrm{d}r.$$

Those two expressions lead to

$$F_2 \leqslant D_2\left(\frac{r_1^2 + r_2^2}{r_2^2 - r_1^2}\right). \qquad \square$$

We consider now the values close to r_1.

LEMMA 10. *There exists $a > 0$ and a constant C_N depending only on N such that if $|r_2 - r_1| \leqslant a r_1$ then*

$$F_1 \leqslant C_N a^2 \rho^{(N-2)/2N} \sqrt{D_1}.$$

PROOF. We split the proof in several parts.

(i) *Minimization of the entropy dissipation*. We define α and β such that

$$M_1 = \int_{r_1}^{\beta} r^{N-1}\,\mathrm{d}r = \int_{\alpha}^{r_1} r^{N-1}\,\mathrm{d}r.$$

From the definition of M_1, notice that $\beta \leqslant r_2$. In the same way we have $\alpha \geqslant r_0$. We want to show that

$$D_1 \geqslant \int_{r_1}^{\beta} r^{N+1}\,\mathrm{d}r - \int_{\alpha}^{r_1} r^{N+1}\,\mathrm{d}r.$$

First we calculate

$$\begin{aligned}
&\int_{r_1}^{r_2} r^{N+1}\bar{f}(r)\,\mathrm{d}r - \int_{r_1}^{\beta} r^{N+1}\,\mathrm{d}r \\
&\quad = \int_{r_1}^{\beta} r^2\bigl[r^{N-1}(\bar{f}(r) - 1)\bigr]\,\mathrm{d}r + \int_{\beta}^{r_2} r^2\bigl[r^{N-1}\bar{f}(r)\bigr]\,\mathrm{d}r \\
&\quad = \int_{\beta}^{r_2} r^2\bigl[r^{N-1}\bar{f}(r)\bigr]\,\mathrm{d}r - \int_{r_1}^{\beta} r^2\bigl[r^{N-1}\bigl(1 - \bar{f}(r)\bigr)\bigr]\,\mathrm{d}r \\
&\quad \geqslant \beta^2\left[\int_{\beta}^{r_2} r^{N-1}\bar{f}(r)\,\mathrm{d}r - \int_{r_1}^{\beta} r^{N-1}\bigl(1 - \bar{f}(r)\bigr)\,\mathrm{d}r\right] \\
&\quad \geqslant \beta^2(M_1 - M_1) = 0.
\end{aligned}$$

In the same way we calculate

$$\int_{r_0}^{r_1} r^{N+1}\bigl(\bar{f}(r) - 1\bigr)\,\mathrm{d}r + \int_{\alpha}^{r_1} r^{N+1}\,\mathrm{d}r$$

$$
\begin{aligned}
&= \int_{r_0}^{\alpha} r^{N+1}\big(\bar f(r) - 1\big)\,\mathrm{d}r + \int_{\alpha}^{r_1} r^{N+1}\bar f(r)\,\mathrm{d}r \\
&\geqslant \alpha^2\bigg[\int_{r_0}^{\alpha} r^{N-1}\big(\bar f(r) - 1\big)\,\mathrm{d}r + \int_{\alpha}^{r_1} r^{N-1}\bar f(r)\,\mathrm{d}r\bigg] \\
&\geqslant 0.
\end{aligned}
$$

Summing those two last inequalities gives the desired result.

(ii) *Taylor expansion of the critical entropy dissipation*. We call critical entropy dissipation the function defined by

$$
D^c = \bigg(\int_{r_1}^{\beta} r^{N+1}\,\mathrm{d}r - \int_{\alpha}^{r_1} r^{N+1}\,\mathrm{d}r\bigg),
$$

where α and β are defined in (i). Then we have

$$
\begin{aligned}
&nM_1 = \beta^N - r_1^N, \\
&nM_1 = r_1^N - \alpha^N, \\
&(n+2)D^c = \beta^{N+2} - 2r_1^{N+2} + \alpha^{N+2},
\end{aligned}
$$

therefore

$$
\begin{aligned}
\frac{D^c}{r_1^{N+2}} = {}& \frac{\alpha+\beta-2r_1}{r_1} + \frac{n+1}{2}\bigg(\bigg(\frac{\beta-r_1}{r_1}\bigg)^2 + \bigg(\frac{\alpha-r_1}{r_1}\bigg)^2\bigg) \\
&+ \mathrm{O}\bigg(\bigg(\frac{\beta-r_1}{r_1}\bigg)^3 + \bigg(\frac{\alpha-r_1}{r_1}\bigg)^3\bigg).
\end{aligned}
$$

Now

$$
\begin{aligned}
\frac{M_1}{r_1^N} &= \frac{\beta-r_1}{r_1} + \frac{n-1}{2}\bigg(\frac{\beta-r_1}{r_1}\bigg)^2 + \mathrm{O}\bigg(\frac{\beta-r_1}{r_1}\bigg)^3 \\
&= \frac{r_1-\alpha}{r_1} - \frac{n-1}{2}\bigg(\frac{r_1-\alpha}{r_1}\bigg)^2 + \mathrm{O}\bigg(\frac{r_1-\alpha}{r_1}\bigg)^3,
\end{aligned}
$$

hence

$$
\begin{aligned}
0 = {}& \frac{\beta+\alpha-2r_1}{r_1} + \frac{n-1}{2}\bigg[\bigg(\frac{\beta-r_1}{r_1}\bigg)^2 + \bigg(\frac{r_1-\alpha}{r_1}\bigg)^2\bigg] \\
&+ \mathrm{O}\bigg(\bigg(\frac{\beta-r_1}{r_1}\bigg)^3 + \bigg(\frac{r_1-\alpha}{r_1}\bigg)^3\bigg).
\end{aligned}
$$

Finally we obtain

$$
\begin{aligned}
\frac{D^c}{r_1^{N+2}} &= \bigg[\bigg(\frac{\beta-r_1}{r_1}\bigg)^2 + \bigg(\frac{r_1-\alpha}{r_1}\bigg)^2\bigg] + \mathrm{O}\bigg(\bigg(\frac{\beta-r_1}{r_1}\bigg)^3 + \bigg(\frac{r_1-\alpha}{r_1}\bigg)^3\bigg) \\
&= 2\bigg(\frac{M_1}{r_1^N}\bigg)^2 + \mathrm{O}\bigg(\bigg(\frac{\beta-r_1}{r_1}\bigg)^3 + \bigg(\frac{r_1-\alpha}{r_1}\bigg)^3\bigg).
\end{aligned}
$$

Hence, there exists $\eta > 0$ and $\delta > 0$ such that

$$D^c \geqslant \frac{\delta}{r_1^{N-2}} M_1^2$$

whenever

$$\left|\frac{\beta - r_1}{r_1}\right| + \left|\frac{r_1 - \alpha}{r_1}\right| \leqslant \eta.$$

(iii) *Final estimation*. From the definition of α, there exists $a > 0$ such that $|(r_1 - \alpha)/r_1| \leqslant \eta$ whenever $|(\beta - r_1)/r_1| \leqslant a$. Remember that $r_2 \leqslant \beta$. Hence if $|r_2 - r_1| \leqslant a r_1$ then

$$\left|\frac{\beta - r_1}{r_1}\right| + \left|\frac{r_1 - \alpha}{r_1}\right| \leqslant \eta,$$

and

$$\begin{aligned} F_1 &\leqslant r_2^2 M_1 \leqslant a^2 \delta \sqrt{D^c}\, r_1^{(N+2)/2} \\ &\leqslant C_N a^2 \sqrt{D_1}\, \rho^{(N+2)/(2N)}. \end{aligned}$$

The first inequality uses the definition to F_1, the second one uses the result of (ii), and the third one uses the definition to r_1 and the result of (i). □

Now we are able to prove the estimate of Proposition 8.

PROOF OF PROPOSITION 8. We fix a and r_2 verifying the properties of Lemma 10. Thanks to Lemmas 9 and 10 we have:

$$\begin{aligned} F &\leqslant F_1 + F_2 \leqslant D_2\left(\frac{1+a}{a}\right) + C_N \rho^{(N+2)/2N} \sqrt{D_1} \\ &\leqslant C_N'\big(D + \rho^{(N+2)/2N} \sqrt{D}\big). \end{aligned}$$

□

4.6. *Hydrodynamic limits of the Fokker–Planck Navier–Stokes system*

We consider the case $x \in \mathbb{T}^N$. We denote

$$\eta\big(n, \rho, (n+\rho)u\big) = (\rho + n)\frac{|u|^2}{2} + \frac{\rho^\gamma}{\gamma - 1} + n \ln \frac{n}{(2\pi)^{N/2}} + \frac{1}{e}.$$

Note that it is an entropy of the system (33).

THEOREM 7. *Let $(f_\varepsilon^0, \rho_\varepsilon^0, u_\varepsilon^0)$ be a family of initial value such that $f_\varepsilon^0 \geqslant 0$, $\rho^0 \geqslant 0$ and there exists $(n^0, \rho^0, (n^0+\rho^0)u^0)$ such that*

$$\int_{\mathbb{R}^N} f_\varepsilon \,\mathrm{d}v \xrightarrow{L^1(\mathbb{T}^N)} n^0, \quad \rho_\varepsilon^0 \xrightarrow{L^1(\mathbb{T}^N)} \rho^0, \quad \int_{\mathbb{R}^N} v f_\varepsilon \,\mathrm{d}v + \rho_\varepsilon^0 u_\varepsilon^0 \xrightarrow{L^1(\mathbb{T}^N)} (n^0 + \rho^0)u^0,$$

with

$$\mathcal{H}\big(f_\varepsilon^0, \rho_\varepsilon^0, u_\varepsilon^0\big) \xrightarrow{L^1(\mathbb{T}^N)} \eta\big(n^0, \rho^0, (n^0+\rho^0)u^0\big),$$

when $\varepsilon \to 0$. We consider entropy solutions f_ε to the rescaled Fokker–Planck Navier–Stokes systems:

$$\begin{aligned}
&\partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon + \frac{\operatorname{div}_v((u_\varepsilon - v) f_\varepsilon - \nabla_v f_\varepsilon)}{\varepsilon} = 0,\\
&\partial_t \rho_\varepsilon + \operatorname{div}_x(\rho_\varepsilon u_\varepsilon) = 0,\\
&\partial_t(\rho_\varepsilon u_\varepsilon) + \operatorname{div}_x(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \nabla_x \rho_\varepsilon^\gamma - \nu_\varepsilon \Delta u_\varepsilon = \frac{(J_\varepsilon - n_\varepsilon u_\varepsilon)}{\varepsilon},\\
&(f_\varepsilon, \rho_\varepsilon, \rho_\varepsilon u_\varepsilon)(t=0) = \left(f_\varepsilon^0, \rho_\varepsilon^0, \rho_\varepsilon^0 u_\varepsilon^0\right),
\end{aligned}$$

where $n_\varepsilon = \int f_\varepsilon(x,v,t)\,\mathrm{d}v$ and $J_\varepsilon = \int v f_\varepsilon(x,v,t)\,\mathrm{d}v$. Assume moreover that

$$0 < \nu_\varepsilon \leqslant \varepsilon \quad \textit{and} \quad \gamma \in (1,2).$$

Then, up to a subsequence, the family $U_\varepsilon = (n_\varepsilon, \rho_\varepsilon, \rho_\varepsilon u_\varepsilon + J_\varepsilon)$ converges weakly in $L^p(0,\overline{T}; L^1_{\mathrm{loc}}(\mathbb{T}^N))$, for every $\overline{T} < \infty$ and $1 \leqslant p < \infty$, to a dissipative solution $U = (n, \rho, (n+\rho)u)$ of the bi-fluid system (33) on $[0,\infty)$, with initial value $U^0 = (n^0, \rho^0, (n^0 + \rho^0)u^0)$. In particular, we have the following special situation.

- *If U is a dissipative test function on $[0,T)$, and there exists $C_0 > 0$ such that*

$$\int_{\mathbb{T}^N} \eta\left(U_\varepsilon^0 | U^0\right) \mathrm{d}x + \int_{\mathbb{T}^N} \left|\mathcal{H}\left(f_\varepsilon^0, \rho_\varepsilon^0, \rho_\varepsilon^0 u_\varepsilon^0\right) - \eta\left(U_\varepsilon^0\right)\right| \mathrm{d}x \leqslant C_0\sqrt{\varepsilon},$$

  then, for every $t < T$, there exists a constant C_t such that

$$\int_{\mathbb{R}^N} \eta(U_\varepsilon | U)(s,x)\,\mathrm{d}x \leqslant C_t\sqrt{\varepsilon} \quad \textit{for any } s \in [0,t]. \tag{80}$$

  Moreover, the whole family U_ε converges strongly to U in $C^0(0,\overline{T}; L^1_{\mathrm{loc}})$.
- *If U^0 is a discontinuous initial value as in Proposition 4 (rarefaction waves) then the whole sequence converges to U^* on $(0,T)$.*

This result can be found in [75] where viscosity is taken into account. See also [37].

PROOF. We first prove a consistency result of the asymptotic system with the kinetic model. Integrating the kinetic equation of the rescaled problem with respect to v, we find

$$\partial_t n_\varepsilon + \operatorname{div}_x J_\varepsilon = 0.$$

Moreover, multiplying the same equation by v and integrating with respect to v, we get

$$\partial_t \int v f_\varepsilon\,\mathrm{d}v = -\operatorname{div}_x \int v\otimes v f_\varepsilon\,\mathrm{d}v + \frac{1}{\varepsilon}\int (u_\varepsilon - v) f_\varepsilon\,\mathrm{d}v$$

and thus

$$\partial_t(J_\varepsilon + \rho_\varepsilon u_\varepsilon) + \operatorname{div}_x\left(\int_{\mathbb{R}^N} v\otimes v f_\varepsilon\,\mathrm{d}v + \rho_\varepsilon u_\varepsilon \otimes u_\varepsilon - \nu_\varepsilon \nabla_x u_\varepsilon + \rho_\varepsilon^\gamma I_N\right) = 0.$$

We can rewrite this system in the form

$$\partial_t U_\varepsilon + \operatorname{div}_x A_\varepsilon = 0,$$

where

$$A_\varepsilon = \left(J_\varepsilon, \rho_\varepsilon u_\varepsilon, \int_{\mathbb{R}^N} v \otimes v f_\varepsilon \,\mathrm{d}v + \rho_\varepsilon u_\varepsilon \otimes u_\varepsilon - \nu_\varepsilon \nabla_x u_\varepsilon + \rho_\varepsilon^\gamma I_N \right).$$

We can then use Proposition 3. We need to control

$$\left| \int_{\mathbb{R}^N} (u_\varepsilon - v) f_\varepsilon \,\mathrm{d}v \right|, \quad \left| \int_{\mathbb{R}^N} (u_\varepsilon \otimes u_\varepsilon - v \otimes v + I) f_\varepsilon \,\mathrm{d}v \right|, \quad \nu_\varepsilon \nabla_x u_\varepsilon,$$

by the dissipation.

1 – We have

$$\begin{aligned} \left| \int (u_\varepsilon - v) f_\varepsilon \,\mathrm{d}v \right| &= \left| \int (u_\varepsilon - v) f_\varepsilon - \nabla_v f_\varepsilon \,\mathrm{d}v \right| \\ &\leqslant \left(\int f_\varepsilon \mathrm{d}v \right)^{1/2} \left(\int \left| (u_\varepsilon - v) f_\varepsilon - \nabla_v f_\varepsilon \right|^2 \frac{1}{f_\varepsilon} \,\mathrm{d}v \right)^{1/2} \end{aligned}$$

and therefore

$$\int_0^t \int_\Omega \left| \int (u_\varepsilon - v) f_\varepsilon \,\mathrm{d}v \right| \mathrm{d}x \,\mathrm{d}s \leqslant C \sqrt{\int_0^t D(f_\varepsilon)\,\mathrm{d}s} \leqslant C\sqrt{\varepsilon}.$$

2 – Next, we write

$$\begin{aligned} &\int (u_\varepsilon \otimes u_\varepsilon - v \otimes v + I_N) f_\varepsilon \,\mathrm{d}v \\ &\quad = \int \left[u_\varepsilon \otimes (u_\varepsilon - v) + (u_\varepsilon - v) \otimes v + I_N \right] f_\varepsilon \,\mathrm{d}v \\ &\quad = \int u_\varepsilon \sqrt{f_\varepsilon} \otimes \left[(u_\varepsilon - v)\sqrt{f}_\varepsilon - 2\nabla_v \sqrt{f}_\varepsilon \right] + u_\varepsilon \otimes 2\sqrt{f}_\varepsilon \nabla_v \sqrt{f}_\varepsilon \,\mathrm{d}v \\ &\qquad + \int \left[(u_\varepsilon - v)\sqrt{f}_\varepsilon - 2\nabla_v \sqrt{f}_\varepsilon \right] \otimes v\sqrt{f}_\varepsilon + 2\sqrt{f}_\varepsilon \nabla_v \sqrt{f}_\varepsilon \otimes v + I_N f_\varepsilon \,\mathrm{d}v \\ &\quad = \int u_\varepsilon \sqrt{f_\varepsilon} \otimes \left[(u_\varepsilon - v)\sqrt{f}_\varepsilon - 2\nabla_v \sqrt{f}_\varepsilon \right] + u_\varepsilon \otimes \nabla_v f_\varepsilon \,\mathrm{d}v \\ &\qquad + \int \left[(u_\varepsilon - v)\sqrt{f}_\varepsilon - 2\nabla_v \sqrt{f}_\varepsilon \right] \otimes v\sqrt{f}_\varepsilon + \nabla_v f_\varepsilon \otimes v + I_N f_\varepsilon \,\mathrm{d}v \\ &\quad = \int u_\varepsilon \sqrt{f_\varepsilon} \otimes \left[(u_\varepsilon - v)\sqrt{f}_\varepsilon - 2\nabla_v \sqrt{f}_\varepsilon \right] \mathrm{d}v \\ &\qquad + \int \left[(u_\varepsilon - v)\sqrt{f}_\varepsilon - 2\nabla_v \sqrt{f}_\varepsilon \right] \otimes v\sqrt{f}_\varepsilon \,\mathrm{d}v \end{aligned}$$

and so

$$\begin{aligned} &\int_0^t \int_\Omega \left| \int (u_\varepsilon \otimes u_\varepsilon - v \otimes v + I_N) f_\varepsilon \,\mathrm{d}v \right| \mathrm{d}x \,\mathrm{d}s \\ &\quad \leqslant \left(\int_0^t \int_\Omega \int \left(|u_\varepsilon|^2 + |v|^2 \right) f_\varepsilon \,\mathrm{d}v \,\mathrm{d}x \,\mathrm{d}s \right)^{1/2} \sqrt{\int_0^t D(f_\varepsilon)\,\mathrm{d}s}. \end{aligned}$$

It only remains to see that the term $\int(|u_\varepsilon|^2 + |v|^2) f_\varepsilon \,\mathrm{d}v$ is bounded uniformly by $\mathcal{H}(f_\varepsilon, \rho_\varepsilon, \rho_\varepsilon u_\varepsilon)$. Using the entropy inequality, we already know that the quantity $\int |v|^2 f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x$ is bounded, so it is enough to check that $\iint (u_\varepsilon - v)^2 f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x$ is bounded. To that purpose, we write

$$\begin{aligned}\int (u_\varepsilon - v)^2 f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x &= \iint (u_\varepsilon - v)\sqrt{f_\varepsilon}\big[(u_\varepsilon - v)\sqrt{f}_\varepsilon - 2\nabla_v\sqrt{f_\varepsilon}\,\big]\,\mathrm{d}v\,\mathrm{d}x \\ &\quad + \iint (u_\varepsilon - v)\nabla_v f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x \\ &\leqslant \left(\int (u_\varepsilon - v)^2 f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x\right)^{1/2}\sqrt{D(f_\varepsilon)} + \int f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x,\end{aligned}$$

which gives

$$\int_0^t\!\!\int (u_\varepsilon - v)^2 f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \leqslant \int_0^t\!\!\int D(f_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s + 2\int_0^t\!\!\int f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s.$$

and yields the result. We can then apply Proposition 3. □

5. Conclusion and open problems

In this chapter, a general theory on hydrodynamic limit has been presented. It showed how some rather general conditions on the structure of the kinetic, and fluid models and compatibility requirements between the two descriptions, can ensure the validity of the hydrodynamic limit. However, it ends, leaving a lot of unsolved problems.

- The case of the hydrodynamic limit of the Boltzmann equation is not treated by this method. The first difficulty is that the limit conservation law, the temperature dependent Euler system, is not admissible on the whole set of values $(\rho, u, T) \in \mathcal{V} = (0,\infty)\times\mathbb{R}^N\times(0,\infty)$ (see Section 2.7.4). Therefore, the method would require some a priori bounds on the solutions of the rescaled kinetic equations. It would be interesting, however, to see in which extent a refined study of the Boltzmann equation, as in [69], is unavoidable.
- The method relies heavily on the validity of the dissipative solutions. It is then necessary to derive uniqueness results for this kind of solutions, at least on special meaningful singular ones. It is not obvious at all if this can be done for any type of singularities. Shocks seems to be at odd with the theory. Even negative results would be interesting to show the limit of the method.
- Boundary conditions involving boundary layers are known to produce a great richness of phenomena (see Sone [87], and also [6,54,90]). This has been completely avoided here.
- In different areas of physics and engineering, kinetic models which do not conserve energy are used. Those models do not have classical nontrivial thermodynamical equilibrium. However, recent results show some rich behavior in large time asymptotic (see [18,17,47]). The derivation of associated fluid models is a great challenge.

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CHAPTER 8

Introduction to Stefan-Type Problems

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Contents

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Abstract

The classical Stefan model is a free boundary problem that represents thermal processes in phase transitions just by accounting for heat-diffusion and exchange of latent heat. The weak and the classical formulations of the basic Stefan system, in one and in several dimensions of space, are here reviewed. The basic model is then improved by accounting for surface tension, for nonequilibrium, and dealing with phase transitions in binary composites, where both heat and mass diffuse. The existence of a weak solution is proved for the initial- and boundary-value problem associated to the basic Stefan model, and also for a problem with phase relaxation and nonlinear heat-diffusion. Some basic analytical notions are also briefly illustrated: convex calculus, maximal monotonicity, accretiveness, and others.

0. Introduction

> *Il n'y a pas des problèmes résolus, il n'y a que des problèmes plus ou moins résolus.*
> (There are no solved problems, there are just problems that are more or less solved.)
>
> Henri Poincaré

The model of solid-liquid transitions that Josef Stefan formulated in 1889 provides a good example for this aphorism of Poincaré. Existence of a solution for that problem was proved by Lev Rubinstein in 1947. Tenths of thousands of papers and a number of meetings have then been devoted to this model and its extensions, and research on this topic is still in full development.[1]

The present work has a twofold purpose: to introduce the basic Stefan problem and some more refined models of (first-order) phase transitions, and also to illustrate some methods for the analysis of associated nonlinear initial- and boundary-value problems. These two aspects might hardly be separated, for the interplay between modelling and analysis is the blood and life of research on Stefan-type problems.

Phase transitions occur in many relevant processes in physics, natural sciences, and engineering: almost every industrial product involves solidification at some stage. Examples include metal casting, steel annealing, crystal growth, thermal welding, freezing of soil, freezing and melting of the earth surface water, food conservation, and others. All of these processes are characterized by two basic phenomena: heat-diffusion and exchange of latent heat of phase transition. A model accounts for this basic behaviour in terms of partial differential equations (shortly, PDEs): this is known as the *Stefan problem*, and was extensively studied in the last half century.

Because of the size of the existing literature, it has been mandatory to operate a drastic selection: important topics like numerical approximation, solid–solid phase transitions, shape-memory alloys, and others will be omitted. Here we shall confine throughout to modelling and analysis of solid-liquid transitions.[2] It is natural to recognize this phenomenon as an example of *free boundary problem* (shortly, FBP), for the evolution of the domains occupied by the phases is not known a priori. This is also labelled as a *moving boundary problem*, for the interface between the two phases evolves in time. Many mathematicians addressed the Stefan problem from this point of view, especially for univariate systems and in the framework of classical function spaces (i.e., C^k).

Phase transitions may also be regarded from a different perspective. Heat diffusion and exchange of latent heat may also be formulated in weak form, since they are accounted for by the energy balance equation, provided that this is meant in the sense of distributions. This leads to the formulation of an initial- and boundary-value problem in a fixed space–time domain for a nonlinear parabolic equation. This nonlinearity is expressed via a *maximal monotone graph*, and the problem may thus be reduced to a *variational inequality.* The natural framework is here provided by the Sobolev spaces.

---

[1] This provides evidence of the richness of Stefan-type problems and of the phenomena they represent, and also confirms the fertility of the academic mind in finding new results to offer to the attention of the community.

[2] In recent years the term *phase transition* has come in use among applied analysts with reference to stationary two-phase systems, too. Here we remain with the more traditional use: for us phase transitions are transitions, namely processes.

The two approaches above are known as the classical and the weak formulation of the Stefan problem. However, rather than being two formulations of the same problem, these represent two alternative models of phase transitions, that turn out to be equivalent only in special cases. The classical model is a genuine FBP, for it is based on the assumption that the phases are separated by an (unknown) smooth interface that also evolves smoothly; this approach allows for the onset of metastable states at the interior of the phases. On the other hand the weak formulation makes no direct reference to any phase interface: this may or may not exist, anyway it does not explicitly occur in the statement of the model. Solid and liquid phases may actually be separated by a set having nonempty interior, a so-called *mushy region*, that represents a fine-length-scale mixture of the two phases. In this respect the weak formulation is more general than the classical one, but it excludes metastable states. These issues are illustrated in Section 1.

The Stefan model is simple to be stated, combines analytical and geometrical aspects, has a suggestive physical substrate, is relevant for a number of applications, and is the prototype of a large class of evolutive FBPs. However it provides an oversimplified picture of (first-order) phase transitions, and is far from accounting for the richness of the physics of this large class of phenomena. In Section 2 we then improve the basic Stefan model by accounting for surface tension and for nonequilibrium at the phase interface. We also consider composite materials, in which the heat equation must be coupled with the equation of mass-diffusion, and the transition temperature depends on the chemical composition. Dealing with coupled diffusion, it seems especially convenient to use an approach based on the entropy balance and on the second principle of thermodynamics. This leads to the formulation of an initial- and boundary-value problem for a parabolic system of equations with two nonlinearities.

In Section 3 we then deal with some methods for the analysis of the weak formulation of the basic Stefan problem in several space dimensions. Several analytical procedures may be applied for that purpose: L^p-techniques, transformation by either space- or time-integration, and semigroup methods provide well-posedness and regularity properties in the framework of Sobolev spaces.

In Section 4 we study a multi-nonlinear extension of the Stefan problem, that accounts for nonlinear heat conduction and phase relaxation. We provide the weak formulation of an initial- and boundary-value problem in the framework of Sobolev spaces, and prove existence of a solution in any time interval, via a procedure that rests upon the notion of saddle point.

Convex calculus is often applied in the analysis of FBPs. In Section 5 we review basic results of that theory, and also illustrate some other analytical tools that are used in this work: maximal monotonicity, accretiveness, De Giorgi's notion of Γ-convergence, and so on. We conclude with a bibliographical note and with a collection of few hundred references on Stefan-type problems – just a small sample from an overwhelming literature. An effort has been done to quote some references also for the most investigated issues, where making a wise selection is hardly possible.

This paper is just meant as an introduction to Stefan-type models of phase transitions. Sections 3 and 4 are devoted to the analysis of nonlinear PDEs, and may be read independently of Sections 1 and 2 that deal with modelling – however the reader should be aware that divorcing analysis from modelling somehow spoils this theory. In the spirit of this Handbook, it seemed appropriate to devote special attention to the analysis of the weak formulation of the basic Stefan problem. For the benefit of the less experienced reader, we provide detailed arguments in Sections 3 and 4, devote some room to illustrate the basic analytical tools in Section 5, and also quote a number of fundamental monographs. A somehow didactical attitude has been maintained throughout, although this author cannot forget the Italian saying "*chi sa fa, chi non sa insegna*".[3]

A part of this paper is based on Chapters II, IV, V of this author's monograph [453], the reader is referred to for a more detailed account. On the other hand, the analysis of Section 4 provides new results.

1. The Stefan model

This first part is mainly devoted to the construction of two alternative formulations of the basic Stefan model of phase transitions, and to illustrate some related problems.[4] First we introduce the main physical assumptions and the weak formulation in several space dimensions, and then state the associated classical formulation. We illustrate how these models are based on partially different physical hypotheses, and compare some of their properties. In this part we deal with a homogeneous material, neglect surface tension, and assume either local stability or local metastability; these restrictions will be dropped in Section 2.

We shall also outline a vector extension of the Stefan model that accounts for processes in ferromagnetic materials having negligible hysteresis, the quasi-steady Stefan problem, the Hele-Shaw model, and the hyperbolic Stefan problem. Finally, a brief historical note updates Section IV.9 of Visintin [453].

1.1. *Weak formulation*

We shall always deal with solid–liquid (and liquid–solid) phase transitions, although our developments also apply to other first-order phase transitions. Here we shall represent phase transitions in an especially simplified way, focusing upon the thermal aspects, that is, heat-diffusion and exchange of latent heat. We shall neglect stress and deformation in the solid, convection in the liquid, change of density.[5]

We shall assume that the process occurs at constant volume, although in experiments usually it is the pressure that is maintained constant. Our developments however take over to systems that are maintained at constant pressure, at the only expense of some minor changes in the terminology: for instance, the term *internal energy* should then be replaced by *enthalpy*.[6]

[3] "Who knows, makes. Who does not know, he teaches."

[4] Throughout this work, we shall just refer to *first-order* phase transitions. These exhibit a latent heat of phase transition, at variance with second-order phase transitions.

[5] These simplifications might hardly be assumed in the case of vapour–liquid systems. This is the main reason for our preference for solid–liquid systems.

[6] See e.g. the discussion of Section 7 of Penrose and Fife [373].

DEFINITION. Let us denote by Ω a bounded domain of the Euclidean space $\mathbf{R}^3$, which is occupied by a homogeneous material capable of attaining two phases, liquid and solid. Let us fix a constant $T > 0$, set $Q := \Omega \times]0, T[$, and use the following notation:

u: density of internal energy – namely, internal energy per unit volume,

θ: relative temperature – namely, difference between the actual absolute (Kelvin) temperature τ and the value τ_E at which a planar solid–liquid interface is at thermodynamic equilibrium,

χ ($\in [-1, 1]$): rescaled liquid fraction: $\chi = -1$ in the solid, $\chi = 1$ in the liquid (the actual liquid fraction is $\frac{1}{2}(\chi + 1)$),

$\vec{q}$: heat flux per unit surface,

$k(\theta, \chi)$: thermal conductivity – a positive-definite 3×3-tensor,

f: intensity of a space-distributed heat source (or sink) – namely, heat either produced or absorbed per unit volume.

One may assume that at a *mesoscopic* length-scale (namely, an intermediate scale between that of macroscopic observations and that of molecular phenomena) just liquid and solid phases may be present, that is, $\chi = \pm 1$ at each point. At the macroscopic length-scale however a so-called *mushy region* (or mushy zone), namely a fine solid–liquid mixture, may appear. This is characterized by $-1 < \chi < 1$, which corresponds to a liquid concentration $0 < (\chi + 1)/2 < 1$.

The energy balance. Let us assume that the density of internal energy u is a known function of the state variables θ and χ:

$$u = \hat{u}(\theta, \chi) \quad \text{in } Q. \tag{1.1.1}$$

This functional dependence is characteristic of the specific material. Under the above physical restrictions, the global energy balance reads

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{q} = f \quad \text{in } \mathcal{D}'(Q)\ (\nabla\cdot := \operatorname{div}). \tag{1.1.2}$$

This equation may just be expected to hold in the sense of distributions, for in general u and $\vec{q}$ will be discontinuous at phase interfaces, as we shall see ahead. We couple this balance with the Fourier conduction law

$$\vec{q} = -k(\theta, \chi) \cdot \nabla\theta \quad \text{in } Q, \tag{1.1.3}$$

the thermal conductivity k being a prescribed positive-definite tensor function. We thus get the global heat equation

$$\frac{\partial u}{\partial t} - \nabla \cdot \big[k(\theta, \chi) \cdot \nabla\theta\big] = f \quad \text{in } \mathcal{D}'(Q). \tag{1.1.4}$$

A phase-temperature relation is then needed to close the system, besides of course appropriate boundary- and initial-conditions.

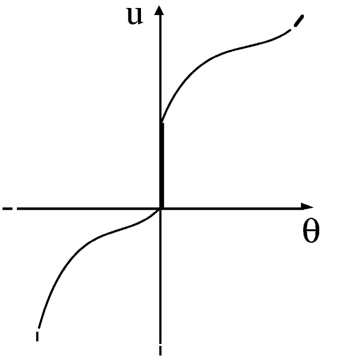

Fig. 1. Constitutive relation between the density of internal energy, u, and the temperature, θ.

The temperature-phase rule. Although Eq. (1.1.2) describes processes that are outside (and possibly far from) equilibrium, here we assume *local thermodynamic equilibrium*. By this we mean that in a neighbourhood of each point the system is governed by the same constitutive relations as at equilibrium.

This hypothesis excludes the occurrence of *undercooled* and *superheated* states, that we shall illustrate ahead. This thus yields the following temperature-phase rule:

$$\theta \geqslant 0 \quad \text{in } Q_1, \qquad \theta \leqslant 0 \quad \text{in } Q_2, \tag{1.1.5}$$

where by Q_1 and Q_2 we denote the subsets of Q that correspond to the liquid and solid phases, respectively. Defining the multi-valued sign function,

$$\begin{aligned} &\operatorname{sign}(\xi) := \{-1\} \quad \text{if } \xi < 0, \\ &\operatorname{sign}(0) := [-1, 1], \quad \operatorname{sign}(\xi) := \{1\} \quad \text{if } \xi > 0, \end{aligned} \tag{1.1.6}$$

the conditions (1.1.5) also read

$$\chi \in \operatorname{sign}(\theta) \quad \text{in } Q. \tag{1.1.7}$$

By eliminating χ from (1.1.1) and (1.1.7), for the density of internal energy and the temperature we get a relation of the form

$$u \in \alpha(\theta) \quad \text{in } Q, \tag{1.1.8}$$

where $\alpha : \mathbf{R} \to 2^{\mathbf{R}}$ is a multi-valued maximal monotone function (cf. Section 5.5); see Figure 1.[7]

Notice that under the hypothesis of local thermodynamic equilibrium $k(\theta, \chi) \cdot \nabla\theta$ is independent of χ; indeed k is determined by χ where $\theta \neq 0$, and $\nabla\theta = \vec{0}$ in the interior of the set where $\theta \neq 0$. Setting $\tilde{k}(\theta) := k(\theta, \alpha(\theta))$, we may then write $\vec{q} = -\tilde{k}(\theta) \cdot \nabla\theta$ in Q.

The system (1.1.5), (1.1.8) must be coupled with an initial condition for u and with boundary conditions either for θ or for the normal component of the heat flux. This constitutes the *weak formulation* of the *two-phase Stefan problem* in several space dimensions.

[7] By 2^A we denote the power set of any set A.

(Traditionally, one speaks of a *two-phase* problem, for the temperature evolution is unknown in both phases.)

Let us denote by $\vec{\nu}$ the outward-oriented unit normal vector on the boundary Γ of Ω. For instance, one may choose a partition $\{\Gamma_D, \Gamma_N\}$ of Γ and assume that

$$\theta = \theta_D \quad \text{on } \Sigma_D := \Gamma_D \times]0, T[, \tag{1.1.9}$$

$$\tilde{k}(\theta) \cdot \frac{\partial \theta}{\partial \nu} = h \quad \text{on } \Sigma_N := \Gamma_N \times]0, T[, \tag{1.1.10}$$

for a prescribed boundary temperature θ_D and a prescribed incoming heat flux h. We may now formulate a problem in the framework of Sobolev spaces.

PROBLEM 1.1.1 (Weak formulation of the multi-dimensional two-phase Stefan problem). Find $\theta \in L^2(0, T; H^1(\Omega))$ and $u \in L^2(Q)$ such that (1.1.8), (1.1.9) are fulfilled and

$$\iint_Q \left[\left(u^0 - u\right) \frac{\partial v}{\partial t} + \left(\tilde{k}(\theta) \cdot \nabla \theta\right) \cdot \nabla v \right] \mathrm{d}x\, \mathrm{d}t = \iint_Q f v \,\mathrm{d}x\, \mathrm{d}t + \iint_{\Sigma_N} h v \,\mathrm{d}S\, \mathrm{d}t$$
$$\forall v \in H^1(Q),\ v = 0 \text{ on } \left(\Omega \times \{T\}\right) \cup \Sigma_D. \tag{1.1.11}$$

Here by $\mathrm{d}S$ we denote the bidimensional Hausdorff measure. Note that by integrating (1.1.11) by parts in space and time we retrieve (1.1.4), (1.1.10) and an initial condition for u. This problem will be studied in Section 3.

1.2. *Classical formulation*

The classical formulation of the Stefan problem is based on two main hypotheses, which are at variance with those that underlie the weak formulation:

(i) no mushy region is either initially present or is formed during the process,

(ii) the liquid and solid phases are separated by a regular surface that also evolves regularly.

On the other hand here the condition of local thermodynamic equilibrium is restricted to the phase interface. Consistently with the hypotheses (i), we assume that all the solid (either initially present or formed during the process) is in the crystalline state, so that it is free of latent heat; this is at variance with the behaviour of amorphous solids like glasses and polymers, see Section 2.2. Let us label quantities relative to the liquid and solid phases by 1 and 2, respectively, assume that the constitutive function $\hat{u}$ is differentiable, and use the following further notation:

Q_i: open subset of Q corresponding to the ith phase, for $i = 1, 2$,

$\mathcal{S} := \partial Q_1 \cap \partial Q_2$: (possibly disconnected) manifold of $\mathbf{R}^4$ representing the space–time points at solid–liquid interfaces,

$\mathcal{S}_t := \mathcal{S} \cap (\Omega \times \{t\})$: configuration of the solid–liquid interface at the instant $t \in [0, T]$,

$C_V := \partial \hat{u}/\partial \theta$ (cf. (1.1.1)): heat capacity (at constant volume) per unit volume – namely, heat needed to increase the temperature of a unit volume by one degree; this equals the product between the mass density and the specific heat,

$L := \partial \hat{u}/\partial \chi$: density of *latent heat* of phase transition – namely, heat needed to melt a unit volume of solid.

We thus get

$$\frac{\partial u}{\partial t} = C_V(\theta,\chi)\frac{\partial \theta}{\partial t} + \frac{L(\theta,\chi)}{2}\frac{\partial \chi}{\partial t} \quad \text{in } \mathcal{D}'(Q). \tag{1.2.1}$$

(Note that $\chi/2$ has a unit jump across the interface.) This relation is set in the sense of distributions, for the phase function χ is discontinuous across phase interfaces. In each phase the energy balance reads $\partial u/\partial t = -\nabla \cdot \vec{q} + f$. Denoting the heat capacity and the thermal conductivity in the phase i by $C_{Vi}(\theta)$ and $k_i(\theta)$, we then retrieve the heat equation in each phase:

$$C_{Vi}(\theta)\frac{\partial \theta}{\partial t} - \nabla \cdot \big[k_i(\theta)\nabla\theta\big] = f \quad \text{in } Q_i \ (i = 1, 2). \tag{1.2.2}$$

The Stefan condition. Let us assume that $\mathcal{S}$ is sufficiently regular and that the temperature θ is continuous across the solid–liquid interface $\mathcal{S}$. At any instant t let us denote by $\vec{n} \in \mathbf{R}^3$ a unit vector field normal to $\mathcal{S}_t$ oriented from the liquid to the solid, by $\vec{q}_i$ the heat flux per unit surface that is either contributed or absorbed by the ith phase through $\mathcal{S}_t$.[8] For instance, let us assume that in a small time interval dt an element dS of the phase interface advances with normal velocity $\vec{v} \cdot \vec{n}$ through the solid phase. Excluding any tangential contribution along the interface, the net heat flux absorbed by dS in dt then equals $\mathrm{d}Q = (\vec{q}_1 \cdot \vec{n} - \vec{q}_2 \cdot \vec{n})\,\mathrm{d}S$. The melting process transforms this heat into an amount of latent heat that is proportional to the volume spanned by dS in dt. Thus

$$\vec{q}_1 \cdot \vec{n} - \vec{q}_2 \cdot \vec{n} = L(\theta)\vec{v} \cdot \vec{n} \quad \text{on } \mathcal{S}. \tag{1.2.3}$$

This equality also holds in case of freezing; in that case both members are negative, and represent the heat released at the solid–liquid interface. In either case, by the Fourier law, (1.2.3) yields the *Stefan condition*

$$k_1(\theta) \cdot \frac{\partial \theta_1}{\partial n} - k_2(\theta) \cdot \frac{\partial \theta_2}{\partial n} = -L(\theta)\vec{v} \cdot \vec{n} \quad \text{on } \mathcal{S}, \tag{1.2.4}$$

where we denote by $\partial\theta_i/\partial n$ the normal derivative of θ on $\mathcal{S}$ relative to the ith phase. If $g \in C^1(Q)$ is such that $\mathcal{S} = \{(x,t) \in Q\colon g(x,t) = 0\}$ and $\nabla g \neq \vec{0}$, then $\nabla g \cdot \vec{v} + \partial g/\partial t = 0$ and $\vec{n}|\nabla g| = \nabla g$ on $\mathcal{S}$ (possibly after inverting the sign of g). The condition (1.2.4) is then equivalent to

$$\big[k_1(\theta) \cdot \nabla\theta_1 - k_2(\theta) \cdot \nabla\theta_2\big] \cdot \nabla g = L(\theta)\frac{\partial g}{\partial t} \quad \text{on } \mathcal{S}. \tag{1.2.5}$$

Metastability. In the framework of the classical formulation of the Stefan problem, we allow for the occurrence of metastable states at the interior of the phases, namely,

undercooling (also called supercooling), i.e., $\theta < 0$ in the liquid, and

superheating, i.e., $\theta > 0$ in the solid.

Nevertheless we assume local thermodynamic equilibrium at the phase interface. For a homogeneous material, neglecting surface tension effects this corresponds to

$$\theta = 0 \quad \text{on } \mathcal{S}. \tag{1.2.6}$$

[8] The moving interface is not a material surface, and only the normal component of its velocity has a physical meaning.

In the Stefan condition (1.2.4) we may then replace $L(\theta)$ by $L(0)$ and $k_i(\theta)$ by $k_i(0)$ $(i = 1, 2)$.

The evolution of the solid–liquid interface is unknown. In principle, this lack of information is compensated by setting *two* quantitative conditions at the *free boundary* $\mathcal{S}$, namely, (1.2.4) and (1.2.6). Appropriate conditions on the initial value of θ and on the initial phase configuration must also be provided, as well as boundary conditions like (1.1.9) and (1.1.10). As an example we consider the following model problem, under natural regularity assumptions on the data f, θ_0, θ_D, h.

PROBLEM 1.2.1 (Classical formulation of the multi-dimensional two-phase Stefan problem). Find $\theta \in C^0(\bar{Q})$ and a partition $\{Q_1, Q_2, \mathcal{S}\}$ of Q such that:

(i) Q_1 and Q_2 are open sets;

(ii) $\mathcal{S} = \partial Q_1 \cap \partial Q_2$ is a regular 3-dimensional manifold, and $\mathcal{S}_t := \mathcal{S} \cap (\Omega \times \{t\})$ is a regular surface, for any $t \in]0, T[$;

(iii) $\partial\theta/\partial t$, $\partial^2\theta/\partial x_i \partial x_j$ (for $i, j \in \{1, 2, 3\}$) exist and are continuous in Q_1 and in Q_2;

(iv) the normal derivative $\partial\theta_i/\partial n$ exists on the respective sides of $\mathcal{S}$;

(v) $\bar{\mathcal{S}} \cap (\Omega \times \{0\})$ is prescribed, and

$$C_{Vi}(\theta)\frac{\partial\theta}{\partial t} - \nabla \cdot \big[k_i(\theta) \cdot \nabla\theta\big] = f \quad \text{in } Q_i \ (i = 1, 2), \tag{1.2.7}$$

$$k_1 \cdot \frac{\partial\theta_1}{\partial n} - k_2 \cdot \frac{\partial\theta_2}{\partial n} = -L\vec{v} \cdot \vec{n} \quad \text{on } \mathcal{S}, \tag{1.2.8}$$

$$\theta = 0 \quad \text{on } \mathcal{S}; \tag{1.2.9}$$

$$\theta = \theta_D \quad \text{on } \Sigma_D, \qquad k(\theta) \cdot \frac{\partial\theta}{\partial\nu} = h \quad \text{on } \Sigma_N, \tag{1.2.10}$$

$$\theta = \theta^0 \quad \text{in } \Omega \times \{0\}. \tag{1.2.11}$$

(Here and elsewhere, $\mathcal{S}$ and $\mathcal{S}_t$ might be disconnected.) If the temperature vanishes identically in one of the phases, one often speaks of a *one-phase* Stefan problem.[9] Besides phase transitions, this problem may represent a number of physical phenomena.[10] If the source term f vanishes identically, the occurrence of undercooled and superheated states may be excluded by assuming natural sign conditions on the initial and boundary data, because of the maximum and minimum principles.

The one-dimensional Stefan problem. Next we deal with a univariate system, e.g. an infinite slab, that we represent by a finite interval $\Omega =]a, b[$. We assume that $a < s^0 < b$, and that for instance the interval $]a, s^0[$ ($]s^0, b[$, resp.) represents the solid (liquid, resp.) phase at $t = 0$. If we exclude the formation of new phases, the solid–liquid interface $\mathcal{S}$ then coincides with the graph of a function $s : [0, T] \to [a, b]$ such that $s(0) = s^0$. Let us

[9] This traditional terminology is slightly misleading, for there are two-phases, although the temperature is unknown just in one of them.

[10] For instance, dissolution of a bubble gas in liquid, see e.g. Friedman [229, part III]; diffusion with chemical reaction, see e.g. Boley [70]; Darcy's filtration through porous media, see e.g. Bear [52,53]; swelling of polymers, see e.g. Astarita and Sarti [32]. These and other examples are illustrated in the rich account of Primicerio [378].

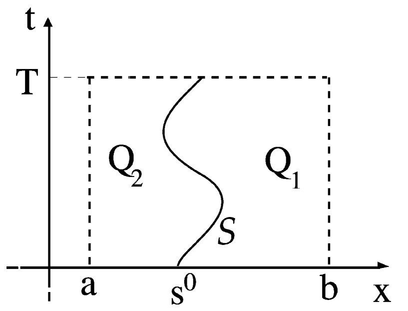

Fig. 2. Evolution of the free boundary in a one-dimensional Stefan problem.

assume that

$$\begin{aligned}
&C_{Vi} \in C^0(\mathbf{R}), \quad k_i \in C^1(\mathbf{R}), \quad C_{Vi}, k_i > 0 \quad (i = 1, 2),\\
&f \in C^0(\bar{Q}), \quad \theta_a, \theta_b \in C^0\big([0, T]\big), \quad \theta_a < 0, \quad \theta_b > 0,\\
&\theta^0 \in C^0\big([a, b]\big), \quad \theta^0 < 0 \quad \text{in }]a, s^0[, \quad \theta^0 > 0 \quad \text{in } \big]s^0, b\big[.
\end{aligned} \tag{1.2.12}$$

The previous equations coupled with natural initial and boundary conditions yield the following problem; cf. Figure 2.

PROBLEM 1.2.2 (Classical formulation of the one-dimensional two-phase Stefan problem). Find $s \in C^0([0, T]) \cap C^1(]0, T[)$ and $\theta \in C^0(\bar{Q})$ such that, setting

$$Q_1 := \big\{(x, t) \in Q \colon x > s(t)\big\}, \qquad Q_2 := \big\{(x, t) \in Q \colon x < s(t)\big\},$$

$\partial\theta/\partial t$, $\partial^2\theta/\partial x^2 \in C^0(Q_i)$ $(i = 1, 2)$, the limits $[k_i(\theta)\partial\theta/\partial x](s(t) \pm 0, t)$ exist for any $t \in]0, T[$, and

$$C_{Vi}(\theta)\frac{\partial\theta}{\partial t} - \frac{\partial}{\partial x}\left(k_i(\theta)\frac{\partial\theta}{\partial x}\right) = f \quad \text{in } Q_i \ (i = 1, 2), \tag{1.2.13}$$

$$\left(k_1(\theta)\frac{\partial\theta}{\partial x}\right)\big(s(t) + 0, t\big) - \left(k_2(\theta)\frac{\partial\theta}{\partial x}\right)\big(s(t) - 0, t\big) = -L(\theta)\frac{\mathrm{d}s}{\mathrm{d}t}(t)$$
$$\text{for } 0 < t < T, \tag{1.2.14}$$

$$\theta\big(s(t), t\big) = 0 \quad \text{for } 0 < t < T, \tag{1.2.15}$$

$$\theta(a, t) = \theta_a(t), \quad \theta(b, t) = \theta_b(t) \quad \text{for } 0 < t < T, \tag{1.2.16}$$

$$s(0) = s^0, \quad \theta(x, 0) = \theta^0(x) \quad \text{for } a < x < b. \tag{1.2.17}$$

1.3. *Comparison between the weak and the classical formulation*

Despite of the terminology, in general the classical and the weak formulations of the Stefan problem (CSP and WSP, resp.) are not different formulations of the same physical model, and rest upon different physical assumptions.

An ideal experiment. The next example looks especially enlightening of the difference between the CSP and the WSP. Let a solid system be initially at a uniform temperature,

$\theta(\cdot,0) = \theta^0 < 0$, and be exposed to a constant and uniform heat source of intensity $\hat{f} = 1$, e.g. by infrared radiation. If the system is thermally insulated, according to the WSP the temperature remains uniform in Ω, and the energy balance (1.1.4) is reduced to the ordinary differential equation $\mathrm{d}u/\mathrm{d}t = 1$, namely,

$$C_V(\theta,\chi)\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{L(\theta)}{2}\frac{\mathrm{d}\chi}{\mathrm{d}t} = 1 \quad \text{in } [0,T]. \tag{1.3.1}$$

Initially the temperature thus increases linearly. As θ vanishes it stops, melting starts without superheating, and χ increases linearly in time from -1 to 1, uniformly throughout Ω. For a time-interval the temperature remains null, and the whole body consists of a mush with increasing liquid content. As χ reaches the value 1, this mush has completed the liquid transformation, and the temperature increases again. Thus, according to the WSP, no phase interface is formed, and phase transition occurs throughout by the transformation process "solid $\to$ mush $\to$ liquid."

The CSP provides a completely different picture: no front of phase transition is formed, and θ just grows linearly in time, so that the solid is indefinitely superheated. Physically this is obviously unrealistic; it would be more reasonable to expect that a liquid phase is *nucleated* as a certain threshold is attained. But this behaviour is not accounted for by the CSP.

Equivalence between the CSP and the WSP. Let us now exclude the occurrence of mushy regions and of metastable states, and assume that the solid–liquid interface is regular and evolves regularly, so that both the weak and the classical formulation apply. The next statement bridges the two models.

PROPOSITION 1.3.1. *Let the pair $(\theta,\mathcal{S})$ fulfill the regularity conditions of Problem* 1.2.1, *and let χ fulfill* (1.1.7). *The system* (1.2.2), (1.2.4) *is then equivalent to the distributional equation*

$$C_V(\theta,\chi)\frac{\partial\theta}{\partial t} + \frac{L(\theta)}{2}\frac{\partial\chi}{\partial t} - \nabla\cdot\big[k(\theta,\chi)\cdot\nabla\theta\big] = f \quad \text{in } \mathcal{D}'(Q). \tag{1.3.2}$$

PROOF. Let us denote by $\vec{\nu} := (\vec{\nu}_x, \nu_t) \in \mathbf{R}^4$ the unit vector field normal to $\mathcal{S}$, oriented towards Q_1, say. As the vector $(\nabla g, \partial g/\partial t)$ is parallel to $\vec{\nu}$, the Stefan condition (1.2.5) also reads

$$\big[k_1(\theta)\cdot\nabla\theta_1 - k_2(\theta)\cdot\nabla\theta_2\big]\cdot\vec{\nu}_x = L(\theta)\nu_t \quad \text{on } \mathcal{S}. \tag{1.3.3}$$

Denoting the duality pairing between $\mathcal{D}'(Q)$ and $\mathcal{D}(Q)$ by $\langle\cdot,\cdot\rangle$, a simple calculation yields

$$\begin{aligned}
&\left\langle C_V(\theta,\chi)\frac{\partial\theta}{\partial t} + \frac{L(\theta)}{2}\frac{\partial\chi}{\partial t} - \nabla\cdot\big[k(\theta,\chi)\cdot\nabla\theta\big], \varphi\right\rangle \\
&\quad= \iint_Q \left\{C_V(\theta,\chi)\frac{\partial\theta}{\partial t}\varphi - \chi\frac{\partial}{\partial t}\frac{L(\theta)\varphi}{2} + \big[k(\theta,\chi)\cdot\nabla\theta\big]\cdot\nabla\varphi\right\}\mathrm{d}x\,\mathrm{d}t \\
&\quad= \iint_{Q\setminus\mathcal{S}} \left\{C_V(\theta,\chi)\frac{\partial\theta}{\partial t} - \nabla\cdot\big[k(\theta,\chi)\cdot\nabla\theta\big]\right\}\varphi\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\pm \int_{\mathcal{S}} \big\{L(\theta)\nu_t - \vec{\nu}_x\cdot\big[k_1(\theta)\cdot\nabla\theta_1 - k_2(\theta)\cdot\nabla\theta_2\big]\big\}\varphi\,\mathrm{d}\mathcal{S} \quad \forall\varphi\in\mathcal{D}(Q).
\end{aligned} \tag{1.3.4}$$

Table 1
Comparison between the classical and the weak formulations of the basic Stefan model, i.e., Problems 1.1.1 and 1.2.1

	Classical formulation (CSP)	Weak formulation (WSP)
Energy balance	heat equation in Q_1, Q_2 Stefan condition on $\mathcal{S}$	energy balance in $\mathcal{D}'(Q)$
Unknowns	θ, $\mathcal{S}$	θ, χ
Local equilibrium condition	$\theta = 0$ on $\mathcal{S}$	$\chi \in \operatorname{sign}(\theta)$ a.e. in Q
Phase characterization	global, via $\mathcal{S}$	local, via $\operatorname{sign}(\theta)$
Mushy regions	excluded	allowed
Metastable states	allowed	excluded
Analytical features	free boundary problem	degenerate PDE
Function spaces	classical C^k-spaces	Sobolev spaces
Well-posedness for $\Omega \subset \mathbf{R}$	for any time	for any time
Well-posedness for $\Omega \subset \mathbf{R}^3$	just for small time	for any time

The selection of the sign of the latter integral depends on the orientation of $\vec{\nu}$.

The system (1.2.2), (1.2.4) is thus equivalent to (1.3.2). (Loosely speaking, in the latter equation the *Dirac-type masses* of the second and third addendum cancel each other.) □

Comparison of analytical properties. The CSP consists of nondegenerate equations set in unknown domains, hence it is a genuine *free boundary problem*. On the other hand in the WSP the domain is fixed but the equation is degenerate.

The one-dimensional CSP is well posed in any time interval, under natural assumptions.[11] On the other hand in several space dimensions in general the CSP has a solution only in a *small* time interval. It is true that under suitable quantitative restrictions on the data a solution exists in any time interval, and may also be very regular.[12] But if one excludes special configurations, in general the classical solution may fail after some time, even if the heat source term f vanishes identically. Actually, discontinuities may occur in the temperature evolution as the topological properties of the phase interface change: e.g., a connected component may split into two components, or conversely the latter may merge into a single one. On the other hand under simple hypotheses the WSP is well posed in any time interval in any number of space dimensions, see Section 3.1, and may also be solved numerically by means of standard techniques. These differences are summarized in Table 1.

1.4. *A Stefan-type problem arising in ferromagnetism*

Phase transitions occur in many physical phenomena. Here we outline a macroscopic model of ferromagnetism without hysteresis that is reminiscent of the Stefan model, although in this case the unknown field is a vector and the equations have a different structure.[13]

[11] See e.g. Meirmanov [331].

[12] See e.g. Borodin [73], Prüss, Saal and Simonett [380]. Compare also with Friedman and Kinderlehrer [236], Nochetto [358].

[13] Problems of this sort were studied e.g. by Bossavit [75–78], Bossavit and Damlamian [79], Visintin [448,449].

Let the domain Ω be occupied by a ferromagnetic material, denote the magnetic field by $\vec{H}$, the magnetization by $\vec{M}$, and the magnetic induction by $\vec{B}$; in Gauss units, $\vec{B} = \vec{H} + 4\pi\vec{M}$. Let us also denote the electric field by $\vec{E}$, the electric displacement by $\vec{D}$, the electric current density by $\vec{J}$, the electric charge density by $\hat{\rho}$, and the speed of light in vacuum by c.

Dealing with electromagnetic processes, in general it is not natural to formulate a boundary-value problem in a Euclidean domain. In fact the exterior evolution may affect the interior process, and it is not easy to account for this interaction at a distance by prescribing appropriate boundary conditions. For this reason we rather set the Maxwell equations in the whole space $\mathbf{R}^3$, and assume different constitutive relations inside and outside Ω. The Ampère–Maxwell, Faraday and Gauss laws respectively read

$$c\nabla \times \vec{H} = 4\pi\vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \text{in } Q_\infty := \mathbf{R}^3 \times]0, T[, \tag{1.4.1}$$

$$c\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{in } Q_\infty \ (\nabla\times := \text{curl}), \tag{1.4.2}$$

$$\nabla \cdot \vec{B} = 0, \quad \nabla \cdot \vec{D} = 4\pi\hat{\rho} \quad \text{in } Q_\infty. \tag{1.4.3}$$

These equations must be coupled with appropriate constitutive relations, with initial conditions for $\vec{D}$ and $\vec{B}$, and with suitable restrictions on the behaviour of $\vec{H}$ and $\vec{E}$ at infinity.[14]

We assume that the magnetic material is surrounded by air, and that $\vec{J}$ equals a prescribed time-dependent field, $\vec{J}_{\text{ext}}$, outside Ω; this might be due for instance to an electric current circulating in an exterior conductor. We extend this field by setting $\vec{J}_{\text{ext}} := \vec{0}$ in Ω. We also assume that a prescribed electromotive force $\vec{E}_{\text{app}}$, that may be due e.g. to an electric generator, is applied to the system. Denoting by σ the electric conductivity, the *Ohm law* then reads

$$\vec{J} = \sigma(\vec{E} + \vec{E}_{\text{app}}) + \vec{J}_{\text{ext}} \quad \text{in } Q_\infty. \tag{1.4.4}$$

We assume that $\sigma = 0$ outside Ω, and that the field $\vec{E}$ does not vary too rapidly in Ω.[15] On the other hand in metals the conductivity σ is very large. In Eq. (1.4.1) in Ω the Ohmic current $\vec{J}$ thus dominates the displacement current $\partial\vec{D}/\partial t$, which may then be neglected; this is named the *eddy-current approximation*. As

$$\vec{D} = \epsilon\vec{E} \quad \text{in } Q, \tag{1.4.5}$$

with a constant electric permittivity ϵ, (1.4.1) then yields

$$\begin{aligned} &c\nabla \times \vec{H} = 4\pi\sigma(\vec{E} + \vec{E}_{\text{app}}) \quad \text{in } Q, \\ &c\nabla \times \vec{H} = 4\pi\vec{J}_{\text{ext}} + \epsilon\frac{\partial \vec{E}}{\partial t} \quad \text{in } Q_\infty \setminus Q. \end{aligned} \tag{1.4.6}$$

By (1.4.2) we may eliminate the field $\vec{E}$, and thus get

[14] In the functional formulation, the behaviour at infinity is implicit in the Sobolev spaces.

[15] This is an a priori assumption on the unknown field $\vec{E}$, and should be verified a posteriori.

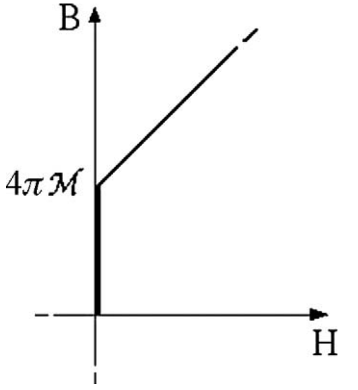

Fig. 3. Constitutive relation between the moduli of the colinear vectors $\vec{H}$ and $\vec{B}$, for an (isotropic) ferromagnetic material with negligible hysteresis.

$$4\pi\sigma\frac{\partial\vec{B}}{\partial t} + c^2\nabla\times\nabla\times\vec{H} = 4\pi c\sigma\nabla\times\vec{E}_{\text{app}} \quad \text{in } Q,$$
$$\epsilon\frac{\partial^2\vec{B}}{\partial t^2} + c^2\nabla\times\nabla\times\vec{H} = 4\pi c\nabla\times\vec{J}_{\text{ext}} \quad \text{in } Q_\infty\setminus Q. \tag{1.4.7}$$

However, as we said, we shall not formulate the problem separately inside and outside the domain Ω. Although ferromagnetic materials exhibit hysteresis, soft iron with high field-saturation and other metals are characterized by so narrow a hysteresis loop, that in first approximation this may be replaced by a maximal monotone graph; see Secttion 5.5. Let us then prescribe the following constitutive relation:

$$\vec{B}\in\vec{H}+4\pi\mathcal{M}\vec{\beta}(\vec{H})\ \big(=:\vec{\mathcal{F}}(\vec{H})\big) \quad \text{in } Q, \qquad \vec{B}=\vec{H} \quad \text{in } Q_\infty\setminus Q, \tag{1.4.8}$$

where $\mathcal{M}$ is a positive constant and $\vec{\beta}$ is the subdifferential of the modulus function:

$$\vec{\beta}(\vec{v}) := \begin{cases} \left\{\dfrac{\vec{v}}{|\vec{v}|}\right\} & \text{if } \vec{v}\neq\vec{0}, \\ \left\{\vec{v}\in\mathbf{R}^3\colon\ |\vec{v}|\leqslant 1\right\} & \text{if } \vec{v}=\vec{0}, \end{cases} \qquad \forall\vec{v}\in\mathbf{R}^3; \tag{1.4.9}$$

cf. Figure 3 (see Section 5.2). In this case, the unmagnetized and magnetically saturated phases are respectively characterized by $\vec{B}=\vec{0}$ and $|\vec{B}|\geqslant 4\pi\mathcal{M}$. In general, the occurrence of a mixed phase characterized by $0<|\vec{B}|<4\pi\mathcal{M}$ in a subdomain of Ω (a sort of *magnetic mushy region)* is not a priori excluded.

More generally, we may assume that $\vec{\mathcal{F}}:\mathbf{R}^3\to 2^{\mathbf{R}^3}$ is a (possibly multi-valued) maximal monotone mapping.

The system (1.4.2)–(1.4.6), (1.4.8), (1.4.9) is a *vector* parabolic–hyperbolic problem. More precisely, it is quasilinear parabolic in Q and semilinear hyperbolic in $Q_\infty\setminus Q$. The former setting may be compared with Problem 1.1.1, namely the weak formulation of the Stefan problem: the vector fields $\vec{H}$, $\vec{M}$, and $\vec{B}$ play similar roles to those of the scalar variables θ, χ and u, respectively.

If the system has planar symmetry, that is, if all variables only depend on two space coordinates (x, y, say), and if the fields $\vec{H}$ and $\vec{B}$ are parallel to the orthogonal z-axis, then they may be represented by their z-components, H and B. In this case in the second-order equation (1.4.7) the operator $\nabla \times \nabla\times$ equals $-\Delta$ (here the bidimensional Laplace operator), and Eqs. (1.4.7) are thus reduced to

$$\begin{aligned}
&4\pi\sigma \frac{\partial B}{\partial t} - c^2 \Delta H = g_1 \quad \text{in } Q, \\
&\epsilon \frac{\partial^2 B}{\partial t^2} - c^2 \Delta H = g_2 \quad \text{in } Q_\infty \setminus Q,
\end{aligned} \tag{1.4.10}$$

for prescribed scalar fields g_1 and g_2.

A vector free boundary problem. If $\vec{\mathcal{F}}$ is multi-valued, then *formally* the system (1.4.2)–(1.4.6), (1.4.8), (1.4.9) is the weak formulation of a free boundary problem. In general it is not obvious a priori that the magnetically saturated and unsaturated phases are separated by an interface, even under regularity hypotheses. However under appropriate restrictions (e.g., planar symmetry) we are reduced to a scalar problem, for which conditions are known that guarantee the existence of an interface. The next statement concerning the free boundary conditions may be compared with Proposition 1.3.1.

PROPOSITION 1.4.1 *(Discontinuity conditions). Let us assume that*:

(i) $\mathcal{S} \subset Q$ *is a smooth* 3*-dimensional manifold, and* $\mathcal{S}_t := \mathcal{S} \cap (\Omega \times \{t\})$ *is a (possibly disconnected) smooth surface, for any* $t \in\,]0, T[$;

(ii) $\vec{B}, \vec{H}, \partial\vec{B}/\partial t, \nabla \times \nabla \times \vec{H} \in L^1(Q \setminus \mathcal{S})^3$;

(iii) *the traces of* $\vec{B}$ *and* $\nabla \times \vec{H}$ *exist on both sides of* $\mathcal{S}$.

