COLLEGE ALGEBRA AND TRIGONOMETRY SECOND EDITION

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PREFACE

We have been very pleased by the widespread acceptance of the first edition of this book. It has been especially gratifying to find other texts adopting some of our innovative ideas. For example, the early introduction of complex numbers, permitting the solution of any quadratic equation, has now become the accepted approach.

The objectives of this edition are

- to improve the chapters on trigonometry by:
  a. eliminating the "wrapping function" to permit a faster pacing of the opening sections;
  b. introducing early the concept of angular measure (in both radians and degrees) to avoid the abrupt switch from theory to application;
  c. guiding the student to use calculators for determining the values of the trigonometric functions;
- to respond to suggestions from instructors and students, thus improving the exposition and the examples;
- to eliminate the need for the student to purchase a study guide, by providing a section in the back of the book that contains worked-out solutions to selected **Review Exercises;**
- to ensure accuracy of the **Answers** section by utilizing computer programs with rational arithmetic to verify the answers;
- to enliven the book by introducing **Features** of interest to both student and instructor (see pages with color marking along the edges);
- to add topics that are currently taught at many schools (for example, linear programming).

We have retained the supportive elements that have become the hallmark of this series:

SPLIT SCREENS Many algebraic procedures are described with the aid of a "split screen" that displays simultaneously both the steps of an algorithm and a worked-out example.

PROGRESS CHECKS At carefully selected places, problems similar to those worked in the text have been inserted (with answers) to enable the student to test his or her understanding of the material just described.

WARNINGS To help eliminate misconceptions and prevent bad mathematical habits, we have inserted numerous **Warnings** (indicated by the symbol shown in the margin) that point out the incorrect practices most commonly found in homework and exam papers.

END-OF-CHAPTER MATERIAL Every chapter contains a summary, including

Terms and Symbols with appropriate page references;

Key Ideas for Review to stress the concepts;

Review Exercises to provide additional practice;

Progress Tests to provide self-evaluation and reinforcement.

ANSWERS The answers to all **Review Exercises** and **Progress Tests** appear in the back of the book.

EXERCISES Abundant, carefully graded exercises provide practice in the mechanical and conceptual aspects of algebra. Exercises requiring a calculator are indicated by the symbol shown in the margin. Answers to odd-numbered exercises appear at the back of the book. Answers to even-numbered exercises appear in the *Instructor's Manual*. The *Instructor's Manual/Test Bank* is available to the instructor on request.

ACKNOWLEDGMENTS

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We would also like to thank Todd Rimmer and Jan Richard, who assisted in checking the manuscript and galleys, and Jacqueline Shapiro, who entered data into computer software for finding roots of polynomial equations and inverses of matrices.

Finally, our grateful thanks to our editors, Wesley Lawton and Ted Buchholz, to our production editors, Ann Colowick and Jennifer Keith, and to the staff of Academic Press for their ongoing commitment to excellence.

TO THE STUDENT

This book was written for you. It gives you every possible chance to succeed—if you use it properly.

We would like to have you think of mathematics as a challenging game—but not as a spectator sport. This wish leads to our primary rule: *Read this textbook with pencil and paper handy.* Every new idea or technique is illustrated by fully worked-out examples. As you read the text, carefully follow the examples and then do the **Progress Checks.** The key to success in a math course is working problems, and the **Progress Checks** are there to provide immediate practice with the material you have just learned.

Your instructor will assign homework from the extensive selection of exercises that follows each section in the book. *Do the assignments regularly, thoroughly, and independently.* By doing lots of problems, you will develop the necessary skills in algebra, and your confidence will grow. Since algebraic techniques and concepts build on previous results, you can't afford to skip any of the work.

To help prevent or eliminate improper habits and to help you avoid the errors that we see each semester as we grade papers, we have interspersed **Warnings** throughout the book. The **Warnings** point out common errors and emphasize the proper method.

There is important review material at the end of each chapter. The **Terms and Symbols** should all be familiar by the time you reach them. If your understanding of a term or symbol is hazy, use the page reference to find the place in the text where it is introduced. Go back and read the definition.

It is possible to become so involved with the details of techniques that you lose track of the broader concepts. The list of **Key Ideas for Review** at the end of each chapter will help you focus on the principal ideas.

The **Review Exercises** at the end of each chapter can be used as part of your preparation for examinations. The section covering each exercise is indicated so that, if needed, you can go back to restudy the material. If you get stuck on a problem, see if the problem that is giving you difficulty or a similar problem is numbered in color, indicating that a worked-out solution appears in the back of the book. You are then ready to try **Progress Test A.** You will soon pinpoint your weak spots and can go back for further review and more exercises in those areas. Only then should you proceed to **Progress Test B.**

We believe that the eventual "payoff" in studying mathematics is an improved ability to tackle practical problems in your field of interest. To that end, this book places special emphasis on word problems, which recent surveys show are often troublesome to students. Since algebra is the basic language of the mathematical techniques used in virtually all fields, the mastery of algebra is well worth your effort.

1

THE FOUNDATIONS OF ALGEBRA

No one would debate that $2 + 2 = 4$, or that $5 + 3 = 3 + 5$. The significance of the statement "$2 + 2 = 4$" lies in the recognition that it is true whether the objects under discussion are apples or ants, cradles or cars. Further, the statement "$5 + 3 = 3 + 5$" indicates that the order of addition is immaterial, and this principle is true for any pair of integers.

These simple examples illustrate the fundamental task of algebra: to abstract those properties that apply to a number system. Of course, the properties depend on the type of numbers we choose to deal with. We will therefore begin with a discussion of the ***real number system*** and its properties, since much of our work in algebra will involve this number system. We will then indicate a correspondence between the real numbers and the points on a real number line and will give a graphical presentation of this correspondence.

The remainder of this chapter is devoted to a review of some fundamentals of algebra: the meaning and use of variables; algebraic expressions and polynomial forms; factoring; and operations with rational expressions or algebraic fractions.

1.1 THE REAL NUMBER SYSTEM

SETS

We will need to use the notation and terminology of sets from time to time. A **set** is simply a collection of objects or numbers, which are called the **elements** or **members** of the set. The elements of a set are written within braces so that the notation

$$A = \{4, 5, 6\}$$

tells us that the set A consists of the numbers 4, 5, and 6. The set

$$B = \{\text{Exxon, Ford, Honeywell}\}$$

consists of the names of these three corporations. We also write $4 \in A$, which we read as "4 is a member of the set A." Similarly, Ford $\in B$ is read as "Ford is a member of the set B," and IBM $\notin B$ is read as "IBM is not a member of the set B."

If every element of a set A is also a member of a set B, then A is a **subset** of B. For example, the set of all robins is a subset of the set of all birds.

EXAMPLE 1

The set C consists of the names of all coins whose denominations are less than 50 cents. We may write C in set notation as follows:

$$C = \{\text{penny, nickel, dime, quarter}\}$$

We see that dime $\in C$ but half dollar $\notin C$. Further, the set $H = \{\text{nickel, dime}\}$ is a subset of C.

PROGRESS CHECK

The set V consists of the vowels in the English alphabet.

(a) Write V in set notation.
(b) Is the letter k a member of V?
(c) Is the letter u a member of V?
(d) List the subsets of V having four elements.

ANSWERS

(a) $V = \{a, e, i, o, u\}$ (b) No (c) Yes
(d) $\{a, e, i, o\}$, $\{e, i, o, u\}$, $\{a, i, o, u\}$, $\{a, e, o, u\}$, $\{a, e, i, u\}$

THE REAL NUMBER SYSTEM

Since much of our work in algebra deals with the real number system, we'll begin with a review of the composition of this number system.

The numbers 1, 2, 3, . . . , used for counting, form the set of **natural numbers.** If we had only these numbers to use to show the profit earned by a company, we would have no way to indicate that the company had no profit or had a loss. To indicate no profit we introduce 0, and for losses we need to introduce negative numbers. The numbers

$$\ldots, -2, -1, 0, 1, 2, \ldots$$

form the set of **integers.** Thus, every natural number is an integer, and the set of natural numbers is seen to be a subset of the set of integers.

When we try to divide two apples equally among four people we find no number in the set of integers that will express how many apples each person should get. We need to introduce the set of **rational numbers,** which are numbers that can be written as a ratio of two integers,

$$\frac{p}{q}, \qquad \text{with } q \text{ not equal to zero}$$

Examples of rational numbers are

$$0 \qquad \frac{2}{3} \qquad -4 \qquad \frac{7}{5} \qquad \frac{-3}{4}$$

By writing an integer n in the form $n/1$, we see that every integer is a rational number. The decimal number 1.3 is also a rational number, since $1.3 = \frac{13}{10}$.

We have now seen three fundamental number systems: the natural number system, the system of integers, and the rational number system. Each later system includes the previous system or systems, and each is more complicated than the one before. However, the rational number system is still inadequate for sophisticated uses of mathematics, since there exist numbers that are not rational, that is, numbers that cannot be written as the ratio of two integers. These are called **irrational numbers.** It can be shown that the number a that satisfies $a \cdot a = 2$ is such a number. The number π, which is the ratio of the circumference of a circle to its diameter, is also such a number.

The decimal form of a rational number will either terminate, as

$$\frac{3}{4} = 0.75 \qquad -\frac{4}{5} = -0.8$$

or will form a repeating pattern, as

$$\frac{2}{3} = 0.666\ldots \quad \frac{1}{11} = 0.090909\ldots \quad \frac{1}{7} = 0.1428571\ldots$$

Remarkably, the decimal form of an irrational number *never* forms a repeating pattern.

The rational and irrational numbers together form the **real number system** (Figure 1).

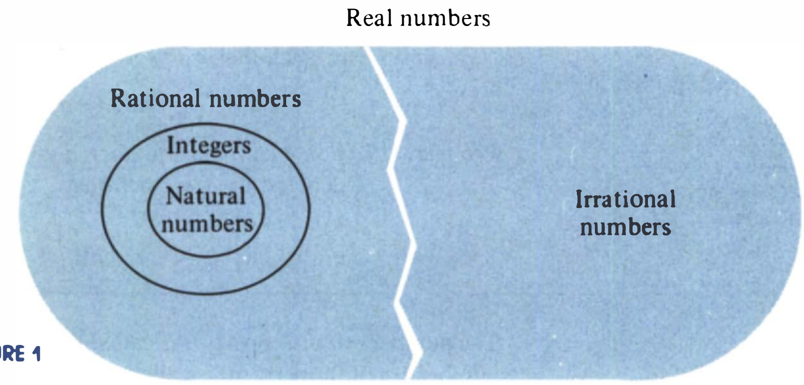


FIGURE 1

PROPERTIES OF THE REAL NUMBERS (Optional)

With respect to the operations of addition and multiplication, the real number system has properties that are fundamental to algebra. The letters a, b, and c will denote real numbers.

Closure	*Property 1*. The sum of a and b, denoted by $a + b$, is a real number. *Property 2*. The product of a and b, denoted by $a \cdot b$ or ab, is a real number.

We say that the set of real numbers is **closed** with respect to the operations of addition and multiplication, since the sum and the product of two real numbers are also real numbers.

Commutative Laws	*Property 3.* $a + b = b + a$	**Commutative law of addition**
	Property 4. $ab = ba$	**Commutative law of multiplication**

That is, we may add or multiply real numbers in any order.

Associative Laws	*Property 5.* $(a + b) + c = a + (b + c)$	**Associative law of addition**
	Property 6. $(ab)c = a(bc)$	**Associative law of multiplication**

That is, when adding or multiplying real numbers we may group them in any order.

Identities

Property 7. There is a unique real number, denoted by 0, such that $a + 0 = 0 + a = a$ for every real number a.

Property 8. There is a unique real number, denoted by 1, such that $a \cdot 1 = 1 \cdot a = a$ for every real number a.

The real number 0 of Property 7 is called the **additive identity;** the real number 1 of Property 8 is called the **multiplicative identity.**

Inverses

Property 9. For every real number a, there is a unique real number, denoted by $-a$, such that

$$a + (-a) = (-a) + a = 0$$

Property 10. For every real number $a \neq 0$, there is a unique real number, denoted by $1/a$, such that

$$a\left(\frac{1}{a}\right) = \left(\frac{1}{a}\right)a = 1$$

The number $-a$ of Property 9 is called the **negative** or **additive inverse** of a. The number $1/a$ of Property 10 is called the **reciprocal** or **multiplicative inverse** of a.

Distributive Laws

Property 11. $a(b + c) = ab + ac$

Property 12. $(a + b)c = ac + bc$

EXAMPLE 2

Specify the property illustrated by each of the following statements.

(a) $2 + 3 = 3 + 2$ (b) $(2 \cdot 3) \cdot 4 = 2 \cdot (3 \cdot 4)$

(c) $2 \cdot \frac{1}{2} = 1$ (d) $2(3 + 5) = 2 \cdot 3 + 2 \cdot 5$

SOLUTION

(a) commutative law of addition
(b) associative law of multiplication
(c) multiplicative inverse
(d) distributive law

EQUALITY

When we say that two numbers are **equal,** we mean that they are identical. Thus, when we write

$$a = b$$

(read "a equals b"), we mean that a and b represent the same number. For example, $2 + 5$ and $4 + 3$ are different ways of writing the number 7, so we can write

$$2 + 5 = 4 + 3$$

Equality satisfies four basic properties.

Properties of Equality

Let a, b, and c be elements of a set.

1. $a = a$ **Reflexive property**
2. If $a = b$, then $b = a$. **Symmetric property**
3. If $a = b$ and $b = c$, then $a = c$. **Transitive property**
4. If $a = b$, then a may be replaced by b in any statement that involves a or b. **Substitution property**

EXAMPLE 3

Specify the property illustrated by each of the following statements.

(a) If $5a - 2 = b$, then $b = 5a - 2$.
(b) If $a = b$ and $b = 5$ then $a = 5$.
(c) If $3(a + 2) = 3a + 6$, and $a = b$, then $3(b + 2) = 3b + 6$.

SOLUTION

(a) symmetric property
(b) transitive property
(c) substitution property

THEOREMS

Using Properties 1–12, the properties of equality, and rules of logic, we can *prove* many other properties of the real numbers.

Theorem 1 If a, b, and c are real numbers, and $a = b$, then
(a) $a + c = b + c$
(b) $ac = bc$

This theorem, which will be used often in working with equations, allows us to add the same number to both sides of an equation and to multiply both sides of an equation by the same number. We will prove Theorem 1a and leave the proof of Theorem 1b as an exercise.

PROOF OF THEOREM 1a

Statement	Reason
$a + c$ is a real number	Closure property
$a + c = a + c$	Reflexive property
$a + c = b + c$	Substitution property with $a = b$

The following theorem is the converse of Theorem 1.

Cancellation Laws

Theorem 2 Let a, b, and c be real numbers.
(a) If $a + c = b + c$, then $a = b$.
(b) If $ac = bc$ and $c \neq 0$, then $a = b$.

Part b of Theorem 2 is often called the **cancellation law of multiplication.** We'll prove this theorem to offer another example of the method to be used.

PROOF OF THEOREM 2b

Statement	Reason
ac, bc are real numbers	Closure property
$\frac{1}{c}$ is a real number	Inverse
$(ac)\left(\frac{1}{c}\right) = (bc)\left(\frac{1}{c}\right)$	Theorem 1b
$a\left(c \cdot \frac{1}{c}\right) = b\left(c \cdot \frac{1}{c}\right)$	Associative law
$a \cdot 1 = b \cdot 1$	Multiplicative inverse
$a = b$	Multiplicative identity

We can restate part a of Theorems 1 and 2 in this way: If a, b, and c are real numbers, then $a + c = b + c$ if and only if $a = b$. The connector "if and only if" is used to indicate that either both statements are true or both statements are false.

Theorem 3 Let a and b be real numbers.
(a) $a \cdot 0 = 0 \cdot a = 0$
(b) If $ab = 0$, then $a = 0$ or $b = 0$.

The real numbers a and b are said to be **factors** of the product ab. Part b of Theorem 3 says that a product of two real numbers can be zero only if at least one of the factors is zero.

The next theorem gives us the usual rules of signs.

Theorem 4 Let a and b be real numbers. Then
(a) $-(-a) = a$
(b) $(-a)(b) = -(ab) = a(-b)$
(c) $(-1)(a) = -a$
(d) $(-a)(-b) = ab$
(e) $-(a + b) = (-a) + (-b)$

It is important to note that $-a$ is not necessarily a negative number. In fact, Theorem 4(a) shows that $-(-3) = 3$.

We next introduce the operations of subtraction and division. If a and b are real numbers, the **difference** between a and b, denoted by $a - b$, is defined by

$$a - b = a + (-b)$$

and the operation is called **subtraction.** Thus,

$$6 - 2 = 6 + (-2) = 4 \qquad 2 - 2 = 0 \qquad 0 - 8 = -8$$

It is easy to show that the distributive laws hold for subtraction, that is,

$$a(b - c) = ab - ac$$
$$(a - b)c = ac - bc$$

If a and b are real numbers and $b \neq 0$, then the **quotient** of a and b, denoted by a/b, is defined by

$$\frac{a}{b} = a \cdot \frac{1}{b}$$

and the operation is called **division.** We also write a/b as $a \div b$ and speak of the **fraction** a over b. The numbers a and b are called the **numerator** and **denominator,** respectively, of the fraction a/b. Observe that we have not defined division by zero, since 0 has no reciprocal.

The following theorem summarizes the familiar properties of fractions.

Theorem 5 Let a, b, c, and d be real numbers with $b \neq 0$, $d \neq 0$. Then

	Example
(a) $\dfrac{a}{b} = \dfrac{c}{d}$ if and only if $ad = bc$	$\dfrac{2}{3} = \dfrac{4}{6}$ since $2 \cdot 6 = 3 \cdot 4$
(b) $\dfrac{a}{b} = \dfrac{ad}{bd} = \dfrac{\frac{a}{d}}{\frac{b}{d}}$	$\dfrac{6}{12} = \dfrac{6 \cdot 3}{12 \cdot 3} = \dfrac{\frac{6}{3}}{\frac{12}{3}}$
(c) $\dfrac{a}{b} + \dfrac{c}{b} = \dfrac{a+c}{b}$	$\dfrac{2}{3} + \dfrac{5}{3} = \dfrac{2+5}{3} = \dfrac{7}{3}$
(d) $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad+bc}{bd}$	$\dfrac{2}{5} + \dfrac{3}{4} = \dfrac{2 \cdot 4 + 5 \cdot 3}{5 \cdot 4} = \dfrac{23}{20}$
(e) $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$	$\dfrac{2}{3} \cdot \dfrac{4}{5} = \dfrac{2 \cdot 4}{3 \cdot 5} = \dfrac{8}{15}$
(f) $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{a}{b} \cdot \dfrac{d}{c}$ (if $c \neq 0$)	$\dfrac{\frac{2}{3}}{\frac{7}{5}} = \dfrac{2}{3} \cdot \dfrac{5}{7} = \dfrac{2 \cdot 5}{3 \cdot 7} = \dfrac{10}{21}$

PROGRESS CHECK

Perform the indicated operations.

(a) $\dfrac{3}{5} + \dfrac{1}{4}$ (b) $\dfrac{5}{2} \cdot \dfrac{4}{15}$ (c) $\dfrac{2}{3} + \dfrac{3}{7}$

ANSWERS

(a) $\dfrac{17}{20}$ (b) $\dfrac{2}{3}$ (c) $\dfrac{23}{21}$

EXERCISE SET 1.1

In Exercises 1–8 write each set by listing its elements within braces.

1. The set of natural numbers from 3 to 7, inclusive.
2. The set of integers between -4 and 2.
3. The set of integers between -10 and -8.
4. The set of natural numbers from -9 to 3, inclusive.
5. The subset of the set $S = \{-3, -2, -1, 0, 1, 2\}$ consisting of the positive integers in S.
6. The subset of the set $S = \{-\frac{2}{3}, -1.1, 3.7, 4.8\}$ consisting of the negative rational numbers in S.

7. The subset of all $x \in S$, $S = \{1, 3, 6, 7, 10\}$, such that x is an odd integer.

8. The subset of all $x \in S$, $S = \{2, 5, 8, 9, 10\}$, such that x is an even integer.

In Exercises 9–22 determine whether the given statement is true (T) or false (F).

9. -14 is a natural number.

10. $-\frac{4}{5}$ is a rational number.

11. $\pi/3$ is a rational number.

12. $1.75/18.6$ is an irrational number.

13. -1207 is an integer.

14. 0.75 is an irrational number.

15. $\frac{4}{5}$ is a real number.

16. 3 is a rational number.

17. 2π is a real number.

18. The sum of two rational numbers is always a rational number.

19. The sum of two irrational numbers is always an irrational number.

20. The product of two rational numbers is always a rational number.

21. The product of two irrational numbers is always an irrational number.

22. The difference of two irrational numbers is always an irrational number.

In Exercises 23–36 the letters represent real numbers. Identify the property or properties of real numbers that justify each statement.

23. $a + x = x + a$

24. $(xy)z = x(yz)$

25. $xyz + xy = xy(z + 1)$

26. $x + y$ is a real number

27. $(a + b) + 3 = a + (b + 3)$

28. $5 + (x + y) = (x + y) + 5$

29. cx is a real number

30. $(a + 5) + b = (a + b) + 5$

31. $uv = vu$

32. $x + 0 = x$

33. $a(bc) = c(ab)$

34. $xy - xy = 0$

35. $5 \cdot \frac{1}{5} = 1$

36. $xy \cdot 1 = xy$

In Exercises 37–40 find a counterexample; that is, find real values for which the statement is false.

37. $a - b = b - a$

38. $\dfrac{a}{b} = \dfrac{b}{a}$

39. $a(b + c) = ab + c$

40. $(a + b)(c + d) = ac + bd$

In Exercises 41–44 indicate the property or properties of equality that justify the statement.

41. If $3x = 5$, then $5 = 3x$.

42. If $x + y = 7$ and $y = 5$, then $x + 5 = 7$.

43. If $2y = z$ and $z = x + 2$, then $2y = x + 2$.

44. If $x + 2y + 3z = r + s$ and $r = x + 1$, then $x + 2y + 3z = x + 1 + s$.

In Exercises 45–49, a, b, and c are real numbers. Use the properties of the real numbers and the properties of equality to prove each theorem.

45. If $a = b$, then $ac = bc$. (Theorem 1b)

46. If $a = b$ and $c \neq 0$, then $\dfrac{a}{c} = \dfrac{b}{c}$.

47. If $a + c = b + c$, then $a = b$. (Theorem 2a)

48. $a(b - c) = ab - ac$

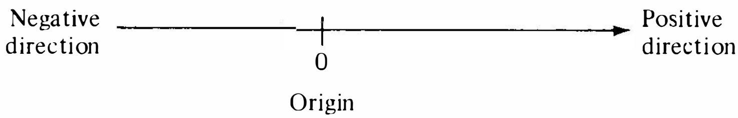
49. Prove that the real number 0 does not have a reciprocal. (*Hint:* Assume $b = 1/0$ is the reciprocal of 0. Supply a reason for each of the following steps.

$$\begin{aligned} 1 &= 0 \cdot \frac{1}{0} \\ &= 0 \cdot b \\ &= 0 \end{aligned}$$

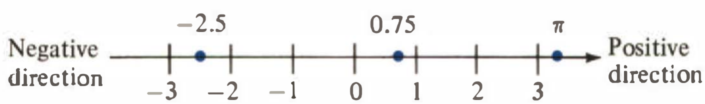
Since this conclusion is impossible, the original assumption must be false.)

1.2 THE REAL NUMBER LINE

There is a simple and very useful geometric interpretation of the real number system. Draw a horizontal straight line; pick a point on this line, label it with the number 0, call it the **origin,** and denote it by O. Designate the side to the right of the origin as the **positive direction** and the side to the left as the **negative direction.**



Next, select a unit of length for measuring distance. With each positive real number r we associate the point that is r units to the right of the origin, and with each negative number $-r$ we associate the point that is r units to the left of the origin. Thus, the set of real numbers is identified with all possible points on a straight line. For every point on the line there is a real number and for every real number there is a point on the line. The line is called the **real number line,** and the number associated with a point is called its **coordinate.** We can now show some points on this line.



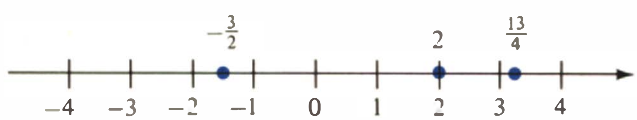
The numbers to the right of zero are called **positive;** the numbers to the left of zero are called **negative.** The positive numbers and zero together are called the **nonnegative** numbers.

We will frequently turn to the real number line to help us picture the results of algebraic computations.

EXAMPLE 1

Draw a real number line and plot the following points: $-\frac{3}{2}$, 2, $\frac{13}{4}$.

SOLUTION



INEQUALITIES

If a and b are real numbers, we can compare their positions on the real number line by using the relations **less than, greater than, less than or equal to,** and **greater than or equal to,** denoted by the **inequality symbols** $<$, $>$, $\leq$, and $\geq$, respectively. Table 1 describes both algebraic and geometric interpretations of the inequality symbols.

TABLE 1

Algebraic Statement	Equivalent Statement	Geometric Statement
$a > 0$	a is positive	a lies to the right of the origin
$a < 0$	a is negative	a lies to the left of the origin
$a > b$	$a - b$ is positive	a lies to the right of b
$a < b$	$a - b$ is negative	a lies to the left of b
$a \geq b$	$a - b$ is zero or positive	a coincides with b or lies to the right of b
$a \leq b$	$a - b$ is zero or negative	a coincides with b or lies to the left of b

Expressions involving inequality symbols, such as $a < b$ and $a \geq b$, are called **inequalities.** We often combine these expressions so that $a \leq b < c$ means both $a \leq b$ and $b < c$. For example, $-5 \leq x < 2$ is equivalent to $-5 \leq x$ and $x < 2$.

PROGRESS CHECK

Verify that the following inequalities are true by using either the "Equivalent Statement" or "Geometric Statement" of Table 1.

(a) $-1 < 3$ (b) $2 \leq 2$ (c) $-2.7 < -1.2$

(d) $-4 < -2 < 0$ (e) $-\frac{7}{2} < \frac{7}{2} < 7$

The real numbers satisfy the following useful properties of inequalities.

Properties of Inequalities

Let a, b, and c be real numbers.

1. One and only one of the following relations holds:

$$a < b, \ a > b, \ a = b \quad \textbf{Trichotomy property}$$

2. If $a < b$ and $b < c$, then $a < c$. **Transitive property**

3. If $a < b$, then $a + c < b + c$.

4. If $a < b$ and $c > 0$, then $ac < bc$. When an inequality is multiplied by a positive number, the sense of the inequality is preserved.

5. If $a < b$ and $c < 0$, then $ac > bc$. When an inequality is multiplied by a negative number, the sense of the inequality is reversed.

EXAMPLE 2

(a) Since $-2 < 4$ and $4 < 5$, then $-2 < 5$.
(b) Since $-2 < 5$, $-2 + 3 < 5 + 3$, or $1 < 8$.
(c) Since $3 < 4$, $3 + (-5) < 4 + (-5)$, or $-2 < -1$.
(d) Since $2 < 5$, $2(3) < 5(3)$, or $6 < 15$.
(e) Since $-3 < 2$, $(-3)(-2) > 2(-2)$, or $6 > -4$.

ABSOLUTE VALUE

Suppose we are interested in the *distances* between the origin and the points labeled 4 and -4 on the real number line. Each of these points is four units from the origin; that is, the *distance is independent of the direction* and is nonnegative (Figure 2).

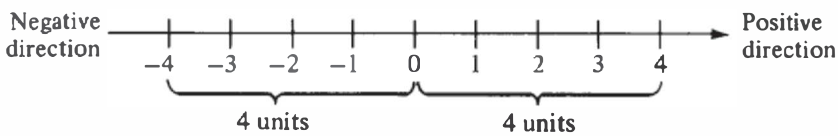


FIGURE 2

When we are interested in the magnitude of a number a, and don't care about the direction or sign, we use the concept of **absolute value,** which we write as $|a|$. The formal definition of absolute value is stated as follows.

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Since distance is independent of direction and is always nonnegative, we can view $|a|$ as the distance from the origin to either point a or point $-a$ on the real number line.

EXAMPLE 3

(a) $|4| = 4$ $\quad |-4| = 4$ $\quad |0| = 0$
(b) The distance on the real number line between the point labeled 3.4 and the origin is $|3.4| = 3.4$. Similarly, the distance between point -2.3 and the origin is $|-2.3| = 2.3$.

In working with the notation of absolute value, it is important to perform the operations within the bars first. Here are some examples.

EXAMPLE 4

(a) $|5 - 2| = |3| = 3$ $\quad$ (b) $|2 - 5| = |-3| = 3$
(c) $|3 - 5| - |8 - 6| = |-2| - |2| = 2 - 2 = 0$
(d) $\dfrac{|4 - 7|}{-6} = \dfrac{|-3|}{-6} = \dfrac{3}{-6} = -\dfrac{1}{2}$

The following properties of absolute value follow from the definition.

Properties of Absolute Value	For all real numbers a and b, **1.** $\lvert a \rvert \geq 0$ **2.** $\lvert a \rvert = \lvert -a \rvert$ **3.** $\lvert a - b \rvert = \lvert b - a \rvert$

We began by showing a use for absolute value in denoting distance from the origin without regard to direction. We will conclude by demonstrating the use of absolute value to denote the distance between *any* two points a and b on the real number line. In Figure 3, the distance between the points labeled 2 and 5 is 3 units and can be obtained by evaluating either $\lvert 5 - 2 \rvert$ or $\lvert 2 - 5 \rvert$. Similarly, the distance between the points labeled -1 and 4 is given by either $\lvert 4 - (-1) \rvert = 5$ or $\lvert -1 - 4 \rvert = 5$. Using the notation $\overline{AB}$ to denote the distance between the points A and B, we provide the following definition.

Distance on the Real Number Line	The **distance** $\overline{AB}$ between points A and B on the real number line, whose coordinates are a and b, respectively, is given by $\overline{AB} = \lvert b - a \rvert$

Property (3) then tells us that $\overline{AB} = \lvert b - a \rvert = \lvert a - b \rvert$. Viewed another way, Property (3) states that the distance between any two points on the real number line is independent of the direction.

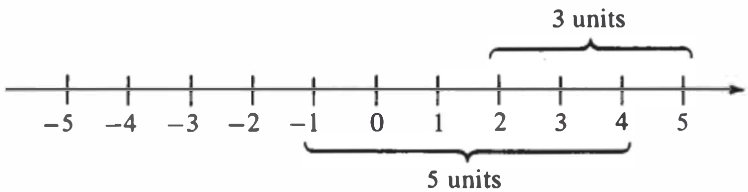


FIGURE 3

EXAMPLE 5

Let points A, B, and C have coordinates -4, -1, and 3, respectively, on the real number line. Find the following distances.

(a) $\overline{AB}$ (b) $\overline{CB}$ (c) $\overline{OB}$

SOLUTION

Using the definition, we have

(a) $\overline{AB} = \lvert -1 - (-4) \rvert = \lvert -1 + 4 \rvert = \lvert 3 \rvert = 3$

(b) $\overline{CB} = \lvert -1 - 3 \rvert = \lvert -4 \rvert = 4$

(c) $\overline{OB} = \lvert -1 - 0 \rvert = \lvert -1 \rvert = 1$

PROGRESS CHECK

The points P, Q, and R on the real number line have coordinates -6, 4, and 6, respectively. Find the following distances.

(a) $\overline{PR}$ (b) $\overline{QP}$ (c) $\overline{PQ}$

ANSWERS

(a) 12 (b) 10 (c) 10

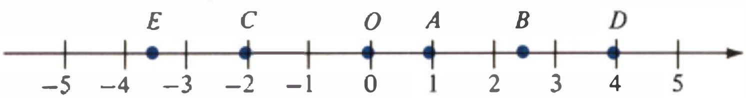
EXERCISE SET 1.2

1. Draw a real number line and plot the following points.
   (a) 4 (b) -2 (c) $\frac{3}{2}$ (d) -3.5
   (e) 0

2. Draw a real number line and plot the following points.
   (a) -5 (b) 4 (c) -3.5 (d) $\frac{7}{2}$
   (e) -4

3. Give the real numbers associated with the points A, B, C, D, O, and E on the real number line below.

4. Represent the following by real numbers.
   (a) a profit of \$10
   (b) a loss of \$20
   (c) a temperature of 20° above zero
   (d) a temperature of 5° below zero



In Exercises 5–10 indicate which of the two given numbers appears first, viewed from left to right, on the real number line.

5. 4, 6 6. $\frac{1}{2}$, 0 7. -2, $\frac{3}{4}$ 8. 0, -4

9. -5, $-\frac{2}{3}$ 10. 4, -5

In Exercises 11–14 indicate the set of numbers on a real number line.

11. The natural numbers less than 8.

12. The natural numbers greater than 4 and less than 10.

13. The integers that are greater than 2 and less than 7.

14. The integers that are greater than -5 and less than or equal to 1.

In Exercises 15–24 express the statement as an inequality.

15. 10 is greater than 9.99.

16. -6 is less than -2.

17. a is nonnegative.

18. b is negative.

19. x is positive.

20. a is between 3 and 7.

21. b is less than or equal to -4.

22. a is between $\frac{1}{2}$ and $\frac{1}{4}$.

23. b is greater than or equal to 5.

24. x is negative.

In Exercises 25–30 give a property of inequalities that justifies the statement.

25. Since $-3 < 1$, then $-1 < 3$.

26. Since $-5 < -1$ and $-1 < 4$, then $-5 < 4$.

27. Since $14 > 9$, then $-14 < -9$.

28. Since $5 > 3$, then $5 \neq 3$.

29. Since $-1 < 6$, then $-3 < 18$.

30. Since $6 > -1$, then 7 is a positive number.

In Exercises 31–44 find the value of the expression.

31. $|2|$ 32. $|-\frac{2}{3}|$ 33. $|1.5|$ 34. $|-0.8|$

35. $-|2|$ 36. $-|-\frac{2}{5}|$ 37. $|2-3|$ 38. $|2-2|$

39. $|2-(-2)|$ 40. $|2|+|-3|$ 41. $\dfrac{|14-8|}{|-3|}$ 42. $\dfrac{|2-12|}{|1-6|}$

43. $\dfrac{|3|-|2|}{|3|+|2|}$ 44. $\dfrac{|3-2|}{|3+2|}$

In Exercises 45–50 the coordinates of points A and B are given. Find $\overline{AB}$.

45. 2, 5 46. -3, 6 47. -3, -1 48. -4, $\frac{11}{2}$

49. $-\frac{4}{5}$, $\frac{4}{5}$ 50. 2, 2

1.3 ALGEBRAIC EXPRESSIONS; POLYNOMIALS

A **variable** is a symbol to which we can assign values. For example, in Section 1.1 we defined a rational number as one that can be written as p/q, where p and q are integers (and q is not zero). The symbols p and q are variables, since we can assign values to them. A variable can be restricted to a particular number system (for example, p and q must be integers) or to a subset of a number system (note that q cannot be zero).

If we invest P dollars at an annual interest rate of 6%, then we will earn $0.06P$ dollars interest per year, and we will have $P + 0.06P$ dollars at the end of the year. We call $P + 0.06P$ an **algebraic expression.** Note that an algebraic expression involves **variables** (in this case P), **constants** (such as 0.06), and **algebraic operations** (such as $+$, $-$, $\times$, $\div$). Virtually everything we do in algebra involves algebraic expressions, sometimes as simple as our example and sometimes very involved.

An algebraic expression takes on a **value** when we assign a specific number to each variable in the expression. Thus, the expression

$$\frac{3m + 4n}{m + n}$$

is **evaluated** when $m = 3$ and $n = 2$ by substitution of these values for m and n:

$$\frac{3(3) + 4(2)}{3 + 2} = \frac{9 + 8}{5} = \frac{17}{5}$$

We often need to write algebraic expressions in which a variable multiplies itself repeatedly. We use the notation of exponents to indicate such repeated multiplication. Thus,

$$a^1 = a \qquad a^2 = a \cdot a \qquad a^n = \underbrace{a \cdot a \cdot \ \ldots \ \cdot a}_{n \text{ factors}}$$

where n is a natural number and a is a real number. We call a the **base** and n the **exponent** and say that a^n is the nth **power** of a. When $n = 1$, we simply write a rather than a^1.

It is convenient to define a^0 for all real numbers $a \neq 0$ by having $a^0 = 1$. We will provide motivation for this seemingly arbitrary definition in Section 1.6.

EXAMPLE 1
Write without using exponents.
(a) $(\frac{1}{2})^3$ (b) $2x^3$ (c) $(2x)^3$ (d) $-3x^2y^3$

SOLUTION
(a) $(\frac{1}{2})^3 = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8}$ (b) $2x^3 = 2\cdot x\cdot x\cdot x$
(c) $(2x)^3 = 2x\cdot 2x\cdot 2x = 8\cdot x\cdot x\cdot x$ (d) $-3x^2y^3 = -3\cdot x\cdot x\cdot y\cdot y\cdot y$

WARNING Note the difference between

$$(-3)^2 = (-3)(-3) = 9$$

and

$$-3^2 = -(3\cdot 3) = -9$$

Later in this chapter we will need an important rule of exponents. Observe that if m and n are natural numbers and a is any real number, then

$$a^m\cdot a^n = \underbrace{a\cdot a\cdot\ldots\cdot a}_{m \text{ factors}} \cdot \underbrace{a\cdot a\cdot\ldots\cdot a}_{n \text{ factors}}$$

Since there are a total of $m + n$ factors on the right side, we conclude that

$$\boxed{a^m a^n = a^{m+n}}$$

EXAMPLE 2
Multiply.
(a) $x^2\cdot x^3$ (b) $(3x)(4x^4)$

SOLUTION
(a) $x^2\cdot x^3 = x^{2+3} = x^5$
(b) $(3x)(4x^4) = 3\cdot 4\cdot x\cdot x^4 = 12x^{1+4} = 12x^5$

PROGRESS CHECK
Multiply.
(a) $x^5\cdot x^2$ (b) $(2x^6)(-2x^4)$

ANSWERS
(a) x^7 (b) $-4x^{10}$

POLYNOMIALS

A polynomial is an algebraic expression of a certain form. Polynomials play an important role in the study of algebra, since many word problems translate into equations or inequalities that involve polynomials. We first study the manipulative and mechanical aspects of polynomials; this knowledge will serve as background for dealing with their applications in later chapters.

Let x denote a variable and let n be a nonnegative integer. The expression ax^n, where a is a constant real number, is called a **monomial in x.** A **polynomial in x** is an expression that is a sum of monomials and has the general form

$$P = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0, \quad a_n \neq 0 \tag{1}$$

Each of the monomials in Equation (1) is called a **term** of P, and $a_0, a_1, \dots, a_n$ are constant real numbers that are called the **coefficients** of the terms of P. Note that a polynomial may consist of just one term; that is, a monomial is also considered to be a polynomial.

EXAMPLE 3

(a) The following expressions are polynomials in x:

$$3x^4 + 2x + 5 \qquad 2x^3 + 5x^2 - 2x + 1 \qquad \frac{3}{2}x^3$$

Notice that we write $2x^3 + 5x^2 + (-2)x + 1$ as $2x^3 + 5x^2 - 2x + 1$.

(b) The following expressions are not polynomials in x:

$$2x^{1/2} + 5 \qquad 3 - \frac{4}{x} \qquad \frac{2x-1}{x-2}$$

Remember that each term of a polynomial in x must be of the form ax^n where a is a real number and n is a nonnegative integer.

The **degree of a monomial in x** is the exponent of x. Thus, the degree of $5x^3$ is 3. A monomial in which the exponent of x is 0 is called a **constant term** and is said to be of **degree zero.** The nonzero coefficient a_n of the term in P with highest degree is called the **leading coefficient** of P and we say that P is a **polynomial of degree n.** A special case is the polynomial all of whose coefficients are zero. Such a polynomial is called the **zero polynomial,** is denoted by 0, and is said to have no degree.

EXAMPLE 4

Given the polynomial

$$P = 2x^4 - 3x^2 + \frac{4}{3}x - 1$$

The terms of P are $\quad 2x^4, \quad 0x^3, \quad -3x^2, \quad \frac{4}{3}x, \quad -1.$

The coefficients of the terms are $\quad 2, \quad 0, \quad -3, \quad \frac{4}{3}, \quad -1.$

The degree of P is 4 and the leading coefficient is 2.

A **monomial in the variables x and y** is an expression of the form ax^my^n, where a is a constant and m and n are nonnegative integers. The number a is called the **coefficient** of the monomial. The **degree of a monomial in x and y** is the sum of the exponents of x and y. Thus, the degree of $2x^3y^2$ is $3 + 2 = 5$. A **polynomial in x and y** is an expression which is a sum of monomials. The **degree of a polynomial in x and y** is the degree of the highest-degree monomial with nonzero coefficient.

EXAMPLE 5

The following are polynomials in x and y:

$2x^2y + y^2 - 3xy + 1$	Degree is 3.
xy	Degree is 2.
$3x^4 + xy - y^2$	Degree is 4.

OPERATIONS WITH POLYNOMIALS

If P and Q are polynomials in x, then the terms cx^r in P and dx^r in Q are said to be **like terms;** that is, like terms have the same exponent in x. For example, given

$$P = 4x^2 + 4x - 1$$

and

$$Q = 3x^3 - 2x^2 + 4$$

then the like terms are $0x^3$ and $3x^3$; $4x^2$ and $-2x^2$; $4x$ and $0x$; -1 and 4.

We define equality of polynomials in the following way.

> Two polynomials are equal if all like terms are identical.

EXAMPLE 6

Find A, B, C, and D if

$$Ax^3 + (A + B)x^2 + Cx + (C - D) = -2x^3 + x + 3$$

SOLUTION

Equating the coefficients of like terms, we have

$$A = -2 \qquad \begin{aligned} A + B &= 0 \\ B &= 2 \end{aligned} \qquad C = 1 \qquad \begin{aligned} C - D &= 3 \\ D &= -2 \end{aligned}$$

If P and Q are polynomials in x, the **sum** $P + Q$ is obtained by forming the sums of all pairs of like terms. The sum of cx^r in P and dx^r in Q is $(c + d)x^r$. Similarly, the **difference** $P - Q$ is obtained by forming the differences, $(c - d)x^r$, of like terms.

EXAMPLE 7

(a) Add $2x^3 + 2x^2 - 3$ and $x^3 - x^2 + x + 2$.
(b) Subtract $2x^3 + x^2 - x + 1$ from $3x^3 - 2x^2 + 2x$.

SOLUTION

(a) Adding the coefficients of like terms,

$$(2x^3 + 2x^2 - 3) + (x^3 - x^2 + x + 2) = 3x^3 + x^2 + x - 1$$

(b) Subtracting the coefficients of like terms,

$$(3x^3 - 2x^2 + 2x) - (2x^3 + x^2 - x + 1) = x^3 - 3x^2 + 3x - 1$$

WARNING

$$(x + 5) - (x + 2) \neq x + 5 - x + 2$$

The coefficient -1 must multiply each term in the parentheses. Thus,

$$-(x + 2) = -x - 2$$

and

$$\begin{aligned}(x + 5) - (x + 2) &= x + 5 - x - 2 \\ &= 3\end{aligned}$$

Multiplication of polynomials is based on the rule for exponents developed earlier in this section,

$$a^m a^n = a^{m+n}$$

and on the distributive laws

$$a(b + c) = ab + ac$$
$$(a + b)c = ac + bc$$

EXAMPLE 8

Multiply $3x^3(2x^3 - 6x^2 + 5)$.

SOLUTION

$$\begin{aligned}3x^3(2x^3 - 6x^2 + 5) &= (3x^3)(2x^3) + (3x^3)(-6x^2) + (3x^3)(5) && \text{Distributive law} \\ &= (3)(2)x^{3+3} + (3)(-6)x^{3+2} + (3)(5)x^3 && a^m a^n = a^{m+n} \\ &= 6x^6 - 18x^5 + 15x^3\end{aligned}$$

EXAMPLE 9

Multiply $(x + 2)(3x^2 - x + 5)$.

SOLUTION

$$
\begin{aligned}
(x+2)(3x^2-x+5) \\
&= x(3x^2-x+5) + 2(3x^2-x+5) && \text{Distributive law} \\
&= 3x^3 - x^2 + 5x + 6x^2 - 2x + 10 && \text{Distributive law and } a^m a^n = a^{m+n} \\
&= 3x^3 + 5x^2 + 3x + 10 && \text{Adding like terms}
\end{aligned}
$$

PROGRESS CHECK

Multiply.

(a) $(x^2+2)(x^2-3x+1)$ (b) $(x^2-2xy+y)(2x+y)$

ANSWERS

(a) $x^4 - 3x^3 + 3x^2 - 6x + 2$ (b) $2x^3 - 3x^2y + 2xy - 2xy^2 + y^2$

The multiplication in Example 9 can be carried out in "long form" as follows.

$$
\begin{array}{rrrrl}
 & 3x^2 & -\ x & +\ 5 & \\
 & & x & +\ 2 & \\
\hline
3x^3 & -\ x^2 & +\ 5x & & = x(3x^2-x+5) \\
 & 6x^2 & -\ 2x & +\ 10 & = 2(3x^2-x+5) \\
\hline
3x^3 & +\ 5x^2 & +\ 3x & +\ 10 & = \text{sum of above lines}
\end{array}
$$

In Example 9 the product of polynomials of degrees one and two is seen to be a polynomial of degree three. From the multiplication process it is easy to derive the following useful rule.

> The degree of the product of two nonzero polynomials is the sum of the degrees of the polynomials.

Products of the form $(2x+3)(5x-2)$ or $(2x+y)(3x-2y)$ occur often, and we can handle them mentally by the familiar method:

$$10x^2 \qquad -6$$

$$(2x+3)(5x-2) = 10x^2 + 11x - 6$$

$$
\begin{array}{r}
15x \\
-4x \\
\hline
\text{Sum} = 11x
\end{array}
$$

PROGRESS CHECK

(a) Multiply $(2x^2 - xy + y^2)(3x + y)$ in long form.

(b) Multiply $(2x-3)(3x-2)$ mentally.

ANSWERS

(a) $6x^3 - x^2y + 2xy^2 + y^3$ (b) $6x^2 - 13x + 6$

A number of special products occur frequently, and it is worthwhile knowing them.

Special Products	
	$(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2$
	$(a - b)^2 = (a - b)(a - b) = a^2 - 2ab + b^2$
	$(a + b)(a - b) = a^2 - b^2$

EXAMPLE 10

Multiply mentally.

(a) $(x + 2)^2$ (b) $(x - 3)^2$ (c) $(x + 4)(x - 4)$

SOLUTION

(a) $(x + 2)^2 = (x + 2)(x + 2) = x^2 + 4x + 4$

(b) $(x - 3)^2 = (x - 3)(x - 3) = x^2 - 6x + 9$

(c) $(x + 4)(x - 4) = x^2 - 16$

EXERCISE SET 1.3

In Exercises 1–6 evaluate the given expression when $r = 2$, $s = 3$, and $t = 4$.

1. $r + 2s + t$
2. rst
3. $\dfrac{rst}{r + s + t}$
4. $(r + s)t$
5. $\dfrac{r + s}{rt}$
6. $\dfrac{r + s + t}{t}$
7. Evaluate $\frac{2}{3}r + 5$ when $r = 12$.
8. Evaluate $\frac{9}{5}C + 32$ when $C = 37$.
9. If P dollars are invested at a simple interest rate of r percent per year for t years, the amount on hand at the end of t years is $P + Prt$. Suppose you invest \$2000 at 8% per year ($r = 0.08$). Find the amount you will have on hand after
   (a) 1 year; (b) $\frac{1}{2}$ year; (c) 8 months.
10. The perimeter of a rectangle is given by the formula $P = 2\ (L + W)$, where L is the length and W is the width of the rectangle. Find the perimeter if
    (a) $L = 2$ feet, $W = 3$ feet;
    (b) $L = \frac{1}{2}$ meter, $W = \frac{1}{4}$ meter.
11. Evaluate $0.02r + 0.314st + 2.25t$ when $r = 2.5$, $s = 3.4$, and $t = 2.81$.
12. Evaluate $10.421x + 0.821y + 2.34xyz$ when $x = 3.21$, $y = 2.42$, and $z = 1.23$.

Evaluate the given expression in Exercises 13–18.

13. $|x| - |x| \cdot |y|$ when $x = -3$, $y = 4$
14. $|x + y| + |x - y|$ when $x = -3$, $y = 2$
15. $\dfrac{|a - 2b|}{2a}$ when $a = 1$, $b = 2$
16. $\dfrac{|x| + |y|}{|x| - |y|}$ when $x = -3$, $y = 4$
17. $\dfrac{-|a - 2b|}{|a + b|}$ when $a = -2$, $b = -1$
18. $\dfrac{|a - b| - 2|c - a|}{|a - b + c|}$ when $a = -2$, $b = 3$, $c = -5$

Carry out the indicated operations in Exercises 19–24.

19. $b^5 \cdot b^2$
20. $x^3 \cdot x^5$
21. $(4y^3)(-5y^6)$
22. $(-6x^4)(-4x^7)$
23. $\left(\frac{3}{2}x^3\right)(-2x)$
24. $\left(-\frac{5}{3}x^6\right)\left(-\frac{3}{10}x^3\right)$

25. Which of the following expressions are *not* polynomials?
(a) $-3x^2 + 2x + 5$ (b) $-3x^2y$
(c) $-3x^{2/3} + 2xy + 5$ (d) $-2x^{-4} + 2xy^3 + 5$

26. Which of the following expressions are *not* polynomials?
(a) $4x^5 - x^{1/2} + 6$ (b) $\frac{2}{5}x^3 + \frac{4}{3}x - 2$
(c) $4x^5y$ (d) $x^{4/3}y + 2x - 3$

In Exercises 27–30 indicate the leading coefficient and the degree of the given polynomial.

27. $2x^3 + 3x^2 - 5$

28. $-4x^5 - 8x^2 + x + 3$

29. $\frac{3}{5}x^4 + 2x^2 - x - 1$

30. $-1.5 + 7x^3 + 0.75x^7$

In Exercises 31–34 find the degree of the given polynomial.

31. $3x^2y - 4x^2 - 2y + 4$

32. $4xy^3 + xy^2 - y^2 + y$

33. $2xy^3 - y^3 + 3x^2 - 2$

34. $\frac{1}{2}x^3y^3 - 2$

35. Find the value of the polynomial $3x^2y^2 + 2xy - x + 2y + 7$ when $x = 2$ and $y = -1$.

36. Find the value of the polynomial $0.02x^2 + 0.3x - 0.5$ when $x = 0.3$.

37. Find the value of the polynomial $2.1x^3 + 3.3x^2 - 4.1x - 7.2$ when $x = 4.1$.

38. Write a polynomial giving the area of a circle of radius r.

39. Write a polynomial giving the area of a triangle of base b and height h.

40. A field consists of a rectangle and a square arranged as shown in Figure 4. What does each of the following polynomials represent?
(a) $x^2 + xy$ (b) $2x + 2y$
(c) $4x$ (d) $4x + 2y$

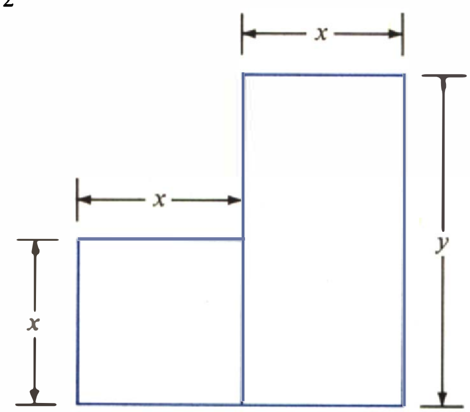


FIGURE 4

41. An investor buys x shares of G.E. stock at \$55 per share, y shares of Exxon stock at \$45 per share, and z shares of A.T.&T. stock at \$60 per share. What does the polynomial $55x + 45y + 60z$ represent?

Perform the indicated operations in Exercises 42–60.

42. $(4x^2 + 3x + 2) + (3x^2 - 2x - 5)$

43. $(2x^2 + 3x + 8) - (5 - 2x + 2x^2)$

44. $4xy^2 + 2xy + 2x + 3 - (-2xy^2 + xy - y + 2)$

45. $(2s^2t^3 - st^2 + st - s + t) - (3s^2t^2 - 2s^2t - 4st^2 - t + 3)$

46. $3xy^2z - 4x^2yz + xy + 3 - (2xy^2z + x^2yz - yz + x - 2)$

47. $a^2bc + ab^2c + 2ab^3 - 3a^2bc - 4ab^3 + 3$

48. $(x + 1)(x^2 + 2x - 3)$

49. $(2 - x)(2x^3 + x - 2)$

50. $(2s - 3)(s^3 - s + 2)$

51. $(-3s + 2)(-2s^2 - s + 3)$

52. $(x^2 + 3)(2x^2 - x + 2)$

53. $(2y^2 + y)(-2y^3 + y - 3)$

54. $(x^2 + 2x - 1)(2x^2 - 3x + 2)$

55. $(a^2 - 4a + 3)(4a^3 + 2a + 5)$

56. $(2a^2 + ab + b^2)(3a - b^2 + 1)$

57. $(-3a + ab + b^2)(3b^2 + 2b + 2)$

58. $5(2x - 3)^2$

59. $2(3x - 2)(3 - x)$

60. $(x - 1)(x + 2)(x + 3)$

61. An investor buys x shares of IBM stock at \$260 per share at Thursday's opening of the stock market. Later in the day, he sells y shares of G&W stock at \$13 per share and z shares of Holiday Inn stock at \$17 per share. Write a polynomial that expresses the money transactions for the day.

62. An artist takes a rectangular piece of cardboard whose sides are x and y and cuts out a square of side $x/2$ (Figure 5) to obtain a mat for a painting. Write a polynomial giving the area of the mat.

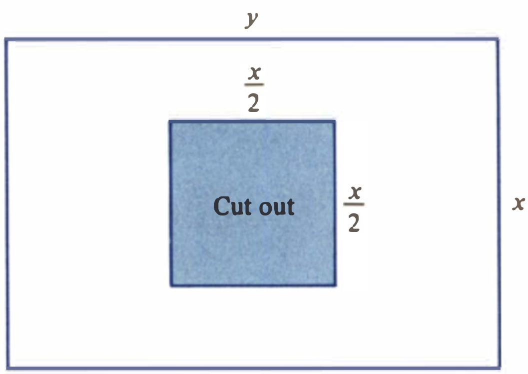


FIGURE 5

In Exercises 63–76 perform the multiplication mentally.

63. $(x - 1)(x + 3)$

64. $(x + 2)(x + 3)$

65. $(2x + 1)(2x + 3)$

66. $(3x - 1)(x + 5)$

67. $(3x - 2)(x - 1)$

68. $(x + 4)(2x - 1)$

69. $(x + y)^2$

70. $(x - 4)^2$

71. $(3x - 1)^2$

72. $(x + 2)(x - 2)$

73. $(2x + 1)(2x - 1)$

74. $(3a + 2b)^2$

75. $(x^2 + y^2)^2$

76. $(x - y)^2$

1.4 FACTORING

Now that we can find the product of two polynomials, let's consider the reverse problem: given a polynomial, can we find factors whose product will yield the given polynomial? This process, known as **factoring,** is one of the basic tools of algebra. In this chapter a polynomial with *integer* coefficients is to be factored as a product of polynomials of lower degree with *integer* coefficients; a polynomial with *rational* coefficients is to be factored as a product of polynomials of lower degree with *rational* coefficients.

We will approach factoring by learning to recognize the situations in which factoring is possible.

COMMON FACTORS

Consider the polynomial

$$x^2 + x$$

Since the factor x is common to both terms, we can write

$$x^2 + x = x(x + 1)$$

EXAMPLE 1

Factor.

(a) $15x^3 - 10x^2$ (b) $4x^2y - 8xy^2 + 6xy$ (c) $2x(x + y) - 5y(x + y)$

SOLUTION

(a) Both 5 and x^2 are common to both terms. Therefore,

$$15x^3 - 10x^2 = 5x^2(3x - 2)$$

(b) Here we see that 2, x, and y are common to all terms. Therefore,

$$4x^2y - 8xy^2 + 6xy = 2xy(2x - 4y + 3)$$

(c) The expression $(x + y)$ is found in both terms. Factoring, we have

$$2x(x + y) - 5y(x + y) = (x + y)(2x - 5y)$$

PROGRESS CHECK

Factor.

(a) $4x^2 - x$ (b) $3x^4 - 9x^2$ (c) $3m(2x - 3y) - n(2x - 3y)$

ANSWERS

(a) $x(4x - 1)$ (b) $3x^2(x^2 - 3)$ (c) $(2x - 3y)(3m - n)$

FACTORING BY GROUPING

It is sometimes possible to discover common factors by first grouping terms. The best way to learn the procedure is by studying some examples.

EXAMPLE 2

Factor.

(a) $2ab + b + 2ac + c$ (b) $2x - 4x^2y - 3y + 6xy^2$

SOLUTION

(a) Group those terms containing b and those terms containing c.

$$\begin{aligned} 2ab + b + 2ac + c &= (2ab + b) + (2ac + c) && \text{Grouping} \\ &= b(2a + 1) + c(2a + 1) && \text{Common factors } b, c \\ &= (2a + 1)(b + c) && \text{Common factor } 2a + 1 \end{aligned}$$

(b)

$$\begin{aligned} 2x - 4x^2y - 3y + 6xy^2 &= (2x - 4x^2y) - (3y - 6xy^2) && \text{Grouping with sign change} \\ &= 2x(1 - 2xy) - 3y(1 - 2xy) && \text{Common factors } 2x, 3y \\ &= (1 - 2xy)(2x - 3y) && \text{Common factor } 1 - 2xy \end{aligned}$$

PROGRESS CHECK

Factor.

(a) $2m^3n + m^2 + 2mn^2 + n$ (b) $2a^2 - 4ab^2 - ab + 2b^3$

ANSWERS
(a) $(2mn + 1)(m^2 + n)$ (b) $(a - 2b^2)(2a - b)$

FACTORING SECOND-DEGREE POLYNOMIALS

To factor a second-degree polynomial, such as

$$x^2 + 5x + 6$$

we first note that the term x^2 can have come only from $x \cdot x$, so we can write two incomplete factors like this:

$$x^2 + 5x + 6 = (x \qquad)(x \qquad)$$

The constant term $+6$ can be the product of either two positive numbers or two negative numbers. Since the middle term $+5x$ is the sum of two other products, both signs must be positive. Thus,

$$x^2 + 5x + 6 = (x + \qquad)(x + \qquad)$$

Finally, the number 6 can be written as the product of two integers in only two ways: $1 \cdot 6$ and $2 \cdot 3$. The first pair gives a middle term of $7x$. The second pair gives the actual middle term, $5x$. So

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

EXAMPLE 3
Factor.
(a) $x^2 - 7x + 10$ (b) $x^2 - 3x - 4$

SOLUTION
(a) Since the constant term is positive and the middle term is negative, we must have two negative signs. Integer pairs whose product is 10 are 1 and 10, and 2 and 5. We find that

$$x^2 - 7x + 10 = (x - 2)(x - 5)$$

(b) Since the constant term is negative, we must have opposite signs. Integer pairs whose product is 4 are 1 and 4, and 2 and 2. We find that

$$x^2 - 3x - 4 = (x + 1)(x - 4)$$

When the leading coefficient of a second-degree polynomial is an integer other than 1, the factoring process becomes more complex. To factor the polynomial $ax^2 + bx + c$, where a, b, and c are integers with $a > 1$, we must have

$$ax^2 + bx + c = (rx + u)(sx + v) = (rs)x^2 + (rv + su)x + uv$$

where r, s, u, and v are integers. Equating the coefficients of like terms, we have

$$rs = a \qquad rv + su = b \qquad uv = c$$

These three equations give candidates for r, s, u, and v. The final choices from among the candidates are determined by trial and error, which is made easier by using mental multiplication.

EXAMPLE 4

Factor $2x^2 - x - 6$.

SOLUTION

The term $2x^2$ can result only from the factors $2x$ and x, so the factors must be of the form

$$2x^2 - x - 6 = (2x \qquad)(x \qquad)$$

The constant term, -6, must be the product of factors of opposite signs, so we may write

$$2x^2 - x - 6 = \begin{cases} (2x + \qquad)(x - \qquad) \\ \text{or} \\ (2x - \qquad)(x + \qquad) \end{cases}$$

The integer factors of 6 are

$$1 \cdot 6 \qquad 6 \cdot 1 \qquad 2 \cdot 3 \qquad 3 \cdot 2$$

By trying these we find that

$$2x^2 - x - 6 = (2x + 3)(x - 2)$$

PROGRESS CHECK

Factor.

(a) $3x^2 - 16x + 21$ (b) $2x^2 + 3x - 9$

ANSWERS

(a) $(3x - 7)(x - 3)$ (b) $(2x - 3)(x + 3)$

WARNING The polynomial $x^2 - 6x$ can be written as

$$x^2 - 6x = x(x - 6)$$

and is then a product of two polynomials of positive degree. Students often fail to consider x to be a "true" factor.

SPECIAL FACTORS

There is a special case of the second-degree polynomial that occurs frequently and factors easily. Given the polynomial $x^2 - 9$, we see that each term is a perfect square, and we can easily verify that

$$x^2 - 9 = (x + 3)(x - 3)$$

The general rule, which holds whenever we are dealing with a difference of two squares, may be stated as follows.

Difference of Two Squares

$$a^2 - b^2 = (a + b)(a - b)$$

EXAMPLE 5

Factor.

(a) $4x^2 - 25$ (b) $9r^2 - 16t^2$

SOLUTION

(a) Since

$$4x^2 - 25 = (2x)^2 - (5)^2$$

we may use the formula for the difference of two squares with $a = 2x$ and $b = 5$. Thus,

$$4x^2 - 25 = (2x + 5)(2x - 5)$$

(b) Since

$$9r^2 - 16t^2 = (3r)^2 - (4t)^2$$

we have $a = 3r$ and $b = 4t$, resulting in

$$9r^2 - 16t^2 = (3r + 4t)(3r - 4t)$$

PROGRESS CHECK

Factor.

(a) $x^2 - 49$ (b) $16x^2 - 9$ (c) $25x^2 - y^2$

ANSWERS

(a) $(x + 7)(x - 7)$ (b) $(4x + 3)(4x - 3)$ (c) $(5x + y)(5x - y)$

The formulas for a sum of two cubes and a difference of two cubes can be verified by multiplying the factors on the right-hand sides of the following equations.

Sum and Difference of Two Cubes

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

These formulas provide a direct means of factoring the sum or difference of two cubes and are used in the same way as the formula for a difference of two squares. Be careful with the placement of plus and minus signs when using these formulas.

EXAMPLE 6
Factor.

(a) $x^3 + 1$ (b) $27m^3 - 64n^3$ (c) $\frac{1}{27}u^3 + 8v^3$

SOLUTION
(a) With $a = x$ and $b = 1$, the formula for the sum of two cubes yields the following result:

$$x^3 + 1 = (x + 1)(x^2 - x + 1)$$

(b) Since

$$27m^3 - 64n^3 = (3m)^3 - (4n)^3$$

we can use the formula for the difference of two cubes with $a = 3m$ and $b = 4n$:

$$27m^3 - 64n^3 = (3m - 4n)(9m^2 + 12mn + 16n^2)$$

(c) Note that

$$\frac{1}{27}u^3 + 8v^3 = \left(\frac{1}{3}u\right)^3 + (2v)^3$$

and then use the formula for the sum of two cubes:

$$\frac{1}{27}u^3 + 8v^3 = \left(\frac{u}{3} + 2v\right)\left(\frac{u^2}{9} - \frac{2}{3}uv + 4v^2\right)$$

COMBINING METHODS

We conclude with problems that combine the various methods of factoring that we have studied. Here is a good rule to follow.

> Always remove common factors before attempting any other factoring techniques.

EXAMPLE 7
Factor.
(a) $2x^3 - 8x$ (b) $3y(y + 3) + 2(y + 3)(y^2 - 1)$

SOLUTION
(a) Removing $2x$ as a common factor, we find that

"NO FUSS" FACTORING FOR SECOND-DEGREE POLYNOMIALS

Factoring involves a certain amount of trial and error, which can become frustrating, especially when the lead coefficient is not 1. You might want to try a rather neat scheme that will greatly reduce the number of candidates.

We'll demonstrate the method for the polynomial

$$4x^2 + 11x + 6 \tag{1}$$

Using the lead coefficient of 4, write the pair of incomplete factors

$$(4x \qquad)(4x \qquad) \tag{2}$$

Next, multiply the coefficient of x^2 and the constant term in (1) to produce $4 \cdot 6 = 24$. Now find two integers whose product is 24 and whose sum is 11, the coefficient of the middle term of (1). It's clear that 8 and 3 will do nicely, so we write

$$(4x + 8)(4x + 3) \tag{3}$$

Finally, within each parenthesis in (3) discard any common divisor. Thus $(4x + 8)$ reduces to $(x + 2)$ and we write

$$(x + 2)(4x + 3) \tag{4}$$

which is the factorization of $4x^2 + 11x + 6$.

Will the method always work? Yes—if you first remove all common factors in the original polynomial. That is, you must first write

$$6x^2 + 15x + 6 = 3(2x^2 + 5x + 2)$$

and apply the method to the polynomial $2x^2 + 5x + 2$.

(For a proof that the method works, see M. A. Autrie and J. D. Austin, "A Novel Way to Factor Quadratic Polynomials," *The Mathematics Teacher* 72, no. 2 [1979].)

We'll use the polynomial $2x^2 - x - 6$ of Example 7 to demonstrate the method when some of the coefficients are negative.

Try the method on these second-degree polynomials.

$$3x^2 + 10x - 8$$
$$6x^2 - 13x + 6$$
$$4x^2 - 15x - 4$$
$$10x^2 + 11x - 6$$

Factoring $ax^2 + bx + c$	**Example: $2x^2 - x - 6$**
Step 1. Use the lead coefficient a to write the incomplete factors $(ax \quad)(ax \quad)$	*Step 1.* The lead coefficient is 2, so we write $(2x \quad)(2x \quad)$
Step 2. Multiply a and c, the coefficients of x^2 and the constant term.	*Step 2.* $a \cdot c = (2)(-6) = -12$
Step 3. Find integers whose product is $a \cdot c$ and whose sum equals b. Write these integers in the incomplete factors of Step 1.	*Step 3.* Two integers whose product is -12 and whose sum is -1 are 3 and -4. We then write $(2x + 3)(2x - 4)$
Step 4. Discard any common factor *within each parenthesis* in Step 3. The result is the desired factorization.	*Step 4.* Reducing $(2x - 4)$ to $(x - 2)$, we have $2x^2 - x - 6 = (2x + 3)(x - 2)$

$$2x^3 - 8x = 2x(x^2 - 4)$$
$$= 2x(x + 2)(x - 2)$$

(b) Removing the common factor $y + 3$, we see that

$$3y(y + 3) + 2(y + 3)(y^2 - 1) = (y + 3)[3y + 2(y^2 - 1)]$$
$$= (y + 3)(3y + 2y^2 - 2)$$
$$= (y + 3)(2y^2 + 3y - 2)$$
$$= (y + 3)(2y - 1)(y + 2)$$

PROGRESS CHECK

Factor.

(a) $x^3 + 5x^2 - 6x$ (b) $2x^3 - 2x^2y - 4xy^2$
(c) $-3x(x + 1) + (x + 1)(2x^2 + 1)$

ANSWERS

(a) $x(x + 6)(x - 1)$ (b) $2x(x + y)(x - 2y)$ (c) $(x + 1)(2x - 1)(x - 1)$

IRREDUCIBLE POLYNOMIALS

Are there polynomials that cannot be written as a product of polynomials of lower degree with integer coefficients? The answer is yes. Examples are the polynomials $x^2 + 1$ and $x^2 + x + 1$. A polynomial is said to be **prime** or **irreducible** if it cannot be written as a product of two polynomials each of positive degree. Thus, $x^2 + 1$ is irreducible over the integers.

EXERCISE SET 1.4

Factor completely.

1. $5x - 15$
2. $\frac{1}{2}x + \frac{1}{4}y$
3. $-2x - 8y$
4. $3x - 6y + 15$
5. $5bc + 25b$
6. $2x^4 + x^2$
7. $-3y^2 - 4y^5$
8. $3abc + 12bc$
9. $3x^2 + 6x^2y - 9x^2z$
10. $9a^3b^3 + 12a^2b - 15ab^2$
11. $x^2 + 4x + 3$
12. $x^2 + 2x - 8$
13. $y^2 - 8y + 15$
14. $y^2 + 7y - 8$
15. $a^2 - 7ab + 12b^2$
16. $x^2 - 49$
17. $y^2 - \frac{1}{9}$
18. $a^2 - 7a + 10$
19. $9 - x^2$
20. $4b^2 - a^2$
21. $x^2 - 5x - 14$
22. $x^2y^2 - 9$
23. $\frac{1}{16} - y^2$
24. $4a^2 - b^2$
25. $x^2 - 6x + 9$
26. $a^2b^2 - \frac{1}{9}$
27. $x^2 - 12x + 20$
28. $x^2 - 8x - 20$
29. $x^2 + 11x + 24$
30. $y^2 - \frac{9}{16}$
31. $2x^2 - 3x - 2$
32. $2x^2 + 7x + 6$
33. $3a^2 - 11a + 6$
34. $4x^2 - 9x + 2$
35. $6x^2 + 13x + 6$
36. $4y^2 - 9$
37. $8m^2 - 6m - 9$
38. $9x^2 + 24x + 16$
39. $10x^2 - 13x - 3$
40. $9y^2 - 16x^2$
41. $6a^2 - 5ab - 6b^2$
42. $4x^2 + 20x + 25$
43. $10r^2s^2 + 9rst + 2t^2$
44. $x^{12} - 1$
45. $16 - 9x^2y^2$
46. $6 + 5x - 4x^2$

47. $8n^2 - 18n - 5$
48. $15 + 4x - 4x^2$
49. $2x^2 - 2x - 12$
50. $3y^2 + 6y - 45$
51. $30x^2 - 35x + 10$
52. $x^4y^4 - x^2y^2$
53. $18x^2m + 33xm + 9m$
54. $25m^2n^3 - 5m^2n$
55. $12x^2 - 22x^3 - 20x^4$
56. $10r^2 - 5rs - 15s^2$
57. $x^4 - y^4$
58. $a^4 - 16$
59. $b^4 + 2b^2 - 8$
60. $4b^4 + 20b^2 + 25$
61. $x^3 + 27y^3$
62. $8x^3 + 125y^3$
63. $27x^3 - y^3$
64. $64x^3 - 27y^3$
65. $a^3 + 8$
66. $8r^3 - 27$
67. $\frac{1}{8}m^3 - 8n^3$
68. $8a^3 - \frac{1}{64}b^3$
69. $(x + y)^3 - 8$
70. $27 + (x + y)^3$
71. $8x^6 - 125y^6$
72. $a^6 + 27b^6$
73. $4(x + 1)(y + 2) - 8(y + 2)$
74. $2(x + 1)(x - 1) + 5(x - 1)$
75. $3(x + 2)^2(x - 1) - 4(x + 2)^2(2x + 7)$
76. $4(2x - 1)^2(x + 2)^3(x + 1) - 3(2x - 1)^5(x + 2)^2(x + 3)$

1.5 RATIONAL EXPRESSIONS

Much of the terminology and many of the techniques of arithmetic fractions carry over to **algebraic fractions,** which are the quotients of algebraic expressions. In particular, we refer to a quotient of two polynomials as a **rational expression.** Our objective in this section is to review the procedures for adding, subtracting, multiplying, and dividing rational expressions. We will then be able to convert a complicated fraction like

$$\frac{1 - \dfrac{1}{x}}{\dfrac{1}{x^2} + \dfrac{1}{x}}$$

into a form that will simplify evaluation of the fraction and facilitate other operations with it.

MULTIPLICATION AND DIVISION OF RATIONAL EXPRESSIONS

The symbols appearing in rational expressions represent real numbers. We may, therefore, apply the rules of arithmetic to rational expressions.

$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$	Multiplication of rational expressions
$\dfrac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$	Division of rational expressions

EXAMPLE 1

Divide $\frac{2x}{y}$ by $\frac{3y^3}{x - 3}$.

SOLUTION

$$\frac{\dfrac{2x}{y}}{\dfrac{3y^3}{x-3}} = \frac{2x}{y}\cdot\frac{x-3}{3y^3} = \frac{2x(x-3)}{3y^4}$$

The basic rule that allows us to simplify rational expressions is the cancellation principle.

Cancellation Principle	$\dfrac{ab}{ac} = \dfrac{b}{c}, \qquad a \neq 0$

This rule results from the fact that $a/a = 1$. Thus,

$$\frac{ab}{ac} = \frac{a}{a}\cdot\frac{b}{c} = 1\cdot\frac{b}{c} = \frac{b}{c}$$

Once again we find that a rule for arithmetic of fractions carries over to rational expressions.

EXAMPLE 2

Simplify.

(a) $\dfrac{x^2-4}{x^2+5x+6}$ (b) $\dfrac{3x^2(y-1)}{y+1} \div \dfrac{6x(y-1)^2}{(y+1)^3}$ (c) $\dfrac{x^2-x-6}{3x-x^2}$

SOLUTION

(a) $$\frac{x^2-4}{x^2+5x+6} = \frac{(x+2)(x-2)}{(x+3)(x+2)} = \frac{x-2}{x+3}, \qquad x \neq -2$$

(b) $$\frac{\dfrac{3x^2(y-1)}{y+1}}{\dfrac{6x(y-1)^2}{(y+1)^3}} = \frac{3x^2(y-1)}{y+1}\cdot\frac{(y+1)^3}{6x(y-1)^2} = \frac{3x^2(y-1)(y+1)^3}{6x(y-1)^2(y+1)}$$

$$= \frac{x(y+1)^2}{2(y-1)}, \qquad y \neq 1, -1$$

(c) $$\frac{x^2-x-6}{3x-x^2} = \frac{(x-3)(x+2)}{x(3-x)} = \frac{(x-3)(x+2)}{-x(x-3)}$$

$$= \frac{x+2}{-x} = -\frac{x+2}{x}, \qquad x \neq 3$$

Note that in Example 2c we wrote $(3-x)$ as $-(x-3)$. This technique is often used to recognize factors that may be canceled.

PROGRESS CHECK

Simplify.

(a) $\dfrac{4-x^2}{x^2-x-6}$ (b) $\dfrac{8-2x}{y} \div \dfrac{x^2-16}{y}$

ANSWERS

(a) $\dfrac{2-x}{x-3}, \quad x \neq -2$ (b) $-\dfrac{2}{x+4}, \quad x \neq 4$

WARNING

(a) Only multiplicative factors can be canceled. Thus,

$$\frac{2x-4}{x} \neq 2-4$$

Since x is *not a multiplicative factor* in the numerator, we may *not* perform cancellation.

(b) Note that

$$\frac{y^2-x^2}{y-x} \neq y-x$$

To simplify correctly, write

$$\frac{y^2-x^2}{y-x} = \frac{(y+x)(y-x)}{y-x} = y+x, \qquad y \neq x$$

ADDITION AND SUBTRACTION OF RATIONAL EXPRESSIONS

Since the variables in rational expressions represent real numbers, the rules of arithmetic for addition and subtraction of fractions apply to rational expressions. When rational expressions have the same denominator, the addition and subtraction rules are as follows.

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

$$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

For example,

$$\frac{2}{x-1} - \frac{4}{x-1} + \frac{5}{x-1} = \frac{2-4+5}{x-1} = \frac{3}{x-1}$$

To add or subtract rational expressions with *different* denominators, we must first rewrite each rational expression as an equivalent one with the same denominator as the others. Although any common denominator will do, we will concentrate on finding the **least common denominator,** or **LCD,** of two or more rational expressions. We now outline the procedure and provide an example.

EXAMPLE 3

Find the LCD of the following three rational expressions:

$$\frac{1}{x^3 - x^2} \qquad \frac{-2}{x^3 - x} \qquad \frac{3x}{x^2 + 2x + 1}$$

SOLUTION

Least Common Denominator	
Step 1. Factor the denominator of each rational expression.	*Step 1.* $\dfrac{1}{x^2(x-1)} \qquad \dfrac{-2}{x(x-1)(x+1)} \qquad \dfrac{3x}{(x+1)^2}$
Step 2. Determine the different factors in the denominators of the rational expressions, and the highest power to which each factor occurs in any denominator.	*Step 2.* factor / highest exponent / final factors x / 2 / x^2 $x-1$ / 1 / $x-1$ $x+1$ / 2 / $(x+1)^2$
Step 3. The product of the factors determined in *Step 2* is the LCD.	*Step 3.* The LCD is $x^2(x-1)(x+1)^2$

PROGRESS CHECK

Find the LCD of the following fractions:

$$\frac{2a}{(3a^2 + 12a + 12)b} \qquad \frac{-7b}{a(4b^2 - 8b + 4)} \qquad \frac{3}{ab^3 + 2b^3}$$

ANSWER

$12ab^3 (a + 2)^2(b - 1)^2$

Equivalent Fractions

The fractions 2/5 and 6/15 are said to be **equivalent,** because we obtain 6/15 by multiplying 2/5 by $1 = 3/3$. We also say that algebraic fractions are **equivalent fractions** if we can obtain one from the other by multiplying both the numerator and denominator by the same expression.

To add rational expressions, we must first determine the LCD and then convert each rational expression into an equivalent fraction with the LCD as its denominator. We can accomplish this conversion by multiplying the fraction by the appropriate equivalent of 1. We now outline the procedure and provide an example.

EXAMPLE 4
Simplify

$$\frac{x+1}{2x^2} - \frac{2}{3x(x+2)}$$

SOLUTION

Addition of Rational Expressions	
Step 1. Find the LCD.	*Step 1*. $$\text{LCD} = 6x^2(x+2)$$
Step 2. Multiply each rational expression by a fraction whose numerator and denominator are the same and consist of all factors of the LCD that are missing in the denominator of the expression.	*Step 2*. $$\frac{x+1}{2x^2}\cdot\frac{3(x+2)}{3(x+2)} = \frac{3x^2+9x+6}{6x^2(x+2)}$$ $$\frac{2}{3x(x+2)}\cdot\frac{2x}{2x} = \frac{4x}{6x^2(x+2)}$$
Step 3. Add the rational expressions. Do not multiply out the denominators since it may be possible to cancel.	*Step 3*. $$\frac{x+1}{2x^2} - \frac{2}{3x(x+2)} = \frac{3x^2+9x+6}{6x^2(x+2)} - \frac{4x}{6x^2(x+2)}$$ $$= \frac{3x^2+5x+6}{6x^2(x+2)}$$

PROGRESS CHECK
Perform the indicated operations.

(a) $\dfrac{x-8}{x^2-4} + \dfrac{3}{x^2-2x}$ (b) $\dfrac{4r-3}{9r^3} - \dfrac{2r+1}{4r^2} + \dfrac{2}{3r}$

ANSWERS
(a) $\dfrac{x-3}{x(x+2)}$, $x \neq 2$ (b) $\dfrac{6r^2+7r-12}{36r^3}$

COMPLEX FRACTIONS

At the beginning of this section we said that we wanted to be able to simplify fractions like

$$\frac{1-\dfrac{1}{x}}{\dfrac{1}{x^2}+\dfrac{1}{x}}$$

This is an example of a **complex fraction,** which is a fractional form with fractions in the numerator or denominator or both.

There are two methods commonly used to simplify complex fractions. Fortunately, we already have all the tools needed, and we can now demonstrate both methods.

EXAMPLE 5
Simplify

$$\frac{1 - \frac{1}{x}}{\frac{1}{x^2} + \frac{1}{x}}$$

SOLUTION

Simplifying Complex Fractions

Method 1	Example
Step 1. Find the LCD of all fractions in the numerator and denominator.	*Step 1*. The LCD of $1/1$, $1/x$, and $1/x^2$ is x^2.
Step 2. Multiply the numerator and denominator by the LCD. Since this is multiplication by 1, the result is an equivalent fraction.	*Step 2*. $\dfrac{x^2\left(1 - \frac{1}{x}\right)}{x^2\left(\frac{1}{x^2} + \frac{1}{x}\right)} = \dfrac{x^2 - x}{1 + x} = \dfrac{x(x-1)}{x+1}$

Method 2	Example
Step 1. Combine the terms in the numerator into a single rational expression.	*Step 1*. $1 - \dfrac{1}{x} = \dfrac{x}{x} - \dfrac{1}{x} = \dfrac{x-1}{x}$
Step 2. Combine the terms in the denominator into a single rational expression.	*Step 2*. $\dfrac{1}{x^2} + \dfrac{1}{x} = \dfrac{1}{x^2} + \dfrac{x}{x^2} = \dfrac{1+x}{x^2}$
Step 3. Apply the rules for division of rational expressions; that is, multiply the numerator by the reciprocal of the denominator.	*Step 3*. $\dfrac{\frac{x-1}{x}}{\frac{1+x}{x^2}} = \dfrac{x-1}{x} \cdot \dfrac{x^2}{1+x} = \dfrac{x(x-1)}{x+1}$

PROGRESS CHECK

Simplify.

(a) $\dfrac{2+\dfrac{1}{x}}{1-\dfrac{2}{x}}$ (b) $\dfrac{\dfrac{a}{b}+\dfrac{b}{a}}{\dfrac{1}{a}-\dfrac{1}{b}}$

ANSWERS

(a) $\dfrac{2x+1}{x-2}$ (b) $-\dfrac{a^2+b^2}{a-b}$

EXERCISE SET 1.5

Perform all possible simplification in Exercises 1–20.

1. $\dfrac{x+4}{x^2-16}$
2. $\dfrac{y^2-25}{y+5}$
3. $\dfrac{x^2-8x+16}{x-4}$
4. $\dfrac{5x^2-45}{2x-6}$
5. $\dfrac{6x^2-x-1}{2x^2+3x-2}$
6. $\dfrac{2x^3+x^2-3x}{3x^2-5x+2}$
7. $\dfrac{2}{3x-6}\div\dfrac{3}{2x-4}$
8. $\dfrac{5x+15}{8}\div\dfrac{3x+9}{4}$
9. $\dfrac{25-a^2}{b+3}\cdot\dfrac{2b^2+6b}{a-5}$
10. $\dfrac{2xy^2}{x+y}\cdot\dfrac{x+y}{4xy}$
11. $\dfrac{x+2}{3y}\div\dfrac{x^2-2x-8}{15y^2}$
12. $\dfrac{3x}{x+2}\div\dfrac{6x^2}{x^2-x-6}$
13. $\dfrac{6x^2-x-2}{2x^2-5x+3}\cdot\dfrac{2x^2-7x+6}{3x^2+x-2}$
14. $\dfrac{6x^2+11x-2}{4x^2-3x-1}\cdot\dfrac{5x^2-3x-2}{3x^2+7x+2}$
15. $(x^2-4)\cdot\dfrac{2x+3}{x^2+2x-8}$
16. $(a^2-2a)\cdot\dfrac{a+1}{6-a-a^2}$
17. $(x^2-2x-15)\div\dfrac{x^2-7x+10}{x^2+1}$
18. $\dfrac{2y^2-5y-3}{y-4}\div(y^2+y-12)$
19. $\dfrac{x^2-4}{x^2+2x-3}\cdot\dfrac{x^2+3x-4}{x^2-7x+10}\cdot\dfrac{x+3}{x^2+3x+2}$
20. $\dfrac{x^2-9}{6x^2+x-1}\cdot\dfrac{2x^2+5x+2}{x^2+4x+3}\cdot\dfrac{x^2-x-2}{x^2-3x}$

In Exercises 21–30 find the LCD.

21. $\dfrac{4}{x},\ \dfrac{x-2}{y}$
22. $\dfrac{x}{x-1},\ \dfrac{x+4}{x+2}$
23. $\dfrac{5-a}{a},\ \dfrac{7}{2a}$
24. $\dfrac{x+2}{x},\ \dfrac{x-2}{x^2}$
25. $\dfrac{2b}{b-1},\ \dfrac{3}{(b-1)^2}$
26. $\dfrac{2+x}{x^2-4},\ \dfrac{3}{x-2}$
27. $\dfrac{4x}{x-2},\ \dfrac{5}{x^2+x-6}$
28. $\dfrac{3}{y^2-3y-4},\ \dfrac{2y}{y+1}$
29. $\dfrac{3}{x+1},\ \dfrac{2}{x},\ \dfrac{x}{x-1}$
30. $\dfrac{4}{x},\ \dfrac{3}{x-1},\ \dfrac{x}{x^2-2x+1}$

In Exercises 31–50 perform the indicated operations and simplify.

31. $\dfrac{8}{a-2}+\dfrac{4}{2-a}$
32. $\dfrac{x}{x^2-4}+\dfrac{2}{4-x^2}$
33. $\dfrac{x-1}{3}+2$
34. $\dfrac{1}{x-1}+\dfrac{2}{x-2}$

35. $\dfrac{1}{a+2}+\dfrac{3}{a-2}$

36. $\dfrac{a}{8b}-\dfrac{b}{12a}$

37. $\dfrac{4}{3x}-\dfrac{5}{xy}$

38. $\dfrac{4x-1}{6x^3}+\dfrac{2}{3x^2}$

39. $\dfrac{5}{2x+6}-\dfrac{x}{x+3}$

40. $\dfrac{x}{x-y}-\dfrac{y}{x+y}$

41. $\dfrac{5x}{2x^2-18}+\dfrac{4}{3x-9}$

42. $\dfrac{4}{r}-\dfrac{3}{r+2}$

43. $\dfrac{1}{x-1}+\dfrac{2x-1}{(x-2)(x+1)}$

44. $\dfrac{2x}{2x+1}-\dfrac{x-1}{(2x+1)(x-2)}$

45. $\dfrac{2x}{x^2+x-2}+\dfrac{3}{x+2}$

46. $\dfrac{2}{x-2}+\dfrac{x}{x^2-x-6}$

47. $\dfrac{2x-1}{x^2+5x+6}-\dfrac{x-2}{x^2+4x+3}$

48. $\dfrac{2x-1}{x^3-4x}-\dfrac{x}{x^2+x-2}$

49. $\dfrac{2x}{x^2-1}+\dfrac{x+1}{x^2+3x-4}$

50. $\dfrac{2x}{x+2}+\dfrac{x}{x-2}-\dfrac{1}{x^2-4}$

In Exercises 51–66 simplify the complex fraction and perform all indicated operations.

51. $\dfrac{1+\dfrac{2}{x}}{1-\dfrac{3}{x}}$

52. $\dfrac{x-\dfrac{1}{x}}{2+\dfrac{1}{x}}$

53. $\dfrac{x+1}{1-\dfrac{1}{x}}$

54. $\dfrac{1-\dfrac{r^2}{s^2}}{1+\dfrac{r}{s}}$

55. $\dfrac{x^2-16}{\dfrac{1}{4}-\dfrac{1}{x}}$

56. $\dfrac{\dfrac{a}{a-b}-\dfrac{b}{a+b}}{a^2-b^2}$

57. $2-\dfrac{1}{1+\dfrac{1}{a}}$

58. $\dfrac{\dfrac{4}{x^2-4}+1}{\dfrac{x}{x^2+x-6}}$

59. $\dfrac{\dfrac{a}{b}-\dfrac{b}{a}}{\dfrac{1}{a}+\dfrac{1}{b}}$

60. $\dfrac{\dfrac{x}{x-2}-\dfrac{x}{x+2}}{\dfrac{2x}{x-2}+\dfrac{x^2}{x-2}}$

61. $3-\dfrac{2}{1-\dfrac{1}{1+x}}$

62. $2+\dfrac{3}{1+\dfrac{2}{1-x}}$

63. $\dfrac{y-\dfrac{1}{1-\dfrac{1}{y}}}{y+\dfrac{1}{1+\dfrac{1}{y}}}$

64. $1-\dfrac{1-\dfrac{1}{y}}{y-\dfrac{1}{y}}$

65. $1-\dfrac{1}{1+\dfrac{1}{1-\dfrac{1}{1+x}}}$

66. $1+\dfrac{1}{1-\dfrac{1}{1+\dfrac{1}{1+x}}}$

1.6 INTEGER EXPONENTS

POSITIVE INTEGER EXPONENTS

In Section 1.3 we defined a^n for a real number a and a positive integer n as

$$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}}$$

and we showed that if m and n are positive integers then $a^m a^n = a^{m+n}$. The method we used to establish this rule was to write out the factors a^m and a^n and count the total number of occurrences of a. The same method can be used to establish the rest of the rules in Table 2 when m and n are positive integers.

TABLE 2

POSITIVE INTEGER EXPONENTS	
Rule	**Examples**
$a^m a^n = a^{m+n}$	$4^5 4^2 = 4^{5+2} = 4^7$ $x^3 x^2 = x^{3+2} = x^5$ $(2y)^3(2y)^5 = (2y)^{3+5} = (2y)^8$
$(a^m)^n = a^{mn}$	$(2^2)^3 = 2^{2\cdot 3} = 2^6$ $(x^4)^3 = x^{4\cdot 3} = x^{12}$ $(a^2)^n = a^{2n}$ $[(x+2)^4]^3 = (x+2)^{4\cdot 3} = (x+2)^{12}$
$(ab)^m = a^m b^m$	$(ab)^4 = a^4 b^4$ $(2x^2y)^4 = 2^4(x^2)^4y^4 = 16x^8y^4$
$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, \quad b \neq 0$	$-\left(\frac{2}{x}\right)^3 = -\frac{2^3}{x^3} = -\frac{8}{x^3}$ $\left(\frac{-x}{2y}\right)^4 = \frac{(-x)^4}{(2y)^4} = \frac{x^4}{2^4y^4} = \frac{x^4}{16y^4}$
If $a \neq 0$, $\frac{a^m}{a^n} = a^{m-n}$ if $m > n$ $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$ if $n > m$ $\frac{a^m}{a^n} = 1$ if $m = n$	$\frac{(-3)^2}{(-3)^3} = \frac{1}{(-3)^{3-2}} = \frac{1}{-3} = -\frac{1}{3}$ $\frac{x^{2n+1}}{x^n} = x^{2n+1-n} = x^{n+1}$ $\frac{y^3}{y^3} = 1$

EXAMPLE 1

Simplify the following.

(a) $(4a^2b^3)(2a^3b)$ (b) $(2x^2y)^4$ (c) $\frac{y^{3k-1}}{2y^{2k}}, \quad y \neq 0$

SOLUTION

(a) $(4a^2b^3)(2a^3b) = 4\cdot 2\cdot a^2a^3b^3b = 8a^5b^4$

(b) $(2x^2y)^4 = 2^4(x^2)^4y^4 = 16x^8y^4$

(c) $\frac{y^{3k-1}}{2y^{2k}} = \frac{y^{3k-1-2k}}{2} = \frac{1}{2}y^{k-1}$

PROGRESS CHECK

Simplify, using only positive exponents.

(a) $(x^3)^4$ (b) $x^4(x^2)^3$ (c) $\frac{a^{14}}{a^8}$

(d) $\frac{-2(x+1)^n}{(x+1)^{2n}}$ (e) $(3ab^2)^3$ (f) $\left(\frac{-ab^2}{c^3}\right)^3$

ANSWERS

(a) x^{12} (b) x^{10} (c) a^6 (d) $-\dfrac{2}{(x+1)^n}$ (e) $27a^3b^6$

(f) $-\dfrac{a^3b^6}{c^9}$

ZERO AND NEGATIVE EXPONENTS

We next expand our rules to include zero and negative exponents when the base is nonzero. We will assume that the previous rules for exponents apply to a^0 and see if this leads us to a definition of a^0. For example, applying the rule $a^m a^n = a^{m+n}$ yields

$$a^m a^0 = a^{m+0} = a^m$$

Dividing both sides by a^m, we obtain $a^0 = 1$. We therefore *define* a^0 for any nonzero real number by

$$\boxed{a^0 = 1}$$

The same approach will lead us to a definition of negative exponents. For consistency, we must have

$$a^m a^{-m} = a^{m-m} = a^0 = 1 \qquad \text{or} \qquad a^m a^{-m} = 1 \tag{1}$$

Division of both sides of Equation (1) by a^m suggests that we define a^{-m} by

$$\boxed{a^{-m} = \frac{1}{a^m}, \qquad a \neq 0}$$

Dividing Equation (1) by a^{-m}, we have

$$\boxed{a^m = \frac{1}{a^{-m}}, \qquad a \neq 0}$$

Thus, a^{-m} is the reciprocal of a^m, and a^m is the reciprocal of a^{-m}. The rule for handling negative exponents can be expressed as follows.

> A nonzero factor moves from numerator to denominator (or from denominator to numerator) by changing the sign of the exponent.

Table 3 summarizes and illustrates these results.

TABLE 3

ZERO AND INTEGER EXPONENTS	
Definition	**Example**
$a^0 = 1, \quad a \neq 0$	$3^0 = 1 \qquad \left(\frac{2}{5}\right)^0 = 1 \qquad 4(xy)^0 = 4$ $\frac{2}{(x-1)^0} = \frac{2}{1} = 2 \qquad -3(y^2+1)^0 = -3$
$a^m = \frac{1}{a^{-m}}, \quad a \neq 0$	$\frac{-2}{(a-1)^{-2}} = -2(a-1)^2 \qquad (2x)^{-3} = \frac{1}{(2x)^3} = \frac{1}{8x^3}$
$a^{-m} = \frac{1}{a^m}, \quad a \neq 0$	$(x^2y^{-3})^{-5} = (x^2)^{-5}(y^{-3})^{-5} = x^{-10}y^{15} = \frac{y^{15}}{x^{10}}$

PROGRESS CHECK

Simplify, using only positive exponents.

(a) $x^{-2}y^{-3}$ (b) $\frac{-3x^4y^{-2}}{9x^{-8}y^6}$ (c) $\left(\frac{x^{-3}}{x^{-4}}\right)^{-1}$

ANSWERS

(a) $\frac{1}{x^2y^3}$ (b) $-\frac{x^{12}}{3y^8}$ (c) $\frac{1}{x}$

WARNING Don't confuse negative numbers and negative exponents.

(a) $2^{-4} = \frac{1}{2^4}$

Note that $2^{-4} \neq -2^4$.

(b) $(-2)^{-3} = \frac{1}{(-2)^3} = \frac{1}{-8} = -\frac{1}{8}$

Note that $(-2)^{-3} \neq \frac{1}{2^3} = \frac{1}{8}$.

EXERCISE SET 1.6

In Exercises 1–6 the right-hand side is incorrect. Find the correct term.

1. $x^2 \cdot x^4 = x^8$
2. $(y^2)^5 = y^7$
3. $\frac{b^6}{b^2} = b^3$
4. $\frac{x^2}{x^6} = x^4$
5. $(2x)^4 = 2x^4$
6. $\left(\frac{4}{3}\right)^4 = \frac{4}{3^4}$

In Exercises 7–64 use the rules for exponents to simplify. Write the answers using only positive exponents.

7. $\left(-\frac{1}{2}\right)^4\left(-\frac{1}{2}\right)^3$
8. $(x^m)^{3m}$
9. $(y^4)^{2n}$
10. $\frac{(-4)^6}{(-4)^{10}}$

11. $-\left(\frac{x}{y}\right)^3$ 12. $-3r^3r^3$ 13. $(x^3)^5 \cdot x^4$ 14. $\frac{x^{12}}{x^8}$

15. $(-2x^2)^5$ 16. $-(2x^2)^5$ 17. $x^{3n} \cdot x^n$ 18. $(-2)^m(-2)^n$

19. $\frac{x^n}{x^{n+2}}$ 20. $\left(\frac{3x^3}{y^2}\right)^5$ 21. $(-5x^3)(-6x^5)$ 22. $(x^2)^3(y^2)^4(x^3)^7$

23. $\frac{(r^2)^4}{(r^4)^2}$ 24. $[(3b+1)^5]^5$ 25. $\left(\frac{3}{2}x^2y^3\right)^n$ 26. $\frac{(-2a^2b)^4}{(-3ab^2)^3}$

27. $(2x+1)^3(2x+1)^7$ 28. $\frac{y^3(y^3)^4}{(y^4)^6}$ 29. $(-2a^2b^3)^{2n}$ 30. $\left(-\frac{2}{3}a^2b^3c^2\right)^3$

31. $2^0 + 3^{-1}$ 32. $(xy)^0 - 2^{-1}$ 33. $\frac{3}{(2x^2+1)^0}$ 34. $(-3)^{-3}$

35. $\frac{1}{3^{-4}}$ 36. x^{-5} 37. $(-x)^3$ 38. $-x^{-5}$

39. $\frac{1}{y^{-6}}$ 40. $(2a)^{-6}$ 41. $5^{-3}5^5$ 42. $4y^5y^{-2}$

43. $(3^2)^{-3}$ 44. $(x^{-2})^4$ 45. $(x^{-3})^{-3}$ 46. $[(x+y)^{-2}]^2$

47. $\frac{2^2}{2^{-3}}$ 48. $\frac{x^8}{x^{-10}}$ 49. $\frac{2x^4y^{-2}}{x^2y^{-3}}$ 50. $(x^4y^{-2})^{-1}$

51. $(3a^{-2}b^{-3})^{-2}$ 52. $\frac{1}{(2xy)^{-2}}$ 53. $\left(-\frac{1}{2}x^3y^{-4}\right)^{-3}$ 54. $\frac{(x^{-2})^2}{(3y^{-2})^3}$

55. $\frac{3a^5b^{-2}}{9a^{-4}b^2}$ 56. $\left(\frac{x^3}{x^{-2}}\right)^2$ 57. $\left(\frac{2a^2b^{-4}}{a^{-3}c^{-3}}\right)^2$ 58. $\frac{2x^{-3}y^2}{x^{-3}y^{-3}}$

59. $(a-2b^2)^{-1}$ 60. $\left(\frac{y^{-2}}{y^{-3}}\right)^{-1}$ 61. $\frac{(a+b)^{-1}}{(a-b)^{-2}}$ 62. $(a^{-1}+b^{-1})^{-1}$

63. $\frac{a^{-1}+b^{-1}}{a^{-1}-b^{-1}}$ 64. $\left(\frac{a}{b}\right)^{-1} + \left(\frac{b}{a}\right)^{-1}$ 65. Show that $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

Evaluate each expression in Exercises 66–69.

66. $(1.20^2)^{-1}$ 67. $[(-3.67)^2]^{-1}$ 68. $\left(\frac{7.65^{-1}}{7.65^2}\right)^2$ 69. $\left(\frac{4.46^2}{4.46^{-1}}\right)^{-1}$

1.7 RATIONAL EXPONENTS AND RADICALS

nTH ROOTS

Consider a square whose area is 25 square centimeters, and whose sides are of length a. We can then write

$$a^2 = 25$$

so that a is a number whose square is 25. We say that a is the **square root** of b if $a^2 = b$. Similarly, we say that a is a **cube root** of b if $a^3 = b$, and, in general, if n is a natural number, we say that

a is an **nth root** of b if $a^n = b$

Thus, 5 is a square root of 25 since $5^2 = 25$, and -2 is a cube root of -8 since $(-2)^3 = -8$.

Since $(-5)^2 = 25$, we conclude that -5 is also a square root of 25. More generally, if $b > 0$ and a is a square root of b, then $-a$ is also a square root of b. If $b < 0$, there is no real number a such that $a^2 = b$, since the square of a real number is always nonnegative. (We'll see in Section 1.8 that mathematicians have created an extended number system in which there is a root when $b < 0$ and n is even.)

The cases are summarized in Table 4.

TABLE 4

b	n	Number of nth roots of b such that $b = a^n$	Form of nth roots	b	Examples
> 0	Even	2	$a, -a$	4	Square roots are 2, -2.
< 0	Even	None	None	-1	No square roots.
> 0	Odd	1	$a > 0$	8	Cube root is 2.
< 0	Odd	1	$a < 0$	-8	Cube root is -2.
0	All	1	0	0	Square root is 0.

We would like to define rational exponents in a manner that will be consistent with the rules for integer exponents. If the rule $(a^m)^n = a^{mn}$ is to hold, then we must have

$$(b^{1/n})^n = b^{n/n} = b$$

But a is an nth root of b if $a^n = b$. Then for every natural number n, we say that

$$\boxed{b^{1/n} \text{ is an } n\text{th root of } b}$$

Principal nth Root

If n is even and b is positive, Table 4 indicates that there are two numbers, a and $-a$, that are nth roots of b. For example,

$$4^2 = 16 \qquad \text{and} \qquad (-4)^2 = 16$$

There are then two candidates for $16^{1/2}$, namely 4 and -4. To avoid ambiguity we say that $16^{1/2} = 4$. That is, if n is even and b is positive, we always *choose* the positive number a such that $a^n = b$ to be the nth root and call a the **principal nth root** of b. Thus, $b^{1/n}$ denotes the principal nth root of b.

EXAMPLE 1

Evaluate.

(a) $144^{1/2}$ (b) $(-8)^{1/3}$ (c) $(-25)^{1/2}$ (d) $-\left(\frac{1}{16}\right)^{1/4}$

SOLUTION

(a) $144^{1/2} = 12$ (b) $(-8)^{1/3} = -2$

(c) $(-25)^{1/2}$ is not a real number (d) $-\left(\frac{1}{16}\right)^{1/4} = -\frac{1}{2}$

RATIONAL EXPONENTS

Now we are prepared to define $b^{m/n}$, where m is an integer (positive or negative), n is a natural number, and $b > 0$ when n is even. We want the rules for exponents to hold for rational exponents as well. That is, we want to have

$$4^{3/2} = 4^{(1/2)(3)} = (4^{1/2})^3 = 2^3 = 8$$

and

$$4^{3/2} = 4^{(3)(1/2)} = (4^3)^{1/2} = (64)^{1/2} = 8$$

To achieve this consistency, we define $b^{m/n}$, for an integer m, a natural number n, and a real number b, by

$$\boxed{b^{m/n} = (b^{1/n})^m = (b^m)^{1/n}}$$

where b must be positive when n is even. With this definition, all the rules of exponents continue to hold when the exponents are rational numbers.

EXAMPLE 2

Simplify.

(a) $(-8)^{4/3}$ (b) $x^{1/2} \cdot x^{3/4}$ (c) $(x^{3/4})^2$ (d) $(3x^{2/3}y^{-5/3})^3$

SOLUTION

(a) $(-8)^{4/3} = [(-8)^{1/3}]^4 = (-2)^4 = 16$

(b) $x^{1/2} \cdot x^{3/4} = x^{1/2 + 3/4} = x^{5/4}$

(c) $(x^{3/4})^2 = x^{(3/4)(2)} = x^{3/2}$

(d) $(3x^{2/3}y^{-5/3})^3 = 3^3 \cdot x^{(2/3)(3)}y^{(-5/3)(3)} = 27x^2y^{-5} = \frac{27x^2}{y^5}, \quad y \neq 0$

PROGRESS CHECK

Simplify. Assume all variables are positive real numbers.

(a) $27^{4/3}$ (b) $(a^{1/2}b^{-2})^{-2}$ (c) $\left(\frac{x^{1/3}y^{2/3}}{z^{5/6}}\right)^{12}$

ANSWERS

(a) 81 (b) $\frac{b^4}{a}$ (c) $\frac{x^4y^8}{z^{10}}$

WHEN IS A PROOF NOT A PROOF?

Books of mathematical puzzles love to include "proofs" that lead to false or contradictory results. Of course, there is always an incorrect step hidden somewhere in the proof. The error may be subtle, but a good grounding in the fundamentals of mathematics will enable you to catch it.

Examine the following "proof."

$$
\begin{aligned}
1 &= 1^{1/2} && (1)\\
&= [(-1)^2]^{1/2} && (2)\\
&= (-1)^{2/2} && (3)\\
&= (-1)^1 && (4)\\
&= -1 && (5)
\end{aligned}
$$

The result is obviously contradictory: we can't have $1 = -1$. Yet each step seems to be legitimate. Did you spot the flaw? The rule

$$(b^m)^{1/n} = b^{m/n}$$

used in going from (2) to (3) doesn't apply when n is even and b is negative. Any time the rules of algebra are abused the results are unpredictable!

RADICALS

The symbol $\sqrt{b}$ is an alternative way of writing $b^{1/2}$; that is, $\sqrt{b}$ denotes the nonnegative square root of b. The symbol $\sqrt{\ }$ is called a **radical sign,** and $\sqrt{b}$ is called the **principal square root** of b. Thus,

$$\sqrt{25} = 5 \qquad \sqrt{0} = 0 \qquad \sqrt{-25} \text{ is undefined}$$

In general, the symbol $\sqrt[n]{b}$ is an alternative way of writing $b^{1/n}$, the principal nth root of b. Of course, we must apply the same restrictions to $\sqrt[n]{b}$ that we established for $b^{1/n}$. In summary,

$$\sqrt[n]{b} = b^{1/n} = a \qquad \text{where } a^n = b$$

with these restrictions:

- if n is even and $b < 0$, $\sqrt[n]{b}$ is not a real number;
- if n is even and $b \geq 0$, $\sqrt[n]{b}$ is the *nonnegative* number a satisfying $a^n = b$.

WARNING Many students are accustomed to writing $\sqrt{4} = \pm 2$. This is incorrect, since the symbol $\sqrt{\ }$ indicates the *principal* square root, which is nonnegative. Get in the habit of writing $\sqrt{4} = 2$. If you want to indicate *all* square roots of 4, write $\pm\sqrt{4} = \pm 2$.

In short, $\sqrt[n]{b}$ is the **radical form** of $b^{1/n}$. We can switch back and forth from one form to the other. For instance,

$$\sqrt[3]{7} = 7^{1/3} \qquad (11)^{1/5} = \sqrt[5]{11}$$

Finally, we treat the radical form of $b^{m/n}$ where m is an integer and n is a natural number as follows.

$$b^{m/n} = (b^m)^{1/n} = \sqrt[n]{b^m}$$

and

$$b^{m/n} = (b^{1/n})^m = (\sqrt[n]{b})^m$$

Thus,

$$7^{2/3} = (7^2)^{1/3} = \sqrt[3]{7^2}$$
$$7^{2/3} = (7^{1/3})^2 = (\sqrt[3]{7})^2$$

EXAMPLE 3

Change from radical form to rational exponent form or vice versa. Assume all variables are nonzero.

(a) $(2x)^{-3/2}, \quad x > 0$ (b) $\dfrac{1}{\sqrt[7]{y^4}}$

(c) $(-3a)^{3/7}$ (d) $\sqrt{x^2 + y^2}$

SOLUTION

(a) $(2x)^{-3/2} = \dfrac{1}{(2x)^{3/2}} = \dfrac{1}{\sqrt{8x^3}}$ (b) $\dfrac{1}{\sqrt[7]{y^4}} = \dfrac{1}{y^{4/7}} = y^{-4/7}$

(c) $(-3a)^{3/7} = \sqrt[7]{-27a^3}$ (d) $\sqrt{x^2 + y^2} = (x^2 + y^2)^{1/2}$

PROGRESS CHECK

Change from radical form to rational exponent form or vice versa. Assume all variables are positive real numbers.

(a) $\sqrt[4]{2rs^3}$ (b) $(x + y)^{5/2}$

(c) $y^{-5/4}$ (d) $\dfrac{1}{\sqrt[4]{m^5}}$

ANSWERS

(a) $(2r)^{1/4}s^{3/4}$ (b) $\sqrt{(x + y)^5}$

(c) $\dfrac{1}{\sqrt[4]{y^5}}$ (d) $m^{-5/4}$

Since radicals are just another way of writing exponents, the properties of radicals can be derived from the properties of exponents.

Properties of Radicals

If n is a natural number, a and b are real numbers, and all radicals denote real numbers, then

(1)	$\sqrt[n]{b^m} = (b^m)^{1/n} = (b^{1/n})^m = (\sqrt[n]{b})^m$	$\sqrt[3]{8^2} = (\sqrt[3]{8})^2$
(2)	$\sqrt[n]{a} \cdot \sqrt[n]{b} = a^{1/n} \cdot b^{1/n} = (ab)^{1/n} = \sqrt[n]{ab}$	$\sqrt{4}\sqrt{9} = \sqrt{36}$
(3)	$\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \dfrac{a^{1/n}}{b^{1/n}} = \left(\dfrac{a}{b}\right)^{1/n} = \sqrt[n]{\dfrac{a}{b}},\ b \neq 0$	$\dfrac{\sqrt[3]{8}}{\sqrt[3]{27}} = \sqrt[3]{\dfrac{8}{27}}$
(4)	$\sqrt[n]{a^n} = \begin{cases} a & \text{if } n \text{ is odd} \\ \lvert a\rvert & \text{if } n \text{ is even} \end{cases}$	$\sqrt{(-4)^2} = \lvert -4\rvert = 4$

Here are some examples using these properties.

EXAMPLE 4

Simplify.

(a) $\sqrt{18}$ (b) $\sqrt[3]{-54}$ (c) $2\sqrt[3]{8x^3y}$ (d) $\sqrt{x^6}$

SOLUTION

(a) $\sqrt{18} = \sqrt{9 \cdot 2} = \sqrt{9}\sqrt{2} = 3\sqrt{2}$

(b) $\sqrt[3]{-54} = \sqrt[3]{(-27)(2)} = \sqrt[3]{-27}\sqrt[3]{2} = -3\sqrt[3]{2}$

(c) $2\sqrt[3]{8x^3y} = 2\sqrt[3]{8}\sqrt[3]{x^3}\sqrt[3]{y} = 2(2)(x)\sqrt[3]{y} = 4x\sqrt[3]{y}$

(d) $\sqrt{x^6} = \sqrt{x^2} \cdot \sqrt{x^2} \cdot \sqrt{x^2} = \lvert x\rvert \cdot \lvert x\rvert \cdot \lvert x\rvert = \lvert x\rvert^3$

WARNING The properties of radicals state that

$$\sqrt{x^2} = \lvert x\rvert$$

It is a common error to write $\sqrt{x^2} = x$, but this leads to the conclusion that $\sqrt{(-6)^2} = -6$. Since the symbol $\sqrt{\ }$ represents the principal or nonnegative square root of a number, the result cannot be negative. It is therefore essential to write $\sqrt{x^2} = \lvert x\rvert$ (and, in fact, $\sqrt[n]{x^n} = \lvert x\rvert$ whenever n is even) unless we know that $x \geq 0$, in which case we can write $\sqrt{x^2} = x$.

Simplifying Radicals

A radical is said to be in **simplified form** when the following conditions are satisfied:

1. $\sqrt[n]{b^m}$ has $m < n$;

2. $\sqrt[n]{b^m}$ has no common factors between m and n;

3. A denominator is free of radicals.

The first two conditions can always be met by using the properties of radicals and by writing radicals in exponent form. For example,

$$\sqrt[3]{x^4} = \sqrt[3]{x^3 \cdot x} = \sqrt[3]{x^3}\,\sqrt[3]{x} = x\sqrt[3]{x}$$

and

$$\sqrt[6]{x^4} = x^{4/6} = x^{2/3} = \sqrt[3]{x^2}$$

The third condition can always be satisfied by multiplying the fraction by a properly chosen form of unity, a process called **rationalizing the denominator.** For example, to rationalize $1/\sqrt{3}$, we proceed as follows.

$$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{\sqrt{3^2}} = \frac{\sqrt{3}}{3}$$

In this connection, a useful formula is

$$(\sqrt{m} + \sqrt{n})\,(\sqrt{m} - \sqrt{n}) = m - n$$

which we will apply in the following examples.

EXAMPLE 5

Rationalize the denominator. Assume all variables denote positive real numbers.

(a) $\sqrt{\dfrac{x}{y}}$ (b) $\dfrac{4}{\sqrt{5} - \sqrt{2}}$ (c) $\dfrac{5}{\sqrt{x} + 2}$ (d) $\dfrac{5}{\sqrt{x + 2}}$

SOLUTION

(a) $\sqrt{\dfrac{x}{y}} = \dfrac{\sqrt{x}}{\sqrt{y}} = \dfrac{\sqrt{x}}{\sqrt{y}} \cdot \dfrac{\sqrt{y}}{\sqrt{y}} = \dfrac{\sqrt{xy}}{\sqrt{y^2}} = \dfrac{\sqrt{xy}}{y}$

(b) $\dfrac{4}{\sqrt{5} - \sqrt{2}} = \dfrac{4}{\sqrt{5} - \sqrt{2}} \cdot \dfrac{\sqrt{5} + \sqrt{2}}{\sqrt{5} + \sqrt{2}}$

$= \dfrac{4(\sqrt{5} + \sqrt{2})}{5 - 2} = \dfrac{4}{3}(\sqrt{5} + \sqrt{2})$

(c) $\dfrac{5}{\sqrt{x} + 2} = \dfrac{5}{\sqrt{x} + 2} \cdot \dfrac{\sqrt{x} - 2}{\sqrt{x} - 2} = \dfrac{5(\sqrt{x} - 2)}{x - 4}, \quad x \neq 4$

(d) $\dfrac{5}{\sqrt{x + 2}} = \dfrac{5}{\sqrt{x + 2}} \cdot \dfrac{\sqrt{x + 2}}{\sqrt{x + 2}} = \dfrac{5\sqrt{x + 2}}{x + 2}$

PROGRESS CHECK

Rationalize the denominator. Assume all radicals denote real numbers.

(a) $\dfrac{-9xy^3}{\sqrt{3xy}}$ (b) $\dfrac{-6}{\sqrt{2} + \sqrt{6}}$ (c) $\dfrac{4}{\sqrt{x} - \sqrt{y}}, \quad x \neq y$

ANSWERS

(a) $-3y^2\sqrt{3xy}$ (b) $\frac{3}{2}(\sqrt{2}-\sqrt{6})$ (c) $\frac{4(\sqrt{x}+\sqrt{y})}{x-y}$

EXAMPLE 6

Write in simplified form. Assume all radicals denote real numbers.

(a) $\sqrt[4]{y^5}$ (b) $\sqrt{\frac{8x^3}{y}}$, $y>0$ (c) $\sqrt[6]{\frac{x^3}{y^2}}$

SOLUTION

(a) $\sqrt[4]{y^5}=\sqrt[4]{y^4\cdot y}=\sqrt[4]{y^4}\,\sqrt[4]{y}=y\,\sqrt[4]{y}$

(b) $$\sqrt{\frac{8x^3}{y}}=\frac{\sqrt{(4x^2)(2x)}}{\sqrt{y}}=\frac{\sqrt{4x^2}\,\sqrt{2x}}{\sqrt{y}}=\frac{2x\sqrt{2x}}{\sqrt{y}}$$
$$=\frac{2x\sqrt{2x}}{\sqrt{y}}\cdot\frac{\sqrt{y}}{\sqrt{y}}=\frac{2x\sqrt{2xy}}{y}$$

(c) $$\sqrt[6]{\frac{x^3}{y^2}}=\frac{\sqrt[6]{x^3}}{\sqrt[6]{y^2}}=\frac{\sqrt{x}}{\sqrt[3]{y}}=\frac{\sqrt{x}}{\sqrt[3]{y}}\cdot\frac{\sqrt[3]{y^2}}{\sqrt[3]{y^2}}$$ This multiplier will produce $\sqrt[3]{y^3}$.
$$=\frac{\sqrt{x}\,\sqrt[3]{y^2}}{y}$$

PROGRESS CHECK

Write in simplified form. All radicals denote real numbers.

(a) $\sqrt{75}$ (b) $\sqrt{\frac{18x^6}{y}}$ (c) $\sqrt[3]{ab^4c^7}$ (d) $\frac{-2xy^3}{\sqrt[4]{32x^3y^5}}$, $x, y>0$

ANSWERS

(a) $5\sqrt{3}$ (b) $\frac{3|x|^3\sqrt{2y}}{y}$ (c) $bc^2\sqrt[3]{abc}$ (d) $-\frac{y}{2}\sqrt[4]{8xy^3}$

Operations with Radicals

We can add or subtract expressions involving exactly the same radical forms. For example,

$$2\sqrt{2}+3\sqrt{2}=5\sqrt{2}$$

since

$$2\sqrt{2}+3\sqrt{2}=(2+3)\sqrt{2}=5\sqrt{2}$$

And

$$3\sqrt[3]{x^2y}-7\sqrt[3]{x^2y}=-4\sqrt[3]{x^2y}$$

EXAMPLE 7

(a) $7\sqrt{5} + 4\sqrt{3} - 9\sqrt{5} = -2\sqrt{5} + 4\sqrt{3}$

(b) $\sqrt[3]{x^2y} - \frac{1}{2}\sqrt{xy} - 3\sqrt[3]{x^2y} + 4\sqrt{xy} = -2\sqrt[3]{x^2y} + \frac{7}{2}\sqrt{xy}$

WARNING

$$\sqrt{9} + \sqrt{16} \neq \sqrt{25}$$

You can perform addition only with identical radical forms. *Adding unlike radicals is one of the most common mistakes made by students in algebra!* You can easily verify that

$$\sqrt{9} + \sqrt{16} = 3 + 4 = 7$$

The product of $\sqrt[n]{a}$ and $\sqrt[m]{b}$ can be readily simplified only when $m = n$. Thus,

$$\sqrt[5]{x^2y} \cdot \sqrt[5]{xy} = \sqrt[5]{x^3y^2}$$

but

$$\sqrt[3]{x^2y} \cdot \sqrt[5]{xy}$$

cannot be readily simplified.

EXAMPLE 8

Multiply and simplify.

(a) $2\sqrt[3]{xy^2} \cdot \sqrt[3]{x^2y^2} = 2\sqrt[3]{x^3y^4} = 2xy\sqrt[3]{y}$

(b) $\sqrt[5]{a^2b}\,\sqrt{ab}\,\sqrt[5]{ab^2} = \sqrt[5]{a^3b^3}\,\sqrt{ab}$

EXERCISE SET 1.7

In Exercises 1–12 simplify, and write the answer using only positive exponents.

1. $16^{3/4}$	2. $(-125)^{-1/3}$	3. $(-64)^{-2/3}$	4. $c^{1/4}c^{-2/3}$
5. $\dfrac{2x^{1/3}}{x^{-3/4}}$	6. $\dfrac{y^{-2/3}}{y^{1/5}}$	7. $\left(\dfrac{x^{3/2}}{x^{2/3}}\right)^{1/6}$	8. $\dfrac{125^{4/3}}{125^{2/3}}$
9. $(x^{1/3}y^2)^6$	10. $(x^6y^4)^{-1/2}$	11. $\left(\dfrac{x^{15}}{y^{10}}\right)^{3/5}$	12. $\left(\dfrac{x^{18}}{y^{-6}}\right)^{2/3}$

In Exercises 13–18 write the expression in radical form.

13. $\left(\dfrac{1}{4}\right)^{2/5}$	14. $x^{2/3}$	15. $a^{3/4}$	16. $(-8x^2)^{2/5}$
17. $(12x^3y^{-2})^{2/3}$	18. $\left(\dfrac{8}{3}x^{-2}y^{-4}\right)^{-3/2}$		

In Exercises 19–24 write the expression in exponent form.

19. $\sqrt[4]{8^3}$ 20. $\sqrt[5]{3^2}$ 21. $\dfrac{1}{\sqrt[5]{(-8)^2}}$ 22. $\dfrac{1}{\sqrt[3]{x^7}}$

23. $\dfrac{1}{\sqrt[4]{\frac{4}{9}a^3}}$ 24. $\sqrt[5]{(2a^2b^3)^4}$

In Exercises 25–33 evaluate the expression.

25. $\sqrt{\dfrac{4}{9}}$ 26. $\sqrt{\dfrac{25}{4}}$ 27. $\sqrt[4]{-81}$ 28. $\sqrt[3]{\dfrac{1}{27}}$

29. $\sqrt{(-5)^2}$ 30. $\sqrt{\left(-\dfrac{1}{3}\right)^2}$ 31. $\sqrt{\left(\dfrac{5}{4}\right)^2}$ 32. $\sqrt{\left(-\dfrac{7}{2}\right)^2}$

33. $(14.43)^{3/2}$

In Exercises 34–36 provide a real value for each variable to demonstrate the result.

34. $\sqrt{x^2} \neq x$ 35. $\sqrt{x^2 + y^2} \neq x + y$ 36. $\sqrt{x}\,\sqrt{y} \neq \sqrt{xy}$

In Exercises 37–56 write the expression in simplified form. (Every variable represents a positive real number.)

37. $\sqrt{48}$ 38. $\sqrt{200}$ 39. $\sqrt[3]{54}$ 40. $\sqrt{x^8}$

41. $\sqrt[3]{y^7}$ 42. $\sqrt[4]{b^{14}}$ 43. $\sqrt[4]{96x^{10}}$ 44. $\sqrt{x^5y^4}$

45. $\sqrt{x^5y^3}$ 46. $\sqrt[3]{24b^{10}c^{14}}$ 47. $\sqrt[4]{16x^8y^5}$ 48. $\sqrt{20x^5y^7z^4}$

49. $\sqrt{\dfrac{1}{5}}$ 50. $\dfrac{4}{3\sqrt{11}}$ 51. $\dfrac{1}{\sqrt{3y}}$ 52. $\sqrt{\dfrac{2}{y}}$

53. $\dfrac{4x^2}{\sqrt{2x}}$ 54. $\dfrac{8a^2b^2}{2\sqrt{2b}}$ 55. $\sqrt[3]{x^2y^7}$ 56. $\sqrt[4]{48x^8y^6z^2}$

In Exercises 57–66 simplify and combine terms.

57. $2\sqrt{3} + 5\sqrt{3}$ 58. $4\sqrt[3]{11} - 6\sqrt[3]{11}$

59. $3\sqrt{x} + 4\sqrt{x}$ 60. $3\sqrt{2} + 5\sqrt{2} - 2\sqrt{2}$

61. $2\sqrt{27} + \sqrt{12} - \sqrt{48}$ 62. $\sqrt{20} - 4\sqrt{45} + \sqrt{80}$

63. $\sqrt[3]{40} + \sqrt{45} - \sqrt[3]{135} + 2\sqrt{80}$ 64. $\sqrt{2abc} - 3\sqrt{8abc} + \sqrt{\dfrac{abc}{2}}$

65. $2\sqrt{5} - (3\sqrt{5} + 4\sqrt{5})$ 66. $2\sqrt{18} - (3\sqrt{12} - 2\sqrt{75})$

In Exercises 67–74 multiply and simplify.

67. $\sqrt{3}(\sqrt{3} + 4)$ 68. $\sqrt{8}(\sqrt{2} - \sqrt{3})$ 69. $3\sqrt[3]{x^2y}\,\sqrt[3]{xy^2}$ 70. $-4\sqrt[5]{x^2y^3}\,\sqrt[5]{x^4y^2}$

71. $(\sqrt{2} - \sqrt{3})^2$ 72. $(\sqrt{8} - 2\sqrt{2})(\sqrt{2} + 2\sqrt{8})$

73. $(\sqrt{3x} + \sqrt{2y})(\sqrt{3x} - 2\sqrt{2y})$ 74. $(\sqrt[3]{2x} + 3)(\sqrt[3]{2x} - 3)$

In Exercises 75–86 rationalize the denominator.

75. $\dfrac{3}{\sqrt{2} + 3}$ 76. $\dfrac{-3}{\sqrt{7} - 9}$ 77. $\dfrac{-2}{\sqrt{3} - 4}$ 78. $\dfrac{3}{\sqrt{x} - 5}$

79. $\dfrac{-3}{3\sqrt{a} + 1}$ 80. $\dfrac{4}{2 - \sqrt{2y}}$ 81. $\dfrac{-3}{5 + \sqrt{5y}}$ 82. $\dfrac{\sqrt{3}}{\sqrt{3} - 5}$

83. $\dfrac{\sqrt{2}+1}{\sqrt{2}-1}$ 84. $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$ 85. $\dfrac{\sqrt{6}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$ 86. $\dfrac{2\sqrt{a}}{\sqrt{2x}+\sqrt{y}}$

In Exercises 87–88 provide real values for x and y and a positive integer value for n to demonstrate the result.

87. $\sqrt{x}+\sqrt{y} \neq \sqrt{x+y}$

88. $\sqrt[n]{x^n+y^n} \neq x+y$

89. Find the step in the following "proof" that is incorrect. Explain.
$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = -1$$

90. Prove that $|ab| = |a|\,|b|$. (*Hint*: Begin with $|ab| = \sqrt{(ab)^2}$.)

1.8 COMPLEX NUMBERS

One of the central problems in algebra is to find solutions to a given polynomial equation. This problem will be discussed in later chapters of this book. For now, observe that there is no real number that satisfies a simple polynomial equation like
$$x^2 = -4$$
since the square of a real number is always nonnegative.

To resolve this problem, mathematicians created a new number system built upon an "imaginary unit" i, defined by $i = \sqrt{-1}$. This number i has the property that when we square both sides of the equation we have $i^2 = -1$, a result that cannot be obtained with real numbers. By definition,

$$\boxed{\begin{aligned} i &= \sqrt{-1} \\ i^2 &= -1 \end{aligned}}$$

We also assume that i behaves according to all the algebraic laws we have already developed (with the exception of the rules for inequalities for real numbers). This allows us to simplify higher powers of i. Thus,
$$i^3 = i^2 \cdot i = (-1)i = -i$$
$$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$$

Now it's easy to simplify i^n when n is any natural number. Since $i^4 = 1$, we simply seek the highest multiple of 4 which is less than or equal to n. For example,
$$i^5 = i^4 \cdot i = (1) \cdot i = i$$
$$i^{27} = i^{24} \cdot i^3 = (i^4)^6 \cdot i^3 = (1)^6 \cdot i^3 = i^3 = -i$$

EXAMPLE 1

Simplify.

(a) i^{51} (b) $-i^{74}$

SOLUTION

(a) $i^{51} = i^{48} \cdot i^3 = (i^4)^{12} \cdot i^3 = (1)^{12} \cdot i^3 = i^3 = -i$

(b) $-i^{74} = -i^{72} \cdot i^2 = -(i^4)^{18} \cdot i^2 = -(1)^{18} \cdot i^2 = -(1)(-1) = 1$

It is easy also to write square roots of negative numbers in terms of i. For example,

$$\sqrt{-25} = i\sqrt{25} = 5i$$

and, in general, we define

$$\boxed{\sqrt{-a} = i\sqrt{a} \qquad \text{for } a > 0}$$

Any number of the form bi, where b is a real number, is called an **imaginary number.**

WARNING

$$\sqrt{-4}\sqrt{-9} \neq \sqrt{36}$$

The rule $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ holds only when $a \geq 0$ and $b \geq 0$. Instead, write

$$\sqrt{-4}\sqrt{-9} = 2i \cdot 3i = 6i^2 = -6$$

Having created imaginary numbers, we next combine real and imaginary numbers. We say that $a + bi$, where a and b are real numbers, is a **complex number.** The number a is called the **real part** of $a + bi$ and b is called the **imaginary part.** The following are examples of complex numbers.

$$3 + 2i \qquad 2 - i \qquad -2i \qquad \frac{4}{5} + \frac{1}{5}i$$

Note that every real number a can be written as a complex number by choosing $b = 0$. Thus,

$$\boxed{a = a + 0i}$$

We see that the real number system is a subset of the complex number system. The desire to find solutions to every quadratic equation has led mathematicians to create a more comprehensive number system, which incorporates all previous number systems.

Will you have to learn still more number systems? The answer, fortunately, is a resounding "No!" We will show in a later chapter that complex numbers are all that we need to provide solutions to any polynomial equation.

EXAMPLE 2

Write as a complex number.

(a) $-\frac{1}{2}$ (b) $\sqrt{-9}$ (c) $-1 - \sqrt{-4}$

SOLUTION

(a) $-\frac{1}{2} = -\frac{1}{2} + 0i$

(b) $\sqrt{-9} = i\sqrt{9} = 3i = 0 + 3i$

(c) $-1 - \sqrt{-4} = -1 - i\sqrt{4} = -1 - 2i$

Don't be disturbed by the word "complex." You already have all the basic tools you will need to tackle this number system. We will next define operations with complex numbers in such a way that the rules for the real numbers and the imaginary unit i continue to hold. We begin with equality and say that two complex numbers are **equal** if their real parts are equal and their imaginary parts are equal; that is,

$$a + bi = c + di \quad \text{if} \quad a = c \text{ and } b = d$$

EXAMPLE 3
Solve the equation $x + 3i = 6 - yi$ for x and y.

SOLUTION
Equating the real parts, we have $x = 6$; equating the imaginary parts, $3 = -y$ or $y = -3$.

Complex numbers are added and subtracted by adding or subtracting the real parts and by adding or subtracting the imaginary parts.

Addition and Subtraction of Complex Numbers

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

Note that the sum or difference of two complex numbers is again a complex number.

EXAMPLE 4
Perform the indicated operations.
(a) $(7 - 2i) + (4 - 3i)$ (b) $14 - (3 - 8i)$

SOLUTION
(a) $(7 - 2i) + (4 - 3i) = (7 + 4) + (-2 - 3)i = 11 - 5i$
(b) $14 - (3 - 8i) = (14 - 3) + 8i = 11 + 8i$

PROGRESS CHECK
Perform the indicated operations.
(a) $(-9 + 3i) + (6 - 2i)$ (b) $7i - (3 + 9i)$

ANSWERS
(a) $-3+i$ (b) $-3-2i$

We now define multiplication of complex numbers in a manner that permits the commutative, associative, and distributive laws to hold, along with the definition $i^2 = -1$. We must have

$$\begin{aligned}(a+bi)(c+di) &= a(c+di) + bi(c+di)\\ &= ac + adi + bci + bdi^2\\ &= ac + (ad+bc)i + bd(-1)\\ &= (ac-bd) + (ad+bc)i\end{aligned}$$

The rule for multiplication is

Multiplication of Complex Numbers

$$(a+bi)(c+di) = (ac-bd) + (ad+bc)i$$

This result is significant because it demonstrates that the product of two complex numbers is again a complex number. It need not be memorized; simply use the distributive law to form all the products and the substitution $i^2 = -1$ to simplify.

EXAMPLE 5
Find the product of $(2-3i)$ and $(7+5i)$.

SOLUTION

$$\begin{aligned}(2-3i)(7+5i) &= 2(7+5i) - 3i(7+5i)\\ &= 14 + 10i - 21i - 15i^2\\ &= 14 - 11i - 15(-1)\\ &= 29 - 11i\end{aligned}$$

PROGRESS CHECK
Find the product.
(a) $(-3-i)(4-2i)$ (b) $(-4-2i)(2-3i)$

ANSWERS
(a) $-14+2i$ (b) $-14+8i$

EXERCISE SET 1.8

Simplify in Exercises 1–9.

1. i^{60}
2. i^{27}
3. i^{83}
4. $-i^{54}$
5. $-i^{33}$
6. i^{-15}
7. i^{-84}
8. $-i^{39}$
9. $-i^{-25}$

In Exercises 10–21 write the number in the form $a + bi$.

10. 2
11. $-\frac{1}{2}$
12. -0.3
13. $\sqrt{-25}$

14. $-\sqrt{-5}$
15. $-\sqrt{-36}$
16. $-\sqrt{-18}$
17. $3-\sqrt{-49}$
18. $-\frac{3}{2}-\sqrt{-72}$
19. $0.3-\sqrt{-98}$
20. $-0.5+\sqrt{-32}$
21. $-2-\sqrt{-16}$

In Exercises 22–26 solve for x and for y.

22. $(x+2)+(2y-1)i=-1+5i$
23. $(3x-1)+(y+5)i=1-3i$
24. $\left(\frac{1}{2}x+2\right)+(3y-2)i=4-7i$
25. $(2y+1)-(2x-1)i=-8+3i$
26. $(y-2)+(5x-3)i=5$

In Exercises 27–42 compute the answer and write it in the form $a+bi$.

27. $2i+(3-i)$
28. $-3i+(2-5i)$
29. $2+3i+(3-2i)$
30. $(3-2i)-\left(2+\frac{1}{2}i\right)$
31. $-3-5i-(2-i)$
32. $\left(\frac{1}{2}-i\right)+\left(1-\frac{2}{3}i\right)$
33. $-2i(3+i)$
34. $3i(2-i)$
35. $i\left(-\frac{1}{2}+i\right)$
36. $\frac{i}{2}\left(\frac{4-i}{2}\right)$
37. $(2-i)(2+i)$
38. $(5+i)(2-3i)$
39. $(-2-2i)(-4-3i)$
40. $(2+5i)(1-3i)$
41. $(3-2i)(2-i)$
42. $(4-3i)(2+3i)$

In Exercises 43–46 evaluate the polynomial x^2-2x+5 for the given complex value of x.

43. $1+2i$
44. $2-i$
45. $1-i$
46. $1-2i$

47. Prove that the commutative law of addition holds for the set of complex numbers.
48. Prove that the commutative law of multiplication holds for the set of complex numbers.
49. Prove that $0+0i$ is the additive identity and $1+0i$ is the multiplicative identity for the set of complex numbers.
50. Prove that $-a-bi$ is the additive inverse of the complex number $a+bi$.
51. Prove the distributive property for the set of complex numbers.
52. For what values of x is $\sqrt{x-3}$ a real number?
53. For what values of y is $\sqrt{2y-10}$ a real number?

TERMS AND SYMBOLS

set (p. 1)
element, member (p. 1)
$\{\ \}$ (p. 1)
$\in$ (p. 1)
$\notin$ (p. 1)
subset (p. 2)
natural numbers (p. 2)
integers (p. 2)
rational numbers (p. 2)
irrational numbers (p. 3)
real number system (p. 3)
equal (p. 5)
factor (p. 7)
real number line (p. 10)
origin (p. 10)
nonnegative (p. 10)
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real part (p. 53)
imaginary part (p. 53)

KEY IDEAS FOR REVIEW

- ☐ A set is simply a collection of objects or numbers.
- ☐ The real number system is composed of the rational and irrational numbers. The rational numbers are those that can be written as the ratio of two integers, p/q, with $q \neq 0$; the irrational numbers cannot be written as a ratio of integers.
- ☐ The real number system satisfies a number of important properties. These are

  closure commutativity associativity
  identities inverses distributivity
- ☐ If two numbers are identical, we say that they are equal. Equality satisfies these basic properties

  reflexive property symmetric property
  transitive property substitution property
- ☐ There is a one-to-one correspondence between the set of all real numbers and the set of all points on the real number line. That is, for every point on the line there is a real number and for every real number there is a point on the line.
- ☐ Algebraic statements using inequality symbols have straightforward geometric interpretations using the real number line. For example, $a < b$ says that a lies to the left of b on the real number line.
- ☐ Inequalities can be operated on in the same manner as statements involving an equals sign, with one important exception. When an inequality is multiplied or divided by a negative number, the sense of the inequality is reversed.
- ☐ Absolute value specifies distance independent of the direction. Three important properties of absolute value are

  $$|a| \geq 0 \qquad |a| = |-a| \qquad |a - b| = |b - a|$$
- ☐ The distance between points A and B whose coordinates are a and b, respectively, is given by

  $$\overline{AB} = |b - a|$$
- ☐ Algebraic expressions of the form

  $$P = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

  are called polynomials.
- ☐ To add (subtract) polynomials, simply add (subtract) like terms. To multiply polynomials, form all possible products, using the rule for exponents: $a^ma^n = a^{m+n}$.
- ☐ A polynomial is said to be factored when it is written as a product of polynomials of lower degree.
- ☐ Most of the rules of arithmetic for handling fractions carry over to rational expressions. For example, the LCD has the same meaning except that we deal with polynomials in factored form rather than with integers.
- ☐ The rules for positive integer exponents also apply to zero and negative integer exponents and to rational exponents.
- ☐ Radical notation is simply another way of writing a rational exponent. That is, $\sqrt[n]{b} = b^{1/n}$.
- ☐ If n is even and b is positive, there are two real numbers a such that $b^{1/n} = a$. Under these circumstances, we insist that the nth root be positive. That is, $\sqrt[n]{b}$ is a positive number if n is even and b is positive. Thus, $\sqrt{16} = 4$.
- ☐ We must write $\sqrt{x^2} = |x|$ to ensure that the result is a positive number.
- ☐ To be in simplified form, a radical must satisfy the following conditions.
  - $\sqrt[n]{x^m}$ has $m < n$.
  - $\sqrt[n]{x^m}$ has no common factors between m and n.
  - The denominator has been rationalized.
- ☐ Complex numbers were created because there are no real numbers that satisfy such simple polynomial equations as $x^2 + 5 = 0$.
- ☐ Using the imaginary unit $i = \sqrt{-1}$, a complex number is of the form $a + bi$, where a and b are real numbers.
- ☐ The real number system is a subset of the complex number system.

REVIEW EXERCISES

Solutions to exercises whose numbers are in color are in the Solutions section in the back of the book.

1.1 In Exercises 1–3 write each set by listing its elements within braces.

1. The set of natural numbers from -5 to 4, inclusive.

2. The set of integers from -3 to -1, inclusive.

3. The subset of $x \in S$, $S = \{0.5, 1, 1.5, 2\}$ such that x is an even integer.

For Exercises 4–7 determine whether the statement is true (T) or false (F).

4. $\sqrt{7}$ is a real number.

5. -35 is a natural number.

6. -14 is not an integer.

7. 0 is an irrational number.

In Exercises 8–11 identify the property of the real number system that justifies the statement. All variables represent real numbers.

8. $(3a) + (-3a) = 0$

9. $(3 + 4)x = 3x + 4x$

10. $2x + 2y + z = 2x + z + 2y$

11. $9x \cdot 1 = 9x$

1.2 In Exercises 12–14 sketch the given set of numbers on a real number line.

12. The negative real numbers.

13. The real numbers x such that $x > 4$.

14. The real numbers x such that $-1 \leq x < 1$.

15. Find the value of $|-3| - |1 - 5|$.

16. Find $\overline{PQ}$ if the coordinates of P and Q are [illegible] and 6, respectively.

1.3 17. A salesperson receives $3.25x + 0.15y$ dollars, where x is the number of hours worked and y is the number of miles driven. Find the amount due the salesperson if $x = 12$ hours and $y = 80$ miles.

18. Which of the following expressions are not polynomials.

(a) $-2xy^2 + x^2y$ (b) $3b^2 + 2b - 6$

(c) $x^{-1/2} + 5x^2 - x$ (d) $7.5x^2 + 3x - \frac{1}{2}x^0$

In Exercises 19 and 20 indicate the leading coefficient and the degree of each polynomial.

19. $-0.5x^7 + 6x^3 - 5$ 20. $2x^2 + 3x^4 - 7x^5$

In Exercises 21–23 perform the indicated operations.

21. $(3a^2b^2 - a^2b + 2b - a) - (2a^2b^2 + 2a^2b - 2b - a)$

22. $x(2x - 1)(x + 2)$ 23. $3x(2x + 1)^2$

1.4 In Exercises 24–29 factor each expression.

24. $2x^2 - 2$

25. $x^2 - 25y^2$

26. $2a^2 + 3ab + 6a + 9b$

27. $4x^2 + 19x - 5$

28. $x^8 - 1$

29. $27r^6 + 8s^6$

1.5 In Exercises 30–33 perform the indicated operations and simplify.

30. $\dfrac{14(y - 1)}{3(x^2 - y^2)} \cdot \dfrac{9(x + y)}{-7xy^2}$ 31. $\dfrac{4 - x^2}{2y^2} \div \dfrac{x - 2}{3y}$

32. $\dfrac{a + b}{a + 2b} \cdot \dfrac{a^2 - 4b^2}{a^2 - b^2}$

33. $\dfrac{x^2 - 2x - 3}{2x^2 - x} \div \dfrac{x^2 - 4x + 3}{3x^3 - 3x^2}$

In Exercises 34–37 find the LCD.

34. $\dfrac{-1}{2x^2}, \dfrac{2}{x^2 - 4}, \dfrac{3}{x - 2}$

35. $\dfrac{4}{x}, \dfrac{5}{x^2 - x}, \dfrac{-3}{(x - 1)^2}$

36. $\dfrac{2}{(x - 1)y}, \dfrac{-4}{y^2}, \dfrac{x + 2}{5(x - 1)^2}$

37. $\dfrac{y - 1}{x^2(y + 1)}, \dfrac{x - 2}{2xy - 2x}, \dfrac{3x}{4y^2 + 8y + 4}$

In Exercises 38–41 perform the indicated operations and simplify.

38. $2 + \dfrac{4}{a^2 - 4}$ 39. $\dfrac{3}{x^2 - 16} - \dfrac{2}{x - 4}$

40. $\dfrac{\dfrac{3}{x + 2} - \dfrac{2}{x - 1}}{x - 1}$ 41. $x^2 + \dfrac{\dfrac{1}{x} + 1}{x - \dfrac{1}{x}}$

1.6–1.7 In Exercises 42–50 simplify, and express the answers using only positive exponents. All variables are positive numbers.

42. $(2a^2b^{-3})^{-3}$ 43. $2(a^2 - 1)^0$ 44. $\left(\dfrac{x^3}{y^{-6}}\right)^{-4/3}$

45. $\dfrac{x^{3+n}}{x^n}$ 46. $\sqrt{80}$ 47. $\dfrac{2}{\sqrt{12}}$

48. $\sqrt{x^7y^5}$ 49. $\sqrt[4]{32x^8y^6}$ 50. $\dfrac{\sqrt{x}}{\sqrt{x} + \sqrt{y}}$

In Exercises 51–52 perform the indicated operations. Simplify the answer.

51. $\sqrt[4]{x^2y^2} + 2\sqrt[4]{x^2y^2}$ 52. $(\sqrt{3} + \sqrt{5})^2$

1.8 53. Solve for x and for y:

$$(x - 2) + (2y - 1)i = -4 + 7i$$

54. Simplify i^{47}.

In Exercises 55–57 perform the indicated operations and write all answers in the form $a + bi$.

55. $2 + (6 - i)$ 56. $(2 + i)^2$

57. $(4 - 3i)(2 + 3i)$

PROGRESS TEST 1A

In Problems 1 and 2 write each set by listing its elements within braces.

1. The set of positive, even integers less than 13.
2. The subset of $x \in S$, $S = \{-1, 2, 3, 5, 7\}$, such that x is a multiple of 3.

In Problems 3 and 4 determine whether the statement is true (T) or false (F).

3. -1.36 is an irrational number.
4. π is equal to $\frac{22}{7}$.

In Problems 5 and 6 identify the property of the real number system that justifies the statement. All variables represent real numbers.

5. $xy(z + 1) = (z + 1)xy$ 6. $(-6)\left(-\frac{1}{6}\right) = 1$

In Problems 7 and 8 sketch the given set of numbers on a real number line.

7. The integers that are greater than -3 and less than or equal to 3.
8. The real numbers x such that $-2 \le x < 1/2$.
9. Find the value of $|2 - 3| - |4 - 2|$.
10. Find $\overline{AB}$ if the coordinates of A and B are -6 and -4, respectively.
11. The area of a region is given by the expression $3x^2 - xy$. Find the area when $x = 5$ meters and $y = 10$ meters.
12. Evaluate the expression $\dfrac{-|y - 2x|}{|xy|}$ when $x = 3$ and $y = -1$.
13. Which of the following expressions are not polynomials?
    (a) x^5 (b) $5x^{-4}y + 3x^2 - y$
    (c) $4x^3 + x$ (d) $2x^2 + 3x^0$

In Problems 14 and 15 indicate the leading coefficient and the degree of each polynomial.

14. $-2.2x^5 + 3x^3 - 2x$ 15. $14x^6 - 2x + 1$

In Problems 16 and 17 perform the indicated operations.

16. $3xy + 2x + 3y + 2 - (1 - y - x + xy)$
17. $(a + 2)(3a^2 - a + 5)$

In Problems 18 and 19 factor each expression.

18. $8a^3b^5 - 12a^5b^2 + 16a^2b$ 19. $4 - 9x^2$

In Problems 20 and 21 perform the indicated operations and simplify.

20. $\dfrac{m^4}{3n^2} \div \left(\dfrac{m^2}{9n} \cdot \dfrac{n}{2m^3}\right)$ 21. $\dfrac{16 - x^2}{x^2 - 3x - 4} \cdot \dfrac{x - 1}{x + 4}$

22. Find the LCD of

$$\frac{-1}{2x^2} \qquad \frac{2}{4x^2 - 4} \qquad \frac{3}{x - 2}$$

In Problems 23 and 24 perform the indicated operations and simplify.

23. $\dfrac{2x}{x^2 - 9} + \dfrac{5}{3x + 9}$ 24. $\dfrac{2 - \dfrac{4}{x + 1}}{x - 1}$

In Problems 25–28 simplify, and express the answers using only positive exponents.

25. $\left(\dfrac{x^{7/2}}{x^{2/3}}\right)^{-6}$ 26. $\dfrac{y^{2n}}{y^{n-1}}$

27. $\dfrac{-1}{(x - 1)^0}$ 28. $(2a^2b^{-1})^2$

In Problems 29–31 perform the indicated operations.

29. $3\sqrt[3]{24} - 2\sqrt[3]{81}$ 30. $(\sqrt{7} - 5)^2$

31. $\dfrac{1}{2}\sqrt{\dfrac{xy}{4}} - \sqrt{9xy}$

32. For what values of x is $\sqrt{2 - x}$ a real number?

In Problems 33 and 34 perform the indicated operations and write all answers in the form $a + bi$.

33. $(2 - i) + (-3 + i)$ 34. $(5 + 2i)(2 - 3i)$

PROGRESS TEST 1B

In Problems 1 and 2 write each set by listing its elements within braces.

1. The set of positive, odd integers less than 10.
2. The subset of $x \in S$, $S = \{0, 15, 12, 24\}$, such that x is divisible by 3.

In Problems 3 and 4 determine whether the statement is true (T) or false (F).

3. 19.6 is a real number.
4. π is equal to 3.14.

In Problems 5 and 6 identify the property of the real number system that justifies the statement. All variables represent real numbers.

5. $a + b + c = c + a + b$
6. $2(3 + x) = 6 + 2x$

In Problems 7 and 8 sketch the given set of numbers on a real number line.

7. The natural numbers that are less than 5.
8. The real numbers x such that $\frac{3}{2} < x < 3$.
9. Find the value of $\dfrac{|2-5| + |1-5|}{|-7|}$.
10. Find $\overline{AB}$ if the coordinates of A and B are -2 and 5, respectively.
11. The area of a trapezoid is given by the formula $A = \frac{1}{2}h(b + b')$. Find the area if $h = 4$ meters, $b = 3$ meters, and $b' = 4$ meters.
12. Evaluate the expression $|x|/|x - y|$ when $x = -2$ and $y = -3$.
13. Which of the following expressions are not polynomials?

    (a) $3x^2 + x^{-1} - 2$ (b) $2x^3 - xy^2 + x$
    (c) $2x^2y^2 + xy - 4$ (d) $x^2y + x^{1/2}y + 2$

In Problems 14 and 15 indicate the leading coefficient and the degree of each polynomial.

14. $-3x^3 + 4x^5$
15. $1.5x^{10} - x^9 + 17x^8$

In Problems 16 and 17 perform the indicated operations.

16. $(2s^2t^3 - st^2 + st - s + t) - (3s^2t^2 - 2s^2t - 4st^2 - t + 3)$
17. $(b + 3)(-3b^2 + 2b + 4)$

In Problems 18 and 19 factor each expression.

18. $5r^3s^4 - 40r^4s^3t$
19. $2x^2 + 7x - 4$

In Problems 20 and 21 perform the indicated operations and simplify.

20. $\dfrac{3x^2(y-1)}{6u^2v^3} + \dfrac{(y-1)^2}{2uv^2}$
21. $\dfrac{x^2 + 7x - 8}{x - x^2} \cdot \dfrac{x}{x^2 + 8x}$
22. Find the LCD of

    $\dfrac{y-1}{x^2(y+1)}$ $\dfrac{x-2}{2xy - 2x}$ $\dfrac{3x}{4y^2 + 8y + 4}$

In Problems 23 and 24 perform the indicated operations and simplify.

23. $\dfrac{-4}{x-1} - \dfrac{3}{1-x} + \dfrac{x}{x-1}$
24. $\dfrac{1-x}{x^2+x} - \dfrac{2x}{x+1}$

In Problems 25–28 simplify, and express the answers using only positive exponents.

25. $\dfrac{4x^{-3}}{x^{-2}}$
26. $(b^2)^5(b^3)^6$
27. $\left(\dfrac{x^8}{y^{12}}\right)^{3/4}$
28. $\dfrac{2(x+2)^0}{-2}$

In Problems 29–31 simplify the given expression.

29. $\sqrt{x^{14}y^{17}}$
30. $\dfrac{-4}{2\sqrt{x} - 2}$
31. $\sqrt[3]{a^3b^5}$
32. For what values of x is $\dfrac{1}{\sqrt{x-2}}$ a real number?

In Problems 33 and 34 perform the indicated operations and write all answers in the form $a + bi$.

33. $(4 - 2i) - \left(2 - \frac{1}{2}i\right)$
34. $(3 - 2i)(2 - i)$

2

EQUATIONS AND INEQUALITIES

A major concern of algebra is the solution of equations. Does a given equation have a solution? Is it possible for an equation to have more than one solution? Is there a procedure for solving an equation? In this chapter we will explore the answers to these questions for polynomial equations of the first and second degree. We will also see that the ability to solve equations enables us to tackle a wide variety of applications and word problems.

Linear inequalities also play an important role in solving word problems. For example, if we are required to combine food products in such a way that a specified minimum or maximum of protein is provided, we need to use inequalities. Many important industries, including steel and petroleum refincries, use computers daily to solve problems that involve thousands of inequalities. The solutions to such problems enable a company to optimize its ''product mix'' and its profitability.

2.1 LINEAR EQUATIONS IN ONE UNKNOWN

SOLVING EQUATIONS

Expressions of the form

$$x - 2 = 0 \qquad x^2 - 9 = 0 \qquad 3(2x - 5) = 3$$

$$2x + 5 = \sqrt{x - 7} \qquad \frac{1}{2x + 3} = 5 \qquad x^3 - 3x^2 = 32$$

are examples of equations in the unknown x. An **equation** states that two algebraic expressions are equal. We refer to these expressions as the **left-hand** and **right-hand sides** of the equation.

Our task is to find values of the unknown for which the equation holds true. These values are called **solutions** or **roots** of the equation, and the set of all solutions is called the **solution set.** For example, 2 is a solution of the equation $3x - 1 = 5$ since $3(2) - 1 = 5$. However, -2 is *not* a solution since $3(-2) - 1 \neq 5$.

The solutions of an equation depend on the number system we are using. For example, the equation $2x - 5$ has no integer solutions but does have a solution among the rational numbers, namely $\frac{5}{2}$. Similarly, the equation $x^2 = -4$ has no solutions among the real numbers but does have solutions if we consider complex numbers, namely $2i$ and $-2i$. The solution sets of these two equations are $\{\frac{5}{2}\}$ and $\{2i, -2i\}$, respectively.

Identities and Conditional Equations

We say that an equation is an **identity** if it is true for every real number for which both sides of the equation are defined. For example, the equation

$$x^2 - 1 = (x + 1)(x - 1)$$

is an identity because it is true for all real numbers; that is, every real number is a solution of the equation. The equation

$$x - 5 = 3$$

is a false statement for all values of x except 8. If, as in that equation, there are real-number values of x for which the sides of the equation, although both defined, are unequal, the equation is called a **conditional equation.**

When we say that we want to "solve an equation," we mean that we want to find *all* solutions or roots. If we can replace an equation with another, simpler equation that has the same solutions, we will have an approach to solving equations. Equations having the same solutions are called **equivalent equations.** For example, $3x - 1 = 5$ and $3x = 6$ are equivalent equations because it can be shown that $\{2\}$ is the solution set of both equations.

There are two important rules that allow us to replace an equation with an equivalent equation.

Equivalent Equations

The solutions of a given equation are not affected by the following operations:

1. addition or subtraction of the same number or expression on both sides of the equation

2. multiplication or division of both sides of the equation by a number other than 0

EXAMPLE 1

Solve $3x + 4 = 13$.

SOLUTION

We apply the preceding rules to this equation. The strategy is to isolate x, so we *subtract 4 from both sides* of the equation.

$$3x + 4 - 4 = 13 - 4$$
$$3x = 9$$

Dividing both sides by 3, we obtain the solution

$$x = 3$$

It is generally a good idea to check by substitution, to make sure that 3 does indeed satisfy the original equation.

$$3x + 4 \stackrel{?}{=} 13$$
$$3(3) + 4 \stackrel{?}{=} 13$$
$$13 \stackrel{\checkmark}{=} 13$$

To be technically accurate, the *solution* of the equation in Example 1 is 3, while $x = 3$ is an equation that is *equivalent* to the original equation. Now that this distinction is understood, we will join in the common usage that says that the equation $3x + 4 = 13$ "has the solution $x = 3$."

When the given equation contains rational expressions, we eliminate fractions by first multiplying by the least common denominator of all of the fractions. This technique is illustrated in Examples 2, 3, and 4.

EXAMPLE 2

Solve the equation $\frac{5}{6}x - \frac{4}{3} = \frac{3}{5}x + 1$.

SOLUTION

We first eliminate fractions by multiplying both sides of the equation by the LCD of all fractions, which is 30.

$$30\left(\frac{5}{6}x - \frac{4}{3}\right) = 30\left(\frac{3}{5}x + 1\right)$$
$$25x - 40 = 18x + 30$$
$$7x = 70$$
$$x = 10$$

The student should verify that $x = 10$ is a solution of the original equation.

PROGRESS CHECK

Solve and check.

(a) $-\frac{2}{3}(x - 5) = \frac{3}{2}(x + 1)$

(b) $\frac{1}{3}x + 2 - 3\left(\frac{x}{2} + 4\right) = 2\left(\frac{x}{4} - 1\right)$

ANSWERS

(a) $\frac{11}{13}$ (b) $-\frac{24}{5}$

SOLVING LINEAR EQUATIONS

The equations we have solved are all of the first degree and involve only one unknown. Such equations are called **first-degree equations in one unknown,** or more simply, **linear equations.** The general form of such equations is

$$ax + b = 0$$

where a and b are any real numbers and $a \neq 0$. Let's see how we would solve this equation.

$$\begin{aligned} ax + b &= 0 \\ ax + b - b &= 0 - b && \text{Subtract } b \text{ from both sides.} \\ ax &= -b \\ \frac{ax}{a} &= \frac{-b}{a} && \text{Divide both sides by } a \neq 0. \\ x &= -\frac{b}{a} \end{aligned}$$

We have thus obtained the following result.

Roots of a Linear Equation	The linear equation $ax + b = 0$, $a \neq 0$, has exactly one solution: $-b/a$.

Sometimes we are led to linear equations in the course of solving other equations. The following example illustrates this situation.

EXAMPLE 3

Solve $\dfrac{5x}{x+3} - 3 = \dfrac{1}{x+3}$.

SOLUTION

The LCD of all fractions is $x + 3$. Multiplying both sides of the equation by $x + 3$ to eliminate fractions, we obtain

$$\begin{aligned} 5x - 3(x + 3) &= 1 \\ 5x - 3x - 9 &= 1 \\ 2x &= 10 \\ x &= 5 \end{aligned}$$

Checking the solution, we have

$$\begin{aligned} \frac{5(5)}{5+3} - 3 &\stackrel{?}{=} \frac{1}{5+3} \\ \frac{25}{8} - 3 &\stackrel{?}{=} \frac{1}{8} \\ \frac{1}{8} &\doteq \frac{1}{8} \end{aligned}$$

We said earlier that multiplication of both sides of an equation by any nonzero number results in an equivalent equation. What happens if we multiply an equation by an expression that contains an unknown? In Example 3 this procedure worked just fine and gave us a solution. But this will not always be so, because the answer we obtain may produce a zero denominator when substituted in the original equation. The following rule must therefore be carefully observed.

Multiplying by an Unknown

When we multiply or divide both sides of an equation by an expression that contains the unknown, the resulting equation might not be equivalent to the original equation. The answer obtained must be substituted in the original equation to verify that it is a solution.

EXAMPLE 4

Solve and check: $\dfrac{8x+1}{x-2} + 4 = \dfrac{7x+3}{x-2}$.

SOLUTION

The LCD of all fractions is $x - 2$. Multiplying both sides of the equation by $x - 2$, we eliminate fractions and obtain

$$8x + 1 + 4(x - 2) = 7x + 3$$
$$8x + 1 + 4x - 8 = 7x + 3$$
$$5x = 10$$
$$x = 2$$

Checking our answer, we find that $x = 2$ is not a solution, since substituting $x = 2$ in the original equation yields a denominator of zero. We conclude that the given equation has no solution.

PROGRESS CHECK

Solve and check.

(a) $\dfrac{3}{x} - 1 = \dfrac{1}{2} - \dfrac{6}{x}$ (b) $-\dfrac{2x}{x+1} = 1 + \dfrac{2}{x+1}$

ANSWERS

(a) $x = 6$ (b) no solution

EXAMPLE 5

Solve the equation $2x + 1 = 2x - 3$.

SOLUTION

Subtracting $2x$ from both sides, we have

$$2x + 1 - 2x = 2x - 3 - 2x$$
$$1 = -3$$

This equivalent equation is a contradiction. Conclusion: not every equation has a solution!

EXERCISE SET 2.1

In Exercises 1–4 determine whether the given statement is true (T) or false (F).

1. $x = -5$ is a solution of $2x + 3 = -7$.
2. $x = \frac{5}{2}$ is a solution of $3x - 4 = \frac{5}{2}$.
3. $x = \frac{6}{4-k}$, $k \neq 4$, is a solution of $kx + 6 = 4x$.
4. $x = \frac{7}{3k}$, $k \neq 0$, is a solution of $2kx + 7 = 5x$.

In Exercises 5–24 solve the given linear equation and check your answer.

5. $3x + 5 = -1$
6. $5r + 10 = 0$
7. $2 = 3x + 4$
8. $\frac{1}{2}s + 2 = 4$
9. $\frac{3}{2}t - 2 = 7$
10. $-1 = -\frac{2}{3}x + 1$
11. $0 = -\frac{1}{2}a - \frac{2}{3}$
12. $4r + 4 = 3r - 2$
13. $-5x + 8 = 3x - 4$
14. $2x - 1 = 3x + 2$
15. $-2x + 6 = -5x - 4$
16. $6x + 4 = -3x - 5$
17. $2(3b + 1) = 3b - 4$
18. $-3(2x + 1) = -8x + 1$
19. $4(x - 1) = 2(x + 3)$
20. $-3(x - 2) = 2(x + 4)$
21. $2(x + 4) - 1 = 0$
22. $3a + 2 - 2(a - 1) = 3(2a + 3)$
23. $-4(2x + 1) - (x - 2) = -11$
24. $3(a + 2) - 2(a - 3) = 0$

Solve for x in Exercises 25–28.

25. $kx + 8 = 5x$
26. $8 - 2kx = -3x$
27. $2 - k + 5(x - 1) = 3$
28. $3(2 + 3k) + 4(x - 2) = 5$

Solve and check in Exercises 29–44.

29. $\frac{x}{2} = \frac{5}{3}$
30. $\frac{3x}{4} - 5 = \frac{1}{4}$
31. $\frac{2}{x} + 1 = \frac{3}{x}$
32. $\frac{5}{a} - \frac{3}{2} = \frac{1}{4}$
33. $\frac{2y - 3}{y + 3} = \frac{5}{7}$
34. $\frac{1 - 4x}{1 - 2x} = \frac{9}{8}$
35. $\frac{1}{x - 2} + \frac{1}{2} = \frac{2}{x - 2}$
36. $\frac{4}{x - 4} - 2 = \frac{1}{x - 4}$
37. $\frac{2}{x - 2} + \frac{2}{x^2 - 4} = \frac{3}{x + 2}$
38. $\frac{3}{x - 1} + \frac{2}{x + 1} = \frac{5}{x^2 - 1}$
39. $\frac{x}{x - 1} - 1 = \frac{3}{x + 1}$
40. $\frac{2}{x - 2} + 1 = \frac{x + 2}{x - 2}$
41. $\frac{4}{b} - \frac{1}{b + 3} = \frac{3b + 2}{b^2 + 2b - 3}$
42. $\frac{3}{x^2 - 2x} + \frac{2x - 1}{x^2 + 2x - 8} = \frac{2}{x + 4}$
43. $\frac{3r + 1}{r + 3} + 2 = \frac{5r - 2}{r + 3}$
44. $\frac{2x - 1}{x - 5} + 3 = \frac{3x - 2}{5 - x}$

In Exercises 45–48 indicate whether the equation is an identity (I) or a conditional equation (C).

45. $x^2 + x - 2 = (x + 2)(x - 1)$

46. $(x - 2)^2 = x^2 - 4x + 2$

47. $2x + 1 = 3x - 1$

48. $3x - 5 = 4x - x - 2 - 3$

In Exercises 49–54 write (T) if the equations within each exercise are all equivalent equations and (F) if they are not equivalent.

49. $2x - 3 = 5 \qquad 2x = 8 \qquad x = 4$

50. $5(x - 1) = 10 \qquad x - 1 = 2 \qquad x = 3$

51. $x(x - 1) = 5x \qquad x - 1 = 5 \qquad x = 6$

52. $x = 5 \qquad x^2 = 25$

53. $3(x^2 + 2x + 1) = -6$
$x^2 + 2x + 1 = -2$
$(x + 1)^2 = -2$

54. $(x + 3)(x - 1) = x^2 - 2x + 1$
$(x + 3)(x - 1) = (x - 1)^2$
$x + 3 = x - 1$

2.2 APPLICATIONS

Many applied problems lead to linear equations that must be solved. The solution procedure described in Section 2.1 was already familiar to you and probably presents no difficulties. The challenge of applied problems is translating words into appropriate algebraic forms. This translation process requires an ability that you can acquire only with practice.

The steps listed here can guide you in solving word problems.

Step 1. Read the problem carefully to understand what is required.
Step 2. Separate what is known from what is to be found.
Step 3. In many problems, the unknown quantity is the answer to a question such as "how much" or "how many." Let an algebraic symbol, say x, represent the unknown.
Step 4. If possible, represent other quantities in the problem in terms of x.
Step 5. Find the relationship in the problem that you can express as an equation (or an inequality).
Step 6. Solve and check.

The words and phrases in Table 1 should prove helpful in translating a word problem into an algebraic expression that can be solved.

EXAMPLE 1

If you pay \$66 for a car radio after receiving a 25% discount, what was the price of the radio before the discount?

SOLUTION

Let x = the price of the radio (in dollars) before the discount. Then

$$0.25x = \text{the amount discounted}$$

and the price of the radio after the discount is given by

$$x - 0.25x$$

TABLE 1

Word or phrase	Algebraic symbol	Example	Algebraic expression
Sum	$+$	Sum of two numbers	$a + b$
Difference	$-$	Difference of two numbers Difference of a number and 3	$a - b$ $x - 3$
Product	$\times$ or $\cdot$	Product of two numbers	$a \cdot b$
Quotient	$\div$ or /	Quotient of two numbers	$\frac{a}{b}$ or a/b
Exceeds More than More of		a exceeds b by 3 a is 3 more than b There are 3 more of a than of b	$a = b + 3$ or $a - 3 = b$
Twice		Twice a number Twice the difference of x and 3 3 more than twice a number 3 less than twice a number	$2x$ $2(x - 3)$ $2x + 3$ $2x - 3$
Is or equals	$=$	The sum of a number and 3 is 15.	$x + 3 = 15$

Hence

$$x - 0.25x = 66$$
$$0.75x = 66$$
$$x = \frac{66}{0.75} = 88$$

The price of the radio was \$88 before the discount.

COIN PROBLEMS

Coin problems are easy to interpret if you keep this in mind: Always distinguish between the *number* of coins and the *value* of the coins. You will also find it helpful to use a chart, as in the following example.

EXAMPLE 2

A purse contains \$3.20 in quarters and dimes. If there are 3 more quarters than dimes, how many coins of each type are there?

SOLUTION

In this problem, we may let the unknown represent either the number of quarters or the number of dimes. We make a choice. Let

n = the number of quarters

Then

$$n - 3 = \text{the number of dimes}$$

since there are 3 more quarters than dimes.

The following table is useful in further analysis of the problem.

	Number of coins	× Number of cents in each coin	= Value in cents
Quarters	n	25	$25n$
Dimes	$n - 3$	10	$10(n - 3)$

We know that

$$\begin{aligned} \text{total value} &= (\text{value of quarters}) + (\text{value of dimes}) \\ 320 &= 25n + 10(n - 3) \\ 320 &= 25n + 10n - 30 \\ 350 &= 35n \\ 10 &= n \end{aligned}$$

Then

$$\begin{aligned} n &= \text{number of quarters} = 10 \\ n - 3 &= \text{number of dimes} = 7 \end{aligned}$$

Now verify that the total value of all the coins is \$3.20.

SIMPLE INTEREST

Interest is the fee charged for borrowing money. In this section we will deal only with simple interest, which assumes the fee to be a fixed percentage r of the amount borrowed. We call the amount borrowed the **principal** and denote it by P.

If the principal P is borrowed at a simple interest rate r, then the interest due at the end of each year is Pr, and the total interest I due at the end of t years is

$$I = Prt$$

Consequently, if S is the total amount owed at the end of t years, then

$$S = P + Prt$$

since both the principal and interest are to be repaid.

The basic formulas that we have derived for simple interest calculations are

$$\boxed{\begin{aligned} I &= Prt \\ S &= P + Prt \end{aligned}}$$

EXAMPLE 3

A part of \$7000 was borrowed at 6% simple annual interest and the remainder at 8%. If the total amount of interest due after 3 years is \$1380, how much was borrowed at each rate?

SOLUTION

Let

$$n = \text{the amount borrowed at } 6\%$$

Then

$$7000 - n = \text{the amount borrowed at } 8\%$$

since the total amount is \$7000. We can display the information in table form using the equation $I = Prt$.

	P ×	r ×	t =	Interest
6% portion	n	0.06	3	$0.18n$
8% portion	$7000 - n$	0.08	3	$0.24(7000 - n)$

Note that we write the rate r in its decimal form, so that 6% = 0.06 and 8% = 0.08.

Since the total interest of \$1380 is the sum of the interest from the two portions, we have

$$\begin{aligned} 1380 &= 0.18n + 0.24(7000 - n) \\ 1380 &= 0.18n + 1680 - 0.24n \\ 0.06n &= 300 \\ n &= 5000 \end{aligned}$$

We conclude that \$5000 was borrowed at 6% and \$2000 was borrowed at 8%.

DISTANCE (UNIFORM MOTION) PROBLEMS

Here is the key to the solution of distance problems.

$$\text{Distance} = \text{rate} \times \text{time}$$

or

$$d = r \cdot t$$

The relationships that permit you to write an equation are sometimes obscured by the words. Here are some questions to ask as you set up a distance problem.

(a) Are there two distances that are equal? (Will two objects have traveled the same distance? Is the distance on a return trip the same as the distance going?)

(b) Is the sum (or difference) of two distances equal to a constant? (When two objects are traveling toward each other, they meet when the sum of the distances traveled by them equals the original distance between them.)

EXAMPLE 4

Two trains leave New York for Chicago. The first train travels at an average speed of 60 miles per hour. The second train, which departs an hour later, travels at an average speed of 80 miles per hour. How long will it take the second train to overtake the first train?

SOLUTION

Since we are interested in the time the second train travels, we choose to let

$$t = \text{the number of hours the second train travels}$$

Then

$$t + 1 = \text{the number of hours the first train travels}$$

since the first train departs one hour earlier.

	Rate $\times$	Time $=$	Distance
First train	60	$t + 1$	$60(t + 1)$
Second train	80	t	$80t$

At the moment the second train overtakes the first, they must both have traveled the *same* distance. Thus,

$$\begin{aligned} 60(t + 1) &= 80t \\ 60t + 60 &= 80t \\ 60 &= 20t \\ 3 &= t \end{aligned}$$

It takes the second train 3 hours to catch up with the first train.

MIXTURE PROBLEMS

One type of mixture problem involves mixing commodities, say two or more types of nuts, to obtain a mixture with a desired value. If the commodities are measured in pounds, the relationships we need are

number of pounds $\times$ price per pound = value of commodity
pounds in mixture = sum of pounds of each commodity
value of mixture = sum of values of individual commodities

EXAMPLE 5

How many pounds of Brazilian coffee worth \$5 per pound must be mixed with 20 pounds of Colombian coffee worth \$4 per pound to produce a mixture worth \$4.20 per pound?

SOLUTION

Let n = number of pounds of Brazilian coffee. We display all the information, using cents in place of dollars.

Type of coffee	Number of pounds $\times$	Price per pound $=$	Value (in cents)
Brazilian	n	500	$500n$
Colombian	20	400	8000
Mixture	$n + 20$	420	$420(n + 20)$

Note that the weight of the mixture equals the sum of the weights of the Brazilian and Colombian coffees that make up the mixture. Since the value of the mixture is the sum of the values of the two types of coffee,

$$\begin{aligned} \text{value of mixture} &= (\text{value of Brazilian}) + (\text{value of Colombian}) \\ 420(n + 20) &= 500n + 8000 \\ 420n + 8400 &= 500n + 8000 \\ 400 &= 80n \\ 5 &= n \end{aligned}$$

We must add 5 pounds of Brazilian coffee to make the required mixture.

WORK PROBLEMS

Work problems typically involve two or more people or machines working on the same task. The key to these problems is to express the *rate of work per unit of time*, whether an hour, a day, a week, or some other unit. For example, if a machine can do a job in 5 days, then

$$\text{rate of machine} = \frac{1}{5} \text{ job per day}$$

If this machine were used for two days, it would perform $\frac{2}{5}$ of the job. In summary:

> If a machine (or person) can complete a job in n days, then
>
> $$\text{Rate of machine (or person)} = \frac{1}{n} \text{ job per day}$$
>
> $$\text{Work done} = \text{Rate} \times \text{Time}$$

EXAMPLE 6

Using a small mower, at 12 noon a student begins to mow a lawn, a job that would take him 9 hours working alone. At 1 P.M. another student, using a tractor, joins him, and they complete the job together at 3 P.M. How many hours would it take to do the job by tractor only?

SOLUTION

Let x = number of hours to do the job by tractor alone. The small mower works from 12 noon to 3 P.M., or 3 hours; the tractor is used from 1 P.M. to 3 P.M., or 2 hours.

All the information can be displayed in table form.

	Rate $\times$	Time $=$	Work done
Small mower	$\frac{1}{9}$	3	$\frac{3}{9}$
Tractor	$\frac{1}{x}$	2	$\frac{2}{x}$

Since

$$\begin{array}{ccccc} \text{work done by} & + & \text{work done by} & = & 1 \text{ whole job} \\ \text{small mower} & & \text{tractor} & & \\ \frac{3}{9} & + & \frac{2}{x} & = & 1 \end{array}$$

To solve, multiply both sides by the LCD, which is $9x$.

$$9x\left(\frac{3}{9} + \frac{2}{x}\right) = 9x \cdot 1$$

$$3x + 18 = 9x$$

$$x = 3$$

Thus, by tractor alone, the job can be done in 3 hours.

LITERAL EQUATIONS

The circumference C of a circle is given by the formula

$$C = 2\pi r$$

where r is the radius of the circle. For every value of r, the formula gives us a value of C. If $r = 20$, we have

$$C = 2\pi(20) = 40\pi$$

It is sometimes convenient to be able to turn a formula around, that is, to be able to solve for a different variable. For example, if we want to express the radius of a circle in terms of the circumference, we have

$$C = 2\pi r$$

$$\frac{C}{2\pi} = \frac{2\pi r}{2\pi} \qquad \text{Dividing by } 2\pi$$

$$\frac{C}{2\pi} = r$$

Now, given a value of C, we can determine a value of r.

EXAMPLE 7

If an amount P is borrowed at the simple annual interest rate r, then the amount S due at the end of t years is

$$S = P + Prt$$

Solve for P.

SOLUTION

$$S = P + Prt$$

$$S = P(1 + rt) \qquad \text{Common factor } P$$

$$\frac{S}{1 + rt} = P \qquad \text{Dividing both sides by } (1 + rt)$$

EXERCISE SET 2.2

In Exercises 1–3 let n represent the unknown. Translate from words to an algebraic expression or equation.

1. The number of blue chips is 3 more than twice the number of red chips.
2. The number of station wagons on a parking lot is 20 fewer than 3 times the number of sedans.
3. Five less than 6 times a number is 26.

In Exercises 4–41 translate from words to an algebraic problem and solve.

4. Janis is 3 years older than her sister. Thirty years from now the sum of their ages will be 111. Find the current ages of the sisters.
5. John is presently 12 years older than Fred. Four years ago John was twice as old as Fred. How old is each now?
6. The larger of two numbers is 3 more than twice the smaller. If their sum is 18, find the numbers.
7. Find three consecutive integers whose sum is 21.
8. A certain number is 5 less than another number. If their sum is 11, find the two numbers.
9. A resort guarantees that the average temperature over the period Friday, Saturday, and Sunday will be exactly 80°F, or else each guest pays only half price for the facilities. If the temperatures on Friday and Saturday were 90°F and 82°F, respectively, what must the temperature be on Sunday so that the resort does not lose half of its revenue?
10. A patient's temperature was taken at 6 A.M., 12 noon, 3 P.M., and 8 P.M. The first, third, and fourth readings were 102.5°, 101.5°, and 102°F, respectively. The nurse forgot to write down the second reading but recorded that the average of the four readings was 101.5°F. What was the second temperature reading?
11. A 12-meter-long steel beam is to be cut into two pieces so that one piece will be 4 meters longer than the other. How long will each piece be?
12. A rectangular field whose length is 10 meters longer

than its width is to be enclosed with exactly 100 meters of fencing material. What are the dimensions of the field?

13. A vending machine contains $3.00 in nickels and dimes. If the number of dimes is 5 more than twice the number of nickels, how many coins of each type are there?

14. A wallet contains $460 in $5, $10, and $20 bills. The number of $5 bills exceeds twice the number of $10 bills by 4, and the number of $20 bills is 6 fewer than the number of $10 bills. How many bills of each type are there?

15. A movie theater charges $3 admission for an adult and $1.50 for a child. If 700 tickets were sold on a particular day and the total revenue received was $1650, how many tickets of each type were sold?

16. A student bought 5-cent, 10-cent, and 15-cent stamps with a total value of $6.70. If the number of 5-cent stamps is 2 more than the number of 10-cent stamps, and the number of 15-cent stamps is 5 more than one-half the number of 10-cent stamps, how many stamps of each denomination did the student obtain?

17. An amateur theater group is converting a classroom to an auditorium for a forthcoming play. The group will sell $3, $5, and $6 tickets, and will receive exactly $503 from the sale of tickets. If the number of $5 tickets is twice the number of $6 tickets, and the number of $3 tickets is 1 more than 3 times the number of $6 tickets, how many tickets of each type are there?

18. To pay for their child's college education, the parents invested $10,000, part in a certificate of deposit paying 8.5% annual interest, the rest in a mutual fund paying 7% annual interest. The annual income from the certificate of deposit is $200 more than the annual income from the mutual fund. How much money was put into each type of investment?

19. A bicycle store is closing out its entire stock of a certain brand of 3-speed and 10-speed models. The profit on a 3-speed bicycle is 11% of the sale price, and the profit on a 10-speed model is 22% of the sale price. If the entire stock will be sold for $16,000 and the profit on the entire stock will be 19%, how much will be obtained from the sale of each type of bicycle?

20. A film shop carrying black-and-white film and color film has $4000 in inventory. The profit on black-and-white film is 12%, and the profit on color film is 21%. If all the film is sold, and if the profit on color film is $150 less than the profit on black-and-white film, how much was invested in each type of film?

21. A firm borrowed $12,000 at a simple annual interest rate of 8% for a period of 3 years. At the end of the first year, the firm found that its needs were reduced. The firm returned a portion of the original loan and retained the remainder until the end of the 3-year period. If the total interest paid was $1760, how much was returned at the end of the first year?

22. A finance company lent a certain amount of money to Firm A at 7% annual interest. An amount $100 less than that lent to Firm A was lent to Firm B at 8%, and an amount $200 more than that lent to Firm A was lent to Firm C at 8.5% for one year. If the total annual income is $126.50, how much was lent to each firm?

23. Two trucks leave Philadelphia for Miami. The first truck to leave travels at an average speed of 50 kilometers per hour. The second truck, which leaves two hours later, travels at an average speed of 55 kilometers per hour. How long will it take the second truck to overtake the first truck?

24. Jackie either drives or bicycles from home to school. Her average speed when driving is 36 miles per hour, and her average speed when bicycling is 12 miles per hour. If it takes her $\frac{1}{2}$ hour less to drive to school than to bicycle, how long does it take her to go to school, and how far is the school from her home?

25. Professors Roberts and Jones, who live 676 miles apart, are exchanging houses and jobs for the summer. They start out for their new locations at exactly the same time, and they meet after 6.5 hours of driving. If their average speeds differ by 4 miles per hour, what are their average speeds?

26. Steve leaves school by moped for spring vacation. Forty minutes later his roommate, Frank, notices that Steve forgot to take his camera, so Frank decides to try to catch up with Steve by car. If Steve's average speed is 25 miles per hour and Frank averages 45 miles per hour, how long does it take Frank to overtake Steve?

27. An express train and a local train start out from the same point at the same time and travel in opposite directions. The express train travels twice as fast as

the local train. If after 4 hours they are 480 kilometers apart, what is the average speed of each train?

28. How many pounds of raisins worth \$1.50 per pound must be mixed with 10 pounds of peanuts worth \$1.20 per pound to produce a mixture worth \$1.40 per pound?

29. How many ounces of Ceylon tea worth \$1.50 per ounce and how many ounces of Formosa tea worth \$2.00 per ounce must be mixed to obtain a mixture of 8 ounces that is worth \$1.85 per ounce?

30. A copper alloy that is 40% copper is to be combined with a copper alloy that is 80% copper to produce 120 kilograms of an alloy that is 70% copper. How many kilograms of each alloy must be used?

31. A vat contains 27 gallons of water and 9 gallons of acetic acid. How many gallons of water must be evaporated if the resulting solution is to be 40% acetic acid?

32. A producer of packaged frozen vegetables wants to market mixed vegetables at \$1.20 per kilogram. How many kilograms of green beans worth \$1.00 per kilogram must be mixed with 100 kilograms of corn worth \$1.30 per kilogram and 90 kilograms of peas worth \$1.40 per kilogram to produce a satisfactory mixture?

33. A certain number is 3 times another. If the difference of their reciprocals is 8, find the numbers.

34. If $\frac{1}{3}$ is subtracted from 3 times the reciprocal of a certain number, the result is $\frac{25}{6}$. Find the number.

35. Computer A can carry out an engineering analysis in 4 hours, while computer B can do the same job in 6 hours. How long will it take to complete the job if both computers work together?

36. Jackie can paint a certain room in 3 hours, Lisa in 4 hours, and Susan in 2 hours. How long will it take to paint the room if they all work together?

37. A senior copy editor together with a junior copy editor can edit a book in 3 days. The junior editor, working alone, would take twice as long to complete the job as the senior editor would require if working alone. How long would it take each editor to complete the job by herself?

38. Hose A can fill a certain vat in 3 hours. After 2 hours of pumping, hose A is turned off. Hose B is then turned on and completes filling the vat in 3 hours. How long would it take hose B alone to fill the vat?

39. A printing shop starts a job at 10 A.M. on press A. Using this press alone, it would take 8 hours to complete the job. At 2 P.M. press B is also turned on, and both presses together finish the job at 4 P.M. How long would it take press B alone to do the job?

40. A boat travels 20 kilometers upstream in the same time that it would take the same boat to travel 30 kilometers downstream. If the rate of the stream is 5 kilometers per hour, find the speed of the boat in still water.

41. An airplane flying against the wind travels 300 miles in the same time that it would take the same plane to travel 400 miles with the wind. If the wind speed is 20 miles per hour, find the speed of the airplane in still air.

In Exercises 42–51 solve for the indicated variable in terms of the remaining variables.

42. $A = Pr$ for r

43. $C = 2\pi r$ for r

44. $V = \frac{1}{3}\pi r^2 h$ for h

45. $F = \frac{9}{5}C + 32$ for C

46. $S = \frac{1}{2}gt^2 + vt$ for v

47. $A = \frac{1}{2}h\,(b + b')$ for b

48. $A = P(1 + rt)$ for r

49. $\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$ for f_2

50. $a = \frac{v_1 - v_0}{t}$ for v_0

51. $S = \frac{a - rL}{L - r}$ for L

2.3 LINEAR INEQUALITIES

Much of the terminology of equations carries over to inequalities. A **solution of an inequality** is a value of the unknown that satisfies the inequality, and the **solution set** is composed of all solutions. The properties of inequalities listed in Section 1.1 enable us to use the same procedures in solving inequalities as in solving equations *with one exception*.

> Multiplication or division of an inequality by a negative number reverses the sense of the inequality.

We will concentrate for now on solving a **linear inequality**, that is, an inequality in which the unknown appears only in the first degree.

EXAMPLE 1

Solve the inequality $2x + 11 \geq 5x - 1$.

SOLUTION

We perform addition and subtraction to collect terms in x just as we did for equations.

$$\begin{aligned} 2x + 11 &\geq 5x - 1 \\ 2x &\geq 5x - 12 \\ -3x &\geq -12 \end{aligned}$$

We now divide both sides of the inequality by -3, a negative number, and therefore reverse the sense of the inequality.

$$\frac{-3x}{-3} \leq \frac{-12}{-3}$$

$$x \leq 4$$

PROGRESS CHECK

Solve the inequality $3x - 2 \geq 5x + 4$.

ANSWER

$x \leq -3$

WARNING Given the inequality

$$-2x \geq -6$$

it is a common error to conclude that dividing by -2 gives $x \leq -3$. Multiplication or division by a negative number changes the sense of the inequality but the *signs* obey the usual rules of algebra. Thus,

$$-2x \geq -6$$

$$\frac{-2x}{-2} \leq \frac{-6}{-2} \qquad \text{Reverse sense of the inequality.}$$

$$x \leq 3$$

There are three methods commonly used to describe subsets of the real numbers: graphs on a real number line, interval notation, and set-builder notation. Since there will be occasions when we want to use each of these schemes, this is a convenient time to introduce them and to apply them to inequalities.

The **graph of an inequality** is the set of all points satisfying the inequality. The graph of the inequality $a \leq x < b$ is shown in Figure 1. The portion of the

FIGURE 1

real number line that is in color is the solution set of the inequality. The circle at point a has been filled in to indicate that a is also a solution of the inequality; the circle at point b has been left open to indicate that b is not a member of the solution set.

An **interval** is a set of numbers on the real number line that form a line segment, a half line, or the entire real number line. The subset shown in Figure 1 would be written in **interval notation** as $[a, b)$, where a and b are the **endpoints** of the interval. A bracket, [or], indicates that the endpoint is included, while a parenthesis, (or), indicates that the endpoint is not included. The interval $[a, b]$ is called a **closed interval** because both endpoints are included. The interval (a, b) is called an **open interval** because neither endpoint is included. Finally, the intervals $[a, b)$ and $(a, b]$ are called **half-open intervals.**

The set of all real numbers satisfying a given property P is written as

$$\{x|x \text{ satisfies property } P\}$$

which is read as "the set of all x such that x satisfies property P." This form, called **set-builder notation,** provides a third means of designating subsets of the real number line. Thus, the interval $[a, b)$ shown in Figure 1 is written as

$$\{x|a \leq x < b\}$$

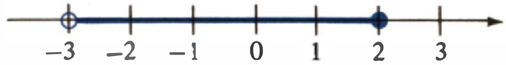
which indicates that x must satisfy the inequalities $x \geq a$ and $x < b$.

EXAMPLE 2

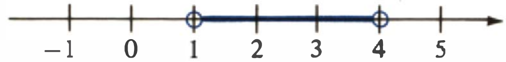
Graph each of the given intervals on a real number line and indicate the same subset of the real number line in set-builder notation.

(a) $(-3, 2]$ (b) $(1, 4)$ (c) $[-4, -1]$

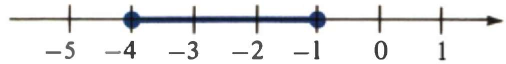
SOLUTION



$\{x|-3 < x \leq 2\}$



$\{x|1 < x < 4\}$



$\{x|-4 \leq x \leq 1\}$

To describe the inequalities $x > 2$ and $x \leq 3$ in interval notation, we need to introduce the symbols ∞ and $-\infty$ (read "infinity" and "minus infinity," respectively). The inequalities $x > 2$ and $x \leq 3$ are then written as $(2, \infty)$ and $(-\infty, 3]$, respectively, in interval notation and would be graphed on a real number line as shown in Figure 2. Note that ∞ and $-\infty$ are symbols (not numbers) indicating that the intervals extend indefinitely. An interval using one of these symbols is called an **infinite interval.** The interval $(-\infty, \infty)$ designates the entire real number line. Square brackets must never be used around ∞ and $-\infty$, since they are not real numbers.

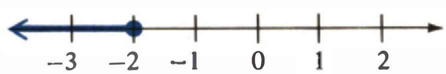
FIGURE 2

EXAMPLE 3

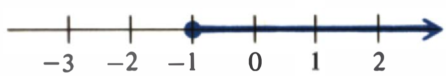
Graph each inequality and write the solution set in interval notation.

(a) $x \leq -2$ (b) $x \geq -1$ (c) $x < 3$

SOLUTION



$(-\infty, -2]$



$[-1, \infty)$

$(-\infty, 3)$

EXAMPLE 4

Solve the inequality

$$\frac{x}{2} - 9 < \frac{1-2x}{3}$$

Graph the solution set, and write the solution set in both interval notation and set-builder notation.

SOLUTION

To clear the inequality of fractions, we multiply both sides by the LCD of all fractions, which is 6.

$$\begin{aligned} 3x - 54 &< 2(1-2x) \\ 3x - 54 &< 2 - 4x \\ 7x &< 56 \\ x &< 8 \end{aligned}$$

We may write the solution set as $\{x|x < 8\}$ or as the infinite interval $(-\infty, 8)$. The graph of the solution set is shown in Figure 3.

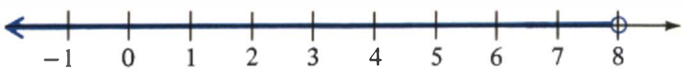


FIGURE 3

EXAMPLE 5

Solve the inequality.

(a) $\dfrac{2(x+1)}{3} < \dfrac{2x}{3} - \dfrac{1}{6}$ (b) $2(x-1) < 2x + 5$

SOLUTION

(a) The LCD of all fractions is 6. Multiplying both sides of the inequality by 6, we obtain

$$\begin{aligned} 4(x+1) &< 4x - 1 \\ 4x + 4 &< 4x - 1 \\ 4 &< -1 \end{aligned}$$

Our procedure has led to a contradiction, indicating that there is no solution to the inequality.

(b) Expanding and simplifying leads to the inequality

$$-2 < 5$$

Since this inequality is true for all real values of x, we conclude that the solution set is the set of all real numbers.

PROGRESS CHECK

Solve, and write the answers in interval notation.

(a) $\dfrac{3x-1}{4} + 1 > 2 + \dfrac{x}{3}$ (b) $\dfrac{2x-3}{2} \geq x + \dfrac{2}{5}$

ANSWERS

(a) $(3, \infty)$ (b) no solution

DOUBLE INEQUALITIES

We can solve double inequalities such as

$$1 < 3x - 2 \leq 7$$

by operating on both inequalities at the same time.

$$3 < 3x \leq 9 \quad \text{Add } +2 \text{ to each member.}$$
$$1 < x \leq 3 \quad \text{Divide each member by 3.}$$

The solution set is the half-open interval $(1, 3]$.

EXAMPLE 6

Solve the inequality $-3 \leq 1 - 2x < 6$, and write the answer in interval notation.

SOLUTION

Operating on both inequalities, we have

$$-4 \leq -2x < 5 \quad \text{Add } -1 \text{ to each member.}$$
$$2 \geq x > -\frac{5}{2} \quad \text{Divide each member by } -2.$$

The solution set is the half-open interval $(-\frac{5}{2}, 2]$.

PROGRESS CHECK

Solve the inequality $-5 < 2 - 3x < -1$, and write the answer in interval notation.

ANSWER

$\left(1, \dfrac{7}{3}\right)$

EXAMPLE 7

A taxpayer may choose to pay a 20% tax on the gross income or a 25% tax on the gross income less \$4000. Above what income level should the taxpayer elect to pay at the 20% rate?

SOLUTION

It we let x = gross income, then the choice available to the taxpayer is

(a) pay at the 20% rate on the gross income, that is, pay $0.20x$, or

(b) pay at the 25% rate on the gross income less $4000, that is, pay

$$0.25(x - 4000)$$

To determine when (a) produces a lower tax than (b), we must solve

$$0.20x < 0.25(x - 4000)$$
$$0.20x < 0.25x - 1000$$
$$-0.05x < -1000$$
$$x > \frac{1000}{0.05} = 20{,}000$$

The taxpayer should choose to pay at the 20% rate if the gross income is more than $20,000.

PROGRESS CHECK

A customer is offered the following choice of telephone services: unlimited local calls at a fixed $20 monthly charge, or a base rate of $8 per month plus $0.06 per message unit. At what level of use does it cost less to choose the unlimited service?

ANSWER

Unlimited service costs less when the anticipated use exceeds 200 message units.

EXERCISE SET 2.3

In Exercises 1–9 express the given inequality in interval notation.

1. $-5 \le x < 1$
2. $-4 < x \le 1$
3. $x > 9$
4. $x \le -2$
5. $-12 \le x \le -3$
6. $x \ge -5$
7. $3 < x < 7$
8. $x < 17$
9. $-6 < x \le -4$

In Exercises 10–18 express the given interval as an inequality.

10. $(-4, 3]$
11. $[5, 8]$
12. $(-\infty, -2]$
13. $(3, \infty)$
14. $[-3, 10)$
15. $(-\infty, 5]$
16. $(-2, -1)$
17. $[0, \infty)$
18. $(-5, 7)$

In Exercises 19–36 solve the inequality and graph the result.

19. $x + 4 < 8$
20. $x + 5 < 4$
21. $x + 3 < -3$
22. $x - 2 \le 5$
23. $x - 3 \ge 2$
24. $x + 5 \ge -1$
25. $2 < a + 3$
26. $-5 > b - 3$
27. $2y < -1$
28. $3x < 6$
29. $2x \ge 0$
30. $-\frac{1}{2}y \ge 4$
31. $2r + 5 < 9$
32. $3x - 2 > 4$
33. $3x - 1 \ge 2$
34. $\frac{-1}{2x + 3} > 0$

35. $\dfrac{4}{5-3x} < 0$ 36. $\dfrac{3}{3x-1} > 0$

Solve the given inequality in Exercises 37–60 and write the solution set in interval notation.

37. $4x + 3 \le 11$
38. $\frac{1}{2}y - 2 \le 2$
39. $\frac{3}{2}x + 1 \ge 4$
40. $-5x + 2 > -8$
41. $4(2x + 1) < 16$
42. $3(3r - 4) \ge 15$
43. $2(x - 3) < 3(x + 2)$
44. $4(x - 3) \ge 3(x - 2)$
45. $3(2a - 1) > 4(2a - 3)$
46. $2(3x - 1) + 4 < 3(x + 2) - 8$
47. $\frac{2}{3}(x + 1) + \frac{5}{6} \ge \frac{1}{2}(2x - 1) + 4$
48. $\frac{1}{4}(3x + 2) - 1 \le -\frac{1}{2}(x - 3) + \frac{3}{4}$
49. $\dfrac{x-1}{3} + \dfrac{1}{5} < \dfrac{x+2}{5} - \dfrac{1}{3}$
50. $\dfrac{x}{5} - \dfrac{1-x}{2} > \dfrac{x}{2} - 3$
51. $3(x + 1) + 6 \ge 2(2x - 1) + 4$
52. $4(3x + 2) - 1 \le -2(x - 3) + 15$
53. $-2 < 4x \le 5$
54. $3 \le 6x < 12$
55. $-4 \le 2x + 2 \le -2$
56. $5 \le 3x - 1 \le 11$
57. $3 \le 1 - 2x < 7$
58. $5 < 2 - 3x \le 11$
59. $-8 < 2 - 5x \le 7$
60. $-10 < 5 - 2x < -5$

In Exercises 61–67 translate from words to an algebraic problem and solve.

61. A student has grades of 42 and 70 in the first two tests of the semester. If an average of 70 is required to obtain a C grade, what is the minimum score the student must achieve on the third exam to obtain a C?

62. A compact car can be rented from firm A for \$160 per week with no charge for mileage or from firm B for \$100 per week plus 20 cents for each mile driven. If the car is driven m miles, for what values of m does it cost less to rent from firm A?

63. An appliance salesperson is paid \$30 per day plus \$25 for each appliance sold. How many appliances must be sold for the salesperson's income to exceed \$130 per day?

64. A pension trust invests \$6000 in a bond that pays 5% simple interest per year. Additional funds are to be invested in a more speculative bond paying 9% simple interest per year, so that the return on the total investment will be at least 6%. What is the minimum amount that must be invested in the more speculative bond?

65. A book publisher spends \$38,000 on editorial expenses and \$12 per book for manufacturing and sales expenses in the course of publishing a psychology textbook. If the book sells for \$25, how many copies must be sold to show a profit?

66. If the area of a right triangle is not to exceed 80 square inches and the base is 10 inches, what values may be assigned to the altitude h?

67. A total of 70 meters of fencing material is available with which to enclose a rectangular area. If the width of the rectangle is 15 meters, what values can be assigned to the length L?

2.4 ABSOLUTE VALUE IN EQUATIONS AND INEQUALITIES

In Section 1.2 we discussed the use of absolute value notation to indicate distance, and we provided this formal definition.

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

The following example illustrates the application of this definition to the solution of equations involving absolute value.

EXAMPLE 1

Solve the equation $|2x - 7| = 11$.

SOLUTION

We apply the definition of absolute value to the two cases.

Case 1. $2x - 7 \geq 0$
With the first part of the definition,

$$|2x - 7| = 2x - 7 = 11$$
$$2x = 18$$
$$x = 9$$

Case 2. $2x - 7 < 0$
With the second part of the definition,

$$|2x - 7| = -(2x - 7) = 11$$
$$-2x + 7 = 11$$
$$x = -2$$

PROGRESS CHECK

Solve each equation and check the solution(s).

(a) $|x + 8| = 9$ (b) $|3x - 4| = 7$

ANSWERS

(a) $1, -17$ (b) $\frac{11}{3}, -1$

When used in inequalities, absolute value notation plays an important and frequently used role in higher mathematics. To solve inequalities involving absolute value, we recall that $|x|$ is the distance between the origin and the point on the real number line corresponding to x. For $a > 0$, the solution set of the inequality $|x| < a$ is then seen to consist of all real numbers whose distance from the origin is less than a, that is, all real numbers in the open interval $(-a, a)$, shown in Figure 4. Similarly, if $|x| > a > 0$, the solution set consists of all real numbers whose

FIGURE 4

distance from the origin is greater than a, that is, all points in the infinite intervals $(-\infty, -a)$ and (a, ∞), shown in Figure 5. Of course, $|x| \leq a$ and $|x| \geq a$ would include the endpoints a and $-a$, and the circles would be filled in.

FIGURE 5

EXAMPLE 2

Solve $|2x - 5| \leq 7$, graph the solution set, and write the solution set in interval notation.

SOLUTION

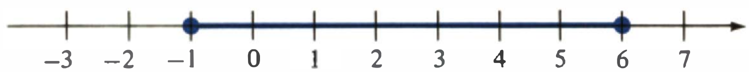
We must solve the equivalent double inequality

$$-7 \le 2x - 5 \le 7$$

$$-2 \le 2x \le 12 \qquad \text{Add } +5 \text{ to each member.}$$

$$-1 \le x \le 6 \qquad \text{Divide each member by 2.}$$

The graph of the solution set is then



Thus, the solution set is the closed interval $[-1, 6]$.

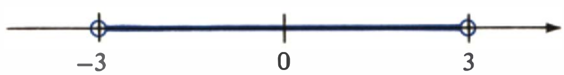
PROGRESS CHECK

Solve each inequality, graph the solution set, and write the solution set in interval notation.

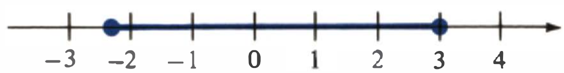
(a) $|x| < 3$ (b) $|3x - 1| \le 8$ (c) $|x| < -2$

ANSWERS

(a) $(-3, 3)$



(b) $\left[-\frac{7}{3}, 3\right]$



(c) No solution. Since $|x|$ is always nonnegative, $|x|$ cannot be less than -2.

EXAMPLE 3

Solve the inequality $|2x - 6| > 4$, write the solution set in interval notation, and graph the solution.

SOLUTION

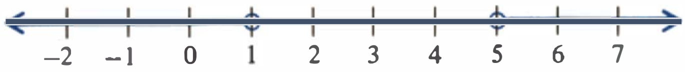
We must solve the equivalent inequalities

$$2x - 6 > 4 \qquad 2x - 6 < -4$$

$$2x > 10 \qquad 2x < 2$$

$$x > 5 \qquad x < 1$$

The solution set consists of the real numbers in the infinite intervals $(-\infty, 1)$ and $(5, \infty)$. The graph of the solution set is then



PROGRESS CHECK

Solve each inequality, write the solution set in interval notation, and graph the solution.

(a) $|5x - 6| > 9$ (b) $|2x - 2| \geq 8$

ANSWERS

(a) $\left(-\infty, -\frac{3}{5}\right), (3, \infty)$

(b) $(-\infty, -3], [5, \infty)$

WARNING Students sometimes write

$$1 > x > 5$$

This is a misuse of the inequality notation since it states that x is simultaneously less than 1 *and* greater than 5, which is impossible. What is usually intended is the pair of infinite intervals $(-\infty, 1)$ and $(5, \infty)$, and the inequalities must be written

$$x < 1 \quad or \quad x > 5$$

EXERCISE SET 2.4

In Exercises 1–9 solve and check.

1. $|x + 2| = 3$
2. $|r - 5| = \frac{1}{2}$
3. $|2x - 4| = 2$
4. $|5y + 1| = 11$
5. $|-3x + 1| = 5$
6. $|2t + 2| = 0$
7. $3|-4x - 3| = 27$
8. $\frac{1}{|x|} = 5$
9. $\frac{1}{|s - 1|} = \frac{1}{3}$

In Exercises 10–15 solve the inequality and graph the solution set.

10. $|x + 3| < 5$
11. $|x + 1| > 3$
12. $|3x + 6| \leq 12$
13. $|4x - 1| > 3$

14. $|3x + 2| \geq -1$

15. $\left|\frac{1}{3} - x\right| < \frac{2}{3}$

In Exercises 16–24 solve the inequality, and write the solution set using interval notation.

16. $|x - 2| \leq 4$

17. $|x - 3| \geq 4$

18. $|2x + 1| < 5$

19. $\frac{|2x - 1|}{4} < 2$

20. $\frac{|3x + 2|}{2} \leq 4$

21. $\frac{|2x + 1|}{3} < 0$

22. $\left|\frac{4}{3x - 2}\right| < 1$

23. $\left|\frac{5 - x}{3}\right| > 4$

24. $\left|\frac{2x + 1}{3}\right| \leq 5$

In Exercises 25 and 26, x and y are real numbers.

25. Prove that $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ (*Hint:* Treat as four cases.)

26. Prove that $|x|^2 = x^2$

27. A machine that packages 100 vitamin pills per bottle can make an error of 2 pills per bottle. If x is the number of pills in a bottle, write an inequality, using absolute value, that indicates a maximum error of 2 pills per bottle. Solve the inequality.

28. The weekly income of a worker in a manufacturing plant differs from \$300 by no more than \$50. If x is the weekly income, write an inequality, using absolute value, that expresses this relationship. Solve the inequality.

2.5 THE QUADRATIC EQUATION

We now turn our attention to equations involving second-degree polynomials. A **quadratic equation** is an equation of the form

$$ax^2 + bx + c = 0, \qquad a \neq 0$$

where a, b, and c are real numbers. In this section we will explore techniques for solving this important class of equations. We will also show that there are several kinds of equations that can be transformed to quadratic equations and then solved.

THE FORMS $ax^2 + c = 0$ AND $a(x + h)^2 + c = 0$

When the quadratic equation $ax^2 + bx + c = 0$ has the coefficient $b = 0$, we have an equation of the form

$$ax^2 + c = 0$$

Solving for x, we have

$$x^2 = -\frac{c}{a}$$

or

$$x = \pm\sqrt{-\frac{c}{a}}$$

That is,

$$x = \sqrt{-\frac{c}{a}} \quad \text{and} \quad x = -\sqrt{-\frac{c}{a}}$$

are solutions of the original equation. Don't try to memorize the form of the solution; in dealing with the form $ax^2 + c = 0$, just follow the usual steps of solving for x^2, as in the following example.

EXAMPLE 1

Solve the equation $3x^2 - 8 = 0$.

SOLUTION

$$\begin{aligned} 3x^2 - 8 &= 0 \\ x^2 &= \frac{8}{3} \\ x &= \pm\sqrt{\frac{8}{3}} = \pm\frac{\sqrt{24}}{3} = \pm\frac{2\sqrt{6}}{3} \end{aligned}$$

The solutions are $\frac{2}{3}\sqrt{6}$ and $-\frac{2}{3}\sqrt{6}$.

Equations of the form

$$a(x + h)^2 + c = 0$$

are also easy to solve. Again, just follow the usual steps as shown in the following example.

EXAMPLE 2

Solve the equation $3(x - 5)^2 - 18 = 0$.

SOLUTION

Solving for x, we have

$$\begin{aligned} 3(x - 5)^2 &= 18 \\ (x - 5)^2 &= 6 \\ x - 5 &= \pm\sqrt{6} \\ x &= 5 \pm \sqrt{6} \end{aligned}$$

The solutions are the real numbers $5 + \sqrt{6}$ and $5 - \sqrt{6}$.

PROGRESS CHECK

Solve the given equation.

(a) $5x^2 + 13 = 0$ (b) $(2x - 7)^2 - 5 = 0$

ANSWERS

(a) $\pm\dfrac{i\sqrt{65}}{5}$ (b) $\dfrac{7 \pm \sqrt{5}}{2}$

We have seen that the solutions of a quadratic equation may be complex numbers whereas the solution of a linear equation is a real number. In addition, quadratic equations appear to have two solutions. We will have more to say about these observations when we study the roots of polynomial equations in a later chapter.

SOLVING BY FACTORING

If we can factor the left-hand side of the quadratic equation

$$ax^2 + bx + c = 0, \qquad a \neq 0$$

into two linear factors, then we can solve the equation quickly. For example, the quadratic equation

$$3x^2 + 5x - 2 = 0$$

can be written as

$$(3x - 1)(x + 2) = 0$$

Since the product of two real numbers can be zero only if one or more of the factors are zero, we can set each factor equal to zero.

$$\begin{aligned} 3x - 1 &= 0 \qquad \text{or} \qquad & x + 2 &= 0 \\ x &= \frac{1}{3} & x &= -2 \end{aligned}$$

The solutions of the given quadratic equation are $\frac{1}{3}$ and -2.

EXAMPLE 3

Solve the equation $2x^2 - 3x - 2 = 0$ by factoring.

SOLUTION

$$\begin{aligned} 2x^2 - 3x - 2 &= 0 \\ (2x + 1)(x - 2) &= 0 \end{aligned}$$

Since the product of the factors is 0, at least one factor must be 0. Setting each factor equal to 0, we have

$$\begin{aligned} 2x + 1 &= 0 \qquad \text{or} \qquad & x - 2 &= 0 \\ x &= -\frac{1}{2} & x &= 2 \end{aligned}$$

EXAMPLE 4
Solve the equation $3x^2 - 4x = 0$ by factoring.

SOLUTION
Factoring, we have

$$3x^2 - 4x = 0$$
$$x(3x - 4) = 0$$

Setting each factor equal to zero,

$$x = 0 \qquad \text{or} \qquad x = \frac{4}{3}$$

EXAMPLE 5
Solve the equation $x^2 + x + 1 = 0$ by factoring.

SOLUTION
The polynomial $x^2 + x + 1$ is irreducible over the integers and even over the reals; that is, it cannot be written as a product

$$(x + r)(x + s)$$

where r and s are real numbers. It can, however, be written in this form if r and s are complex numbers. Since it is not easy to find these factors, we will next develop solution techniques that are more powerful than factoring.

PROGRESS CHECK
Solve each of the given equations by factoring.
(a) $4x^2 - x = 0$ (b) $3x^2 - 11x - 4 = 0$ (c) $2x^2 + 4x + 1 = 0$

ANSWERS
(a) $0, \frac{1}{4}$ (b) $-\frac{1}{3}, 4$ (c) cannot be factored over the reals

---

COMPLETING THE SQUARE

We have shown that a quadratic equation of the form

$$a(x + h)^2 + c = 0 \tag{1}$$

where a, h, and c are constants, is easily solved (see Example 2). A technique known as **completing the square** permits us to rewrite *any* quadratic equation in the form of Equation (1). Beginning with the expression $x^2 + dx$, we seek a constant h^2 to complete the square so that

$$x^2 + dx + h^2 = (x + h)^2$$

Expanding and solving, we have

$$x^2 + dx + h^2 = x^2 + 2hx + h^2$$
$$dx = 2hx$$
$$h = \frac{d}{2}$$

so h^2 is the square of half the coefficient of x.

EXAMPLE 6

Complete the square for each of the following.

(a) $x^2 - 6x$ (b) $x^2 + 3x$

SOLUTION

(a) The coefficient of x is -6, so $h^2 = (-\frac{6}{2})^2 = 9$. Then

$$(x^2 - 6x + 9) = (x - 3)^2$$

(b) The coefficient of x is 3, and $h^2 = (3/2)^2 = 9/4$. Then

$$x^2 + 3x + \frac{9}{4} = \left(x + \frac{3}{2}\right)^2$$

We are now in a position to use this method to solve a quadratic equation.

EXAMPLE 7

Solve the quadratic equation $2x^2 - 10x + 1 = 0$ by completing the square.

SOLUTION

We now outline and explain each step of the process.

Completing the Square	
Step 1. Rewrite the equation with the constant term on the right-hand side.	*Step 1.* $2x^2 - 10x \quad = -1$
Step 2. Factor out the coefficient a of x^2.	*Step 2.* $2(x^2 - 5x \quad) = -1$
Step 3. Complete the square $$x^2 + dx + h^2 = (x + h)^2$$ where $h^2 = (d/2)^2$. Balance the equation by adding ah^2 to the right-hand side. Simplify.	*Step 3.* $$2\left(x^2 - 5x + \frac{25}{4}\right) = -1 + \frac{25}{2}$$ $$2\left(x - \frac{5}{2}\right)^2 = \frac{23}{2}$$
Step 4. Solve for x.	*Step 4.* $$\left(x - \frac{5}{2}\right)^2 = \frac{23}{4}$$ $$x - \frac{5}{2} = \pm\frac{\sqrt{23}}{2}$$ $$x = \frac{5 \pm \sqrt{23}}{2}$$

PROGRESS CHECK

Solve by completing the square.

(a) $x^2 - 3x + 2 = 0$ (b) $3x^2 - 4x + 2 = 0$

ANSWERS

(a) 1, 2 (b) $\dfrac{2 \pm i\sqrt{2}}{3}$

THE QUADRATIC FORMULA

We can apply the method of completing the square to the general quadratic equation

$$ax^2 + bx + c = 0, \qquad a > 0$$

Following the steps of the method, we have

$ax^2 + bx \qquad = -c$	Move constant term to right-hand side.
$a\left(x^2 + \frac{b}{a}x \qquad\right) = -c$	Factor out the coefficient of x^2
$a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right] = a\left(\frac{b}{2a}\right)^2 - c$	Complete the square and balance the equation.
$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - c$	Simplify.
$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$	
$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$	Solve for x.

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{or} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

By applying the method of completing the square to the standard form of the quadratic equation, we have derived a *formula* that gives us the roots or solutions for *any* quadratic equation in one variable.

Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \qquad a > 0$$

EXAMPLE 8

Solve $2x^2 - 3x - 3 = 0$ by use of the quadratic formula.

SOLUTION

Since $a = 2$, $b = -3$, and $c = -3$, we have

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)\,(-3)}}{2(2)} \\
&= \frac{3 \pm \sqrt{33}}{4}
\end{aligned}
$$

EXAMPLE 9

Solve $-5x^2 + 3x = 2$ by the quadratic formula.

SOLUTION

We first rewrite the given equation as $5x^2 - 3x + 2 = 0$ so that $a > 0$ and the right-hand side equals 0. Then $a = 5$, $b = -3$, and $c = 2$. Substituting in the quadratic formula, we have

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(5)(2)}}{2(5)} \\
&= \frac{3 \pm \sqrt{-31}}{10} = \frac{3 \pm i\sqrt{31}}{10}
\end{aligned}
$$

PROGRESS CHECK

Solve by use of the quadratic formula.

(a) $x^2 - 8x = -10$ (b) $4x^2 - 2x + 1 = 0$

ANSWERS

(a) $4 \pm \sqrt{6}$ (b) $\dfrac{1 \pm i\sqrt{3}}{4}$

WARNING There are a number of errors that students make in using the quadratic formula.

(a) To solve $x^2 - 3x = -4$, you must write the equation in the form

$$x^2 - 3x + 4 = 0$$

to properly identify a, b, and c. Note that $b = -3$, *not* 3.

(b) The quadratic formula is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note that

$$x \neq -b \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

since the term $-b$ must also be divided by $2a$.

Now that you have a formula that works for *any* quadratic equation, you may be tempted to use it all the time. However, if you see an equation of the form

$$x^2 = 15$$

it is certainly easier to supply the answer immediately: $x = \pm\sqrt{15}$. Similarly, if you are faced with

$$x^2 + 3x + 2 = 0$$

it is faster to solve if you see that

$$x^2 + 3x + 2 = (x + 1)(x + 2)$$

The method of completing the square is generally not used for solving quadratic equations once you have learned the quadratic formula. The *technique* of completing the square is helpful in a variety of applications, and we will use it in a later chapter when we graph second-degree equations.

THE DISCRIMINANT

By analyzing the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

we can learn a great deal about the roots of the quadratic equation

$$ax^2 + bx + c = 0, \qquad a > 0$$

The key to the analysis is the **discriminant** $b^2 - 4ac$ found under the radical.

- If $b^2 - 4ac$ is negative, we have the square root of a negative number, and the roots of the quadratic equation are complex numbers.
- If $b^2 - 4ac$ is positive, we have the square root of a positive number, and the roots of the quadratic equation will be real numbers.
- If $b^2 - 4ac = 0$, then $x = -b/2a$, which we call a **double root** or **repeated root** of the quadratic equation. For example, if $x^2 - 10x + 25 = 0$, then the discriminant is 0 and $x = 5$. But

$$x^2 - 10x + 25 = (x - 5)(x - 5) = 0$$

We call $x = 5$ a double root because the factor $(x - 5)$ is a double factor of $x^2 - 10x + 25 = 0$. This hints at the importance of the relationship between roots and factors, a relationship that we will explore in a later chapter on roots of polynomial equations.

If the roots of the quadratic equation are real and a, b, and c are rational numbers, the discriminant enables us to determine whether the roots are rational or irrational. Since $\sqrt{k}$ is a rational number only if k is a perfect square, we see that the quadratic formula produces a rational result only if $b^2 - 4ac$ is a perfect square. We summarize as follows.

The quadratic equation $ax^2 + bx + c = 0$, $a > 0$, has exactly two roots, the nature of which are determined by the discriminant $b^2 - 4ac$.

Discriminant		Roots
Negative		Two complex roots
0		A double root
Positive		Two real roots
a, b, c rational	A perfect square	Rational roots
	Not a perfect square	Irrational roots

EXAMPLE 10

Without solving, determine the nature of the roots of the quadratic equation $3x^2 - 4x + 6 = 0$.

SOLUTION

We evaluate $b^2 - 4ac$ using $a = 3$, $b = -4$, and $c = 6$:

$$b^2 - 4ac = (-4)^2 - 4(3)(6) = 16 - 72 = -56$$

The discriminant is negative, so the equation has two complex roots.

EXAMPLE 11

Without solving, determine the nature of the roots of the equation

$$2x^2 - 7x = -1$$

SOLUTION

We rewrite the equation in the standard form

$$2x^2 - 7x + 1 = 0$$

and then substitute $a = 2$, $b = -7$, and $c = 1$ in the discriminant. Thus,

$$b^2 - 4ac = (-7)^2 - 4(2)(1) = 49 - 8 = 41$$

The discriminant is positive and is not a perfect square; thus, the roots are real, unequal, and irrational.

PROGRESS CHECK

Without solving, determine the nature of the roots of the quadratic equation by using the discriminant.

(a) $4x^2 - 20x + 25 = 0$ (b) $5x^2 - 6x = -2$
(c) $10x^2 = x + 2$ (d) $x^2 + x - 1 = 0$

ANSWERS

(a) a real, double root (b) 2 complex roots
(c) 2 real, rational roots (d) 2 real, irrational roots

FORMS LEADING TO QUADRATICS

Certain types of equations can be transformed into quadratic equations, which can be solved by the methods discussed in this section. One form that leads to a quadratic equation is the **radical equation,** such as

$$x - \sqrt{x - 2} = 4$$

which is solved in Example 12. To solve the equation, we isolate the radical and raise both sides to a suitable power. The following is the key to the solution of such equations.

> If P and Q are algebraic expressions, then the solution set of the equation
> $$P = Q$$
> is a subset of the solution set of the equation
> $$P^n = Q^n$$
> where n is a natural number.

This suggests that we can solve radical equations if we observe a precaution.

> If both sides of an equation are raised to the same power, the solutions of the resulting equation must be checked to see that they satisfy the original equation.

EXAMPLE 12

Solve $x - \sqrt{x - 2} = 4$

SOLUTION

Solving Radical Equations	
Step 1. When possible, isolate the radical on one side of the equation.	*Step 1.* $x - 4 = \sqrt{x-2}$
Step 2. Raise both sides of the equation to a suitable power to eliminate the radical.	*Step 2.* Squaring both sides, we have $x^2 - 8x + 16 = x - 2$
Step 3. Solve for the unknown.	*Step 3.* $x^2 - 9x + 18 = 0$ $(x-3)(x-6) = 0$ $x = 3 \qquad x = 6$
Step 4. Check each solution by substituting in the *original* equation.	*Step 4.* checking $x = 3$: $3 - \sqrt{3-2} \stackrel{?}{=} 4$, $3 - 1 \stackrel{?}{=} 4$, $2 \neq 4$ checking $x = 6$: $6 - \sqrt{6-2} \stackrel{?}{=} 4$, $6 - \sqrt{4} \stackrel{?}{=} 4$, $4 = 4$

We conclude that 6 is a solution of the original equation and 3 is not a solution of the original equation. We say that 3 is an **extraneous solution** that was introduced when we raised each side of the original equation to the second power.

PROGRESS CHECK

Solve $x - \sqrt{1-x} = -5$.

ANSWER

-3

The equation in the next example contains more than one radical. Solving this equation will require that we square both sides *twice*.

EXAMPLE 13

Solve $\sqrt{2x-4} - \sqrt{3x+4} = -2$.

SOLUTION

Before squaring, rewrite the equation so that a radical is on each side of the equation.

$$\sqrt{2x-4} = \sqrt{3x+4} - 2$$

$$2x - 4 = (3x + 4) - 4\sqrt{3x+4} + 4 \qquad \text{Square both sides.}$$

$$-x - 12 = -4\sqrt{3x+4} \qquad \text{Isolate the radical.}$$

$$x^2 + 24x + 144 = 16(3x + 4) \qquad \text{Square both sides.}$$

$$x^2 - 24x + 80 = 0$$

$$(x - 20)(x - 4) = 0$$

$$x = 20 \qquad x = 4$$

Verify that both 20 and 4 are solutions of the original equation.

PROGRESS CHECK

Solve $\sqrt{5x-1} - \sqrt{x+2} = 1$.

ANSWER

2

Although the equation

$$x^4 - x^2 - 2 = 0$$

is not a quadratic in the unknown x, it is a quadratic in the unknown x^2:

$$(x^2)^2 - (x^2) - 2 = 0$$

This may be seen more clearly by replacing x^2 with a new unknown u such that $u = x^2$. Substituting, we have

$$u^2 - u - 2 = 0$$

which is a quadratic equation in the unknown u. Solving, we find

$$(u + 1)(u - 2) = 0$$

$$u = -1 \quad \text{or} \quad u = 2$$

Since $x^2 = u$, we must next solve the equations

$$x^2 = -1 \quad \text{and} \quad x^2 = 2$$

$$x = \pm i \qquad\qquad x = \pm\sqrt{2}$$

The original equation has four solutions: i, $-i$, $\sqrt{2}$, and $-\sqrt{2}$.

The technique we have used is called a **substitution of variable.** Although simple in concept, this is a powerful method that is commonly used in calculus.

PROGRESS CHECK

Indicate an appropriate substitution of variable and solve each of the following equations.

(a) $3x^4 - 10x^2 - 8 = 0$ (b) $4x^{2/3} + 7x^{1/3} - 2 = 0$

(c) $\dfrac{2}{x^2} + \dfrac{1}{x} - 10 = 0$ (d) $\left(1 + \dfrac{2}{x}\right)^2 - 8\left(1 + \dfrac{2}{x}\right) + 15 = 0$

ANSWERS

(a) $u = x^2$; ± 2, $\pm\dfrac{i\sqrt{6}}{3}$ (b) $u = x^{1/3}$; $\dfrac{1}{64}$, -8

(c) $u = \dfrac{1}{x}$; $-\dfrac{2}{5}$, $\dfrac{1}{2}$ (d) $u = 1 + \dfrac{2}{x}$; 1, $\dfrac{1}{2}$

EXERCISE SET 2.5

In Exercises 1–10 solve the given equation.

1. $3x^2 - 27 = 0$
2. $4x^2 - 64 = 0$
3. $5y^2 - 25 = 0$
4. $6x^2 - 12 = 0$
5. $(2r + 5)^2 = 8$
6. $(3x - 4)^2 = -6$
7. $(3x - 5)^2 - 8 = 0$
8. $(4t + 1)^2 - 3 = 0$
9. $9x^2 + 64 = 0$
10. $81x^2 + 25 = 0$

In Exercises 11–24 solve by factoring.

11. $x^2 - 3x + 2 = 0$
12. $x^2 - 6x + 8 = 0$
13. $x^2 + x - 2 = 0$
14. $3r^2 - 4r + 1 = 0$
15. $x^2 + 6x = -8$
16. $x^2 + 6x + 5 = 0$
17. $y^2 - 4y = 0$
18. $2x^2 - x = 0$
19. $2x^2 - 5x = -2$
20. $2s^2 - 5s - 3 = 0$
21. $t^2 - 4 = 0$
22. $4x^2 - 9 = 0$
23. $6x^2 - 5x + 1 = 0$
24. $6x^2 - x = 2$

In Exercises 25–36 solve by completing the square.

25. $x^2 - 2x = 8$
26. $t^2 - 2t = 15$
27. $2r^2 - 7r = 4$
28. $9x^2 + 3x = 2$
29. $3x^2 + 8x = 3$
30. $2y^2 + 4y = 5$
31. $2y^2 + 2y = -1$
32. $3x^2 - 4x = -3$
33. $4x^2 - x = 3$
34. $2x^2 + x = 2$
35. $3x^2 + 2x = -1$
36. $3u^2 - 3u = -1$

In Exercises 37–48 solve by the quadratic formula.

37. $2x^2 + 3x = 0$
38. $2x^2 + 3x + 3 = 0$
39. $5x^2 - 4x + 3 = 0$
40. $2x^2 - 3x - 2 = 0$
41. $5y^2 - 4y + 5 = 0$
42. $x^2 - 5x = 0$
43. $3x^2 + x - 2 = 0$
44. $2x^2 + 4x - 3 = 0$
45. $3y^2 - 4 = 0$
46. $2x^2 + 2x + 5 = 0$
47. $4u^2 + 3u = 0$
48. $4x^2 - 1 = 0$

In Exercises 49–58 solve by any method.

49. $2x^2 + 2x - 5 = 0$
50. $2t^2 + 2t + 3 = 0$
51. $3x^2 + 4x - 4 = 0$
52. $x^2 + 2x = 0$
53. $2x^2 + 5x + 4 = 0$
54. $2r^2 - 3r + 2 = 0$
55. $4u^2 - 1 = 0$
56. $x^2 + 2 = 0$
57. $4x^3 + 2x^2 + 3x = 0$
58. $4s^3 + 4s^2 - 15s = 0$

In Exercises 59–64 solve for the indicated variable in terms of the remaining variables.

59. $a^2 + b^2 = c^2$, for b
60. $s = \dfrac{1}{2}gt^2$, for t
61. $V = \dfrac{1}{3}\pi r^2 h$, for r
62. $A = \pi r^2$, for r
63. $s = \dfrac{1}{2}gt^2 + vt$, for t
64. $F = g\dfrac{m_1 m_2}{d^2}$, for d

Without solving, determine the nature of the roots of each quadratic equation in Exercises 65–80.

65. $x^2 - 2x + 3 = 0$
66. $3x^2 + 2x - 5 = 0$
67. $4x^2 - 12x + 9 = 0$
68. $2x^2 + x + 5 = 0$
69. $-3x^2 + 2x + 5 = 0$
70. $-3y^2 + 2y - 5 = 0$
71. $3x^2 + 2x = 0$
72. $4x^2 + 20x + 25 = 0$
73. $2r^2 = r - 4$
74. $3x^2 = 5 - x$
75. $3x^2 + 6 = 0$
76. $4x^2 - 25 = 0$
77. $6r = 3r^2 + 1$
78. $4x = 2x^2 + 3$
79. $12x = 9x^2 + 4$
80. $4s^2 = -4s - 1$

In Exercises 81–84 find a value or values of k for which the quadratic has a double root.

81. $kx^2 - 4x + 1 = 0$
82. $2x^2 + 3x + k = 0$
83. $x^2 - kx - 2k = 0$
84. $kx^2 - 4x + k = 0$

In Exercises 85–92 find the solution set.

85. $x + \sqrt{x+5} = 7$
86. $x - \sqrt{13-x} = 1$
87. $2x + \sqrt{x+1} = 8$
88. $3x - \sqrt{1+3x} = 1$
89. $\sqrt{3x+4} - \sqrt{2x+1} = 1$
90. $\sqrt{4-4x} - \sqrt{x+4} = 3$
91. $\sqrt{2x-1} + \sqrt{x-4} = 4$
92. $\sqrt{5x+1} + \sqrt{4x-3} = 7$

In Exercises 93–100 indicate an appropriate substitution of variable and solve each of the equations.

93. $3x^4 + 5x^2 - 2 = 0$
94. $2x^6 + 15x^3 - 8 = 0$
95. $\dfrac{6}{x^2} + \dfrac{1}{x} - 2 = 0$
96. $\dfrac{2}{x^4} - \dfrac{3}{x^2} - 9 = 0$
97. $2x^{2/5} + 5x^{1/5} + 2 = 0$
98. $3x^{4/3} - 4x^{2/3} - 4 = 0$
99. $2\left(\dfrac{1}{x} + 1\right)^2 - 3\left(\dfrac{1}{x} + 1\right) - 20 = 0$
100. $3\left(\dfrac{1}{x} - 2\right)^2 + 2\left(\dfrac{1}{x} - 2\right) - 1 = 0$

In Exercises 101 and 102 provide a proof of the stated theorem.

101. If r_1 and r_2 are the roots of the equation $ax^2 + bx + c = 0$, then (a) $r_1 r_2 = c/a$ and (b) $r_1 + r_2 = -b/a$.
102. If a, b, and c are rational numbers, and the discriminant of the equation $ax^2 + bx + c = 0$ is positive, then the quadratic has either two rational roots or two irrational roots.

In Exercises 103–109 use the theorems of Exercise 101 to find a value or values of k that will satisfy the indicated condition.

103. $kx^2 + 3x + 5 = 0$; sum of the roots is 6.
104. $2x^2 - 3kx - 2 = 0$; sum of the roots is -3.
105. $3x^2 - 10x + 2k = 0$; product of the roots is -4.
106. $2kx^2 + 5x - 1 = 0$; product of the roots is $\frac{1}{2}$.
107. $2x^2 - kx + 9 = 0$; one root is double the other.
108. $3x^2 - 4x + k = 0$; one root is triple the other.
109. $6x^2 - 13x + k = 0$; one root is the reciprocal of the other.

2.6 APPLICATIONS OF QUADRATIC EQUATIONS

As your knowledge of mathematical techniques and ideas grows, you will become capable of solving an ever wider variety of applied problems. In Section 2.2 we explored many types of word problems that lead to linear equations. We can now tackle a group of applied problems that lead to quadratic equations.

One word of caution: It is possible to arrive at a solution that is meaningless. For example, a negative solution that represents hours worked or the age of an individual is meaningless and must be rejected.

EXAMPLE 1

The larger of two positive numbers exceeds the smaller by 2. If the sum of the squares of the two numbers is 74, find the two numbers.

SOLUTION

If we let

$$x = \text{the larger number}$$

then

$$x - 2 = \text{the smaller number}$$

The sum of the squares of the numbers is 74.

$$\begin{aligned}
(\text{larger number})^2 + (\text{smaller number})^2 &= 74 \\
x^2 + (x-2)^2 &= 74 \\
x^2 + x^2 - 4x + 4 &= 74 \\
2x^2 - 4x - 70 &= 0 \\
x^2 - 2x - 35 &= 0 \\
(x+5)(x-7) &= 0 \\
x &= 7 \qquad \text{Reject } x = -5.
\end{aligned}$$

The numbers are then 7 and $(7 - 2) = 5$. Verify that the sum of the squares is indeed 74.

EXAMPLE 2

The length of a pool is 3 times its width, and the pool is surrounded by a grass walk 4 feet wide. If the total area covered and enclosed by the walk is 684 square feet, find the dimensions of the pool.

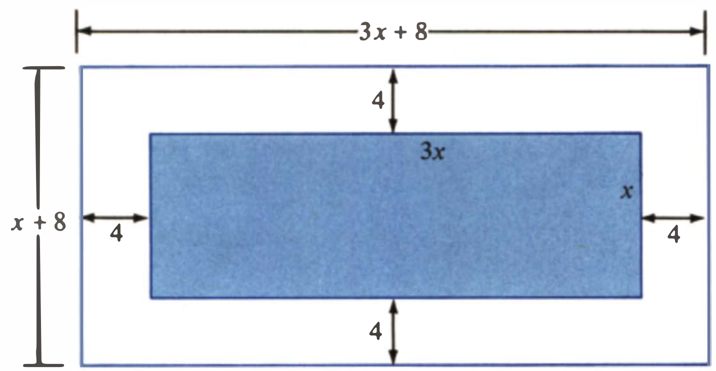


FIGURE 6

SOLUTION

A diagram such as Figure 6 is useful in solving geometric problems. If we let x = width of the pool, then $3x$ = length of the pool, and the region enclosed by the walk has length $3x + 8$ and width $x + 8$. The total area is the product of the length and width, so

$$\text{length} \times \text{width} = 684$$
$$(3x + 8)(x + 8) = 684$$
$$3x^2 + 32x + 64 = 684$$
$$3x^2 + 32x - 620 = 0$$
$$(3x + 62)(x - 10) = 0$$
$$x = 10 \qquad \text{Reject } x = -\frac{62}{3}.$$

The dimensions of the pool are 10 feet by 30 feet.

EXAMPLE 3

Working together, two cranes can unload a ship in 4 hours. The slower crane, working alone, requires 6 hours more than the faster crane to do the job. How long does it take each crane to do the job by itself?

SOLUTION

Let x = number of hours for the faster crane to do the job. Then $x + 6$ = number of hours for the slower crane to do the job. The rate of the faster crane is $1/x$, the portion of the whole job that it completes in 1 hour; similarly, the rate of the slower crane is $1/(x + 6)$. We display this information in a table.

	Rate	×	Time	=	Work done
Faster crane	$\frac{1}{x}$		4		$\frac{4}{x}$
Slower crane	$\frac{1}{x+6}$		4		$\frac{4}{x+6}$

When the two cranes work together, we must have

$$\begin{pmatrix}\text{work done by}\\ \text{fast crane}\end{pmatrix} + \begin{pmatrix}\text{work done by}\\ \text{slow crane}\end{pmatrix} = 1 \text{ whole job}$$

or

$$\frac{4}{x} + \frac{4}{x+6} = 1$$

To solve, we multiply by the LCD, $x(x + 6)$, obtaining

$$4(x + 6) + 4x = x^2 + 6x$$
$$0 = x^2 - 2x - 24$$
$$0 = (x + 4)(x - 6)$$
$$x = -4 \qquad \text{or} \qquad x = 6$$

The solution $x = -4$ is rejected, because it makes no sense to speak of negative hours of work. Then

$x = 6$ is the number of hours in which the fast crane can do the job alone.

$x + 6 = 12$ is the number of hours in which the slow crane can do the job alone.

EXERCISE SET 2.6

1. Working together, computers A and B can complete a data-processing job in 2 hours. Computer A working alone can do the job in 3 hours less than computer B working alone. How long does it take each computer to do the job by itself?
2. A graphic designer and her assistant working together can complete an advertising layout in 6 days. The assistant working alone could complete the job in 16 more days than the designer working alone. How long would it take each person to do the job alone?
3. A roofer and his assistant working together can finish a roofing job in 4 hours. The roofer working alone could finish the job in 6 hours less than the assistant working alone. How long would it take each person to do the job alone?
4. A 16-by-20-inch mounting board is used to mount a photograph. How wide a uniform border is there if the photograph occupies $\frac{3}{5}$ of the area of the mounting board?
5. The length of a rectangle exceeds twice its width by 4 feet. If the area of the rectangle is 48 square feet, find the dimensions.
6. The length of a rectangle is 4 centimeters less than twice its width. Find the dimensions if the area of the rectangle is 96 square centimeters.
7. The area of a rectangle is 48 square centimeters. If the length and width are each increased by 4 centimeters, the area of the newly formed rectangle is 120 square centimeters. Find the dimensions of the original rectangle.
8. The base of a triangle is 2 feet more than twice its altitude, If the area is 12 square feet, find the dimensions.
9. Find the width of a strip that has been mowed around a rectangular field 60 feet by 80 feet if $\frac{1}{2}$ the lawn has not yet been mowed.
10. The sum of the reciprocals of two consecutive numbers is $\frac{7}{12}$. Find the numbers.
11. The sum of a number and its reciprocal is $\frac{26}{5}$. Find the number.
12. The difference of a number and its reciprocal is $\frac{35}{6}$. Find the number.
13. The smaller of two numbers is 4 less than the larger. If the sum of their squares is 58, find the numbers.
14. The sum of the reciprocals of two consecutive odd numbers is $\frac{8}{15}$. Find the numbers.
15. The sum of the reciprocals of two consecutive even numbers is $\frac{7}{24}$. Find the numbers.
16. A number of students rented a car for $160 for a one-week camping trip. If another student had joined the original group, each person's share of expenses would have been reduced by $8. How many students were in the original group?
17. An investor placed an order totaling $1200 for a certain number of shares of a stock. If the price of each share of stock were $2 more, the investor would get 30 shares less for the same amount of money. How many shares did the investor buy?
18. A fraternity charters a bus for a ski trip at a cost of $360. When 6 more students join the trip, each person's cost decreases by $2. How many students were in the original group of travelers?
19. A salesman worked a certain number of days to earn $192. If he had been paid $8 more per day, he would have earned the same amount of money in 2 fewer days. How many days did he work?
20. A freelance photographer worked a certain number of days for a newspaper to earn $480. If she had been paid $8 less per day, she would have earned the same amount in 2 more days. What was her daily rate of pay?

2.7 SECOND-DEGREE INEQUALITIES

To solve a second-degree inequality, such as

$$x^2 - 2x > 15$$

we rewrite the inequality in the form

$$x^2 - 2x - 15 > 0$$

or, after factoring,

$$(x + 3)(x - 5) > 0$$

With the right-hand side equal to 0, this inequality requires that the product of the two factors, which represent real numbers, be positive. That means that both factors must have the same sign. We must therefore analyze the *signs* of $(x + 3)$ and $(x - 5)$.

In any situation like this we are interested in knowing all values of x for which the general expression $ax + b$ will be positive and those values for which it will be negative. Since $ax + b = 0$ when $x = -b/a$, we see that

> The linear factor $ax + b$ equals 0 at the **critical value** $x = -b/a$ and has opposite signs to the left and right of the critical value on a number line.

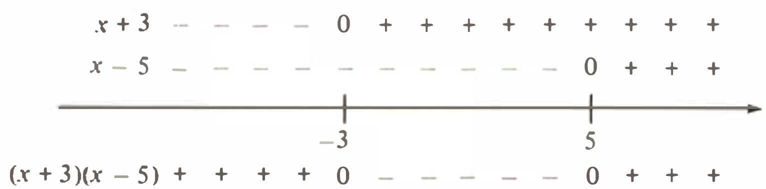


FIGURE 7

A practical means for solving such problems as the current example is illustrated in Figure 7. Since the critical values occur where $x + 3 = 0$ and $x - 5 = 0$, the values -3 and $+5$ are displayed on a real number line. The rows above the real number line display the *signs* of the factors $x + 3$ and $x - 5$ for all real values of x. The row below the real number line displays the *signs* of the product $(x + 3)(x - 5)$. The product is positive when the factors have the same sign, is negative when the factors are of opposite sign, and is zero when either factor is zero. The row below the real number line shows the solution set of the inequality $(x + 3)(x - 5) > 0$ to be

$$\{x | x < -3 \text{ or } x > 5\}$$

which consists of the real numbers in the open intervals $(-\infty, -3)$ and $(5, \infty)$. The solution set is shown in Figure 8.

FIGURE 8

EXAMPLE 1

Solve the inequality $x^2 \leq -3x + 4$ and graph the solution set on a real number line.

SOLUTION

We rewrite the inequality and factor.

$$x^2 \leq -3x + 4$$
$$x^2 + 3x - 4 \leq 0$$
$$(x - 1)(x + 4) \leq 0$$

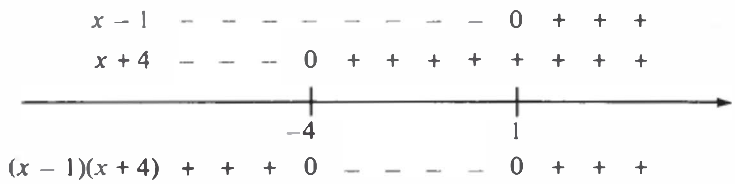


FIGURE 9

We seek values of x for which the factors $(x - 1)$ and $(x + 4)$ have opposite signs or are zero. The critical values occur where $x - 1 = 0$ and where $x + 4 = 0$, that is, at $+1$ and -4. Figure 9 gives an analysis of the signs of the factors $x - 1$ and $x + 4$ as well as the signs of their product, $(x - 1)(x + 4)$. We see that the solution set consists of all real numbers

$$\{x \mid -4 \leq x \leq 1\}$$

which is the closed interval $[-4, 1]$, shown in Figure 10.

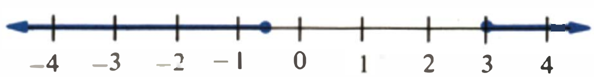
FIGURE 10

PROGRESS CHECK

Solve the inequality $2x^2 \geq 5x + 3$ and graph the solution set on a real number line.

ANSWER

$\left\{x \mid x \leq -\frac{1}{2} \text{ or } x \geq 3\right\}$



Although

$$\frac{ax + b}{cx + d} < 0$$

is not a second-degree inequality, the solution of this inequality is the same as the solution of the inequality

$$(ax + b)(cx + d) < 0$$

since both inequalities require that the two expressions composing them have different signs.

EXAMPLE 2

Solve the inequality $\dfrac{y+1}{2-y} \leq 0$.

SOLUTION

Figure 11 gives an analysis of the signs of $y + 1$ and $2 - y$. The critical values occur where $y + 1 = 0$ and where $2 - y = 0$, that is, at -1 and $+2$. The bottom row shows the signs of the quotient $(y + 1)/(2 - y)$, from which we see that the solution set is $\{y | y \leq -1 \text{ or } y > 2\}$ or all real numbers in the intervals $(-\infty, -1]$, $(2, \infty)$. Note that $y = 2$ would result in division by 0 and must be excluded.

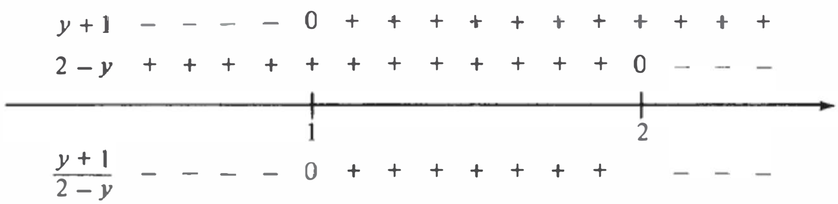


FIGURE 11

PROGRESS CHECK

Solve the inequality $\dfrac{2x-3}{1-2x} \geq 0$.

ANSWER

$\left\{x \middle| \dfrac{1}{2} < x \leq \dfrac{3}{2}\right\}$ or $\left(\dfrac{1}{2}, \dfrac{3}{2}\right]$

EXAMPLE 3

Solve the inequality $(x - 2)(2x + 5)(3 - x) < 0$.

SOLUTION

Although this is a third-degree inequality, the same approach will work. Figure 12 gives an analysis of the signs of $x - 2$, $2x + 5$, and $3 - x$. The product of three factors is negative when there is an odd number of negative factors. The solution set is then

$$\left\{x \mid -\frac{5}{2} < x < 2 \text{ or } x > 3\right\} \quad \text{or} \quad \left(\frac{5}{2}, 2\right), (3, \infty)$$

$x-2$	−	−	−	−	−	−	−	−	−	0	+	+	+	+	+	+	+	+
$2x+5$	−	−	−	−	0	+	+	+	+	+	+	+	+	+	+	+	+	+
$3-x$	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	−	−	−
					$-\frac{5}{2}$					2					3			
$(x-2)(2x+5)(3-x)$		+	+	+	0	−	−	−	−	0	+	+	+	+	0	−	−	−

FIGURE 12

PROGRESS CHECK

Solve the inequality $(2y-9)(6-y)(y+5) \geq 0$.

ANSWER

$$\left\{y \mid y \leq -5 \text{ or } \frac{9}{2} \leq y \leq 6\right\} \quad \text{or} \quad (-\infty, -5], \left[-\frac{9}{2}, 6\right]$$

EXAMPLE 4

Solve the inequality $x^2 + 1 > 0$.

SOLUTION

This inequality is equivalent to

$$x^2 > -1$$

For any real number x, we know that x^2 is nonnegative. Therefore, the solution set consists of all real numbers.

To solve the inequality

$$x^2 + x + 1 < 0$$

we need to factor the polynomial $x^2 + x + 1$. We saw in Section 2.5, however, that this polynomial is irreducible over the reals. We will be better equipped to solve more difficult inequalities after we have studied methods for solving polynomial equations in a later chapter.

EXERCISE SET 2.7

Determine the solution set of each inequality.

1. $x^2 + 5x + 6 > 0$
2. $x^2 + 3x - 4 \leq 0$
3. $2x^2 - x - 1 < 0$
4. $3x^2 - 4x - 4 \geq 0$

5. $4x - 2x^2 < 0$

6. $r^2 + 4r \geq 0$

7. $\dfrac{x+5}{x+3} \leq 0$

8. $\dfrac{x-6}{x+4} \geq 0$

9. $\dfrac{2r+1}{r-3} \leq 0$

10. $\dfrac{x-1}{2x-3} \geq 0$

11. $\dfrac{3s+2}{2s-1} \geq 0$

12. $\dfrac{4x+5}{x^2} \leq 0$

13. $(x+2)(3x-2)(x-1) > 0$

14. $(x-4)(2x+5)(2-x) \leq 0$

Indicate the solution set of each inequality on a real number line.

15. $x^2 + x - 6 > 0$

16. $x^2 - 3x - 10 \geq 0$

17. $2x^2 - 3x - 5 < 0$

18. $3x^2 - 4x - 4 \leq 0$

19. $\dfrac{2r+3}{2r-1} < 0$

20. $\dfrac{3x+2}{2x-3} \geq 0$

21. $\dfrac{x-1}{x+1} \geq 0$

22. $\dfrac{2x-1}{x+2} \leq 0$

23. $6x^2 + 8x + 2 \geq 0$

24. $2x^2 + 5x + 2 \leq 0$

25. $(y-3)(2-y)(2y+4) \geq 0$

26. $(2x+5)(3x-2)(x+1) < 0$

27. $(x-3)(1+2x)(3x+5) > 0$

28. $(1-2x)(2x+1)(x-3) \leq 0$

In Exercises 29–32 find the values of x for which the given expression has real values.