For any $t \in [0, T]$ *let us denote by* $\vec{\nu} \in \mathbf{R}^3$ *a unit vector field normal to* $\mathcal{S}_t$, *by* $v := \vec{v} \cdot \vec{\nu}$ *the (normal) speed of* $\mathcal{S}_t$, *and by* $[\![\cdot]\!]$ *the difference between the traces on the two sides of* $\mathcal{S}_t$. *Let us also assume that*:

(iv) $\vec{\nu} \times [\![\vec{H}]\!] = \vec{0}$ *a.e. on* $\mathcal{S}$.[16]

Then Eq. (1.4.7)$_1$ *in the sense of distributions is equivalent to the same equation pointwise in* $Q \setminus \mathcal{S}$, *coupled with the Rankine–Hugoniot-type condition*

$$4\pi\sigma v [\![\vec{B}]\!] = c^2 \vec{\nu} \times [\![\nabla \times \vec{H}]\!] \quad \textit{a.e. on } \mathcal{S}. \tag{1.4.11}$$

This statement may be checked via a similar argument to that of Proposition 1.3.1, that we omit. Moreover, the Gauss law $\nabla \cdot \vec{B} = 0$ and the identity $\nabla \cdot (\nabla \times \vec{H}) = 0$ in the sense of distributions entail that

$$\vec{\nu} \cdot [\![\vec{B}]\!] = 0, \quad \vec{\nu} \cdot [\![\nabla \times \vec{H}]\!] = 0 \quad \text{a.e. on } \mathcal{S}. \tag{1.4.12}$$

1.5. *Other Stefan-type problems*

The quasi-steady Stefan problem and the Hele–Shaw problem. If either the heat capacity C_V is very small or the temperature evolves very slowly, then one may replace the heat

[16] By the Ampère law (1.4.1), this assumption is equivalent to the absence of surface currents.

equation (1.1.4) by the quasi-stationary equation

$$-\nabla \cdot \big[k(\theta, \chi) \cdot \nabla\theta\big] = f \quad \text{in } Q_i \ (i = 1, 2). \tag{1.5.1}$$

The energy balance (1.1.4) is then reduced to

$$\frac{L(\theta)}{2}\frac{\partial\chi}{\partial t} - \nabla \cdot \big[k(\theta, \chi) \cdot \nabla\theta\big] = f \quad \text{in } \mathcal{D}'(Q), \tag{1.5.2}$$

in place of (1.1.4). Of course as an initial condition here one must just specify $\chi(\cdot, 0)$. This also applies to material diffusion in heterogeneous systems, for the time-scale of mass-diffusion is rather small (much smaller than that of heat-diffusion); see Section 2.3.

This setting is also known as the *Hele–Shaw problem*, since in the two-dimensional case it represents the evolution of a Hele–Shaw cell. This consists of two slightly separated parallel plates partially filled with a viscous fluid. If some fluid is injected into the cell with a syringe the fluid expands, and the evolution of the pressure p may be represented by

$$\begin{cases} \dfrac{\partial\chi}{\partial t} - \nabla \cdot \big[k(p, \chi) \cdot \nabla p\big] = f & \text{in } \mathcal{D}'(Q), \\ \chi \in \operatorname{sign}(p) & \text{a.e. in } Q, \end{cases} \tag{1.5.3}$$

with $f \geqslant 0$.[17] This model may also represent the industrial process of *electrochemical machining*, by which a metal body is either machined or formed by using it as an anode in an electrolytic cell.[18] A rather different setting is obtained if the fluid is extracted from the Hele–Shaw cell. In this case $f \leqslant 0$ and $p \leqslant 0$, and the condition $(1.5.3)_2$ must be replaced by

$$\chi \in \operatorname{sign}(-p) \quad \text{a.e. in } Q. \tag{1.5.4}$$

This problem is known as the *inverse Hele–Shaw problem*, since it is equivalent to a *backward* Hele–Shaw problem, and is ill-posed.[19]

The hyperbolic Stefan problem. It is well known that the heat equation represents instantaneous propagation of heat, at variance with one of the main issues of the Einstein theory of relativity. As most of applications of the Stefan problem do not involve relativistic velocities, this shortcoming has no practical relevance. Anyway this drawback may be eliminated by replacing the Fourier conduction law (1.1.3) by a suitable relaxation dynamics. Here we illustrate four alternatives, and refer to Joseph and Preziosi [281,282] for a detailed review of conduction laws and associated *heat waves* (in the linear setting). For the sake of simplicity, throughout this discussion we shall assume that k is a positive constant scalar.[20]

[17] See e.g. Ambrose [21], Antontsev, Meirmanov and Yurinsky [27], DiBenedetto and Friedman [176], Elliott and Janovsky [192], Escher and Simonett [195,196], Lacey et al. [303], Howison [277], Kim [288,289], Richardson [386,387], Saffman and Taylor [416].

[18] See e.g. Alexiades and Cannon [7], Elliott [189], McGeough [325], McGeough and Rasmussen [326].

[19] See e.g. DiBenedetto and Friedman [176], Nie and Tian [351].

[20] Here the relevant assumption is the independence of χ, for that on θ may be treated via the *Kirchhoff transformation* (3.1.15).

(i) In alternative to (1.1.3), after [120] one may use the *Cattaneo law*

$$\tau \frac{\partial \vec{q}}{\partial t} + \vec{q} = -k\nabla\theta \quad \text{in } Q, \tag{1.5.5}$$

where τ is a relaxation constant. The parabolic system (1.1.4), (1.1.8) is accordingly replaced by the quasilinear hyperbolic system

$$\begin{cases} \tau \dfrac{\partial^2 u}{\partial t^2} + \dfrac{\partial u}{\partial t} - k\Delta\theta = \tau \dfrac{\partial f}{\partial t} + f & \text{in } \mathcal{D}'(Q), \\ u \in \alpha(\theta) & \text{in } Q, \end{cases} \tag{1.5.6}$$

still with α a maximal monotone function $\mathbf{R} \to 2^{\mathbf{R}}$. The analysis of this problem is rather challenging. However, usually τ is so small that for most applications the Fourier approximation is acceptable.[21]

(ii) One may also insert a further relaxation term into the Cattaneo law (1.5.5):

$$\tau \frac{\partial \vec{q}}{\partial t} + \vec{q} = -k\nabla\theta - k_1 \frac{\partial \nabla\theta}{\partial t} \quad \text{in } Q. \tag{1.5.7}$$

By coupling this equation with the energy balance law (1.1.2), we get a third-order differential equation:

$$\tau \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - k\Delta\theta - k_1 \frac{\partial}{\partial t}\Delta\theta = \tau \frac{\partial f}{\partial t} + f \quad \text{in } Q, \tag{1.5.8}$$

that must then be coupled with the inclusion (1.1.8). Here one may prove well-posedness of a weak formulation for an associated boundary- and initial-value problem.

(iii) After Gurtin and Pipkin [261] one may also consider a dynamics with memory:

$$\vec{q}(x,t) = -\int_0^{+\infty} h(s)\nabla\theta(x,t-s)\,\mathrm{d}s \quad \text{for } (x,t) \in Q, \tag{1.5.9}$$

for a prescribed positive-definite, decreasing and integrable kernel $h(s)$. This equation is more general than the Cattaneo law, that is retrieved for $h(s) = (k/\tau)\exp\{-s/\tau\}$. The analysis of the integro-differential problem that is obtained by coupling the energy balance law (1.1.2) with the inclusion (1.1.8) and with (1.5.9) exhibits difficulties comparable to those of the quasilinear hyperbolic system (1.5.6).

(iv) On the other hand, if after Coleman and Gurtin [131] one also allows for the occurrence of a Dirac mass δ_0 in the kernel h, by assuming

$$\vec{q}(x,t) = -\frac{k_1}{\tau}\nabla\theta(x,t) + \left(\frac{k_1}{\tau^2} - \frac{k}{\tau}\right)\int_0^{+\infty} \mathrm{e}^{-s/\tau}\nabla\theta(x,t-s)\,\mathrm{d}s \\ \text{for } (x,t) \in Q \tag{1.5.10}$$

one then retrieves the (more feasible) third-order equation (1.5.7).

[21] See e.g. Colli and Recupero [141]. The physical aspects of hyperbolic and parabolic models are discussed e.g. in Herrera and Pavón [265]. Incidentally notice that, although the occurrence of a relaxation term like $\tau\partial\vec{q}/\partial t$ might be expected to have effects just on a (short) transient, in [265, p. 122] it is maintained that "transient phenomena may affect the way in which the system leaves the equilibrium, thereby affecting the future of the system even for time scales much larger than the relaxation time."

Initial- and boundary-value problems for the above equations were analysed in several works.[22]

We represented phase transitions in an extremely simplified way, neglecting physically relevant aspects like stress and deformation in the solid, convection in the liquid, change of density, microforces, and so on.[23] In Section 2 we shall address several other extensions of the Stefan model.

1.6. *Historical note*

In 1831 Lamé and Clapeyron [306] formulated what seems to be the first model of phase transition.[24] However the basic mathematical model of this phenomenon is traditionally named after the Austrian physicist Josef Stefan, who in 1889 studied the melting of the polar ices, dealing with several aspects of the one- and two-phase problem in a single dimension of space, see [430].

The first result of existence of a solution (for a large class of data) is be due to L. Rubinstein, who formulated the one-dimensional two-phase Stefan problem in 1947 in terms of a system of integral equations, and proved existence and uniqueness of a solution in a small time interval [406–408]. Further integral formulations of the one-dimensional Stefan problem were then studied.[25]

Several techniques were used to prove well-posedness, results of approximation, regularity, asymptotic behaviour, and other properties.[26] Many physically motivated generalizations were also investigated, and a fairly satisfactory understanding of a large class of single-dimensional problems was thus achieved.

The early research on the Stefan problem concentrated on the classical formulation of the univariate model. The introduction of weak formulations for nonlinear partial differential equations in the 1950s provided the key tool for the extension of the Stefan problem to the multi-dimensional setting in the early 1960s. The first results in this direction were achieved by Kamenomostskaya [283] and Oleĭnik [364].[27] Although these pioneering works were followed by an extensive research, for some time this new trend was somehow controversial, since for some researchers just the classical formulation was the genuine mathematical model of phase transitions. This also prompted the study of the regularity of the weak solution.

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22 See e.g. Aizicovici and Barbu [5], Barbu [48], Colli and Grasselli [134–137], Friedman [235], Showalter and Walkington [422].

23 The convection in phase transitions was studied e.g. in Cannon and DiBenedetto [113], Cannon, DiBenedetto and Knightly [114,115], Casella [118], DiBenedetto and Friedman [177], DiBenedetto and O'Leary [178], Hoffmann and Starovoitov [273], Rodrigues [398], Rodrigues and Urbano [399,400], Wang [462], Xu and Shillor [473]. Thermodynamic theories of phase transitions in presence of microforces were developed e.g. in Bonetti and Frémond [72], Frémond [224], Fried and Gurtin [226–228], Gurtin [256].

24 The process of technical solidification is older. The first cast objects (in copper) date back to more than 6000 years ago. . . .

25 See Evans [199], Sestini [420], Friedman [229], Kolodner [295], Jiang [280], and others.

26 See e.g. Cannon and Hill [116], Friedman [230, Chapter 8] and [231–233], Fasano and Primicerio [210,211], Fasano, Primicerio and Kamin [215], Rubinstein, Fasano and Primicerio [411], Schaeffer [418], and others.

27 See also Ladyženskaja, Solonnikov and Ural'ceva [305, Section V.9].

Results on the regularity of the solid–liquid interface for the multi-dimensional one-phase Stefan model were obtained by reformulating the problem as a variational inequality by means of a variable transformation due to Baiocchi [44,45], Duvaut [184], and Frémond [223].[28] Under appropriate restrictions Friedman and Kinderlehrer [236], Caffarelli [96,97], Kinderlehrer and Nirenberg [290,291] proved that the weak solution is also classical. Continuity of the temperature was showed by Caffarelli and Friedman [99] for the multi-dimensional one-phase problem. An analogous result was obtained by DiBenedetto [174,175], Ziemer [477], and Caffarelli and L.C. Evans [98] for the two-phase problem. In [323] Matano proved that any weak solution of a one-phase Stefan problem in an exterior region eventually becomes a classical solution after a finite time, and that the shape of the free boundary approaches that of a growing sphere as $t \to +\infty$.

In 1979 Meirmanov [327,328] proved the existence of the classical solution of the multi-dimensional two-phase Stefan problem in a small time interval; see also [329,331]. An analogous result was also shown by Hanzawa [263] for the one-phase problem by means of the Nash–Moser regularity theory.[29]

Mushy regions were first investigated for the one-dimensional Stefan problem by Atthey [38] (who introduced that denomination), Lacey and Tayler [304], Fasano and Primicerio [214], Meirmanov [329,330], Primicerio [379], and others. After the introduction of weak solutions, these regions were also studied in several space dimensions by Andreucci [23], Bertsch, De Mottoni and Peletier [62], Bertsch and Klaver [63], Götz and Zaltzman [247,248], Lacey and Herraiz [301,302], Rogers and Berger [403] (see also Berger, Brezis and Rogers [60]), and in several other papers. See also the survey [205] of Fasano.[30]

Free boundary problems. We already pointed out that the Stefan problems is a free boundary problem (FBP). Many other FBPs were formulated and studied in the last decades. Examples also include more general models of phase transitions, see Section 2. Free boundaries also occur as fronts between saturated and unsaturated regions in filtration through porous media, between plastic and elastic phases in continuous mechanics, between conducting and superconducting phases in electromagnetism, just to mention few cases. Relevant examples also come from reaction–diffusion, fluid dynamics, biomathematics, and so on.

Since the early years, the research on Stefan-type problems stimulated and was paralleled by that on other FBPs. Several of these problems are of industrial interest, and offer opportunities of collaboration among mathematicians, physicists, engineers, material

[28] This transformation is illustrated in Section 3.3.

[29] The existence of regular viscosity solutions was proved by Athanasopoulos, Caffarelli and Salsa, see [33–36,100]. Regularity results were also obtained by DiBenedetto and Vespri [180], Koch [294], Borodin [73], Bizhanova [64], Bizhanova and Rodrigues [65], Bizhanova and Solonnikov [66], Prüss, Saal and Simonett [380], and others.

[30] The development of the mathematical analysis of phase transitions would hardly have been conceivable without the achievements of mathematical-physicists and applied scientists. We just select a small sample from a huge literature: Cahn [106–108], Cahn and Hilliard [109], Collins and Levine [146], Frémond [224], Fried and Gurtin [226–228], Gurtin [250–258], Gurtin and Soner [262], Hilliard [268], Langer [308,309], Mullins and Sekerka [346,347], Penrose and Fife [373,374], Romano [404], Wang et al. [461].

scientists, biologists, and other researchers. A large community of mathematicians, engineers and applicative scientists spread over the world has been formulating and studying those problems for many years, and have regularly been meeting in major conferences. The proceedings of those conferences provide a comprehensive picture of the development of research on FBPs in the last decades; see the item (V) of the Bibliographical Note in Section 6.

2. More general models of phase transitions

As we saw, the classical and the weak formulations of the Stefan problem are both based on a number of simplifying assumptions. Even under favourable circumstances, these models should then be regarded just as first approximations of melting and freezing processes, both from the qualitative and quantitative viewpoint. Nevertheless the Stefan model is the basis for the construction of more refined models of phase transitions, since heat-diffusion and exchange of latent heat underlie any (first-order) phase transition.

In this second part we illustrate some physically justified extensions of the Stefan model. We amend the basic Stefan problem by inserting the *Gibbs–Thomson law*, and derive the latter by defining a suitable free energy functional. We then replace the equilibrium conditions (1.1.7) and (1.2.6) (for the weak and classica formulations of the Stefan model, respectively) by a kinetic law, that accounts for decay towards local equilibrium.

Next we concentrate our attention upon phase transitions in binary alloys. First, we outline a model that is often used in engineering, that essentially consists in coupling heat and mass-diffusion, and point out some physical and mathematical drawbacks. We then introduce an alternative and more satisfactory model, in which the constitutive laws are formulated consistently with the second principle, along the lines of the theory of nonequilibrium thermodynamics.

Finally, we outline the phase-field model and the Cahn–Hilliard equation for phase separation, and relate models set at different length-scales by means of De Giorgi's notion of Γ-limit (cf. Section 5.8).

2.1. *The Gibbs–Thomson law*

Undercooling and superheating. So far we dealt with phase transitions in pure materials, assuming local thermodynamic equilibrium at the solid–liquid interface, and neglecting surface tension effects. If these restrictions are dropped, then the interface (relative) temperature, θ, need not vanish. In the framework of the classical formulation, the interface condition (1.2.6) is actually replaced by a more general law of the form

$$\theta = \theta_{\text{s.t.}} + \theta_{\text{n.e.}} + \theta_{\text{imp.}} \quad \text{on } \mathcal{S}. \tag{2.1.1}$$

The first term on the right accounts for *surface tension*, and is proportional to the mean curvature of the solid–liquid interface. The second contribution is related to deviations from local thermodynamic equilibrium, and depends on the rate of phase transition. The third one accounts for the presence of secondary components (so-called *impurities)*. The two latter corrections are especially relevant for applications to metallurgy and to other engineering processes.

By the continuity of the temperature at the solid–liquid interface, Eq. (2.1.1) entails the onset of undercooling and/or superheating in the interior of the phases, so that here the temperature-phase rule (1.1.5) necessarily fails. Next we shall examine the above three terms separately.

The Gibbs–Thomson law. First we deal with the term $\theta_{\text{s.t.}}$. Let us assume that at any instant t the solid–liquid interface $\mathcal{S}_t$ is a surface of class C^2, denote by κ its *mean curvature* (assumed positive for a convex solid phase). The interface condition (1.2.6) may then be replaced by the *Gibbs–Thomson law*

$$\theta = -\frac{2\sigma\tau_E}{L}\kappa \quad \text{on } \mathcal{S}. \tag{2.1.2}$$

The quantities τ_E and L were already introduced in Section 1.1. σ is known as the coefficient of *surface tension* (or *capillarity*), and is equal to the surface density of the free energy at the solid–liquid interface, see Section 2.5. For the sake of simplicity, we shall assume that L and σ are constant.

For water at atmospheric pressure at about 0 °C, $2\sigma\tau_E/L$ is of the order of 10^{-5} cm, so that the deviation from the null temperature is significant just for mesoscopic curvature radii. The effects of the Gibbs–Thomson law are nevertheless perceivable also at the macroscopic length-scale, for it accounts for the undercooling prior to solid nucleation.[31]

Contact angle condition. The curvature condition (2.1.2) may be represented by a second-order elliptic equation for the (local) parametric formulation of the solid–liquid interface $\mathcal{S}$. For any $t \in]0, T[$ it is natural to associate to this equation a condition at the line of contact between $\mathcal{S}_t$ and the boundary Γ of the domain Ω. For any $(x, t) \in \bar{\mathcal{S}} \cap (\Gamma \times]0, T[)$, let us denote by $\omega(x, t)$ the angle formed by the normal to $\mathcal{S}_t$, oriented towards the liquid phase $\Omega_1(t)$, and the outward normal to Ω at x. We thus prescribe the *contact angle condition*

$$\cos\omega = \frac{\sigma_S - \sigma_L}{\sigma} \quad \text{on } \bar{\mathcal{S}} \cap \big(\Gamma \times]0, T[\big), \tag{2.1.3}$$

where σ_L and σ_S (here also assumed to be constant) are equal to the surface density of free energy at a surface separating the liquid and solid phases, resp., from an external material. Of course (2.1.3) makes sense only if

$$|\sigma_S - \sigma_L| \leqslant \sigma. \tag{2.1.4}$$

We may now formulate the *Stefan–Gibbs–Thomson Problem*, or (*Stefan Problem with Surface Tension*) just by replacing (1.2.9) by (2.1.2) and (2.1.3) in the formulation of Problem 1.2.1.[32]

In general this problem cannot have a solution for large time, for discontinuities may occur at the solid–liquid interface, just as for the classical formulation of the basic Stefan

[31] For some materials this may even be of the order of hundreds of degrees. See e.g. the monographs quoted in the item (VI) of the Bibliographical Note in Section 6.

[32] If the heat capacity $C_V = \partial u/\partial\theta$ vanishes in both phases, then this is often referred to as the *Mullins–Sekerka problem*. This especially applies to material diffusion in heterogeneous systems.

problem. However several properties are known to hold for small-time evolution,[33] and for the weak solution for any time.[34] An anisotropic variant representing crystal growth was also studied.[35]

Free energy. Let us define the perimeter functional P as in (5.7.2). In the framework of a mesoscopic model, the *(Helmholtz) free energy* of a solid–liquid system may be represented as follows, but for an additive contribution that depends on the temperature field:[36]

$$\Phi_{\theta,\sigma}(\chi) := \begin{cases} \sigma P(\chi) + \dfrac{\sigma_L - \sigma_S}{2} \displaystyle\int_\Gamma \gamma_0 \chi \, \mathrm{d}\Gamma - \dfrac{L}{2\tau_E} \displaystyle\int_\Omega \theta \chi \, \mathrm{d}x \\ \quad \forall \chi \in \mathrm{Dom}(P), \\ +\infty \qquad \forall \chi \in L^1(\Omega) \setminus \mathrm{Dom}(P). \end{cases} \tag{2.1.5}$$

By Proposition 5.7.1, whenever the inequality (2.1.4) is satisfied,[37] $\Phi_{\theta,\sigma}$ is lower semicontinuous and has at least one absolute minimizer. Moreover, whenever $\theta \in L^p(\Omega)$ for some $p > 3$, by Theorem 5.7.2 any relative minimizer of $\Phi_{\theta,\sigma}$ fulfills the Gibbs–Thomson law (2.1.2) and the contact angle condition (2.1.3).

Limit as $\sigma \to 0$. On the macroscopic length-scale $\sigma = 0$. It is easily seen that, as $\sigma \to 0$ (whence $\sigma_S - \sigma_L \to 0$ by (2.1.4)), the functional $\Phi_{\theta,\sigma}$ Γ-converges in the sense of De Giorgi (cf. Section 5.8) to

$$\Phi_\theta(\chi) := \begin{cases} -\dfrac{L}{2\tau_E} \displaystyle\int_\Omega \theta \chi \, \mathrm{d}x & \text{if } |\chi| \leqslant 1 \text{ a.e. in } \Omega, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.1.6}$$

This functional is convex and lower semicontinuous in $L^1(\Omega)$, and its minimization is clearly equivalent to the temperature-phase rule (1.1.5). In Section 2.5 we shall further discuss the form of the free energy functional at different length-scales.

Surface tension plays an important role in several phase transition phenomena. For instance it accounts for *phase nucleation*, see e.g.[38] Capillarity effects are also relevant for *crystal growth*.[39]

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[33] See e.g. Chen, Hong and Yi [127], Chen and Reitich [128], Escher, Prüss and Simonett [198], Mucha [342, 343], Radkevitch [382,383].

[34] See e.g. Garcke and Sturzenhecker [238], Luckhaus, [313], Luckhaus and Sturzenhecker [316], Röger [401, 402], Savaré [417].

[35] See e.g. Amar and Pomeau [22], Giga and Rybka [243], Gurtin and Matias [259], Herring [266], Rybka [413–415].

[36] It would be more precise, but rather cumbersome, to denote this functional by $\Phi_{\theta,\sigma,\sigma_L-\sigma_S}$. By γ_0 we denote the (continuous) trace operator $\mathrm{BV}(\Omega) \to L^1(\Gamma)$.

[37] Incidentally note that the condition (2.1.4) does not hold for all materials; for instance, for gold in contact with its vapour it fails. This means that solid and vapour should always be separated by a monoatomic liquid layer; see Chalmers [124, p. 85]. In this case (2.1.3) is meaningless, and actually $\bar{S} \cap (\Gamma \times]0, T[) = \emptyset$.

[38] See e.g. Visintin [451–454,456].

[39] See e.g. Almgren, Taylor and Wang [15], Almgren and Wang [16], Cahn and Taylor [110,439], Crank and Ockendon [150], Eck, Knabner and Korotov [187], Gurtin and Matias [259], Giga and Rybka [243], Ishii and Soner [279], Rybka [413–415], Taylor [438].

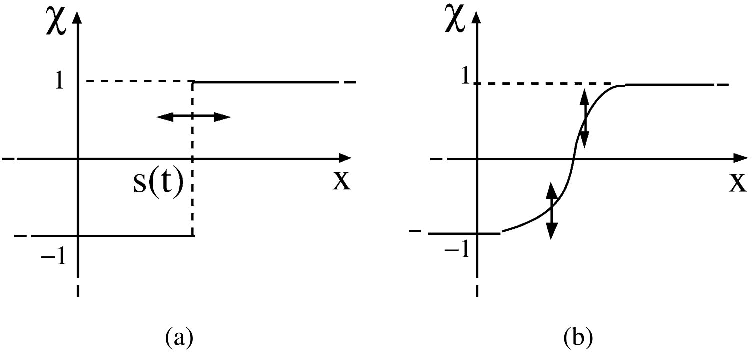

Fig. 4. *Kinetic undercooling* and *superheating* in part (a). *Phase relaxation* in part (b).

2.2. *Kinetic undercooling and phase relaxation*

In this section we represent the decay of a liquid–solid system towards local equilibrium via two basic modes of evolution: a kinetic law at the solid–liquid interface, and so-called phase relaxation. These modes are respectively associated with the classical and weak formulations of the basic Stefan problem, cf. Sections 1.1 and 1.2. We also reformulate this process from the point of view of *nonequilibrium thermodynamics*, in which the local formulation of the second principle plays a central role.

First mode: Directional solidification (or Columnar Growth or Kinetic Undercooling). A close inspection of the process of solidification shows that this is driven by undercooling; see e.g. the monographs quoted in the item (VI) of the Bibliographical Note in Section 6. We shall assume that melting is also driven by superheating, consistently with the symmetry of the representation of these phenomena that characterizes the Stefan model.[40]

In the framework of the classical formulation in a univariate system, we may replace the equilibrium condition $\theta(s(t),t) = 0$, cf. (1.2.6), by the *kinetic law*

$$\frac{\mathrm{d}s}{\mathrm{d}t}(t) + \gamma\big(\theta\big(s(t),t\big)\big) = 0, \tag{2.2.1}$$

for a *kinetic function* γ that depends on the material; cf. Figure 4(a). By replacing (1.2.9) with (2.2.1) in Problem 1.1.2, one gets the *one-dimensional two-phase Stefan problem with kinetic law*.

In the metallurgical literature, this mode of solidification is named *directional solidification*, and the corresponding undercooling is often referred to as *kinetic undercooling;* see e.g. Visintin [446].

[40] In general solidification is more relevant and exhibits a richer phenomenology than melting, as it is confirmed by the wealthy of morphologies that appear for instance in crystal growth. This asymmetry between solidification (or rather crystallization) and melting stems from the process of nucleation and growth: building the crystal structure is harder than destroying it.

For many materials $\gamma : \mathbf{R} \to \mathbf{R}$ may be assumed to be continuous and strictly increasing, with $\gamma(0) = 0$. In several cases one may also deal with the corresponding linearized law

$$\frac{\mathrm{d}s}{\mathrm{d}t}(t) + c\theta\big(s(t), t\big) = 0, \tag{2.2.2}$$

where c is a positive constant. However we shall see that this does not apply to all substances.

Second mode: Equiaxed solidification (or phase relaxation). Dealing with the weak formulation of the Stefan problem, next we replace the condition of local equilibrium "$\chi \in \operatorname{sign}(\theta)$ in Q," cf. (1.1.7), by a nonequilibrium law. As this inclusion also reads $\operatorname{sign}^{-1}(\chi) \ni \theta$ in Q, it is natural to consider the relaxation law

$$a\frac{\partial \chi}{\partial t} + \operatorname{sign}^{-1}(\chi) \ni \theta \quad \text{in } Q, \tag{2.2.3}$$

or equivalently,

$$\begin{cases} -1 \leqslant \chi \leqslant 1 & \text{in } Q, \\ \left(a\dfrac{\partial \chi}{\partial t} - \theta\right)(\chi - v) \leqslant 0 & \forall v \in [-1, 1], \text{ in } Q, \end{cases} \tag{2.2.4}$$

for some relaxation coefficient a; cf. Figure 4(b). For materials that are characterized by an increasing kinetic function γ, we may replace the right-hand side of (2.2.3) by $\gamma(\theta)$. In the metallurgical literature this mode of phase transition is referred to as *equiaxed solidification*.[41]

Comparison of the two modes. The laws (2.2.1) and (2.2.3) describe different evolution modes, although both represent relaxation towards local equilibrium. Equation (2.2.1) accounts for motion of the interface separating two pure phases, without formation of any mushy region. On the other hand, the second mode represents phase transition by formation of a mushy region, and (2.2.3) describes the evolution of the liquid concentration in that zone. From an analytical viewpoint, these two modes are naturally associated with the classical and weak formulations of the Stefan problem, respectively. The extension of the first mode to several space dimensions actually requires a revision of the mathematical model.

Directional and equiaxed growth are the basic modes of solidification, and may also combine to form a hybrid mode. For instance, in casting metal an equiaxed zone is at first formed in contact with the wall of the mould, and gives soon raise to a columnar region that advances towards the interior. Solid nucleation also occurs in the bulk, and an equiaxed solid phase grows in the remainder of the liquid. Eventually the two solid phases impinge on, and occupy the whole volume; see Figure 5. These physical aspects are illustrated e.g. in Flemings [222, Chapter 5], Kurz and Fisher [299, Section 1.1.2].

Glass formation. As we anticipated, for some materials the kinetic function is not monotone. For steel, polymers, and materials capable of forming a glass, the viscosity

[41] See e.g. the monographs quoted in the item (VI) of the Bibliographical Note in Section 6.

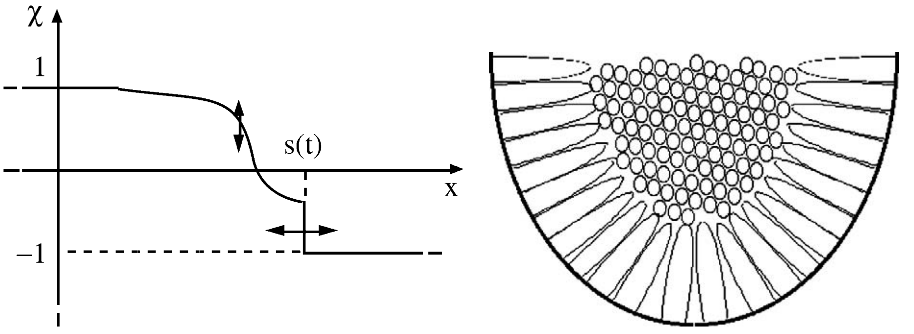

Fig. 5. Part (a) illustrates how *columnar and equiaxed solidification* may interact in a univariate system. Part (b) represents the grain structure of a crystal that grew from an undercooled liquid in a vessel: the solid columns advanced from the border, and impinged on the equiaxed grains which formed in the bulk.

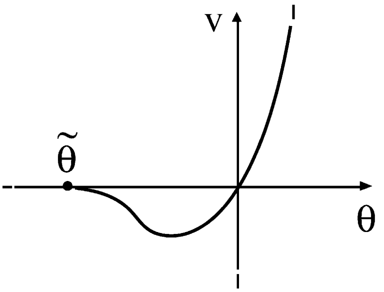

Fig. 6. Kinetic function for an amorphous material, e.g. a polymer.

increases so much with the undercooling, that the mobility of particles in their migration to reach the crystal sites is strongly impaired.

Although a glass apparently behaves like a solid, it has the fine-scale structure of a highly-viscous undercooled liquid, and indeed retains a large part of the latent heat of phase transition. Its crystal structure is largely uncomplete, and the material is accordingly said to be *amorphous*. For these materials the kinetic function has the qualitative behaviour of Figure 6. A glass is formed by quenching (i.e., very rapidly cooling) the liquid material to a temperature below $\tilde{\theta}$. The glass will eventually crystallize, but this may easily need geological time-scales.[42] A similar process occurs in the austenite-pearlite transformation in eutectoid carbon steel.[43]

The entropy balance. We shall represent the density of internal energy, u, as a convex and lower semicontinuous function of the density of entropy, s, and of the phase function, χ.

[42] Phase transitions in polymers and related industrial processes were studied e.g. by Andreucci et al. [24], Astarita and Sarti [32], Fasano [207], Fasano and Mancini [208], Fasano, Meyer and Primicerio [209].

[43] See e.g. Agarwal and Brimacombe [4], Cahn [106], Hawboldt, Chau and Brimacombe [264], Scheil [419] for an outline of the phenomenon, and Brokate and Sprekels [91, Chapter 8], Hömberg [275], Verdi and Visintin [444], Visintin [450] for its mathematical analysis.

Thus $u = \hat{u}(s, \chi)$, i.e. more explicitly[44]

$$u(x,t) = \hat{u}\big(s(x,t), \chi(x,t)\big) \quad \text{for } (x,t) \in Q.$$

By the very definition of the absolute (Kelvin) temperature, τ, the function $\hat{u}$ is differentiable w.r.t. s and

$$\tau = \frac{\partial \hat{u}}{\partial s}(s, \chi) \quad (=: \hat{\tau}(s, \chi)).$$

As $\hat{\tau} > 0$, the entropy may equivalently be represented as a concave function of u and χ, that is, $s = \hat{s}(u, \chi)$. Because of the constraint $-1 \leqslant \chi \leqslant 1$, the functions $\hat{u}$ and $\hat{s}$ cannot be differentiable at $\chi = \pm 1$. We shall accordingly use the notion of (partial) subdifferential, see Section 5.2. The differential notation is however too convenient for being dropped without a second thought, especially considering that $\hat{u}$ and $\hat{s}$ may be differentiable where $-1 < \chi < 1$; we shall actually assume them to be so. We shall thus write differential formulas only for the restriction to these values of χ. For instance, we define the potential λ by setting

$$\mathrm{d}u = \tau \,\mathrm{d}s + \lambda \,\mathrm{d}\chi, \quad \mathrm{d}s = \frac{1}{\tau}\,\mathrm{d}u - \frac{\lambda}{\tau}\,\mathrm{d}\chi \quad \text{where } -1 < \chi < 1. \tag{2.2.5}$$

More precisely, distinguishing between the potential λ and its functional representations $\hat{\lambda}_1(s, \chi)$ and $\hat{\lambda}_2(u, \chi)$, and denoting the partial subdifferential w.r.t. χ by ∂_χ, we have[45]

$$\hat{\lambda}_1(s, \chi) \in \partial_\chi \hat{u}(s, \chi), \quad -\frac{\hat{\lambda}_2(u, \chi)}{\tau} \in \partial_\chi \hat{s}(u, \chi) \quad \text{for } -1 \leqslant \chi \leqslant 1. \tag{2.2.6}$$

The energy balance (1.1.2) and (2.2.5) yield the entropy balance equation

$$\begin{aligned}\frac{\partial s}{\partial t} = \frac{1}{\tau}\frac{\partial u}{\partial t} - \frac{\lambda}{\tau}\frac{\partial \chi}{\partial t} &= -\nabla \cdot \frac{\vec{q}}{\tau} + \vec{q} \cdot \nabla \frac{1}{\tau} - \frac{\lambda}{\tau}\frac{\partial \chi}{\partial t} + \frac{f}{\tau} \\ &= -\nabla \cdot \vec{j}_s + \pi + \frac{f}{\tau} \quad \text{in } Q,\end{aligned} \tag{2.2.7}$$

where we set

$$\vec{j}_s := \frac{\vec{q}}{\tau}: \quad \text{entropy flux (per unit surface),} \tag{2.2.8}$$

$$\pi := \vec{q} \cdot \nabla \frac{1}{\tau} - \frac{\lambda}{\tau}\frac{\partial \chi}{\partial t}: \quad \text{entropy production rate (per unit volume).} \tag{2.2.9}$$

Thus

$$\pi := \vec{J} \cdot \vec{G}, \quad \text{where } \vec{J} := \left(\vec{q}, \frac{\partial \chi}{\partial t}\right), \ \vec{G} := \left(\nabla \frac{1}{\tau}, -\frac{\lambda}{\tau}\right). \tag{2.2.10}$$

The quantity f/τ is the rate of entropy production per unit volume, due to an external source or sink of heat. By the local formulation of the second principle of thermodynamics,[46]

[44] By this "hat notation" we shall distiguish between the physical field, $u = u(x,t)$, and the function that represents how it depends on other variables, $u = \hat{u}(s, \chi)$.

[45] Here the symbol of inclusion is needed, for the subdifferential may be multivalued, see Section 5.2.

[46] See e.g. the monographs quoted in the item (VII) of the Bibliographical Note in Section 6.

$$\pi \geqslant 0, \quad \pi = 0 \text{ at local equilibrium}$$
$$\text{(\textit{Clausius–Duhem inequality}).} \tag{2.2.11}$$

Because of the arbitrariness of $\vec{G}$, assuming a linear dependence between $\vec{J}$ and $\vec{G}$, by this inequality we infer[47] the following linearized conduction and phase relaxation laws:

$$\vec{q} = K \cdot \nabla \frac{1}{\tau}, \quad \text{that is,} \quad \vec{q} = -\frac{K}{\tau^2} \cdot \nabla\tau, \tag{2.2.12}$$

$$a\frac{\partial \chi}{\partial t} = -\frac{\lambda}{\tau}, \quad \text{that is,} \quad a\frac{\partial \chi}{\partial t} \in \partial_\chi \hat{s}; \tag{2.2.13}$$

here the tensor K and the scalar a are positive-definite functions of the state variables. These two equations may respectively be compared with the Fourier law (1.1.3) and with the phase relaxation dynamics (2.2.3). Thus

$$\pi \geqslant 0; \quad \pi = 0 \quad \Leftrightarrow \quad \nabla\tau = \vec{0}, \quad \lambda = 0. \tag{2.2.14}$$

Linearization. The occurrence of the term $1/\tau$ in the above formulas raises the need of granting that $\tau > 0$. This has been a source of technical difficulties in the analysis, that were overcome only at the expense of a certain effort.[48] That achievement is valuable in itself, and a result which allows for extreme temperatures is clearly of interest. However, one might wonder whether in practice the risk of getting τ close to zero is physically significant, and if so whether it is legitimate to extrapolate our models to those temperatures. Actually, constitutive relations typically just have a limited range of validity. This leads us to introduce a simplified model, that we shall study in Section 4.

Let us first define the function[49]

$$\varphi := u - \tau_E s, \quad \text{that is,} \quad \varphi = \hat{\varphi}(u, \chi) := u - \tau_E \hat{s}(u, \chi), \tag{2.2.15}$$

and notice that by (2.2.5)

$$\mathrm{d}\varphi = \mathrm{d}u - \frac{\tau_E}{\tau}\,\mathrm{d}u - \tau_E \frac{\partial \hat{s}}{\partial \chi}\,\mathrm{d}\chi = \frac{\theta}{\tau}\,\mathrm{d}u - \tau_E \frac{\partial \hat{s}}{\partial \chi}\,\mathrm{d}\chi. \tag{2.2.16}$$

Thus $\partial\hat{\varphi}/\partial\chi = -\tau_E \partial\hat{s}/\partial\chi$ for $-1 < \chi < 1$, and more generally in terms of partial subdifferentials $\partial_\chi \hat{\varphi} = -\tau_E \partial_\chi \hat{s}$ for $-1 \leqslant \chi \leqslant 1$.

The energy balance (1.1.2) and the entropy balance (2.2.7) yield the balance of the function φ:

$$\begin{aligned} \frac{\partial \varphi}{\partial t} = \frac{\partial u}{\partial t} - \tau_E \frac{\partial s}{\partial t} &= -\nabla \cdot \vec{q} + f + \tau_E \nabla \cdot \frac{\vec{q}}{\tau} - \tau_E \pi - \frac{\tau_E}{\tau} f \\ &= -\nabla \cdot \left(\frac{\theta}{\tau}\vec{q}\right) - \tau_E \pi + \frac{\theta}{\tau} f, \end{aligned} \tag{2.2.17}$$

[47] In Section 2.4 we shall illustrate this derivation in a more general framework.

[48] See Sprekels and Zheng [429], Zheng [476].

[49] This potential is named *available free energy* and tends to a minimum as equilibrium is approached in a source-free isolated system, cf. (2.2.17) below; see Müller and Weiss [345, Chapter 7]. In [255,257] Gurtin referred to it as *a Gibbs function*; actually, this function was first introduced by Gibbs dealing with uniform fields. It may also be noticed that relaxation towards thermal equilibrium is much faster than other relaxation processes, so that the difference between φ and ψ is not quantitatively relevant.

with π defined as in (2.2.9):

$$\tau_E \pi := \vec{q} \cdot \nabla \frac{\tau_E}{\tau} - \frac{\lambda \tau_E}{\tau} \frac{\partial \chi}{\partial t} = -\vec{q} \cdot \nabla \frac{\theta}{\tau} - \lambda \left(1 - \frac{\theta}{\tau}\right) \frac{\partial \chi}{\partial t}. \tag{2.2.18}$$

Next we linearize the system (2.2.17) and (2.2.18) w.r.t. θ. For any function σ, let us set $\sigma(\rho) = \mathrm{o}(\rho)$ whenever $\sigma(\rho)/\rho \to 0$ as $\rho \to 0$. As

$$\frac{\theta}{\tau} = \frac{\theta}{\tau_E} + \mathrm{o}\left(\frac{\theta}{\tau_E}\right),$$

$$\nabla \frac{\theta}{\tau} = \tau_E \frac{\nabla \theta}{\tau^2} = \frac{\nabla \theta}{\tau_E}\left(1 + \frac{\theta}{\tau_E}\right)^{-2} = \frac{\nabla \theta}{\tau_E}\left[1 - 2\frac{\theta}{\tau_E} + \mathrm{o}\left(\frac{\theta}{\tau_E}\right)\right],$$

we get

$$\mathrm{d}\varphi = \left[\frac{\theta}{\tau_E} + \mathrm{o}\left(\frac{\theta}{\tau_E}\right)\right] \mathrm{d}u - \tau_E \frac{\partial \hat{s}}{\partial \chi} \mathrm{d}\chi, \tag{2.2.19}$$

$$\frac{\partial \varphi}{\partial t} = -\nabla \cdot \left\{\left[\frac{\theta}{\tau_E} + \mathrm{o}\left(\frac{\theta}{\tau_E}\right)\right]\vec{q}\right\} - \tau_E \pi + \left[\frac{\theta}{\tau_E} + \mathrm{o}\left(\frac{\theta}{\tau_E}\right)\right] f, \tag{2.2.20}$$

$$\tau_E \pi = -\vec{q} \cdot \frac{\nabla \theta}{\tau_E}\left[1 - 2\frac{\theta}{\tau_E} + \mathrm{o}\left(\frac{\theta}{\tau_E}\right)\right] - \lambda \left[1 - \frac{\theta}{\tau_E} + \mathrm{o}\left(\frac{\theta}{\tau_E}\right)\right] \frac{\partial \chi}{\partial t}. \tag{2.2.21}$$

Notice that in these formulas the terms in square brackets are all positive.

Neglecting infinitesima, we thus get

$$d\varphi = \frac{\theta}{\tau_E} \mathrm{d}u - \tau_E \frac{\partial \hat{s}}{\partial \chi} \mathrm{d}\chi, \tag{2.2.22}$$

$$\frac{\partial \varphi}{\partial t} = -\nabla \cdot \left(\frac{\theta}{\tau_E} \vec{q}\right) - \tau_E \pi + \frac{\theta}{\tau_E} f, \tag{2.2.23}$$

$$\tau_E \pi = -\vec{q} \cdot \frac{\nabla \theta}{\tau_E}\left(1 - 2\frac{\theta}{\tau_E}\right) - \lambda \left(1 - \frac{\theta}{\tau_E}\right) \frac{\partial \chi}{\partial t}. \tag{2.2.24}$$

2.3. *Phase transitions in heterogeneous systems*

In this section we extend the Stefan model to phase transitions in mixtures of two materials. The diffusion of heat is here coupled with that of the constituents, so that we have a system of equations instead of a single parabolic equation. In this case the interface is characterized by a discontinuity not only of the heat flux, but also of the mass flux and of the composition of the mixture. We shall first introduce a classical formulation, and then derive a weak one.[50] Some physical and mathematical drawbacks will also arise, and

[50] This procedure is opposite to that we followed in Sections 1.1 and 1.2, that however might be applied here, too.

these will induce us to reformulate this phenomenon by a different model in the next section.

Mass diffusion. We confine ourselves to a composite of two constituents. We might also deal with a larger number of species, but even in this simple setting we shall encounter some difficulties in the analysis. More precisely, we shall deal with a *binary alloy*, that is, a homogeneous mixture of two substances, that are soluble in each other in all proportions in each phase, outside a critical range of temperatures. Here *homogeneity* means that the constituents are intermixed on the atomic length-scale to form a single phase, either solid or liquid. We shall regard one of the two components as the *solute*, for instance that with the lower solid–liquid equilibrium temperature, and the other one as the *solvent*. We shall also use the following notation:

c: concentration of the solute (per unit volume),

$\vec{j}_c$: flux of the solute (per unit surface),

D_1 (D_2, resp.): mass diffusivity of the solute in the liquid (in the solid, resp.).

Although the coefficient D_2 is much smaller than D_1, it need not vanish.[51]

In a simplified formulation, we assume that the specific heat and the heat conductivity may depend on the temperature and on the phase, but not on the (solute) concentration. We also assume that the mass diffusivity may depend on the concentration and on the phase, but not on the temperature. Thus

$$C_{Vi} = C_{Vi}(\theta), \quad k_i = k_i(\theta), \quad D_i = D_i(c) \quad \text{for } i = 1, 2. \tag{2.3.1}$$

If the two constituents have different temperatures of phase transition, then that of the mixture depends on the concentration. The latent heat is then a prescribed function of the temperature: $L = L(\theta)$.

As for pure substances, the heat equation is here fulfilled in the interior of each phase

$$C_{Vi}(\theta)\frac{\partial \theta}{\partial t} - \nabla \cdot \big[k_i(\theta) \cdot \nabla\theta\big] = f \quad \text{in } Q_i \ (i = 1, 2), \tag{2.3.2}$$

and is complemented by the Stefan condition at the solid–liquid interface $\mathcal{S}$:

$$k_1(\theta) \cdot \frac{\partial \theta_1}{\partial n} - k_2(\theta) \cdot \frac{\partial \theta_2}{\partial n} = -L(\theta)\vec{v} \cdot \vec{n} \quad \text{on } \mathcal{S}. \tag{2.3.3}$$

The principle of mass conservation, $\partial c/\partial t + \nabla \cdot \vec{j}_c = 0$, and the *Fick law*, $\vec{j}_c = -D_i(c)\nabla c$, yield the *equation of mass-diffusion* in each phase:

$$\frac{\partial c}{\partial t} - \nabla \cdot \big[D_i(c)\nabla c\big] = 0 \quad \text{in } Q_i \ (i = 1, 2). \tag{2.3.4}$$

Let us introduce some further notation:

$\vec{j}_{ci}$: mass flux (per unit surface) across $\mathcal{S}$ contributed by the phase i,

c_i: limit of c on $\mathcal{S}$ from the phase i,

[51] Moreover the liquid diffusivity D_1 is much smaller than the heat conductivity k in either phase. One might also assume that $D_1 = D_2 = 0$, as in the *Mullins–Sekerka problem*, where however the capillarity is also accounted for.

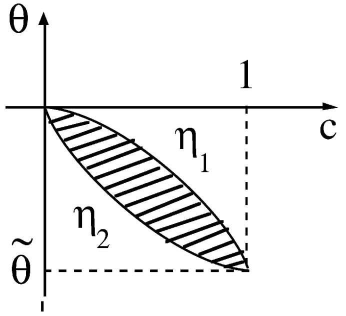

Fig. 7. The graph of η_1 and η_2 (respectively named *liquidus* and *solidus*) represent states of stable thermodynamic equilibrium at the solid–liquid interface for a noneutectic composite. The states outside the lens-shaped region (and with $0 \leqslant c \leqslant 1$) are also stable, whereas those inside are either metastable or unstable.

$\vec{v}$: (normal) velocity of $\mathcal{S}_t$,
$\vec{n} \in \mathbf{R}^3$: unit vector field normal to $\mathcal{S}_t$ oriented from the liquid to the solid.
By mass conservation we have

$$\vec{j}_{c2} \cdot \vec{n} - \vec{j}_{c1} \cdot \vec{n} = (c_2 - c_1)\vec{v} \cdot \vec{n} \quad \text{on } \mathcal{S}. \tag{2.3.5}$$

The Fick law then yields another discontinuity condition:

$$D_1(c_1)\frac{\partial c_1}{\partial n} - D_2(c_2)\frac{\partial c_2}{\partial n} = (c_2 - c_1)\vec{v} \cdot \vec{n} \quad \text{on } \mathcal{S}. \tag{2.3.6}$$

The reader will notice the analogy between the balance laws (2.3.2) and (2.3.4) in the interior of the phases, and the difference between the discontinuity conditions (2.3.3) and (2.3.6) at the solid–liquid interface: the field c is discontinuous across $\mathcal{S}$, at variance with θ. Actually, the concentration, c, should be compared with the density of internal energy, u, rather than with the temperature, θ. Ahead we shall introduce a further field, w, that is continuous at the solid–liquid interface, and plays an analogous role to that of θ.

Phase separation. At the solid–liquid interface, the temperature and the concentration fulfill an equilibrium relation of the form

$$\theta = \eta_1(c_1) = \eta_2(c_2) \quad \text{on } \mathcal{S}, \tag{2.3.7}$$

where η_1 and η_2 are known functions. Their graphs are traditionally named *liquidus* and *solidus*, for obvious reasons. For a noneutectic composite,[52] we may also assume that

$$\begin{aligned} &\eta_i \in C^1\big([0,1]\big), \quad \eta_i' < 0 \ (i = 1, 2), \quad \eta_1 > \eta_2 \text{ in }]0,1[, \\ &\eta_1(0) = \eta_2(0) = 0, \qquad \eta_1(1) = \eta_2(1) \ (=: \tilde{\theta}) < 0 \quad \text{(see Figure 7)}. \end{aligned} \tag{2.3.8}$$

At local thermodynamic equilibrium, the temperature-phase rule (1.1.5) is here replaced by a *temperature-concentration-phase rule*:

$$\theta \geqslant \eta_1(c) \quad \text{in } Q_1, \qquad \theta \leqslant \eta_2(c) \quad \text{in } Q_2. \tag{2.3.9}$$

[52] A composite is named a *eutectic* if $\eta_1(\bar{c}) = \eta_2(\bar{c})$ for some *eutectic concentration* $\bar{c} \in]0, 1[$.

The states where $\eta_2(c) < \theta < \eta_1(c)$ are in either metastable or unstable thermodynamic equilibrium. Whenever the variables are forced to attain those intermediate values, e.g. by rapid cooling of a liquid system, one or more nuclei of the secondary phase are formed, and grow until the two phases have attained the respective concentrations of equilibrium: $c_i = \eta_i^{-1}(\theta)$ $(i = 1, 2)$. Under isothermal conditions, this process of *phase separation* (also known as *spinodal decomposition*) is represented by the *Cahn–Hilliard equation*, that we briefly illustrate in Section 2.5.

Next we introduce a classical formulation, that extends that of the basic Stefan model, cf. Problem 1.2.1.

PROBLEM 2.3.1 (Classical formulation of the multi-dimensional problem of phase transition in binary alloys). Find $\theta, c : Q \to \mathbf{R}$ and a partition $\{Q_1, Q_2, \mathcal{S}\}$ of Q such that:

(i) Q_1 and Q_2 are open sets;
(ii) $\mathcal{S} \subset Q$ is a regular 3-dimensional manifold, and $\mathcal{S}_t := \mathcal{S} \cap (\Omega \times \{t\})$ is a regular surface, for any $t \in \,]0, T[$;
(iii) θ, c, $\partial\theta/\partial t$, $\partial c/\partial t$, $\partial^2\theta/\partial x_i \partial x_j$, $\partial^2 c/\partial x_i \partial x_j$ (for $i, j \in \{1, 2, 3\}$) exist and are continuous in Q_1 and in Q_2;
(iv) the normal derivatives $\partial\theta_i/\partial n$ and $\partial c_i/\partial n$ exist on the respective sides of $\mathcal{S}$;
(v) Eqs. (2.3.2)–(2.3.4), (2.3.6), (2.3.7) are fulfilled;
(vi) θ and c attain prescribed values on $\Omega \times \{0\}$ and on $\Gamma_D \times \,]0, T[$;
(vii) the normal derivatives $\partial\theta/\partial\nu$ and $\partial c/\partial\nu$ attain prescribed values on $\Gamma_N \times \,]0, T[$;
(viii) $\bar{\mathcal{S}} \cap (\Omega \times \{0\})$ is also prescribed.

Here the occurrence of metastable states is not excluded, just as for Problem 1.2.1.

A transformation of variable. In view of deriving a weak formulation of Problem 2.3.1, let us introduce the new variable[53]

$$w := \eta_i(c) \left(\in [\tilde{\theta}, 0]\right) \quad \text{in } Q_i \ (i = 1, 2), \tag{2.3.10}$$

so that by (2.3.7) and (2.3.9)

$$w \text{ is continuous across } \mathcal{S}, \qquad w = \theta \quad \text{on } \mathcal{S}, \tag{2.3.11}$$

$$\theta \geqslant w \quad \text{in } Q_1, \qquad \theta \leqslant w \quad \text{in } Q_2. \tag{2.3.12}$$

Setting $\zeta_i := \eta_i^{-1}$ for $i = 1, 2$, we have

$$c = \zeta_i(w), \quad \nabla c = \zeta_i'(w)\nabla w \quad \text{in } Q_i \ (i = 1, 2). \tag{2.3.13}$$

Let us also set

$$\widetilde{D}_i(w) := -D_i\big(\zeta_i(w)\big)\zeta_i'(w) \ (> 0) \quad \forall w \in [\tilde{\theta}, 0] \ (i = 1, 2), \tag{2.3.14}$$

so that the Fick law also reads

$$\vec{j}_c := -D_i(c)\nabla c = \widetilde{D}_i(w)\nabla w \quad \text{in } Q_i \ (i = 1, 2). \tag{2.3.15}$$

[53] In Section 2.4 we shall see that ∇w is proportional to $-\nabla\mu$, where by μ we denote the relative *chemical potential*, namely the difference between the chemical potentials of the two components.

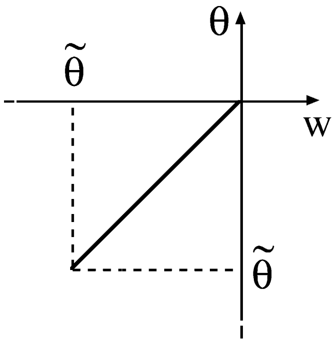

Fig. 8. Constitutive relation between the variables θ and w at the solid–liquid interface. As $0 \leqslant c \leqslant 1$, only $\tilde{\theta} \leqslant w \leqslant 0$ is physically meaningful. The liquid phase is characterized by $w < \theta$, the solid phase by $w > \theta$.

Defining the phase function χ as above (i.e., $\chi = 1$ in the liquid, $\chi = -1$ in the solid), the phase rule (1.1.7) is here replaced by

$$\chi \in \operatorname{sign}(\theta - w) \quad \text{in } Q \quad (\text{cf. Figure 8}). \tag{2.3.16}$$

Let us now set

$$\begin{aligned} &C_V(\theta,\chi) := C_{V1}(\theta)\frac{1+\chi}{2} + C_{V2}(\theta)\frac{1-\chi}{2},\\ &k(\theta,\chi) := k_1(\theta)\frac{1+\chi}{2} + k_2(\theta)\frac{1-\chi}{2},\\ &\widetilde{D}(w,\chi) := \widetilde{D}_1(w)\frac{1+\chi}{2} + \widetilde{D}_2(w)\frac{1-\chi}{2},\\ &\quad \forall(\theta,w,\chi) \in \mathbf{R} \times [\tilde{\theta}, 0] \times [-1,1]. \end{aligned} \tag{2.3.17}$$

By Proposition 1.3.1, the heat equations (2.3.2) and the Stefan condition (2.3.3) may be expressed in weak form by the single equation

$$C_V(\theta,\chi)\frac{\partial\theta}{\partial t} + \frac{L(\theta)}{2}\frac{\partial\chi}{\partial t} - \nabla\cdot\big[k(\theta,\chi)\cdot\nabla\theta\big] = 0 \quad \text{in } \mathcal{D}'(Q). \tag{2.3.18}$$

That argument yields the analogous statement for mass-diffusion.

PROPOSITION 2.3.1. *Let the pair $(u, \mathcal{S})$ fulfill the regularity conditions of Problem 2.3.1, define w as in (2.3.10), and set $\chi := -1$ in Q_2, $\chi := 1$ in Q_1. The system (2.3.4), (2.3.6) is then equivalent to*

$$\frac{\partial c}{\partial t} + \nabla\cdot\big[\widetilde{D}(w,\chi)\nabla w\big] = 0 \quad \textit{in } \mathcal{D}'(Q). \tag{2.3.19}$$

Notice that by (2.3.13)

$$c = \zeta_1(w)\frac{1+\chi}{2} + \zeta_2(w)\frac{1-\chi}{2} \quad \text{in } Q. \tag{2.3.20}$$

Thus c is a decreasing function of w, and Eq. (2.3.19) is *forward parabolic*.

Table 2
Comparison between heat and mass-diffusion in binary alloys. By $[\![v_i]\!] := v_1 - v_2$ we denote the jump of a quantity v across the solid–liquid interface $\mathcal{S}$

	Heat diffusion	Mass diffusion
continuous variable	temperature θ	new variable w
discontinuous variable	internal energy u	solute concentration c
equation in each phase	$C_{Vi}(\theta)\dfrac{\partial\theta}{\partial t} - \nabla\cdot\big[k_i(\theta)\nabla\theta\big] = f$	$\dfrac{\partial c}{\partial t} - \nabla\cdot\big[D_i(c)\nabla c\big] = 0$
jump condition across $\mathcal{S}$	$\left[\!\!\left[k_i(\theta)\dfrac{\partial\theta_i}{\partial n}\right]\!\!\right] = -L(\theta)\vec{v}\cdot\vec{n}$	$\left[\!\!\left[D_i(c)\dfrac{\partial c_i}{\partial n}\right]\!\!\right] = (c_2 - c_1)\vec{v}\cdot\vec{n}$
equation in $\mathcal{D}'(Q)$	$\dfrac{\partial u}{\partial t} - \nabla\cdot\big[k(\theta,\chi)\nabla\theta\big] = f$	$\dfrac{\partial c}{\partial t} + \nabla\cdot\big[\widetilde{D}(w,\chi)\nabla w\big] = 0$

Equations (2.3.16), (2.3.18), (2.3.19) and (2.3.20) for the unknown functions θ, c, w, χ, coupled with appropriate initial and boundary conditions, constitute the weak formulation of the problem of phase transition in binary alloys. We derived these equations from the classical formulation; alternatively one might also derive them directly from the laws of heat and mass conservation set in the whole space–time domain Q.

Linearized constitutive laws. If the solute concentration c is *small* (as it often occurs in practice), it is possible to linearize the functions η_1 and η_2, that is, to replace (2.3.7) by

$$\theta = \eta_i'(0)c_i =: -\frac{1}{r_i}c_i \quad \text{on } \mathcal{S}\ (i = 1, 2), \tag{2.3.21}$$

with $0 \leqslant r_2 < r_1$; cf. Figure 9. By setting

$$w := -\frac{1}{r_i}c_i\ (\leqslant 0) \quad \text{in } Q_i\ (i = 1, 2), \tag{2.3.22}$$

from (2.3.20) we thus get

$$c = -r_1 w\frac{\chi + 1}{2} - r_2 w\frac{1 - \chi}{2} \quad \text{in } Q. \tag{2.3.23}$$

Although the linearization only applies for small values of c, here the range of c is assumed to be the whole $\mathbf{R}^+$, which corresponds to $w \leqslant 0$.

A nonparabolic system of equations. The model above has extensively been used by material scientists and engineers, and their numerical approximation provided quantitatively acceptable results.[54] However, as far as this author knows, even existence of a weak solution is not known for this problem in the multi-variate setting, in spite of the simplifications that are inherent in this model. Actually Problem 2.3.1 does not seem prone to analysis. The equations of heat and mass-diffusion (2.3.18) and (2.3.19) are separately parabolic;

[54] See e.g. the monographs quoted in the item (VI) of the Bibliographical Note in Section 6. This model was also investigated by mathematicians, see e.g. Alexiades and Cannon [7], Alexiades and Solomon [8], Alexiades, Solomon and Wilson [9,425,464,465], Bermudez and Saguez [61], Crowley [151], Crowley and Ockendon [152], Fix [219], Ockendon and Tayler [363], Tayler [437].

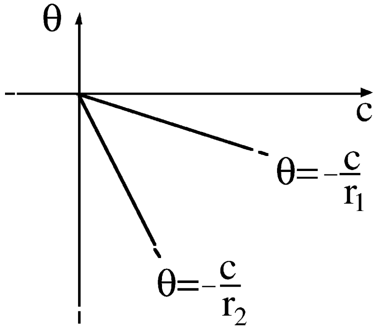

Fig. 9. Linearized *liquidus* and *solidus* curves.

however, coupled with (2.3.16), as a system they miss this property, for the multi-valued mapping

$$\mathbf{R}^2 \to 2^{\mathbf{R}^2} : \begin{pmatrix} \theta \\ w \end{pmatrix} \mapsto \begin{pmatrix} u \\ c \end{pmatrix} = \begin{pmatrix} C_V(\theta, \chi)\theta + L(\theta)\,\mathrm{sign}(\theta - w) \\ \zeta_1(w)\frac{1+\mathrm{sign}(\theta-w)}{2} + \zeta_2(w)\frac{1-\mathrm{sign}(\theta-w)}{2} \end{pmatrix} \tag{2.3.24}$$

fails to be monotone. This is easily checked, as this property fails for the discontinuous part of that mapping:

$$\mathbf{R}^2 \to 2^{\mathbf{R}^2} : \begin{pmatrix} \theta \\ w \end{pmatrix} \mapsto \begin{pmatrix} L(\theta)\,\mathrm{sign}(\theta - w) \\ \frac{1}{2}[\zeta_1(w) - \zeta_2(w)]\,\mathrm{sign}(\theta - w) \end{pmatrix}. \tag{2.3.25}$$

This analytical issue has a physical counterpart: the model of this section does not account for cross effects between heat and mass-diffusion: a temperature gradient induces a mass flux (*Soret effect*), and in turn a gradient of chemical potential causes a heat flux (*Dufour effect*).[55] Although in several cases the omitted terms do not seem to be quantitatively very significant, their absence impairs the analytical structure of the problem. Another physical drawback of this model was pointed out in Alexiades and Cannon [7], Alexiades, Wilson and Solomon [9].

These physical and mathematical drawbacks are overcome by the theory of *nonequilibrium thermodynamics*, that we illustrate in the next section. There the constitutive relations are dictated by the second principle, rather than being just extrapolated from the uncoupled heat and mass-diffusion, as above.

2.4. *Approach via nonequilibrium thermodynamics*

The physical and mathematical drawbacks that emerged in the last section induce us to represent phase transitions in binary alloys via an alternative and more successful approach, that also applies to more general heterogeneous systems. This is based on the following main elements:

(i) the first principle of thermodynamics and the principle of mass conservation (i.e., two balance laws),

[55] See e.g. the monographs quoted in the item (VII) of the Bibliographical Note in Section 6.

(ii) a constitutive relation for the entropy density (namely, the *Gibbs formula*),
(iii) three further constitutive relations, the so-called *phenomenological laws*, that are consistent with a local formulation of the second principle of thermodynamics, and include a dynamics of phase-relaxation.

The first two issues lead to the formulation of a doubly-nonlinear second-order system of PDEs. The third requirement accounts for dissipation, which in analytical terms corresponds to the forward parabolicity of the problem. This approach is based on the theory of nonequilibrium thermodynamics.[56]

Although one might deal with any number of constituents and also allow for chemical reactions, see e.g. Luckhaus and Visintin [317], in this simplified presentation we still confine ourselves to a nonreacting (noneutectic) binary system. As we did in Section 2.2, but at variance with the model of Section 2.3 and with previous works, here we also account for phase nonequilibrium by including the phase function among the state variables.

Balance laws and Gibbs formula. In view of extending the derivation of the entropy balance of Section 2.2 to binary alloys, let us define some further notation:
μ: difference between the *chemical potential* of the two constituents,
$\vec{j}_u$: flux of energy (per unit surface), due to flux of heat and mass,
h: intensity of a prescribed energy source or sink, due to injection or extraction either of heat or mass.

In the absence of chemical reactions, the principles of energy and mass conservation yield

$$\frac{\partial u}{\partial t} = -\nabla \cdot \vec{j}_u + h \quad \text{in } Q, \tag{2.4.1}$$

$$\frac{\partial c}{\partial t} = -\nabla \cdot \vec{j}_c \quad \text{in } Q. \tag{2.4.2}$$

We shall assume that the internal energy is a prescribed convex function of the entropy density, s, of the solute concentration, c, and of the phase function, χ; that is, $u = \hat{u}(s, c, \chi)$. The specific form of the function $\hat{u}$ obviously depends on the constituents.