29. $\sqrt{(x-2)(x+1)}$

30. $\sqrt{(2x+1)(x-3)}$

31. $\sqrt{2x^2 + 7x + 6}$

32. $\sqrt{2x^2 + 3x + 1}$

33. A manufacturer of solar heaters finds that when x units are made and sold, the profit (in thousands of dollars) is given by $x^2 - 50x - 5000$. For what values of x will the firm show a loss?

34. A ball thrown directly upward from level ground at an initial velocity of 40 feet per second attains a height d given by $d = 40t - 16t^2$ after t seconds. During what time interval is the ball at a height of at least 16 feet?

TERMS AND SYMBOLS

equation (p. 61)
left-hand side (p. 61)
right-hand side (p. 61)
solution (p. 61)
root (p. 61)
solution set (p. 61)
equivalent equation (p. 62)
first-degree equation in one unknown (p. 64)
linear equation (p. 64)
graph of the solution set (p. 78)
interval (p. 78)
interval notation (p. 78)
open interval (p. 78)
closed interval (p. 78)
half-open interval (p. 78)
∞, $-\infty$ (p. 79)
infinite interval (p. 79)
quadratic equation (p. 87)
completing the square (p. 90)
quadratic formula (p. 92)
discriminant (p. 94)
double root (p. 94)
repeated root (p. 94)
radical equation (p. 96)
extraneous solution (p. 97)
substitution of variable (p. 98)
second-degree inequality (p. 104)
critical value (p. 104)

KEY IDEAS FOR REVIEW

- ☐ To solve an equation, we generally form a succession of simpler, equivalent equations.
- ☐ In the process of solving an equation, we may add to or subtract from both sides of the equation any number or expression. We may also multiply both sides by any nonzero number. If we multiply the equation by an expression containing a variable, the answers must be substituted in the original equation to verify that they are solutions.
- ☐ The linear equation $ax + b = 0$, $a \neq 0$, has precisely one solution: $-b/a$.
- ☐ When solving linear inequalities, remember that multiplication or division by a negative number reverses the sense of the inequality.
- ☐ The solution set of a linear inequality can be indicated by a graph on the real number line, by set-builder notation, or by interval notation.
- ☐ The quadratic equation $ax^2 + bx + c = 0$, $a > 0$, always has two solutions, which are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $b = 0$ or if the quadratic can be factored, then faster solution methods are available.

☐ The solutions or roots of a quadratic equation may be complex numbers. The expression $b^2 - 4ac$, called the discriminant, which appears under the radical of the quadratic formula, permits the nature of the roots to be analyzed without solving the equation.

☐ Radical equations often can be transformed into quadratic equations. Since the process involves raising both sides of an equation to a power, the answers must be checked to see that they satisfy the original equation.

☐ The method called *substitution of variable* can be used to transform certain equations into quadratics. This technique is a handy tool and will be used in other chapters of this book.

☐ If a second-degree inequality can be written in the factored form

$$(ax + b)(cx + d) < 0$$

or

$$(ax + b)(cx + d) > 0$$

then the solution set is easily found. First, determine the intervals on the real number line in which each factor is positive and the intervals in which each is negative. If the product of the factors is negative (< 0), then the solution set consists of the intervals in which the factors are opposite in sign; if the product is positive (> 0), the solution set consists of the intervals in which the factors are of like sign.

REVIEW EXERCISES

Solutions to exercises whose numbers are in color are in the Solutions section in the back of the book.

2.1 In Exercises 1–4 solve for x.

1. $3x - 5 = 3$ 2. $2(2x - 3) - 3(x + 1) = -9$

3. $\dfrac{2 - x}{3 - x} = 4$ $k - 2x = 4kx$

2.2 5. The width of a rectangle is 4 centimeters less than twice its length. If the perimeter is 12 centimeters, find the measurement of each side.

6. A donation box contains coins consisting of dimes and quarters. The number of dimes is 4 more than twice the number of quarters. If the total value of the coins is \$2.65, how many coins of each type are there?

7. It takes 4 hours for a bush pilot in Australia to pick up mail at a remote village and return to home base. If the average speed going is 150 miles per hour and the average speed returning is 100 miles per hour, how far from the home base is the village?

8. Copying machines A and B, working together, can prepare enough copies of the annual report for the Board of Directors in 2 hours. Machine A, working alone, would require 3 hours to do the job. How long would it take machine B to do the job by itself?

2.1 9. Indicate whether the statement is true (T) or false (F): The equation $3x^2 = 9$ is an identity.

10. Indicate whether the statement is true (T) or false (F): $x = 3$ is a solution of the equation $3x - 1 = 10$.

2.3 11. Solve and graph $3 \le 2x + 1$.

12. Solve and graph $-4 < -2x + 1 \le 10$.

In Exercises 13–15 solve the inequality and express the solution set in interval notation.

13. $2(a + 5) > 3a + 2$ 14. $\dfrac{-1}{2x - 5} < 0$

15. $\dfrac{2x}{3} + \dfrac{1}{2} \ge \dfrac{x}{2} - 1$

2.4 16. Solve $|3x + 2| = 7$ for x.

17. Solve and graph $|4x - 1| = 5$.

18. Solve and graph $|2x + 1| > 7$.

19. Solve $|2 - 5x| < 1$ and write the solution in interval notation.

20. Solve $|3x - 2| \ge 6$ and write the solution in interval notation.

2.5 21. Solve $x^2 - x - 20 = 0$ by factoring.

22. Solve $6x^2 - 11x + 4 = 0$ by factoring.

23. Solve $x^2 - 2x + 6 = 0$ by completing the square.

24. Solve $2x^2 - 4x + 3 = 0$ by the quadratic formula.

25. Solve $3x^2 + 2x - 1 = 0$ by the quadratic formula.

In Exercises 26–28 solve for x.

26. $49x^2 - 9 = 0$ 27. $kx^2 - 3\pi = 0$

28. $x^2 + x = 12$

In Exercises 29–31 determine the nature of the roots of the quadratic equation without solving.

29. $3r^2 = 2r + 5$ 30. $4x^2 + 20x + 25 = 0$

31. $6y^2 - 2y = -7$

In Exercises 32–35 solve the given equation.

32. $\sqrt{x} + 2 = x$ 33. $\sqrt{x+3} + \sqrt{2x-3} = 6$

34. $x^4 - 4x^2 + 3 = 0$

35. $\left(1 - \frac{2}{x}\right)^2 - 8\left(1 - \frac{2}{x}\right) + 15 = 0$

2.6 36. A charitable organization rented an auditorium for a meeting at a cost of \$420 and split the cost among the attendees. If 10 additional persons had attended the meeting, the cost per person would have decreased by \$1. How many persons actually attended?

2.7 37. Find the values of x for which $\sqrt{2x^2 - x - 6}$ has real values.

38. Write the solution set of the inequality $x^2 + 4x - 5 \geq 0$ in interval notation.

39. Write the solution set for $\frac{2x+1}{x+5} \geq 0$ in interval notation.

40. Write the solution set for

$$(3 - x)(2x + 3)(x + 2) < 0$$

in interval notation.

PROGRESS TEST 2A

In Problems 1 and 2 solve for y.

1. $5 - 4y = 2$ 2. $\frac{2 + 5y}{3y - 1} = 6$

3. One side of a triangle is 2 meters shorter than the base, and the other side is 3 meters longer than half the base. If the perimeter is 15 meters, find the length of each side.

4. A trust fund invested a certain amount of money at 6.5% simple annual interest, a second amount \$200 more than the first amount at 7.5%, and a third amount \$300 more than twice the first amount at 9%. If the total annual income from these investments is \$1962, how much was invested at each rate?

5. Indicate whether the statement is true (T) or false (F): The equation $(2x - 1)^2 = 4x^2 - 4x + 1$ is an identity.

6. Solve $-1 \leq 2x + 3 < 5$ and graph the solution set.

In Problems 7 and 8 solve the inequality and express the solution set in interval notation.

7. $3(2a - 1) - 4(a + 2) \leq 4$ 8. $-2 \leq 2 - x \leq 6$

9. Solve $|4x - 1| = 9$.

10. Solve $|2x - 1| \leq 5$ and graph the solution set.

11. Solve $|1 - 3x| < 5$ and write the solution in interval notation.

12. Solve $x^2 - 5x = 14$ by factoring.

13. Solve $5x^2 - x + 4 = 0$ by completing the square.

14. Solve $12x^2 + 5x - 3 = 0$ by the quadratic formula.

In Problems 15 and 16 solve for x.

15. $(2x - 5)^2 + 9 = 0$ 16. $2 + \frac{1}{x} - \frac{3}{x^2} = 0$

In Problems 17 and 18 determine the nature of the roots of the quadratic equation without solving.

17. $6x^2 + x - 2 = 0$ 18. $3x^2 - 2x = -6$

In Problems 19 and 20 solve the given equation.

19. $x - \sqrt{4 - 3x} = -8$ 20. $3x^4 + 5x^2 - 2 = 0$

21. The area of a rectangle is 96 square meters. If the length and the width are each increased by 2 meters, the area of the newly formed rectangle is 140 square meters. Find the dimensions of the original rectangle.

22. Find the values of x for which $\sqrt{3x^2 - 4x + 1}$ has real values.

In Problems 23 and 24 write the solution set in interval notation.

23. $-2x^2 + 3x - 1 \leq 0$

24. $(x - 1)(2 - 3x)(x + 2) \leq 0$

PROGRESS TEST 2B

In Problems 1 and 2 solve for x.

1. $3(2x + 5) = 5 - (3x - 1)$ 2. $3x - k^2 = -kx$

3. An alloy that is 60% silver is to be combined with an alloy that is 80% silver to produce 120 ounces of an alloy that is 75% silver. How many ounces of each alloy must be used?

4. Solve $\dfrac{ax + b}{cx + b} = d$ for x.

5. Indicate whether the statement is true (T) or false (F): $x = -1$ is a solution of the equation $\dfrac{x - 1}{x + 1} = 0$.

6. Solve $-9 \leq 1 - 5x \leq -4$ and graph the solution set.

In Problems 7 and 8 solve the inequality and express the solution set in interval notation.

7. $\dfrac{x}{4} - \dfrac{1}{2} \leq \dfrac{1}{2} - x$ 8. $\dfrac{-2}{3 - x} \geq 0$

9. Solve $|1 - 3x| = 7$.

10. Solve $\dfrac{|x - 4|}{2} \geq 1$ and graph the solution set.

11. Solve $|5x + 2| > 3$ and write the solution set in interval notation.

12. Solve $6x^2 + 13x - 5 = 0$ by factoring.

13. Solve $2x^2 - 5x + 2 = 0$ by completing the square.

14. Solve $3x^2 - x = -7$ by the quadratic formula.

In Problems 15 and 16 solve for x.

15. $(x - 3)^2 + 2 = 0$ 16. $k_1 + \dfrac{k_2}{x^2} = k_3$

In Problems 17 and 18 determine the nature of the roots of the quadratic equation without solving.

17. $6z^2 - 4z = -2$ 18. $4y^2 - 20y + 25 = 0$

In Problems 19 and 20 solve the given equation.

19. $\sqrt{x - 1} - \sqrt{3x - 2} = -1$

20. $\dfrac{8}{x^{4/3}} + \dfrac{9}{x^{2/3}} + 1 = 0$

21. If the price of a large candy bar rose by 10 cents, a buyer would receive 2 fewer candy bars for \$6.00 than she does at the current price. What is the current price?

22. Find the values of x for which $x/\sqrt{2x - 1}$ has real values.

In Problems 23 and 24 write the solution set in interval notation.

23. $\dfrac{x^2}{x + 5} \leq 0$

24. $(3x - 2)(x + 4)(1 - x) > 0$

3

FUNCTIONS

What is the effect of increased fertilization on the growth of an azalea? If the minimum wage is increased, what will be the impact on the number of unemployed workers? When a submarine dives, can we calculate the water pressure against the hull at a given depth?

Each of the questions posed above seeks a relationship between phenomena. The search for relationships, or correspondence, is a central activity in our attempts to understand the universe; it is used in mathematics, engineering, the physical and biological sciences, the social sciences, and business and economics.

The concept of a function has been developed as a means of organizing and assisting in the study of relationships. Since graphs are powerful means of exhibiting relationships, we begin with a study of the Cartesian, or rectangular, coordinate system. We will then formally define a function and will offer a number of ways of viewing the function concept. Function notation will be introduced to provide a convenient means of writing functions.

We will also explore some special types of functional relationships (increasing and decreasing functions) and will see that variation can be viewed as a functional relationship.

3.1 THE RECTANGULAR COORDINATE SYSTEM

In Chapter 1 we associated the system of real numbers with points on the real number line. That is, we saw that there is a one-to-one correspondence between the system of real numbers and points on the real number line.

We will now develop an analogous way to handle points in a plane. We begin by drawing a pair of perpendicular lines intersecting at a point O called the **origin.** One of the lines, called the ***x*-axis,** is usually drawn in a horizontal position. The other line, called the ***y*-axis,** is usually drawn vertically.

If we think of the x-axis as a real number line, we may mark off some convenient unit of length, with positive numbers to the right of the origin and negative numbers to the left of the origin. Similarly, we may think of the y-axis as a real number line. Again, we may mark off a convenient unit of length (usually the same as the unit of length on the x-axis) with the upward direction representing positive numbers and the downward direction negative numbers. The x and y axes are called **coordinate axes,** and together they constitute a **rectangular** or **Cartesian coordinate system.** The coordinate axes divide the plane into four **quadrants,** which we label I, II, III, and IV as in Figure 1.

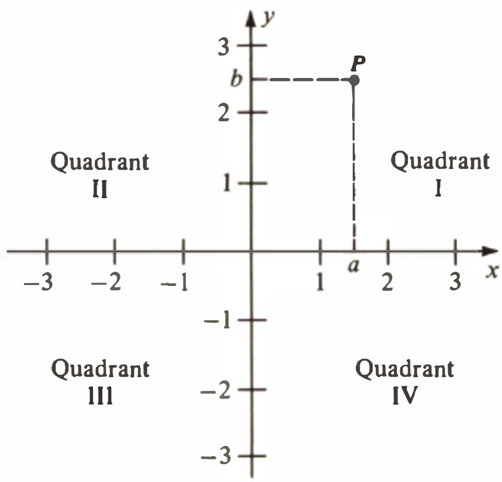


FIGURE 1

By using the coordinate axes, we can outline a procedure for labeling a point P in the plane. From P, draw a perpendicular to the x-axis and note that it meets the x-axis at $x = a$. Now draw a perpendicular from P to the y-axis and note that it meets the y-axis at $y = b$. We say that the **coordinates** of P are given by the **ordered pair** (a, b). The term "ordered pair" means that the order is significant; that is, the ordered pair (a, b) is different from the ordered pair (b, a).

The first number of the ordered pair (a, b) is sometimes called the **abscissa** or ***x*-coordinate** of P. The second number is called the **ordinate** or ***y*-coordinate** of P.

We have now developed a procedure for associating with each point P in the plane a unique ordered pair of real numbers (a, b). We usually write the point P as $P(a, b)$. Conversely, every ordered pair of real numbers (a, b) determines a unique point P in the plane. The point P is located at the intersection of the lines perpendicular to the x-axis and to the y-axis at the points on the axes having coordinates a and b, respectively. We have thus established a one-to-one correspondence between the set of all points in the plane and the set of all ordered pairs of real numbers.

We have indicated a number of points in Figure 2. Note that all points on the x-axis have a y-coordinate of 0 and all points on the y-axis have an x-coordinate

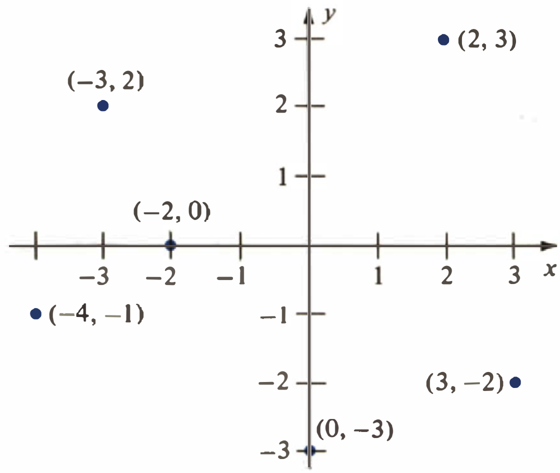


FIGURE 2

of 0. It is important to observe that the x-coordinate of a point P is the distance of P from the y-axis; the y-coordinate is its distance from the x-axis. The point $(2, 3)$ in Figure 2 is 2 units from the y-axis and 3 units from the x-axis.

THE DISTANCE FORMULA

There is a useful formula that gives the distance PQ between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$. In Figure 3a we have shown the x-coordinate of a point as the distance of the point from the y-axis, and the y-coordinate as its distance from the x-axis. Thus we labeled the horizontal segments x_1 and x_2 and the vertical segments y_1 and y_2. In Figure 3b we use the lengths from Figure 3a to indicate that $\overline{PR} = x_2 - x_1$ and $\overline{QR} = y_2 - y_1$. Since triangle PRQ is a right triangle, we can apply the Pythagorean theorem.

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

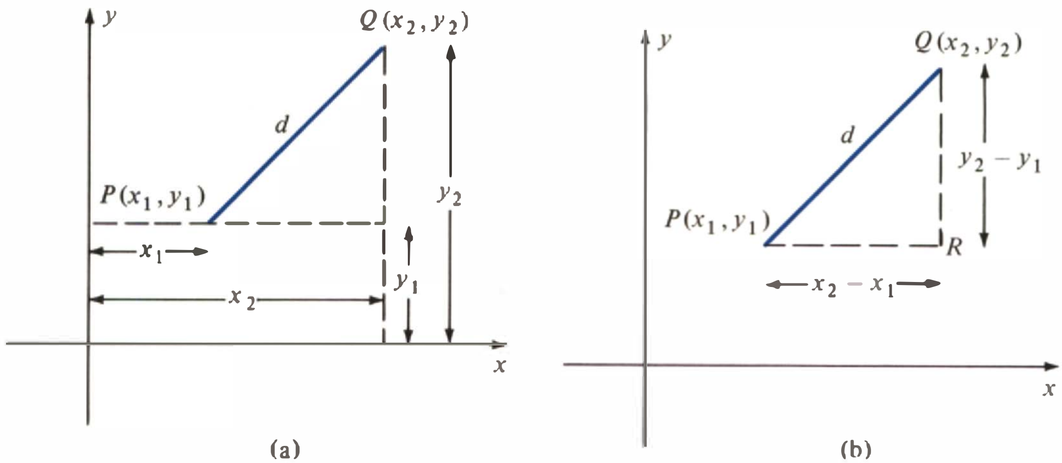


FIGURE 3

Although the points in Figure 3 are both in quadrant I, the same result will be obtained for any two points. Since distance cannot be negative, we have

The Distance Formula The distance $\overline{PQ}$ between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the plane is

$$\overline{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

It is also clear from the distance formula that $\overline{PQ} = \overline{QP}$.

EXAMPLE 1
Find the distance between the points $P(-2, -3)$ and $Q(1, 2)$.

SOLUTION
Using the distance formula, we have

$$\overline{PQ} = \sqrt{[1 - (-2)]^2 + [2 - (-3)]^2} = \sqrt{3^2 + 5^2} = \sqrt{34}$$

PROGRESS CHECK
Find the distance between the points $P(-3, 2)$ and $Q(4, -2)$.

ANSWER
$\sqrt{65}$

EXAMPLE 2
Show that the triangle with vertices $A(-2, 3)$, $B(3, -2)$, and $C(6, 1)$ is a right triangle.

SOLUTION

It is a good idea to draw a diagram as in Figure 4. We compute the lengths of the three sides.

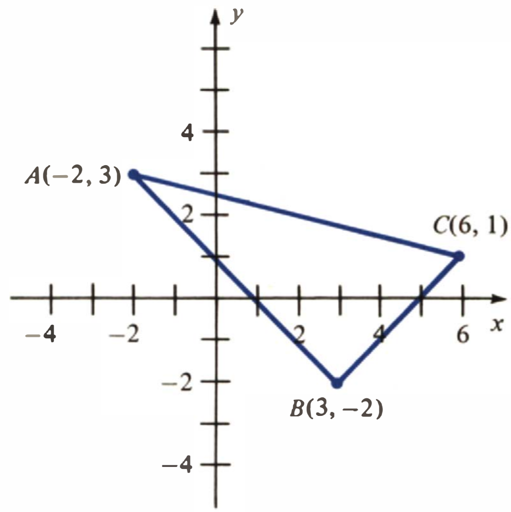


FIGURE 4

$$\overline{AB} = \sqrt{(3+2)^2 + (-2-3)^2} = \sqrt{50}$$
$$\overline{BC} = \sqrt{(6-3)^2 + (1+2)^2} = \sqrt{18}$$
$$\overline{AC} = \sqrt{(6+2)^2 + (1-3)^2} = \sqrt{68}$$

If the Pythagorean theorem holds, then triangle ABC is a right triangle. We see that

$$(\overline{AC})^2 = (\overline{AB})^2 + (\overline{BC})^2 \quad \text{since } 68 = 50 + 18$$

and we conclude that triangle ABC is a right triangle whose hypotenuse is AC.

GRAPHS OF EQUATIONS

By the **graph of an equation in two variables** x and y we mean the set of all points $P(x, y)$ whose coordinates satisfy the given equation. We say that the ordered pair (a, b) is a **solution** of the equation if substitution of a for x and b for y yields a true statement.

To graph $y = x^2 - 4$, an equation in the variables x and y, we note that we can obtain solutions by assigning arbitrary values to x and computing corresponding values of y. Thus, if $x = 3$, then $y = 3^2 - 4 = 5$, and the ordered pair $(3, 5)$ is a solution of the equation. Table 1 shows a number of solutions. We next plot the points corresponding to these ordered pairs. Since the equation has an infinite number of solutions, the plotted points represent only a portion of the graph. We assume that the curve behaves nicely between the plotted points and connect these points by a smooth curve (Figure 5). We must plot enough points to feel reasonably certain of the outline of the curve.

TABLE 1

x	-3	-2	-1	0	1	2	$\frac{5}{2}$
y	5	0	-3	-4	-3	0	$\frac{9}{4}$

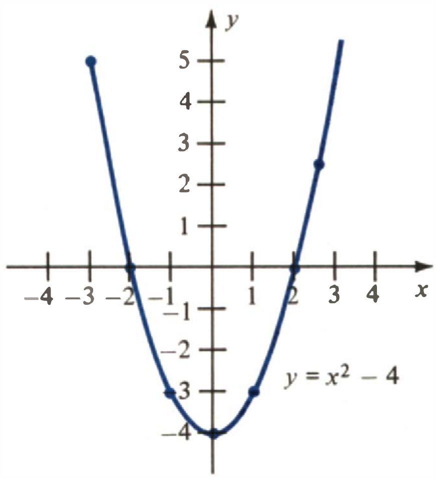


FIGURE 5

The abscissa of a point at which a graph meets the x-axis is called an **x-intercept.** Since the graph in Figure 5 meets the x-axis at the points (2, 0) and $(-2, 0)$, we see that 2 and -2 are the x-intercepts. Similarly, we define the **y-intercept** as the ordinate of a point at which the graph meets the y-axis. In Figure 5 the y-intercept is -4. Intercepts are often easy to calculate and are useful in sketching a graph.

EXAMPLE 3

Sketch the graph of the equation $y = 2x + 1$. Determine the x- and y-intercepts, if any.

SOLUTION

We form a short table of values and sketch the graph in Figure 6. The graph appears to be a straight line that intersects the x-axis at $(-\frac{1}{2}, 0)$ and the y-axis at (0, 1). The x-intercept is $-\frac{1}{2}$ and the y-intercept is 1. Alternatively, we can find

x	-2	-1	0	1	2
y	-3	-1	1	3	5

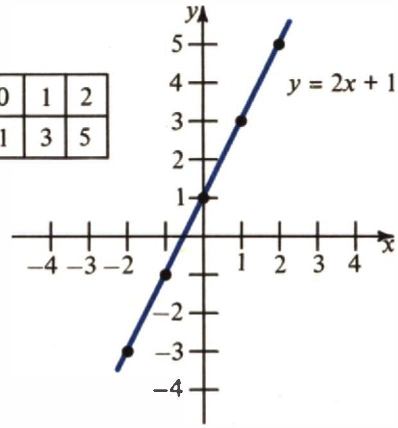


FIGURE 6

the y-intercept algebraically by letting $x = 0$ so that

$$y = 2x + 1 = 2(0) + 1 = 1$$

and the x-intercept by letting $y = 0$ so that

$$\begin{aligned} y &= 2x + 1 \\ 0 &= 2x + 1 \\ x &= -\tfrac{1}{2} \end{aligned}$$

SYMMETRY

If we folded the graph of Figure 7a along the x-axis, the top and bottom portions would exactly match, which is what we intuitively mean when we speak of symmetry about the x-axis. We would like to develop a means of testing for symmetry that doesn't rely upon examining a graph. We can then use information about symmetry to help in sketching the graph.

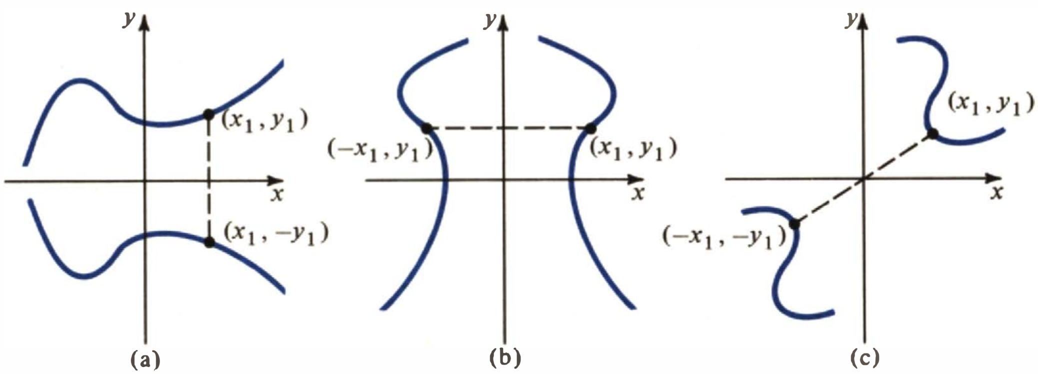


FIGURE 7

Returning to Figure 7a, we see that every point (x_1, y_1) on the portion of the curve above the x-axis is reflected in a point $(x_1, -y_1)$ that lies on the portion of the curve below the x-axis. Similarly, using the graph of Figure 7b, we can argue that symmetry about the y-axis occurs if, for every point (x_1, y_1) on the curve, $(-x_1, y_1)$ also lies on the curve. Finally, using the graph sketched in Figure 7c, we see that symmetry about the origin occurs if, for every point (x_1, y_1) on the curve, $(-x_1, -y_1)$ also lies on the curve. We now summarize these results.

Tests for Symmetry

The graph of an equation is **symmetric with respect to the**
(i) **x-axis** if replacing y with $-y$ results in an equivalent equation;
(ii) **y-axis** if replacing x with $-x$ results in an equivalent equation;
(iii) **origin** if replacing x with $-x$ and y with $-y$ results in an equivalent equation.

EXAMPLE 4

Use intercepts and symmetry to assist in graphing the equations.
(a) $y = 1 - x^2$ (b) $x = y^2 + 1$

SOLUTION

(a) To determine the intercepts, set $x = 0$ to yield $y = 1$ as the y-intercept. Setting $y = 0$, we have $x^2 = 1$ or $x = \pm 1$ as the x-intercepts.

To test for symmetry, replace x with $-x$ in the equation $y = 1 - x^2$ to obtain

$$y = 1 - (-x)^2 = 1 - x^2$$

Since the equation is unaltered, the curve is symmetric with respect to the y-axis. Now, replacing y with $-y$, we have

$$-y = 1 - x^2$$

which is *not* equivalent to the original equation. The curve is therefore not symmetric with respect to the x-axis. Finally, replacing x with $-x$ and y with $-y$ repeats the last result and shows that the curve is not symmetric with respect to the origin.

We can now form a table of values for $x \geq 0$ and use symmetry with respect to the y-axis to help sketch the graph of the equation (see Figure 8a).

(b) The y-intercepts occur where $x = 0$. Since this leads to the equation $y^2 = -1$, which has no real roots, there are no y-intercepts. Setting $y = 0$, we have $x = 1$ as the x-intercept.

Replacing x with $-x$ in the equation $x = y^2 + 1$ gives us

$$-x = y^2 + 1$$

x	0	1	2	3
y	1	0	−3	−8

y	0	1	2	3
x	1	2	5	10

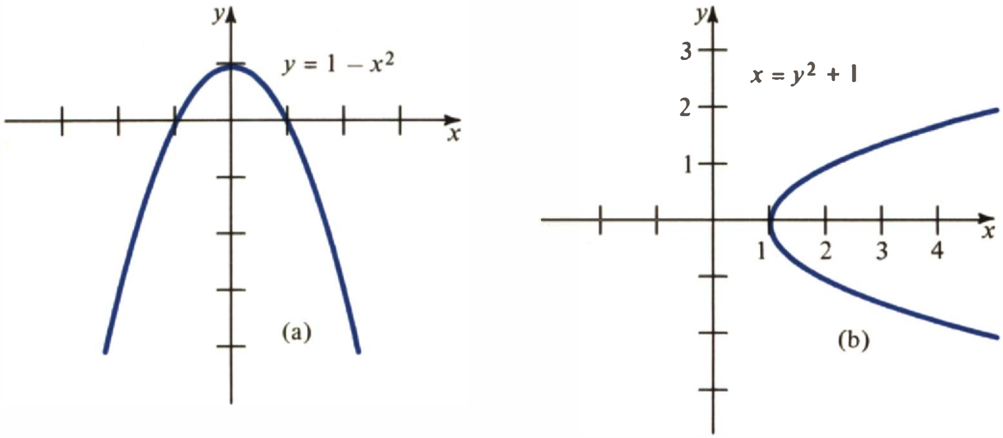


FIGURE 8

which is *not* an equivalent equation. The curve is therefore not symmetric with respect to the y-axis. Replacing y with $-y$, we find that

$$x = (-y)^2 + 1 = y^2 + 1$$

which is the same as the original equation. Thus, the curve is symmetric with respect to the x-axis. Replacing x with $-x$ and y with $-y$ also results in the equation

$$-x = y^2 + 1$$

and demonstrates that the curve is not symmetric with respect to the origin. We next form the table of values shown in Figure 8b by assigning nonnegative values to y and calculating the corresponding values of x from the equation; symmetry enables us to sketch the lower half of the graph without plotting points.

Solving the given equation for y yields $y = \pm\sqrt{x-1}$, which confirms the symmetry about the x-axis. We can think of the upper half of Figure 8b as the graph of the equation $y = \sqrt{x-1}$ and the lower half as the graph of the equation $y = -\sqrt{x-1}$.

EXAMPLE 5

Without sketching the graph, determine symmetry with respect to the x-axis, the y-axis, and the origin.

(a) $x^2 + 4y^2 - y = 1$ (b) $xy = 5$ (c) $y^2 = \dfrac{x^2+1}{x^2-1}$

SOLUTION

(a) Replacing x with $-x$ in the equation, we have

$$(-x)^2 + 4y^2 - y = 1$$
$$x^2 + 4y^2 - y = 1$$

Since the equation is unaltered, the curve is symmetric with respect to the y-axis. Next, replacing y with $-y$, we have

$$x^2 + 4(-y)^2 - (-y) = 1$$
$$x^2 + 4y^2 + y = 1$$

which is *not* an equivalent equation. Replacing x with $-x$ and y with $-y$ repeats the last result. The curve is therefore not symmetric with respect to either the x-axis or the origin.

(b) Replacing x with $-x$, we have $-xy = 5$, which is *not* an equivalent equation. Replacing y with $-y$, we again have $-xy = 5$. Thus the curve is not symmetric with respect to either axis. However, replacing x with $-x$ and y with $-y$ gives us

$$(-x)(-y) = 5$$

which is equivalent to $xy = 5$. We conclude that the curve is symmetric with respect to the origin.

(c) Since x and y both appear to the second power only, all tests will lead to an equivalent equation. The curve is therefore symmetric with respect to both axes and the origin.

PROGRESS CHECK

Without graphing, determine symmetry with respect to the coordinate axes and the origin.

(a) $x^2 - y^2 = 1$ (b) $x + y = 10$ (c) $y = x + \dfrac{1}{x}$

ANSWERS

(a) Symmetric with respect to the x-axis, the y-axis, and the origin.
(b) Not symmetric with respect to either axis or the origin.
(c) Symmetric with respect to the origin only.

Note that in Example 5c and in (a) of the last Progress Check, the curves are symmetric with respect to both the x- and y-axes, as well as the origin. In fact, we have the following general rule.

> A curve that is symmetric with respect to both coordinate axes is also symmetric with respect to the origin. However, a curve that is symmetric with respect to the origin need not be symmetric with respect to the coordinate axes.

The curve in Figure 7c illustrates the last point. The curve is symmetric with respect to the origin but not with respect to the coordinate axes.

EXERCISE SET 3.1

In each of Exercises 1 and 2 plot the given points on the same coordinate axes.

1. $(2, 3)$, $(-3, -2)$, $\left(-\frac{1}{2}, \frac{1}{2}\right)$, $\left(0, \frac{1}{4}\right)$, $\left(-\frac{1}{2}, 0\right)$, $(3, -2)$
2. $(-3, 4)$, $(5, -2)$, $(-1, -3)$, $\left(-1, \frac{3}{2}\right)$, $(0, 1.5)$

In Exercises 3–8 find the distance between each pair of points.

3. $(5, 4)$, $(2, 1)$
4. $(-4, 5)$, $(-2, 3)$
5. $(-1, -5)$, $(-5, -1)$
6. $(-3, 0)$, $(2, -4)$
7. $\left(\frac{2}{3}, \frac{3}{2}\right)$, $(-2, -4)$
8. $\left(-\frac{1}{2}, 3\right)$, $\left(-1, -\frac{3}{4}\right)$

In Exercises 9–12 find the length of the shortest side of the triangle determined by the three given points.

9. $A(6, 2)$, $B(-1, 4)$, $C(0, -2)$
10. $P(2, -3)$, $Q(4, 4)$, $R(-1, -1)$
11. $R\left(-1, \frac{1}{2}\right)$, $S\left(-\frac{3}{2}, 1\right)$, $T(2, -1)$
12. $F(-5, -1)$, $G(0, 2)$, $H(1, -2)$

In Exercises 13–16 determine if the given points form a right triangle. (*Hint:* A triangle is a right triangle if and only if the lengths of the sides satisfy the Pythagorean theorem.)

13. $(1, -2)$, $(5, 2)$, $(2, 1)$
14. $(2, -3)$, $(-1, -1)$, $(3, 4)$
15. $(-4, 1)$, $(1, 4)$, $(4, -1)$
16. $(1, -1)$, $(-6, 1)$, $(1, 2)$

In Exercises 17–20 show that the points lie on the same line. (*Hint:* Three points are collinear if and only if the sum of the lengths of two sides equals the length of the third side.)

17. $(-1, 2)$, $(1, 1)$, $(5, -1)$
18. $(-1, -4)$, $(1, 10)$, $(0, 3)$
19. $(-1, 2)$, $(1, 5)$, $\left(-2, \frac{1}{2}\right)$
20. $(-1, -5)$, $(1, 1)$, $(-2, -8)$
21. Find the perimeter of the quadrilateral whose vertices are $(-2, -1)$, $(-4, 5)$, $(3, 5)$, $(4, -2)$.
22. Show that the points $(-2, -1)$, $(2, 2)$, $(5, -2)$ are the vertices of an isosceles triangle.
23. Show that the points $(9, 2)$, $(11, 6)$, $(3, 5)$, and $(1, 1)$ are the vertices of a parallelogram.
24. Show that the point $(-1, 1)$ is the midpoint of the line segment whose endpoints are $(-5, -1)$ and $(3, 3)$.
25. The points $A(2, 7)$, $B(4, 3)$, and $C(x, y)$ determine a right triangle whose hypotenuse is AB. Find x and y. (*Hint:* There is more than one answer.)
26. The points $A(2, 6)$, $B(4, 6)$, $C(4, 8)$, and $D(x, y)$ form a rectangle. Find x and y.

In Exercises 27–32 determine the intercepts and sketch the graph of the given equation.

27. $y = 2x + 4$
28. $y = -2x + 5$
29. $y = \sqrt{x}$
30. $y = \sqrt{x - 1}$
31. $y = |x + 3|$
32. $y = 2 - |x|$

In Exercises 33–38 determine the intercepts and use symmetry to assist in sketching the graph of the given equation.

33. $y = 3 - x^2$
34. $y = 3x - x^2$
35. $y = x^3 + 1$
36. $x = y^3 - 1$
37. $x = y^2 - 1$
38. $y = 3x$

Without graphing, determine whether each curve is symmetric with respect to the x-axis, the y-axis, the origin, or none of these.

39. $3x + 2y = 5$
40. $y = 4x^2$
41. $y^2 = x - 4$
42. $x^2 - y = 2$
43. $y^2 = 1 + x^3$
44. $y = (x - 2)^2$
45. $y^2 = (x - 2)^2$
46. $y^2x + 2x = 4$
47. $y^2x + 2x^2 = 4x^2y$
48. $y^3 = x^2 - 9$
49. $y = \dfrac{x^2 + 4}{x^2 - 4}$
50. $y = \dfrac{1}{x^2 + 1}$
51. $y^2 = \dfrac{x^2 + 1}{x^2 - 1}$
52. $4x^2 + 9y^2 = 36$
53. $xy = 4$

3.2 FUNCTIONS AND FUNCTION NOTATION

The equation

$$y = 2x + 3$$

assigns a value to y for every value of x. If we let X denote the set of values that we can assign to x, and let Y denote the set of values that the equation assigns to y, we can show the correspondence schematically as in Figure 9. The equation can be thought of as a rule defining the correspondence between the sets X and Y.

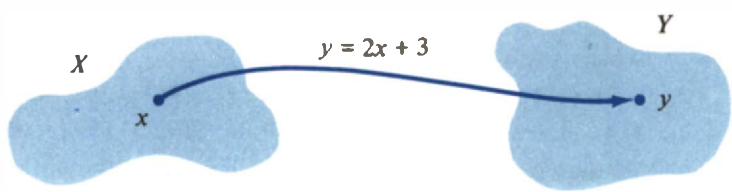


FIGURE 9

We are particularly interested in the situation where, for each element x in X, there corresponds one and only one element y in Y; that is, the rule assigns exactly one y for a given x. This type of correspondence plays a fundamental role in mathematics and is given a special name.

Function, Domain, Image, and Range

A **function** is a rule that, for each x in a set X, assigns exactly one y in a set Y. The element y is called the **image** of x. The set X is called the **domain** of the function and the set of all images is called the **range** of the function.

We can think of the rule defined by the equation $y = 2x + 3$ as a function machine (see Figure 10). Each time we drop a value of x from the domain into the input hopper, exactly one value of y falls out of the output chute. If we drop in $x = 5$, the function machine follows the rule and produces $y = 13$. Since we are free to choose the values of x that we drop into the machine, we call x the **independent variable;** the value of y that drops out depends upon the choice of x, so y is called the **dependent variable.** We say that the dependent variable is

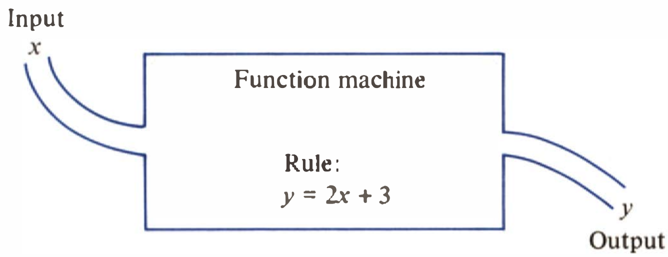


FIGURE 10

a function of the independent variable; that is, *the output is a function of the input*.

Let's look at a few schematic presentations. The correspondence in Figure 11a is a function; for each x in X there is exactly one corresponding value of y in Y. The fact that y_1 is the image of both x_1 and x_2 does not violate the definition of a function. However, the correspondence in Figure 11b is not a function, since x_1 has two images, y_1 and y_2, assigned to it, thus violating the definition of a function.

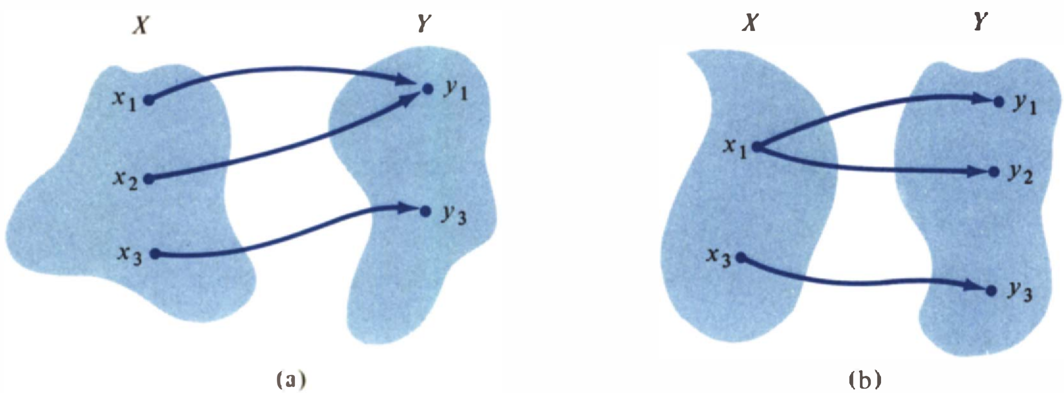


FIGURE 11

VERTICAL LINE TEST

There is a simple graphic way to test whether a correspondence determines a function. When we draw vertical lines on the graph of Figure 12a, we see that no vertical line intersects the graph at more than one point. This means that the correspondence used in sketching the graph assigns exactly one y-value for each x-value and therefore determines y as a function of x. When we draw vertical lines on the graph of Figure 12b, however, some vertical lines intersect the graph at two points. Since the correspondence graphed in Figure 12b assigns the values y_1 and y_2 to x_1, it does not determine y as a function of x. Thus, *not every equation or correspondence in the variables x and y determines y as a function of x*.

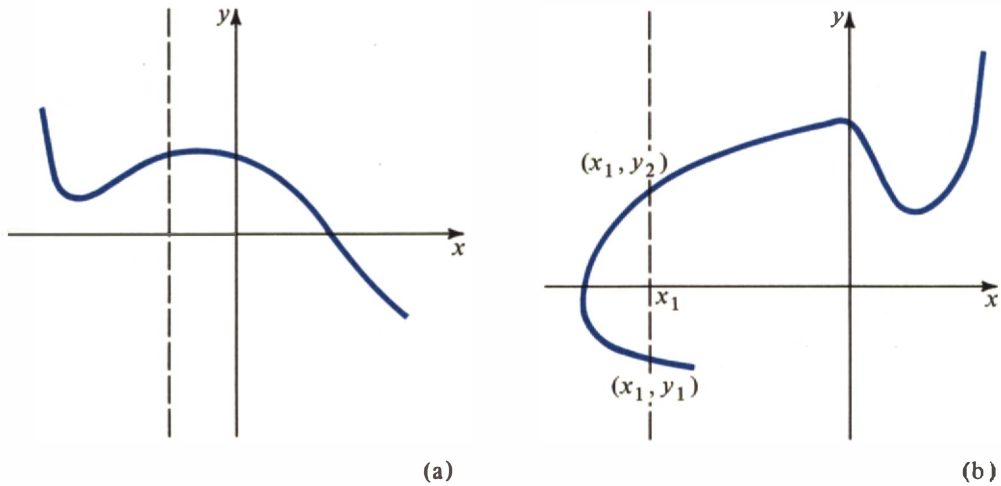
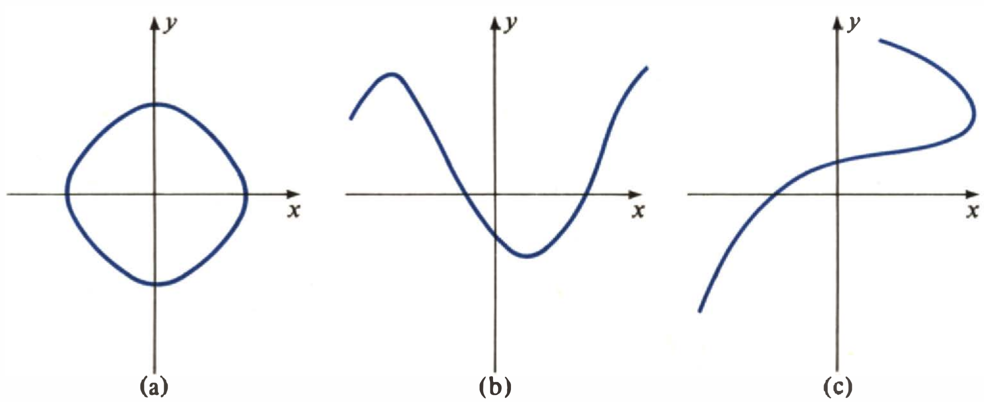


FIGURE 12

Vertical Line Test	A graph determines y as a function of x if and only if no vertical line meets the graph at more than one point.

EXAMPLE 1

Which of the following graphs determine y as a function of x?



SOLUTION

(a) Not a function. Some vertical lines meet the graph in more than one point.
(b) A function. Passes the vertical line test.
(c) Not a function. Fails the vertical line test.

DOMAIN AND RANGE

We have defined the domain of a function as the set of values assumed by the independent variable. In more advanced courses in mathematics, the domain may include complex numbers. In this book we will restrict the domain of a function to those real numbers for which the image is also a real number, and we say that the function **is defined at** such values. When a function is defined by an equation, we must always be alert to two potential problems.

(a) *Division by zero.* For example, the domain of the function

$$y = \frac{2}{x-1}$$

is the set of all real numbers other than $x = 1$. When $x = 1$, the denominator is 0, and division by 0 is not defined.

(b) *Even roots of negative numbers.* For example, the function

$$y = \sqrt{x-1}$$

is defined only for $x \geq 1$, since we exclude the square root of negative numbers. Hence the domain of the function consists of all real numbers $x \geq 1$.

The range of a function is, in general, not as easily determined as is the domain. The range is the set of all y-values that occur in the correspondence; that is, it is the set of all outputs of the function. For our purposes, it will suffice to determine the range by examining the graph.

EXAMPLE 2

Graph the equation $y = \sqrt{x}$. If the correspondence determines a function, find the domain and range.

SOLUTION

We obtain the graph of the equation by plotting points and connecting them to form a smooth curve. Applying the vertical line test to the graph as shown in Figure 13, we see that the equation determines a function. The domain of the function is the set $\{x|x \geq 0\}$ and the range is the set $\{y|y \geq 0\}$.

x	0	1	4	9
y	0	1	2	3

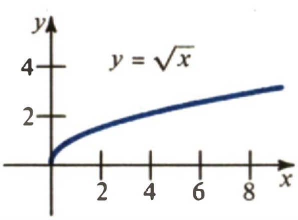


FIGURE 13

PROGRESS CHECK

Graph the equation $y = x^2 - 4$, $-3 \leq x \leq 3$. If the correspondence determines a function, find the domain and range.

ANSWER

The graph is that portion of the curve shown in Figure 5 that lies between $x = -3$ and $x = 3$. The domain is $\{x|-3 \leq x \leq 3\}$; the range is $\{y|-4 \leq y \leq 5\}$.

FUNCTION NOTATION

It is customary to designate a function by a letter of the alphabet, such as f, g, F, or C. We then denote the output corresponding to x by $f(x)$, which is read "f of x." Thus,

$$f(x) = 2x + 3$$

specifies a rule f for determining an output $f(x)$ for a given value of x. To find $f(x)$ when $x = 5$, we simply substitute 5 for x and obtain

$$f(5) = 2(5) + 3 = 13$$

The notation $f(5)$ is a convenient way of specifying "the value of the function f that corresponds to $x = 5$." The symbol f represents the function or rule; the notation $f(x)$ represents the output produced by the rule. For convenience, however, we will at times join in the common practice of designating the function f by $f(x)$.

EXAMPLE 3

(a) If $f(x) = 2x^2 - 2x + 1$, find $f(-1)$.

(b) If $f(t) = 3t^2 - 1$, find $f(2a)$.

SOLUTION

(a) We substitute -1 for x.
$f(-1) = 2(-1)^2 - 2(-1) + 1 = 5$

(b) We substitute $2a$ for t.
$f(2a) = 3(2a)^2 - 1 = 3(4a^2) - 1 = 12a^2 - 1$

PROGRESS CHECK

(a) If $f(u) = u^3 + 3u - 4$, find $f(-2)$.

(b) If $f(t) = t^2 + 1$, find $f(t - 1)$.

ANSWERS

(a) -18 (b) $t^2 - 2t + 2$

EXAMPLE 4

Let the function f be defined by $f(x) = x^2 - 1$. Find

(a) $f(-2)$ (b) $f(a)$ (c) $f(a + h)$ (d) $f(a + h) - f(a)$

SOLUTION

(a) $f(-2) = (-2)^2 - 1 = 4 - 1 = 3$

(b) $f(a) = a^2 - 1$

(c) $f(a + h) = (a + h)^2 - 1 = a^2 + 2ah + h^2 - 1$

(d)
$$\begin{aligned} f(a + h) - f(a) &= (a + h)^2 - 1 - (a^2 - 1) \\ &= a^2 + 2ah + h^2 - 1 - a^2 + 1 \\ &= 2ah + h^2 \end{aligned}$$

WARNING

(a) Note that $f(a + 3) \neq f(a) + f(3)$. Function notation is not to be confused with the distributive law.

(b) Note that $f(x^2) \neq f \cdot x^2$. The use of parentheses in function notation does not imply multiplication.

EXAMPLE 5

A newspaper makes this offer to its advertisers: The first column inch will cost \$40, and each subsequent column inch will cost \$30. If T is the total cost of running an ad whose length is n column inches, and the minimum space is 1 column inch,

(a) express T as a function of n;
(b) find T when $n = 4$.

SOLUTION

(a) The equation

$$\begin{aligned} T &= 40 + 30(n - 1) \\ &= 10 + 30n \end{aligned}$$

gives the correspondence between n and T. In function notation,

$$T(n) = 10 + 30n, \quad n \geq 1$$

(b) When $n = 4$,

$$T(4) = 10 + 30(4) = 130$$

EXERCISE SET 3.2

In Exercises 1–6 graph the equation. If the graph determines y as a function of x, find the domain and use the graph to determine the range of the function.

1. $y = 2x - 3$
2. $y = x^2 + x, \quad -2 \leq x \leq 1$
3. $x = y + 1$
4. $x = y^2 - 1$
5. $y = \sqrt{x - 1}$
6. $y = |x|$

In Exercises 7–12 determine the domain of the function defined by the given rule.

7. $f(x) = \sqrt{2x - 3}$
8. $f(x) = \sqrt{5 - x}$
9. $f(x) = \dfrac{1}{\sqrt{x - 2}}$
10. $f(x) = \dfrac{-2}{x^2 + 2x - 3}$
11. $f(x) = \dfrac{\sqrt{x - 1}}{x - 2}$
12. $f(x) = \dfrac{x}{x^2 - 4}$

In Exercises 13–16 find the number (or numbers) whose image is 2.

13. $f(x) = 2x - 5$
14. $f(x) = x^2$
15. $f(x) = \dfrac{1}{x - 1}$
16. $f(x) = \sqrt{x - 1}$

Given the function f defined by $f(x) = 2x^2 + 5$, determine the following.

17. $f(0)$
18. $f(-2)$
19. $f(a)$
20. $f(3x)$
21. $3f(x)$
22. $-f(x)$

Given the function g defined by $g(x) = x^2 + 2x$, determine the following.

23. $g(-3)$
24. $g\left(\dfrac{1}{x}\right)$
25. $\dfrac{1}{g(x)}$
26. $g(-x)$
27. $g(a + h)$
28. $\dfrac{g(a + h) - g(a)}{h}$

Given the function F defined by $F(x) = \dfrac{x^2 + 1}{3x - 1}$, determine the following.

29. $F(-2.73)$
30. $F(16.11)$
31. $\dfrac{1}{F(x)}$
32. $F(-x)$
33. $2F(2x)$
34. $F(x^2)$

Given the function r defined by $r(t) = \dfrac{t - 2}{t^2 + 2t - 3}$, determine the following.

35. $r(-8.27)$
36. $r(2.04)$
37. $r(2a)$
38. $2r(a)$
39. $r(a + 1)$
40. $r(1 + h)$
41. If x dollars are borrowed at 7% simple annual interest, express the interest I at the end of 4 years as a function of x.
42. Express the area A of an equilateral triangle as a function of the length s of its side.
43. Express the diameter d of a circle as a function of its circumference C.
44. Express the perimeter P of a square as a function of its area A.

3.3 GRAPHS OF FUNCTIONS

We have used the graph of an equation to help us find out whether or not the equation determines a function. It is not surprising, therefore, that the **graph of a function** f is defined as the graph of the equation $y = f(x)$. For example, the graph of the function f defined by the rule $f(x) = \sqrt{x}$ is the graph of the equation $y = \sqrt{x}$, which was sketched in Figure 13.

"SPECIAL" FUNCTIONS AND THEIR GRAPHS

There are a number of ''special'' functions that a mathematics instructor is likely to use to demonstrate a point. The instructor will sketch the graph of the function, since the graph shows at a glance many characteristics of the function. For example, information about symmetry, domain, and range is available from a graph. In fact, we have already used the graphs of some of these functions to illustrate these characteristics.

You should become thoroughly acquainted with the following functions and their graphs. For each function we will form a table of values, sketch the graph of the function, and discuss symmetry, domain, and range.

$f(x) = x$ Identity function

The domain of f is the set of all real numbers. We form a table of values and use it to sketch the graph of $y = x$ in Figure 14. The graph is symmetric with respect to the origin (note that $-y = -x$ is equivalent to $y = x$). The range of f is seen to be the set of all real numbers.

$f(x) = -x$ Negation function

The domain of f is the set of all real numbers. A table of values is used to sketch the graph of $y = -x$ in Figure 15. The graph is symmetric with respect to the origin (note that $-y = x$ is equivalent to $y = -x$). The range of f is seen to be the set of all real numbers.

x	-2	-1	0	1	2
y	-2	-1	0	1	2

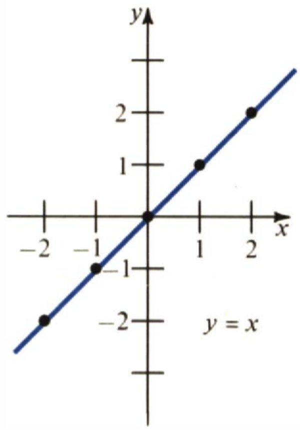


FIGURE 14

x	-2	-1	0	1	2
y	2	1	0	-1	-2

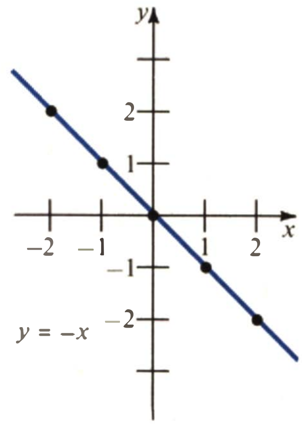


FIGURE 15

$f(x) = |x|$ Absolute value function

The domain of f is the set of all real numbers. A table of values allows us to sketch the graph in Figure 16. The graph is symmetric with respect to the y-axis. Since the graph always lies above the x-axis, the range of f is the set of all nonnegative real numbers.

$f(x) = c$ Constant function

The domain of f is the set of all real numbers. In fact, the value of f is the same for all values of x (see Figure 17). The range of f is the set $\{c\}$. The graph is symmetric with respect to the y-axis (note that $y = c$ is unaltered when x is replaced by $-x$).

x	-2	-1	0	1	2
y	2	1	0	1	2

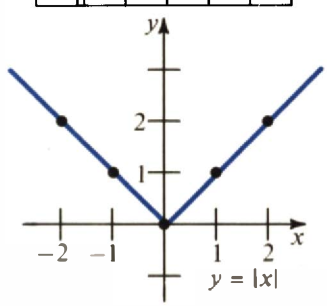


FIGURE 16

x	-2	-1	0	1	2
y	c	c	c	c	c

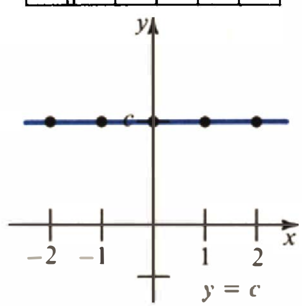


FIGURE 17

$f(x) = x^2$ Squaring function

The domain of f is the set of all real numbers. The graph in Figure 18 is called a **parabola** and illustrates the general shape of all second-degree polynomials. The graph of f is symmetric with respect to the y-axis (note that $y = (-x)^2 = x^2$). Since $y \geq 0$ for all values of x, the range is the set of all nonnegative real numbers.

$f(x) = \sqrt{x}$ Square root function

Since $\sqrt{x}$ is not defined for $x < 0$, the domain is the set of nonnegative real numbers. The graph in Figure 19 always lies above the x-axis, so the range of f is $\{y | y \geq 0\}$, that is, the set of all nonnegative real numbers. The graph is not symmetric with respect to either axis or the origin.

$f(x) = x^3$ Cubing function

The domain is the set of all real numbers. Since the graph in Figure 20 extends indefinitely both upward and downward with no gaps, the range is also the set of all real numbers. The graph is symmetric with respect to the origin (note that $-y = (-x)^3 = -x^3$ is equivalent to $y = x^3$).

x	-2	-1	0	1	2
y	4	1	0	1	4

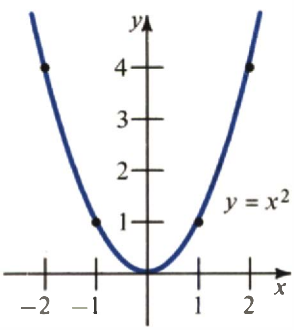


FIGURE 18

x	0	1	4	9
y	0	1	2	3

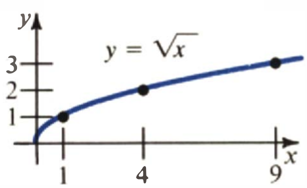


FIGURE 19

x	-2	-1	0	1	2
y	-8	-1	0	1	8

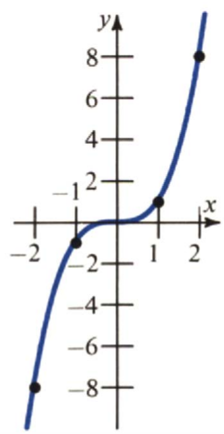


FIGURE 20

PIECEWISE-DEFINED FUNCTIONS

Thus far we have defined each function by means of an equation. A function can also be defined by a table, by a graph, or by several equations. When a function is defined in different ways over different parts of its domain, it is said to be a **piecewise-defined function.** We illustrate this idea by several examples.

EXAMPLE 1
Sketch the graph of the function f defined by

$$f(x) = \begin{cases} x^2 & \text{if } -2 \le x \le 2 \\ 2x + 1 & \text{if } \quad x > 2 \end{cases}$$

SOLUTION
We form a table of points to be plotted, being careful to use the first equation when $-2 \le x \le 2$ and the second equation when $x > 2$.

x	-2	-1	0	1	2	3	4	5
y	4	1	0	1	4	7	9	11

Note that the graph in Figure 21 has a gap. Also note that the point (2, 5) has been marked with an open circle to indicate that it is not on the graph of the function.

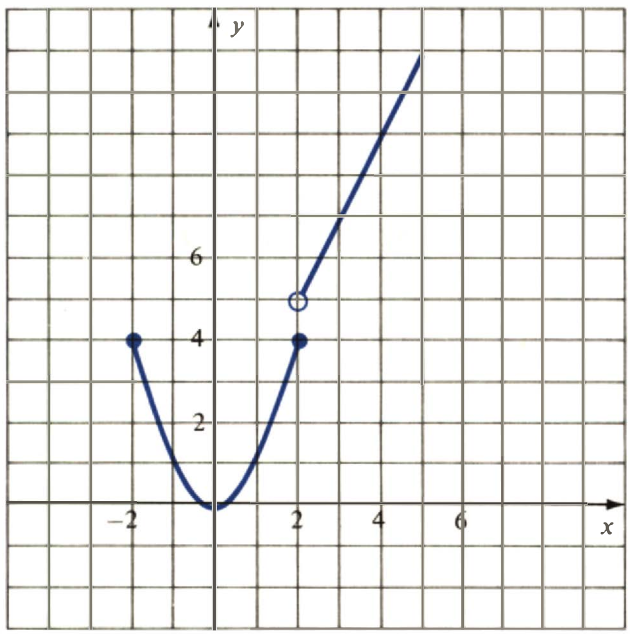


FIGURE 21

EXAMPLE 2
Sketch the graph of the function $f(x) = |x + 1|$.

SOLUTION

We apply the definition of absolute value to obtain

$$y = |x + 1| = \begin{cases} x + 1 & \text{if } x + 1 \geq 0 \\ -(x + 1) & \text{if } x + 1 < 0 \end{cases}$$

or

$$y = \begin{cases} x + 1 & \text{if } x \geq -1 \\ -x - 1 & \text{if } x < -1 \end{cases}$$

From this example it is easy to see that a function involving absolute value will usually be a piecewise-defined function. As usual, we form a table of values, being careful to use $y = x + 1$ when $x \geq -1$ and $y = -x - 1$ when $x < -1$. It is a good idea to include the value $x = -1$ in the table.

x	-3	-2	-1	0	1	2	3
y	2	1	0	1	2	3	4

The points are joined by a smooth curve (Figure 22), which consists of two rays or half-lines intersecting at $(-1, 0)$.

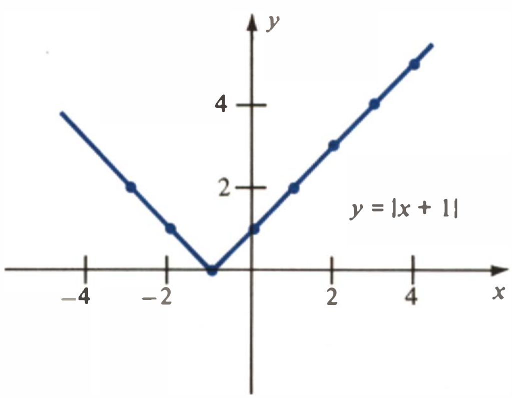


FIGURE 22

EXAMPLE 3

The commission earned by a door-to-door cosmetics salesperson is determined as shown in the accompanying table.

(a) Express the commission C as a function of sales s.
(b) Find the commission if the weekly sales are \$425.
(c) Sketch the graph of the function.

Weekly sales	Commission
less than \$300	20% of sales
\$300 or more but less than \$400	\$60 + 35% of sales over \$300
\$400 or more	\$95 + 60% of sales over \$400

SOLUTION

(a) The function C can be described by three equations.

$$C(s) = \begin{cases} 0.20s & \text{if } s < 300 \\ 60 + 0.35(s - 300) & \text{if } 300 \leq s < 400 \\ 95 + 0.60(s - 400) & \text{if } s \geq 400 \end{cases}$$

(b) When $s = 425$, we must use the third equation and substitute to determine $C(425)$.

$$\begin{aligned} C(425) &= 95 + 0.60(425 - 400) \\ &= 95 + 0.60(25) \\ &= 110 \end{aligned}$$

The commission on sales of \$425 is \$110.

(c) The graph of the function C consists of three line segments (Figure 23).

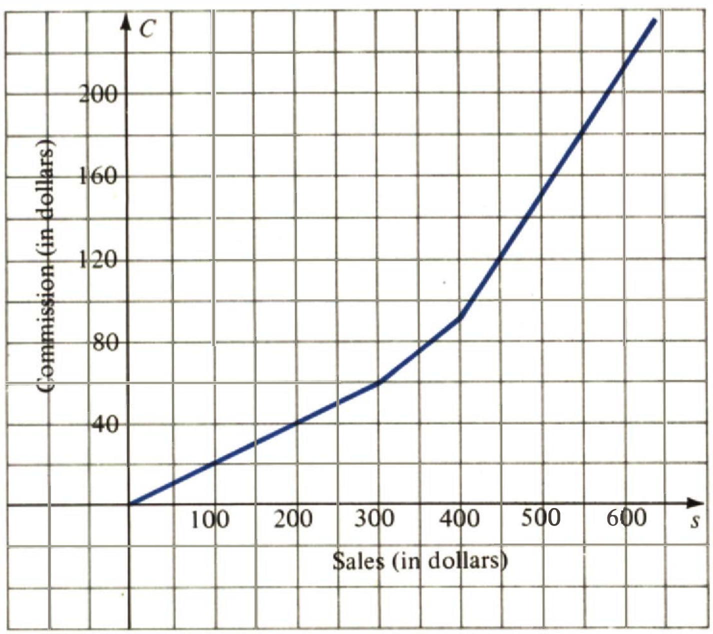


FIGURE 23

INCREASING AND DECREASING FUNCTIONS

When we apply the terms *increasing* and *decreasing* to the graph of a function, we assume that we are viewing the graph from left to right. The straight line of Figure 24a is increasing, since the values of y increase as we move from left to right; similarly, the graph in Figure 24b is decreasing, since the values of y decrease as we move from left to right.

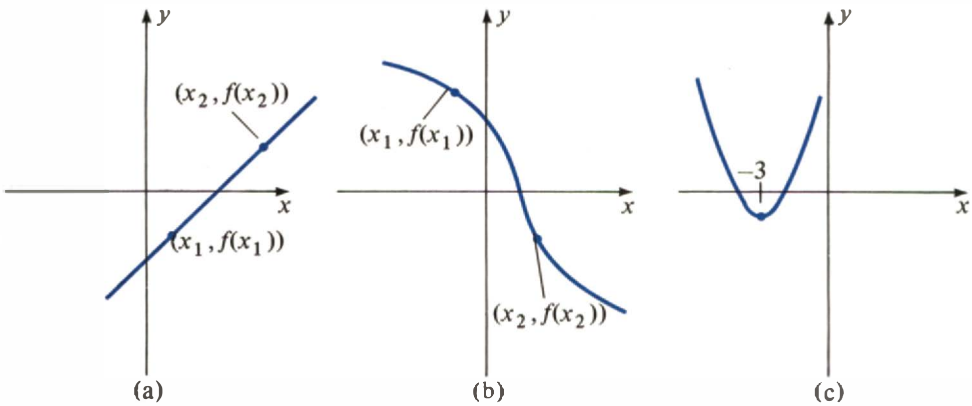


FIGURE 24

One portion of the graph pictured in Figure 24c is decreasing and another is increasing. Since this is the most common situation, we define increasing and decreasing on an interval.

> If x_1 and x_2 are in the interval $[a, b]$ in the domain of a function f, then
> f is **increasing** on $[a, b]$ if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$
> f is **decreasing** on $[a, b]$ if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$
> f is **constant** on $[a, b]$ if $f(x_1) = f(x_2)$ for all x_1, x_2

Returning to Figure 24c, note that the function is decreasing when $x \leq -3$ and increasing when $x \geq -3$; that is, the function is decreasing on the interval $(-\infty, -3]$ and increasing on the interval $[-3, \infty)$. The graph shows that the function has a minimum value at the point $x = -3$. Finding such points is very useful in sketching graphs and is an important technique taught in calculus courses.

It is important to become accustomed to the notation used in Figure 24, where the y-coordinate at the point $x = x_1$ is denoted by $f(x_1)$.

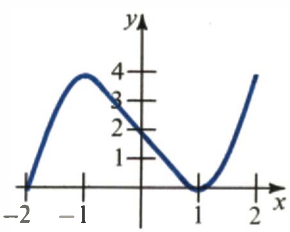


FIGURE 25

EXAMPLE 4

Use the graph of the function $f(x) = x^3 - 3x + 2$, shown in Figure 25, to determine where the function is increasing and where it is decreasing.

SOLUTION

From the graph we see that there are turning points at $(-1, 4)$ and at $(1, 0)$. We conclude that

f is increasing on the intervals $(-\infty, -1]$ and $[1, \infty)$
f is decreasing on the interval $[-1, 1]$

EXAMPLE 5
The function f is defined by

$$f(x) = \begin{cases} |x| & \text{if} \quad x \leq 2 \\ -3 & \text{if} \quad x > 2 \end{cases}$$

Use a graph to find the values of x for which the function is increasing, decreasing, and constant.

SOLUTION
Note that the piecewise-defined function f is composed of the absolute value function when $x \leq 2$ and a constant function when $x > 2$. We can therefore sketch the graph of f immediately as shown in Figure 26. From the graph in Figure 26 we determine that

f is increasing on the interval $[0, 2]$
f is decreasing on the interval $(-\infty, 0]$
f is constant and has value -3 on the interval $(2, \infty)$

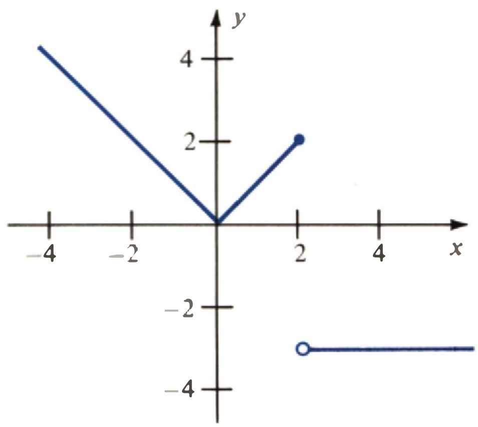


FIGURE 26

PROGRESS CHECK
The function f is defined by

$$f(x) = \begin{cases} 2x + 1 & \text{if} \quad x < -1 \\ 0 & \text{if} \quad -1 \leq x \leq 3 \\ -2x + 1 & \text{if} \quad x > 3 \end{cases}$$

Use a graph to find the values of x for which the function is increasing, decreasing, and constant.

ANSWERS

increasing on the interval $(-\infty, -1)$; constant on $[-1, 3]$; decreasing on $(3, \infty)$.

POLYNOMIAL FUNCTIONS

The polynomial function of first degree

$$f(x) = ax + b$$

is called a **linear function.** We have already graphed a number of such functions in this chapter: $f(x) = 2x + 1$ (Figure 6, page 119), $f(x) = x$ (Figure 14, page 131), and $f(x) = -x$ (Figure 15, page 131). In each case the graph appeared to be a straight line. We will prove in the next section that the graph of every linear function is indeed a straight line.

The polynomial function of second degree

$$f(x) = ax^2 + bx + c, \qquad a \neq 0$$

is called a **quadratic function.** We have graphed a few quadratic functions: $f(x) = x^2 - 4$ (Figure 5, page 118), $f(x) = 1 - x^2$ (Figure 8a, page 121), and $f(x) = x^2$ (Figure 18, page 132). The graph of the quadratic function is called a parabola and will be studied in detail in a later chapter. For now, we offer an example for which a, b, and c are all nonzero.

EXAMPLE 6

Sketch the graph of $f(x) = 2x^2 - 4x + 3$.

SOLUTION

We need to graph $y = 2x^2 - 4x + 3$. We form a table of values, plot the corresponding points, and connect them by a smooth curve, as shown in Figure 27.

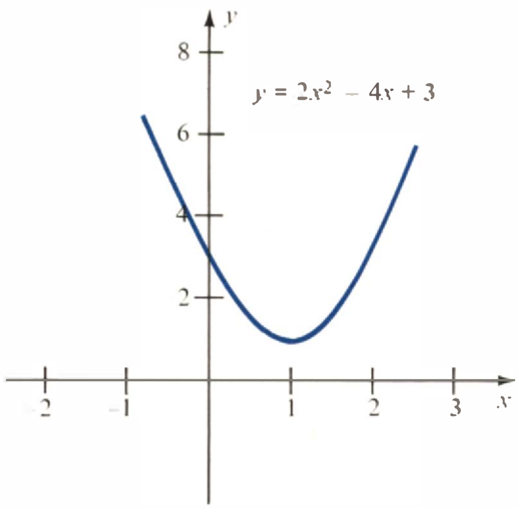


FIGURE 27

An investigation of polynomials of any degree reveals that they are all functions. The graphs of polynomials of degree greater than 2 are always smooth curves; their shapes, however, are not easily determined. The exercises at the end of this section are intended to help you gain experience with the graphs of polynomials. We will take another look at this topic in a later chapter after learning more about the roots of polynomial equations. You should be warned, however, that it is very difficult to graph polynomial functions accurately without results obtained by methods taught in calculus courses.

EXERCISE SET 3.3

In Exercises 1–16 sketch the graph of the function and state where it is increasing, decreasing, and constant.

1. $f(x) = 3x + 1$
2. $f(x) = 3 - 2x$
3. $f(x) = x^2 + 1$
4. $f(x) = x^2 - 4$
5. $f(x) = 9 - x^2$
6. $f(x) = 4x - x^2$
7. $f(x) = |2x + 1|$
8. $f(x) = |1 - x|$
9. $f(x) = \begin{cases} 2x, & x > -1 \\ -x, - 1, & x \leq -1 \end{cases}$
10. $f(x) = \begin{cases} x + 1, & x > 2 \\ 1, & -1 \leq x \leq 2 \\ -x + 1, & x < -1 \end{cases}$
11. $f(x) = \begin{cases} x, & x < 2 \\ 2, & x \geq 2 \end{cases}$
12. $f(x) = \begin{cases} -x, & x \leq -2 \\ x^2, & -2 < x \leq 2 \\ -x, & 3 \leq x \leq 4 \end{cases}$
13. $f(x) = \begin{cases} -x^2, & -3 < x < 1 \\ 0, & 1 \leq x \leq 2 \\ -3x, & x > 2 \end{cases}$
14. $f(x) = \begin{cases} 2 & \text{if } x \text{ is an integer} \\ -1 & \text{if } x \text{ is not an integer} \end{cases}$
15. $f(x) = \begin{cases} -2, & x < -2 \\ -1, & -2 \leq x \leq -1 \\ 1, & x > -1 \end{cases}$
16. $f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 3, & x = 1 \end{cases}$

In Exercises 17–24 sketch the graphs of the given functions on the same coordinate axes.

17. $f(x) = x^2, \quad g(x) = 2x^2, \quad h(x) = \frac{1}{2}x^2$
18. $f(x) = \frac{1}{2}x^2, \quad g(x) = \frac{1}{3}x^2, \quad h(x) = \frac{1}{4}x^2$
19. $f(x) = 2x^2, \quad g(x) = -2x^2$
20. $f(x) = x^2 - 2, \quad g(x) = 2 - x^2$
21. $f(x) = x^3, \quad g(x) = 2x^3$
22. $f(x) = \frac{1}{2}x^3, \quad g(x) = \frac{1}{4}x^3$
23. $f(x) = x^3, \quad g(x) = -x^3$
24. $f(x) = -2x^3, \quad g(x) = -4x^3$

25. The telephone company charges a fee of \$6.50 per month for the first 100 message units and an additional fee of \$0.06 for each of the next 100 message units. A reduced rate of \$0.05 is charged for each message unit after the first 200 units. Express the monthly charge C as a function of the number of message units u.

26. The annual dues of a union are as shown in the table.

Employee's annual salary	Annual dues
less than \$8000	\$60
\$8000 or more but less than \$15,000	\$60 + 1% of the salary in excess of \$8000
\$15,000 or more	\$130 + 2% of the salary in excess of \$15,000

Express the annual dues d as a function of the salary S.

27. A tour operator who runs charter flights to Rome has established the following pricing schedule. For a group of no more than 100 people, the round trip fare per person is \$300, with a minimum rental of \$30,000 for the plane. For a group of more than 100, the fare per person for all passengers is reduced by \$1 for each passenger in excess of 100. Write the tour operator's total revenue R as a function of the number of people x in the group.

28. A firm packages and ships 1-pound jars of instant coffee. The cost C of shipping is 40 cents for the first pound and 25 cents for each additional pound.
    (a) Write C as a function of the weight w (in pounds) for $0 < w \leq 30$.
    (b) What is the cost of shipping a package containing 24 jars of instant coffee?

29. The daily rates of a car rental firm are \$14 plus \$0.08 per mile.
    (a) Express the cost C of renting a car as a function of the number of miles m traveled.
    (b) What is the domain of the function?
    (c) How much would it cost to rent a car for a 100-mile trip?

30. In a wildlife preserve, the population P of eagles depends on the population x of its basic food supply, rodents. Suppose that P is given by

$$P(x) = 0.002x + 0.004x^2$$

Find the eagle population when the rodent population is
(a) 500; (b) 2000.

3.4 LINEAR FUNCTIONS

In the last section we said that the polynomial function of first degree

$$f(x) = ax + b$$

is called a linear function, and we observed that the graph of such a function *appears* to be a straight line. In this section we will look at the property of a straight line that differentiates it from all other curves. We will then develop equations for the straight line, and we will show that the graph of a linear function is indeed a straight line.

SLOPE OF THE STRAIGHT LINE

In Figure 28 we have drawn a straight line L that is not vertical. We have indicated the distinct points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on L. The increments or changes $x_2 - x_1$ and $y_2 - y_1$ in the x- and y-coordinates, respectively, from P_1 to P_2 are also indicated. Note that the increment $x_2 - x_1$ cannot be zero, because L is not vertical.

If $P_3(x_3, y_3)$ and $P_4(x_4, y_4)$ are another pair of points on L, the increments $x_4 - x_3$ and $y_4 - y_3$ will, in general, be different from the increments obtained by using P_1 and P_2. However, since triangles P_1AP_2 and P_3BP_4 are similar, the corresponding sides are in proportion; that is, the ratios

$$\frac{y_4 - y_3}{x_4 - x_3} \quad \text{and} \quad \frac{y_2 - y_1}{x_2 - x_1}$$

are the same. This ratio is called the **slope of the line** L and is denoted by m.

Slope of a Line

The slope of a line that is not vertical is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

where $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are any two points on the line.

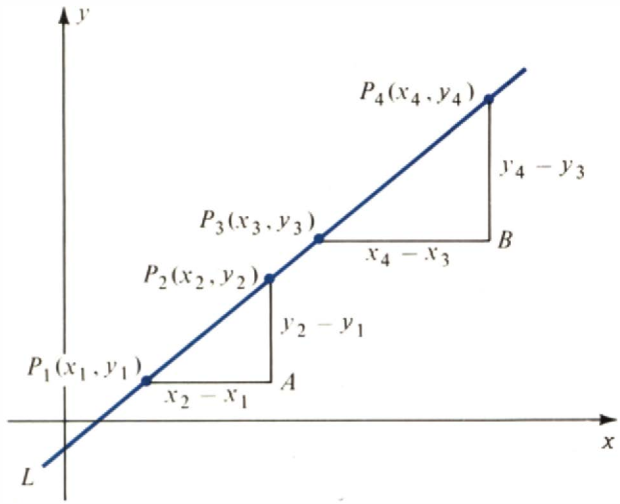


FIGURE 28

For a vertical line, $x_1 = x_2$, so $x_2 - x_1 = 0$. Since we cannot divide by 0, we say that a vertical line has no slope.

The property of constant slope characterizes the straight line; that is, no other curve has this property. In fact, to define slope for a curve other than a straight line is not a trivial task; it requires use of the concept of limit, which is fundamental to calculus.

EXAMPLE 1

Find the slope of the line that passes through the points (4, 2), (1, −2).

SOLUTION

We may choose either point as (x_1, y_1) and the other as (x_2, y_2). Our choice is

$$(x_1, y_1) = (4, 2) \quad \text{and} \quad (x_2, y_2) = (1, -2)$$

Then

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 2}{1 - 4} = \frac{-4}{-3} = \frac{4}{3}$$

The student should verify that reversing the choice of P_1 and P_2 produces the same result for the slope m. We may choose either point as P_1 and the other as P_2, but we must use this choice consistently once it has been made.

Slope is a means of measuring the steepness of a line. That is, slope specifies the number of units we must move up or down to reach the line after moving **one** unit to the left or right of the line. In Figure 29 we have displayed several lines with positive and negative slopes. We can summarize this way.

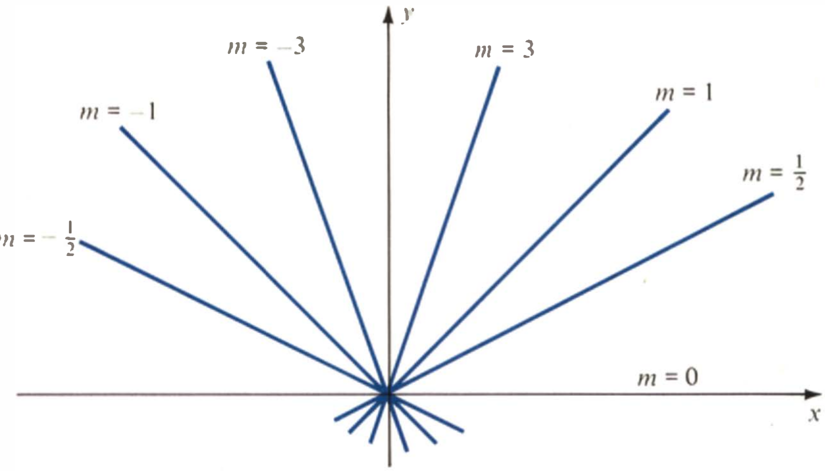


FIGURE 29

Let m be the slope of a line L.
(a) When $m > 0$, the line is the graph of an increasing function.
(b) When $m < 0$, the line is the graph of a decreasing function.
(c) When $m = 0$, the line is the graph of a constant function.
(d) Slope does not exist for a vertical line, and a vertical line is not the graph of a function.

EQUATIONS OF THE STRAIGHT LINE

We can apply the concept of slope to develop two important forms of the equation of a straight line. In Figure 30 the point $P_1(x_1, y_1)$ lies on a line L whose slope is m. If $P(x, y)$ is any other point on L, then we may use P and P_1 to compute m; that is,

$$m = \frac{y - y_1}{x - x_1}$$

which can be written in the form

$$y - y_1 = m(x - x_1)$$

Since (x_1, y_1) satisfies this equation, every point on L satisfies this equation. Conversely, any point satisfying this equation must lie on the line L, since there is only one line through $P_1(x_1, y_1)$ with slope m. This equation is called the **point-slope form** of a line.

Point-Slope Form

$$y - y_1 = m(x - x_1)$$

is an equation of the line with slope m that passes through the point (x_1, y_1).

THE PIRATE TREASURE (PART I)

Five pirates traveling with a slave found a chest of gold coins. The pirates agreed to divide the coins among themselves the following morning.

During the night Pirate 1 awoke and, not trusting his fellow pirates, decided to remove his share of the coins. After dividing the coins into five equal lots, he found that one coin remained. The pirate took his lot and gave the remaining coin to the slave to ensure his silence.

Later that night Pirate 2 awoke and decided to remove his share of the coins. After dividing the remaining coins into five equal lots, he found one coin left over. The pirate took his lot and gave the extra coin to the slave.

That same night the process was repeated by Pirates 3, 4, and 5. Each time there remained one coin, which was given to the slave.

In the morning these five compatible pirates divided the remaining coins into five equal lots. Once again a single coin remained.

Question: What is the minimum number of coins there could have been in the chest? (For help, see Part II on page 147.)

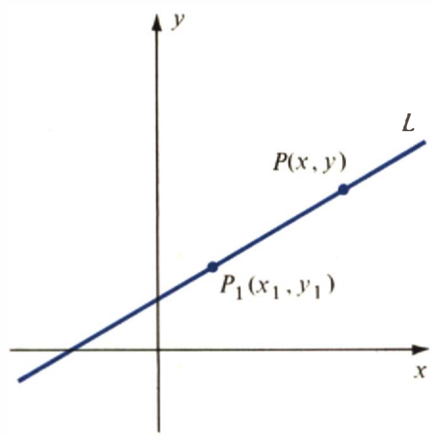


FIGURE 30

EXAMPLE 2

Find an equation of the line that passes through the points (6, −2) and (−4, 3).

SOLUTION

We first find the slope. We let $(x_1, y_1) = (6, -2)$ and $(x_2, y_2) = (-4, 3)$; then

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - (-2)}{-4 - 6} = \frac{5}{-10} = -\frac{1}{2}$$

Next, the point-slope form is used with $m = -\frac{1}{2}$ and $(x_1, y_1) = (6, -2)$.

$$y - y_1 = m(x - x_1)$$

$$y - (-2) = -\frac{1}{2}(x - 6)$$

$$y = -\frac{1}{2}x + 1$$

The student should verify that using the point $(-4, 3)$ and $m = -\frac{1}{2}$ in the point-slope form will yield the same equation.

PROGRESS CHECK

Find an equation of the line that passes through the points $(-5, 0)$ and $(2, -5)$.

ANSWER

$y = -\frac{5}{7}x - \frac{25}{7}$

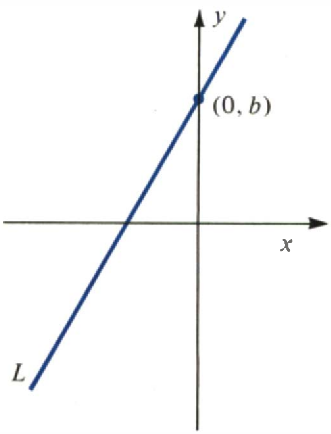


FIGURE 31

There is another form of the equation of the straight line that is very useful. In Figure 31 the line L meets the y-axis at the point $(0, b)$ and is assumed to have slope m. Then we can let $(x_1, y_1) = (0, b)$ and use the point-slope form:

$$y - y_1 = m(x - x_1)$$
$$y - b = m(x - 0)$$
$$y = mx + b$$

Recalling that b is the y-intercept, we call this equation the **slope-intercept form** of the line.

Slope-Intercept Form	The graph of the equation $$y = mx + b$$ is a straight line with slope m and y-intercept b.

The last result leads to the important conclusion mentioned in the introduction to this section. Since the graph of $y = mx + b$ is the graph of the function $f(x) = mx + b$, we have shown that the *graph of a linear function is a straight line*.

EXAMPLE 3

Find the slope and y-intercept of the line $y - 3x + 1 = 0$.

SOLUTION

The equation must be placed in the form $y = mx + b$. Solving for y gives

$$y = 3x - 1$$

and we find that $m = 3$ is the slope and $b = -1$ is the y-intercept.

PROGRESS CHECK

Find the slope and y-intercept of the line $2y + x - 3 = 0$.

ANSWER

slope $= m = -\frac{1}{2}$; y-intercept $= b = \frac{3}{2}$

HORIZONTAL AND VERTICAL LINES

In Figure 32a we have drawn a horizontal line through the point (a, b). Every point on this line has the form (x, b), since the y-coordinate remains constant. If $P(x_1, b)$ and $Q(x_2, b)$ are any two distinct points on the line, then the slope is

$$m = \frac{b - b}{x_2 - x_1} = 0$$

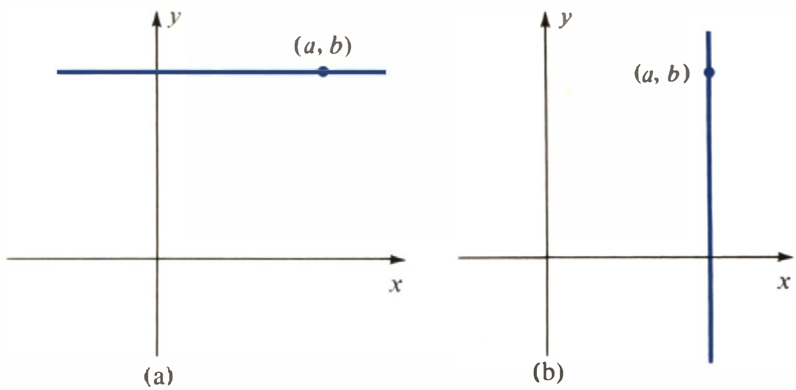


FIGURE 32

We have established the following.

Horizontal Lines

The equation of the horizontal line through the point (a, b) is

$$y = b$$

The slope of a horizontal line is 0.

In Figure 32b every point on the vertical line through the point (a, b) has the form (a, y), since the x-coordinate remains constant. The slope computation using any two points $P(a, y_1)$ and $Q(a, y_2)$ on the line produces

$$m = \frac{y_2 - y_1}{a - a} = \frac{y_2 - y_1}{0}$$

Since we cannot divide by 0, slope is not defined for a vertical line.

Vertical Lines

The equation of the vertical line through the point (a, b) is

$$x = a$$

A vertical line has no slope.

EXAMPLE 4
Find the equations of the horizontal and vertical lines through $(-4, 7)$.

SOLUTION
The horizontal line has the equation $y = 7$. The vertical line has the equation $x = -4$.

WARNING Don't confuse "no slope" and "zero slope." A horizontal line has zero slope. A vertical line has no slope; in other words, its slope is undefined.

PARALLEL AND PERPENDICULAR LINES

The concept of slope of a line can be used to determine when two lines are parallel or perpendicular. Since parallel lines have the same "steepness," we intuitively recognize that they must have the same slope.

Two lines with slopes m_1 and m_2 are parallel if and only if

$$m_1 = m_2$$

The criterion for perpendicular lines can be stated in this way.

Two lines with slopes m_1 and m_2 are perpendicular if and only if

$$m_2 = -\frac{1}{m_1}$$

These two theorems do not apply to vertical lines, since the slope of a vertical line is undefined. The proofs of these theorems are geometric in nature and are outlined in Exercises 54 and 56.

EXAMPLE 5
Given the line $y = 3x - 2$, find an equation of the line passing through the point $(-5, 4)$ that is (a) parallel to the given line; (b) perpendicular to the given line.

THE PIRATE TREASURE (PART II)

First, note that any number that is a multiple of 5 can be written in the form $5n$, where n is an integer. Since the number of coins found in the chest by Pirate 1 was one more than a multiple of 5, we can write the original number of coins C in the form $C = 5n + 1$, where n is a positive integer. Now, Pirate 1 removed his lot of n coins and gave one to the slave. The remaining coins can be calculated as

$$5n + 1 - (n + 1) = 4n$$

and since this is also one more than a multiple of five, we can write $4n = 5p + 1$, where p is a positive integer. Repeating the process, we have the following sequence of equations.

$C = 5n + 1$	found by Pirate 1
$4n = 5p + 1$	found by Pirate 2
$4p = 5q + 1$	found by Pirate 3
$4q = 5r + 1$	found by Pirate 4
$4r = 5s + 1$	found by Pirate 5
$4s = 5t + 1$	found next morning

Solving for s in the last equation and substituting successively in the preceding equations leads to the requirement that

$$1024n - 3125t = 2101 \tag{1}$$

where n and t are positive integers. Equations such as this, which require integer solutions, are called Diophantine equations, and there is an established procedure for solving them that is studied in courses in number theory.

You might want to try to solve Equation (1) using a computer program. Since

$$n = \frac{3125t + 2101}{1024}$$

you can substitute successive integer values for t until you produce an integer result for n. The accompanying BASIC program does just that.

```
10  FOR K = 1 TO
    3200
20  X = (3125*K+
    2101)/1024
30  I = INT(X)
40  IF X = I THEN
    GO TO 60
50  NEXT K
60  PRINT "MINI-
    MUM NUMBER
    OF COINS=";
    5*I + 1
70  END
```

SOLUTION

We first note that the line $y = 3x - 2$ has slope $m_1 = 3$.

(a) Every line parallel to the line $y = 3x - 2$ must have slope $m_2 = m_1 = 3$. We therefore seek a line with slope 3 that passes through the point $(-5, 4)$. Using the point-slope formula, we have

$$y - y_1 = m(x - x_1)$$
$$y - 4 = 3(x + 5)$$
$$y = 3x + 19$$

(b) Every line perpendicular to the line $y = 3x - 2$ has slope $m_2 = -1/m_1 = -1/3$. The line we seek has slope $-\frac{1}{3}$ and passes through the point $(-5, 4)$. We can again apply the point-slope formula to obtain

$$y - y_1 = m(x - x_1)$$

$$y - 4 = -\frac{1}{3}(x + 5)$$

$$y = -\frac{1}{3}x + \frac{7}{3}$$

The three lines are shown in Figure 33.

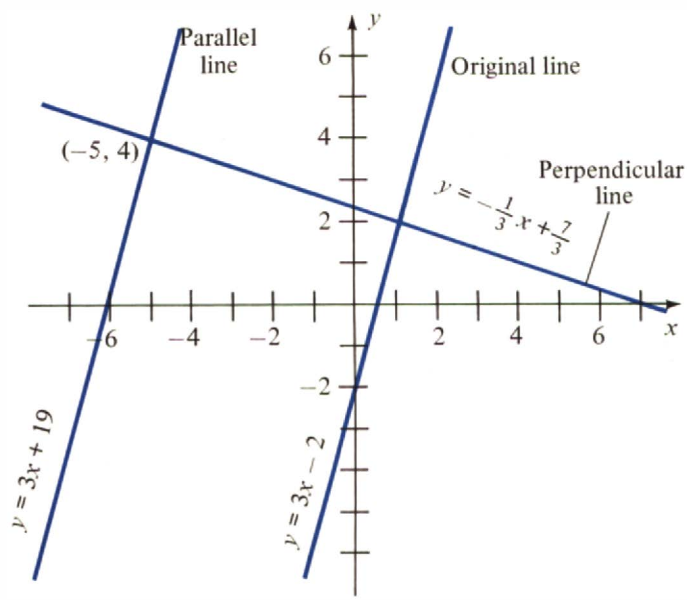


FIGURE 33

GENERAL FIRST-DEGREE EQUATION

The general first-degree equation in x and y can always be written in the form

$$Ax + By + C = 0$$

where A, B, and C are constants and A and B are not both zero. We can rewrite this equation as

$$By = -Ax - C$$

If $B \neq 0$, the equation becomes

$$y = -\frac{A}{B}x - \frac{C}{B}$$

which we recognize as having a straight-line graph with slope $-A/B$ and y-intercept $-C/B$. If $B = 0$, the original equation becomes $Ax + C = 0$, whose graph is a vertical line.

The General First-Degree Equation

- The graph of the general first-degree equation

$$Ax + By + C = 0$$

is a straight line.
- If $B = 0$, the graph is a vertical line.
- If $A = 0$, the graph is a horizontal line.

EXERCISE SET 3.4

In Exercises 1–6 determine the slope of the line through the given points. State whether the line is the graph of an increasing function, a decreasing function, or a constant function.

1. $(2, 3), (-1, -3)$
2. $(1, 2), (-2, 5)$
3. $(-2, 3), (0, 0)$
4. $(2, 4), (-3, 4)$
5. $\left(\frac{1}{2}, 2\right), \left(\frac{3}{2}, 1\right)$
6. $(-4, 1), (-1, -2)$
7. Use slopes to show that the points $A(-1, -5)$, $B(1, -1)$, and $C(3, 3)$ are collinear (lie on the same line).
8. Use slopes to show that the points $A(-3, 2)$, $B(3, 4)$, $C(5, -2)$, and $D(-1, -4)$ are the vertices of a parallelogram.

In Exercises 9–12 determine an equation of the line with the given slope m that passes through the given point.

9. $m = 2$, $(-1, 3)$
10. $m = -\frac{1}{2}$, $(1, -2)$
11. $m = 3$, $(0, 0)$
12. $m = 0$, $(-1, 3)$

In Exercises 13–18 determine an equation of the line through the given points.

13. $(2, 4), (-3, -6)$
14. $(-3, 5), (1, 7)$
15. $(0, 0), (3, 2)$
16. $(-2, 4), (3, 4)$
17. $\left(-\frac{1}{2}, -1\right), \left(\frac{1}{2}, 1\right)$
18. $(-8, -4), (3, -1)$

In Exercises 19–24 determine an equation of the line with the given slope m and the given y-intercept b.

19. $m = 3$, $b = 2$
20. $m = -3$, $b = -3$
21. $m = 0$, $b = 2$
22. $m = -\frac{1}{2}$, $b = \frac{1}{2}$
23. $m = \frac{1}{3}$, $b = -5$
24. $m = -2$, $b = -\frac{1}{2}$

In Exercises 25–30 determine the slope m and y-intercept b of the given line.

25. $3x + 4y = 5$
26. $2x - 5y + 3 = 0$
27. $y - 4 = 0$
28. $x = -5$
29. $3x + 4y + 2 = 0$
30. $x = -\frac{1}{2}y + 3$

In Exercises 31–36 write an equation of (a) the horizontal line passing through the given point and (b) the vertical line passing through the given point.

31. $(-6, 3)$
32. $(-5, -2)$
33. $(-7, 0)$
34. $(0, 5)$
35. $(9, -9)$
36. $\left(-\frac{3}{2}, 1\right)$

In Exercises 37–40 determine the slope of (a) every line that is parallel to the given line and (b) every line that is perpendicular to the given line.

37. $y = -3x + 2$
38. $2y - 5x + 4 = 0$
39. $3y = 4x - 1$
40. $5y + 4x = -1$

In Exercises 41–44 determine an equation of the line through the given point that (a) is parallel to the given line; (b) is perpendicular to the given line.

41. $(1, 3)$; $y = -3x + 2$
42. $(-1, 2)$; $3y + 2x = 6$
43. $(-3, 2)$; $3x + 5y = 2$
44. $(-1, -3)$; $3y + 4x - 5 = 0$

45. The Celsius (C) and Fahrenheit (F) temperature scales are related by a linear equation. Water boils at 212°F or 100°C, and freezes at 32°F or 0°C.
    (a) Write a linear equation expressing F in terms of C.
    (b) What is the Fahrenheit temperature when the Celsius temperature is 20°?

46. The college bookstore sells a textbook that costs \$10 for \$13.50, and a textbook that costs \$12 for \$15.90. If the markup policy of the bookstore is linear, write a linear function that relates sales price S and cost C. What is the cost of a book that sells for \$22?

47. An appliance manufacturer finds that it had sales of \$200,000 five years ago and sales of \$600,000 this year. If the growth in sales is assumed to be linear, what will the sales amount be five years from now?

48. A product that cost \$2.50 three years ago sells for \$3 this year. If price increases are assumed to be linear, how much will the product cost six years from now?

49. Find a real number c such that $P(-2, 2)$ is on the line $3x + cy = 4$.

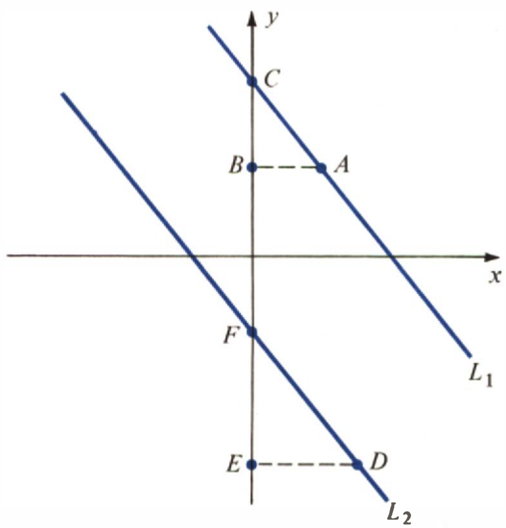
50. Find a real number c such that the line $cx - 5y + 8 = 0$ has x-intercept 4.

51. If the points $(-2, -3)$ and $(-1, 5)$ are on the graph of a linear function f, find $f(x)$.

52. If $f(1) = 4$ and $f(-1) = 3$ and the function f is linear, find $f(x)$.

53. Prove that the linear function $f(x) = ax + b$ is an increasing function if $a > 0$ and is a decreasing function if $a < 0$.

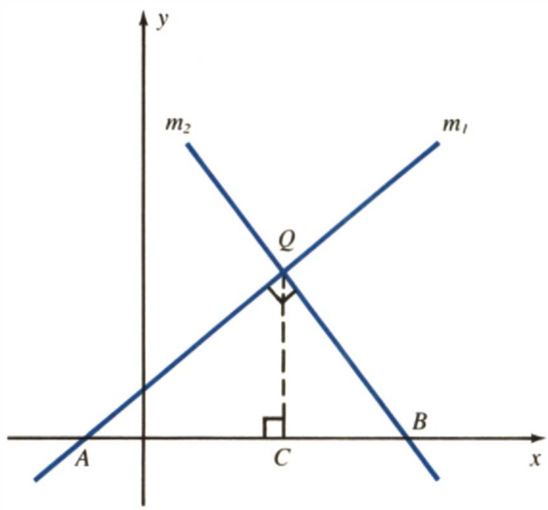
54. In the accompanying figure, lines L_1 and L_2 are parallel. Points A and D are selected on lines L_1 and L_2, respectively. Lines parallel to the x-axis are constructed through A and D that intersect the y-axis at points B and E. Supply a reason for each of the steps in the following proof.



(a) Angles ABC and DEF are equal.
(b) Angles ACB and DFE are equal.
(c) Triangles ABC and DEF are similar.
(d) $\dfrac{\overline{CB}}{\overline{BA}} = \dfrac{\overline{FE}}{\overline{ED}}$
(e) $m_1 = \dfrac{\overline{CB}}{\overline{BA}}, \quad m_2 = \dfrac{\overline{FE}}{\overline{ED}}$

(f) $m_1 = m_2$

(g) Parallel lines have the same slope.



55. Prove that if two lines have the same slope, they are parallel.

56. In the accompanying figure, lines perpendicular to each other, with slopes m_1 and m_2, intersect at a point Q. A perpendicular from Q to the x-axis intersects the x-axis at the point C. Supply a reason for each of the steps in the following proof.

(a) Angles CAQ and BQC are equal.

(b) Triangles ACQ and BCQ are similar.

(c) $\dfrac{\overline{CQ}}{\overline{AC}} = \dfrac{\overline{CB}}{\overline{CQ}}$

(d) $m_1 = \dfrac{\overline{CQ}}{\overline{AC}}, \quad m_2 = \dfrac{\overline{CQ}}{\overline{CB}}$

(e) $m_2 = -\dfrac{1}{m_1}$

57. Prove that if two lines have slopes m_1 and m_2 such that $m_2 = -1/m_1$, the lines are perpendicular.

58. If x_1 and x_2 are the abscissas of two points on the graph of the function $y = f(x)$, show that the slope m of the line connecting the two points can be written as

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

3.5 DIRECT AND INVERSE VARIATION (Optional)

DIRECT VARIATION

Two functional relationships occur so frequently that they are given distinct names. They are direct and inverse variation. We say that two quantities **vary directly** if an increase in one causes a proportional increase in the other. In the table

x	1	2	3	4
y	3	6	9	12

we see that an increase in x causes a proportional increase in y. If we look at the ratios y/x, we see that

$$\frac{y}{x} = \frac{3}{1} = \frac{6}{2} = \frac{9}{3} = \frac{12}{4} = 3$$

or $y = 3x$. The ratio y/x remains constant for all values of y and $x \neq 0$. This is an example of the **principle of direct variation.**

Principle of Direct Variation If y varies directly as x, then $y = kx$ for some constant k.

As another example, when we say that y varies directly as the square of x, we mean that $y = kx^2$ for some constant k. The constant k is called the **constant of variation.**

EXAMPLE 1

Suppose that y varies directly as the cube of x and that $y = 24$ when $x = -2$. Write the appropriate equation, solve for the constant of variation k, and use this k to relate the variables.

SOLUTION

From the principle of direct variation, we know that the functional relationship is

$$y = kx^3 \qquad \text{for some constant } k$$

Substituting the values $y = 24$ and $x = -2$, we have

$$24 = k \cdot (-2)^3 = -8k$$
$$k = -3$$

Thus,

$$y = -3x^3$$

PROGRESS CHECK

(a) If P varies directly as the square of V, and $P = 64$ when $V = 16$, find the constant of variation.
(b) The circumference C of a circle varies directly as the radius r. If $C = 25.13$ when $r = 4$, express C as a function of r; that is, use the constant of variation to relate the variables C and r.

ANSWERS

(a) $\frac{1}{4}$ (b) $C = 6.2825r$

INVERSE VARIATION

Two quantities are said to **vary inversely** if an increase in one causes a proportional decrease in the other. In the table

x	1	2	3	4
y	24	12	8	6

we see that an increase in x causes a proportional decrease in y. If we look at the product xy, we see that

$$xy = 1 \cdot 24 = 2 \cdot 12 = 3 \cdot 8 = 4 \cdot 6 = 24$$

or

$$y = \frac{24}{x}$$

In general, the **principle of inverse variation** is as follows.

Principle of Inverse Variation	If y varies inversely as x, then $y = \frac{k}{x}$ for some constant k.

Once again, k is called the constant of variation.

EXAMPLE 2

Suppose that y varies inversely as x^2 and that $y = 10$ when $x = 10$. Write the appropriate equation, solve for the constant of variation k, and use this k to relate the variables.

SOLUTION

The functional relationship is

$$y = \frac{k}{x^2} \qquad \text{for some constant } k$$

Substituting $y = 10$ and $x = 10$, we have

$$10 = \frac{k}{(10)^2} = \frac{k}{100}$$
$$k = 1000$$

Thus,

$$y = \frac{1000}{x^2}$$

PROGRESS CHECK

If v varies inversely as the cube of w, and $v = 2$ when $w = -2$, find the constant of variation.

ANSWER

-16

JOINT VARIATION

An equation of variation can involve more than two variables. We say that a quantity **varies jointly** as two or more other quantities if it varies directly as their product.

EXAMPLE 3

Express the following statement as an equation: P varies jointly as R, S, and the square of T.

SOLUTION

Since P must vary directly as RST^2, we have $P = kRST^2$ for some constant k.

EXAMPLE 4

A snow removal firm finds that the annual profit P varies jointly as the number of available plows p and the square of the total inches of snowfall s and inversely as the price per gallon of gasoline g. If the profit is \$15,000 when the snowfall is 6 inches, 5 plows are used, and the price of gasoline is \$1.50 per gallon, express the profit P as a function of s, p, and g.

SOLUTION

We are given that

$$P = k\frac{ps^2}{g}$$

for some constant k. To determine k, we substitute $P = 15{,}000$, $p = 5$, $s = 6$, and $g = 1.5$. Thus,

$$15{,}000 = k\frac{(5)(6)^2}{1.5} = 120k$$

$$k = \frac{15{,}000}{120} = 125$$

Thus,

$$P = 125\frac{ps^2}{g}$$

EXERCISE SET 3.5

1. In the following table, y varies directly with x.

x	2	3	4	6	8	12		
y	8	12	16	24			80	120

   (a) Find the constant of variation.
   (b) Write an equation showing that y varies directly with x.
   (c) Fill the blanks in the table.

2. In the accompanying table, y varies inversely with x.

x	1	2	3	6	9	12	15	18		
y	6	3	2	1	$\frac{2}{3}$				$\frac{1}{4}$	$\frac{1}{10}$

   (a) Find the constant of variation.
   (b) Write an equation showing that y varies inversely with x.
   (c) Fill the blanks in the table.

3. If y varies directly as x, and $y = -\frac{1}{4}$ when $x = 8$,
   (a) find the constant of variation;
   (b) find y when $x = 12$.

4. If C varies directly as the square of s, and $C = 12$ when $s = 6$,
   (a) find the constant of variation;
   (b) find C when $s = 9$.
5. If s varies directly as the square of t, and $s = 10$ when $t = 10$,
   (a) find the constant of variation;
   (b) find s when $t = 5$.
6. If V varies as the cube of T, and $V = 16$ when $T = 4$,
   (a) find the constant of variation;
   (b) find V when $T = 6$.
7. If y varies inversely as x, and $y = -\frac{1}{2}$ when $x = 6$,
   (a) find the constant of variation;
   (b) find y when $x = 12$.
8. If V varies inversely as the square of p, and $V = \frac{2}{3}$ when $p = 6$,
   (a) find the constant of variation;
   (b) find V when $p = 8$.
9. If K varies inversely as the cube of r, and $K = 8$ when $r = 4$,
   (a) find the constant of variation;
   (b) find K when $r = 5$.
10. If T varies inversely as the cube of u, and $T = 2$ when $u = 2$,
    (a) find the constant of variation;
    (b) find T when $u = 5$.
11. If M varies directly as the square of r and inversely as the square of s, and $M = 4$ when $r = 4$ and $s = 2$,
    (a) write the appropriate equation relating M, r, and s;
    (b) find M when $r = 6$ and $s = 5$.
12. If f varies jointly as u and v, and $f = 36$ when $u = 3$ and $v = 4$,
    (a) write the appropriate equation connecting f, u, and v;
    (b) find f when $u = 5$ and $v = 2$.
13. If T varies jointly as p and the cube of v, and inversely as the square of u, and $T = 24$ when $p = 3$, $v = 2$, and $u = 4$,
    (a) write the appropriate equation connecting T, p, v, and u;
    (b) find T when $p = 2$, $v = 3$, and $u = 36$.
14. If A varies jointly as the square of b and the square of c, and inversely as the cube of d, and $A = 18$ when $b = 4$, $c = 3$, $d = 2$,
    (a) write the appropriate equation relating A, b, c, and d;
    (b) find A when $b = 9$, $c = 4$, and $d = 3$.
15. The distance s an object falls from rest in t seconds varies directly as the square of t. If an object falls 144 feet in 3 seconds,
    (a) how far does it fall in 5 seconds?
    (b) how long does it take to fall 400 feet?
16. In a certain state the income tax paid by a person varies directly as the income. If the tax is \$20 per month when the monthly income is \$1600, find the tax due when the monthly income is \$900.
17. The resistance R of a conductor varies inversely as the area A of its cross section. If $R = 20$ ohms when $A = 8$ square centimeters, find R when $A = 12$ square centimeters.
18. The pressure P of a certain enclosed gas varies directly as the temperature T and inversely as the volume V. Suppose that 300 cubic feet of gas exert a pressure of 20 pounds per square foot when the temperature is 500° K (absolute temperature measured on the Kelvin scale). What is the pressure of this gas when the temperature is lowered to 400° K and the volume is increased to 500 cubic feet?
19. The intensity of illumination I from a source of light varies inversely as the square of the distance d from the source. If the intensity is 200 candlepower when the source is 4 feet away,
    (a) what is the intensity when the source is 6 feet away?
    (b) how close should the source be to provide an intensity of 50 candlepower?
20. The weight of a body in space varies inversely as the square of its distance from the center of the earth. If a body weighs 400 pounds on the surface of the earth, how much does it weigh 1000 miles from the surface of the earth? (Assume that the radius of the earth is 4000 miles.)
21. The equipment cost of a printing job varies jointly as the number of presses and the number of hours that the presses are run. When 4 presses are run for 6 hours, the equipment cost is \$1200. If the equipment cost for 12 hours of running is \$3600, how many presses are being used?
22. The current I in a wire varies directly as the electromotive force E and inversely as the resistance R. In a

wire whose resistance is 10 ohms, a current of 36 amperes is obtained when the electromotive force is 120 volts. Find the current produced when $E = 220$ volts and $R = 30$ ohms.

23. The illumination from a light source varies directly as the intensity of the source and inversely as the square of the distance from the source. If the illumination is 50 candlepower per square foot on a screen 2 feet away from a light source whose intensity is 400 candlepower, what is the illumination 4 feet away from a source whose intensity is 3840 candlepower?

24. If f varies directly as u and inversely as the square of v, what happens to f if both u and v are doubled?

3.6 COMBINING FUNCTIONS; INVERSE FUNCTIONS

Functions such as

$$f(x) = x^2 \qquad g(x) = x - 1$$

can be combined by the usual operations of addition, subtraction, multiplication, and division. Using these functions f and g, we can form

$$(f + g)(x) = f(x) + g(x) = x^2 + x - 1$$
$$(f - g)(x) = f(x) - g(x) = x^2 - (x - 1) = x^2 - x + 1$$
$$(f \cdot g)(x) = f(x) \cdot g(x) = x^2(x - 1) = x^3 - x^2$$
$$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{x^2}{x - 1}$$

In each case, we have combined two functions f and g to form a new function. Note, however, that the domain of the new function need not be the same as the domain of either of the original functions. The function formed by division in the above example has as its domain the set of all real numbers x except $x = 1$, since we cannot divide by 0. On the other hand, the original functions $f(x) = x^2$ and $g(x) = x - 1$ are both defined at $x = 1$.

EXAMPLE 1

Given $f(x) = x - 4$, $g(x) = x^2 - 4$, find the following.

(a) $(f + g)(x)$ (b) $(f - g)(x)$ (c) $(f \cdot g)(x)$

(d) $\left(\frac{f}{g}\right)(x)$ (e) the domain of $\left(\frac{f}{g}\right)(x)$

SOLUTION

(a) $(f + g)(x) = f(x) + g(x) = x - 4 + x^2 - 4 = x^2 + x - 8$

(b) $(f - g)(x) = f(x) - g(x) = x - 4 - (x^2 - 4) = -x^2 + x$

(c) $(f \cdot g)(x) = f(x) \cdot g(x) = (x - 4)(x^2 - 4) = x^3 - 4x^2 - 4x + 16$

(d) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x - 4}{x^2 - 4}$

(e) The domain of $(f/g)(x)$ must exclude values of x for which $x^2 - 4 = 0$. Thus, the domain consists of the set of all real numbers except 2 and -2.

PROGRESS CHECK

Given $f(x) = 2x^2$, $g(x) = x^2 - 5x + 6$, find the following.

(a) $(f+g)(x)$ (b) $(f-g)(x)$ (c) $(f \cdot g)(x)$

(d) $\left(\frac{f}{g}\right)(x)$ (e) the domain of $\left(\frac{f}{g}\right)(x)$

ANSWERS

(a) $3x^2 - 5x + 6$ (b) $x^2 + 5x - 6$

(c) $2x^4 - 10x^3 + 12x^2$ (d) $\dfrac{2x^2}{x^2 - 5x + 6}$

(e) The set of all real numbers except 2 and 3.

COMPOSITE FUNCTION

There is another important way in which two functions f and g can be combined to form a new function. In Figure 34 the function f assigns the value y in set Y to x in set X; then function g assigns the value z in set Z to y in Y. The net effect of this combination of f and g is a new function h, called the **composite function of g and f, $g \circ f$,** which assigns z in Z to x in X. We write the new function as

$$h(x) = (g \circ f)(x) = g[f(x)]$$

which is read "g of f of x."

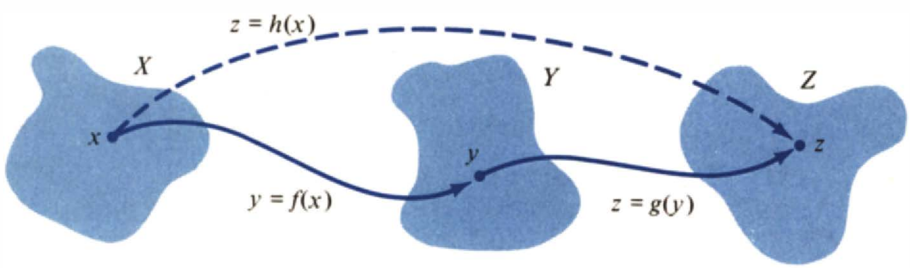


FIGURE 34

EXAMPLE 2

Given $f(x) = x^2$, $g(x) = x - 1$, find the following.

(a) $f[g(3)]$ (b) $g[f(3)]$ (c) $f[g(x)]$ (d) $g[f(x)]$

SOLUTION

(a) We begin by evaluating $g(3)$:

$$g(x) = x - 1$$
$$g(3) = 3 - 1 = 2$$

Therefore,

$$f[g(3)] = f(2)$$

Since

$$f(x) = x^2$$

then

$$f(2) = 2^2 = 4$$

Thus,

$$f[g(3)] = 4$$

(b) Beginning with $f(3)$, we have

$$f(3) = 3^2 = 9$$

Then we find by substituting $f(3) = 9$ that

$$g[f(3)] = g(9) = 9 - 1 = 8$$

(c) Since $g(x) = x - 1$, we make the substitution

$$f[g(x)] = f(x-1) = (x-1)^2 = x^2 - 2x + 1$$

(d) Since $f(x) = x^2$, we make the substitution

$$g[f(x)] = g(x^2) = x^2 - 1$$

Note that $f[g(x)] \neq g[f(x)]$.

PROGRESS CHECK

Given $f(x) = x^2 - 2x$, $g(x) = 3x$, find the following.

(a) $f[g(-1)]$	(b) $g[f(-1)]$	(c) $f[g(x)]$
(d) $g[f(x)]$	(e) $(f \circ g)(2)$	(f) $(g \circ f)(2)$

ANSWERS

(a) 15	(b) 9	(c) $9x^2 - 6x$
(d) $3x^2 - 6x$	(e) 24	(f) 0

ONE-TO-ONE FUNCTIONS

An element in the range of a function may correspond to more than one element in the domain of the function. In Figure 35 we see that y in Y corresponds to both x_1 and x_2 in X. If we demand that every element in the domain be assigned to a *different* element of the range, then the function is called **one-to-one**. More formally:

> A function f is one-to-one if $f(a) = f(b)$ only when $a = b$.

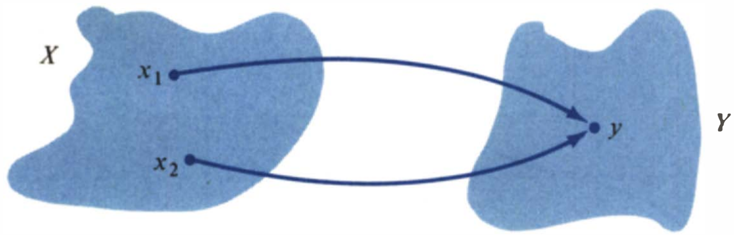


FIGURE 35

There is a simple means of determining if a function f is one-to-one by examining the graph of the function. In Figure 36a we see that a horizontal line meets the graph in more than one point. Thus, $f(a) = f(b)$ although $a \neq b$; hence the function is not one-to-one. On the other hand, no horizontal line meets the graph in Figure 36b in more than one point; the graph thus determines a one-to-one function. In summary, we have the following test.

Horizontal Line Test	If no horizontal line meets the graph of a function in more than one point, then the function is one-to-one.

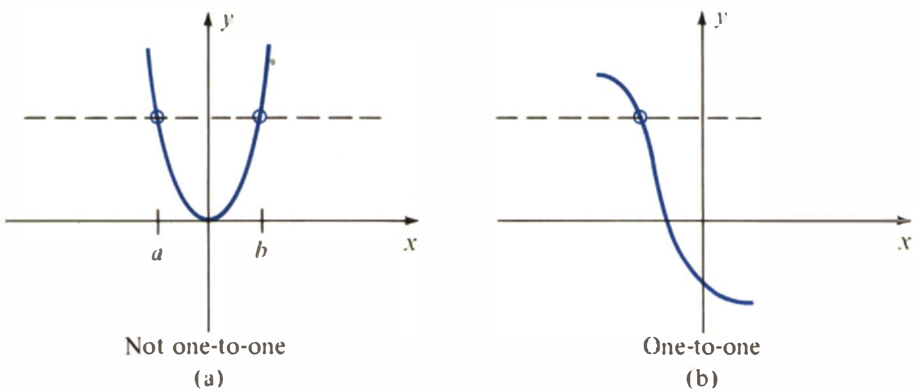


FIGURE 36

EXAMPLE 3

Which of the graphs in Figure 37 are graphs of one-to-one functions?

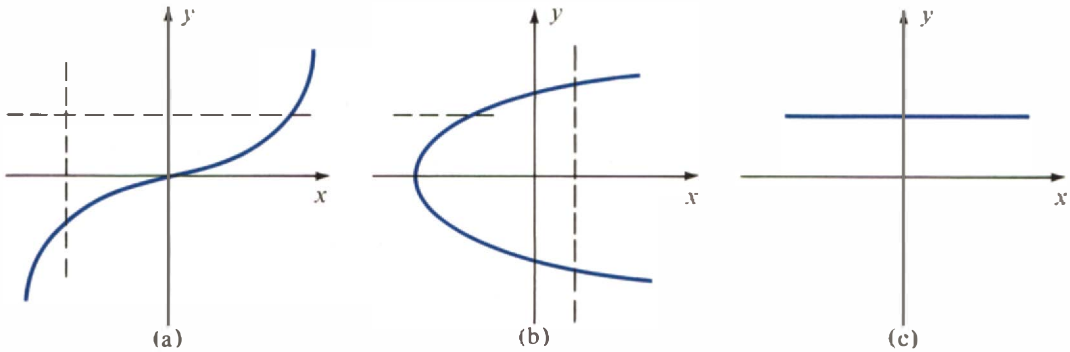


FIGURE 37

SOLUTION

(a) No *vertical* line meets the graph in more than one point; hence, it is the graph of a function. No *horizontal* line meets the graph in more than one point; hence, it is the graph of a one-to-one function.

(b) No *horizontal* line meets the graph in more than one point. But *vertical* lines do meet the graph in more than one point. It is therefore not the graph of a function and consequently cannot be the graph of a one-to-one function.

(c) No *vertical* line meets the graph in more than one point; hence, it is the graph of a function. But a *horizontal* line does meet the graph in more than one point. This is the graph of a function but not of a one-to-one function.

PROGRESS CHECK

Which of the graphs in Figure 38 are graphs of one-to-one functions?

ANSWER

(b)

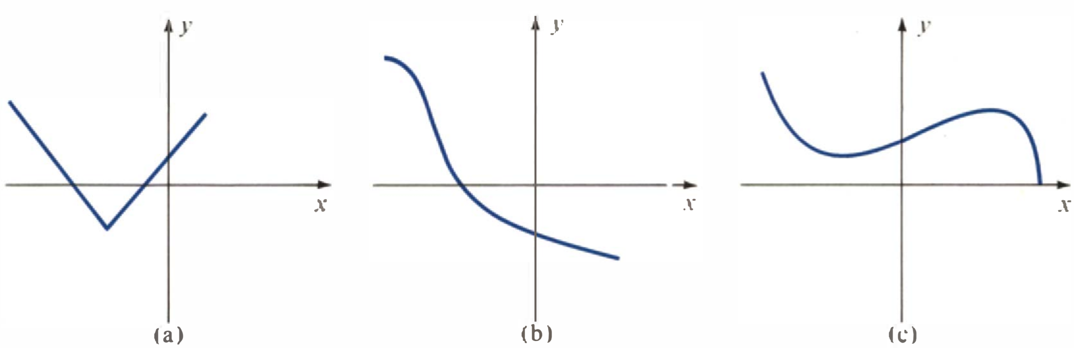


FIGURE 38

INVERSE FUNCTIONS

Suppose the function f in Figure 39a is a one-to-one function and that $y = f(x)$. Since f is one-to-one, we know that the correspondence is unique; that is, x in X is the *only* element of the domain for which $y = f(x)$. It is then possible to define a function g (Figure 39b) with domain Y and range X that reverses the correspondence, that is,

$$g(y) = x \qquad \text{for every } x \text{ in } X$$

If we substitute $y = f(x)$, we have

$$g[f(x)] = x \qquad \text{for every } x \text{ in } X \tag{1}$$

Substituting $g(y) = x$ in the equation $f(x) = y$ yields

$$f[g(y)] = y \qquad \text{for every } y \text{ in } Y \tag{2}$$

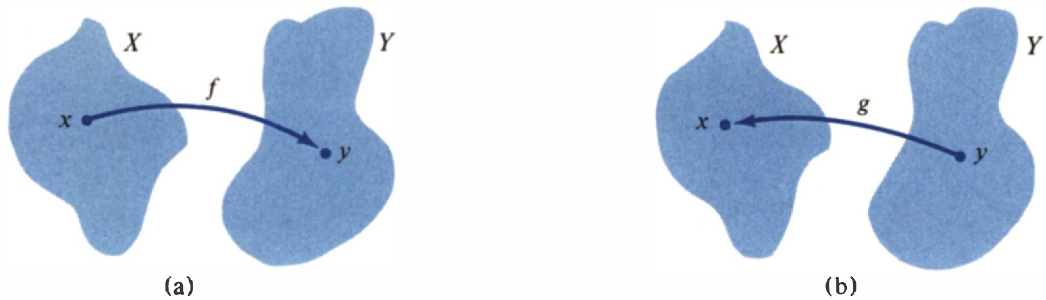


FIGURE 39

The functions f and g of Figure 39 are therefore seen to satisfy the properties of Equations (1) and (2). Such functions are called inverse functions.

Inverse Functions

If f is a one-to-one function with domain X and range Y, then the function g with domain Y and range X satisfying

$$g[f(x)] = x \quad \text{for every } x \text{ in } X$$
$$f[g(y)] = y \quad \text{for every } y \text{ in } Y$$

is called an **inverse function** of f.

It is not difficult to show that the inverse of a one-to-one function is unique (see Exercise 61).

Since the inverse (reciprocal) $1/x$ of a real number $x \neq 0$ can be written as x^{-1}, it is natural to write the inverse of a function f as f^{-1}. Thus we have

$$f^{-1}[f(x)] = x \quad \text{for every } x \text{ in } X$$
$$f[f^{-1}(y)] = y \quad \text{for every } y \text{ in } Y$$

See Figure 40 for a graphical representation.

In the following chapter we will study a very important class of inverse functions, the exponential and logarithmic functions. Always remember that *we can define the inverse function of f only if f is one-to-one.*

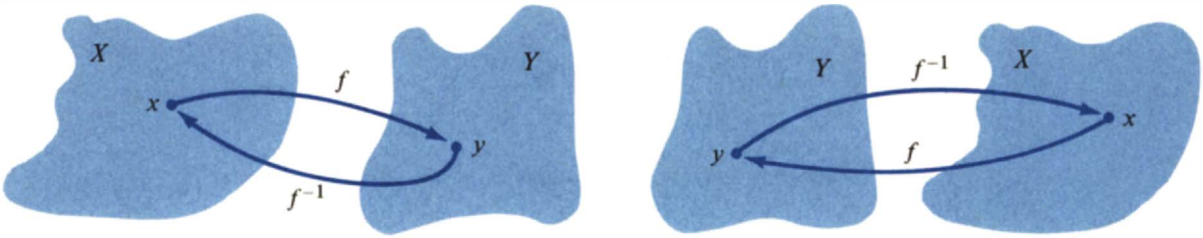


FIGURE 40

EXAMPLE 4

Let f be the function defined by

$$f(x) = x^2 - 4, \qquad x \geq 0$$

Verify that the inverse of f is given by

$$f^{-1}(x) = \sqrt{x + 4}$$

SOLUTION

We must verify that $f[f^{-1}(x)] = x$ and $f^{-1}[f(x)] = x$. Thus,

$$\begin{aligned} f[f^{-1}(x)] &= f(\sqrt{x + 4}) \\ &= (\sqrt{x + 4})^2 - 4 \\ &= x + 4 - 4 = x \end{aligned}$$

and

$$\begin{aligned} f^{-1}[f(x)] &= f^{-1}(x^2 - 4) \\ &= \sqrt{(x^2 - 4) + 4} \\ &= \sqrt{x^2} = |x| \end{aligned}$$

Since $x \geq 0$,

$$f^{-1}[f(x)] = |x| = x$$

We have verified that the equations defining inverse functions hold, and we conclude that the inverse of f is as given. The student should verify (a) that the domain of f is the set of nonnegative real numbers and the range of f is the set of all real numbers in the interval $[-4, \infty)$; (b) that the domain of f^{-1} is the range of f and the range of f^{-1} is the domain of f.

We may also think of the function f defined by $y = f(x)$ as the set of all ordered pairs $(x, f(x))$, where x assumes all values in the domain of f. Since the inverse function reverses the correspondence, the function f^{-1} is the set of all ordered pairs $(f(x), x)$, where $f(x)$ assumes all values in the range of f. With this approach, we see that the graphs of inverse functions are related in a distinct manner. First, note that the points (a, b) and (b, a) in Figure 41a are located symmetrically with respect to the graph of the line $y = x$. That is, if we fold the paper along the line $y = x$, the two points will coincide. And if (a, b) lies on the graph of a function f, then (b, a) must lie on the graph of f^{-1}. Thus, the graphs of a pair of inverse functions are reflections of each other about the line $y = x$. In Figure 41b we have sketched the graphs of the functions from Example 4 on the same coordinate axes to demonstrate this interesting relationship.

It is sometimes possible to find an inverse by algebraic methods, as is shown by the following example.

EXAMPLE 5

Find the inverse function of $f(x) = 2x - 3$.

SOLUTION

By definition, $f[f^{-1}(x)] = x$. Then we must have

$$f[f^{-1}(x)] = 2[f^{-1}(x)] - 3 = x$$

$$f^{-1}(x) = \frac{x + 3}{2}$$

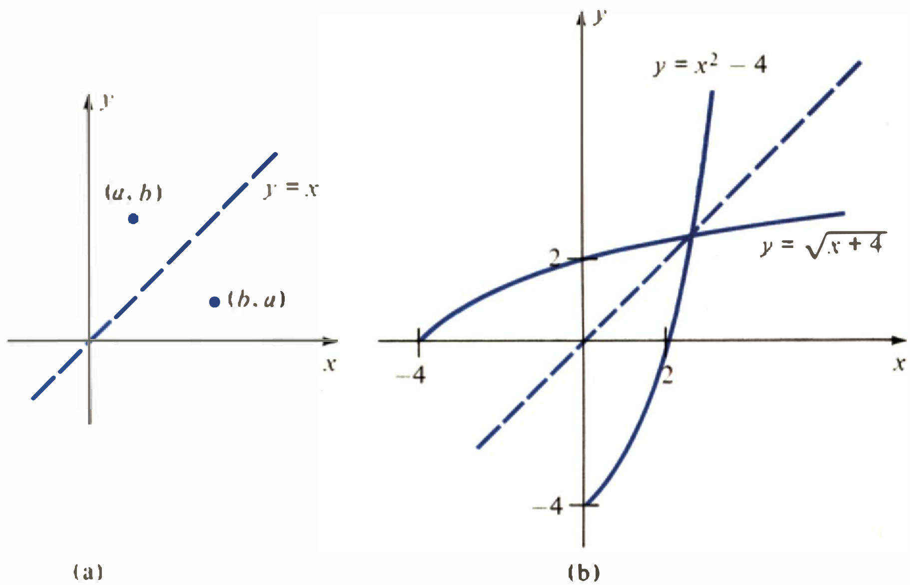


FIGURE 41

We then verify that $f^{-1}[f(x)] = x$:

$$f^{-1}[f(x)] = \frac{2x - 3 + 3}{2} = x$$

PROGRESS CHECK

Given $f(x) = 3x + 5$, find f^{-1}.

ANSWER

$f^{-1}(x) = \dfrac{x - 5}{3}$

WARNING

(a) In general, $f^{-1}(x) \neq \dfrac{1}{f(x)}$. If $g(x) = x - 1$, then

$$g^{-1}(x) \neq \frac{1}{x - 1}$$

Use the methods of this section to show that

$$g^{-1}(x) = x + 1$$

(b) The inverse function notation is *not* to be thought of as a power.

EXERCISE SET 3.6

In Exercises 1–10 $f(x) = x^2 + 1$ and $g(x) = x - 2$. Determine the following.

1. $(f + g)(x)$
2. $(f + g)(2)$
3. $(f - g)(x)$
4. $(f - g)(3)$
5. $(f \cdot g)(x)$
6. $(f \cdot g)(-1)$
7. $\left(\dfrac{f}{g}\right)(x)$
8. $\left(\dfrac{f}{g}\right)(-2)$
9. the domains of f and of g
10. the domains of $\dfrac{f}{g}$ and of $\dfrac{g}{f}$

In Exercises 11–18 $f(x) = 2x + 1$ and $g(x) = 2x^2 + x$. Determine the following.

11. $(f \circ g)(x)$
12. $(g \circ f)(x)$
13. $(f \circ g)(2)$
14. $(g \circ f)(3)$
15. $(f \circ g)(x + 1)$
16. $(f \circ f)(-2)$
17. $(g \circ f)(x - 1)$
18. $(g \circ g)(x)$

In Exercises 19–24 $f(x) = x^2 + 4$ and $g(x) = \sqrt{x + 2}$. Determine the following.

19. $(f \circ g)(x)$
20. $(g \circ f)(x)$
21. $(f \circ f)(-1)$
22. the domain of $(f \circ g)(x)$
23. the domain of $(g \circ f)(x)$
24. the domain of $(g \circ g)(x)$

In Exercises 25–28 determine $(f \circ g)(x)$ and $(g \circ f)(x)$.

25. $f(x) = x - 1$, $g(x) = x + 2$
26. $f(x) = \sqrt{x + 1}$, $g(x) = x + 2$
27. $f(x) = \dfrac{1}{x + 1}$, $g(x) = \dfrac{1}{x - 1}$
28. $f(x) = \dfrac{x + 1}{x - 1}$, $g(x) = x$

In Exercises 29–38 write the given function $h(x)$ as a composite of two functions f and g so that $h(x) = (f \circ g)(x)$. (There may be more than one answer.)

29. $h(x) = x^2 + 3$

30. $h(x) = \dfrac{1}{x+2}$

31. $h(x) = (3x + 2)^8$

32. $h(x) = (x^3 + 2x^2 + 1)^{15}$

33. $h(x) = (x^3 - 2x^2)^{1/3}$

34. $h(x) = \left(\dfrac{x^2 + 2x}{x^3 - 1}\right)^{3/2}$

35. $h(x) = |x^2 - 4|$

36. $h(x) = |x^2 + x| - 4$

37. $h(x) = \sqrt{4 - x}$

38. $h(x) = \sqrt{2x^2 - x + 2}$

In Exercises 39–44 verify that $g = f^{-1}$ for the given functions f and g by showing that $f[g(x)] = x$ and $g[f(x)] = x$.

39. $f(x) = 2x + 4 \qquad g(x) = \dfrac{1}{2}x - 2$

40. $f(x) = 3x - 2 \qquad g(x) = \dfrac{1}{3}x + \dfrac{2}{3}$

41. $f(x) = 2 - 3x \qquad g(x) = -\dfrac{1}{3}x + \dfrac{2}{3}$

42. $f(x) = x^3 \qquad g(x) = \sqrt[3]{x}$

43. $f(x) = \dfrac{1}{x} \qquad g(x) = \dfrac{1}{x}$

44. $f(x) = \dfrac{1}{x-2} \qquad g(x) = \dfrac{1}{x} + 2$

In Exercises 45–52 find $f^{-1}(x)$. Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same coordinate axes.

45. $f(x) = 2x + 3$

46. $f(x) = 3x - 4$

47. $f(x) = 3 - 2x$

48. $f(x) = \dfrac{1}{2}x + 1$

49. $f(x) = \dfrac{1}{3}x - 5$

50. $f(x) = 2 - \dfrac{1}{5}x$

51. $f(x) = x^3 + 1$

52. $f(x) = \dfrac{1}{x+1}$

In Exercises 53–60 use the horizontal line test to determine whether the given function is a one-to-one function.

53. $f(x) = 2x - 1$

54. $f(x) = 3 - 5x$

55. $f(x) = x^2 - 2x + 1$

56. $f(x) = x^2 + 4x + 4$

57. $f(x) = -x^3 + 1$

58. $f(x) = x^3 - 2$

59. $f(x) = \begin{cases} 2x, & x \le -1 \\ x^2, & -1 < x \le 0 \\ 3x - 1, & x > 0 \end{cases}$

60. $f(x) = \begin{cases} x^2 - 4x + 4, & x \le 2 \\ x, & x > 2 \end{cases}$

61. Prove that a one-to-one function can have at most one inverse function. (*Hint:* Assume that the functions g and h are both inverses of the function f. Show that $g(x) = h(x)$ for all real values x in the range of f.)

62. Prove that the linear function $f(x) = ax + b$ is a one-to-one function if $a \neq 0$, and is not a one-to-one function if $a = 0$.

63. Find the inverse of the linear function $f(x) = ax + b$, $a \neq 0$.

TERMS AND SYMBOLS

origin (p. 114)
x-axis (p. 114)
y-axis (p. 114)
coordinate axes (p. 114)
rectangular coordinate system (p. 114)
Cartesian coordinate system (p. 114)
quadrant (p. 114)
coordinates of a point (p. 114)
ordered pair (p. 114)
abscissa (p. 114)
x-coordinate (p. 114)
ordinate (p. 114)
y-coordinate (p. 114)
distance formula (p. 116)
graph of an equation in two variables (p. 117)
solution of an equation in two variables (p. 117)
x-intercept (p. 118)
y-intercept (p. 118)
symmetry (p. 119)
symmetry with respect to the x-axis (p. 120)
symmetry with respect to the y-axis (p. 120)
symmetry with respect to the origin (p. 120)
function (p. 124)
image (p. 124)

domain (p. 124)
range (p. 124)
independent variable (p. 124)
dependent variable (p. 124)
vertical line test (p. 126)
$f(x)$ (p. 127)
graph of a function (p. 130)
increasing function (p. 136)
decreasing function (p. 136)
constant function (p. 136)
linear function (p. 138)
quadratic function (p. 138)
parabola (p. 138)
polynomial function (p. 139)
slope (p. 141)
point-slope form (p. 143)
slope-intercept form (p. 144)
general first-degree equation (p. 149)
direct variation (p. 151)
constant of variation (p. 152)
inverse variation (p. 152)
joint variation (p. 153)
composite function (p. 157)
$f[g(x)]$ (p. 157)
$f \circ g$ (p. 157)
one-to-one function (p. 158)
horizontal line test (p. 159)
inverse function (p. 161)
f^{-1} (p. 161)

KEY IDEAS FOR REVIEW

- ☐ In a rectangular coordinate system, every ordered pair of real numbers (a, b) corresponds to a point in the plane, and every point in the plane corresponds to an ordered pair of real numbers.
- ☐ The distance $\overline{PQ}$ between points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by the distance formula

$$\overline{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- ☐ An equation in two variables can be graphed by plotting points that satisfy the equation and joining the points to form a smooth curve.
- ☐ A function is a rule that assigns exactly one element y of a set Y to each element x of a set X. The domain is the set of inputs, and the range is the set of outputs.
- ☐ A graph represents a function if no vertical line meets the graph in more than one point.
- ☐ The domain of a function is the set of all real numbers for which the function is defined. Beware of division by zero and of even roots of negative numbers.
- ☐ Function notation gives the definition of the function and also the value or expression at which to evaluate the function. If the function f is defined by $f(x) = x^2 + 2x$, then the notation $f(3)$ denotes the result of replacing the independent variable x by 3 wherever it appears:

$$f(x) = x^2 + 2x$$
$$f(3) = 3^2 + 2(3) = 15$$

- ☐ To graph $f(x)$, simply graph $y = f(x)$.
- ☐ An equation is not the only way to define a function. Sometimes a function is defined by a table or chart, or by several equations. Moreover, not every equation determines a function.
- ☐ As we move from left to right, the graph of an increasing function rises and the graph of a decreasing function falls. The graph of a constant function neither rises nor falls; it is horizontal.
- ☐ The graph of a function can have holes or gaps, and can be defined in "pieces."
- ☐ Polynomials in one variable are all functions and have "smooth" curves as their graphs.
- ☐ The graph of the linear function $f(x) = ax + b$ is a straight line.
- ☐ Any two points on a line can be used to find its slope m:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- ☐ Positive slope indicates that a line is rising; negative slope indicates that a line is falling.
- ☐ The slope of a horizontal line is 0; the slope of a vertical line is undefined.
- ☐ The point-slope form of a line is $y - y_1 = m(x - x_1)$.
- ☐ The slope-intercept form of a line is $y = mx + b$.
- ☐ The equation of the horizontal line through the point (a, b) is $y = b$; the equation of the vertical line through the point (a, b) is $x = a$.
- ☐ Parallel lines have the same slope.
- ☐ The slopes of perpendicular lines are negative reciprocals of each other.
- ☐ The graphs of the linear function $f(x) = ax + b$ and of the general first-degree equation $Ax + By = C$ are always straight lines.
- ☐ Direct and inverse variation are functional relationships.
- ☐ We say that y varies directly as x if $y = kx$ for some constant k; we say that y varies inversely as x if $y = k/x$ for some constant k.
- ☐ Joint variation is a term for direct variation involving more than two quantities.

- ☐ Functions can be combined by the usual operations of addition, subtraction, multiplication, and division. However, the domain of the resulting function need not correspond with the domain of either of the original functions.
- ☐ A composite function is a function of a function.
- ☐ We say a function is one-to-one if every element of the range corresponds to precisely one element of the domain.
- ☐ No horizontal line meets the graph of a one-to-one function in more than one point.
- ☐ The inverse of a function, f^{-1}, reverses the correspondence defined by the function f. The domain of f becomes the range of f^{-1}, and the range of f becomes the domain of f^{-1}.
- ☐ The inverse of a function f is defined only if f is a one-to-one function.

REVIEW EXERCISES

Solutions to exercises whose numbers are in color are in the Solutions section in the back of the book.

3.1 1. Find the distance between the points $(-4, -6)$ and $(2, -1)$.

2. Find the length of the longest side of the triangle whose vertices are $A(3, -4)$, $B(-2, -6)$, and $C(-1, 2)$.

In Exercises 3 and 4 sketch the graph of the given equation by forming a table of values.

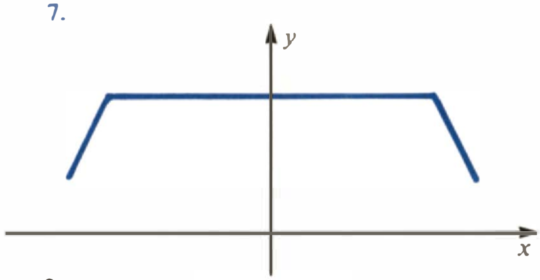
3. $y = 1 - |x|$
4. $y = \sqrt{x - 2}$

In Exercises 5 and 6 analyze the given equation for symmetry with respect to the x-axis, y-axis, and origin.

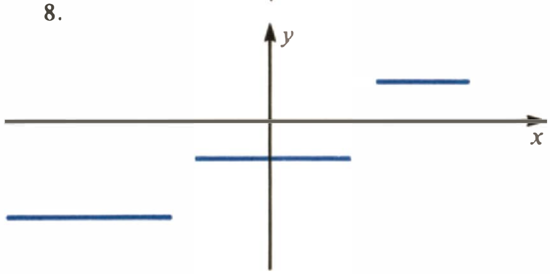
5. $y^2 = 1 - x^3$
6. $y^2 = \dfrac{x^2}{x^2 - 5}$

3.2 In Exercises 7 and 8 state if the graph determines y to be a function of x.

7.



8.



In Exercises 9 and 10 determine the domain of the given function.

9. $f(x) = \sqrt{3x - 5}$
10. $f(x) = \dfrac{x}{x^2 + 2x + 1}$

11. If $f(x) = \sqrt{x - 1}$, find a real number whose image is 15.

12. If $f(t) = t^2 + 1$, find a real number whose image is 10.

In Exercises 13–15 $f(x) = x^2 - x$. Evaluate the following.

13. $f(-3)$
14. $f(y - 1)$
15. $\dfrac{f(2 + h) - f(2)}{h}$, $h \neq 0$

3.3 Exercises 16–19 refer to the function f defined by

$$f(x) = \begin{cases} x - 1, & x \leq -1 \\ x^2, & -1 < x \leq 2 \\ -2, & x > 2 \end{cases}$$

16. Sketch the graph of the function f.
17. Determine where the function f is increasing, decreasing, and constant.
18. Evaluate $f(-4)$.
19. Evaluate $f(4)$.

3.4 In Exercises 20–25 the points A and B have coordinates $(-4, -6)$ and $(-1, 3)$, respectively.

20. Find the slope of the line through A and B.
21. Find an equation of the line through the points A and B.
22. Find an equation of the line through A that is parallel to the y-axis.

23. Find an equation of the horizontal line through B.

24. Find an equation of the line through A that is parallel to the line $2x - y - 3 = 0$.

25. Find an equation of the line through B that is perpendicular to the line $2y + x - 5 = 0$.

3.5 26. If R varies directly as q, and if $R = 20$ when $q = 5$, find R when $q = 40$.

27. If S varies inversely as the cube of t, and if $S = 8$ when $t = -1$, find S when $t = -2$.

28. P varies jointly as q and r and inversely as the square of t, and $P = -3$ when $q = 2$, $r = -3$, and $t = 4$. Find P when $q = -1$, $r = \frac{1}{2}$, and $t = 4$.

3.6 In Exercises 29–34 $f(x) = x + 1$ and $g(x) = x^2 - 1$. Determine the following.

29. $(f + g)(x)$

30. $(f \cdot g)(-1)$

31. $\left(\frac{f}{g}\right)(x)$

32. the domain of $\left(\frac{f}{g}\right)(x)$

33. $(g \circ f)(x)$

34. $(f \circ g)(2)$

In Exercises 35–38 $f(x) = \sqrt{x} - 2$ and $g(x) = x^2$. Determine the following.

35. $(f \circ g)(x)$

36. $(g \circ f)(x)$

37. $(f \circ g)(-2)$

38. $(g \circ f)(-2)$

In Exercises 39 and 40 $f(x) = 2x + 4$ and $g(x) = \frac{x}{2} - 2$.

39. Prove that f and g are inverse functions of each other.

40. Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the same coordinate axes.

PROGRESS TEST 3A

1. Find the perimeter of the triangle whose vertices are $(2, 5)$, $(-3, 1)$, and $(-3, 4)$.

2. Use symmetry to assist in sketching the graph of the equation $y = 2x^2 - 1$.

3. Analyze the equation $y = 1/x^3$ for symmetry with respect to the axes and origin.

4. Determine the domain of the function

$$f(x) = \frac{1}{\sqrt{x} - 1}$$

5. If $f(x) = \sqrt{x - 1}$, find a real number whose image is 4.

6. If $f(x) = 2x^2 + 3$, find $f(2t)$.

Problems 7–9 refer to the function f defined by

$$f(x) = \begin{cases} 0, & x < -2 \\ |x|, & -2 \le x \le 3 \\ x^2 - x, & x > 3 \end{cases}$$

7. Determine where the function f is increasing, decreasing, and constant.

8. Evaluate $f(-5)$.

9. Evaluate $f(-2)$.

10. Find an equation of the line through the points $(-3, 5)$ and $(-5, 2)$.

11. Find an equation of the vertical line through the point $(-3, 4)$.

12. Find the slope m and y-intercept b of the line whose equation is $2y - x = 4$.

13. Find an equation of the line through the point $(4, -1)$ that is parallel to the x-axis.

14. Find an equation of the line that passes through the point $(-2, 3)$ and is perpendicular to the line

$$y - 3x - 2 = 0$$

15. If h varies directly as the cube of r, and $h = 2$ when $r = -\frac{1}{2}$, find h when $r = 4$.

16. T varies jointly as a and the square of b and inversely as the cube of c, and $T = 64$ when $a = -1$, $b = \frac{1}{2}$, and $c = 2$. Find T when $a = 2$, $b = 4$, and $c = -1$.

In Problems 17–19 $f(x) = 1/(x - 1)$ and $g(x) = x^2$. Find the following.

17. $(f - g)(2)$

18. $\left(\frac{g}{f}\right)(x)$

19. $(g \circ f)(3)$

20. Prove that $f(x) = -3x + 1$ and $g(x) = -\frac{1}{3}(x - 1)$ are inverse functions of each other.

PROGRESS TEST 3B

1. Find the length of the shorter diagonal of the parallelogram whose vertices are $(-3, 2)$, $(-5, -4)$, $(3, -4)$, and $(5, 2)$.
2. Use symmetry to assist in sketching the graph of the equation $y^2 = -4x + 4$.
3. Analyze the equation $x^2 - xy + 2 = 0$ for symmetry with respect to the axes and the origin.
4. Determine the domain of the function
$$f(x) = \frac{x^2}{16 - x^2}$$
5. If $f(x) = x^2 - 2x$, find a real number whose image is -1.
6. If $f(x) = \sqrt{x} - 1$, find $f(4)$.

Problems 7–9 refer to the function f defined by

$$f(x) = \begin{cases} x^2 - 1, & x \le -3 \\ 10, & -3 < x \le 3 \\ \sqrt{x}, & x > 3 \end{cases}$$

7. Determine where the function f is increasing, decreasing, and constant.
8. Evaluate $f(2)$.
9. Evaluate $f(-5)$.
10. Find the slope of the line through the points $(-2, -3)$ and $(-4, 6)$.
11. Find an equation of the horizontal line through the point $(-6, -5)$.
12. Find the y-intercept of the line through the points $(4, -3)$ and $(-1, 2)$.
13. Determine the slope of every line that is perpendicular to the line $6y - 2x = 5$.
14. Determine an equation of the line that passes through the point $(3, -2)$ and is parallel to the line
$$3y + x - 4 = 0$$
15. If A varies inversely as the square of b, and if $A = -2$ when $b = 4$, find A when $b = 3$.
16. R varies jointly as x and the square root of y and inversely as the square of z, and $R = \frac{3}{8}$ when $x = 2$, $y = 9$, and $z = 4$. Find the constant of variation.

In Problems 17–19 $f(x) = 1/\sqrt{x + 1}$ and $g(x) = x - 1$. Find the following.

17. $(f \circ g)(3)$
18. $(f + g)(1)$
19. $(f \circ g)(x)$
20. Prove that $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1}{x}$ are inverse functions.

4

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Thus far in our study of algebra we have dealt primarily with functions that are polynomials, or sums, differences, products, quotients, or powers of polynomials. In this chapter we introduce a new type of function, the exponential function, and its inverse, the logarithmic function.

Exponential functions arise in nature and are useful in chemistry, biology, and economics, as well as in mathematics and engineering. We will study applications of exponential functions in calculating such quantities as compound interest and the growth rate of bacteria in a culture medium.

Logarithms can be viewed as another way of writing exponents. Historically, logarithms have been used to simplify calculations; in fact, the slide rule, a device long used by engineers, is based on logarithmic scales. In today's world of inexpensive hand calculators, the need for manipulating logarithms is reduced. The section on computing with logarithms will provide enough background to allow you to use this powerful tool but omits some of the detail found in older textbooks.

4.1 EXPONENTIAL FUNCTIONS

The function $f(x) = 2^x$ is very different from any of the functions we have worked with thus far. Previously we defined functions by using the basic algebraic operations (addition, subtraction, multiplication, division, powers, and roots). However, $f(x) = 2^x$ has a variable in the exponent and doesn't fall into the class of algebraic functions. Rather, it is our first example of an exponential function.

An **exponential function** has the form

$$f(x) = a^x$$

where $a > 0$, $a \neq 1$. The constant a is called the **base,** and the independent variable x may assume any real value.

GRAPHS OF EXPONENTIAL FUNCTIONS

The best way to become familiar with exponential functions is to sketch their graphs.

EXAMPLE 1
Sketch the graph of $f(x) = 2^x$.

SOLUTION
We let $y = 2^x$, and we form a table of values of x and y. Then we plot these points and sketch the smooth curve as in Figure 1. Note that the x-axis is a horizontal asymptote.

In a sense, we have cheated in our definition of $f(x) = 2^x$ and in sketching the graph in Figure 1. Since we have not explained the meaning of 2^x when x is irrational, we have no right to plot values such as $2^{\sqrt{2}}$. For our purposes, however, it will be adequate to think of $2^{\sqrt{2}}$ as the value we approach by taking successively closer approximations to $\sqrt{2}$, such as $2^{1.4}$, $2^{1.41}$, $2^{1.414}$, ... A precise definition is given in more advanced mathematics courses, where it is also shown that the laws of exponents hold for irrational exponents.

We now look at $f(x) = a^x$ when $0 < a < 1$.

EXAMPLE 2
Sketch the graph of $f(x) = (\frac{1}{2})^x = 2^{-x}$.

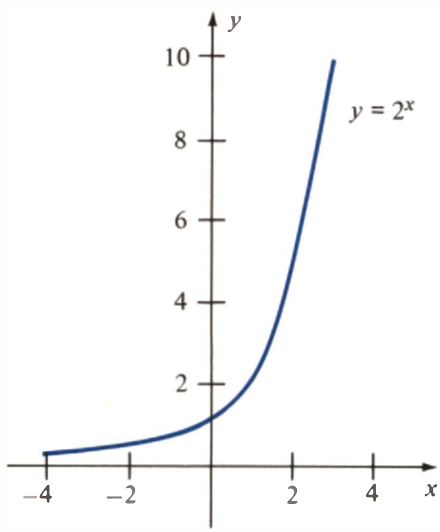


FIGURE 1

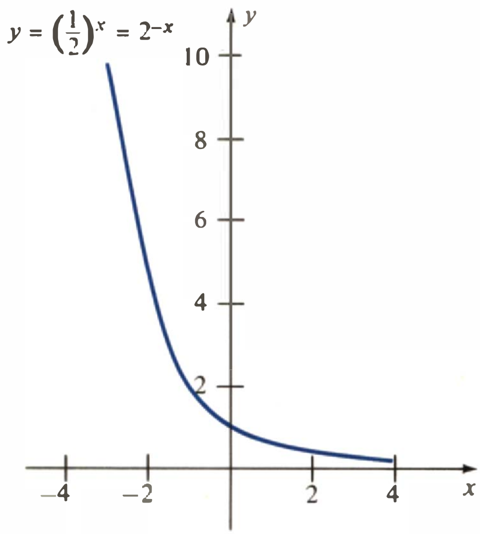


FIGURE 2

SOLUTION

We form a table, plot points, and sketch the graph shown in Figure 2. Note that the graph of $y = 2^{-x}$ is a reflection about the y-axis of the graph of $y = 2^x$.

In Figure 3 we have sketched the graphs of

$$f(x) = 2^x \qquad g(x) = 3^x \qquad h(x) = \left(\frac{1}{2}\right)^x \qquad k(x) = \left(\frac{1}{3}\right)^x$$

on the same coordinate axes to provide additional examples of the graphs of exponential functions.

PROPERTIES OF THE EXPONENTIAL FUNCTIONS

The graphs in Figure 3 illustrate the following important properties of the exponential functions. (Recall that the definition of the exponential function $f(x) = a^x$ requires that $a > 0$ and $a \neq 1$.)

Properties of the Exponential Functions

- The graph of $f(x) = a^x$ always passes through the point (0, 1), since $a^0 = 1$.
- The domain of $f(x) = a^x$ consists of the set of all real numbers; the range is the set of all positive real numbers.
- If $a > 1$, a^x is an increasing function; if $a < 1$, a^x is a decreasing function.
- If $a < b$, then $a^x < b^x$ for all $x > 0$, and $a^x > b^x$ for all $x < 0$. Note in Figure 3 that $y = 3^x$ lies above $y = 2^x$ when $x > 0$ and below it when $x < 0$.

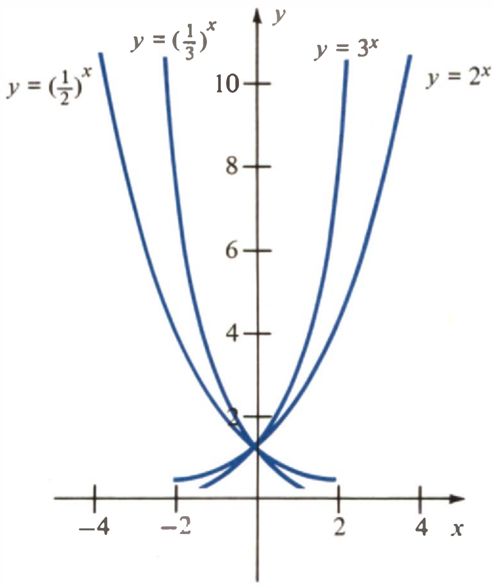


FIGURE 3

Since a^x is either increasing or decreasing, it never assumes the same value twice. (Recall that $a \neq 1$.) This leads to a useful conclusion.

> If $a^u = a^v$, then $u = v$.

The graphs of a^x and b^x intersect only at $x = 0$. This observation provides us with the following result.

> If $a^u = b^u$ for $u \neq 0$, then $a = b$.

EXAMPLE 3

Solve for x.

(a) $3^{10} = 3^{5x}$ (b) $2^7 = (x - 1)^7$ (c) $3^{3x} = 9^{x-1}$

SOLUTION

(a) Since $a^u = a^v$ implies $u = v$, we have

$$10 = 5x$$
$$2 = x$$

(b) Since $a^u = b^u$ implies $a = b$, we have

$$2 = x - 1$$
$$3 = x$$

(c) $$3^{3x} = 9^{x-1} = (3^2)^{x-1} = 3^{2x-2}$$

Since $a^u = a^v$ implies $u = v$, we have

$$3x = 2x - 2$$
$$x = -2$$

PROGRESS CHECK

Solve for x.

(a) $2^8 = 2^{x+1}$ (b) $4^{2x+1} = 4^{11}$ (c) $8^{x+1} = 2$

ANSWERS

(a) 7 (b) 5 (c) $-\frac{2}{3}$

THE NUMBER e

There is an irrational number that was first designated by the letter e by the Swiss mathematician Leonhard Euler (1707–1783). The number e is the value approached by the expression

$$\left(1 + \frac{1}{m}\right)^m$$

as m gets larger. The procedure for studying the behavior of this expression as m gets larger and larger is developed in calculus courses. We will simply evaluate this expression for different values of m, as shown in Table 1.

TABLE 1

m	1	2	10	100	1000	10,000	100,000	1,000,000
$\left(1 + \frac{1}{m}\right)^m$	2.0	2.25	2.5937	2.7048	2.7169	2.7181	2.7182	2.71828

The function $f(x) = e^x$ is called the **natural exponential function;** we assume that this is the function referred to when someone speaks of "the exponential function." The graphs of $f(x) = e^x$ and $f(x) = e^{-x}$ are shown in Figure 4. Since $e \approx 2.71828$, the graph of $y = e^x$ falls between the graphs of $y = 2^x$ and $y = 3^x$. Table I in the Tables Appendix lists values for e^x and e^{-x}.

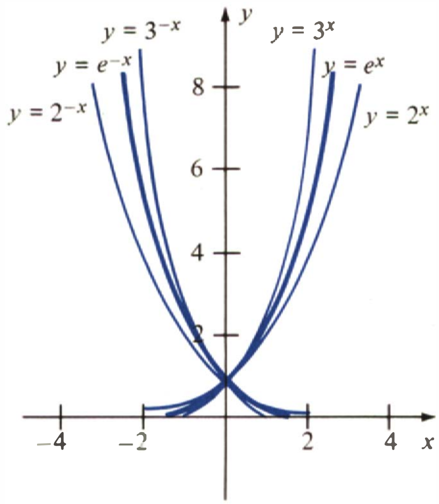


FIGURE 4

APPLICATIONS

Exponential functions occur in a wide variety of applied problems. We will look at problems dealing with population growth, such as predicting the growth of bacteria in a culture medium; radioactive decay, such as determining the half-life of strontium 90; and the interest earned when an interest rate is compounded.

Exponential Growth

The function Q defined by

$$Q(t) = q_0 e^{kt}, \qquad k > 0$$

in which the variable t represents time, is called an **exponential growth model;** k is a constant and t is the independent variable. We may think of Q as the quantity of a substance available at any given time t. Note that when $t = 0$ we have

$$Q(0) = q_0 e^0 = q_0$$

which says that q_0 is the initial quantity. (It is customary to use the subscript 0 to denote an initial value.) The constant k is called the **growth constant.**

EXAMPLE 4

The number of bacteria in a culture after t hours is described by the exponential growth model

$$Q(t) = 50e^{0.7t}$$

(a) Find the initial number of bacteria, q_0, in the culture.
(b) How many bacteria are in the culture after 10 hours?

SOLUTION

(a) To find q_0 we need to evaluate $Q(t)$ at $t = 0$:

$$Q(0) = 50e^{0.7(0)} = 50e^0 = 50 = q_0$$

Thus, there are initially 50 bacteria in the culture.
(b) The number of bacteria in the culture after 10 hours is given by

$$Q(10) = 50e^{0.7(10)} = 50e^{7} = 50(1096.6) = 54{,}830$$

Thus, there are 54,830 bacteria after 10 hours. (The value $e^7 = 1096.6$ can be found by using Table I in the Tables Appendix; it can also be found by using a calculator with a "y^x" key, with $y = e = 2.71828$ and $x = 7$.)

PROGRESS CHECK
The number of bacteria in a culture after t minutes is described by the exponential growth model $Q(t) = q_0e^{0.005t}$. If there were 100 bacteria present initially, how many bacteria will be present after 1 hour has elapsed?

ANSWER
135

Exponential Decay

The model defined by the function

$$Q(t) = q_0e^{-kt}, \qquad k > 0$$

is called an **exponential decay model;** k is a constant, called the **decay constant,** and t is the independent variable denoting time. Here is an application of this model.

EXAMPLE 5
A substance has a decay rate of 5% per hour. If 500 grams are present initially, how much of the substance remains after 4 hours?

SOLUTION
The general equation of an exponential decay model is

$$Q(t) = q_0e^{-kt}$$

In our model, $q_0 = 500$ grams (since the quantity available initially is 500 grams), and $k = 0.05$ (since the decay rate is 5% per hour). After 4 hours

$$Q(4) = 500e^{-0.05(4)} = 500e^{-0.2} = 500(0.8187) = 409.4$$

(The value $e^{-0.2} = 0.8187$ is obtained from Table I in the Tables Appendix). Thus, there remain 409.4 grams of the substance.

PROGRESS CHECK
The number of grams Q of a certain radioactive substance present after t seconds is given by the exponential decay model $Q(t) = q_0e^{-0.4t}$. If 200 grams of the substance are present initially, find how much remains after 6 seconds.

ANSWER
18.1 grams

Compound Interest

We begin by recalling the definition of simple interest. If the principal P is invested at a simple annual interest rate r, then the amount or sum S that we will have after t years is given by

$$S = P + Prt$$

In many business transactions the interest that is added to the principal at regular time intervals also earns interest. This is called the **compound interest** process.

The time period between successive additions of interest is known as the **conversion period.** If interest is compounded quarterly, the conversion period is three months; if interest is compounded semiannually, the conversion period is six months.

Suppose now that a principal P is invested at an annual interest rate r, compounded k times a year. Then each conversion period lasts $t = 1/k$ years. Thus, the amount S_1 at the end of the first conversion period is

$$\begin{aligned} S_1 &= P + Prt \\ &= P + P \cdot r \cdot \frac{1}{k} = P\left(1 + \frac{r}{k}\right) \end{aligned}$$

The amount S_2 at the end of the second conversion period is

$$\begin{aligned} S_2 &= S_1 + S_1rt \\ &= P\left(1 + \frac{r}{k}\right) + P\left(1 + \frac{r}{k}\right) \cdot r \cdot \frac{1}{k} \\ &= \left[P\left(1 + \frac{r}{k}\right)\right]\left(1 + \frac{r}{k}\right) \\ S_2 &= P\left(1 + \frac{r}{k}\right)^2 \end{aligned}$$

In this way, we see that the amount S_n after n conversion periods is given by

$$S_n = P\left(1 + \frac{r}{k}\right)^n$$

which is usually written

$$\boxed{S = P(1 + i)^n}$$

where $i = r/k$. Table IV in the Tables Appendix gives values of $(1 + i)^n$ for a number of values of i and n. Accurate results can also be obtained by using a calculator with a "y^x" key.

EXAMPLE 6

Suppose that \$6000 is invested at an annual interest rate of 8%. What will the value of the investment be after 3 years if

(a) interest is compounded quarterly?

(b) interest is compounded semiannually?

SOLUTION

(a) We are given $P = 6000$, $r = 0.08$, $k = 4$, and $n = 12$ (since there are 4 conversion periods per year for 3 years). Thus,

$$i = \frac{r}{k} = \frac{0.08}{4} = 0.02$$

and

$$S = P(1 + i)^n = 6000(1 + 0.02)^{12}$$

Table IV in the Tables Appendix, with $i = r/k = 0.02 = 2\%$ and $n = 12$, yields

$$S = 6000(1.26824179) = 7609.45$$

Thus, the sum at the end of the three-year period will be \$7609.45.

(b) We have $P = 6000$, $r = 0.08$, $k = 2$, and $n = 6$ (since there are 2 conversion periods per year for 3 years). Then

$$i = \frac{r}{k} = \frac{0.08}{2} = 0.04$$

$$S = P(1 + i)^n = 6000(1 + 0.04)^6$$

Table IV in the Tables Appendix, with $i = 0.04 = 4\%$ and $n = 6$, yields

$$S = 6000(1.26531902) = 7591.91$$

The sum at the end of the three-year period will be \$7591.91, which is \$17.54 less than the interest earned when compounding is quarterly rather than semiannual.

PROGRESS CHECK

Suppose that \$5000 is invested at an annual interest rate of 6% compounded semiannually. What is the value of the investment after 12 years?

ANSWER

\$10,163.97

WARNING When using Table IV, be certain that n is the total number of conversion periods and i is the interest rate per conversion period. For example, an interest rate of 18% compounded monthly for 2 years leads to $n = 24$ and $i = 1\frac{1}{2}\%$.

Continuous Compounding

When P, r, and t are held fixed and the frequency of compounding is increased, the return on the investment is increased. We wish to determine the effect of making the number of conversions per year larger and larger.

Suppose a principal P is invested at an annual rate r, compounded k times per year. After t years, the number of conversions is $n = tk$. Then the value of the investment after t years is

$$S = P\left(1 + \frac{r}{k}\right)^{tk}$$

Letting $m = k/r$, we can rewrite this equation as

$$S = P\left(1 + \frac{1}{m}\right)^{tmr}$$

or

$$S = P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt}$$

If the number of conversions k per year gets larger and larger, then m gets larger and larger. Since we saw in Table 1 of this chapter that the expression

$$\left(1 + \frac{1}{m}\right)^{m}$$

gets closer and closer to e as m gets larger and larger, we conclude that

$$\boxed{S = Pe^{rt} \qquad (1)}$$

As the number of conversions increases, so does the value of the investment. But there is a limit, or bound, to this value, and it is given by Equation (1). We say that Equation (1) represents the result of **continuous compounding.**

EXAMPLE 7

Suppose that \$20,000 is invested at an annual interest rate of 7% compounded continuously. What is the value of the investment after 4 years?

SOLUTION

We have $P = 20{,}000$, $r = 0.07$, and $t = 4$, and we substitute in Equation (1):

$$\begin{aligned} S &= Pe^{rt} \\ &= 20{,}000e^{0.07(4)} = 20{,}000e^{0.28} \\ &= 20{,}000(1.3231) \qquad \text{from Table I, Tables Appendix, or a calculator} \\ &= 26{,}462 \end{aligned}$$

The sum available after 4 years is \$26,462.

PROGRESS CHECK

Suppose that \$10,000 is invested at an annual interest rate of 10% compounded continuously. What is the value of the investment after 6 years?

ANSWER

\$18,221

By solving Equation (1) for P, we can determine the principal P that must be invested at continuous compounding to have a certain amount S at some future time. The values of e^{-x} from Table I in the Tables Appendix will be used in this connection.

EXAMPLE 8

Suppose that a principal P is to be invested at continuous compound interest of 8% per year to yield \$10,000 in 5 years. Approximately how much should be invested?

SOLUTION

Using Equation (1) with $S = 10{,}000$, $r = 0.08$, and $t = 5$, we have

$$\begin{aligned} S &= Pe^{rt} \\ 10{,}000 &= Pe^{0.08(5)} = Pe^{0.40} \\ P &= \frac{10{,}000}{e^{0.40}} \\ &= 10{,}000e^{-0.40} \\ &= 10{,}000(0.6703) \qquad \text{from Table I, Tables Appendix, or a calculator} \\ &= 6703 \end{aligned}$$

Thus, approximately \$6703 should be invested initially.

PROGRESS CHECK

Approximately how much money should a 35-year-old woman invest now at continuous compound interest of 10% per year to obtain the sum of \$20,000 upon her retirement at age 65?

ANSWER

\$996

EXERCISE SET 4.1

In Exercises 1–12 sketch the graph of the given function f.

1. $f(x) = 4^x$
2. $f(x) = 4^{-x}$
3. $f(x) = 10^x$
4. $f(x) = 10^{-x}$
5. $f(x) = 2^{x+1}$
6. $f(x) = 2^{x-1}$
7. $f(x) = 2^{|x|}$
8. $f(x) = 2^{-|x|}$
9. $f(x) = 2^{2x}$
10. $f(x) = 3^{-2x}$
11. $f(x) = e^{x+1}$
12. $f(x) = e^{-2x}$

In Exercises 13–20 solve for x.

13. $2^x = 2^3$
14. $2^{x-1} = 2^4$
15. $3^x = 9^{x-2}$
16. $2^x = 8^{x+2}$
17. $2^{3x} = 4^{x+1}$
18. $3^{4x} = 9^{x-1}$
19. $e^{x-1} = e^3$
20. $e^{x-1} = 1$

In Exercises 21–24 solve for a.

21. $(a + 1)^x = (2a - 1)^x$
22. $(2a + 1)^x = (a + 4)^x$
23. $(a + 1)^x = (2a)^x$
24. $(2a + 3)^x = (3a + 1)^x$

In Exercises 25–29 use Table I in the Tables Appendix to evaluate e^x and e^{-x}.

25. The number of bacteria in a culture after t hours is described by the exponential growth model $Q(t) = 200e^{0.25t}$
    (a) What is the initial number of bacteria in the culture?
    (b) Find the number of bacteria in the culture after 20 hours.
    (c) Use Table I in the Tables Appendix to complete the following table.

t	1	4	8	10
Q				

26. The number of bacteria in a culture after t hours is described by the exponential growth model $Q(t) = q_0e^{0.01t}$. If there were 400 bacteria present initially, how many bacteria are present after 2 *days?*
27. At the beginning of 1975 the world population was approximately 4 billion. Suppose that the population is described by an exponential growth model, and that the rate of growth is 2% per year. Give the approximate world population in the year 2000.
28. The number of grams of potassium 42 present after t hours is given by the exponential decay model $Q(t) = q_0e^{-0.055t}$. If 400 grams of the substance were present initially, how much remains after 10 hours?
29. A radioactive substance has a decay rate of 4% per hour. If 1000 grams were present initially, how much of the substance remains after 10 hours?

In Exercises 30–33 use Table IV in the Tables Appendix, or a calculator, to assist in the computations.

30. An investor purchases a \$12,000 savings certificate paying 10% annual interest compounded semiannually. Find the amount received when the savings certificate is redeemed at the end of 8 years.
31. The parents of a newborn infant place \$10,000 in an investment that pays 8% annual interest compounded quarterly. What sum is available at the end of 18 years to finance the child's college education?
32. A widow is offered a choice of two investments. Investment A pays 8% annual interest compounded quarterly, and investment B pays 9% compounded annually. Which investment will yield a greater return?
33. A firm intends to replace its present computer in 5 years. The treasurer suggests that \$25,000 be set aside in an investment paying 12% compounded monthly. What sum will be available for the purchase of the new computer?

In Exercises 34–38 use Tables I and IV in the Tables Appendix, or a calculator, to assist in the computations.

34. If \$5000 is invested at an annual interest rate of 9% compounded continuously, how much is available after 5 years?
35. If \$100 is invested at an annual interest rate of 5.5% compounded continuously, how much is available after 10 years?
36. A principal P is to be invested at continuous compound interest of 9% to yield \$50,000 in 20 years. What is the approximate value of P to be invested?
37. A 40-year-old executive plans to retire at age 65. How much should be invested at 12% annual interest compounded continuously to provide the sum of \$50,000 upon retirement?
38. Investment A offers 8% annual interest compounded

semiannually, and investment B offers 8% annual interest compounded continuously. If \$1000 were invested in each, what would be the approximate difference in value after 10 years?

In Exercises 39 and 40 use a calculator to determine which number is greater.

39. $2^{\pi}, \pi^2$

40. $3^{\pi}, \pi^3$

4.2 LOGARITHMIC FUNCTIONS

LOGARITHMS AS EXPONENTS

We have previously noted that $f(x) = a^x$ is an increasing function if $a > 1$ and a decreasing function if $0 < a < 1$. It is thus clear that no horizontal line can meet the graph of $f(x) = a^x$ in more than one point. We conclude that the exponential function is a one-to-one function.

In Figure 5a, we see the function $f(x) = 2^x$ assigning values in the set Y for various values of x in the domain X. Since $f(x) = 2^x$ is a one-to-one function, it makes sense to seek a function f^{-1} that will return the values of the range of f back to their origin as in Figure 5b. That is,

$$f \text{ maps 3 into 8,} \quad f^{-1} \text{ maps 8 into 3}$$
$$f \text{ maps 4 into 16,} \quad f^{-1} \text{ maps 16 into 4}$$

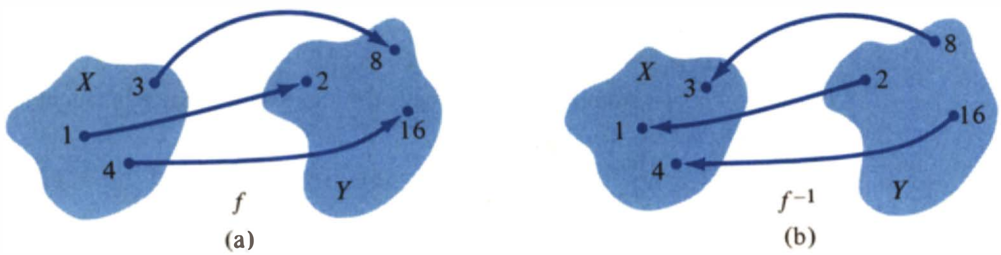


FIGURE 5

and so on. Since 2^x is always positive, we see that the domain of f^{-1} is the set of all positive real numbers. Its range is the set of all real numbers.

The function f^{-1} of Figure 5b has a special name, the **logarithmic function base 2,** which we write as $\log_2$. It is also possible to generalize and define the logarithmic function as the inverse of the exponential function with any base a such that $a > 0$ and $a \neq 1$.

Logarithmic Function Base a	$y = \log_a x$ if and only if $x = a^y$

When no base is indicated, the notation $\log x$ is interpreted to mean $\log_{10} x$.

The notation **ln *x*** is used to indicate logarithms to the base e. Since $\ln x$ is the inverse of the natural exponential function e^x, it is called the **natural logarithm of *x*.**

Natural Logarithm	$\ln x = \log_e x$

The exponential form $x = a^y$ and the logarithmic form $y = \log_a x$ are two ways of expressing the same relationship among x, y, and a. Further, it is always possible to convert from one form to the other. The natural question, then, is why bother to create a logarithmic form when we already have an equivalent exponential form. One reason is to allow us to switch an equation from the form $x = a^y$ to a form in which y is a function of x. We will also demonstrate that the logarithmic function has some very useful properties.

EXAMPLE 1

Write in exponential form.

(a) $\log_3 9 = 2$ (b) $\log_2 \frac{1}{8} = -3$

(c) $\log_{16} 4 = \frac{1}{2}$ (d) $\ln 7.39 = 2$

SOLUTION

We change from the logarithmic form $\log_a x = y$ to the equivalent exponential form $a^y = x$.

(a) $3^2 = 9$ (b) $2^{-3} = \frac{1}{8}$ (c) $16^{1/2} = 4$ (d) $e^2 = 7.39$

PROGRESS CHECK

Write in exponential form.

(a) $\log_4 64 = 3$ (b) $\log_{10}\left(\frac{1}{10,000}\right) = -4$

(c) $\log_{25} 5 = \frac{1}{2}$ (d) $\ln 0.3679 = -1$

ANSWERS

(a) $4^3 = 64$ (b) $10^{-4} = \frac{1}{10,000}$

(c) $25^{1/2} = 5$ (d) $e^{-1} = 0.3679$

EXAMPLE 2

Write in logarithmic form.

(a) $36 = 6^2$ (b) $7 = \sqrt{49}$ (c) $\frac{1}{16} = 4^{-2}$ (d) $0.1353 = e^{-2}$

SOLUTION
Since $y = \log_a x$ if and only if $x = a^y$, the logarithmic forms are

(a) $\log_6 36 = 2$ (b) $\log_{49} 7 = \frac{1}{2}$

(c) $\log_4 \frac{1}{16} = -2$ (d) $\ln 0.1353 = -2$

PROGRESS CHECK
Write in logarithmic form.
(a) $64 = 8^2$ (b) $6 = 36^{1/2}$
(c) $\frac{1}{7} = 7^{-1}$ (d) $20.09 = e^3$

ANSWERS
(a) $\log_8 64 = 2$ (b) $\log_{36} 6 = \frac{1}{2}$

(c) $\log_7 \frac{1}{7} = -1$ (d) $\ln 20.09 = 3$

LOGARITHMIC EQUATIONS

Logarithmic equations can often be solved by changing them to equivalent exponential forms. Here are some straightforward examples; more challenging problems will be handled in Section 4.5.

EXAMPLE 3
Solve for x.
(a) $\log_3 x = -2$ (b) $\log_x 81 = 4$
(c) $\log_5 125 = x$ (d) $\ln x = \frac{1}{2}$

SOLUTION
(a) Using the equivalent exponential form, we have

$$x = 3^{-2} = \frac{1}{9}$$

(b) Changing to the equivalent exponential form, we have

$$x^4 = 81 = 3^4$$

Since $a^u = b^u$ implies $a = b$, we have

$$x = 3$$

(c) In exponential form we have

$$5^x = 125$$

Writing 125 to the base 5, we have

$$5^x = 5^3$$

and since $a^u = a^v$ implies $u = v$, we conclude that

$$x = 3$$

(d) The equivalent exponential form is

$$x = e^{1/2} \approx 1.65$$

which we obtain from Table I in the Tables Appendix or by using a calculator with a "y^x" key.

PROGRESS CHECK

Solve for x.

(a) $\log_x 1000 = 3$ (b) $\log_2 x = 5$ (c) $x = \log_7 \dfrac{1}{49}$

ANSWERS

(a) 10 (b) 32 (c) -2

LOGARITHMIC IDENTITIES

If $f(x) = a^x$, then $f^{-1}(x) = \log_a x$. Recall that inverse functions have the property that

$$f[f^{-1}(x)] = x \quad \text{and} \quad f^{-1}[f(x)] = x$$

Substituting $f(x) = a^x$ and $f^{-1}(x) = \log_a x$, we have

$$\begin{aligned} f[f^{-1}(x)] &= x \qquad & f^{-1}[f(x)] &= x \\ f(\log_a x) &= x & f^{-1}(a^x) &= x \\ a^{\log_a x} &= x & \log_a a^x &= x \end{aligned}$$

These two identities are useful in simplifying expressions and should be remembered.

$$\boxed{\begin{aligned} a^{\log_a x} &= x \\ \log_a a^x &= x \end{aligned}}$$

MEASURING AN EARTHQUAKE

Richter Scale Readings

Here's what you can anticipate from earthquakes of various Richter scale readings.

2.0	not noticed
4.5	some damage in a very limited area
6.0	hazardous; serious damage with destruction of buildings in a limited area
7.0	felt over a wide area with significant damage
8.0	great damage
8.7	maximum recorded

The great San Francisco earthquake of 1906 is estimated to have had a Richter scale reading of 8.3.

Radio and television newscasts often describe earthquakes in this way: "A minor earthquake in China registered 3.0 on the Richter scale," or, "A major earthquake in Chile registered 8.0 on the Richter scale." From statements like this we know that 3.0 is a "low" value and 8.0 is a "high" value. But just what is the Richter scale?

On the Richter scale, the magnitude R of an earthquake is defined as

$$R = \log \frac{I}{I_0}$$

where I_0 is a constant that represents a standard intensity and I is the intensity of the earthquake being measured. The Richter scale is a means of measuring a given earthquake against a "standard earthquake" of intensity I_0.

What does 3.0 on the Richter scale mean? Substituting $R = 3$ in the above equation, we have

$$3 = \log \frac{I}{I_0}$$

or, in the equivalent exponential form,

$$1000 = \frac{I}{I_0}$$

Solving for I, we arrive at the equation

$$I = 1000\, I_0$$

which states that an earthquake with a Richter scale reading of 3.0 is 1000 times as intense as the standard! No wonder, then, that an earthquake registering 8.0 on the Richter scale is serious: it has an intensity 100,000,000 times that of the standard!

The following pair of identities can be established by converting to the equivalent exponential form.

$$\log_a a = 1$$
$$\log_a 1 = 0$$

EXAMPLE 4

Evaluate.

(a) $8^{\log_8 5}$ (b) $\log_{10} 10^{-3}$ (c) $\log_7 7$ (d) $\log_4 1$

SOLUTION

(a) 5 (b) -3 (c) 1 (d) 0

PROGRESS CHECK

Evaluate.

(a) $\log_3 3^4$ (b) $6^{\log_6 9}$ (c) $\log_5 1$ (d) $\log_8 8$

ANSWERS

(a) 4 (b) 9 (c) 0 (d) 1

GRAPHS OF THE LOGARITHMIC FUNCTIONS

To sketch the graph of a logarithmic function, we convert to the equivalent exponential form. For example, to sketch the graph of $y = \log_2 x$, we form a table of values for the equivalent exponential equation $x = 2^y$.

y	-3	-2	-1	0	1	2	3
$x = 2^y$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

We can now plot these points and sketch a smooth curve, as in Figure 6. Note that the y-axis is a vertical asymptote. We have included the graph of $y = 2^x$ to illustrate that the graphs of a pair of inverse functions are reflections of each other about the line $y = x$.

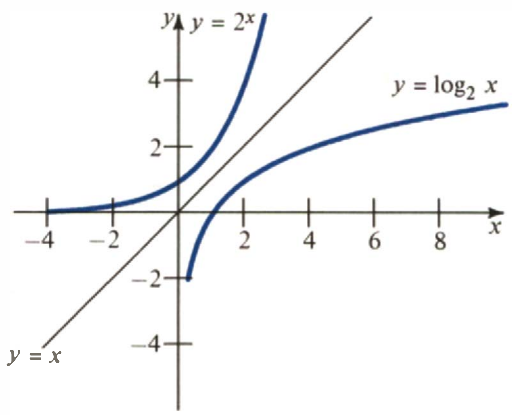


FIGURE 6

EXAMPLE 5

Sketch the graphs of $y = \log_3 x$ and $y = \log_{1/3} x$ on the same coordinate axes.

SOLUTION

The graphs are shown in Figure 7. Practical applications of logarithms generally involve a base $a > 1$.

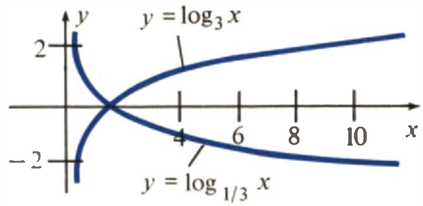


FIGURE 7

PROPERTIES OF LOGARITHMIC FUNCTIONS

The graphs in Figures 6 and 7 illustrate the following important properties of logarithmic functions.

Properties of Logarithmic Functions

- The point (1, 0) lies on the graph of the function $f(x) = \log_a x$ for all real numbers $a > 0$. This is another way of saying $\log_a 1 = 0$.
- The domain of $f(x) = \log_a x$ is the set of all positive real numbers; the range is the set of all real numbers.
- When $a > 1$, $f(x) = \log_a x$ is an increasing function; when $0 < a < 1$, $f(x) = \log_a x$ is a decreasing function.

These results are in accord with what we anticipate for a pair of inverse functions. As expected, the domain of the logarithmic function is the range of the corresponding exponential function, and vice versa.

Since $\log_a x$ is either increasing or decreasing, the same value cannot be assumed more than once. Thus:

If $\log_a u = \log_a v$, then $u = v$.

Since the graphs of $\log_a u$ and $\log_b u$ intersect only at $u = 1$, we have the following rule:

If $\log_a u = \log_b u$ and $u \neq 1$, then $a = b$.

EXAMPLE 6

Solve for x.

(a) $\log_5(x + 1) = \log_5 25$ (b) $\log_{x-1} 31 = \log_5 31$

SOLUTION

(a) Since $\log_a u = \log_a v$ implies $u = v$, then

$$x + 1 = 25$$
$$x = 24$$

(b) Since $\log_a u = \log_b u$, $u \neq 1$, implies $a = b$,

$$x - 1 = 5$$
$$x = 6$$

PROGRESS CHECK

Solve for x.

(a) $\log_2 x^2 = \log_2 9$ (b) $\log_7 14 = \log_{2x} 14$

ANSWERS

(a) $3, -3$ (b) $\frac{7}{2}$

EXERCISE SET 4.2

In Exercises 1–12 write each equation in exponential form.

1. $\log_2 4 = 2$
2. $\log_5 125 = 3$
3. $\log_9 \frac{1}{81} = -2$
4. $\log_{64} 4 = \frac{1}{3}$
5. $\ln 20.09 = 3$
6. $\ln \frac{1}{7.39} = -2$
7. $\log_{10} 1000 = 3$
8. $\log_{10} \frac{1}{1000} = -3$
9. $\ln 1 = 0$
10. $\log_{10} 0.01 = -2$
11. $\log_3 \frac{1}{27} = -3$
12. $\log_{125} \frac{1}{5} = -\frac{1}{3}$

In Exercises 13–26 write each equation in logarithmic form.

13. $25 = 5^2$
14. $27 = 3^3$
15. $10{,}000 = 10^4$
16. $\frac{1}{100} = 10^{-2}$
17. $\frac{1}{8} = 2^{-3}$
18. $\frac{1}{27} = 3^{-3}$
19. $1 = 2^0$
20. $1 = e^0$
21. $6 = \sqrt{36}$
22. $2 = \sqrt[3]{8}$
23. $64 = 16^{3/2}$
24. $81 = 27^{4/3}$
25. $\frac{1}{3} = 27^{-1/3}$
26. $\frac{1}{2} = 16^{-1/4}$

In Equations 27–44 solve for x.

27. $\log_5 x = 2$
28. $\log_{16} x = \frac{1}{2}$
29. $\log_{25} x = -\frac{1}{2}$
30. $\log_{1/2} x = 3$
31. $\ln x = 2$
32. $\ln x = -3$
33. $\ln x = -\frac{1}{2}$
34. $\log_4 64 = x$
35. $\log_5 \frac{1}{25} = x$
36. $\log_x 4 = \frac{1}{2}$
37. $\log_x \frac{1}{8} = -\frac{1}{3}$
38. $\log_3(x - 1) = 2$
39. $\log_5(x + 1) = 3$
40. $\log_2(x - 1) = \log_2 10$
41. $\log_{x+1} 24 = \log_3 24$
42. $\log_3 x^3 = \log_3 64$
43. $\log_{x+1} 17 = \log_4 17$
44. $\log_{3x} 18 = \log_4 18$

In Exercises 45–64 evaluate the given expression.

45. $3^{\log_3 6}$	46. $2^{\log_2(2/3)}$	47. $e^{\ln 2}$	48. $e^{\ln 1/2}$
49. $\log_5 5^3$	50. $\log_4 4^{-2}$	51. $\log_8 8^{1/2}$	52. $\log_{64} 64^{-1/3}$
53. $\log_7 49$	54. $\log_7 \sqrt{7}$	55. $\log_5 5$	56. $\ln e$
57. $\ln 1$	58. $\log_4 1$	59. $\log_2 \frac{1}{4}$	60. $\log_{16} 4$
61. $\log 10{,}000$	62. $\log 0.001$	63. $\ln e^2$	64. $\ln e^{-2/3}$

In Exercises 65–72 sketch the graph of the given function.

65. $f(x) = \log_4 x$	66. $f(x) = \log_{1/2} x$	67. $f(x) = \log 2x$	68. $f(x) = \frac{1}{2} \log x$
69. $f(x) = \ln \frac{x}{2}$	70. $f(x) = \ln 3x$	71. $f(x) = \log_3(x - 1)$	72. $f(x) = \log_3(x + 1)$

4.3 FUNDAMENTAL PROPERTIES OF LOGARITHMS

There are three fundamental properties of logarithms that have made them a powerful computational aid.

Property 1. $\log_a(x \cdot y) = \log_a x + \log_a y$

Property 2. $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$

Property 3. $\log^a x_n = n \log_a x,\quad n$ a real number

These properties can be proved by using equivalent exponential forms. To prove the first property, $\log_a(x \cdot y) = \log_a x + \log_a y$, we let

$$\log_a x = u \quad \text{and} \quad \log_a y = v$$

Then the equivalent exponential forms are

$$a^u = x \quad \text{and} \quad a^v = y$$

Multiplying the left-hand and right-hand sides of these equations, we have

$$a^u \cdot a^v = x \cdot y$$

or

$$a^{u+v} = x \cdot y$$

Substituting a^{u+v} for $x \cdot y$ in $\log_a(x \cdot y)$, we have

$$\begin{aligned} \log_a(x \cdot y) &= \log_a(a^{u+v}) \\ &= u + v \qquad \text{since } \log_a a^x = x \end{aligned}$$

Substituting for u and v, we find that

$$\log_a(x \cdot y) = \log_a x + \log_a y$$

Properties 2 and 3 can be established in much the same way.

It is these properties that originally made the study of logarithms worthwhile. Note that the more complex operations of multiplication and division are converted to addition and subtraction and that exponentiation is converted to multiplication. We will first demonstrate these properties, and in the next section we will apply them to realistic computational problems.

EXAMPLE 1

(a) $\log_{10}(225 \times 478) = \log_{10} 225 + \log_{10} 478$

(b) $\log_8\left(\frac{422}{735}\right) = \log_8 422 - \log_8 735$

(c) $\log_2(2)^5 = 5 \log_2 2 = 5 \cdot 1 = 5$

(d) $\log_a\left(\frac{x \cdot y}{z}\right) = \log_a x + \log_a y - \log_a z$

PROGRESS CHECK

Write in terms of simpler logarithmic forms.

(a) $\log_4(1.47 \times 22.3)$ (b) $\log_a \frac{x-1}{\sqrt{x}}$

ANSWERS

(a) $\log_4 1.47 + \log_4 22.3$ (b) $\log_a(x-1) - \frac{1}{2}\log_a x$

SIMPLIFYING LOGARITHMS

The next example illustrates rules that speed the handling of logarithmic forms.

EXAMPLE 2

Write $\log_a \frac{(x-1)^{-2}(y+2)^3}{\sqrt{x}}$ in terms of simpler logarithmic forms.

SOLUTION

Simplifying Logarithms	
Step 1. Rewrite the expression so that each factor has a positive exponent.	*Step 1.* $\log_a \dfrac{(x-1)^{-2}(y+2)^3}{\sqrt{x}}$ $= \log_a \dfrac{(y+2)^3}{(x-1)^2\sqrt{x}}$
Step 2. Apply Property 1 and Property 2 for multiplication and division of logarithms. Each factor in the numerator will yield a term with a plus sign. Each factor in the denominator will yield a term with a minus sign.	*Step 2.* $= \log_a(y+2)^3 - \log_a(x-1)^2 - \log_a \sqrt{x}$
Step 3. Apply Property 3 to simplify.	*Step 3.* $= 3\log_a(y+2) - 2\log_a(x-1) - \dfrac{1}{2}\log_a x$

PROGRESS CHECK

Simplify $\log_a \dfrac{(2x-3)^{1/2}(y+2)^{2/3}}{z^4}$.

ANSWER

$\dfrac{1}{2}\log_a(2x-3) + \dfrac{2}{3}\log_a(y+2) - 4\log_a z$

EXAMPLE 3

If $\log_a 1.5 = r$, $\log_a 2 = s$, and $\log_a 5 = t$, find the following.

(a) $\log_a 7.5$ (b) $\log_a \sqrt{10}$

SOLUTION

(a) Since

$$7.5 = 1.5 \times 5$$

then

$$\begin{aligned} \log_a 7.5 &= \log_a(1.5 \times 5) \\ &= \log_a 1.5 + \log_a 5 && \text{Property 1} \\ &= r + t && \text{Substitution} \end{aligned}$$

(b) Write this as

$$\begin{aligned}
\log_a(10)^{1/2} &= \frac{1}{2}\log_a 10 && \text{Property 3}\\
&= \frac{1}{2}\log_a(2\cdot 5) && \\
&= \frac{1}{2}[\log_a 2 + \log_a 5] && \text{Property 1}\\
&= \frac{1}{2}(s+t) && \text{Substitution}
\end{aligned}$$

PROGRESS CHECK

If $\log_a 2 = 0.43$ and $\log_a 3 = 0.68$, find the following.

(a) $\log_a 18$ (b) $\log_a \sqrt[3]{\dfrac{9}{2}}$

ANSWERS

(a) 1.79 (b) 0.31

WARNING

(a) Note that

$$\log_a(x+y) \neq \log_a x + \log_a y$$

Property 1 tells us that

$$\log_a(x\cdot y) = \log_a x + \log_a y$$

Don't try to apply this property to $\log_a(x+y)$, which cannot be simplified.

(b) Note that

$$\log_a x^n \neq (\log_a x)^n$$

By Property 3,

$$\log_a x^n = n\log_a x$$

We can also apply the properties of logarithms to combine terms involving logarithms.

EXAMPLE 4

Write as a single logarithm:

$$3\log_a x - \frac{1}{2}\log_a(x-1)$$

SOLUTION

$$\begin{aligned} 3 \log_a x - \frac{1}{2}\log_a(x-1) &= \log_a x^3 - \log_a(x-1)^{1/2} && \text{Property 3} \\ &= \log_a \frac{x^3}{(x-1)^{1/2}} && \text{Property 2} \\ &= \log_a \frac{x^3}{\sqrt{x-1}} \end{aligned}$$

PROGRESS CHECK

Write as a single logarithm:

$$\frac{1}{3}[\log_a(2x-1) - \log_a(2x-5)] + 4\log_a x$$

ANSWER

$\log_a x^4 \sqrt[3]{\dfrac{2x-1}{2x-5}}$

WARNING

(a) Note that

$$\frac{\log_a x}{\log_a y} \neq \log_a x - \log_a y$$

Property 2 tells us that

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

Don't try to apply this property to $(\log_a x)/(\log_a y)$, which cannot be simplified.

(b) The expressions

$$\log_a x + \log_b x$$

and

$$\log_a x - \log_b x$$

cannot be simplified. Logarithms with different bases do not readily combine except in special cases.

CHANGE OF BASE

Sometimes it is convenient to be able to write a logarithm that is given in terms of a base a in terms of another base b, that is, to convert $\log_a x$ to $\log_b x$. (As always, we must require a and b to be positive real numbers other than 1.) For example, some calculators can compute $\log x$ but not $\ln x$, or vice versa.

To compute $\log_b x$ given $\log_a x$, let $y = \log_b x$. The equivalent exponential form is then

$$b^y = x$$

Taking logarithms to the base a of both sides of this equation, we have

$$\log_a b^y = \log_a x$$

We now apply the fundamental properties of logarithms developed earlier in this section. By Property 3,

$$y \log_a b = \log_a x$$

Solving for y gives us

$$y = \frac{\log_a x}{\log_a b}$$

Since $y = \log_b x$, we have

Change of Base Formula

$$\log_b x = \frac{\log_a x}{\log_a b}$$

EXAMPLE 5

A calculator has a key labeled "log" (for $\log_{10}$) but doesn't have a key labeled "ln." The calculator is used to find that

$$\log 27 = 1.4314$$
$$\log e \approx \log 2.7183 = 0.4343$$

Find ln 27.

SOLUTION

We use the change of base formula

$$\log_b x = \frac{\log_a x}{\log_a b}$$

with $b = e$, $a = 10$, and $x = 27$. Then

$$\begin{aligned} \log_e 27 = \ln 27 &= \frac{\log 27}{\log e} \\ &= \frac{1.4314}{0.4343} = 3.2959 \end{aligned}$$

Note that for any positive number k, the conversion from $\log k$ to $\ln k$

involves division by the constant $\log e = 0.4343$. A calculator that has a "log" key can thereby be used efficiently to find natural logarithms.

PROGRESS CHECK

Given $\log 16 = 1.2041$ and $\log 5 = 0.6990$, find $\log_5 16$.

ANSWER

1.7226

EXERCISE SET 4.3

In Exercises 1–20 write each expression in terms of simpler logarithmic forms.

1. $\log_{10}(120 \times 36)$
2. $\log_6\left(\frac{187}{39}\right)$
3. $\log_3(3^4)$
4. $\log_3(4^3)$
5. $\log_a(2xy)$
6. $\ln(4xyz)$
7. $\log_a\left(\frac{x}{yz}\right)$
8. $\ln\left(\frac{2x}{y}\right)$
9. $\ln x^5$
10. $\log_3 y^{2/3}$
11. $\log_a(x^2y^3)$
12. $\log_a(xy)^3$
13. $\log_a\sqrt{xy}$
14. $\log_a\sqrt[3]{xy^4}$
15. $\ln(x^2y^3z^4)$
16. $\log_a(xy^3z^2)$
17. $\ln\left(\sqrt{x}\sqrt[3]{y}\right)$
18. $\ln\sqrt[3]{xy^2}\sqrt[4]{z}$
19. $\log_a\left(\frac{x^2y^3}{z^4}\right)$
20. $\ln\frac{x^4y^2}{z^{1/2}}$

In Exercises 21–30 $\log 2 = 0.30$, $\log 3 = 0.47$, and $\log 5 = 0.70$. Find the following.

21. $\log 6$
22. $\log \frac{2}{3}$
23. $\log 9$
24. $\log \sqrt{5}$
25. $\log 12$
26. $\log \frac{6}{5}$
27. $\log \frac{15}{2}$
28. $\log 0.3$
29. $\log \sqrt{7.5}$
30. $\log \sqrt[4]{30}$

In Exercises 31–44 write each expression as a single logarithm.

31. $2 \log x + \frac{1}{2} \log y$
32. $3 \log_a x - 2 \log_a z$
33. $\frac{1}{3} \ln x + \frac{1}{3} \ln y$
34. $\frac{1}{3} \ln x - \frac{2}{3} \ln y$
35. $\frac{1}{3} \log_a x + 2 \log_a y - \frac{3}{2} \log_a z$
36. $\frac{2}{3}\log_a x + \log_a y - 2 \log_a z$
37. $\frac{1}{2}(\log_a x + \log_a y)$
38. $\frac{2}{3}(4 \ln x - 5 \ln y)$
39. $\frac{1}{3}(2 \ln x + 4 \ln y) - 3 \ln z$
40. $\ln x - \frac{1}{2}(3 \ln x + 5 \ln y)$
41. $\frac{1}{2} \log_a(x - 1) - 2 \log_a(x + 1)$
42. $2 \log_a(x + 2) - \frac{1}{2}(\log_a y + \log_a z)$
43. $3 \log_a x - 2 \log_a(x - 1) + \frac{1}{2} \log_a\sqrt[3]{x + 1}$
44. $4 \ln(x - 1) + \frac{1}{2} \ln(x + 1) - 3 \ln y$

The key labeled "ln" on a calculator is used to compute ln 10 = 2.3026, ln 6 = 1.7918, and ln 3 = 1.0986. In Exercises 45–50 use the first value to find the required value.

45. $\ln 17 = 2.8332$; find $\log 17$

46. $\ln 22 = 3.0910$; find $\log_6 22$

47. $\ln 141 = 4.9488$; find $\log_3 141$

48. $\ln 78 = 4.3567$; find $\log_6 78$

49. $\ln 245 = 5.5013$; find $\log 245$

50. $\ln 7 = 1.9459$; find $\log_3 7$

4.4 COMPUTING WITH LOGARITHMS (Optional)

We indicated earlier that logarithms can be used to simplify complex calculations. In this section we will demonstrate the power of logarithms in computational work.

We will use 10 as the base for our computations with logarithms because 10 is the base of our number system. We call logarithms to the base 10 **common logarithms.**

We begin with the observation that any positive real number can be written as a product of a number c, $1 \leq c < 10$, and an integer power of 10, say 10^k. This format is often referred to as **scientific notation.** Here are some examples.

$$643 = 6.43 \times 10^2 \qquad 4629 = 4.629 \times 10^3$$
$$754{,}000 = 7.54 \times 10^5 \qquad 1.76 = 1.76 \times 10^0$$
$$0.0423 = 4.23 \times 10^{-2} \qquad 0.0000926 = 9.26 \times 10^{-5}$$

Let's begin with the number 643 expressed in scientific notation, that is,

$$643 = 6.43 \times 10^2$$

We next take the logarithm of each side of this equation (to the base 10) and then apply the properties of logarithms.

$$\begin{aligned}
\log 643 &= \log(6.43 \times 10^2) & \\
&= \log 6.43 + \log 10^2 & \text{Property 1} \\
&= \log 6.43 + 2 \log 10 & \text{Property 3} \\
&= \log 6.43 + 2 & \log 10 = 1
\end{aligned}$$

We can generalize this result dealing with the logarithm of a number that is written in scientific notation.

> If x is a positive real number and $x = c \cdot 10^k$, then
> $$\log x = \log c + k$$

The number $\log c$ is called the **mantissa** and the integer k the **characteristic** of the number x. Since

$$x = c \cdot 10^k \qquad \text{where } 1 \leq c < 10$$

and since the function $f(x) = \log x$ is an increasing function, we see that

$$\log 1 \leq \log c < \log 10$$

or

$$0 \leq \log c < 1$$

We conclude that the mantissa is always a number between 0 and 1.

Table II in the Tables Appendix can be used to approximate the common logarithm of any three-digit number between 1.00 and 9.99 at intervals of 0.01. The next example shows how to proceed.

EXAMPLE 1

Find the following.

(a) $\log 73.5$ (b) $\log 0.00451$.

SOLUTION

(a) Since $73.5 = 7.35 \times 10^1$, the characteristic of 73.5 is 1. Using Table II in the Tables Appendix, we find that $\log 7.35$ is 0.8663 (approximately). Then

$$\begin{aligned}\log 73.5 &= \log(7.35 \times 10^1) = \log 7.35 + \log 10^1 \\ &= 0.8663 + 1 = 1.8663\end{aligned}$$

(b) Since $0.00451 = 4.51 \times 10^{-3}$, the characteristic of 0.00451 is -3. From Table II we have

$$\log 0.00451 = 0.6542 - 3$$

Here we have a positive mantissa and a negative characteristic. For reasons that will be clear later, we always leave the answer in this form.

PROGRESS CHECK

Find the following.

(a) $\log 69{,}700$ (b) $\log 0.000697$ (c) $\log 0.697$

ANSWERS

(a) 4.8432 (b) $0.8432 - 4$ (c) $0.8432 - 1$

WARNING Note that

$$\log 0.00547 = 0.7380 - 3$$

but

$$\log 0.00547 \neq -3.7380$$

since this is algebraically incorrect. In fact, $0.7380 - 3 = -2.2620$.

Table II in the Tables Appendix can also be used in the reverse process, that is, to find x if $\log x$ is known. Since the entries in the body of Table II are numbers between 0 and 1, we must first write the number $\log x$ in the form

$$\log x = \log c + k$$

where $\log c$ is the mantissa, $0 \le \log c < 1$, and k, the characteristic, is an integer. (This is why we insisted in Example 1b that the number be left in the form $0.6542 - 3$.)

EXAMPLE 2

Find x if

(a) $\log x = 2.8351$ (b) $\log x = -6.6478$

SOLUTION

(a) We must write $\log x$ in the form

$$\log x = \log c + k, \quad 0 \le \log c < 1$$

so

$$\log x = 2.8351 = \underbrace{0.8351}_{\log c} + \underbrace{2}_{k}$$

We seek the mantissa 0.8351 in the body of Table II in the Tables Appendix and find that it corresponds to log 6.84. Since the characteristic $k = 2$,

$$x = 6.84 \times 10^2 = 684$$

(b) We have to proceed carefully to ensure that the mantissa is between 0 and 1. If we add and subtract 7, we have

$$\begin{aligned}\log x &= (7 - 6.6478) - 7\\ &= \underbrace{0.3522}_{\log c} \underbrace{- 7}_{k}\end{aligned}$$

We seek the mantissa 0.3522 in the body of Table II in the Tables Appendix and find

$$0.3522 = \log 2.25$$

Since the characteristic $k = -7$, we have

$$x = 2.25 \times 10^{-7} = 0.000000225$$

PROGRESS CHECK

Find x for the following.

(a) $\log x = 3.8457$ (b) $\log x = 0.6201 - 2$ (c) $\log x = -2.0487$

ANSWERS

(a) 7010 (b) 0.0417 (c) 0.00894

DATING THE LATEST ICE AGE

$$
\begin{aligned}
Q(t) &= q_0 e^{-kt} \\
0.254 q_0 &= q_0 e^{-0.00012t} \\
0.254 &= e^{-0.00012t} \\
\ln 0.254 &= \ln e^{-0.00012t} \\
-1.3704 &= -0.00012t \\
t &= 11{,}420
\end{aligned}
$$

All organic forms of life contain radioactive carbon 14. In 1947 the chemist Willard Libby (who won the Nobel prize in 1960) found that the percentage of carbon 14 in the atmosphere equals the percentage found in the living tissues of all organic forms of life. When an organism dies, it stops replacing carbon 14 in its living tissues. Yet the carbon 14 continues decaying at the rate of 0.012% per year. By measuring the amount of carbon 14 in the remains of an organism, it is possible to estimate fairly accurately when the organism died.

In the late 1940s radiocarbon dating was used to date the last ice sheet to cover the North American and European continents. Remains of trees in the Two Creeks Forest in northern Wisconsin were found to have lost 74.6% of their carbon 14 content. The remaining carbon 14, therefore, was 25.4% of the original quantity q_0 that was present when the descending ice sheet felled the trees. The accompanying computations use the general equation of an exponential decay model to find the age t of the wood. Conclusion: The latest ice age occurred approximately 11,420 years before the measurements were taken.

The following example shows how to use logarithms to simplify computations.

EXAMPLE 3

Approximate 478×0.0345 by using logarithms.

SOLUTION

If

$$N = 478 \times 0.0345$$

then

$$
\begin{aligned}
\log N &= \log(478 \times 0.0345) \\
&= \log 478 + \log 0.0345 \qquad \text{Property 1}
\end{aligned}
$$

Using Table II in the Tables Appendix, we find

$$
\begin{aligned}
\log 478 &= 2.6794 \\
\underline{\log 0.0345} &= \underline{0.5378 - 2} \\
\log N &= 3.2172 - 2 \qquad \text{Adding the logarithms} \\
&= 1.2172 = \underbrace{0.2172}_{\log c} + \underbrace{1}_{k}
\end{aligned}
$$

Looking in the body of Table II, we find that the mantissa 0.2172 does not appear. However, 0.2175 does appear and corresponds to log 1.65. Thus,

$$N \approx 1.65 \times 10^1 = 16.5$$

Inexpensive calculators have reduced the importance of logarithms as a computational device. Still, many calculators cannot, for example, handle $\sqrt[5]{14.2}$ directly. If you know how to compute with logarithms and if you combine this knowledge with a calculator that can handle logarithms, you can enhance the power of your calculator. Many additional applications of logarithms occur in more advanced mathematics, especially in calculus.

EXAMPLE 4

Approximate $\dfrac{\sqrt{47.4}}{(2.3)^3}$ by using logarithms.

SOLUTION

If

$$N = \frac{\sqrt{47.4}}{(2.3)^3}$$

then

$$\log N = \frac{1}{2} \log 47.4 - 3 \log 2.3$$

$$\frac{1}{2}\log 47.4 = \frac{1}{2}(1.6758) = 0.8379$$

$$3 \log 2.3 = 3(0.3617) = 1.0851$$

$$\log N = 0.8379 - 1.0851 = -0.2472 \quad \text{Subtracting the logarithms}$$

or

$$\log N = \underbrace{0.7528}_{\log c} \underbrace{- 1}_{k} \qquad \text{Adding and subtracting 1}$$

From the body of Table II in the Tables Appendix, we find $0.7528 \approx \log 5.66$, so

$$N \approx 5.66 \times 10^{-1} = 0.566$$

PROGRESS CHECK

Approximate by logarithms: $\dfrac{(4.64)^{3/2}}{\sqrt{7.42 \times 165}}$

ANSWER

0.286

Problems in compound interest provide us with an opportunity to demonstrate the power of logarithms in computational work. In Section 4.1 we showed that the amount S available when a principal P is invested at an annual interest rate r compounded k times a year is given by

$$S = P(1 + i)^n$$

where $i = r/k$, and n is the number of conversion periods.

EXAMPLE 5

If \$1000 is left on deposit at an interest rate of 8% per year compounded quarterly, how much money is in the account at the end of 6 years?

SOLUTION

We have $P = 1000$, $r = 0.08$, $k = 4$, and $n = 24$ (since there are 24 quarters in 6 years). Thus,

$$S = P(1 + i)^n = 1000\left(1 + \frac{0.08}{4}\right)^{24}$$
$$= 1000(1 + 0.02)^{24} = 1000(1.02)^{24}$$

Then

$$\log S = \log 1000 + 24 \log 1.02$$
$$= 3 + 24(0.0086) = 3.2064$$

From the body of Table II in the Tables Appendix, we find $0.2064 \approx \log 1.61$, so

$$S \approx 1.61 \times 10^3 = 1610$$

The account contains \$1610 (approximately) at the end of 6 years.

PROGRESS CHECK

If \$1000 is left on deposit at an interest rate of 6% per year compounded semiannually, approximately how much is in the account at the end of 4 years?

ANSWER

\$1267

EXERCISE SET 4.4

In Exercises 1–8 write each number in scientific notation.

1. 2725	2. 493	3. 0.0084	4. 0.000914
5. 716,000	6. 527,600,000	7. 296.2	8. 32.767

In Exercises 9–20 compute the logarithm by using Table II or Table III in the Tables Appendix.

9. log 3.56	10. ln 3.2	11. log 37.5	12. log 85.3
13. ln 4.7	14. ln 60	15. log 74	16. log 4230
17. log 48,200	18. log 7,890,000	19. log 0.342	20. log 0.00532

In Exercises 21–32 use Table II or Table III in the Tables Appendix to find x.

21. $\log x = 0.4014$	22. $\ln x = -0.5108$	23. $\ln x = 1.0647$	24. $\log x = 2.7332$

25. $\ln x = 2.0669$
26. $\log x = 0.1903 - 2$
27. $\log x = 0.4099 - 1$
28. $\log x = 0.7024 - 2$
29. $\log x = 0.7832 - 4$
30. $\log x = 0.9320 - 2$
31. $\log x = -1.6599$
32. $\log x = -3.9004$

In Exercises 33–43 find an approximate answer by using logarithms.

33. $(320)(0.00321)$
34. $(8780)(2.13)$
35. $\dfrac{679}{321}$
36. $\dfrac{88.3}{97.2}$
37. $(3.19)^4$
38. $(42.3)^3(71.2)^2$
39. $\dfrac{(87.3)^2(0.125)^3}{(17.3)^3}$
40. $\sqrt[3]{(66.9)^4(0.781)^2}$
41. $\dfrac{\sqrt{7870}}{(46.3)^4}$
42. $\dfrac{(7.28)^{2/3}}{\sqrt[3]{(87.3)(16.2)^4}}$
43. $\dfrac{(32.870)(0.00125)}{(12.8)(124{,}000)}$

44. The period T (in seconds) of a simple pendulum of length L (in feet) is given by the formula

$$T = 2\pi\sqrt{\frac{L}{g}}$$

Using common logarithms, find the approximate value of T if $L = 4.72$ feet, $g = 32.2$, and $\pi = 3.14$.

45. Use logarithms to find the approximate amount that accumulates if \$6000 is invested for 8 years in a bank paying 7% interest per year compounded quarterly?

46. Use logarithms to find the approximate sum if \$8000 is invested for 6 years in a bank paying 8% interest per year compounded monthly.

47. If \$10,000 is invested at 7.8% interest per year compounded semiannually, what sum is available after 5 years?

48. Which of the following offers will yield a greater return: 8% annual interest compounded annually, or 7.75% annual interest compounded quarterly?

49. Which of the following offers will yield a greater return: 9% annual interest compounded annually, or 8.75% annual interest compounded quarterly?

50. The area of a triangle whose sides are a, b, and c in length is given by the formula

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a+b+c)$. Use logarithms to find the approximate area of a triangle whose sides are 12.86 feet, 13.72 feet, and 20.3 feet.

4.5 EXPONENTIAL AND LOGARITHMIC EQUATIONS

Some exponential equations, such as $2^x = 8$, are easily solved. Here is a useful approach that will often work on tougher problems.

- To solve an exponential equation, take logarithms of both sides of the equation.
- To solve a logarithmic equation, form a single logarithm on one side of the equation, and then convert the equation to the equivalent exponential form.

EXAMPLE 1

Solve $3^{2x-1} = 17$.

SOLUTION

Taking logarithms to the base 10 of both sides of the equation, we have

$$\log 3^{2x-1} = \log 17$$
$$(2x - 1)\log 3 = \log 17 \qquad \text{Property 3}$$
$$2x - 1 = \frac{\log 17}{\log 3}$$
$$2x = 1 + \frac{\log 17}{\log 3}$$
$$x = \frac{1}{2} + \frac{\log 17}{2 \log 3}$$

If a numerical value is required, Table II in the Tables Appendix, or a calculator, can be used to approximate log 17 and log 3. Also, note that we could have taken logarithms to *any* base in solving this equation.

PROGRESS CHECK

Solve $2^{x+1} = 3^{2x-3}$

ANSWER

$$\frac{\log 2 + 3 \log 3}{2 \log 3 - \log 2}$$

EXAMPLE 2

Solve $\log x = 2 + \log 2$.

SOLUTION

If we rewrite the equation in the form

$$\log x - \log 2 = 2$$

then we can apply Property 2 to form a single logarithm:

$$\log \frac{x}{2} = 2$$

Now we convert to the equivalent exponential form:

$$\frac{x}{2} = 10^2 = 100$$

$$x = 200$$

PROGRESS CHECK

Solve $\log x - \frac{1}{2} = -\log 3$

ANSWER

$\sqrt{10}/3$

EXAMPLE 3
Solve $\log_2 x = 3 - \log_2(x + 2)$.

SOLUTION
Rewriting the equation with a single logarithm, we have

$$\begin{aligned} \log_2 x + \log_2(x+2) &= 3 \\ \log_2[x(x+2)] &= 3 && \text{Why?} \\ x(x+2) &= 2^3 = 8 && \text{Equivalent exponential form} \\ x^2 + 2x - 8 &= 0 \\ (x-2)(x+4) &= 0 && \text{Factoring} \\ x = 2 \quad &\text{or} \quad x = -4 \end{aligned}$$

The "solution" $x = -4$ must be rejected, since the original equation contains $\log_2 x$ and the domain of $f(x) = \log_a x$ is the set of positive real numbers.

PROGRESS CHECK
Solve $\log_3(x - 8) = 2 - \log_3 x$.

ANSWER
$x = 9$

EXAMPLE 4
World population is increasing at an annual rate of 2.5%. If we assume an exponential growth model, in how many years will the population double?

SOLUTION
The exponential growth model

$$Q(t) = q_0 e^{0.025t}$$

describes the population Q as a function of time t. Since the initial population is $Q(0) = q_0$, we seek the time t required for the population to double or become $2q_0$. We wish to solve the equation

$$Q(t) = 2q_0 = q_0 e^{0.025t}$$

for t. We then have

$$\begin{aligned} 2q_0 &= q_0 e^{0.025t} \\ 2 &= e^{0.025t} && \text{Dividing by } q_0 \\ \ln 2 &= \ln e^{0.025t} && \text{Taking natural logs of both sides} \\ &= 0.025t && \text{Since } \ln e^x = x \\ t &= \frac{\ln 2}{0.025} = \frac{0.6931}{0.025} = 27.7 \end{aligned}$$

or approximately 28 years.

EXAMPLE 5

A trust fund invests \$8000 at an annual interest rate of 8% compounded continuously. How long does it take for the initial investment to grow to \$12,000?

SOLUTION

By Equation (1) of Section 4.1,

$$S = Pe^{rt}$$

We have $S = 12{,}000$, $P = 8000$, and $r = 0.08$, and we must solve for t. Thus,

$$12{,}000 = 8000e^{0.08t}$$
$$\frac{12{,}000}{8000} = e^{0.08t}$$
$$e^{0.08t} = 1.5$$

Taking natural logarithms of both sides, we have

$$0.08t = \ln 1.5$$
$$t = \frac{\ln 1.5}{0.08} \approx \frac{0.4055}{0.08} \qquad \text{from Table III}$$
$$\approx 5.07$$

It takes approximately 5.07 years for the initial \$8000 to grow to \$12,000.

EXERCISE SET 4.5

In Exercises 1–31 solve for x.

1. $5^x = 18$
2. $2^x = 24$
3. $2^{x-1} = 7$
4. $3^{x-1} = 12$
5. $3^{2x} = 46$
6. $2^{2x-1} = 56$
7. $5^{2x-5} = 564$
8. $3^{3x-2} = 23.1$
9. $3^{x-1} = 2^{2x+1}$
10. $4^{2x-1} = 3^{2x+3}$
11. $2^{-x} = 15$
12. $3^{-x+2} = 103$
13. $4^{-2x+1} = 12$
14. $3^{-3x+2} = 2^{-x}$
15. $e^x = 18$
16. $e^{x-1} = 2.3$
17. $e^{2x+3} = 20$
18. $e^{-3x+2} = 40$
19. $\log x + \log 2 = 3$
20. $\log x - \log 3 = 2$
21. $\log_x(3 - 5x) = 1$
22. $\log_x(8 - 2x) = 2$
23. $\log x + \log(x - 3) = 1$
24. $\log x + \log(x + 21) = 2$
25. $\log(3x + 1) - \log(x - 2) = 1$
26. $\log(7x - 2) - \log(x - 2) = 1$
27. $\log_2 x = 4 - \log_2(x - 6)$
28. $\log_2(x - 4) = 2 - \log_2 x$
29. $\log_2(x + 4) = 3 - \log_2(x - 2)$
30. $y = \dfrac{e^x + e^{-x}}{2}$
31. $y = \dfrac{e^x - e^{-x}}{2}$

32. Suppose that world population is increasing at an annual rate of 2%. If we assume an exponential growth model, in how many years will the population double?

33. Suppose that the population of a certain city is increasing at an annual rate of 3%. If we assume an exponential growth model, in how many years will the population triple?

34. The population P of a certain city t years from now is given by

$$P = 20{,}000e^{0.05t}$$

How many years from now will the population be 50,000?

35. Potassium 42 has a decay rate of approximately 5.5% per hour. Assuming an exponential decay model, find the number of hours it will take for the original quantity of potassium 42 to be halved.

36. Consider an exponential decay model given by

$$Q = q_0e^{-0.4t}$$

where t is in weeks. How many weeks does it take for Q to decay to $\frac{1}{4}$ of its original amount?

37. How long does it take an amount of money to double if it is invested at a rate of 8% per year compounded semiannually?

38. At what rate of annual interest, compounded semiannually, should a certain amount of money be invested so that it will double in 8 years?

39. The number N of radios that an assembly line worker can assemble daily after t days of training is given by

$$N = 60 - 60e^{-0.04t}$$

After how many days of training does the worker assemble 40 radios daily?

40. The quantity Q (in grams) of a radioactive substance that is present after t days of decay is given by

$$Q = 400e^{-kt}$$

If $Q = 300$ when $t = 3$, find k, the decay rate.

41. A person on an assembly line produces P items per day after t days of training, where

$$P = 400(1 - e^{-t})$$

How many days of training will it take this person to be able to produce 300 items per day?

42. Suppose that the number N of mopeds sold when x thousands of dollars are spent on advertising is given by

$$N = 4000 + 1000 \ln(x + 2)$$

How much advertising money must be spent to sell 6000 mopeds?

TERMS AND SYMBOLS

exponential function (p. 171)
base (p. 171)
a^x (p. 171)
e (p. 175)
exponential growth model (p. 176)
growth constant (p. 176)
exponential decay model (p. 177)
decay constant (p. 177)
compound interest (p. 178)
conversion period (p. 178)
continuous compounding (p. 180)
logarithmic function (p. 183)
$\log_a x$ (p. 183)
$\ln x$ (p. 183)
natural logarithm (p. 183)
common logarithm (p. 198)
scientific notation (p. 198)
mantissa (p. 198)
characteristic (p. 198)

KEY IDEAS FOR REVIEW

- ☐ An exponential function has a variable in the exponent and has a base that is a positive constant.
- ☐ The graph of the exponential function $f(x) = a^x$, where $a > 0$ and $a \neq 1$,
  - passes through the points $(0, 1)$ and $(1, a)$ for any value of x;
  - is increasing if $a > 1$ and decreasing if $0 < a < 1$.
- ☐ The domain of the exponential function is the set of all real numbers; the range is the set of all positive numbers.
- ☐ If $a^x = a^y$, then $x = y$ (assuming $a > 0$, $a \neq 1$).
- ☐ If $a^x = b^x$ for all $x \neq 0$, then $a = b$ (assuming $a > 0$, $b > 0$).
- ☐ Exponential functions play a key role in the following important applications:
  - Exponential growth model: $Q(t) = q_0e^{kt}$, $k > 0$
  - Exponential decay model: $Q(t) = q_0e^{-kt}$, $k > 0$
  - Compound interest: $S = P(1 + i)^n$
  - Continuous compounding: $S = Pe^{rt}$

- The logarithmic function $\log_a x$ is the inverse of the function a^x.
- The logarithmic form $y = \log_a x$ and the exponential form $x = a^y$ are two ways of expressing the same relationship. In short, logarithms are exponents. Consequently, it is always possible to convert from one form to the other.
- The following identities are useful in simplifying expressions and in solving equations.

$$a^{\log_a x} = x \qquad \log_a a = 1$$
$$\log_a a^x = x \qquad \log_a 1 = 0$$

- The graph of the logarithmic function $f(x) = \log_a x$, where $x > 0$,
  - passes through the points (1, 0) and (a, 1) for any $a > 0$;
  - is increasing if $a > 1$ and decreasing if $0 < a < 1$.
- The domain of the logarithmic function is the set of all positive real numbers; the range is the set of all real numbers.
- If $\log_a x = \log_a y$, then $x = y$.
- If $\log_a x = \log_b x$ and $x \neq 1$, then $a = b$.
- The fundamental properties of logarithms are as follows.

Property 1. $\log_a(xy) = \log_a x + \log_a y$

Property 2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

Property 3. $\log_a x^n = n \log_a x$

- The fundamental properties of logarithms, used in conjunction with tables of logarithms, are a powerful tool in performing calculations. It is these properties that make the study of logarithms worthwhile.
- The change of base formula is

$$\log_b x = \frac{\log_a x}{\log_a b}$$

REVIEW EXERCISES

Solutions to exercises whose numbers are in color are in the Solutions section in the back of the book.

4.1 1. Sketch the graph of $f(x) = (\frac{1}{3})^x$. Label the point $(-1, f(-1))$.

2. Solve $2^{2x} = 8^{x-1}$ for x.

3. Solve $(2a + 1)^x = (3a - 1)^x$ for a.

4. The sum of \$8000 is invested in a certificate paying 12% annual interest compounded semiannually. What sum is available at the end of 4 years?

4.2 In Exercises 5–8 write each logarithmic form in exponential form and vice versa.

5. $27 = 9^{3/2}$

6. $\log_{64} 8 = \frac{1}{2}$

7. $\log_2 \frac{1}{8} = -3$

8. $6^0 = 1$

In Exercises 9–12 solve for x.

9. $\log_x 16 = 4$

10. $\log_5 \frac{1}{125} = x - 1$

11. $\ln x = -4$

12. $\log_3(x + 1) = \log_3 27$

In Exercises 13–16 evaluate the given expression.

13. $\log_3 3^5$

14. $\ln e^{-1/3}$

15. $\log_3\left(\frac{1}{3}\right)$

16. $e^{\ln 3}$

17. Sketch the graph of $f(x) = \log_3 x + 1$.

4.3 In Exercises 18–21 write the given expression in terms of simpler logarithmic forms.

18. $\log_a \frac{\sqrt{x-1}}{2x}$

19. $\log_a \frac{x(2-x)^2}{(y+1)^{1/2}}$

20. $\ln(x + 1)^4(y - 1)^2$

21. $\log \sqrt[5]{\frac{y^2 z}{z+3}}$

In Exercises 22–25 use the values $\log 2 = 0.30$, $\log 3 = 0.50$, and $\log 7 = 0.85$ to evaluate the given expression.

22. $\log 14$

23. $\log 3.5$

24. $\log \sqrt{6}$

25. $\log 0.7$

In Exercises 26–29 write the given expression as a single logarithm.

26. $\frac{1}{3}\log_a x - \frac{1}{2}\log_a y$

27. $\frac{4}{3}[\log x + \log(x-1)]$

28. $\ln 3x + 2\left(\ln y - \frac{1}{2}\ln z\right)$

29. $2\log_a(x+2) - \frac{3}{2}\log_a(x+1)$

In Exercises 30 and 31 use the values $\log 32 = 1.5$, $\log 8 = 0.9$, and $\log 5 = 0.7$ to find the requested value.

30. $\log_8 32$ 31. $\log_5 32$

4.4 In Exercises 32–35 write the given number in scientific notation.

32. 476.5 33. 0.098

34. 26,475 35. 77.67

In Exercises 36–38 use logarithms to calculate the value of the given expression.

36. $(0.765)(32.4)^2$ 37. $\sqrt{62.3}$

38. $\dfrac{2.1}{(32.5)^{5/2}}$

39. A substance is known to have a decay rate of 6% per hour. Approximately how many hours are required for the remaining quantity to be half of the original quantity?

4.5 In Exercises 40–42 solve for x.

40. $2^{3x-1} = 14$

41. $2\log x - \log 5 = 3$

42. $\log(2x-1) = 2 + \log(x-2)$

PROGRESS TEST 4A

1. Sketch the graph of $f(x) = 2^{x+1}$. Label the point $(1, f(1))$.
2. Solve $(\frac{1}{2})^x = (\frac{1}{4})^{2x+1}$

In Problems 3 and 4 convert from logarithmic form to exponential form or vice versa.

3. $\log_3 \frac{1}{9} = -2$ 4. $64 = 16^{3/2}$

In Problems 5 and 6 solve for x.

5. $\log_x 27 = 3$ 6. $\log_6\left(\frac{1}{36}\right) = 3x + 1$

In Problems 7 and 8 evaluate the given expression.

7. $\ln e^{5/2}$ 8. $\log_5 \sqrt{5}$

In Problems 9 and 10 write the given expression in terms of simpler logarithmic forms.

9. $\log_a \dfrac{x^3}{y^2 z}$ 10. $\log \dfrac{x^2\sqrt{2y-1}}{y^3}$

In Problems 11 and 12 use the values $\log 2.5 = 0.4$ and $\log 2 = 0.3$ to evaluate the given expression.

11. $\log 5$ 12. $\log 2\sqrt{2}$

In Problems 13 and 14 write the given expression as a single logarithm.

13. $2\log x - 3\log(y+1)$

14. $\frac{2}{3}[\log_a(x+3) - \log_a(x-3)]$

In Problems 15 and 16 write the given number in scientific notation.

15. 0.000273 16. 5.972

In Problems 17 and 18 use logarithms to evaluate the given expression.

17. $\dfrac{72.9}{(39.4)^2}$ 18. $\sqrt[3]{0.0176}$

19. The number of bacteria in a culture is described by the exponential growth model

$$Q(t) = q_0 e^{0.02t}$$

Approximately how many hours are required for the number of bacteria to double?

In Problems 20 and 21 solve for x.

20. $\log x - \log 2 = 2$

21. $\log_4(x-3) = 1 - \log_4 x$

PROGRESS TEST 4B

1. Sketch the graph of $f(x) = (\frac{1}{2})^{x-1}$. Label the point $(0, f(0))$.
2. Solve $(a+3)^x = (2a-5)^x$ for a.

In Problems 3 and 4 convert from logarithmic form to exponential form and vice versa.

3. $\dfrac{1}{1000} = 10^{-3}$ 4. $\log_3 1 = 0$

In Problems 5 and 6 solve for x.

5. $\log_2(x - 1) = -1$ 6. $\log_{2x} 27 = \log_3 27$

In Problems 7 and 8 evaluate the given expression.

7. $\log_3 3^{10}$ 8. $e^{\ln 4}$

In Problems 9 and 10 write the given expression in terms of simpler logarithmic forms.

9. $\log_a(x - 1)(y + 3)^{5/4}$ 10. $\ln \sqrt{xy}\,\sqrt[4]{2z}$

In Problems 11 and 12 use the values $\log 2.5 = 0.4$, $\log 2 = 0.3$, and $\log 6 = 0.75$ to evaluate the given expression.

11. $\log 7.5$ 12. $\log 36$

In Problems 13 and 14 write the given expression as a single logarithm.

13. $\dfrac{3}{5}\ln(x - 1) + \dfrac{2}{5}\ln y - \dfrac{1}{5}\ln z$

14. $\log \dfrac{x}{y} - \log \dfrac{y}{x}$

In Problems 15 and 16 write the given number in scientific notation.

15. 22,684,321 16. 0.297

In Problems 17 and 18 use logarithms to evaluate the given expression.

17. $(0.295)(31.7)^3$ 18. $\dfrac{\sqrt{42.9}}{(3.75)^2(747)}$

19. Suppose that \$500 is invested in a certificate at an annual interest rate of 12% compounded monthly. What is the value of the investment after 6 months?

In Problems 20 and 21 solve for x.

20. $\log_x(x + 6) = 2$

21. $\log(x - 9) = 1 - \log x$

5

THE TRIGONOMETRIC FUNCTIONS

The word *trigonometry* derives from the Greek, meaning "measurement of triangles." The conventional approach to the subject matter of trigonometry deals with relationships among the sides and angles of a triangle, reflecting the important applications of trigonometry in such fields as navigation and surveying.

The modern approach to trigonometry emphasizes the function concept that was introduced in Chapter 3. This has become the accepted approach since it demonstrates the unifying influence of the function concept.

In this chapter we will define this important class of functions and will discuss their fundamental properties and graphs. We will devote the next chapter to the study of triangles and their applications.

5.0 REVIEW OF GEOMETRY

We need to recall various facts about the circle from plane geometry. A line segment joining the center of a circle to any point on the circle is called a **radius.** Since every point on the circle is the same distance from the center, the radii of a circle are all equal. Thus, in Figure 1, $\overline{OP} = \overline{OQ}$. A **chord** of a circle is a line

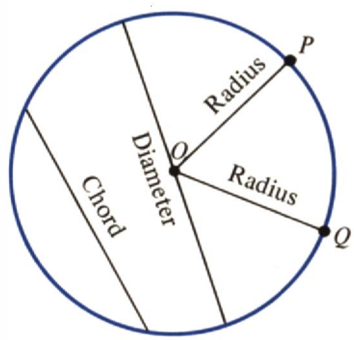


FIGURE 1

segment joining any two points on the circle; a **diameter** of a circle is a chord that passes through the center of the circle. Note that the length of a diameter is twice that of a radius.

The **circumference** C of a circle is the distance around the circle and is given by

$$C = 2\pi r$$

where r is the radius of the circle. The constant π is then seen to be the ratio of the circumference of a circle to the length of its diameter and has an approximate value of 3.14159. The **area** A of a circle of radius r is given by

$$A = \pi r^2$$

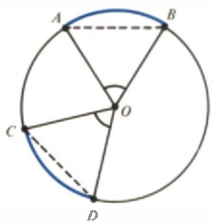


FIGURE 2

An **arc** of a circle is simply a part of a circle. The arc $\widehat{AB}$ of Figure 2 consists of the two endpoints A and B and the set of all points on the circle that are between A and B and are shown in color.

A **central angle** has its vertex at the center of the circle, and its sides are radii of the circle. We define the measure of a central angle to be the same as that of the arc it intercepts. Thus, in Figure 2, the measurement of $\angle AOB$ and $\widehat{AB}$ are the same. We can then show that equal arcs determine or subtend equal chords. If arc $\widehat{AB}$ = arc $\widehat{CD}$ in Figure 2, then, by definition, $\measuredangle AOB = \measuredangle COD$. Since $\overline{AO} = \overline{BO} = \overline{CO} = \overline{DO}$ are all radii, it follows that triangles AOB and COD are congruent. Hence, $\overline{CD} = \overline{AB}$. The converse can be proven in a similar manner. Thus,

Equal arcs determine equal chords.
Equal chords determine equal arcs.

5.1 ANGLES AND THEIR MEASUREMENT

DEFINITION OF AN ANGLE

In geometry you frequently dealt with angles formed by the sides of a triangle. In trigonometry we need to introduce a more general concept of angle. The angles we use will be much less restricted; not only are they unlimited in magnitude, but they can be either positive or negative as well.

An **angle** is the geometric shape formed by two rays or half-lines with a common endpoint. For our purposes, it is convenient to think of an angle as the result of a rotation of a ray about its endpoint. In Figure 3 the **initial side** is rotated about its endpoint at O until it coincides with the **terminal side** to form the angle α.

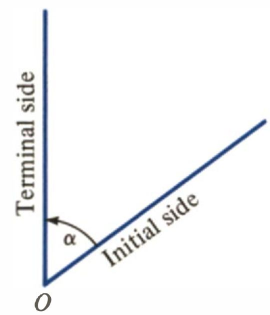


FIGURE 3

We will frequently place an angle on a rectangular coordinate system in **standard position** so that the initial side coincides with the positive x-axis and the rotation occurs about the endpoint at the origin. In Figure 4a the angle α results from a rotation in the counterclockwise direction, and we say that α is a **positive angle**. In Figure 4b the ray has been rotated in a clockwise direction to form the angle β, and we say that β is a **negative angle.**

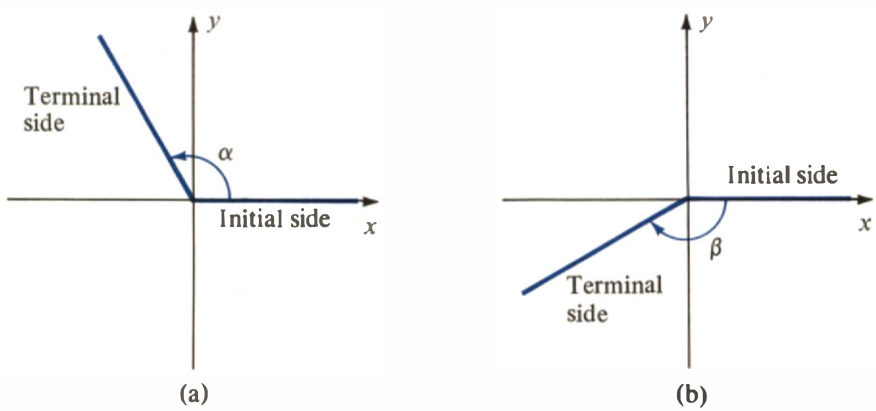


FIGURE 4

If an angle is in standard position and its terminal side coincides with a coordinate axis, the angle is called a **quadrantal angle;** otherwise, the angle is said to lie in the same quadrant as its terminal side. Figure 5 displays an angle in each of the four quadrants. Note that the quadrant designation depends only upon the quadrant in which the terminal side lies and not upon the direction of rotation.

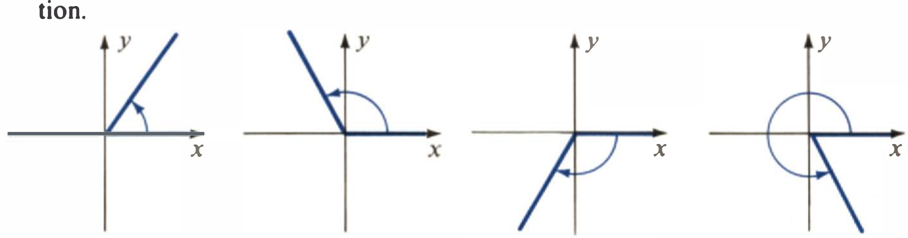


FIGURE 5

THE UNIT CIRCLE

Before we talk about angle measurement, it is helpful to define the **unit circle** as a circle of radius one whose center is at the origin of a rectangular coordinate system (Figure 6a). A point $P(x,y)$ is on the unit circle if and only if the distance $\overline{OP} = 1$. Using the distance formula,

$$\overline{OP} = \sqrt{(x-0)^2 + (y-0)^2} = 1$$

Squaring both sides, we conclude

> The equation of the unit circle is
> $$x^2 + y^2 = 1$$

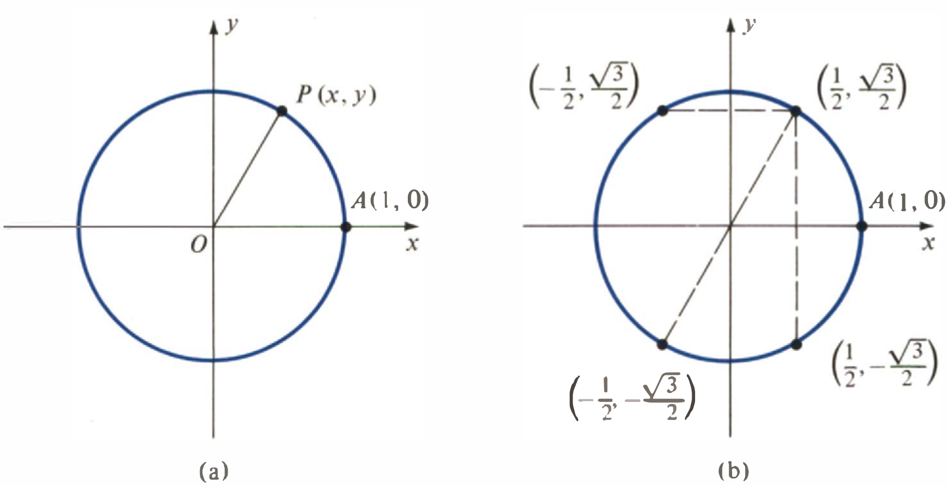


FIGURE 6

Using the methods of Section 3.1, we find that the unit circle is symmetric with respect to the x-axis, the y-axis, and the origin. These symmetries will prove to be very useful. For example, you can easily verify that the point $(1/2, \sqrt{3}/2)$ lies on the unit circle. Figure 6b shows the coordinates of various other points that can be obtained from the symmetries of the circle. (See Exercises 61 and 62.)

ANGULAR MEASUREMENT: DEGREES AND RADIANS

In geometry, you used **degree measure** of an angle. This measure is the result of an arbitrary (but convenient) selection made thousands of years ago.

> **Degree Measure of an Angle**
> An angle formed by one complete revolution of an initial side back to its starting position has a measure of 360 degrees (written 360°).

Since

$$360° = 1 \text{ revolution}$$

it follows that the **right angle** obtained by one-fourth of a complete revolution has a measure of 90°. An angle between 0° and 90° is called an **acute angle;** an angle between 90° and 180° is called an **obtuse angle.**

There is a second unit of angular measure, called **radian measure,** that is often used in mathematics and has certain advantages over degree measure. To introduce this unit, we place an angle θ in standard position and include a unit circle as in Figure 7. The terminal side of the angle θ intersects the unit circle at the point P and the arc $\overset{\frown}{AP}$ is of length t. We then say that θ is an **angle of t radians.** We are, in effect, measuring the angle θ by the length of the arc that θ intercepts on the unit circle. (We will later show that it is possible to find the radian measure of an angle by using *any* circle.)

Radian Measure of an Angle

The radian measure of an angle θ is the length t of the arc that θ intercepts on the unit circle. We write

$$\theta = t \text{ radians} \quad \text{or} \quad \theta = t$$

If the arc is measured in the counterclockwise direction, then t is positive. If the arc is measured in the clockwise direction, then t is negative.

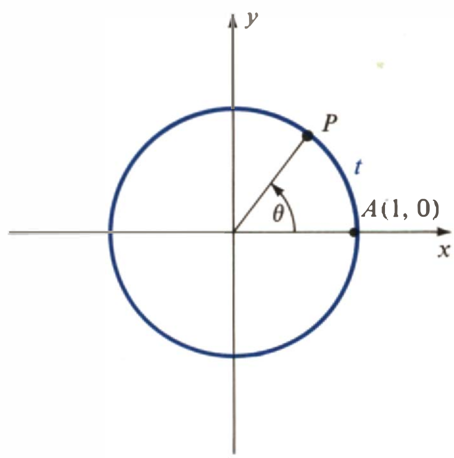


FIGURE 7

Since we will be dealing with arcs on a unit circle, it is convenient to establish a standard notation and terminology. The letter A will always denote the point $A(1, 0)$ as in Figure 7. If the arc $\overset{\frown}{AP}$ on the unit circle is of length t, we will say that P is the **unit circle point** corresponding to t or determined by t.