Consistently with the discussion of Section 2.2, because of the constraints $0 \leqslant c \leqslant 1$ and $-1 \leqslant \chi \leqslant 1$, we may assume $\hat{u}$ to be differentiable for $(c, \chi) \in \,]0, 1[\, \times \,]-1, 1[$, but not on the boundary of this set. Let us extent $\hat{u}$ with value $+\infty$ for $(c, \chi) \notin \,]0, 1[\,\times\,]-1, 1[$. We may thus assume this function to be differentiable for (c, χ) in this open rectangle, but not on its boundary. We must then deal with the partial subdifferentials $\partial_c \hat{u}$, $\partial_\chi \hat{u}$ (see Section 5.2), which coincide with the respective derivatives only in $]0, 1[\,\times\,]-1, 1[$. Reminding the definition of the absolute temperature, $\tau = \partial_s \hat{u}(s, c, \chi)$, for any selection

$$\mu \in \partial_c \hat{u}(s, c, \chi), \qquad \lambda \in \partial_\chi \hat{u}(s, c, \chi), \tag{2.4.3}$$

we thus have

$$\begin{aligned} &u = \hat{u}(s, c, \chi), \\ &\mathrm{d}u = \tau\, \mathrm{d}s + \mu\, \mathrm{d}c + \lambda\, \mathrm{d}\chi \quad \forall (s, c, \chi) \in (\mathrm{Dom}\, \hat{u})^0. \end{aligned} \tag{2.4.4}$$

[56] See e.g. the monographs quoted in the item (VII) of the Bibliographical Note in Section 6. The mathematical aspects of this formulation were studied e.g. in Alexiades, Wilson and Solomon [9], Donnelly [182], Luckhaus [314], Luckhaus and Visintin [317].

As $\partial_s \hat{u}(s, c, \chi) = \tau > 0$, this constitutive relation may also be written in the equivalent form $s = \hat{s}(u, c, \chi)$. Moreover (2.4.4) is equivalent to the *Gibbs-type formula*

$$
\begin{aligned}
&s = \hat{s}(u, c, \chi),\\
&\mathrm{d}s = \frac{1}{\tau}\,\mathrm{d}u - \frac{\mu}{\tau}\,\mathrm{d}c - \frac{\lambda}{\tau}\,\mathrm{d}\chi \quad \forall (u, c, \chi) \in (\mathrm{Dom}\,\hat{s})^0,
\end{aligned}
\tag{2.4.5}
$$

and more generally

$$
\begin{aligned}
&\frac{1}{\tau} \in \partial_u \hat{s}(u, c, \chi), \quad -\frac{\mu}{\tau} \in \partial_c \hat{s}(u, c, \chi), \quad -\frac{\lambda}{\tau} \in \partial_\chi \hat{s}(u, c, \chi)\\
&\quad \forall (u, c, \chi) \in \mathrm{Dom}\,\hat{s}.
\end{aligned}
\tag{2.4.6}
$$

Consistently with a basic postulate of *nonequilibrium thermodynamics*, we assume that (2.4.4) and (2.4.5) also apply to systems that are not too far from equilibrium. Actually, the limits of validity of the whole theory strongly depend on those of the Gibbs formula (2.4.5).

Entropy balance. Let us multiply (2.4.1) by $1/\tau$ and (2.4.2) by $-\mu/\tau$. By (2.4.6) we thus get the *entropy balance* equation

$$
\begin{aligned}
\frac{\partial s}{\partial t} &= \frac{1}{\tau}\frac{\partial u}{\partial t} - \frac{\mu}{\tau}\frac{\partial c}{\partial t} - \frac{\lambda}{\tau}\frac{\partial \chi}{\partial t}\\
&= -\frac{1}{\tau}\nabla \cdot \vec{j}_u + \frac{h}{\tau} + \frac{\mu}{\tau}\nabla \cdot \vec{j}_c - \frac{\lambda}{\tau}\frac{\partial \chi}{\partial t}\\
&= -\nabla \cdot \frac{\vec{j}_u - \mu \vec{j}_c}{\tau} + \vec{j}_u \cdot \nabla \frac{1}{\tau} - \vec{j}_c \cdot \nabla \frac{\mu}{\tau} - \frac{\lambda}{\tau}\frac{\partial \chi}{\partial t} + \frac{h}{\tau}\\
&= -\nabla \cdot \vec{j}_s + \pi + \frac{h}{\tau} \quad \text{in } Q,
\end{aligned}
\tag{2.4.7}
$$

where we set

$$
\vec{j}_s := \frac{\vec{j}_u - \mu \vec{j}_c}{\tau}: \quad \text{entropy flux (per unit surface)}, \tag{2.4.8}
$$

$$
\begin{aligned}
&\pi := \vec{j}_u \cdot \nabla \frac{1}{\tau} - \vec{j}_c \cdot \nabla \frac{\mu}{\tau} - \frac{\lambda}{\tau}\frac{\partial \chi}{\partial t}:\\
&\quad \text{entropy production rate (per unit volume).}
\end{aligned}
\tag{2.4.9}
$$

The quantity h/τ is the rate at which entropy is either provided to the system or extracted from it by an external source or sink of heat. Note that $\vec{j}_u = \vec{q} + \mu \vec{j}_c$, where $\vec{q}$ is the heat flux; (2.4.8) and (2.4.9) then also read

$$
\vec{j}_s = \frac{\vec{q}}{\tau}, \qquad \pi = \vec{q} \cdot \nabla \frac{1}{\tau} - \frac{\vec{j}_c}{\tau} \cdot \nabla \mu - \frac{\lambda}{\tau}\frac{\partial \chi}{\partial t}. \tag{2.4.10}
$$

According to the local formulation of the second principle of thermodynamics, the entropy production rate is pointwise nonnegative, and vanishes only at equilibrium. This is tantamount to the *Clausius–Duhem inequality*:

$$\begin{aligned}&\pi \geqslant 0 \quad \text{for any process, and}\\ &\pi = 0 \quad \text{only if} \quad \nabla\tau = \nabla\mu = \vec{0} \quad \text{and} \quad \partial\chi/\partial t = 0.\end{aligned} \tag{2.4.11}$$

Moreover, $\pi = 0$ ($\pi > 0$, resp.) corresponds to a reversible (irreversible, resp.) process. We must then formulate constitutive laws consistent with (2.4.11).

Phenomenological laws. Let us introduce some further definitions:
$\vec{J} := (\vec{j}_u, \vec{j}_c, \partial\chi/\partial t)$: *generalized fluxes*,
$z := (1/\tau, -\mu/\tau, -\lambda/\tau)$ ($\in \mathrm{Dom}(s^*)$): dual state variables, $z' := (1/\tau, -\mu/\tau)$,
$\vec{G} := (\nabla(1/\tau), -\nabla(\mu/\tau), -\lambda/\tau)$: *generalized forces*.[57]

Along the lines of nonequilibrium thermodynamics, we assume that the generalized fluxes are functions of the dual state variables and of the generalized forces, via constitutive relations called *phenomenological laws*,

$$\vec{J} = \vec{F}(z, \vec{G}) \quad \forall z \in \mathrm{Dom}(s^*)\big(\subset \mathbf{R}^+ \times \mathbf{R}^2\big), \ \forall \vec{G} \in \big(\mathbf{R}^3\big)^2 \times \mathbf{R}, \tag{2.4.12}$$

that must be consistent with the second principle, cf. (2.4.11). The mapping $\vec{F}$ must thus be positive-definite w.r.t. $\vec{G}$. Close to thermodynamic equilibrium, namely, for small generalized forces, one may also assume that this dependence is linear. As the first two components of $\vec{J}$ and $\vec{G}$ are vectors and the third one is a scalar, the above linearized relations uncouple, because of the *Curie principle*: "generalized forces cannot have more elements of symmetry than the generalized fluxes that they produce."[58] Thus

$$\begin{pmatrix} \vec{j}_u \\ \vec{j}_c \end{pmatrix} = \mathcal{L}(z) \cdot \begin{pmatrix} \nabla\frac{1}{\tau} \\ -\nabla\frac{\mu}{\tau} \end{pmatrix} \ \big(= \mathcal{L}(z) \cdot \nabla z'\big), \tag{2.4.13}$$

$$a(z)\frac{\partial\chi}{\partial t} = -\frac{\lambda}{\tau}, \quad \text{that is,} \quad a(z)\frac{\partial\chi}{\partial t} \in \partial_\chi \hat{s}. \tag{2.4.14}$$

(In (2.4.13) the dot denotes the rows by columns product of a tensor of $(\mathbf{R}^3)^{2\times 2}$ by a vector of $(\mathbf{R}^3)^2$.) Consistently with (2.4.11), for any z the tensor $\mathcal{L}(z)$ is assumed to be positive-definite, and $a(z) > 0$. A fundamental result of nonequilibrium thermodynamics due to Onsager states that the tensor $\mathcal{L}(z)$ is symmetric:

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix}, \quad \mathcal{L}_{12}(z) = \mathcal{L}_{21}(z) \ \big(\in \mathbf{R}^3\big) \quad \forall z \in \mathrm{Dom}(s^*). \tag{2.4.15}$$

The phenomenological laws (2.4.13) and (2.4.14) may then be represented in gradient form, for a suitable potential Φ:

$$\vec{J} = \nabla\Phi(z, \vec{G}) \quad \forall z \in \mathrm{Dom}(s^*), \ \forall \vec{G} \in \big(\mathbf{R}^3\big)^2 \times \mathbf{R}, \tag{2.4.16}$$

where by $\nabla\Phi$ we denote the gradient w.r.t. the second argument, $\vec{G}$.

[57] Notice that $-\lambda/\tau$ occurs as a dual state variable and also as a generalized force. See Section 5.2 for the definition of the conjugate concave function s^*.
[58] See e.g. the monographs quoted in the item (VII) of the Bibliographical Note in Section 6.

In conclusion, we have represented processes in two-phase composites by the quasilinear parabolic system (2.4.1), (2.4.2), (2.4.6), (2.4.13), (2.4.14). This system is doubly nonlinear, and the techniques of Alt and Luckhaus [17], DiBenedetto and Showalter [179], and others may be used.

A transformation of the state variables. The vector of state variables $z := (1/\tau, -\mu/\tau, -\lambda/\tau)$ may equivalently be replaced by $\tilde{z} := (1/\tau, \mu, -\lambda/\tau)$; let us also set $\tilde{z}' := (1/\tau, \mu)$. As the corresponding transformation $\nabla z' \to \nabla \tilde{z}'$ is linear and $\vec{j}_u = \vec{q} + \mu \vec{j}_c$, by a simple calculation the linearized phenomenological laws (2.4.13) may equivalently be reformulated in terms of these new variables:

$$\begin{pmatrix} \vec{q} \\ \vec{j}_c \end{pmatrix} = \mathcal{M}(\tilde{z}) \cdot \begin{pmatrix} \nabla \frac{1}{\tau} \\ -\frac{1}{\tau} \nabla \mu \end{pmatrix} \left(=: \widetilde{\mathcal{M}}(\tilde{z}) \cdot \nabla \tilde{z}'\right). \tag{2.4.17}$$

(The latter equality represents the definition of the tensor $\widetilde{\mathcal{M}}$.) Like $\mathcal{L}$, the tensor $\mathcal{M}$ is also positive-definite for any $\tilde{z}$. By the symmetry of $\mathcal{L}$ it is easily checked that $\mathcal{M}$ is also symmetric, at variance with $\widetilde{\mathcal{M}}$. These conclusions may also be attained by applying the above argument based on the second principle to (2.4.10), instead of (2.4.9).

If the tensor $\mathcal{M}(\tilde{z})$ is diagonal and depends continuously on its arguments, then by the developments of the final part of Section 2.2 we retrieve Fourier- and Fick-type laws:

$$\begin{aligned} \vec{q} &= \mathcal{M}_{11}(\tilde{z}) \nabla \frac{1}{\tau} = -\frac{1}{\tau_E^2} \mathcal{M}_{11} \tilde{z} \nabla \theta \left[1 - 2\frac{\theta}{\tau_E} + \mathrm{o}\left(\frac{\theta}{\tau_E}\right)\right], \\ \vec{j}_c &= -\frac{1}{\tau} \mathcal{M}_{22}(\tilde{z}) \nabla \mu = -\frac{1}{\tau_E} \mathcal{M}_{22}(\tilde{z}) \nabla \mu \left[1 - \frac{\theta}{\tau_E} + \mathrm{o}\left(\frac{\theta}{\tau_E}\right)\right], \end{aligned} \tag{2.4.18}$$

with $\mathcal{M}_{11}$ and $\mathcal{M}_{22}$ positive scalars (more generally, positive-definite 3×3-tensors). On the other hand, a nondiagonal tensor $\mathcal{M}$ would also account for the Soret and Dufour cross effects.

As $\mathcal{M}_{22}$ and $\widetilde{D}_i(w)$ (cf. (2.3.14)) are both positive, by comparing (2.3.15) with (2.4.18) we see that ∇w is proportional to $-\nabla \mu$.

REMARKS. (i) It is possible to define a function analogous to (2.2.15), with the further dependence on c:

$$\varphi := u - \tau_E s, \quad \text{that is,} \quad \varphi = \hat{\varphi}(u, c, \chi) := u - \tau_E \hat{s}(u, c, \chi). \tag{2.4.19}$$

Along the lines of (2.2.15)–(2.2.21), one may reformulate the entropy balance in terms of φ, θ, μ and λ, and then linearize the state variables $1/\tau$, $-\lambda/\tau$, and $-\mu/\tau$ in a neighbourhood of $\tau = \tau_E$.

(ii) An approach based on nonequilibrium thermodynamics may also be applied to other coupled phenomena with phase transition, e.g. thermal and electromagnetic processes in a ferromagnetic body (with negligible hysteresis), see Visintin [449].

2.5. *Diffuse-interface models and length-scales*

In this section we introduce the Landau–Ginzburg representation of the free energy of a two-phase system, the associated Cahn–Hilliard and Allen–Cahn dynamics, and the

Penrose–Fife and phase-field models. As these models are set at a finer length-scale than the (macroscopic) scale that we considered in Section 1, we relate these free energy functionals and that of Section 2.1 by means of De Giorgi's notion of Γ-limit.[59]

Double wells. Along the lines of the Landau–Ginzburg theory of phase transitions, see e.g. Landau and Lifshitz [307], here we fix two positive parameters b and ε, and represent the free energy of a solid–liquid system by a functional of the form

$$F_{\theta,\varepsilon}(\chi) := \int_\Omega \left(\varepsilon b |\nabla\chi|^2 + \frac{1}{\varepsilon}\bigl(1-\chi^2\bigr)^2 - \frac{L}{2\tau_E}\theta\chi \right) \mathrm{d}x \quad \forall\chi \in H^1(\Omega), \tag{2.5.1}$$

plus a constant that may depend on the temperature field. By the direct method of the calculus of variations,[60] for any $\varepsilon > 0$ this functional has an absolute minimizer.

The terms $\varepsilon b|\nabla\chi|^2$ and the *double well* potential $(1-\chi^2)^2/\varepsilon$ compete for the minimization of $F_{\theta,\varepsilon}(\chi)$. As the second term is minimized by $\chi = \pm 1$, a temperature having nonuniform sign may induce sharp variations of χ between -1 to 1; but high gradients of χ are penalized by the first term. Compromising between these two exhigences, for small ε any relative minimizer of $F_{\theta,\varepsilon}(\chi)$ attains values that are close to ± 1 in the whole Ω, but for thin transition layers. The actual physical value of the coefficients b, ε is so small that the layer thickness is typically of the order of nanometers.

The functional (2.5.1) is Fréchet differentiable, and its functional derivative reads

$$DF_{\theta,\varepsilon}(\chi) = -2\varepsilon b\Delta\chi + \frac{4}{\varepsilon}\chi\bigl(\chi^2-1\bigr) - \frac{L\theta}{2\tau_E} \quad \forall\chi \in H^1(\Omega). \tag{2.5.2}$$

Because of the nonconvexity, a stationary point of $F_{\theta,\varepsilon}$ may either be an absolute minimizer, or a relative minimizer (namely, the absolute minimizer of the restriction of $F_{\theta,\varepsilon}$ to some neighbourhood of that point), or a saddle point, or even a relative maximizer. These points may respectively be interpreted as states of stable, metastable, and (for the two latter cases) unstable equilibrium.[61]

Two relaxation dynamics. The phase function χ may be regarded as an *order parameter;* the same applies to the solute concentration in alloys, the magnetization in ferromagnetics, the polarization in ferroelectrics, and so on. One may distinguish between phenomena in which for an isolated system the integral of the order parameter is conserved, and those in which it is not. Phase separation in alloys belongs to the first class, phase transition in solid–liquid systems to the second one.

Conserved-integral dynamics. Along the lines of Hohenberg and Halperin [274], in the first case one typically represents processes by a relaxation dynamics of the form

$$a\frac{\partial\chi}{\partial t} - \nabla\cdot\bigl\{K\cdot\nabla\bigl[DF_{\theta,\varepsilon}(\chi)\bigr]\bigr\} = 0 \qquad \text{in } Q; \tag{2.5.3}$$

[59] See e.g. the monographs quoted in the item (XIV) of the Bibliographical Note in Section 6.

[60] See e.g. Braides and Defranceschi [82], Buttazzo, Giaquinta and Hildebrandt [94], Carbone and De Arcangelis [117], Dacorogna [154], Dal Maso [155]), Evans and Gariepy [201], Giusti [245].

[61] The notion of metastability is (implicitly) referred to a time-scale. A (nonabsolute) relative minimizer will appear as stable at a sufficiently fine time-scale, and as unstable at a sufficiently long time-scale. Steel, polymers and glasses are examples of these stable-looking relative minimizers, up to geological time-scales.

here a is a positive relaxation coefficient, and K is a positive-definite tensor that may depend on the state variables. For instance let us consider a binary alloy and denote by c the concentration of one component; in this case $L = 0$. By coupling Eq. (2.5.3) with the homogeneous boundary condition $\{K \cdot \nabla[DF_{\theta,\varepsilon}(c)]\} \cdot \nu = 0$, one then sees that $\int_\Omega c(x,t)\,\mathrm{d}x$ is constant in time. After Cahn [107,108], Cahn and Hilliard [109], for isothermal processes one thus gets the *Cahn–Hilliard equation* of phase separation:

$$a\frac{\partial c}{\partial t} + \nabla \cdot \left\{ K \cdot \nabla \left[-2\varepsilon b \Delta c + \frac{4}{\varepsilon} c\left(c^2 - 1\right) \right] \right\} = \lambda \quad \text{in } Q, \tag{2.5.4}$$

where λ is the (unknown) Lagrange multiplier associated to the constraint on the integral of χ. The phenomenon of phase separation and the analytical properties of the Cahn–Hilliard equation were studied in a large literature.[62]

Nonconserved-integral dynamics. On the other hand for solid–liquid systems the typical dynamics of the phase function χ (and more generally that of nonconserved order parameters) reads

$$a\frac{\partial \chi}{\partial t} + DF_{\theta,\varepsilon}(\chi) = 0 \quad \text{in } Q. \tag{2.5.5}$$

For $F_{\theta,\varepsilon}(\chi)$ as in (2.5.1), this yields the *Allen–Cahn* (or *Landau–Ginzburg) equation*, see Allen and Cahn [13]:

$$a\frac{\partial \chi}{\partial t} - 2\varepsilon b \Delta \chi + \frac{4}{\varepsilon}\chi\left(\chi^2 - 1\right) = \frac{L\theta}{2\tau_E} \quad \text{in } Q. \tag{2.5.6}$$

The Penrose–Fife and phase-field models. So far we dealt with the representation of the free energy functional and with its dynamics. For nonisothermal processes equation (2.5.6) must be coupled with the energy balance. As we saw, the theory of nonequilibrium thermodynamics leads one to formulate the Fourier law as the proportionality between the heat flux and $-\nabla(1/\tau)$, cf. (2.2.12). An approach of this sort was proposed by Penrose and Fife in [373,374], and then studied in many works.[63]

By linearizing $1/\tau$ in a neighbourhood of $1/\tau_E$ we have $\nabla(1/\tau) \simeq -\nabla\theta/\tau_E^2$, and the Fourier law is reduced to the form (1.1.3). By coupling this law with the energy balance (1.1.2) and with the free energy dynamics (2.5.6), one obtains the so-called *phase-field* model, which was first proposed by Fix [219–221] and Collins and Levine [146], and was then extensively studied by Caginalp and others.[64]

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62 See e.g. Alikakos, Bates and Chen [10], Bates and Fife [50,51], Blowey and Elliott [69], Caginalp [104], Chen [126], Chen, Hong and Yi [127], Elliott [190], Elliott and Garke [191], Elliott and Zheng [194], Escher and Simonett [197], Kessler et al. [287], Novick-Cohen and Segel [361], Pego [372], Rappaz and Scheid [385]. See also Alt and Pawlow [18–20], Fabrizio, Giorgi and Morro [204], for the extension to nonisothermal processes.

63 See e.g. Bonetti et al. [71], Chen and Fife [125], Colli and Laurençot [139], Colli and Plotnikov [140], Colli and Sprekels [142,143], Fife [217], Hilliard [268], Kenmochi and Kubo [286], Miranville, Yin and Showalter [333], Sprekels and Zheng [429], Wang et al. [461], Zheng [476].

64 See e.g. Aizicovici and Barbu [5], Caginalp [101–104], Caginalp and Xie [105], Colli, Gilardi and Grasselli [132], Colli et al. [133], Fried and Gurtin [226–228], Miranville, Yin and Showalter [333], Krejčí, Rocca and Sprekels [297], Novick-Cohen [359,360], Plotnikov and Starovoitov [375].

A limit procedure. As we already pointed out, the parameter ε defines a nanoscopic length-scale. It is then of interest to study the limit of the free energy $F_{\theta,\varepsilon}$ as ε vanishes, in order to retrieve a model at a larger length-scale. The notion of *Γ-limit* in the sense of De Giorgi, see Section 5.8, is especially appropriate to represent the asymptotic behaviour of the absolute minimizers of $F_{\theta,\varepsilon}$, see Proposition 5.8.2.[65]

PROPOSITION 2.5.1 *(Γ-limit)*[66]. *Let $\theta \in L^1(\Omega)$ and $\sigma = 4\sqrt{b}/3$. As $\varepsilon \to 0$, the family of functionals $\{F_{\theta,\varepsilon}\}$ Γ-converges to $\Phi_{\theta,\sigma}$ in $L^1(\Omega)$ (here with $\sigma_L = \sigma_S$), cf.* (2.1.5).

This result directly follows from Theorem 5.8.4. By Corollary 5.8.3 the latter statement entails that, if u_ε is an absolute minimizer of the functional $F_{\theta,\varepsilon}$ for any ε, then there exists a state $u \in L^1(\Omega)$ such that, as ε vanishes along a suitable sequence (not relabelled),

$$u_\varepsilon \to u \quad \text{in } L^1(\Omega), \qquad F_{\theta,\varepsilon}(u_\varepsilon) \to \Phi_{\theta,\sigma}(u) = \inf \Phi_{\theta,\sigma}. \tag{2.5.7}$$

Macroscopic-mesoscopic, and microscopic length-scales. We represented the free energy of solid–liquid systems at three length-scales, see Table 3:

(i) At the *macroscopic* scale the functional Φ_θ is convex, cf. (2.1.6), and processes may be described by the weak formulation of the Stefan problem, i.e., Problem 1.1.1. At this length-scale a mushy region may appear, corresponding to the condition $|\chi| < 1$ a.e.. The solid–liquid interface may accordingly be either sharp or diffuse.

(ii) At the *mesoscopic* scale $\sigma \ll 1$ the functional $\Phi_{\theta,\sigma}$ is nonconvex, cf. (2.1.5), and evolution may be represented by the Stefan–Gibbs–Thomson problem. Here $|\chi| = 1$ a.e., and thus one distinguishes solid from liquid parts, also in the mushy region. In other terms, what at the macroscopic scale appears as a mushy region is here resolved in its liquid and solid constituents.

(iii) At the *microscopic* scale $\varepsilon \ll \sigma$ the functional $F_{\theta,\varepsilon}$ is also nonconvex, cf. (2.5.1), and evolution may be described by the phase-field model. Here χ varies smoothly: the interface is represented by a nanoscopic transition layer, and may thus be regarded as diffuse. In this case $|\chi| < 1$ a.e., but intermediate values of χ represent a transition layer rather than a mushy region.

For instance, length-scales of the order of the millimeter, of the micrometer and of the nanometer may loosely be labelled as macroscopic, mesoscopic and microscopic, respectively. The process of *zooming out* from the microscopic to the mesoscopic scale is here represented by the Γ-limit as $\varepsilon \to 0$. On the other hand the Γ-limit as $\sigma \to 0$ accounts for the passage from the mesoscopic to the macroscopic scale.

For evolution problems the asymptotic analysis is more delicate. A large literature was devoted to the models of Cahn–Hilliard, Allen–Cahn, Mullins–Sekerka, Stefan–Gibbs–Thomson, in particular to establish asymptotic relations among them.[67]

65 The analysis of the limit behaviour of the relative minimizers would also be of interest, but needs a different approach; see e.g. Dal Maso and Modica [156,157].

66 See Luckhaus and Modica [315], Modica [335,336] for related results.

67 See e.g. Alikakos, Bates and Chen [10], Alikakos, Bates, Chen and Fusco [11], Alikakos, Fusco and Karali [12], Caginalp [103,104], Escher and Simonett [197], Evans, Soner and Souganidis [202], Garroni and

Table 3
Comparison among some properties of the solid–liquid interface at different length-scales

Macroscopic scale	Mesoscopic scale	Microscopic scale
Stefan model	Stefan–Gibbs–Thomson model	Phase-field model
Sharp/diffuse interface	Sharp interface	Diffuse interface
Convex free energy	Nonconvex free energy	Nonconvex free energy
$\lvert\chi\rvert \leqslant 1$	$\lvert\chi\rvert = 1$	$\lvert\chi\rvert \leqslant 1$
No $\nabla\chi$ in free energy	$\sigma \int_\Omega \lvert\nabla\chi\rvert$	$\varepsilon b \int_\Omega \lvert\nabla\chi\rvert^2 \,\mathrm{d}x$
$\chi \in L^\infty(\Omega)$	$\chi \in L^\infty(\Omega) \cap \mathrm{BV}(\Omega)$	$\chi \in L^\infty(\Omega) \cap H^1(\Omega)$

Discussion. The Landau–Ginzburg potential $F_{\theta,\varepsilon}$ may look rather appealing to the mathematical analyst: its principal part is quadratic in the derivatives, the nonconvexity is confined to the null-order term and is smooth. This functional is Frechét differentiable, and the differential part of the derivative is linear, so that the Euler equation is semilinear. The Cahn–Hilliard and Allen–Cahn relaxation dynamics are easily formulated, and are also semilinear. What better?

On the other hand, its Γ-limit as $\varepsilon \to 0$, namely the functional $\Phi_{\theta,\sigma}$, cf. (2.1.5), misses all these nice features. Its principal part is nonquadratic (even worse: it has critical growth of degree 1), and is not integrable: it is just a Borel measure. Here the nonconvexity is highly nonsmooth: it is the characteristic constraint, namely $|\chi| = 1$. The functional $\Phi_{\theta,\sigma}$ is nondifferentiable, and the associated dynamics is also nontrivial: it consists of the *mean curvature flow* with forcing term.[68] What worse?

All elements seem to indicate that $F_{\theta,\varepsilon}$ should be preferred to $\Phi_{\theta,\sigma}$. In favour of the latter there are however two features: $\Phi_{\theta,\sigma}(\chi)$ is more appropriate for mesoscopic models, and it may be discretized by means of a coarser mesh, for it is set at a larger length-scale than $F_{\theta,\varepsilon}$.[69]

3. Analysis of the weak formulation of the Stefan model

In Section 1.1 we represented the weak formulation of the basic Stefan model as an initial- and boundary-value problem for the quasilinear parabolic system

$$\begin{cases} u \in \alpha(\theta), \\ \dfrac{\partial u}{\partial t} - \nabla \cdot \big[k(\theta) \cdot \nabla\theta\big] = f \end{cases} \quad \text{in } Q, \tag{3.0.1}$$

Niethammer [239], Luckhaus and Sturzenhecker [316], Krejčí, Rocca and Sprekels [297] Miranville, Yin and Showalter [333], Niethammer [352,353], Plotnikov and Starovoitov [375], Röger [402], Soner [427,428], Stoth [432], and the detailed review of Soner [426].

68 See e.g. the monographs Almgren and Wang [16], Buttazzo and Visintin [95], Damlamian, Spruck and Visintin [163], Evans and Spruck [203], Giga [242] and references therein.

69 One might also question the use of a continuous model at a microscopic (actually, nanoscopic) length-scale: is it really justified to apply differential calculus at a scale at which the discrete structure of matter starts becoming perceivable?

for a multi-valued maximal monotone function α, cf. Problem 1.1.1. In this part we illustrate a number of classical methods that may be used for the analysis of this problem. Specifically, we deal with:

(i) approximation by implicit time-discretization (Section 3.1),
(ii) a priori estimates in L^2-spaces (Section 3.1),
(iii) passage to the limit via compactness, monotonicity and lower semicontinuity techniques (Section 3.1),
(iv) a contraction procedure in $L^1(\Omega)$ (Section 3.2),
(v) a priori estimates in $L^\infty(\Omega)$ (Section 3.2),
(vi) a priori estimates in $L^q(\Omega)$ (Section 3.2),
(vii) a variable transformation via time-integration (Section 3.3),
(viii) a variable transformation via inversion of the elliptic operator (Section 3.3),
(ix) nonlinear semigroups of contractions in $H^{-1}(\Omega)$ and in $L^1(\Omega)$ (Section 3.4).[70]

3.1. *L^2-techniques*

In this section we prove the existence of a solution of the weak formulation of the basic Stefan problem in any prescribed time interval, show its structural stability, and derive some regularity results. This gives us the opportunity to illustrate some basic techniques for the analysis of quasilinear parabolic equations in Sobolev spaces.[71]

As above we shall assume that Ω is a bounded domain of $\mathbf{R}^3$ of Lipschitz class, denote its boundary by Γ, fix any $T > 0$, and set $Q := \Omega \times]0, T[$, $\Sigma := \Gamma \times]0, T[$. We also fix a subset Γ_D of Γ of positive bidimensional Hausdorff measure, and set $\Gamma_N := \Gamma \setminus \Gamma_D$,

$$V := \{v \in H^1(\Omega): \gamma_0 v = 0 \text{ on } \Gamma_D\},$$

where γ_0 denotes the *trace operator*.[72] This is a Hilbert space equipped with the customary H^1-norm, which by the Friedrichs-Poincaré inequality is equivalent to

$$\|v\|_V := \left(\int_\Omega |\nabla v|^2 \,\mathrm{d}x \right)^{1/2}.$$

We shall identify the space $L^2(\Omega)$ with its dual $L^2(\Omega)'$. As V is a dense subspace of $L^2(\Omega)$, the dual space $L^2(\Omega)'$ may in turn be identified with a subspace of V'. This yields the *Hilbert triplet*

$$V \subset L^2(\Omega) = L^2(\Omega)' \subset V', \quad \text{with dense and compact injections.} \tag{3.1.2}$$

We shall denote by $\langle \cdot,\cdot \rangle$ the duality pairing between V' and V, and define the linear, continuous and coercive operator

$$A : V \to V', \quad \langle Au, v \rangle := \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x \quad \forall u, v \in V,$$

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[70] This part rests upon the classical theory of linear and nonlinear PDEs, see e.g. the monographs quoted in the items (X)–(XIII) of the Bibliographical Note in Section 6.

[71] See e.g. the monographs quoted in the item (XIII) of the Bibliographical Note in Section 6, Alt and Luckhaus [17], Brezis [83], Damlamian [158], Damlamian and Kenmochi [161], and many others.

[72] See e.g. Lions and Magenes [312].

whence $Au = -\Delta u$ in $\mathcal{D}'(\Omega)$. We also assume that[73]

$$\varphi : \mathbf{R} \to \widetilde{\mathbf{R}} \ \big(:= \mathbf{R} \cup \{+\infty\}\big)$$
$$\text{is lower semicontinuous and convex, and } \varphi \not\equiv +\infty, \tag{3.1.3}$$

denote its convex conjugate function by φ^*, and notice that $\partial\varphi^* = (\partial\varphi)^{-1}$, cf. (5.2.11). Finally, we fix any

$$u^0 \in L^2(\Omega), \qquad f \in L^2(0, T; V'), \tag{3.1.4}$$

and introduce our weak formulation.

PROBLEM 3.1.1. Find $u \in L^2(Q)$ and $\theta \in L^2(0, T; V)$ such that

$$u(\theta - v) \geqslant \varphi(\theta) - \varphi(v) \quad \forall v \in \mathrm{Dom}(\varphi), \text{ a.e. in } Q, \tag{3.1.5}$$

$$\begin{gathered} \iint_Q \left[\big(u^0 - u\big) \frac{\partial v}{\partial t} + \nabla\theta \cdot \nabla v \right] \mathrm{d}x\, \mathrm{d}t = \int_0^T \langle f, v \rangle\, \mathrm{d}t \\ \forall v \in L^2(0, T; V) \cap H^1\big(0, T; L^2(\Omega)\big), \quad v(\cdot, T) = 0. \end{gathered} \tag{3.1.6}$$

Interpretation. By Proposition 5.2.5 the variational inequality (3.1.5) is tantamount to the inclusion

$$u \in \partial\varphi(\theta) \quad \text{a.e. in } Q. \tag{3.1.7}$$

The variational equation (3.1.6) yields

$$\frac{\partial u}{\partial t} + A\theta = f \quad \text{in } V' \text{ a.e. in }]0, T[, \tag{3.1.8}$$

whence $\partial u/\partial t = f - A\theta \in L^2(0, T; V')$. Thus $u \in H^1(0, T; V')$, and by integrating (3.1.6) by parts in time we get

$$u|_{t=0} = u^0 \quad \text{in } V' \text{ (in the sense of the traces of } H^1(0, T; V')). \tag{3.1.9}$$

In turn (3.1.8) and (3.1.9) yield (3.1.6). In view of interpreting Eq. (3.1.8), let us now take

$$g \in L^2(Q), \qquad h \in L^2\big(\Gamma_N \times]0, T[\big), \tag{3.1.10}$$

and define $f \in L^2(0, T; V')$ by setting

$$\begin{gathered} \big\langle f(t), v \big\rangle := \int_\Omega g(x, t) v(x)\, \mathrm{d}x + \int_{\Gamma_N} h(x, t) \gamma_0 v(x)\, \mathrm{d}\sigma \\ \forall v \in V, \text{ for a.a. } t \in]0, T[. \end{gathered} \tag{3.1.11}$$

In this case Eq. (3.1.8) then yields the differential equation

$$\frac{\partial u}{\partial t} - \Delta\theta = g \quad \text{in } \mathcal{D}'(Q), \tag{3.1.12}$$

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[73] This function φ should not be confused with that of Section 2.2.

whence $\Delta\theta = \partial u/\partial t - g \in H^{-1}(0,T;L^2(\Omega))$. Denoting by $\partial/\partial\nu$ the external normal derivative, the trace $\partial\theta/\partial\nu$ is then an element of $H^{-1}(0,T;H^{1/2}_{00}(\Gamma_N)')$.[74] By partial integration of (3.1.6) we then retrieve the Neumann condition

$$\frac{\partial\theta}{\partial\nu} = h \quad \text{in } H^{-1}\big(0,T;H^{1/2}_{00}(\Gamma_N)'\big). \tag{3.1.13}$$

Moreover the definition of V obviously yields the Dirichlet condition

$$\gamma_0\theta = 0 \quad \text{on } \Gamma_D \times]0,T[. \tag{3.1.14}$$

In conclusion, if f is as in (3.1.10) and (3.1.11), then (3.1.6) is a weak formulation of the system (3.1.12)–(3.1.14).

REMARK. Let k be a continuous, bounded and positive function $\mathbf{R} \to \mathbf{R}$. If we replace $\Delta\theta$ by $\nabla\cdot[k(\theta)\nabla\theta]$ in Eq. (3.1.12), then by the *Kirchhoff transformation*

$$\mathcal{K}: V \to V : \theta \mapsto \tilde\theta := \int_0^\theta k(\xi)\,\mathrm{d}\xi \tag{3.1.15}$$

we get $\nabla\cdot[k(\theta)\nabla\theta] = \Delta\tilde\theta$. As $\mathcal{K}$ is invertible, the inclusion (3.1.7) may then be replaced by $u \in \partial\varphi(\mathcal{K}^{-1}(\tilde\theta))$ a.e. in Q, which is also of the form $u \in \partial\tilde\varphi(\tilde\theta)$, for a lower semicontinuous and convex function $\tilde\varphi : \mathbf{R} \to \tilde{\mathbf{R}}$. A formulation like Problem 3.1.1 for the unknown pair $(u,\tilde\theta)$ is thus retrieved in this case, too.

THEOREM 3.1.1 *(Existence). Assume that the hypotheses* (3.1.3) *and* (3.1.4) *are satisfied. If*

$$\exists L, M > 0\colon \forall(\xi,\eta) \in \operatorname{graph}(\partial\varphi), \quad |\xi| \leqslant L|\eta| + M, \tag{3.1.16}$$

$$\varphi^*\big(u^0\big) \in L^1(\Omega), \tag{3.1.17}$$

then Problem 3.1.1 *has a solution such that* $u \in L^\infty(0,T;L^2(\Omega))$.

PROOF. (i) *Approximation*. We shall prove existence of a solution by means of a classic procedure: time-discretization, derivation of a priori estimates, and passage to the limit.[75] We fix any $m \in \mathbf{N}$, set

$$\begin{aligned} &k := \frac{T}{m}, \quad u_m^0 := u^0, \\ &f_m^n := \frac{1}{k}\int_{(n-1)k}^{nk} f(\tau)\,\mathrm{d}\tau \quad \text{in } V', \ \text{for } n = 1,\dots,m, \end{aligned} \tag{3.1.18}$$

and approximate our problem by the following implicit time-discretization scheme.

74 By $H^{1/2}_{00}(\Gamma_N)$ we denote the Hilbert space of the restrictions to Γ_N of the functions of $H^{1/2}(\Gamma)$, that vanish a.e. in $\Gamma \setminus \Gamma_N$, $H^{1/2}(\Gamma)$ being the space of the traces of the functions of $H^1(\Omega)$. See e.g. Lions and Magenes [312, vol. I] and other monographs on Sobolev spaces, that are quoted in the item (XII) of the Bibliographical Note in Section 6.

75 See e.g. the discussion in the introduction of Lions [311].

PROBLEM 3.1.1$_m$. Find $u_m^n \in L^2(\Omega)$ and $\theta_m^n \in V$ for $n = 1, \dots, m$, such that for $n = 1, \dots, m$

$$\frac{u_m^n - u_m^{n-1}}{k} + A\theta_m^n = f_m^n \quad \text{in } V', \tag{3.1.19}$$

$$u_m^n \in \partial\varphi\big(\theta_m^n\big) \quad \text{a.e. in } \Omega. \tag{3.1.20}$$

In view of solving this problem step by step, let us fix any $n \in \{1, \dots, m\}$, assume that $u_m^{n-1} \in L^2(\Omega)$ is known, and define the lower semicontinuous, convex, coercive functional

$$J_m^n(v) := \int_\Omega \Big[\varphi(v) + \frac{k}{2}|\nabla v|^2 - u_m^{n-1} v\Big]\,\mathrm{d}x - k\big\langle f_m^n, v\big\rangle \quad \forall v \in V.$$

By the direct method of the calculus of variations, this functional has a minimizer $\theta_m^n \in V$. Hence $0 \in \partial J_m^n(\theta_m^n)$ in V', whence by Theorem 5.2.3

$$0 \in \partial\varphi\big(\theta_m^n\big) + kA\theta_m^n - u_m^{n-1} - kf_m^n \quad \text{in } V',\ \forall n \in \{1, \dots, m\}. \tag{3.1.21}$$

This inclusion is equivalent to the system (3.1.19) and (3.1.20).

(ii) *A priori estimates*. First notice that by (3.1.20) and (5.2.11), for $n = 1, \dots, m$

$$\theta_m^n \in (\partial\varphi)^{-1}\big(u_m^n\big) = \partial\varphi^*\big(u_m^n\big) \quad \text{a.e. in } \Omega,$$

whence

$$\theta_m^n\big(u_m^n - u_m^{n-1}\big) \geqslant \varphi^*\big(u_m^n\big) - \varphi^*\big(u_m^{n-1}\big) \quad \text{a.e. in } \Omega. \tag{3.1.22}$$

Let us now multiply (3.1.19) by $k\theta_m^n$ and sum for $n = 1, \dots, \ell$, for any $\ell \in \{1, \dots, m\}$. By (3.1.22) this yields

$$\begin{aligned} &\int_\Omega \big[\varphi^*\big(u_m^\ell\big) - \varphi^*\big(u^0\big)\big]\,\mathrm{d}x + k\sum_{n=1}^{\ell}\int_\Omega \big|\nabla\theta_m^n\big|^2\,\mathrm{d}x \\ &\quad \leqslant k\sum_{n=1}^{\ell}\big\| f_m^n\big\|_{V'}\big\|\theta_m^n\big\|_V \leqslant \|f\|_{L^2(0,T;V')}\Bigg(k\sum_{n=1}^{\ell}\big\|\theta_m^n\big\|_V^2\Bigg)^{1/2}. \end{aligned} \tag{3.1.23}$$

By (3.1.16) it is easy to check that

$$\varphi^*(\eta) \geqslant \frac{1}{2L}(\eta - M)^2 \quad \forall \eta \in \mathbf{R} \text{ such that } |\eta| \geqslant M. \tag{3.1.24}$$

The inequality (3.1.23) then yields

$$\max_{n=1,\dots,m}\big\|u_m^n\big\|_{L^2(\Omega)}, k\sum_{n=1}^{m}\big\|\theta_m^n\big\|_V^2 \leqslant C_1. \tag{3.1.25}$$

(By $C_1, C_2, \dots$ we shall denote several constants that are independent of m.)

For any family $\{v_m^n\}_{n=0,\dots,m}$ of functions $\Omega \to \mathbf{R}$, let us now set

$$\begin{aligned} &v_m := \text{piecewise linear time-interpolate of } v_m^0, \dots, v_m^m, \ \text{a.e. in } \Omega, \\ &\bar v_m(\cdot, t) := v_m^n \quad \text{a.e. in } \Omega, \forall t \in \big](n-1)h, nh\big[, \ \text{for } n = 1, \dots, m. \end{aligned} \tag{3.1.26}$$

Defining $u_m, \bar{u}_m, \bar{\theta}_m, \bar{f}_m$ in this way, the system (3.1.19), (3.1.20) and the estimates (3.1.25) also read

$$\frac{\partial u_m}{\partial t} + A\bar{\theta}_m = \bar{f}_m \quad \text{in } V', \text{ a.e. in }]0,T[, \tag{3.1.27}$$

$$\bar{u}_m \in \partial\varphi(\bar{\theta}_m) \quad \text{a.e. in } Q, \tag{3.1.28}$$

$$\|u_m\|_{L^\infty(0,T;L^2(\Omega))}, \|\bar{\theta}_m\|_{L^2(0,T;V)} \leqslant C_2. \tag{3.1.29}$$

Hence $A\bar{\theta}_m$ is uniformly bounded in $L^2(0,T;V')$, and by comparing the terms of (3.1.27) we get

$$\|u_m\|_{H^1(0,T;V')} \leqslant C_3. \tag{3.1.30}$$

(iii) *Limit procedure*. By the uniform estimates (3.1.29) and (3.1.30), there exist θ, u such that, possibly taking $m \to \infty$ along a subsequence (not relabelled),

$$\bar{\theta}_m \to \theta \quad \text{weakly in } L^2(0,T;V), \tag{3.1.31}$$

$$u_m \to u \quad \text{weakly star in } L^\infty\big(0,T;L^2(\Omega)\big) \cap H^1(0,T;V'). \tag{3.1.32}$$

By passing to the limit in (3.1.27) we then get (3.1.8). In view of proving (3.1.5), first notice that by the compactness of the injection $V \subset L^2(\Omega)$ and by Sobolev-space interpolation[76]

$$L^\infty\big(0,T;L^2(\Omega)\big) \cap H^1(0,T;V') \subset C^0\big([0,T];V'\big) \quad \text{with compact injection}, \tag{3.1.33}$$

so that by (3.1.32)

$$u_m \to u \quad \text{strongly in } C^0\big([0,T];V'\big). \tag{3.1.34}$$

Hence $\|\bar{u}_m - u_m\|_{V'} \to 0$ uniformly in $]0,T[$, and we get

$$\iint_Q \bar{u}_m\bar{\theta}_m \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \langle \bar{u}_m - u_m, \bar{\theta}_m\rangle \,\mathrm{d}t + \iint_Q u_m\bar{\theta}_m \,\mathrm{d}x\,\mathrm{d}t \to \iint_Q u\theta \,\mathrm{d}x\,\mathrm{d}t. \tag{3.1.35}$$

By (3.1.20), (3.1.31), (3.1.32), (3.1.35), applying Corollary 5.5.5 we then get (3.1.5).

Finally, by (3.1.34) the initial condition for u (cf. (3.1.18)) is preserved in the limit. □

REMARK. By the self-adjointness of the operator $-\Delta$, existence of a solution might also be proved without exploiting the compactness of the injection $V \subset L^2(\Omega)$, by a procedure that is only based on the convexity and lower semicontinuity of the function φ.[77]

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[76] See e.g. Lions and Magenes [312].

[77] See e.g. Visintin [453, Section II.3].

Next we show that our problem is structurally stable.

THEOREM 3.1.2 *(Weakly-continuous dependence on the data). Let us assume that* $\{\varphi_n\}$, $\{f_n\}$, $\{u_n^0\}$ *are sequences that fulfill the assumptions of Theorem* 3.1.1, *with* L *and* M *independent of* n *in* (3.1.16). *Let us also assume that the sequence* $\{\varphi_n^*(u_n^0)\}$ *is bounded in* $L^1(\Omega)$, *and that*

$$\varphi_n \to \varphi \quad \text{uniformly in } \mathbf{R}, \tag{3.1.36}$$

$$f_n \to f \quad \text{weakly in } L^2(0,T;V'), \tag{3.1.37}$$

$$u_n^0 \to u^0 \quad \text{weakly in } L^2(\Omega). \tag{3.1.38}$$

For any n, *let* (u_n, θ_n) *be a solution of the corresponding Problem* $3.1.1_n$.[78] *Then there exist* θ *and* u *such that, as* $n \to \infty$ *along a suitable sequence* (*not relabelled*),

$$u_n \to u \quad \text{weakly star in } L^\infty\big(0,T;L^2(\Omega)\big) \cap H^1(0,T;V'), \tag{3.1.39}$$

$$\theta_n \to \theta \quad \text{weakly in } L^2(0,T;V). \tag{3.1.40}$$

This entails that (u,θ) *is a solution of Problem* 3.1.1.

We shall see that the solution of Problem 3.1.1 is unique, so that it is not necessary to extract any subsequence in (3.1.39) and (3.1.40).

OUTLINE OF THE PROOF.[79] By multiplying Eq. $(3.1.8)_n$ by θ_n and then using the procedure of the proof of Theorem 3.1.1, one may easily derive uniform estimates like (3.1.29) and (3.1.30). One can then pass to the limit as above as $n \to \infty$ along a subsequence, and show that (u,θ) is the solution of Problem 3.1.1 by mimicking the argument of Theorem 3.1.1. □

Regularity. Several regularity properties may be proved for the solution of Problem 3.1.1, by deriving further a priori estimates on the approximate solution. Next we just illustrate three examples, all based on the symmetry of the elliptic operator (i.e., Δ).

PROPOSITION 3.1.3 *(First regularity result). If the assumptions of Theorem* 3.1.2 *are satisfied and*

$$\begin{aligned} &\exists c > 0\colon \forall(\xi_i,\eta_i) \in \operatorname{graph}(\partial\varphi)\ (i=1,2),\\ &(\xi_1-\xi_2)(\eta_1-\eta_2) \geqslant c(\xi_1-\xi_2)^2, \end{aligned} \tag{3.1.41}$$

then Problem 3.1.1 *has a solution such that*

$$\theta \in H^r\big(0,T;L^2(\Omega)\big) \quad \forall r < \tfrac{1}{2}. \tag{3.1.42}$$

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[78] By this we denote the problem that is obtained from Problem 3.1.1 by replacing the functions $u, \theta, \ldots$ by $u_n, \theta_n, \ldots$. We shall use this sort of notation several repeatedly.

[79] See Visintin [453, Section II.3] for details.

OUTLINE OF THE PROOF.[80] The first part of the proof of Theorem 3.1.1 yields (3.1.29) and (3.1.30). In view of deriving a further uniform estimate, let us first fix any $h \in]0, T[$ and define the shift operator $(\tau_h v)(t) := v(t + h)$ for any $t \in \mathbf{R}$ and any function $v : \mathbf{R} \to \mathbf{R}$. Let us then set $\theta_m(\cdot, t) := \theta_m(\cdot, 0)$ for any $t < 0$, multiply the approximate equation (3.1.27) by $\theta_m - \tau_{-h}\theta_m$, and then integrate with respect to t. This procedure provides a uniform estimate for θ_m in $H^r(0, T; L^2(\Omega))$. □

PROPOSITION 3.1.4 *(Second regularity result). If the assumptions of Theorem 3.1.2 and (3.1.41) are satisfied and moreover*[81]

$$\theta^0 := \partial\varphi^*\big(u^0\big) \in V, \tag{3.1.43}$$

$$f \in L^2(Q) + W^{1,1}(0, T; V'), \tag{3.1.44}$$

then Problem 3.1.1 has a solution such that

$$u \in L^\infty\big(0, T; L^2(\Omega)\big), \qquad \theta \in H^1\big(0, T; L^2(\Omega)\big) \cap L^\infty(0, T; V). \tag{3.1.45}$$

OUTLINE OF THE PROOF.[82] By (3.1.41) we have

$$\int_\Omega \big(u_m^n - u_m^{n-1}\big)\big(\theta_m^n - \theta_m^{n-1}\big)\,\mathrm{d}x \geqslant c \int_\Omega \big(\theta_m^n - \theta_m^{n-1}\big)^2\,\mathrm{d}x \quad \text{for } n = 1, \dots, \ell. \tag{3.1.46}$$

Multiplying (3.1.19) by $\theta_m^n - \theta_m^{n-1}$ and summing for $n = 1, \dots, \ell$, for any $\ell \in \{1, \dots, m\}$, one then gets uniform estimates on u_m and θ_m that yield (3.1.45) in the limit. □

PROPOSITION 3.1.5 *(Third regularity result). If (3.1.3), (3.1.16), (3.1.41) are satisfied and*

$$f = f_1 + f_2, \quad \sqrt{t}\, f_1 \in L^2(Q), \quad \sqrt{t}\,\frac{\partial f_2}{\partial t} \in L^1(0, T; V'), \tag{3.1.47}$$

then Problem 3.1.1 has a solution such that

$$\sqrt{t}\,\frac{\partial\theta}{\partial t} \in L^2(Q), \qquad \sqrt{t}\,\theta \in L^\infty(0, T; V). \tag{3.1.48}$$

Hence $\theta \in H^1(\delta, T; L^2(\Omega)) \cap L^\infty(\delta, T; V)$ for any $\delta > 0$, without any hypothesis for the initial datum u^0.

OUTLINE OF THE PROOF.[83] The further regularity (3.1.48) stems from an estimation procedure that follows the lines of the argument that we used above to derive (3.1.25); the main difference is that here one multiplies (3.1.19) by $nk(\theta_m^n - \theta_m^{n-1})$, instead of $\theta_m^n - \theta_m^{n-1}$.

[80] See Visintin [453, Section II.2] for details.
[81] Notice that (3.1.41) entails that the mapping $\partial\varphi^*$ is single-valued.
[82] See Visintin [453, Section II.2] for details.
[83] See Visintin [453, Section II.2] for details.

Notice that

$$\sum_{n=1}^{\ell}\langle A\theta_m^n, nk(\theta_m^n - \theta_m^{n-1})\rangle \geqslant \frac{\ell k}{2}\int_\Omega |\nabla\theta_m^\ell|^2\,\mathrm{d}x - \frac{k}{2}\sum_{n=0}^{\ell-1}\int_\Omega |\nabla\theta_m^n|^2\,\mathrm{d}x,$$

and the latter sum is uniformly estimated because of (3.1.25). One thus gets a uniform estimate for θ_m that corresponds to (3.1.48). □

By a judicious choice of the test function, one may also prove results of local regularity in space and time, see e.g. Visintin [453, Section II.4].

3.2. L^1- and L^∞-techniques

In this section we prove the well-posedness of Problem 3.1.1 by using an L^1-contraction technique.[84] We then show the essential boundedness of the solution via cut-off and approximation procedures.

An L^1-result. Next we prove that the solution of Problem 3.1.1 depends monotonically and Lipschitz-continuously on the data in the L^1-metric. This technique is at the basis of the semigroup approach that we shall illustrate in Section 3.4.

THEOREM 3.2.1 *(Monotone and L^1-Lipschitz-continuous dependence on the data). Assume that the assumptions of Theorem 3.1.1 are satisfied. For $i = 1, 2$, let*

$$u_i^0 \in L^2(\Omega), \quad f_i \in L^2(0,T;V'), \quad f_1 - f_2 \in L^1(Q), \tag{3.2.1}$$

and (u_i, θ_i) be a solution of the corresponding Problem 3.1.1. Setting $\tilde{u} := u_1 - u_2$, $\tilde{u}^0 := u_1^0 - u_2^0$, $\tilde{f} := f_1 - f_2$, we then have[85]

$$\begin{aligned}&\int_\Omega \tilde{u}^+(x,t)\,\mathrm{d}x\\ &\quad\leqslant \int_\Omega (\tilde{u}^0)^+(x)\,\mathrm{d}x + \int_0^t \mathrm{d}\tau \int_\Omega \tilde{f}^+(x,\tau)\,\mathrm{d}x \quad \text{for a.a. } t \in\,]0,T[,\end{aligned} \tag{3.2.2}$$

$$\begin{aligned}&\int_\Omega |\tilde{u}(x,t)|\,\mathrm{d}x\\ &\quad\leqslant \int_\Omega |\tilde{u}^0(x)|\,\mathrm{d}x + \int_0^t \mathrm{d}\tau \int_\Omega |\tilde{f}(x,\tau)|\,\mathrm{d}x \quad \text{for a.a. } t \in\,]0,T[.\end{aligned} \tag{3.2.3}$$

PROOF. (i) At first we assume that

$$\begin{aligned}&\partial\varphi \text{ is Lipschitz-continuous and fulfills (3.1.41);}\\ &u_i^0 \text{ and } f_i \text{ fulfill (3.1.43) and (3.1.44), for } i = 1, 2;\end{aligned} \tag{3.2.4}$$

[84] See also e.g. Bénilan [54], Damlamian [158,160].

[85] By ξ^+ we denote the positive part of any real number ξ.

afterwards we shall drop these restrictions. By Proposition 3.1.4 then $\theta_i, u_i \in H^1(0,T; L^2(\Omega))$ for $i = 1, 2$. Let us also define the Heaviside graph H and its *Yosida regularized function* $H_\lambda := \lambda^{-1}[I-(I+\lambda H)^{-1}]$ (by I we denote the identity operator) for any $\lambda > 0$:

$$H(\eta) := \begin{cases} \{0\} & \text{if } \eta < 0, \\ [0,1] & \text{if } \eta = 0, \\ \{1\} & \text{if } \eta > 0, \end{cases} \qquad H_\lambda(\eta) := \begin{cases} 0 & \text{if } \eta < 0, \\ \lambda\eta & \text{if } 0 \leqslant \eta \leqslant \lambda, \\ 1 & \text{if } \eta > \lambda. \end{cases} \tag{3.2.5}$$

Let us then write (3.1.8) for $i = 1, 2$, take the difference between these equations, multiply it by $H_\lambda(\tilde{\theta})$, and integrate it in Ω. As

$$\int_\Omega \nabla\tilde{\theta} \cdot \nabla H_\lambda(\tilde{\theta})\,\mathrm{d}x = \int_\Omega H'_\lambda(\tilde{\theta})|\nabla\tilde{\theta}|^2\,\mathrm{d}x \geqslant 0 \quad \text{a.e. in }]0,T[,$$

we get

$$\int_\Omega \frac{\partial\tilde{u}}{\partial t} H_\lambda(\tilde{\theta})\,\mathrm{d}x \leqslant \int_\Omega \tilde{f} H_\lambda(\tilde{\theta})\,\mathrm{d}x \leqslant \int_\Omega \tilde{f}^+\,\mathrm{d}x \quad \text{a.e. in }]0,T[. \tag{3.2.6}$$

Let us then pass to the limit as $\lambda \to 0^+$. Note that

$$H_\lambda(\tilde{\theta}) \to \psi := \begin{cases} 0 & \text{where } \tilde{\theta} \leqslant 0, \\ 1 & \text{where } \tilde{\theta} > 0 \end{cases} \quad \text{a.e. in } Q,$$

so that $\psi \in H(\tilde{\theta})$ a.e. in Q. Moreover $H(\tilde{\theta}) = H(\tilde{u})$ a.e. in Q, for by the auxiliary assumption (3.2.4) $\partial\varphi$ and $(\partial\varphi)^{-1}$ are both monotone and single-valued. Hence $\psi \in H(\tilde{u})$ a.e. in Q. By the Lebesgue dominated-convergence theorem, we thus get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \tilde{u}^+\,\mathrm{d}x &= \int_\Omega \frac{\partial}{\partial t}\tilde{u}^+\,\mathrm{d}x = \int_\Omega \frac{\partial\tilde{u}}{\partial t}\psi\,\mathrm{d}x = \int_\Omega \frac{\partial\tilde{u}}{\partial t}\lim_{\lambda\to 0} H_\lambda(\tilde{\theta})\,\mathrm{d}x \\ &= \lim_{\lambda\to 0}\int_\Omega \frac{\partial\tilde{u}}{\partial t} H_\lambda(\tilde{\theta})\,\mathrm{d}x \leqslant \int_\Omega \tilde{f}^+\,\mathrm{d}x \quad \text{a.e. in }]0,T[. \end{aligned} \tag{3.2.7}$$

This yields (3.2.2) by time integration.

(ii) Let us now drop the auxiliary hypothesis (3.2.4), and approximate φ, u_i^0, f_i ($i = 1, 2$) by means of sequences $\{\varphi_n\}$, $\{u_{in}^0\}$, $\{f_{in}\}$ that fulfill (3.2.4) for any n. For any n the inequality (3.2.2) thus holds for the difference of the corresponding solutions, $\tilde{u}_n$. By Theorem 3.1.2,

$$\tilde{u}_n \to \tilde{u} \quad \text{weakly in } L^2(\Omega),\ \forall t \in [0,T],$$

whence, by the convexity of the positive-part mapping,

$$\liminf_{n\to\infty}\int_\Omega (\tilde{u}_n)^+(x,t)\,\mathrm{d}x \geqslant \int_\Omega \tilde{u}^+(x,t)\,\mathrm{d}x \quad \forall t \in [0,T]. \tag{3.2.8}$$

By writing (3.2.2) for $\tilde{u}_n$ and then passing to the inferior limit as $n \to \infty$, we then get (3.2.2) for $\tilde{u}$. The inequality (3.2.3) is finally obtained by exchanging u_1 and u_2 in (3.2.2), and then adding the two inequalities. □

COROLLARY 3.2.2. *Under the assumptions of Theorem 3.1.1, the solution of Problem 3.1.1 is unique and depends monotonically and Lipschitz-continuously on the data u^0 and f.*

The technique of Theorem 3.2.1 also allows one to derive some results of time regularity, that here we omit.[86]

L^∞-results. L^∞-estimates may be derived in two ways, multiplying the equation either by a *cut-off* of the solution, or by a power of the solution and then letting the exponent diverge. The second procedure also provides L^q-estimates for any fixed $q > 2$, but, at variance with the first one, it does not need the uniqueness of the solution.

PROPOSITION 3.2.3 *(Maximum and minimum principles). Assume that (3.1.3), (3.1.4), (3.1.16) and (3.1.17) hold. If*

$$\exists M > 0, \exists \theta^0 \in L^1(\Omega): \theta^0 \in \partial\varphi^*\big(u^0\big),$$

$$\theta^0 \leqslant M \quad (\theta^0 \geqslant -M,\ \textit{resp.})\ \textit{a.e. in}\ \Omega, \tag{3.2.9}$$

$$f \leqslant 0 \quad (f \geqslant 0, \textit{resp.}) \quad \textit{in the sense of}\ \mathcal{D}'(Q), \tag{3.2.10}$$

then the solution of Problem 3.1.1 is such that

$$\theta \leqslant M \quad (\theta \geqslant -M, \textit{resp.}) \quad \textit{a.e. in}\ Q. \tag{3.2.11}$$

PROOF. By Theorem 3.1.1 and Corollary 3.2.2, Problem 3.1.1 has one and only one solution. Let us assume that $\theta^0 \leqslant M$, $f \leqslant 0$. For any measurable selection b of $\partial\varphi^*$ $(= (\partial\varphi)^{-1})$, let us also set

$$\Phi(v) := \int_0^v \big[b(\xi) - M\big]^+ \,\mathrm{d}\xi \ (\geqslant 0) \quad \forall v \in \mathrm{Dom}(\partial\varphi^*);$$

notice that this integral is independent of the selection. Let us then multiply (3.1.8) by $(\theta - M)^+$ $(\in L^2(0, T; V))$, and integrate in time. Note that (3.2.9) and (3.2.10), respectively, yield

$$\Phi\big(u^0\big) = 0 \quad \text{a.e. in}\ \Omega, \qquad \big\langle f, (\theta - M)^+\big\rangle \,\mathrm{d}\tau \leqslant 0 \quad \text{a.e. in}\ ]0, T[.$$

Moreover by Proposition 5.2.7

$$\int_\Omega \Phi(u)\,\mathrm{d}x \in W^{1,1}(0, T), \qquad \left\langle \frac{\partial u}{\partial t}, (\theta - M)^+ \right\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Phi(u)\,\mathrm{d}x.$$

We thus get

$$\int_\Omega \Phi\big(u(x, T)\big)\,\mathrm{d}x + \int_0^T \mathrm{d}t \int_\Omega \big|\nabla(\theta - M)^+\big|^2 \,\mathrm{d}x \leqslant 0,$$

whence $\nabla(\theta - M)^+ = 0$ a.e. in Q. As $\theta \in V$ this yields $(\theta - M)^+ = 0$ a.e. in Q, namely $\theta \leqslant M$.

The case of $\theta^0 \geqslant -M$ and $f \geqslant 0$ may be dealt with similarly, using $-(\theta - M)^-$ in place of $(\theta - M)^+$. □

[86] See e.g. Visintin [453, Section II.3].

PROPOSITION 3.2.4 *(L^q-estimate). Assume that (3.1.3) and (3.1.16) are fulfilled, and that for some $q > 2$*

$$u^0 \in L^q(\Omega), \qquad f \in L^q(Q) \cap L^2(0,T;V'). \tag{3.2.12}$$

Then the solution of Problem 3.1.1 is such that $u \in L^\infty(0,T;L^q(\Omega))$, and

$$\begin{aligned}&\big\|u(\cdot,t)\big\|_{L^q(\Omega)} \leqslant \big[1 + \big(q\|f\|_{L^q(Q)}\big)^{1/q} + \big\|u^0\big\|_{L^q(\Omega)}\big]\exp\big(t\|f\|_{L^q(Q)}\big)\\ &\quad \forall t \in \,]0,T].\end{aligned} \tag{3.2.13}$$

PROOF. By Theorem 3.1.1 and Corollary 3.2.2, Problem 3.1.1 has one and only one solution. Let us at first assume that $\partial\varphi$ is Lipschitz-continuous; afterwards we shall drop this auxiliary hypothesis. Let us also fix any $M > 0$, and set

$$\begin{aligned}&\alpha_q(v) = |v|^{q-2}v, \quad \beta_M(v) := \min\big\{\max\{v,-M\},M\big\} \quad \forall v \in \mathbf{R},\\ &\gamma_{qM} = \alpha_q \circ \beta_M \circ \partial\varphi \quad \text{in } \mathbf{R}, \qquad u_M = \beta_M(u) \quad \text{a.e. in } Q.\end{aligned}$$

Hence $\alpha_q(u_M) = \gamma_{qM}(\theta)$ by (3.1.7). Note that the function γ_{qM} is Lipschitz-continuous and nondecreasing. Let us multiply Eq. (3.1.8) by $\alpha_q(u_M) = \gamma_{qM}(\theta)$ ($\in L^2(0,T;V)$) and integrate in time. Note that $\langle A\theta, \gamma_{qM}(\theta)\rangle = \int_\Omega \nabla\theta \cdot \nabla\gamma_{qM}(\theta)\,\mathrm{d}x \geqslant 0$. By the Schwarz–Hölder inequality we then get

$$\begin{aligned}&\int_\Omega \big(\big|u_M(x,t)\big|^q - \big|u^0(x)\big|^q\big)\,\mathrm{d}x\\ &\quad = q\int_0^t \left\langle \frac{\partial u}{\partial t}, \alpha_q(u_M)\right\rangle \mathrm{d}\tau\\ &\quad \leqslant q\|f\|_{L^q(\Omega\times]0,t[)}\big\||u_M|^{q-1}\big\|_{L^{q/(q-1)}(\Omega\times]0,t[)}\\ &\quad = q\|f\|_{L^q(Q)}\left(\int_0^t \mathrm{d}\tau\int_\Omega \big|u_M(x,\tau)\big|^q\,\mathrm{d}x\right)^{(q-1)/q}\\ &\quad \leqslant q\|f\|_{L^q(Q)}\int_0^t \mathrm{d}\tau\int_\Omega \big(1 + \big|u_M(x,\tau)\big|^q\big)\,\mathrm{d}x \quad \forall t \in \,]0,T].\end{aligned} \tag{3.2.14}$$

The Gronwall Lemma 3.2.5 yields

$$\int_\Omega \big|u_M(x,t)\big|^q\,\mathrm{d}x \leqslant \left(q\|f\|_{L^q(Q)} + \int_\Omega \big|u^0(x)\big|^q\,\mathrm{d}x\right)\exp\big(qt\|f\|_{L^q(Q)}\big),$$

for any $t \in \,]0,T]$. By passing to the limit as $M \to +\infty$ and then taking the qth root of both members, we finally get (3.2.13).