ANGLE CONVERSION

We can obtain a better grasp of radian measure by developing a simple relationship between the degree measure and the radian measure of an angle. An angle in standard position that traces a complete revolution in the counterclockwise direction has, by definition, a measure of 360°. This angle intercepts an arc on the unit circle that corresponds to its circumference and the arc must therefore be of length $C = 2\pi r = 2\pi$ since $r = 1$. Thus, 2π radians = 360° or

$$\boxed{\pi \text{ radians} = 180^\circ}$$

This relationship enables us to transform angular measure from radians to degrees and vice versa. From the equation just given we obtain

$$\boxed{1 \text{ radian} = \left(\frac{180}{\pi}\right)^\circ \quad \text{and} \quad 1^\circ = \frac{\pi}{180} \text{ radians}}$$

EXAMPLE 1

Convert 120° to radian measure.

SOLUTION

Since $1^\circ = \pi/180$ radians, we must have

$$120^\circ = 120\left(\frac{\pi}{180}\right) = \frac{2\pi}{3} \text{ radians}$$

PROGRESS CHECK

Convert the following from degree to radian measure.

(a) -210° (b) 390°

ANSWERS

(a) $-\dfrac{7\pi}{6}$ radians (b) $\dfrac{13\pi}{6}$ radians

---

EXAMPLE 2

Convert $2\pi/3$ radians to degree measure.

SOLUTION

Since 1 radian = $180/\pi$ degrees, we have

$$\frac{2\pi}{3} \text{ radians} = \frac{2\pi}{3}\left(\frac{180}{\pi}\right)^\circ = 120^\circ$$

PROGRESS CHECK

Convert the following from radian measure to degrees.

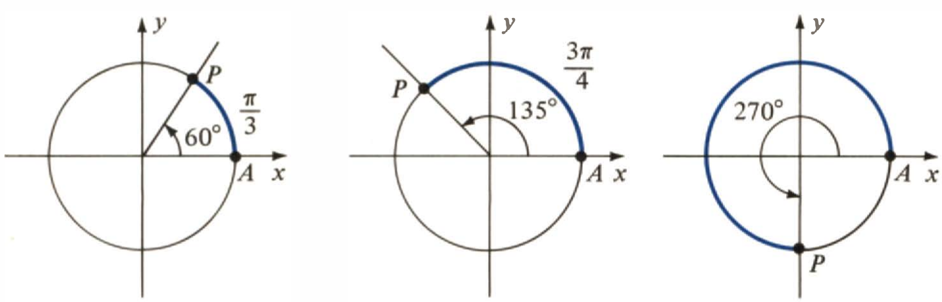
(a) $\dfrac{9\pi}{2}$ radians (b) $-\dfrac{4\pi}{3}$ radians

ANSWERS
(a) 810° (b) −240°

There are certain angles that we will use frequently in the examples and exercises throughout this chapter. It will prove helpful if you take the time now to verify the conversions shown in Table 1; you will then see how easy it is to switch between degree and radian measure for these values. Figure 8 displays some angles in standard position and shows both the degree measure and the radian measure.

TABLE 1

Radians	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$	2π
Degrees	30°	45°	60°	90°	180°	270°	360°



(a) $\theta = 60° = \frac{\pi}{3}$ radians (b) $\theta = 135° = \frac{3\pi}{4}$ radians (c) $\theta = 270° = \frac{3\pi}{2}$ radians

FIGURE 8

EXAMPLE 3
If the angle θ is in standard position, determine the quadrant in which the angle lies.
(a) $\theta = 200°$ (b) $\theta = 7\pi/4$ radians

SOLUTION
Figure 9 shows the quadrantal angles in standard position.
(a) Since $\theta = 200°$ is between 180° and 270°, θ lies in quadrant III.
(b) Since $\theta = 7\pi/4$ radians is between $3\pi/2$ and 2π, θ lies in quadrant IV.

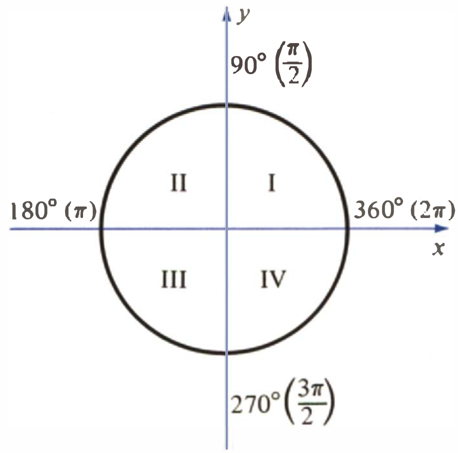
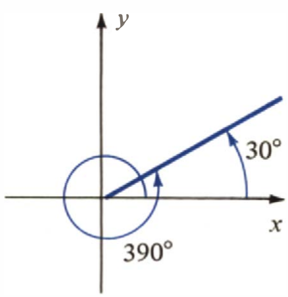


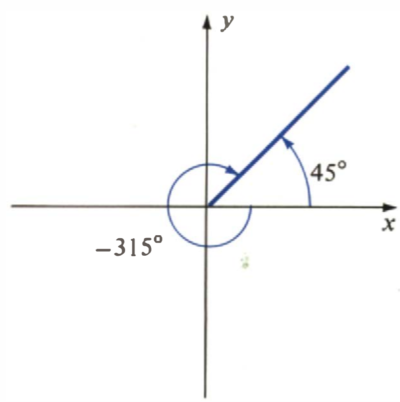
FIGURE 9

COTERMINAL ANGLES

Figure 10a displays an angle of 30° in standard position. Since a revolution of 360° returns to the same position, different angles in standard position may have the same terminal side. For instance, the angles of 30° and 390° shown in Figure 10a have the same terminal side and are said to be **coterminal angles.** Similarly, the angles of 45° and $-315°$ shown in Figure 10b are coterminal angles.



(a)



(b)

FIGURE 10

It is intuitively clear that any number of revolutions in either direction will return to the same terminal side. Since a revolution requires 360°, or 2π radians, we can write this result in a mathematical shorthand as follows:

Coterminal Angles

An angle θ in standard position is coterminal with every angle of the form

$$\theta + 360°n \qquad \text{(degree measure)}$$

or

$$\theta + 2\pi n \qquad \text{(radian measure)}$$

where n is an integer.

EXAMPLE 4

Find a first quadrant angle that is coterminal with an angle of (a) 410° (b) $-5\pi/3$ radians

SOLUTION

(a) With $\theta = 410°$ and $n = -1$, we have

$$\theta + 360°n = 410° - 360° = 50°$$

(b) With $\theta = -5\pi/3$ radians and $n = 1$, we have

$$\theta + 2\pi n = -\frac{5\pi}{3} + 2\pi = \frac{\pi}{3} \text{ radians}$$

PROGRESS CHECK

Show that each pair of angles is coterminal.

(a) $-265°$ and 95° (b) $22\pi/3$ and $4\pi/3$ radians

THE CENTRAL ANGLE FORMULA

The radian measure of an angle can be found by using a circle other than a unit circle. In Figure 11 the central angle θ subtends an arc of length t on the unit circle and an arc of length s on a circle of radius r. By definition, $\theta = t$. Since the ratio

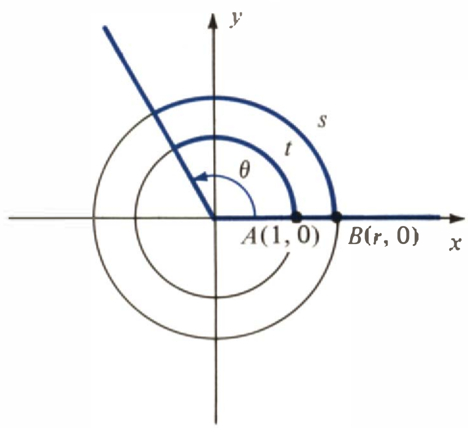


FIGURE 11

of the arcs is the same as the ratio of the radii, we have

$$\frac{t}{s} = \frac{1}{r}$$

or

$$t = \frac{s}{r}$$

Since $\theta = t$, we have the following useful result.

Central Angle Formula

If a central angle θ subtends an arc of length s on a circle of radius r, then the radian measure of θ is given by

$$\theta = \frac{s}{r}$$

and the length of the arc s is given by

$$s = r\theta$$

Note that when $s = r$, $\theta = 1$ radian. This leads to an alternative definition of radian measure.

Radian Measure

An angle of **one radian** subtends an arc on a circle whose length equals the length of the radius of the circle.

WARNING The formula

$$\theta = \frac{s}{r}$$

can only be applied if the angle θ is in radian measure.

EXAMPLE 5

A central angle θ subtends an arc of length 12 inches on a circle whose radius is 6 inches. Find the radian measure of the central angle.

SOLUTION

We have $s = 12$ and $r = 6$, so that

$$\theta = \frac{s}{r} = \frac{12}{6} = 2 \text{ radians}$$

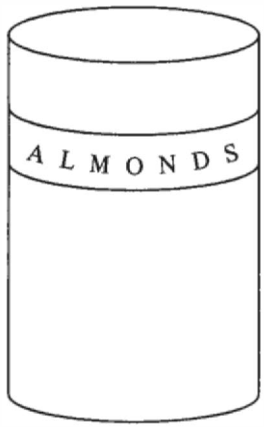
In Example 5 we used the formula $\theta = s/r$ to find that the radian measure of θ is given by

$$\theta = \frac{12 \text{ inches}}{6 \text{ inches}} = 2$$

This shows that the *radian measure of an angle is dimensionless*. We may therefore treat the radian measure of an angle as a real number. We will make use of this result throughout the next section.

EXAMPLE 6

A designer has to place the word ALMONDS on a can using equally spaced letters (see Figure 12a). For good visibility, the letters must cover a sector of the circle having a 90° central angle. If the base of the can is a circle of radius 2 inches (see Figure 12b), what is the maximum width of each letter?



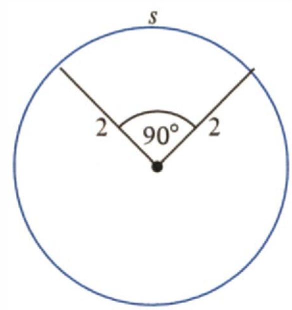


FIGURE 12

SOLUTION

Since $\theta = 90° = \pi/2$ radians, the arc has length

$$s = r\theta = \frac{\pi}{2}(2) = \pi$$

Each of the seven letters can occupy 1/7 of this arc, or $\pi/7$ inches, which is about 0.45 inches.

EXERCISE SET 5.1

If the angle θ is in standard position, determine the quadrant in which the angle lies.

1. $\theta = 313°$
2. $\theta = 182°$
3. $\theta = 14°$
4. $\theta = 227°$
5. $\theta = 141°$
6. $\theta = -167°$
7. $\theta = -345°$
8. $\theta = 555°$
9. $\theta = 618°$
10. $\theta = -428°$
11. $\theta = -195°$
12. $\theta = 730°$
13. $\theta = \frac{7\pi}{8}$
14. $\theta = \frac{-3\pi}{5}$
15. $\theta = \frac{-8\pi}{3}$
16. $\theta = \frac{3\pi}{8}$
17. $\theta = \frac{13\pi}{3}$
18. $\theta = \frac{9\pi}{5}$

Convert from degree measure to radian measure.

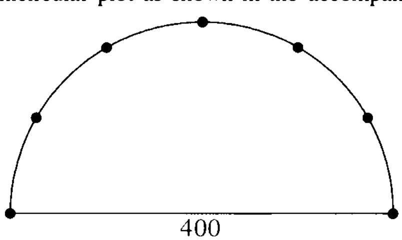
19. $30°$
20. $200°$
21. $-150°$
22. $-330°$
23. $75°$
24. $570°$
25. $-450°$
26. $-570°$
27. $135°$
28. $405°$
29. $120°$
30. $90°$
31. $45.22°$
32. $196.54°$

Convert from radian measure to degree measure.

33. $\frac{\pi}{4}$
34. $\frac{\pi}{3}$
35. $\frac{3\pi}{2}$
36. $\frac{5\pi}{6}$
37. $\frac{-\pi}{2}$
38. $\frac{-7\pi}{12}$
39. $\frac{4\pi}{3}$
40. 3π
41. $\frac{5\pi}{2}$
42. -5π
43. $\frac{-5\pi}{3}$
44. $\frac{9\pi}{2}$
45. 1.72
46. 24.98

For each pair of angles, write T if they are coterminal and F if they are not coterminal.

47. $30°, 390°$
48. $50°, -310°$
49. $45°, -45°$
50. $120°, \frac{14\pi}{3}$
51. $\frac{\pi}{2}, \frac{7\pi}{2}$
52. $-60°, 760°$
53. If a central angle θ subtends an arc of length 4 centimeters on a circle of radius 7 centimeters, find the approximate measure of θ in radians and in degrees.
54. Find the length of an arc subtended by a central angle of $\pi/5$ radians on a circle of radius 6 inches.
55. Find the radius of a circle if a central angle of $2\pi/3$ radians subtends an arc of 4 meters.
56. In a circle of radius 150 centimeters, what is the length of an arc subtended by a central angle of $45°$?
57. A subcompact car uses a tire whose radius is 13 inches. How far has the car moved when the tire completes one rotation? How many rotations are completed when the tire has traveled one mile? (Assume $\pi \approx 3.14$.)
58. A builder intends placing 7 equally spaced homes on a semicircular plot as shown in the accompanying figure. If the circle has a diameter of 400 feet, what is the distance between any two adjacent homes? (Use $\pi \approx 3.14$.)



59. How many ribs are there in an umbrella if the length of each rib is 1.5 feet and the arc between two adjacent ribs measures $3\pi/10$ feet?
60. A microcomputer accepts both $5\frac{1}{4}$-inch and 8-inch floppy disks. If both disks are divided into 8 sectors, find the ratio of the arc length of a sector of the 8-inch disk to the arc length of a sector of the $5\frac{1}{4}$-inch disk.
61. If the point (a, b) is on the unit circle, show that $(a, -b)$, $(-a, b)$, and $(-a, -b)$ also lie on the unit circle.
62. If the point (a, b) is on the unit circle, show that (b, a), $(b, -a)$, $(-b, a)$, and $(-b, -a)$ also lie on the unit circle.

5.2 THE SINE, COSINE, AND TANGENT FUNCTIONS

Figure 13a displays an angle θ whose radian measure is t. Focusing on the unit circle point P determined by t, we assume the coordinates of P to be (x, y). There are six **trigonometric functions** that are defined by the coordinates of the point P. In this section we will discuss the **sine, cosine,** and **tangent** functions, which are written as **sin, cos,** and **tan,** respectively. The remaining three functions are reciprocals of these and will be described in Section 5.5. We now define the first three trigonometric functions.

DEFINITION

Trigonometric Functions If $P(x, y)$ is the unit circle point determined by the real number t, then

$$\sin t = y$$

$$\cos t = x$$

$$\tan t = \frac{y}{x}, \quad x \neq 0$$

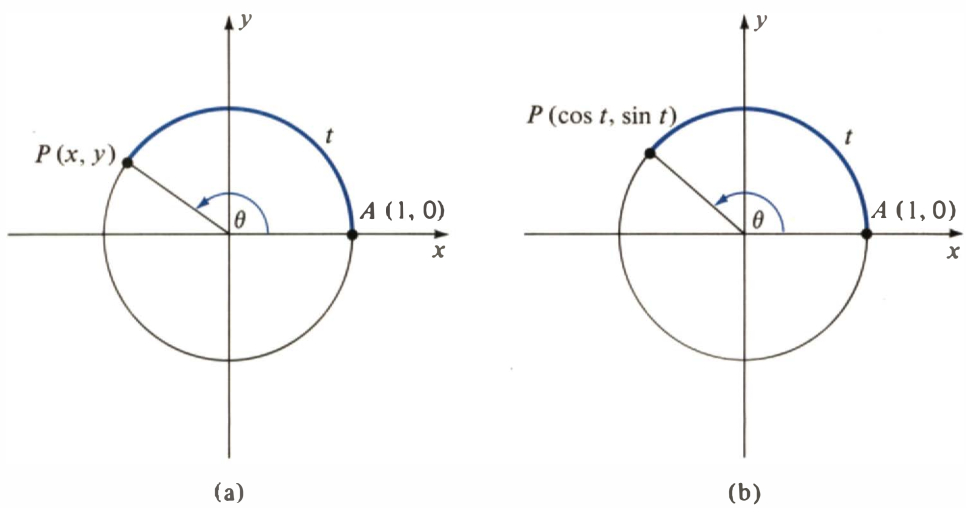


FIGURE 13

The trigonometric functions are seen to be functions of a real number t whose values are determined by the coordinates of the unit circle point P corresponding to t as in Figure 13b.

It is convenient to speak of the trigonometric functions of an angle. To do so, we simply associate the angle with its radian measure.

If an angle θ has a radian measure of t, then

$$\sin \theta = \sin t$$

$$\cos \theta = \cos t$$

$$\tan \theta = \tan t$$

From here on we will use Greek letters (such as α, β, γ, and θ) to represent *angles* and Roman letters (such as s, t, u, and v) to represent arc lengths or *real numbers*. In higher mathematics as well as in engineering and computer science, it is generally more convenient to deal with sine, cosine, and tangent as functions of real numbers, and we will stress this approach throughout the remainder of this chapter. We will work with angular measure whenever it is convenient and to make sure that you are comfortable with both approaches.

We stated earlier that an angle in standard position is said to lie in the same quadrant as its terminal side. Similarly, if a real number t determines the unit circle point P as shown in Figure 13, we say that t lies in the same quadrant as the point P.

EXAMPLE 1

The coordinates of the unit circle point P determined by the real number t are $(-3/4, -\sqrt{7}/4)$. Determine

(a) the quadrant in which t lies

(b) $\sin t$, $\cos t$, and $\tan t$

SOLUTION

(The student is urged to first verify that the coordinates $(-3/4, -\sqrt{7}/4)$ satisfy the equation of the unit circle.)

(a) The point P is sketched in Figure 14 and lies in the third quadrant. Then, by definition, t lies in the third quadrant.

(b) From the definitions of the trigonometric functions we have

$$\sin t = y = -\frac{\sqrt{7}}{4}$$

$$\cos t = x = -\frac{3}{4}$$

$$\tan t = \frac{y}{x} = \frac{\sqrt{7}}{3}$$

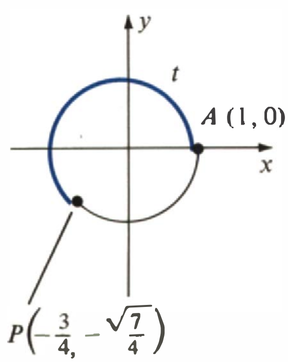


FIGURE 14

DOMAIN AND RANGE

From the definitions, we see immediately that the domain of the sine and cosine functions is the set of all real numbers. The tangent function, however, is not defined when $x = 0$. The unit circle points where $x = 0$ correspond to arcs of length $t = \pi/2$ and $t = 3\pi/2$. From our earlier work with coterminal angles we see that $x = 0$ whenever

$$t = \frac{\pi}{2} + 2\pi n \quad \text{and} \quad t = \frac{3\pi}{2} + 2\pi n$$

or, more compactly, when $t = \pi/2 + \pi n$ for all integers n. We conclude that the tangent function is not defined for these values.

Domain of the Trigonometric Functions	sin t: all real values of t cos t: all real values of t tan t: all real values of t such that $t \neq \frac{\pi}{2} + \pi n$

Turning again to the definitions, we note that the values of sin t and cos t correspond to coordinates of points on the unit circle. Since the coordinates of a point on the unit circle cannot exceed 1 in absolute value, we must have

$$|\sin t| \leq 1 \quad \text{and} \quad |\cos t| \leq 1$$

Range of Sine and Cosine	The range of the sine and cosine functions is given by $-1 \leq \sin t \leq 1 \qquad -1 \leq \cos t \leq 1$

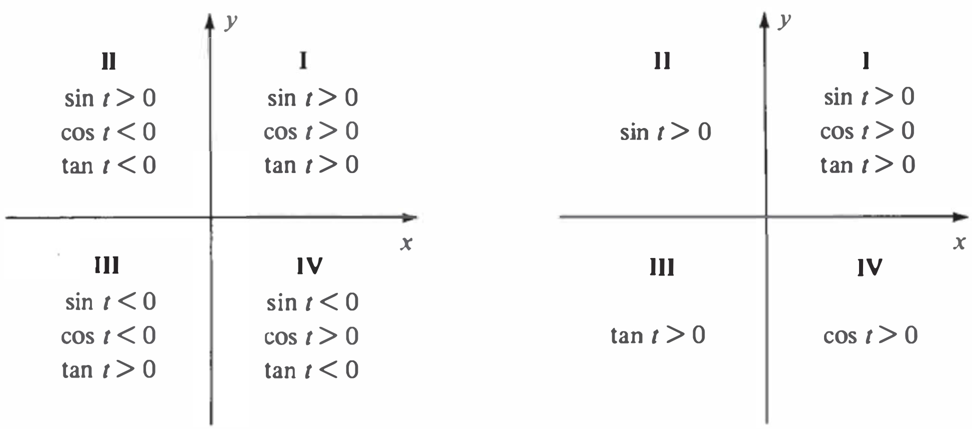
We will show in Section 5.4 that the range of the tangent function consists of the set of all real numbers. The student is invited to use the definition of tan t to show that this is a reasonable conclusion.

PROPERTIES OF THE TRIGONOMETRIC FUNCTIONS

Signs

In mathematics, whenever we define a new quantity or function, we then proceed to investigate its properties. We will spend the rest of this section determining some simple properties of the trigonometric functions.

The *signs* of the trigonometric functions in each of the four quadrants are shown in Figure 15a. These follow immediately from the definitions. For example, since both the x- and y-coordinates of any point in the third quadrant are negative, the sine and cosine functions both have negative values if t is in the third quadrant. The tangent will be positive in the third quadrant since it is the ratio y/x of two negative values.



(a) (b)

FIGURE 15

Figure 15b shows where each of the trigonometric functions is positive. If you remember this, you can then determine by inference the quadrants in which the trigonometric functions are negative.

EXAMPLE 2

Determine the quadrant in which t lies in each of the following.

(a) $\sin t > 0$ and $\tan t < 0$ (b) $\sin t < 0$ and $\cos t > 0$

SOLUTION

(a) $\sin t > 0$ in quadrants I and II; $\tan t < 0$ in quadrants II and IV. Both conditions therefore apply only in quadrant II.

(b) $\sin t < 0$ in quadrants III and IV; $\cos t > 0$ in quadrants I and IV. Both conditions therefore apply only in quadrant IV.

PROGRESS CHECK

Determine the quadrant in which t lies in each of the following.

(a) $\cos t < 0$ and $\tan t > 0$ (b) $\cos t < 0$ and $\sin t > 0$

ANSWERS

(a) quadrant III (b) quadrant II

Negative Arguments

We can use the symmetries of the unit circle to find $\sin(-t)$ and $\cos(-t)$. In Figure 16 we see that arcs of lengths t and $-t$ correspond to points P and P', for which the x-coordinates are the same and the y-coordinates are opposite in sign. Thus

$$\sin t = y \quad \text{and} \quad \sin(-t) = -y$$

or $\sin(-t) = -\sin t$. Similarly,

$$\cos t = x = \cos(-t) \quad \text{and} \quad \tan(-t) = -\frac{y}{x} = -\tan t$$

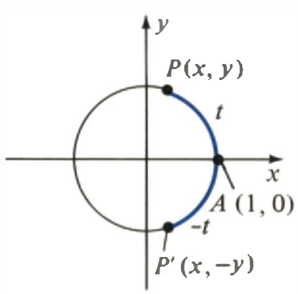


FIGURE 16

In summary,

$$\boxed{\begin{aligned} \sin(-t) &= -\sin t \\ \cos(-t) &= \cos t \\ \tan(-t) &= -\tan t \end{aligned}}$$

where t is any real number in the domain of the function.

EXAMPLE 3

(a) Given $\sin(\pi/6) = 1/2$, find $\sin(-\pi/6)$.

(b) Given $\cos 45° = \sqrt{2}/2$, find $\cos(-45°)$.

SOLUTION

(a) $\sin\left(-\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$

(b) $\cos(-45°) = \cos(45°) = \frac{\sqrt{2}}{2}$

PROGRESS CHECK

Given $\tan(\pi/4) = 1$, find $\tan(-45°)$.

ANSWER

-1

Symmetries of the Circle (Optional)

The following theorem can be combined with earlier results to extend our ability to find values of the trigonometric functions.

> If $P(a, b)$ and $P'(a', b')$ are the unit circle points corresponding to the real numbers t and $t \pm \pi/2$, respectively, then either
>
> (i) $(a', b') = (-b, a)$, or
>
> (ii) $(a', b') = (b, -a)$.

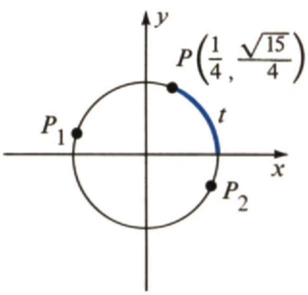


FIGURE 17

The proof is outlined in Exercises 38–40. We will use this result in the next example.

EXAMPLE 4

In Figure 17 the unit circle point $P(1/4, \sqrt{15}/4)$ corresponds to the real number t. Find

(a) $\sin\left(t + \frac{\pi}{2}\right)$ (b) $\cos\left(t - \frac{\pi}{2}\right)$

SOLUTION

In Figure 17, unit circle points P_1 and P_2 correspond to $t + \pi/2$ and to $t - \pi/2$, respectively. From the symmetries of the circle, their coordinates are

$$t + \frac{\pi}{2}: P_1\left(-\frac{\sqrt{15}}{4}, \frac{1}{4}\right) \qquad t - \frac{\pi}{2}: P_2\left(\frac{\sqrt{15}}{4}, -\frac{1}{4}\right)$$

Then

$$\sin\left(t + \frac{\pi}{2}\right) = \frac{1}{4} \quad \text{and} \quad \cos\left(t - \frac{\pi}{2}\right) = \frac{\sqrt{15}}{4}$$

PROGRESS CHECK

If the unit circle point $P(3/5, -4/5)$ corresponds to the real number t, find

(a) $\sin\left(t - \frac{\pi}{2}\right)$ (b) $\cos\left(t + \frac{\pi}{2}\right)$ (c) $\cos\left(-t + \frac{\pi}{2}\right)$

ANSWERS

(a) $-\frac{3}{5}$ (b) $\frac{4}{5}$ (c) $-\frac{4}{5}$

IDENTITIES

Trigonometry often involves the use of **identities,** that is, equations that are true for *all* values in the domain of the variable. Identities are useful in simplifying equations and in providing alternative forms for computations. We can now establish two fundamental identities of trigonometry.

Since the coordinates (x, y) of every point on the unit circle satisfy the equation $x^2 + y^2 = 1$, we may substitute $x = \cos t$ and $y = \sin t$ to obtain

$$(\cos t)^2 + (\sin t)^2 = 1$$

Expressions of the form $(\sin t)^n$ occur so frequently that a special notation is used:

$$\sin^n t = (\sin t)^n \quad \text{when } n \neq -1$$

Using this notation and reordering the terms, the identity becomes

$$\boxed{\sin^2 t + \cos^2 t = 1}$$

Of course, we may also use this identity in the alternative forms

$$\sin^2 t = 1 - \cos^2 t$$
$$\cos^2 t = 1 - \sin^2 t$$

Since $\tan t = y/x$, $x \neq 0$, we may substitute $\sin t = y$ and $\cos t = x$ to obtain

$$\boxed{\tan t = \frac{\sin t}{\cos t}}$$

for all values of t in the domain of the tangent function.

EXAMPLE 5

If $\cos t = 3/5$ and t is in quadrant IV, find $\sin t$ and $\tan t$.

SOLUTION

Using the identity $\sin^2 t + \cos^2 t = 1$, we have

$$\sin^2 t + \left(\frac{3}{5}\right)^2 = 1$$

$$\sin^2 t = 1 - \frac{9}{25} = \frac{16}{25}$$

$$\sin t = \pm\frac{4}{5}$$

Since t is in quadrant IV, $\sin t$ must be negative so that $\sin t = -4/5$. Then

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{4}{5}}{\frac{3}{5}} = -\frac{4}{3}$$

PROGRESS CHECK

If $\sin t = 12/13$ and t is in quadrant II, find the following.

(a) $\cos t$ (b) $\tan t$

ANSWERS

(a) $-\dfrac{5}{13}$ (b) $-\dfrac{12}{5}$

WARNING Don't confuse

$$\sin t^2 \quad \text{and} \quad \sin^2 t$$

We have defined $\sin^2 t$ by

$$\sin^2 t = (\sin t)^2$$

which indicates that we find $\sin t$ and then square the result. But $\sin t^2$ indicates that we are to square t and *then* find the sine of the argument t^2.

EXAMPLE 6

Show that $1 + \tan^2 x = 1/\cos^2 x$.

SOLUTION

We will use the trigonometric identities to transform the left-hand side of the equation into the right-hand side. Since $\tan x = \sin x/\cos x$, we have

$$\begin{aligned} 1 + \tan^2 x &= 1 + \frac{\sin^2 x}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \end{aligned}$$

Since $\cos^2 x + \sin^2 x = 1$,

$$1 + \tan^2 x = \frac{1}{\cos^2 x}$$

PROGRESS CHECK

Use identities to transform the expression $(\tan t)(\cos t) + \sin t$ to $2 \sin t$.

WARNING You cannot verify an identity by checking to see that it "works" for one or more values of the variable as these values could turn out to be solutions to a conditional equation (see Section 2.1). You must show that an equation is true for *all* values in the domain of its variable to prove that it is an identity.

EXERCISE SET 5.2

In Exercises 1–8 determine the quadrant in which t or θ lies.

1. $t = \dfrac{4\pi}{3}$
2. $t = -\dfrac{\pi}{6}$
3. $\theta = 150^\circ$
4. $\theta = 215^\circ$
5. $\theta = -190^\circ$
6. $t = \dfrac{3\pi}{4}$
7. $t = -\dfrac{7\pi}{6}$
8. $t = \dfrac{10\pi}{6}$

In Exercises 9–16 the unit circle point P corresponds to the real number t. Find $\sin t$, $\cos t$, and $\tan t$.

9. $P\left(-\dfrac{3}{5}, \dfrac{4}{5}\right)$
10. $P\left(-\dfrac{4}{5}, \dfrac{3}{5}\right)$
11. $P\left(\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2}\right)$
12. $P\left(-\dfrac{1}{2}, -\dfrac{\sqrt{3}}{2}\right)$
13. $P\left(-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$
14. $P\left(\dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}\right)$
15. $P\left(\dfrac{\sqrt{15}}{4}, -\dfrac{1}{4}\right)$
16. $P\left(\dfrac{1}{4}, \dfrac{\sqrt{15}}{4}\right)$

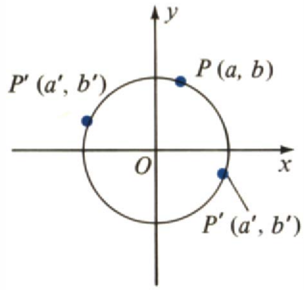
In Exercises 17–22 find the quadrant in which t lies if the following conditions hold.

17. $\sin t > 0$ and $\cos t < 0$
18. $\sin t < 0$ and $\tan t > 0$
19. $\cos t < 0$ and $\tan t > 0$
20. $\tan t < 0$ and $\sin t > 0$
21. $\sin t < 0$ and $\cos t < 0$
22. $\tan t < 0$ and $\cos t < 0$

In Exercises 23–34 find the value of the trigonometric function when t is replaced by $-t$. [For example, given $\sin t = \frac{1}{2}$, find $\sin(-t)$].

23. $\tan t = \dfrac{3}{2}$
24. $\sin t = 1$
25. $\tan t = 1$
26. $\cos t = -1$
27. $\sin t = \dfrac{\sqrt{2}}{2}$
28. $\cos t = \dfrac{\sqrt{3}}{2}$
29. $\cos t = -\dfrac{\sqrt{3}}{2}$
30. $\sin t = -\dfrac{1}{2}$
31. $\tan t = \sqrt{3}$
32. $\sin t = \dfrac{1}{2}$
33. $\sin t = \dfrac{\sqrt{3}}{2}$
34. $\tan t = \dfrac{\sqrt{3}}{3}$
35. If the unit circle point $(3/5, 4/5)$ corresponds to the real number t, use the symmetries of the circle to find the coordinates of the unit circle point corresponding to the real number
    (a) $t + \pi$ (b) $t - \pi/2$
    (c) $-t$ (d) $-t - \pi$
36. If the unit circle point $(-4/5, -3/5)$ corresponds to the real number t, use the symmetries of the circle to find the coordinates of the unit circle point corresponding to the real number
    (a) $t - \pi$ (b) $t + \pi/2$
    (c) $-t$ (d) $-t + \pi$
37. The unit circle point $P(a, b)$ corresponding to the real number t lies in quadrant II. Find the values of
    (a) $\sin(t + \pi/2)$ (b) $\cos(t + \pi/2)$

In Exercises 38–40 the unit circle points $P(a, b)$ and $P'(a', b')$ correspond to the real numbers t and $t \pm \pi/2$, respectively, as shown in the following figure.



38. Show that $b'/a' = -a/b$. (*Hint:* Show that the lines OP and OP' are perpendicular and then determine their slopes.)

39. Show that $b' = \pm a$ and $a' = \pm b$. (*Hint:* The radii OP and OP' are equal in length. Use the distance formula combined with the result of Exercise 38 above to substitute alternately for b' and for a'.)

40. Show that either (i) $(a', b') = (-b, a)$ or (ii) $(a', b') = (b, -a)$. (*Hint:* Begin with the result of Exercise 39 and apply the result of Exercise 38.)

In Exercises 41–48 use trigonometric identities to find the indicated value under the given conditions.

41. $\sin t = \frac{3}{5}$ and t is in quadrant II; find $\tan t$.

42. $\tan t = -\frac{3}{4}$ and t is in quadrant II; find $\cos t$.

43. $\cos t = -\frac{5}{13}$ and t is in quadrant III; find $\sin t$.

44. $\sin t = -\frac{5}{13}$ and t is in quadrant III; find $\tan t$.

45. $\cos t = \frac{4}{5}$ and $\sin t < 0$; find $\sin t$.

46. $\tan t = \frac{12}{5}$ and $\cos t < 0$; find $\sin t$.

47. $\sin t = -\frac{3}{5}$ and $\tan t < 0$; find $\cos t$.

48. $\tan t = -\frac{5}{12}$ and $\sin t > 0$; find $\sin t$.

In Exercises 49–58 use trigonometric identities to transform the first expression into the second.

49. $(\tan t)(\cos t)$, $\sin t$

50. $\dfrac{\cos t}{\sin t}$, $\dfrac{1}{\tan t}$

51. $\dfrac{1 - \sin^2 t}{\sin t}$, $\dfrac{\cos t}{\tan t}$

52. $(\tan t)(\sin t) + \cos t$, $\dfrac{1}{\cos t}$

53. $\cos t\left(\dfrac{1}{\cos t} - \cos t\right)$, $\sin^2 t$

54. $\dfrac{1 - \cos^2 t}{\sin t}$, $\sin t$

55. $\dfrac{1 - \cos^2 t}{\cos^2 t}$, $\tan^2 t$

56. $\dfrac{\cos^2 t}{1 - \sin t}$, $1 + \sin t$

57. $(\sin t - \cos t)^2$, $1 - 2(\sin t)(\cos t)$

58. $\dfrac{1}{1 - \sin t} + \dfrac{1}{1 + \sin t}$, $\dfrac{2}{\cos^2 t}$

5.3 VALUES OF SINE, COSINE, AND TANGENT

"SPECIAL VALUES"

The definitions of sine, cosine, and tangent indicate that their values depend only upon the coordinates of the unit circle point P corresponding to the real number t. In general, it is not easy to find these coordinates. The simplest cases occur when the point P lies on a coordinate axis.

EXAMPLE 1

Find $\sin t$, $\cos t$, and $\tan t$ for

(a) $t = 0$ (b) $t = \pi/2$ (c) $t = \pi$ (d) $t = 3\pi/2$

SOLUTION

From Figure 18, the coordinates of the unit circle point corresponding to the values of t are

$$t = 0\text{: } (1, 0) \qquad t = \frac{\pi}{2}\text{: } (0, 1) \qquad t = \pi\text{: } (-1, 0) \qquad t = \frac{3\pi}{2}\text{: } (0, -1)$$

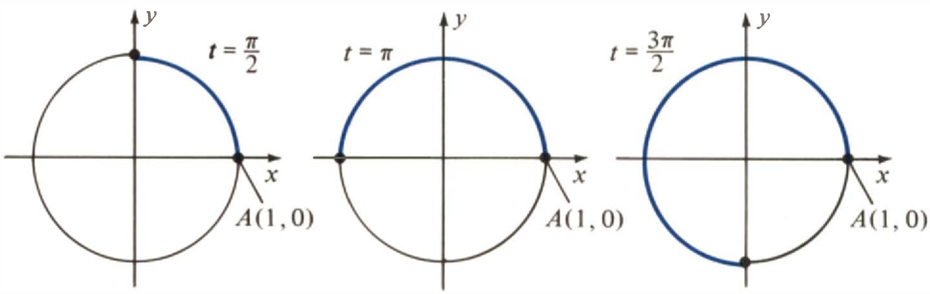


FIGURE 18

Applying the definitions of sine, cosine, and tangent, we have

$$\sin 0 = 0 \qquad \sin\frac{\pi}{2} = 1 \qquad \sin \pi = 0 \qquad \sin\frac{3\pi}{2} = -1$$

$$\cos 0 = 1 \qquad \cos\frac{\pi}{2} = 0 \qquad \cos \pi = -1 \qquad \cos\frac{3\pi}{2} = 0$$

$$\tan 0 = 0 \qquad \tan\frac{\pi}{2} = \text{undefined} \qquad \tan \pi = 0 \qquad \tan\frac{3\pi}{2} = \text{undefined}$$

Note that $\tan \pi/2$ and $\tan 3\pi/2$ are both undefined, since $\tan t = y/x$ is undefined when $x = 0$.

PROGRESS CHECK

Repeat Example 1, replacing t by the degree measure of the corresponding angle θ, $0 \leq \theta \leq 360°$.

Another special value occurs when $t = \pi/4$; that is, when $\theta = 45°$. We will apply a geometric argument in the next example.

EXAMPLE 2

Find the values of $\sin \pi/4$, $\cos \pi/4$, and $\tan \pi/4$.

SOLUTION

In Figure 19, the point P determined by the arc $\overset{\frown}{AP}$ of length $\pi/4$ is seen to bisect the arc $\overset{\frown}{AQ}$ of length $\pi/2$. Therefore P must lie on the line whose equation is $y = x$. We can then designate the coordinates of P as (a, a). Since the coordinates of any point on the unit circle satisfy the condition

$$x^2 + y^2 = 1$$

we have

$$a^2 + a^2 = 1$$
$$2a^2 = 1$$
$$a = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$$

Since P is in the first quadrant, we conclude that the coordinates of P are $(\sqrt{2}/2, \sqrt{2}/2)$ and that

$$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\tan\frac{\pi}{4} = 1$$

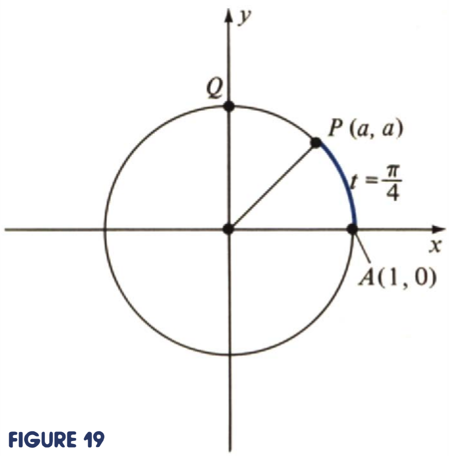


FIGURE 19

Finally, we will tackle the special value of $\pi/6$, or 30°. In Figure 20a, arc $\widehat{AB}$ has length $\pi/6$ on a unit circle, so $\measuredangle AOB$ has a measure of 30°. We let (a, b) be the coordinates of the point B. To determine the values of a and b, we locate the point D in quadrant IV so that arc $\widehat{AD}$ is also of length $\pi/6$. Then $\measuredangle DOA$ also has a measure of 30° and, by the symmetries of the circle, the coordinates of point D are $(a, -b)$. We complete the figure by drawing the line BD as in Figure 20a.

We now turn to the tools of plane geometry applied to the triangles shown in Figure 20b. Since OB and OD are radii of a circle, $\overline{OB} = \overline{OD}$ and $\triangle BOD$ is isosceles. But an isosceles triangle whose vertex angle measures 60° must be an equilateral triangle. Therefore $\overline{BD} = 1$ and $b = 1/2$. Since $a^2 + b^2 = 1$, $a = \sqrt{3}/2$. Finally, we have

$$\sin\frac{\pi}{6} = \frac{1}{2} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$$

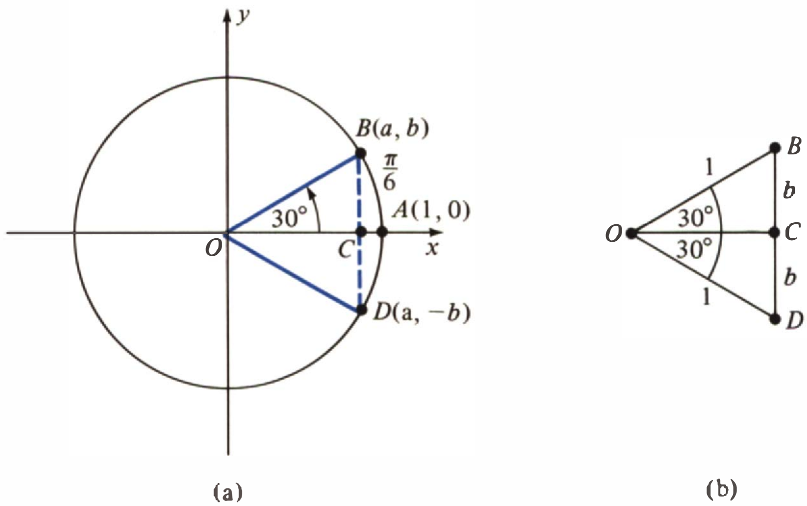


FIGURE 20

A similar argument (see Exercise 95) applied to the special value of $\pi/3$, or 60°, would show the coordinates of the point B to be $(1/2, \sqrt{3}/2)$. Table 2 lists the values of the sine, cosine, and tangent functions for certain frequently used real numbers t in the interval $[0, 2\pi]$. There is no entry for tan $\pi/2$ or tan $3\pi/2$ because the tangent function is not defined for these values.

TABLE 2

t	Unit circle point P	$\sin t$	$\cos t$	$\tan t$
0	(1, 0)	0	1	0
$\frac{\pi}{6}$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	(0, 1)	1	0	—
π	(−1, 0)	0	−1	0
$\frac{3\pi}{2}$	(0, −1)	−1	0	—

PERIODIC FUNCTIONS

In Section 5.1 we defined coterminal angles as angles that have the same terminal side. We can extend this concept to the unit circle point determined by a real number t. Since the circumference of the unit circle is 2π, we have the following result.

> For any real number t, the real numbers
> $$t + 2\pi n$$
> determine the same unit circle point for all integer values of n.

Since the values of the trigonometric functions depend only upon the unit circle point determined by t, we can reach the following conclusion.

> For any real number t,
> $$\sin (t + 2\pi n) = \sin t \quad \text{and} \quad \cos (t + 2\pi n) = \cos t$$
> for all integer values of n.

The cyclical behavior of sine and cosine is characteristic of functions that are called **periodic,** and the least number for which the cyclical behavior is exhibited is called the **period** of the function. It is not difficult to show that the period of the sine and cosine functions is 2π (see Exercises 93 and 94); that is, 2π is the smallest positive real value of p such that

$$\sin(t + p) = \sin t \quad \text{and} \quad \cos(t + p) = \cos t$$

We will discuss the periodicity of the tangent function in Section 5.4.

GENERAL VALUES OF sin t AND cos t

We have thus far found sin t and cos t for some very special values of the argument t. If you want to find the value of a trigonometric function for an arbitrary value of t, you can use a calculator or you can refer to a table of values.

Calculators

Many inexpensive calculators now have keys labeled *sin, cos,* and *tan* that can provide the values of the trigonometric functions. The procedure for using these calculators varies slightly for each manufacturer. In general, the steps are

Step 1. Set a selector switch to radians or degrees. This switch is often marked RAD/DEG or DRG.

Step 2. Enter the input in radians or degrees, corresponding to the switch setting.

Step 3. Depress the appropriate key for sine, cosine, or tangent.

If you have yet to purchase a calculator that can handle trigonometric functions, here are some of the things you should look for.

(Courtesy of Texas Instruments, Inc.)

- Arguments can be entered in either degrees or radians.
- Arguments are unrestricted in size. (We will show you later in this section how to work with a calculator that restricts the arguments; it is easier, however, to use a calculator that permits any argument.)
- Inverse functions are available. (We will discuss this topic at the end of this chapter.)

Do not be surprised if the answers you obtain when using a calculator differ slightly from the answers in this book. Most calculators use some type of approximation method (see Exercises 85–90) that may not give results identical with those in our tables.

EXAMPLE 3

Using the instruction booklet for your calculator, verify the following approximations.

(a) $\sin 47.45° \approx 0.7367$ (b) $\sin 5.763 \approx -0.4970$

(c) $\tan 5.11 \approx -2.3811$ (d) $\cos(-15°) \approx 0.9659$

(e) $\tan 48° \approx 1.1106$ (f) $\cos 6.83 \approx 0.8542$

SOLUTION

Illustrated here are the keys you might have to press to solve part (a):

DEG	47.45	SIN

The result would then be displayed.

Tables of Values

Tables have long been the customary way to find the values of the trigonometric functions. If you examine Table V in the Tables Appendix, you will find the values of sin t and cos t for $0 \leq t \leq 1.57$ in increments of 0.01, which corresponds (approximately) to $0 \leq t \leq \pi/2$. We are now prepared to show that this limited table is adequate to enable you to find sin t or cos t for *any* real value of t.

First, we note that if t is negative we can use the identities

$$\sin(-t) = -\sin t \quad \text{and} \quad \cos(-t) = \cos t$$

to provide us with a positive value of the argument.

Next, we observe that the periodic nature of the trigonometric functions provides us with an approach if $t > 2\pi$. We need only subtract an appropriate multiple of 2π until the remaining value is between 0 and 2π.

Finally, we need to find sin t and cos t when $\pi/2 < t < 2\pi$. In Figure 21 we illustrate the cases in which the unit circle point $P(x, y)$, determined by t, lies in quadrants II, III, and IV. We define the **reference number** t' associated with t as the shortest arc of the unit circle between P and the x-axis. Clearly, if P is not on a coordinate axis, then the reference number t' is less than $\pi/2$; that is,

$$0 < t' < \pi/2$$

The real number t' determines the unit circle point $P'(x', y')$ in the first quadrant. By the symmetries of the unit circle, in all three cases we have

$$x' = |x| \quad \text{and} \quad y' = |y|$$

Then

$$\sin t = y \quad \text{and} \quad \sin t' = y' = |y|$$
$$\cos t = x \quad \text{and} \quad \cos t' = x' = |x|$$

so sin t and sin t' *differ only in sign* and cos t and cos t' *differ only in sign*. If we can find sin t' and cos t' from a table of values of t' in the interval $[0, \pi/2]$, we need only attach the proper *sign* to find sin t and cos t according to the quadrant in which t lies. This procedure is known as the **Reference Number Rule** and is outlined in Example 4.

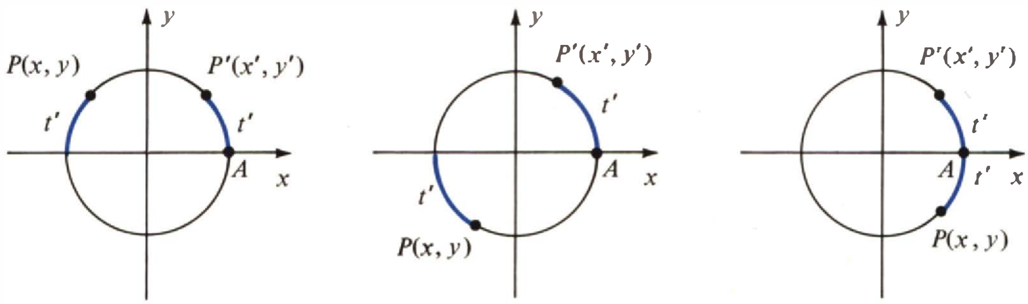


FIGURE 21

EXAMPLE 4

Find $\cos 2\pi/3$.

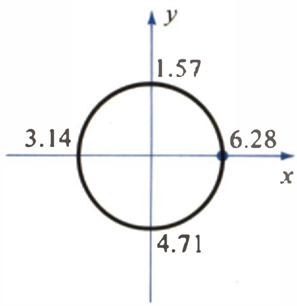
SOLUTION

<table>
<tr><th colspan="2">Reference Number Rule</th></tr>
<tr><td>Step 1. Find the reference number t' associated with t.
<table>
<tr><th>t</th><th>Quadrant</th><th>t'</th></tr>
<tr><td>$\pi/2 < t < \pi$</td><td>II</td><td>$t' = \pi - t$</td></tr>
<tr><td>$\pi < t < 3\pi/2$</td><td>III</td><td>$t' = t - \pi$</td></tr>
<tr><td>$3\pi/2 < t < 2\pi$</td><td>IV</td><td>$t' = 2\pi - t$</td></tr>
</table></td>
<td>Step 1. The argument $t = 2\pi/3$ is in quadrant II. Thus,
$$\begin{aligned} t' &= \pi - t \\ &= \pi - 2\pi/3 \\ &= \pi/3 \end{aligned}$$</td></tr>
<tr><td>Step 2. Obtain the value of the required trigonometric function from Table V in the Tables Appendix.</td>
<td>Step 2. Since $\pi/3$ is a "special value," we know that
$$\cos \pi/3 = 0.5$$</td></tr>
<tr><td>Step 3. Assign the appropriate sign according to the quadrant in which t lies and the trigonometric function being sought.</td>
<td>Step 3. Since $\cos t$ is negative in quadrant II, we have
$$\cos 2\pi/3 = -\cos \pi/3 = -0.5$$</td></tr>
</table>

In Example 4 the argument t was given in terms of π and led to a "special value" of t'. For less convenient values of t you will find Figure 22 helpful in determining the quadrant in which t lies.

EXAMPLE 5

Find $\sin 3.62$ using the Reference Number Rule and Table V in the Tables Appendix.



Quadrantal Arcs

FIGURE 22

SOLUTION

Step 1. Since

$$\pi < 3.62 < 3\pi/2$$

the argument t is in quadrant III and, using $\pi \approx 3.14$,

$$t' = t - \pi \approx 3.62 - 3.14 = 0.48$$

Step 2. From Table V,

$$\sin 0.48 = 0.4618$$

Step 3. Since sine is negative in quadrant III, we have

$$\sin 3.62 = -\sin 0.48 = -0.4618$$

PROGRESS CHECK

Use $\pi \approx 3.14$ and Table V to find

(a) $\tan 5.96$ (b) $\sin 3.79$ (c) $\cos 2.68$

ANSWER

(a) -0.3314 (b) -0.6052 (c) -0.8961

Example 5 shows that the Reference Number Rule is cumbersome when the argument t is not expressed in terms of π. Not only must you determine the quadrant in which t lies and use a table, but the use of a two-place approximation for π will lead to inaccurate results. The method is fine for values of t that lead to "special values." You will find, however, that using a calculator is a much more practical way to obtain values of the trigonometric functions for arbitrary arguments.

To deal with degree measure, we use a scheme analogous to the Reference Number Rule. The **reference angle** θ' associated with the angle θ is the acute angle formed by the terminal side of θ and the x-axis. If θ lies in quadrant I, then θ is itself an acute angle and $\theta' = \theta$. The other cases are illustrated in Figure 23.

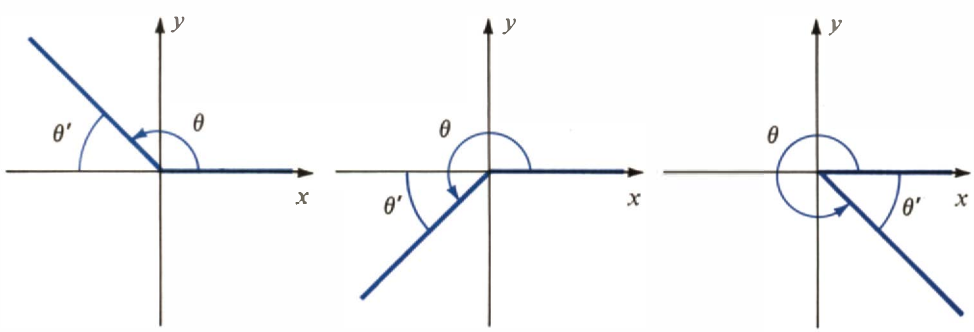


FIGURE 23

The procedure for finding $\sin\theta$ and $\cos\theta$ when θ is in the interval $[90°, 360°]$ is called the **Reference Angle Rule** and is identical to the Reference Number Rule with these exceptions:

- replace π by the degree measure of 180°
- use Table VI in the Tables Appendix instead of Table V

Examination of Table VI indicates that **minutes** and **seconds** are used as subdivisions of a degree according to these definitions.

1 degree = 60 minutes (written 60′)
1 minute = 60 seconds (written 60″)

EXAMPLE 6
Find sin 200°.

SOLUTION

Reference Angle Rule

Step 1. Find the reference angle θ' associated with θ.	*Step 1.* Since the terminal side of an angle of 200° is in quadrant III, we have

θ	Quadrant	θ'
$90° < \theta < 180°$	II	$\theta' = 180° - \theta$
$180° < \theta < 270°$	III	$\theta' = \theta - 180°$
$270° < \theta < 360°$	IV	$\theta' = 360° - \theta$

$$\begin{aligned}\theta' &= \theta - 180°\\ &= 200° - 180°\\ &= 20°\end{aligned}$$

Step 2. Obtain the value of the required trigonometric function from Table VI in the Tables Appendix.	*Step 2.* $\sin 20° = 0.3420$
Step 3. Assign the appropriate sign according to the quadrant in which θ lies and the trigonometric function being sought.	*Step 3.* Since sine is negative in quadrant III, we have $\sin 200° = -\sin 20° = -0.3420$

Of course, a calculator that provides the values of the trigonometric functions for *any* degree measure of an angle is more efficient than using the Reference Angle Rule and Table VI. You should note that the *calculator* entry, 32.5°, corresponds to the *table* entry of 32° 30′; that is, the calculator uses decimal format to represent a fractional part of a degree.

EXAMPLE 7

Find tan 611°20′ by (a) using a calculator and (b) using the Reference Angle Rule.

SOLUTION

(a) Using a calculator, the value 611.333 would be entered since 20′ is 20/60, or 1/3 of a degree. The key sequence would be

DEG	611.333	TAN

The calculator will display an (approximate) answer of 2.9600.

(b) We first reduce the angle 611°20′ to an angle in the interval (0, 360°) by subtracting 360°, leaving $\theta = 251°20'$. Using the Reference Angle Rule, we note that the angle θ is in quadrant III. The reference angle θ' is given by

$$\theta' = \theta - 180° = 251°20' - 180° = 71°20'$$

From Table VI in the Tables Appendix, tan 71°20′ = 2.9600. Finally, we note that tangent is positive in quadrant III and 2.9600 is indeed the answer.

PROGRESS CHECK

For each angle θ

1. use the Reference Angle Rule to find cos θ;
2. convert the angle to decimal form and use a calculator to find cos θ.

(a) 143°40′ (b) 345°10′

ANSWERS

(a) −0.8056 (b) 0.9667

EXERCISE SET 5.3

In Exercises 1–12 replace each given real number s by a real number t, $0 \leq t < 2\pi$, so that s and t determine the same unit circle point.

1. 4π
2. $\frac{13\pi}{2}$
3. $\frac{15\pi}{7}$
4. $-\frac{25\pi}{4}$
5. $-\frac{21\pi}{2}$
6. $-\frac{11\pi}{2}$
7. $\frac{41\pi}{6}$
8. $\frac{11\pi}{2}$
9. -9π
10. 7π
11. $\frac{27\pi}{5}$
12. $-\frac{22\pi}{3}$

In Exercises 13–20 find a positive and a negative value of t, $|t| < 2\pi$, that determine the unit circle point P whose coordinates are given.

13. $P(-1, 0)$
14. $P(0, -1)$
15. $P\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
16. $P\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
17. $P\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
18. $P\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
19. $P\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
20. $P\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

In Exercises 21–36, for each given real number t find (a) the coordinates of the unit circle point P determined by t; and (b) the values of $\sin t$, $\cos t$, and $\tan t$.

21. 5π	22. $\frac{5\pi}{2}$	23. $-\frac{\pi}{4}$	24. $-\frac{3\pi}{2}$
25. $\frac{5\pi}{4}$	26. 8π	27. $\frac{4\pi}{3}$	28. $\frac{2\pi}{3}$
29. $-\frac{2\pi}{3}$	30. $-\frac{19\pi}{3}$	31. $\frac{19\pi}{6}$	32. $\frac{17\pi}{6}$
33. $-\frac{5\pi}{6}$	34. $-\frac{11\pi}{6}$	35. $\frac{19\pi}{3}$	36. $\frac{25\pi}{3}$

In Exercises 37–48 use Table V in the Tables Appendix to find each of the following. (Use $\pi \approx 3.14$.)

37. $\cos 1.12$	38. $\sin 0.48$	39. $\tan(-1.39)$	40. $\sin 4.86$
41. $\tan 3.44$	42. $\cos(-4.79)$	43. $\sin(-5.28)$	44. $\tan 6.05$
45. $\cos(-2.91)$	46. $\sin 2.43$	47. $\tan(-3.27)$	48. $\cos 1.72$

49–60. Repeat Exercises 37–48 using a calculator to find the value of the required trigonometric function.

In Exercises 61–72 use Table VI in the Tables Appendix to find each of the following.

61. $\tan 155°$	62. $\cos(-305°)$	63. $\sin 232°$	64. $\sin(-147°)$
65. $\cos 257°$	66. $\tan 290°$	67. $\cos 136°$	68. $\sin 345°$
69. $\tan 19°10'$	70. $\cos 470°50'$	71. $\sin(-197°30')$	72. $\tan(-105°40')$

73–84. Repeat Exercises 61–72 using a calculator to find the value of the required trigonometric function.

In Exercises 85–90 use a calculator and the polynomial approximations

$$\sin t \approx t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040}$$

$$\cos t \approx t - \frac{t^2}{2} + \frac{t^4}{24} - \frac{t^6}{720}$$

to find each of the following.

85. $\sin 0.80$	86. $\cos 1.10$	87. $\sin(-0.20)$	88. $\cos(-0.75)$
89. $\tan 0.1$	90. $\tan(-1.2)$		

91. Using the polynomial approximation for $\sin t$ given above, show that sine is an odd function; that is, $\sin(-t) = -\sin t$.

92. Using the polynomial approximation for $\cos t$ given above, show that cosine is an even function; that is, $\cos(-t) = \cos t$.

93. Prove that the period of the sine function is 2π. [*Hint:* Assume $\sin(t + c) = \sin t$, $0 < c < 2\pi$, for all t. By letting $t = 0$, show that $\sin c = 0$ and, consequently, that $c = \pi$. Finally, conclude that $\sin(t + \pi) = \sin t$ does not hold for $t = \pi/2$.]

94. Prove that the period of the cosine function is 2π.

95. Show that the unit circle point P determined by the real number $t = \pi/3$ has coordinates $(1/2, \sqrt{3}/2)$. [*Hint:* If P has coordinates (a, b), then the unit circle point P' corresponding to $t = 2\pi/3$ has coordinates $(-a, b)$ and $\overset{\frown}{AP} = \overset{\frown}{PP'}$.]

5.4 GRAPHS OF SINE, COSINE, AND TANGENT

In the last section we used the periodic property of the trigonometric functions to reduce the number of entries in Tables V and VI in the Tables Appendix to a reasonable length. We will now take advantage of the same periodic property in sketching graphs of the trigonometric functions.

SINE AND COSINE

If we can graph the sine and cosine functions over the interval $[0, 2\pi]$, we can then repeat the graph for every interval of length 2π. As usual, we form a table of values, plot the corresponding points on a ty coordinate system, and sketch a smooth curve. We can make use of the results of the last section to provide us with values for plotting, as in Table 3.

TABLE 3

t	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
$\sin t$	0	0.50	0.71	0.87	1	0.71	0	−0.71	−1	−0.71	0
$\cos t$	1	0.87	0.71	0.50	0	−0.71	−1	−0.71	0	0.71	1

We have used the approximations $\sqrt{2} \approx 1.414$ and $\sqrt{3} \approx 1.732$. With the values in the table we can sketch $y = \sin t$ over the interval $[0, 2\pi]$, as in Figure 24.

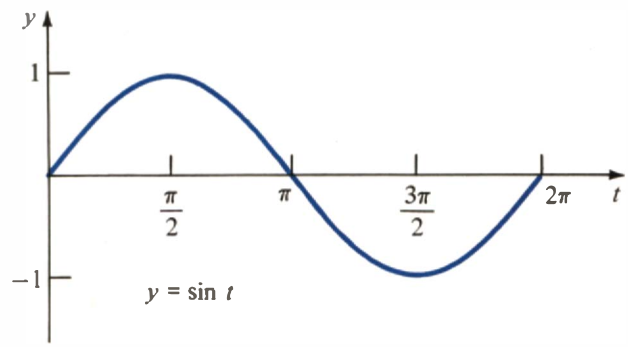


FIGURE 24

Repeating for adjacent intervals of length 2π yields the graph in Figure 25.

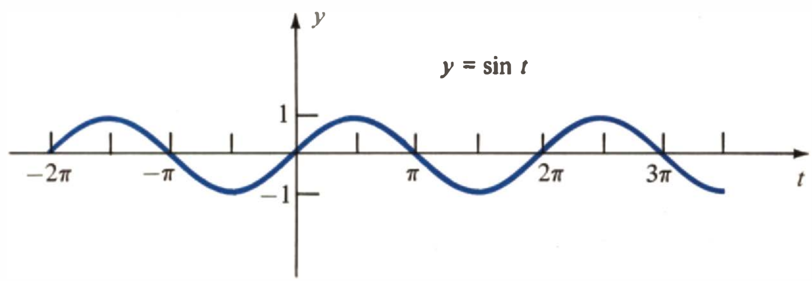


FIGURE 25

Turning to the cosine function, we can use values given in Table 3 to sketch the graph of $y = \cos t$ as in Figure 26.

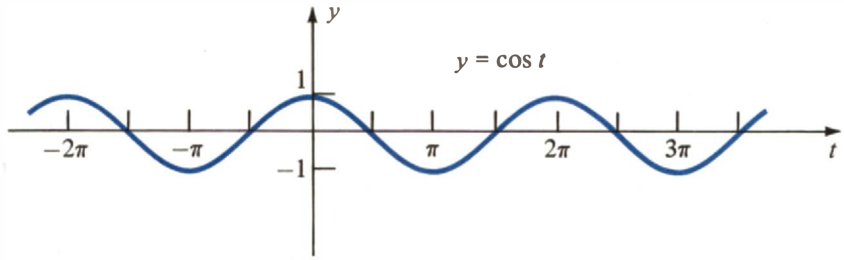


FIGURE 26

TANGENT

To graph the tangent function, we first establish that $\tan(t + \pi) = \tan t$ for all real values of t. If $P(x, y)$ is any point on the unit circle, then $P'(-x, -y)$ also lies on the unit circle (Figure 27), and arc $\overparen{PP'}$ is of length π. If the unit circle point $P(x, y)$ corresponds to the real number t, then

$$\tan t = \frac{y}{x}$$

and

$$\tan(t + \pi) = \frac{-y}{-x} = \frac{y}{x}$$

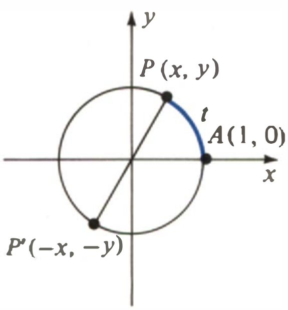


FIGURE 27

so that $\tan(t + \pi) = \tan t$. It is easy to show that there is no real number c, $0 < c < \pi$, such that $\tan(t + c) = \tan t$ for all real numbers t. Hence, the tangent function has period π.

We can use the identities

$$\tan t = \frac{\sin t}{\cos t} \quad \text{and} \quad \tan(-t) = -\tan t$$

to establish the entries in Table 4. For example,

$$\tan 0 = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0$$

$$\tan \frac{\pi}{6} = \frac{\sin \frac{\pi}{6}}{\cos \frac{\pi}{6}} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{3} \approx 0.58$$

$$\tan \frac{\pi}{3} = \frac{\sin \frac{\pi}{3}}{\cos \frac{\pi}{3}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} \approx 1.73$$

TABLE 4

t	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
tan t		-1.73	-1	-0.58	0	0.58	1	1.73	

Since tan t is undefined at $\pi/2$ and at $-\pi/2$, we need to carefully consider the behavior of the graph *near* these values of t. As t increases from 0 toward $\pi/2$, the x-coordinate of the unit circle point $P(x, y)$ corresponding to t gets closer and closer to 0. Since tan $t = y/x$, arbitrarily small values of x produce arbitrarily large values for the quotient y/x. We say that tan t **increases without bound** as t approaches $\pi/2$. Similarly, as t decreases from 0 toward $-\pi/2$, tan t grows smaller and smaller. Accordingly, we say that tan t **decreases without bound** as t approaches $-\pi/2$. These considerations lead us to the graph of tan t shown in Figure 28. The vertical, dashed lines are called **vertical asymptotes.**

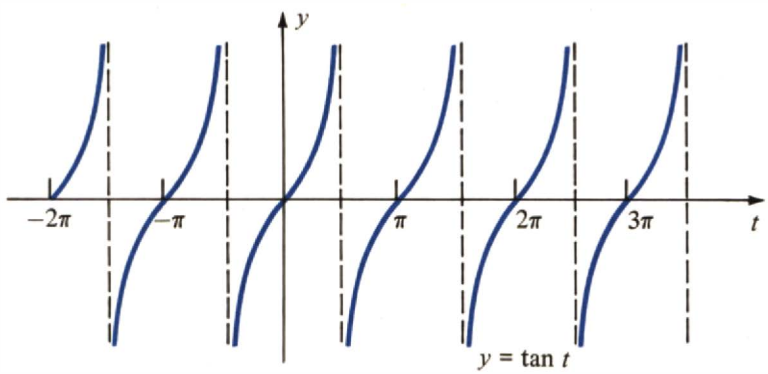


FIGURE 28

EVEN AND ODD FUNCTIONS

A function f for which $f(-x) = f(x)$ is said to be an **even function;** if $f(-x) = -f(x)$, then f is called an **odd function.** From our earlier work with negative values, we see that sine and tangent are odd functions, while cosine is an even function.

It is easy to see from the definitions that the graph of an even function is always symmetric about the y-axis, while the graph of an odd function is symmetric with respect to the origin. An example of an even function whose graph you know is $f(x) = x^2$, while $f(x) = x^3$ is a good example of an odd function. Now that we have sketched the graphs of sin t, cos t, and tan t in Figures 24, 25, and 27, respectively, we have visual verification that sine and tangent are odd functions and cosine is an even function.

RANGE OF THE TRIGONOMETRIC FUNCTIONS

From the graphs of the sine and cosine functions it is clear that both functions assume values between -1 and $+1$. Examination of the graph of the tangent function again shows that $\tan t$ is not bounded. These conclusions concerning the range of the trigonometric functions are listed in Table 5, along with the domain and period of each function.

TABLE 5

	$\sin t$	$\cos t$	$\tan t$
Domain	all t	all t	$t \neq \pi/2 + n\pi$, n an integer
Range	$-1 \leq y \leq 1$	$-1 \leq y \leq 1$	all real numbers
Period	2π	2π	π

EXAMPLE 1
Sketch the graph of $f(t) = 1 + \sin t$.

SOLUTION
Rather than form a table of values and plot points, we simply note that the y-coordinate of $f(t) = 1 + \sin t$ is one unit larger than that of $\sin t$ for each value of t. In Figure 29 we have sketched $\sin t$ with dashed lines and $f(t) = 1 + \sin t$ with a solid line.

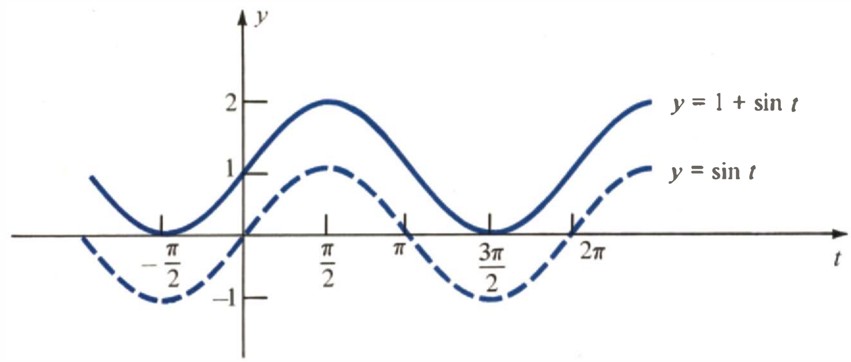


FIGURE 29

EXAMPLE 2
Sketch the graph of $f(t) = \sin t + \cos t$.

SOLUTION
Again, rather than plot points, we note that the y-coordinate of $f(t) = \sin t + \cos t$ is simply the sum of the y-coordinates of $\sin t$ and $\cos t$ for each value of t. In

Figure 30 we have sketched the graphs of sin t and cos t with dashed lines, formed the sum of the y-coordinates geometrically, and then sketched a smooth curve through the resulting points.

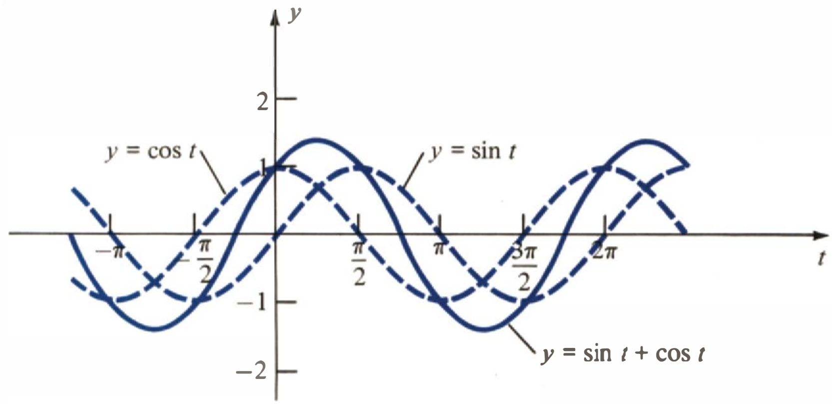


FIGURE 30

GRAPHS: AMPLITUDE, PERIOD, AND PHASE SHIFT

We next seek to sketch the graph of $f(x) = A \sin(Bx + C)$, where A, B, and C are real numbers and $B > 0$. Note that we are now using the familiar symbol "x" to indicate the independent variable, rather than the symbol "t" used until now. Of course, any symbol can be used to denote a variable; however, the symbol "x" used here is not to be confused with the x-coordinate of the unit circle point $P(x, y)$ corresponding to an arc of length t. The results that we obtain throughout this section will also apply to the form $A \cos(Bx + C)$.

Amplitude

Since the sine function has a maximum value of $+1$ and a minimum value of -1, it is clear that the function $f(x) = A \sin x$ has a maximum value of $|A|$ and a minimum value of $-|A|$. If we define the **amplitude** of a periodic function as half the difference of the maximum and minimum values, we see that the amplitude of $f(x) = A \sin x$ is $[|A| - (-|A|)]/2 = |A|$.

The amplitude of $f(x) = A \sin x$ is $|A|$.

The multiplier A acts as a vertical "stretch" factor when $|A| > 1$, and as a vertical "shrinkage" factor when $|A| < 1$. These remarks hold for both $y = A \sin x$ and $y = A \cos x$. Here are some examples.

EXAMPLE 3
Sketch the graphs of $y = 2\sin x$ and $y = \frac{1}{2}\sin x$ on the same coordinate axes.

SOLUTION
The graph of $y = 2 \sin x$ has an amplitude of 2; the maximum value of y is $+2$ and the minimum is -2. Similarly, the amplitude of $y = \frac{1}{2} \sin x$ is $\frac{1}{2}$ and the graph has a maximum value of $+\frac{1}{2}$ and a minimum of $-\frac{1}{2}$ (Figure 31).

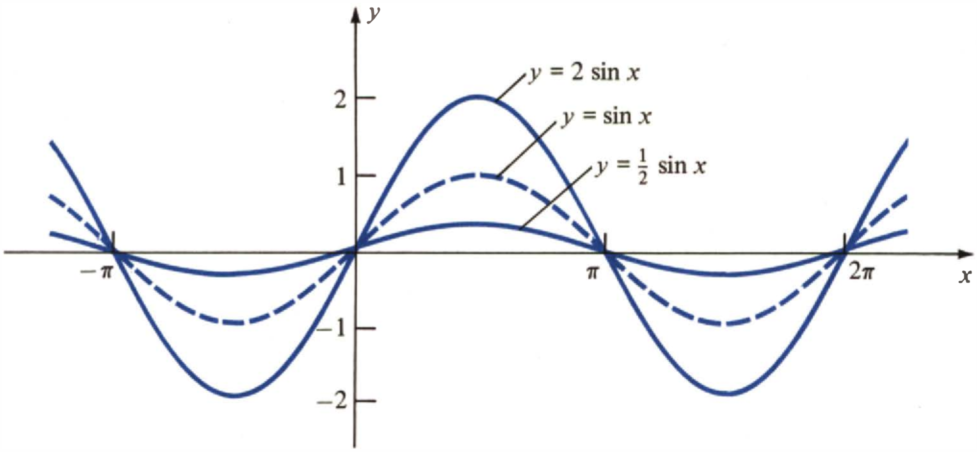


FIGURE 31

EXAMPLE 4

Sketch the graph of $f(x) = -3 \cos x$.

SOLUTION

The amplitude is 3, and $y = -3 \cos x$ has maximum and minimum values of $+3$ and -3, respectively. Since $A = -3$, each y-coordinate will be that of $\cos x$ multiplied by -3.

The graph of $y = -3 \cos x$ shown in Figure 32 is said to be a **reflection** about the x-axis of the graph of $y = 3 \cos x$.

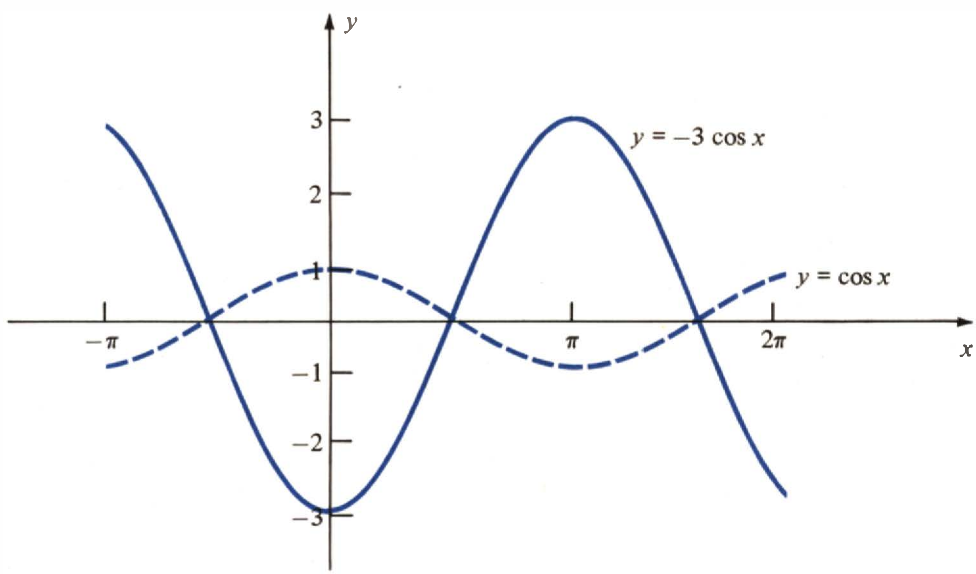


FIGURE 32

Period

The key to sketching the graph of $f(x) = \sin Bx$, $B > 0$, lies in determining the period of the function. Table 6 shows values of $\sin x$ and $\sin 2x$ for selected values of x in the interval $[0, 2\pi]$. These values were used in sketching the graphs shown in Figure 33. Since $y = \sin x$ has period 2π, the graph shows that the sine function completes one cycle or wave as x varies from 0 to 2π. However, the graph of $y = \sin 2x$ completes *two* cycles as x varies from 0 to 2π.

TABLE 6

x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
$\sin x$	0	$\sqrt{2}/2$	1	$\sqrt{2}/2$	0	$-\sqrt{2}/2$	-1	$-\sqrt{2}/2$	0
$\sin 2x$	0	1	0	-1	0	1	0	-1	0

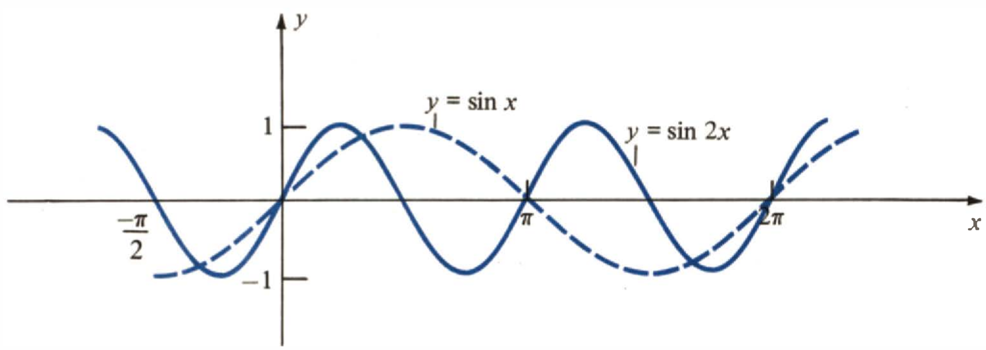


FIGURE 33

In general, $\sin Bx$ will complete B cycles over the interval $[0, 2\pi]$, so that a cycle is completed as x varies from 0 to $2\pi/B$. We conclude:

> The period of $f(x) = \sin Bx$, $B > 0$, is $\dfrac{2\pi}{B}$.

The multiplier B acts as a horizontal "stretch" factor if $0 < B < 1$ and as a horizontal "shrinkage" factor if $B > 1$.

EXAMPLE 5

Sketch the graph of $f(x) = 2 \cos \frac{1}{2}x$.

SOLUTION

Since $B = \frac{1}{2}$, the period is $2\pi/\frac{1}{2} = 4\pi$. The graph will complete a cycle every 4π units. Note that the amplitude is 2, which provides us with maximum and minimum values of 2 and -2, respectively. The graph is shown in Figure 34.

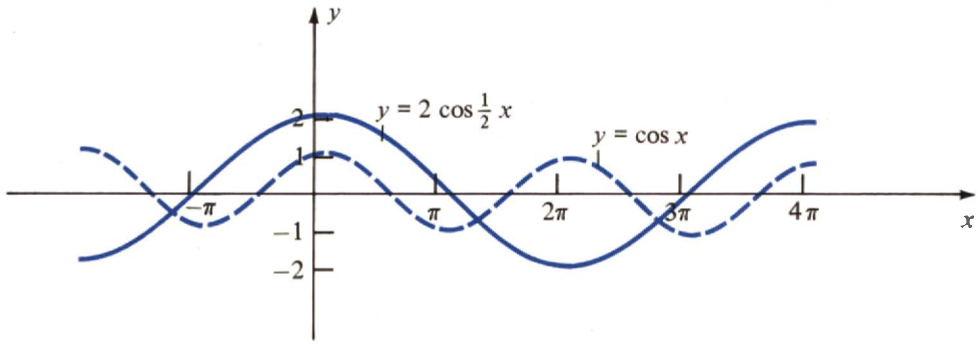


FIGURE 34

Phase Shift

Let us examine the behavior of the function $f(x) = A\sin(Bx + C)$, $B > 0$. Since $y = \sin x$ completes a cycle as x varies from 0 to 2π, the function f will complete a cycle as $Bx + C$ varies from 0 to 2π. Solving the equations

$$Bx + C = 0 \qquad Bx + C = 2\pi$$

we have

$$x = -\frac{C}{B} \qquad x = \frac{2\pi - C}{B} = \frac{2\pi}{B} - \frac{C}{B}$$

The number $-C/B$ is called the **phase shift** and indicates that the graph of the function is shifted right $-C/B$ units if $-C/B > 0$ and is shifted left if $-C/B$ is negative.

> The phase shift of
>
> $$f(x) = A\sin(Bx + C),\ B > 0$$
>
> is $-C/B$.

Note that the amplitude of f is $|A|$ and the period is $2\pi/B$; that is, the introduction of a phase shift has not altered our earlier results.

EXAMPLE 6

Sketch the graph of $f(x) = 3\sin(2x - \pi)$.

SOLUTION

Graphing $f(x) = A\sin(Bx + C)$	
Step 1. Determine A, B, and C.	*Step 1*. Since $f(x) = 3\sin(2x - \pi) = A\sin(Bx + C)$ $A = 3$, $B = 2$, and $C = -\pi$.

Step 2. Determine the amplitude, period, and phase shift.

Step 2.

$$\text{amplitude} = |A| = 3$$

$$\text{period} = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi$$

$$\text{phase shift} = -\frac{C}{B} = \frac{\pi}{2}$$

(or: $2x - \pi = 0$ yields $x = \dfrac{\pi}{2}$ as the phase shift)

Step 3. Analyze the effect of the phase shift on the point (0, 0).

Step 3. A phase shift of $\pi/2$ causes the cycle to "begin" at $(\pi/2, 0)$ rather than at $(0, 0)$.

Step 4. Use the period to determine the values of x at which a cycle is completed.

Step 4. Adding and subtracting the period of π to the phase shift of $\pi/2$, we have

$$x = \frac{\pi}{2} - \pi = -\frac{\pi}{2}$$

$$x = \frac{\pi}{2} + \pi = \frac{3\pi}{2}$$

The graph completes a cycle in the intervals

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad \text{and} \quad \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$$

Step 5. Using the amplitude, sketch the graph.

Step 5. Recalling that the amplitude is 3, see the graph sketched in Figure 35.

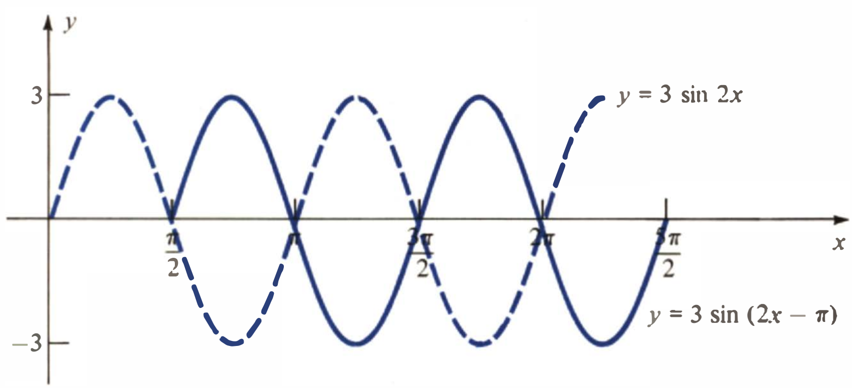


FIGURE 35

PROGRESS CHECK

If $f(x) = 2\cos(2x + \pi/2)$, find the amplitude, period, and phase shift of f. Sketch the graph of the function.

ANSWER

amplitude = 2 period = π phase shift = $-\pi/4$ (or shift left $\pi/4$) (Figure 36)

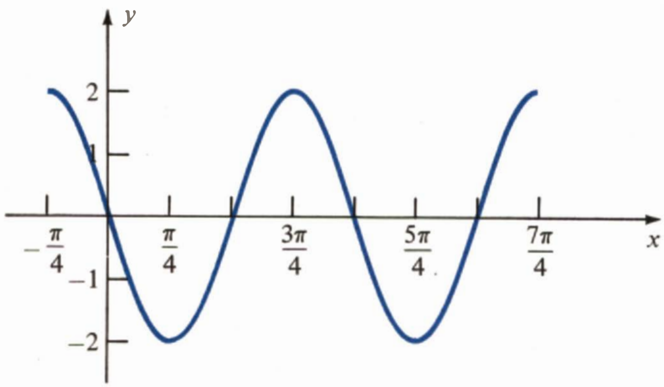


FIGURE 36

EXAMPLE 7

Rewrite the equations

(a) $y = \frac{1}{2}\sin(-x + \pi)$ (b) $y = -2\cos(-2x - \pi)$

as equivalent equations with $B > 0$.

SOLUTION

(a) We rewrite the original equation as

$$y = \frac{1}{2}\sin(-x + \pi)$$
$$= \frac{1}{2}\sin[-(x - \pi)]$$

Since $\sin(-t) = -\sin t$, we have

$$y = -\frac{1}{2}\sin(x - \pi)$$

where $B > 0$.

(b) We rewrite the original equation as

$$y = -2\cos(-2x - \pi)$$
$$= -2\cos[-(2x + \pi)]$$

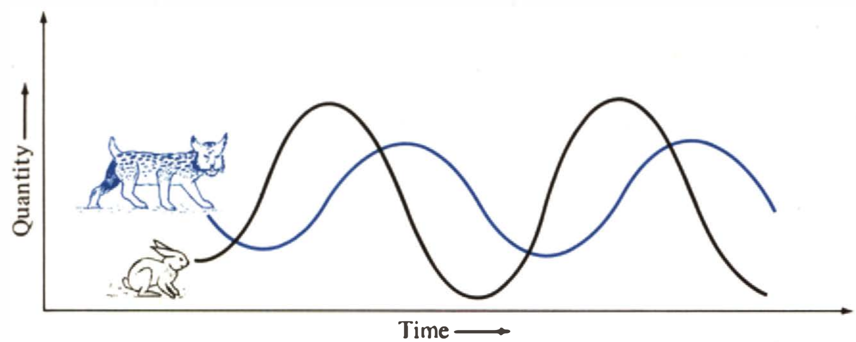
Since $\cos(-t) = \cos(t)$, we have

$$y = -2\cos(2x + \pi)$$

PREDATOR-PREY INTERACTION

In the natural world we frequently find that two plant or animal species interact in their environment in such a manner that one species (the prey) serves as the primary food supply for the second species (the predator). Examples of such interaction are the relationships between trees (prey) and insects (predators) and between rabbits (prey) and lynxes (predators). As the population of the prey increases, the additional food supply results in an increase in the population of the predators. More predators consume more food, so the population of the prey will decrease, which, in turn, will lead to a decrease in the population of the predators. The reduction in the predator population results in an increase in the number of prey and the cycle will start all over again.

The accompanying figure, adapted from *Mathematics: Ideas and Applications*, by Daniel D. Benice, Academic Press, 1978 (used with pennission), shows the interaction between lynx and rabbit populations. Both curves demonstrate periodic behavior and can be described by trigonometric functions.



EXERCISE SET 5.4

In Exercises 1–6 sketch the graph of each given function.

1. $f(t) = 1 + \cos t$
2. $f(t) = -1 + \sin t$
3. $f(t) = \sin t - \cos t$
4. $f(t) = \sin(-t) + \cos t$
5. $f(t) = t + \sin t$
6. $f(t) = -t + \cos t$
7. Verify that $\sin(-t) = -\sin t$ by using the graph of the sine function.
8. Verify that $\cos(-t) = \cos t$ by using the graph of the cosine function.

Determine the amplitude and period and sketch the graph of each of the following functions.

9. $f(x) = 3 \sin x$
10. $f(x) = \frac{1}{4} \cos x$
11. $f(x) = \cos 4x$
12. $f(x) = \sin \frac{x}{4}$
13. $f(x) = -2 \sin 4x$
14. $f(x) = -\cos \frac{x}{4}$
15. $f(x) = 2 \cos \frac{x}{3}$
16. $f(x) = 4 \sin 4x$
17. $f(x) = \frac{1}{4} \sin \frac{x}{4}$
18. $f(x) = \frac{1}{2} \cos \frac{x}{4}$
19. $f(x) = -3 \cos 3x$
20. $f(x) = -2 \sin 3x$

For each given function, determine the amplitude, period, and phase shift. Sketch the graph of the function.

21. $f(x) = 2\sin(x - \pi)$

22. $f(x) = \frac{1}{2}\cos\left(x + \frac{\pi}{2}\right)$

23. $f(x) = 3\cos(2x - \pi)$

24. $f(x) = 4\sin\left(x + \frac{\pi}{4}\right)$

25. $f(x) = \frac{1}{3}\sin\left(3x + \frac{3\pi}{4}\right)$

26. $f(x) = 2\cos\left(2x + \frac{\pi}{2}\right)$

27. $f(x) = 2\cos\left(\frac{x}{4} - \pi\right)$

28. $f(x) = 6\sin\left(\frac{x}{2} + \frac{\pi}{2}\right)$

Use the identities $\sin(-t) = -\sin t$ and $\cos(-t) = \cos t$ to rewrite each equation as an equivalent equation with $B > 0$.

29. $y = -2\sin(-2x + \pi)$

30. $y = 4\cos\left(-\frac{x}{2} + \frac{\pi}{2}\right)$

31. $y = 3\cos\left(-\frac{x}{3} + \frac{2\pi}{3}\right)$

32. $y = -5\sin(-2x - \pi)$

5.5 SECANT, COSECANT, AND COTANGENT

We stated earlier in this chapter that there are six trigonometric functions and that the remaining three functions are reciprocals of sine, cosine, and tangent. These functions are called the **secant, cosecant,** and **cotangent** and are written as **sec, csc,** and **cot,** respectively. We now formally define these functions.

Definition of sec *t*, csc *t*, and cot *t*

$$\sec t = \frac{1}{\cos t}, \quad \cos t \neq 0$$

$$\csc t = \frac{1}{\sin t}, \quad \sin t \neq 0$$

$$\cot t = \frac{1}{\tan t}, \quad \tan t \neq 0$$

By using these definitions, we can apply the results that we have obtained for sine, cosine, and tangent to these new functions.

EXAMPLE 1
Find $\sec \pi/3$.

SOLUTION
Since $\cos \pi/3 = \frac{1}{2}$ (from Table 2 in Section 5.3) we see that

$$\sec\frac{\pi}{3} = \frac{1}{\cos\frac{\pi}{3}} = \frac{1}{\frac{1}{2}} = 2$$

EXAMPLE 2
Find the real number t, $0 \le t \le \pi/2$, such that $\cot t = \sqrt{3}$.

SOLUTION

We seek the real number t such that

$$\tan t = \frac{1}{\cot t} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Thus, $t = \pi/6$ since (from Table 2 in Section 5.3) $\tan \pi/6 = \sqrt{3}/3$.

We know that a real number and its reciprocal have the same sign; that is, if $x > 0$, then $1/x > 0$ and if $x < 0$, then $1/x < 0$. From this, we can immediately extend our conclusions (see Figure 15b) concerning the signs of the trigonometric functions in each quadrant (Figure 37). You do not have to memorize these; simply associate each function with its reciprocal.

II $\sin t > 0$ $\csc t > 0$	**I** All positive
III $\tan t > 0$ $\cot t > 0$	**IV** $\cos t > 0$ $\sec t > 0$

FIGURE 37

EXAMPLE 3

Find the quadrant in which t lies if $\sin t > 0$ and $\sec t < 0$.

SOLUTION

If $\sec t < 0$, then $\cos t < 0$. We know that sine is positive in quadrants I and II, cosine is negative in quadrants II and III (Figure 37). Both conditions are satisfied in quadrant II.

PROGRESS CHECK

Find the quadrant in which t lies if $\tan t < 0$ and $\csc t < 0$.

ANSWER

quadrant IV

EXAMPLE 4

Find t if $\sin t = \sqrt{3}/2$ and $\sec t < 0$.

SOLUTION

Since sec $t < 0$, we have cos $t < 0$. Then t must lie in quadrant II, since sine is positive and cosine is negative only in quadrant II (Figure 37). Finally, we know (from Table 2 in Section 5.3) that sin $\pi/3 = \sqrt{3}/2$. Thus, $\pi/3$ is the reference number of t; that is,

$$\pi - t = \frac{\pi}{3}$$
$$t = \frac{2\pi}{3}$$

PROGRESS CHECK

Find t if cos $t = -\frac{1}{2}$ and cot $t > 0$.

ANSWER

$4\pi/3$

EXAMPLE 5

Use a calculator to find csc 0.72.

SOLUTION

Most calculators do not have function keys for secant, cosecant, and cotangent. To find csc 0.72, we can use the calculator to find sin 0.72 and then compute the reciprocal. The typical key sequence is

$\boxed{\text{RAD}}$ $\boxed{0.72}$ $\boxed{\text{SIN}}$ $\boxed{1/x}$

where $1/x$ indicates the key for finding a reciprocal. The answer displayed is 1.517.

GRAPHS OF SECANT, COSECANT, AND COTANGENT

We can also employ the definition of cosecant to aid in sketching the graph of the function. Since csc $t = 1/\sin t$, we compute the reciprocal of the y-coordinate of sin t at a point to determine the y-coordinate of csc t at that point. Of course, we cannot form a reciprocal when sin $t = 0$, that is, when $t = n\pi$, where n is an integer. The situation at these values of t is analogous to that of the tangent function when $t = \pi/2 + n\pi$. We conclude that the graph of csc t has vertical asymptotes when $t = n\pi$, for all integer values of n. In Figure 38 we have sketched the graph of the sine function with dashed lines, to aid in sketching the reciprocal values of the y-coordinates for the cosecant function.

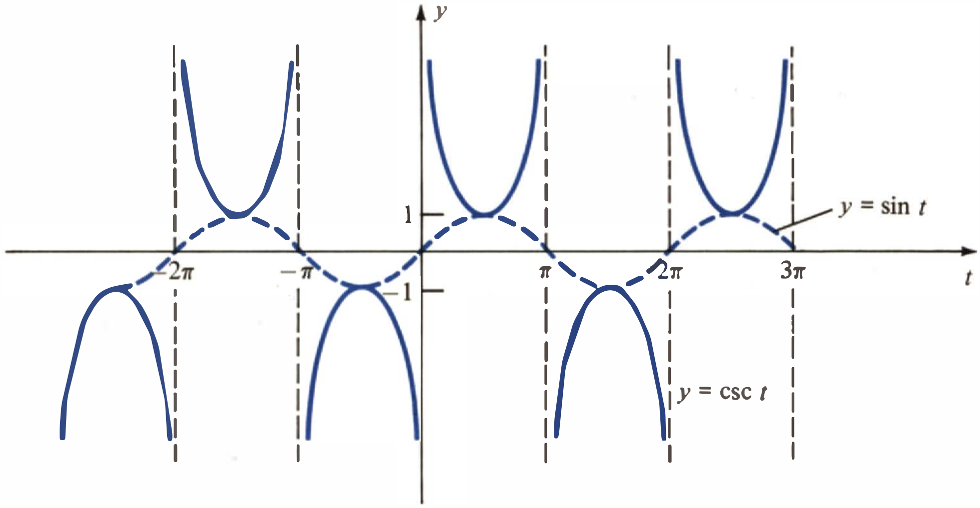


FIGURE 38

A similar approach yields the graphs of sec t and cot t shown in Figures 39 and 40.

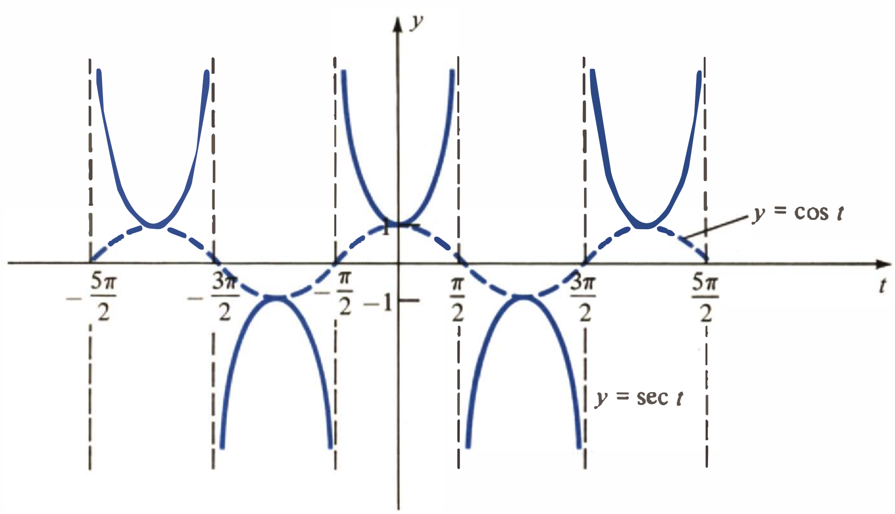


FIGURE 39

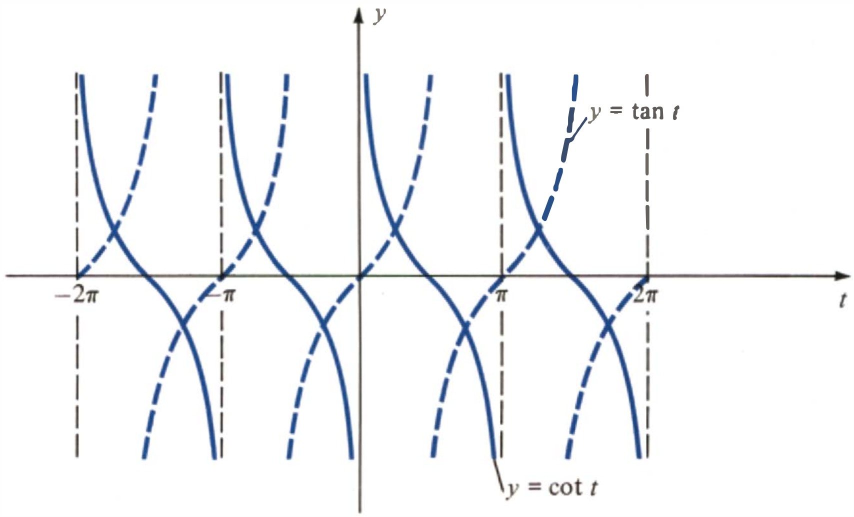


FIGURE 40

Table 7 summarizes the significant properties of the trigonometric functions.

TABLE 7

	Positive in Quadrant	$-t$	Period	Domain	Range
sin	I, II	$-\sin t$	2π	all real numbers	$[-1,1]$
cos	I, IV	$\cos t$	2π	all real numbers	$[-1, 1]$
tan	I, III	$-\tan t$	π	$t \neq \frac{\pi}{2} + n\pi$	$(-\infty, \infty)$
csc	I, II	$-\csc t$	2π	$t \neq n\pi$	$(-\infty, -1], [1, \infty)$
sec	I, IV	$\sec t$	2π	$t \neq \frac{\pi}{2} + n\pi$	$(-\infty, -1], [1, \infty)$
cot	I, III	$-\cot t$	π	$t \neq n\pi$	$(-\infty, \infty)$

EXERCISE SET 5.5

Use the definitions of secant, cosecant, and cotangent to determine sec t, csc t, and cot t for each of the following values of t.

1. $\frac{\pi}{3}$
2. $\frac{\pi}{6}$
3. $\frac{\pi}{4}$
4. $\frac{\pi}{2}$
5. $\frac{5\pi}{6}$
6. $\frac{4\pi}{3}$
7. $\frac{3\pi}{2}$
8. $\frac{7\pi}{4}$

9. $\frac{3\pi}{4}$ 10. $\frac{11\pi}{6}$ 11. $\frac{5\pi}{4}$ 12. $\frac{7\pi}{6}$

Determine the value(s) of t, $0 \leq t \leq 2\pi$, that satisfy each of the following.

13. $\sec t = 1$
14. $\sec t = -1$
15. $\csc t = -2$
16. $\csc t = 0$
17. $\cot t = 1$
18. $\cot t = \sqrt{3}$
19. $\cot t = -1$
20. $\cot t = \frac{\sqrt{3}}{3}$
21. $\sec t = \sqrt{2}$
22. $\csc t = -\sqrt{2}$
23. $\cot t = -\sqrt{3}$
24. $\csc t = 2\frac{\sqrt{3}}{3}$

Find the quadrant in which t lies if the following conditions hold.

25. $\sec t < 0$, $\sin t < 0$
26. $\tan t < 0$, $\sec t < 0$
27. $\csc t > 0$, $\sec t < 0$
28. $\sin t < 0$, $\cot t > 0$
29. $\sec t < 0$, $\cot t > 0$
30. $\cot t < 0$, $\sin t > 0$
31. $\sec t < 0$, $\csc t < 0$
32. $\csc t < 0$, $\cot t > 0$

Determine the value of t, $0 \leq t < 2\pi$, that satisfies each of the following.

33. $\sin t = 1/2$, $\sec t < 0$
34. $\tan t = \sqrt{3}$, $\csc t < 0$
35. $\sec t = -2$, $\csc t > 0$
36. $\csc t = -2$, $\cot t > 0$
37. $\csc t = -\sqrt{2}$, $\sec t < 0$
38. $\sec t = \sqrt{2}$, $\cot t > 0$
39. $\cot t = -1$, $\sec t < 0$
40. $\cot t = \sqrt{3}$, $\csc t < 0$

Use Table V in the Tables Appendix to find each of the following. (Assume $\pi \approx 3.14$ to find the reference number.)

41. cot 3.37
42. sec 0.48
43. csc(−4.68)
44. csc 2.48
45. sec 1.26
46. cot(−1.82)

5.6 THE INVERSE TRIGONOMETRIC FUNCTIONS

Inverse functions were introduced in Section 3.6 and were used to define logarithmic functions in Section 4.2. We have seen that if f is a one-to-one function whose domain is the set X and whose range is the set Y, then the inverse function f^{-1} reverses the correspondence; that is,

$$f^{-1}(y) = x$$

if and only if

$$f(x) = y \quad \text{for all } x \in X$$

Using this definition, we saw that the following identities characterize inverse functions.

$$f^{-1}[f(x)] = x \quad \text{for all } x \text{ in } X$$
$$f[f^{-1}(y)] = y \quad \text{for all } y \text{ in } Y$$

If we attempt to find an inverse of the sine function, we have an immediate problem. Since sine is a periodic function, it is not a one-to-one function and has no inverse. However, we can resolve this problem by defining a function that agrees with the sine function but over a restricted domain. That is, we would like to find an interval such that $y = \sin x$ is one-to-one and y assumes all values between -1 and $+1$ over this interval. If we define the function f by

$$f(x) = \sin x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

then f takes on the same values as the sine function over the interval $[-\pi/2, \pi/2]$ and assumes all real values in the interval $[-1, 1]$. The graph of $\sin x$ over the interval $[-\pi/2, \pi/2]$ shows that f is an increasing function and is therefore one-to-one. Consequently, f has an inverse, and we are led to the following definition.

Inverse Sine Function

The inverse sine function, denoted by **arcsin** or $\mathbf{sin^{-1}}$, is defined by

$$\sin^{-1} y = x \quad \text{if and only if} \quad \sin x = y$$

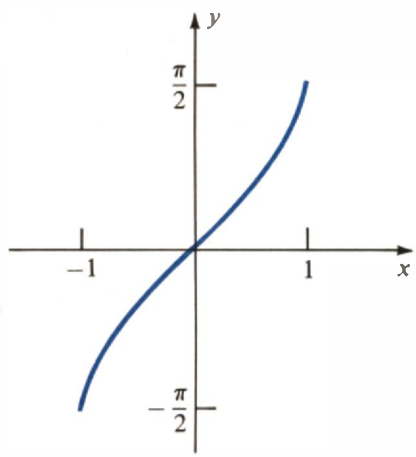
where $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$.

Note that $-1 \le y \le 1$, so the domain of the inverse sine function is the set of all real numbers in the interval $[-1, 1]$.

WARNING When we defined $\sin^n t = (\sin t)^n$ we said that this definition does not hold when $n = -1$, allowing us to reserve the notation $\sin^{-1}$ for the inverse sine function. Therefore, $\sin^{-1} y$ is not to be confused with $1/\sin y$; specifically,

$$\sin^{-1} y \ne \frac{1}{\sin y}$$

y	x
$-\frac{\pi}{2}$	-1
$-\frac{\pi}{3}$	$-\frac{\sqrt{3}}{2} \approx -0.87$
$-\frac{\pi}{6}$	$-\frac{1}{2}$
0	0
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2} \approx 0.87$
$\frac{\pi}{2}$	1



$y = \arcsin x = \sin^{-1} x$

FIGURE 41

The notations arcsin and $\sin^{-1}$ are both in common use and we will therefore employ both notations. Note that if $x = \arcsin y$, then $\sin x = y$. On a unit circle, x determines an *arc whose sine is* y, which is the origin of the notation arcsin y. Although this notation has the advantage of avoiding the possible confusion noted in the preceding warning, the $\sin^{-1}$ notation has become more popular in recent years.

We would like to sketch the graph of $y = \sin^{-1} x$. (Since x and y are simply symbols for variables, we have reverted to the usual practice of letting x be the independent variable.) The graph, of course, is the same as that of $\sin y = x$, with the restriction that $-\pi/2 \leq y \leq \pi/2$. We form a table of values and sketch the graph in Figure 41. Note that for a given value of x, x and $\sin^{-1} x$ are both positive or both negative.

EXAMPLE 1

Find (a) $\arcsin \frac{1}{2}$ (b) $\arcsin(-1)$.

SOLUTION

(a) If $y = \arcsin \frac{1}{2}$, then $\sin y = \frac{1}{2}$ where y is restricted to the interval $[-\pi/2, \pi/2]$. Thus, $y = \pi/6$ is the *only* correct answer.
(b) If $y = \arcsin(-1)$, then $\sin y = -1$ where $-\pi/2 \leq y \leq \pi/2$. Thus, $-\pi/2$ is the *only* correct answer.

EXAMPLE 2

Evaluate $\sin^{-1}\left(\cos \dfrac{\pi}{4}\right)$.

SOLUTION

Since $\cos \dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$, we have

$$\sin^{-1}\left(\cos \frac{\pi}{4}\right) = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

We let

$$y = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

Then

$$\sin y = \frac{\sqrt{2}}{2} \quad \text{where} \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

$$y = \frac{\pi}{4}$$

which is the *only* solution.

PROGRESS CHECK

Find (a) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ (b) $\arcsin\left(\tan\frac{5\pi}{4}\right)$.

ANSWERS

(a) $-\pi/3$ (b) $\pi/2$

We may use a similar approach to define the inverse cosine function. If we define the function f by

$$f(x) = \cos x,\ 0 \le x \le \pi$$

then f agrees with the cosine function over the interval $[0, \pi]$, assumes all real values in the interval $[-1, 1]$, and is a decreasing function. Consequently, f is a one-to-one function and has an inverse.

Inverse Cosine Function The inverse cosine function, denoted by **arccos** or $\mathbf{cos^{-1}}$, is defined by

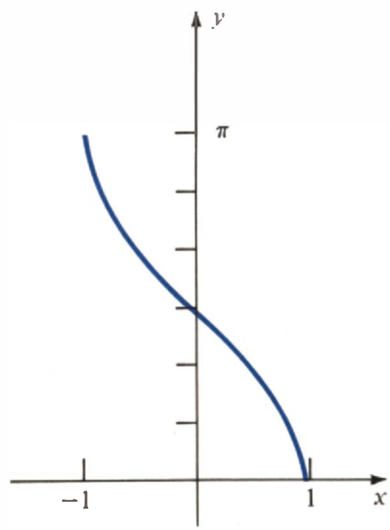
$$\cos^{-1} y = x \quad \text{if and only if} \quad \cos x = y$$

where $0 \le x \le \pi$.

Since $-1 \le y \le 1$, the domain of the inverse cosine function is the set of all real numbers in the interval $[-1, 1]$.

To sketch the graph of $y = \cos^{-1} x$ we sketch the graph of $\cos y = x$ as in Figure 42. Note that $\cos^{-1} x$ is always positive.

y	x
0	1
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2} \approx 0.87$
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	$-\frac{1}{2}$
$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2} \approx -0.87$
π	-1



$y = \arccos x = \cos^{-1} x$

FIGURE 42

EXAMPLE 3
Find (a) $\cos^{-1}(-\frac{1}{2})$ (b) $\arccos(\sin \pi/2)$.

SOLUTION
(a) If $y = \cos^{-1}(-\frac{1}{2})$, then $\cos y = -\frac{1}{2}$ where y is restricted to the interval $[0, \pi]$. Consequently, $y = 2\pi/3$ is the *only* correct answer.
(b) Since $\sin \pi/2 = 1$, we let $y = \arccos(1)$. Then $\cos y = 1$ where $0 \leq y \leq \pi$. Therefore, $y = 0$ is the *only* correct answer.

If we restrict the tangent function to the interval $[-\pi/2, \pi/2]$, we can define the inverse tangent function.

Inverse Tangent Function

The inverse tangent function, denoted by **arctan** or **tan^{-1}**, is defined by

$$\tan^{-1} y = x \quad \text{if and only if} \quad \tan x = y$$

where $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.

Note that the domain of the inverse tangent function is the set of all real numbers.

Proceeding as before, we sketch the graph of $y = \tan^{-1} x$ in Figure 43.

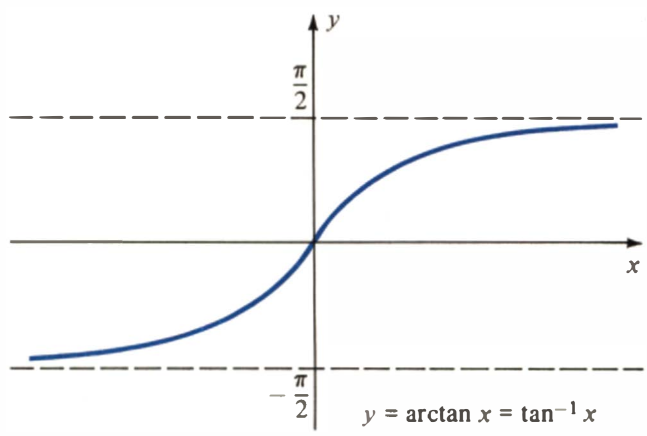


FIGURE 43

EXAMPLE 4
Find $\tan^{-1} \sqrt{3}$.

SOLUTION
If $y = \tan^{-1} \sqrt{3}$, then $\tan y = \sqrt{3}$. Since $-\pi/2 < y < \pi/2$, we must have $y = \pi/3$.

PROGRESS CHECK
Find $\tan^{-1}(-1)$.

ANSWER
$-\pi/4$

EXAMPLE 5
Find cos(arctan 4/3) without using tables or a calculator.

SOLUTION
If we let $x = \arctan 4/3$, then $\tan x = 4/3$ and $0 \leq x < \pi/2$. Using trigonometric identities,

$$\tan x = \frac{\sin x}{\cos x} = \frac{4}{3}$$

$$3 \sin x = 4 \cos x \qquad \text{Clearing fractions}$$

$$9 \sin^2 x = 16 \cos^2 x \qquad \text{Squaring both sides}$$

$$9(1 - \cos^2 x) = 16 \cos^2 x \qquad \sin^2 x = 1 - \cos^2 x$$

$$\cos^2 x = \frac{9}{25}$$

$$\cos x = \pm\frac{3}{5}$$

Since $x \in [0, \pi/2]$, we conclude that $\cos x = 3/5$.

PROGRESS CHECK
Without using tables or a calculator, find $\cot\left(\sin^{-1} -\frac{5}{13}\right)$.

ANSWER
$-12/5$

CALCULATORS

The values of the inverse trigonometric functions can be found by using a calculator. For example, to find arcsin 0.86, you would enter the following functions on most calculators:

$\boxed{\text{RAD}}$ $\boxed{.86}$ $\boxed{\text{INV}}$ $\boxed{\text{SIN}}$

By pressing INV before SIN, you are requesting the inverse sine function rather than the sine function. The answer displayed is 1.035 and will always obey the same restrictions that we have defined for each of the inverse trigonometric functions.

EXAMPLE 6
Using a calculator, find (a) $\tan^{-1} 4.256$ (b) $\cot(\arccos 0.627)$.

SOLUTION
(a) The key sequence

$\boxed{\text{RAD}}\ \boxed{4.256}\ \boxed{\text{INV}}\ \boxed{\text{TAN}}$

provides the answer 1.340 in the display.
(b) The key sequence

$\boxed{\text{RAD}}\ \boxed{0.627}\ \boxed{\text{INV}}\ \boxed{\text{COS}}$

produces the answer 0.8931. To evaluate cot 0.8931, we use the key sequence

$\boxed{0.8931}\ \boxed{\text{TAN}}\ \boxed{1/x}$

and obtain the answer 0.8049 in the display.

PROGRESS CHECK
Use a calculator to find (a) $\sin^{-1}(-0.725)$ (b) $\sec(\arcsin -0.429)$.

ANSWERS
(a) -0.8110 (b) 1.107

EXACT SOLUTIONS

The inverse trigonometric functions can be used to provide exact expressions for the solutions of certain equations. The next pair of examples illustrates this point.

EXAMPLE 7
Find all solutions of the equation $3 \sin x = 1$ that are in the interval $[0, \pi/2]$.

SOLUTION
Solving for $\sin x$, we have

$$\sin x = \frac{1}{3}$$

which we can then write as

$$x = \arcsin \frac{1}{3}$$

This is an *exact* expression for the solution. Using a calculator (or a table), we find that 0.3398 is an *approximate* value of x that satisfies the original equation. Since sine is an increasing function in the interval $[0, \pi/2]$, there can be at most, one solution.

EXAMPLE 8

Find the solutions of the equation $5\cos^2 x - 3 = 0$ that are in the interval $[0, \pi]$.

SOLUTION

We treat the equation as a quadratic in $\cos x$. Then

$$5\cos^2 x = 3$$
$$\cos x = \pm\sqrt{\frac{3}{5}} = \pm\frac{\sqrt{15}}{5}$$

We may then write

$$x = \arccos\left(\frac{\sqrt{15}}{5}\right) \quad \text{or} \quad x = \arccos\left(-\frac{\sqrt{15}}{5}\right)$$

These are exact expressions for the solutions. Numerical approximations can be obtained using Table V in the Tables Appendix or a calculator. The student is urged to verify that

$$x \approx 0.6847 \quad \text{and} \quad x \approx 2.4568$$

are appropriate solutions of the original equation.

PROGRESS CHECK

Find the solutions of the equation $2\sin^2 x + 2\sin x - 1 = 0$ that are in the interval $[-\pi/2, \pi/2]$.

ANSWER

$\arcsin(-\frac{1}{2} \pm \frac{1}{2}\sqrt{3})$

WARNING It is important to remember that the range of each of the inverse trigonometric functions is a subset of the domain of the corresponding trigonometric function. Given the equation

$$t = \sin^{-1}\left(-\frac{1}{2}\right)$$

students often write $t = 7\pi/6$, which is incorrect since t must lie in the interval $[-\pi/2, \pi/2]$. The only correct answer is $-\pi/6$.

EXERCISE SET 5.6

In Exercises 1–18 evaluate the given expression.

1. $\sin^{-1}\left(-\frac{1}{2}\right)$
2. $\arccos\left(\frac{\sqrt{3}}{2}\right)$
3. $\arctan \sqrt{3}$
4. $\tan^{-1} 0$

5. $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$
6. $\cos^{-1}(-1)$
7. $\arccos\left(-\frac{\sqrt{3}}{2}\right)$
8. $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$
9. $\sin^{-1}(-1)$
10. $\arctan 1$
11. $\cos^{-1} 0$
12. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
13. $\cos^{-1} 1$
14. $\arcsin\left(\frac{\sqrt{2}}{2}\right)$
15. $\arctan(-1)$
16. $\sin^{-1} 0$
17. $\cos^{-1}\left(-\frac{1}{2}\right)$
18. $\arcsin\left(\frac{1}{2}\right)$

In Exercises 19–24 use Table V in the Tables Appendix to approximate the given expression.

19. $\sin^{-1}(0.3709)$
20. $\arctan(1.398)$
21. $\cos^{-1}(-0.7648)$
22. $\tan^{-1}(-3.010)$
23. $\arcsin(0.9636)$
24. $\arccos(-0.921)$

25–30. Repeat Exercises 19–24 using a calculator that has a key marked INV or an equivalent notation.

In Exercises 31–46 evaluate the given expression.

31. $\sin(\arctan 1)$
32. $\cos\left(\arcsin -\frac{1}{2}\right)$
33. $\tan^{-1}\left(\cos \frac{\pi}{2}\right)$
34. $\sin^{-1}(\sin 0.62)$
35. $\cos^{-1}\left(\sin \frac{9\pi}{4}\right)$
36. $\tan(\sin^{-1} 0)$
37. $\cos^{-1}\left(\cos \frac{2\pi}{3}\right)$
38. $\sin^{-1}\left(\cos \frac{\pi}{6}\right)$

In Exercises 39–44 use the inverse trigonometric functions to express the solutions of the given equation exactly.

39. $7 \sin^2 x - 1 = 0,\ x \in [-\pi/2, \pi/2]$
40. $6 \cos^2 y - 5 = 0,\ y \in [0, \pi]$
41. $12 \cos^2 x - \cos x - 1 = 0,\ x \in [0, \pi]$
42. $2 \tan^2 t + 4 \tan t - 3 = 0,\ t \in [-\pi/2, \pi/2]$
43. $9 \sin^2 t - 12 \sin t + 4 = 0,\ t \in [-\pi/2, \pi/2]$
44. $3 \cos^2 x - 7 \cos x - 6 = 0,\ x \in [0, \pi]$

In Exercises 45 and 46 provide a value for x to show that the equation is not an identity.

45. $\sin^{-1} x = \frac{1}{\sin x}$
46. $(\sin^{-1} x)^2 + (\cos^{-1} x)^2 = 1$

TERMS AND SYMBOLS

angle (p. 214)
initial side of an angle (p. 214)
terminal side of an angle (p. 214)
standard position of an angle (p. 215)
positive angle (p. 215)
negative angle (p. 215)
quadrantal angle (p. 215)
unit circle (p. 216)
degree measure (p. 216)
right angle (p. 217)
acute angle (p. 217)
obtuse angle (p. 217)
radian measure (p. 217)
angle of t radians (p. 217)
unit circle point (p. 217)
coterminal angles (p. 220)
trigonometric functions (p. 225)
sine (sin) (p. 225)
cosine (cos) (p. 225)
tangent (tan) (p. 225)
identities (p. 230)
circular functions (p. 230)
trigonometric identity (p. 230)
periodic function (p. 237)
period (p. 237)
reference number (p. 239)
Reference Number Rule (p. 239)
reference angle (p. 241)
Reference Angle Rule (p. 242)
minutes (p. 242)
seconds (p. 242)
increases without bound (p. 247)
decreases without bound (p. 247)
vertical asymptotes (p. 247)
even function (p. 247)
odd function (p. 247)
amplitude (p. 249)
reflection (p. 250)
phase shift (p. 252)
secant (sec) (p. 256)
cosecant (csc) (p. 256)
cotangent (cot) (p. 256)
arcsin ($\sin^{-1}$) (p. 262)
arccos ($\cos^{-1}$) (p. 264)
arctan ($\tan^{-1}$) (p. 265)

KEY IDEAS FOR REVIEW

- ☐ An angle may be measured in either degrees or in radians. The two forms of measure are related by the equation π radians = 180°.
- ☐ Every real number t determines a unit circle point P measured along the circle from $A(1, 0)$. If t is positive, the arc is measured in the counterclockwise direction; if t is negative, the arc is measured in the clockwise direction.
- ☐ If the unit circle point P corresponds to the real number t, then it also corresponds to every real number of the form $t + 2\pi n$ where n is an integer.
- ☐ The trigonometric functions sine, cosine, and tangent are defined in terms of the rectangular coordinates of a unit circle point $P(x, y)$ determined by a real number t:

$$\sin t = y$$
$$\cos t = x$$
$$\tan t = y/x,\ x \neq 0$$

- ☐ A trigonometric function of an angle is the same as the trigonometric function of the arc on the unit circle that the angle intercepts.
- ☐ The signs of the trigonometric functions in each of the quadrants follow from the definitions and are displayed in Figure 37.
- ☐ Sine and tangent are odd functions and cosine is an even function. That is,

$$\sin(-t) = -\sin t$$
$$\cos(-t) = \cos t$$
$$\tan(-t) = -\tan t$$

- ☐ The trigonometric functions are all periodic. The period of the sine and cosine functions is 2π; the period of the tangent function is π.
- ☐ Standard tables of the values of the trigonometric functions (see Table V in the Tables Appendix) display the independent variable t from 0 to $\pi/2$. If it is desired to find $\sin t'$ where $\pi/2 < t' < 2\pi$, the Reference Number Rule is used to determine a real number t, $0 \leq t < \pi/2$, such that $|\sin t| = |\sin t'|$. The appropriate sign is then assigned depending on the quadrant of t'.
- ☐ The Reference Angle Rule is analogous to the Reference Number Rule. It enables us to find the value of a trigonometric function of an angle greater than 90° by using Table VI in the Tables Appendix, which gives the values for angles between 0° and 90°.
- ☐ To sketch the graph of $f(x) = A \sin(Bx + C)$, note that
  (i) the amplitude is $|A|$;
  (ii) the period is $2\pi/B$;
  (iii) the phase shift is $-C/B$.
  The same observations hold for $f(x) = A \cos(Bx + C)$.
- ☐ The secant, cosecant, and cotangent functions are defined as the reciprocals of cosine, sine, and tangent, respectively.
- ☐ To define the inverse trigonometric functions, it is necessary to restrict the domain of the trigonometric functions to ensure that the result is a one-to-one function.