If $\partial\varphi$ is not Lipschitz-continuous, one can approximate it via Yosida approximation and then apply Theorem 3.1.2. □

LEMMA 3.2.5 *(Gronwall).*[87] *Let $g, a, b : [0,T[\, \to \mathbf{R}$ be continuous functions, with a nondecreasing and $b \geqslant 0$. If*

$$g(t) \leqslant a(t) + \int_0^t b(\tau)g(\tau)\,\mathrm{d}\tau \quad \forall t \in [0,T[, \tag{3.2.15}$$

[87] See e.g. Walter [460, Section I.1].

then

$$g(t) \leqslant a(t) \exp\left(\int_0^t b(\tau)\, \mathrm{d}\tau \right) \quad \forall t \in [0, T[. \tag{3.2.16}$$

COROLLARY 3.2.6. *Assume that* (3.1.3) *and* (3.1.16) *are fulfilled, and that*

$$u^0 \in L^\infty(\Omega), \qquad f \in L^\infty(Q) \cap L^2(0, T; V'). \tag{3.2.17}$$

Then the solution of Problem 3.1.1 *is such that* $u \in L^\infty(Q)$.

PROOF. It suffices to apply Proposition 3.2.4 for any $q > 2$, and then to pass to the limit as $q \to +\infty$ in (3.2.13). □

3.3. *Two integral transformations*

In this section we discuss two natural transformations of Problem 3.1.1. By integrating Eq. (3.1.8) in time, we eliminate the time derivative that acts on u. Because of (3.1.7), Problem 3.1.1 may then be formulated as a variational inequality. A similar conclusion may be attained by applying the operator A^{-1} to (3.1.8). Actually, as it is the case for several parabolic problems, these two transformations are essentially equivalent, and yield analogous regularity properties. The theory of variational inequalities has extensively been applied to PDEs.[88]

Time-integral transformation. The transformation that here we illustrate was independently introduced by Duvaut [184,185] and Frémond [223]. This technique was inspired by an integral transformation, that Baiocchi introduced for a free boundary problem arising in porous medium filtration, see Baiocchi [44], and Baiocchi and Capelo [45]. Let us set

$$z(\cdot, t) := \int_0^t \theta(\cdot, \tau)\, \mathrm{d}\tau, \quad F(t) := \int_0^t f(\tau)\, \mathrm{d}\tau + u^0 \quad \forall t \in [0, T], \tag{3.3.1}$$

and note that, by integrating (3.1.8) in time and coupling it with (3.1.7), we get

$$\partial\varphi\left(\frac{\partial z}{\partial t}\right) + Az \ni F \quad \text{in } V',\ \forall t \in [0, T]. \tag{3.3.2}$$

By definition of subdifferential, cf. (5.2.5), this inclusion is equivalent to the following variational inequality:

$$\left\langle Az - F, \frac{\partial z}{\partial t} - v \right\rangle + \int_\Omega \left[\varphi\left(\frac{\partial z}{\partial t}\right) - \varphi(v) \right] \mathrm{d}x \leqslant 0$$
$$\forall v \in V,\ \text{a.e. in }]0, T[. \tag{3.3.3}$$

[88] See e.g. the monographs quoted in the item (XI) of the Bibliographical Note in Section 6.

Notice that, as $z(\cdot, 0) = 0$ a.e. in Ω,

$$\int_0^{\tilde{t}} \left\langle Az - F, \frac{\partial z}{\partial t} \right\rangle \mathrm{d}t$$
$$= \frac{1}{2} \int_\Omega \left|\nabla z(\cdot, \tilde{t})\right|^2 \mathrm{d}x + \int_0^{\tilde{t}} \left\langle \frac{\partial F}{\partial t}, z \right\rangle \mathrm{d}t - \left\langle F(\tilde{t}), z(\cdot, \tilde{t}) \right\rangle \quad \forall \tilde{t} \in]0, T].$$

Although in terms of the new variable z the regularity that is prescribed for u in Problem 3.1.1 corresponds to $z \in H^1(0, T; V)$, here we reformulate the variational inequality (3.3.3) under weaker regularity requirements for z. Let us first assume that $f \in L^2(0, T; V')$ and $u^0 \in V'$, so that

$$F \in W^{1,1}(0, T; V'). \tag{3.3.4}$$

PROBLEM 3.3.1. Find $z \in L^\infty(0, T; V) \cap H^1(0, T; L^2(\Omega))$ such that

$$\frac{1}{2} \int_\Omega \left|\nabla z(\cdot, \tilde{t})\right|^2 \mathrm{d}x + \int_0^{\tilde{t}} \left\langle \frac{\partial F}{\partial t}, z \right\rangle \mathrm{d}t - \left\langle F(\tilde{t}), z(\cdot, \tilde{t}) \right\rangle$$
$$+ \int_0^{\tilde{t}} \mathrm{d}t \int_\Omega \left[\varphi\left(\frac{\partial z}{\partial t}\right) - \varphi(v)\right] \mathrm{d}x \leqslant \int_0^{\tilde{t}} \langle Az - F, v \rangle \,\mathrm{d}t \tag{3.3.5}$$
$$\forall v \in V, \text{ for a.a. } \tilde{t} \in]0, T[,$$

$$z(\cdot, 0) = 0 \quad \text{a.e. in } \Omega. \tag{3.3.6}$$

THEOREM 3.3.1 *(Existence and uniqueness). If* (3.1.3), (3.1.4), (3.1.16), (3.1.17) *are satisfied and*

$$\exists \hat{L}, \hat{M} > 0\colon \forall (\xi, \eta) \in \operatorname{graph}(\partial\varphi), |\eta| \leqslant \hat{L}|\xi| + \hat{M}, \tag{3.3.7}$$

then Problem 3.3.1 *has one and only one solution.*

OUTLINE OF THE PROOF.[89] Let us set $z_m^0 := 0$ a.e. in Ω, $F_m^n := F(nk)$ in V' for any $n \in \{1, \dots, m\}$, and

$$\hat{J}_m^n(v) := \int_\Omega \left[k\varphi\left(\frac{v - z_m^{n-1}}{k}\right) + \frac{1}{2}|\nabla v|^2\right] \mathrm{d}x - \left\langle F_m^n, v \right\rangle \quad \forall v \in V.$$

These functionals are convex, lower semicontinuous and coercive, and may thus be minimized recursively. Each of them has a (unique) minimum point z_m^n, which thus solves the implicit time-discretization scheme:

$$\partial\varphi\left(\frac{z_m^n - z_m^{n-1}}{k}\right) + Az_m^n \ni F_m^n \quad \text{in } V', \text{ for } n = 1, \dots, m. \tag{3.3.8}$$

Let us define the time-interpolate functions z_m as in (3.1.26). Multiplying (3.3.8) by $z_m^n - z_m^{n-1}$, one may easily derive a uniform estimate for z_m in $L^\infty(0, T; V) \cap$

[89] See Visintin [453, Section II.5].

$H^1(0, T; L^2(\Omega))$. Therefore there exists z such that, possibly extracting a subsequence,

$$z_m \to z \quad \text{weakly star in } L^\infty(0, T; V) \cap H^1\big(0, T; L^2(\Omega)\big).$$

By passing to the limit in the approximate variational inequality, by lower semicontinuity one then obtains (3.3.5). The proof of uniqueness is straightforward. □

Inversion of the Laplace operator. In view of introducing our second integral transformation, let us first revisit the functional setting of Problem 3.1.1. We shall use the inverse operator $A^{-1} : V' \to V$. For instance, if $g \in L^2(\Omega)$, $h \in L^2(\Gamma_N)$ and

$$\langle f, v\rangle = \int_\Omega g(x)v(x)\,\mathrm{d}x + \int_{\Gamma_N} h(x)\gamma_0 v(x)\,\mathrm{d}\sigma \quad \forall v \in V,$$

then

$$u = A^{-1} f \quad \Leftrightarrow \quad u \in V, \quad -\Delta u = g \text{ a.e. in } \Omega, \quad \frac{\partial u}{\partial \nu} = h \text{ a.e. on } \Gamma_N, \tag{3.3.9}$$

for in this case the normal trace $\partial u/\partial \nu$ is an element of $L^2(\Gamma_N)$.

By applying the operator A^{-1} to (3.1.8), we have

$$A^{-1}\frac{\partial u}{\partial t} + \theta = A^{-1} f =: F \quad \text{in } V, \text{ a.e. in }]0, T[. \tag{3.3.10}$$

By coupling this equation with (3.1.7) we then get the inclusion

$$A^{-1}\frac{\partial u}{\partial t} + \partial\varphi^*(u) \ni F \quad \text{in } V, \text{ a.e. in }]0, T[, \tag{3.3.11}$$

which is equivalent to the following variational inequality:

$$\left\langle A^{-1}\frac{\partial u}{\partial t} - F, u - v\right\rangle + \int_\Omega \big[\varphi^*(u) - \varphi^*(v)\big]\,\mathrm{d}x \leqslant 0$$
$$\forall v \in L^2(\Omega), \text{ a.e. in }]0, T[. \tag{3.3.12}$$

Assuming (3.1.4), namely

$$u^0 \in V', \qquad F \in L^2(0, T; V), \tag{3.3.13}$$

we can now introduce a further weak formulation of our problem.

PROBLEM 3.3.2. Find $u \in L^2(Q) \cap H^1(0, T; V')$ such that (3.3.12) is satisfied, and

$$u|_{t=0} = u^0 \quad \text{in } V' \text{ (in the sense of the traces of } H^1(0, T; V')). \tag{3.3.14}$$

THEOREM 3.3.2 *(Existence and uniqueness). If* (3.1.3), (3.1.4), (3.1.16), (3.1.17) *and* (3.3.13) *are satisfied, then Problem* 3.3.1 *has one and only one solution.*

OUTLINE OF THE PROOF. [90] One may approximate the inclusion (3.3.11) via implicit time-discretization, and then derive a priori estimates by multiplying the approximate equation by the approximate solution u_m^n. This yields a uniform estimate for the linear interpolate function u_m in $L^2(Q) \cap H^1(0,T;V')$. Hence a suitable subsequence of $\{u_m\}$ weakly converges in this space. By passing to the limit in the approximate variational inequality, Eq. (3.3.12) follows by lower semicontinuity. The proof of uniqueness is straightforward. □

3.4. *Semigroup techniques*

In this section we apply to Problem 3.1.1 methods of the theory of nonlinear semigroups of contractions in Hilbert and Banach spaces, cf. Section 5.6. We shall see that in this framework the spaces $H^{-1}(\Omega)$ and $L^1(\Omega)$ play special roles.[91] In the first case we shall retrieve the method of inversion of the Laplace operator that we just illustrated, whereas in the second one we shall exploit the L^1-contraction procedure of Theorem 3.2.1.

Change of pivot space. Here we continue our discussion on the inversion of the operator A, under the assumption that Γ_D has positive $(N-1)$-dimensional Hausdorff measure. Let us first notice that the bilinear forms

$$(u,v)_V := \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x \quad \forall u,v \in V, \tag{3.4.1}$$

$$(u,v)_{V'} := \left(A^{-1}u, A^{-1}v\right)_V = \left\langle A^{-1}u, v\right\rangle \quad \forall u,v \in V' \tag{3.4.2}$$

are scalar products in the Hilbert spaces V and V', respectively, and that

$$(Au,v)_{V'} := \langle u,v\rangle = \int_\Omega uv\,\mathrm{d}x \quad \forall u \in V,\ \forall v \in L^2(\Omega). \tag{3.4.3}$$

Let us now denote the space V' by $\mathcal{H}$, in order to avoid any possible confusion with the dual spaces that we are going to introduce, and define the *Riesz operator*

$$\mathcal{R} : \mathcal{H} \to \mathcal{H}', \quad \langle \mathcal{R}u, v\rangle = (u,v)_{\mathcal{H}} \quad \forall u,v \in \mathcal{H}.$$

As $L^2(\Omega) \subset \mathcal{H}$ with continuous and dense injection, we can identify $\mathcal{H}'$ with a subspace of $L^2(\Omega)'$. This yields

$$\mathcal{R}L^2(\Omega) \subset \mathcal{R}\mathcal{H} = \mathcal{H}' \subset L^2(\Omega)' \quad \text{with dense and compact injections.}$$

The space $\mathcal{H}$ is thus identified with $\mathcal{R}^{-1}(\mathcal{H}')$, and accordingly plays the role of *pivot space*. This approach is at variance from the more usual procedure of identifying $L^2(\Omega)$ with its dual, cf. (3.1.2). Henceforth we shall omit to display the operator $\mathcal{R}$.

[90] See Visintin [453, Section II.5].

[91] See e.g. Bénilan [55], Bénilan and Crandall [57], Bénilan, Crandall and Pazy [58], Bénilan, Crandall and Sacks [59], Berger, Brezis, and Rogers [60], Brezis [83,84], Brezis and Pazy [88], Crandall and Pierre [148], Magenes, Verdi and Visintin [321], Rogers and Berger [403].

By (3.4.3), in the space $\mathcal{H}$ the variational equation (3.1.8) also reads

$$\left(\frac{\partial u}{\partial t}, v\right)_{\mathcal{H}} + \int_{\Omega} \theta v \, \mathrm{d}x = (f, v)_{\mathcal{H}} \quad \forall v \in \mathcal{H}, \tag{3.4.4}$$

which is equivalent to (3.3.10). In this way we have thus retrieved Problem 3.3.2.

This technique was studied by Lions [311, Section 2.3].

L^2-semigroups. Whenever the mapping $\partial\varphi^*$ is nonlinear, the operator $u \mapsto -\Delta\partial\varphi^*(u)$ is nonmonotone in $L^2(\Omega)$. Next we shall see that, if properly defined, this operator is maximal and cyclically monotone in $\mathcal{H}$, in the sense of Section 5.5. Let us first assume that

$$\exists c, \widehat{M} > 0\colon \forall v \in \mathbf{R}^3, \quad \varphi^*(v) \leqslant c|v|^2 + \widehat{M}, \tag{3.4.5}$$

so that $\varphi^*(v) \in L^1(\Omega)$ for any $v \in L^2(\Omega)$. One may see that this condition is equivalent to (3.3.7). Let us then define the (possibly multivalued) operator

$$\begin{cases} \Lambda_2 : \mathrm{Dom}(\Lambda_2) := L^2(\Omega) \subset \mathcal{H} \to 2^{\mathcal{H}} : \\ u \mapsto A\partial\varphi^*(u) := \left\{A\theta\colon \theta \in V, \theta \in \partial\varphi^*(u) \text{ a.e. in } \Omega\right\}. \end{cases} \tag{3.4.6}$$

Note that, for any $u \in L^2(\Omega)$ and any $\theta \in V$, $\theta \in \partial\varphi^*(u)$ a.e. in Ω if and only if

$$(A\theta, u - v)_{\mathcal{H}} = \int_{\Omega} \theta(u - v) \, \mathrm{d}x \geqslant \int_{\Omega} \varphi^*(u) \, \mathrm{d}x - \int_{\Omega} \varphi^*(v) \, \mathrm{d}x \quad \forall v \in L^2(\Omega).$$

By this variational inequality Λ_2 coincides with the subdifferential of the proper, convex, and lower semicontinuous functional

$$\mathcal{H} \to \mathbf{R} : v \mapsto \begin{cases} \displaystyle\int_{\Omega} \varphi^*(v) \, \mathrm{d}x & \text{if } v \in L^2(\Omega), \\ +\infty & \text{if } v \in \mathcal{H} \setminus L^2(\Omega). \end{cases} \tag{3.4.7}$$

By Theorem 5.5.3, the operator Λ_2 is then maximal and cyclically monotone. One may then apply the classical theory of *semigroups of nonlinear contractions* in Hilbert spaces, see Section 5.6, to the equation

$$\frac{\partial u}{\partial t} + \Lambda_2(u) = f \quad \text{in } \mathcal{H}, \text{ a.e. in }]0, T[. \tag{3.4.8}$$

In this way one retrieves Theorem 3.3.2 and several other results. In fact this semigroup approach is essentially equivalent to the inversion of the Laplace operator.

L^1-semigroups. The L^1-contraction technique that we used in the proof of Theorem 3.2.1 suggests investigating the accretiveness of the multi-valued operator $u \mapsto -\Delta\partial\varphi^*(u)$ in $L^1(\Omega)$, in the framework of the theory of nonlinear semigroups of contractions in Banach spaces. We still assume (3.3.7), but there we also require that $\Gamma_D = \Gamma$, namely that the homogeneous Dirichlet condition is fulfilled on the whole boundary.[92] We then define the

[92] More general boundary conditions are considered e.g. in Bénilan [55], Magenes, Verdi and Visintin [321].

operator

$$\begin{cases} \Lambda_1 : \mathrm{Dom}(\Lambda_1) \subset L^1(\Omega) \to L^1(\Omega) : u \mapsto -\Delta\partial\varphi^*(u) := \\ \{-\Delta\theta \in L^1(\Omega) \colon \theta \in L^1(\Omega),\ \gamma_0\theta = 0 \text{ on } \Gamma,\ \theta \in \partial\varphi^*(u) \text{ a.e. in } \Omega\}. \end{cases} \tag{3.4.9}$$

For any

$$u^0 \in L^1(\Omega), \qquad f \in L^1(Q), \tag{3.4.10}$$

we now reformulate the weak Stefan problem as follows.

PROBLEM 3.4.1. Find a continuous vector-valued function $u : [0, T] \to L^1(\Omega)$, that is absolutely continuous in $]0, T[$ and such that

$$\frac{\partial u}{\partial t} + \Lambda_1(u) \ni f \quad \text{in } L^1(\Omega), \text{ a.e. in }]0, T[, \tag{3.4.11}$$

$$u(\cdot, 0) = u^0 \quad \text{a.e. in } \Omega. \tag{3.4.12}$$

In general this problem has no solution. Actually, the occurrence of phase interfaces is not consistent with the condition $u(\cdot, t) \in \mathrm{Dom}(\Lambda_1)$, for $\Delta\theta = \Delta\partial\varphi^*(u)$ is a nonintegrable Borel measure whenever the Stefan condition (1.2.4) is fulfilled. We shall then investigate a weaker notion of solution.

LEMMA 3.4.1.[93] *Assume that α is a maximal monotone mapping $\mathbf{R} \to 2^{\mathbf{R}}$ such that $\alpha(0) \ni 0$. Let $p \in [1, +\infty[$ and set $p' = p/(p-1)$ if $p \neq 1$, $p' = +\infty$ if $p = 1$. If*

$$\begin{aligned} &u \in L^p(\Omega), \quad \Delta u \in L^p(\Omega), \quad \gamma_0 u = 0 \quad \textit{a.e. on } \Gamma, \\ &h \in L^{p'}(\Omega), \quad h \in \alpha(u) \quad \textit{a.e. in } \Omega, \end{aligned} \tag{3.4.13}$$

then $-\int_\Omega h\Delta u \,\mathrm{d}x \geqslant 0$.

THEOREM 3.4.2. *Assume that (3.1.3), (3.1.16) and (3.3.7) hold, and that $\Gamma_D = \Gamma$. The operator Λ_1 is then T- and m-accretive in $L^1(\Omega)$, that is,*

$$\begin{aligned} &\forall u_i \in \mathrm{Dom}(\Lambda_1),\ \forall -\Delta\theta_i \in \Lambda_1(u_i)\ (i = 1, 2),\ \forall\lambda > 0, \\ &\|(u_1 - u_2)^+\|_{L^1(\Omega)} \leqslant \|[u_1 - u_2 - \lambda\Delta(\theta_1 - \theta_2)]^+\|_{L^1(\Omega)}, \end{aligned} \tag{3.4.14}$$

$$\forall\lambda > 0,\ \forall f \in L^1(\Omega),\ \exists u \in \mathrm{Dom}(\Lambda_1) \colon u + \lambda\Lambda_1(u) \ni f \textit{ a.e. in } \Omega. \tag{3.4.15}$$

PROOF. (i) In view of proving (3.4.14), let us first fix any $u_i \in \mathrm{Dom}(\Lambda_1)$, and select any $-\Delta\theta_i \in \Lambda_1(u_i)$ for $i = 1, 2$. Let us then set

$$h(x) := \begin{cases} 1 & \text{if either } u_1(x) > u_2(x) \text{ or } \theta_1(x) > \theta_2(x), \\ 0 & \text{otherwise} \end{cases} \qquad \text{for a.a. } x \in \Omega.$$

[93] See Brezis and Strauss [89, p. 566], where this result is stated in more general form, for a class of unbounded m-accretive operators in $L^1(\Omega)$ that fulfill a maximum principle.

Note that h is measurable and, defining the Heaviside graph H as in (3.2.5), $h \in H(u_1-u_2) \cap H(\theta_1-\theta_2)$ a.e. in Ω. Lemma 3.4.1 then yields $-\int_\Omega h\Delta(\theta_1-\theta_2)\,\mathrm{d}x \geqslant 0$. Hence

$$\begin{aligned}\int_\Omega \big[u_1-u_2-\lambda\Delta(\theta_1-\theta_2)\big]^+\,\mathrm{d}x &\geqslant \int_\Omega \big[u_1-u_2-\lambda\Delta(\theta_1-\theta_2)\big]h\,\mathrm{d}x\\ &\geqslant \int_\Omega (u_1-u_2)h\,\mathrm{d}x = \int_\Omega (u_1-u_2)^+\,\mathrm{d}x,\end{aligned}$$

that is (3.4.14). As an analogous statement is fulfilled with the negative part in place of the positive part, we get

$$\begin{aligned}&\forall u_i \in \mathrm{Dom}(\varLambda_1),\ \forall -\Delta\theta_i \in \varLambda_1(u_i)\ (i=1,2),\ \forall\lambda>0,\\ &\|u_1-u_2\|_{L^1(\Omega)} \leqslant \big\|u_1-u_2+\lambda\Delta(\theta_1-\theta_2)\big\|_{L^1(\Omega)};\end{aligned}\tag{3.4.16}$$

namely, $\varLambda_1$ is accretive in $L^1(\Omega)$.

(ii) Next we prove (3.4.15). Let us first assume that $f \in L^2(\Omega)$. The functional

$$J : V \to \widetilde{\mathbf{R}} : v \mapsto \int_\Omega \left(\varphi(v) + \frac{\lambda}{2}|\nabla v|^2 - fv\right)\mathrm{d}x$$

is convex, lower semicontinuous, and coercive; hence it has a minimum point $\theta \in V$. Thus $\partial J(\theta) \ni 0$ in V', that is,

$$u := f + \lambda\Delta\theta \in \partial\varphi(\theta) \quad \text{in } V';$$

hence $u \in L^2(\Omega)$, by (3.1.16). For any $f \in L^2(\Omega)$ thus there exists a pair $(\theta, u) \in V \times L^2(\Omega)$ such that $\Delta\theta \in L^2(\Omega)$ and

$$\theta \in \partial\varphi^*(u), \quad u - \lambda\Delta\theta = f \quad \text{a.e. in } \Omega. \tag{3.4.17}$$

Let us now consider any $f \in L^1(\Omega)$ and any sequence $\{f_n\}$ in $L^2(\Omega)$ that converges to f strongly in $L^1(\Omega)$. For any n let (θ_n, u_n) solve $(3.4.17)_n$. By the accretiveness of $\varLambda_1$, $\{u_n\}$ is a Cauchy sequence in $L^1(\Omega)$; hence it converges to some u strongly in this space. By comparing the terms of $(3.4.17)_n$, we see that the sequence $\{\Delta\theta_n\}$ is then uniformly bounded in $L^1(\Omega)$. Possibly extracting a subsequence, θ_n then converges to some θ strongly in $L^1(\Omega)$. Therefore, possibly extracting further subsequences, θ_n and u_n converge a.e. in Ω. By passing to the limit in $(3.4.17)_n$ a.e. in Ω, we then infer (3.4.17) for the pair (θ, u). □

Theorems 3.4.2 and 5.6.1 yield the next statement.

THEOREM 3.4.3. *Assume that* (3.1.3), (3.1.16) *and* (3.3.7) *are satisfied and that* $\Gamma_D = \Gamma$. *For any* $u^0 \in L^1(\Omega)$ *and any* $f \in L^1(Q)$, *Problem* 3.4.1 *then has one and only one mild solution* (*in the sense of Section* 5.6).

This solution depends Lipschitz-continuously and monotonically on the data.

REMARKS. (i) If $f \in \mathrm{BV}(0,T;L^1(\Omega))$ and $u^0 \in \mathrm{Dom}(\varLambda_1)$, then $u : [0,T] \to L^1(\Omega)$ is Lipschitz-continuous. However, consistently with our previous remark, u need not be a *strong solution*, for the space $L^1(\Omega)$ does not fulfill the *Radon–Nikodým property*, cf. Section 5.6.

(ii) So far we studied quasilinear parabolic P.D.E.s containing a single nonlinear term. One may also deal with *doubly nonlinear equations*, namely equations that contain two nonlinear functions, for instance of the form

$$\frac{\partial}{\partial t}\alpha(u) - \nabla \cdot \vec{\gamma}(\nabla u) \ni f \quad \text{in } Q, \tag{3.4.18}$$

$$\alpha\left(\frac{\partial u}{\partial t}\right) - \nabla \cdot \vec{\gamma}(\nabla u) \ni f \quad \text{in } Q, \tag{3.4.19}$$

with α and $\vec{\gamma}$ given (possibly multi-valued) maximal monotone mappings. In general these two equations are not mutually equivalent. For instance an equation of the form (3.4.18) is met in nonequilibrium thermodynamics: the Gibbs relation and the phenomenological laws provide the two nonlinearities, see the system (2.4.1), (2.4.2), (2.4.6), (2.4.13), (2.4.14). Equations of the form (3.4.19) arise in a number of diffusion problems, with α equal to the subdifferential of a so-called *dissipation potential*, see e.g. Germain [241].

4. Phase relaxation with nonlinear heat diffusion

In this part we deal with an initial- and boundary-value problem for a quasilinear (actually, *multi-nonlinear)* parabolic P.D.E., that represents phase transition coupled with nonlinear heat-diffusion and with phase relaxation;[94] cf. Section 2.2. We provide the weak formulation of an initial- and boundary-value problem in the framework of Sobolev spaces, and prove existence of a solution in any prescribed time interval.

This part includes some elements of novelty. A rather general constitutive relation is assumed between internal energy, temperature, and phase; this also allows for (possibly nonlinear) dependence of the heat capacity on the phase. Existence of an approximate solution is here proved via a saddle point formulation. Compactness by strict convexity (cf. Section 5.4) is applied in the limit procedure, besides more standard techniques of compactness, convexity and lower semicontinuity. This approach might also be extended to coupled heat- and mass-diffusion, as well as to other models.[95]

4.1. *Weak formulation*

In this section we formulate our problem in the framework of Sobolev spaces.

Let the sets Ω, Γ, Q, the space V, the operator A, and the duality pairing $\langle\cdot,\cdot\rangle$ be defined as in Section 3.1. Let us assume that the following functions are also given:

$$\begin{aligned} &\varphi : \mathbf{R}^2 \to \widetilde{\mathbf{R}}\ \big(:= \mathbf{R} \cup \{+\infty\}\big) \text{ proper, convex and lower semicontinuous,} \\ &\mathrm{Dom}(\varphi) = \mathbf{R} \times [-1, 1], \end{aligned} \tag{4.1.1}$$

[94] This mode of phase transition was studied in many works, see e.g. Bénilan, Blanchard and Ghidouche [56], Blanchard, Damlamian and Ghidouche [67], Blanchard and Ghidouche [68], Frémond and Visintin [225], Visintin [446,447] and [453, Chapter V].

[95] See Visintin [459]. This part rests upon the classical theory of linear and nonlinear PDEs, see e.g. the monographs quoted in the items (XI), (XII), (XIII) of the Bibliographical Note in Section 6.

$$\begin{aligned}
&\Phi : \mathbf{R} \times \mathbf{R}^3 \to \mathbf{R} \quad \text{such that} \\
&\Phi(\cdot, \vec{\xi}) \text{ is continuous } \forall \vec{\xi} \in \mathbf{R}^3, \\
&\Phi(\theta, \cdot) \text{ is convex and lower semicontinuous } \forall \theta \in \mathbf{R}.
\end{aligned} \tag{4.1.2}$$

We shall deal with the system

$$(\theta, r) \in \partial\varphi(u, \chi), \tag{4.1.3}$$

$$\vec{q} \in -\partial_2\Phi(\theta, \nabla\theta), \tag{4.1.4}$$

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{q} = g, \tag{4.1.5}$$

$$a\frac{\partial \chi}{\partial t} + r = 0, \tag{4.1.6}$$

in Q, coupled with initial conditions for u and χ, and with boundary conditions for u and $\vec{q}$. Here a is a positive constant. By $\partial_2\Phi$ we denote the partial subdifferential of Φ w.r.t. the second argument, here $\nabla\theta$; see Section 5.2. The inclusion (4.1.3) accounts for a dependence of the heat capacity on the phase.[96] In the next section we shall see that the system (4.1.3)–(4.1.6) is consistent with the second principle of thermodynamics, provided that the function φ represents the function of (2.2.15), cf. (2.2.20) and (2.2.21).[97]

Let us assume that

$$u^0, \chi^0 \in L^2(\Omega), \quad \big(u^0, \chi^0\big) \in \mathrm{Dom}(\partial\varphi) \quad \text{a.e. in } \Omega, \tag{4.1.7}$$

$$f \in L^2(0, T; V'), \tag{4.1.8}$$

and introduce our weak formulation.

PROBLEM 4.1.1. Find $u, \theta, \chi, r, \vec{q}$ such that

$$\begin{aligned}
&u, r \in L^2(Q), \quad \chi \in H^1\big(0, T; L^2(\Omega)\big), \\
&\quad \theta \in L^2(0, T; V), \quad \vec{q} \in L^2(Q)^3,
\end{aligned} \tag{4.1.9}$$

$$(u, \chi) \in \mathrm{Dom}(\varphi) \quad \text{and} \quad \forall(\tilde{u}, \tilde{\chi}) \in \mathrm{Dom}(\varphi),$$

$$\theta(u - \tilde{u}) + r(\chi - \tilde{\chi}) \geqslant \varphi(u, \chi) - \varphi(\tilde{u}, \tilde{\chi}) \quad \text{a.e. in } Q, \tag{4.1.10}$$

$$\vec{q} \cdot (\vec{\xi} - \nabla\theta) \geqslant \Phi(\theta, \nabla\theta) - \Phi(\theta, \vec{\xi}) \quad \forall \vec{\xi} \in L^2(\Omega)^3, \text{ a.e. in } Q, \tag{4.1.11}$$

$$\begin{aligned}
&\iint_Q \left[\big(u^0 - u\big)\frac{\partial v}{\partial t} - \vec{q} \cdot \nabla v\right] \mathrm{d}x\, \mathrm{d}t = \int_0^T \langle f, v\rangle \,\mathrm{d}t \\
&\quad \forall v \in H^1\big(0, T; L^2(\Omega)\big) \cap L^2(0, T; V), v(\cdot, T) = 0,
\end{aligned} \tag{4.1.12}$$

$$a\frac{\partial \chi}{\partial t} + r = 0 \quad \text{a.e. in } Q, \tag{4.1.13}$$

$$\chi(\cdot, 0) = \chi^0 \quad \text{a.e. in } \Omega. \tag{4.1.14}$$

[96] In Section 1 we had no difficulty in dealing with this dependence under the hypothesis of local equilibrium, for in that case the phase is determined by the temperature. On the other hand, if that hypothesis is dropped, the analysis is less obvious.

[97] This constant factor is here included in φ, in order to simplify the lay-out of formulas. This rescaling is immaterial for the present analysis.

If f is defined as in (3.1.11), this problem is the weak formulation of an initial- and boundary-value problem for the system (4.1.3)–(4.1.6). The system (4.1.13) and (4.1.14) is trivially integrated: $\chi(\cdot,t) = \chi^0 - \frac{1}{a}\int_0^t r(\cdot,\tau)\,\mathrm{d}\tau$ a.e. in Q.

REMARK. Setting

$$U = (u,\chi), \quad V = (\theta, r), \quad \gamma(V,\nabla V) = \big(\partial_2 \Phi(\theta,\nabla\theta), r\big), \quad F = (f,0)$$
$$\Lambda(\vec{z}_1, z_2) = (-\nabla\cdot\vec{z}_1, z_2) \quad \forall(\vec{z}_1,z_2) \in \mathbf{R}^3 \times \mathbf{R}, \tag{4.1.15}$$

the system (4.1.3)–(4.1.6) reads as a doubly nonlinear system:

$$V \in \partial\varphi(U), \quad W \in \gamma(V,\nabla V), \quad \frac{\partial U}{\partial t} + \Lambda W = F. \tag{4.1.16}$$

Because of the multiple nonlinearity of this problem, here the results of Section 3 are not directly applicable. Nevertheless those techniques are at the basis of the theorem of existence, that we prove in the next section.[98]

4.2. *Existence of a weak solution*

In view of stating our existence result, let us first define the partial conjugate $\psi : \mathbf{R} \times [-1,1] \to \widetilde{\mathbf{R}}$ $(:= \mathbf{R} \cup \{+\infty\})$ of the function φ w.r.t. u, cf. (5.2.3):

$$\psi(\theta,\chi) := \sup_{u\in\mathbf{R}}\big[u\theta - \varphi(u,\chi)\big] \quad \forall(\theta,\chi) \in \mathbf{R}\times[-1,1]. \tag{4.2.1}$$

By Theorem 5.3.3,

$$\begin{aligned} &\psi(\cdot,\chi) \text{ is convex and lower semicontinuous } \forall\chi \in [-1,1],\\ &\psi(\theta,\cdot) \text{ is concave and upper semicontinuous } \forall\theta \in \mathbf{R}. \end{aligned} \tag{4.2.2}$$

THEOREM 4.2.1 *(Existence). Assume that* (4.1.7)–(4.1.8) *are satisfied, and that*

$$\begin{aligned} &\textit{the function } \psi \textit{ may be represented in the form}\\ &\psi(\theta,r) = \psi_1(\theta) + \psi_2(\theta,r) \quad \forall(\theta,\chi)\in\mathbf{R}\times[-1,1],\\ &\textit{where } \psi_1 \textit{ is strictly convex and everywhere finite},\\ &\psi_2(\cdot,\chi) \textit{ is convex and lower semicontinuous } \forall\chi\in[-1,1],\\ &\psi_2(\theta,\cdot) \textit{ is concave and upper semicontinuous } \forall\theta\in\mathbf{R}, \end{aligned} \tag{4.2.3}$$

$$\exists a_1, a_2 > 0\colon\ \forall(u,\chi)\in \mathrm{Dom}(\varphi), \quad \varphi(u,\chi) \geqslant a_1|u|^2 - a_2, \tag{4.2.4}$$

$$\begin{aligned} &\exists a_3,\dots,a_6 > 0\colon\ \forall(\theta,\vec{\xi}) \in \mathbf{R}\times\mathbf{R}^3,\\ &a_3|\vec{\xi}|^2 - a_4 \leqslant \Phi(\theta,\vec{\xi}) \leqslant a_5|\vec{\xi}|^2 + a_6, \end{aligned} \tag{4.2.5}$$

$$\exists a_7 > 0\colon\ \forall\theta\in\mathbf{R}, \quad \Phi(\theta,\vec{0}) \leqslant a_7. \tag{4.2.6}$$

Then Problem 4.1.1 *has a solution such that* $u \in L^\infty(0,T;L^2(\Omega))$.

[98] Boundary- and initial-value problems for doubly-nonlinear PDEs were studied in many works; see e.g. Alt and Luckhaus [17], Colli and Visintin [145], DiBenedetto and Showalter [179], Damlamian, Kenmochi and Sato [162], Visintin [453, Chapter III], and references therein.

The reader will notice that in (4.2.3) the function ψ_1 is independent of χ, so that strict convexity is only assumed for the dependence of ψ_1 on θ. We do not assume ψ to be strictly concave w.r.t. the phase variable χ, for this would exclude the occurrence of a sharp interface between the phases. See the remarks after the proof of this theorem.

PROOF. We proceed via approximation by time-discretization, derivation of a priori estimates, and passage to the limit, as we did for Theorem 3.1.1.

(i) *Approximation*. Let us fix any $m \in \mathbf{N}$, set

$$k := \frac{T}{m}, \quad u_m^0 := u^0, \quad \chi_m^0 := \chi^0 \quad \text{a.e. in } \Omega,$$
$$f_m^n := \frac{1}{k}\int_{(n-1)k}^{nk} f(\tau)\,\mathrm{d}\tau \quad \text{in } V', \text{ for } n = 1, \ldots, m, \tag{4.2.7}$$

fix any $\theta^0 \in L^2(\Omega)$, and introduce the following approximate problem.[99]

PROBLEM 4.1.1$_m$. Find $u_m^n, \theta_m^n, \chi_m^n, r_m^n, \vec{q}_m^{\,n}$ for $n = 1, \ldots, m$ such that

$$u_m^n, \chi_m^n, r_m^n \in L^2(\Omega), \quad \theta_m^n \in V, \quad \vec{q}_m^{\,n} \in L^2(\Omega)^3, \tag{4.2.8}$$
$$\left(\theta_m^n, r_m^n\right) \in \partial\varphi\left(u_m^n, \chi_m^n\right) \quad \text{a.e. in } \Omega, \tag{4.2.9}$$
$$\vec{q}_m^{\,n} \in -\partial_2\Phi\left(\theta_m^{n-1}, \nabla\theta_m^n\right) \quad \text{a.e. in } \Omega, \tag{4.2.10}$$
$$\int_\Omega \left(\frac{u_m^n - u_m^{n-1}}{k} v - \vec{q}_m^{\,n} \cdot \nabla v\right) \mathrm{d}x = \left\langle f_m^n, v\right\rangle \quad \forall v \in V, \tag{4.2.11}$$
$$a\frac{\chi_m^n - \chi_m^{n-1}}{k} + r_m^n = 0 \quad \text{a.e. in } \Omega. \tag{4.2.12}$$

Let us define the *double subdifferential* $\tilde{\partial}\psi(\theta,\chi) := (\partial_\theta\psi(\theta,\chi), \partial_\chi(-\psi)(\theta,\chi))$ as in (5.3.17). By Theorem 5.3.3, the inclusion (4.2.9) is equivalent to the system

$$\left(u_m^n, r_m^n\right) \in \tilde{\partial}\psi\left(\theta_m^n, \chi_m^n\right), \quad \text{i.e.,} \quad \begin{cases} u_m^n \in \partial_\theta\psi\left(\theta_m^n, \chi_m^n\right), \\ r_m^n \in \partial_\chi(-\psi)\left(\theta_m^n, \chi_m^n\right) \end{cases}$$
$$\text{a.e. in } \Omega. \tag{4.2.13}$$

We shall prove the existence of a solution of Problem 4.1.1$_m$ recursively. Let us fix any $n \in \{1, \ldots, m\}$, assume that $u_m^{n-1}, \theta_m^{n-1}, \chi_m^{n-1}$ are known, define the closed and convex set $X := V \times L^\infty(\Omega; [1, 1])$ and set

$$J_m^n(\theta,\chi) := \int_\Omega \left[\psi(\theta,\chi) - \frac{a}{2}\chi^2 - u_m^{n-1}\theta + a\chi_m^{n-1}\chi + k\Phi\left(\theta_m^{n-1}, \nabla\theta\right)\right] \mathrm{d}x$$
$$- k\left\langle f_m^n, \theta\right\rangle \quad \forall(\theta,\chi) \in X. \tag{4.2.14}$$

Notice that

[99] It is necessary to select a value for θ^0, because this function occurs in (4.2.10) for $n = 1$. Nevertheless this function is not prescribed among the data, and it turns out that it is immaterial which value is selected, for the function θ need not have a trace for $t = 0$.

$$\begin{aligned}&V\to\mathbf{R}:\theta\mapsto J_m^n(\theta,\chi)\\&\text{is convex and lower semicontinuous,}\ \ \forall\chi\in L^\infty\big(\Omega;[1,1]\big),\end{aligned}\tag{4.2.15}$$

$$\begin{aligned}&L^\infty\big(\Omega;[1,1]\big)\to\mathbf{R}:\chi\mapsto J_m^n(\theta,\chi)\\&\text{is concave and upper semicontinuous,}\ \ \forall\theta\in V;\end{aligned}\tag{4.2.16}$$

moreover by (4.2.5) the functional J_m^n is coercive w.r.t. θ, that is,

$$J_m^n(\theta,\chi)\to+\infty\quad\text{as }\|\theta\|_V\to+\infty,\ \forall\chi\in\big(\Omega;[-1,1]\big).\tag{4.2.17}$$

(The coerciveness of J_m^n w.r.t. χ makes no sense, for this variable is confined to the interval $[-1,1]$.) By Theorem 5.3.2, the functional J_m^n has then a saddle point in X, namely

$$\begin{aligned}&\exists\big(\theta_m^n,\chi_m^n\big)\in X\quad\text{such that}\\&J_m^n\big(\theta_m^n,\chi\big)\leqslant J_m^n\big(\theta,\chi_m^n\big)\quad\forall(\theta,\chi)\in X.\end{aligned}\tag{4.2.18}$$

Hence $\tilde\partial J_m^n(\theta_m^n,\chi_m^n)\ni(0,0)$, that is

$$\partial_\theta J_m^n\big(\theta_m^n,\chi_m^n\big)\ni 0\quad\text{in }V',\tag{4.2.19}$$

$$\partial_\chi\big(-J_m^n\big)\big(\theta_m^n,\chi_m^n\big)\ni 0\quad\text{a.e. in }\Omega.\tag{4.2.20}$$

For a suitable selection of the fields

$$u_m^n\in\partial_\theta\psi\big(\theta_m^n,\chi_m^n\big),\quad r_m^n\in\partial_\chi(-\psi)\big(\theta_m^n,\chi_m^n\big),\quad \vec q_m^{\,n}\in-\partial_2\Phi\big(\theta_m^{n-1},\nabla\theta_m^n\big),$$

a.e. in Ω, the inclusions (4.2.19) and (4.2.20) yield (4.2.11) and (4.2.12). The functions $u_m^n,\theta_m^n,\chi_m^n,r_m^n,\vec q_m^{\,n}$ thus solve Problem 4.1.1$_m$.

(ii) *A priori estimates*. In view of deriving the balance of the function φ, let us first notice that by (4.2.9)

$$\begin{aligned}&\sum_{n=1}^{\ell}\big[\big(u_m^n-u_m^{n-1}\big)\theta_m^n+\big(\chi_m^n-\chi_m^{n-1}\big)r_m^n\big]\\&\quad\geqslant\sum_{n=1}^{\ell}\big[\varphi\big(u_m^n,\chi_m^n\big)-\varphi\big(u_m^{n-1},\chi_m^{n-1}\big)\big]\\&\quad=\varphi\big(u_m^\ell,\chi_m^\ell\big)-\varphi\big(u^0,\chi^0\big)\quad\text{a.e. in }\Omega.\end{aligned}\tag{4.2.21}$$

Moreover, by the inclusion (4.2.10) and by the hypotheses (4.2.5), (4.2.6),

$$\begin{aligned}-\vec q_m^{\,n}\cdot\nabla\theta_m^n&\geqslant\Phi\big(\theta_m^{n-1},\nabla\theta_m^n\big)-\Phi\big(\theta_m^{n-1},\vec 0\big)\\&\geqslant a_3\big|\nabla\theta_m^n\big|^2-a_4-a_7\quad\text{a.e. in }\Omega,\forall n.\end{aligned}\tag{4.2.22}$$

Let us now take $v=k\theta_m^n$ in (4.2.11), multiply (4.2.12) by $\chi_m^n-\chi_m^{n-1}$, and sum these formulas for $n=1,\dots,\ell$, for any $\ell\in\{1,\dots,m\}$. By (4.2.21) we get

$$\begin{aligned}&\int_\Omega\big[\varphi\big(u_m^\ell,\chi_m^\ell\big)-\varphi\big(u^0,\chi^0\big)\big]\,\mathrm{d}x-k\sum_{n=1}^{\ell}\int_\Omega\vec q_m^{\,n}\cdot\nabla\theta_m^n\,\mathrm{d}x\\&\quad+ak\sum_{n=1}^{\ell}\int_\Omega\left|\frac{\chi_m^n-\chi_m^{n-1}}{k}\right|^2\mathrm{d}x\leqslant k\sum_{n=1}^{\ell}\big\langle f_m^n,\theta_m^n\big\rangle,\end{aligned}\tag{4.2.23}$$

and then by (4.2.22)

$$\int_\Omega \big[\varphi\big(u_m^\ell, \chi_m^\ell\big) - \varphi\big(u^0, \chi^0\big)\big]\,\mathrm{d}x + a_3 k \sum_{n=1}^{\ell} \int_\Omega \big|\nabla\theta_m^n\big|^2\,\mathrm{d}x$$
$$+ ak \sum_{n=1}^{\ell} \int_\Omega \left|\frac{\chi_m^n - \chi_m^{n-1}}{k}\right|^2 \mathrm{d}x \leqslant (a_4 + a_7)\ell k|\Omega| + k\sum_{n=1}^{\ell} \big\|f_m^n\big\|_{V'} \big\|\theta_m^n\big\|_V. \tag{4.2.24}$$

By (4.2.4) we then get

$$\max_{n=1,\dots,m} \big\|u_m^n\big\|_{L^2(\Omega)},\ k\sum_{n=1}^{m}\big\|\theta_m^n\big\|_V^2,\ k\sum_{n=1}^{m}\left\|\frac{\chi_m^n - \chi_m^{n-1}}{k}\right\|_{L^2(\Omega)}^2 \leqslant C_1. \tag{4.2.25}$$

(By $C_1, C_2, \dots$ we shall denote constants independent of m.) Using the notation (3.1.26) and setting $\tau_k v(t) = v(t-k)$ for any function v of time, next we write the system (4.2.9)–(4.2.12) and the estimates (4.2.25) in terms of the time-interpolate functions:

$$(\bar\theta_m, \bar r_m) \in \partial\varphi(\bar u_m, \bar\chi_m) \quad \text{a.e. in } Q, \tag{4.2.26}$$

$$\bar{\vec q}_m \in -\partial_2\Phi(\tau_k\bar\theta_m, \nabla\bar\theta_m) \quad \text{a.e. in } Q, \tag{4.2.27}$$

$$\frac{\partial u_m}{\partial t} + \nabla\cdot\bar{\vec q}_m = \bar f_m \quad \text{in } V', \tag{4.2.28}$$

$$a\frac{\partial \chi_m}{\partial t} + \bar r_m = 0 \quad \text{a.e. in } Q, \tag{4.2.29}$$

$$\|u_m\|_{L^\infty(0,T;L^2(\Omega))},\ \|\bar\theta_m\|_{L^2(0,T;V)},\ \|\chi_m\|_{H^1(0,T;L^2(\Omega))} \leqslant C_2. \tag{4.2.30}$$

By (4.2.27) and by the second inequality of (4.2.5), we then have

$$\|\bar{\vec q}_m\|_{L^2(Q)^3} \leqslant C_3, \quad \text{whence } \|\nabla\cdot\bar{\vec q}_m\|_{L^2(0,T;V')} \leqslant C_4. \tag{4.2.31}$$

By comparing the terms of the approximate equation (4.2.28), we thus get

$$\|u_m\|_{H^1(0,T;V')} \leqslant C_5. \tag{4.2.32}$$

(iii) *Limit procedure*. By the above uniform estimates, there exist $u, \theta, \chi, r, \vec q$ such that, possibly taking $m \to \infty$ along a subsequence,[100]

$$u_m \to u \quad \text{weakly star in } L^\infty\big(0,T;L^2(\Omega)\big) \cap H^1(0,T;V'), \tag{4.2.33}$$

$$\chi_m \to \chi \quad \text{weakly star in } L^\infty(Q) \cap H^1\big(0,T;L^2(\Omega)\big), \tag{4.2.34}$$

$$\theta_m, \bar\theta_m \to \theta \quad \text{weakly in } L^2(0,T;V), \tag{4.2.35}$$

$$r_m, \bar r_m \to r \quad \text{weakly in } L^2(Q), \tag{4.2.36}$$

$$\vec q_m, \bar{\vec q}_m \to \vec q \quad \text{weakly in } L^2(Q)^3. \tag{4.2.37}$$

[100] Dealing with spaces of (vector-valued) time-dependent functions, the weak (star) convergence of the linear interpolates is equivalent to that of the piecewise-constant interpolates, and the two limits coincide, provided that no time differentiability is involved. The convergences (4.2.33) and (4.2.34) thus entail $\bar u_m \to u$ weakly star in $L^\infty(0,T;L^2(\Omega))$ and $\bar\chi_m \to \chi$ weakly star in $L^\infty(Q)$.

Possibly taking $m \to \infty$ along a further subsequence, we may also assume that

$$\chi_m(\cdot, T) \to \chi(\cdot, T) \quad \text{weakly star in } L^\infty(\Omega). \tag{4.2.38}$$

By passing to the limit in (4.2.28) and (4.2.29) we then get (4.1.12) and (4.1.13). Because of (4.2.34) the initial condition of χ is preserved in the limit. We are thus just left with the proof of (4.1.10) and (4.1.11), that we shall perform in the next steps.

(iv) *Proof of* (4.1.10). By (3.1.33), the convergence (4.2.33) yields

$$u_m \to u \quad \text{strongly in } C^0([0, T]; V'). \tag{4.2.39}$$

As $\bar{u}_m - u_m \to 0$ in $L^2(0, T; V')$, by (4.2.35) we then get

$$\iint_Q \bar{u}_m \bar{\theta}_m \,\mathrm{d}x\,\mathrm{d}t = \int_0^T {}_{V'}\langle \bar{u}_m - u_m, \bar{\theta}_m\rangle_V \,\mathrm{d}t + \iint_Q u_m \bar{\theta}_m \,\mathrm{d}x\,\mathrm{d}t \to \iint_Q u\theta \,\mathrm{d}x\,\mathrm{d}t. \tag{4.2.40}$$

Moreover by (4.1.13), (4.2.29) and (4.2.38)

$$\begin{aligned}
\limsup_{m\to\infty} \iint_Q \bar{r}_m \bar{\chi}_m \,\mathrm{d}x\,\mathrm{d}t &= -a \liminf_{m\to\infty} \iint_Q \frac{\partial \chi_m}{\partial t} \bar{\chi}_m \,\mathrm{d}x\,\mathrm{d}t \\
&= -\frac{a}{2} \liminf_{m\to\infty} \int_\Omega \left(|\chi_m(x, T)|^2 - |\chi^0|^2\right) \mathrm{d}x \\
&\leqslant -\frac{a}{2} \int_\Omega \left(|\chi(x, T)|^2 - |\chi^0|^2\right) \mathrm{d}x \\
&= -a \iint_Q \frac{\partial \chi}{\partial t} \chi \,\mathrm{d}x\,\mathrm{d}t = \iint_Q r\chi \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{4.2.41}$$

Note that the inclusion (4.2.9) also reads

$$\begin{aligned}
&\iint_Q \left[\bar{\theta}_m(\bar{u}_m - \tilde{u}) + \bar{r}_m(\bar{\chi}_m - \tilde{\chi})\right] \mathrm{d}x\,\mathrm{d}t \geqslant \iint_Q \left[\varphi(\bar{u}_m, \bar{\chi}_m) - \varphi(\tilde{u}, \tilde{\chi})\right] \mathrm{d}x\,\mathrm{d}t \\
&\quad \forall (\tilde{u}, \tilde{\chi}) \in L^2(Q) \quad \text{such that } (\tilde{u}, \tilde{\chi}) \in \mathrm{Dom}(\varphi), \text{ a.e. in } Q.
\end{aligned} \tag{4.2.42}$$

By (4.2.40) and (4.2.41), passing to the limit in the latter inequality (more precisely, the superior limit in the left side and the inferior limit in the right side), we get

$$\begin{aligned}
&\iint_Q \left[\theta(u - \tilde{u}) + r(\chi - \tilde{\chi})\right] \mathrm{d}x\,\mathrm{d}t \geqslant \iint_Q \left[\varphi(u, \chi) - \varphi(\tilde{u}, \tilde{\chi})\right] \mathrm{d}x\,\mathrm{d}t \\
&\quad \forall (\tilde{u}, \tilde{\chi}) \in L^2(Q)^2 \colon\ (\tilde{u}, \tilde{\chi}) \in \mathrm{Dom}(\varphi), \text{ a.e. in } Q,
\end{aligned} \tag{4.2.43}$$

and this is tantamount to (4.1.10).

(v) *Strong convergence of* $\bar{\theta}_m$. We claim that

$$\theta_m \to \theta \quad \text{strongly in } L^1(Q). \tag{4.2.44}$$

First note that, by Rockafellar's Theorem 5.2.3,

$$\partial_\theta[\psi_1 + \psi_2](\theta, \chi) = \partial_\theta \psi_1(\theta) + \partial_\theta \psi_2(\theta, \chi).$$

The analogous property for ∂_χ is trivial, as ψ_1 is independent of χ. Regarding $\psi_1(\theta)$ as a function of (θ, χ) that is independent of χ, and defining $\tilde\partial$ as in (5.3.17), we thus have $\tilde\partial(\psi_1 + \psi_2) = \tilde\partial\psi_1 + \tilde\partial\psi_2$. The hypothesis (4.2.3) and the inclusion (4.2.9) then yield

$$(\bar u_m, \bar r_m) \in \tilde\partial\psi_1(\bar\theta_m) + \tilde\partial\psi_2(\bar\theta_m, \bar\chi_m) \quad \text{a.e. in } Q, \forall m;$$

thus there exist $\bar u_{1m}, \bar u_{2m} \in L^2(Q)$ such that

$$\begin{cases} (\bar u_m, \bar r_m) = (\bar u_{1m}, 0) + (\bar u_{2m}, \bar r_m), \\ (\bar u_{1m}, 0) \in \tilde\partial\psi_1(\bar\theta_m), \\ (\bar u_{2m}, \bar r_m) \in \tilde\partial\psi_2(\bar\theta_m, \bar\chi_m) \end{cases} \quad \text{a.e. in } Q,\ \forall m. \tag{4.2.45}$$

By Theorem 5.3.4 the operators $\tilde\partial\psi_1$ and $\tilde\partial\psi_2$ are maximal monotone. We can thus apply Proposition 5.5.4, or rather the Remark (i) that follows that statement, taking $(\bar u_i, 0)$, (θ, χ), $\tilde\partial\psi_i$ in place of u_i, u^*, β_i, respectively (for $i = 1, 2$). We thus infer that

$$(u_1, 0) \in \tilde\partial\psi_1(\theta), \quad \text{namely} \quad u_1 \in \partial\psi_1(\theta) \quad \text{a.e. in } Q. \tag{4.2.46}$$

Let us also recall (4.2.33) and (4.2.35). By the strict convexity of ψ_1 (cf. (4.2.3)), by (4.2.40) and by Proposition 5.4.3, the claim (4.2.44) then follows.

(vi) *An auxiliary inequality*. We claim that

$$\limsup_{m\to\infty} -\int_0^t d\tau \int_\Omega \bar{\vec q}_m \cdot \nabla\bar\theta_m \, dx \leqslant -\int_0^t d\tau \int_\Omega \vec q \cdot \nabla\theta \, dx \quad \forall t \in\,]0, T]. \tag{4.2.47}$$

In view of proving this inequality, notice that (4.2.23) also reads

$$\begin{aligned} &\int_\Omega \big[\varphi(u_m, \chi_m)(\cdot, t) - \varphi\big(u^0, \chi^0\big)\big]\, dx - \int_0^t d\tau \int_\Omega \bar{\vec q}_m \cdot \nabla\bar\theta_m \, dx \\ &\quad + a\int_0^t d\tau \int_\Omega \left|\frac{\partial\chi_m}{\partial t}\right|^2 dx \leqslant \int_0^t \langle \bar f_m, \bar\theta_m\rangle\, d\tau \quad \text{for a.e. } t \in\,]0, T[. \end{aligned} \tag{4.2.48}$$

Moreover by (4.1.10) and by Proposition 5.2.7

$$\int_0^t \left\langle \frac{\partial u}{\partial\tau}, \theta\right\rangle d\tau + \int_0^t d\tau \int_\Omega \frac{\partial\chi}{\partial\tau} r \, dx = \int_\Omega \big[\varphi(u, \chi)(\cdot, t) - \varphi\big(u^0, \chi^0\big)\big]\, dx \quad \text{for a.e. } t \in\,]0, T[. \tag{4.2.49}$$

Note that Eq. (4.1.5) holds in $L^2(0, T; V')$, cf. (4.1.12). Let us next multiply (4.1.5) by θ, multiply (4.1.13) by $\frac{1}{a}\partial\chi/\partial t$, sum these formulas, and integrate in time. By (4.2.49) this yields

$$\begin{aligned} &\int_\Omega \big[\varphi(u, \chi)(\cdot, t) - \varphi\big(u^0, \chi^0\big)\big]\, dx - \int_0^t d\tau \int_\Omega \vec q \cdot \nabla\theta \, dx \\ &\quad + a\int_0^t d\tau \int_\Omega \left|\frac{\partial\chi}{\partial t}\right|^2 dx = \int_0^t \langle f, \theta\rangle\, d\tau \quad \text{for a.e. } t \in\,]0, T[. \end{aligned} \tag{4.2.50}$$

Notice that, by the lower semicontinuity of the convex integral functionals,

$$\liminf_{m\to\infty}\left(\int_\Omega \varphi(u_m,\chi_m)(\cdot,t)\,\mathrm{d}x + a\int_0^t \mathrm{d}\tau \int_\Omega \left|\frac{\partial \chi_m}{\partial t}\right|^2 \mathrm{d}x\right) \geqslant \int_\Omega \varphi(u,\chi)(\cdot,t)\,\mathrm{d}x + a\int_0^t \mathrm{d}\tau \int_\Omega \left|\frac{\partial \chi}{\partial t}\right|^2 \mathrm{d}x \quad \forall t \in\,]0,T]. \tag{4.2.51}$$

The formulas (4.2.48), (4.2.50), (4.2.51) yield (4.2.47).

(vii) *Proof of* (4.1.11). For any $\vec{\xi} \in L^2(\Omega)^3$,

$$\iint_Q \bar{\vec{q}}_m \cdot (\vec{\xi} - \nabla\bar{\theta}_m)\,\mathrm{d}x\,\mathrm{d}t \geqslant \iint_Q \big[\Phi(\tau_k\bar{\theta}_m, \nabla\bar{\theta}_m) - \Phi(\tau_k\bar{\theta}_m, \vec{\xi})\big]\,\mathrm{d}x\,\mathrm{d}t. \tag{4.2.52}$$

Moreover, by (4.1.2), (4.2.44) and by the lower semicontinuity of the integral functional associated to $\Phi(\theta,\cdot)$,

$$\liminf_{m\to\infty}\iint_Q \Phi(\tau_k\bar{\theta}_m, \nabla\bar{\theta}_m)\,\mathrm{d}x\,\mathrm{d}t \geqslant \iint_Q \Phi(\theta,\nabla\theta)\,\mathrm{d}x\,\mathrm{d}t, \tag{4.2.53}$$

$$\iint_Q \Phi(\tau_k\bar{\theta}_m, \vec{\xi})\,\mathrm{d}x\,\mathrm{d}t \to \iint_Q \Phi(\theta,\vec{\xi})\,\mathrm{d}x\,\mathrm{d}t. \tag{4.2.54}$$

By (4.2.47), (4.2.53) and (4.2.54), by passing to the superior limit in the left side and to the inferior limit in the right side of (4.2.52), we then get

$$\iint_Q \vec{q}\cdot(\vec{\xi} - \nabla\theta)\,\mathrm{d}x\,\mathrm{d}t \geqslant \iint_Q \big[\Phi(\theta,\nabla\theta) - \Phi(\theta,\vec{\xi})\big]\,\mathrm{d}x\,\mathrm{d}t \tag{4.2.55}$$

for any $\vec{\xi} \in L^2(\Omega)^3$, that is (4.1.11). □

Modelling remarks. (i) In Problem 4.1.1 we formulated the constitutive law (4.1.3) in terms of the convex potential φ. This relation is equivalent to

$$u \in \partial_\theta \psi(\theta,\chi), \qquad r \in \partial_\chi(-\psi)(\theta,\chi). \tag{4.2.56}$$

Next we show that for the two-phase system it is equivalent to construct the functions ψ and φ. Let us assume that the constitutive relation between u and θ is known in each phase; that is,

$$u \in \partial\psi_s(\theta) \quad \text{in the solid}, \qquad u \in \partial\psi_\ell(\theta) \quad \text{in the liquid}, \tag{4.2.57}$$

for given convex functions ψ_s and ψ_ℓ. This suggests to set $\psi(\theta,-1) := \psi_s(\theta)$, $\psi(\theta,1) := \psi_\ell(\theta)$, and to extend $\psi(\theta,\chi)$ by linear interpolation:

$$\psi(\theta,\chi) := \psi_s(\theta)\frac{1-\chi}{2} + \psi_\ell(\theta)\frac{1+\chi}{2} \quad \forall(\theta,\chi) \in \mathbf{R}\times[-1,1]. \tag{4.2.58}$$

This function is not globally convex w.r.t. the pair (θ,χ); actually, it is convex in θ and linear (hence concave) in χ. By Theorem 5.3.3, a convex function $\varphi(u,\chi)$ is then retrieved by partially conjugating $\psi(\theta,\chi)$ w.r.t. θ, and in turn ψ is the partial conjugate of φ w.r.t. u:

$$\varphi(u,\chi) := \sup_{\theta\in\mathbf{R}}\big[u\theta - \psi(\theta,\chi)\big] \quad \forall(u,\chi)\in\mathbf{R}\times[-1,1],$$
$$\psi(\theta,\chi) := \sup_{u\in\mathbf{R}}\big[u\theta - \varphi(u,\chi)\big] \quad \forall(\theta,\chi)\in\mathbf{R}\times[-1,1]. \tag{4.2.59}$$

This is equivalent to the dual interpolation procedure. Setting $\varphi_s := \psi_s^*$ and $\varphi_\ell := \psi_\ell^*$, the prescribed relations (4.2.57) indeed also read

$$\theta\in\partial\varphi_s(u) \quad \text{in the solid,} \qquad \theta\in\partial\varphi_\ell(u) \quad \text{in the liquid.} \tag{4.2.60}$$

By interpolation a nonconvex function $\tilde{\varphi}$ is obtained:

$$\tilde{\varphi}(u,\chi) := \varphi_s(u)\frac{1-\chi}{2} + \varphi_\ell(u)\frac{1+\chi}{2} \quad \forall(u,\chi)\in\mathbf{R}\times[-1,1]. \tag{4.2.61}$$

(ii) The equality (4.2.50) accounts for the balance of the function φ (here rescaled by the factor τ_E), cf. (2.2.15) and (2.2.17). The first term is the total variation of $\int_\Omega \varphi\,\mathrm{d}x$ in the time interval $[0, \ell k]$. The opposite of the second and the third terms represent the (nonnegative) amount of φ that is *dissipated* in that time interval. The second member is the contribution of the heat source (or sink) f. The function φ is the potential of (2.2.15) rescaled by the constant factor τ_E, and this balance accounts for the consistency of this model with the second principle of thermodynamics, as we saw in Section 2.2.

(iii) If the function ψ were strictly convex w.r.t. χ, then $\psi_\chi(\theta,\chi)$ would depend on χ continuously for $0<\chi<1$ for any θ. This would exclude the occurrence of sharp interfaces between the phases, so that Problem 4.1.1 would represent heat-diffusion with phase transition smoothed out in a temperature interval.[101] In this case the above argument might also be simplified, for Corollary 5.4.2 would also entail the strong convergence of χ_m in $L^1(Q)$, without the need of the hypothesis (4.2.3).

One might prove several further results for Problem 4.1.1. For instance, the solution depends weakly-continuously on the data in the sense of Theorem 3.1.2. In presence of a composite material, one might also homogenize this problem along the lines of [458].[102] On the other hand the uniqueness of the solution does not seem obvious, because of the multiple nonlinearity of the problem. However, if (4.1.14) is reduced to the linear Fourier law $\vec{q} = -k\cdot\nabla\theta$, after time integration the uniqueness may be proved along the lines of Section 3.3.

5. Convexity and other analytical tools

The analysis of Stefan-type problems requires several tools of linear and nonlinear functional analysis. In this appendix we briefly review basic notions of convex analysis, maximal monotone and accretive operators, nonlinear semigroups of contractions in Banach spaces, Γ-convergence, and others. We review some definitions, and state few results that are referred to in the remainder of this survey. We just display some of the most simple proofs. For an appropriate treatment we refer to the literature that is quoted in the respective sections.

[101] This is physically acceptable for several materials, e.g. of organic origin.

[102] The homogenization of phase transitions was also studied e.g. by Ansini, Braides and Chiadò Piat [26], Bossavit and Damlamian [79], Damlamian [159], Rodrigues [393], Visintin [458].

5.1. *Convex and lower semicontinuous functions*

In this and in the next two sections we outline some notions and properties of convex analysis.[103]

We assume throughout that B is a real Banach space equipped with the norm $\|\cdot\|$.[104] By means of the pairing $\langle\cdot,\cdot\rangle$, we put B in duality with its dual space B^*, by equipping these spaces respectively with the weak and the weak star topology. In this way B will play the role of the dual of B^*, even if B is not reflexive. For any function $F : B \to \widetilde{\mathbf{R}} := \mathbf{R} \cup \{+\infty\}$ let us set

$$\mathrm{Dom}(F) := \big\{v \in B : F(v) < +\infty\big\}: \quad (\textit{effective})\ \textit{domain} \text{ of } F, \tag{5.1.1}$$

$$\mathrm{epi}(F) := \big\{(v,a) \in B \times \mathbf{R}\colon\ F(v) \leqslant a\big\}: \quad \textit{epigraph} \text{ of } F. \tag{5.1.2}$$

Let us also define the *indicator function* of any set $K \subset B$:

$$I_K : B \to \widetilde{\mathbf{R}} : v \mapsto \begin{cases} 0 & \text{if } v \in K, \\ +\infty & \text{if } v \notin K. \end{cases} \tag{5.1.3}$$

This definition allows one to reformulate constrained minimization problems as unconstrained ones, for

$$u = \inf\nolimits_K F \quad \Leftrightarrow \quad u = \inf\nolimits_B (F + I_K). \tag{5.1.4}$$

Any set $K \subset B$ is said to be *convex* if either it is empty or

$$\lambda v_1 + (1-\lambda) v_2 \in K \quad \forall v_1, v_2 \in K,\ \forall \lambda \in]0,1[. \tag{5.1.5}$$

A function $F : B \to \widetilde{\mathbf{R}}$ is said to be *convex* if

$$\begin{aligned} &F\big(\lambda v_1 + (1-\lambda) v_2\big) \leqslant \lambda F(v_1) + (1-\lambda) F(v_2) \\ &\quad \forall v_1, v_2 \in B,\ \forall \lambda \in]0,1[, \end{aligned} \tag{5.1.6}$$

with obvious conventions for the arithmetical operations in $\widetilde{\mathbf{R}}$. If the inequality (5.1.6) is strict for any $v_1 \neq v_2$, the function F is said to be *strictly convex*. The function F is said to be *lower semicontinuous* if the set $\{v \in B\colon\ F(v) \leqslant a\}$ is closed for any $a \in \mathbf{R}$. F is said to be *proper* if it is not identically equal to $+\infty$.

PROPOSITION 5.1.1.

(i) *A function $F : B \to \widetilde{\mathbf{R}}$ is convex* (*lower semicontinuous*, *resp.*) *if and only if* $\mathrm{epi}(F)$ *is convex* (*closed*, *resp.*).

(ii) *A set $K \subset B$ is convex* (*closed*, *resp.*) *if and only if I_K is convex* (*lower semicontinuous*, *resp.*).

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[103] See e.g. the monographs quoted in the item (VIII) of the Bibliographical Note in Section 6.

[104] This theory might also be developed in the more general framework of topological vector spaces, see e.g. Ekeland and Temam [188], Moreau [339]. One may also deal with complex spaces, just replacing the duality pairing by its real part.

PROPOSITION 5.1.2.

(i) *If* $\{F_i : B \to \widetilde{\mathbf{R}}\}_{i\in I}$ *is a family of convex* (*lower semicontinuous*, *resp.*) *functions*, *then their upper hull* $F : v \mapsto \sup_{i\in I} F_i(v)$ *is convex* (*lower semicontinuous*, *resp.*).

(ii) *If* $\{K_i\}_{i\in I}$ *is a family of convex* (*closed*, *resp.*) *subsets of* B, *then their intersection* $\bigcap_{i\in I} K_i$ *is convex* (*closed*, *resp.*).

Let us denote by $\Gamma(B)$ the class of functions $F : B \to \widetilde{\mathbf{R}}$ that are the upper hull of a family of continuous and affine functions $B \to \mathbf{R}$. This consists of the class $\Gamma_0(B)$ of proper, convex, lower semicontinuous functions, and of the function identically equal to $+\infty$.

By part (ii) of Proposition 5.1.2, for any set $K \subset B$ the intersection of the convex and closed subsets of B that contain K is convex and closed; this is named the *closed convex hull* of K, and is denoted by $\overline{\mathrm{co}}(K)$. Similarly, let us consider any function $F : B \to\ ]-\infty, +\infty]$ that has a convex lower bound. By part (i) of Proposition 5.1.2, the upper hull of all affine lower bounds of F is convex and lower semicontinuous; this is the largest lower bound of F in $\Gamma(B)$, and is named the Γ*-regularized function* of F. Its epigraph coincides with the closed convex hull of the epigraph of F.

5.2. *Legendre–Fenchel transformation and subdifferential*

Let $F : B \to \widetilde{\mathbf{R}}$ be a proper function. The function

$$F^* : B^* \to \widetilde{\mathbf{R}} : u^* \mapsto \sup_{u\in B}\bigl\{\langle u^*, u\rangle - F(u)\bigr\} \tag{5.2.1}$$

is called the (convex) *conjugate function* of F. If F^* is proper, its conjugate function

$$F^{**} : B \to \widetilde{\mathbf{R}} : u \mapsto \sup_{u^*\in B^*}\bigl\{\langle u^*, u\rangle - F^*(u^*)\bigr\} \tag{5.2.2}$$

is called the *biconjugate function* of F. (Notice that we defined F^{**} on B rather than the bidual space B^{**}.) If the function F depends on two (or more) variables, one may also introduce the *partial conjugate* function w.r.t. any of these variables. For instance, if $F : B^2 \to \tilde{\mathbf{R}}$, then its partial conjugate w.r.t. the first variable reads

$$G : B^* \times B \to \widetilde{\mathbf{R}} : (u^*, w) \mapsto \sup_{u\in B}\bigl\{\langle u^*, u\rangle - F(u, w)\bigr\}. \tag{5.2.3}$$

THEOREM 5.2.1. *For any proper function* $F : B \to \widetilde{\mathbf{R}}$ *such that* F^* *is also proper*,

$$\begin{aligned} &F^* \in \Gamma(B^*); \quad F^{**} \leqslant F; \\ &F^{**} = F \quad \Leftrightarrow \quad F \in \Gamma(B); \quad (F^*)^{**} = F^*. \end{aligned} \tag{5.2.4}$$

Moreover, F^{**} *coincides with the* Γ*-regularized function of* F (*Fenchel–Moreau theorem*).

The conjugacy transformation $F \mapsto F^*$ *is a bijection between* $\Gamma_0(B)$ *and* $\Gamma_0(B^*)$.

We define the *subdifferential* $\partial F : \mathrm{Dom}(F) \subset B \to 2^{B^*}$ (the power set) of any proper function $F : B \to \widetilde{\mathbf{R}}$ as follows:

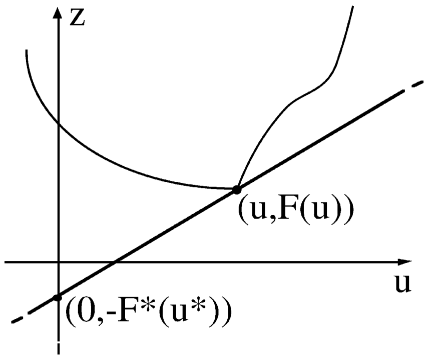

Fig. 10. The drawn subtangent straight-line is characterized by the equation $z = \langle u^*, v - u\rangle + F(u)$, or equivalently $z = \langle u^*, v\rangle - F^*(u^*)$, for any $u^* \in \partial F(u)$.

$$\partial F(u) := \bigl\{u^* \in B^*\colon \langle u^*, u - v\rangle \geqslant F(u) - F(v),\ \forall v \in B\bigr\}$$
$$\forall u \in \mathrm{Dom}(F), \tag{5.2.5}$$

cf. Figure 10. Dually, we define $\partial F^* : \mathrm{Dom}(F^*) \subset B^* \to 2^B$:

$$\partial F^*(u^*) := \bigl\{u \in B\colon \langle u, u^* - v^*\rangle \geqslant F^*(u^*) - F^*(v^*),\ \forall v^* \in B^*\bigr\}$$
$$\forall u^* \in \mathrm{Dom}(F^*). \tag{5.2.6}$$

Note that $\partial F(u) = \emptyset$ is not excluded, so that one may also take the subdifferential of an either nonconvex or nonlower-semicontinuous function at any point of its domain. We also set

$$\partial F(u) := \emptyset \quad \forall u \in B \setminus \mathrm{Dom}(F), \qquad \partial F^*(u^*) := \emptyset \quad \forall u^* \in B^* \setminus \mathrm{Dom}(F^*).$$

If the function F depends on two or more variables, one may also introduce the *partial subdifferential* w.r.t. one of its arguments, extending the notion of partial derivative. For instance, if $F : B^2 \to \widetilde{\mathbf{R}}$, then the partial subdifferential $\partial_u F(u, w)$ is defined as in (5.2.5), by *freezing* the dependence on the argument w.

PROPOSITION 5.2.2. *Let $F : B \to \widetilde{\mathbf{R}}$. Then for any $u \in B$ and any $u^* \in B^*$:*

$$F(u) + F^*(u^*) \geqslant \langle u^*, u\rangle, \tag{5.2.7}$$
$$u^* \in \partial F(u) \quad \Leftrightarrow \quad F(u) + F^*(u^*) = \langle u^*, u\rangle, \tag{5.2.8}$$
$$u^* \in \partial F(u) \quad \Rightarrow \quad u \in \partial F^*(u^*), \tag{5.2.9}$$
$$\bigl[F(u) = F^{**}(u),\ u \in \partial F^*(u^*)\bigr] \quad \Rightarrow \quad u^* \in \partial F(u), \tag{5.2.10}$$
$$F \in \Gamma_0(B) \quad \Rightarrow \quad \partial F^* = (\partial F)^{-1}. \tag{5.2.11}$$

(5.2.7) follows from the definition of F^*. By taking the supremum over all test functions v in (5.2.5), we get $F(u) + F^*(u^*) \leqslant \langle u^*, u\rangle$ whenever $u^* \in \partial F(u)$; (5.2.7) then entails the equality. The opposite implication directly follows from the definition of F^*. (5.2.8) is thus established. The statements (5.2.9)–(5.2.11) are easily proved by means of (5.2.8).

THEOREM 5.2.3 *(Rockafellar). Let $F_1, F_2 : B \to \widetilde{\mathbf{R}}$. Then*

$$\partial F_1(u) + \partial F_2(u) \subset \partial(F_1 + F_2)(u) \quad \forall u \in \mathrm{Dom}(F_1) \cap \mathrm{Dom}(F_2). \tag{5.2.12}$$

The opposite inclusion holds whenever F_1 and F_2 are both convex and lower semicontinuous, and either F_1 or F_2 is continuous at some point $u_0 \in \mathrm{Dom}(F_1) \cap \mathrm{Dom}(F_2)$.

PROPOSITION 5.2.4. *Let $F : B \to \widetilde{\mathbf{R}}$ be convex and proper. Then F is locally Lipschitz-continuous at the interior of* $\mathrm{Dom}(F)$, *and there $\partial F \neq \emptyset$.*

EXAMPLE. Let B be a real Hilbert space, denote it by H, and identify it with its dual space, so that the duality pairing coincides with the scalar product: $\langle u, v\rangle = (u, v)$ for any $u, v \in H$. Let also $1 \leqslant p < +\infty$ and set $p' = p/(p-1)$ if $p > 1$, $p' = \infty$ if $p = 1$. Let us consider the functional $F_p(u) = \|u\|^p/p$ for any $u \in H$. If $p > 1$, then

$$\partial F_p(u) = \|u\|^{p-2}u \quad \forall u \in H, \qquad F_p^*(v) = \frac{1}{p'}\|v\|^{p'} \quad \forall v \in H. \tag{5.2.13}$$

On the other hand for $p = 1$,

$$\begin{aligned} &\partial F_1(u) = \big\{\|u\|^{-1}u\big\} \quad \forall u \in H \setminus \{0\}, \qquad \partial F_1(0) = \big\{v \in H\colon\ \|v\| \leqslant 1\big\}, \\ &F_1^*(v) = 0 \quad \text{if } \|u\| \leqslant 1, \qquad F_1^*(v) = +\infty \quad \text{otherwise.} \end{aligned} \tag{5.2.14}$$

In particular, if $H = \mathbf{R}$ then ∂F_1 coincides with the multi-valued sign function:

$$\begin{aligned} &\mathrm{sign}(u) := \{-1\} \quad \text{if } u < 0, \qquad \mathrm{sign}(0) := [-1, 1], \\ &\mathrm{sign}(u) := \{1\} \quad \text{if } u > 0. \end{aligned} \tag{5.2.15}$$

Here (5.2.7) and (5.2.8) read

$$\left.\begin{aligned} &\frac{1}{p}\|u\|^p + \frac{1}{p'}\|v\|^{p'} \geqslant (u, v), \\ &v = \|u\|^{p-2}u \quad \Leftrightarrow \quad \frac{1}{p}\|u\|^p + \frac{1}{p'}\|v\|^{p'} = (u, v) \end{aligned}\right. \qquad \forall u, v \in H. \tag{5.2.16}$$

(For $H = \mathbf{R}$, the former inequality is the classical Young inequality.) A similar example applies if B is a Banach space, but this requires the introduction of the notion of *duality mapping*.

PROPOSITION 5.2.5. *For any proper, convex, lower semicontinuous function $F : B \to \widetilde{\mathbf{R}}$, any $u \in B$ and any $u^* \in B^*$, the following statements are mutually equivalent*:

$$u^* \in \partial F(u), \tag{5.2.17}$$

$$u \in \partial F^*(u^*), \tag{5.2.18}$$

$$u \in \mathrm{Dom}(F), \quad \langle u^*, u - v\rangle \geqslant F(u) - F(v) \quad \forall v \in B, \tag{5.2.19}$$

$$u^* \in \mathrm{Dom}(F^*), \quad \langle u, u^* - v^*\rangle \geqslant F^*(u^*) - F^*(v^*) \quad \forall v^* \in B^*, \tag{5.2.20}$$

$$\langle u, u^*\rangle \geqslant F(u) + F^*(u^*), \tag{5.2.21}$$

$$\langle u, u^*\rangle = F(u) + F^*(u^*). \tag{5.2.22}$$

The equivalence between (5.2.17) and (5.2.18) follows from (5.2.10). The inclusions (5.2.17) and (5.2.18) are respectively equivalent to the *variational inequalities* (5.2.19) and (5.2.20), by the definitions of ∂F and ∂F^*. The inequality (5.2.21) is equivalent to (5.2.19) by the definition of F^*. Finally, (5.2.21) is equivalent to (5.2.22) because of (5.2.7).

The next statement is just a particular case of the latter proposition.

COROLLARY 5.2.6. *For any* (*nonempty*) *closed convex set* $K \subset B$, *any* $u \in B$ *and any* $u^* \in B^*$, *the following statements are mutually equivalent*:

$$u^* \in \partial I_K(u), \tag{5.2.23}$$

$$u \in \partial I_K^*(u^*), \tag{5.2.24}$$

$$u \in K, \quad \langle u^*, u - v\rangle \geqslant 0 \quad \forall v \in K, \tag{5.2.25}$$

$$\langle u, u^* - v^*\rangle \geqslant I_K^*(u^*) - I_K^*(v^*) \quad \forall v^* \in B^*, \tag{5.2.26}$$

$$u \in K, \quad \langle u, u^*\rangle \geqslant I_K^*(u^*), \tag{5.2.27}$$

$$u \in K, \quad \langle u, u^*\rangle = I_K^*(u^*). \tag{5.2.28}$$

The next statement is often applied.

PROPOSITION 5.2.7. *Let* B^* *be a Banach space*, $F : B \to \widetilde{\mathbf{R}}$ *be convex and lower semi-continuous*, *and* $p \in [1, +\infty[$. *If*

$$u \in W^{1,p}(0, T; B), \qquad w \in L^{p'}(0, T; B^*), \tag{5.2.29}$$

$$w \in \partial F(u) \quad \textit{a.e. in }]0, T[, \tag{5.2.30}$$

then

$$F(u) \in W^{1,1}(0, T), \qquad \frac{\mathrm{d}}{\mathrm{d}t} F(u) = \left\langle w, \frac{\mathrm{d}u}{\mathrm{d}t}\right\rangle \quad \textit{a.e. in }]0, T[. \tag{5.2.31}$$

5.3. *Saddle points*

Let U, V be nonempty subsets of two real topological vector spaces X_1, X_2 (resp.), and $L : U \times V \to \mathbf{R}$. (We assume that this function is finite in order to simplify the presentation.) Note that

$$\inf_{u\in U} L(u, \bar v) \leqslant \sup_{v\in V} L(\bar u, v) \quad \forall(\bar u, \bar v) \in U \times V. \tag{5.3.1}$$

A point $(\bar u, \bar v) \in U \times V$ is called a *saddle point* of L whenever the opposite inequality is fulfilled, or equivalently

$$L(\bar u, v) \leqslant L(u, \bar v) \quad \forall(u, v) \in U \times V. \tag{5.3.2}$$

This is also equivalent to the so-called *min-max* equality:

$$\min_{u\in U} \sup_{v\in V} L(u, v) = \max_{v\in V} \inf_{u\in U} L(u, v). \tag{5.3.3}$$

In view of the next two statements, we shall say that a function $f : X \to \mathbf{R}$ is *quasi-convex* if for any $a \in \mathbf{R}$ the sublevel set $\{v \in X\colon\ f(v) \leqslant a\}$ is convex, and that it is *quasi-concave* if $-f$ is quasi-convex. Obviously, any convex (concave, resp.) function is quasi-convex (quasi-concave, resp.), but the converse may fail: for instance, any nondecreasing real function is quasi-convex.

The next statement is often used in the study of saddle points.

THEOREM 5.3.1 *(Fan inequality). Let K be a compact convex subset of a real topological vector space X, and $\varphi : K^2 \to \mathbf{R}$ be such that*

$$\varphi(\cdot, y) \text{ is lower semicontinuous } \forall y \in K, \tag{5.3.4}$$

$$\varphi(x, \cdot) \text{ is quasi-concave } \forall x \in K. \tag{5.3.5}$$

Then

$$\min_{x\in K} \sup_{y\in K} \varphi(x, y) \leqslant \sup_{y\in K} \varphi(y, y). \tag{5.3.6}$$

THEOREM 5.3.2 *(Existence of a saddle point – Von Neumann – Sion). Let U and V be nonempty compact convex subsets of two real topological vector spaces X_1 and X_2 (resp.), and $L : U \times V \to \mathbf{R}$ be such that*

$$\begin{aligned} &L(\cdot, v) \text{ is quasi-convex and lower semicontinuous } \forall v \in V,\\ &L(u, \cdot) \text{ is quasi-concave and upper semicontinuous } \forall u \in U. \end{aligned} \tag{5.3.7}$$

Then L has a saddle point, and more precisely

$$\min_{u\in U} \max_{v\in V} L(u, v) = \max_{v\in V} \min_{u\in U} L(u, v). \tag{5.3.8}$$

We just show that, under the strengthened hypotheses that $L(\cdot, v)$ is convex and $L(u, \cdot)$ is concave, this statement follows from Theorem 5.3.1. Let us first set

$$\varphi\big((\tilde u, \tilde v), (u, v)\big) := L(\tilde u, v) - L(u, \tilde v) \quad \forall (\tilde u, \tilde v), (u, v) \in K := U \times V,$$

and notice that the hypotheses of Fan's theorem are fulfilled. There exists then $(\bar u, \bar v) \in K$ such that

$$\begin{aligned} \varphi\big((\bar u, \bar v), (u, v)\big) &= \min_{(\tilde u, \tilde v)\in K} \sup_{(u,v)\in K} \varphi\big((\tilde u, \tilde v), (u, v)\big)\\ &\leqslant \sup_{(u,v)\in K} \varphi\big((u, v), (u, v)\big) = 0. \end{aligned} \tag{5.3.9}$$

Thus $(\bar u, \bar v)$ fulfills (5.3.3), namely it is a saddle point of L. Actually, by the compactness of U and V, in this case the min-max equality has the more precise form (5.3.8).

The function L may be allowed to attain the values $\pm\infty$, but then some care must be paid in defining lower and upper semicontinuity.[105]

[105] See e.g. Rockafellar [388], [389, Section 33].

THEOREM 5.3.3. *Let B_1, B_2 be two real Banach spaces, $F \in \Gamma_0(B_1 \times B_2)$, and L be the partial conjugate of F w.r.t. u*:

$$L(u^*, w) = \sup_{u \in B_1} \big\{ \langle u^*, u \rangle - F(u, w) \big\} \quad \forall (u^*, w) \in B_1^* \times B_2. \tag{5.3.10}$$

Then: (i)

$$L(\cdot, w) \text{ is convex and lower semicontinuous } \forall w \in B_2, \tag{5.3.11}$$

$$L(u^*, \cdot) \text{ is concave } \forall u^* \in B_1^*. \tag{5.3.12}$$

(*In general $L(u^*, \cdot)$ need not be upper semicontinuous.*)

(ii) *Moreover, let B_1 be reflexive, and F be coercive with respect to u locally uniformly with respect to w, in the sense that*

$$\begin{aligned} &\forall \text{ bounded } S \subset B_2, \ \forall M > 0, \\ &\big\{ u \in B_1 \colon F(u, w) \leqslant M, \ \forall w \in S \big\} \text{ is bounded.} \end{aligned} \tag{5.3.13}$$

Then $L(u^*, \cdot)$ is upper semicontinuous for any $u^* \in B_1^*$.

(iii) *Under the above hypotheses, for any $(u, w) \in B_1 \times B_2$ and any $(u^*, w^*) \in B_1^* \times B_2^*$,*

$$\begin{cases} u^* \in \partial_u F(u, w), \\ w^* \in \partial_w F(u, w) \end{cases} \Leftrightarrow \quad \begin{cases} u \in \partial_{u^*} L(u^*, w), \\ w^* \in \partial_w (-L)(u^*, w). \end{cases} \tag{5.3.14}$$

(iv) *Conversely, if $L : B_1^* \times B_2 \to \mathbf{R}$ fulfills (5.3.11) and (5.3.12), then the mapping*

$$G(u, w) := \sup_{u^* \in B_1^*} \big\{ \langle u^*, u \rangle - L(u^*, w) \big\} \quad \forall (u, w) \in B_1 \times B_2 \tag{5.3.15}$$

is convex and lower semicontinuous. Moreover, if L fulfills (5.3.10) then $F = G$.

PARTIAL PROOF. (i) The statement (5.3.11) directly follows from part (i) of Proposition 5.1.2.

In view of proving (5.3.12), let us fix any $u^* \in B_1^*$, any $u', u'' \in B_1$, any $w', w'' \in B_2$ and any $\lambda \in]0, 1[$. By the convexity of F we have

$$\begin{aligned} &L\big(u^*, \lambda w' + (1 - \lambda) w''\big) \\ &\quad \geqslant \big\langle u^*, \lambda u' + (1 - \lambda) u'' \big\rangle - F\big(\lambda u' + (1 - \lambda) u'', \lambda w' + (1 - \lambda) w''\big) \\ &\quad \geqslant \lambda\big(\langle u^*, u' \rangle - F(u', w')\big) + (1 - \lambda)\big(\langle u^*, u'' \rangle - F(u'', w'')\big). \end{aligned}$$

By taking the supremum with respect to u' and u'', we then get

$$L\big(u^*, \lambda w' + (1 - \lambda) w''\big) \geqslant \lambda L(u^*, w') + (1 - \lambda) L(u^*, w'').$$

The property (5.3.12) has thus been proved.

(ii) Let us fix any sequence $\{w_n\}$ in B_2 that weakly converges to some $w \in B_2$; $\{w_n\}$ is necessarily bounded. If $M := \limsup_{n \to \infty} L(u^*, w_n) = -\infty$ then trivially $L(u^*, w) \geqslant M$. If instead $M > -\infty$ then there exists a sequence $\{u_n\}$ in B_1 such that for n large enough

$$\langle u^*, u_n \rangle - F(u_n, w_n) \geqslant L(u^*, w_n) + 1/n \quad \forall n \in \mathbf{N} \quad \text{if } M < +\infty, \tag{5.3.16}$$

$$\langle u^*, u_n\rangle - F(u_n, w_n) \geqslant n \quad \forall n \in \mathbf{N} \quad \text{if } M = +\infty.$$

By (5.3.13) the sequence $\{u_n\}$ is confined to a bounded subset of the reflexive space B_1. Hence there exists $u \in B_1$ such that, as n diverges along a further subsequence (not relabelled), $u_n \to u$ weakly in B_1. Passing to the limit in (5.3.16) on this subsequence, by the lower semicontinuity of F we then get

$$\langle u^*, u\rangle - F(u, w) \geqslant \limsup_{n\to\infty}\bigl(\langle u^*, u_n\rangle - F(u_n, w_n)\bigr) \geqslant M.$$

Thus $L(u^*, w) \geqslant M$.

(iii) For the proof of (5.3.14) see e.g. Barbu and Precupanu [49, p. 135], Rockafellar [388], [389, p. 395].

(iv) The convexity and lower semicontinuity of G follow from Proposition 5.1.2. As $G(\cdot, w)$ is the biconjugate function of $(\cdot, w)$, by Theorem 5.2.1 we conclude that $F = G$. □

For any function $L : U \times V \to \mathbf{R}$, let us define the *double subdifferential*

$$\tilde{\partial} L : U \times V \to 2^{B_1^*} \times 2^{B_2^*} : (u, v) \mapsto \bigl(\partial_u L(u, v), \partial_v[-L(u, v)]\bigr). \tag{5.3.17}$$

This definition is especially convenient if L is convex-concave, cf. (5.3.18) below.

Incidentally, note that any (u, v) is a saddle point of L if and only if $(0, 0) \in \tilde{\partial} L(u, v)$.

THEOREM 5.3.4.[106] *Let U and V be nonempty, closed, convex subsets of two real Banach spaces B_1 and B_2 (respect.), with at least one of them reflexive. Let $L : U \times V \to \mathbf{R}$ be such that*

$$\begin{aligned} &L(\cdot, v) \text{ is convex and lower semicontinuous } \forall v \in V,\\ &L(u, \cdot) \text{ is concave and upper semicontinuous } \forall u \in U. \end{aligned} \tag{5.3.18}$$

The operator $\tilde{\partial} L$ is then maximal monotone.[107]

5.4. *Compactness by strict convexity*

Let K be a closed subset of $\mathbf{R}^N$. A point $\xi \in K$ is said *extremal* for K if

$$\xi = \lambda\xi' + (1-\lambda)\xi'' \in K, \quad \xi', \xi'' \in K,\ 0 < \lambda < 1 \quad \Rightarrow \quad \xi = \xi' = \xi''. \tag{5.4.1}$$

Let Ω be a domain of $\mathbf{R}^N$. A multi-valued mapping $K : \Omega \to 2^{\mathbf{R}^M}$ is said to be measurable if there exists a sequence of measurable single-valued functions $\{k_m : \Omega \to \mathbf{R}^M\}$ such that $\bigcup_{m\in\mathbf{N}} k_m(x)$ is dense in $K(x)$ for a.a. $x \in \Omega$.[108]

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[106] See e.g. Barbu and Precupanu [49, p. 137], Rockafellar [388], [389, p. 396], [390].

[107] See Section 5.6 for the definition of the latter notion.

[108] See e.g. Castaing and Valadier [119, Section III.2], Ioffe and Tihomirov [278, Section 8.1].

THEOREM 5.4.1 [109]. *$K : \Omega \to 2^{\mathbf{R}^M}$ be measurable, and $K(x)$ be closed and convex for a.a. $x \in \Omega$. If*

$$v_n \to v \quad \textit{weakly in } L^1(\Omega)^M, \tag{5.4.2}$$

$$v_n(x) \in K(x) \quad \textit{for a.a. } x \in \Omega,\ \forall n, \tag{5.4.3}$$

$$v(x) \textit{ is an } \text{extremal point} \textit{ of } K(x) \textit{ for a.a. } x \in \Omega, \tag{5.4.4}$$

then

$$v_n \to v \quad \textit{strongly in } L^1(\Omega)^M. \tag{5.4.5}$$

By and large, the rational behind this result is that if a sequence of function converges weakly in $L^1(\Omega)^M$ without oscillating around the limit value, then it also converges strongly. For instance, this applies to any L^1-weakly vanishing sequence of nonnegative scalar functions. (On the other hand, it fails in L^p-spaces with $p > 1$.)

COROLLARY 5.4.2. *If $\varphi : \mathbf{R}^M \to \widetilde{\mathbf{R}}$ is strictly convex, lower semicontinuous, and*

$$u_n \to u \quad \textit{weakly in } L^1(\Omega)^M, \tag{5.4.6}$$

$$\int_\Omega \varphi(u_n)\,\mathrm{d}x \to \int_\Omega \varphi(u)\,\mathrm{d}x \ (< +\infty), \tag{5.4.7}$$

then

$$u_n \to u \quad \textit{strongly in } L^1(\Omega)^M, \tag{5.4.8}$$

$$\varphi(u_n) \to \varphi(u) \quad \textit{strongly in } L^1(\Omega). \tag{5.4.9}$$

OUTLINE OF THE PROOF. By (5.4.7) and by the convexity of φ, it is not difficult to see that $\varphi(u_n) \to \varphi(u)$ weakly in $L^1(\Omega)$. Note that $(u, \varphi(u))$ is an extremal point of $K := \operatorname{epi}(\varphi) \subset \mathbf{R}^{M+1}$ a.e. in Ω. It then suffices to apply Theorem 5.4.1 taking $v_n = (u_n, \varphi(u_n))$. □

PROPOSITION 5.4.3. *In Corollary 5.4.2 the assumption (5.4.7) holds whenever*

$$w_n := \partial\varphi(u_n) \to w := \partial\varphi(u) \quad \textit{weakly in } L^1(\Omega)^M, \tag{5.4.10}$$

$$\int_\Omega u_n \cdot w_n\,\mathrm{d}x \to \int_\Omega u \cdot w\,\mathrm{d}x. \tag{5.4.11}$$

($\partial\varphi$ *is single-valued, because of the hypothesis of strict convexity.*)

PROOF. The Fenchel property yields

$$\begin{aligned} &\int_\Omega \varphi(u_n)\,\mathrm{d}x + \int_\Omega \varphi^*(w_n)\,\mathrm{d}x = \int_\Omega u_n \cdot w_n\,\mathrm{d}x, \\ &\int_\Omega \varphi(u)\,\mathrm{d}x + \int_\Omega \varphi^*(w)\,\mathrm{d}x = \int_\Omega u \cdot w\,\mathrm{d}x. \end{aligned} \tag{5.4.12}$$

[109] See Visintin [445], [453, Section X.1].

By the lower semicontinuity of these integral functionals and by (5.4.11), we then infer (5.4.7) (and $\int_\Omega \varphi^*(w_n)\,\mathrm{d}x \to \int_\Omega \varphi^*(w)\,\mathrm{d}x$). □

Dually, if the conjugate function φ^* is strictly convex, then (5.4.6), (5.4.10) and (5.4.11) entail the strong L^1-convergence of the sequence $\{w_n\}$.

5.5. *Maximal monotone operators*

In this section we briefly illustrate the notion of maximal monotone operator in a real Banach space B.[110]

An operator $A : B \to 2^{B^*}$ is said *monotone* if, setting $\operatorname{graph}(A) := \{(u, u^*)\colon u^* \in A(u)\}$,

$$\langle u^* - v^*, u - v\rangle \geqslant 0 \quad \forall (u, u^*), (v, v^*) \in \operatorname{graph}(A). \tag{5.5.1}$$

This operator is said *maximal monotone* if it is monotone and its graph is not properly included in that of any other monotone operator $B \to 2^{B^*}$. It is said *cyclically monotone* if

$$\begin{aligned}&\forall m \in \mathbf{N}\ (m \geqslant 2),\ \forall \big\{(u_i, u_i^*)\big\}_{i=1,\dots,m} \subset \operatorname{graph}(A),\\ &\sum_{i=1}^{m} \langle u_i^*, u_i - u_{i-1}\rangle \geqslant 0 \quad (\text{setting } u_0 = u_m).\end{aligned} \tag{5.5.2}$$

(For $m = 2$ the inequality (5.5.1) is obviously retrieved.) By the Hausdorff maximal chain theorem (a consequence of the Zorn lemma), it is easy to see that any monotone operator $A : B \to 2^{B^*}$ can be extended to a maximal monotone operator. The inverse of a monotone operator is defined as the operator that has the inverse graph: $u \in A^{-1}(u^*)$ if and only if $u^* \in A(u)$. If A is maximal monotone, then the inverse operator A^{-1} is also maximal monotone.

THEOREM 5.5.1. *Let B be reflexive and $A : B \to 2^{B^*}$ be maximal monotone. If*

$$\frac{\langle w, v\rangle}{\|v\|} \to +\infty \quad \text{as } w \in A(v),\ \|v\| \to \infty, \tag{5.5.3}$$

then for any $f \in B^*$ there exists $u \in B$ such that $A(u) \ni f$.

The latter theorem extends the next result to Banach spaces.

THEOREM 5.5.2 *(Minty and Browder). Let H be a real Hilbert space. A monotone operator $A : H \to 2^H$ is maximal monotone if and only if the mapping $A + \lambda I$ is surjective for some* (*equivalently, for any*) $\lambda > 0$.

THEOREM 5.5.3 *(Rockafellar). For any $F \in \Gamma_0(B)$ the operator ∂F is maximal monotone.*

An operator $A : B \to 2^{B^*}$ is maximal monotone and cyclically monotone if and only if $A = \partial F$ for some proper lower semicontinuous convex function $F : B \to \widetilde{\mathbf{R}}$.

[110] See e.g. the monographs quoted in the item (IX) of the Bibliographical Note in Section 6.

PROPOSITION 5.5.4. *Let* $\alpha_1, \alpha_2 : B \to 2^{B^*}$ *be two maximal monotone operators*, $\{u_{1n}\}$, $\{u_{2n}\}$ *be two sequences in* B, *and* $\{u_n^*\}$ *be a sequence in* B^*. *If*

$$u_n^* \in \alpha_1(u_{1n}) \cap \alpha_2(u_{2n}) \quad \forall n \in \mathbf{N}, \tag{5.5.4}$$

$$u_{in} \to u_i \quad \textit{weakly in } B, \textit{ for } i = 1, 2, \tag{5.5.5}$$

$$u_n^* \to u^* \quad \textit{weakly star in } B^*, \tag{5.5.6}$$

$$\liminf_{n\to\infty} \langle u_n^*, u_{1n} + u_{2n}\rangle \leqslant \langle u^*, u_1 + u_2\rangle, \tag{5.5.7}$$

then $u^* \in \alpha_1(u_1) \cap \alpha_2(u_2)$.

PROOF. By (5.5.4),

$$\langle u_n^* - v_i^*, u_{in} - v_i\rangle \geqslant 0 \quad \forall (v_i, v_i^*) \in \operatorname{graph}(\alpha_i), \text{ for } i = 1, 2;$$

by adding these two inequalities we get

$$\langle u_n^*, u_{1n} + u_{2n}\rangle - \sum_{i=1,2} \big(\langle u_n^*, v_i\rangle + \langle v_i^*, u_{in} - v_i\rangle\big) \geqslant 0.$$

By passing to the inferior limit as $n \to \infty$, the hypotheses (5.5.5)–(5.5.7) yield

$$\langle u^*, u_1 + u_2\rangle - \sum_{i=1,2} \big(\langle u^*, v_i\rangle + \langle v_i^*, u_i - v_i\rangle\big) \geqslant 0.$$

By selecting either $v_1 = u_1$ or $v_2 = u_2$, we then obtain

$$\langle u^* - v_i^*, u_i - v_i\rangle \geqslant 0 \quad \forall (v_i, v_i^*) \in \operatorname{graph}(\alpha_i), \text{ for } i = 1, 2,$$

namely $u^* \in \alpha_1(u_1) \cap \alpha_2(u_2)$. □

REMARKS.

(i) The latter statement also admits a dual formulation. In fact, denoting by β_i the inverse of the operator α_i ($i = 1, 2$), the hypothesis (5.5.4) and the thesis respectively also read

$$u_{1n} \in \beta_1(u_n^*), \quad u_{2n} \in \beta_2(u_n^*) \quad \forall n \in \mathbf{N}, \tag{5.5.8}$$

$$u_1 \in \beta_1(u^*), \qquad u_2 \in \beta_2(u^*). \tag{5.5.9}$$

(ii) Although we stated Proposition 5.5.4 for two operators, the further extension to an either finite or even countable family of maximal monotone operators is straightforward. On the other hand for $\alpha_1 = \alpha_2$ we get the next statement, that is often applied in the analysis of nonlinear problems.

COROLLARY 5.5.5. *Let* $\alpha : B \to 2^{B^*}$ *be a maximal monotone operator and* $\{(u_n, u_n^*)\}$ *be a sequence in* $B \times B^*$. *If*

$$u_n^* \in \alpha(u_n) \quad \forall n \in \mathbf{N}, \tag{5.5.10}$$

$$u_n \to u \quad \textit{weakly in } B, \tag{5.5.11}$$

$$u_n^* \to u^* \quad \text{weakly star in } B^*, \tag{5.5.12}$$

$$\liminf_{n\to\infty}\langle u_n^*, u_n\rangle \leqslant \langle u^*, u\rangle, \tag{5.5.13}$$

then $u^* \in \alpha(u)$.

5.6. *M-accretive operators and semigroups*

In this section we illustrate nonlinear semigroups of contractions in Banach spaces.[111]

After dealing with (multi-valued) operators $B \to B^*$, next we consider a different notion of monotonicity for operators that map B to itself. Of course this distinction makes sense only if B is not a Hilbert space. An operator $A : B \to 2^B$ is said *accretive* if

$$\begin{aligned}&\forall u_i \in \mathrm{Dom}(A),\ \forall v_i \in A(u_i)\ (i = 1, 2),\ \forall \lambda > 0,\\ &\|u_1 - u_2\| \leqslant \big\|u_1 - u_2 + \lambda(v_1 - v_2)\big\|.\end{aligned} \tag{5.6.1}$$

A is said *m-accretive* if it is accretive and $I + \lambda A$ is surjective for some $\lambda > 0$ (equivalently, for any $\lambda > 0$). By the Minty and Browder Theorem 5.5.2, an operator that acts on a Hilbert space is m-accretive if and only if it is maximal monotone.

Cauchy problem. Let $A : B \to 2^B$, $T > 0$, $f \in L^1(0, T; B)$, $u^0 \in \overline{\mathrm{Dom}}(A)$ (the strong closure of the domain of A), and consider the equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} + A(u) \ni f \quad \text{in }]0, T[. \tag{5.6.2}$$

A function $u :]0, T[\to B$ is called a *strong solution* of this equation if

(i) u is absolutely continuous on any interval $[a, b] \subset]0, T[$, and strongly differentiable a.e. in $]0, T[$,

(ii) $u \in \mathrm{Dom}(A)$ a.e. in $]0, T[$, and

(iii) Eq. (5.6.2) is fulfilled a.e. in $]0, T[$.

On the other hand, $u :]0, T[\to B$ is called a *mild solution* of (5.6.2) if there exists a sequence $\{(u_n, f_n)\}$ such that u_n is a strong solution of the same equation with f_n in place of f for any n, and

$$\begin{aligned}&u_n \to u \quad \text{in } B,\ \text{locally uniformly in }]0, T[,\\ &f_n \to f \quad \text{strongly in } L^1(0, T; B).\end{aligned} \tag{5.6.3}$$

These notions are easily extended to the Cauchy and periodic problems associated with (5.6.2):

$$\text{(CP)} \quad \begin{cases}\dfrac{\mathrm{d}u}{\mathrm{d}t} + A(u) \ni f & \text{in }]0, T[,\\ u(0) = u^0,\end{cases} \tag{5.6.4}$$

$$\text{(PP)} \quad \begin{cases}\dfrac{\mathrm{d}u}{\mathrm{d}t} + A(u) \ni f & \text{in }]0, T[,\\ u(0) = u(T).\end{cases} \tag{5.6.5}$$

Here u is also assumed to be continuous at $t = 0$ for (CP), at $t = 0, T$ for (PP).

[111] See e.g. see e.g. the monographs quoted in the item (X) of the Bibliographical Note in Section 6.

THEOREM 5.6.1. *Let B be a Banach space and $A : B \to 2^B$ be an m-accretive operator. Then*:

(i) *If $f \in L^1(0, T; B)$ and $u^0 \in \overline{\mathrm{Dom}}(A)$, then the Cauchy problem* (CP) *has a mild solution.*

(ii) *If $f_i \in L^1(0, T; B)$, $u_i^0 \in \overline{\mathrm{Dom}}(A)$, and u_i is a corresponding mild solution of* (CP) *for $i = 1, 2$, then*

$$\begin{aligned}&\left\| u_1(t) - u_2(t) \right\| \\ &\quad \leqslant \left\| u_1^0 - u_2^0 \right\| + \int_0^t \left\| f_1(s) - f_2(s) \right\| \mathrm{d}s \quad \forall t \in [0, T].\end{aligned} \tag{5.6.6}$$

(*The mild solution is thus unique*.)

(iii) *If the mild solution of* (CP) *is absolutely continuous on any interval $[a, b] \subset]0, T[$ and is strongly differentiable a.e. in $]0, T[$, then it is a strong solution.*

(iv) *If $f :]0, T[\to B$ has bounded variation and $u^0 \in \mathrm{Dom}(A)$, then the mild solution of* (CP) *is Lipschitz-continuous in* $[0, T]$.

(v) *If the operator $A - aI$ is accretive for some constant $a > 0$, then the periodic problem* (PP) *has one and only one mild solution.*

A Banach space B is said to have the *Radon–Nikodým property* if any Lipschitz-continuous mapping $[0, 1] \to B$ is strongly differentiable a.e. in $]0, T[$. This holds if either B is reflexive or it is separable and has a pre-dual. For instance, this applies to all reflexive Banach spaces and to ℓ^1, but neither to $L^1(\Omega)$ nor to $L^\infty(\Omega)$.[112] By parts (iii) and (iv) of Theorem 5.6.1, the mild solution of the Cauchy problem (CP) is a strong solution whenever B has the *Radon–Nikodým property*, $f :]0, T[\to B$ has bounded variation and $u^0 \in \mathrm{Dom}(A)$.

Let us now assume that $f \equiv 0$. For any $u^0 \in \overline{\mathrm{Dom}}(A)$ let u be the mild solution of (CP), and set $S(t)u^0 := u(t)$ for any $t \geqslant 0$. The mapping $t \mapsto S(t)$ is then a *continuous semigroup of contractions*, for it is a continuous semigroup and

$$\left\| S(t)u_1^0 - S(t)u_2^0 \right\| \leqslant \left\| u_1^0 - u_2^0 \right\| \quad \forall t \geqslant 0, \forall u_1^0, u_2^0 \in \overline{\mathrm{Dom}}(A). \tag{5.6.7}$$

For any Lipschitz-continuous operator $F : B \to B$, the above results are easily extended to the operator $\tilde{A} := A + F$. In this case, denoting by ω the Lipschitz constant of F, $t \mapsto S(t)$ is a *continuous semigroup of ω-contractions*, for (5.6.7) is replaced by

$$\left\| S(t)u_1^0 - S(t)u_2^0 \right\| \leqslant e^{\omega t} \left\| u_1^0 - u_2^0 \right\| \quad \forall t \geqslant 0,\ \forall u_1^0, u_2^0 \in \overline{\mathrm{Dom}}(A). \tag{5.6.8}$$

T-accretiveness. A Banach space B is called a *Banach lattice* if it is an ordered set such that any finite nonempty subset admits infimum and supremum, and, setting $|u| := \sup\{u, -u\}$ and $u \leqslant v$ if $u = \inf\{u, v\}$, it satisfies the following conditions, for any $u, v, w \in B$:

(i) if $u \leqslant v$ then $u + w \leqslant v + w$,

(ii) if $u \leqslant v$ and $\alpha > 0$, then $\alpha u \leqslant \alpha v$,

(iii) if $u \leqslant v$ then $-v \leqslant -u$,

[112] See, e.g., Bénilan [54], Diestel and Uhl [181, Chapter III], Kufner, John and Fučík [298, Section 2.22.5].

(iv) if $|u| \leqslant |v|$ then $\|u\| \leqslant \|v\|$.

For any $u \in B$, let us set

$$u^+ := \sup\{u, 0\} \quad \text{and} \quad u^- := \sup\{-u, 0\}.$$

An operator $A : B \to 2^B$ is then said to be *T-accretive* if

$$\begin{aligned} &\forall u_i \in \mathrm{Dom}(A), \forall v_i \in A(u_i)\ (i = 1, 2),\ \forall \lambda > 0, \\ &\left\|(u_1 - u_2)^+\right\| \leqslant \left\|\left[u_1 - u_2 + \lambda(v_1 - v_2)\right]^+\right\|. \end{aligned} \tag{5.6.9}$$

Any T-accretive operator in B is also accretive whenever

$$\|u^+\| \leqslant \|v^+\|, \quad \|u^-\| \leqslant \|v^-\| \quad \Rightarrow \quad \|u\| \leqslant \|v\| \quad \forall u, v \in B. \tag{5.6.10}$$

THEOREM 5.6.2. *If B is a Banach lattice and A is m- and T-accretive, then the mild solution of the Cauchy problem* (CP) *depends monotonically on the data. That is, if u_i is the mild solution corresponding to u_i^0, f_i $(i = 1, 2)$ and $u_1^0 \leqslant u_2^0$, $f_1 \leqslant f_2$, then $u_1 \leqslant u_2$.*

5.7. *Perimeter and curvature*

In this section we state a result about sets of finite perimeter in the sense of Caccioppoli.[113]

For any measurable function $v : \Omega \to \mathbf{R}$, let us first define the *total variation functional*

$$\int_\Omega |\nabla v| := \sup\left\{\int_\Omega v\nabla \cdot \vec{\eta}\colon\ \vec{\eta} \in C_0^1(\Omega)^N,\ |\vec{\eta}| \leqslant 1 \text{ in } \Omega\right\}. \tag{5.7.1}$$

The domain of this operator in $L^1(\Omega)$ is thus the space $\mathrm{BV}(\Omega)$. Let us also set

$$P(v) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla v|\ (\leqslant +\infty) & \text{if } |v| = 1 \text{ a.e. in } \Omega, \\ +\infty & \text{otherwise.} \end{cases} \tag{5.7.2}$$

If $v \in \mathrm{Dom}(P)$ then $P(v)$ is the *perimeter* in Ω in the sense of Caccioppoli of the set $\Omega^+ = \{x \in \Omega\colon\ v(x) = 1\}$. Whenever Ω^+ is of Lipschitz class, this perimeter coincides with the bidimensional Hausdorff measure of $\partial\Omega^+$.

Let us now fix any $g \in L^1(\Omega)$, any constants $a > 0$ and b, and set

$$\Phi(v) := \begin{cases} a\displaystyle\int_\Omega |\nabla v| + b\int_\Gamma \gamma_0 v \,\mathrm{d}\sigma + \int_\Omega g v \,\mathrm{d}x & \forall v \in \mathrm{Dom}(P), \\ +\infty & \forall v \in L^1(\Omega) \setminus \mathrm{Dom}(P). \end{cases} \tag{5.7.3}$$

This operator is well-defined, for the trace operator γ_0 maps $\mathrm{BV}(\Omega)$ to $L^1(\Gamma)$ (and is continuous).

PROPOSITION 5.7.1.[114] *Under the above assumptions, the functional Φ is lower semicontinuous with respect to the strong topology of $L^1(\Omega)$ if and only if $|b| \leqslant a$. In that case Φ has an* (*in general nonunique*) *absolute minimizer.*

[113] See e.g. the monographs quoted in the item (XV) of the Bibliographical Note in Section 6.

[114] See e.g. Massari and Pepe [322].

The existence of a minimizer of Φ then follows by the *direct method of the calculus of variations*.[115]

THEOREM 5.7.2 *(Gibbs–Thomson law and contact angle condition). Let* $|b| \leqslant a$, $g \in W^{1,p}(\Omega)$ *with* $p > 3$, $u \in L^1(\Omega)$ *and*

$$\liminf \frac{\Phi(v) - \Phi(u)}{\|v - u\|_{L^1(\Omega)}} \geqslant 0 \quad \text{as } v \to u \text{ in } L^1(\Omega). \tag{5.7.4}$$

Then:

(i) *The boundary* S *in* Ω *of the set* $\Omega^+ := \{x \in \Omega\colon u(x) = 1\}$ *is a surface of class* $C^{1,(p-N)/2p}$.

(ii) *Denoting by* $\vec{n}$ *the unit normal vector to the surface* S *oriented towards* Ω^+, $\kappa := \frac{1}{2}\nabla_S \cdot \vec{n} \in L^p(S)$.[116] *Moreover, equipping* S *with the two-dimensional Hausdorff measure and denoting by* γ_0 *the trace operator* $W^{1,p}(\Omega) \to W^{1-1/p,p}(S)$,

$$\kappa = \gamma_0 g \quad \text{a.e. on } S. \tag{5.7.5}$$

(iii) *If* Γ *is of class* C^1, *then, denoting by* ω *the angle between* $\vec{n}$ *and the outward normal vector to* Γ, *and equipping* $S \cap \Gamma$ *with the one-dimensional Hausdorff measure*,

$$\cos\omega = b/a \quad \text{a.e. on } S \cap \Gamma. \tag{5.7.6}$$

Part (i) follows from a classic result of Almgren [14]. Parts (ii) and (iii) may be proved by representing S locally in Cartesian form, and then letting the first variation of Φ vanish for any local Cartesian perturbation of the interface.[117]

5.8. *Γ-convergence*

In this section we state De Giorgi's notion of Γ-convergence, and some basic results of this theory.[118]

Let (X, d) be a metric space, f_n ($n \in \mathbf{N}$) and f be functions $X \to \mathbf{R} \cup \{\pm\infty\}$. If for some $u \in X$

(i) for any sequence $\{u_n\}$ in X,

$$\text{if } u_n \to u \text{ then } \liminf_{n\to\infty} f_n(u_n) \geqslant f(u), \tag{5.8.1}$$

(ii) there exists a sequence $\{u_n\}$ in X such that

$$u_n \to u \text{ and } \limsup_{n\to\infty} f_n(u_n) \leqslant f(u), \tag{5.8.2}$$

[115] See e.g. Braides and Defranceschi [82], Buttazzo, Giaquinta and Hildebrandt [94], Carbone and De Arcangelis [117], Dacorogna [154], Dal Maso [155], Evans and Gariepy [201], Giusti [245].

[116] κ is the tangential divergence of $\vec{n}/2$ over S, in the sense of $H^{-1}(S)$, say. Thus κ equals the mean curvature of S.

[117] See e.g. Visintin [453, Section VI.4].

[118] See e.g. the pioneering papers De Giorgi [168], De Giorgi and Franzoni [169], and the monographs quoted in the item (XIV) of the Bibliographical Note in Section 6.

then we say that f_n *Γ-converges to* f (in (X,d)) at u, and write $f_n(u) \xrightarrow{\Gamma} f(u)$. If this occurs for any $u \in X$, then we say that f_n *Γ-converges to* f, and write $f_n \xrightarrow{\Gamma} f$. The function f is then lower semicontinuous.

PROPOSITION 5.8.1 *(Compactness). Let (X,d) be a separable metric space*[119] *and $\{f_n\}$ be a sequence of functions $X \to \widetilde{\mathbf{R}}$. A subsequence of $\{f_n\}$ then Γ-converges to some function $f : X \to \widetilde{\mathbf{R}}$.*

PROPOSITION 5.8.2 *(Minimization). Let (X,d) be a metric space, and $\{f_n\}$ is a sequence of functions $X \to \widetilde{\mathbf{R}}$ such that f_n Γ-converges to f. If a sequence $\{u_n\} \subset X$ and $u \in X$ are such that*

$$f_n(u_n) \leqslant \inf f_n + \frac{1}{n} \quad \forall n, \qquad u_n \to u \quad \text{in } X, \tag{5.8.3}$$

then $f_n(u_n) \to f(u)$ and $f(u) = \inf f$. (Thus $\inf f_n \to \inf f$.)

PROOF. By (5.8.1), $f(u) \leqslant \liminf_{n\to\infty} f_n(u_n)$. Moreover, for any $v \in X$, there exists a sequence $\{v_n\} \subset X$ such that $v_n \to v$ in X and $f_n(v_n) \to f(v)$. As by (5.8.3), $f_n(u_n) \leqslant f(v_n) + \frac{1}{n}$ for any n, we get

$$f(u) \leqslant \liminf_{n\to\infty} f_n(u_n) \leqslant \lim_{n\to\infty} f_n(v_n) = f(v) \quad \forall v \in X.$$

Thus $f(u) = \inf f$. If $\{\tilde{u}_n\}$ is the sequence prescribed by (5.8.2), then

$$f(u) \geqslant \limsup_{n\to\infty} f_n(\tilde{u}_n) \geqslant \limsup_{n\to\infty}(\inf f_n) \geqslant (\text{by (5.8.3)}) \limsup_{n\to\infty} f_n(u_n).$$

Thus $f_n(u_n) \to f(u)$. □

The two latter propositions entail the next statement, that shows that the notion Γ-convergence is especially appropriate for the study of the limit behaviour of minimization problems.

COROLLARY 5.8.3. *Let (X,d) be a separable metric space, $\{f_n\}$ be a sequence of functions $X \to \widetilde{\mathbf{R}}$, and $\{u_n\}$ be a compact sequence of X such that $f_n(u_n) = \inf f_n$ for any n. Then there exist f and u such that, as $n \to \infty$ along a suitable sequence (not relabelled),*

$$f_n \xrightarrow{\Gamma} f, \quad u_n \to u \quad \text{in } X, \qquad f_n(u_n) \to f(u) = \inf f. \tag{5.8.4}$$

The next statement has been applied to several models of multi-phase systems. It may be noticed that in this case the intersection between the domain of the sequence of functionals, $H^1(\Omega) \cap L^4(\Omega)$, and that of the Γ-limit, $\{v \in \mathrm{BV}(\Omega)\colon |v| = 1 \text{ a.e. in } \Omega\}$, is reduced to the two constant functions $v \equiv \pm 1$.

[119] I.e., a space equipped with a countable basis of open sets, in the topology induced by the metric.

THEOREM 5.8.4.[120] *Let Ω be a Lipschitz domain of $\mathbf{R}^N$, and for any $v \in L^1(\Omega)$ set* (*cf.* (5.7.1))

$$f_n(v) := \begin{cases} \displaystyle\int_\Omega \left(\frac{1}{n}|\nabla v|^2 + n\big(v^2 - 1\big)^2\right) \mathrm{d}x & \text{if } v \in H^1(\Omega) \cap L^4(\Omega), \\ +\infty & \text{otherwise}, \end{cases} \tag{5.8.5}$$

$$f(v) := \begin{cases} \displaystyle\frac{4}{3}\int_\Omega |\nabla v| & \text{if } |v| = 1 \text{ a.e. in } \Omega, \\ +\infty & \text{otherwise}. \end{cases} \tag{5.8.6}$$

The sequence f_n then Γ-converges to f in $L^1(\Omega)$. (*Note that $f = \frac{8}{3}P$, cf.* (5.7.2).)

Half of this proof is not difficult, and allows us to justify the occurrence of the constant $4/3$ in (5.8.6). Setting $a(y) := 2y^3/3 - 2y$ for any $y \in \mathbf{R}$, by the obvious inequality $b^2 + c^2 \geqslant 2bc$, for any sequence $\{u_n\}$ in $H^1(\Omega)$ we have

$$\begin{aligned} f_n(u_n) &= \int_\Omega \left(\frac{1}{n}|\nabla u_n|^2 + n\big(u_n^2 - 1\big)^2\right) \mathrm{d}x \\ &\geqslant 2\int_\Omega |\nabla u_n||u_n^2 - 1| \, \mathrm{d}x = \int_\Omega \big|\nabla a(u_n)\big| \, \mathrm{d}x. \end{aligned}$$

Notice also that, if $u_n \to u$ in $L^1(\Omega)$ and $\liminf_{n\to\infty} f_n(u_n) < +\infty$, then $|u| = 1$ a.e. in Ω. Hence

$$\liminf_{n\to\infty} f_n(u_n) \geqslant \begin{cases} \displaystyle\int_\Omega \big|\nabla a(u)\big| & \text{if } |u| = 1 \text{ a.e. in } \Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover, as $|a(\pm 1)| = 4/3$,

$$\int_\Omega \big|\nabla a(u)\big| = \big|a(\pm 1)\big| \int_\Omega |\nabla u| = \frac{4}{3}\int_\Omega |\nabla u| \quad \text{if } |u| = 1 \text{ a.e. in } \Omega.$$

We thus checked (5.8.1). The construction of a *recovery sequence* fulfilling (5.8.2) is less obvious and is here omitted.

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[120] See Modica and Mortola [337].

6. Bibliography

Bibliographical note. A large mathematical literature has been devoted to the analysis of the Stefan problem and of its generalizations. Below we loosely gather a number of monographs and proceedings under few headings, mainly for convenience of reference within this survey. These references concern Stefan-type and free boundary problems, as well as some topics of the analysis of nonlinear PDEs and calculus of variations.[121]

(I) Monographs on the Stefan problem and related models:

Alexiades and Solomon [8], Avdonin [43], Brokate and Sprekels [91], Datzeff [166], Elliott and Ockendon [193], Fasano [206], Gupta [249], Gurtin [256], Hill [267], Meirmanov [331], Romano [404], Rubinstein [410], Visintin [453], Yamaguchi and Nogi [474].

(II) Surveys on the Stefan problem and related models:

Andreucci, Herrero and Velázquez [25], Damlamian [160], Danilyuk [164], Fasano [205], Fasano and Primicerio [213], Magenes [318,320], Niezgódka [355], Oleĭnik, Primicerio and Radkevich [365], Primicerio [377,378], Rodrigues [395,397], Rubinstein [409], Tarzia [434], Visintin [455,457].

(III) Monographs on free boundary problems:

Baiocchi and Capelo [45], Caffarelli and Salsa [100], Chipot [129], Crank [149], Diaz [172], Duvaut and Lions [186], Friedman [234], Monakhov [338], Kinderlehrer and Stampacchia [292], Monakhov [338], Naumann [348], Nečas and Hlaváček [350], Pukhnachev [381], Radkevich and Melikulov [384], Rubinstein and Martuzans [412], Steinbach [431].

(IV) Collections of references on free boundary problems:

Cannon [112], Cryer [153], Tarzia [435,436], Wilson, Solomon and Trent [467].

(V) Proceedings and collective books on free boundary problems and applications:

Albrecht, Collatz and Hoffmann [6], Antontsev, Diaz and Shmarev [28], Antontsev, Hoffmann and Khludnev [29], Argoul, Frémond and Nguyen [30], Athanasopoulos, Makrakis and Rodrigues [37], Bossavit, Damlamian and Frémond [80], Brown and Davis [93], Chadam and Rasmussen [121–123], Colli, Kenmochi and Sprekels [138], Colli, Verdi and Visintin [144], Diaz, Herrero, Liñán and Vázquez [173], Fasano and Primicerio [212], Figueiredo, Rodrigues and Santos [218], Friedman and Spruck [237], Gurtin and McFadden [260], Hoffmann and Sprekels [271,272], Kenmochi [285], Magenes [319], Miranville [332], Miranville, Yin and Showalter [333], Neittanmäki [354], Niezgódka and Pawlow [356], Niezgódka and Strzelecki [357], Ockendon and Hodgkins [362], Rodrigues [396], Wilson, Solomon and Boggs [466], Wrobel and Brebbia [470,471], Wrobel, Brebbia and Sarler [472].

(VI) Monographs on physical and engineering aspects of phase transitions:

Abraham [2], Abeyaratne and Knowles [1], Brice [90], Chalmers [124], Christian [130], Doremus [183], Flemings [222], Kassner [284], Kurz and Fisher [299], Pamplin [366], Papon, Leblond and Meijer [369], Skripov [424], Turnbull [441], Ubbelohde [442], Woodruff [468].

[121] An exercise like this exposes the author to the risk of omissions, even if he has no aim of completeness. Facing so large a literature, more than a risk this is a certainty.

(VII) Monographs on nonequilibrium thermodynamics:

Astarita [31], Callen [111], De Groot [170], De Groot and Mazur [171], Frémond [224], Glansdorff and Prigogine [246], Kondepudi and Prigogine [296], Lavenda [310], Müller [344], Müller and Weiss [345], Prigogine [376], Woods [469].

(VIII) Some monographs on convex analysis:

Attouch [39], Aubin [41], Aubin and Ekeland [42], Barbu and Precupanu [49], Borwein and Lewis [74], Castaing and Valadier [119], Ekeland and Temam [188], Hiriart-Urruty and Lemarechal [269,270], Hörmander [276], Ioffe and Tihomirov [278], Kusraev and Kutateladze [300], Moreau [339], Rockafellar [389,391], Rockafellar and Wets [392], van Tiel [440], Willem [463].

(IX) Some monographs on maximal monotone operators:

Barbu [46,47], Barbu and Precupanu [49], Brezis [86], Browder [92], Lions [311], Pascali and Sburlan [370], Showalter [421].

(X) Some monographs on nonlinear semigroups of contractions:

Barbu [46,47], Bénilan [54], Bénilan, Crandall and Pazy [58], Brezis [86], Da Prato [165], Miyadera [334], Morosanu [341], Pavel [371].

(XI) Some monographs on variational inequalities and applications to PDEs:

Baiocchi and Capelo [45], Barbu and Precupanu [49], Brezis [84–86], Duvaut and Lions [186], Ekeland and Temam [188], Friedman [234], Gajewski, Gröger and Zacharias [240], Kinderlehrer and Stampacchia [292], Kluge [293], Lions [311], Naumann [348], Nečas and Hlaváček [350], Panagiotopoulos [367,368], Pascali and Sburlan [370], Rodrigues [394], Showalter [421], Vaĭnberg [443], Zeidler [475].

(XII) Some monographs on Sobolev spaces:

Adams [3], Attouch, Buttazzo and Michaille [40], Baiocchi and Capelo [45], Brezis [87], Dautray and Lions [167], Evans [200], Evans and Gariepy [201], Kufner, John and Fučík [298], Lions and Magenes [312], Maz'ja [324], Nečas [349], Tartar [433], Zeidler [475], Ziemer [478].

(XIII) Some monographs dealing with quasilinear parabolic equations in Sobolev spaces:

Barbu [46,47], Brezis [85,86], Friedman [230], Ladyženskaja, Solonnikov and Ural'ceva [305], Lions [311], Naumann [348], Roubíček [405], Showalter [421], Zeidler [475].

(XIV) Some monographs on Γ-convergence:

Attouch [39], Braides [81], Braides and Defranceschi [82], Carbone and De Arcangelis [117], Dal Maso [155].

(XV) Some monographs on sets of finite perimeter:

Almgren [14], Colombini, De Giorgi and Piccinini [147], Evans and Gariepy [201], Federer [216], Giusti [244], Morgan [340], Simon [423].

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CHAPTER 9

The KdV Equation

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Contents

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1. Historical background

In 1844 the Scottish John Scott Russell was the first to observe the solitary waves. Russell was watching a boat being drawn along the Edinburgh–Glasgow canal by a pair of horses [4,15,70]. He observed that when the boat stopped suddenly, the bow wave continued forward along the canal without any change of its form or speed. Russell observed a large protrusion of water slowly traveling on the Edinburgh–Glasgow canal without change in shape for eight miles an hour, but he lost it after two miles. He called the bulge of water, that he observed, a "great wave of translation" [4]. The wave was traveling along the channel of water for a long period of time while still retaining its original identity. This single humped wave of bulge of water is now called solitary waves or solitons [1,3,2,5]. Russell was convinced that his discovery is of great significance, but unfortunately was ignored for many years. The solitons – localized, highly stable waves that retain its identity (shape and speed), upon interaction – was discovered *experimentally* by Russell [15,30]. The remarkable discovery motivated Russell to conduct physical laboratory experiments to emphasize his observance and to study these solitary waves. He empirically derived the relation

$$c^2 = g(h + a), \tag{1}$$

that determines the speed c of the solitary wave, where a is the maximum amplitude above the water surface, h is the finite depth and g is the acceleration of gravity. The solitary waves are therefore called gravity waves.

In 1895, Diederik Johannes Korteweg (1848–1941) together with his Ph.D student, Gustav de Vries (1866–1934) derived *analytically* a nonlinear partial differential equation, well known now as the KdV equation [34]. The KdV equation is used to model the disturbance of the surface of shallow water in the presence of solitary waves. The KdV equation is a generic model for the study of weakly nonlinear long waves, incorporating leading order nonlinearity and dispersion [47]. Also, it describes surface waves of long wavelength and small amplitude on shallow water [78,48,49,80]. The KdV equation in its simplest form is given by

$$u_t + auu_x + u_{xxx} = 0. \tag{2}$$

This equation incorporates two competing effects: nonlinearity represented by uu_x, and linear dispersion represented by u_{xxx}. Nonlinearity tends to localize the wave while dispersion spreads it out [11,12]. The balance between these two weak nonlinearity and dispersion explains the formulation of solitons that consist of single humped waves. The equilibrium between these two effects is stable [32–39,41,40,42].