REVIEW EXERCISES

Solutions to exercises whose numbers are in color are in the Solutions section in the back of the book.

5.1 In Exercises 1–4 convert from degree measure to radian measure or from radian measure to degree measure.

1. $-60°$
2. $\dfrac{3\pi}{2}$
3. $-\dfrac{5\pi}{12}$
4. $45°$

In Exercises 5–7 determine if the pair of angles are coterminal.

5. $100°, \frac{5\pi}{9}$

6. $\frac{4\pi}{3}, 480°$

7. $\frac{5\pi}{4}, -135°$

8. If a central angle θ subtends an arc of length 14 centimeters on a circle whose radius is 10 centimeters, find the radian measure of θ.

9. A central angle of $2\pi/3$ radians subtends an arc of length $5\pi/2$ centimeters. Find the radius of the circle.

5.2 In Exercises 10–13 determine the quadrant in which t or θ lies.

10. $t = \frac{11\pi}{6}$

11. $\theta = -220°$

12. $\theta = 490°$

13. $t = -\frac{11\pi}{3}$

In Exercises 14–17 replace each given real number t by t', $0 \le t' < 2\pi$, so that t and t' determine the same unit circle point.

14. $\frac{9\pi}{2}$

15. $-\frac{15\pi}{2}$

16. -6π

17. $\frac{23\pi}{3}$

In Exercises 18–22 the unit circle point $(4/5, -3/5)$ corresponds to the real number t. Use the symmetries of the circle to find the rectangular coordinates corresponding to the given real number.

18. $(t - \pi)$

19. $\left(t + \frac{\pi}{2}\right)$

20. $(-t)$

21. $\left(t - \frac{\pi}{2}\right)$

22. $(-t - \pi)$

In Exercises 23–26 find the quadrant in which t lies if the following conditions hold.

23. $\tan t < 0$ and $\sin t < 0$

24. $\sin t < 0$ and $\cos t > 0$

25. $\sin(-t) > 0$ and $\tan t > 0$

26. $\sin(-t) < 0$ and $\cos(-t) > 0$

In Exercises 27–30 use the trigonometric identities

$$\sin^2 t + \cos^2 t = 1 \qquad \tan t = \frac{\sin t}{\cos t}$$

to find the indicated value under the given conditions.

27. $\cos t = \frac{3}{5}$ and t is in quadrant IV; find $\cot t$.

28. $\sin t = -\frac{4}{5}$ and $\tan t > 0$; find $\sec t$.

29. $\sin t = \frac{12}{13}$ and $\cos t < 0$; find $\tan t$.

30. $\cos t = -\frac{5}{13}$ and $\tan t < 0$; find $\csc t$.

In Exercises 31 and 32 use the trigonometric identities to transform the first expression into the second.

31. $(\sin t)(\sec t)$, $\tan t$

32. $\frac{\sin t}{\cos^2 t}$, $(\tan t)(\sec t)$

5.3 In Exercises 33–36 determine the value of the indicated trigonometric function, without the use of tables or a calculator.

33. $\sin \frac{2\pi}{3}$

34. $\sec\left(-\frac{5\pi}{4}\right)$

35. $\tan \frac{5\pi}{6}$

36. $\csc\left(-\frac{\pi}{6}\right)$

In Exercises 37–40 find a value of t, $0 \le t \le 2\pi$, satisfying the given conditions.

37. $\sin t = -\frac{\sqrt{2}}{2}$, t in quadrant III

38. $\cos t = \frac{\sqrt{3}}{2}$, t in quadrant IV

39. $\cot t = \frac{\sqrt{3}}{3}$, t in quadrant I

40. $\sec t = -2$, t in quadrant II

In Exercises 41 and 42 use a calculator (or Table V in the Tables Appendix) to evaluate the given expression.

41. $\cos 3.71 - \sin 1.44$

42. $\tan(-2.74)$

5.4 In Exercises 43 and 44 sketch the graph of the given function.

43. $f(x) = 1 - \sin x$

44. $f(x) = 2 \sin\left(\frac{x}{2} + \pi\right)$

In Exercises 45–47 determine the amplitude, period, and phase shift of each given function.

45. $f(x) = -\cos(2x - \pi)$

46. $f(x) = 4 \sin\left(-x + \frac{\pi}{2}\right)$

47. $f(x) = -2 \sin\left(\frac{x}{3} + \frac{\pi}{3}\right)$

5.5 In Exercises 48 and 49 determine the value of t, $0 \leq t \leq 2\pi$, that satisfies the given conditions.

48. $\cos t = 1$, $\sec t < 0$

49. $\sec t = -\sqrt{2}$, $\csc t > 0$

5.6 In Exercises 50–53 evaluate the given expression.

50. $\arcsin\left(-\frac{1}{2}\right)$

51. $\tan(\cos^{-1} 1)$

52. $\tan(\tan^{-1} 5)$

53. Use the inverse cosine function to express the exact solutions of the equation

$$5 \cos^2 x - 4 = 0$$

PROGRESS TEST 5A

In Problems 1–3 convert from degree measure to radian measure or from radian measure to degree measure.

1. $\frac{5\pi}{3}$

2. $-200°$

3. $75°$

In Problems 4 and 5 find an angle θ, $0 \leq \theta < 360°$, that is coterminal with the given angle.

4. $-25°$

5. $\frac{17\pi}{4}$

6. If a central angle θ subtends an arc of length 12 inches on a circle whose radius is 15 inches, find the radian measure of θ.

In Problems 7 and 8 replace the given real number t by t', $0 \leq t' < 2\pi$, so that t and t' determine the same unit circle point.

7. $\frac{19\pi}{3}$

8. -22π

In Problems 9 and 10 find the rectangular coordinates of the unit circle point determined by the given real number t.

9. $\frac{29\pi}{6}$

10. $-\frac{\pi}{3}$

In Problems 11–13 the unit circle point $P(-5/13, 12/13)$ corresponds to the real number t. Use the symmetries of the circle to find the rectangular coordinates of the point corresponding to the given real number.

11. $t + \pi$

12. $t - \frac{\pi}{2}$

13. $-t$

In Problems 14 and 15 find the reference angle of the given angle.

14. $160°$

15. $\frac{7\pi}{4}$

In Problems 16 and 17 determine the value of the indicated trigonometric function without the use of the tables or a calculator.

16. $\cos\left(\frac{7\pi}{3}\right)$

17. $\csc\left(-\frac{2\pi}{3}\right)$

In Problems 18 and 19 find a value of $t \in [0, 2\pi]$ satisfying the given conditions.

18. $\tan t = 1$, t in quadrant III

19. $\sec t = \sqrt{2}$, t in quadrant IV

In Problems 20 and 21 use the trigonometric identities

$$\sin^2 t + \cos^2 t = 1, \quad \tan t = \frac{\sin t}{\cos t}$$

to find the indicated value under the given conditions.

20. $\cos t = -\frac{12}{13}$ and $\tan t > 0$; find $\sin t$.

21. $\sin t = \frac{3}{5}$ and t is in quadrant II; find $\sec t$.

22. Use the trigonometric identities given for Problems 20 and 21 to transform

$$1 - \tan x \quad \text{to} \quad \frac{\cos x - \sin x}{\cos x}$$

In Problems 23 and 24 use Table V in the Tables Appendix or a calculator to evaluate the given expression.

23. $\tan(-3.68)$

24. $\cos 1.15 - \sin 0.72$

25. Sketch the graph of the function f defined by $f(x) = x + \cos x$.

In Problems 26 and 27 determine the amplitude, period, and phase shift of each given function.

26. $f(x) = -2\cos(\pi - x)$

27. $f(x) = 2\sin\left(\frac{x}{2} - \frac{\pi}{2}\right)$

In Problems 28 and 29 evaluate the given expression without the use of tables or a calculator.

28. $\tan^{-1}(-\sqrt{3})$

29. $\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)$

30. Use the inverse tangent function to express the exact solutions of the equation

$$6\tan^2 x - 13\tan x + 6 = 0$$

PROGRESS TEST 5B

In Problems 1–3 convert from degree measure to radian measure or from radian measure to degree measure.

1. $-135°$

2. $\frac{3\pi}{4}$

3. $-\frac{5\pi}{6}$

In Problems 4 and 5 find an angle θ, $0 \leq \theta < 360°$, that is coterminal with the given angle.

4. $430°$

5. $-\frac{2\pi}{3}$

6. A central angle of 100° subtends an arc of length $7\pi/3$ centimeters. Find the radius of the circle.

In Problems 7 and 8 replace the given real number t by t', $0 \leq t' < 2\pi$, so that t and t' determine the same unit circle point.

7. -14π

8. $\frac{51\pi}{5}$

In Problems 9 and 10 find the rectangular coordinates of the unit circle point determined by the given real number t.

9. $\frac{23\pi}{6}$

10. $-\frac{3\pi}{4}$

In Problems 11–13 the unit circle point $P(-4/5, -3/5)$ corresponds to the real number t. Use the symmetries of the circle to find the rectangular coordinates of the point corresponding to the given real number.

11. $-t$

12. $t + \frac{\pi}{2}$

13. $-t + \pi$

In Problems 14 and 15 find the reference angle of the given angle.

14. $\frac{9\pi}{16}$

15. $345°$

In Problems 16 and 17 determine the value of the indicated trigonometric function without the use of the tables or a calculator.

16. $\tan\left(\frac{7\pi}{4}\right)$

17. $\sin\left(-\frac{3\pi}{2}\right)$

In Problems 18 and 19 find a value of $t \in [0, 2\pi]$ satisfying the given conditions.

18. $\sin t = \frac{\sqrt{3}}{2}$, t in quadrant I

19. $\sec t = -2$, t in quadrant II

In Problems 20 and 21 use the trigonometric identities

$$\sin^2 t + \cos^2 t = 1 \qquad \tan t = \frac{\sin t}{\cos t}$$

to find the indicated value under the given conditions.

20. $\sin t = -\dfrac{5}{13}$ and $\tan t < 0$; find $\tan t$.

21. $\cos t = \dfrac{3}{5}$ and $\cot t < 0$; find $\cot t$.

22. Use the trigonometric identities of Problems 20 and 21 to transform $\sec^2 t \cot t$ to $\csc t$.

In Problems 23 and 24 use Table V in the Tables Appendix or a calculator to evaluate the given expression.

23. $\sin(2.45)$

24. $\tan(-1.25) + \cos 1.67$

25. Sketch the graph of the function f defined by

$$f(x) = \sin x + \sin \frac{x}{2}$$

In Problems 26 and 27 determine the amplitude, period, and phase shift of each given function.

26. $f(x) = 4 \sin(3x - \pi)$

27. $f(x) = -\dfrac{1}{2} \cos\left(2x + \dfrac{\pi}{2}\right)$

In Problems 28 and 29 evaluate the given expression without the use of tables or a calculator.

28. $\sin^{-1}\left(\cos \dfrac{\pi}{3}\right)$

29. $\tan\left(\cos^{-1} \dfrac{\sqrt{2}}{2}\right)$

30. Use the inverse sine function to express the exact solutions of the equation $5 \sin^2 x - 2 \sin x - 3 = 0$.

6

TRIGONOMETRY: MEASURING TRIANGLES

In the previous chapter we discussed trigonometry in terms of functions of angles and real numbers. This approach has the advantage of illustrating the centrality of the function concept in much of modern mathematical thinking.

We now turn to the more traditional approach to trigonometry, which revolves about the measurement of triangles. We will show that it is possible to define the trigonometric functions in terms of the angles and sides of a right triangle. This will then give us an opportunity to explore a wide variety of applications that clearly demonstrate the usefulness of trigonometry in such fields as surveying and navigation.

We will conclude by examining the law of sines and the law of cosines, two important rules that can be employed when dealing with an oblique triangle, that is, a triangle that does not contain a right angle.

6.1 RIGHT TRIANGLE TRIGONOMETRY

We are now prepared to show that the trigonometric functions of an acute angle are related to the ratios of the sides of a right triangle. In Figure 1a we display a right triangle with sides a and b, hypotenuse r, and an acute angle θ. We can place this triangle on a Cartesian coordinate system with θ in standard position (Figure 1b). We can draw a unit circle and let $N(x, y)$ denote the point of intersection of the circle and the hypotenuse OP. If we drop the perpendicular NM as indicated, we see that the triangles OMN and OQP are similar. The corresponding sides must then be proportional so that

$$\frac{\overline{MN}}{\overline{ON}} = \frac{\overline{QP}}{\overline{OP}} \quad \text{and} \quad \frac{\overline{OM}}{1} = \frac{\overline{OQ}}{\overline{OP}}$$

Since $\overline{OP} = r$, we obtain by substitution

$$\frac{y}{1} = \frac{b}{r} \quad \text{and} \quad \frac{x}{1} = \frac{a}{r}$$

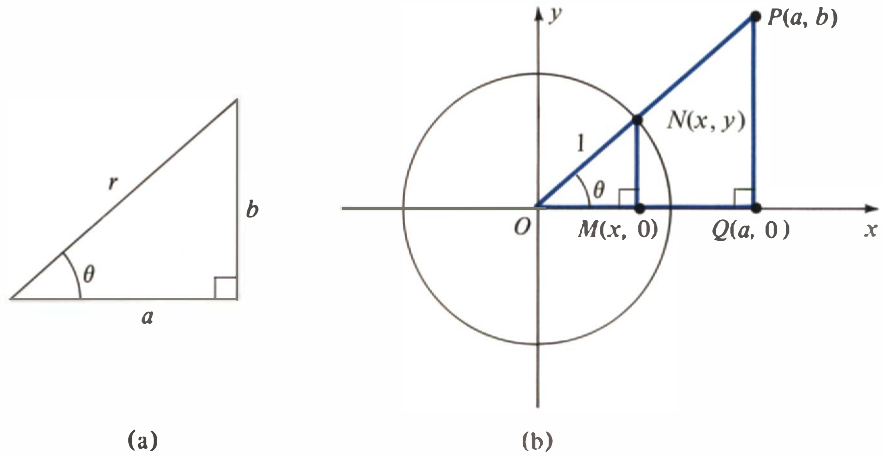


FIGURE 1

By definition, $\sin\,\theta = y$ and $\cos\,\theta = x$. Substituting, we have

$$\sin\,\theta = y = \frac{b}{r}$$

$$\cos\,\theta = x = \frac{a}{r}$$

$$\tan\,\theta = \frac{\sin\,\theta}{\cos\,\theta} = \frac{b}{a}$$

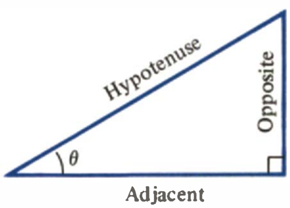


FIGURE 2

If we denote the sides a and b of the right triangle in Figure 1a as the adjacent and opposite sides relative to the angle θ (see Figure 2), then this last result expresses the trigonometric functions as ratios of the lengths of the sides of the right triangle.

$$\sin\,\theta = \frac{\text{side opposite } \theta}{\text{hypotenuse}} \qquad \csc\,\theta = \frac{\text{hypotenuse}}{\text{side opposite } \theta}$$

$$\cos\,\theta = \frac{\text{side adjacent to } \theta}{\text{hypotenuse}} \qquad \sec\,\theta = \frac{\text{hypotenuse}}{\text{side adjacent to } \theta}$$

$$\tan\,\theta = \frac{\text{side opposite } \theta}{\text{side adjacent to } \theta} \qquad \cot\,\theta = \frac{\text{side adjacent to } \theta}{\text{side opposite } \theta}$$

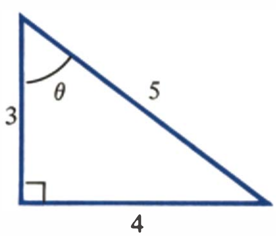


FIGURE 3

EXAMPLE 1

Find the values of the trigonometric functions of the angle θ in Figure 3.

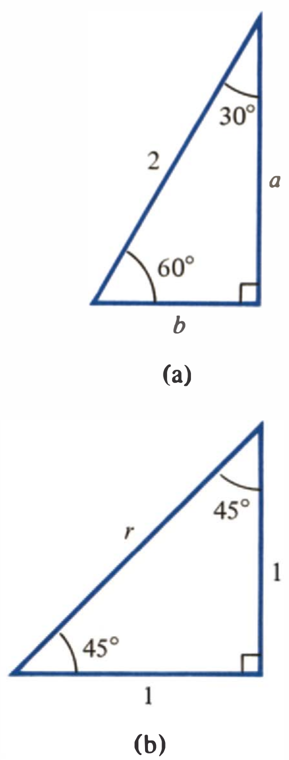


FIGURE 4

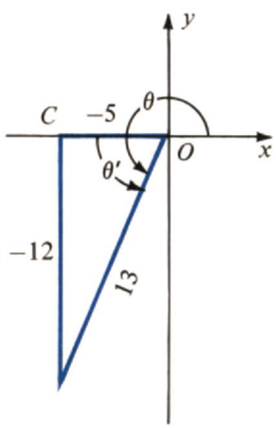


FIGURE 5

SOLUTION

$$\sin\theta = \frac{4}{5} \qquad \csc\theta = \frac{5}{4}$$

$$\cos\theta = \frac{3}{5} \qquad \sec\theta = \frac{5}{3}$$

$$\tan\theta = \frac{4}{3} \qquad \cot\theta = \frac{3}{4}$$

EXAMPLE 2

Use the values of the trigonometric functions to find the following.

(a) The sides of a 30°–60°–90° right triangle whose hypotenuse is of length 2.

(b) The hypotenuse of an isosceles right triangle whose sides are of length 1.

SOLUTION

(a) (See Figure 4a.) Since $\cos 60° = \frac{1}{2}$, we have

$$\cos 60° = \frac{1}{2} = \frac{b}{2} \quad \text{or} \quad b = 1$$

Similarly, we can establish that

$$\sin 60° = \frac{\sqrt{3}}{2} = \frac{a}{2} \quad \text{or} \quad a = \sqrt{3}$$

The student is urged to verify these results using the 30° angle and to verify that these values of a and b satisfy the Pythagorean theorem.

The results for a 30°–60°–90° right triangle can also be obtained by a geometric argument starting with an equilateral triangle whose sides are of length 2. (See Exercise 40.)

(b) (See Figure 4b.) We know that $\sin 45° = \sqrt{2}/2$, so

$$\sin 45° = \frac{\sqrt{2}}{2} = \frac{1}{r} \quad \text{or} \quad r = \frac{2}{\sqrt{2}} = \sqrt{2}$$

Of course, we could have obtained r directly by using the Pythagorean theorem.

EXAMPLE 3

Find $\sec\theta$ if the point $P(-5, -12)$ lies on the terminal side of θ.

SOLUTION

(See Figure 5.) We construct a perpendicular from P to the x-axis to form right triangle PCO and use the Pythagorean theorem to find $\overline{OP} = 13$. Then

$$\sec\theta' = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{13}{5}$$

Since θ is in the third quadrant, by the Reference Angle Rule,

$$\sec\theta = -\sec\theta' = -\frac{13}{5}$$

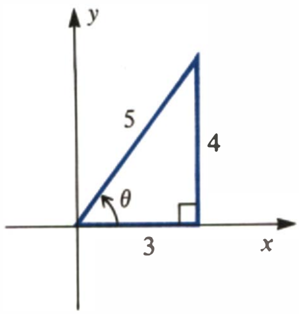


FIGURE 6

In the last chapter we worked on various problems involving the inverse trigonometric functions. Right triangle trigonometry provides us with a faster, simpler approach to many of these problems. We illustrate by repeating Example 5 of Section 5.6.

EXAMPLE 4
Find cos(arctan 4/3) without using tables or a calculator.

SOLUTION
We let $\theta = \arctan 4/3$ so that $\tan\theta = 4/3$ and $0 \le \theta \le \pi/2$. The angle θ in Figure 6 satisfies these conditions. Then we see that

$$\cos\left(\arctan\frac{4}{3}\right) = \cos\theta = \frac{3}{5}$$

SOLVING A TRIANGLE

The expression "to solve a triangle" is used to indicate that we seek all parts of the triangle, that is, the length of each side and the measure of each angle. For any right triangle, given any two sides, or given one side and an acute angle, it is always possible to solve the triangle. We will standardize the notation as shown in Figure 7 so that (a) the acute angles are labeled α and β, the right angle is labeled γ, and (b) the sides opposite angles α, β, and γ are labeled a, b, and c, respectively. In solving a triangle, we will restrict ourselves to the sine, cosine, and tangent functions since these are the trigonometric functions available on calculators.

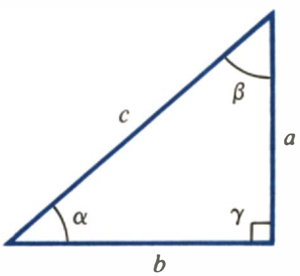


FIGURE 7

EXAMPLE 5
In triangle ABC, $\gamma = 90°$, $\beta = 27°$, and $b = 8.6$. Find approximate values for the remaining parts of the triangle.

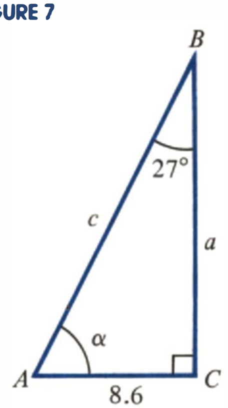


FIGURE 8

SOLUTION
We begin by labeling a right triangle as in Figure 8. Since the sum of the angles of a triangle is 180°, we see that $\alpha = 63°$. Using the trigonometric functions of angle β we have

$$\sin 27° = \frac{8.6}{c} \quad \text{and} \quad \tan 27° = \frac{8.6}{a}$$

Solving for a and c yields

$$c = \frac{8.6}{\sin 27°} \qquad a = \frac{8.6}{\tan 27°}$$

Using a calculator, $\sin 27° = 0.4540$ and $\tan 27° = 0.5095$, so

$$c = 8.6/0.4540 \approx 18.9$$
$$a = 8.6/0.5095 \approx 16.9$$

PROGRESS CHECK

In triangle ABC, $\gamma = 90°$, $\alpha = 64°$, and $b = 24.7$. Solve the triangle.

ANSWERS

$\beta = 26°$ $\quad a = 50.6 \quad c = 56.3$

EXAMPLE 6

In triangle ABC, $\gamma = 90°$, $a = 22.5$, and $b = 12.8$. Find approximate values for the remaining parts of the triangle.

SOLUTION

Figure 9 displays the parts of the triangle. Using angle β we have

$$\tan \beta = \frac{12.8}{22.5} = 0.5689$$

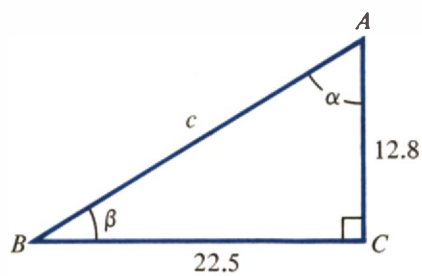


FIGURE 9

Using a calculator, we find that $\beta \approx 29.63°$, or $29°38'$. (The closest entry in Table VI in the Tables Appendix is $29°40'$.) Since the sum of the angles is $180°$, we must have $\alpha \approx 60°22'$. Alternatively,

$$\tan \alpha = \frac{22.5}{12.8} = 1.7578$$

also yields $\alpha \approx 60°22'$.

Finally, c can be found by the Pythagorean theorem or by trigonometry.

$$\sin \beta = \sin 29°38' = \frac{12.8}{c}$$
$$c = \frac{12.8}{0.4944} \approx 25.9$$

PROGRESS CHECK

In triangle ABC, $\gamma = 90°$, $a = 17.4$, and $b = 38.2$. Solve the triangle.

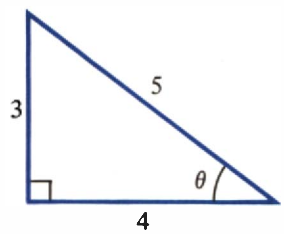
ANSWERS

$\alpha = 24°30'$ $\beta = 65°30'$ $c = 42$

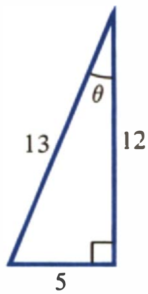
EXERCISE SET 6.1

Find the values of the trigonometric functions of the angle θ in each of the following right triangles.

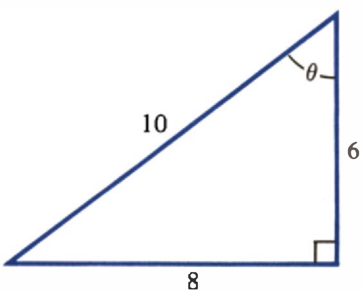
1.



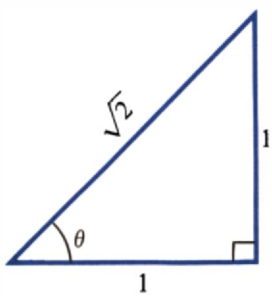
2.



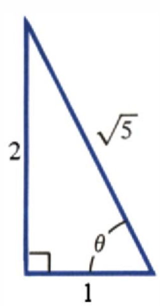
3.



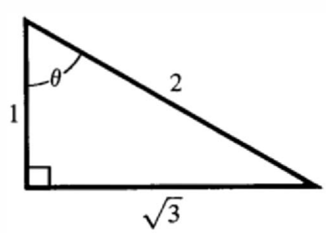
4.

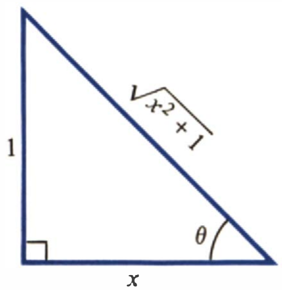


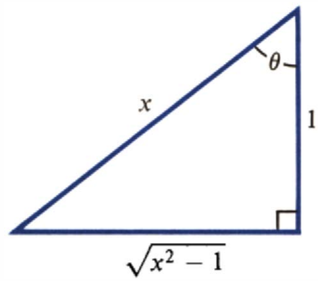
5.



6.



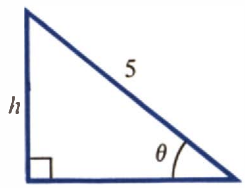
7.


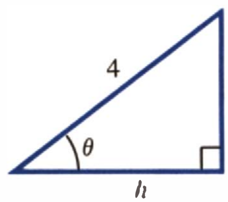
8.


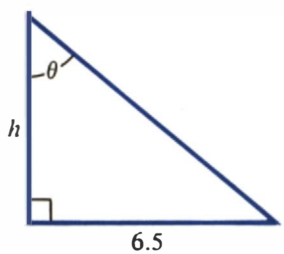
Find the values of the trigonometric functions of the angle θ if the point P lies on the terminal side of θ.

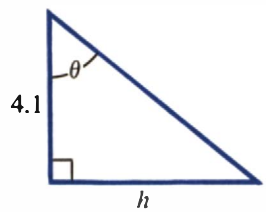
9. $P(-5, 12)$
10. $P(3, -4)$
11. $P(-1, -1)$
12. $P(1, 2)$
13. $P(-8, 6)$
14. $P(12, 5)$
15. $P(12, -5)$
16. $P(-1, \sqrt{3})$
17. $P(-12, -5)$
18. $P(-3, 4)$
19. $P(-2, 1)$
20. $P(-2, -1)$

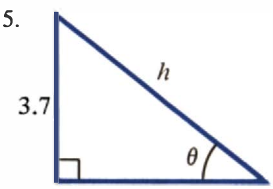
In each of the following right triangles, express the length h as a trigonometric function of the angle θ.

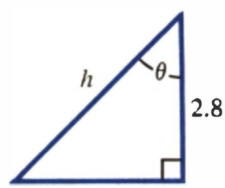
21.


22.


23.


24.


25.


26.


In triangle ABC, $\gamma = 90°$. Find the required parts of the triangle in each of the following.

27. $a = 12$, $b = 16$; find α.
28. $a = 5$, $b = 15$; find β.
29. $b = 40$, $\beta = 40°$; find c.
30. $a = 22$, $\alpha = 36°$; find b.
31. $a = 75$, $\beta = 22°$; find b.
32. $b = 60$, $\alpha = 53°$; find c.
33. $a = 25$, $\beta = 42°30'$; find c.
34. $b = 50$, $\alpha = 36°20'$; find a.

Evaluate the given expression without using tables or a calculator.

35. $\tan\left(\sin^{-1} -\frac{5}{13}\right)$

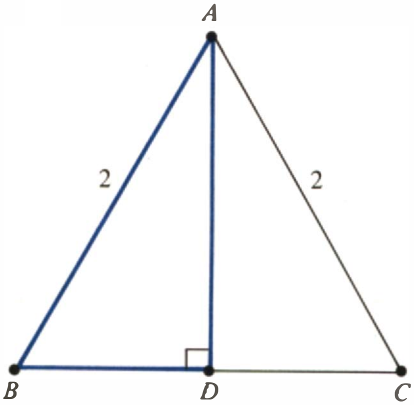
36. $\sin\left(\arctan -\frac{12}{5}\right)$

37. $\cos\left(\sin^{-1} \frac{4}{5}\right)$

38. $\cos\left(\arcsin -\frac{2}{3}\right)$

39. $\tan\left(\cos^{-1} -\frac{3}{5}\right)$

40. In the accompanying figure, ABC is an equilateral triangle whose sides are of length 2, and AD is the perpendicular from A to side BC. Show that
(a) triangle ABD is a 30°–60°–90° right triangle;
(b) sides BD and AD are of lengths 1 and $\sqrt{3}$, respectively.



6.2 APPLICATIONS OF RIGHT TRIANGLE TRIGONOMETRY

Many applied problems involve right triangles. We are now prepared to use our ability in solving triangles to tackle a variety of interesting problems.

EXAMPLE 1

A ladder leaning against a building makes an angle of 35° with the ground. If the bottom of the ladder is 5 meters from the building, how long is the ladder? To what height does it rise along the building?

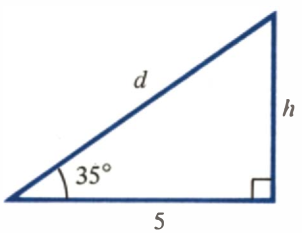


FIGURE 10

SOLUTION

In Figure 10 we seek the length d of the ladder and the height h along the building. Using right triangle trigonometry,

$$\cos 35° = \frac{5}{d} \qquad \text{and} \qquad \tan 35° = \frac{h}{5}$$

$$d = \frac{5}{\cos 35°} \qquad\qquad h = 5 \tan 35°$$

$$d = \frac{5}{0.8192} \qquad\qquad h = 5(0.7002)$$

$$d \approx 6.1 \text{ meters} \qquad \text{and} \qquad h \approx 3.5 \text{ meters}$$

PROGRESS CHECK

The string of a kite makes an angle of 32°30′ with the ground. If 125 meters of string have been let out, how high is the kite?

ANSWER
67 meters

ELEVATION AND DEPRESSION

There are two technical terms that will occur frequently in our word problems. The **angle of elevation** is the angle between the horizontal and the line of sight. In Figure 11a, θ is the angle of elevation of the top T of a tree from a point x meters from the base of the tree.

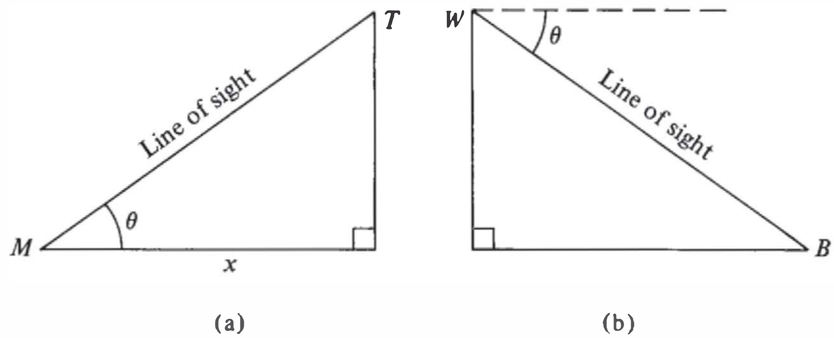


FIGURE 11

The **angle of depression** is the angle between the horizontal and the line of sight when looking down. In Figure 11b, θ is the angle of depression of a boat B as seen from a watchtower W.

EXAMPLE 2
A vendor of balloons inadvertently releases a balloon, which rises straight up. A child standing 50 feet from the vendor watches the balloon rise. When the angle of elevation of the balloon reaches 44°, how high is the balloon?

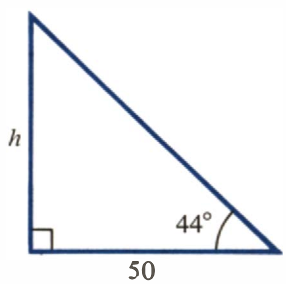


FIGURE 12

SOLUTION
We seek the height h in Figure 12. Thus,

$$\tan 44° = \frac{h}{50}$$

$$h = 50 \tan 44°$$

$$h = 50(0.9657) \approx 48$$

The balloon has risen approximately 48 feet.

EXAMPLE 3
A forest ranger is in a tower 65 feet above the ground. If the ranger spots a fire at an angle of depression of 6°40′, how far is the fire from the base of the tower (assuming level terrain)?

SOLUTION

We need to find the distance d in Figure 13. Since $\theta + 6°40' = 90°$, $\theta = 83°20'$, then

$$\tan \theta = \frac{d}{65}$$

$$d = 65 \tan 83°20'$$

$$d = 65(8.5555) \approx 556$$

The fire is approximately 556 feet from the base of the tower.

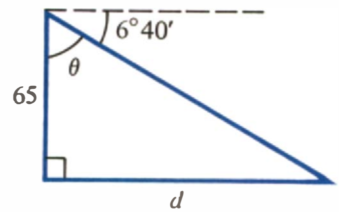


FIGURE 13

EXAMPLE 4

A mathematics professor walks toward the university clock tower on the way to her office, and decides to find the height of the clock above ground. She determines the angle of elevation to be 30° and, after proceeding an additional 60 feet toward the base of the tower, finds the angle of elevation to be 40°. What is the height of the clock tower?

SOLUTION

This problem is somewhat more sophisticated since it involves more than one right triangle. In Figure 14 we seek to determine h. From triangle ACD,

$$\tan 30° = \frac{h}{d + 60} \quad \text{or} \quad h = (d + 60)(\tan 30°)$$

and from triangle ACB,

$$\tan 40° = \frac{h}{d} \quad \text{or} \quad h = d \tan 40°$$

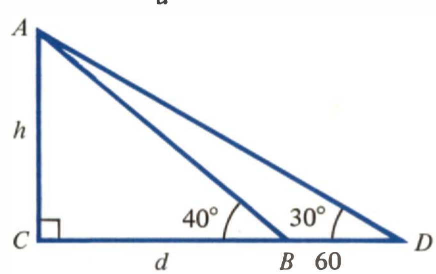


FIGURE 14

Equating the two expressions for h yields

$$(d + 60)(\tan 30^\circ) = d \tan 40^\circ$$
$$60 \tan 30^\circ = d(\tan 40^\circ - \tan 30^\circ)$$
$$d = \frac{60 \tan 30^\circ}{\tan 40^\circ - \tan 30^\circ} \approx 132 \text{ feet}$$
$$h = d \tan 40^\circ \approx (132)(0.8391) \approx 110.8$$

The height of the clock tower is approximately 110.8 feet.

NAVIGATION AND SURVEYING

In navigation and surveying, directions are often given by **bearings,** which specify an acute angle and its direction from the north–south line. In Figure 15a the bearing of point B from point A is N 40° E, that is, 40° east of north; in Figure 15b the bearing of point B from point A is S 60° W; and in Figure 15c it is S 20° E.

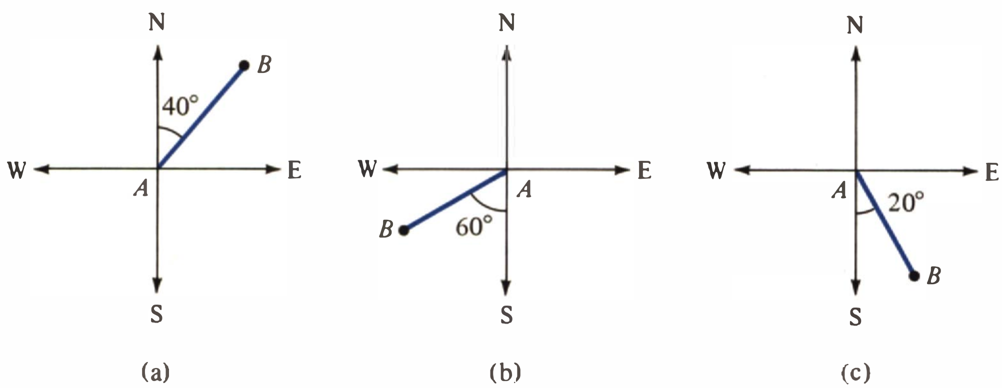


FIGURE 15

EXAMPLE 5

A ship leaves port at 10 A.M. and heads due east at a rate of 22 miles per hour. At 11 A.M. the course is changed to S 52° E. Find the distance and bearing of the ship from the dock at noon.

SOLUTION

The situation is depicted in Figure 16. We find angle $\beta = 38°$. From right triangle BCE,

$$\cos \beta = \frac{e}{22} \quad \text{or} \quad e = 22 \cos 38^\circ \approx 17.3 \text{ miles}$$

$$\sin \beta = \frac{b}{22} \quad \text{or} \quad b = 22 \sin 38^\circ \approx 13.5 \text{ miles}$$

We now know two sides of right triangle ACE, namely

$$\overline{AC} = 22 + e \approx 22 + 17.3 = 39.3$$
$$\overline{CE} = b \approx 13.5$$

We can now solve triangle ACE to obtain

$$\tan \alpha \approx \frac{13.5}{39.3} \quad \text{or} \quad \alpha \approx 19°$$

From triangle ACE,

$$\sin \alpha = \frac{b}{d} = \frac{13.5}{d}$$

$$d = \frac{13.5}{\sin 19°} \approx 41.5 \text{ miles}$$

The ship is 41.5 miles from port at a bearing of S 71° E.

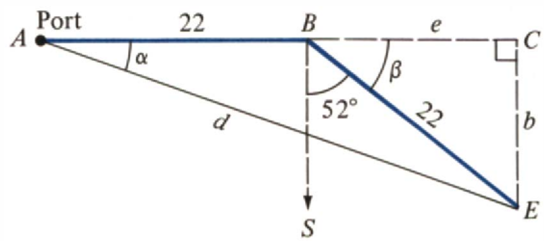
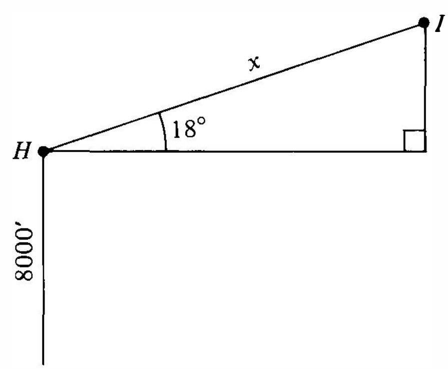


FIGURE 16

EXERCISE SET 6.2

1. A ladder 20 feet in length touches a wall at a point 16 feet above the ground. Find the angle the ladder makes with the ground.
2. A monument is 550 feet high. What is the length of the shadow cast by the monument when the sun is 64° above the horizon?
3. Find the angle of elevation of the sun when a tower 45 meters in height casts a horizontal shadow 25 meters in length.
4. A technician positioned on an oil-drilling rig 120 feet above the water spots a boat at an angle of depression of 16°. How far is the boat from the rig?
5. A mountainside hotel is located 8000 feet above sea level. From the hotel, a trail leads farther up the mountain to an inn at an elevation of 10,400 feet. If the trail has an angle of inclination of 18° (that is, the angle of elevation of the inn from the hotel is 18°), find the distance along the trail from the hotel to the inn.



6. A hill is known to be 200 meters high. A surveyor standing on the ground finds the angle of elevation of the top of the hill to be 42°50′. Find the distance from the surveyor to a point directly below the top of the hill. (Ignore the height of the surveyor.)

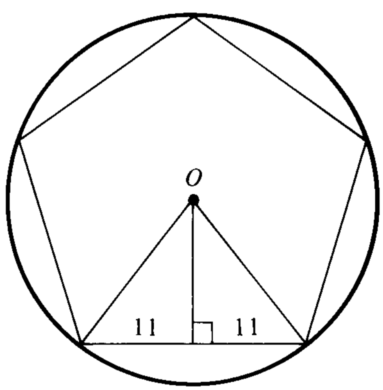
7. An observer is 425 meters from a launching pad when a rocket is launched vertically. If the angle of elevation of the rocket at its apogee (highest point) is 66°20′, how high does the rocket rise?

8. An airplane pilot wants to climb from an altitude of 6000 feet to an altitude of 16,000 feet. If the plane climbs at an angle of 9° with a constant speed of 22,000 feet per minute, how long will it take to reach the increased altitude?

9. A rectangle is 16 inches long and 13 inches wide. Find the measures of the angles formed by a diagonal with the sides.

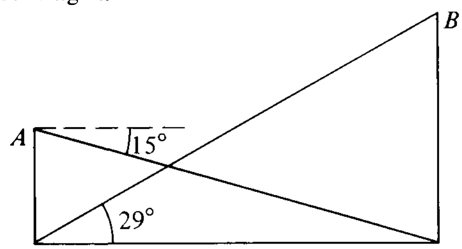
10. The sides of an isosceles triangle are 15, 15, and 26 centimeters. Find the measures of the angles of the triangle. (*Hint:* The altitude of an isosceles triangle bisects the base.)

11. The side of a regular pentagon is 22 centimeters. Find the radius of the circle circumscribed about the pentagon. (*Hint:* The radii from the center of the circumscribed circle to any two adjacent vertices of the regular pentagon form an isosceles triangle. The altitude of an isosceles triangle bisects the base.)



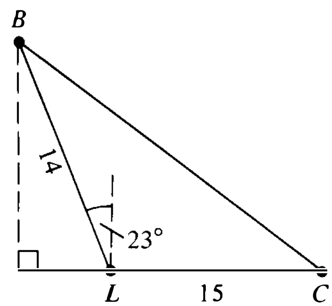
12. To determine the width of a river, markers are placed at each side of the river in line with the base of a tower that rises 23.4 meters above the ground. From the top of the tower, the angles of depression of the markers are 58°20′ and 11°40′. Find the width of the river.

13. The angle of elevation of the top of building B from the base of building A is 29°. From the top of building A, the angle of depression of the base of building B is 15°. If building B is 110 feet high, find the height of building A.



14. A ship leaves port at 2 P.M. and heads due east at a rate of 40 kilometers per hour. At 4 P.M. the course is changed to N 32° E. Find the distance and bearing of the ship from the dock at 6 P.M.

15. An attendant in a lighthouse receives a request for aid from a stalled craft located 15 miles due east of the lighthouse. The attendant contacts a second boat located 14 miles from the lighthouse at a bearing of N 23° W. What is the distance of the rescue ship from the stalled craft?



6.3 LAW OF COSINES

In Section 6.1 we studied the trigonometry of a right triangle. In this and the next section we will examine an **oblique triangle,** a triangle that does not contain a right angle.

We can always solve an oblique triangle by dropping a perpendicular as in Figures 17a and 17b and treating the resulting right triangles ADC and BDC. It is, however, worthwhile to perform the analysis in a general way. This yields two results, known as the *law of sines* and the *law of cosines*. We now state and prove the law of cosines, maintaining the notation of the last section; thus, the angles of triangle ABC are denoted by α, β, and γ, with opposite sides a, b, and c, respectively.

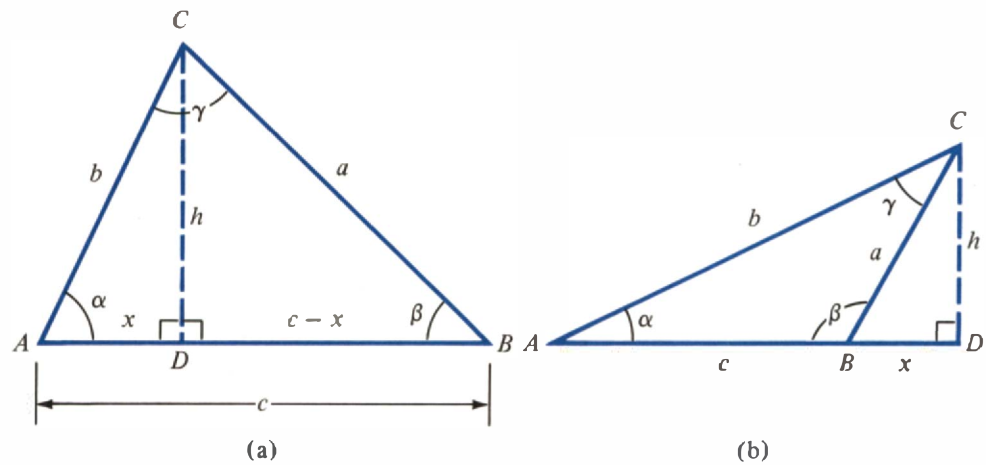


FIGURE 17

The Law of Cosines

In triangle ABC,

$$a^2 = b^2 + c^2 - 2bc \cos \alpha \tag{1}$$
$$b^2 = a^2 + c^2 - 2ac \cos \beta \tag{2}$$
$$c^2 = a^2 + b^2 - 2ab \cos \gamma \tag{3}$$

The student is urged to note the *pattern* of the three forms of the law of cosines as an aid in their memorization.

To prove the law of cosines, we deal with the cases shown in Figure 17.

Case 1. The angles of triangle ABC are all acute (Figure 17a). We construct the perpendicular CD to side AB. Applying the Pythagorean theorem to right triangles BDC and ADC, we have

$$\begin{aligned} a^2 &= h^2 + (c - x)^2 \\ &= h^2 + c^2 - 2cx + x^2 \\ &= (h^2 + x^2) + c^2 - 2cx \\ &= b^2 + c^2 - 2cx \end{aligned} \tag{4}$$

The last step results from application of the Pythagorean theorem to right triangle *ADC*. Also,

$$\cos \alpha = \frac{x}{b} \quad \text{or} \quad x = b \cos \alpha$$

which we then substitute in Equation (4) to yield

$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$

This establishes the desired result of Equation (1).

Case 2. Triangle *ABC* has an obtuse angle β (Figure 17b). We construct the perpendicular *CD* to side *AB*. The Pythagorean theorem can be applied to right triangle *BDC* to give

$$a^2 = h^2 + x^2 \tag{5}$$

Next, we use the trigonometry of the right triangle *ADC* to obtain

$$\sin \alpha = \frac{h}{b} \quad \text{or} \quad h = b \sin \alpha$$

$$\cos \alpha = \frac{c + x}{b} \quad \text{or} \quad x = b \cos \alpha - c$$

Substituting for h and x in Equation (5) we have

$$\begin{aligned} a^2 &= b^2 \sin^2 \alpha + (b \cos \alpha - c)^2 \\ &= b^2 \sin^2 \alpha + b^2 \cos^2 \alpha - 2bc \cos \alpha + c^2 \\ &= b^2(\sin^2 \alpha + \cos^2 \alpha) - 2bc \cos \alpha + c^2 \\ &= b^2 + c^2 - 2bc \cos \alpha \quad (\text{Since } \sin^2 \alpha + \cos^2 \alpha = 1) \end{aligned}$$

Once again, this is the desired result of Equation (1).

We have thus established the first form of the law of cosines for both cases. A similar argument can be used to establish the other two forms, given in Equations (2) and (3).

Examination of the law of cosines shows that it can be used in the following circumstances.

Applying the Law of Cosines

The law of cosines may be used when

(a) three sides of a triangle are known (SSS), or

(b) two sides of a triangle are known and the measure of the angle formed by those sides is known (SAS).

The law of cosines involves a good deal of computation. A calculator is great for easing the burden; not only will you be able to evaluate the cosine and inverse cosine functions, but you will also be able to effortlessly perform the arithmetic computations.

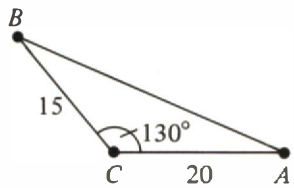


FIGURE 18

EXAMPLE 1

Find the length of the third side of the triangle shown in Figure 18.

SOLUTION

We are given two sides and the included angle (SAS), so the law of cosines can be used:

$$
\begin{aligned}
c^2 &= a^2 + b^2 - 2ab\cos\gamma \\
&= 15^2 + 20^2 - 2(15)(20)\cos 130^\circ \\
&\approx 225 + 400 - 600(-0.6428) \\
c^2 &\approx 1010.7 \\
c &\approx 31.8
\end{aligned}
$$

EXAMPLE 2

Highway engineers who are to dig a tunnel through a small mountain wish to determine the length of the tunnel. Points A and B are chosen as the endpoints of the tunnel. Then a point C is selected from which the distances to A and B are found to be 190 feet and 230 feet, respectively. If angle ACB measures 48°, find the approximate length of the tunnel.

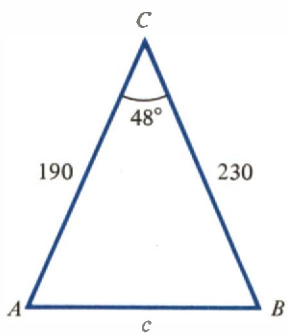


FIGURE 19

SOLUTION

The known information is displayed in Figure 19. Applying the law of cosines,

$$
\begin{aligned}
c^2 &= a^2 + b^2 - 2ab\cos\gamma \\
&= 230^2 + 190^2 - 2(230)(190)\cos 48^\circ \\
c^2 &\approx 30{,}518 \\
c &\approx 175 \text{ feet}
\end{aligned}
$$

EXAMPLE 3

Find the approximate measure of the angles of triangle ABC if $a = 150$, $b = 100$, and $c = 75$.

SOLUTION

Substituting in the equation

$$
\begin{aligned}
a^2 &= b^2 + c^2 - 2bc\cos\alpha \\
150^2 &= 100^2 + 75^2 - 2(100)(75)\cos\alpha \\
22{,}500 &= 10{,}000 + 5625 - 15{,}000\cos\alpha \\
\cos\alpha &= -0.4583
\end{aligned}
$$

Since $\cos\alpha$ is negative, angle α must lie in the second quadrant and is an obtuse angle. Using a calculator, enter

DEG	.4583	+/−	INV	COS

The display shows an answer of 117.28°. Converting to degrees and minutes, we have

$$\alpha \approx 117°17'$$

Similarly,

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac \cos \beta \\ 100^2 &= 150^2 + 75^2 - 2(150)(75) \cos \beta \\ 10{,}000 &= 22{,}500 + 5625 - 22{,}500 \cos \beta \\ \cos \beta &= 0.8056 \\ \beta &\approx 36°20' \end{aligned}$$

Finally, we may easily determine γ since the sum of the angles of a triangle is 180°.

$$\gamma \approx 180° - (117°17' + 36°20') \approx 26°23'$$

The student should verify this result by substituting in the equation

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

We conclude this section by reminding you of a useful result from plane geometry.

> In triangle ABC, if $a < b$, then $\alpha < \beta$; that is, the smaller angle lies opposite the smaller side.

This theorem provides you with a means to perform a quick check as to whether your computational results are reasonable. You should always verify that the angles and sides correspond, that is, the smallest angle is opposite the smallest side and the largest angle is opposite the largest side.

EXERCISE SET 6.3

In Exercises 1–10 use the law of cosines to approximate the required part of triangle ABC.

1. $a = 10$, $b = 15$, $c = 21$; find β.
2. $a = 5$, $b = 12$, $c = 15$; find γ.
3. $a = 25$, $c = 30$, $\beta = 28°30'$; find b.
4. $b = 20$, $c = 13$, $\alpha = 19°10'$; find a.
5. $a = 10$, $b = 12$, $\gamma = 108°$; find c.
6. $a = 30$, $c = 40$, $\beta = 122°$; find b.
7. $b = 6$, $a = 7$, $\gamma = 68°$; find α.
8. $a = 6$, $b = 15$, $c = 16$; find β.
9. $a = 9$, $b = 12$, $c = 15$; find γ.
10. $a = 11$, $c = 15$, $\beta = 33°$; find γ.

11. The sides of a parallelogram measure 25 centimeters and 40 centimeters, and the longer diagonal measures 50 centimeters. Find the approximate measure of the smaller angle of the parallelogram.

12. The sides of a parallelogram measure 40 inches and 70 inches, and one of the angles is 108°. Find the approximate length of each diagonal of the parallelogram.

13. A ship leaves port at 9 A.M. and travels due west at a rate of 15 miles per hour. At 11 A.M. the ship changes direction to S 32° W. What is the distance and bearing of the ship from port at 1 P.M.?

14. A ship leaves from port A intending to travel direct to port B, a distance of 25 kilometers. After traveling 12 kilometers the captain finds that his course has been in error by 10°. How far is the ship from port B?

15. Two trains leave Pennsylvania Station in New York City at 2 P.M. and travel in directions that differ by 55°. If the trains travel at constant rates of 50 miles per hour and 80 miles per hour, respectively, what is the distance between them at 2:30 P.M.?

16. Hurricane David has left a telephone pole in a nonvertical position. Workmen place a 30-foot ladder at a point 10 feet from the base of the pole. If the ladder touches the pole at a point 26 feet up the pole, find the angle the pole makes with the ground.

17. Find the approximate perimeter of triangle ABC if $a = 20$, $b = 30$, and $\gamma = 37°$.

18. A hill makes an angle of 10° with the horizontal. An antenna 50 feet in height is erected at the top of the hill and a guy wire is run to a point 30 feet from the base of the antenna. What is the length of the guy wire?

19. Prove that if ABC is a right triangle, the law of cosines reduces to the Pythagorean theorem.

20. Prove the following in triangle ABC.
(a) $a^2 + b^2 + c^2$
$$= 2(bc \cos \alpha + ac \cos \beta + ab \cos \gamma)$$
(b) $\dfrac{\cos \alpha}{a} + \dfrac{\cos \beta}{b} + \dfrac{\cos \gamma}{c} = \dfrac{a^2 + b^2 + c^2}{2abc}$

21. Prove that if
$$\frac{\cos \beta}{a} = \frac{\cos \alpha}{b}$$
triangle ABC is either a right triangle or an isosceles triangle.

6.4 LAW OF SINES

In the last section we applied the law of cosines to an oblique triangle. That law derives its name from the appearance of the cosine function in its statement.

We will now state and prove the law of sines, which also applies to an oblique triangle. Not surprisingly, the law of sines involves the sine function. Once again, we denote the angles of triangle ABC by α, β, and γ, with opposite sides a, b, and c, respectively.

The Law of Sines

In triangle ABC,
$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

The two cases are illustrated in Figure 20.

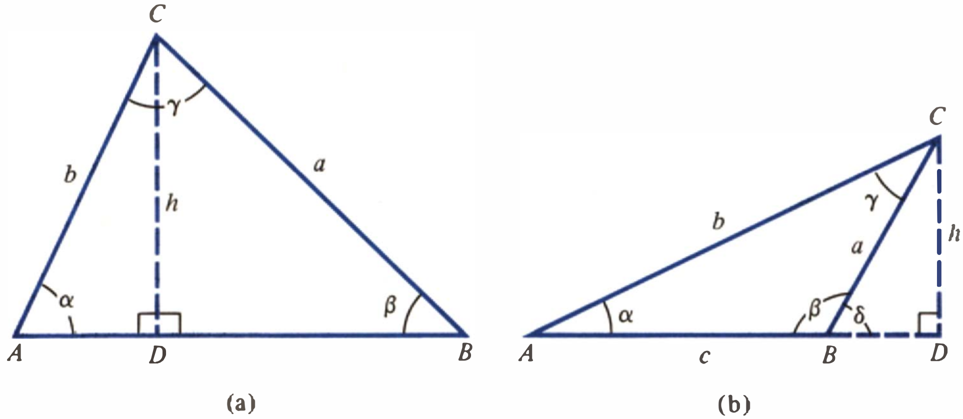


FIGURE 20

Case 1. The angles of triangle ABC are all acute (Figure 20a). We construct the perpendicular CD to side AB. Then triangles ADC and BDC are both right triangles, and we can apply trigonometry of a right triangle to obtain

$$\sin \alpha = \frac{h}{b} \quad \text{or} \quad h = b \sin \alpha$$

$$\sin \beta = \frac{h}{a} \quad \text{or} \quad h = a \sin \beta$$

Equating the expressions for h yields

$$b \sin \alpha = a \sin \beta$$

which can be written in the convenient form

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}$$

Case 2. Triangle ABC has an obtuse angle β (Figure 20b). We construct the perpendicular CD to side AB. Applying right triangle trigonometry to triangles ADC and BDC, and noting that $\delta = 180° - \beta$, we obtain

$$\sin \alpha = \frac{h}{b} \quad \text{or} \quad h = b \sin \alpha$$

$$\sin \delta = \sin(180° - \beta) = \frac{h}{a} \quad \text{or} \quad h = a \sin(180° - \beta)$$

Equating the expressions for h yields

$$b \sin \alpha = a \sin(180° - \beta)$$

Since sine is positive in both the first and second quadrants, the Reference Angle Rule tells us that

$$\sin(180° - \beta) = \sin \beta$$

Substituting, we again obtain

$$b \sin \alpha = a \sin \beta$$

or

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}$$

To complete the proof of the law of sines we need only drop a perpendicular from A to BC and use a similar argument to show that

$$\frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

The law of sines then follows from the transitive property of equality.

The law of sines can be used in the following circumstances.

Applying the Law of Sines

The law of sines may be used when the known parts of a triangle are

(a) one side and two angles (SAA), or

(b) two sides and an angle opposite one of these sides (SSA).

Remember that if two angles of a triangle are known, we can immediately determine the third angle. Here is an example.

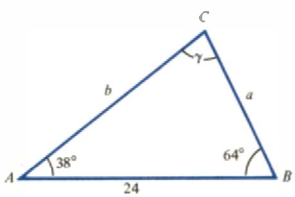


FIGURE 21

EXAMPLE 1

In triangle ABC, $\alpha = 38°$, $\beta = 64°$, and $c = 24$. Find approximate values for the remaining parts of the triangle.

SOLUTION

(See Figure 21.) Since α and β are known,

$$\gamma = 180° - (\alpha + \beta) = 180° - (38° + 64°) = 78°$$

Applying the law of sines,

$$\frac{a}{\sin \alpha} = \frac{c}{\sin \gamma}$$

$$\frac{a}{\sin 38°} = \frac{24}{\sin 78°}$$

$$a = \frac{24 \sin 38°}{\sin 78°} \approx \frac{24(0.6157)}{(0.9781)} \approx 15.1$$

Similarly, from

$$\frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

we obtain

$$b \approx 22.1$$

UNIQUE AND AMBIGUOUS CASES

When the given parts of a triangle are two sides and an angle opposite one of them, the situation is not straightforward since a *unique* triangle is not always determined. In Figure 22 we have constructed angle α and side b and then used a compass to construct a side of length a with an endpoint at C. In Figure 22a no triangle exists satisfying the given conditions; Figure 22b shows that we may obtain a right triangle; Figure 22c illustrates the possibility that two triangles will satisfy the given conditions; Figure 22d shows that precisely one acute triangle may be possible.

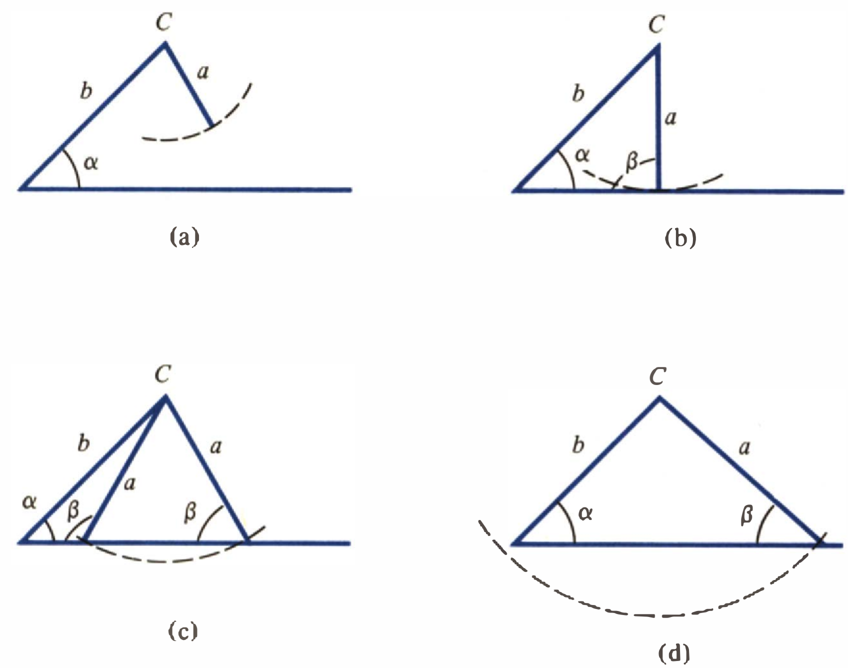


FIGURE 22

In Exercise 23 you will be asked to prove a number of inequalities that determine which of the four cases applies to a given set of conditions. In practice, we prefer to have you go ahead with the law of sines and let the results lead you to the appropriate answer.

Assume that sides a and b and angle α of triangle ABC are known and that we use the law of sines to determine angle β. These are the results that correspond to the possibilities of Figure 22.

(a) $\sin\beta > 1$. Since $|\sin\theta| \leq 1$ for all θ, there is no angle β satisfying the given conditions. This corresponds to the illustration in Figure 22a.

(b) $\sin\beta = 1$. Then $\beta = 90°$ and the given parts determine a unique right triangle (Figure 22b).

(c) $0 < \sin\beta < 1$. There are two possible choices for β, which is why this is called the **ambiguous case.** Since the sine function is positive in quadrants I and II, one choice will be an acute angle and one will be an obtuse angle (Figure 22c).

(d) $0 < \sin\beta < 1$. There are two possible choices for β but the obtuse angle does not form a triangle (Figure 22d). This case is signaled by $\alpha + \beta$ exceeding $180°$.

Here are several illustrations of the law of sines when two sides and an angle opposite one of these sides are known.

EXAMPLE 2

In triangle ABC, $\alpha = 60°$, $a = 5$, and $b = 7$. Find angle β.

SOLUTION

Using the law of sines,

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

$$\sin\beta = \frac{b\sin\alpha}{a} = \frac{7\sin 60°}{5} \approx 1.2$$

Since the sine function has a maximum value of 1, there is no angle β such that $\sin\beta = 1.2$. Hence, there is no triangle with the given parts. This example corresponds to Figure 22a.

EXAMPLE 3

In triangle ABC, $a = 5$, $b = 8$, and $\alpha = 22°$. Find the remaining angles of the triangle.

SOLUTION

Using the law of sines,

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

$$\sin\beta = \frac{b\sin\alpha}{a} = \frac{8\sin 22°}{5} \approx 0.5994$$

Using tables or a calculator, we find that $\beta \approx 36°50'$. Thus, the angles are (approximately) $\alpha = 22°$, $\beta = 36°50'$, and $\gamma = 121°10'$.

However, the angle $\beta = 180° - 36°50' = 143°10'$ also satisfies the requirement that $\sin\beta = 0.5994$. Therefore, another satisfactory triangle has angles $\alpha = 22°$, $\beta = 143°10'$, and $\gamma = 14°50'$.

This is an example of the ambiguous case, and corresponds to Figure 22c.

EXAMPLE 4

In triangle ABC, $a = 9$, $b = 6$, and $\alpha = 35°$. Find angles β and γ.

SOLUTION

We again apply the law of sines.

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}$$

$$\sin \beta = \frac{b \sin \alpha}{a} = \frac{6 \sin 35°}{9} \approx 0.3824$$

Using tables or a calculator yields $\beta \approx 22°30'$. A triangle satisfying the given conditions has $\alpha = 35°$, $\beta = 22°30'$, and $\gamma = 122°30'$.

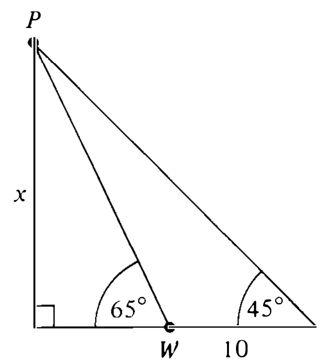
The angle $\beta = 180° - 22°30' = 157°30'$ also satisfies the requirement that $\sin \beta = 0.3824$. But this "solution" must be rejected since $\alpha + \beta > 180°$.

This example corresponds to Figure 22d.

EXERCISE SET 6.4

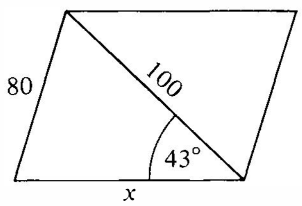
In Exercises 1–12 use the law of sines to approximate the required part(s) of triangle ABC. Give both solutions if more than one triangle satisfies the given conditions.

1. $\alpha = 25°$, $\beta = 82°$, $a = 12.4$; find b.
2. $\alpha = 74°$, $\gamma = 36°$, $c = 6.8$; find a.
3. $\beta = 23°$, $\gamma = 47°$, $a = 9.3$; find c.
4. $\alpha = 46°$, $\beta = 88°$, $c = 10.5$; find b.
5. $\alpha = 42°20'$, $\gamma = 78°40'$, $b = 20$; find a.
6. $\beta = 16°30'$, $\gamma = 84°40'$, $a = 15$; find c.
7. $\alpha = 65°$, $a = 25$, $b = 30$; find β.
8. $\beta = 32°$, $b = 20$, $c = 14$; find α and γ.
9. $\gamma = 30°$, $a = 12.6$, $c = 6.3$; find b.
10. $\beta = 64°$, $a = 10$, $b = 8$; find c.
11. $\gamma = 45°$, $b = 7$, $c = 6$; find a.
12. $\alpha = 64°$, $a = 11$, $b = 12$; find β and γ.
13. Points A and B are chosen on opposite sides of a rock quarry. A point C is 160 meters from B, and the measures of angles BAC and ABC are found to be 95° and 47°, respectively. Find the width of the quarry.
14. A tunnel is to be dug between points A and B on opposite sides of a hill. A point C is chosen that is 150 meters from A and 180 meters from B. If angle ABC measures 54°, find the length of the tunnel.
15. A ski lift 750 meters in length rises to the top of a mountain at an angle of inclination of 40°. A second lift is to be built whose base is in the same horizontal plane as the initial lift. If the angle of elevation of the second lift is 45°, what is the length of the second lift?
16. A tree leans away from the sun at an angle of 9° from the vertical. The tree casts a shadow 20 meters in length when the angle of elevation of the sun is 62°. Find the height of the tree.
17. A ship is sailing due north at a rate of 22 miles per hour. At 2 P.M. a lighthouse is seen at a bearing of N 15° W. At 4 P.M., the bearing of the same lighthouse is S 65° W. Find the distance of the ship from the lighthouse at 2 P.M.
18. A plane leaves airport A and flies at a bearing of N 32° E. A few moments later, the plane is spotted from airport B at a bearing of N 56° W. If airport B lies 15 miles due east of airport A, find the distance of the plane from airport B at the moment it is spotted.
19. A guy wire attached to the top of a vertical pole has an angle of inclination of 65° with the ground. From a point 10 meters farther from the pole, the angle of elevation of the top of the pole is 45°. Find the height of the pole.

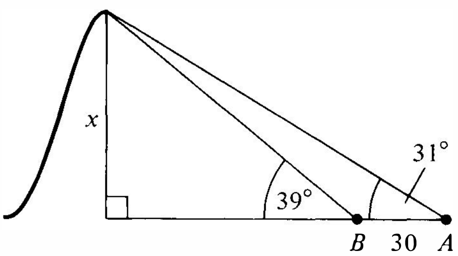


20. At 5 P.M. a sailor on board a ship sailing at a rate of 18 miles per hour spots an island due east of the ship. The ship maintains a bearing of N 26° E. At 6 P.M. the sailor finds the bearing of the island to be S 37° E. Find the distance of the island from the ship at 6 P.M.

21. The short side of a parallelogram and the shorter diagonal measure 80 centimeters and 100 centimeters, respectively. If the angle between the longer side and the shorter diagonal is 43°, find the length of the longer side.



22. An archaeological mound is discovered in a jungle in Central America. To determine the height of the mound, a point A is chosen from which the angle of elevation of the top of the mound is found to be 31°. A second point B is chosen on a line with A and the base of the mound, 30 meters closer to the base of the mound. If the angle of elevation of the top of the mound from point B is 39°, find the height of the mound.



23. In a triangle, sides of length a and b and an angle α are given. Prove the following.
(a) If $b \sin \alpha > a$, there is no triangle with the given parts.
(b) If $b \sin \alpha = a$, the parts determine a right triangle.
(c) If $b \sin \alpha < a < b$, there are two triangles with the given parts.
(d) If $b \leq a$, there is one acute triangle with the given parts.

TERMS AND SYMBOLS

angle of elevation (p. 283)
angle of depression (p. 283)
bearing (p. 285)
oblique triangle (p. 288)
law of cosines (p. 288)
law of sines (p. 292)
ambiguous case of the law of sines (p. 295)

KEY IDEAS FOR REVIEW

- ☐ Right triangle trigonometry relates a trigonometric function of an angle θ of a right triangle to the ratio of the lengths of two of its sides as follows:

$$\sin \theta = \frac{\text{side opposite } \theta}{\text{hypotenuse}}$$

$$\cos \theta = \frac{\text{side adjacent to } \theta}{\text{hypotenuse}}$$

$$\tan \theta = \frac{\text{side opposite to } \theta}{\text{side adjacent to } \theta}$$

- ☐ Right triangle trigonometry can be used to solve a wide variety of applied problems.
- ☐ The law of cosines and the law of sines are useful in solving problems that involve an oblique triangle. The derivation of these laws is accomplished by using right triangle trigonometry.

REVIEW EXERCISES

Solutions to exercises whose numbers are in color are in the Solutions section in the back of the book.

6.1 In Exercises 1–5 express the required trigonometric function as a ratio of the given parts of the right triangle ABC with $\gamma = 90°$.

1. $a = 5$, $b = 12$; find $\sin \alpha$.
2. $a = 3$, $c = 5$; find $\tan \beta$.
3. $a = 4$, $b = 7$; find $\sec \alpha$.
4. $b = 8$, $c = 10$; find $\cot \alpha$.
5. $b = 4$, $c = 7$; find $\sec \beta$.

In Exercises 6–9 the point P lies on the terminal side of the angle θ. Find the value of the required trigonometric function without using tables or a calculator.

6. $P(-\sqrt{3}, 1)$; $\csc \theta$
7. $P(\sqrt{2}, -\sqrt{2})$; $\cot \theta$
8. $P(-1, -\sqrt{3})$; $\cos \theta$
9. $P(\sqrt{2}, \sqrt{2})$; $\sin \theta$

In Exercises 10–13 find the required part of triangle ABC with $\gamma = 90°$. Use Table VI in the Tables Appendix, or a calculator.

10. $a = 50$, $b = 60$; find α.
11. $a = 40$, $\beta = 20°$; find b.
12. $a = 20$, $\alpha = 52°$; find c.
13. $b = 15$, $\alpha = 25°$; find c.

6.2

14. A ladder 6 meters in length leans against a vertical wall. If the ladder makes an angle of 65° with the ground, find the height that the ladder reaches above the ground.
15. Find the angle of elevation of the sun when a tree 25 meters in height casts a horizontal shadow 10 meters in length.
16. A rectangle is 22 centimeters long and 16 centimeters wide. Find the measure of the smaller angle formed by the diagonal with a side.

6.3
6.4 In Exercises 17-20 use the law of cosines or the law of sines to approximate the required part of triangle ABC.

17. $a = 12$, $b = 7$, $c = 15$; find α.
18. $a = 20$, $b = 15$, $\alpha = 55°$; find β.
19. $a = 10$, $\alpha = 38°$, $\beta = 22°$; find c.
20. $b = 8$, $c = 12$, $\alpha = 35°$; find a.

PROGRESS TEST 6A

In Problems 1–3 ABC is a right triangle with $\gamma = 90°$. Express the required trigonometric function as a ratio of the given parts of the triangle.

1. $a = 7$, $b = 5$; $\tan \alpha$
2. $b = 5$, $c = 15$; $\sec \alpha$
3. $a = 5$, $c = 13$; $\cot \beta$

In problems 4–7 the point P lies on the terminal side of the angle θ. Find the value of the required trigonometric function without using tables or a calculator.

4. $P(-\sqrt{2}, \sqrt{2})$; $\cot \theta$
5. $P(0, -5)$; $\sin \theta$
6. $P(2, 2\sqrt{3})$; $\sec \theta$
7. $P(-1, -3/2)$; $\cos \theta$

In Problems 8–10 use Table VI in the Tables Appendix, or a calculator, to find the required part of triangle ABC with $\gamma = 90°$.

8. $a = 25$, $c = 30$; find α.
9. $b = 20$, $\alpha = 32°$; find c.
10. $a = 15$, $b = 20$; find β.

In Problems 11 and 12 find the required part of triangle ABC.

11. $a = 2$, $b = 4$, $c = 5$; find α.
12. $b = 10$, $\alpha = 15°$, $\beta = 28°$; find c.
13. From the top of a hill 100 meters in height, the angle of depression of the entrance to a castle is 36°. Find the distance of the castle from the base of the hill.

PROGRESS TEST 6B

In Problems 1–3 ABC is a right triangle with $\gamma = 90°$. Express the required trigonometric function as a ratio of the given parts of the triangle.

1. $a = 6$, $b = 8$; $\csc \beta$
2. $a = 7$, $b = 6$; $\cot \alpha$
3. $b = 4$, $c = 5$; $\sin \alpha$

In Problems 4–7 the point P lies on the terminal side of the angle θ. Find the value of the required trigonometric function without using tables or a calculator.

4. $P(-3, 0)$; $\csc \theta$
5. $P(2, 2\sqrt{3})$; $\csc \theta$
6. $P(-\sqrt{2}, -\sqrt{2})$; $\tan \theta$
7. $P(2, -1)$; $\sin \theta$

In Problems 8–10 use Table VI in the Tables Appendix, or a calculator, to find the required part of triangle ABC with $\gamma = 90°$.

8. $a = 5$, $\beta = 61°$; find c.
9. $b = 6$, $c = 15$; find α.
10. $a = 7$, $b = 10$; find c.

In Problems 11 and 12 find the required part of triangle ABC.

11. $b = 10$, $c = 13$, $\alpha = 54°$; find β.
12. $a = 5$, $c = 9$, $\beta = 36°$; find b.
13. A surveyor finds the angle of elevation of the top of a tree to be 42°. If the surveyor is 75 feet from the base of the tree, find the height of the tree.

7

ANALYTIC TRIGONOMETRY

Much of the language and terminology of algebra carries over to trigonometry. For example, we have seen that algebraic expressions involve variables, constants, and algebraic operations. **Trigonometric expressions** involve these same elements but also permit trigonometric functions of variables and constants. They also allow algebraic operations upon these trigonometric functions. Thus,

$$x + \sin x \qquad \sin x + \tan x \qquad \frac{1 - \cos x}{\sec^2 x}$$

are all examples of trigonometric expressions.

The distinction between an identity and an equation also carries over to trigonometry. Thus, a **trigonometric identity** is true for all real values in the domain of the variable, but a **trigonometric equation** is true only for certain values called **solutions.** (Note that the solutions of a trigonometric equation may be expressed as real numbers or as angles.) As usual, the set of all solutions of a trigonometric equation is called the **solution set.**

7.1 TRIGONOMETRIC IDENTITIES

FUNDAMENTAL IDENTITIES

In Section 5.2 we established the identity

$$\boxed{\sin^2 t + \cos^2 t = 1 \qquad (1)}$$

If $\cos t \neq 0$, we may divide both sides of Equation (1) by $\cos^2 t$ to obtain

$$\frac{\sin^2 t}{\cos^2 t} + \frac{\cos^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}$$

or

$$\boxed{\tan^2 t + 1 = \sec^2 t \qquad (2)}$$

Similarly, if $\sin t \neq 0$, dividing Equation (1) by $\sin^2 t$ yields

$$\frac{\sin^2 t}{\sin^2 t} + \frac{\cos^2 t}{\sin^2 t} = \frac{1}{\sin^2 t}$$

or

$$\cot^2 t + 1 = \csc^2 t \tag{3}$$

Observe that $\tan t$ and $\cot t$ are undefined for exactly those values of t for which $\cos t$ and $\sin t$ are 0, respectively. It follows that the identities (2) and (3) are true for all values of t for which the trigonometric expressions are defined.

The two identities that we have just established, together with the identities discussed in Sections 5.2 and 5.5, are called the **fundamental identities.** Since we will use these eight identities throughout this chapter, it is essential that you know and recognize them in their various forms as shown in Table 1.

TABLE 1

Fundamental Identity	Alternate Form(s)
$\tan t = \dfrac{\sin t}{\cos t}$	
$\cot t = \dfrac{\cos t}{\sin t}$	
$\csc t = \dfrac{1}{\sin t}$	$\sin t = \dfrac{1}{\csc t}$
$\sec t = \dfrac{1}{\cos t}$	$\cos t = \dfrac{1}{\sec t}$
$\cot t = \dfrac{1}{\tan t}$	$\tan t = \dfrac{1}{\cot t}$
$\sin^2 t + \cos^2 t = 1$	$\sin^2 t = 1 - \cos^2 t$
	$\cos^2 t = 1 - \sin^2 t$
$\tan^2 t + 1 = \sec^2 t$	$\tan^2 t = \sec^2 t - 1$
$\cot^2 t + 1 = \csc^2 t$	$\cot^2 t = \csc^2 t - 1$

In Section 5.2 we saw that trigonometric identities can be used to simplify a trigonometric expression. Here is another example, in which we use the identities developed in this section.

EXAMPLE 1
Simplify the expression $\sin^2 x + \sin^2 x \tan^2 x$.

SOLUTION

We begin by noting that $\sin^2 x$ appears in both terms, which suggests that we factor.

$$\begin{aligned}
\sin^2 x + \sin^2 x \tan^2 x &= \sin^2 x(1 + \tan^2 x) && \text{Factoring} \\
&= \sin^2 x \sec^2 x && 1 + \tan^2 x = \sec^2 x \\
&= \frac{\sin^2 x}{\cos^2 x} && \sec x = \frac{1}{\cos x} \\
&= \tan^2 x && \frac{\sin x}{\cos x} = \tan x
\end{aligned}$$

PROGRESS CHECK

Simplify the expression $\dfrac{\csc \theta}{1 + \cot^2 \theta}$.

ANSWER

$\sin \theta$

TRIGONOMETRIC IDENTITIES

The fundamental identities can be employed to prove or, more properly, to verify various trigonometric identities. The principal reasons for including this topic are (a) to improve your skills in recognizing and using the fundamental identities, and (b) to sharpen your reasoning processes. There are also times in calculus and applied mathematics when simplification of a trigonometric expression may enable us to see a relationship that would otherwise be obscured. Finally, in computer applications it is much more efficient to evaluate a simple trigonometric expression than an involved one.

The preferred method of verifying an identity is to transform one side of the equation into the other. We will use this method whenever practical, recognizing that it is also acceptable to transform each side independently with the hope of arriving at the same expression.

Unfortunately, we cannot outline a rigid set of steps that will "work" to transform one side into the other; in fact, there are often many ways to tackle a given identity. Each of the next four examples demonstrates a different technique (highlighted in italics) for working on trigonometric identities. If you should make a false start and find yourself trying something that doesn't appear to be working, start again and try another approach. With practice your skills will improve.

EXAMPLE 2

Verify the identity $\cos x \tan x \csc x = 1$.

SOLUTION

It is often helpful to write all of the trigonometric functions in terms of sine and cosine. The student should supply a reason for each step.

$$\cos x \tan x \csc x = \cos x \frac{\sin x}{\cos x} \frac{1}{\sin x}$$
$$= 1$$

PROGRESS CHECK

Verify the identity $\sin x \sec x = \tan x$.

EXAMPLE 3

Verify the identity $\dfrac{1}{1 - \sin x} + \dfrac{1}{1 + \sin x} = 2 \sec^2 x$.

SOLUTION

Another useful technique is to begin with the more complicated expression and complete the indicated operations. We will begin with the left-hand side and will combine the fractions.

$$\frac{1}{1 - \sin x} + \frac{1}{1 + \sin x} = \frac{1 + \sin x + 1 - \sin x}{(1 - \sin x)(1 + \sin x)}$$
$$= \frac{2}{1 - \sin^2 x} = \frac{2}{\cos^2 x}$$
$$= 2 \sec^2 x$$

PROGRESS CHECK

Verify the identity $\cos x + \tan x \sin x = \sec x$.

EXAMPLE 4

Verify the identity $\sin \alpha - \sin^2 \alpha = \dfrac{1 - \sin \alpha}{\csc \alpha}$.

SOLUTION

Factoring will sometimes help to simplify an expression. The student should supply a reason for each step.

$$\sin \alpha - \sin^2 \alpha = \sin \alpha(1 - \sin \alpha)$$
$$= \frac{1 - \sin \alpha}{\csc \alpha}$$

PROGRESS CHECK

Verify the identity $\dfrac{\sin^2 y - 1}{1 - \sin y} = -1 - \sin y$.

EXAMPLE 5

Verify the identity $\dfrac{\cos \theta}{1 - \sin \theta} = \sec \theta + \tan \theta$.

SOLUTION

Multiplying the numerator and denominator of a rational expression by the same quantity is a useful technique. Of course, this quantity should be selected carefully. In this example, multiplying the denominator $1 - \sin\theta$ by $1 + \sin\theta$ will produce $1 - \sin^2\theta = \cos^2\theta$. (Similarly, should $\sec x - 1$ appear in a denominator, you might try multiplying by $\sec x + 1$ to obtain $\sec^2 x - 1 = \tan^2 x$.)

The student should supply a reason for each of the following steps.

$$\begin{aligned}
\frac{\cos\theta}{1-\sin\theta} &= \frac{\cos\theta}{1-\sin\theta}\cdot\frac{1+\sin\theta}{1+\sin\theta}\\
&= \frac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta}\\
&= \frac{\cos\theta(1+\sin\theta)}{\cos^2\theta}\\
&= \frac{1+\sin\theta}{\cos\theta}\\
&= \frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}\\
&= \sec\theta+\tan\theta
\end{aligned}$$

PROGRESS CHECK

Verify the identity $\dfrac{1+\cos t}{\sin t}+\dfrac{\sin t}{1+\cos t}=2\csc t$.

We said earlier that the preferred way of verifying an identity is to transform one side of the equation into the other. At times, both sides may involve complicated expressions and this approach may not be practical. We can then try to transform each side of the equation into the same expression, being careful to use only procedures that are reversible. Here is an example.

EXAMPLE 6

Verify the identity $\dfrac{\cot u - \tan u}{\sin u \cos u} = \csc^2 u - \sec^2 u$.

SOLUTION

Beginning with the left-hand side we have

$$\begin{aligned}
\frac{\cot u - \tan u}{\sin u\cos u} &= \frac{\dfrac{\cos u}{\sin u}-\dfrac{\sin u}{\cos u}}{\sin u\cos u}\\
&= \frac{\cos^2 u - \sin^2 u}{\sin^2 u\cos^2 u}
\end{aligned}$$

We then transform the right-hand side of the equation by writing all trigonometric functions in terms of sine and cosine.

$$\csc^2 u - \sec^2 u = \frac{1}{\sin^2 u} - \frac{1}{\cos^2 u}$$
$$= \frac{\cos^2 u - \sin^2 u}{\sin^2 u \cos^2 u}$$

We have successfully transformed both sides of the equation into the same expression. Since all the steps are reversible, we have verified the identity.

PROGRESS CHECK

Verify the identity $\dfrac{\sin x + \cos x}{\tan^2 x - 1} = \dfrac{\cos^2 x}{\sin x - \cos x}$.

EXERCISE SET 7.1

Verify each of the following identities.

1. $\csc \gamma - \cos \gamma \cot \gamma = \sin \gamma$
2. $\cot x \sec x = \csc x$
3. $\sec v + \tan v = \dfrac{1 + \sin v}{\cos v}$
4. $\cos \theta + \tan \theta \sin \theta = \sec \theta$
5. $\sin \alpha \sec \alpha = \tan \alpha$
6. $\sec \beta - \cos \beta = \sin \beta \tan \beta$
7. $3 - \sec^2 x = 2 - \tan^2 x$
8. $1 - 2 \sin^2 t = 2 \cos^2 t - 1$
9. $\dfrac{\sec^2 y}{\tan y} = \tan y + \cot y$
10. $\dfrac{\sin x + \cos x}{\cos x} = 1 + \tan x$
11. $\dfrac{\sin u}{\csc u} + \dfrac{\cos u}{\sec u} = 1$
12. $\dfrac{\tan^2 \alpha}{1 + \sec \alpha} = \sec \alpha - 1$
13. $\dfrac{\sec^2 \theta - 1}{\sec^2 \theta} = \sin^2 \theta$
14. $\sin^4 x + 2 \sin^2 x \cos^2 x + \cos^4 x = 1$
15. $\cos \gamma + \cos \gamma \tan^2 \gamma = \sec \gamma$
16. $\dfrac{1}{\tan u + \cot u} = \cos u \sin u$
17. $\dfrac{\sec w \sin w}{\tan w + \cot w} = \sin^2 w$
18. $(1 - \cos^2 \beta)(1 + \cot^2 \beta) = 1$
19. $(\sin \alpha + \cos \alpha)^2 + (\sin \alpha - \cos \alpha)^2 = 2$
20. $\dfrac{1 + \tan^2 u}{\csc^2 u} = \tan^2 u$
21. $\sec^2 v + \cos^2 v = \dfrac{\sec^4 v + 1}{\sec^2 v}$
22. $\sin^2 \theta - \tan^2 \theta = -\tan^2 \theta \sin^2 \theta$
23. $\dfrac{\sin^2 \alpha}{1 + \cos \alpha} = 1 - \cos \alpha$
24. $\cot x \sin^2 x = \cos x(1 - \sin x)$
25. $\dfrac{\cos t}{1 + \sin t} = \dfrac{1 - \sin t}{\cos t}$
26. $\dfrac{\sin \beta}{1 + \cos \beta} + \dfrac{1 + \cos \beta}{\sin \beta} = 2 \csc \beta$
27. $\csc^2 \theta - \dfrac{\cos^2 \theta}{\sin^2 \theta} = 1$
28. $\dfrac{\cos^2 u}{1 - \sin u} = 1 + \sin u$
29. $\dfrac{\cot y}{1 + \cot^2 y} = \sin y \cos y$
30. $\dfrac{1 + \tan^2 x}{\tan^2 x} = \csc^2 x$

31. $\cos(-t)\csc(-t) = -\cot t$

32. $\sin(-\theta)\sec(-\theta) = -\tan\theta$

33. $\dfrac{\sec x + \csc x}{1 + \tan x} = \csc x$

34. $\dfrac{\sec u}{\sec u - 1} = \dfrac{1}{1 - \cos u}$

35. $\dfrac{1 + \tan x}{1 + \cot x} = \dfrac{\sec x}{\csc x}$

36. $(\tan u + \sec u)^2 = \dfrac{1 + \sin u}{1 - \sin u}$

37. $\dfrac{1 - \sin t}{1 + \sin t} = (\sec t - \tan t)^2$

38. $2\csc^2\theta - \csc^4\theta = 1 - \cot^4\theta$

39. $\dfrac{\sin^2 w}{\cos^4 w + \cos^2 w \sin^2 w} = \tan^2 w$

40. $\dfrac{\sin z + \tan z}{1 + \cos z} = \tan z$

41. $\dfrac{\sec\gamma - \csc\gamma}{\sec\gamma + \csc\gamma} = \dfrac{\tan\gamma - 1}{\tan\gamma + 1}$

42. $\dfrac{\cot x - 1}{1 - \tan x} = \dfrac{\csc x}{\sec x}$

43. $\dfrac{\tan\gamma - \sin\gamma}{\tan\gamma} = \dfrac{\sin^2\gamma}{1 + \cos\gamma}$

44. $\cos^4 u - \sin^4 u = \cos^2 u - \sin^2 u$

45. $\dfrac{\csc x}{1 + \csc x} - \dfrac{\csc x}{1 - \csc x} = 2\sec^2 x$

46. $\sin^3\theta + \cos^3\theta = (1 - \sin\theta\cos\theta)(\sin\theta + \cos\theta)$

Show that each of the following equations is not an identity by finding a value of the variable for which the equation is not true.

47. $\sin x = \sqrt{1 - \cos^2 x}$

48. $\tan x = \sqrt{\sec^2 x - 1}$

49. $(\sin t + \cos t)^2 = \sin^2 t + \cos^2 t$

50. $\sin\theta + \cos\theta = \sec\theta + \csc\theta$

51. $\sqrt{\cos^2 x} = \cos x$

52. $\sqrt{\cot^2 x} = \cot x$

7.2 THE ADDITION FORMULAS

The identities that we verified in the examples and exercises of Section 7.1 were themselves of no special significance; we were primarily interested in having you practice manipulation with the fundamental identities. There are, however, many trigonometric identities that are indeed of importance; these identities are called **trigonometric formulas.** Such formulas are used so frequently that it is probably best for you to memorize them. We will develop these formulas in a logical sequence so that you will be able to derive them yourself should you wish to verify that your memorization is correct.

Our first objective is to develop the **addition formula** for $\cos(s + t)$ where s and t are any real numbers. It happens that it is easier to begin with $\cos(s - t)$, which demonstrates that the mathematician may at times have to take a circuitous route to establish a result!

For convenience, we assume that s, t, and $s - t$ are all positive and less than 2π. We let P, Q, and R be the unit circle points determined by the real numbers s, t, and $s - t$ as in Figure 1. Then $\overset{\frown}{AP} = s$, $\overset{\frown}{AQ} = t$, $\overset{\frown}{AR} = s - t$, and by the definitions of sine and cosine, the coordinates of the points can be written as

$$P(\cos s, \sin s) \quad Q(\cos t, \sin t) \quad R(\cos(s - t), \sin(s - t))$$

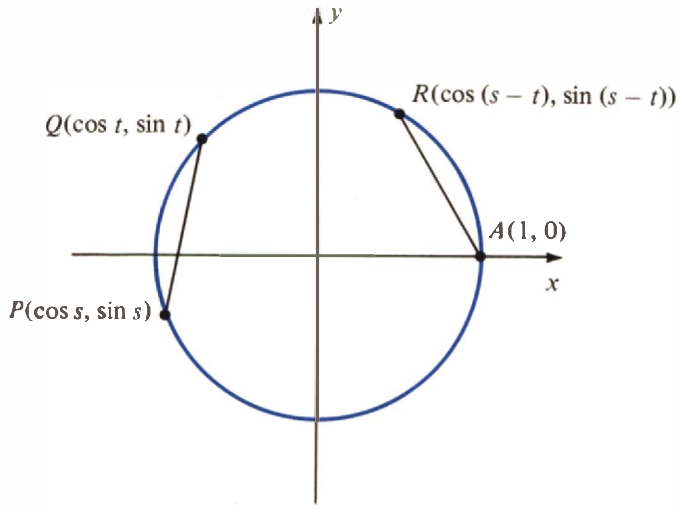


FIGURE 1

Since the arcs $\widehat{QP}$ and $\widehat{AR}$ are both of length $s - t$, the chords QP and AR are also of equal length. By the distance formula, we have

$$\overline{AR} = \overline{QP}$$
$$\sqrt{[\cos(s-t) - 1]^2 + [\sin(s-t)]^2} = \sqrt{(\cos s - \cos t)^2 + (\sin s - \sin t)^2}$$

Squaring both sides and rearranging terms, we have

$$\sin^2(s-t) + \cos^2(s-t) - 2\cos(s-t) + 1$$
$$= \sin^2 s + \cos^2 s + \sin^2 t + \cos^2 t - 2\cos s \cos t - 2\sin s \sin t$$

Since each of the expressions $\sin^2(s-t) + \cos^2(s-t)$, $\sin^2 s + \cos^2 s$, and $\sin^2 t + \cos^2 t$ equals 1, we have

$$2 - 2\cos(s-t) = 2 - 2\cos s \cos t - 2\sin s \sin t$$

Solving for $\cos(s-t)$ yields the formula

$$\boxed{\cos(s-t) = \cos s \cos t + \sin s \sin t \qquad (1)}$$

Now it is easy to obtain the addition formula for $\cos(s+t)$. By writing

$$s + t = s - (-t)$$

we have

$$\cos(s+t)) = \cos(s-(-t))$$
$$= \cos s \cos(-t) + \sin s \sin(-t)$$

Since $\cos(-t) = \cos t$ and $\sin(-t) = -\sin t$,

$$\boxed{\cos(s + t) = \cos s \cos t - \sin s \sin t \qquad (2)}$$

EXAMPLE 1
Find cos 15° without the use of tables or a calculator.

SOLUTION
Since $15° = 45° - 30°$, we may use the formula for $\cos(s - t)$ to obtain

$$\begin{aligned}\cos 15° &= \cos(45° - 30°)\\ &= \cos 45° \cos 30° + \sin 45° \sin 30°\\ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}\\ &= \frac{\sqrt{6} + \sqrt{2}}{4}\end{aligned}$$

PROGRESS CHECK
Solve Example 1 using $15° = 60° - 45°$.

EXAMPLE 2
Find the exact value of $\cos(5\pi/12)$.

SOLUTION
We note that $5\pi/12 = 2\pi/12 + 3\pi/12 = \pi/6 + \pi/4$. Then

$$\begin{aligned}\cos\left(\frac{5\pi}{12}\right) &= \cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right)\\ &= \cos \frac{\pi}{6} \cos \frac{\pi}{4} - \sin \frac{\pi}{6} \sin \frac{\pi}{4}\\ &= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

PROGRESS CHECK
Solve Example 2 using the identity $5\pi/12 = 9\pi/12 - 4\pi/12$.

COFUNCTIONS

Before tackling $\sin(s + t)$, we first establish the following important functional relationships.

$$\cos\left(\frac{\pi}{2} - t\right) = \sin t \tag{3}$$

$$\sin\left(\frac{\pi}{2} - t\right) = \cos t \tag{4}$$

$$\tan\left(\frac{\pi}{2} - t\right) = \cot t \tag{5}$$

Functions satisfying the properties of the identities (3) and (4) are called **cofunctions.** Thus, sine and cosine are cofunctions. So, too, the tangent and cotangent functions are cofunctions, as are secant and cosecant. This is the origin of the prefix *co* in *co*sine, *co*secant, and *co*tangent.

Applying the difference formula for cosine to the left-hand side of Equation (3),

$$\begin{aligned}\cos\left(\frac{\pi}{2} - t\right) &= \cos\frac{\pi}{2}\cos t + \sin\frac{\pi}{2}\sin t\\ &= 0 \cdot \cos t + 1 \cdot \sin t\\ &= \sin t\end{aligned}$$

which establishes Equation (3). Replacing t with $\frac{\pi}{2} - t$ in this identity yields

$$\begin{aligned}\cos\left[\frac{\pi}{2} - \left(\frac{\pi}{2} - t\right)\right] &= \sin\left(\frac{\pi}{2} - t\right)\\ \cos t &= \sin\left(\frac{\pi}{2} - t\right)\end{aligned}$$

which establishes Equation (4). The third identity follows from the definition of tangent and from Equations (3) and (4):

$$\tan\left(\frac{\pi}{2} - t\right) = \frac{\sin\left(\frac{\pi}{2} - t\right)}{\cos\left(\frac{\pi}{2} - t\right)} = \frac{\cos t}{\sin t} = \cot t$$

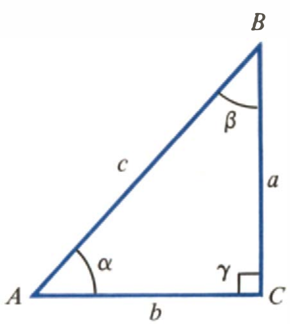


FIGURE 2

EXAMPLE 3

Use trigonometry of the right triangle to show that sine and cosine are cofunctions.

SOLUTION

In right triangle ABC, angle $\gamma = 90°$ (Figure 2). Then $\sin\alpha = a/c = \cos\beta$. But angles α and β are complementary; that is, $\alpha + \beta = 90°$. Thus $\sin\alpha = \cos(90° - \alpha)$ and $\cos\beta = \sin(90° - \beta)$, which establishes that they are cofunctions.

We are now prepared to prove the following.

$$\sin(s+t) = \sin s \cos t + \cos s \sin t \tag{6}$$

$$\sin(s-t) = \sin s \cos t - \cos s \sin t \tag{7}$$

We supply the steps for a proof of Equation (6); the student should supply a reason for each step.

$$\begin{aligned}
\sin(s+t) &= \cos\left[\frac{\pi}{2} - (s+t)\right] \\
&= \cos\left[\left(\frac{\pi}{2} - s\right) - t\right] \\
&= \cos\left(\frac{\pi}{2} - s\right) \cos t + \sin\left(\frac{\pi}{2} - s\right) \sin t \\
&= \sin s \cos t + \cos s \sin t
\end{aligned}$$

The student should now prove Equation (7) by using

$$\sin(s-t) = \sin[s + (-t)]$$

We conclude with the addition formulas for the tangent function.

$$\tan(s+t) = \frac{\tan s + \tan t}{1 - \tan s \tan t} \tag{8}$$

$$\tan(s-t) = \frac{\tan s - \tan t}{1 + \tan s \tan t} \tag{9}$$

Again, we supply the steps for a proof of Equation (8) and will let the student supply a reason for each step.

$$\begin{aligned}
\tan(s+t) &= \frac{\sin(s+t)}{\cos(s+t)} \\
&= \frac{\sin s \cos t + \cos s \sin t}{\cos s \cos t - \sin s \sin t} \\
&= \frac{\left(\dfrac{\sin s}{\cos s} \cdot \dfrac{\cos t}{\cos t}\right) + \left(\dfrac{\cos s}{\cos s} \cdot \dfrac{\sin t}{\cos t}\right)}{\left(\dfrac{\cos s}{\cos s} \cdot \dfrac{\cos t}{\cos t}\right) - \left(\dfrac{\sin s}{\cos s} \cdot \dfrac{\sin t}{\cos t}\right)} \\
&= \frac{\tan s + \tan t}{1 - \tan s \tan t}
\end{aligned}$$

The student should now prove Equation (9) by using

$$\tan(s-t) = \tan[s + (-t)]$$

EXAMPLE 4

Show that $\sin(x + 3\pi/2) = -\cos x$.

SOLUTION

Using the addition formula,

$$\begin{aligned}\sin\left(x + \frac{3\pi}{2}\right) &= \sin x \cos \frac{3\pi}{2} + \cos x \sin \frac{3\pi}{2} \\ &= \sin x \cdot 0 + \cos x \cdot (-1) \\ &= -\cos x\end{aligned}$$

PROGRESS CHECK

Verify that $\tan(x - \pi) = \tan x$.

EXAMPLE 5

Given $\sin \alpha = -4/5$, with α an angle in quadrant III, and $\cos \beta = -5/13$, with β an angle in quadrant II, use the addition formula to find $\sin(\alpha + \beta)$ and the quadrant in which $\alpha + \beta$ lies.

SOLUTION

The addition formula

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

requires that we know $\sin \alpha$, $\cos \alpha$, $\sin \beta$, and $\cos \beta$. Using the fundamental identity $\sin^2 \alpha + \cos^2 \alpha = 1$, we have

$$\cos^2 \alpha = 1 - \sin^2 \alpha = 1 - \frac{16}{25} = \frac{9}{25}$$

Taking the square root of both sides, we must have $\cos \alpha = -3/5$ since α is in quadrant III. Similarly,

$$\sin^2 \beta = 1 - \cos^2 \beta = 1 - \frac{25}{169} = \frac{144}{169}$$

Taking the square root of both sides, we must have $\sin \beta = 12/13$ since β is in quadrant II. Thus,

$$\begin{aligned}\sin(\alpha + \beta) &= \left(-\frac{4}{5}\right)\left(-\frac{5}{13}\right) + \left(-\frac{3}{5}\right)\left(\frac{12}{13}\right) \\ &= \frac{20}{65} - \frac{36}{65} = -\frac{16}{65}\end{aligned}$$

Since $\sin(\alpha + \beta)$ is negative, $\alpha + \beta$ lies in either quadrant III or quadrant IV. However, the sum of an angle that lies in quadrant III and an angle that lies in quadrant II cannot lie in quadrant III. Thus, $\alpha + \beta$ lies in quadrant IV.

COMPUTING SINE AND COSINE

We can make use of the trigonometric formulas to generate a table of sine and cosine values. Suppose we have determined that

$$\sin 1° = 0.01745 \quad \cos 1° = 0.99985 \tag{1}$$

We can then write

$$\sin(1° + \alpha) = \sin 1° \cos \alpha + \cos 1° \sin \alpha$$
$$\cos(1° + \alpha) = \cos 1° \cos \alpha - \sin 1° \sin \alpha$$

Substituting for sin 1° and cos 1° from Equation (1),

$$\sin(1° + \alpha) = 0.01745 \cos \alpha + 0.99985 \sin \alpha \tag{2}$$
$$\cos(1° + \alpha) = 0.99985 \cos \alpha - 0.01745 \sin \alpha \tag{3}$$

Now, if we let $\alpha = 1°$, Equations (2) and (3) can be used to calculate sin 2° and cos 2°. We can then repeat the process with $\alpha = 2°$ to calculate sin 3° and cos 3°, and so on. Since this is an iterative procedure well suited for a computer, we are providing a program in BASIC that will calculate sine and cosine values from 2° to 90° in increments of 1°.

```
10 LET S1 = 0.01745
20 LET C1 = 0.99985
30 PRINT
   ''DEGREES'',
   ''SIN'',''COS''
40 PRINT ''1'',
   S1, C1
50 LET S2 = S1
60 LET C2 = C1
70 FOR I = 2 TO 90
80   LET S3 = S2
90   LET S2 =
     (S1 * C2) +
     (C1 * S2)
100  LET C2 =
     (C1 * C2) -
     (S1 * S3)
110  PRINT I, S2,
     C2
120 NEXT I
130 END
```

PROGRESS CHECK

Given $\cos \alpha = -4/5$, with α in quadrant III, and $\cos \beta = 3/5$, with β in quadrant I, find $\cos(\alpha - \beta)$ and the quadrant in which $\alpha - \beta$ lies.

ANSWER

$-24/25$, quadrant II

EXERCISE SET 7.2

Exercises 1–6 display conditional equations. To show that they are *not* identities, find a pair of values of s and t for which each equation is not true.

1. $\cos(s - t) = \cos s - \cos t$
2. $\sin(s + t) = \sin s + \sin t$
3. $\sin(s - t) = \sin s - \sin t$
4. $\cos(s + t) = \cos s + \cos t$
5. $\tan(s + t) = \tan s + \tan t$
6. $\tan(s - t) = \tan s - \tan t$

In Exercises 7–22 use the addition formulas to find exact values.

7. $\cos\left(\dfrac{\pi}{6} + \dfrac{\pi}{4}\right)$
8. $\sin\left(\dfrac{\pi}{6} - \dfrac{\pi}{4}\right)$

9. $\sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$

10. $\cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

11. $\cos(30° + 180°)$

12. $\tan(60° + 300°)$

13. $\tan(300° - 60°)$

14. $\sin(270° - 45°)$

15. $\sin 11\pi/12$ (*Hint*: $11\pi/12 = \pi/6 + 3\pi/4$)

16. $\tan 7\pi/12$ (*Hint*: $7\pi/12 = \pi/4 + \pi/3$)

17. $\cos 7\pi/12$ (*Hint*: $7\pi/12 = 5\pi/6 - \pi/4$)

18. $\tan 75°$ (*Hint*: $75° = 135° - 60°$)

19. $\sin 7\pi/6$

20. $\cos 5\pi/6$

21. $\tan 15°$

22. $\tan 165°$

In Exercises 23–28 write the given expression in terms of cofunctions of complementary angles.

23. $\sin 47°$

24. $\cos 78°$

25. $\tan \pi/6$

26. $\tan 84°$

27. $\cos \pi/3$

28. $\sin 72°30'$

29. If $\sin t = -3/5$, with t in quadrant III, find $\sin(\pi/2 - t)$.

30. If $\cos t = -5/13$, with t in quadrant II, find $\sin(t - \pi)$.

31. If $\tan \theta = 4/3$ and angle θ lies in quadrant III, find $\tan(\theta + \pi/4)$.

32. If $\sec \theta = 5/3$ and angle θ lies in quadrant I, find $\sin(\theta + \pi/6)$.

33. If $\cos t = 0.4$, with t in quadrant IV, find $\tan(t + \pi)$.

34. If $\sec \alpha = 1.2$ and angle α lies in quadrant IV, find $\tan(\alpha - \pi)$.

35. If $\sin s = 3/5$ and $\cos t = -12/13$, with s in quadrant II and t in quadrant III, find $\sin(s + t)$.

36. If $\sin s = -4/5$ and $\csc t = 13/5$, with s in quadrant IV and t in quadrant II, find $\cos(s - t)$.

37. If $\cos \alpha = 5/13$ and $\tan \beta = -2$, with angle α in quadrant I and angle β in quadrant II, find
$$\tan(\alpha + \beta)$$

38. If $\sec \alpha = 5/3$ and $\cot \beta = 15/8$, with angle α in quadrant IV and angle β in quadrant III, find
$$\tan(\alpha - \beta)$$

Prove each of the following identities by transforming the left-hand side of the equation into the expression on the right-hand side.

39. $\sin 2\alpha = 2 \sin \alpha \cos \alpha$

40. $\cos 2t = \cos^2 t - \sin^2 t$

41. $\tan 2\alpha = \dfrac{2 \tan \alpha}{1 - \tan^2 \alpha}$

42. $\sin(x + y)\sin(x - y) = \sin^2 x - \sin^2 y$

43. $\cos(x - y) \cos(x + y) = \cos^2 x - \sin^2 x$

44. $\dfrac{\sin(s + t)}{\sin(s - t)} = \dfrac{\tan s + \tan t}{\tan s - \tan t}$

45. $\csc(t + \pi/2) = \sec t$

46. $\tan(\alpha + 90°) = -\cot \alpha$

47. $\tan(x + \pi/4) = \dfrac{1 + \tan x}{1 - \tan x}$

48. $\csc(t - \pi) = -\csc t$

49. $\cot(s - t) = \dfrac{1 + \tan s \tan t}{\tan s - \tan t}$

50. $\cot(u + v) = \dfrac{\cot u \cot v - 1}{\cot u + \cot v}$

51. $\sin(s + t) + \sin(s - t) = 2 \sin s \cos t$

52. $\cos(s + t) + \cos(s - t) = 2 \cos s \cos t$

53. $\dfrac{\sin(x + h) - \sin x}{h} = \sin x\left(\dfrac{\cos h - 1}{h}\right) + \cos x\left(\dfrac{\sin h}{h}\right)$

54. $\dfrac{\cos(x + h) - \cos x}{h} = \cos x\left(\dfrac{\cos h - 1}{h}\right) - \sin x\left(\dfrac{\sin h}{h}\right)$

7.3 DOUBLE- AND HALF-ANGLE FORMULAS

DOUBLE-ANGLE FORMULAS

Our initial objective in this section is to derive expressions for sin $2t$, cos $2t$, and tan $2t$ in terms of trigonometric functions of t. We will establish the following **double-angle formulas.**

$$\sin 2t = 2 \sin t \cos t \quad (1)$$

$$\cos 2t = \cos^2 t - \sin^2 t \quad (2)$$

$$\tan 2t = \frac{2 \tan t}{1 - \tan^2 t} \quad (3)$$

These formulas are used quite often and you might want to memorize them. However, the derivations are so straightforward that you can always return to them to verify the results.

To establish Equation (1), we simply rewrite $2t$ as $(t + t)$ and use the addition formula.

$$\begin{aligned}\sin 2t &= \sin(t + t)\\ &= \sin t \cos t + \cos t \sin t\\ &= 2 \sin t \cos t\end{aligned}$$

We proceed in the same manner to prove Equation (2).

$$\begin{aligned}\cos 2t &= \cos(t + t)\\ &= \cos t \cos t - \sin t \sin t\\ &= \cos^2 t - \sin^2 t\end{aligned}$$

Using the addition formula for the tangent function yields a proof of Equation (3).

$$\begin{aligned}\tan 2t &= \tan(t + t)\\ &= \frac{\tan t + \tan t}{1 - \tan t \tan t}\\ &= \frac{2 \tan t}{1 - \tan^2 t}\end{aligned}$$

EXAMPLE 1

If $\cos t = -3/5$ and t is in quadrant II, evaluate sin $2t$ and cos $2t$. In which quadrant does $2t$ lie?

SOLUTION

We first find $\sin t$ by use of the fundamental identity $\sin^2 t + \cos^2 t = 1$. Thus,

$$\sin^2 t + \frac{9}{25} = 1$$

$$\sin^2 t = \frac{16}{25}$$

Since t is in quadrant II, $\sin t$ must be positive. Therefore,

$$\sin t = \frac{4}{5}$$

Applying the double-angle formulas with $\cos t = -3/5$, $\sin t = 4/5$, yields

$$\sin 2t = 2 \sin t \cos t = 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) = -\frac{24}{25}$$

$$\cos 2t = \cos^2 t - \sin^2 t = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}$$

Since $\sin 2t$ and $\cos 2t$ are both negative, we may conclude that $2t$ lies in quadrant III.

PROGRESS CHECK

If $\sin \theta = 5/13$ and θ is in quadrant I, evaluate $\sin 2\theta$ and $\tan 2\theta$.

ANSWER

$\sin 2\theta = \frac{120}{169}$, $\tan 2\theta = \frac{120}{119}$

EXAMPLE 2

Express $\sin 3t$ in terms of $\sin t$ and $\cos t$.

SOLUTION

We write $3t$ as $(2t + t)$. Then

$$\begin{aligned}\sin 3t &= \sin(2t + t)\\ &= \sin 2t \cos t + \cos 2t \sin t\\ &= 2 \sin t \cos t \cos t + (\cos^2 t - \sin^2 t) \sin t\\ &= 2 \sin t \cos^2 t + \sin t \cos^2 t - \sin^3 t\\ &= 3 \sin t \cos^2 t - \sin^3 t\end{aligned}$$

PROGRESS CHECK

Express $\cos 3t$ in terms of $\sin t$ and $\cos t$.

ANSWER

$\cos 3t = 4 \cos^3 t - 3 \cos t$

If we begin with the formula for $\cos 2t$ and use the fundamental identity $\cos^2 t = 1 - \sin^2 t$, we obtain

$$\begin{aligned}\cos 2t &= \cos^2 t - \sin^2 t\\ &= (1 - \sin^2 t) - \sin^2 t\\ &= 1 - 2 \sin^2 t\end{aligned}$$

Similarly, replacing $\sin^2 t$ by $1 - \cos^2 t$ yields

$$\begin{aligned}\cos 2t &= \cos^2 t - \sin^2 t\\ &= \cos^2 t - (1 - \cos^2 t)\\ &= 2\cos^2 t - 1\end{aligned}$$

We then have three useful formulas for $\cos 2t$.

$$\cos 2t = \cos^2 t - \sin^2 t \qquad (4)$$

$$\cos 2t = 1 - 2\sin^2 t \qquad (5)$$

$$\cos 2t = 2\cos^2 t - 1 \qquad (6)$$

EXAMPLE 3

Verify the identity $\dfrac{1 - \cos 2\alpha}{2 \sin \alpha \cos \alpha} = \tan \alpha$.

SOLUTION

Substituting $\cos 2\alpha = 1 - 2\sin^2 \alpha$, we have

$$\begin{aligned}\frac{1 - \cos 2\alpha}{2\sin\alpha\cos\alpha} &= \frac{1 - (1 - 2\sin^2\alpha)}{2\sin\alpha\cos\alpha}\\ &= \frac{2\sin^2\alpha}{2\sin\alpha\cos\alpha}\\ &= \frac{\sin\alpha}{\cos\alpha}\\ &= \tan\alpha\end{aligned}$$

PROGRESS CHECK

Verify the identity $\dfrac{1 + \cos 2\theta}{\sin 2\theta} = \cot \theta$.

WARNING Note that

$$\frac{\sin 2t}{2} \neq \sin t$$

From Equation (1),

$$\frac{\sin 2t}{2} = \frac{2\sin t \cos t}{2} = \sin t \cos t$$

HALF-ANGLE FORMULAS

If we begin with the alternative forms for $\cos 2t$ given in Equations (5) and (6), we can obtain the following expressions for $\sin^2 t$ and $\cos^2 t$. These expressions are often used in calculus.

$$\sin^2 t = \frac{1 - \cos 2t}{2} \tag{7}$$

$$\cos^2 t = \frac{1 + \cos 2t}{2} \tag{8}$$

Since the identities in Equations (7) and (8) hold for all values of t, they must hold when we replace t by $t/2$. This yields the pair of equations

$$\sin^2 \frac{t}{2} = \frac{1 - \cos t}{2}$$

$$\cos^2 \frac{t}{2} = \frac{1 + \cos t}{2}$$

Solving, we have

$$\sin \frac{t}{2} = \pm\sqrt{\frac{1 - \cos t}{2}} \tag{9}$$

$$\cos \frac{t}{2} = \pm\sqrt{\frac{1 + \cos t}{2}} \tag{10}$$

The appropriate sign to use in Equations (9) and (10) depends on the quadrant in which $t/2$ is located. Thus, $\sin t/2$ is positive if $t/2$ lies in quadrant I or II; similarly, we choose the positive root for $\cos t/2$ in Equation (10) if $t/2$ lies in quadrant I or IV.

Using the identity

$$\tan \frac{t}{2} = \frac{\sin \frac{t}{2}}{\cos \frac{t}{2}}$$

we obtain

$$\tan \frac{t}{2} = \pm\sqrt{\frac{1 - \cos t}{1 + \cos t}} \tag{11}$$

Formulas (9), (10), and (11) are known as the **half-angle formulas.**

TRIGONOMETRY AND THE PYTHAGOREAN THEOREM

The Pythagorean Theorem can be derived by using trigonometry of the right triangle. In the accompanying figure, ABC is a right triangle, and CD is perpendicular to the hypotenuse AB of length c. Using triangle ABC, you can verify that

$$\sin \alpha = \frac{a}{c} \quad \text{and} \quad \cos \alpha = \frac{b}{c} \tag{1}$$

Now, from right triangle ACD,

$$\overline{AD} = b \cos \alpha \tag{2}$$

Noting that $\beta = 90° - \alpha$ and using right triangle BCD,

$$\overline{BD} = a \cos(90° - \alpha) = a \sin \alpha \tag{3}$$

since $\cos(90° - \alpha) = \sin \alpha$. We can now use Equations (2) and (3) to sum

$$c = \overline{BD} + \overline{AD} = a \sin \alpha + b \cos \alpha$$

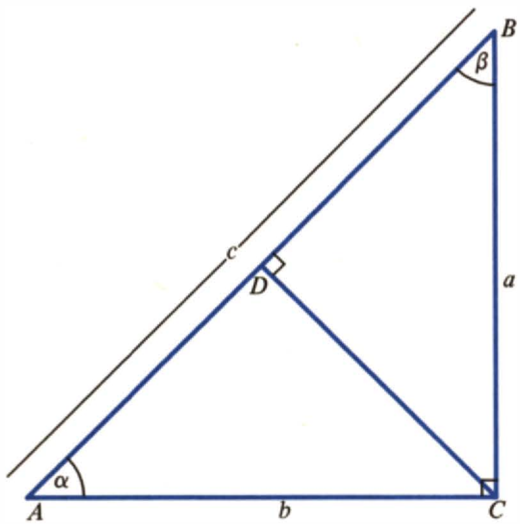
and, substituting from Equation (1),

$$c = \frac{a^2}{c} + \frac{b^2}{c}$$

or

$$c^2 = a^2 + b^2$$

This, of course, is a statement of the Pythagorean Theorem.



EXAMPLE 4

Find the exact values of sin 22.5° and cos 112.5°.

SOLUTION

Applying the half-angle formulas with 22.5° = 45°/2 yields

$$
\begin{aligned}
\sin 22.5^\circ &= \sin \frac{45^\circ}{2} \\
&= \sqrt{\frac{1 - \cos 45^\circ}{2}} && \text{Choose the positive square root for an angle in quadrant I.} \\
&= \sqrt{\frac{1 - \sqrt{2}/2}{2}} \\
&= \frac{\sqrt{2 - \sqrt{2}}}{2}
\end{aligned}
$$

$$
\begin{aligned}
\cos 112.5^\circ &= \cos \frac{225^\circ}{2} \\
&= -\sqrt{\frac{1 + \cos 225^\circ}{2}} && \text{Choose the negative square root since cosine is negative in quadrant II.} \\
&= -\sqrt{\frac{1 - \cos 45^\circ}{2}} \\
&= -\sqrt{\frac{1 - \sqrt{2}/2}{2}} \\
&= -\frac{\sqrt{2 - \sqrt{2}}}{2}
\end{aligned}
$$

PROGRESS CHECK

Use the half-angle formulas to evaluate tan $\pi/8$.

ANSWER

$\sqrt{2} - 1$

EXAMPLE 5

If sin $\theta = -3/5$ and θ is in quadrant III, evaluate cos $\theta/2$.

SOLUTION

We first evaluate cos θ by using the identity

$$\cos^2 \theta = 1 - \sin^2 \theta = 1 - \frac{9}{25} = \frac{16}{25}$$

Since θ is in quadrant III, $\cos\theta$ is negative. Thus, $\cos\theta = -4/5$. We can now employ the half-angle formula

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$
$$= \pm\sqrt{\frac{1-\frac{4}{5}}{2}}$$
$$= \pm\frac{\sqrt{10}}{10}$$

Since $180° < \theta < 270°$, we see that $90° < \theta/2 < 135°$. Thus, $\theta/2$ is in quadrant II and $\cos\theta/2$ is negative. We conclude that $\cos\theta/2 = -\sqrt{10}/10$.

PROGRESS CHECK

If $\tan\alpha = 3/4$ and α is in quadrant III, evaluate $\tan\alpha/2$.

ANSWER

-3

EXERCISE SET 7.3

Use the given conditions to determine the value of the specified trigonometric function.

1. $\sin u = 3/5$ and u is in quadrant II; find $\cos 2u$.
2. $\cos x = -5/13$ and x is in quadrant III; find $\sin 2x$.
3. $\sec\alpha = -2$ and α is in quadrant II; find $\sin 2\alpha$.
4. $\tan\theta = 4/3$ and θ is in quadrant I; find $\cos 2\theta$.
5. $\csc t = -17/8$ and t is in quadrant IV; find $\tan 2t$.
6. $\cot\beta = 3/4$ and β is in quadrant III; find $\cot 2\beta$.
7. $\sin 2\alpha = -4/5$ and 2α is in quadrant IV; find $\sin 4\alpha$.
8. $\sec 5x = -13/12$ and $5x$ is in quadrant III; find $\tan 10x$.
9. $\cos(\theta/2) = 8/17$ and $\theta/2$ is acute; find $\cos\theta$.
10. $\csc(t/4) = -13/5$ and $t/4$ is in quadrant IV; find $\cos(t/2)$.
11. $\sin 42° = 0.67$; find $\cos 84°$.
12. $\cos 77° = 0.22$; find $\cos 154°$.