In 1965, Norman J. Zabusky (1929–) and Martin D. Kruskal (1925–2006) investigated *numerically* the nonlinear interaction of a large solitary-wave overtaking a smaller one, and the recurrence of initial states [79]. They discovered that solitary waves undergo nonlinear interaction following the KdV equation. Further, the waves emerge from this interaction retaining its original shape and amplitude, and therefore conserved energy and mass. The only effect of the interaction was a phase shift. The remarkable discovery, that solitary waves retain their identities and that their character resembles particle like behavior, motivated Zabusky and Kruskal [79] to call these solitary waves *solitons*. Zabusky and Kruskal [79] marked the birth of soliton, a name intended to signify particle like quantities [64,65, 62,63]. The interaction of two solitons emphasized the reality of the preservation of shapes and speeds and of the steady pulse like character of solitons [70], therefore the collision of KdV solitons is considered elastic.

A soliton can be defined as a solution of a nonlinear partial differential equation that exhibits the following properties [64,65,62,63,66–77]:

(i) the solution should demonstrate a wave of permanent form;
(ii) the solution is localized, which means that the solution either decays exponentially to zero such as the solitons provided by the KdV equation, or converges to a constant at infinity such as the solitons given by the Sine–Gordon equation;
(iii) the soliton interacts with other solitons preserving its character.

There are many types of traveling waves that are of particular interest in solitary wave theory. Three of these types are: the *solitary waves*, which are localized traveling waves, asymptotically zero at large distances [6–8,10,9], the *periodic solutions*, and the *kink waves* which rise or descend from one asymptotic state to another [50–55]. Another type is the *peakons* that are peaked solitary wave solutions [8,59]. In this case, the traveling wave solutions are smooth except for a peak at a corner of its crest. Peakons are the points at which spatial derivative changes sign [61,78] so that peakons have a finite jump in first derivative of the solution $u(x, t)$. *Cuspons* are other forms of solitons where solution exhibits cusps at their crests [59–61]. Unlike peakons where the derivatives at the peak differ only by a sign, the derivatives at the jump of a cuspon diverges. It is important to note that the soliton solution $u(x, t)$, along with its derivatives, tends to zero as $|x| \to \infty$.

In 1993, Rosenau and Hyman [56] discovered a new class of solitons that are termed *compactons*, which are solitons with compact spatial support such that each compacton is a soliton confined to a finite core. Compactons are defined by solitary waves with the remarkable soliton property that after colliding with other compactons, they reemerge with the same coherent shape [56]. These particle like waves exhibit elastic collision that are similar to the soliton interaction associated with completely integrable PDEs supporting an infinite number of conservation laws. It was found that a compacton is a solitary wave with a compact support where the nonlinear dispersion confines it to a finite core, therefore the exponential wings vanish.

The genuinely nonlinear dispersive $K(n, n)$ equations, a family of nonlinear KdV like equations is of the form

$$u_t + a(u^n)_x + (u^n)_{xx} = 0, \quad a > 0,\ n > 1, \tag{3}$$

which supports compact solitary traveling structures for $a > 0$. The existence and stability of the compact entities was examined by many authors such as in [42,54–56,68–77].

The definitions given so far for compactons are [70]:

(i) compactons are solitons with finite wavelength;
(ii) compactons are solitary waves with compact support;
(iii) compactons are solitons free of exponential tails;
(iv) compactons are solitons characterized by the absence of infinite wings;
(v) compactons are robust soliton-like solutions.

Two important features of compactons structures are observed, namely:

(i) unlike the standard KdV soliton where $u(\xi) \to 0$ as $\xi \to \infty$, the compacton is characterized by the absence of the exponential tails or wings, where $u(\xi)$ does not tend to 0 as $\xi \to \infty$;
(ii) unlike the standard KdV soliton where width narrows as the amplitude increases, the width of the compacton is independent of the amplitude.

It is important to note that Eq. (3) with $+a$ is called the focusing branch of the $K(n,n)$ equations. The equation

$$u_t - a(u^n)_x + (u^n)_{xx} = 0, \quad a > 0,\ n > 1, \tag{4}$$

is called the defocussing branch of the $K(n,n)$ equations. The studies in [54–56,71–76] and many of the references therein revealed that Eq. (4) supports solutions with solitary patterns having cusps or infinite slopes. Further, it was shown that while compactons are the essence of the focusing branch ($+a$), spikes, peaks and cusps are the hallmark of the defocussing branch ($-a$). This in turn means that the focusing branch (3) and the defocussing branch (4) represent two different models, each leading to a different physical structure. The remarkable discovery of compactons has led to an intense study over the last few years. The study of compactons may give insight into many scientific processes [42] such as the super deformed nuclei, preformation of cluster in hydrodynamic models, the fission of liquid drops and inertial fusion. The stability analysis has shown that compacton solutions are stable, where the stability condition is satisfied for arbitrary values of the nonlinearity parameter. The stability of the compactons solutions was investigated by means of both linear stability and by Lyapunov stability criteria. Moreover, the compactons are nonanalytic solutions whereas classical solitons are analytic solutions. The points of nonanalyticity at the edge of the compacton correspond to points of genuine nonlinearity of the differential equation [56]. Compactons such as drops do not possess infinite wings, hence they interact among themselves only across short distances. *Solitons* and *compactons* with and without exponential wings respectively, are termed by using the suffix-*on* to indicate that it has the property of a particle, such as phonon, and photon [64,65,62,63,66].

Solitons play a prevalent role in propagation of light in fibers. optical switching in slab waveguides, surface waves in nonlinear dielectrics, optical bistability, and propagation through excitable media. A great deal of research work has been invested in recent years for the study of the soliton concept. Various methods, analytic and numerical, were applied to study several evolution equations. The inverse scattering method, tanh method, Darboux transformation, Jacobi elliptic functions method, sine-cosine method, Bắcklund transformation techniques, and the F-expansion methods are among the methods used. Hirota [24–29] constructed the N-soliton solutions of the evolution equation by reducing it to the bilinear form. The bilinear formalism established by Hirota [24–29] was a very helpful tool in the study of the nonlinear equations and it was the most suitable for computer algebra. The Hirota bilinear formalism was extensively used in the literature such as in [17,19,18,20–23,31–33,80] and the references therein.

1.1. *Preliminary profile solution*

The KdV equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{5}$$

has a large variety of solutions. The solutions propagate at speed c while retaining its identity. We usually introduce the new variable $\xi = x - ct$, so that

$$u(x,t) = u(\xi). \tag{6}$$

The soliton solution is spatially localized solution, hence $u', u'', u''' \to 0$ as $\xi \to \pm\infty$, $\xi = x - ct$.

Using (6) into (5) gives

$$-cu' + 6uu' + u''' = 0. \tag{7}$$

Integrating (7) gives

$$-cu + 3u^2 + u'' = 0, \tag{8}$$

where constant of integration is taken to be zero. Multiplying (8) by $2f'$ and integrating the resulting equation we find

$$(u')^2 = cu^2 - 2u^3, \tag{9}$$

or equivalently

$$\frac{\mathrm{d}u}{\sqrt{cu^2 - 2u^3}} = \mathrm{d}\xi. \tag{10}$$

Using a change of variable

$$u = \frac{c}{2}\,\mathrm{sech}^2(\mu\xi), \tag{11}$$

that will give the soliton solution

$$u(x,t) = \frac{c}{2}\,\mathrm{sech}^2\,\frac{\sqrt{c}}{2}(x - ct). \tag{12}$$

It is obvious that $u(x,t)$ in (12), along with its derivatives, tends to zero as $\xi \to \infty$. We can also show that

$$u(x,t) = -\frac{c}{2}\,\mathrm{csch}^2\left(\frac{\sqrt{c}}{2}(x - ct)\right), \tag{13}$$

is also a solution for the KdV equation.

It is also interesting to solve this equation by using Bäcklund transformation, where we introduce a function v such that $u = v_x$. This will convert the KdV equation to

$$v_{xt} + 6v_x v_{xx} + v_{xxxx} = 0, \tag{14}$$

where by integrating with respect to x we obtain

$$v_t + 3(v_x)^2 + v_{xxx} = 0. \tag{15}$$

The last equation is called the potential KdV equation; by using the wave variable $\xi = x - ct$, we can easily obtain the solutions

$$\begin{aligned} v &= \sqrt{c}\tanh\left(\frac{\sqrt{c}}{2}(x - ct)\right),\\ v &= \sqrt{c}\coth\left(\frac{\sqrt{c}}{2}(x - ct)\right). \end{aligned} \tag{16}$$

Recall that $u = v_x$, hence we obtain

$$\begin{aligned} u(x,t) &= \frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right), \\ u(x,t) &= -\frac{c}{2}\operatorname{csch}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right). \end{aligned} \tag{17}$$

2. The family of the KdV equations

The KdV equations appear in three or more order forms. In what follows, we present a brief summary of these forms. The complete analysis of each form will be addressed in the forthcoming sections.

2.1. *Third-order KdV equations*

The family of third order Korteweg–de Vries is of the form

$$u_t + P(u)u_x + u_{xxx} = 0, \tag{18}$$

where $u(x,t)$ is a function of space x and time variable t. Constants can be used as coefficients of $P(u)u_x$ and u_{xxx}, but these constants can be usually scaled out. The nonlinear term $P(u)$ appears in the following forms

$$P(u) = \begin{cases} au, \\ au^2, \\ au^n, \\ u_x, \\ au^n - bu^{2n}. \end{cases} \tag{19}$$

(i) For $P(u) = \pm 6u$ we obtain the standard KdV equation

$$u_t \pm 6uu_x + u_{xxx} = 0, \tag{20}$$

where the factor ± 6 is appropriate for complete integrability. This means that the KdV equation has N-soliton solutions as will be discussed later.

(ii) For $P(u) = 6u^2$, Eq. (18) is called the modified KdV (mKdV) equation and given by

$$u_t + 6u^2u_x + u_{xxx} = 0. \tag{21}$$

The mKdV equation is completely integrable and has infinitely many conserved quantities. This equation appears in electric circuits and multi-component plasmas. The modified KdV equation gives algebraic solitons solutions in the form of a rational function [58]. Stability and instability conditions of algebraic solitons have been investigated thoroughly in [1,3,2,4,5].

(iii) For $P(u) = au^n$, Eq. (18) is called the generalized KdV (gKdV) equation [13] and given by

$$u_t + au^nu_x + u_{xxx} = 0, \quad n \geqslant 3. \tag{22}$$

(iv) For $P(u) = u_x$, Eq. (18) is called the potential KdV equation given by

$$u_t + (u_x)^2 + u_{3x} = 0. \tag{23}$$

It is to be noted that this equation can be obtained from the standard KdV equation by setting $u = v_x$ and integrating the resulting equation with respect to x.

(v) For $P(u) = au^n - bu^{2n}$ we obtain a generalized KdV equation with two power nonlinearities of the form

$$u_t + (au^n - bu^{2n})u_x + u_{xxx} = 0. \tag{24}$$

This last equation describes the propagation of nonlinear long acoustic-type waves [78]. The function f', where $f = (\frac{a}{n+1}u^{n+1} - \frac{b}{2n+1}u^{2n+1})$ is regarded as a nonlinear correction to the limiting long-wave phase speed c. If the amplitude is not supposed to be small, Eq. (24) serves as an approximate model for the description of weak dispersive effects on the propagation of nonlinear waves along a characteristic direction. It is to be noted that for $n = 1$, Eq. (24) is the well-known Gardner equation [16] that is also called the combined KdV-mKdV equation.

Attention has been focused on equations like (24) in [78] and the references therein due to its appearance in many branches of physics. The main focus of these works was the solitary wave solutions, collapsing solitons, algebraic solitons, and solitary wave instability. Algebraic solitons decay to zero at infinity or approach nonzero boundary values at an algebraic rate [33].

2.2. *Fifth-order KdV equations*

The most well-known fifth-order KdV equations appear in the form

$$u_t + \alpha u^2 u_x + \beta u_x u_{xx} + \gamma u u_{3x} + u_{5x} = 0, \tag{25}$$

where α, β, and γ are arbitrary nonzero and real parameters, and $u = u(x, t)$ is a sufficiently smooth function. Because the parameters α, β, and γ are arbitrary and take different values, this will drastically change the characteristics of the fKdV equation (25). Lots of forms of the fKdV equation can be constructed by changing the real values of the parameters. This equation includes, for specific values of α, β, and γ, the Lax equation [36], Kaup–Kupershmidt (KP) equation [32,35], Sawada–Kotera (SK) equation [57], and Ito equation [31] that will be discussed later.

2.3. *Higher-order KdV equation*

Higher-order KdV equations of the seventh-order and ninth-order are of the form

$$u_t + 6uu_x + u_{3x} - u_{5x} + \alpha u_{7x} = 0. \tag{26}$$

and

$$u_t + 6uu_x + u_{3x} - u_{5x} + \alpha u_{7x} + \beta u_{9x} = 0, \tag{27}$$

respectively. Model equation (26) was examined in [14,43,53] for discussing the structural stability of the KdV solitons under a singular perturbation. The seventh-order and the ninth-order equations have infinitely many conservation laws, most of them are polynomials that depend on u and its derivatives.

2.4. *The $K(n,n)$ equation*

It is interesting to note that a KdV-like equation was introduced by Rosenau et al. [56] and given by

$$u_t + a(u^n)_x + b(u^n)_{xxx} = 0, \tag{28}$$

where the balance between the nonlinear convection term $(u^n)_x$ and the genuinely dispersion term $(u^n)_{xxx}$ gives rise to the so-called *compacton*, solitary wave with compact support and without tails or wings. Eq. (28) was thoroughly investigated in [42,54–56, 66–77] and some of the references therein.

3. The methods

As stated before, several methods were implemented in the literature to handle nonlinear evolution equations. For single soliton solutions, several methods, such as the tanh method [44–46], the tanh-coth method [66,67], the sine-cosine method [70], the pseudo spectral method [44], the inverse scattering method [1], Hirota's bilinear method, the truncated Painlevé expansion, and others are used. The tanh–coth method and the sine–cosine method have been applied for a wide variety of nonlinear problems and will be used in this work. However, the Hirota bilinear formalism [24–29] and a simplified version of this method [19,18] will be used to address the concept of multiple soliton solutions. The main features of the tanh–coth method, sine–cosine method and the Hirota formalism will be presented.

3.1. *The tanh–coth method*

A wave variable $\xi = x - ct$ converts a PDE

$$P(u, u_t, u_x, u_{xx}, u_{xxx}, \ldots) = 0, \tag{29}$$

to an ODE

$$Q(u, u', u'', u''', \ldots) = 0. \tag{30}$$

Eq. (30) is then integrated as long as all terms contain derivatives where integration constants are considered zeros.

The standard tanh method is developed by Malfliet [44] where the tanh is used as a new variable, since all derivatives of a tanh are represented by a tanh itself. Introducing a new independent variable

$$Y = \tanh(\mu\xi), \quad \xi = x - ct, \tag{31}$$

leads to the change of derivatives:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\xi} &= \mu\big(1-Y^2\big)\frac{\mathrm{d}}{\mathrm{d}Y},\\
\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} &= -2\mu^2 Y\big(1-Y^2\big)\frac{\mathrm{d}}{\mathrm{d}Y} + \mu^2\big(1-Y^2\big)^2\frac{\mathrm{d}^2}{\mathrm{d}Y^2},\\
\frac{\mathrm{d}^3}{\mathrm{d}\xi^3} &= 2\mu^3\big(1-Y^2\big)\big(3Y^2-1\big)\frac{\mathrm{d}}{\mathrm{d}Y} - 6\mu^3 Y\big(1-Y^2\big)^2\frac{\mathrm{d}^2}{\mathrm{d}Y^2} + \mu^3\big(1-Y^2\big)^3\frac{\mathrm{d}^3}{\mathrm{d}Y^3},\\
\frac{\mathrm{d}^4}{\mathrm{d}\xi^4} &= -8\mu^4 Y\big(1-Y^2\big)\big(3Y^2-2\big)\frac{\mathrm{d}}{\mathrm{d}Y} + 4\mu^4\big(1-Y^2\big)^2\big(9Y^2-2\big)\frac{\mathrm{d}^2}{\mathrm{d}Y^2}\\
&\quad - 12\mu^4 Y\big(1-Y^2\big)^3\frac{\mathrm{d}^3}{\mathrm{d}Y^3} + \mu^4\big(1-Y^2\big)^4\frac{\mathrm{d}^4}{\mathrm{d}Y^4}.
\end{aligned} \tag{32}$$

The tanh–coth method [66,67] admits the use of the finite expansion

$$u(\mu\xi) = S(Y) = \sum_{k=0}^{M} a_k Y^k + \sum_{k=1}^{M} b_k Y^{-k}, \tag{33}$$

where M is a positive integer, in most cases, that will be determined. For noninteger M, a transformation formula is used to overcome this difficulty. Expansion (33) reduces to the standard tanh method [44–46] for $b_k = 0$, $1 \leqslant k \leqslant M$. Substituting (33) into the reduced ODE results in an algebraic equation in powers of Y.

To determine the parameter M, we usually balance the linear terms of highest order in the resulting equation with the highest order nonlinear terms. We then collect all coefficients of powers of Y in the resulting equation where these coefficients have to vanish. This will give a system of algebraic equations involving the parameters a_k, b_k, μ, and c. Having determined these parameters we obtain an analytic solution $u(x,t)$ in a closed form.

3.2. *The sine–cosine method*

Proceeding as in the tanh–coth method, Eq. (30) is integrated as long as all terms contain derivatives where integration constants are considered zeros. The sine–cosine method admits the use of the solutions in the forms

$$u(x,t) = \lambda\cos^{\beta}(\mu\xi), \quad |\xi| \leqslant \frac{\pi}{2\mu}, \tag{34}$$

and

$$u(x,t) = \lambda\sin^{\beta}(\mu\xi), \quad |\xi| \leqslant \frac{\pi}{\mu}, \tag{35}$$

where λ, μ, and β are parameters that will be determined, μ and c are the wave number and the wave speed respectively. Equations (34) and (35) give

$$(u^n)'' = -n^2\mu^2\beta^2\lambda^n\cos^{n\beta}(\mu\xi) + n\mu^2\lambda^n\beta(n\beta-1)\cos^{n\beta-2}(\mu\xi), \tag{36}$$

and

$$(u^n)'' = -n^2\mu^2\beta^2\lambda^n \sin^{n\beta}(\mu\xi) + n\mu^2\lambda^n\beta(n\beta-1)\sin^{n\beta-2}(\mu\xi). \tag{37}$$

Substituting (36) or (37) into (30) gives a trigonometric equation of $\cos^R(\mu\xi)$ or $\sin^R(\mu\xi)$ terms. The parameters are then determined by first balancing the exponents of each pair of cosine or sine to determine R. We next collect all coefficients of the same power in $\cos^k(\mu\xi)$ or $\sin^k(\mu\xi)$, where these coefficients have to vanish. This gives a system of algebraic equations among the unknowns β, λ and μ that will be determined. The solutions proposed in (34) and (35) follow immediately.

The algorithms described above certainly work well for a large class of very interesting nonlinear equations. The main advantage of the tanh–coth method and the sine–cosine method, presented above, is that the great capability of reducing the size of computational work compared to existing techniques such as the pseudo spectral method, the inverse scattering method, Hirota's bilinear method, and the truncated Painlevé expansion.

3.3. *Hirota's bilinear method*

A well-known third method, namely, the Hirota [24–29] bilinear form, will be employed to handle specific integrable KdV forms. The method is widely used especially to handle the multi-solitons solutions of many evolution equations. Hirota introduced the bilinear differential operators

$$D_t^m D_x^n(a\cdot b) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n a(x,t)b(x',t')|_{x=x',t=t'}. \tag{38}$$

In what follows, we express some of the bilinear differentials operators:

$$\begin{aligned}
&D_x(a\cdot b) = a_x b - a b_x,\\
&D_x^2(a\cdot b) = a_{2x}b - 2a_x b_x + a b_{2x},\\
&D_x D_t(a\cdot b) = D_x(a_t b - a b_t) = a_{xt}b - a_t b_x - a_x b_t + a b_{xt},\\
&D_x D_t(a\cdot a) = 2(a a_{xt} - a_x a_t),\\
&D_x^4(a\cdot b) = a_{4x}b - 4a_{3x}b_x + 6a_{2x}b_{2x} - 4a_x b_{3x} + a b_{4x},\\
&D^n(a\cdot a) = 0, \quad \text{for } n \text{ is odd.}
\end{aligned} \tag{39}$$

The solution of the canonical KdV equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{40}$$

can be expressed by

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}\log f, \tag{41}$$

where $f(x,t)$ is given by the perturbation expansion

$$f(x,t) = 1 + \sum_{n=1}^{\infty}\epsilon^n f_n(x,t), \tag{42}$$

where ϵ is a formal expansion parameter. For the one-soliton solution we set

$$f(x,t) = 1 + \epsilon f_1, \tag{43}$$

and for the two-soliton solution we set

$$f(x,t) = 1 + \epsilon f_1 + \epsilon^2 f_2, \tag{44}$$

and so on. The functions $f_1, f_2, f_3, \ldots$ can be determined by using the Hirota bilinear formalism or by direct substitution of (42) into the appropriate equation as will be seen later. In [19,18], a simplified form of the Hirota's bilinear formalism was introduced to minimize the cumbersome work of Hirota's method. The simplified approach in [19,18] will be examined in a forthcoming section.

It is important to note that other methods will be used as well. Each method will be presented properly at specific sections. Before we begin our discussion, it is normal to give a brief idea about conservation laws of the KdV equation.

4. Conservation laws

An equation of the form

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0, \tag{45}$$

where T and X are the density and the flux respectively and neither one involves derivatives with respect to t, is called a conservation law [1,3,2,13]. This means that T and X may depend on $x, t, u, u_x, \ldots$ but not u_t. Considering the canonical form of this equation by

$$u_t - 6uu_x + u_{xxx} = 0. \tag{46}$$

This equation is in conservation form [13] where

$$T = u, \qquad X = u_{xx} - 3u^2. \tag{47}$$

This in turn gives the first conservation law

$$\int_{-\infty}^{\infty} u \,\mathrm{d}x = \text{constant}. \tag{48}$$

Multiplying (46) by u yields

$$\frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right) + \frac{\partial}{\partial x}\left(uu_{xx} - \frac{1}{2}(u_x)^2 - 2u^3\right) = 0, \tag{49}$$

that gives the second law of conservation laws

$$\int_{-\infty}^{\infty} u^2 \,\mathrm{d}x = \text{constant}. \tag{50}$$

Multiplying (46) by $3u^2$ gives

$$3u^2(u_t - 6uu_x + u_{xxx}) = 0. \tag{51}$$

Multiplying the partial derivative of (46) with respect to x by u_x gives

$$u_x\big(u_{xt} - 6(u_x)^2 - 6uu_{xx} + u_{xxxx}\big) = 0. \tag{52}$$

Adding the last two quantities yields

$$\begin{aligned}&\frac{\partial}{\partial t}\left(u^3 - \frac{1}{2}(u_x)^2\right)\\ &\quad + \frac{\partial}{\partial x}\left(-\frac{9}{2}u^4 + 3u^2u_{xx} - 6u(u_x)^2 + u_xu_{xxx} - \frac{1}{2}(u_{xx})^2\right) = 0.\end{aligned} \tag{53}$$

This gives the third conservation law of the KdV equation

$$\int_{-\infty}^{\infty}\left(u^3 - \frac{1}{2}(u_x)^2\right)\mathrm{d}x = \text{constant}. \tag{54}$$

However, it was revealed in [1,3,2,12,13,24–28] that there is an infinite set of conservation laws for the KdV equation.

The existence of conservation laws has been considered as an indication of the integrability of the KdV. There is an infinite set of independent conservation laws for the KdV equation. The first five conservation laws of this set are

$$\begin{aligned}&\int_{-\infty}^{\infty} u\,\mathrm{d}x = \text{constant},\\ &\int_{-\infty}^{\infty} u^2\,\mathrm{d}x = \text{constant},\\ &\int_{-\infty}^{\infty}\left(u^3 - \frac{1}{2}(u_x)^2\right)\mathrm{d}x = \text{constant},\\ &\int_{-\infty}^{\infty}\big(5u^4 + 10u(u_x)^2 + (u_{xx})^2\big)\,\mathrm{d}x = \text{constant},\\ &\int_{-\infty}^{\infty}\big(21u^5 + 105u^2(u_x)^2 + 21u(u_{xx})^2 + (u_{xxx})^2\big)\,\mathrm{d}x = \text{constant},\end{aligned} \tag{55}$$

where each conservation law includes a higher power of u than the preceding law.

5. The KdV equation

As stated before this equation is given by

$$u_t + auu_x + u_{xxx} = 0. \tag{56}$$

Substituting the wave variable $\xi = x - ct$, c is the wave speed, into (56) and integrating one we obtain

$$-cu + \frac{a}{2}u^2 + u'' = 0. \tag{57}$$

5.1. *Using the tanh–coth method*

Notice that the parameter M is defined in (33) and (32). This means that the highest power of u^2 is $2M$, and for u'' is $M+2$ obtained by using (33) and (32) respectively. Balancing the nonlinear term u^2 with the highest order derivative u'' gives

$$2M = M + 2, \tag{58}$$

that gives

$$M = 2. \tag{59}$$

The tanh–coth method allows us to use the substitution

$$u(x,t) = S(Y) = a_0 + a_1 Y + a_2 Y^2 + b_1 Y^{-1} + b_2 Y^{-2}. \tag{60}$$

Substituting (60) into (57), collecting the coefficients of each power of $Y^i, 0 \leqslant i \leqslant 8$, setting each coefficient to zero, and solving the resulting system of algebraic equations we obtain the following sets of solutions

(i) *First set*

$$a_0 = \frac{3c}{a}, \quad a_1 = a_2 = b_1 = 0, \quad b_2 = -\frac{3c}{a}, \quad \mu = \frac{1}{2}\sqrt{c}, \quad c > 0. \tag{61}$$

(ii) *Second set*

$$a_0 = -\frac{c}{a}, \quad a_1 = a_2 = b_1 = 0, \quad b_2 = \frac{3c}{a}, \quad \mu = \frac{1}{2}\sqrt{-c}, \quad c < 0. \tag{62}$$

(iii) *Third set*

$$a_0 = \frac{3c}{a}, \quad a_1 = b_1 = b_2 = 0, \quad a_2 = -\frac{3c}{a}, \quad \mu = \frac{1}{2}\sqrt{c}, \quad c > 0. \tag{63}$$

(iv) *Fourth set*

$$a_0 = -\frac{c}{a}, \quad a_1 = b_1 = b_2 = 0, \quad a_2 = \frac{3c}{a}, \quad \mu = \frac{1}{2}\sqrt{-c}, \quad c < 0. \tag{64}$$

In view of these results, we obtain the following soliton solutions

$$\begin{aligned} u_1(x,t) &= \frac{3c}{a}\operatorname{sech}^2\left[\frac{\sqrt{c}}{2}(x-ct)\right], \quad c > 0, \\ u_2(x,t) &= -\frac{c}{a}\left(1 - 3\tanh^2\left[\frac{\sqrt{-c}}{2}(x-ct)\right]\right), \quad c < 0. \end{aligned} \tag{65}$$

In addition, the traveling wave solutions

$$\begin{aligned} u_3(x,t) &= -\frac{3c}{a}\operatorname{csch}^2\left[\frac{\sqrt{c}}{2}(x-ct)\right], \quad c > 0, \\ u_4(x,t) &= -\frac{c}{a}\left(1 - 3\coth^2\left[\frac{\sqrt{-c}}{2}(x-ct)\right]\right), \quad c < 0, \end{aligned} \tag{66}$$

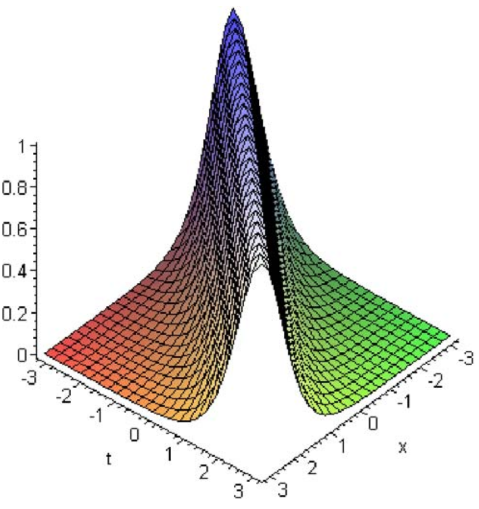

Fig. 1. Graph of a soliton that has an infinite support.

follow immediately. Figure 1 shows a graph of a one-soliton solution characterized by infinite wings or infinite tails. This shows that $u \to 0$ as $\xi \to \pm\infty$, $\xi = x - ct$.

It is obvious that the physical structures of the obtained solutions in (65) depend mainly on the sign of the wave speed c, we therefore obtain the following plane periodic solutions:

$$\begin{aligned}
u_5(x,t) &= \frac{3c}{a}\csc^2\left[\frac{\sqrt{-c}}{2}(x-ct)\right], \quad c<0,\\
u_6(x,t) &= -\frac{c}{a}\left(1+3\cot^2\left[\frac{\sqrt{c}}{2}(x-ct)\right]\right), \quad c>0,\\
u_7(x,t) &= \frac{3c}{a}\sec^2\left[\frac{\sqrt{-c}}{2}(x-ct)\right], \quad c<0,\\
u_8(x,t) &= -\frac{c}{a}\left(1+3\tan^2\left[\frac{\sqrt{c}}{2}(x-ct)\right]\right), \quad c>0.
\end{aligned} \tag{67}$$

5.2. *Using the sine–cosine method*

Substituting (34) into (57) yields

$$\begin{aligned}
&-c\lambda\cos^\beta(\mu\xi) + \frac{a}{2}\lambda^2\cos^{2\beta}(\mu\xi),\\
&-\lambda\mu^2\beta^2\cos^\beta(\mu\xi) + \lambda\mu^2\beta(\beta-1)\cos^{\beta-2}(\mu\xi) = 0.
\end{aligned} \tag{68}$$

Equation (68) is satisfied only if the following system of algebraic equations holds

$$\begin{aligned}
&\beta - 1 \neq 0,\\
&2\beta = \beta - 2,
\end{aligned}$$

$$\mu^2\beta^2\lambda = -c\lambda,$$
$$\frac{a}{2}\lambda^2 = -c\lambda\mu^2\beta(\beta-1), \tag{69}$$

which leads to

$$\beta = -2,$$
$$\mu = \frac{1}{2}\sqrt{-c},$$
$$\lambda = \frac{3c}{a}. \tag{70}$$

The results in (70) can be easily obtained if we also use the sine method (35). Moreover, the last results give the solutions $u_1(x,t)$, $u_3(x,t)$, $u_7(x,t)$ and $u_9(x,t)$ that are obtained before. It is interesting to point out that the sine–cosine method does not always provide the same solutions as those given by the tanh–coth method.

5.3. *Multiple-soliton solutions of the KdV equation*

In this section, we will examine multiple-solitons solutions of the standard KdV equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{71}$$

As stated before, Hirota [24–29] introduced a method to determine exact solutions of nonlinear PDEs. A necessary condition for the direct method to be applicable is that the PDE can be brought into a bilinear form [19,18]. Hirota proposed a bilinear form where it was shown that soliton solutions are just polynomials of exponentials as will be seen later. Finding bilinear forms for nonlinear PDEs, if they exist at all, is highly nontrivial [19,18]. Considering $u(x,t) = 2(\ln(f))_{xx}$, the *bilinear form* for the KdV equation is

$$B(f,f) = \left(D_x^4 + D_x D_t\right) f \cdot f = 0.$$

Hereman et al. [19,18] introduced a simplified version of Hirota method, where exact solitons can be obtained by solving a perturbation scheme using a symbolic manipulation package, and without any need to use bilinear forms. In what follows, we summarize the main steps of the simplified version of Hirota's method.

The simplified version of Hirota method introduces the change of dependent variable

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2} = 2\frac{f f_{2x} - (f_x)^2}{f^2}, \tag{72}$$

to carry out (71) into a quadratic equation

$$\left[f(f_{xt} + f_{4x})\right] - \left[f_x f_t + 4 f_x f_{3x} - 3 f_{2x}^2\right] = 0. \tag{73}$$

Equation (73) can be decomposed into linear operator L and nonlinear operator N defined by

$$L = \frac{\partial^2}{\partial x \partial t} + \frac{\partial^4}{\partial x^4},$$
$$N(f,f) = -f_x f_t - 4 f_x f_{3x} + 3 f_{2x} f_{2x}. \tag{74}$$

We next assume that $f(x,t)$ has a perturbation expansion of the form

$$f(x,t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f_n(x,t), \tag{75}$$

where ϵ is a nonsmall formal expansion parameter. Following Hirota's method and the simplified version introduced in [19,18], we substitute (75) into (74) and equate to zero the powers of ϵ:

$$\begin{aligned}
&O(\epsilon^0): && B(1\cdot 1) = 0,\\
&O(\epsilon^1): && B(1\cdot f_1 + f_1 \cdot 1) = 0,\\
&O(\epsilon^2): && B(1\cdot f_2 + f_1\cdot f_1 + f_2\cdot 1) = 0,\\
&O(\epsilon^3): && B(1\cdot f_3 + f_1\cdot f_2 + f_2\cdot f_1 + f_3\cdot 1) = 0,\\
&O(\epsilon^4): && B(1\cdot f_4 + f_1\cdot f_3 + f_2\cdot f_2 + f_3\cdot f_1 + f_4\cdot 1) = 0,\\
&\vdots\\
&O(\epsilon^n): && B\left(\sum_{j=0}^{n} f_j\cdot f_{n-j}\right) = 0,
\end{aligned}$$

where the bilinear form B is defined above. It is to be noted that the previous scheme is the same for every bilinear operator B. This in turn means

$$O(\epsilon^1): \quad Lf_1 = 0, \tag{76}$$

$$O(\epsilon^2): \quad Lf_2 = -N(f_1, f_1), \tag{77}$$

$$O(\epsilon^3): \quad Lf_3 = -f_1 Lf_2 - N(f_1, f_2) - N(f_2, f_1), \tag{78}$$

$$\begin{aligned} O(\epsilon^4): \quad Lf_4 = &- f_1 Lf_3 - f_2 Lf_2 - f_3 Lf_1 - N(f_1, f_3)\\ &- N(f_2, f_2) - N(f_2, f_1), \end{aligned} \tag{79}$$

$$\vdots \tag{80}$$

$$O(\epsilon^n): \quad Lf_n = -\sum_{j=1}^{n-1}\big[f_j Lf_{n-1} + N(f_j, f_{n-j})\big] = 0.$$

The N-soliton solution is obtained from

$$f_1 = \sum_{i=1}^{N} \exp(\theta_i), \tag{81}$$

where

$$\theta_i = k_i x - c_i t, \tag{82}$$

where k_i and c_i are arbitrary constants, k_i is called the wave number. Substituting (81) into (76) gives the dispersion relation

$$c_i = k_i^3, \tag{83}$$

and in view of this result we obtain

$$\theta_i = k_i x - k_i^3 t. \tag{84}$$

This means that

$$f_1 = \exp(\theta_1) = \exp\big(k_1(x - k_1^2 t)\big), \tag{85}$$

obtained by using $N = 1$ in (81).

Consequently, for the one-soliton solution, we set

$$f = 1 + \exp(\theta_1) = 1 + \exp\big(k_1(x - k_1^2 t)\big), \tag{86}$$

where we set $\epsilon = 1$. Recall that $u(x,t) = 2(\ln f)_{xx}$, therefore the one soliton solution is therefore

$$u(x,t) = \frac{2k_1^2 \exp(k_1(x - k_1^2 t))}{(1 + \exp(k_1(x - k_1^2 t))^2}, \tag{87}$$

or equivalently

$$u(x,t) = \frac{k_1^2}{2} \operatorname{sech}^2\left[\frac{k_1}{2}(x - k_1^2 t)\right]. \tag{88}$$

Setting $k_1 = \sqrt{c}$ in (88) gives the one-soliton solution obtained above in (65) by using the tanh–coth and the sine–cosine methods. Another soliton solution is obtained in (65).

To determine the two-soliton solution, we first set $N = 2$ in (81) to get

$$f_1 = \exp(\theta_1) + \exp(\theta_2). \tag{89}$$

To determine f_2, we substitute (81) into (77) to evaluate the right-hand side and equate it with the left-hand side to obtain

$$f_2 = \sum_{1 \leqslant i < j \leqslant N} a_{ij} \exp(\theta_i + \theta_j), \tag{90}$$

where the phase factor a_{ij} is given by

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \tag{91}$$

and θ_i and θ_j are given above in (82). For the two-soliton solution we use $1 \leqslant i < j \leqslant 2$, and therefore we obtain

$$f = 1 + \exp(\theta_1) + \exp(\theta_2) + a_{12} \exp(\theta_1 + \theta_2), \tag{92}$$

where the phase factor a_{12} is given by

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}. \tag{93}$$

This in turn gives

$$f = 1 + e^{k_1(x - k_1^2 t)} + e^{k_2(x - k_2^2 t)} + \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} e^{(k_1 + k_2)x - (k_1^3 + k_2^3)t}. \tag{94}$$

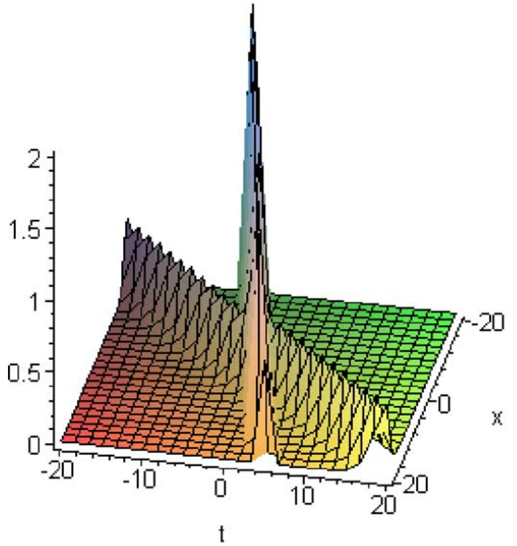

Fig. 2. A two-soliton solution graph.

To determine the two-soliton solution explicitly, we use (72) for the function f in (94).

Similarly, we can determine f_3. Proceeding as before, we therefore set

$$\begin{aligned} f_1(x,t) &= \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3), \\ f_2(x,t) &= a_{12}\exp(\theta_1+\theta_2) + a_{23}\exp(\theta_2+\theta_2) + a_{13}\exp(\theta_1+\theta_3), \end{aligned} \tag{95}$$

and accordingly we have

$$\begin{aligned} f(x,t) = 1 &+ \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3) \\ &+ a_{12}\exp(\theta_1+\theta_2) + a_{23}\exp(\theta_2+\theta_3) + a_{13}\exp(\theta_1+\theta_3) \\ &+ f_3(x,t). \end{aligned} \tag{96}$$

Figure 2 shows a two-soliton solution.

Substituting (96) into (78) and proceeding as before we find

$$f_3 = b_{123}\exp(\theta_1+\theta_2+\theta_3), \tag{97}$$

where

$$b_{123} = a_{12}a_{13}a_{23} = \frac{(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^2}, \tag{98}$$

and θ_1, θ_2 and θ_3 are given above in (82). For the three-soliton solution we use $1 \leqslant i < j \leqslant 3$, we therefore obtain

$$\begin{aligned} f = 1 &+ \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3) + a_{12}\exp(\theta_1+\theta_2) \\ &+ a_{13}\exp(\theta_1+\theta_3) + a_{23}\exp(\theta_2+\theta_3) + b_{123}\exp(\theta_1+\theta_2+\theta_3), \end{aligned} \tag{99}$$

where

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad 1 \leqslant i < j \leqslant 3, \quad b_{123} = a_{12}a_{13}a_{23}. \tag{100}$$

This in turn gives

$$\begin{aligned} f &= 1 + e^{k_1(x-k_1^2 t)} + e^{k_2(x-k_2^2 t)} + e^{k_3(x-k_3^2 t)} \\ &\quad + \frac{(k_1-k_2)^2}{(k_1+k_2)^2} e^{(k_1+k_2)x-(k_1^3+k_2^3)t} + \frac{(k_1-k_3)^2}{(k_1+k_3)^2} e^{(k_1+k_3)x-(k_1^3+k_3^3)t} \\ &\quad + \frac{(k_2-k_3)^2}{(k_2+k_3)^2} e^{(k_2+k_3)x-(k_2^3+k_3^3)t} \\ &\quad + \frac{(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^2} e^{(k_1+k_2+k_3)x-(k_1^3+k_2^3+k_3^3)t}. \end{aligned} \tag{101}$$

To determine the three-solitons solution explicitly, we use (72) for the function f in (101).

Similarly, we can determine f_4. Substituting (81) into (80) and proceeding as before we get

$$f_4 = c_{1234} \exp(\theta_1 + \theta_2 + \theta_3 + \theta_4), \tag{102}$$

where

$$\begin{aligned} c_{1234} &= a_{12}a_{13}a_{14}a_{23}a_{24}a_{34}, \\ &= \frac{(k_1-k_2)^2(k_1-k_3)^2(k_1-k_4)^2(k_2-k_3)^2(k_2-k_4)^2(k_3-k_4)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_1+k_4)^2(k_2+k_3)^2(k_2+k_4)^2(k_3+k_4)^2}, \end{aligned} \tag{103}$$

and θ_i, $1 \leqslant i \leqslant 4$, are given above in (82).

For the four-soliton solution we use $1 \leqslant i < j \leqslant 4$ to obtain

$$\begin{aligned} f &= 1 + \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3) + \exp(\theta_4) \\ &\quad + a_{12}\exp(\theta_1+\theta_2) + a_{13}\exp(\theta_1+\theta_3) + a_{14}\exp(\theta_1+\theta_4) \\ &\quad + a_{23}\exp(\theta_2+\theta_3) + a_{24}\exp(\theta_2+\theta_4) + a_{34}\exp(\theta_3+\theta_4) \\ &\quad + b_{123}\exp(\theta_1+\theta_2+\theta_3) + b_{134}\exp(\theta_1+\theta_3+\theta_4) \\ &\quad + b_{124}\exp(\theta_1+\theta_2+\theta_4) + b_{234}\exp(\theta_2+\theta_3+\theta_4) \\ &\quad + c_{1234}\exp(\theta_1+\theta_2+\theta_3+\theta_4). \end{aligned} \tag{104}$$

Figure 3 shows a three soliton solution.

It is important to note that

$$\begin{aligned} &a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad 1 \leqslant i < j \leqslant 4, \\ &b_{123} = a_{12}a_{13}a_{23}, \quad b_{134} = a_{13}a_{14}a_{34}, \quad b_{124} = a_{12}a_{14}a_{24}, \\ &b_{234} = a_{23}a_{24}a_{34}, \quad c_{1234} = a_{12}a_{13}a_{14}a_{23}a_{24}a_{34}. \end{aligned} \tag{105}$$

Figure 4 shows two-soliton and three-soliton solutions.

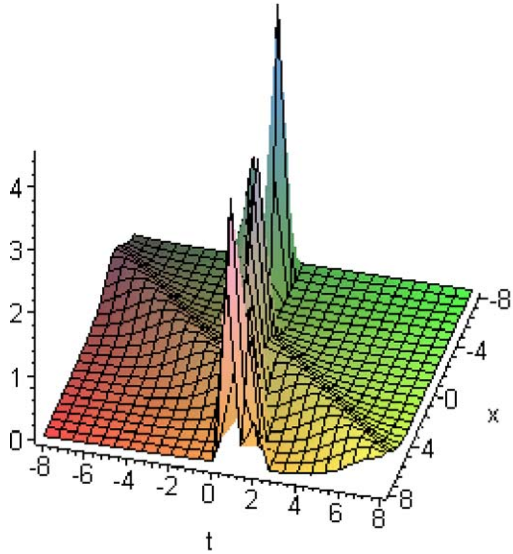

Fig. 3. A three-soliton solution graph.

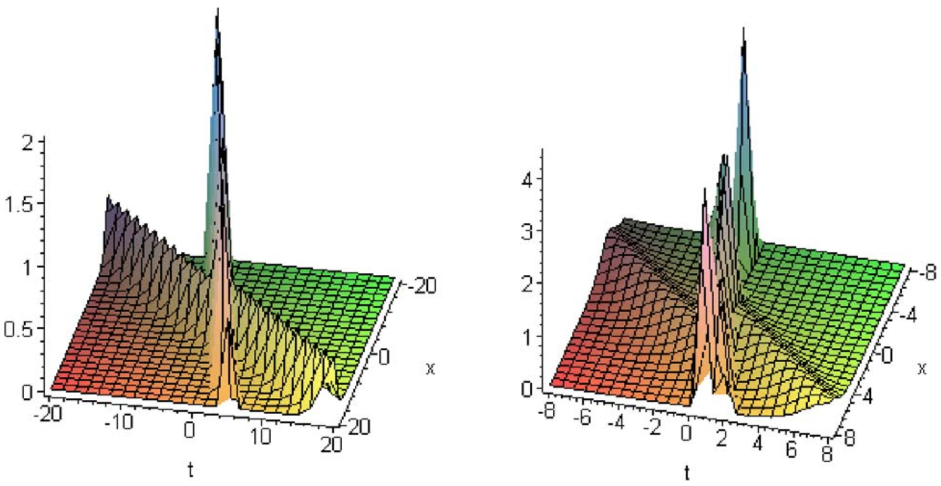

Fig. 4. Graphs of two-soliton (left) and three-soliton (right) solutions.

This in turn gives

$$
\begin{aligned}
f = {} & 1 + e^{k_1(x-k_1^2 t)} + e^{k_2(x-k_2^2 t)} + e^{k_3(x-k_3^2 t)} + e^{k_4(x-k_4^2 t)} \\
& + a_{12} e^{(k_1+k_2)x-(k_1^3+k_2^3)t} + a_{13} e^{(k_1+k_3)x-(k_1^3+k_3^3)t} \\
& + a_{14} e^{(k_1+k_4)x-(k_1^3+k_4^3)t} + a_{23} e^{(k_2+k_3)x-(k_2^3+k_3^3)t}
\end{aligned}
$$

$$+ a_{24}e^{(k_2+k_4)x-(k_2^3+k_4^3)t} + a_{34}e^{(k_3+k_4)x-(k_3^3+k_4^3)t}$$
$$+ b_{123}e^{(k_1+k_2+k_3)x-(k_1^3+k_2^3+k_3^3)t} + b_{134}e^{(k_1+k_3+k_4)x-(k_1^3+k_3^3+k_4^3)t}$$
$$+ b_{124}e^{(k_1+k_2+k_4)x-(k_1^3+k_2^3+k_4^3)t} + b_{234}e^{(k_2+k_3+k_4)x-(k_2^3+k_3^3+k_4^3)t}$$
$$+ c_{1234}e^{(k_1+k_2+k_3+k_4)x-(k_1^3+k_2^3+k_3^3+k_4^3)t}. \tag{106}$$

To determine the four-solitons solution explicitly, we use (72) for the function f in (106).

In summary, the multi-soliton solutions of the KdV equation can be formally constructed as:

(i) one-soliton solution:

$$f = 1 + e^{\theta_1},$$

(ii) two-soliton solution:

$$f = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2},$$

(iii) three-soliton solution:

$$f = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3}$$
$$+ a_{12}a_{13}a_{23}e^{\theta_1+\theta_2+\theta_3},$$

and so on.

Three facts can be confirmed here:

(i) the first is that soliton solutions are just polynomials of exponentials as emphasized by Hirota [24–29],

(ii) the three-soliton solution and the higher level soliton solution as well, do not contain any new free parameters other than a_{ij} derived for the two-soliton solution, and

(iii) every solitonic equation that has generic $N = 3$ soliton solutions, then it has also soliton solutions for any $N \geqslant 4$ [20–23].

To work with explicit solutions, we set $k_i = i$ for example to obtain the following functions

$$f = 1 + e^{x-t},$$
$$f = 1 + e^{x-t} + e^{2(x-4t)} + \frac{1}{9}e^{3(x-3t)},$$
$$f = 1 + e^{x-t} + e^{2(x-4t)} + e^{3(x-9t)} + \frac{1}{9}e^{3(x-3t)} + \frac{1}{4}e^{4(x-7t)} + \frac{1}{25}e^{5(x-7t)}$$
$$+ \frac{1}{900}e^{6(x-6t)},$$
$$f = 1 + e^{x-t} + e^{2(x-4t)} + e^{3(x-9t)} + e^{4(x-16t)} + \frac{1}{9}e^{3(x-3t)}$$
$$+ \frac{1}{4}e^{4(x-7t)} + \frac{9}{25}e^{5(x-13t)} + \frac{2}{25}e^{5(x-7t)} + \frac{1}{9}e^{6(x-12t)}$$
$$+ \frac{1}{49}e^{7(x-13t)} + \frac{1}{450}e^{6(x-6t)} + \frac{1}{225}e^{(7x-73t)}$$
$$+ \frac{9}{4900}e^{(8x-92t)} + \frac{1}{11025}e^{9(x-11t)} + \frac{1}{1102500}e^{10(x-10t)}, \tag{107}$$

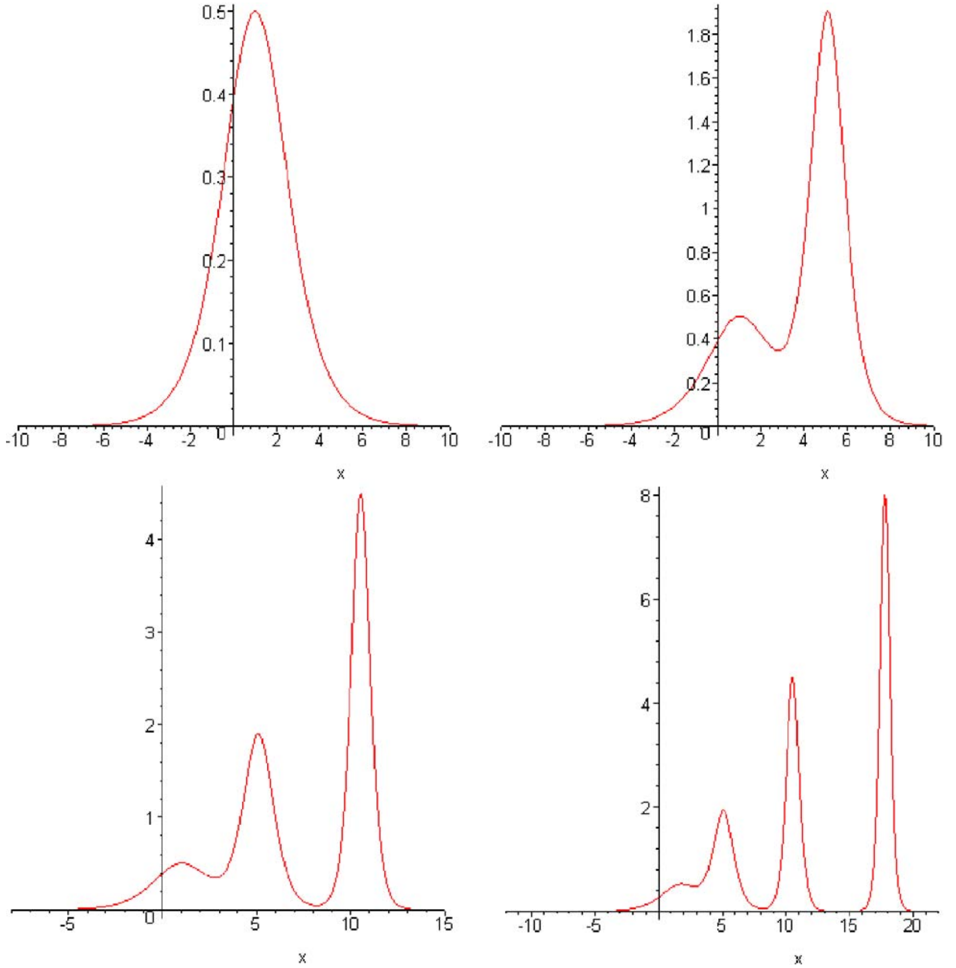

Fig. 5. One-soliton, two-soliton, three-soliton, and four-soliton solutions. Notice that u is plotted against x for fixed t.

for the one, two, three, and four-soliton solutions respectively. The corresponding solitons solutions can be easily obtained by inserting $f(x,t)$ from (107) into

$$u(x,t) = 2\big(\ln(f)\big)_{xx}, \tag{108}$$

to obtain the related soliton solutions. Figure 5 shows graphs of one, two, three and four soliton for fixed t.

6. The modified KdV equation

It is interesting to point out that for the canonical modified KdV (mKdV) equation

$$u_t - 6u^2 u_x + u_{xxx} = 0, \tag{109}$$

the conservation laws are given by u, u^2, $u^4 + (u_x)^2$. Recall that a conservation law is the relation

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0,$$

where T and X are the density and flux respectively. This in turn gives the first three conservation laws:

$$\begin{aligned}
&T_1 = u, \qquad X_1 = 2u^3 + u_{xx};\\
&T_2 = \frac{1}{2}u^2, \qquad X_2 = \frac{3}{2}u^4 + uu_{xx} - \frac{1}{2}u_x^2;\\
&T_3 = \frac{1}{4}u^4 - \frac{1}{4}u_x^2, \qquad X_3 = u^6 + u^3 u_{xx} - 3u^2 u_x^2 - \frac{1}{2}u_x u_{xxx} + \frac{1}{4}u_{xx}^2.
\end{aligned}$$

The mKdV equation is used to describe nonlinear wave propagation in systems with polarity symmetry. This equation is used in electrodynamics, wave propagation in size quantized films, and in elastic media. The mKdV equation is integrable and can be solved by the inverse scattering method [13].

As stated before this equation is given by

$$u_t + au^2 u_x + u_{xxx} = 0. \tag{110}$$

Substituting the wave variable $\xi = x - ct$ into (110) and integrating once we obtain

$$-cu + \frac{a}{3}u^3 + u'' = 0. \tag{111}$$

6.1. *Using the tanh–coth method*

Balancing the nonlinear term u^3 with the highest order derivative u'' gives

$$3M = M + 2, \tag{112}$$

that gives

$$M = 1. \tag{113}$$

The tanh–coth method allows us to use the substitution

$$u(x,t) = S(Y) = a_0 + a_1 Y + b_1 Y^{-1}. \tag{114}$$

Substituting (114) into (111), collecting the coefficients of each power of Y^i, $0 \leqslant i \leqslant 6$, setting each coefficient to zero, and solving the resulting system of algebraic equations we obtain the following sets of solutions

(i) *First set*

$$a_0 = a_1 = 0, \quad b_1 = \sqrt{\frac{3c}{a}}, \quad M = \sqrt{-\frac{c}{2}}, \quad c < 0. \tag{115}$$

(ii) *Second set*

$$a_0 = b_1 = 0, \quad a_1 = \sqrt{\frac{3c}{a}}, \quad M = \sqrt{-\frac{c}{2}}, \quad c < 0. \tag{116}$$

(iii) *Third set*

$$a_0 = 0, \quad a_1 = b_1 = \frac{1}{2}\sqrt{\frac{3c}{a}}, \quad M = \frac{1}{2}\sqrt{-\frac{c}{2}}, \quad c < 0. \tag{117}$$

(iv) *Fourth set*

$$a_0 = 0, \quad a_1 = -b_1 = -\frac{1}{2}\sqrt{-\frac{3c}{2a}}, \quad M = \frac{1}{2}\sqrt{c}, \quad c > 0. \tag{118}$$

This in turn gives the following soliton and kink solutions

$$\begin{aligned} u_1(x,t) &= \sqrt{\frac{6c}{a}}\,\mathrm{sech}\big[\sqrt{c}\,(x-ct)\big], \quad c > 0,\ a > 0, \\ u_2(x,t) &= \sqrt{\frac{3c}{a}}\,\tanh\left[\sqrt{-\frac{c}{2}}(x-ct)\right], \quad c < 0,\ a < 0, \end{aligned} \tag{119}$$

respectively, and the following traveling wave solutions

$$\begin{aligned} u_3(x,t) &= \sqrt{\frac{3c}{a}}\,\coth\left[\sqrt{-\frac{c}{2}}(x-ct)\right], \quad c < 0,\ a < 0, \\ u_4(x,t) &= \sqrt{\frac{6c}{a}}\,\sec\big[\sqrt{-c}\,(x-ct)\big], \quad c < 0,\ a < 0, \\ u_5(x,t) &= 2\sqrt{-\frac{3c}{2a}}\,\mathrm{csch}\big[\sqrt{c}\,(x-ct)\big], \quad c > 0,\ a < 0. \end{aligned} \tag{120}$$

6.2. *Using the sine–cosine method*

Substituting (34) into (111) yields

$$\begin{aligned} &-c\lambda\cos^{\beta}(\mu\xi) + \frac{a}{3}\lambda^3\cos^{3\beta}(\mu\xi) \\ &-\lambda\mu^2\beta^2\cos^{\beta}(\mu\xi) + \lambda\mu^2\beta(\beta-1)\cos^{\beta-2}(\mu\xi) = 0, \end{aligned} \tag{121}$$

Equation (121) is satisfied only if the following system of algebraic equations holds:

$$\begin{aligned} &\beta - 1 \neq 0, \\ &3\beta = \beta - 2, \\ &\mu^2\beta^2\lambda = -c\lambda, \\ &\frac{a}{3}\lambda^3 = -c\lambda\mu^2\beta(\beta-1), \end{aligned} \tag{122}$$

which leads to

$$\beta = -1,\quad \mu = \sqrt{-c},\quad \lambda = \frac{6c}{a}. \tag{123}$$

The results in (123) can be easily obtained if we also use the sine method (35). This in turn gives the periodic solutions for $c < 0$, $a < 0$:

$$u(x,t) = \sqrt{\frac{6c}{a}}\sec\left[\sqrt{-c}(x-ct)\right], \quad c<0,\ a<0,$$
$$u(x,t) = \sqrt{\frac{6c}{a}}\csc\left[\sqrt{-c}(x-ct)\right], \quad c<0,\ a<0. \tag{124}$$

However, for $c > 0$, $a > 0$, we obtain the soliton solution

$$u(x,t) = \sqrt{\frac{6c}{a}}\operatorname{sech}\left[\sqrt{c}(x-ct)\right]. \tag{125}$$

6.3. *Multiple-solitons of the mKdV equation*

In this section, we will examine multiple-solitons solutions of the modified KdV (mKdV) equation

$$u_t - 6u^2u_x + u_{xxx} = 0. \tag{126}$$

In this section, the simplified version of Hirota method, that was introduced by Hereman et al. [19,18], where exact solitons can be obtained by solving a perturbation scheme using a symbolic manipulation package and without any need to use bilinear forms will be used. Our approach will combine the simplified version in [19,18] and the method used introduced by in [20–23]. Hietarinta [20–23] first introduced the function

$$F(x,t) = \frac{f(x,t)}{g(x,t)}, \quad g(x,t) \neq 0. \tag{127}$$

The bilinear form for the mKdV equation is

$$\left(D_t + D_x^3\right)f\cdot g = 0, \qquad D_x^2(f\cdot f + g\cdot g) = 0.$$

The solution of the mKdV equation is assumed to be of the form

$$u(x,t) = \frac{\partial \log F(x,t)}{\partial x} = \frac{gf_x - fg_x}{gf}. \tag{128}$$

We next assume that $f(x,t)$ and $g(x,t)$ have perturbation expansions of the form

$$f(x,t) = 1 + \sum_{n=1}^{\infty}\epsilon^n f_n(x,t),$$
$$g(x,t) = 1 + \sum_{n=1}^{\infty}\epsilon_1^n g_n(x,t), \tag{129}$$

where ϵ and ϵ_1 are nonsmall formal expansion parameters. Following [19,18,20–23], we define the N-soliton solution

$$f_1 = \sum_{i=1}^{N} \epsilon \exp(\theta_i),$$
$$g_1 = \sum_{i=1}^{N} \epsilon_1 \exp(\theta_i), \tag{130}$$

where

$$\theta_i = k_i x - c_i t, \tag{131}$$

where k_i and c_i are arbitrary constants, k_i is called the wave number.

To obtain the one-soliton solution, we set $N = 1$ into (130), and by using (129) we find

$$f(x,t) = 1 + \epsilon f_1(x,t),$$
$$g(x,t) = 1 + \epsilon_1 g_1(x,t), \tag{132}$$

and hence

$$u(x,t) = \frac{\partial \log F(x,t)}{\partial x} = \frac{\partial}{\partial x} \log\left(\frac{1+\epsilon f_1}{1+\epsilon_1 g_1}\right). \tag{133}$$

This is a solution of (126) if

$$\epsilon_1 = -\epsilon. \tag{134}$$

Moreover, this shows that the dispersion relation is

$$c_i = k_i^3, \tag{135}$$

and in view of this result we obtain

$$\theta_i = k_i x - k_i^3 t. \tag{136}$$

The obtained results give a new definition to (129) to be of the form

$$f(x,t) = +\sum_{n=1}^{\infty} \epsilon^n f_n(x,t),$$
$$g(x,t) = 1 + \sum_{n=1}^{\infty} (-1)^n \epsilon^n g_n(x,t), \tag{137}$$

and as a result we obtain

$$f_1 = \exp(\theta_1) = \exp\bigl(k_1(x - k_1^2 t)\bigr),$$
$$g_1 = -\exp(\theta_1) = -\exp\bigl(k_1(x - k_1^2 t)\bigr). \tag{138}$$

Consequently, we find

$$F = \frac{1+f_1}{1+g_1} = \frac{1+\exp(k_1(x-k_1^2 t))}{1-\exp(k_1(x-k_1^2 t))}. \tag{139}$$

This in turn gives the one-soliton solution

$$u(x,t)=\frac{2k_1\exp(k_1(x-k_1^2t))}{1-\exp(k_1(x-k_1^2t))}. \tag{140}$$

To determine the two-soliton solution, we first set $N=2$ in (130) to get

$$\begin{aligned} f_1&=\exp(\theta_1)+\exp(\theta_2),\\ g_1&=-\exp(\theta_1)-\exp(\theta_2). \end{aligned} \tag{141}$$

To determine f_2, we set

$$\begin{aligned} f_2&=\sum_{1\leqslant i<j\leqslant N} a_{ij}\exp(\theta_i+\theta_j),\\ g_2&=\sum_{1\leqslant i<j\leqslant N} b_{ij}\exp(\theta_i+\theta_j) \end{aligned} \tag{142}$$

This in turn gives

$$\begin{aligned} f&=1+\exp(\theta_1)+\exp(\theta_2)+a_{12}\exp(\theta_1+\theta_2),\\ g&=1-\exp(\theta_1)-\exp(\theta_2)+b_{12}\exp(\theta_1+\theta_2). \end{aligned} \tag{143}$$

Substituting (143) into the mKdV equation, we find that (143) is a solution if a_{12} and b_{12}, and therefore a_{ij} and b_{ij}, are equal and given by

$$a_{ij}=b_{ij}=\frac{(k_i-k_j)^2}{(k_i+k_j)^2}, \tag{144}$$

where θ_i and θ_j are given above in (82). For the two-soliton solution we use $1\leqslant i<j\leqslant 2$ to get

$$\begin{aligned} f&=1+e^{k_1(x-k_1^2t)}+e^{k_2(x-k_2^2t)}+\frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{(k_1+k_2)x-(k_1^3+k_2^3)t},\\ g&=1-e^{k_1(x-k_1^2t)}-e^{k_2(x-k_2^2t)}+\frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{(k_1+k_2)x-(k_1^3+k_2^3)t}. \end{aligned} \tag{145}$$

To determine the two-solitons solution explicitly, we use

$$\begin{aligned} &u(x,t)\\ &\quad=\frac{\partial}{\partial x}\left(\log\left[\left\{1+e^{k_1(x-k_1^2t)}+e^{k_2(x-k_2^2t)}+\frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{(k_1+k_2)x-(k_1^3+k_2^3)t}\right\}^{-1}\right.\right.\\ &\qquad\left.\left.\times\left\{1-e^{k_1(x-k_1^2t)}-e^{k_2(x-k_2^2t)}+\frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{(k_1+k_2)x-(k_1^3+k_2^3)t}\right\}^{-1}\right]\right). \end{aligned} \tag{146}$$

For the three-soliton solution, we follow the discussion presented before. To determine f_3 we set

$$
\begin{aligned}
f_1(x,t) &= \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3),\\
f_2(x,t) &= a_{12}\exp(\theta_1+\theta_2) + a_{23}\exp(\theta_2+\theta_3) + a_{13}\exp(\theta_1+\theta_3),\\
g_1(x,t) &= -\exp(\theta_1) - \exp(\theta_2) - \exp(\theta_3),\\
g_2(x,t) &= a_{12}\exp(\theta_1+\theta_2) + a_{23}\exp(\theta_2+\theta_3) + a_{13}\exp(\theta_1+\theta_3),
\end{aligned}
\tag{147}
$$

and accordingly we have

$$
\begin{aligned}
f(x,t) = 1 &+ \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3)\\
&+ a_{12}\exp(\theta_1+\theta_2) + a_{23}\exp(\theta_2+\theta_3) + a_{13}\exp(\theta_1+\theta_3)\\
&+ f_3(x,t),\\
g(x,t) = 1 &- \exp(\theta_1) - \exp(\theta_2) - \exp(\theta_3)\\
&+ a_{12}\exp(\theta_1+\theta_2) + a_{23}\exp(\theta_2+\theta_3) + a_{13}\exp(\theta_1+\theta_3)\\
&+ g_3(x,t).
\end{aligned}
\tag{148}
$$

Substituting (148) into (126) and proceeding as before we find

$$
\begin{aligned}
f_3 &= b_{123}\exp(\theta_1+\theta_2+\theta_3),\\
g_3 &= -b_{123}\exp(\theta_1+\theta_2+\theta_3)
\end{aligned}
\tag{149}
$$

where

$$
b_{123} = a_{12}a_{13}a_{23} = \frac{(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^2},
\tag{150}
$$

and θ_1, θ_2 and θ_3 are given before. For the three-soliton solution we use $1 \leqslant i < j \leqslant 3$, we therefore obtain

$$
\begin{aligned}
f = 1 &+ \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3)\\
&+ a_{12}\exp(\theta_1+\theta_2) + a_{13}\exp(\theta_1+\theta_3) + a_{23}\exp(\theta_2+\theta_3)\\
&+ b_{123}\exp(\theta_1+\theta_2+\theta_3),\\
g = 1 &- \exp(\theta_1) - \exp(\theta_2) - \exp(\theta_3)\\
&+ a_{12}\exp(\theta_1+\theta_2) + a_{13}\exp(\theta_1+\theta_3) + a_{23}\exp(\theta_2+\theta_3)\\
&- b_{123}\exp(\theta_1+\theta_2+\theta_3),
\end{aligned}
\tag{151}
$$

where

$$
\begin{aligned}
&a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \quad a_{13} = \frac{(k_1-k_3)^2}{(k_1+k_3)^2}, \quad a_{23} = \frac{(k_2-k_3)^2}{(k_2+k_3)^2},\\
&b_{123} = a_{12}a_{13}a_{23}.
\end{aligned}
\tag{152}
$$

This in turn gives

$$\begin{aligned}
f(x,t) &= 1 + \mathrm{e}^{k_1(x-k_1^2 t)} + \mathrm{e}^{k_2(x-k_2^2 t)} + \mathrm{e}^{k_3(x-k_3^2 t)} \\
&\quad + \frac{(k_1-k_2)^2}{(k_1+k_2)^2}\mathrm{e}^{(k_1+k_2)x-(k_1^3+k_2^3)t} + \frac{(k_1-k_3)^2}{(k_1+k_3)^2}\mathrm{e}^{(k_1+k_3)x-(k_1^3+k_3^3)t} \\
&\quad + \frac{(k_2-k_3)^2}{(k_2+k_3)^2}\mathrm{e}^{(k_2+k_3)x-(k_2^3+k_3^3)t} \\
&\quad + \frac{(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^2}\mathrm{e}^{(k_1+k_2+k_3)x-(k_1^3+k_2^3+k_3^3)t}, \\
g(x,t) &= 1 - \mathrm{e}^{k_1(x-k_1^2 t)} - \mathrm{e}^{k_2(x-k_2^2 t)} - \mathrm{e}^{k_3(x-k_3^2 t)} \\
&\quad + \frac{(k_1-k_2)^2}{(k_1+k_2)^2}\mathrm{e}^{(k_1+k_2)x-(k_1^3+k_2^3)t} + \frac{(k_1-k_3)^2}{(k_1+k_3)^2}\mathrm{e}^{(k_1+k_3)x-(k_1^3+k_3^3)t} \\
&\quad + \frac{(k_2-k_3)^2}{(k_2+k_3)^2}\mathrm{e}^{(k_2+k_3)x-(k_2^3+k_3^3)t} \\
&\quad - \frac{(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^2}\mathrm{e}^{(k_1+k_2+k_3)x-(k_1^3+k_2^3+k_3^3)t}. \qquad (153)
\end{aligned}$$

The three-soliton solution is therefore given by

$$u(x,t) = \frac{\partial}{\partial x}\left(\ln\left(\frac{f(x,t)}{g(x,t)}\right)\right), \qquad (154)$$

where $f(x,t)$ and $g(x,t)$ are given in (153).

Similarly, we can determine f_4. Substituting (130) into (126) and proceeding as before we get

$$f_4 = g_4 = c_{1234}\exp(\theta_1+\theta_2+\theta_3+\theta_4), \qquad (155)$$

where

$$\begin{aligned}
c_{1234} &= a_{12}a_{13}a_{14}a_{23}a_{24}a_{34}, \\
&= \frac{(k_1-k_2)^2(k_1-k_3)^2(k_1-k_4)^2(k_2-k_3)^2(k_2-k_4)^2(k_3-k_4)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_1+k_4)^2(k_2+k_3)^2(k_2+k_4)^2(k_3+k_4)^2}, \qquad (156)
\end{aligned}$$

and θ_i, $1 \leqslant i \leqslant 4$ are given above in (131).

For the four-soliton solution we use $1 \leqslant i < j \leqslant 4$, we therefore obtain

$$\begin{aligned}
f(x,t) &= 1 + \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3) + \exp(\theta_4) \\
&\quad + a_{12}\exp(\theta_1+\theta_2) + a_{13}\exp(\theta_1+\theta_3) + a_{14}\exp(\theta_1+\theta_4) \\
&\quad + a_{23}\exp(\theta_2+\theta_3) + a_{24}\exp(\theta_2+\theta_4) + a_{34}\exp(\theta_3+\theta_4) \\
&\quad + b_{123}\exp(\theta_1+\theta_2+\theta_3) + b_{134}\exp(\theta_1+\theta_3+\theta_4) \\
&\quad + b_{124}\exp(\theta_1+\theta_2+\theta_4) + b_{234}\exp(\theta_2+\theta_3+\theta_4) \\
&\quad + c_{1234}\exp(\theta_1+\theta_2+\theta_3+\theta_4),
\end{aligned}$$

$$
\begin{aligned}
g(x,t) = 1 &- \exp(\theta_1) - \exp(\theta_2) - \exp(\theta_3) - \exp(\theta_4) \\
&+ a_{12}\exp(\theta_1+\theta_2) + a_{13}\exp(\theta_1+\theta_3) + a_{14}\exp(\theta_1+\theta_4) \\
&+ a_{23}\exp(\theta_2+\theta_3) + a_{24}\exp(\theta_2+\theta_4) + a_{34}\exp(\theta_3+\theta_4) \\
&- b_{123}\exp(\theta_1+\theta_2+\theta_3) - b_{134}\exp(\theta_1+\theta_3+\theta_4) \\
&- b_{124}\exp(\theta_1+\theta_2+\theta_4) - b_{234}\exp(\theta_2+\theta_3+\theta_4) \\
&+ c_{1234}\exp(\theta_1+\theta_2+\theta_3+\theta_4), \qquad (157)
\end{aligned}
$$

where

$$
\begin{aligned}
&a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \quad a_{13} = \frac{(k_1-k_3)^2}{(k_1+k_3)^2}, \quad a_{14} = \frac{(k_1-k_4)^2}{(k_1+k_4)^2}, \\
&a_{23} = \frac{(k_2-k_3)^2}{(k_2+k_3)^2}, \quad a_{24} = \frac{(k_2-k_4)^2}{(k_2+k_4)^2}, \quad a_{34} = \frac{(k_3-k_4)^2}{(k_3+k_4)^2}, \\
&b_{123} = a_{12}a_{13}a_{23}, \quad b_{134} = a_{13}a_{14}a_{34}, \\
&b_{124} = a_{12}a_{14}a_{24}, \quad b_{234} = a_{23}a_{24}a_{34}, \\
&c_{1234} = a_{12}a_{13}a_{14}a_{23}a_{24}a_{34}. \qquad (158)
\end{aligned}
$$

This in turn gives

$$
\begin{aligned}
f(x,t) = 1 &+ e^{k_1(x-k_1^2t)} + e^{k_2(x-k_2^2t)} + e^{k_3(x-k_3^2t)} + e^{k_4(x-k_4^2t)} \\
&+ a_{12}e^{(k_1+k_2)x-(k_1^3+k_2^3)t} + a_{13}e^{(k_1+k_3)x-(k_1^3+k_3^3)t} \\
&+ a_{14}e^{(k_1+k_4)x-(k_1^3+k_4^3)t} + a_{23}e^{(k_2+k_3)x-(k_2^3+k_3^3)t} \\
&+ a_{24}e^{(k_2+k_4)x-(k_2^3+k_2^3)t} + a_{34}e^{(k_3+k_4)x-(k_3^3+k_2^3)t} \\
&+ b_{123}e^{(k_1+k_2+k_3)x-(k_1^3+k_2^3+k_3^3)t} + b_{134}e^{(k_1+k_3+k_4)x-(k_1^3+k_3^3+k_4^3)t} \\
&+ b_{124}e^{(k_1+k_2+k_4)x-(k_1^3+k_2^3+k_4^3)t} + b_{234}e^{(k_2+k_3+k_4)x-(k_2^3+k_3^3+k_4^3)t} \\
&+ c_{1234}e^{(k_1+k_2+k_3+k_4)x-(k_1^3+k_2^3+k_3^3+k_4^3)t}, \qquad (159)
\end{aligned}
$$

$$
\begin{aligned}
g(x,t) = 1 &- e^{k_1(x-k_1^2t)} - e^{k_2(x-k_2^2t)} - e^{k_3(x-k_3^2t)} - e^{k_4(x-k_4^2t)} \\
&+ a_{12}e^{(k_1+k_2)x-(k_1^3+k_2^3)t} + a_{13}e^{(k_1+k_3)x-(k_1^3+k_3^3)t} \\
&+ a_{14}e^{(k_1+k_4)x-(k_1^3+k_4^3)t} + a_{23}e^{(k_2+k_3)x-(k_2^3+k_3^3)t} \\
&+ a_{24}e^{(k_2+k_4)x-(k_2^3+k_2^3)t} + a_{34}e^{(k_3+k_4)x-(k_3^3+k_2^3)t} \\
&- b_{123}e^{(k_1+k_2+k_3)x-(k_1^3+k_2^3+k_3^3)t} - b_{134}e^{(k_1+k_3+k_4)x-(k_1^3+k_3^3+k_4^3)t} \\
&- b_{124}e^{(k_1+k_2+k_4)x-(k_1^3+k_2^3+k_4^3)t} - b_{234}e^{(k_2+k_3+k_4)x-(k_2^3+k_3^3+k_4^3)t} \\
&+ c_{1234}e^{(k_1+k_2+k_3+k_4)x-(k_1^3+k_2^3+k_3^3+k_4^3)t}.
\end{aligned}
$$

To determine the four-soliton solution explicitly, we use (128) for the functions f and g in (159).

If we set $k_i = i$ for example, we obtain the following functions

$$
\begin{aligned}
f &= 1 + \mathrm{e}^{x-t},\\
f &= 1 + \mathrm{e}^{x-t} + \mathrm{e}^{2(x-4t)} + \frac{1}{9}\mathrm{e}^{3(x-3t)},\\
f &= 1 + \mathrm{e}^{x-t} + \mathrm{e}^{2(x-4t)} + \mathrm{e}^{3(x-9t)} + \frac{1}{9}\mathrm{e}^{3(x-3t)} + \frac{1}{4}\mathrm{e}^{4(x-7t)}\\
&\quad + \frac{1}{25}\mathrm{e}^{5(x-7t)} + \frac{1}{900}\mathrm{e}^{6(x-6t)},
\end{aligned}
\tag{160}
$$

$$
\begin{aligned}
f &= 1 + \mathrm{e}^{x-t} + \mathrm{e}^{2(x-4t)} + \mathrm{e}^{3(x-9t)} + \mathrm{e}^{4(x-16t)} + \frac{1}{9}\mathrm{e}^{3(x-3t)} + \frac{1}{4}\mathrm{e}^{4(x-7t)}\\
&\quad + \frac{9}{25}\mathrm{e}^{5(x-13t)} + \frac{1}{25}\mathrm{e}^{5(x-7t)} + \frac{1}{9}\mathrm{e}^{6(x-12t)} + \frac{1}{49}\mathrm{e}^{7(x-13t)}\\
&\quad + \frac{1}{900}\mathrm{e}^{6(x-6t)} + \frac{1}{225}\mathrm{e}^{(7x-73t)} + \frac{9}{4900}\mathrm{e}^{(8x-92t)}\\
&\quad + \frac{1}{11025}\mathrm{e}^{9(x-11t)} + \frac{1}{1102500}\mathrm{e}^{10(x-10t)},
\end{aligned}
\tag{161}
$$

and

$$
\begin{aligned}
g &= 1 - \mathrm{e}^{x-t},\\
g &= 1 - \mathrm{e}^{x-t} - \mathrm{e}^{2(x-4t)} + \frac{1}{9}\mathrm{e}^{3(x-3t)},\\
g &= 1 - \mathrm{e}^{x-t} - \mathrm{e}^{2(x-4t)} - \mathrm{e}^{3(x-9t)}\\
&\quad + \frac{1}{9}\mathrm{e}^{3(x-3t)} + \frac{1}{4}\mathrm{e}^{4(x-7t)} + \frac{1}{25}\mathrm{e}^{5(x-7t)} + \frac{1}{900}\mathrm{e}^{6(x-6t)},\\
g &= 1 - \mathrm{e}^{x-t} - \mathrm{e}^{2(x-4t)} - \mathrm{e}^{3(x-9t)} - \mathrm{e}^{4(x-16t)} + \frac{1}{9}\mathrm{e}^{3(x-3t)} + \frac{1}{4}\mathrm{e}^{4(x-7t)}\\
&\quad + \frac{9}{25}\mathrm{e}^{5(x-13t)} + \frac{1}{25}\mathrm{e}^{5(x-7t)} + \frac{1}{9}\mathrm{e}^{6(x-12t)} + \frac{1}{49}\mathrm{e}^{7(x-13t)}\\
&\quad - \frac{1}{900}\mathrm{e}^{6(x-6t)} - \frac{1}{225}\mathrm{e}^{(7x-73t)} - \frac{9}{4900}\mathrm{e}^{(8x-92t)}\\
&\quad - \frac{1}{11025}\mathrm{e}^{9(x-11t)} + \frac{1}{1102500}\mathrm{e}^{10(x-10t)},
\end{aligned}
\tag{162}
$$

for the one, two, three, and four-soliton solutions respectively. The corresponding solitons solutions can be easily obtained by inserting $f(x,t)$ and $g(x,t)$ into

$$
u(x,t) = \frac{\partial}{\partial x}\left(\ln\left(\frac{f(x,t)}{g(x,t)}\right)\right),
\tag{163}
$$

to obtain the related solitons solutions.

7. The potential KdV equation

In this section we study the potential KdV equation

$$u_t + au_x^2 + u_{3x} = 0, \tag{164}$$

that can be converted to the ODE

$$-cu' + a(u')^2 + u''' = 0, \tag{165}$$

by using the wave variable $\xi = x - ct$. The potential KdV equation will be handled by the tanh–coth method and other hyperbolic functions methods.

7.1. *Using the tanh–coth method*

Balancing the nonlinear term $(u')^2$ with the highest order derivative u''' gives

$$(M + 1)^2 = M + 3, \tag{166}$$

that gives

$$M = 1, -2. \tag{167}$$

Case (i): We first consider the case where $M = 1$. The tanh–coth method allows us to use the substitution

$$u(x, t) = S(Y) = a_0 + a_1 Y + b_1 Y^{-1}. \tag{168}$$

Substituting (168) into (165), proceeding as before and solving the resulting system of algebraic equations we obtain the following sets of solutions

(i) *First set*

$$\begin{aligned} &a_0 = R, \quad R \text{ is an arbitrary constant}, \quad a_1 = \frac{3\sqrt{c}}{a}, \\ &b_1 = 0, \quad M = \frac{\sqrt{c}}{2}, \quad c > 0. \end{aligned} \tag{169}$$

(ii) *Second set*

$$\begin{aligned} &a_0 = R, \quad R \text{ is an arbitrary constant}, \quad a_1 = 0, \quad b_1 = \frac{3\sqrt{c}}{a}, \\ &M = \frac{\sqrt{c}}{2}, \quad c > 0. \end{aligned} \tag{170}$$

(iii) *Third set*

$$\begin{aligned} &a_0 = R, \quad R \text{ is an arbitrary constant}, \quad a_1 = \frac{3\sqrt{c}}{2a}, \\ &b_1 = \frac{3\sqrt{c}}{2a}, \quad M = \frac{\sqrt{c}}{4}, \quad c > 0. \end{aligned} \tag{171}$$

This in turn gives the following kink solution for $c > 0$

$$u_1(x,t) = R + \frac{3\sqrt{c}}{a}\tanh\left[\frac{\sqrt{c}}{2}(x-ct)\right], \tag{172}$$

and the traveling wave solution

$$u_2(x,t) = R + \frac{3\sqrt{c}}{a}\coth\left[\frac{\sqrt{c}}{2}(x-ct)\right]. \tag{173}$$

However, the third set gives $u_2(x,t)$ upon using hyperbolic identities. Moreover, for $c < 0$, the obtained solutions are complex.

Case (ii): We next consider the case where $M = -2$. The tanh–coth method allows us to use the substitution

$$u(x,t) = S(Y) = 1\Big/\big(a_0 + a_1 Y + a_2 Y^2 + b_1 Y^{-1} + b_2 Y^{-2}\big). \tag{174}$$

Substituting (174) into (165), proceeding as before we found that $a_2 = b_2 = 0$. Therefore we substitute

$$u(x,t) = S(Y) = 1/\big(a_0 + a_1 Y + b_1 Y^{-1}\big), \tag{175}$$

into (165) to obtain the following sets of solutions

(i) *First set*

$$a_0 = b_1 = 0, \quad a_1 = \frac{a}{3\sqrt{c}}, \quad M = \frac{\sqrt{c}}{2}, \quad c > 0. \tag{176}$$

(ii) *Second set*

$$a_0 = a_1 = 0, \quad b_1 = \frac{a}{3\sqrt{c}}, \quad M = \frac{\sqrt{c}}{2}, \quad c > 0. \tag{177}$$

(iii) *Third set*

$$\begin{aligned} &a_1 = R, \quad R \text{ is an arbitrary constant,} \\ &a_0 = \sqrt{\frac{3cR^2 - aR\sqrt{c}}{3c}}, \quad b_1 = 0, \quad M = \frac{\sqrt{c}}{2}, \quad c > 0. \end{aligned} \tag{178}$$

(iv) *Fourth set*

$$\begin{aligned} &b_1 = R, \quad R \text{ is an arbitrary constant,} \\ &a_0 = \sqrt{\frac{3cR^2 - aR\sqrt{c}}{3c}}, \quad a_1 = 0, \quad M = \frac{\sqrt{c}}{2}, \quad c > 0. \end{aligned} \tag{179}$$

(v) *Fifth set*

$$a_0 = 0, \quad a_1 = b_1 = \frac{a}{6\sqrt{c}}, \quad M = \frac{\sqrt{c}}{4}, \quad c > 0. \tag{180}$$

(vi) *Sixth set*

$$
\begin{aligned}
&a_1 = b_1 = R, \quad R \text{ is an arbitrary constant},\\
&a_0 = \sqrt{\frac{12cR^2 - 2aR\sqrt{c}}{3c}}, \quad M = \frac{\sqrt{c}}{4}, \quad c > 0.
\end{aligned} \tag{181}
$$

This in turn gives the following solutions for $c > 0$

$$
\begin{aligned}
u_3(x,t) &= \frac{3\sqrt{c}}{a}\tanh\left[\frac{\sqrt{c}}{2}(x-ct)\right],\\
u_4(x,t) &= \frac{3\sqrt{c}}{a}\coth\left[\frac{\sqrt{c}}{2}(x-ct)\right],\\
u_5(x,t) &= 1\Bigg/\left(\sqrt{\frac{3cR^2 - aR\sqrt{c}}{3c}} + R\tanh\left[\frac{\sqrt{c}}{2}(x-ct)\right]\right),\\
u_6(x,t) &= 1\Bigg/\left(\sqrt{\frac{3cR^2 - aR\sqrt{c}}{3c}} + R\coth[\frac{\sqrt{c}}{2}(x-ct)]\right).
\end{aligned} \tag{182}
$$

The solutions u_3 and u_4 are the same as u_1 and u_2 when we set $R = 0$. However, the last two sets give $u_4(x,t)$ and $u_6(x,t)$ respectively. Moreover, for $c < 0$, the obtained solutions are complex.

7.2. *Other methods*

Two ansatze, namely, the tanh–sech, and the coth–csch ansatze will be used to handle nonlinear equations in general, and the potential KdV equation in particular.

The tanh–sech ansatz The tanh–sech ansatz is of the form

$$u(x,t) = R + L\tanh\big[\mu(x-ct)\big] + M\operatorname{sech}\big[\mu(x-ct)\big], \tag{183}$$

where R, L, M, and μ are parameters that will be determined by direct substitution.

The coth–csch ansatz The coth–csch ansatz is of the form

$$u(x,t) = R + L\coth\big[\mu(x-ct)\big] + M\operatorname{csch}\big[\mu(x-ct)\big], \tag{184}$$

where R, L, M, and μ are parameters that will be determined by direct substitution.

7.3. *Using the tanh–sech ansatz*

Substituting the tanh–sech ansatz (183) into (165), and proceeding as before we obtain two sets of parameters given by

$$
\begin{aligned}
&R = \text{any arbitrary constant},\\
&L = \frac{3\sqrt{c}}{a},\\
&M = 0,\\
&\mu = \sqrt{c},
\end{aligned}
\tag{185}
$$

and

$$
\begin{aligned}
&R = \text{any arbitrary constant},\\
&L = \frac{3\sqrt{c}}{a}, \quad c > 0,\\
&M = \mathrm{i}\frac{3\sqrt{c}}{a}, \quad \mathrm{i}^2 = -1,\\
&\mu = \sqrt{c},
\end{aligned}
\tag{186}
$$

where $c > 0$.

The result in (185) gives the tanh solutions obtained above. The result (186) gives the complex solution

$$u(x,t) = R + \frac{3}{a}\sqrt{c}\tanh\bigl[\sqrt{c}(x-ct)\bigr] + \mathrm{i}\frac{3}{a}\sqrt{c}\,\mathrm{sech}\bigl[\sqrt{c}(x-ct)\bigr], \tag{187}$$

However, for $c < 0$, we obtain the solution

$$u(x,t) = R - \frac{3}{a}\sqrt{-c}\tan\bigl[\sqrt{-c}(x-ct)\bigr] - \frac{3}{a}\sqrt{-c}\sec\bigl[\sqrt{-c}(x-ct)\bigr]. \tag{188}$$

7.4. *Using the coth–csch ansatz*

Substituting the coth–csch ansatz (184) into (165), we obtain two sets of parameters given by

$$
\begin{aligned}
&R = \text{any arbitrary constant},\\
&L = \frac{3\sqrt{c}}{a},\\
&M = 0,\\
&\mu = \sqrt{c},
\end{aligned}
\tag{189}
$$

and

$$
\begin{aligned}
&R = \text{any arbitrary constant},\\
&L = \frac{3\sqrt{c}}{a}, \quad c > 0,
\end{aligned}
$$

$$M = \frac{3}{a}\sqrt{c},$$
$$\mu = \sqrt{c}, \tag{190}$$

where $c > 0$.

The result in (189) gives the coth solution obtained above. The result (190) gives the kink solution

$$u(x,t) = R + \frac{3\sqrt{c}}{a}\coth\big[\sqrt{c}(x-ct)\big] + \frac{3\sqrt{c}}{a}\operatorname{csch}\big[\sqrt{c}(x-ct)\big]. \tag{191}$$

However, for $c < 0$, we obtain the periodic solutions

$$u(x,t) = R + \frac{3\sqrt{-c}}{a}\cot\big[\sqrt{-c}(x-ct)\big] + \frac{3\sqrt{-c}}{a}\csc\big[\sqrt{-c}(x-ct)\big]. \tag{192}$$

7.5. *Multiple-solitons of the potential KdV equation*

In this section, we will examine multiple-soliton solutions of the potential KdV equation

$$u_t + 3(u_x)^2 + u_{xxx} = 0. \tag{193}$$

We closely follow our approach presented before. We therefore introduce the change of dependent variable

$$u(x,t) = 2\frac{\partial \ln f(x,t)}{\partial x} = 2\frac{f_x}{f}, \tag{194}$$

to carry out (193) into

$$f_{xt} - f_x f_t + 3(f_{xx})^2 - 4f_{xxx} f_t + f_{xxxx} = 0. \tag{195}$$

Eq. (195) can be decomposed into linear operator L and nonlinear operator N defined by

$$L = \frac{\partial^2}{\partial x \partial t} + \frac{\partial}{\partial x^4},$$
$$N(f,f) = -f_x f_t + 3f_x f_x - 4f_{xxx} f_t. \tag{196}$$

We next assume that $f(x,t)$ has a perturbation expansion of the form

$$f(x,t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f_n(x,t), \tag{197}$$

where ϵ is a nonsmall formal expansion parameter.

The N-soliton solution is obtained from

$$f_1 = \sum_{i=1}^{N} \exp(\theta_i), \tag{198}$$

where

$$\theta_i = k_i x - c_i t, \tag{199}$$

where k_i and c_i are arbitrary constants. Substituting (198) into (193) gives the dispersion relation

$$c_i = k_i^3, \tag{200}$$

and in view of this result we obtain

$$\theta_i = k_i x - k_i^3 t. \tag{201}$$

This means that

$$f_1 = \exp(\theta_1) = \exp\bigl(k_1(x - k_1^2 t)\bigr), \tag{202}$$

obtained by using $N = 1$ in (198).

Consequently, for the one-soliton solution, we set

$$f = 1 + \exp(\theta_1) = 1 + \exp\bigl(k_1(x - k_1^2 t)\bigr), \tag{203}$$

where we set $\epsilon = 1$. The one soliton solution is therefore

$$u(x,t) = \frac{2k_1 \exp(k_1(x - k_1^2 t))}{1 + \exp(k_1(x - k_1^2 t)}, \tag{204}$$

obtained upon using (194).

To determine the two-soliton solution, we first set $N = 2$ in (198) to get

$$f_1 = \exp(\theta_1) + \exp(\theta_2). \tag{205}$$

To determine f_2, we proceed as before to obtain

$$f_2 = \sum_{1 \leqslant i < j \leqslant N} a_{ij} \exp(\theta_i + \theta_j), \tag{206}$$

where

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \tag{207}$$

and θ_i and θ_j are given above in (82). For the two-soliton solution we use $1 \leqslant i < j \leqslant 2$, and therefore we obtain

$$f = 1 + \exp(\theta_1) + \exp(\theta_2) + a_{12} \exp(\theta_1 + \theta_2), \tag{208}$$

where

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}. \tag{209}$$

This in turn gives

$$f = 1 + e^{k_1(x - k_1^2 t)} + e^{k_2(x - k_2^2 t)} + \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} e^{(k_1 + k_2)x - (k_1^3 + k_2^3)t}. \tag{210}$$

To determine the two-soliton solutions explicitly, we use (194) for the function f in (210).

Similarly, we can determine f_3. Proceeding as before, we therefore set

$$f_1(x,t) = \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3),$$
$$f_2(x,t) = a_{12}\exp(\theta_1+\theta_2) + a_{23}\exp(\theta_2+\theta_2) + a_{13}\exp(\theta_1+\theta_3), \tag{211}$$

and accordingly we have

$$\begin{aligned} f(x,t) = 1 &+ \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3) \\ &+ a_{12}\exp(\theta_1+\theta_2) + a_{23}\exp(\theta_2+\theta_3) + a_{13}\exp(\theta_1+\theta_3) \\ &+ f_3(x,t). \end{aligned} \tag{212}$$

Substituting (212) into (193) and proceeding as before we find

$$f_3 = b_{123}\exp(\theta_1+\theta_2+\theta_3), \tag{213}$$

where

$$b_{123} = a_{12}a_{13}a_{23} = \frac{(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^2}. \tag{214}$$

and θ_1, θ_2 and θ_3 are given above in (199). For the three-soliton solution we use $1 \leqslant i < j \leqslant 3$, we therefore obtain

$$\begin{aligned} f = 1 &+ \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3) \\ &+ a_{12}\exp(\theta_1+\theta_2) + a_{13}\exp(\theta_1+\theta_3) + a_{23}\exp(\theta_2+\theta_3) \\ &+ b_{123}\exp(\theta_1+\theta_2+\theta_3). \end{aligned} \tag{215}$$

To determine the three-solitons solution explicitly, we use (194) for the function f in (215). For the four-solitons solution, we proceed as before, hence we skip details.

8. The generalized KdV equation

The generalized KdV (gKdV) equation [13] is given by

$$u_t + au^n u_x + u_{xxx} = 0, \tag{216}$$

that can be converted to the ODE

$$-cu + \frac{a}{n+1}u^{n+1} + u'' = 0, \tag{217}$$

upon using the wave variable $\xi = x - ct$ and integrating once. The gKdV equation is not integrable, therefore N-soliton solutions do not exist.

8.1. *Using the tanh–coth method*

Balancing the nonlinear term u^{n+1} with the highest order derivative u'' gives

$$M = \frac{2}{n}. \tag{218}$$

To obtain closed form solutions, M should be an integer. To achieve this goal we use the transformation

$$u(x,t) = v^{1/n}(x,t). \tag{219}$$

The transformation (219) carries Eq. (217) into the ODE

$$-cn^2(n+1)v^2 + an^2v^3 + n(n+1)vv'' + (1-n^2)(v')^2 = 0. \tag{220}$$

Balancing vv'' with v^3 gives $M = 2$. Based on this, the tanh–coth method admits the use of the substitution

$$u(x,t) = S(Y) = a_0 + a_1Y + a_2Y^2 + b_1Y^{-1} + b_2Y^{-2}. \tag{221}$$

Substituting (221) into (217), collecting the coefficients of each power of Y^i, $0 \leqslant i \leqslant 12$, setting each coefficient to zero, and solving the resulting system of algebraic equations we obtain $a_1 = b_1 = 0$ and the following sets of solutions for $c > 0$

(i) *First set*

$$\begin{aligned} a_0 &= \frac{c(n+1)(n+2)}{2a}, \quad a_2 = 0, \quad b_2 = -\frac{c(n+1)(n+2)}{2a}, \\ M &= \frac{n}{2}\sqrt{c}. \end{aligned} \tag{222}$$

(ii) *Second set*

$$\begin{aligned} a_0 &= \frac{c(n+1)(n+2)}{2a}, \quad b_2 = 0, \quad a_2 = -\frac{c(n+1)(n+2)}{2a}, \\ M &= \frac{n}{2}\sqrt{c}. \end{aligned} \tag{223}$$

(iii) *Third set*

$$\begin{aligned} a_0 &= \frac{c(n+1)(n+2)}{4a}, \quad a_2 = b_2 = -\frac{c(n+1)(n+2)}{8a}, \\ M &= \frac{n}{4}\sqrt{c}. \end{aligned} \tag{224}$$

In view of these results, and noting that $u(x,t) = v^{1/n}(x,t)$, we obtain the following soliton solution

$$u_1(x,t) = \left\{\frac{c(n+1)(n+2)}{2a}\,\mathrm{sech}^2\left[\frac{n}{2}\sqrt{c}(x-ct)\right]\right\}^{1/n}, \tag{225}$$

Table 1
Solutions $u(x,t)$ of the generalized KdV equation for $n \geqslant 1$, $c > 0$

n	a_0	μ	Solution $u(x,t)$
1	$\frac{3c}{a}$	$\frac{1}{2}\sqrt{c}$	$u_1(x,t) = \{\frac{3c}{a}\operatorname{sech}^2[\frac{1}{2}\sqrt{c}(x-ct)]\}^{1/1}$
			$u_2(x,t) = \{-\frac{3c}{a}\operatorname{csch}^2[\frac{1}{2}\sqrt{c}(x-ct)]\}^{1/1}$
2	$\frac{6c}{a}$	$\sqrt{c}$	$u_1(x,t) = \{\frac{6c}{a}\operatorname{sech}^2[\sqrt{c}(x-ct)]\}^{1/2}$
			$u_2(x,t) = \{-\frac{6c}{a}\operatorname{csch}^2[\sqrt{c}(x-ct)]\}^{1/2}$
3	$\frac{10c}{a}$	$\frac{3}{2}\sqrt{c}$	$u_1(x,t) = \{\frac{10c}{a}\operatorname{sech}^2[\frac{3}{2}\sqrt{c}(x-ct)]\}^{1/3}$
			$u_2(x,t) = \{-\frac{10c}{a}\operatorname{csch}^2[\frac{3}{2}\sqrt{c}(x-ct)]\}^{1/3}$
4	$\frac{15c}{a}$	$2\sqrt{c}$	$u_1(x,t) = \{\frac{15c}{a}\operatorname{sech}^2[2\sqrt{c}(x-ct)]\}^{1/4}$
			$u_2(x,t) = \{\frac{15c}{a}\operatorname{csch}^2[2\sqrt{c}(x-ct)]\}^{1/4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

and the solutions

$$
\begin{aligned}
u_2(x,t) &= \left\{-\frac{c(n+1)(n+2)}{2a}\operatorname{csch}^2\left[\frac{n}{2}\sqrt{c}(x-ct)\right]\right\}^{1/n}, \\
u_3(x,t) &= \left\{\Gamma\left(2-\tanh^2\left[\frac{n}{4}\sqrt{c}(x-ct)\right]-\coth^2\left[\frac{n}{4}\sqrt{c}(x-ct)\right]\right)\right\}^{1/n},
\end{aligned}
\tag{226}
$$

where

$$
\Gamma = \frac{c(n+1)(n+2)}{8a}. \tag{227}
$$

As stated before, the sign of the wave speed c plays an important role on the physical structures of the obtained solutions. This means that for $c < 0$, we obtain the following plane periodic solutions

$$
\begin{aligned}
u_4(x,t) &= \left\{\frac{c(n+1)(n+2)}{2a}\sec^2\left[\frac{n}{2}\sqrt{-c}(x-ct)\right]\right\}^{1/n}, \\
u_5(x,t) &= \left\{\frac{c(n+1)(n+2)}{2a}\csc^2\left[\frac{n}{2}\sqrt{-c}(x-ct)\right]\right\}^{1/n}, \\
u_6(x,t) &= \left\{\Gamma\left(2+\tan^2\left[\frac{n}{4}\sqrt{-c}(x-ct)\right]+\cot^2\left[\frac{n}{4}\sqrt{-c}\,(x-ct)\right]\right)\right\}^{1/n}.
\end{aligned}
\tag{228}
$$

In Table 1, we list a variety of solutions of the generalized KdV equation for $n \geqslant 1$, $c > 0$.

8.2. *Using the sine–cosine method*

Substituting the cosine assumption (34) into (217) yields

$$\begin{aligned} &-c\lambda \cos^{\beta}(\mu\xi) + \frac{a}{n+1}\lambda^{n+1}\cos^{(n+1)\beta}(\mu\xi) \\ &\quad - \lambda\mu^2\beta^2\cos^{\beta}(\mu\xi) + \lambda\mu^2\beta(\beta-1)\cos^{\beta-2}(\mu\xi) = 0, \end{aligned} \tag{229}$$

Equation (229) is satisfied only if the following system of algebraic equations holds:

$$\begin{aligned} &\beta - 1 \neq 0, \\ &(n+1)\beta = \beta - 2, \\ &\mu^2\beta^2\lambda = -c\lambda, \\ &\frac{a}{n+1}\lambda^{n+1} = -\lambda\mu^2\beta(\beta-1), \end{aligned} \tag{230}$$

which leads to

$$\begin{aligned} &\beta = -\frac{2}{n}, \\ &\mu = \frac{n}{2}\sqrt{-c}, \quad c < 0, \\ &\lambda = \left(\frac{c(n+1)(n+2)}{2a}\right)^{1/n}. \end{aligned} \tag{231}$$

The last results in (231) gives the soliton solutions $u_1(x,t)$ and the traveling wave solutions obtained above by using the tanh–coth method. However, using the sine–cosine method does not require the use of a transformation formula as is the case of the tanh method when M is not an integer.

9. The Gardner equation

The Gardner equation, or the combined KdV-mKdV equation, reads

$$u_t + 2auu_x - 3bu^2u_x + u_{xxx} = 0, \quad a, b > 0, \tag{232}$$

where $u(x,t)$ is the amplitude of the relevant wave mode. The KdV equation was complemented with a higher-order cubic nonlinear term of the form u^2u_x to obtain the Gardner equation (232). Equation (232) is completely integrable, like the KdV equation, by the inverse scattering method. It was found, as we will discuss later, that soliton solutions exist only for $b > 0$. Gardner equation is widely used in various branches of physics, such as plasma physics, fluid physics, quantum field theory. The Gardner equation has been widely used to model nonlinear phenomena in plasma and solid state physics and in quantum field theory.

It was found that the tanh–coth method gives the same results as the tanh method. Accordingly, we will apply the tanh method for simplicity reasons.

9.1. *Using the tanh method*

The Gardner equation (232) can be converted to the ODE

$$-cu + au^2 - bu^3 + u'' = 0, \tag{233}$$

by using the wave variable $\xi = x - ct$, integrating the ODE and setting the constant of integration to zero.

Balancing u^3 with u'' gives

$$M = 1. \tag{234}$$

The tanh method allows us to use the finite expansion

$$u(x,t) = S(Y) = a_0 + a_1 Y. \tag{235}$$

Substituting (235) into (233), collecting the coefficients of each power of Y, and using any symbolic computation program such as *Mathematica* we obtain

$$\begin{aligned}
a_0 &= \frac{a}{3b},\\
a_1 &= \pm\frac{a}{3b},\\
\mu &= \frac{a}{3\sqrt{2b}},\\
c &= \frac{2a^2}{9b}.
\end{aligned} \tag{236}$$

This in turn gives the kink solution

$$u(x,t) = \frac{a}{3b}\left(1 \pm \tanh\left(\frac{a}{3\sqrt{2b}}\left(x - \frac{2a^2}{9b}t\right)\right)\right), \tag{237}$$

and the traveling wave solution

$$u(x,t) = \frac{a}{3b}\left(1 \pm \coth\left(\frac{a}{3\sqrt{2b}}\left(x - \frac{2a^2}{9b}t\right)\right)\right). \tag{238}$$

The last results emphasize the fact that Gardner equation has real solutions only for $b > 0$. However, for $b < 0$, the complex solutions

$$u(x,t) = \frac{a}{3b}\left(1 \pm i\tan\left(\frac{a}{3\sqrt{-2b}}\left(x - \frac{2a^2}{9b}t\right)\right)\right), \tag{239}$$

$$u(x,t) = \frac{a}{3b}\left(1 \pm i\cot\left(\frac{a}{3\sqrt{-2b}}\left(x - \frac{2a^2}{9b}t\right)\right)\right), \tag{240}$$

follow immediately.

It was found in [68] that other hyperbolic functions methods can handle Gardner equation to give more traveling wave solutions. In what follows we examine these schemes.

9.2. *A cosh ansatz*

We first assume that

$$u(x,t) = \frac{\alpha}{1 + \lambda \cosh(\mu(x - ct))}, \tag{241}$$

where α, λ and μ are parameters that will be determined. Substituting (241) into (232), and collecting the coefficients of like hyperbolic functions, we find

$$\begin{aligned}
&- c + a\alpha - b\alpha^2 - 2\lambda^2\mu^2 = 0,\\
&- 2c\lambda + a\alpha\lambda - \lambda\mu^2 = 0,\\
&- c\lambda^2 + \lambda^2\mu^2 = 0.
\end{aligned} \tag{242}$$

Solving this system gives

$$\begin{aligned}
&\alpha = \frac{3c}{a},\\
&\lambda = \pm\frac{\sqrt{4a^2 - 18bc}}{2a}, \quad 2a^2 > 9bc,\\
&\mu = \sqrt{c},
\end{aligned} \tag{243}$$

where c is left as a free parameter, $c > 0$.

Substituting (243) into (241) gives the soliton solution

$$u(x,t) = \frac{6c}{2a \pm \sqrt{4a^2 - 18bc}\cosh[\sqrt{c}(x - ct)]}. \tag{244}$$

9.3. *A sinh ansatz*

We next use the sinh ansatz

$$u(x,t) = \frac{\alpha}{1 + \lambda \sinh(\mu(x - ct))}, \tag{245}$$

where α, λ and μ are parameters that will be determined. Substituting (245) into (232), and collecting the coefficients of like sinh functions, we find

$$\begin{aligned}
&- c + a\alpha - b\alpha^2 + 2\lambda^2\mu^2 = 0,\\
&- 2c\lambda + a\alpha\lambda - \lambda\mu^2 = 0,\\
&- c\lambda^2 + \lambda^2\mu^2 = 0.
\end{aligned} \tag{246}$$

Solving this system gives

$$\begin{aligned}
&\alpha = \frac{3c}{a},\\
&\lambda = \pm\frac{\sqrt{18bc - 4a^2}}{2a}, \quad 9bc > 2a^2,\\
&\mu = \sqrt{c},
\end{aligned} \tag{247}$$

where c is left as a free parameter, $c > 0$.

Substituting (247) into (245) yields

$$u(x,t) = \frac{6c}{(2a \pm \sqrt{18bc - 4a^2}\sinh[\sqrt{c}(x-ct)])}. \tag{248}$$

9.4. *A sech ansatz*

We now use the ansatz

$$u(x,t) = \frac{\operatorname{sech}(\mu(x-ct))}{1 + \lambda \operatorname{sech}(\mu(x-ct))}, \tag{249}$$

where λ and μ are parameters that will be determined. Substituting (249) into (232), and proceeding as before we find

$$\begin{aligned}
\lambda &= \frac{3b \pm \sqrt{9b^2 + 16a^2}}{4a},\\
\mu &= \pm\frac{\sqrt{-9b \pm 3\sqrt{9b^2 + 16a^2}}}{6},\\
c &= \frac{-3b \pm \sqrt{9b^2 + 16a^2}}{12}.
\end{aligned} \tag{250}$$

Substituting (250) into (249) gives the soliton solution

$$\begin{aligned}
u(x,t) = {} & \left[\operatorname{sech}\left(\pm\frac{\sqrt{-9b \pm 3\sqrt{9b^2+16a^2}}}{6}\left(x - \frac{-3b \pm \sqrt{9b^2+16a^2}}{12}t\right)\right)\right]\\
& \times \left[1 + \frac{3b \pm \sqrt{9b^2+16a^2}}{4a}\operatorname{sech}\left(\pm\frac{\sqrt{-9b \pm 3\sqrt{9b^2+16a^2}}}{6}\right.\right.\\
& \left.\left.\times\left(x - \frac{-3b \pm \sqrt{9b^2+16a^2}}{12}t\right)\right)\right]^{-1}.
\end{aligned} \tag{251}$$

9.5. *A csch ansatz*

In a manner parallel to the previous discussion we use the csch ansatz

$$u(x,t) = \frac{\operatorname{csch}(\mu(x-ct))}{1 + \lambda \operatorname{csch}(\mu(x-ct))}, \tag{252}$$

where λ and μ are parameters that will be determined. Substituting (252) into (232), and proceeding as before we find

$$\begin{aligned}
\lambda &= \frac{3b - \sqrt{9b^2 - 16a^2}}{4a},\\
\mu &= \frac{\sqrt{9b + 3\sqrt{9b^2 - 16a^2}}}{6},\\
c &= \frac{3b + \sqrt{9b^2 - 16a^2}}{12}.
\end{aligned} \tag{253}$$

Substituting (253) into (252) yields

$$u(x,t)=\left[\operatorname{csch}\left(\frac{\sqrt{9b+3\sqrt{9b^2-16a^2}}}{6}\left(x-\frac{3b+\sqrt{9b^2-16a^2}}{12}t\right)\right)\right]\times\left[1+\frac{3b-\sqrt{9b^2-16a^2}}{4a}\times\operatorname{csch}\left(\frac{\sqrt{9b+3\sqrt{9b^2-16a^2}}}{6}(x-\frac{3b+\sqrt{9b^2-16a^2}}{12}t)\right)\right]^{-1}. \tag{254}$$

9.6. *A csch–coth ansatz*

We now introduce the ansatz

$$u(x,t)=\alpha+\lambda\operatorname{csch}\big(\mu(x-ct)\big)+\eta\coth\big(\mu(x-ct)\big), \tag{255}$$

where α, λ, η and μ are parameters that will be determined. Substituting (255) into (232), collecting the coefficients of the resulting hyperbolic functions and equating it to zero we find

$$\alpha=\frac{a}{3b},\quad \lambda=\eta=\frac{\sqrt{3(a^2-3bc)}}{3b},\quad \mu=\sqrt{\frac{2(a^2-3bc)}{3b}}. \tag{256}$$

It is clear that real solutions exist only if $a^2>3bc$.

Substituting (256) into (255) yields the solutions

$$u(x,t)=\frac{a}{3b}+\frac{\sqrt{3(a^2-3bc)}}{3b}\operatorname{csch}\left(\sqrt{\frac{2(a^2-3bc)}{3b}}(x-ct)\right)+\frac{\sqrt{3(a^2-3bc)}}{3b}\coth\left(\sqrt{\frac{2(a^2-3bc)}{3b}}(x-ct)\right). \tag{257}$$

9.7. *A sech–tanh ansatz*

We close our analysis by applying the ansatz

$$u(x,t)=\alpha+\lambda\operatorname{sech}\big(\mu(x-ct)\big)+\eta\tanh\big(\mu(x-ct)\big), \tag{258}$$

where α, λ, η and μ are parameters that will be determined. Substituting (258) into (232), and proceeding as before we find

$$\alpha=\frac{a}{3b},\quad \lambda=\frac{\sqrt{3(3bc-a^2)}}{3b},$$

$$\eta = \frac{\sqrt{3(a^2-3bc)}}{3b},$$
$$\mu = \sqrt{\frac{2(a^2-3bc)}{3b}}. \tag{259}$$

Substituting (259) into (258) yields the solutions

$$u(x,t) = \frac{a}{3b} + \frac{\sqrt{3(3bc-a^2)}}{3b}\operatorname{sech}\left(\sqrt{\frac{2(a^2-3bc)}{3b}}(x-ct)\right) + \frac{\sqrt{3(a^2-3bc)}}{3b}\tanh\left(\sqrt{\frac{2(a^2-3bc)}{3b}}(x-ct)\right). \tag{260}$$

It is worth noting that for the constraint $a^2 > 3bc$, the last solution is an imaginary solution given by

$$u(x,t) = \frac{a}{3b} + \mathrm{i}\frac{\sqrt{3(a^2-3bc)}}{3b}\operatorname{sech}\left(\sqrt{\frac{2(a^2-3bc)}{3b}}(x-ct)\right) + \frac{\sqrt{3(a^2-3bc)}}{3b}\tanh\left(\sqrt{\frac{2(a^2-3bc)}{3b}}(x-ct)\right), \quad \mathrm{i}^2 = -1. \tag{261}$$

10. Generalized KdV equation with two power nonlinearities

This section is concerned with the generalized KdV equation with two power nonlinearities of the form

$$u_t + (au^n - bu^{2n})u_x + u_{xxx} = 0. \tag{262}$$

This equation describes the propagation of nonlinear long acoustic-type waves [77]. The function f', where $f = (\frac{a}{n+1}u^{n+1} - \frac{b}{2n+1}u^{2n+1})$ is regarded as a nonlinear correction to the limiting long-wave phase speed c. If the amplitude is not supposed to be small, Eq. (262) serves as an approximate model for the description of weak dispersive effects on the propagation of nonlinear waves along a characteristic direction [78]. It is to be noted that for $n = 1$, Eq. (262) is the well-known Gardner equation that is also called the combined KdV-mKdV equation.

10.1. *Using the tanh method*

We first apply the tanh method presented above on the generalized KdV equation with power-like nonlinearity

$$u_t + (au^n - bu^{2n})u_x + u_{xxx} = 0, \tag{263}$$

that can be converted to the ODE

$$-cu + \frac{a}{n+1}u^{n+1} - \frac{b}{2n+1}u^{2n+1} + u'' = 0, \tag{264}$$

upon using the wave variable $\xi = x - ct$ and integrating once. Balancing u^{2n+1} with u'' in (264) we find

$$M + 2 = (2n+1)M, \tag{265}$$

so that

$$M = \frac{1}{n}. \tag{266}$$

To get analytic closed solution, M should be an integer, therefore we set the transformation

$$u = v^{1/n}. \tag{267}$$

Using (267) into (264) we find

$$\begin{aligned} &-cn^2(2n+1)(n+1)v^2 + an^2(2n+1)v^3 - bn^2(n+1)v^4 \\ &\quad + n(2n+1)(n+1)vv'' + (1-n^2)(2n+1)(v')^2 = 0. \end{aligned} \tag{268}$$

Balancing vv'' with v^4 gives $M = 1$. The tanh method presents the finite expansion

$$v(\xi) = a_0 + a_1 Y. \tag{269}$$

Substituting (269) into (268), collecting the coefficients of Y, and solving the resulting system we find the following set of solutions

$$\begin{aligned} a_0 &= \frac{a(2n+1)}{2b(n+2)}, \\ a_1 &= \pm\frac{a(2n+1)}{2b(n+2)}, \\ \mu &= \pm\frac{an}{2(n+2)}\sqrt{\frac{2n+1}{b(n+1)}}, \end{aligned} \tag{270}$$

In view of this we obtain the kink solutions

$$\begin{aligned} v_1(x,t) = \frac{a(2n+1)}{2b(n+2)}\left(1 \pm \tanh\left[\frac{an}{2(n+2)}\sqrt{\frac{2n+1}{b(n+1)}}\right.\right. \\ \left.\left.\times\left(x - \frac{a^2(2n+1)}{b(n+1)(n+2)^2}t\right)\right]\right), \end{aligned} \tag{271}$$

and the traveling wave solution

$$\begin{aligned} v_2(x,t) = \frac{a(2n+1)}{2b(n+2)}\left(1 \pm \coth\left[\frac{an}{2(n+2)}\sqrt{\frac{2n+1}{b(n+1)}}\right.\right. \\ \left.\left.\times\left(x - \frac{a^2(2n+1)}{b(n+1)(n+2)^2}t\right)\right]\right), \end{aligned} \tag{272}$$

and by using (267) we obtain the kinks solutions for the generalized KdV equation (263) by

$$u_1(x,t) = \left\{\frac{a(2n+1)}{2b(n+2)}\left(1 \pm \tanh\left[\frac{an}{2(n+2)}\sqrt{\frac{2n+1}{b(n+1)}}\right.\right.\right.$$
$$\left.\left.\left.\times\left(x - \frac{a^2(2n+1)}{b(n+1)(n+2)^2}t\right)\right]\right)\right\}^{1/n}, \tag{273}$$

and the traveling wave solution

$$u_2(x,t) = \left\{\frac{a(2n+1)}{2b(n+2)}\left(1 \pm \coth\left[\frac{an}{2(n+2)}\sqrt{\frac{2n+1}{b(n+1)}}\right.\right.\right.$$
$$\left.\left.\left.\times\left(x - \frac{a^2(2n+1)}{b(n+1)(n+2)^2}t\right)\right]\right)\right\}^{1/n}. \tag{274}$$

It is interesting to point out that for $n = 1$, the functions $u_1(x,t)$ and $u_2(x,t)$ are the solutions for the Gardner or the so called KdV-mKdV equation.

10.2. *Using the sine–cosine method*

Substituting the cosine assumption or the sine assumption as presented before, the method works only if $a = 0$ or $b = 0$. In either case, Eq. (263) will be reduced to the generalized KdV equation that was investigated in the previous section.

10.3. *Other hyperbolic functions methods*

It was found in [77] that other hyperbolic functions method can handle Eq. (263) effectively where solitons solutions can be obtained. To achieve this goal, we assume that

$$u(x,t) = \left(\frac{\alpha}{1+\lambda f(\mu\xi)}\right)^{1/n}, \quad \xi = x - ct, \tag{275}$$

where α, λ and μ are parameters that will be determined, and $f(\mu\xi)$ takes anyone of the hyperbolic functions.

Using cosh ansatz We first start our analysis by setting $f(\mu\xi) = \cosh(\mu\xi)$, hence we set

$$u(x,t) = \left(\frac{\alpha}{1+\lambda\cosh(\mu\xi)}\right)^{1/n}, \quad \xi = x - ct. \tag{276}$$

Substituting (276) into (264), collecting the coefficients of like powers of the hyperbolic functions and equating it to zero we obtain

$$\alpha = (n+2)(n+1)\frac{c}{a},$$
$$\mu = n\sqrt{c}, c > 0,$$
$$\lambda = \pm\sqrt{1 - \frac{n^3+5n^2+8n+4}{2n+1} \times \frac{bc}{a^2}}. \tag{277}$$

In view of (277), we obtain the solitons solutions

$$u_3(x,t) = \left(\frac{(n+2)(n+1)c}{a \pm a\sqrt{1 - \frac{n^3+5n^2+8n+4}{2n+1} \times \frac{bc}{a^2}} \cosh\big(n\sqrt{c}\,(x-ct)\big)}\right)^{1/n}, \quad c > 0. \tag{278}$$

However, for $c < 0$, we obtain the solutions

$$u_4(x,t) = \left(\frac{(n+2)(n+1)c}{a \pm a\sqrt{1 - \frac{n^3+5n^2+8n+4}{2n+1} \times \frac{bc}{a^2}} \cos(n\sqrt{-c}(x-ct))}\right)^{1/n}, \quad c < 0. \tag{279}$$

Using sinh ansatz We next use $f(\mu\xi) = \sinh(\mu\xi)$, hence we set

$$u(x,t) = \left(\frac{\alpha}{1+\lambda\sinh(\mu\xi)}\right)^{1/n}, \quad \xi = x - ct. \tag{280}$$

Substituting (280) into (264), collecting the coefficients of like powers of the hyperbolic functions and equating it to zero we obtain

$$\alpha = (n+2)(n+1)\frac{c}{a},$$
$$\mu = n\sqrt{c}, c > 0,$$
$$\lambda = \pm\sqrt{\frac{n^3+5n^2+8n+4}{2n+1} \times \frac{bc}{a^2} - 1}. \tag{281}$$

In view of (281), we obtain the solutions

$$u_5(x,t) = \left(\frac{(n+2)(n+1)c}{a \pm a\sqrt{\frac{n^3+5n^2+8n+4}{2n+1} \times \frac{bc}{a^2} - 1} \sinh(n\sqrt{c}\,(x-ct))}\right)^{1/n}. \tag{282}$$

However, for $c < 0$, we find the solutions

$$u_6(x,t) = \left(\frac{(n+2)(n+1)c}{a \pm a\sqrt{1 - \frac{n^3+5n^2+8n+4}{2n+1} \frac{bc}{a^2}} \sin(n\sqrt{-c}\,(x-ct))}\right)^{1/n}, \quad c < 0. \tag{283}$$

Using tanh and coth ansatze We next use $f(\mu\xi) = \tanh(\mu\xi)$ or $f(\mu\xi) = \coth(\mu\xi)$, hence we set

$$u(x,t) = \left(\frac{\alpha}{1+\lambda\tanh(\mu\xi)}\right)^{1/n}, \quad \xi = x - ct. \tag{284}$$

It was found that the tanh ansatz and the coth ansatz work only for $n = 1$ where Eq. (262) will be reduced to the well-known combined KdV-mKdV equation. Substituting (284) into (264), collecting the coefficients of like powers of the hyperbolic functions and equating it to zero we obtain

$$\begin{aligned}
\alpha &= \frac{6bc-a^2}{ab},\\
\mu &= \frac{1}{2}\sqrt{\frac{a^2-4bc}{2b}}, \quad a^2 > 4bc,\\
\lambda &= \pm\frac{\sqrt{3(a^2-4bc)}}{a}.
\end{aligned} \tag{285}$$

In view of (285), we obtain the solutions

$$u_7(x,t) = \left(\frac{6bc-a^2}{ab \pm b\sqrt{3(a^2-4bc)}\tanh(\frac{1}{2}\sqrt{\frac{a^2-4bc}{2b}}\,(x-ct))}\right)^{1/n}. \tag{286}$$

In a like manner, we can determine other traveling wave solutions in the form

$$u_8(x,t) = \left(\frac{6bc-a^2}{ab \pm b\sqrt{3(a^2-4bc)}\coth(\frac{1}{2}\sqrt{\frac{a^2-4bc}{2b}}\,(x-ct))}\right)^{1/n}. \tag{287}$$

However, for $a^2 < 4bc$, we obtain the solutions that blow up at its domain of validity

$$u_9(x,t) = \left(\frac{6bc-a^2}{ab \pm b\sqrt{3(4bc-a^2)}\tan(\frac{1}{2}\sqrt{\frac{4bc-a^2}{2b}}\,(x-ct))}\right)^{1/n}, \tag{288}$$

$$u_{10}(x,t) = \left(\frac{6bc-a^2}{ab \pm b\sqrt{3(4bc-a^2)}\cot(\frac{1}{2}\sqrt{\frac{4bc-a^2}{2b}}\,(x-ct))}\right)^{1/n}. \tag{289}$$

Using sech and csch ansatze We next consider

$$f(\mu\xi) = \operatorname{sech}(\mu\xi) \quad \text{or} \quad f(\mu\xi) = \operatorname{csch}(\mu\xi),$$

hence we set

$$u(x,t) = \left(\frac{\alpha}{1+\lambda\operatorname{sech}(\mu\xi)}\right)^{1/n}, \quad \xi = x - ct. \tag{290}$$

The sech and the csch ansatze work only for $n = 1$ where Eq. (56) is the well-known combined KdV-mKdV equation. Substituting (290) into (264), collecting the coefficients of like powers of the hyperbolic functions and equating it to zero we obtain

$$\alpha = \frac{5a^2 + \sqrt{a^2(25a^2 - 96bc)}}{8ab},$$
$$\mu = \frac{\sqrt{2b(5a^2 - 16bc + \sqrt{a^2(25a^2 - 96bc)})}}{8b},$$
$$\lambda = \pm\frac{\sqrt{2(a^2 - 6bc)(\sqrt{a^2(25a^2 - 96bc)} - 24bc + 7a^2)}}{2(a^2 - 6bc)}. \tag{291}$$

In view of (291), we obtain the solitons solutions

$$u_{11}(x,t) = \left(\frac{(5a^2 + \sqrt{a^2(25a^2 - 96bc)})/(8ab)}{1 \pm \lambda\,\mathrm{sech}(\mu(x - ct))}\right)^{1/n}. \tag{292}$$

On the other hand, using the csch instead of the sech, and proceeding as before we obtain

$$u_{12}(x,t) = \left(\frac{(5a^2 + \sqrt{a^2(25a^2 - 96bc)})/(8ab)}{1 \pm \lambda\,\mathrm{csch}(\mu(x - ct))}\right)^{1/n}, \tag{293}$$

where λ and μ are defined above in (291).

11. Fifth-order KdV equation

The fifth-order KdV equation appears in many forms. In this section we will study a class of fifth-order KdV equation (fKdV) of the form [19,18,31,32,34–36,57]

$$u_t + \alpha u^2 u_x + \beta u_x u_{xx} + \gamma u u_{3x} + u_{5x} = 0, \tag{294}$$

where α, β, and γ are arbitrary nonzero and real parameters, and $u = u(x,t)$ is a sufficiently-often differentiable function. The tanh–coth method will be used to study this equation in a manner parallel to the preceding discussions. The multiple-soliton solutions will be investigated as well by using the sense of Hirota's bilinear formalism. The fKdV equation (294) describes motions of long waves in shallow water under gravity and in a one-dimensional nonlinear lattice, and has wide applications in quantum mechanics and nonlinear optics. It is well known that wave phenomena of plasma media and fluid dynamics are modeled by kink shaped tanh solution or by bell shaped sech solutions.

The parameters α, β, and γ are arbitrary that will drastically change the characteristics of the fKdV equation (294). Many forms of the fKdV equation can be constructed by changing these parameters. However, four well known forms of the fKdV that are of particular interest in the literature. These forms are:

(i) The Lax equation [36] is given by

$$u_t + 30u^2 u_x + 20u_x u_{xx} + 10uu_{3x} + u_{5x} = 0, \tag{295}$$

and characterized by

$$\beta = 2\gamma, \qquad \alpha = \frac{3}{10}\gamma^2. \tag{296}$$

(ii) The Sawada–Kotera (SK) [57] equations is given by

$$u_t + 5u^2u_x + 5u_xu_{xx} + 5uu_{3x} + u_{5x} = 0 \tag{297}$$

and characterized by

$$\beta = \gamma, \qquad \alpha = \frac{1}{5}\gamma^2. \tag{298}$$

(iii) The Kaup–Kupershmidt (KK) [32,35] equation

$$u_t + 20u^2u_x + 25u_xu_{xx} + 10uu_{3x} + u_{5x} = 0 \tag{299}$$

is characterized by

$$\beta = \frac{5}{2}\gamma, \qquad \alpha = \frac{1}{5}\gamma^2. \tag{300}$$

(iv) The Ito equation [31]

$$u_t + 2u^2u_x + 6u_xu_{xx} + 3uu_{3x} + u_{5x} = 0, \tag{301}$$

is characterized by

$$\beta = 2\gamma, \qquad \alpha = \frac{2}{9}\gamma^2. \tag{302}$$

It was found that the Lax, SK and KK equations belong to the completely integrable hierarchy of higher-order KdV equations. These three equations have infinite sets of conservation laws [19,18]. However, the Ito equation is not completely integrable but has a limited number of special conservation laws.

11.1. *Using the tanh–coth method*

We begin our analysis by rewriting (294) as

$$u_t + \frac{\alpha}{3}(u^3)_x + \gamma(uu_{xx})_x + \frac{\beta-\gamma}{2}\big((u_x)^2\big)_x + u_{5x} = 0, \tag{303}$$

that can be converted to the ODE

$$-cu + \frac{\alpha}{3}\mu u^3 + \gamma\mu^3uu'' + \frac{\beta-\gamma}{2}\mu^3(u')^2 + \mu^5u^{(\mathrm{iv})} = 0, \tag{304}$$

upon using the wave variable $\xi = \mu x - ct$ and integrating once. Balancing the terms $u^{(\mathrm{iv})}$ with u^3 in (264) we find

$$M + 4 = 3M, \tag{305}$$

so that $M = 2$. Using the tanh–coth method presented above we set

$$u(\xi) = a_0 + a_1Y + a_2Y^2 + b_1Y^{-1} + b_2Y^{-2}. \tag{306}$$

Substituting (306) into (304), collecting the coefficients of Y, and solving the resulting system we find the following four sets of solutions [75]

(1) The *first set* of parameters is given by

$$\begin{aligned}
&a_0 = -\frac{2}{3}a_2, \quad a_1 = b_1 = b_2 = 0,\\
&c = -\frac{2}{3}\beta\mu^3 a_2 - 24\mu^5,\\
&\alpha = -\frac{6\mu^2(60\mu^2 + \beta a_2 + 2\gamma a_2)}{a_2^2}.
\end{aligned} \tag{307}$$

(2) The *second set* of parameters is given by

$$\begin{aligned}
&a_0 = A, \quad A \text{ is a constant}, \quad a_1 = b_1 = b_2 = 0, \quad a_2 = -\frac{60\mu^2}{\beta+\gamma},\\
&c = \frac{\mu[\gamma(\beta+\gamma)^2 a_0^2 - 80\gamma\mu^2(\beta+\gamma)a_0 + 80\mu^4(2\beta + 17\gamma)]}{10(\beta+\gamma)},\\
&\alpha = \frac{\gamma(\beta+\gamma)}{10},
\end{aligned} \tag{308}$$

(3) The *third set* of parameters is given by

$$\begin{aligned}
&a_0 = -\frac{2}{3}a_2, \quad a_1 = b_1 = 0, \quad b_2 = a_2,\\
&c = -\frac{32}{3}\beta\mu^3 a_2 - 384\mu^5,\\
&\alpha = -\frac{6\mu^2(60\mu^2 + \beta a_2 + 2\gamma a_2)}{a_2^2}.
\end{aligned} \tag{309}$$

(4) The *fourth set* of parameters is given by

$$\begin{aligned}
&a_0 = A, \quad A \text{ is a constant}, \quad a_1 = b_1 = 0, \quad a_2 = -\frac{60\mu^2}{\beta+\gamma},\\
&b_2 = -\frac{60\mu^2}{\beta+\gamma},\\
&c = \frac{\mu[\gamma(\beta+\gamma)^2 a_0^2 - 80\gamma\mu^2(\beta+\gamma)a_0 + 320\mu^4(8\beta - 7\gamma)]}{10(\beta+\gamma)},\\
&\alpha = \frac{\gamma(\beta+\gamma)}{10}.
\end{aligned} \tag{310}$$

11.2. *The first criterion*

The first and the third sets of parameters are expressed in terms of μ and a_2. It is normal to examine the result obtained for α from these sets where we find

$$\alpha = -\frac{6\mu^2(60\mu^2 + (\beta + 2\gamma)a_2)}{a_2^2}, \tag{311}$$

that gives

$$\alpha a_2^2 + 6\mu^2(\beta + 2\gamma)a_2 + 360\mu^4 = 0. \tag{312}$$

This quadratic equation has real solutions only if

$$\big(6\mu^2(\beta + 2\gamma)\big)^2 \geqslant 1440\alpha\mu^4, \tag{313}$$

that gives the first criterion, that we are seeking, given by

$$\alpha \leqslant \frac{(\beta + 2\gamma)^2}{40}. \tag{314}$$

The criterion (314) enables us to use several real values for α, even for fixed values of the parameters β and γ. In what follows we will derive solitons solutions for all forms of fifth-order KdV that were presented above.