Use the half-angle formulas to find exact values for each of the following.

13. $\sin 15°$
14. $\cos 75°$
15. $\tan\pi/8$
16. $\sec 5\pi/8$
17. $\csc 165°$
18. $\cot 7\pi/12$

Use the given conditions to determine the exact value of the specified trigonometric function.

19. $\sin\theta = -4/5$ and θ is in quadrant IV; find $\cos\theta/2$.
20. $\cos\theta = 3/5$ and θ is in quadrant I; find $\sin\theta/2$.
21. $\sec t = -3$ and t is in quadrant II; find $\sin t/2$.
22. $\tan x = 4/3$ and x is in quadrant III; find $\cos x/2$.
23. $\cot\beta = 3/4$ and β is in quadrant III; find $\tan\beta/2$.
24. $\csc\alpha = 13/5$ and α is in quadrant II; find $\tan\alpha/2$.
25. $\cos 4x = 1/3$ and $4x$ is in quadrant IV; find $\cos 2x$.
26. $\sec 6\alpha = -13/12$ and α is in quadrant III; find $\sin 3\alpha$.

Verify the given identities.

27. $\sin 50x = 2 \sin 25x \cos 25x$

28. $(\sin \theta + \cos \theta)^2 = 1 + \sin 2\theta$

29. $\tan 2y = \dfrac{2 \cot y}{\csc^2 y - 2}$

30. $2 \sin^2 2t + \cos 4t = 1$

31. $\sin 4\alpha = 4 \sin \alpha \cos^3 \alpha - 4 \sin^3 \alpha \cos \alpha$

32. $\cos 4\beta = 1 - 8 \sin^2 \beta \cos^2 \beta$

33. $\cos 2u = \dfrac{1 - \tan^2 u}{1 + \tan^2 u}$

34. $\sin 2\theta = \dfrac{2 \tan \theta}{1 + \tan^2 \theta}$

35. $\sin \dfrac{t}{2} \cos \dfrac{t}{2} = \dfrac{\sin t}{2}$

36. $\tan \dfrac{y}{2} = \csc y - \cot y$

37. $\sin \alpha - \cos \alpha \tan \dfrac{\alpha}{2} = \tan \dfrac{\alpha}{2}$

38. $\dfrac{1 - \cos 2\beta}{1 + \cos 2\beta} = \tan^2 \beta$

39. $\cos^4 x - \sin^4 x = \cos 2x$

40. $\dfrac{\sin 2t}{\sin t} - \dfrac{\cos 2t}{\cos t} = \sec t$

41. $\dfrac{2 \tan \alpha}{1 + \tan^2 \alpha} = \sin 2\alpha$

42. $\cos^2 \dfrac{x}{2} = \dfrac{\tan x + \sin x}{2 \tan x}$

43. $\sec 2t = \dfrac{\sec^2 t}{2 - \sec^2 t}$

44. $\cos 2t + \cot 2t = \cot 2t(\sin t + \cos t)^2$

45. $\tan \dfrac{t}{2} = \dfrac{1 - \cos t}{\sin t}$

46. $\tan \dfrac{t}{2} = \dfrac{\sin t}{1 + \cos t}$

7.4 THE PRODUCT-SUM FORMULAS

The **product-sum formulas** derived in this section are of use in calculus and in other courses in higher mathematics. They are not as important as the formulas that appeared in Sections 7.2 and 7.3 and need not be memorized. Rather, you should be aware of these formulas so that you can look them up when needed.

The following formulas express a product as a sum.

$$\sin s \cos t = \frac{\sin(s + t) + \sin(s - t)}{2} \tag{1}$$

$$\cos s \sin t = \frac{\sin(s + t) - \sin(s - t)}{2} \tag{2}$$

$$\cos s \cos t = \frac{\cos(s + t) + \cos(s - t)}{2} \tag{3}$$

$$\sin s \sin t = \frac{\cos(s - t) - \cos(s + t)}{2} \tag{4}$$

To prove Equation (1), we begin with the right-hand side of the equation.

$$\begin{aligned}\frac{\sin(s+t)+\sin(s-t)}{2}\\ &= \frac{(\sin s \cos t + \cos s \sin t) + (\sin s \cos t - \cos s \sin t)}{2}\\ &= \frac{2 \sin s \cos t}{2}\\ &= \sin s \cos t\end{aligned}$$

The proof of Equations (2), (3), and (4) are very similar.

EXAMPLE 1
Express $\sin 4x \cos 3x$ as a sum or a difference.

SOLUTION
Applying Equation (1) we obtain

$$\begin{aligned}\sin 4x \cos 3x &= \frac{\sin(4x+3x)+\sin(4x-3x)}{2}\\ &= \frac{\sin 7x + \sin x}{2}\end{aligned}$$

PROGRESS CHECK
Express $\sin 5x \sin 2x$ as a sum or as a difference.

ANSWER
$\frac{1}{2}(\cos 3x - \cos 7x)$

EXAMPLE 2
Evaluate the product $\cos(5\pi/8) \cos(3\pi/8)$ by a product-sum formula.

SOLUTION
Using Equation (3) we have

$$\begin{aligned}\cos\frac{5\pi}{8}\cos\frac{3\pi}{8} &= \frac{1}{2}\left[\cos\left(\frac{5\pi}{8}+\frac{3\pi}{8}\right)+\cos\left(\frac{5\pi}{8}-\frac{3\pi}{8}\right)\right]\\ &= \frac{1}{2}\left[\cos\pi+\cos\frac{\pi}{4}\right]\\ &= \frac{1}{2}\left[-1+\frac{\sqrt{2}}{2}\right]\\ &= \frac{\sqrt{2}-2}{4}\end{aligned}$$

PROGRESS CHECK
Evaluate $\cos(\pi/3) \sin(\pi/6)$ by a product-sum formula.

ANSWER
1/4

The following formulas express a sum as a product.

$$\sin s + \sin t = 2 \sin \frac{s+t}{2} \cos \frac{s-t}{2} \tag{5}$$

$$\sin s - \sin t = 2 \cos \frac{s+t}{2} \sin \frac{s-t}{2} \tag{6}$$

$$\cos s + \cos t = 2 \cos \frac{s+t}{2} \cos \frac{s-t}{2} \tag{7}$$

$$\cos s - \cos t = -2 \sin \frac{s+t}{2} \sin \frac{s-t}{2} \tag{8}$$

To prove the identity in Equation (5), begin with the right-hand side and apply Equation (1). Then

$$\begin{aligned} 2 \sin \frac{s+t}{2} \cos \frac{s-t}{2} &= \frac{1}{2}\left[\sin\left(\frac{s+t}{2} + \frac{s-t}{2}\right) + \sin\left(\frac{s+t}{2} - \frac{s-t}{2}\right)\right] \\ &= \sin s + \sin t \end{aligned}$$

This establishes Equation (5).

EXAMPLE 3
Express $\sin 5x - \sin 3x$ as a product.

SOLUTION
Using Equation (6) we have

$$\begin{aligned} \sin 5x - \sin 3x &= 2 \cos \frac{5x+3x}{2} \sin \frac{5x-3x}{2} \\ &= 2 \cos 4x \sin x \end{aligned}$$

PROGRESS CHECK
Express $\cos 6x + \cos 2x$ as a product.

ANSWER
$2 \cos 4x \cos 2x$

EXAMPLE 4
Evaluate $\cos (5\pi/12) - \cos (\pi/12)$ by using a sum–product formula.

SOLUTION
Using Equation (8), we have

$$\cos\frac{5\pi}{12} - \cos\frac{\pi}{12} = -2\sin\frac{\pi}{4}\sin\frac{\pi}{6}$$
$$= -2\left(\frac{\sqrt{2}}{2}\right)\frac{1}{2} = -\frac{\sqrt{2}}{2}$$

PROGRESS CHECK

Evaluate $\sin 2\pi/3$ by using a sum–product formula.

ANSWER

$\sqrt{3}/2$

EXERCISE SET 7.4

Express each product as a sum or difference.

1. $2\sin 5\alpha\cos\alpha$
2. $-3\cos 6x\sin 2x$
3. $\sin 3x\sin(-2x)$
4. $\cos 7t\cos(-3t)$
5. $-2\cos 2\theta\cos 5\theta$
6. $\sin\frac{5\theta}{2}\sin\frac{\theta}{2}$
7. $\cos(\alpha+\beta)\cos(\alpha-\beta)$
8. $-\sin 2u\cos 4u$

Evaluate each product by using a product-sum formula.

9. $\cos\frac{7\pi}{8}\sin\frac{5\pi}{8}$
10. $\cos\frac{\pi}{3}\cos\frac{\pi}{6}$
11. $\sin 120^\circ\cos 60^\circ$
12. $\sin\frac{13\pi}{12}\sin\frac{11\pi}{12}$

Express each sum or difference as a product.

13. $\sin 5x + \sin x$
14. $\cos 8t - \cos 2t$
15. $\cos 2\theta + \cos 6\theta$
16. $\sin 5\alpha - \sin 7\alpha$
17. $\sin(\alpha+\beta) + \sin(\alpha-\beta)$
18. $\cos\frac{x}{2} - \cos\frac{3x}{2}$
19. $\sin 7x - \sin 3x$
20. $\cos 5\theta + \cos 3\theta$

Evaluate each sum by using a sum-product formula.

21. $\cos 75^\circ + \cos 15^\circ$
22. $\sin\frac{5\pi}{12} + \sin\frac{\pi}{12}$
23. $\cos\frac{3\pi}{4} - \cos\frac{\pi}{4}$
24. $\sin\frac{13\pi}{12} - \sin\frac{5\pi}{12}$

Verify the identities in Exercises 25–34.

25. $\sin 40^\circ + \sin 20^\circ = \sin 10^\circ$
26. $\cos 70^\circ - \cos 10^\circ = -\sin 40^\circ$
27. $\dfrac{\sin 5\theta - \sin 3\theta}{\cos 3\theta - \cos 5\theta} = \cot 4\theta$
28. $\dfrac{\cos 5x - \cos x}{\sin 7x + \sin x} = -\tan 3x$
29. $\dfrac{\sin t - \sin s}{\cos t - \cos s} = -\cot\dfrac{s+t}{2}$
30. $\dfrac{\sin s + \sin t}{\cos s + \cos t} = \tan\dfrac{s+t}{2}$
31. $\dfrac{\sin 50^\circ - \sin 10^\circ}{\cos 50^\circ - \cos 10^\circ} = -\sqrt{3}$
32. $2\sin\left(\theta+\frac{\pi}{4}\right)\sin\left(\theta-\frac{\pi}{4}\right) = -\cos 2\theta$
33. $\dfrac{\cot x - \tan x}{\cot x + \tan x} = \cos 2x$
34. $\cos 6t\cos 2t + \sin^2 4x = \cos^2 2x$
35. Express $(\sin ax)(\cos bx)$ as a sum.
36. Express $(\cos ax)(\cos bx)$ as a sum.
37. Prove the product-sum formulas given in Equations (2) through (4).
38. Prove the product-sum formulas given in Equations (6) through (8).

7.5 TRIGONOMETRIC EQUATIONS

Thus far, this chapter has dealt exclusively with trigonometric identities. We now seek to solve trigonometric equations that are not true for all values of the variable but may be true for some values.

We have seen that algebraic equations may have just one or two solutions. The situation is quite different with trigonometric equations since the periodic nature of the trigonometric functions assures us that if there is a solution, there are an infinite number of solutions. To handle this complication, we simply seek all solutions t such that $0 \leq t < 2\pi$. Then for every integer value of n, $t + 2\pi n$ is also a solution. The following example illustrates this convenient means for writing the solution set.

EXAMPLE 1

Find all solutions of the equation $\cos t = 0$.

SOLUTION

The only values in the interval $[0, 2\pi)$ for whic $\cos t = 0$ are $\pi/2$ and $3\pi/2$. Then every solution is included among those values of t such that

$$t = \frac{\pi}{2} + 2\pi n \quad \text{or} \quad t = \frac{3\pi}{2} + 2\pi n, \quad n \text{ an integer}$$

Since $\frac{3\pi}{2} = \frac{\pi}{2} + \pi$, the solution set can be written in the more compact form

$$t = \frac{\pi}{2} + \pi n, \quad n \text{ an integer}$$

Each of the following examples illustrates a technique (highlighted in italics) for solving trigonometric equations.

EXAMPLE 2

Find all solutions of the equation $2\cos^2 t - \cos t - 1 = 0$ in the interval $[0, 2\pi)$.

SOLUTION

Factoring provides the key for solving many trigonometric equations. If we can write the equation in the form $P(x)Q(x) = 0$, we can then find the solutions by setting $P(x) = 0$ and $Q(x) = 0$. Of course, P and Q will themselves generally contain trigonometric functions.

Factoring the left side of the equation yields

$$(2\cos t + 1)(\cos t - 1) = 0$$

Setting each factor equal to 0, we have

$$2\cos t + 1 = 0 \quad \text{or} \quad \cos t - 1 = 0$$

so that

$$\cos t = -\frac{1}{2} \quad \text{or} \quad \cos t = 1$$

We were asked to find solutions of the original equation in the interval $[0, 2\pi)$. In this interval, the solutions of $\cos t = -\frac{1}{2}$ are $t = 2\pi/3$ and $t = 4\pi/3$; the only solution of $\cos t = 1$ is $t = 0$. The solutions of the original equation in the interval $[0, 2\pi)$ are

$$t = \frac{2\pi}{3}, \; t = \frac{4\pi}{3}, \quad \text{and} \quad t = 0$$

PROGRESS CHECK

Find all solutions of the equation $2\sin^2 t - 3\sin t + 1 = 0$ in the interval $[0, 2\pi)$.

ANSWER

$\frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}$

EXAMPLE 3

Find all solutions of the equation $\tan\theta\cos^2\theta - \tan\theta = 0$.

SOLUTION

Factoring the left side yields

$$\tan\theta(\cos^2\theta - 1) = 0$$

Setting each factor equal to 0,

$$\tan\theta = 0 \quad \text{or} \quad \cos^2\theta = 1$$

so that

$$\tan\theta = 0, \; \cos\theta = 1, \quad \text{or} \quad \cos\theta = -1$$

These equations yield the following solutions in the interval $[0, 2\pi)$.

$$\begin{aligned} \tan\theta = 0&: \quad \theta = 0 \quad \text{or} \quad \theta = \pi \\ \cos\theta = 1&: \quad \theta = 0 \\ \cos\theta = -1&: \quad \theta = \pi \end{aligned}$$

The solutions of the original equation are

$$\theta = 0 + 2\pi n \quad \text{and} \quad \theta = \pi + 2\pi n, \quad n \text{ an integer}$$

which can be expressed more compactly as

$$\theta = \pi n, \quad n \text{ an integer}$$

In degree measure, the solution is

$$\theta = 180^\circ n, \quad n \text{ an integer}$$

EXAMPLE 4

Find all solutions of the equation $\sin 2\theta - 3\sin\theta = 0$ in the interval $[0, 2\pi)$.

SOLUTION

Using trigonometric identities to simplify an equation can help in solving the equation. The substitution

$$\sin 2\theta = 2\sin\theta\cos\theta$$

yields

$$2\sin\theta\cos\theta - 3\sin\theta = 0$$
$$\sin\theta(2\cos\theta - 3) = 0$$
$$\sin\theta = 0 \quad \text{or} \quad 2\cos\theta - 3 = 0$$
$$\sin\theta = 0 \quad \text{or} \quad \cos\theta = \frac{3}{2}$$

The equation $\cos\theta = 3/2$ has no solutions; the solutions of $\sin\theta = 0$ are $\theta = 0$ and $\theta = \pi$. The solutions of the original equation are

$$\theta = 0 \quad \text{and} \quad \theta = \pi$$

or, in degree measure,

$$\theta = 0° \quad \text{and} \quad \theta = 180°$$

PROGRESS CHECK

Find all solutions of the equation $\cos 2\theta + \cos\theta = 0$.

ANSWER

$\frac{\pi}{3} + 2\pi n,\ \pi + 2\pi n,\ \frac{5\pi}{3} + 2\pi n$

or

$60° + 360°n,\ 180° + 360°n,\ 300° + 360°n$

EXAMPLE 5

Find all solutions of the equation $\cos 3x = 0$ in the interval $[0, 2\pi)$.

SOLUTION

Equations involving multiple angles can often be solved by using a substitution of variable. We are given

$$\cos 3x = 0, \qquad 0 \le x < 2\pi$$

Substituting $t = 3x$, we obtain

$$\cos t = 0, \qquad 0 \le \frac{t}{3} < 2\pi$$

or

$$\cos t = 0, \qquad 0 \le t < 6\pi$$

Note that we seek solutions of $\cos t = 0$ in the interval $[0, 6\pi)$ rather than $[0, 2\pi)$. The solutions are then

$$t = \frac{\pi}{2} \quad \frac{3\pi}{2} \quad \frac{5\pi}{2} \quad \frac{7\pi}{2} \quad \frac{9\pi}{2} \quad \frac{11\pi}{2}$$

Since $x = t/3$ we obtain

$$x = \frac{\pi}{6} \quad \frac{\pi}{2} \quad \frac{5\pi}{6} \quad \frac{7\pi}{6} \quad \frac{3\pi}{2} \quad \frac{11\pi}{6}$$

WARNING When you perform a substitution of variable, you must remember to go back and to express the answers in terms of the original variable.

Substitution of variable is a powerful tool. The equation

$$4 \sin^2 x + 3 \sin x - 1 = 0$$

can be viewed as a quadratic in u

$$4u^2 + 3u - 1 = 0$$

by substituting $u = \sin x$. Here is another example.

EXAMPLE 6

Find all solutions of the equation

$$3 \tan^2 x + \tan x - 1 = 0$$

in the interval $[0, \pi)$.

SOLUTION

The equation does not yield to the method of factoring. However, it can be viewed as a quadratic equation in $\tan x$. That is, if we substitute $t = \tan x$ we obtain

$$3t^2 + t - 1 = 0$$

which is a quadratic in t. By the quadratic formula,

$$t = \frac{-1 \pm \sqrt{13}}{6}$$

Solving for t,

$$t = 0.4343 \quad \text{and} \quad t = -0.7676$$

Since $\tan x = t$, we must have

$$\tan x = 0.4343 \quad \text{and} \quad \tan x = -0.7676$$

so that

$$x = \tan^{-1} 0.4343 \quad \text{and} \quad x = \tan^{-1}(-0.7676)$$

are exact expressions for the solutions of the original equation. To obtain numerical values, we can use a calculator to find that

$$x \approx 0.41 \quad \text{and} \quad x \approx -0.65$$

The calculator has provided us with solutions in the interval $[-\pi/2, \pi/2]$ since this is the range of the inverse tangent function. We were, however, instructed to find solutions in the interval $[0, \pi)$. Since -0.65 is not in this interval, we use the fact that the period of the tangent function is π to obtain

$$x \approx 0.65 + \pi = 2.49$$

as an acceptable solution in addition to $x = 0.41$.

EXERCISE SET 7.5

Find all solutions of the given equation in the interval $[0, 2\pi)$. Express the answers in both radian measure and degree measure.

1. $2 \sin \theta - 1 = 0$
2. $2 \cos \theta + 1 = 0$
3. $\cos \alpha + 1 = 0$
4. $\cot \gamma + 1 = 0$
5. $4 \cos^2 \alpha = 3$
6. $\tan^2 \theta = 3$
7. $3 \tan^2 \alpha = 1$
8. $2 \cos^2 \alpha - 1 = 0$
9. $2 \sin^2 \beta = \sin \beta$
10. $\sin \alpha = \cos \alpha$
11. $2 \cos^2 \theta - 3 \cos \theta + 1 = 0$
12. $2 \sin^2 \theta - \sin \theta - 1 = 0$
13. $\sin 5\theta = 1$
14. $\tan 3\beta = -\sqrt{3}$
15. $2 \sin^2 \alpha - 3 \cos \alpha = 0$
16. $\csc 2\theta = 2$
17. $2 \cos^2 \theta - 1 = \sin \theta$
18. $\cos^2 2\alpha = 1/4$
19. $\sin^2 \beta + 3 \cos \beta - 3 = 0$
20. $2 \cos^2 \theta \tan \theta - \tan \theta = 0$

Find all the solutions of the given equation.

21. $3 \tan^2 x - 1 = 0$
22. $2 \sin^2 y - 1 = 0$
23. $3 \cot^2 \theta - 1 = 0$
24. $1 - 4 \cos^2 t = 0$
25. $\sec 2u - 2 = 0$
26. $\tan 3x - 1 = 0$
27. $\sin 4x = 0$
28. $\cos 5t = -1$
29. $4 \cos^2 2t - 3 = 0$
30. $\csc^2 2x - 2 = 0$
31. $\sin 2t + 2 \cos t = 0$
32. $\sin 2t + 3 \cos t = 0$
33. $\cos 2t + \sin t = 0$
34. $2 \cos 2t + 2 \sin t = 0$
35. $\tan^2 x - \tan x = 0$
36. $\sec^2 x - 3 \sec x + 2 = 0$
37. $2 \sin^2 x + 3 \sin x - 2 = 0$
38. $2 \cos^2 x - 5 \cos x - 3 = 0$

Find the approximate solutions of the given equations in the interval $[0, 2\pi)$ by using Table V in the Tables Appendix, or a calculator.

39. $5 \sin^2 x - \sin x - 2 = 0$
40. $\sec^2 y - 5 \sec y + 6 = 0$
41. $3 \tan^2 u + 5 \tan u + 1 = 0$
42. $\cos^2 t - 2 \sin t + 3 = 0$

7.6 TRIGONOMETRY AND COMPLEX NUMBERS

THE COMPLEX PLANE

We associate the complex number $a + bi$ with the point in the plane whose coordinates are (a,b). Figure 3 illustrates the geometric representation of several complex numbers. Conversely, every point (a, b) in the plane represents a complex number, $a + bi$. When a rectangular coordinate system is used to represent complex numbers, it is called a **complex plane** and the x- and y-axes are called the **real axis** and the **imaginary axis**, respectively.

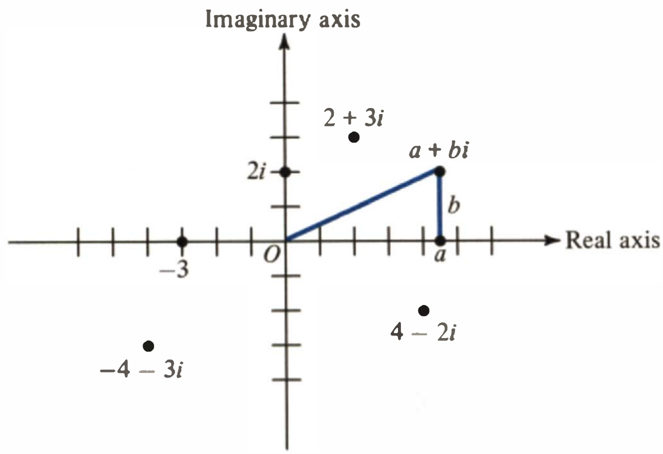


FIGURE 3

We can extend the concept of absolute value to complex numbers in a natural manner. Since $|x|$ represents the distance on a real number line from the origin to a point that corresponds to x, it would be consistent to define the **absolute value** $|a + bi|$ as the distance from the origin to the point corresponding to $a + bi$. Applying the distance formula (see Figure 4) we are led to the following definition.

The absolute value of a complex number $a + bi$ is denoted by $|a + bi|$ and is defined by

$$|a + bi| = \sqrt{a^2 + b^2}$$

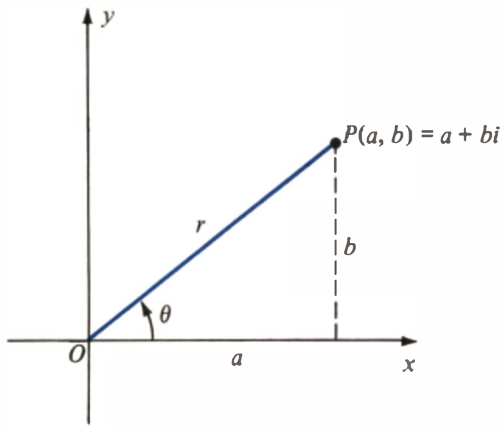


FIGURE 4

EXAMPLE 1

Find the absolute value of each of the following complex numbers.
(a) $2 - 3i$ (b) $4i$ (c) -2

SOLUTION

Applying the definition of absolute value,
(a) $|2 - 3i| = \sqrt{4 + 9} = \sqrt{13}$ (b) $|4i| = \sqrt{0 + 16} = 4$
(c) $|-2| = \sqrt{4 + 0} = 2$

The representation of a complex number as a point in a coordinate plane can be used to link complex numbers with trigonometry of the right triangle. In Figure 4, $a + bi$ is any nonzero complex number, and we consider the line segment OP to be the terminal side of an angle θ in standard position. Using trigonometry of the right triangle, we see that

$$a = r \cos \theta \quad \text{and} \quad b = r \sin \theta$$

We may then write

$$a + bi = (r \cos \theta) + (r \sin \theta)i$$

or

$$\boxed{a + bi = r(\cos \theta + i \sin \theta) \qquad (1)}$$

where $r = \overline{OP} = |a + bi| = \sqrt{a^2 + b^2}$. If $a + bi = 0$, then $r = 0$, and θ may assume any value.

Equation (1) is known as the **trigonometric form** or **polar form** of a complex number. Since we have an infinite number of choices for the angle θ, the polar form of a complex number is not unique. We call r the **modulus** and θ the **argument** of the complex number $r(\cos\theta + i\sin\theta)$. If $0 \le \theta < 360°$, then θ is called the **principal argument.**

EXAMPLE 2

Write the complex number $-2 + 2i$ in trigonometric form.

SOLUTION

The geometric representation is shown in Figure 5. The modulus of $-2 + 2i$ is

$$r = |-2 + 2i| = \sqrt{4 + 4} = 2\sqrt{2}$$

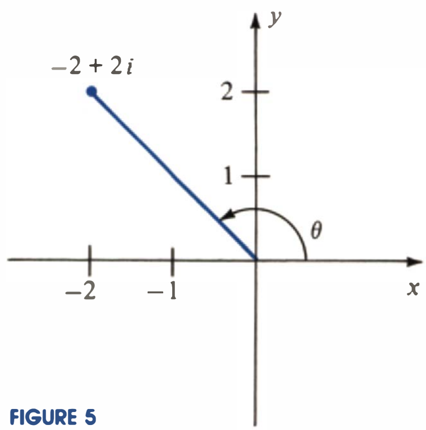


FIGURE 5

The principal argument θ is an angle in the second quadrant such that

$$\tan\theta = \frac{2}{-2} = -1$$

Thus, $\theta = 135°$, and using the trigonometric form of a complex number of Equation (1), we have

$$-2 + 2i = 2\sqrt{2}(\cos 135° + i\sin 135°)$$

PROGRESS CHECK

Write the complex number $1 - \sqrt{3}i$ in trigonometric form.

ANSWER

$2(\cos 300° + i\sin 300°)$

EXAMPLE 3

Write the complex number $2\sqrt{3}(\cos 150° + i\sin 150°)$ in the form $a + bi$.

SOLUTION

We need only substitute $\cos 150° = \sqrt{3}/2$ and $\sin 150° = -\frac{1}{2}$. Thus,

$$2\sqrt{3}(\cos 150° + i \sin 150°) = 2\sqrt{3}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$
$$= 3 - \sqrt{3}i$$

PROGRESS CHECK

Write the complex number $\sqrt{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$ in the form $a + bi$.

ANSWER

$1 + i$

Why have we introduced the trigonometric form of a complex number? Because multiplication and division of complex numbers is very simple when this form is used. If $r_1(\cos \theta_1 + i \sin \theta_1)$ and $r_2(\cos \theta_2 + i \sin \theta_2)$ are any two complex numbers, the rules for their multiplication and division are

$$r_1(\cos \theta_1 + i \sin \theta_1) \cdot r_2(\cos \theta_2 + i \sin \theta_2) = r_1r_2[\cos(\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)] \quad (2)$$

$$\frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)] \quad (3)$$

Note that the rule for multiplication requires the multiplication of the moduli and addition of the arguments. To prove this we see that

$$r_1(\cos \theta_1 + i \sin \theta_1) \cdot r_2(\cos \theta_2 + i \sin \theta_2)$$
$$= r_1r_2[(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)]$$
$$= r_1r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$$

where the last step results from the addition formulas.

The rule for division requires the division of moduli and the subtraction of the arguments. The proof is left as an exercise.

EXAMPLE 4

Find the product of the complex numbers $1 + i$ and $-2i$ (a) by writing the numbers in trigonometric form and (b) by multiplying the numbers algebraically.

SOLUTION

(a) The trigonometric forms of these complex numbers are

$$1 + i = \sqrt{2}(\cos 45° + i \sin 45°)$$

and

$$-2i = 2(\cos 270° + i \sin 270°)$$

Multiplying, we have

$$\begin{aligned}
\sqrt{2}(\cos 45° + i \sin 45°) \cdot 2(\cos 270° + i \sin 270°) \\
&= 2\sqrt{2}(\cos 315° + i \sin 315°) \\
&= 2\sqrt{2}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) \\
&= 2 - 2i
\end{aligned}$$

(b) Multiplying algebraically,

$$(1 + i)(-2i) = -2i - 2i^2 = -2i + 2 = 2 - 2i$$

PROGRESS CHECK

Express the complex numbers $1 + \sqrt{3}i$ and $1 - \sqrt{3}i$ in trigonometric form and find their product.

ANSWER

$2(\cos 60° + i \sin 60°)$; $2(\cos 300° + i \sin 300°)$; 4

DE MOIVRE'S THEOREM

Since exponentiation is repeated multiplication, we are led to anticipate a simple result when a complex number in trigonometric form is raised to a power. The theorem that states this result is credited to Abraham De Moivre, a French mathematician. In this theorem $r(\cos \theta + i \sin \theta)$ is a complex number and n is a natural number.

De Moivre's Theorem

$$[r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta)$$

We can verify the theorem for some values of n. Thus, by Equation (2),

$$\begin{aligned}
[r(\cos \theta + i \sin \theta)]^2 &= r(\cos \theta + i \sin \theta) \cdot r(\cos \theta + i \sin \theta) \\
&= r^2[\cos(\theta + \theta) + i \sin(\theta + \theta)] \\
&= r^2(\cos 2\theta + i \sin 2\theta)
\end{aligned}$$

which is precisely what we obtain by using De Moivre's theorem. If we multiply again by $r(\cos \theta + i \sin \theta)$ and again apply Equation (2), we have

$$\begin{aligned}
[r(\cos \theta + i \sin \theta)]^3 &= r^2(\cos 2\theta + i \sin 2\theta) \cdot r(\cos \theta + i \sin \theta) \\
&= r^3(\cos 3\theta + i \sin 3\theta)
\end{aligned}$$

Thus, De Moivre's theorem seems "reasonable." A rigorous *proof* requires the application of the method of mathematical induction, which will be discussed in a later chapter.

EXAMPLE 5
Evaluate $(1 - i)^{10}$.

SOLUTION
Writing $1 - i$ in trigonometric form we have

$$1 - i = \sqrt{2}(\cos 315° + i \sin 315°)$$

and

$$(1 - i)^{10} = [\sqrt{2}(\cos 315° + i \sin 315°)]^{10}$$

Applying De Moivre's theorem,

$$\begin{aligned}(1 - i)^{10} &= (\sqrt{2})^{10}[\cos 3150° + i \sin 3150°]\\ &= 32[\cos 270° + i \sin 270°]\\ &= 32[0 + i(-1)] = -32i\end{aligned}$$

PROGRESS CHECK
Evaluate $(\sqrt{3} + i)^6$.

ANSWER
-64

Recall that a real number a is said to be an nth root of the real number b if $a^n = b$ for a positive integer n. In an analogous manner, we say that the complex number u is an **nth root** of the nonzero complex number z if $u^n = z$. If we express u and z in trigonometric form as

$$u = s(\cos \phi + i \sin \phi) \qquad z = r(\cos \theta + i \sin \theta) \tag{4}$$

we can then apply De Moivre's theorem to obtain

$$u^n = s^n(\cos n\phi + i \sin n\phi) = r(\cos \theta + i \sin \theta) \tag{5}$$

Since the two complex numbers u^n and z are equal, they are represented by the same point in the complex plane. Hence, the moduli must be equal, since the modulus is the distance of the point from the origin. Therefore, $s^n = r$ or

$$s = \sqrt[n]{r}$$

Since $z \neq 0$, we know that $r \neq 0$. We may therefore divide Equation (5) by r to obtain

$$\cos n\phi + i \sin n\phi = \cos \theta + i \sin \theta$$

By the definition of equality of complex numbers, we must have

$$\cos n\phi = \cos \theta \qquad \sin n\phi = \sin \theta$$

Since both sine and cosine are periodic functions with period 2π, we conclude that

$$n\phi = \theta + 2\pi k$$

or

$$\phi = \frac{\theta + 2\pi k}{n}$$

where k is an integer. Substituting for s and for ϕ in the trigonometric form of u given in Equation (4) yields

The nth Roots of a Complex Number

The n distinct roots of $r(\cos \theta + i \sin \theta)$ are given by

$$\sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i \sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

where $k = 0, 1, 2, \ldots, n - 1$.

Note that when k exceeds $n - 1$, we repeat a previous root. For example, when $k = n$, the angle is

$$\frac{\theta + 2\pi n}{n} = \frac{\theta}{n} + 2\pi = \frac{\theta}{n}$$

which is the same result that is obtained when $k = 0$.

EXAMPLE 6

Find the cube roots of $-8i$.

SOLUTION

In trigonometric form,

$$-8i = 8(\cos 270° + i \sin 270°)$$

We then have $r = 8$, $\theta = 270°$, and $n = 3$.
The cube roots are then

$$\sqrt[3]{8}\left[\cos\left(\frac{270° + 360°k}{3}\right) + i \sin\left(\frac{270° + 360°k}{3}\right)\right]$$

for $k = 0, 1, 2$. Substituting for each value of k we have

$$2(\cos 90° + i \sin 90°) = 2i$$
$$2(\cos 210° + i \sin 210°) = -\sqrt{3} - i$$
$$2(\cos 330° + i \sin 330°) = \sqrt{3} - i$$

When $z = 1$, we call the n distinct nth roots the ***n*th roots of unity.**

EXAMPLE 7
Find the four fourth roots of unity.

SOLUTION
In trigonometric form,

$$1 = 1(\cos 0° + i \sin 0°)$$

so that $r = 1$, $\theta = 0°$, and $n = 4$. The fourth roots are then given by

$$\sqrt[4]{1}\left[\cos\left(\frac{0° + 360°k}{4}\right) + i \sin\left(\frac{0° + 360°k}{4}\right)\right]$$

for $k = 0, 1, 2, 3$. Substituting these values for k yields

$$\cos 0° + i \sin 0° = 1$$
$$\cos 90° + i \sin 90° = i$$
$$\cos 180° + i \sin 180° = -1$$
$$\cos 270° + i \sin 270° = -i$$

It is easy to verify that each of these answers is indeed a fourth root of unity.

PROGRESS CHECK
Find the two square roots of $\sqrt{3}/2 - \frac{1}{2}i$. Express the answers in trigonometric form.

ANSWER
$\cos 165° + i \sin 165°$, $\cos 345° + i \sin 345°$

EXERCISE SET 7.6

Find the absolute value of each given complex number.

1. $3 - 2i$
2. $-7 + 6i$
3. $1 + i$
4. $\frac{1}{2} + \frac{1}{2}i$
5. $-6 - 2i$
6. $3 - i$

Express the given complex number in trigonometric form.

7. $3 - 3i$
8. $2 + 2i$
9. $\sqrt{3} - i$
10. $-2 - 2\sqrt{3}i$
11. $-1 + i$
12. $-2i$
13. -4
14. $3i$

Convert the given complex number from trigonometric form to the algebraic form $a + bi$.

15. $4(\cos 180° + i \sin 180°)$
16. $\frac{1}{2}\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)$
17. $\sqrt{2}(\cos 135° + i \sin 135°)$
18. $2(\cos 120° + i \sin 120°)$
19. $5\left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}\right)$
20. $4(\cos 240° + i \sin 240°)$

Find the product of the given complex numbers. Express the answers in trigonometric form.

21. $2(\cos 150° + i \sin 150°) \cdot 3(\cos 210° + i \sin 210°)$
22. $3(\cos 120° + i \sin 120°) \cdot 3(\cos 150° + i \sin 150°)$
23. $2(\cos 10° + i \sin 10°) \cdot (\cos 320° + i \sin 320°)$
24. $3(\cos 230° + i \sin 230°) \cdot 4(\cos 250° + i \sin 250°)$

Express the given complex numbers in trigonometric form, compute the product, and write the answer in the form $a + bi$.

25. $1 - i, 2i$
26. $-\sqrt{3} + i, -2$
27. $-2 + 2\sqrt{3}i, 3 + 3i$
28. $1 - \sqrt{3}i, 1 + \sqrt{3}i$
29. $5, -2 - 2i$
30. $-4i, -3i$

Use De Moivre's theorem to express the given number in the form $a + bi$.

31. $(-2 + 2i)^6$
32. $(\sqrt{3} - i)^{10}$
33. $(1 - i)^9$
34. $(-1 + \sqrt{3}i)^{10}$
35. $(-1 - i)^7$
36. $(-\sqrt{2} + \sqrt{2}i)^6$

Find the indicated roots of the given complex number. Express the answer in the indicated form.

37. The fourth roots of -16; algebraic form $a + bi$.
38. The square roots of -25; trigonometric form.
39. The square roots of $1 - \sqrt{3}i$; trigonometric form.
40. The four fourth roots of unity; algebraic form.

In Exercises 41–44 determine all roots of the given equation.

41. $x^3 + 8 = 0$
42. $x^3 + 125 = 0$
43. $x^4 - 16 = 0$
44. $x^4 + 16 = 0$

45. Prove $\dfrac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)} = \dfrac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2)]$.

TERMS AND SYMBOLS

trigonometric expression (p. 301)
trigonometric identity (p. 301)
trigonometric equation (p. 301)
fundamental identities (p. 302)
trigonometric formulas (p. 307)
addition formulas (p. 307)
cofunctions (p. 310)
double-angle formulas (p. 315)
half-angle formulas (p. 318)
product-sum formulas (p. 322)
complex plane (p. 331)
real axis (p. 331)
imaginary axis (p. 331)
absolute value of a complex number (p. 331)
trigonometric form (p. 332)
polar form (p. 332)
modulus (p. 333)
argument (p. 333)
principal argument (p. 333)
De Moivre's theorem (p. 335)
*n*th roots of a complex number (p. 336)
*n*th roots of unity (p. 337)

KEY IDEAS FOR REVIEW

☐ A trigonometric identity is true for all real values in the domain of the variable. The fundamental identities are those trigonometric identities that occur so frequently that they must be remembered and recognized.

☐ The fundamental identities can be used to verify other trigonometric identities. The techniques commonly used to verify identities include the following.
- Write all of the trigonometric functions in terms of sine and cosine.
- Factor.
- Complete the indicated operations, especially when this involves the sum of fractional expressions.
- Multiply the numerator and denominator of a fractional expression by the same quantity to produce a simpler product such as $1 - \sin^2 \theta$, $1 - \cos^2 \theta$, or $\sec^2 \theta - 1$.

☐ The most useful of the trigonometric formulas are the following.

Addition Formulas

$$\sin(s + t) = \sin s \cos t + \cos s \sin t$$

$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$

$$\tan(s + t) = \frac{\tan s + \tan t}{1 - \tan s \tan t}$$

Double-Angle Formulas

$$\sin 2t = 2 \sin t \cos t$$

$$\cos 2t = \cos^2 t - \sin^2 t$$

$$\tan 2t = \frac{2 \tan t}{1 - \tan^2 t}$$

Half-Angle Formulas

$$\sin \frac{t}{2} = \pm \sqrt{\frac{1 - \cos t}{2}}$$

$$\cos \frac{t}{2} = \pm \sqrt{\frac{1 + \cos t}{2}}$$

$$\tan \frac{t}{2} = \pm \sqrt{\frac{1 - \cos t}{1 + \cos t}}$$

☐ Since the trigonometric functions are periodic, a trigonometric equation has either no solutions or an infinite number of solutions.

☐ The complex number $a + bi$ can be associated with the point $P(a,b)$. The trigonometric or polar form of the complex number $a + bi$ is given by

$$a + bi = r(\cos \theta + i \sin \theta)$$

where r is the length of the line segment $\overline{OP}$ and θ is the measure of the angle in standard position whose terminal side is $\overline{OP}$.

☐ The trigonometric form of a complex number is useful since multiplication and division of complex numbers take on simple forms. In particular, exponentiation of complex numbers is handled by De Moivre's theorem, which states

$$[r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta)$$

☐ The complex number u is an nth root of the complex number z if $u^n = z$. De Moivre's theorem can be used to find a formula for determining u.

REVIEW EXERCISES

Solutions to exercises whose numbers are in color are in the Solutions section in the back of the book.

7.1 In Exercises 1–3 verify the given identity.

1. $\sin \sigma \sec \sigma + \tan \sigma = 2 \tan \sigma$
2. $\dfrac{\cos^2 x}{1 - \sin x} = 1 + \sin x$
3. $\sin \alpha + \sin \alpha \cot^2 \alpha = \csc \alpha$

7.2 In Exercises 4–7 determine the exact value of the given expression by using the addition formulas.

4. $\sin\left(\dfrac{\pi}{6} + \dfrac{\pi}{4}\right)$
5. $\cos(45° + 90°)$
6. $\tan\left(\dfrac{\pi}{3} + \dfrac{\pi}{4}\right)$
7. $\sin \dfrac{7\pi}{12}$

In Exercises 8–11 write the given expression in terms of cofunctions of complementary angles.

8. $\csc 15°$
9. $\cos 23°$
10. $\sin \dfrac{\pi}{8}$
11. $\tan \dfrac{2\pi}{7}$
12. If $\cos \sigma = -12/13$ and $0 \le \sigma \le 180°$, find $\sin(\pi - \sigma)$.
13. If $\sec \sigma = 10/8$ and σ lies in quadrant IV, find $\csc(\sigma + \pi/3)$.
14. If $\sin t = -3/5$ and $W(t)$ is in quadrant III, find $\tan(t + \pi)$.
15. If $\cos \alpha = -12/13$ and $\tan \beta = -5/2$, with angles α and β in quadrant II, find $\tan(\alpha + \beta)$.
16. If $\sin x = 3/5$ and $\csc y = 13/12$, with x in quadrant II and y in quadrant I, find $\cos(x - y)$.

7.3

17. If $\csc u = -5/4$ and u is in quadrant IV, find $\cos 2u$.
18. If $\tan \sigma = -3/4$ and $0 \le \sigma \le 180°$, find $\sin 2\sigma$.
19. If $\sin 2t = 3/5$ and $2t$ is in quadrant I, find $\sin 4t$.
20. If $\sin \sigma = 0.5$ and $\pi/2 \le \sigma \le \pi$, find $\sin 2\sigma$.
21. If $\cos(\sigma/2) = 12/13$ and σ is acute, find $\sin \sigma$.

22. If $\sin \alpha = -3/5$ and α is in quadrant III, find $\cos(\alpha/2)$.

23. If $\cot t = -4/3$ and t is in quadrant IV, find $\tan(t/2)$.

24. If $\cos 4x = 2/3$ and $4x$ is in quadrant IV, find $\cos 2x$.

25. Find the exact value of $\cos 15°$ by using a half-angle formula.

26. Find the exact value of $\sin \pi/8$ by using a half-angle formula.

27. Find the exact value of $\tan 112.5°$ by using a half-angle formula.

In Exercises 28–30 verify the given identity.

28. $\cos 30x = 1 - 2\sin^2 15x$

29. $\dfrac{1}{2}\sin 2y = \dfrac{\sin y}{\sec y}$

30. $\tan \dfrac{\alpha}{2} = \dfrac{(1 - \cos \alpha)}{\sin \alpha}$

7.4 31. Express $\sin \dfrac{3\alpha}{2} \sin \dfrac{\alpha}{2}$ as a sum or difference.

32. Express $\cos 3x - \cos x$ as a product.

33. Evaluate $\sin 75° \sin 15°$ by using a product–sum formula.

34. Evaluate $\cos \dfrac{3\pi}{4} + \cos \dfrac{\pi}{4}$ by using a sum–product formula.

7.5 In Exercises 35–37 find all solutions of the given equation in the interval $[0, 2\pi)$. Express the answers in radian measure.

35. $2\cos^2 \alpha - 1 = 0$

36. $2 \sin \sigma \cos \sigma = 0$

37. $\sin 2t - \sin t = 0$

In Exercises 38–40 find all solutions of the given equation. Express the answers in degree measure.

38. $\cos^2 \alpha - 2\cos \alpha = 0$

39. $\tan 3x + 1 = 0$

40. $4\sin^2 2t = 3$

7.6 In Exercises 41–43 determine the absolute value of the given complex number.

41. $2 - i$ 42. $-3 + 2i$ 43. $-4 - 5i$

In Exercises 44–47 convert from trigonometric to algebraic form and vice versa.

44. $-3 + 3i$

45. $2(\cos 90° + i \sin 90°)$

46. $\sqrt{2}(\cos 315° + i \sin 315°)$

47. -2

In Exercises 48–50 find the indicated product or quotient. Express the answer in trigonometric form.

48. $4(\cos 22° + i \sin 22°) \cdot 6(\cos 15° + i \sin 15°)$

49. $\dfrac{5(\cos 71° + i \sin 71°)}{3(\cos 50° + i \sin 50°)}$

50. $2(\cos 210° + i \sin 210°) \cdot (\cos 240° + i \sin 240°)$

In Exercises 51 and 52 use De Moivre's theorem to express the given number in the form $a + bi$.

51. $(3 - 3i)^5$

52. $[2(\cos 90° + i \sin 90°)]^3$

53. Express the two square roots of -9 in trigonometric form.

54. Determine all roots of the equation $x^3 - 1 = 0$.

PROGRESS TEST 7A

1. Verify the identity $4 - \tan^2 x = 5 - \sec^2 x$.

In Problems 2 and 3 determine exact values of the given expressions by using the addition formulas.

2. $\cos(270° + 30°)$ 3. $\tan\left(\dfrac{\pi}{4} - \dfrac{\pi}{3}\right)$

4. Write $\sin 47°$ in terms of its cofunction.

5. If $\cos \theta = 4/5$ and θ lies in quadrant IV, find $\sin(\theta - \pi)$.

6. If $\sin x = -5/13$ and $\tan y = 8/3$ with angles x and y in quadrant III, find $\tan(x - y)$.

7. If $\sin v = -12/13$ and v is in quadrant IV, find $\cos 2v$.

8. If $\cos 2\alpha = -4/5$ and 2α is in quadrant II, find $\cos 4\alpha$.

9. If $\csc \alpha = -2$ and α is in quadrant III, find $\cos(\alpha/2)$.

10. Find the exact value of $\tan 15°$ by using a half-angle formula.

11. Verify the identity $\sin \frac{x}{4} = 2 \sin \frac{x}{8} \cos \frac{x}{8}$.

12. Express $\sin 2x + \sin 3x$ as a product.

13. Express $\sin 150° - \sin 30°$ by using a sum-product formula.

14. Find all solutions of the equation $4 \sin^2 \alpha = 3$ in the interval $[0, 2\pi)$. Express the answers in radian measure.

15. Find all solutions of the equation $\sin^2 \theta - \cos^2 \theta = 0$ and express the answers in degree measure.

In Problems 16 and 17 find the indicated product or quotient. Express the answer in trigonometric form.

16. $\frac{1}{2}(\cos 14° + i \sin 14°) \cdot 10(\cos 72° + i \sin 72°)$

17. $\dfrac{3(\cos 85° + i \sin 85°)}{6(\cos 8° + i \sin 8°)}$

18. Use De Moivre's theorem to express $\left[\frac{1}{5}(\cos 120° + i \sin 120°)\right]^4$ in the form $a + bi$.

19. Express the three cube roots of -27 in trigonometric form.

PROGRESS TEST 7B

1. Verify the identity $\dfrac{\tan u + \cot u}{\sec u \sin u} = \csc^2 u$.

In Problems 2 and 3 determine exact values of the given expression by using the addition formulas.

2. $\csc(180° - 30°)$

3. $\sin \frac{7\pi}{12}$

4. Write $\tan 71°$ in terms of its cofunction.

5. If $\sin t = -5/13$ and t lies in quadrant III, find $\sec(t + \pi/4)$.

6. If $\cos \alpha = -0.6$ and $\csc \beta = 5/4$ with angles α and β in quadrant II, find $\sin(\alpha - \beta)$.

7. If $\sec \theta = 5/4$ and $0 \leq \theta \leq 180°$, find $\sin 2\theta$.

8. If $\sin \theta/2 = 3/5$ and θ is acute, find $\sin 2\theta$.

9. If $\sin 6x = -12/13$ and $6x$ is in quadrant IV, find $\cos 3x$.

10. Find the exact value of $\sin \pi/8$ by using a half-angle formula.

11. Verify the identity $\sec 2t = \dfrac{1 + \tan^2 t}{1 - \tan^2 t}$.

12. Express $\sin \pi/4 \cos \pi/3$ as a sum or difference.

13. Express $\cos 75° \cos 15°$ by using a product-sum formula.

14. Find all solutions of the equation $\tan^2 x + \tan x = 0$ in the interval $[0, 2\pi)$. Express the answers in radian measure.

15. Find all solutions of the equation $2 \sin^2 \alpha - \sin \alpha - 1 = 0$ and express the answers in degree measure.

In Problems 16 and 17 find the indicated product or quotient. Express the answer in trigonometric form.

16. $(\cos 125° + i \sin 125°) \cdot 5(\cos 125° + i \sin 125°)$

17. $\dfrac{\frac{1}{2}(\cos 67° + i \sin 67°)}{\frac{1}{4}(\cos 12° + i \sin 12°)}$

18. Use De Moivre's theorem to express $(-2i)^6$ in the form $a + bi$.

19. Determine all roots of the equation $x^3 + 1 = 0$.

8

ANALYTIC GEOMETRY: THE CONIC SECTIONS

In 1637 the great French philosopher and scientist René Descartes developed an idea that the nineteenth-century British philosopher John Stuart Mill described as "the greatest single step ever made in the progress of the exact sciences." Descartes combined the techniques of algebra with those of geometry and created a new field of study called **analytic geometry.** Analytic geometry enables us to apply algebraic methods and equations to the solution of problems in geometry and, conversely, to obtain geometric representations of algebraic equations.

We will first develop a formula for the coordinates of the midpoint of a line segment. We will then use the distance and midpoint formulas as tools to illustrate the usefulness of analytic geometry by proving a number of general theorems from plane geometry.

The power of the methods of analytic geometry is also very well demonstrated, as we shall see in this chapter, in a study of the conic sections. We will find in the course of that study that (a) a geometric definition can be converted into an algebraic equation and (b) an algebraic equation can be classified by the type of graph it represents.

8.1 ANALYTIC GEOMETRY

We have previously seen that the length d of the line segment joining points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

It is also possible to obtain a formula for the coordinates (x, y) of the midpoint P of the line segment whose endpoints are P_1 and P_2 (see Figure 1). Passing lines through P and P_2 parallel to the y-axis and a line through P_1 parallel to the x-axis results in the similar right triangles P_1AP and P_1BP_2. Using the fact that corresponding sides of similar triangles are in proportion, we can write

$$\frac{\overline{P_1P_2}}{\overline{P_2B}} = \frac{\overline{P_1P}}{\overline{PA}}$$

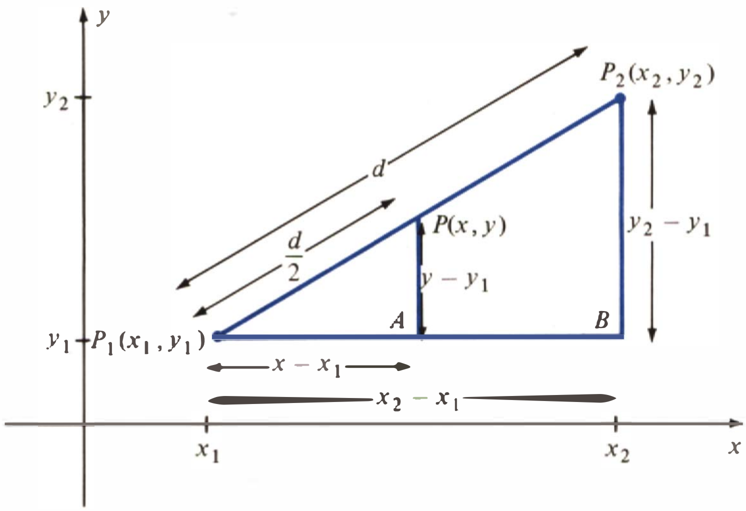


FIGURE 1

Since P is the midpoint of P_1P_2, the length of P_1P is $d/2$, so

$$\frac{d}{y_2 - y_1} = \frac{\frac{d}{2}}{y - y_1}$$

Solving for y, we have

$$y = \frac{y_1 + y_2}{2}$$

Similarly,

$$\frac{\overline{P_1P_2}}{\overline{P_1B}} = \frac{\overline{P_1P}}{\overline{P_1A}} \quad \text{or} \quad \frac{d}{x_2 - x_1} = \frac{\frac{d}{2}}{x - x_1}$$

We solve for x to obtain

$$x = \frac{x_1 + x_2}{2}$$

We have established the following formula.

The Midpoint Formula

If $P(x,y)$ is the midpoint of the line segment whose endpoints are $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, then

$$x = \frac{x_1 + x_2}{2} \qquad y = \frac{y_1 + y_2}{2}$$

EXAMPLE 1

Find the midpoint of the line segment whose endpoints are $P_1(3, 4)$ and $P_2(-2, -6)$.

SOLUTION

If $P(x, y)$ is the midpoint, then

$$x = \frac{x_1 + x_2}{2} = \frac{3 + (-2)}{2} = \frac{1}{2}$$

$$y = \frac{y_1 + y_2}{2} = \frac{4 + (-6)}{2} = -1$$

Thus, the midpoint is $(\frac{1}{2}, -1)$.

PROGRESS CHECK

Find the midpoint of the line segment whose endpoints are given.

(a) $(0, -4), (-2, -2)$ (b) $(-10, 4), (7, -5)$

ANSWERS

(a) $(-1, -3)$ (b) $(-\frac{3}{2}, -\frac{1}{2})$

The formulas for distance, midpoint of a line segment, and slope of a line are sufficient to allow us to demonstrate the beauty and power of analytic geometry. With these tools, we can prove theorems from plane geometry by placing the figures on a rectangular coordinate system.

EXAMPLE 2

Prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and has length equal to one-half the third side.

SOLUTION

We place the triangle OAB in a convenient location, namely, with one vertex at the origin and one side on the positive x-axis (Figure 2). If Q and R are the midpoints of OB and AB, then, by the midpoint formula, the coordinates of Q and R are

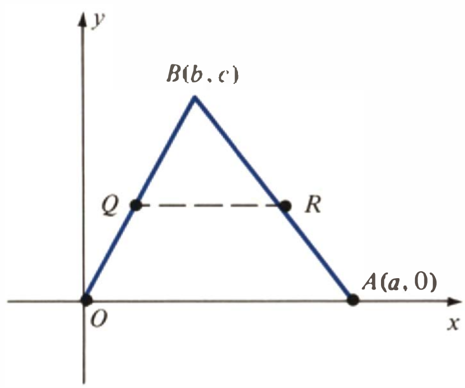


FIGURE 2

$$Q\left(\frac{b}{2}, \frac{c}{2}\right) \qquad R\left(\frac{a+b}{2}, \frac{c}{2}\right)$$

We see that the line joining Q and R has slope 0, since the difference of the y-coordinates is

$$\frac{c}{2} - \frac{c}{2} = 0$$

But side OA also has slope 0, which proves that QR is parallel to OA.

Applying the distance formula to $\overline{QR}$, we have

$$\begin{aligned}\overline{QR} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{\left(\frac{a+b}{2} - \frac{b}{2}\right)^2 + \left(\frac{c}{2} - \frac{c}{2}\right)^2} \\ &= \sqrt{\left(\frac{a}{2}\right)^2} = \frac{a}{2}\end{aligned}$$

Since $\overline{OA}$ has length a, we have shown that $\overline{QR}$ is one-half of $\overline{OA}$.

PROGRESS CHECK

Prove that the midpoint of the hypotenuse of a right triangle is equidistant from all three vertices. (*Hint:* Place the triangle so that two legs coincide with the positive x- and y-axes. Find the coordinates of the midpoint of the hypotenuse by the midpoint formula. Finally, compute the distance from the midpoint to each vertex by the distance formula.)

EXERCISE SET 8.1

In Exercises 1–12 find the midpoint of the line segment whose endpoints are given.

1. $(2, 6), (3, 4)$
2. $(1, 1), (-2, 5)$
3. $(2, 0), (0, 5)$
4. $(-3, 0), (-5, 2)$
5. $(-2, 1), (-5, -3)$
6. $(2, 3), (-1, 3)$
7. $(0, -4), (0, 3)$
8. $(1, -3), (3, 2)$
9. $(-1, 3), (-1, 6)$
10. $(3, 2), (0, 0)$
11. $(1, -1), (-1, 1)$
12. $(2, 4)\ (2, -4)$
13. Prove that the medians from the equal angles of an isosceles triangle are of equal length. (*Hint:* Place the triangle so that its vertices are at the points $A(-a, 0)$, $B(a, 0)$, and $C(0, b)$.)
14. Show that the midpoints of the sides of a rectangle are the vertices of a rhombus (a quadrilateral with four equal sides). (*Hint:* Place the rectangle so that its vertices are at the points $(0, 0)$, $(a, 0)$, $(0, b)$, and (a, b).)
15. Prove that a triangle with two equal medians is isosceles.
16. Show that the sum of the squares of the lengths of the medians of a triangle equals three-fourths the sum of the squares of the lengths of the sides. (*Hint:* Place the triangle so that its vertices are at the points $(-a, 0)$, $(b, 0)$, and $(0, c)$.)
17. Prove that the diagonals of a rectangle are equal in length. (*Hint:* Place the rectangle so that its vertices are at the points $(0, 0)$, $(a, 0)$, $(0, b)$, and (a, b).)

8.2 THE CIRCLE

The conic sections provide us with an outstanding opportunity to illustrate the double-edged power of analytic geometry. We will see that a geometric figure defined as a set of points can often be described analytically by an algebraic equation; conversely, we can start with an algebraic equation and use graphing procedures to study the properties of the curve.

First, let's see how the term conic section originates. If we pass a plane through a cone at various angles, as shown in Figure 3, the intersections are called **conic sections.** In exceptional cases the intersection of a plane and a cone is a point, a line, or a pair of lines.

Let's begin with the geometric definition of a circle.

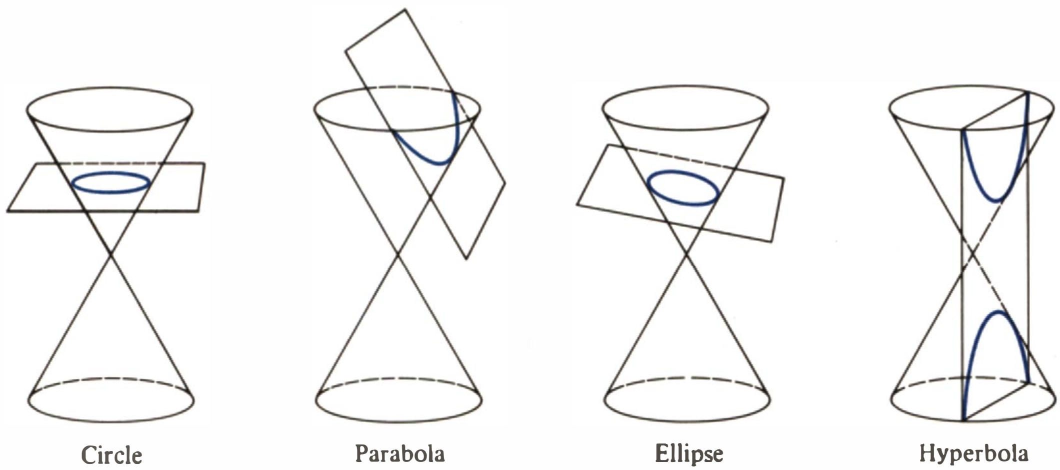


FIGURE 3

> A **circle** is the set of all points in a plane that are at a given distance from a fixed point. The fixed point is called the **center** of the circle and the given distance is called the **radius.**

Using the methods of analytic geometry, we place the center at a point (h, k) as in Figure 4. If $P(x, y)$ is a point on the circle, then, by the distance formula, the distance from P to the center (h, k) is

$$\sqrt{(x-h)^2 + (y-k)^2}$$

Since this distance is equal to the radius r, we can write

$$\sqrt{(x-h)^2 + (y-k)^2} = r$$

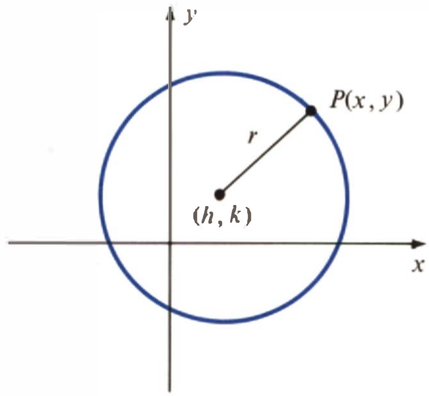


FIGURE 4

Squaring both sides provides us with an important fo
circle.

Standard Form of the Equation of a Circle

$$(x - h)^2 + (y - k)^2 = r^2$$

is the **standard form of the equation of the circle** with center at (h, k) and radius r.

EXAMPLE 1
Write the equation of the circle with center at $(2, -5)$ and radius 3.

SOLUTION
Substituting $h = 2$, $k = -5$, and $r = 3$ in the equation

$$(x - h)^2 + (y - k)^2 = r^2$$

yields

$$(x - 2)^2 + (y + 5)^2 = 9$$

EXAMPLE 2
Find the center and radius of the circle whose equation is

$$(x + 1)^2 + (y - 3)^2 = 4$$

SOLUTION
Since the standard form is

$$(x - h)^2 + (y - k)^2 = r^2$$

we must have

$$x - h = x + 1 \qquad y - k = y - 3 \qquad r^2 = 4$$

Solving, we find that

$$h = -1 \qquad k = 3 \qquad r = 2$$

The center is at $(-1, 3)$ and the radius is 2.

PROGRESS CHECK

Find the center and radius of the circle whose equation is

$$\left(x - \frac{1}{2}\right)^2 + (y + 5)^2 = 15$$

ANSWER

center $\left(\frac{1}{2}, -5\right)$, radius $\sqrt{15}$

GENERAL FORM

When we are given the equation of a circle in the **general form**

$$Ax^2 + Ay^2 + Dx + Ey + F = 0, \qquad A \neq 0$$

in which the coefficients of x^2 and y^2 are the same, we may rewrite the equation in standard form. The process involves completing the square in each variable and is essentially the process we studied in Section 2.5. Recall that if we have the expression

$$x^2 + bx$$

we add $(b/2)^2$ to form

$$x^2 + bx + \frac{b^2}{4} = \left(x + \frac{b}{2}\right)^2$$

For example, starting with the expressions

$$x^2 + 4x \quad \text{and} \quad y^2 - 10y$$

we would complete the squares in this way:

$$x^2 + 4x + 4 = (x + 2)^2 \quad \text{and} \quad y^2 - 10y + 25 = (y - 5)^2$$

After completing the squares, we can write the equation in standard form, determine the center and radius, and easily sketch the graph.

EXAMPLE 3

Write the equation of the circle $2x^2 + 2y^2 - 12x + 16y - 31 = 0$ in standard form.

SOLUTION

Grouping the terms in x and y and factoring produces

$$2(x^2 - 6x) + 2(y^2 + 8y) = 31$$

Completing the square in both x and y, we have

$$\begin{aligned} 2(x^2 - 6x + 9) + 2(y^2 + 8y + 16) &= 31 + 18 + 32 \\ 2(x - 3)^2 + 2(y + 4)^2 &= 81 \end{aligned}$$

Note that the quantities 18 and 32 were added to the right-hand side because each factor is multiplied by 2. The last equation can be written as

$$(x - 3)^2 + (y + 4)^2 = \frac{81}{2}$$

This is the standard form of the equation of the circle with center at $(3, -4)$ and radius $9\sqrt{2}/2$.

PROGRESS CHECK

Write the equation of the circle $4x^2 + 4y^2 - 8x + 4y = 103$ in standard form, and determine the center and radius.

ANSWER

$(x - 1)^2 + \left(y + \frac{1}{2}\right)^2 = 27$, center $\left(1, -\frac{1}{2}\right)$, radius $\sqrt{27}$

EXAMPLE 4

Write the equation $3x^2 + 3y^2 - 6x + 15 = 0$ in standard form.

SOLUTION

Regrouping, we have

$$3(x^2 - 2x) + 3y^2 = -15$$

We then complete the square in x and y:

$$\begin{aligned} 3(x^2 - 2x + 1) + 3y^2 &= -15 + 3 \\ 3(x - 1)^2 + 3y^2 &= -12 \\ (x - 1)^2 + y^2 &= -4 \end{aligned}$$

Since $r^2 = -4$ is an impossible situation, the graph of the equation is not a circle. Note that the left-hand side of the equation in standard form is a sum of squares and is therefore nonnegative, while the right-hand side is negative. Thus, there are no real values of x and y that satisfy the equation. This is an example of an equation that does not have a graph!

PROGRESS CHECK

Write the equation $x^2 + y^2 - 12y + 36 = 0$ in standard form, and analyze its graph.

ANSWER

The standard form is $x^2 + (y - 6)^2 = 0$. The equation is that of a "circle" with center at (0, 6) and radius of 0. The "circle" is actually the point (0, 6).

EXAMPLE 5

Find an equation of the circle that has its center at $C(-1, 2)$ and that passes through the point $P(3, 4)$.

SOLUTION

Since the distance from the center to any point on the circle determines the radius, we can use the distance formula to find

$$r = \overline{PC} = \sqrt{20}$$

Then we can write the equation of the circle in standard form as

$$(x + 1)^2 + (y - 2)^2 = 20$$

EXERCISE SET 8.2

In Exercises 1–8 write an equation of the circle with center at (h, k) and radius r.

1. $(h, k) = (2, 3)$, $r = 2$
2. $(h, k) = (-3, 0)$, $r = 3$
3. $(h, k) = (-2, -3)$, $r = \sqrt{5}$
4. $(h, k) = (2, -4)$, $r = 4$
5. $(h, k) = (0, 0)$, $r = 3$
6. $(h, k) = (0, -3)$, $r = 2$
7. $(h, k) = (-1, 4)$, $r = 2\sqrt{2}$
8. $(h, k) = (2, 2)$, $r = 2$

In Exercises 9–16 find the center and radius of the circle with the given equation.

9. $(x - 2)^2 + (y - 3)^2 = 16$
10. $(x + 2)^2 + y^2 = 9$
11. $(x - 2)^2 + (y + 2)^2 = 4$
12. $\left(x + \frac{1}{2}\right)^2 + (y - 2)^2 = 8$
13. $(x + 4)^2 + \left(y + \frac{3}{2}\right)^2 = 18$
14. $x^2 + (y - 2)^2 = 4$
15. $\left(x - \frac{1}{3}\right)^2 + y^2 = \frac{1}{9}$
16. $(x - 1)^2 + \left(y - \frac{1}{2}\right)^2 = 3$

In Exercises 17–24 write the equation of the given circle in standard form and determine its center and radius.

17. $x^2 + y^2 + 4x - 8y + 4 = 0$
18. $x^2 + y^2 - 2x + 6y - 15 = 0$
19. $2x^2 + 2y^2 - 6x - 10y + 6 = 0$
20. $2x^2 + 2y^2 + 8x - 12y - 8 = 0$
21. $2x^2 + 2y^2 - 4x - 5 = 0$
22. $4x^2 + 4y^2 - 2y - 7 = 0$
23. $3x^2 + 3y^2 - 12x + 18y + 15 = 0$
24. $4x^2 + 4y^2 + 4x + 4y - 4 = 0$

In Exercises 25–36 write the given equation in standard form, and determine if the graph of the equation is a circle, a point, or neither.

25. $x^2 + y^2 - 6x + 8y + 7 = 0$
26. $x^2 + y^2 + 4x + 6y + 5 = 0$
27. $x^2 + y^2 + 3x - 5y + 7 = 0$
28. $x^2 + y^2 - 4x - 6y - 13 = 0$
29. $2x^2 + 2y^2 - 12x - 4 = 0$
30. $2x^2 + 2y^2 + 4x - 4y + 25 = 0$

31. $2x^2 + 2y^2 - 6x - 4y - 2 = 0$

32. $2x^2 + 2y^2 - 10y + 6 = 0$

33. $3x^2 + 3y^2 + 12x - 4y - 20 = 0$

34. $x^2 + y^2 + x + y = 0$

35. $4x^2 + 4y^2 + 12x - 20y + 38 = 0$

36. $4x^2 + 4y^2 - 12x - 36 = 0$

37. Find the area of the circle whose equation is
$$x^2 + y^2 - 2x + 4y - 4 = 0$$

38. Find the circumference of the circle whose equation is
$$x^2 + y^2 - 6x + 8 = 0$$

39. Show that the circles whose equations are
$$x^2 + y^2 - 4x + 9y - 3 = 0$$
and
$$3x^2 + 3y^2 - 12x + 27y - 27 = 0$$
are concentric.

40. Find an equation of the circle that has its center at $(3, -1)$ and that passes through the point $(-2, 2)$.

41. Find an equation of the circle that has its center at $(-5, 2)$ and that passes through the point $(-3, 4)$.

42. The two points $(-2, 4)$ and $(4, 2)$ are the endpoints of the diameter of a circle. Write the equation of the circle in standard form.

43. The two points $(3, 5)$ and $(7, -3)$ are the endpoints of the diameter of a circle. Write the equation of the circle in standard form.

8.3 THE PARABOLA

We begin our study of the parabola with the geometric definition.

Given a fixed point (called the **focus**) and a fixed line (called the **directrix**), a **parabola** is the set of all points each of which is equidistant from the point and from the line.

In Figure 5 each point P on the parabola is equidistant from the focus F and the directrix L, that is, $\overline{PF} = \overline{PQ}$. The line through the focus that is perpendicular to the directrix is called the **axis of the parabola** (or simply the **axis**), and the parabola is seen to be symmetric with respect to the axis. The point V (Figure 5), where the parabola intersects its axis, is called the **vertex** of the parabola. The vertex, then, is the point from which the parabola opens. Note that the vertex is the point on the parabola that is closest to the directrix.

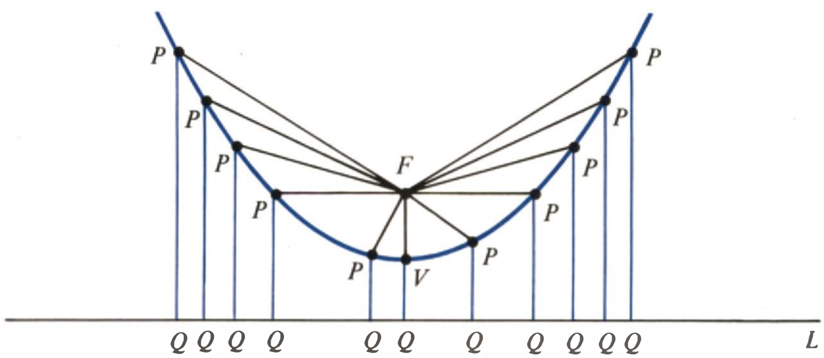


FIGURE 5

VERTEX AT ORIGIN

We can apply the methods of analytic geometry to find an equation of the parabola. We choose the y-axis as the axis of the parabola and the origin as the vertex (Figure 6). Since the vertex is on the parabola, it is equidistant from the focus and the directrix. Thus, if the coordinates of the focus F are $(0, p)$, then the equation of the directrix is $y = -p$. We then let $P(x, y)$ be any point on the parabola, and we equate the distance from P to the focus F and the distance from P to the directrix L. Using the distance formula, we find

$$\overline{PF} = \overline{PQ}$$
$$\sqrt{(x-0)^2 + (y-p)^2} = \sqrt{(x-x)^2 + (y+p)^2}$$

Squaring both sides, we have

$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2$$
$$x^2 = 4py$$

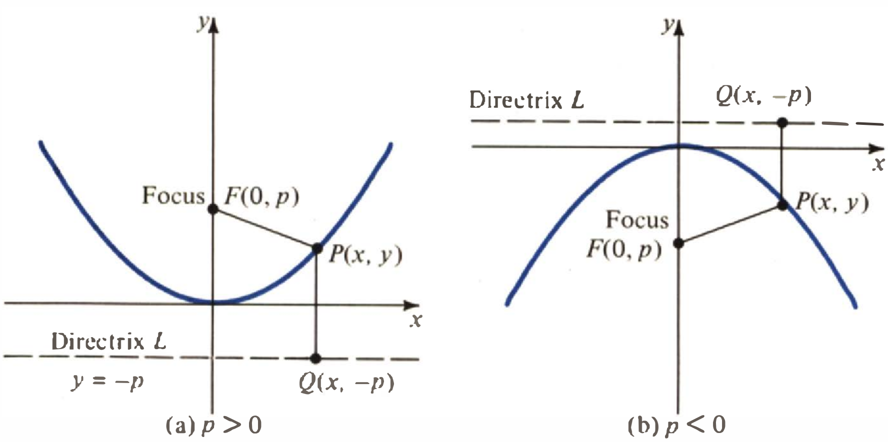


FIGURE 6

We have obtained an important form of the equation of a parabola.

Standard Form of the Equation of a Parabola

$$x^2 = 4py$$

is the standard form of the equation of a parabola whose vertex is at the origin, whose focus is at $(0, p)$, and whose axis is vertical.

Conversely, it can be shown that the graph of the equation $x^2 = 4py$ is a parabola. Note that substituting $-x$ for x leaves the equation unchanged, verifying symmetry with respect to the y-axis. If $p > 0$, the parabola opens upward as shown in Figure 6a; if $p < 0$, the parabola opens downward as shown in Figure 6b.

EXAMPLE 1

Determine the focus and directrix of the parabola $x^2 = 8y$, and sketch its graph.

SOLUTION

The equation of the parabola is of the form

$$x^2 = 4py = 8y$$

so $p = 2$. The equation of the directrix is $y = -p = -2$, and the focus is at $(0, p) = (0, 2)$. Since $p > 0$, the parabola opens upward. The graph of the parabola is shown in Figure 7.

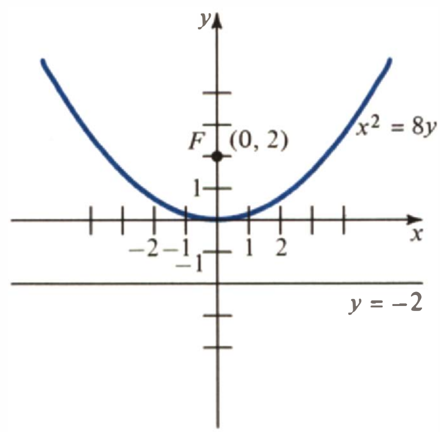


FIGURE 7

PROGRESS CHECK

Determine the focus and directrix of the parabola $x^2 = -3y$.

ANSWER

focus at $\left(0, -\dfrac{3}{4}\right)$, directrix $y = \dfrac{3}{4}$

EXAMPLE 2

Find the equation of the parabola with vertex at $(0, 0)$ and focus at $(0, -\frac{3}{2})$.

SOLUTION

Since the focus is at $(0, p)$, we have $p = -\frac{3}{2}$. The equation of the parabola is

$$x^2 = 4py = 4\left(-\frac{3}{2}y\right) = -6y$$

PROGRESS CHECK

Find the equation of the parabola with vertex at (0, 0) and focus at (0, 3).

ANSWER

$x^2 = 12y$

If we place the parabola as shown in Figure 8, we can proceed as above to obtain the following result.

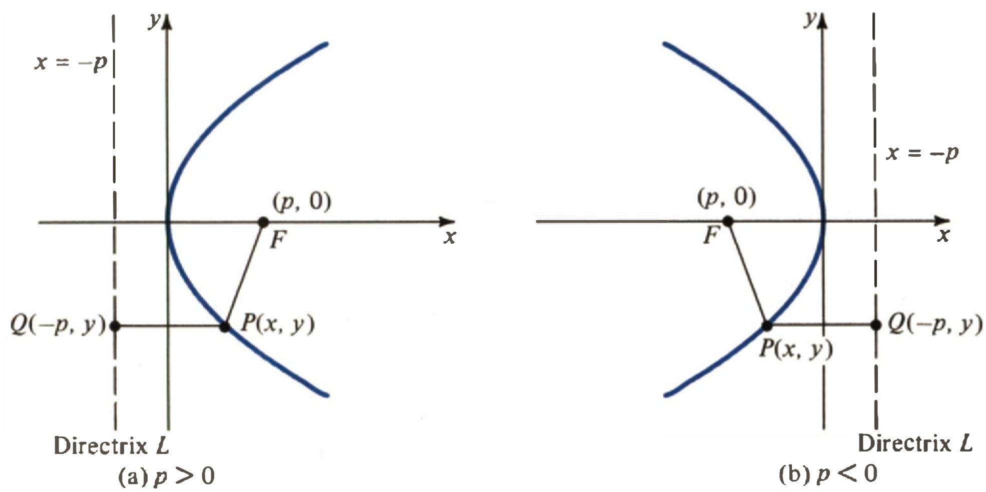


FIGURE 8

Standard Form of the Equation of a Parabola	$y^2 = 4px$ is the standard form of the equation of a parabola whose vertex is at the origin, whose focus is at $(p, 0)$, and whose axis is horizontal.

Note that substituting $-y$ for y leaves this equation unchanged, verifying symmetry with respect to the x-axis. If $p > 0$, the parabola opens to the right as shown in Figure 8a; if $p < 0$, the parabola opens to the left as shown in Figure 8b.

EXAMPLE 3

Find the equation of the parabola with vertex at (0, 0) and directrix $x = \frac{1}{2}$.

SOLUTION

The directrix is $x = -p$, so $p = -\frac{1}{2}$. The equation of the parabola is then

$$y^2 = 4px = -2x$$

EXAMPLE 4

Find the equation of the parabola that has the x-axis as its axis, that has vertex at (0, 0), and that passes through the point (−2, 3).

SOLUTION

Since the axis of the parabola is the x-axis, the equation of the parabola is $y^2 = 4px$. The parabola passes through the point (−2, 3), so the coordinates of this point must satisfy the equation of the parabola. Thus,

$$\begin{aligned} y^2 &= 4px \\ (3)^2 &= 4p(-2) \\ 4p &= -\frac{9}{2} \end{aligned}$$

and the equation of the parabola is

$$y^2 = 4px = -\frac{9}{2}x$$

PROGRESS CHECK

Find the equation of the parabola that has the y-axis as its axis, that has vertex at (0, 0), and that passes through the point (1, −2).

ANSWER

$x^2 = -\frac{1}{2}y$

VERTEX AT (*h, k*)

It is also possible to determine an equation of the parabola when the vertex is at some arbitrary point (h, k). The form of the equation depends on whether the axis of the parabola is parallel to the x-axis or to the y-axis. The situations are summarized in Table 1. Note that if the point (h, k) is the origin, then $h = k = 0$, and we arrive at the equations we derived previously, $x^2 = 4py$ and $y^2 = 4px$. Thus, in all cases, the sign of the constant p determines the direction in which the parabola opens. An equation of a parabola can always be written in the standard forms shown in Table 1 by completing the square, as in Example 6.

DEVICES WITH A PARABOLIC SHAPE

The properties of the parabola are used in the design of some important devices. For example, by rotating a parabola about its axis, we obtain a **parabolic reflector,** a shape used in the headlight of an automobile. In the accompanying figure, the light source (the bulb) is placed at the focus of the parabola. The headlight is coated with a reflecting material, and the rays of light bounce back in lines that are parallel to the axis of the parabola. This permits a headlight to disperse light in front of the auto where it is needed.

A reflecting telescope reverses the use of these same properties. Here, the rays of light from a distant star, which are nearly parallel to the axis of the parabola, are reflected by the mirror to the focus (see accompanying figure). The eyepiece is placed at the focus, where the rays of light are gathered.

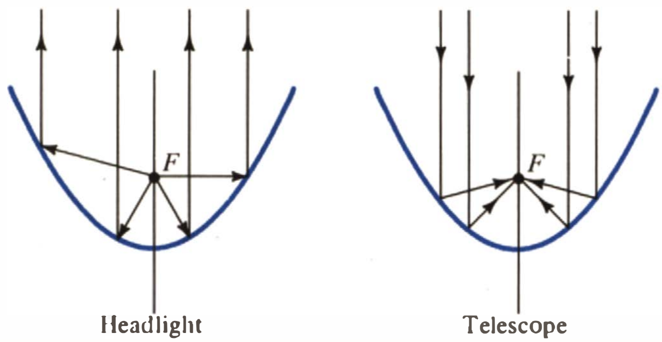


TABLE 1

Standard Forms of the Equations of the Parabola			
Equation	Vertex	Axis	Direction
$(x-h)^2 = 4p(y-k)$	(h, k)	$x = h$	Up if $p > 0$ Down if $p < 0$
$(y-k)^2 = 4p(x-h)$	(h, k)	$y = k$	Right if $p > 0$ Left if $p < 0$

Note that these changes in the equations of the parabola are similar to the change that occurs in the equation of the circle when the center is moved from the origin to a point (h, k). In both cases, x is replaced by $x - h$ and y is replaced by $y - k$.

EXAMPLE 5

Determine the vertex, axis, and direction of the graph of the parabola

$$\left(x - \frac{1}{2}\right)^2 = -3(y + 4)$$

SOLUTION

Comparison of the equation with the standard form

$$(x-h)^2 = 4p(y-k)$$

yields $h = \frac{1}{2}$, $k = -4$, $p = -\frac{3}{4}$. The axis of the parabola is always found by setting the square term equal to 0.

$$\left(x-\frac{1}{2}\right)^2 = 0$$

$$x = \frac{1}{2}$$

Thus, the vertex is at $(h, k) = (\frac{1}{2}, -4)$, the axis is $x = \frac{1}{2}$, and the parabola opens downward since $p < 0$.

PROGRESS CHECK

Determine the vertex, axis, and direction of the graph of the parabola

$$3(y+1)^2 = 12\left(x-\frac{1}{3}\right)$$

ANSWER

vertex $\left(\frac{1}{3}, -1\right)$, axis $y = -1$, opens to the right

---

EXAMPLE 6

Locate the vertex and the axis of symmetry of each of the given parabolas. Sketch the graph.

(a) $x^2 + 2x - 2y - 3 = 0$ (b) $y^2 - 4y + x + 1 = 0$

SOLUTION

(a) We complete the square in x:

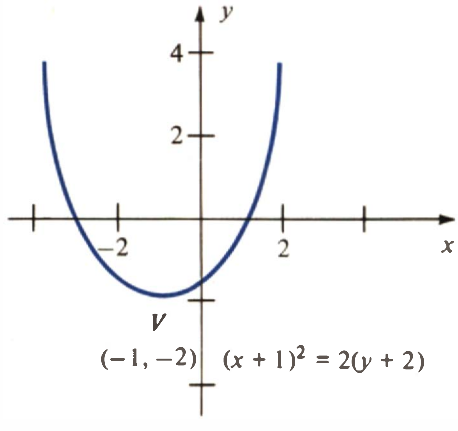
$$x^2 + 2x = 2y + 3$$
$$x^2 + 2x + 1 = 2y + 3 + 1$$
$$(x+1)^2 = 2(y+2)$$

The vertex of the parabola is at $(-1, -2)$; the axis is $x = -1$. See Figure 9a.

(b) We complete the square in y:

$$y^2 - 4y = -x - 1$$
$$y^2 - 4y + 4 = -x - 1 + 4$$
$$(y-2)^2 = -(x-3)$$

The vertex of the parabola is at $(3, 2)$; the axis is $y = 2$. See Figure 9b.



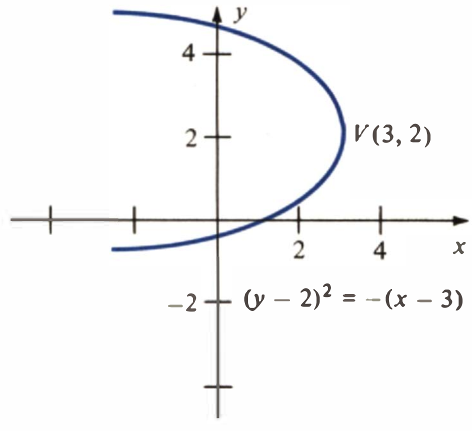


FIGURE 9

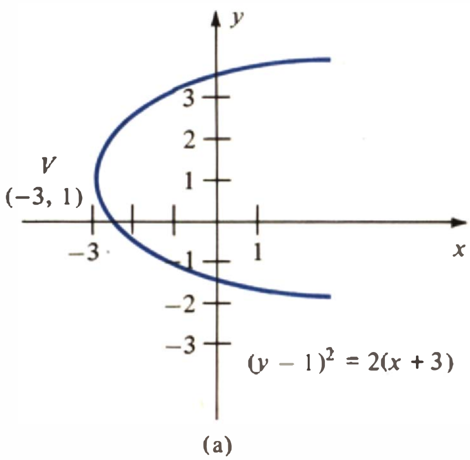
PROGRESS CHECK

Write the equation of the parabola in standard form. Locate the vertex and the axis, and sketch the graph.

(a) $y^2 - 2y - 2x - 5 = 0$ (b) $x^2 - 2x + 2y - 1 = 0$

ANSWERS

(a) $(y-1)^2 = 2(x+3)$, vertex $(-3, 1)$, axis $y = 1$, graph in Figure 10a
(b) $(x-1)^2 = -2(y-1)$, vertex $(1, 1)$, axis $x = 1$, graph in Figure 10b



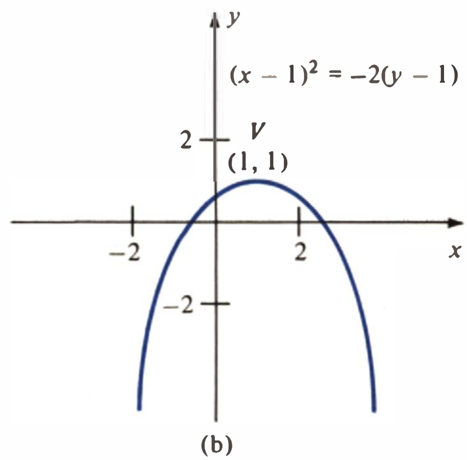


FIGURE 10

EXERCISE SET 8.3

In Exercises 1–8 determine the focus and directrix of the given parabola, and sketch the graph.

1. $x^2 = 4y$
2. $x^2 = -4y$
3. $y^2 = 2x$
4. $y^2 = -\frac{3}{2}x$
5. $x^2 + 5y = 0$
6. $2y^2 - 3x = 0$
7. $y^2 - 12x = 0$
8. $x^2 - 9y = 0$

In Exercises 9–20 determine the equation of the parabola that has its vertex at the origin and that satisfies the given conditions.

9. focus at $(1, 0)$.
10. focus at $(0, -3)$
11. directrix $x = -\frac{3}{2}$
12. directrix $y = \frac{5}{2}$
13. Axis is the x-axis, and parabola passes through the point $(2, 1)$.
14. Axis is the y-axis, and parabola passes through the point $(4, -2)$.
15. Axis is the x-axis, and $p = -\frac{5}{4}$.
16. Axis is the y-axis, and $p = 2$.
17. focus at $(-1, 0)$ and directrix $x = 1$
18. focus at $(0, -\frac{5}{2})$ and directrix $y = \frac{5}{2}$
19. Axis is the x-axis, and parabola passes through the point $(4, 2)$.
20. Axis is the y-axis, and parabola passes through the point $(2, 4)$.

In Exercises 21–34 write the equation in standard form. Determine the vertex, axis, and direction of each parabola.

21. $x^2 - 2x - 3y + 7 = 0$
22. $x^2 + 4x + 2y - 2 = 0$
23. $y^2 - 8y + 2x + 12 = 0$
24. $y^2 + 6y - 3x + 12 = 0$
25. $x^2 - x + 3y + 1 = 0$
26. $y^2 + 2y - 4x - 3 = 0$
27. $y^2 - 10y - 3x + 24 = 0$
28. $x^2 + 2x - 5y - 19 = 0$
29. $x^2 - 3x - 3y + 1 = 0$
30. $y^2 + 4y + x + 3 = 0$
31. $y^2 + 6y + \frac{1}{2}x + 7 = 0$
32. $x^2 + 2x - 3y + 19 = 0$
33. $x^2 + 2x + 2y + 3 = 0$
34. $y^2 - 6y + 2x + 17 = 0$

In Exercises 35–40 determine the vertex, axis, and direction of each parabola. Sketch the graph.

35. $x^2 - 4x - 2y + 2 = 0$
36. $y^2 + 2x - 4y + 6 = 0$
37. $2x^2 + 16x + y + 34 = 0$
38. $2x^2 - y + 3 = 0$
39. $y^2 + 2x + 2 = 0$
40. $y^2 + 3x - 2y - 5 = 0$

8.4 THE ELLIPSE AND HYPERBOLA

THE ELLIPSE

The geometric definition of an ellipse is as follows.

> Given two fixed points (called **foci**), an **ellipse** is the set of all points for which the sum of the distances from the fixed points is a constant.

The ellipse is in standard position if the two fixed points are on either the x-axis or the y-axis and are equidistant from the origin. Thus, if F_1 and F_2 are the foci of the ellipse in Figure 11 and P and Q are points on the ellipse, then

$$\overline{F_1P} + \overline{F_2P} = c \quad \text{and} \quad \overline{F_1Q} + \overline{F_2Q} = c$$

where c is a constant.

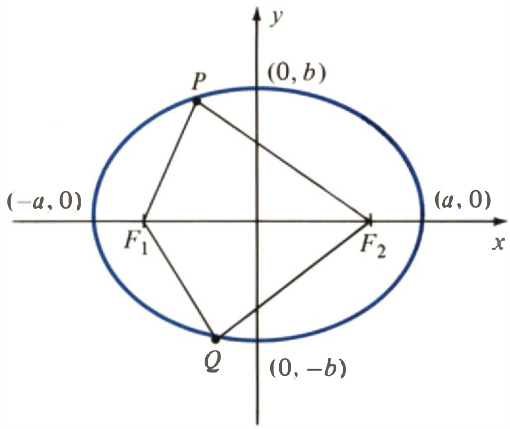


FIGURE 11

The equation of an ellipse in standard position can be shown to be as follows (see Exercise 35).

Standard Form of the Equation of an Ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Note that the equation indicates that the graph will be symmetric with respect to the x-axis, the y-axis, and the origin.

If we let $x = 0$ in the standard form, we find $y = \pm b$; if we let $y = 0$, we find $x = \pm a$. Thus, the ellipse whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

has intercepts $(\pm a, 0)$ and $(0, \pm b)$.

EXAMPLE 1

Find the intercepts and sketch the graph of the ellipse whose equation is

$$\frac{x^2}{4}+\frac{y^2}{25}=1$$

SOLUTION

Setting $x = 0$ and solving for y yields the y-intercepts ± 5; setting $y = 0$ and solving for x yields the x-intercepts ± 2. The graph is then easily sketched as in Figure 12.

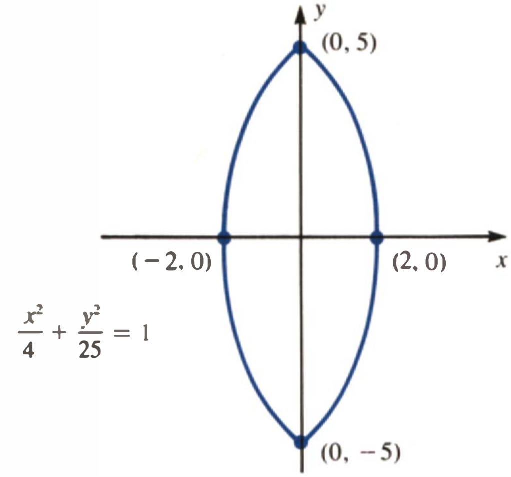


FIGURE 12

EXAMPLE 2

Write the equation of the ellipse in standard form and determine the intercepts.

(a) $4x^2 + 3y^2 = 12$ (b) $9x^2 + y^2 = 10$

SOLUTION

(a) Dividing by 12 to make the right-hand side equal to 1, we have

$$\frac{x^2}{3} + \frac{y^2}{4} = 1$$

The x-intercepts are $(\pm\sqrt{3}, 0)$; the y-intercepts are $(0, \pm 2)$.

(b) Dividing by 10 we have

$$\frac{9x^2}{10} + \frac{y^2}{10} = 1$$

But this is *not* standard form. However, if we write

$$\frac{9x^2}{10} \quad \text{as} \quad \frac{x^2}{\frac{10}{9}}$$

then

$$\frac{x^2}{\frac{10}{9}} + \frac{y^2}{10} = 1$$

is the standard form of an ellipse. The intercepts are

$$\left(\frac{\pm\sqrt{10}}{3}, 0\right) \quad \text{and} \quad (0, \pm\sqrt{10})$$

PROGRESS CHECK

Write the equation of each ellipse in standard form and determine the intercepts.

(a) $2x^2 + 3y^2 = 6$ (b) $3x^2 + y^2 = 5$

ANSWERS

(a) $\frac{x^2}{3} + \frac{y^2}{2} = 1$; $(\pm\sqrt{3}, 0)$, $(0, \pm\sqrt{2})$

(b) $\frac{x^2}{\frac{5}{3}} + \frac{y^2}{5} = 1$; $\left(\frac{\pm\sqrt{15}}{3}, 0\right)$, $(0, \pm\sqrt{5})$

THE HYPERBOLA

The hyperbola is the last of the conic sections we will study.

Given two fixed points (called **foci**), a **hyperbola** is the set of all points for which the difference of the distances from the two fixed points is a constant.

The hyperbola is in standard position if the two fixed points are on either the x-axis or the y-axis and are equidistant from the origin. The equations of the hyperbolas in standard position can be shown to be as follows (see Exercise 36).

Standard Forms of the Equation of a Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{foci on the } x\text{-axis} \tag{1}$$

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad \text{foci on the } y\text{-axis} \tag{2}$$

These equations indicate that the graphs are symmetric with respect to the x-axis, the y-axis, and the origin.

Letting $y = 0$, we see that the x-intercepts of the graph of Equation (1) are $x = \pm a$. Letting $x = 0$, we find there are no y-intercepts since the equation $y^2 = -b^2$ has no real roots (Figure 13a). Similarly, the graph of Equation (2) has y-intercepts of $\pm a$ and no x-intercepts (Figure 13b).

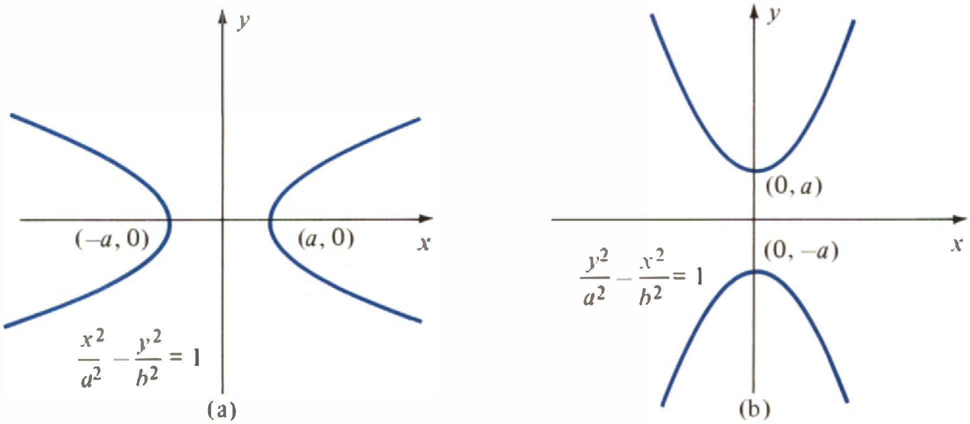


FIGURE 13

EXAMPLE 3

Find the intercepts and sketch the graph of each equation.

(a) $\frac{x^2}{9} - \frac{y^2}{4} = 1$ (b) $\frac{y^2}{4} - \frac{x^2}{3} = 1$

SOLUTION

(a) When $y = 0$, we have $x^2 = 9$ or $x = \pm 3$. The intercepts are (3, 0) and $(-3, 0)$. With the assistance of a few plotted points, we can sketch the graph (Figure 14a).

(b) When $x = 0$, we have $y^2 = 4$ or $y = \pm 2$. The intercepts are (0, 2) and $(0, -2)$. Plotting a few points, we can sketch the graph (Figure 14b).

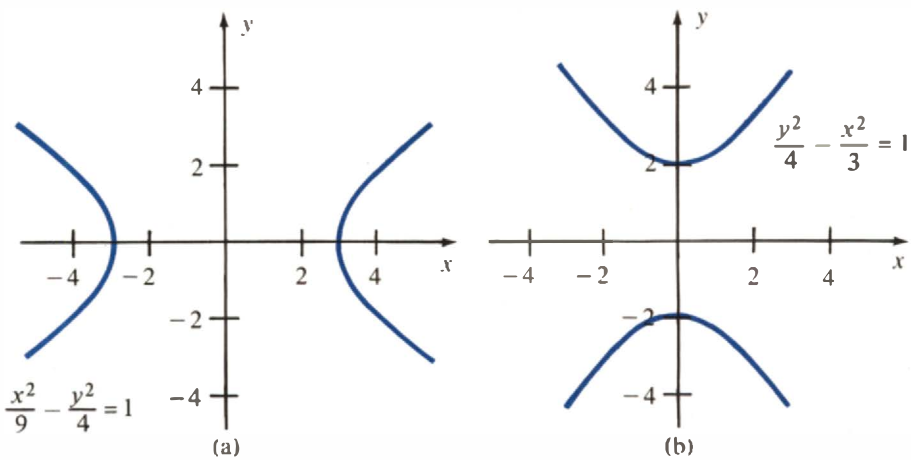


FIGURE 14

EXAMPLE 4

Write the equation of the hyperbola $9x^2 - 5y^2 = 10$ in standard form and determine the intercepts.

SOLUTION

Dividing by 10, we have

$$\frac{9x^2}{10} - \frac{y^2}{2} = 1$$

Rewriting the equation in standard form, we have

$$\frac{x^2}{\frac{10}{9}} - \frac{y^2}{2} = 1$$

The x-intercepts are $\pm\sqrt{10}/3$; there are no y-intercepts.

PROGRESS CHECK

Write the equation of the hyperbola in standard form and determine the intercepts.

(a) $2x^2 - 5y^2 = 6$ (b) $4y^2 - x^2 = 5$

ANSWERS

(a) $\frac{x^2}{3} - \frac{y^2}{\frac{6}{5}} = 1$; $(\pm\sqrt{3}, 0)$ (b) $\frac{y^2}{\frac{5}{4}} - \frac{x^2}{5} = 1$; $\left(0, \frac{\pm\sqrt{5}}{2}\right)$

ASYMPTOTES OF THE HYPERBOLA

There is a way of sketching the graph of a hyperbola without having to plot any points on the curve itself. Given the equation of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

in standard form, we plot the four points $(a, \pm b)$, $(-a, \pm b)$ as in Figure 15 and draw the diagonals of the rectangle formed by the four points. The hyperbola opens from the intercepts $(\pm a, 0)$ and *approaches the lines formed by the diagonals of the rectangle*. We call these lines the **asymptotes** of the hyperbola. Since one asymptote passes through the points $(0, 0)$ and (a, b), its equation is

$$y = \frac{b}{a}x$$

Similarly, the equation of the other asymptote is found to be

$$y = -\frac{b}{a}x$$

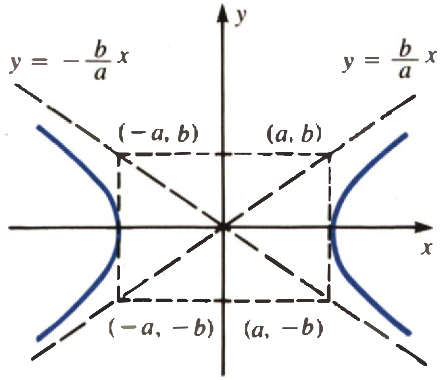


FIGURE 15

Of course, a similar discussion can be carried out for the standard form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

In this case, the four points $(\pm b, \pm a)$ determine the rectangle, and the equations of the asymptotes are

$$y = \pm\frac{a}{b}x$$

EXAMPLE 5
Using asymptotes, sketch the graph of the equation

$$\frac{y^2}{4} - \frac{x^2}{9} = 1$$

SOLUTION
The points $(\pm 3, \pm 2)$ form the vertices of the rectangle. See Figure 16. Using the fact that $(0, \pm 2)$ are intercepts, we can sketch the graph opening from these points and approaching the asymptotes.

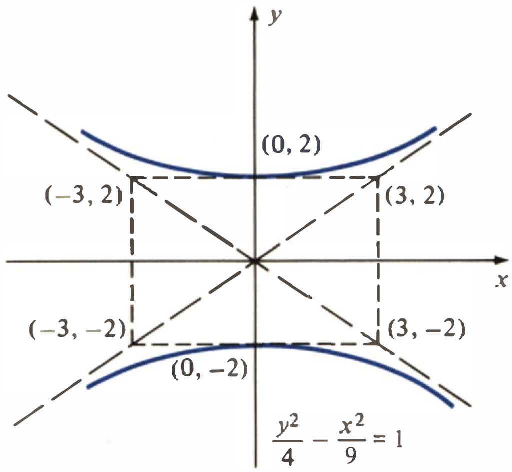


FIGURE 16

EXERCISE SET 8.4

In Exercises 1–6 find the intercepts and sketch the graph of the ellipse.

1. $\frac{x^2}{25} + \frac{y^2}{4} = 1$
2. $\frac{x^2}{4} + \frac{y^2}{16} = 1$
3. $\frac{x^2}{8} + \frac{y^2}{4} = 1$
4. $\frac{x^2}{12} + \frac{y^2}{18} = 1$
5. $\frac{x^2}{16} + \frac{y^2}{25} = 1$
6. $\frac{x^2}{1} + \frac{y^2}{3} = 1$

In Exercises 7–16 write the equation of the ellipse in standard form and determine the intercepts.

7. $4x^2 + 9y^2 = 36$
8. $16x^2 + 9y^2 = 144$
9. $4x^2 + 16y^2 = 16$
10. $25x^2 + 4y^2 = 100$
11. $4x^2 + 16y^2 = 4$
12. $8x^2 + 4y^2 = 32$
13. $8x^2 + 6y^2 = 24$
14. $5x^2 + 6y^2 = 50$
15. $36x^2 + 8y^2 = 9$
16. $5x^2 + 4y^2 = 45$

In Exercises 17–22 find the intercepts and sketch the graph of the hyperbola.

17. $\frac{x^2}{25} - \frac{y^2}{16} = -1$
18. $\frac{y^2}{9} - \frac{x^2}{4} = 1$
19. $\frac{x^2}{36} - \frac{y^2}{1} = 1$
20. $\frac{y^2}{49} - \frac{x^2}{25} = 1$
21. $\frac{x^2}{6} - \frac{y^2}{8} = 1$
22. $\frac{y^2}{8} - \frac{x^2}{10} = -1$

In Exercises 23–28 write the equation of the hyperbola in standard form and determine the intercepts.

23. $16x^2 - y^2 = 64$
24. $4x^2 - 25y^2 = 100$
25. $4y^2 - 4x^2 = 1$
26. $2x^2 - 3y^2 = 6$
27. $4x^2 - 5y^2 = 20$
28. $25y^2 - 16x^2 = 400$

In Exercises 29–34 use the asymptotes and intercepts to sketch the graph of the hyperbola.

29. $16x^2 - 9y^2 = 144$
30. $16y^2 - 25x^2 = 400$
31. $y^2 - x^2 = 9$
32. $25x^2 - 9y^2 = 225$
33. $\frac{x^2}{25} - \frac{y^2}{36} = 1$
34. $y^2 - 4x^2 = 4$

35. Derive the standard form of the equation of the ellipse from the geometric definition of an ellipse. (*Hint:* In Figure 11, let $P(x, y)$ be any point on the ellipse and let $F_1(-c, 0)$ and $F_2(c, 0)$ be the foci. Note that the point $B(a, 0)$ lies on the ellipse and that $\overline{BF_1} + \overline{BF_2} = 2a$. Thus, the sum of the distances $\overline{PF_1} + \overline{PF_2}$ must also equal $2a$. Use the distance formula, simplify, and substitute $b^2 = a^2 - c^2$.)

36. Derive the standard form of the equation of the hyperbola from the geometric definition of a hyperbola. (*Hint:* Proceed in a manner similar to that of Exercise 35.)

8.5 IDENTIFYING THE CONIC SECTIONS

Each of the conic sections we have studied in this chapter has at least one axis of symmetry. We studied circles and parabolas whose axes of symmetry were the coordinate axes or lines parallel to them. Although the only ellipses and hyperbolas we have studied are ones that have the coordinate axes as their axes of symmetry, the method of completing the square, which we used for the circle and the parabola, allows us to transform the **general equation of a conic section**

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

into standard form. This transformation is very helpful in sketching the graph of the conic. Identifying the conic section from the general equation is also easy (see Table 2).

TABLE 2

$Ax^2 + Cy^2 + Dx + Ey + F = 0$	Conic Section	Remarks
$A = 0$ or $C = 0$	Parabola	Second degree in one variable, first degree in the other.
$A = C\ (\neq 0)$	Circle	Coefficients A and C are the same. *Caution:* Complete the square to obtain the standard form and check that radius $r > 0$.
$A \neq C$ $AC > 0$	Ellipse	A and C are unequal but have the same sign. *Caution:* Complete the square and check that the right-hand side is a positive constant.
$AC < 0$	Hyperbola	A and C have opposite signs.

EXAMPLE 1
Identify the conic section.

(a) $3x^2 + 3y^2 - 2y = 4$ (b) $3x^2 - 9y^2 + 2x - 4y = 7$

(c) $2x^2 + 3y^2 - 2x + 6y + \frac{1}{2} = 0$ (d) $3y^2 - 4x + 17y = -10$

SOLUTION
(a) Since the coefficients of x^2 and y^2 are the same, the graph will be a circle if the standard form yields $r > 0$. Completing the square, we have

$$3x^2 + 3\left(y - \frac{1}{3}\right)^2 = \frac{13}{3}$$

which is the equation of a circle.

(b) Since the coefficients of x^2 and y^2 are of opposite sign, the graph is a hyperbola.

(c) The coefficients of x^2 and y^2 are unequal but of like sign. We must complete the square in both x and y to obtain standard form.

$$2(x^2 - x) + 3(y^2 + 2y) = -\frac{1}{2}$$

$$2\left(x^2 - x + \frac{1}{4}\right) + 3(y^2 + 2y + 1) = -\frac{1}{2} + \frac{1}{2} + 3$$

$$2\left(x - \frac{1}{2}\right)^2 + 3(y + 1)^2 = 3$$

Since the right-hand side is positive, the graph is an ellipse.

(d) The graph is a parabola since the equation is of the second degree in y and of the first degree in x.

PROGRESS CHECK
Identify the conic section.

(a) $\frac{x^2}{5} - 3y^2 - 2x + 2y - 4 = 0$ (b) $x^2 - 2y - 3x = 2$

(c) $x^2 + y^2 - 4x - 6y = -11$ (d) $4x^2 + 3y^2 + 6x - 10 = 0$

ANSWERS
(a) hyperbola (b) parabola (c) circle (d) ellipse

A summary of the characteristics of the conic sections is given in Table 3.

TABLE 3

Curve and Standard Equation	Characteristics	Example
Circle $(x-h)^2+(y-k)^2=r^2$	Center: (h, k) Radius: r	$(x-2)^2+(y+4)^2=25$ Center: $(2, -4)$ Radius: 5
Parabola $(x-h)^2=4p(y-k)$	Vertex: (h, k) Axis: $x=h$ $p>0$: Opens up $p<0$: Opens down	$(x+1)^2=2(y-3)$ Vertex: $(-1, 3)$ Axis: $x=-1$ Opens up
or		
$(y-k)^2=4p(x-h)$	Vertex: (h, k) Axis: $y=k$ $p>0$: Opens right $p<0$: Opens left	$(y+4)^2=-3(x+5)$ Vertex: $(-5, -4)$ Axis: $y=-4$ Opens left
Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	Intercepts: $(\pm a, 0)$, $(0, \pm b)$	$\frac{x^2}{4}+\frac{y^2}{6}=1$ Intercepts: $(\pm 2, 0)$, $(0, \pm\sqrt{6})$
Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	Intercepts: $(\pm a, 0)$ Asymptotes: $y=\pm\frac{b}{a}x$ Opens to left and right	$\frac{x^2}{4}-\frac{y^2}{9}=1$ Intercepts: $(\pm 2, 0)$ Asymptotes: $y=\pm\frac{3}{2}x$ Opens to left and right
or		
$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$	Intercepts: $(0, \pm a)$ Asymptotes: $y=\pm\frac{a}{b}x$ Opens up and down	$\frac{y^2}{9}-\frac{x^2}{4}=1$ Intercepts: $(0, \pm 3)$ Asymptotes: $y=\pm\frac{3}{2}x$ Opens up and down

EXERCISE SET 8.5

Identify the conic section.

1. $2x^2+y-x+3=0$
2. $4y^2-x^2+2x-3y+5=0$
3. $4x^2+4y^2-2x+3y-4=0$
4. $3x^2+6y^2-2x+8=0$
5. $36x^2-4y^2+x-y+2=0$
6. $x^2+y^2-6x+4y+13=0$
7. $16x^2+4y^2-2y+3=0$
8. $2y^2-3x+y+4=0$

9. $x^2 + y^2 - 4x - 2y + 8 = 0$

10. $x^2 + y^2 - 2x - 2y + 6 = 0$

11. $4x^2 + 9y^2 - x + 2 = 0$

12. $3x^2 + 3y^2 - 3x + y = 0$

13. $4x^2 - 9y^2 + 2x + y + 3 = 0$

14. $x^2 + y^2 + 6x - 2y + 10 = 0$

15. $x^2 + y^2 - 4x + 4 = 0$

16. $4x^2 + y^2 = 32$

TERMS AND SYMBOLS

analytic geometry (p. 343)
midpoint formula (p. 344)
conic sections (p. 347)
circle (p. 347)
center of a circle (p. 347)
radius of a circle (p. 347)
standard form of the equation of a circle (p. 348)
general form of the equation of a circle (p. 349)
parabola (p. 352)
focus (p. 352)
directrix (p. 352)
axis of a parabola (p. 352)
vertex (p. 352)
parabolic reflector (p. 357)
standard forms of the equation of a parabola (p. 357)
ellipse (p. 360)
foci of an ellipse (p. 360)
standard form of the equation of an ellipse (p. 361)
hyperbola (p. 363)
foci of a hyperbola (p. 363)
standard forms of the equation of a hyperbola (p. 363)
asymptotes of a hyperbola (p. 365)
general equation of a conic section (p. 368)

KEY IDEAS FOR REVIEW

- ☐ The midpoint of the line segment joining the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ has coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

- ☐ Theorems from plane geometry can be proved using the methods of analytic geometry. In general, place the given geometric figure in a convenient position relative to the origin and axes. The distance formula, the midpoint formula, and the computation of slope are the tools to apply in proving a theorem.
- ☐ The conic sections represent the possible intersections of a plane and a cone. The conic sections are the circle, parabola, ellipse, and hyperbola. (In special cases these may be reduced to a point, a line, or two lines.)
- ☐ Each conic section has a geometric definition, which can be used to derive a second-degree equation in two variables whose graph corresponds to the conic.
- ☐ A second-degree equation in x and y can be converted from general form to standard form by completing the square in each variable. It is much simpler to sketch the graph of an equation when it is written in standard form.
- ☐ It is often possible to distinguish the various conic sections even when the equation is given in general form.

REVIEW EXERCISES

Solutions to exercises whose numbers are in color are in the Solutions section in the back of the book.

8.1 In Exercises 1–3 find the midpoint of the line segment whose endpoints are given.

1. $(-5, 4), (3, -6)$
2. $(-2, 0), (-3, 5)$
3. $(2, -7), (-3, -2)$
4. Find the coordinates of the point P_2 if $(2, 2)$ are the coordinates of the midpoint of the line segment joining $P_1(-6, -3)$ and P_2.
5. Use the distance formula to show that $P_1(-1, 2)$, $P_2(4, 3)$, $P_3(1, -1)$, and $P_4(-4, -2)$ are the coordinates of a parallelogram.
6. Show that the points $A(-8, 4)$, $B(5, 3)$, and $C(2, -2)$ are the vertices of a right triangle.
7. Find an equation of the perpendicular bisector of the line segment joining the points $A(-4, -3)$ and $B(1, 3)$. (The perpendicular bisector passes through the midpoint of AB and is perpendicular to AB.)

8.2 8. Write an equation of the circle with center at $(-5, 2)$ and a radius of 4.

9. Write an equation of the circle with center at $(-3, -3)$ and a radius of 2.

In Exercises 10–15 determine the center and radius of the circle with the given equation.

10. $(x-2)^2 + (y+3)^2 = 9$

11. $\left(x+\frac{1}{2}\right)^2 + (y-4)^2 = \frac{1}{9}$

12. $x^2 + y^2 + 4x - 6y = -10$

13. $2x^2 + 2y^2 - 4x + 4y = -3$

14. $x^2 + y^2 - 6y + 3 = 0$

15. $x^2 + y^2 - 2x - 2y = 8$

8.3 In Exercises 16 and 17 determine the vertex and axis of the given parabola. Sketch the graph.

16. $(y+5)^2 = 4\left(x - \frac{3}{2}\right)$

17. $(x-1)^2 = 2 - y$

In Exercises 18–23 determine the vertex, axis, and direction of the given parabola.

18. $y^2 + 3x + 9 = 0$

19. $y^2 + 4y + x + 2 = 0$

20. $2x^2 - 12x - y + 16 = 0$

21. $x^2 + 4x + 2y + 5 = 0$

22. $y^2 - 2y - 4x + 1 = 0$

23. $x^2 + 6x + 4y + 9 = 0$

8.4 In Exercises 24–29 write the given equation in standard form and determine the intercepts.

24. $9x^2 - 4y^2 = 36$

25. $9x^2 + y^2 = 9$

26. $5x^2 + 7y^2 = 35$

27. $9x^2 - 16y^2 = 144$

28. $3x^2 + 4y^2 = 9$

29. $3y^2 - 5x^2 = 20$

In Exercises 30 and 31 use the intercepts and asymptotes of the hyperbola to sketch the graph.

30. $4x^2 - 4y^2 = 1$

31. $9y^2 - 4x^2 = 36$

8.5 In Exercises 32–35 identify the conic section.

32. $2y^2 + 6y - 3x + 2 = 0$

33. $6x^2 - 7y^2 - 5x + 6y = 0$

34. $2x^2 + y^2 + 12x - 2y + 17 = 0$

35. $9x^2 + 4y^2 = -36$

PROGRESS TEST 8A

1. Find the midpoint of the line segment whose endpoints are $(2, 4)$ and $(-2, 4)$.
2. Find the coordinates of the point P if $(-3, 3)$ are the coordinates of the midpoint of the line segment joining P and $Q(-5, 4)$.
3. By computing slopes, show that the points $A(-3, -1)$, $B(-5, 4)$, $C(2, 6)$, and $D(4, 1)$ determine a parallelogram.
4. Write an equation of the circle of radius 6 whose center is at $(2, -3)$.

In Problems 5 and 6 determine the center and radius of the circle.

5. $x^2 + y^2 - 2x + 4y = -1$
6. $x^2 - 4x + y^2 = 1$

In Problems 7 and 8 determine the vertex and axis of the parabola. Sketch the graph.

7. $x^2 + 6x + 2y + 7 = 0$
8. $y^2 - 4x - 4y + 8 = 0$

In Problems 9 and 10 determine the vertex, axis, and direction of the parabola.

9. $x^2 - 6x + 2y + 5 = 0$
10. $y^2 + 8y - x + 14 = 0$

In Problems 11–13 write the given equation in standard form and determine the intercepts.

11. $x^2 + 4y^2 = 4$
12. $4y^2 - 9x^2 = 36$
13. $4x^2 - 4y^2 = 1$
14. Use the intercepts and asymptotes of the hyperbola $9x^2 - y^2 = 9$ to sketch its graph.

In Problems 15 and 16 identify the conic section.

15. $x^2 + y^2 + 2x - 2y - 2 = 0$
16. $x^2 + 9y^2 - 4x + 6y + 4 = 0$

PROGRESS TEST 8B

1. Find the midpoint of the line segment whose endpoints are $(-5, -3)$ and $(4, 1)$.
2. Find the coordinates of the point A if $(-2, -\frac{1}{2})$ are the coordinates of the midpoint of the line segment joining A and $B(3, -2)$.

3. Show that the diagonals of the quadrilateral whose vertices are $P(-3, 1)$, $Q(-1, 4)$, $R(5, 0)$, and $S(3, -3)$ are equal.
4. Write an equation of the circle of radius 5 whose center is at $(-2, -5)$.

In Problems 5 and 6 determine the center and radius of the circle.

5. $x^2 + y^2 + 6x - 4y = -4$
6. $4x^2 + 4y^2 - 4x - 8y = 35$

In Problems 7 and 8 determine the vertex and axis of the parabola. Sketch the graph.

7. $2y^2 - 8y - x + 30 = 0$
8. $9x^2 + 18y - 6x + 7 = 0$

In Problems 9 and 10 determine the vertex, axis, and direction of the parabola.

9. $y^2 - 4y - 3x + 1 = 0$
10. $4x^2 - 4x - 8y - 23 = 0$

In Problems 11–13 write the given equation in standard form and determine the intercepts.

11. $5x^2 + 9y^2 = 25$
12. $7x^2 + 6y^2 = 21$
13. $y^2 - 3x^2 = 9$
14. Use the intercepts and asymptotes of the hyperbola $4y^2 - x^2 = 1$ to sketch its graph.

In Problems 15 and 16 identify the conic section.

15. $5y^2 - 4x^2 - 6x + 2 = 0$
16. $3x^2 - 5x + 6y = 3$

9

SYSTEMS OF EQUATIONS AND INEQUALITIES

Many problems in business and engineering require the solution of systems of equations and inequalities. In fact, systems of linear equations and inequalities occur with such frequency that mathematicians and computer scientists have devoted considerable energy to devising methods for their solution. With the aid of large-scale computers it is possible to solve systems involving thousands of equations or inequalities, a task that previous generations would not have dared tackle.

We begin with the study of the methods of substitution and elimination, methods that are applicable to all types of systems. We then introduce graphical methods for solving systems of linear inequalities and apply these techniques to linear programming problems, a type of optimization problem.

9.1 SYSTEMS OF EQUATIONS

A pile of 9 coins consists of nickels and quarters. If the total value of the coins is $1.25, how many of each type of coin are there?

This type of word problem was handled in earlier chapters by using one variable. A more natural way to approach this problem is to let

$$x = \text{the number of nickels}$$

and

$$y = \text{the number of quarters}$$

that is, to use two variables. The requirements can then be expressed as

$$\begin{aligned} x + \quad y &= \quad 9 \\ 5x + 25y &= 125 \end{aligned}$$

This is an example of a **system of equations,** and we seek values of x and y that satisfy *both* equations. An ordered pair (a, b) such that $x = a$, $y = b$ satisfies both equations is called a **solution** of the system. Thus,

$$x = 5 \qquad y = 4$$

is a solution because substituting in the equations of the system gives

$$\begin{aligned} 5 + \quad 4 &= \quad 9 \\ 5(5) + 25(4) &= 125 \end{aligned}$$

SOLVING BY GRAPHING

The coordinates of every point on the graph of an equation must satisfy the equation. If we sketch the graphs of a pair of equations on the same coordinate axes, it follows that the *points of intersection* must satisfy *both* equations. Thus we have a graphical means of solving a system of equations.