It is important to point out that the solitons solutions obtained by using the first two sets of parameters are examined and reported in [75]. It is normal here to examine the new solitons solutions that will be derived from the third set given in (309).

11.3. *Using the first criterion*

The Lax equation Lax [36] considered the case where $\beta = 20$ and $\gamma = 10$. Using criterion (314) then $\alpha \leqslant 40$. Consequently, Lax considered $\alpha = 30$ that meets the first criterion. We first determine a_2 by using (311). Substituting these parameters in the third set (309) gives

$$\begin{aligned}
&a_2 = b_2 = -2\mu^2, -6\mu^2,\\
&a_1 = b_1 = 0,\\
&a_0 = \frac{4}{3}\mu^2, 4\mu^2,\\
&c = \frac{128}{3}\mu^5, 896\mu^5.
\end{aligned} \tag{315}$$

This in turn gives the solutions

$$\begin{aligned}
u_1(x,t) = \frac{4}{3}\mu^2 - 2\mu^2\tanh^2\left(\mu x - \frac{128}{3}\mu^5 t\right)&\\
- 2\mu^2\coth^2\left(\mu x - \frac{128}{3}\mu^5 t\right)&,
\end{aligned} \tag{316}$$

$$u_2(x,t) = 4\mu^2 - 6\mu^2\tanh^2(\mu x - 896\mu^5 t) - 6\mu^2\coth^2(\mu x - 896\mu^5 t), \tag{317}$$

where μ is a nonzero real parameter.

The Sawada–Kotera (SK) equation Sawada and Kotera [57] investigated the fKdV equation for $\beta = 5$ and $\gamma = 5$, hence given by

$$u_t + 5u^2u_x + 5u_xu_{xx} + 5uu_{3x} + u_{5x} = 0, \tag{318}$$

so that $\alpha = 5$ that justifies the first criterion (314).

Following the previous section, we first determine a_2 by using (311). Substituting these values for α, β and γ in the third set (309) we find

$$\begin{aligned}
&a_2 = b_2 = -6\mu^2, -12\mu^2,\\
&a_1 = b_1 = 0,\\
&a_0 = 4\mu^2, 8\mu^2,\\
&c = -64\mu^5, 256\mu^5.
\end{aligned} \tag{319}$$

This in turn gives the two solutions

$$u_1(x,t) = 4\mu^2 - 6\mu^2\tanh^2(\mu x + 64\mu^5 t) - 6\mu^2\coth^2(\mu x + 64\mu^5 t), \tag{320}$$

and

$$\begin{aligned}
u_2(x,t) = {}& 8\mu^2 - 12\mu^2\tanh^2(\mu x - 256\mu^5 t)\\
&- 12\mu^2\coth^2(\mu x - 256\mu^5 t),
\end{aligned} \tag{321}$$

where μ is a nonzero real free parameter.

The Kaup–Kupershmidt (KK) equation Kaup and Kupershmidt [32,35] studied the case where $\beta = 25$ and $\gamma = 10$ that justifies criterion (314). The KK equation is given by

$$u_t + 20u^2u_x + 25u_xu_{xx} + 10uu_{3x} + u_{5x} = 0. \tag{322}$$

Proceeding as before we find

$$\begin{aligned}
&a_2 = b_2 = -\frac{3}{2}\mu^2, -12\mu^2,\\
&a_1 = b_1 = 0,\\
&a_0 = \mu^2, 8\mu^2,\\
&c = 16\mu^5, 2816\mu^5.
\end{aligned} \tag{323}$$

This in turn gives the solutions

$$u_1(x,t) = \mu^2 - \frac{3}{2}\mu^2\tanh^2(\mu x - 16\mu^5 t) - \frac{3}{2}\mu^2\coth^2(\mu x - 16\mu^5 t), \tag{324}$$

$$\begin{aligned}
u_2(x,t) = {}& 8\mu^2 - 12\mu^2\tanh^2(\mu x - 2816\mu^5 t)\\
&- 12\mu^2\coth^2(\mu x - 2816\mu^5 t).
\end{aligned} \tag{325}$$

The Ito equation Ito [31] used $\beta = 6$ and $\gamma = 3$. Using the criteria set in (314) then $\alpha \leqslant 3.6$. Consequently, $\alpha = 2$ was used in the Ito equation given by

$$u_t + 2u^2 u_x + 6u_x u_{xx} + 3uu_{3x} + u_{5x} = 0. \tag{326}$$

Substituting these parameters in the first set we find

$$\begin{aligned} &a_2 = b_2 = -6\mu^2, -30\mu^2, \\ &a_1 = b_1 = 0, \\ &a_0 = 4\mu^2, 20\mu^2, \\ &c = 0, 1536\mu^5. \end{aligned} \tag{327}$$

This in turn gives the solutions

$$\begin{aligned} u_1(x,t) = {} & 20\mu^2 - 30\mu^2 \tanh^2(\mu x - 1536\mu^5 t) \\ & - 30\mu^2 \coth^2(\mu x - 1536\mu^5 t), \end{aligned} \tag{328}$$

and the solutions

$$u_2(x) = 4\mu^2 - 6\mu^2 \tanh^2(\mu x) - 6\mu^2 \coth^2(\mu x). \tag{329}$$

Notice that the second set of solutions is independent of time t. Unlike the other forms of the fKdV equations where we obtained two pairs of distinct traveling wave solutions, the Ito equation gave two solutions where the second solution does not depend on time t.

11.4. *The second criterion*

In (310), we derived the following set

$$\begin{aligned} &a_0 = A, \quad A \text{ is a constant}, \quad a_1 = b_1 = 0, \quad a_2 = -\frac{60\mu^2}{\beta+\gamma}, \quad b_2 = -\frac{60\mu^2}{\beta+\gamma}, \\ &c = \frac{\mu[\gamma(\beta+\gamma)^2 a_0^2 - 80\gamma\mu^2(\beta+\gamma)a_0 + 320\mu^4(8\beta - 7\gamma)]}{10(\beta+\gamma)}, \\ &\alpha = \frac{\gamma(\beta+\gamma)}{10} \end{aligned} \tag{330}$$

as a fourth set of values for the parameters $a_0, a_1, a_2, b_1, b_2, c$ and α. It is obvious from this set that, unlike the first set where we have an infinite values for α defined by an inequality, instead we have a unique value for α for fixed values of β and γ as shown above. This fixed value for α is only justified for Lax and SK equations. A modification for values of α should be set for KK and Ito equations to obtain solutions for variants of these equations. It is obvious that only one soliton solution will be obtained for Lax and the SK equations.

11.5. *Using the second criterion*

The Lax equation Lax [36] considered the case where $\beta = 20$ and $\gamma = 10$. Using (330) we find

$$\begin{aligned}
&a_0 = a_0, \quad a_0 \text{ is an arbitrary constant,}\\
&a_1 = b_1 = 0,\\
&a_2 = b_2 = -2\mu^2,\\
&c = 2\mu\big(48\mu^4 - 40a_0\mu^2 + 15a_0^2\big),
\end{aligned} \tag{331}$$

where μ is left as a free parameter. This in turn gives the solution

$$\begin{aligned}
u(x,t) = a_0 &- 2\mu^2 \tanh^2\big(\mu x - 2\mu\big(48\mu^4 - 40a_0\mu^2 + 15a_0^2\big)t\big)\\
&- 2\mu^2 \coth^2\big(\mu x - 2\mu\big(48\mu^4 - 40a_0\mu^2 + 15a_0^2\big)t\big).
\end{aligned} \tag{332}$$

Selecting $a_0 = \mu^2$ we obtain the solutions

$$u(x,t) = \mu^2 - 2\mu^2 \tanh^2(\mu x - 46\mu^5 t) - 2\mu^2 \coth^2(\mu x - 46\mu^5 t). \tag{333}$$

The SK equation Substituting $\beta = 5$ and $\gamma = 5$ in the set (330) we find

$$\begin{aligned}
&a_0 = a_0, \quad a_0 \text{ is an arbitrary constant,}\\
&a_2 = b_2 = -6\mu^2,\\
&a_1 = b_1 = 0,\\
&c = \mu\big(16\mu^4 - 40a_0\mu^2 + 5a_0^2\big).
\end{aligned} \tag{334}$$

This in turn gives the solution

$$\begin{aligned}
u(x,t) = a_0 &- 6\mu^2 \tanh^2\big(\mu x - \mu\big(16\mu^4 - 40a_0\mu^2 + 5a_0^2\big)t\big)\\
&- 6\mu^2 \coth^2\big(\mu x - \mu\big(16\mu^4 - 40a_0\mu^2 + 5a_0^2\big)t\big).
\end{aligned} \tag{335}$$

Selecting $a_0 = \mu^2$ gives the solution

$$u(x,t) = \mu^2 - 6\mu^2 \tanh^2(\mu x + 19\mu^5 t) - 6\mu^2 \coth^2(\mu x + 19\mu^5 t). \tag{336}$$

11.6. *Multiple-solitons of the fifth-order KdV equation*

In this section, we will examine multiple-solitons solution of the fifth-order KdV equation. As stated before, Hirota [24–29] proposed a bilinear form where it was shown that soliton solutions are just polynomials of exponentials.

Hereman et al. [18] introduced a simplified version of Hirota method, where exact solitons can be obtained by solving a perturbation scheme using a symbolic manipulation package, and without any need to use bilinear forms. In what follows, we summarize the main steps of the simplified version of Hirota's method. To achieve our goal, we follow the approach used in [18].

We first substitute

$$u(x,t) = R\frac{\partial^2 \ln f(x,t)}{\partial x^2} = R\frac{ff_{2x} - (f_x)^2}{f^2}, \tag{337}$$

into (294), where the auxiliary function $f = 1 + \exp(\theta), \theta = kx - wt$, and solving the equation we get

$$\begin{aligned} \alpha &= \frac{\gamma^2 + \gamma\beta}{10}, \\ R &= \frac{60}{\gamma + \beta}. \end{aligned} \tag{338}$$

The Lax equation For Lax equation, $R = 2$, therefor we use the transformation

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2} = 2\frac{ff_{2x} - (f_x)^2}{f^2}, \tag{339}$$

that will carry out the Lax equation (295) into a cubic equation in f given by

$$\begin{aligned} &f^2(f_{xt} + f_{6x}) - f(f_x f_t + 6f_x f_{5x} - 5f_{2x}f_{4x}) \\ &\quad + 10\big(f_x^2 f_{4x} - 2f_x f_{2x} f_{3x} + f_{2x}^3\big) = 0, \end{aligned} \tag{340}$$

that can be decomposed into linear operator and two nonlinear operators.

Proceeding as before, we assume that $f(x,t)$ has a perturbation expansion of the form

$$f(x,t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f_n(x,t), \tag{341}$$

where ϵ is a nonsmall formal expansion parameter. Following the simplified version of Hirota's method [18], we substitute (341) into (340) and equate to zero the powers of ϵ.

The N-soliton solution is obtained from

$$f_1 = \sum_{i=1}^{N} \exp(\theta_i) = \sum_{i=1}^{N} \exp(k_i x - c_i t), \tag{342}$$

where

$$\theta_i = k_i x - c_i t, \tag{343}$$

where k_i and c_i are arbitrary constants. Substituting (342) into (340), and equate the coefficients of ϵ^1 to zero, we obtain the dispersion relation

$$c_i = k_i^5, \tag{344}$$

and in view of this result we obtain

$$\theta_i = k_i x - k_i^5 t. \tag{345}$$

This means that

$$f_1 = \exp(\theta_1) = \exp\big(k_1\big(x - k_1^4 t\big)\big), \tag{346}$$

obtained by using $N = 1$ in (342).

Consequently, for the one-soliton solution, we set

$$f = 1 + \exp(\theta_1) = 1 + \exp\bigl(k_1\bigl(x - k_1^4 t\bigr)\bigr), \tag{347}$$

where we set $\epsilon = 1$. The one soliton solution is therefore

$$u(x,t) = \frac{2k_1^2 \exp(k_1(x - k_1^4 t))}{(1 + \exp(k_1(x - k_1^4 t)))^2}, \tag{348}$$

or equivalently

$$u(x,t) = \frac{k_1^2}{2} \operatorname{sech}^2\left[\frac{k_1}{2}\bigl(x - k_1^4 t\bigr)\right]. \tag{349}$$

To determine the two-soliton solution, we first set $N = 2$ in (342) to get

$$f_1 = \exp(\theta_1) + \exp(\theta_2). \tag{350}$$

To determine f_2, we set

$$f_2 = \sum_{1 \leqslant i < j \leqslant N} a_{ij} \exp(\theta_i + \theta_j), \tag{351}$$

and therefore we substitute

$$f = 1 + \exp(\theta_1) + \exp(\theta_2) + a_{12} \exp(\theta_1 + \theta_2), \tag{352}$$

into (340) and proceed as before to obtain the phase factor a_{12} by

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \tag{353}$$

and hence

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad 1 \leqslant i < j \leqslant N. \tag{354}$$

This in turn gives

$$f = 1 + \mathrm{e}^{k_1(x - k_1^4 t)} + \mathrm{e}^{k_2(x - k_2^4 t)} + \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} \mathrm{e}^{(k_1 + k_2)x - (k_1^5 + k_2^5)t}. \tag{355}$$

To determine the two-soliton solutions explicitly, we use (339) for the function f in (355).

Similarly, we can determine f_3. Proceeding as before, we therefore set

$$\begin{aligned} f_1(x,t) &= \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3), \\ f_2(x,t) &= a_{12} \exp(\theta_1 + \theta_2) + a_{23} \exp(\theta_2 + \theta_2) + a_{13} \exp(\theta_1 + \theta_3), \end{aligned} \tag{356}$$

and accordingly we have

$$\begin{aligned} f(x,t) = {} & 1 + \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3) \\ & + a_{12} \exp(\theta_1 + \theta_2) + a_{23} \exp(\theta_2 + \theta_3) + a_{13} \exp(\theta_1 + \theta_3) \\ & + f_3(x,t). \end{aligned} \tag{357}$$

Substituting (357) into (340) and proceeding as before we find

$$f_3 = b_{123}\exp(\theta_1 + \theta_2 + \theta_3), \tag{358}$$

where

$$b_{123} = a_{12}a_{13}a_{23} = \frac{(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^2}, \tag{359}$$

and θ_1, θ_2 and θ_3 are given above in (345). For the three-soliton solution we use $1 \leqslant i < j \leqslant 3$, we therefore obtain

$$\begin{aligned} f &= 1 + \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3) \\ &\quad + a_{12}\exp(\theta_1+\theta_2) + a_{13}\exp(\theta_1+\theta_3) + a_{23}\exp(\theta_2+\theta_3) \\ &\quad + b_{123}\exp(\theta_1+\theta_2+\theta_3), \end{aligned} \tag{360}$$

where

$$\begin{aligned} a_{12} &= \frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \quad a_{13} = \frac{(k_1-k_3)^2}{(k_1+k_3)^2}, \\ a_{23} &= \frac{(k_2-k_3)^2}{(k_2+k_3)^2}, \quad b_{123} = a_{12}a_{13}a_{23}. \end{aligned} \tag{361}$$

This in turn gives

$$\begin{aligned} f &= 1 + \mathrm{e}^{k_1(x-k_1^4 t)} + \mathrm{e}^{k_2(x-k_2^4 t)} + \mathrm{e}^{k_3(x-k_3^4 t)} \\ &\quad + \frac{(k_1-k_2)^2}{(k_1+k_2)^2}\mathrm{e}^{(k_1+k_2)x-(k_1^5+k_2^5)t} + \frac{(k_1-k_3)^2}{(k_1+k_3)^2}\mathrm{e}^{(k_1+k_3)x-(k_1^5+k_3^5)t} \\ &\quad + \frac{(k_2-k_3)^2}{(k_2+k_3)^2}\mathrm{e}^{(k_2+k_3)x-(k_2^5+k_3^5)t} \\ &\quad + \frac{(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^2}\mathrm{e}^{(k_1+k_2+k_3)x-(k_1^5+k_2^5+k_3^5)t}. \end{aligned} \tag{362}$$

To determine the three-solitons solution explicitly, we use (339) for the function f in (362). The higher level soliton solution can be obtained in a parallel manner. This indeed requires a tedious work.

As stated before, the Lax equation is characterized by

$$\beta = 2\gamma, \qquad \alpha = \frac{3}{10}\gamma^2, \tag{363}$$

where γ is any arbitrary constant, then the transformation (339) can be generalized to

$$u = \frac{20}{\gamma}\big(\ln\big(f(x,t)\big)\big)_{xx}, \tag{364}$$

that works for every γ.

We again summarize the three facts presented before:

(i) the first is that soliton solutions are just polynomials of exponentials as emphasized by Hirota [24–29], and

(ii) the three-soliton solution and the higher level soliton solution as well, do not contain any new free parameters other than a_{ij} derived for the two-soliton solution.
(iii) every solitonic equation that has generic $N = 3$ soliton solutions, then it has also soliton solutions for any $N \geqslant 4$ [20–23].

The Sawada–Kotera equation For the SK equation, $R = 6$, therefor the transformation

$$u(x,t) = 6\frac{\partial^2 \ln f(x,t)}{\partial x^2} = 6\frac{f f_{2x} - (f_x)^2}{f^2}, \tag{365}$$

that will carry out the SK equation (297) into a quadratic equation in f given by

$$f(f_{xt} + f_{6x}) + (15 f_{2x} f_{4x} - 10 f_{3x}^2 - 6 f_x f_{5x} - f_x f_t) = 0, \tag{366}$$

that can be decomposed into linear operator and a nonlinear operator.

Following the discussions introduced before, we assume that $f(x,t)$ has a perturbation expansion of the form

$$f(x,t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f_n(x,t). \tag{367}$$

Substituting (367) into (366) and equate to zero [18] the powers of ϵ.

The N-soliton solution is obtained from

$$f_1 = \sum_{i=1}^{N} \exp(\theta_i) = \sum_{i=1}^{N} \exp(k_i x - c_i t), \tag{368}$$

where

$$\theta_i = k_i x - c_i t. \tag{369}$$

Substituting (368) into (366) and equate the coefficients of ϵ^1 to zero, we obtain the dispersion relation

$$c_i = k_i^5, \tag{370}$$

and in view of this result we obtain

$$\theta_i = k_i x - k_i^5 t. \tag{371}$$

This means that

$$f_1 = \exp(\theta_1) = \exp\bigl(k_1\bigl(x - k_1^4 t\bigr)\bigr), \tag{372}$$

obtained by using $N = 1$ in (368).

Consequently, for the one-soliton solution, we set

$$f = 1 + \exp(\theta_1) = 1 + \exp\bigl(k_1\bigl(x - k_1^4 t\bigr)\bigr), \tag{373}$$

where we set $\epsilon = 1$. The one soliton solution is therefore

$$u(x,t) = \frac{6k_1^2 \exp(k_1(x - k_1^4 t))}{(1 + \exp(k_1(x - k_1^4 t)))^2}, \tag{374}$$

or equivalently

$$u(x,t) = \frac{3}{2}k_1^2 \operatorname{sech}^2\left[\frac{k_1}{2}\left(x - k_1^4 t\right)\right]. \tag{375}$$

To determine the two-soliton solution, we first set $N = 2$ in (368) to get

$$f_1 = \exp(\theta_1) + \exp(\theta_2). \tag{376}$$

To determine f_2, we set

$$f_2 = \sum_{1 \leqslant i < j \leqslant N} a_{ij} \exp(\theta_i + \theta_j), \tag{377}$$

and therefore we substitute

$$f = 1 + \exp(\theta_1) + \exp(\theta_2) + a_{12} \exp(\theta_1 + \theta_2), \tag{378}$$

into (366) and proceed as before to obtain the phase factor a_{12} by

$$a_{12} = \frac{(k_1 - k_2)^2 (k_1^2 - k_1 k_2 + k_2^2)}{(k_1 + k_2)^2 (k_1^2 + k_1 k_2 + k_2^2)}, \tag{379}$$

and hence

$$a_{ij} = \frac{(k_i - k_j)^2 (k_i^2 - k_i k_j + k_j^2)}{(k_i + k_j)^2 (k_i^2 + k_i k_j + k_j^2)}, \quad 1 \leqslant i < j \leqslant N. \tag{380}$$

This in turn gives

$$\begin{aligned} f = {} & 1 + \mathrm{e}^{k_1(x - k_1^4 t)} + \mathrm{e}^{k_2(x - k_2^4 t)} \\ & + \frac{(k_1 - k_2)^2 (k_1^2 - k_1 k_2 + k_2^2)}{(k_1 + k_2)^2 (k_1^2 + k_1 k_2 + k_2^2)} \mathrm{e}^{(k_1 + k_2)x - (k_1^5 + k_2^5)t}. \end{aligned} \tag{381}$$

To determine the two-solitons solution explicitly, we use (365) for the function f in (381).

Similarly, we can determine f_3. Proceeding as before, we therefore set

$$\begin{aligned} & f_1(x,t) = \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3), \\ & f_2(x,t) = a_{12} \exp(\theta_1 + \theta_2) + a_{23} \exp(\theta_2 + \theta_2) + a_{13} \exp(\theta_1 + \theta_3), \end{aligned} \tag{382}$$

and accordingly we have

$$\begin{aligned} f(x,t) = {} & 1 + \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3) \\ & + a_{12} \exp(\theta_1 + \theta_2) + a_{23} \exp(\theta_2 + \theta_3) + a_{13} \exp(\theta_1 + \theta_3) \\ & + f_3(x,t). \end{aligned} \tag{383}$$

Substituting (383) into (366) and proceeding as before we find

$$f_3 = b_{123} \exp(\theta_1 + \theta_2 + \theta_3), \tag{384}$$

where

$$b_{123} = a_{12} a_{13} a_{23}. \tag{385}$$

For the three-soliton solution we use $1 \leqslant i < j \leqslant 3$, we proceed as before to get

$$\begin{aligned} f &= 1 + e^{k_1(x-k_1^4 t)} + e^{k_2(x-k_2^4 t)} + e^{k_3(x-k_3^4 t)} \\ &\quad + a_{12} e^{(k_1+k_2)x-(k_1^5+k_2^5)t} + a_{13} e^{(k_1+k_3)x-(k_1^5+k_3^5)t} + a_{23} e^{(k_2+k_3)x-(k_2^5+k_3^5)t} \\ &\quad + b_{123} e^{(k_1+k_2+k_3)x-(k_1^5+k_2^5+k_3^5)t} \end{aligned} \tag{386}$$

where a_{ij} and b_{123} are defined above in (380) and (385) respectively. To determine the three-solitons solution explicitly, we use (366) for the function f in (386). The higher level soliton solution can be obtained in a parallel manner.

As stated before, the Sawada–Kotera equation is characterized by

$$\beta = \gamma, \qquad \alpha = \frac{1}{5}\gamma^2, \tag{387}$$

where γ is any arbitrary constant, then the transformation (365) can be generalized to

$$u = \frac{30}{\gamma}\big(\ln\big(f(x,t)\big)\big)_{xx}, \tag{388}$$

that works for every γ.

The Kaup–Kupershmidt equation For the KK equation we only summarize the work in [19,18], where the transformation

$$u(x,t) = \frac{3}{2}\frac{\partial^2 \ln f(x,t)}{\partial x^2} = \frac{3}{2}\frac{f f_{2x} - (f_x)^2}{f^2}, \tag{389}$$

is used.

The dispersion relation is given by

$$\theta_i = k_i x - k_i^5 t. \tag{390}$$

For the one-soliton solution the following function

$$f = 1 + \exp(\theta_1) + \frac{1}{16}\exp(2\theta_1), \tag{391}$$

so that the one soliton solution is

$$u(x,t) = \frac{24k1^2 e^{(k1(-tk1^4+x))}(16 + 4e^{(k1(-tk1^4+x))} + e^{(2k1(-tk1^4+x))})}{(16 + 16e^{(k1(-tk1^4+x))} + e^{(2k1(-tk1^4+x))})^2}. \tag{392}$$

For the two-soliton solution it was found that

$$\begin{aligned} f &= 1 + \exp(\theta_1) + \exp(\theta_2) + \frac{1}{16}\exp(2\theta_1) + \frac{1}{16}\exp(2\theta_2) \\ &\quad + a_{12}\exp(\theta_1+\theta_2) + b_{12}\big[\exp(2\theta_1+\theta_2) + \exp(\theta_1+2\theta_2)\big] \\ &\quad + b_{12}^2\exp(2\theta_1+2\theta_2), \end{aligned} \tag{393}$$

where

$$a_{12} = \frac{2k_1^4 - k_1^2 k_2^2 + 2k_2^4}{2(k_1+k_2)^2(k_1^2 + k_1 k_2 + k_2^2)}, \tag{394}$$

and

$$b_{12} = \frac{(k_1 - k_2)^2(k_1^2 - k_1k_2 + k_2^2)}{16(k_1 + k_2)^2(k_1^2 + k_1k_2 + k_2^2)}. \tag{395}$$

It is obvious that the two soliton solution $u(x, t)$ can be obtained by substituting (393) into (389). For the higher level solitons solution, it becomes more complicated and readers are advised to read [18].

It is well known that the Kaup–Kupershmidt equation is characterized by

$$\beta = \frac{5}{2}\gamma, \qquad \alpha = \frac{1}{5}\gamma^2, \tag{396}$$

where γ is any arbitrary constant, then the transformation (389) can be generalized to

$$u = \frac{15}{\gamma}\big(\ln\big(f(x, t)\big)\big)_{xx}, \tag{397}$$

that works for every γ.

12. Seventh-order KdV equation

The seventh-order KdV equation (sKdV) [14,43,53]

$$u_t + 6uu_x + u_{3x} - u_{5x} + \alpha u_{7x} = 0, \tag{398}$$

where α is a nonzero constant, and $u = u(x, t)$ is a sufficiently often differentiable function. The sech method used in [14,43] will be used to study this equation. The sKdV equation (398) has been introduced by Pomeau et al. [53], and then investigated by [13,43], for discussing the structural stability of the KdV equation under singular perturbation. The sKdV equation possesses the dispersion term u_{3x} and two higher order dispersion terms, namely, u_{5x} and u_{7x}. Moreover, Eq. (398) has three polynomial type conserved quantities given by:

$$\begin{aligned}
I_1 &= \int_{-\infty}^{\infty} u\,\mathrm{d}x, \\
I_2 &= \int_{-\infty}^{\infty} u^2\,\mathrm{d}x, \\
I_3 &= \int_{-\infty}^{\infty} \left(-u^3 + \frac{1}{2}(u_x)^2 - \frac{1}{2}(u_{xx})^2 + \frac{1}{2}\alpha(u_{3x})^2\right)\mathrm{d}x.
\end{aligned} \tag{399}$$

12.1. *The sech method*

We begin our analysis by rewriting (398) as

$$-cu + 3u^2 + u'' - u^{(\mathrm{iv})} + \alpha u^{(\mathrm{vi})} = 0, \tag{400}$$

by using the wave variable $\xi = \mu(x - ct)$ and integrating once. Balancing the terms $u^{(vi)}$ with u^2 in (400) we find

$$M + 6 = 2M, \tag{401}$$

so that $M = 6$. Following [14,43], we assume that the solution is of the form

$$u(\xi) = a_0 + a_1 \operatorname{sech}^6(\mu\xi). \tag{402}$$

Substituting (402) into (400), collecting the coefficients of sech^j, and solving the resulting system we find the following two sets of solutions

(1) The *first set* of parameters is given by

$$\begin{aligned} &a_0 = 0, \quad a_1 = \frac{86625}{591361}, \quad c = \frac{180000}{591361}, \\ &\mu = \frac{5}{\sqrt{1538}}, \quad \alpha = \frac{769}{2500}. \end{aligned} \tag{403}$$

(2) The *second set* of parameters is given by

$$\begin{aligned} &a_0 = -\frac{60000}{591361}, \quad a_1 = \frac{86625}{591361}, \quad c = -\frac{180000}{591361}, \\ &\mu = \frac{5}{\sqrt{1538}}, \quad \alpha = \frac{769}{2500}. \end{aligned} \tag{404}$$

This in turn gives the traveling solitary wave solutions

$$u_1(x,t) = \frac{86625}{591361} \operatorname{sech}^6\left(\frac{5}{\sqrt{1538}}\left(x - \frac{180000}{591361}t\right)\right), \tag{405}$$

and

$$u_2(x,t) = -\frac{60000}{591361} + \frac{86625}{591361} \operatorname{sech}^6\left(\frac{5}{\sqrt{1538}}\left(x + \frac{180000}{591361}t\right)\right). \tag{406}$$

In addition, we obtain the following traveling wave solutions

$$u_3(x,t) = -\frac{86625}{591361} \operatorname{csch}^6\left(\frac{5}{\sqrt{1538}}\left(x - \frac{180000}{591361}t\right)\right), \tag{407}$$

and

$$u_4(x,t) = -\frac{60000}{591361} - \frac{86625}{591361} \operatorname{csch}^6\left(\frac{5}{\sqrt{1538}}\left(x + \frac{180000}{591361}t\right)\right). \tag{408}$$

It is interesting to point out that these traveling solitary wave solutions exist only if the signs of the coefficients of the are opposite. Moreover, the solutions exist only for fixed value of α given before in (403).

However, if the coefficients of the terms u_{3x} and u_{5x} have identical positive signs, we obtain periodic solutions that include $\sec^6(\mu\xi)$. In this case, we assume that the solution is of the form

$$u(\xi) = a_0 + a_1 \sec^6(\mu\xi). \tag{409}$$

Substituting (409) into (400), collecting the coefficients of $\sec^j$, and solving the resulting system we find the following two sets of solutions

(1) The *first set* of parameters is given by

$$a_0 = 0, \quad a_1 = -\frac{86625}{591361}, \quad c = -\frac{180000}{591361},$$
$$\mu = \frac{5}{\sqrt{1538}}, \quad \alpha = \frac{769}{2500}. \tag{410}$$

(2) The *second set* of parameters is given by

$$a_0 = \frac{60000}{591361}, \quad a_1 = -\frac{86625}{591361}, \quad c = \frac{180000}{591361},$$
$$\mu = \frac{5}{\sqrt{1538}}, \quad \alpha = \frac{769}{2500}. \tag{411}$$

This in turn gives the solutions

$$u_5(x,t) = -\frac{86625}{591361}\sec^6\left(\frac{5}{\sqrt{1538}}\left(x + \frac{180000}{591361}t\right)\right), \tag{412}$$
$$u_6(x,t) = -\frac{86625}{591361}\csc^6\left(\frac{5}{\sqrt{1538}}\left(x + \frac{180000}{591361}t\right)\right), \tag{413}$$
$$u_7(x,t) = \frac{60000}{591361} - \frac{86625}{591361}\sec^6\left(\frac{5}{\sqrt{1538}}\left(x - \frac{180000}{591361}t\right)\right), \tag{414}$$

and

$$u_8(x,t) = \frac{60000}{591361} - \frac{86625}{591361}\csc^6\left(\frac{5}{\sqrt{1538}}\left(x - \frac{180000}{591361}t\right)\right). \tag{415}$$

13. Ninth-order KdV equation

The ninth-order KdV equation (nKdV)

$$u_t + 6uu_x + u_{3x} - u_{5x} + \alpha u_{7x} + \beta u_{9x} = 0, \tag{416}$$

where α and β are arbitrary nonzero constants, and u is a sufficiently often differentiable function. The sech method will be used again to study this equation. The nKdV equation possesses the dispersion term u_{3x} and three higher order dispersion terms, namely, u_{5x}, u_{7x} and u_{9x} and possesses polynomial type conserved quantities.

13.1. *The sech method*

We begin our analysis by rewriting (416) as

$$-cu + 3u^2 + u'' - u^{(\mathrm{iv})} + \alpha u^{(\mathrm{vi})} + \beta u^{(\mathrm{viii})} = 0, \tag{417}$$

by using the wave variable $\xi = \mu(x - ct)$ and integrating once. Balancing the terms $u^{(\mathrm{viii})}$ with u^2 in (417) we find

$$M + 8 = 2M, \tag{418}$$

so that $M = 8$. Following our discussion above, we assume that the solution is of the form

$$u(\xi) = a_0 + a_1 \operatorname{sech}^8(\mu\xi). \tag{419}$$

Substituting (419) into (417) and proceeding as before we find the following two sets of solutions

(1) The *first set* of parameters is given by

$$\begin{aligned} &a_0 = 0, \quad a_1 = \frac{3816888075}{22609585952}, \quad c = \frac{249120900}{706549561}, \\ &\mu = \frac{1}{4}\sqrt{\frac{5649}{26581}}, \quad \alpha = \frac{212648}{506527}, \quad \beta = -\frac{11304792976}{180266374449}. \end{aligned} \tag{420}$$

(2) The *second set* of parameters is given by

$$\begin{aligned} &a_0 = -\frac{83040300}{706549561}, \quad a_1 = \frac{3816888075}{22609585952}, \quad c = -\frac{249120900}{706549561}, \\ &\mu = \frac{1}{4}\sqrt{\frac{5649}{26581}}, \quad \alpha = \frac{212648}{506527}, \quad \beta = -\frac{11304792976}{180266374449}. \end{aligned} \tag{421}$$

This in turn gives the traveling solitary wave solutions

$$u_1(x,t) = \frac{3816888075}{22609585952} \operatorname{sech}^8\left(\frac{1}{4}\sqrt{\frac{5649}{26581}}\left(x - \frac{249120900}{706549561}t\right)\right), \tag{422}$$

$$u_2(x,t) = \frac{3816888075}{22609585952} \operatorname{csch}^8\left(\frac{1}{4}\sqrt{\frac{5649}{26581}}\left(x - \frac{249120900}{706549561}t\right)\right), \tag{423}$$

$$\begin{aligned} u_3(x,t) = &-\frac{83040300}{706549561} \\ &+ \frac{3816888075}{22609585952} \operatorname{sech}^8\left(\frac{1}{4}\sqrt{\frac{5649}{26581}}\left(x + \frac{249120900}{706549561}t\right)\right), \end{aligned} \tag{424}$$

and

$$\begin{aligned} u_4(x,t) = &-\frac{83040300}{706549561} \\ &+ \frac{3816888075}{22609585952} \operatorname{csch}^8\left(\frac{1}{4}\sqrt{\frac{5649}{26581}}\left(x + \frac{249120900}{706549561}t\right)\right). \end{aligned} \tag{425}$$

The obtained traveling solitary wave solutions exist only if the signs of the coefficients of the terms u_{3x} and u_{5x} are opposite. Moreover, the solutions exist only for specific values of α and β obtained above in (420).

However, if the coefficients of the terms u_{3x} and u_{5x} have identical positive signs we obtain periodic solutions that include $\sec^8(\mu\xi)$. To achieve our goal, we assume that the solution is of the form

$$u(\xi) = a_0 + a_1 \sec^8(\mu\xi). \tag{426}$$

Substituting (426) into (417) and proceeding as before we find the following two sets of solutions

(1) The *first set* of parameters is given by

$$a_0 = 0, \quad a_1 = -\frac{3816888075}{22609585952}, \quad c = -\frac{249120900}{706549561},$$
$$\mu = \frac{1}{4}\sqrt{\frac{5649}{26581}}, \quad \alpha = \frac{212648}{506527}, \quad \beta = \frac{11304792976}{180266374449}. \tag{427}$$

(2) The *second set* of parameters is given by

$$a_0 = \frac{83040300}{706549561}, \quad a_1 = -\frac{3816888075}{22609585952}, \quad c = \frac{249120900}{706549561},$$
$$\mu = \frac{1}{4}\sqrt{\frac{5649}{26581}}, \quad \alpha = \frac{212648}{506527}, \quad \beta = \frac{11304792976}{180266374449}. \tag{428}$$

This in turn gives the solutions

$$u_5(x,t) = -\frac{3816888075}{22609585952}\sec^8\left(\frac{1}{4}\sqrt{\frac{5649}{26581}}\left(x + \frac{249120900}{706549561}t\right)\right), \tag{429}$$

$$u_6(x,t) = -\frac{3816888075}{22609585952}\csc^8\left(\frac{1}{4}\sqrt{\frac{5649}{26581}}\left(x + \frac{249120900}{706549561}t\right)\right), \tag{430}$$

$$\begin{aligned} u_7(x,t) = {} & \frac{83040300}{706549561} \\ & - \frac{3816888075}{22609585952}\sec^8\left(\frac{1}{4}\sqrt{\frac{5649}{26581}}\left(x - \frac{249120900}{706549561}t\right)\right), \end{aligned} \tag{431}$$

and

$$\begin{aligned} u_8(x,t) = {} & \frac{83040300}{706549561} \\ & - \frac{3816888075}{22609585952}\csc^8\left(\frac{1}{4}\sqrt{\frac{5649}{26581}}\left(x - \frac{249120900}{706549561}t\right)\right). \end{aligned} \tag{432}$$

14. The coupled KdV or the Hirota–Satsuma equations

Hirota and Satsuma [29] proposed a coupled KdV equation which describes interactions of two long waves with different dispersion relations. The Hirota–Satsuma equations are

$$\begin{aligned} u_t &= \frac{1}{2}u_{xxx} + 3uu_x - 6vv_x, \\ v_t &= -v_{xxx} - 3uv_x. \end{aligned} \tag{433}$$

If $v = 0$, Eq. (433) reduces to the KdV equation. In this section we will use the tanh–coth method and the simplified version of the Hirota bilinear formalism to handle the Hirota–

Satsuma system. The following three conserved densities

$$I_1 = u,$$
$$I_2 = u^2 - 2v^2,$$
$$I_3 = \frac{3}{2}\left(u^3 - \frac{1}{2}(u_x)^2\right) - 3\left(uv^2 - (v_x)^2\right), \tag{434}$$

were confirmed.

14.1. *Using the tanh–coth method*

Using the wave variable $\xi = x - ct$, system (433) is converted to

$$-cu - \frac{1}{2}u'' - \frac{3}{2}u^2 + 3v^2 = 0,$$
$$-cv' + v''' + 3uv' = 0. \tag{435}$$

Balancing the nonlinear term u^2 with the highest order derivative u'' in the first equation of the couple gives

$$2M = M + 2, \tag{436}$$

that gives

$$M = 2. \tag{437}$$

Substituting for u from the first equation into the second equation, and balancing the nonlinear term v^2v' with the highest order derivative v''' in the second equation of the couple gives

$$M_1 + 3 = 2M_1 + M_1 + 1, \tag{438}$$

that gives

$$M_1 = 1. \tag{439}$$

The tanh–coth method allows us to use the substitution

$$u(x,t) = S(Y) = a_0 + a_1Y^2 + a_2Y^{-2},$$
$$v(x,t) = S_1(Y) = b_0 + b_1Y + b_2Y^{-1}, \tag{440}$$

where we found that $u(x,t)$ does not include Y or Y^{-1} terms. Substituting (440) into (435), collecting the coefficients of each power of Y^i, $0 \leqslant i \leqslant 8$, setting each coefficient to zero, and solving the resulting system of algebraic equations we obtain the following sets of solutions

(i) *First set*

$$a_0 = \frac{c}{3} + \frac{2}{3}\lambda^2, \quad a_1 = -2\lambda^2, \quad a_2 = 0,$$
$$b_0 = 0, \quad b_1 = \frac{1}{\sqrt{2}}c, \quad b_2 = 0,$$
$$\mu = \lambda, \tag{441}$$

where

$$\lambda = \frac{(2+\sqrt{10})c}{2}. \tag{442}$$

(ii) *Second set*

$$a_0 = \frac{c}{3} + \frac{2}{3}\lambda^2, \quad a_1 = 0, \quad a_2 = -2\lambda^2,$$
$$b_0 = 0, \quad b_1 = 0, \quad b_2 = \frac{1}{\sqrt{2}}c,$$
$$\mu = \lambda, \tag{443}$$

(iii) *Third set*

$$a_0 = \frac{c}{3} - \frac{1}{3}\lambda^2, \quad a_1 = -\frac{1}{2}\lambda^2, \quad a_2 = -\frac{1}{2}\lambda^2,$$
$$b_0 = 0, \quad b_1 = \frac{1}{2\sqrt{2}}c, \quad b_2 = \frac{1}{2\sqrt{2}}c,$$
$$\mu = \lambda. \tag{444}$$

In view of these results we obtain the following sets of solutions

$$u_1(x,t) = \frac{c}{3} + \frac{2}{3}\lambda^2 - 2\lambda^2 \tanh^2\big[\lambda(x-ct)\big],$$
$$v_1(x,t) = \frac{1}{\sqrt{2}}c \tanh\big[\lambda(x-ct)\big]. \tag{445}$$
$$u_2(x,t) = \frac{c}{3} + \frac{2}{3}\lambda^2 - 2\lambda^2 \coth^2\big[\lambda(x-ct)\big],$$
$$v_2(x,t) = \frac{1}{\sqrt{2}}c \coth\big[\lambda(x-ct)\big] \tag{446}$$

and

$$u_3(x,t) = \frac{c}{3} - \frac{1}{3}\lambda^2 - \frac{1}{2}\lambda^2\left(\tanh^2\left[\frac{1}{2}\lambda(x-ct)\right] + \coth^2\left[\frac{1}{2}\lambda(x-ct)\right]\right),$$
$$v_3(x,t) = \frac{1}{2\sqrt{2}}c\left(\coth\left[\frac{1}{2}\lambda(x-ct)\right] + \coth\left[\frac{1}{2}\lambda(x-ct)\right]\right). \tag{447}$$

14.2. *Multiple-soliton solutions of the Hirota–Satsuma system*

In this section, we will examine multiple-soliton solutions of the Hirota–Satsuma system

$$\begin{aligned} u_t &= \frac{1}{2}u_{xxx} + 3uu_x - 6vv_x, \\ v_t &= -v_{xxx} - 3uv_x. \end{aligned} \tag{448}$$

Hirota [29] introduced the dependent variable transformation

$$\begin{aligned} u(x,t) &= 2\frac{\partial^2 \ln f(x,t)}{\partial x^2} = 2\frac{ff_{2x} - (f_x)^2}{f^2}, \\ v(x,t) &= \frac{g}{f}, \end{aligned} \tag{449}$$

that will convert (448) into the bilinear forms

$$\begin{aligned} &D_x\left(D_t - \frac{1}{2}D_x^3\right) f \cdot f = -3g^2, \\ &\left(D_t + D_x^3\right) f \cdot g = 0. \end{aligned} \tag{450}$$

We next assume that $f(x,t)$ and $g(x,t)$ have the perturbation expansions

$$\begin{aligned} f(x,t) &= 1 + \sum_{n=0}^{\infty} \epsilon^n f_n(x,t), \\ g(x,t) &= \sum_{n=0}^{\infty} \sigma^n g_n(x,t), \end{aligned} \tag{451}$$

where ϵ and σ are nonsmall formal expansion parameter. Following Hirota's method and the simplified version in [18] we first set

$$\begin{aligned} f_1(x,t) &= \sum_{i=1}^{N} \exp(2\theta_i), \\ g_1(x,t) &= \sum_{i=1}^{N} \exp(\theta_i), \end{aligned} \tag{452}$$

where

$$\theta_i = k_i x - c_i t, \tag{453}$$

where k_i and c_i are arbitrary constants. Substituting (452) into (448) gives the dispersion relation

$$c_i = k_i^3, \tag{454}$$

and in view of this result we obtain

$$\theta_i = k_i x - k_i^3 t. \tag{455}$$

This means that

$$
\begin{aligned}
&f_1(x,t) = \exp(2\theta_1) = \exp\bigl(2k_1\bigl(x - k_1^2 t\bigr)\bigr),\\
&g_1(x,t) = \exp(\theta_1) = \exp\bigl(k_1\bigl(x - k_1^2 t\bigr)\bigr),\\
&\epsilon = \frac{1}{8k_1^4},\\
&\sigma = 1
\end{aligned}
\tag{456}
$$

obtained by using $N = 1$ in (452). In what follows we list the solutions obtained by Hirota [29], and more details can be found there. For the one-soliton solution, it was found that

$$
\begin{aligned}
&f = 1 + \frac{1}{8k_1^4}\exp(2\theta_1),\\
&g = \exp(\theta_1).
\end{aligned}
\tag{457}
$$

The one soliton solution is therefore

$$
u(x,t) = 2(\ln f)_{xx}, \qquad v(x,t) = g/f. \tag{458}
$$

For the two-soliton solution it was found that

$$
\begin{aligned}
f &= 1 + \frac{1}{8k_1^4}\mathrm{e}^{2\theta_1} + \frac{1}{8k_2^4}\mathrm{e}^{2\theta_2}\\
&\quad + \frac{2}{(k_1+k_2)^2(k_1^2+k_2^2)}\mathrm{e}^{\theta_1+\theta_2} + \frac{(k_1-k_2)^4}{64k_1^4k_2^4(k_1+k_2)^4}\mathrm{e}^{2(\theta_1+\theta_2)},\\
g &= \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \frac{1}{8k_1^4}\frac{(k_1-k_20^2}{(k_1+k_2)^2}\mathrm{e}^{2\theta_1+\theta_2} + \frac{1}{8k_2^4}\frac{(k_1-k_20^2}{(k_1+k_2)^2}\mathrm{e}^{\theta_1+2\theta_2}.
\end{aligned}
\tag{459}
$$

The two-soliton solution is obtained by substituting (459) into (458).

It is interesting to point out that Hirota and Satsuma [29] derived the one and two-soliton solutions only and used this to suggest the existence of the N-soliton solutions.

14.3. *Multiple-soliton solutions by another method*

However, Tam et al. [60] applied a slightly different approach and derived entirely new one, two and three-soliton solutions to the Hirota–Satsuma system.

In [60] the dependent variable transformation

$$
\begin{aligned}
&u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2} = 2\frac{ff_{2x} - (f_x)^2}{f^2},\\
&v(x,t) = \frac{g}{f},
\end{aligned}
\tag{460}
$$

was applied to convert (448) into the bilinear forms

$$D_x\left(D_t - \frac{1}{2}D_x^3\right) f \cdot f = -3g^2 + Cf^2,$$
$$\big(D_t + D_x^3\big) f \cdot g = 0, \tag{461}$$

where C in an integration constant. For $C = 0$ we obtain the bilinear forms (450). In [60], $C = 3$ was used to convert the last bilinear form to

$$D_x\left(D_t - \frac{1}{2}D_x^3\right) f \cdot f = 3\big(f^2 - g^2\big),$$
$$\big(D_t + D_x^3\big) f \cdot g = 0. \tag{462}$$

It was obtained after some tests and guesses that for the one-soliton solution

$$f = 1 + \exp(\theta_1) + \frac{1}{32}\big(4 + k_1^4\big)\exp(2\theta_1),$$
$$g = 1 + \frac{1}{2}\big(2 + k_1^4\big)\exp(\theta_1) + \frac{1}{32}\big(4 + k_1^4\big)\exp(2\theta_1). \tag{463}$$

This result is distinct from that obtained in [29] given in (457). Consequently, the one soliton solution is therefore

$$u(x,t) = 2(\ln f)_{xx}, \qquad v(x,t) = g/f. \tag{464}$$

For the two-soliton solution it was found that, after correcting some of the coefficients in [60]

$$\begin{aligned} f &= 1 + \exp(\theta_1) + \exp(\theta_2) + A_1\exp(2\theta_1) + A_2\exp(2\theta_2) + A_3\exp(\theta_1+\theta_2) \\ &\quad + A_4\exp(2\theta_1+\theta_2) + A_5\exp(\theta_1+2\theta_2) + A_6\exp\big(2(\theta_1+\theta_2)\big), \\ g &= 1 + \frac{1}{2}\big(2 + k_1^4\big)\exp(\theta_1) + \frac{1}{2}\big(2 + k_2^4\big)\exp(\theta_2) + B_1\exp(2\theta_1) + B_2\exp(2\theta_2) \\ &\quad + B_3\exp(\theta_1+\theta_2) + B_4\exp(2\theta_1+\theta_2) + B_5\exp(\theta_1+2\theta_2) \\ &\quad + B_6\exp\big(2(\theta_1+\theta_2)\big), \end{aligned} \tag{465}$$

where

$$\begin{aligned} A_i &= \frac{1}{32}\big(4 + k_i^4\big), \quad i = 1, 2, \\ A_3 &= \frac{2(k_1^4 + k_2^4) + k_1^4k_2^4}{2(k_1+k_2)^2(k_1^2+k_2^2)}, \\ A_{j+3} &= \frac{1}{32}\big(4 + k_j^4\big)\frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \quad j = 1, 2, \\ A_6 &= A_1A_2\frac{(k_1-k_2)^4}{(k_1+k_2)^4}, \\ B_i &= \frac{1}{32}\big(4 + k_i^4\big), \quad i = 1, 2, \end{aligned}$$

$$
\begin{aligned}
B_3 &= \frac{(k_1^8 + k_2^8) - k_1^4 k_2^4 + 2(k_1^4 + k_2^4)}{2(k_1 + k_2)^2 (k_1^2 + k_2^2)}, \\
B_4 &= \frac{1}{2}\left(2 + k_2^4\right) \times \frac{1}{32}\left(4 + k_1^4\right) \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \\
B_5 &= \frac{1}{2}\left(2 + k_1^4\right) \times \frac{1}{32}\left(4 + k_2^4\right) \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \\
B_6 &= A_6.
\end{aligned} \tag{466}
$$

The two-soliton solution is obtained by substituting (465) into (464). The explicit three-soliton solution can be found in [60]. Because the three-soliton solutions were obtained, this clearly indicates that the N-soliton solutions, $N \geqslant 3$ exist for the coupled KdV equations.

15. Compactons and the $K(n, n)$ equation

It is well-known that the nonlinear term uu_x in the standard KdV equation

$$u_t + \alpha u u_x + u_{xxx} = 0, \tag{467}$$

causes the steepening of wave form. At the same time, the dispersion effect term u_{xxx} in Eq. (467) makes the wave form spread [64,65,62,63]. Due to the balance between the weak nonlinearity and dispersion, solitons exist. Soliton has been defined by Wadati [62,63] and others as a nonlinear wave that has the following properties:

(1) A localized wave propagates without change of its properties (shape, velocity, etc.),
(2) Localized waves are stable against mutual collisions and retain their identities. This means that the nonlinear KdV equation (467) with linear dispersion admits solitary waves that are infinite in extent or localized waves with exponential tails. For the Sine–Gordon equation, the kink solution converges to a constant at infinity.

However, the convection term in the genuinely nonlinear dispersive $K(n, n)$ equation

$$u_t + a(u^n)_x + (u^n)_{xxx} = 0, \quad n > 1, \tag{468}$$

is nonlinear. Moreover, the dispersion effect term $(u^n)_{xxx}$ in this equation is genuinely nonlinear. It is formally derived by Rosenau and Hyman [56] that the delicate interaction between nonlinear convection with genuine nonlinear dispersion generates solitary waves with exact compact support that are called *compactons*. The pioneering study conducted by Rosenau and Hyman [56] revealed that Eq. (468) generates compactly supported solutions with nonsmooth fronts. In fact compactons are solitons with finite wavelength, waves with a compact support or solitons free of exponential wings. Unlike soliton that narrows as the amplitude increases, the compacton's width is independent of the amplitude. Compactons such as drops do not possess infinite wings, hence they interact among themselves only across short distances. In modern physics, a suffix-on is used to indicate the particle property [62], for example phonon, photon, and soliton. For this reason, the solitary wave with compact support is called *compacton* to indicate that it has the property of a particle.

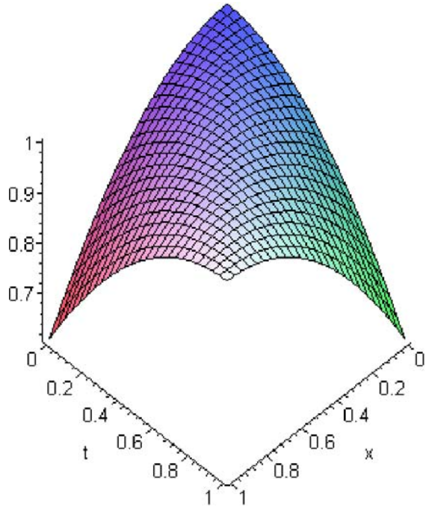

Fig. 6. Graph of a compacton: soliton confined to a finite core free of exponential wings.

The idea of compact localized solutions of nonlinear dispersion has gained a considerable amount of interest. Significant studies have been developed in [56,66–77], and many of the references therein to confirm that purely nonlinear dispersion can cause a deep qualitative change in the genuinely nonlinear phenomenon. It was derived that the compactons are nonanalytic [56] solutions, whereas classical solitons are analytic solutions. The points of nonanalyticity at the edge of the compacton correspond to points of genuine nonlinearity for the differential equation and introduce singularities in the associated dynamical system for the traveling waves.

The $K(n,n)$ equation (468) cannot be derived from a first order Lagrangian except for $n = 1$, and it does not possess the usual conservation laws of energy that KdV equation (467) possessed. The stability analysis has shown that compacton solutions are stable, where the stability condition is satisfied for arbitrary values of the nonlinearity parameter. The stability of the compactons solutions was investigated by means of both linear stability and by Lyapunov stability criteria as well. Compactons were proved to collide elastically and vanish outside a finite core region. Two important features of compactons structures are observed:

(1) The compacton is a soliton characterized by the absence of exponential wings,
(2) The width of the compacton is independent of the amplitude.

The compacton concept has been studied by many analytical and numerical methods. There are many algorithms such as the pseudo spectral method, the tri-Hamiltonian operators, the finite difference method, and many others. Figure 6 shows a graph of a compacton.

It was shown in [33] that in a continuous system, compact breathers are exact cosine solutions with strict bounded support. However, in lattices, the core region of the compact breathers can be described by a cosine shape while the tail region decays according to the super exponential law $e^{-a\exp(bn)}$, where a and b are constants that depend on the

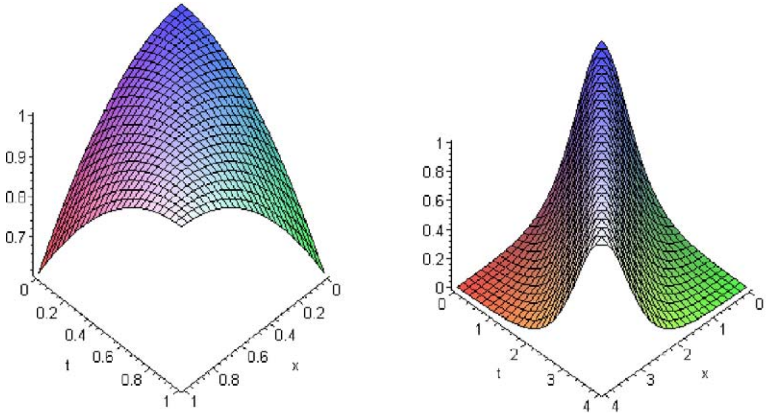

Fig. 7. The compacton graph (left) and the soliton graph without and with infinite wings respectively.

model Hamiltonian. A certain degree of harmonicity in the substrate potential is required to stabilize the breather compacton solution.

The compactons discovery motivated a considerable work to make compactons be practically realized in scientific applications, such as the super deformed nuclei, preformation of cluster in hydrodynamic models, the fission of liquid drops (nuclear physics), inertial fusion and others.

Moreover, recall that solitary wave solutions may be expressed in terms of $\operatorname{sech}^{\alpha}$, or $\arctan(e^{\alpha(x-ct)})$. However, the compactons solutions may be expressed in terms of trigonometric functions raised to an exponent. The cusps or infinite slopes solutions of the defocussing branches, where $a < 0$, are expressed in terms of hyperbolic functions raised to an exponent. Figure 7 shows a compacton (left) and a soliton (right).

The pseudo spectral method and the tri-Hamiltonian operators, among other methods, were used to handle the $K(n, n)$ equation. However, in this section we will use the tanh–coth method to handle the $K(n, n)$ equation.

15.1. *The $K(n, n)$ equation*

The $K(n, n)$ equation

$$u_t + a(u^n)_x + (u^n)_{xxx} = 0, \quad n > 1, \tag{469}$$

has been introduced by Rosenau and Hyman in [56]. To determine the traveling-type wave solution $u(x, t)$ of Eq. (469) we use the wave variable $\xi = (x - ct)$, and integrate the resulting ODE to transform (469) into an ODE

$$-cu + au^n + (u^n)'' = 0, \tag{470}$$

or equivalently

$$-cu + au^n + nu^{n-1}u'' + n(n-1)u^{n-2}(u')^2 = 0. \tag{471}$$

Balancing the terms $u^{n-1}u''$ and u gives

$$(n-1)M + 4 + M - 2 = M, \tag{472}$$

so that

$$M = -\frac{2}{n-1}. \tag{473}$$

To obtain a closed form analytic solution, the parameter M should be an integer. To achieve this goal we use a transformation formula

$$u(x,t) = v^{-1/(n-1)}(x,t). \tag{474}$$

This formula carries (471) into

$$-c(n-1)^2 v^3 + a(n-1)^2 v^2 - bn(n-1)vv'' + bn(2n-1)(v')^2 = 0. \tag{475}$$

Balancing the terms v^3 and vv'' we find

$$3M = M + M + 2, \tag{476}$$

that gives $M = 2$. The tanh–coth method allows us to use the substitution

$$v(x,t) = S(Y) = a_0 + a_1 Y + a_2 Y^2 + \frac{b_1}{Y} + \frac{b_2}{Y^2}. \tag{477}$$

Substituting (477) into (475), collecting the coefficients of each power of Y, and solving the resulting system of algebraic equations we obtain the following three sets:

(i) The *first* set

$$\begin{aligned} &a_0 = \frac{a(n+1)}{2cn}, \quad a_1 = b_1 = b_2 = 0, \quad a_2 = -\frac{a(n+1)}{2cn}, \\ &\mu = \frac{n-1}{2n}\sqrt{-a}, \end{aligned} \tag{478}$$

and

(ii) The *second* set

$$\begin{aligned} &a_0 = \frac{a(n+1)}{2cn}, \quad a_1 = b_1 = a_2 = 0, \quad b_2 = -\frac{a(n+1)}{2cn}, \\ &\mu = \frac{n-1}{2n}\sqrt{-a}, \end{aligned} \tag{479}$$

Noting that $u = v^{-1/(n-1)}$, we first obtain the solitary patterns solutions

$$u_1(x,t) = \left\{-\frac{2cn}{a(n+1)}\sinh^2\left[\frac{n-1}{2n}\sqrt{-a}\,(x-ct)\right]\right\}^{1/(n-1)}, \tag{480}$$

$$u_2(x,t) = \left\{\frac{2cn}{a(n+1)}\cosh^2\left[\frac{n-1}{2n}\sqrt{-a}\,(x-ct)\right]\right\}^{1/(n-1)}, \tag{481}$$

for $a < 0$, where $\xi = x - ct$.

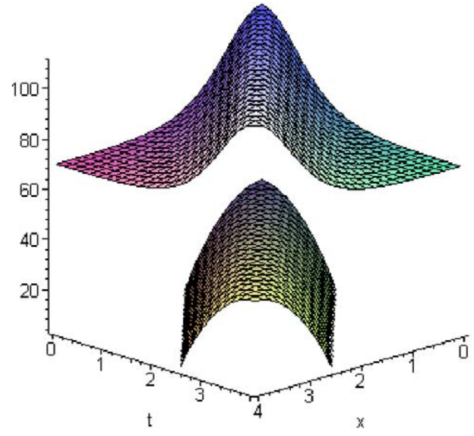

Fig. 8. The upper graph is a soliton with infinite support, and the lower graph is a compacton confined to a finite core free of exponential tails.

However, for $a > 0$ we obtain the compactons solutions

$$u_3(x,t) = \begin{cases} \{\frac{2cn}{a(n+1)} \sin^2[\frac{n-1}{2n}\sqrt{a}(x-ct)]\}^{1/(n-1)}, & |\xi| \leqslant \frac{\pi}{\mu}, \\ 0 & \text{otherwise,} \end{cases} \tag{482}$$

and

$$u_4(x,t) = \begin{cases} \{\frac{2cn}{a(n+1)} \cos^2[\frac{n-1}{2n}\sqrt{a}(x-ct)]\}^{1/(n-1)}, & |\xi| \leqslant \frac{\pi}{2\mu}, \\ 0 & \text{otherwise.} \end{cases} \tag{483}$$

The last results are in consistent with the results that other researchers obtained by using different approaches. Figure 8 shows a graph of a soliton (upper) and a compacton (lower).

15.2. *Variant of the $K(n,n)$ equation*

A variant of the $K(n,n)$ equation of the form

$$u_t + a(u^{n+1})_x + \left[u(u^n)_{xx}\right]_x = 0, \quad a > 0,\ n \geqslant 1, \tag{484}$$

was investigated by Rosenau [54,55]. This model emerges in nonlinear lattices and was used to describe the dispersion of dilute suspensions for $n = 1$. Equation (484) was considered as a variant of the KdV equation or of the K(n,n) equation.

To determine the traveling-type wave solution $u(x,t)$ of Eq. (484) we use the wave variable $\xi = (x - ct)$, and integrate the resulting ODE to transform (484) into an ODE

$$-cu + au^{n+1} + u(u^n)'' = 0, \tag{485}$$

or equivalently

$$-cu + au^{n+1} + nu^n u'' + n(n-1)u^{n-1}(u')^2 = 0. \tag{486}$$

Balancing the terms $u^n u''$ and u gives

$$nM + M + 2 = M, \tag{487}$$

so that

$$M = -\frac{2}{n}. \tag{488}$$

To obtain a closed form analytic solution, the parameter M should be an integer. To achieve this goal we use a transformation formula

$$u(x,t) = v^{-1/n}(x,t). \tag{489}$$

This formula carries (486) into

$$-cv^3 + av^2 - vv'' + 2(v')^2 = 0. \tag{490}$$

Balancing the terms v^3 and vv'' we find

$$3M = M + M + 2, \tag{491}$$

that gives $M = 2$. The tanh–coth method allows us to use the substitution

$$v(x,t) = S(Y) = a_0 + a_1 Y + a_2 Y^2 + \frac{b_1}{Y} + \frac{b_2}{Y^2}. \tag{492}$$

Substituting (492) into (490), collecting the coefficients of each power of Y, and solving the resulting system of algebraic equations we obtain the following three sets:

(i) The *first* set

$$a_0 = \frac{a}{2c}, \quad a_1 = b_1 = b_2 = 0, \quad a_2 = -\frac{a}{2c}, \quad \mu = \frac{\sqrt{-a}}{2}, \tag{493}$$

and

(ii) The *second* set

$$a_0 = \frac{a}{2c}, \quad a_1 = b_1 = a_2 = 0, \quad b_2 = -\frac{a}{2c}, \quad \mu = \frac{\sqrt{-a}}{2}. \tag{494}$$

Noting that $u = v^{-1/n}$, we first obtain the solitary patterns solutions

$$u_1(x,t) = \left\{\frac{2c}{a}\sinh^2\left[\frac{\sqrt{a}}{2}(x-ct)\right]\right\}^{1/n}, \tag{495}$$

$$u_2(x,t) = -\left\{\frac{2c}{a}\cosh^2\left[\frac{\sqrt{a}}{2}(x-ct)\right]\right\}^{1/n}, \tag{496}$$

for $a < 0$, where $\xi = x - ct$.

For $a > 0$, the following compactons solutions

$$u(x,t) = \begin{cases} \{\frac{2c}{a}\sin^2[\frac{\sqrt{a}}{2}(x-ct)]\}^{1/n}, & |x-ct| \leqslant \frac{\pi}{\mu}, \\ 0 & \text{otherwise,} \end{cases} \tag{497}$$

$$u(x,t) = \begin{cases} \{\frac{2c}{a}\cos^2[\frac{\sqrt{a}}{2}(x-ct)]\}^{1/n}, & |x-ct| \leqslant \frac{\pi}{2\mu}, \\ 0 & \text{otherwise.} \end{cases} \tag{498}$$

16. Compacton-like solutions

In this section we develop compacton-like solutions for the modified KdV equation

$$u_t + au^2u_x + u_{xxx} = 0, \tag{499}$$

where $a > 0$ is a constant. We first set the compacton-like solutions of Eq. (499) in the form

$$u(x,t) = M\frac{\cos^2 k(x-4k^2t)}{1-\frac{2}{3}\cos^2 k(x-4k^2t)}, \tag{500}$$

or in the form

$$u(x,t) = N\frac{\sin^2 k(x-4k^2t)}{1-\frac{2}{3}\sin^2 k(x-4k^2t)}, \tag{501}$$

where M and N are constants. Substituting (500) and (501) into (499) and solving straightforwardly for M and N we obtain

$$M = N = \frac{4k}{3}\sqrt{\frac{2}{a}}. \tag{502}$$

Substituting (502) into (500) and (501) gives the following set of compacton-like solutions

$$u(x,t) = \begin{cases} \frac{4k}{3}\sqrt{\frac{2}{a}}\,\frac{\cos^2 k(x-4k^2t)}{1-\frac{2}{3}\cos^2 k(x-4k^2t)}, & |(x-4k^2t)| \leqslant \frac{\pi}{2k}, \\ 0 & \text{otherwise}, \end{cases} \tag{503}$$

and

$$u(x,t) = \begin{cases} \frac{4k}{3}\sqrt{\frac{2}{a}}\,\frac{\sin^2 k(x-4k^2t)}{1-\frac{2}{3}\sin^2 k(x-4k^2t)}, & |(x-4k^2t)| \leqslant \frac{\pi}{k}, \\ 0 & \text{otherwise}. \end{cases} \tag{504}$$

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