EXAMPLE 1

Solve the system of equations by graphing.

$$\begin{aligned} x^2 + y^2 &= 25 \\ x + y &= -1 \end{aligned}$$

SOLUTION

The graphs of the equations are a circle and a line, as shown in Figure 1. The points of intersection are seen to be $(-4, 3)$ and $(3, -4)$. The solutions of the system are $x = -4$, $y = 3$ and $x = 3$, $y = -4$.

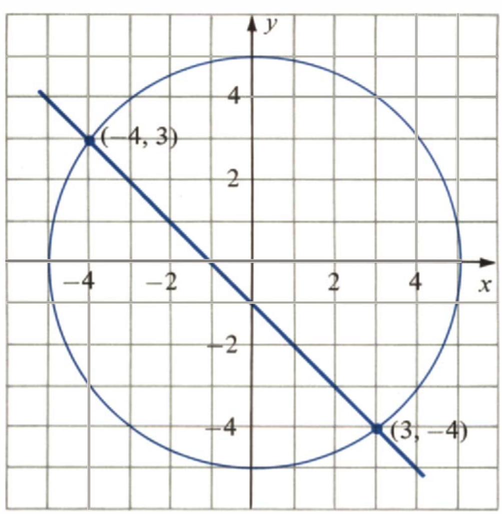


FIGURE 1

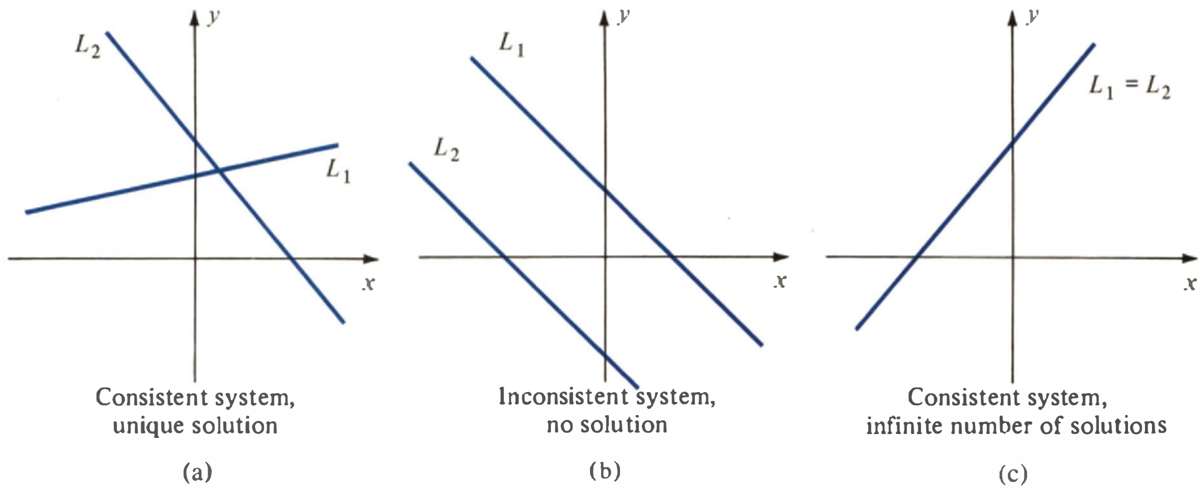


FIGURE 2

It is possible for a system of equations to have no solutions. Surprisingly, a system of equations may even have an infinite number of solutions. The following terminology is used to distinguish these situations.

Consistent and Inconsistent Systems

- A **consistent** system of equations has one or more solutions.
- An **inconsistent** system of equations has no solutions.

A system consisting only of equations that are of the first degree in x and y is called a **system of linear equations** or simply a **linear system.** When we graph a linear system of two equations on the same set of coordinate axes, there are three possibilities:

1. The two lines intersect at a point (Figure 2a). The system is consistent and has a unique solution, the point of intersection.

2. The two lines are parallel (Figure 2b). Since the lines do not intersect, the linear system is inconsistent.

3. The equations are different forms of the same line (Figure 2c). The system is consistent and has an infinite number of solutions, namely, all points on the line.

The method of graphing has severe limitations since the accuracy of the solution depends on the accuracy of the graph. The algebraic methods that follow avoid this limitation.

SOLVING BY SUBSTITUTION

If we can use one of the equations of a system to express one variable in terms of the other variable, then we can *substitute* this expression in the other equation.

EXAMPLE 2

Solve the system of equations.

$$\begin{aligned} x^2 + y^2 &= 25 \\ x + y &= -1 \end{aligned}$$

SOLUTION

From the second equation, we have

$$y = -1 - x$$

Substituting for y in the first equation, we have

$$\begin{aligned} x^2 + (-1 - x)^2 &= 25 \\ x^2 + 1 + 2x + x^2 &= 25 \\ 2x^2 + 2x - 24 &= 0 \\ x^2 + x - 12 &= 0 \\ (x + 4)(x - 3) &= 0 \end{aligned}$$

which yields $x = -4$ and $x = 3$. Substituting these values for x in the equation $x + y = -1$, we obtain the corresponding values of y.

$$x = -4:\quad \begin{aligned} -4 + y &= -1 \\ y &= 3 \end{aligned} \qquad\qquad x = 3:\quad \begin{aligned} 3 + y &= -1 \\ y &= -4 \end{aligned}$$

The solutions are the same as those obtained when we solved this same system by graphing (Example 1).

PROGRESS CHECK

Solve the system of equations.

(a) $\begin{aligned} x^2 + 3y^2 &= 12 \\ x + 3y &= 6 \end{aligned}$ (b) $\begin{aligned} x^2 + y^2 &= 34 \\ x - y &= 2 \end{aligned}$

ANSWERS

(a) $x = 3,\ y = 1;\ x = 0,\ y = 2$

(b) $x = -3,\ y = -5;\ x = 5,\ y = 3$

WARNING The expression for x or y obtained from an equation *must not be substituted in the same equation.* From the first equation of the system

$$\begin{aligned} x + 2y &= -1 \\ 3x^2 + y &= 2 \end{aligned}$$

we obtain

$$x = -1 - 2y$$

Substituting *(incorrectly)* in the same equation would result in

$$\begin{aligned} (-1 - 2y) + 2y &= -1 \\ -1 &= -1 \end{aligned}$$

The substitution $x = -1 - 2y$ must be made in the *second* equation.

EXAMPLE 3

Solve the system of equations.

(a) $\begin{aligned} x^2 - 2x - y + 3 &= 0 \\ x + y - 1 &= 0 \end{aligned}$ (b) $\begin{aligned} x + 4y &= 10 \\ -2x - 8y &= -20 \end{aligned}$

SOLUTION

(a) Solving the second equation for y, we have

$$y = 1 - x$$

and substituting in the first equation yields

$$\begin{aligned} x^2 - 2x - (1 - x) + 3 &= 0 \\ x^2 - x + 2 &= 0 \end{aligned}$$

Since the discriminant of this quadratic equation is negative, the equation has no real roots. But any solution of the system of equations must satisfy this quadratic equation. We can therefore conclude that the system is inconsistent. The graphs of the equations are a parabola and a line that do not intersect (see Figure 3).

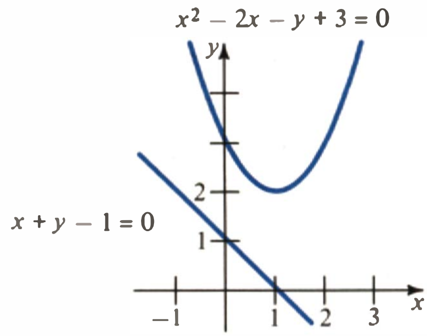


FIGURE 3

(b) Solving the first equation for x, we have

$$x = 10 - 4y$$

and substituting in the second equation gives

$$\begin{aligned} -2(10-4y)-8y &= -20 \\ -20+8y-8y &= -20 \\ -20 &= -20 \end{aligned}$$

The substitution procedure has resulted in an identity, indicating that any solution of the first equation will also satisfy the second equation. Since there are an infinite number of ordered pairs $x = a$, $y = b$ satisfying the first equation, the system is consistent and has an infinite number of solutions.

PROGRESS CHECK

Solve by substitution.

(a) $\begin{aligned} 3x - y &= 7 \\ -9x + 3y &= -22 \end{aligned}$ (b) $\begin{aligned} -5x + 2y &= -4 \\ \frac{5}{2}x - y &= 2 \end{aligned}$

ANSWERS

(a) no solution (b) any point on the line $-5x + 2y = -4$

EXERCISE SET 9.1

In Exercises 1–10 find approximate solutions of the given system by graphing.

1. $\begin{aligned} x+y&=1 \\ x-y&=3 \end{aligned}$
2. $\begin{aligned} x-y&=1 \\ x+y&=5 \end{aligned}$
3. $\begin{aligned} 3x-y&=4 \\ 6x-2y&=-8 \end{aligned}$
4. $\begin{aligned} x^2+4y^2&=32 \\ x+2y&=0 \end{aligned}$
5. $\begin{aligned} xy&=-4 \\ 4x-y&=8 \end{aligned}$
6. $\begin{aligned} 4x^2+y^2&=4 \\ x^2-y^2&=9 \end{aligned}$
7. $\begin{aligned} 4x^2+9y^2&=72 \\ 4x-3y^2&=0 \end{aligned}$
8. $\begin{aligned} 2y^2-x^2&=-1 \\ 4y^2+x^2&=25 \end{aligned}$
9. $\begin{aligned} x^2+y^2&=1 \\ y^2-3x^2&=5 \end{aligned}$
10. $\begin{aligned} 3x^2+8y^2&=21 \\ x^2+4y^2&=10 \end{aligned}$

In Exercises 11–20 solve the system of equations by the method of substitution.

11. $\begin{aligned} x+y&=1 \\ x-y&=3 \end{aligned}$
12. $\begin{aligned} x+2y&=8 \\ 3x-4y&=4 \end{aligned}$
13. $\begin{aligned} x^2+y^2&=13 \\ 2x-y&=4 \end{aligned}$
14. $\begin{aligned} x^2+4y^2&=32 \\ x+2y&=0 \end{aligned}$
15. $\begin{aligned} y^2-x&=0 \\ y-4x&=-3 \end{aligned}$
16. $\begin{aligned} xy&=-4 \\ 4x-y&=8 \end{aligned}$
17. $\begin{aligned} x^2-2x+y^2&=3 \\ 2x+y&=4 \end{aligned}$
18. $\begin{aligned} 4x^2+y^2&=4 \\ x-y&=3 \end{aligned}$
19. $\begin{aligned} xy&=1 \\ x-y+1&=0 \end{aligned}$
20. $\begin{aligned} \frac{1}{2}x-\frac{3}{2}y&=4 \\ \frac{3}{2}x+y&=1 \end{aligned}$

9.2 SOLVING BY ELIMINATION

When we solve a system of equations by graphing, we must estimate the coordinates of the point of intersection. If we require the answers to be accurate to, say, five decimal places, it is clear that graphing will not suffice. The method of substitution provides us with exact answers but suffers from the disadvantage that it is difficult to program for use in a computer.

The **method of elimination** overcomes these difficulties. The strategy of the method is to obtain an equation that has just one variable and is easily solved. The procedure is illustrated in the following example.

EXAMPLE 1

Solve by elimination.

$$\begin{aligned} 4x^2 + 9y^2 &= 36 \\ -9x^2 + 18y^2 &= 4 \end{aligned}$$

SOLUTION

Method of Elimination	
Step 1. Multiply each equation by a constant so that the coefficients of either x or y will differ only in sign.	*Step 1*. Multiply the first equation by -2 and the second equation by 1 so that the coefficients of y will be -18 and 18: $$\begin{aligned} -8x^2 - 18y^2 &= -72 \\ -9x^2 + 18y^2 &= 4 \end{aligned}$$
Step 2. Add the equations. The resulting equation will contain (at most) one variable.	*Step 2*. $-17x^2 = -68$
Step 3. Solve the resulting equation in one variable.	*Step 3*. $x^2 = 4$ $x = \pm 2$
Step 4. Substitute in either of the *original* equations to solve for the second variable.	*Step 4*. Substitute $x = 2$ in the first equation of the original system: $$\begin{aligned} 4x^2 + 9y^2 &= 36 \\ 4(2)^2 + 9y^2 &= 36 \\ y &= \pm\frac{2}{3}\sqrt{5} \end{aligned}$$ Substituting $x = -2$ yields the same values for y.
Step 5. Check in both equations.	*Step 5*. Verify that the solutions $$x = 2,\quad y = \frac{2}{3}\sqrt{5}$$ $$x = 2,\quad y = -\frac{2}{3}\sqrt{5}$$ $$x = -2,\quad y = \frac{2}{3}\sqrt{5}$$ $$x = -2,\quad y = -\frac{2}{3}\sqrt{5}$$ satisfy both equations.

Note that in Step 2 we have "eliminated" y, which is why we call this the method of elimination. The graphs are the hyperbola and ellipse shown in Figure 4.

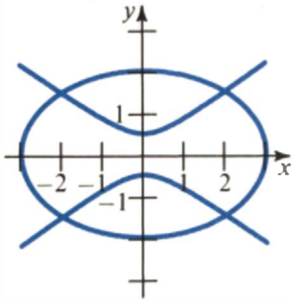


FIGURE 4

PROGRESS CHECK

Find the real solutions of the system.

$$\begin{aligned} x^2 - 4x + y^2 - 4y &= 1 \\ x^2 - 4x \qquad + y &= -5 \end{aligned}$$

ANSWER

$x = 2$, $y = -1$ (The parabola is tangent to the circle.)

EXAMPLE 2

Solve by elimination.

(a) $\begin{aligned} 2x^2 - 3y^2 &= 9 \\ x^2 + y^2 &= 4 \end{aligned}$ (b) $\begin{aligned} 5x + 6y &= 4 \\ -10x - 12y &= -8 \end{aligned}$

SOLUTION

(a) Adding -2 times the second equation to the first equation yields

$$-5y^2 = 1 \quad \text{or} \quad y^2 = -\frac{1}{5}$$

Since this equation has no solutions, the graphs of the given system do not intersect, and the system is inconsistent (see Figure 5).

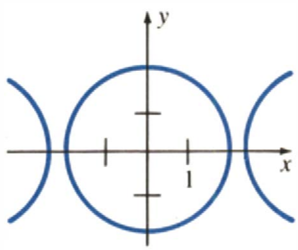


FIGURE 5

(b) Multiplying the first equation by 2, we have

$$\begin{array}{r} 10x + 12y = 8 \\ -10x - 12y = -8 \\ \hline 0x + 0y = 0 \end{array}$$

We conclude that the equations represent the same line and that the solution set consists of all points on the line $5x + 6y = 4$.

PROGRESS CHECK

Solve by elimination.

(a) $\begin{aligned} x - y &= 2 \\ 3x - 3y &= -6 \end{aligned}$ (b) $\begin{aligned} 4x + 6y &= 3 \\ -2x - 3y &= -\frac{3}{2} \end{aligned}$

ANSWERS

(a) no solution (b) all points on the line $4x + 6y = 3$

EXERCISE SET 9.2

In Exercises 1–10 solve the system of equations by the method of elimination.

1. $x + 2y = 1$
   $5x + 2y = 13$
2. $x - 4y = -7$
   $2x + 3y = -8$
3. $25y^2 - 16x^2 = 400$
   $9y^2 - 4x^2 = 36$
4. $x^2 - y^2 = 3$
   $x^2 + y^2 = 5$
5. $4x^2 + 9y^2 = 72$
   $4x - 3y^2 = 0$
6. $x^2 + y^2 + 2y = 9$
   $y - 2x = 4$
7. $3x - y = 4$
   $6x - 2y = -8$
8. $2x + 3y = -2$
   $-3x - 5y = 4$
9. $2y^2 - x^2 = -1$
   $4y^2 + x^2 = 25$
10. $x^2 + 4y^2 = 25$
    $4x^2 + y^2 = 25$

In Exercises 11–18 determine whether the system is consistent (C) or inconsistent (I). If the system is consistent, find all solutions.

11. $2x + 2y = 6$
    $3x + 3y = 6$
12. $2x + y = 2$
    $3x - y = 8$
13. $y^2 - 8x^2 = 9$
    $y^2 + 3x^2 = -31$
14. $4y^2 + 3x^2 = 24$
    $3y^2 - 2x^2 = 35$
15. $3x + 3y = 9$
    $2x + 2y = -6$
16. $x - 4y = -7$
    $2x - 8y = -4$
17. $3x - y = 18$
    $\frac{3}{2}x - \frac{1}{2}y = 9$
18. $2x + y = 6$
    $x + \frac{1}{2}y = 3$

In Exercises 19–23 use a pair of equations to solve the given problem.

19. A pile of 34 coins worth \$4.10 consists of nickels and quarters. Find the number of each type of coin.
20. Car A can travel 20 kilometers per hour faster than car B. If car A travels 240 kilometers in the same time that car B travels 200 kilometers, what is the speed of each car?
21. How many pounds of nuts worth \$2.10 per pound and how many pounds of raisins worth \$0.90 per pound must be mixed to obtain a mixture of two pounds that is worth \$1.62 per pound?
22. A part of \$8000 was invested at an annual interest of 7% and the remainder at 8%. If the total interest received at the end of one year is \$590, how much was invested at each rate?
23. The sum of the squares of the sides of a rectangle is 100 square meters. If the area of the rectangle is 48 square meters, find the length of each side of the rectangle.

9.3 APPLICATIONS

In earlier sections of this chapter we saw that many of the word problems we had previously solved by using one variable could be recast as a system of linear equations. There are, in addition, many word problems that are difficult to handle with one variable but are easily formulated by using two variables.

EXAMPLE 1

If 3 sulfa pills and 4 penicillin pills cost 69 cents, whereas 5 sulfa pills and 2 penicillin pills cost 73 cents, what is the cost of each type of pill?

SOLUTION

Using two variables, we let

x = the cost of each sulfa pill
y = the cost of each penicillin pill

Then

$$3x + 4y = 69$$

$$5x + 2y = 73$$

We multiply the second equation by -2 and add to eliminate y:

$$\begin{array}{rcr} 3x + 4y & = & 69 \\ -10x - 4y & = & -146 \\ \hline -7x \quad & = & -77 \\ x & = & 11 \end{array}$$

Substituting in the first equation, we have

$$3(11) + 4y = 69$$

$$4y = 36$$

$$y = 9$$

Each sulfa pill costs 11 cents and each penicillin pill costs 9 cents. (Could you have set up this problem using only one variable? Not easily!)

EXAMPLE 2

Swimming downstream, a swimmer can cover 2 kilometers in 15 minutes. The return trip upstream requires 20 minutes. What is the rate of the swimmer and of the current in kilometers per hour? (The rate of the swimmer is the speed at which he would swim if there were no current.)

SOLUTION

Let

x = the rate of the swimmer (in km per hour)

y = the rate of the current (in km per hour)

For swimming downstream, the rate of the current is added to the rate of the swimmer, so $x + y$ is the rate downstream. Similarly, $x - y$ is the rate for swimming upstream. We display the information we have, expressing time in hours.

	Rate $\times$	Time $=$	Distance
Downstream	$x + y$	$\frac{1}{4}$	$\frac{1}{4}(x + y)$
Upstream	$x - y$	$\frac{1}{3}$	$\frac{1}{3}(x - y)$

Since distance upstream = distance downstream = 2 kilometers,

$$\frac{1}{4}(x + y) = 2$$

$$\frac{1}{3}(x - y) = 2$$

or, equivalently,

$$x + y = 8$$

$$x - y = 6$$

Solving, we have

$$x = 7 \qquad \text{rate of the swimmer}$$

$$y = 1 \qquad \text{rate of the current}$$

Thus, the rate of the swimmer is 7 kilometers per hour, and the rate of the current is 1 kilometer per hour. (The student is urged to verify the solution.)

EXAMPLE 3

The sum of a two-digit number and its units digit is 64, and the sum of the number and its tens digit is 62. Find the number.

SOLUTION

The basic idea in solving digit problems is to note that if we let

$$t = \text{tens digit}$$

and

$$u = \text{units digit}$$

then

$$10t + u = \text{the two-digit number}$$

Then "the sum of a two-digit number and its units digit is 64" translates into

$$(10t + u) + u = 64 \quad \text{or} \quad 10t + 2u = 64$$

Also, "the sum of the number and its tens digit is 62" becomes

$$(10t + u) + t = 62 \quad \text{or} \quad 11t + u = 62$$

Solving, we find that $t = 5$ and $u = 7$ (verify this), so the number we seek is 57.

APPLICATIONS IN BUSINESS AND ECONOMICS: BREAK-EVEN ANALYSIS

An important problem faced by a manufacturer is that of determining the **level of production,** that is, the number of units of the product to be manufactured during a given time period—a day, a week, or a month. Suppose that

$$C = 400 + 2x \tag{1}$$

is the total cost (in thousands of dollars) of producing x units of the product and that

$$R = 4x \tag{2}$$

is the total revenue (in thousands of dollars) when x units of the product are sold. In this example, after setting up production at a cost of \$400,000, the manufacturer has an additional cost of \$2000 to make each unit [Equation (1)], and a revenue of \$4000 is earned from the sale of each unit [Equation (2)]. If all units that are manufactured are sold, the total profit P is the difference between total revenue and total cost:

$$\begin{aligned} P &= R - C \\ &= 4x - (400 + 2x) \\ &= 2x - 400 \end{aligned}$$

The value of x for which $R = C$, so that the profit is zero, is called the **break-even point.** When that many units of the product have been produced and sold the manufacturer neither makes money nor loses money. To find the break-even

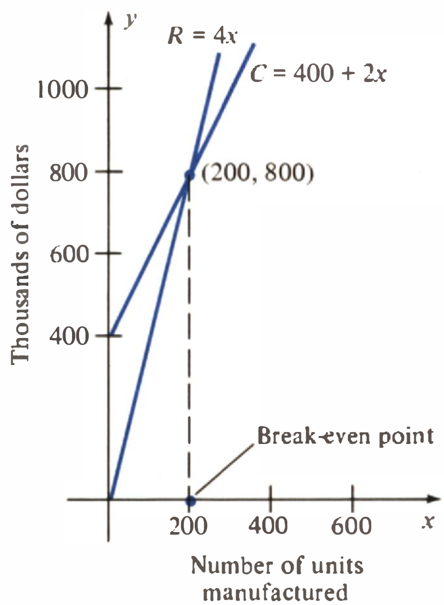


FIGURE 6

point, we set $R = C$. Using Equations (1) and (2), we obtain

$$400 + 2x = 4x$$
$$x = 200$$

Thus, the break-even point is 200 units.

The break-even point can also be obtained graphically as follows. Observe that Equations (1) and (2) are linear equations and therefore equations of straight lines. The break-even point is the x-coordinate of the point where the two lines intersect. Figure 6 shows the lines and their point of intersection (200, 800). When 200 units of the product are made, the cost ($800,000) is exactly equal to the revenue, and the profit is $0. If $x > 200$, then $R > C$, so the manufacturer is making a profit. If $x < 200$, $R < C$ and the manufacturer is losing money.

PROGRESS CHECK

A producer of photographic developer finds that the total weekly cost of producing x liters of developer is given (in dollars) by $C = 550 + 0.40x$. The manufacturer sells the product at $0.50 per liter.

(a) What is the total revenue received when x liters of developer are sold?
(b) Find the break-even point graphically.
(c) What is the total revenue received at the break-even point?

ANSWERS

(a) $R = 0.50x$ (b) 5500 liters (c) $2750

APPLICATIONS IN BUSINESS AND ECONOMICS: SUPPLY AND DEMAND

A manufacturer of a product is free to set any price p (in dollars) for each unit of the product. Of course, if the price is too high, not enough people will buy the product; if the price is too low, so many people will rush to buy the product that the producer will not be able to satisfy demand. Thus, in setting price, the manufacturer must take into consideration the demand for the product.

Let S be the number of units that the manufacturer is willing to supply at the price p; S is called the **supply.** Generally, the value of S will increase as p increases; that is, the manufacturer is willing to supply more of the product as the price p increases. Let D be the number of units of the product that consumers are willing to buy at the price p; D is called the **demand.** Generally, the value of D will decrease as p increases; that is, consumers are willing to buy fewer units of the product as the price rises. For example, suppose that S and D are given by

$$S = 2p + 3 \qquad (3)$$
$$D = -p + 12 \qquad (4)$$

Equations (3) and (4) are linear equations, so they are equations of straight lines (see Figure 7). The price at which supply S and demand D are equal is called the

equilibrium price. At this price, every unit that is supplied is purchased. Thus there is neither a surplus nor a shortage. In Figure 7 the equilibrium price is $p = 3$. At this price, the number of units supplied equals the number of units demanded and is found by substituting in Equation (3): $S = 2(3) + 3 = 9$. This value can also be obtained by finding the ordinate at the point of intersection in Figure 7.

If we are in an economic system in which there is pure competition, the law of supply and demand states that the selling price of a product will be its equilibrium price. That is, if the selling price were higher than the equilibrium price, consumers' reduced demand would leave the manufacturer with an unsold surplus. To sell this surplus, the manufacturer would be forced to reduce the selling price. If the selling price were below the equilibrium price, the increased demand would cause a shortage of the product, leading the manufacturer to raise the selling price. Of course, in actual practice, the marketplace does not operate under pure competition. Also deeper mathematical analysis of economic systems requires the use of more sophisticated equations.

EXAMPLE 4

Suppose that supply and demand for ball-point pens are given by

$$S = p + 5$$
$$D = -p + 7$$

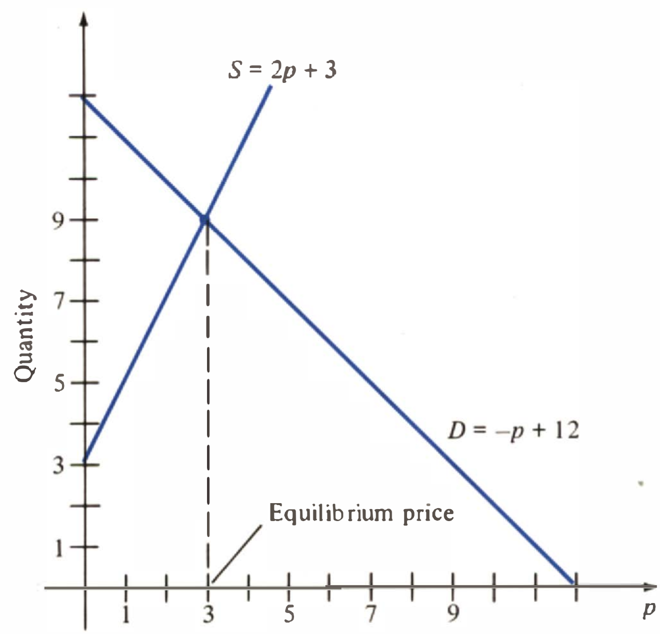


FIGURE 7

(a) Find the equilibrium price.
(b) Find the number of pens sold at that price.

SOLUTION

(a) Figure 8 illustrates the graphical solution. The equilibrium price is $p = 1$. Algebraic methods will, of course, yield the same solution.
(b) When $p = 1$, the number of pens sold is $S = 1 + 5 = 6$, the value of the ordinate at the point of intersection.

PROGRESS CHECK

Suppose that supply and demand for radios are given by

$$S = 3p + 120$$
$$D = -p + 200$$

(a) Find the equilibrium price.
(b) Find the number of radios sold at that price.

ANSWERS

(a) 20 (b) 180

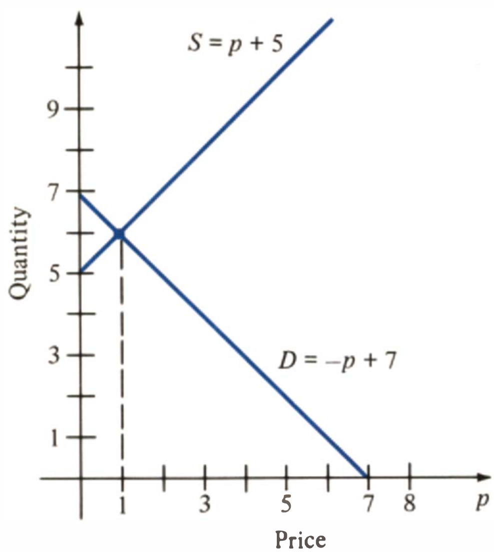


FIGURE 8

EXERCISE SET 9.3

1. A pile of 40 coins consists of nickels and dimes. If the value of the coins is \$2.75, how many of each type of coin are there?

2. An automatic vending machine in the post office, which charges no more than a clerk, sells a packet of 27 ten-cent and twenty-cent stamps for \$3. How many of each type of stamp are in the packet?

3. A photography store sells sampler A, consisting of 6 rolls of color film and 4 rolls of black and white film for \$21. It also sells sampler B, consisting of 4 rolls of color film and 6 rolls of black and white film for \$19. What is the cost per roll of each type of film?

4. A hardware store sells power pack A, consisting of four D cells and two C cells for \$1.70, and power pack B, consisting of six D cells and four C cells for \$2.80. What is the price of each cell?

5. A fund manager invested \$6000 in two types of bonds, A and B. Bond A, which is safer than bond B, pays annual interest of 8 percent, whereas bond B pays annual interest of 10 percent. If the total annual return on both investments is \$520, how much was invested in each type of bond?

6. A trash removal company carries waste material in sealed containers weighing 4 kilograms and 3 kilograms. On a certain trip the company carries 30 containers weighing a total of 100 kilograms. How many of each type of container are there?

7. A paper firm makes rolls of paper 12 inches wide and 15 inches wide by cutting a sheet that is 180 inches wide. Suppose that a total of 14 rolls of paper are to be cut without any waste. How many of each type of roll will be made?

8. An animal-feed producer mixes two types of grain, A and B. Each unit of grain A contains 2 grams of fat and 80 calories, and each unit of grain B contains 3 grams of fat and 60 calories. If the producer wants the final product to provide 18 grams of fat and 480 calories, how much of each type of grain should be used?

9. A supermarket mixes coffee that sells for \$1.20 per pound with coffee that sells for \$1.80 per pound to obtain 24 pounds of coffee selling for \$1.60 per pound. How much of each type of coffee should be used?

10. An airplane flying against the wind covers a distance of 3000 kilometers in 6 hours. The return trip, with the aid of the wind, takes 5 hours. What is the speed of the airplane in still air, and what is the speed of the wind?

11. A cyclist traveling against the wind covers a distance of 45 miles in 4 hours. The return trip, with the aid of the wind, takes 3 hours. What is the speed of the cyclist in still air, and what is the speed of the wind?

12. The sum of a two-digit number and its units digit is 20, and the sum of the number and its tens digit is 16. Find the number.

13. The sum of the digits of a two-digit number is 7. If the digits are reversed, the resulting number exceeds the given number by 9. Find the number.

14. The sum of the units digit and three times the tens digit of a two-digit number is 14, and the sum of the tens digit and twice the units digit is 18. Find the number.

15. A health food shop mixes nuts and raisins into a snack pack. How many pounds of nuts, selling for \$2.00 per pound, and how many pounds of raisins, selling for \$1.50 per pound, must be mixed to produce a 50-pound mixture selling for \$1.80 per pound?

16. A movie theater charges \$3.00 admission for an adult and \$1.50 for a child. On a particular day 600 tickets were sold and the total revenue received was \$1350. How many tickets of each type were sold?

17. A moped dealer selling a model A and a model B moped has \$18,000 in inventory. The profit on a model A moped is 12%, and the profit on a model B moped is 18%. If the profit on the entire stock would be 16%, how much was invested in each model?

18. The cost of sending a telegram is determined as follows: there is a flat charge for the first 10 words and a uniform rate for each additional word. Suppose that an 18-word telegram costs \$1.94 and a 22-word telegram costs \$2.16. Find the cost of the first 10 words and the rate for each additional word.

19. A certain epidemic disease is treated by a combination of the drugs Epiline I and Epiline II. Suppose that each unit of Epiline I contains 1 milligram of factor X and 2

milligrams of factor Y, while each unit of Epiline II contains 2 milligrams of factor X and 3 milligrams of factor Y. Successful treatment of the disease calls for 13 milligrams of factor X and 22 milligrams of factor Y. How many units of Epiline I and Epiline II should be administered to a patient?

20. **(Break-even analysis)** An animal feed manufacturer finds that the weekly cost of making x kilograms of feed is given (in dollars) by $C = 2000 + 0.50x$ and that the revenue received from selling the feed is given by $R = 0.75x$.
    (a) Find the break-even point graphically.
    (b) What is the total revenue at the break-even point?

21. **(Break-even analysis)** A small manufacturer of a new solar device finds that the annual cost of making x units is given (in dollars) by $C = 24{,}000 + 55x$. Each device sells for \$95.
    (a) What is the total revenue received when x devices are sold?
    (b) Find the break-even point graphically.
    (c) What is the total revenue received at the break-even point?

22. **(Supply and demand)** A manufacturer of calculators finds that the supply and demand are given by

$$S = 0.5p + 0.5$$
$$D = -2p + 8$$

    (a) Find the equilibrium price.
    (b) What is the number of calculators sold at this price?

23. **(Supply and demand)** A manufacturer of mopeds finds that the supply and demand are given by

$$S = 2p + 10$$
$$D = -p + 22$$

    (a) Find the equilibrium price.
    (b) What is the number of mopeds sold at this price?

24. Find the dimensions of a rectangle with an area of 30 square feet and a perimeter of 22 feet.

25. Find two numbers whose product is 20 and whose sum is 9.

26. Find two numbers the sum of whose squares is 65 and whose sum is 11.

9.4 SYSTEMS OF LINEAR EQUATIONS IN THREE UNKNOWNS

The method of substitution and the method of elimination can both be applied to systems of linear equations in three unknowns and, more generally, to systems of linear equations in any number of unknowns. There is yet another method, ideally suited for computers, which we will now apply to solving linear systems in three unknowns.

GAUSSIAN ELIMINATION AND TRIANGULAR FORM

In solving equations, we found it convenient to transform an equation into an equivalent equation having the same solution set. Similarly, we can attempt to transform a system of equations into another system, called an **equivalent system,** that has the same solution set. In particular, the objective of **Gaussian elimination** is to transform a linear system into an equivalent system in triangular form, such as

$$\begin{aligned} 3x - y + 3z &= -11 \\ 2y + z &= 2 \\ 2z &= -4 \end{aligned}$$

A linear system is in **triangular form** when the only nonzero coefficient of x appears in the first equation, the only nonzero coefficients of y appear in the first and second equations, and so on.

Note that when a linear system is in triangular form, the last equation immediately yields the value of an unknown. In our example, we see that

$$\begin{aligned} 2z &= -4 \\ z &= -2 \end{aligned}$$

Substituting $z = -2$ in the second equation yields

$$\begin{aligned} 2y + (-2) &= 2 \\ y &= 2 \end{aligned}$$

Finally, substituting $z = -2$ and $y = 2$ in the first equation yields

$$\begin{aligned} 3x - (2) + 3(-2) &= -11 \\ 3x &= -3 \\ x &= -1 \end{aligned}$$

This process of **back-substitution** thus allows us to solve a linear system quickly when it is in triangular form.

The challenge, then, is to find a means of transforming a linear system into triangular form. We now offer (without proof) a list of operations that transform a system of linear equations into an equivalent system.

1. Interchange any two equations.
2. Multiply an equation by a nonzero constant.
3. Replace an equation with the sum of itself plus a constant times another equation.

Using these operations, we can now demonstrate the method of Gaussian elimination.

EXAMPLE 1

Solve the linear system:

$$\begin{aligned} 2y - z &= -5 \\ x - 2y + 2z &= 9 \\ 2x - 3y + 3z &= 14 \end{aligned}$$

SOLUTION

Gaussian Elimination	
Step 1. (a) If necessary, interchange equations to obtain a nonzero coefficient for x in the first equation.	*Step 1.* (a) Interchanging the first two equations yields $$\begin{aligned} x - 2y + 2z &= 9 \\ 2y - z &= -5 \\ 2x - 3y + 3z &= 14 \end{aligned}$$

(b) Replace the second equation with the sum of itself and an appropriate multiple of the first equation, which will result in a zero coefficient for x.	(b) The coefficient of x in the second equation is already 0.
(c) Replace the third equation with the sum of itself and an appropriate multiple of the first equation, which will result in a zero coefficient for x.	(c) Replace the third equation with the sum of itself and -2 times the first equation. $$\begin{aligned} x - 2y + 2z &= 9 \\ 2y - z &= -5 \\ y - z &= -4 \end{aligned}$$
Step 2. Apply the procedures of Step 1 to the second and third equations.	*Step 2.* Replace the third equation with the sum of itself and $-\frac{1}{2}$ times the second equation. $$\begin{aligned} x - 2y + 2z &= 9 \\ 2y - z &= -5 \\ -\frac{1}{2}z &= -\frac{3}{2} \end{aligned}$$
Step 3. The system is now in triangular form. The solution is obtained by back-substitution.	*Step 3.* From the third equation, $$-\frac{1}{2}z = -\frac{3}{2}$$ $$z = 3$$ Substituting this value of z in the second equation, we have $$2y - (3) = -5$$ $$y = -1$$ Substituting for y and for z in the first equation, we obtain $$x - 2(-1) + 2(3) = 9$$ $$x + 8 = 9$$ $$x = 1$$ The solution is $x = 1$, $y = -1$, $z = 3$.

PROGRESS CHECK

Solve by Gaussian elimination.

(a)
$$\begin{aligned} 2x - 4y + 2z &= 1 \\ 3x + y + 3z &= 5 \\ x - y - 2z &= -8 \end{aligned}$$

(b)
$$\begin{aligned} -2x + 3y - 12z &= -17 \\ 3x - y - 15z &= 11 \\ -x + 5y + 3z &= -9 \end{aligned}$$

ANSWERS

(a) $x = -\frac{3}{2}, y = \frac{1}{2}, z = 3$ (b) $x = 5, y = -1, z = \frac{1}{3}$

CONSISTENT AND INCONSISTENT SYSTEMS

The graph of a linear equation in three unknowns is a plane in three-dimensional space. A system of three linear equations in three unknowns corresponds to three planes (Figure 9). If the planes intersect in a point P (Figure 9a), the coordinates

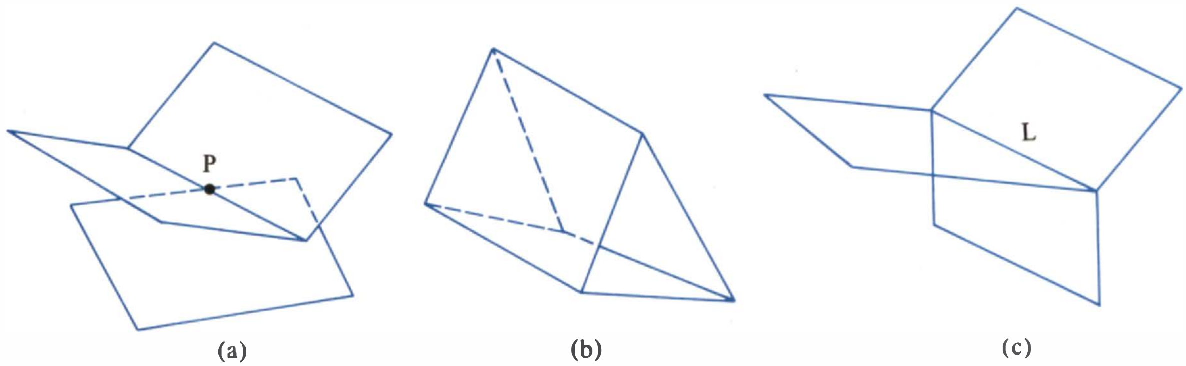


FIGURE 9

of the point P are a solution of the system and can be found by Gaussian elimination. The cases of no solution and of an infinite number of solutions are signaled as follows.

Consistent and Inconsistent Systems

- If Gaussian elimination results in an equation of the form

$$0x + 0y + 0z = c, \quad c \neq 0$$

then the system is inconsistent (Figure 9b).

- If Gaussian elimination results in no equation of the type above but results in an equation of the form

$$0x + 0y + 0z = 0$$

then the system is consistent and has an infinite number of solutions (Figure 9c).

- Otherwise, the system is consistent and has a unique solution.

EXAMPLE 2

Solve the linear system:

$$\begin{aligned} x - 2y + \ \ 2z &= -4 \\ x + \ \ y - \ \ 7z &= \ \ 8 \\ -x - 4y + 16z &= -20 \end{aligned}$$

SOLUTION

Replacing the second equation with itself minus the first equation, and replacing the third equation with itself plus the first equation, we have

$$\begin{aligned} x - 2y + 2z &= -4 \\ 3y - 9z &= 12 \\ -6y + 18z &= -24 \end{aligned}$$

Replacing the third equation of this system with itself plus 2 times the second equation results in the system

$$\begin{aligned} x - 2y + 2z &= -4 \\ 3y - 9z &= 12 \\ 0x + 0y + 0z &= 0 \end{aligned}$$

in which the last equation indicates that the system is consistent and has an infinite number of solutions. If we solve the second equation of the last system for y, we have

$$y = 3z + 4$$

Then, solving the first equation for x, we have

$$\begin{aligned} x &= 2y - 2z - 4 \\ &= 2(3z + 4) - 2z - 4 \qquad \text{Substituting for } y \\ &= 4z + 4 \end{aligned}$$

The equations

$$\begin{aligned} x &= 4z + 4 \\ y &= 3z + 4 \end{aligned}$$

yield a solution of the original system for every real value of z. For example, if $z = 0$, then $x = 4$, $y = 4$, $z = 0$ satisfies the original system; if $z = -2$, then $x = -4$, $y = -2$, $z = -2$ is another solution.

PROGRESS CHECK

(a) Verify that the linear system

$$\begin{aligned} x - 2y + z &= 3 \\ 2x + y - 2z &= -1 \\ -x - 8y + 7z &= 5 \end{aligned}$$

is consistent.

(b) Verify that the linear system

$$\begin{aligned} 2x + y + 2z &= 1 \\ x - 4y + 7z &= -4 \\ x - y + 3z &= -1 \end{aligned}$$

has an infinite number of solutions.

EXERCISE SET 9.4

In Exercises 1–18 solve by Gaussian elimination. Indicate if the system is inconsistent or has an infinite number of solutions.

1. $\begin{aligned} x + 2y + 3z &= -6 \\ 2x - 3y - 4z &= 15 \\ 3x + 4y + 5z &= -8 \end{aligned}$

2. $\begin{aligned} 2x + 3y + 4z &= -12 \\ x - 2y + z &= -5 \\ 3x + y + 2z &= 1 \end{aligned}$

3. $\begin{aligned} x + y + z &= 1 \\ x + y - 2z &= 3 \\ 2x + y + z &= 2 \end{aligned}$

4. $\begin{aligned} 2x - y + z &= 3 \\ x - 3y + z &= 4 \\ -5x \qquad - 2z &= -5 \end{aligned}$

5. $\begin{aligned} x + y + z &= 2 \\ x - y + 2z &= 3 \\ 3x + 5y + 2z &= 6 \end{aligned}$

6. $\begin{aligned} x + y + z &= 0 \\ x + y \qquad &= 3 \\ y + z &= 1 \end{aligned}$

7. $\begin{aligned} x + 2y + z &= 7 \\ x + 2y + 3z &= 11 \\ 2x + y + 4z &= 12 \end{aligned}$

8. $\begin{aligned} 4x + 2y - z &= 5 \\ 3x + 3y + 6z &= 1 \\ 5x + y - 8z &= 8 \end{aligned}$

9. $\begin{aligned} x + y + z &= 2 \\ x + 2y + z &= 3 \\ x + y - z &= 2 \end{aligned}$

10. $\begin{aligned} x + y - z &= 2 \\ x + 2y + z &= 3 \\ x + y + 4z &= 3 \end{aligned}$

11. $\begin{aligned} 2x + y + 3z &= 8 \\ -x + y + z &= 10 \\ x + y + z &= 12 \end{aligned}$

12. $\begin{aligned} 2x \qquad - 3z &= 4 \\ x + 4y - 5z &= -6 \\ 3x + 4y - z &= -2 \end{aligned}$

13. $\begin{aligned} x + 3y + 7z &= 1 \\ 3x - y - 5z &= 9 \\ 2x + y + z &= 4 \end{aligned}$

14. $\begin{aligned} 2x - y + z &= 2 \\ 3x + y + 2z &= 3 \\ x + y - z &= -1 \end{aligned}$

15. $\begin{aligned} x - 2y + 3z &= -2 \\ x - 5y + 9z &= 4 \\ 2x - y \qquad &= 6 \end{aligned}$

16. $\begin{aligned} x + 2y - 2z &= 8 \\ 5y - z &= 6 \\ -2x + y + 3z &= -2 \end{aligned}$

17. $\begin{aligned} x - 2y + z &= -5 \\ 2x \qquad + z &= -10 \\ y - z &= 15 \end{aligned}$

18. $\begin{aligned} 2y - 3z &= 4 \\ x \qquad + 2z &= -2 \\ x - 8y + 14z &= -18 \end{aligned}$

19. A special low-calorie diet consists of dishes A, B, and C. Each unit of A has 2 grams of fat, 1 gram of carbohydrate, and 3 grams of protein. Each unit of B has 1 gram of fat, 2 grams of carbohydrate, and 1 gram of protein. Each unit of C has 1 gram of fat, 2 grams of carbohydrate, and 3 grams of protein. The diet must provide exactly 10 grams of fat, 14 grams of carbohydrate, and 18 grams of protein. How much of each dish should be used?

20. A furniture manufacturer makes chairs, coffee tables, and dining room tables. Each chair requires 2 minutes of sanding, 2 minutes of staining, and 4 minutes of varnishing. Each coffee table requires 5 minutes of sanding, 4 minutes of staining, and 3 minutes of varnishing. Each dining room table requires 5 minutes of sanding, 4 minutes of staining, and 6 minutes of varnishing. The sanding bench is available 6 hours per day, the staining bench 5 hours per day, and the varnishing bench 6 hours per day. How many of each type of furniture can be made if all facilities are used to capacity?

21. A manufacturer produces 12-inch, 16-inch, and 19-inch television sets that require assembly, testing, and packing. Each 12-inch set requires 45 minutes to assemble, 30 minutes to test, and 10 minutes to package. Each 16-inch set requires 1 hour to assemble, 45 minutes to test, and 15 minutes to package. Each 19-inch set requires $1\frac{1}{2}$ hours to assemble, 1 hour to test, and 15 minutes to package. If the assembly line operates for $17\frac{3}{4}$ hours per day, the test facility is used for $12\frac{1}{2}$ hours per day, and the packing equipment is used for $3\frac{3}{4}$ hours per day, how many of each type of set can be produced?

9.5 SYSTEMS OF LINEAR INEQUALITIES

GRAPHING LINEAR INEQUALITIES

When we draw the graph of a linear equation, say

$$y = 2x - 1$$

we can readily see that the graph of the line divides the plane into two regions, which we call **half-planes** (see Figure 10). If, in the equation $y = 2x - 1$, we replace the equals sign with any of the symbols $<$, $>$, $\leq$, or $\geq$, we have a **linear**

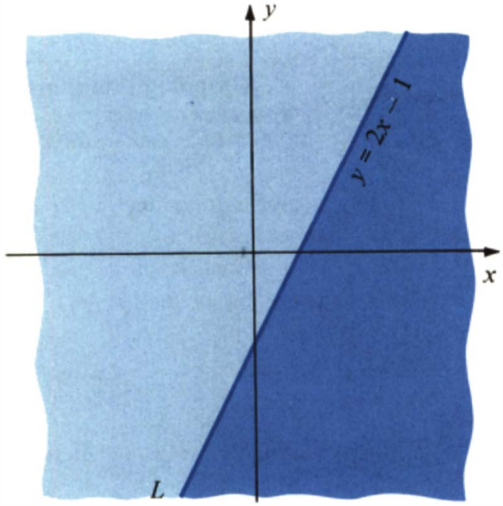


FIGURE 10

inequality in two variables. By the **graph of a linear inequality** such as

$$y < 2x - 1$$

we mean the set of all points whose coordinates satisfy the inequality. Thus, the point (4, 2) lies on the graph of $y < 2x - 1$, since the substitution

$$2 < (2)(4) - 1$$
$$2 < 7$$

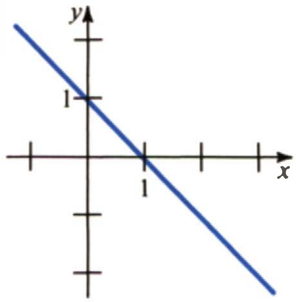
shows that $x = 4$, $y = 2$ satisfies the inequality. The point (1, 5), however, does *not* lie on the graph of $y < 2x - 1$, because the statement

$$5 < (2)(1) - 1$$
$$5 < 1$$

is not true. Since the coordinates of every point on the line L in Figure 10 satisfy the *equation* $y = 2x - 1$, we readily see that the coordinates of those points in the half-plane below the line must satisfy the *inequality* $y < 2x - 1$. Similarly, the coordinates of those points in the half-plane above the line must satisfy the *inequality* $y > 2x - 1$. This observation suggests that the graph of a linear inequality in two variables is a half-plane, and it leads to a straightforward method for graphing linear inequalities.

EXAMPLE 1
Sketch the graph of the inequality $x + y \geq 1$.

SOLUTION

Graphing Linear Inequalities	
Step 1. Replace the inequality sign with an equals sign and plot the line. (a) If the inequality is $\leq$ or $\geq$, plot a solid line (points on the line will satisfy the inequality). (b) If the inequality is $<$ or $>$, plot a dashed line (points on the line will not satisfy the inequality).	*Step 1*. $x + y = 1$ 
Step 2. Choose any point that is not on the line as a test point. If the origin is not on the line, it is the most convenient choice.	*Step 2*. Choose (0, 0) as a test point.
Step 3. Substitute the coordinates of the test point in the inequality. (a) If the test point satisfies the inequality, the coordinates of every point in the half-plane that contains the test point will satisfy the inequality.	*Step 3*. Substituting (0, 0) in $x + y \geq 1$ gives $0 + 0 \geq 1 \quad (?)$ $0 \geq 1$ which is false.
(b) If the test point does not satisfy the inequality, the half-plane on the other side of the line contains all the points satisfying the inequality.	Since (0, 0) is in the half-plane below the line and does not satisfy the inequality, all the points above the line will satisfy the inequality. See Figure 11.

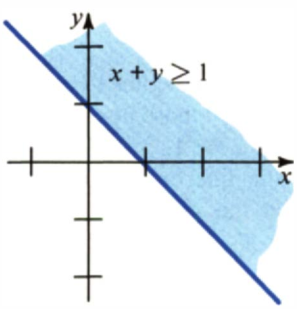


FIGURE 11

EXAMPLE 2

Sketch the graph of the inequality $2x - 3y > 6$.

SOLUTION

We first graph the line $2x - 3y = 6$. We draw a dashed or broken line to indicate that $2x - 3y = 6$ is not part of the graph (see Figure 12). Since (0, 0) is not on the line, we can use it as a test point:

$$2x - 3y > 6$$
$$2(0) - 3(0) > 6 \quad (?)$$
$$0 - 0 > 6 \quad (?)$$
$$0 > 6$$

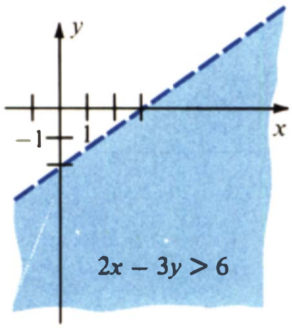


FIGURE 12

The last statement is false. Since (0, 0) is in the half-plane above the line, the graph consists of the half-plane below the line.

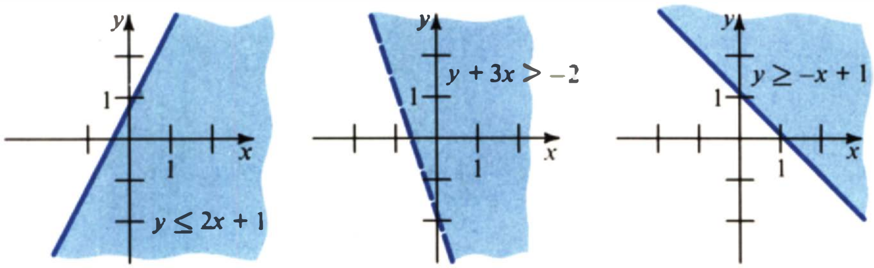
PROGRESS CHECK

Graph the inequalities.

(a) $y \leq 2x + 1$ (b) $y + 3x > -2$ (c) $y \geq -x + 1$

ANSWERS

(a) (b) (c)



EXAMPLE 3

Graph the inequalities.

(a) $y > x$ (b) $2x \geq 5$

SOLUTION

(a) Since the origin lies on the line $y = x$, we choose another test point, say (0, 1), which is above the line. Since (0, 1) does satisfy the inequality, the graph of the inequality is the half-plane above the line. See Figure 13a.

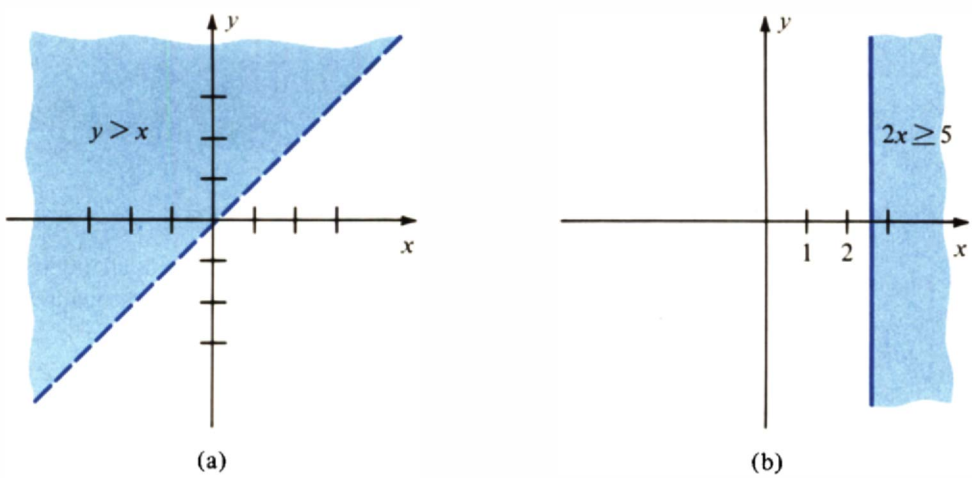


FIGURE 13

(b) The graph of $2x = 5$ is a vertical line, and the graph of $2x \geq 5$ is the half-plane to the right of the line and also the line itself. See Figure 13b.

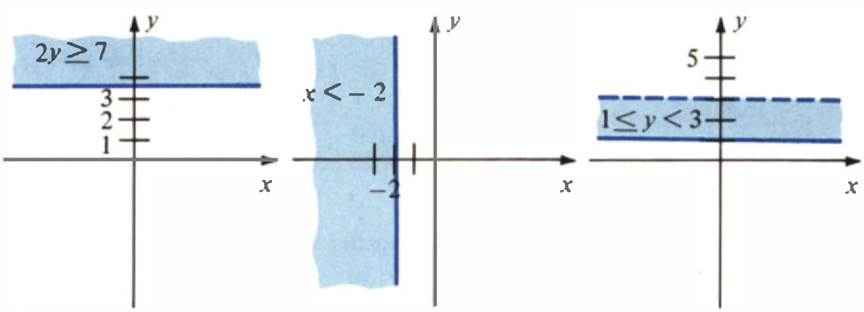
PROGRESS CHECK

Graph the inequalities.

(a) $2y \geq 7$ (b) $x < -2$ (c) $1 \leq y < 3$

ANSWERS

(a) (b) (c)



SYSTEMS OF LINEAR INEQUALITIES

We can also consider **systems of linear inequalities** in two variables, x and y. Examples of such systems are

$$\begin{aligned} 2x - 3y &> 6 \\ x + 2y &< 2 \end{aligned} \qquad \begin{aligned} 2x - 5y &\leq 12 \\ 2x + y &\leq 18 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

The **solution of a system of linear inequalities** consists of all ordered pairs (a, b) such that the substitution $x = a$, $y = b$ satisfies *all* the inequalities. Thus, the ordered pair $(2, 1)$ is a solution of the system

$$\begin{aligned} 2x - 3y &\leq 2 \\ x + y &\leq 6 \end{aligned}$$

because the substitution $x = 2$, $y = 1$ satisfies both inequalities:

$$(2)(2) - (3)(1) = 1 \leq 2$$
$$2 + 1 = 3 \leq 6$$

We can graph the solution set of a system of linear inequalities by graphing the solution set of each inequality and marking that portion of the graph that satisfies *all* the inequalities.

EXAMPLE 4

Graph the solution set of the system:

$$\begin{aligned} 2x - 3y &\leq 2 \\ x + y &\leq 6 \end{aligned}$$

SOLUTION

In Figure 14 we have graphed the solution set of each of the inequalities. The cross-hatched region indicates those points that satisfy both inequalities and is therefore the solution set of the system of inequalities.

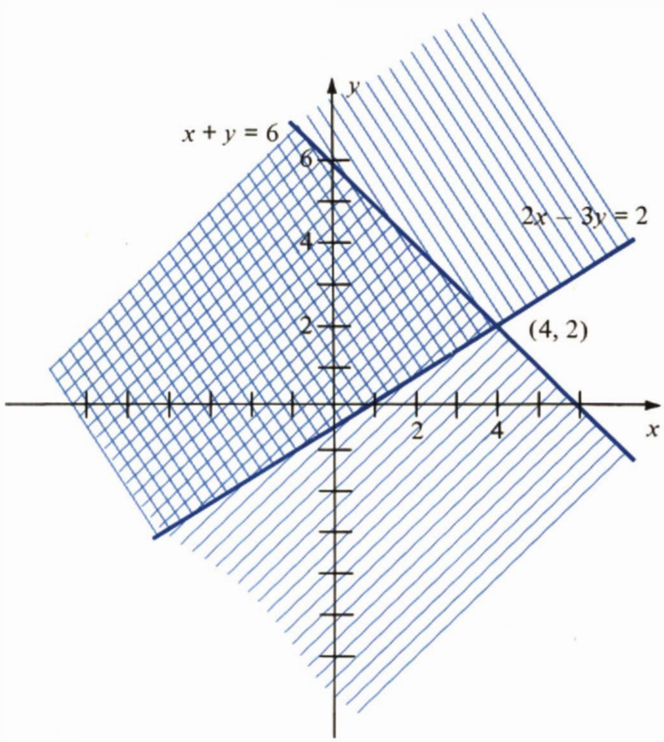


FIGURE 14

EXAMPLE 5

Graph the solution set of the system:

$$\begin{aligned} x + y &< 2 \\ 2x + 3y &\geq 9 \\ x &\geq 1 \end{aligned}$$

SOLUTION

See Figure 15. Since there are no points satisfying *all* the inequalities, we conclude that the system is inconsistent and has no solutions.

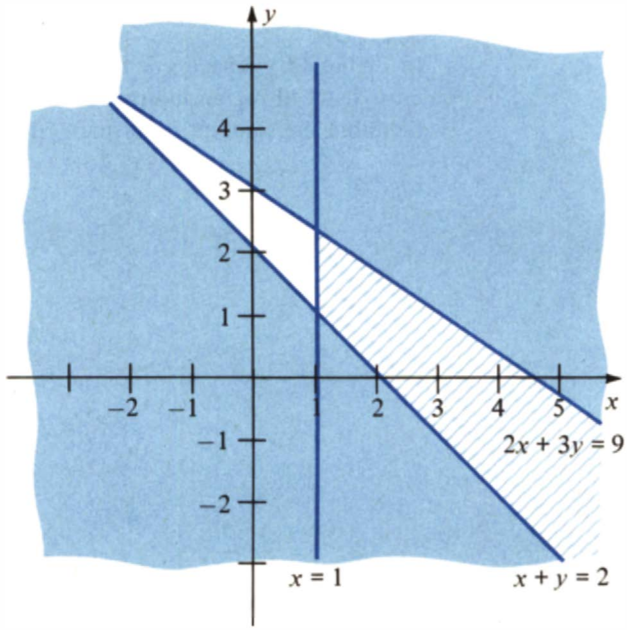


FIGURE 15

PROGRESS CHECK

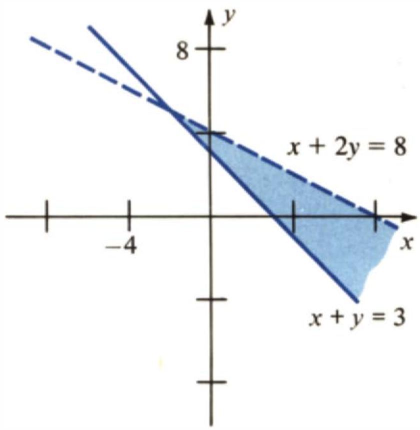
Graph the solution set of the given system.

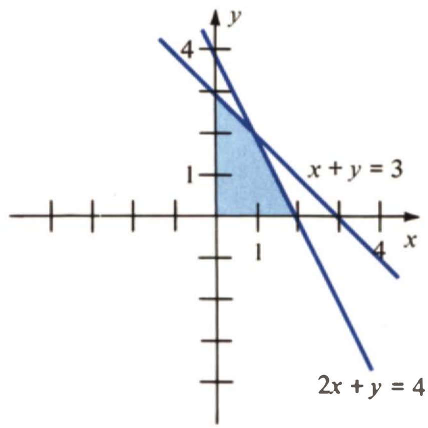
(a) $$\begin{aligned} x + y &\geq 3 \\ x + 2y &< 8 \end{aligned}$$

(b) $$\begin{aligned} 2x + y &\leq 4 \\ x + y &\leq 3 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

ANSWERS

(a) (b)





EXAMPLE 6

A dietitian at a university is planning a menu for a meal to consist of two primary foods, A and B, whose nutritional contents are shown in the table. The dietitian insists that the meal provide at most 12 units of fat, at least 2 units of carbohydrate, and at least 1 unit of protein. If x and y represent the number of grams of food types A and B, respectively, write a system of linear inequalities expressing the restrictions. Graph the solution set.

	Nutritional Content in Units per Gram		
	Fat	Carbohydrate	Protein
A	2	2	0
B	3	1	1

SOLUTION

The number of units of fat contained in the meal is $2x + 3y$, so x and y must satisfy the inequality

$$2x + 3y \leq 12 \quad \text{fat requirement}$$

Similarly, the requirements for carbohydrate and protein result in the inequalities

$$2x + y \geq 2 \quad \text{carbohydrate requirement}$$
$$y \geq 1 \quad \text{protein requirement}$$

Of course, we must also have $x \geq 0$, since negative quantities of food type A would make no sense. The system of linear inequalities is then

$$\begin{aligned} 2x + 3y &\leq 12 \\ 2x + y &\geq 2 \\ x &\geq 0 \\ y &\geq 1 \end{aligned}$$

and the graph is shown in Figure 16.

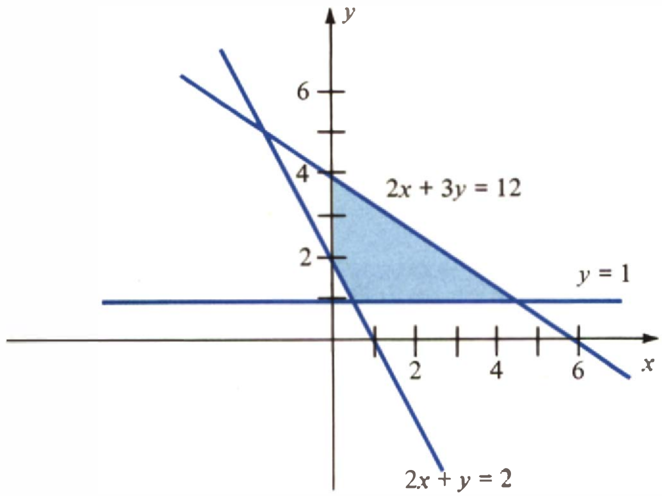


FIGURE 16

EXERCISE SET 9.5

Graph the solution set of the given inequality in the following exercises.

1. $y \leq x + 2$
2. $y \geq x + 3$
3. $y > x - 4$
4. $y < x - 5$
5. $y \leq 4 - x$
6. $y \geq 2 - x$
7. $y > x$
8. $y \leq 2x$
9. $3x - 5y > 15$
10. $2y - 3x < 12$
11. $x \leq 4$
12. $3x > -2$
13. $y > -3$
14. $5y \leq 25$
15. $x > 0$
16. $y < 0$
17. $-2 \leq x \leq 3$
18. $-6 < y < -2$
19. A steel producer makes two types of steel, regular and special. A ton of regular steel requires 2 hours in the open-hearth furnace, and a ton of special steel requires 5 hours. Let x and y denote the number of tons of regular and special steel, respectively, made per day. If the open-hearth furnace is available at most 15 hours per day, write an inequality that must be satisfied by x and y. Graph this inequality.
20. A patient is placed on a diet that restricts caloric intake to 1500 calories per day. The patient plans to eat x ounces of cheese, y slices of bread, and z apples on the first day of the diet. If cheese contains 100 calories per ounce, bread 110 calories per slice, and apples 80 calories each, write an inequality that must be satisfied by x, y, and z.

Graph the solution set of the system of linear inequalities.

21. $\begin{aligned} 2x - y &\le 3 \\ 2x + 3y &\ge -3 \end{aligned}$

22. $\begin{aligned} x - y &\le 4 \\ 2x + y &\ge 6 \end{aligned}$

23. $\begin{aligned} 3x - y &\ge -7 \\ 3x + y &\le -2 \end{aligned}$

24. $\begin{aligned} 3x - 2y &> 1 \\ 2x + 3y &\le 18 \end{aligned}$

25. $\begin{aligned} 3x - 2y &\ge -4 \\ 2x - y &\le 5 \\ y &\ge 1 \end{aligned}$

26. $\begin{aligned} 2x - y &\ge -3 \\ x + y &\le 5 \\ y &\ge 1 \end{aligned}$

27. $\begin{aligned} 2x - y &\le 5 \\ x + 2y &\ge 1 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$

28. $\begin{aligned} -x + 3y &\le 2 \\ 4x + 3y &\le 18 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$

29. $\begin{aligned} 3x + y &\le 6 \\ x - 2y &\le -1 \\ x &\ge 2 \end{aligned}$

30. $\begin{aligned} x - y &\ge -2 \\ x + y &\ge -5 \\ y &\ge 0 \end{aligned}$

31. $\begin{aligned} 3x - 2y &\le -6 \\ 8x + 3y &\le 24 \\ 5x + 4y &\ge 20 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$

32. $\begin{aligned} 2x + 3y &\ge 18 \\ x + 3y &\ge 12 \\ 4x + 3y &\ge 24 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$

33. A farmer has 10 quarts of milk and 15 quarts of cream, which he will use to make ice cream and yogurt. Each quart of ice cream requires 0.4 quart of milk and 0.2 quart of cream, and each quart of yogurt requires 0.2 quart of milk and 0.4 quart of cream. Graph the set of points representing the possible production of ice cream and of yogurt.

34. A coffee packer uses Jamaican and Colombian coffee to prepare a mild blend and a strong blend. Each pound of mild blend contains $\frac{1}{2}$ pound of Jamaican coffee and $\frac{1}{2}$ pound of Colombian coffee, and each pound of the strong blend requires $\frac{1}{4}$ pound of Jamaican coffee and $\frac{3}{4}$ pound of Colombian coffee. The packer has available 100 pounds of Jamaican coffee and 125 pounds of Colombian coffee. Graph the set of points representing the possible production of the two blends.

35. A trust fund of \$100,000 that has been established to provide university scholarships must adhere to certain restrictions.

- No more than half of the fund may be invested in common stocks.
- No more than \$35,000 may be invested in preferred stocks.
- No more than \$60,000 may be invested in all types of stocks.
- The amount invested in common stocks may not be more than twice the amount invested in preferred stocks.

Graph the solution set representing the possible investments in common and preferred stocks.

36. An institution serves a luncheon consisting of two dishes, A and B, whose nutritional content in grams per unit served is given in the accompanying table.

	Fat	Carbohydrate	Protein
A	1	1	2
B	2	1	6

The meal is to provide no more than 10 grams of fat, no more than 7 grams of carbohydrate, and at least 6 grams of protein. Graph the solution set of possible quantities of dishes A and B.

9.6 LINEAR PROGRAMMING (Optional)

Let's pose the following problem:

A lot is zoned for an apartment building to consist of no more than 40 apartments, totaling no more than 45,000 square feet. A builder is planning to construct 1-bedroom apartments, each of which will require 1000 square feet and will rent for \$200 per month, and 2-bedroom apartments, each of which will utilize 1500 square feet and will rent for $280 per month. If all available apartments can be rented, how many apartments of each type should be built to maximize the builder's monthly rental revenue?

If we let x denote the number of 1-bedroom units and y denote the number of 2-bedroom units, the accompanying table displays the information given in the problem.

	Number of units	Square feet	Rental
1-bedroom	x	1,000	$200
2-bedroom	y	1,500	280
Total	40	45,000	z

Using the methods of the previous section, we can translate the **constraints** or requirements on the variables x and y into a system of inequalities. The total number of apartments is $x + y$, so we have

$$x + y \leq 40 \quad \text{number of units constraint}$$

Since each 1-bedroom apartment occupies 1000 square feet of space, x apartments will occupy $1000x$ square feet of space. Similarly, the 2-bedroom apartments will require $1500y$ square feet of space. The total amount of space needed is $1000x + 1500y$, so we must have

$$1000x + 1500y \leq 45{,}000 \quad \text{square footage constraint}$$

Moreover, since x and y denote the number of apartments to be built, we must have $x \geq 0$, $y \geq 0$. Thus, we have obtained the following system of inequalities:

$$\begin{aligned} x + \quad\; y &\leq 40 && \text{number of units constraint} \\ 1000x + 1500y &\leq 45{,}000 && \text{square footage constraint} \\ x &\geq 0 && \text{need for number of apartments} \\ y &\geq 0 && \quad \text{to be nonnegative} \end{aligned}$$

We can graph the solution set of this system of linear inequalities as in Figure 17.

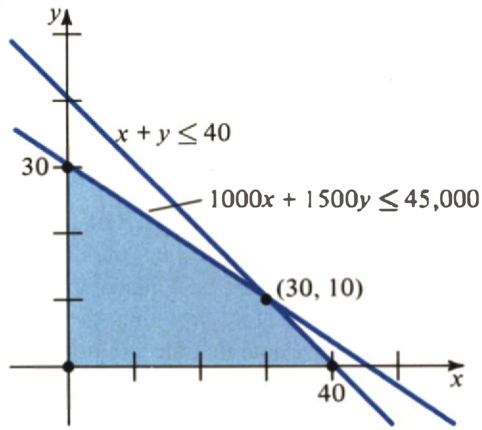


FIGURE 17

But the problem as stated asks that we *maximize* the monthly rental

$$z = 200x + 280y$$

a requirement that we have never before seen in a mathematical problem of this sort! It is this requirement to **optimize,** that is, to seek a maximum or a minimum value of a linear expression, that characterizes a linear programming problem.

Linear Programming Problem

A **linear programming problem** seeks the optimal (either the largest or the smallest) value of a linear expression called the **objective function** while satisfying constraints that can be formulated as a system of linear inequalities.

Returning to our apartment builder, we can state the linear programming problem in this way:

$$\begin{aligned} \text{maximize } z &= 200x + 280y \\ \text{subject to } \quad x + y &\leq 40 \\ 1000x + 1500y &\leq 45{,}000 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

Then the coordinates of each point of the solution set shown in Figure 17 are a **feasible solution;** that is, the coordinates give us ordered pairs (a, b) that satisfy the system of linear inequalities. But which points provide us with values of x and y that maximize the rental income z? For example, the points (40, 0) and (15, 20) are feasible solutions, yielding these results for z:

x	y	$z = 200x + 280y$
40	0	8000
15	20	8600

Clearly, building 15 one-bedroom and 20 two-bedroom units yields a higher rental revenue than building 40 one-bedroom units, but is there a solution that will yield a still higher value for z?

Before providing the key to solving linear programming problems, we first must note that the solution set is bounded by straight lines, and we use the term **vertex** to denote an intersection point of any two boundary lines. We are then ready to state the following theorem.

Fundamental Theorem of Linear Programming If a linear programming problem has an optimal solution, that solution occurs at a vertex of the set of feasible solutions.

With this result, the builder need only examine the vertices of the solution set of Figure 17, rather than considering each of the infinite number of feasible solutions—a bewildering task! We then evaluate the objective function $z = 200x + 280y$ for the coordinates of the vertices (0, 0), (0, 30), (40, 0), and (30, 10).

x	y	$z = 200x + 280y$
0	0	0
0	30	8400
40	0	8000
30	10	8800

Since the largest value of z is 8800 and this value corresponds to $x = 30$, $y = 10$, the builder finds that the optimal strategy is to build 30 one-bedroom and 10 two-bedroom units.

We can now illustrate the steps in solving a linear programming problem.

EXAMPLE 1

Solve the linear programming problem

$$\begin{aligned} &\text{minimize } z = x - 4y \\ &\text{subject to} \quad x + 2y \le 10 \\ &\qquad -x + 4y \le 8 \\ &\qquad x \ge 0 \\ &\qquad y \ge 1 \end{aligned}$$

SOLUTION

Linear Programming

Step 1. Sketch the solution set of the system of linear inequalities.

Step 1.

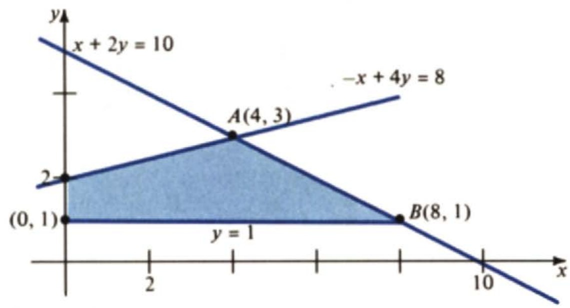


FIGURE 18

Step 2. Determine all vertices of the solution set.

Step 2. The vertices (0, 1) and (0, 2) are the y-intercepts of the lines whose equations are $y = 1$ and $-x + 4y = 8$, respectively. The vertex B in Figure 18 is the intersection of the lines $y = 1$ and $x + 2y = 10$ and is seen to be (8, 1). The vertex A of Figure 18 is the intersection of the lines whose equations are

$$-x + 4y = 8$$

and

$$x + 2y = 10$$

Solving the system of equations (try elimination) yields the vertex $A(4, 3)$.

Step 3. Evaluate the objective function for the coordinates of each vertex.

Step 3.

Vertex	x	y	$z = x - 4y$
(0, 1)	0	1	−4
(0, 2)	0	2	−8
(8, 1)	8	1	4
(4, 3)	4	3	−8

Step 4. The point or points providing the optimal value of the objective function are solutions of the linear programming problem.

Step 4. The minimal value of the objective function is −8, which occurs at the vertices (0, 2) and (4, 3). Thus, $x = 0$, $y = 2$ and $x = 4$, $y = 3$ are both solutions of the linear programming problem.

Linear programming problems occur in real-life situations with great frequency. In certain industries these problems can involve thousands of variables and hundreds of constraints. Obviously, the method of graphical solution we presented for two variables cannot be used. A solution method known as the simplex algorithm was first devised by George Dantzig in 1947. Despite the sophistication of this approach, the number of calculations required becomes unmanageably large for hand computation for even relatively small numbers of constraints. Fortunately, the discovery of the simplex algorithm occurred at the time electronic computers made their initial appearance. Since then industries such as oil refining and steel production have used linear programming to determine the optimum use of their facilities.

EXERCISE SET 9.6

In Exercises 1–8 find the minimum value and the maximum value of the linear expression, subject to the given constraints. Indicate coordinates of the vertices at which the minimum and maximum values occur.

1. $x - \frac{1}{2}y$ subject to

$$\begin{aligned} 3x - y &\geq 1 \\ x &\geq 0 \\ x &\leq 5 \\ y &\geq 0 \end{aligned}$$

2. $2x + y$ subject to

$$\begin{aligned} x + y &\leq 4 \\ x &\geq 1 \\ y &\geq 2 \end{aligned}$$

3. $\frac{1}{2}x - 2y$ subject to

$$\begin{aligned} x + 2y &\leq 6 \\ -2x + 3y &\leq 2 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

4. $0.2x + 0.8y$ subject to

$$\begin{aligned} x + 3y &\leq 8 \\ x - 4y &\geq 1 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

5. $2x - y$ subject to

$$\begin{aligned} -x + y &\leq 0 \\ 3x + 4y &\geq 6 \\ x &\leq 4 \end{aligned}$$

6. $x + 3y$ subject to

$$\begin{aligned} 2x + y &\geq 2 \\ 4x + 5y &\leq 40 \\ x &\geq 0 \\ y &\geq 1 \\ y &\leq 6 \end{aligned}$$

7. $2x - y$ subject to

$$\begin{aligned} -x + 2y &\leq 8 \\ x + 2y &\geq 12 \\ 5x + 2y &\leq 44 \\ x &\geq 3 \end{aligned}$$

8. $y - x$ subject to

$$\begin{aligned} -5x + 2y &\leq 10 \\ 5x + 6y &\leq 50 \\ 5x + y &\leq 20 \\ x &\geq 0 \\ y &\geq 1 \end{aligned}$$

9. A firm has budgeted \$1500 for display space at a toy show. Two types of display booths are available: "preferred space" costs \$18 per square foot, with a minimum rental of 60 square feet, and "regular space" costs \$12 per square foot, with a minimum rental of 30 square feet. It is estimated that there will be 120 visitors for each square foot of "preferred space" and 60 visitors for each square foot of "regular space." How should the firm allot its budget to maximize the number of potential clients that will visit the booths?

10. A company manufactures an eight-bit computer and a sixteen-bit computer. To meet existing orders, it must schedule at least 50 eight-bit computers for the next production cycle and can produce no more than 150 eight-bit computers. The manufacturing facilities are adequate to produce no more than 300 sixteen-bit computers, but the total number of computers that can be produced cannot exceed 400. The profit on each eight-bit computer is \$310; on each sixteen-bit computer the profit is \$275. Find the number of computers of each type that should be manufactured to maximize profit.

11. Swift Truckers is negotiating a contract with Better Spices, which uses two sizes of containers: large, 4-cubic-foot containers weighing 10 pounds and small, 2-cubic-foot containers weighing 8 pounds. Swift Truckers will use a vehicle that can handle a maximum load of 3280 pounds and a cargo size of up to 1000 cubic feet. The firms have agreed on a shipping rate of 50 cents for each large container and 30 cents for each small container. How many containers of each type should Swift place on a truck to maximize income?

12. A bakery makes both yellow cake and white cake. Each pound of yellow cake requires $\frac{1}{4}$ pound of flour and $\frac{1}{4}$ pound of sugar; each pound of white cake requires $\frac{1}{3}$ pound of flour and $\frac{1}{5}$ pound of sugar. The baker finds that 100 pounds of flour and 80 pounds of sugar are available. If yellow cake sells for \$3 per pound and white cake sells for \$2.50 per pound, how many pounds of each cake should the bakery produce to maximize income, assuming that all cakes baked can be sold?

13. A shop sells a mixture of Java and Colombian coffee beans for \$4 per pound. The shopkeeper has allocated \$1000 for buying fresh beans and finds that he must pay \$1.50 per pound for Java beans and \$2 per pound for Colombian beans. In a satisfactory mixture the weight of Colombian beans will be at least twice and no more than four times the weight of the Java beans. How many pounds of each type of coffee bean should be ordered to maximize the profit if all the mixture can be sold?

14. A pension fund plans to invest up to \$50,000 in U.S. Treasury bonds yielding 12% interest per year and corporate bonds yielding 15% interest per year. The fund manager is told to invest a minimum of \$25,000 in the Treasury bonds and a minimum of \$10,000 in the corporate bonds, with no more than $\frac{1}{4}$ of the total investment to be in corporate bonds. How much should the manager invest in each type of bond to achieve a maximum amount of annual interest? What is the maximum interest?

15. A farmer intends to plant crops A and B on all or part of a 100-acre field. Seed for crop A costs \$6 per acre, and labor and equipment costs \$20 per acre. For crop B, seed costs \$9 per acre, and labor and equipment costs \$15 per acre. The farmer cannot spend more than \$810 for seed and \$1800 for labor and equipment. If the income per acre is \$150 for crop A and \$175 for crop B, how many acres of each crop should be planted to maximize total income?

16. The farmer in Exercise 15 finds that a worldwide surplus in crop B reduces the income to \$140 per acre while the income for crop A remains steady at \$150 per acre. How many acres of each crop should be planted to maximize total income?

17. In preparing food for the college cafeteria, a dietitian will combine Volume Pack A and Volume Pack B. Each pound of Volume Pack A costs \$2.50 and contains 4 units of carbohydrate, 3 units of protein, and 5 units of fat. Each pound of Volume Pack B costs \$1.50 and contains 3 units of carbohydrate, 4 units of protein, and 1 unit of fat. If minimum monthly requirements are 60 units of carbohydrates, 52 units of protein, and 42 units of fat, how many pounds of each food pack will the dietitian use to minimize costs?

18. A lawn service uses a riding mower that cuts a 5000-square-foot area per hour and a smaller mower that cuts a 3000-square-foot area per hour. Surprisingly, each mower uses $\frac{1}{2}$ gallon of gasoline per hour. Near the end of a long summer day, the supervisor finds that both mowers are empty and that there remains 0.6 gallon of gasoline in the storage cans. To conclude the day at a sensible point, at least 4000 square feet of lawn must still be mowed. If the cost of operating the riding mower is \$9 per hour and the cost of operating the smaller mower is \$5 per hour, how much of the remaining gasoline should be allocated to each mower to do the job at the least possible cost?

TERMS AND SYMBOLS

system of equations (p. 375)
solution of a system of equations (p. 375)
consistent system (p. 377)
inconsistent system (p. 377)
linear system (p. 377)
method of substitution (p. 377)
method of elimination (p. 381)
equivalent system (p. 391)
Gaussian elemination (p. 391)
triangular form (p. 391)
back-substitution (p. 392)
half-plane (p. 396)
linear inequality in two variables (p. 397)
graph of a linear inequality (p. 397)
system of linear inequalities (p. 400)
solution of a system of linear inequalities (p. 400)
constraints (p. 406)
optimize (p. 407)
linear programming problem (p. 407)
objective function (p. 407)
feasible solution (p. 407)
vertex (p. 408)

KEY IDEAS FOR REVIEW

- ☐ The method of substitution involves solving an equation for one variable and substituting the result in another equation.
- ☐ The method of elimination involves multiplying an equation by a nonzero constant so that, when the equation is added to a second equation, a variable drops out.
- ☐ A consistent system of equations has one or more real solutions; an inconsistent system has no real solutions.
- ☐ The graph of a pair of linear equations in two variables is two straight lines, which may either (a) intersect in a point, (b) be parallel, or (c) be the same line. If the two straight lines intersect, the coordinates of the point of intersection are a solution of the system of linear equations. If the lines do not intersect, the system is inconsistent.
- ☐ With any method of solution, it is possible to detect the special cases when lines are parallel or reduce to the same line.
- ☐ It is often easier and more natural to set up word problems by using two or more variables.
- ☐ Gaussian elimination is a systematic way of transforming a linear system to triangular form. A linear system in triangular form is easily solved by back-substitution.
- ☐ The solution of a system of linear inequalities can be found graphically as the region satisfying all the inequalities.
- ☐ To solve a linear programming problem, it is only necessary to consider the vertices of the region of feasible solutions.

REVIEW EXERCISES

Solutions to exercises whose numbers are in color are in the Solutions section in the back of the book.

9.1 In Exercises 1 and 2 solve the given system by graphing.

1. $\begin{aligned} 2x + 3y &= 2 \\ 4x + 5y &= 3 \end{aligned}$

2. $\begin{aligned} y^2 &= x - 1 \\ x + y &= 7 \end{aligned}$

In Exercises 3–8 solve the given system by the method of substitution.

3. $\begin{aligned} -x + 6y &= -11 \\ 2x + 5y &= 5 \end{aligned}$

4. $\begin{aligned} 2x - 4y &= -14 \\ -x - 6y &= -5 \end{aligned}$

5. $\begin{aligned} 2x + y &= 0 \\ x - 3y &= \frac{7}{4} \end{aligned}$

6. $\begin{aligned} x^2 + y^2 &= 25 \\ x + 3y &= 5 \end{aligned}$

7. $\begin{aligned} x^2 - 4y^2 &= 9 \\ y - 2x &= 0 \end{aligned}$

8. $\begin{aligned} y^2 &= 4x \\ y^2 + x - 2y &= 12 \end{aligned}$

9.2 In Exercises 9–14 solve the given system by the method of elimination.

9. $\begin{aligned} x + 4y &= 17 \\ 2x - 3y &= -21 \end{aligned}$

10. $\begin{aligned} 5x - 2y &= 14 \\ -x - 3y &= 4 \end{aligned}$

11. $\begin{aligned} -3x + y &= -13 \\ 2x - 3y &= 11 \end{aligned}$

12. $\begin{aligned} 7x - 2y &= -20 \\ 3x - y &= -9 \end{aligned}$

13. $\begin{aligned} y^2 &= 2x - 1 \\ x - y &= 2 \end{aligned}$

14. $\begin{aligned} x^2 + y^2 &= 9 \\ y &= x^2 + 3 \end{aligned}$

9.3 15. The sum of a two-digit number and its tens digit is 49. If we reverse the digits of the number, the resulting number is 9 more than the original number. Find the number.

16. The sum of the digits of a two-digit number is 9. The sum of the number and its units digit is 74. Find the number.

17. Five pounds of hamburger and 4 pounds of steak cost \$22, and 3 pounds of hamburger and 7 pounds of steak cost \$28.15. Find the cost per pound of hamburger and of steak.

18. An airplane flying with a tail wind can complete a journey of 3500 kilometers in 5 hours. Flying the reverse direction, the plane completes the same trip in 7 hours. What is the speed of the plane in still air?

19. A manufacturer of faucets finds that the supply S and demand D are related to price p as follows:

$$\begin{aligned} S &= 3p + 2 \\ D &= -2p + 17 \end{aligned}$$

Find the equilibrium price and the number of faucets sold at that price.

20. An auto repair shop finds that its monthly expenditure (in dollars) is given by $C = 4025 + 9x$, where x is the total number of hours worked by all employees. If the revenue received (in dollars) is given by $R = 16x$, find the break-even point in number of work hours, and the total revenue received at that point.

9.4 In Exercises 21–24 use Gaussian elimination to solve the given linear system.

21. $$\begin{aligned} -3x - y + z &= 12 \\ 2x + 5y - 2z &= -9 \\ -x + 4y + 2z &= 15 \end{aligned}$$

22. $$\begin{aligned} 3x + 2y - z &= -8 \\ 2x + 3z &= 5 \\ x - 4y &= -4 \end{aligned}$$

23. $$\begin{aligned} 5x - y + 2z &= 10 \\ -2x + 3y - z &= -7 \\ 3x + 2z &= 7 \end{aligned}$$

24. $$\begin{aligned} x + 4y &= 4 \\ -x + 3z &= -4 \\ 2x + 2y - z &= \frac{41}{6} \end{aligned}$$

In Exercises 25–28 solve by any method.

25. $$\begin{aligned} 2x + 3y &= 6 \\ 3x - y &= -13 \end{aligned}$$

26. $$\begin{aligned} x + 2y &= 0 \\ -x + 4y &= 5 \end{aligned}$$

27. $$\begin{aligned} 2x + 3y - z &= -4 \\ x - 2y + 2z &= -6 \\ 2x - 3z &= 5 \end{aligned}$$

28. $$\begin{aligned} 2x + 2y - 3z &= -4 \\ 3y - z &= -4 \\ 4x - y + z &= 4 \end{aligned}$$

9.5 In Exercises 29–34 graph the solution set of the linear inequality or system of linear inequalities.

29. $x - 2y \le 5$

30. $2x + y > 4$

31. $$\begin{aligned} 2x + 3y &\le 2 \\ x - y &\ge 1 \end{aligned}$$

32. $$\begin{aligned} x - 2y &\ge 4 \\ 2x - y &\le 2 \end{aligned}$$

33. $$\begin{aligned} 2x + 3y &\le 6 \\ x &\ge 0 \\ y &\ge 1 \end{aligned}$$

34. $$\begin{aligned} 2x + y &\le 4 \\ 2x - y &\le 3 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

9.6 In Exercises 35 and 36 solve the given linear programming problem.

35. maximize $z = 5y - x$
subject to
$$\begin{aligned} 8y - 3x &\le 36 \\ 6x + y &\le 30 \\ y &\ge 1 \\ x &\ge 0 \end{aligned}$$

36. minimize $z = x + 4y$
subject to
$$\begin{aligned} 4x - y &\ge 8 \\ 4x + y &\le 24 \\ 5y + 4x &\ge 32 \end{aligned}$$

PROGRESS TEST 9A

1. Solve the linear system by graphing:

$$\begin{aligned} 3x - y &= -17 \\ x + 2y &= -1 \end{aligned}$$

In Problems 2 and 3 solve the given system by the method of substitution.

2. $\begin{aligned} 2x + y &= 4 \\ 3x - 2y &= -15 \end{aligned}$

3. $\begin{aligned} y^2 &= 5x \\ y^2 - x^2 &= 6 \end{aligned}$

In Problems 4 and 5 solve the given linear system by the method of elimination.

4. $\begin{aligned} x - 2y &= 7 \\ 3x + 4y &= -9 \end{aligned}$

5. $\begin{aligned} x^2 + y^2 &= 25 \\ 4x^2 - y^2 &= 20 \end{aligned}$

6. The sum of the digits of a two-digit number is 11. If the sum of the number and its tens digit is 41, find the number.
7. An elegant men's shop is having a post-Christmas sale. All shirts are reduced to one low price, and all ties are reduced to an even lower price. A customer purchases 3 ties and 7 shirts, paying \$135. Another customer selects 5 ties and 3 shirts and pays \$95. What is the sale price of each tie and of each shirt?
8. A school cafeteria manager finds that the weekly cost of operation is \$1375 plus \$1.25 for every meal served. If the average meal produces a revenue of \$2.50, find the number of meals served that results in zero profit and zero loss.
9. Solve by Gaussian elimination:

$$\begin{aligned} 3x + 2y - z &= -4 \\ x - y + 3z &= 12 \\ 2x - y - 2z &= -20 \end{aligned}$$

Solve Problems 10 and 11 by any method.

10. $\begin{aligned} -3x + 2y &= -1 \\ 6x - y &= -1 \end{aligned}$

11. $\begin{aligned} 3x + y - 2z &= 8 \\ 3y - 4z &= 14 \\ 3x + \frac{1}{2}y + z &= 1 \end{aligned}$

In Problems 12 and 13 graph the solution set of the system of linear inequalities.

12. $\begin{aligned} x - 2y &\le 1 \\ 3x + 2y &\ge 4 \end{aligned}$

13. $\begin{aligned} 2x + y &\le 10 \\ -x + 3y &\le 12 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$

PROGRESS TEST 9B

1. Solve the system by graphing:

$$\begin{aligned} x^2 + 3y^2 &= 12 \\ x + 3y &= 6 \end{aligned}$$

In Problems 2 and 3 solve the given linear system by the method of substitution.

2. $\begin{aligned} 3x + y &= 1 \\ x - \frac{1}{3}y &= 1 \end{aligned}$

3. $\begin{aligned} 2x - 3y &= 1 \\ 3x - 2y &= 1 \end{aligned}$

In Problems 4 and 5 solve the given system by the method of elimination.

4. $\begin{aligned} -2x + 4y &= 5 \\ -x + 3y &= 2 \end{aligned}$

5. $\begin{aligned} x^2 - y^2 &= 9 \\ x^2 + y^2 &= 41 \end{aligned}$

6. The sum of the digits of a two-digit number is 14. The difference between the number and that obtained by reversing the digits of the number is 18. Find the number.
7. A motorboat can travel 60 kilometers downstream in 3 hours, and the return trip requires 4 hours. What is the rate of the current?
8. Suppose that supply and demand for a particular tennis racket is related to price p by

$$\begin{aligned} S &= 5p + 1 \\ D &= -2p + 43 \end{aligned}$$

Find the equilibrium price and the number of rackets sold at this price.

9. Solve by Gaussian elimination:

$$\begin{aligned} x + 2z &= 7 \\ 3y + 4z &= -10 \\ -2x + y - 2z &= -14 \end{aligned}$$

In Problems 10 and 11 solve by any method.

10. $\begin{aligned} x - 2y &= 1 \\ 3x + 2y &= 1 \end{aligned}$

11. $\begin{aligned} 3x + y - 7z &= -4 \\ 2x - 2y - z &= 9 \\ -2x + y + 3z &= -4 \end{aligned}$

In Problems 12 and 13 graph the solution set of the system of linear inequalities.

12. $\begin{aligned} 2x - 3y &\ge 6 \\ 3x + y &\le 3 \end{aligned}$

13. $\begin{aligned} 2x + y &\le 4 \\ 2x - 5y &\le 5 \\ y &\ge 1 \end{aligned}$

10

MATRICES AND DETERMINANTS

The material on matrices and determinants presented in this chapter serves as an introduction to linear algebra, a mathematical subject that is used in the natural sciences, business and economics, and the social sciences. Since matrix methods may require millions of numerical computations, computers have played an important role in expanding the use of matrix techniques to a wide variety of practical problems.

Our study of matrices and determinants will focus on their application to the solution of systems of linear equations. We will see that the method of Gaussian elimination studied in the previous chapter can be neatly implemented using matrices. We will show that matrix notation provides a convenient means for writing linear systems and that the inverse of a matrix enables us to solve such a system. Determinants will also provide us with an additional technique, known as Cramer's rule, for the solution of certain linear systems.

It should be emphasized that this material is a very brief introduction to matrices and determinants. Their properties and applications are both extensive and important.

10.1 MATRICES AND LINEAR SYSTEMS

DEFINITIONS

We have already studied several methods for solving a linear system such as

$$\begin{aligned} 2x + 3y &= -7 \\ 3x - y &= 17 \end{aligned}$$

This system can be displayed by a **matrix,** which is simply a rectangular array of mn real numbers arranged in m horizontal rows and n vertical columns. The numbers are called the **entries** or **elements** of the matrix and are enclosed within brackets. Thus,

$$A = \begin{array}{c} \begin{bmatrix} 2 & 3 & -7 \\ 3 & -1 & 17 \end{bmatrix} \begin{array}{l} \leftarrow \\ \leftarrow \end{array} \text{rows} \\ \uparrow \qquad \uparrow \qquad \uparrow \\ \text{columns} \end{array}$$

is a matrix consisting of two rows and three columns, whose entries are obtained from the two given equations. In general, a matrix of m rows and n columns is said to be of **dimension m by n**, written $m \times n$. The matrix A is seen to be of dimension 2×3. If the numbers of rows and columns of a matrix are both equal to n, the matrix is called a **square matrix** of **order** n.

EXAMPLE 1

(a)
$$A = \begin{bmatrix} -1 & 4 \\ 0.1 & -2 \end{bmatrix}$$

is a 2×2 matrix. Since matrix A has two rows and two columns, it is a square matrix of order 2.

(b)
$$B = \begin{bmatrix} 4 & -5 \\ -2 & 1 \\ 3 & 0 \end{bmatrix}$$

has three rows and two columns and is a 3×2 matrix.

(c)
$$C = [-8 \quad 6 \quad 1]$$

is a 1×3 matrix and is called a **row matrix** because it has precisely one row.

(d)
$$D = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

is a 2×1 matrix and is called a **column matrix** because it has precisely one column.

SUBSCRIPT NOTATION

There is a convenient way of denoting a general $m \times n$ matrix, using "double subscripts."

$$\begin{array}{cl}
A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} & \begin{array}{l} \leftarrow \text{first row} \\ \leftarrow \text{second row} \\ \\ \leftarrow i\text{th row} \\ \\ \leftarrow m\text{th row} \end{array} \\
\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \text{first} & \text{second} & j\text{th} & n\text{th} \\ \text{column} & \text{column} & \text{column} & \text{column} \end{array} &
\end{array}$$

Thus, a_{ij} is the entry in the ith row and jth column of the matrix A. It is customary to write $A = [a_{ij}]$ to indicate that a_{ij} is the entry in row i and column j of matrix A.

EXAMPLE 2

Let

$$A = \begin{bmatrix} 3 & -2 & 4 & 5 \\ 9 & 1 & 2 & 0 \\ -3 & 2 & -4 & 8 \end{bmatrix}$$

Matrix A is of dimension 3×4. The element a_{12} is found in the first row and second column and is seen to be -2. Similarly, we see that $a_{31} = -3$, $a_{33} = -4$, and $a_{34} = 8$.

PROGRESS CHECK

Let

$$B = \begin{bmatrix} 4 & 8 & 1 \\ 2 & -5 & 3 \\ -8 & 6 & -4 \\ 0 & 1 & -1 \end{bmatrix}$$

Find the following:

(a) b_{11} (b) b_{23} (c) b_{31} (d) b_{42}

ANSWERS

(a) 4 (b) 3 (c) -8 (d) 1

COEFFICIENT AND AUGMENTED MATRICES

If we begin with the system of linear equations

$$\begin{aligned} 2x + 3y &= -7 \\ 3x - y &= 17 \end{aligned}$$

the matrix

$$\begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix}$$

in which the first column is formed from the coefficients of x and the second column is formed from the coefficients of y, is called the **coefficient matrix.** The matrix

$$\left[\begin{array}{cc|c} 2 & 3 & -7 \\ 3 & -1 & 17 \end{array}\right]$$

which includes a column consisting of the right-hand sides of the equations, separated from the other columns by a dashed line, is called the **augmented matrix.**

EXAMPLE 3

Write a system of linear equations that corresponds to the augmented matrix

$$\left[\begin{array}{rrr|r} -5 & 2 & -1 & 15 \\ 0 & -2 & 1 & -7 \\ \frac{1}{2} & 1 & -1 & 3 \end{array}\right]$$

SOLUTION

We attach the unknown x to the first column, the unknown y to the second column, and the unknown z to the third column. The resulting system is

$$\begin{aligned} -5x + 2y - z &= 15 \\ -2y + z &= -7 \\ \tfrac{1}{2}x + y - z &= 3 \end{aligned}$$

Now that we have seen how a matrix can be used to represent a system of linear equations, we next proceed to show how routine operations on that matrix can yield the solution of the system. These "matrix methods" are simply a clever streamlining of the methods already studied.

In the previous chapter we used three elementary operations to transform a system of linear equations into triangular form. When applying the same procedures to a matrix, we speak of rows, columns, and elements instead of equations, variables, and coefficients. The three elementary operations that yield an equivalent system now become the **elementary row operations.**

Elementary Row Operations

The following elementary row operations transform an augmented matrix into an equivalent system.

1. Interchange any two rows.
2. Multiply each element of any row by a constant $k \neq 0$.
3. Replace each element of a given row with the sum of itself plus k times the corresponding element of any other row.

GAUSSIAN ELIMINATION

The method of Gaussian elimination introduced in the previous chapter can now be restated in terms of matrices. By use of elementary row operations, we seek to transform an augmented matrix into a matrix for which $a_{ij} = 0$ when $i > j$. The resulting matrix will have the following appearance for a system of three linear equations in three unknowns.

$$\left[\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right]$$

Since this matrix represents a linear system in triangular form, back-substitution will provide a solution of the original system. We will illustrate the process with an example.

EXAMPLE 4

Solve the system

$$\begin{aligned} x - y + 4z &= 4 \\ 2x + 2y - z &= 2 \\ 3x - 2y + 3z &= -3 \end{aligned}$$

SOLUTION

We describe and illustrate the steps of the procedure.

Gaussian Elimination	
Step 1. Form the augmented matrix.	*Step 1.* The augmented matrix is $\left[\begin{array}{rrr\|r} 1 & -1 & 4 & 4 \\ 2 & 2 & -1 & 2 \\ 3 & -2 & 3 & -3 \end{array}\right]$
Step 2. If necessary, interchange rows to make sure that a_{11}, the first element of the first row, is nonzero. We call a_{11} the **pivot element** and row 1 the **pivot row.**	*Step 2.* We see that $a_{11} = 1 \neq 0$. The pivot element, a_{11}, is shown in color.
Step 3. Arrange to have 0 as the first element of every row below row 1. To do so, replace each row after row 1 with the sum of itself and an appropriate multiple of row 1.	*Step 3.* To make $a_{21} = 0$, replace row 2 with the sum of itself and -2 times row 1; to make $a_{31} = 0$, replace row 3 with the sum of itself and -3 times row 1. $\left[\begin{array}{rrr\|r} 1 & -1 & 4 & 4 \\ 0 & 4 & -9 & -6 \\ 0 & 1 & -9 & -15 \end{array}\right]$
Step 4. Repeat the process defined by Steps 2 and 3, allowing row 2, row 3, and so on to play the role of the first row. Thus, each row in turn serves as the pivot row.	*Step 4.* Since $a_{22} = 4 \neq 0$, it will serve as the next pivot element and is shown in color. To make $a_{32} = 0$, replace row 3 with the sum of itself and $-\frac{1}{4}$ times row 2. $\left[\begin{array}{rrr\|r} 1 & -1 & 4 & 4 \\ 0 & 4 & -9 & -6 \\ 0 & 0 & -\frac{27}{4} & -\frac{27}{2} \end{array}\right]$
Step 5. The corresponding linear system is in triangular form. Solve by back-substitution.	*Step 5.* The third row of the final matrix yields $-\frac{27}{4}z = -\frac{27}{2}$, $z = 2$

Substituting $z = 2$, we obtain from the second row of the final matrix

$$\begin{aligned} 4y - 9z &= -6 \\ 4y - 9(2) &= -6 \\ y &= 3 \end{aligned}$$

Substituting $y = 3$, $z = 2$, we obtain from the first row of the final matrix

$$\begin{aligned} x - y + 4z &= 4 \\ x - 3 + 4(2) &= 4 \\ x &= -1 \end{aligned}$$

The solution is $x = -1$, $y = 3$, $z = 2$.

PROGRESS CHECK

Solve the linear system by matrix methods.

$$\begin{aligned} 2x + 4y - z &= 0 \\ x - 2y - 2z &= 2 \\ -5x - 8y + 3z &= -2 \end{aligned}$$

ANSWER

$x = 6, \quad y = -2, \quad z = 4$

Note that we have described the process of Gaussian elimination in a manner that will apply to any augmented matrix that is $n \times (n + 1)$; that is, Gaussian elimination may be used on any system of n linear equations in n unknowns that has a unique solution.

It is also permissible to perform elementary row operations in clever ways to simplify the arithmetic. For instance, you may wish to interchange rows, or to multiply a row by a constant to obtain a pivot element equal to 1. We will illustrate these ideas with an example.

EXAMPLE 5

Solve by matrix methods.

$$\begin{aligned} 2y + 3z &= 4 \\ 4x + y + 8z + 15w &= -14 \\ x - y + 2z &= 9 \\ -x - 2y - 3z - 6w &= 10 \end{aligned}$$

SOLUTION

We begin with the augmented matrix and perform a sequence of elementary row operations. The pivot element is shown in color.

$$\left[\begin{array}{rrrr|r} 0 & 2 & 3 & 0 & 4 \\ 4 & 1 & 8 & 15 & -14 \\ 1 & -1 & 2 & 0 & 9 \\ -1 & -2 & -3 & -6 & 10 \end{array}\right]$$

Augmented matrix.
Note that $a_{11} = 0$.

$$\left[\begin{array}{rrrr|r} 1 & -1 & 2 & 0 & 9 \\ 4 & 1 & 8 & 15 & -14 \\ 0 & 2 & 3 & 0 & 4 \\ -1 & -2 & -3 & -6 & 10 \end{array}\right]$$

Interchanged rows 1 and 3 so that $a_{11} = 1$.

$$\left[\begin{array}{rrrr|r} 1 & -1 & 2 & 0 & 9 \\ 0 & 5 & 0 & 15 & -50 \\ 0 & 2 & 3 & 0 & 4 \\ 0 & -3 & -1 & -6 & 19 \end{array}\right]$$

To make $a_{21} = 0$, replaced row 2 with the sum of itself and -4 times row 1.
To make $a_{41} = 0$, replaced row 4 with the sum of itself and row 1.

$$\left[\begin{array}{rrrr|r} 1 & -1 & 2 & 0 & 9 \\ 0 & 1 & 0 & 3 & -10 \\ 0 & 2 & 3 & 0 & 4 \\ 0 & -3 & -1 & -6 & 19 \end{array}\right]$$

Multiplied row 2 by $\frac{1}{5}$ so that $a_{22} = 1$.

$$\left[\begin{array}{rrrr|r} 1 & -1 & 2 & 0 & 9 \\ 0 & 1 & 0 & 3 & -10 \\ 0 & 0 & 3 & -6 & 24 \\ 0 & 0 & -1 & 3 & -11 \end{array}\right]$$

To make $a_{32} = 0$, replaced row 3 with the sum of itself and -2 times row 2.
To make $a_{42} = 0$, replaced row 4 with the sum of itself and 3 times row 2.

$$\left[\begin{array}{rrrr|r} 1 & -1 & 2 & 0 & 9 \\ 0 & 1 & 0 & 3 & -10 \\ 0 & 0 & -1 & 3 & -11 \\ 0 & 0 & 3 & -6 & 24 \end{array}\right]$$

Interchanged rows 3 and 4 so that the next pivot will be $a_{33} = -1$.

$$\left[\begin{array}{rrrr|r} 1 & -1 & 2 & 0 & 9 \\ 0 & 1 & 0 & 3 & -10 \\ 0 & 0 & -1 & 3 & -11 \\ 0 & 0 & 0 & 3 & -9 \end{array}\right]$$

To make $a_{43} = 0$, replaced row 4 with the sum of itself and 3 times row 3.

The last row of the matrix indicates that

$$\begin{aligned} 3w &= -9 \\ w &= -3 \end{aligned}$$

The remaining variables are found by back-substitution.

Third row of final matrix	Second row of final matrix	First row of final matrix
$-z + 3w = -11$	$y + 3w = -10$	$x - y + 2z = 9$
$-z + 3(-3) = -11$	$y + 3(-3) = -10$	$x - (-1) + 2(2) = 9$
$z = 2$	$y = -1$	$x = 4$

The solution is $x = 4$, $y = -1$, $z = 2$, $w = -3$.

Gauss–Jordan Elimination

There is an important variant of Gaussian elimination known as **Gauss–Jordan elimination.** The objective is to transform a linear system into a form that yields a solution without back-substitution. For a 3×3 system that has a unique solution, the final matrix and equivalent linear system will look like this.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{array}\right] \qquad \begin{aligned} x + 0y + 0z &= c_1 \\ 0x + y + 0z &= c_2 \\ 0x + 0y + z &= c_3 \end{aligned}$$

The solution is then seen to be $x = c_1$, $y = c_2$, and $z = c_3$.

The execution of the Gauss–Jordan method is essentially the same as that of Gaussian elimination except that

- the pivot elements are always required to be equal to 1, and
- all elements in a column, other than the pivot element, are forced to be 0.

These objectives are accomplished by the use of elementary row operations, as illustrated in the following example.

EXAMPLE 6

Solve the linear system by the Gauss–Jordan method.

$$\begin{aligned} x - 3y + 2z &= 12 \\ 2x + y - 4z &= -1 \\ x + 3y - 2z &= -8 \end{aligned}$$

SOLUTION

We begin with the augmented matrix. At each stage, the pivot element is shown in color and is used to force all elements in that column (other than the pivot element itself) to be zero.

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 12 \\ 2 & 1 & -4 & -1 \\ 1 & 3 & -2 & -8 \end{array}\right] \qquad \text{Pivot element is } a_{11}.$$

$$\left[\begin{array}{rrr|r} 1 & -3 & 2 & 12 \\ 0 & 7 & -8 & -25 \\ 0 & 6 & -4 & -20 \end{array}\right]$$

To make $a_{21} = 0$, replaced row 2 with the sum of itself and -2 times row 1.
To make $a_{31} = 0$, replaced row 3 with the sum of itself and -1 times row 1.

$$\left[\begin{array}{rrr|r} 1 & -3 & 2 & 12 \\ 0 & 1 & -4 & -5 \\ 0 & 6 & -4 & -20 \end{array}\right]$$

Replaced row 2 with the sum of itself and -1 times row 3 to yield the next pivot, $a_{22} = 1$.

$$\left[\begin{array}{rrr|r} 1 & 0 & -10 & -3 \\ 0 & 1 & -4 & -5 \\ 0 & 0 & 20 & 10 \end{array}\right]$$

To make $a_{12} = 0$, replaced row 1 with the sum of itself and 3 times row 2.
To make $a_{32} = 0$, replaced row 3 with the sum of itself and -6 times row 2.

$$\left[\begin{array}{rrr|r} 1 & 0 & -10 & -3 \\ 0 & 1 & -4 & -5 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

Multiplied row 3 by $\frac{1}{20}$ so that $a_{33} = 1$.

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

To make $a_{13} = 0$, replaced row 1 with the sum of itself and 10 times row 3.
To make $a_{23} = 0$, replaced row 2 with the sum of itself and 4 times row 3.

We can see the solution directly from the final matrix: $x = 2$, $y = -3$, and $z = \frac{1}{2}$.

EXERCISE SET 10.1

In Exercises 1–6 state the dimension of each matrix.

1. $\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 2 & 3 & -1 \end{bmatrix}$

3. $\begin{bmatrix} 4 & 2 & 3 \\ 5 & -1 & 4 \\ 2 & 3 & 6 \\ -8 & -1 & 2 \end{bmatrix}$

4. $\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$

5. $\begin{bmatrix} 4 & 2 & 1 \\ 3 & 1 & 5 \\ -4 & -2 & 3 \end{bmatrix}$

6. $\begin{bmatrix} 3 & -1 & 2 & 6 \\ 2 & 8 & 4 & 1 \end{bmatrix}$

7. Given

$$A = \begin{bmatrix} 3 & -4 & -2 & 5 \\ 8 & 7 & 6 & 2 \\ 1 & 0 & 9 & -3 \end{bmatrix}$$

find

(a) a_{12} (b) a_{22} (c) a_{23} (d) a_{34}

8. Given

$$B = \begin{bmatrix} -5 & 6 & 8 \\ 4 & 1 & 3 \\ 0 & 2 & -6 \\ -3 & 9 & 7 \end{bmatrix}$$

find

(a) b_{13} (b) b_{21} (c) b_{33} (d) b_{42}

In Exercises 9–12 write the coefficient matrix and the augmented matrix for the given linear system.

9. $\begin{aligned} 3x - 2y &= 12 \\ 5x + y &= -8 \end{aligned}$

10. $\begin{aligned} 3x - 4y &= 15 \\ 4x - 3y &= 12 \end{aligned}$

11. $\begin{aligned} \tfrac{1}{2}x + y + z &= 4 \\ 2x - y - 4z &= 6 \\ 4x + 2y - 3z &= 8 \end{aligned}$

12. $\begin{aligned} 2x + 3y - 4z &= 10 \\ -3x + y \phantom{{}-4z} &= 12 \\ 5x - 2y + z &= -8 \end{aligned}$

In Exercises 13–16 write the linear system whose augmented matrix is given.

13. $\left[\begin{array}{cc|c} \frac{3}{2} & 6 & -1 \\ 4 & 5 & 3 \end{array}\right]$

14. $\left[\begin{array}{cc|c} 4 & 0 & 2 \\ -7 & 8 & 3 \end{array}\right]$

15. $\left[\begin{array}{ccc|c} 1 & 1 & 3 & -4 \\ -3 & 4 & 0 & 8 \\ 2 & 0 & 7 & 6 \end{array}\right]$

16. $\left[\begin{array}{ccc|c} 4 & 8 & 3 & 12 \\ 1 & -5 & 3 & -14 \\ 0 & 2 & 7 & 18 \end{array}\right]$

In Exercises 17–20 the augmented matrix corresponding to a linear system has been transformed to the given matrix by elementary row operations. Find a solution of the original linear system.

17. $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right]$

18. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 5 \end{array}\right]$

19. $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & -4 \end{array}\right]$

20. $\left[\begin{array}{ccc|c} 1 & -4 & 2 & -4 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & 5 \end{array}\right]$

In Exercises 21–30 solve the given linear system by applying Gaussian elimination to the augmented matrix.

21. $\begin{aligned} x - 2y &= -4 \\ 2x + 3y &= 13 \end{aligned}$

22. $\begin{aligned} 2x + y &= -1 \\ 3x - y &= -7 \end{aligned}$

23. $\begin{aligned} x + y + z &= 4 \\ 2x - y + 2z &= 11 \\ x + 2y + z &= 3 \end{aligned}$

24. $\begin{aligned} x - y + z &= -5 \\ 3x + y + 2z &= -5 \\ 2x - y - z &= -2 \end{aligned}$

25. $\begin{aligned} 2x + y - z &= 9 \\ x - 2y + 2z &= -3 \\ 3x + 3y + 4z &= 11 \end{aligned}$

26. $\begin{aligned} 2x + y - z &= -2 \\ -2x - 2y + 3z &= 2 \\ 3x + y - z &= -4 \end{aligned}$

27. $\begin{aligned} -x - y + 2z &= 9 \\ x + 2y - 2z &= -7 \\ 2x - y + z &= -9 \end{aligned}$

28. $\begin{aligned} 4x + y - z &= -1 \\ x - y + 2z &= 3 \\ -x + 2y - z &= 0 \end{aligned}$

29. $\begin{aligned} x + y - z + 2w &= 0 \\ 2x + y \phantom{{}-z} - w &= -2 \\ 3x \phantom{{}+y} + 2z \phantom{{}+2w} &= -3 \\ -x + 2y \phantom{{}-z} + 3w &= 1 \end{aligned}$

30. $\begin{aligned} 2x + y \phantom{{}+2z} - 3w &= -7 \\ 3x \phantom{{}+y} + 2z + w &= 0 \\ -x + 2y \phantom{{}+2z} + 3w &= 10 \\ -2x - 3y + 2z - w &= 7 \end{aligned}$

31–40. Solve the linear systems of Exercises 21–30 by applying Gauss–Jordan elimination to the augmented matrix.

10.2 MATRIX OPERATIONS AND APPLICATIONS (Optional)

After defining a new type of mathematical entity, it is useful to define operations using this entity. It is common practice to begin with a definition of equality.

Equality of Matrices

Two matrices are **equal** if they are of the same dimension and their corresponding entries are equal.

EXAMPLE 1

Solve for all unknowns.

$$\begin{bmatrix} -2 & 2x & 9 \\ y-1 & 3 & -4s \end{bmatrix} = \begin{bmatrix} z & 6 & 9 \\ -4 & r & 7 \end{bmatrix}$$

SOLUTION

Equating corresponding elements, we must have

$$\begin{aligned} -2 &= z &\quad \text{or}\quad & z = -2 \\ 2x &= 6 & \text{or}\quad & x = 3 \\ y - 1 &= -4 & \text{or}\quad & y = -3 \\ 3 &= r & \text{or}\quad & r = 3 \\ -4s &= 7 & \text{or}\quad & s = -\tfrac{7}{4} \end{aligned}$$

Matrix addition can be performed only when the matrices are of the same dimension.

Matrix Addition The **sum** of two $m \times n$ matrices A and B is the $m \times n$ matrix obtained by adding the corresponding elements of A and B.

EXAMPLE 2

Given the following matrices,

$$A = [2 \quad -3 \quad 4] \qquad B = [5 \quad 3 \quad 2]$$

$$C = \begin{bmatrix} 1 & 6 & -1 \\ -2 & 4 & 5 \end{bmatrix} \qquad D = \begin{bmatrix} 16 & 2 & 9 \\ 4 & -7 & -1 \end{bmatrix}$$

find (if possible)

(a) $A + B$ (b) $A + D$ (c) $C + D$

SOLUTION

(a) Since A and B are both 1×3 matrices, they can be added, giving

$$A + B = [2 + 5 \quad -3 + 3 \quad 4 + 2] = [7 \quad 0 \quad 6]$$

(b) Matrices A and D are not of the same dimension and cannot be added.

(c) C and D are both 2×3 matrices. Thus,

$$C + D = \begin{bmatrix} 1 + 16 & 6 + 2 & -1 + 9 \\ -2 + 4 & 4 + (-7) & 5 + (-1) \end{bmatrix} = \begin{bmatrix} 17 & 8 & 8 \\ 2 & -3 & 4 \end{bmatrix}$$

Matrices are a natural way of writing the information displayed in a table. For example, Table 1 displays the current inventory of the Quality TV Company

TABLE 1

TV Sets	Boston	Miami	Chicago
17″	140	84	25
19″	62	17	48

at its various outlets. The same data is displayed by the matrix

$$S = \begin{bmatrix} 140 & 84 & 25 \\ 62 & 17 & 48 \end{bmatrix}$$

in which we understand the columns to represent the cities and the rows to represent the sizes of the television sets. If the matrix

$$M = \begin{bmatrix} 30 & 46 & 15 \\ 50 & 25 & 60 \end{bmatrix}$$

specifies the number of sets of each size received at each outlet the following month, then the matrix

$$T = S + M = \begin{bmatrix} 170 & 130 & 40 \\ 112 & 42 & 108 \end{bmatrix}$$

gives the revised inventory.

Suppose the salespeople at each outlet are told that half of the revised inventory is to be placed on sale. To determine the number of sets of each size to be placed on sale, we need to multiply each element of the matrix T by 0.5. When working with matrices, we call a real number such as 0.5 a **scalar** and define **scalar multiplication** as follows.

Scalar Multiplication To multiply a matrix A by a scalar c, multiply each entry of A by c.

EXAMPLE 3

The matrix Q

$$\begin{array}{cc} & \begin{array}{ccc} \text{Regular} & \text{Unleaded} & \text{Premium} \end{array} \\ Q = & \begin{bmatrix} 130 & 250 & 60 \\ 110 & 180 & 40 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{City A} \\ \text{City B} \end{array}$$

shows the quantities (in thousands of gallons) of the principal types of gasolines stored by a refiner at two different locations. It is decided to increase the quantity of each type of gasoline stored at each site by 10%. Use scalar multiplication to determine the desired inventory levels.

SOLUTION

To increase each entry of matrix Q by 10%, we compute the scalar product $1.1Q$.

$$1.1Q = 1.1\begin{bmatrix} 130 & 250 & 60 \\ 110 & 180 & 40 \end{bmatrix}$$

$$= \begin{bmatrix} 1.1(130) & 1.1(250) & 1.1(60) \\ 1.1(110) & 1.1(180) & 1.1(40) \end{bmatrix} = \begin{bmatrix} 143 & 275 & 66 \\ 121 & 198 & 44 \end{bmatrix}$$

We denote $A + (-1)B$ by $A - B$ and refer to this as the **difference** of A and B.

Matrix Subtraction	The difference of two $m \times n$ matrices A and B is the $m \times n$ matrix obtained by subtracting each entry of B from the corresponding entry of A.

EXAMPLE 4

Using the matrices C and D of Example 2, find $C - D$.

SOLUTION

By definition,

$$C - D = \begin{bmatrix} 1 - 16 & 6 - 2 & -1 - 9 \\ -2 - 4 & 4 - (-7) & 5 - (-1) \end{bmatrix} = \begin{bmatrix} -15 & 4 & -10 \\ -6 & 11 & 6 \end{bmatrix}$$

MATRIX MULTIPLICATION

We will use the Quality TV Company again, this time to help us arrive at a definition of matrix multiplication. Suppose

$$\begin{array}{cc} & \begin{array}{ccc} \text{Boston} & \text{Miami} & \text{Chicago} \end{array} \\ S = & \begin{bmatrix} 60 & 85 & 70 \\ 40 & 100 & 20 \end{bmatrix} \begin{array}{c} 17'' \\ 19'' \end{array} \end{array}$$

is a matrix representing the supply of television sets at the end of the year. Further, suppose the cost of each 17-inch set is \$80 and the cost of each 19-inch set is \$125. To find the total cost of the inventory at each outlet, we need to multiply the number of 17-inch sets by \$80, multiply the number of 19-inch sets by \$125, and sum the two products. If we let

$$C = [80 \quad 125]$$

be the cost matrix, we seek to define the product

$$[80 \quad 125]\begin{bmatrix} 60 & 85 & 70 \\ 40 & 100 & 20 \end{bmatrix}$$

so that the result will be a matrix displaying the total cost at each outlet. To find the total cost at the Boston outlet, we need to calculate

$$(80)(60) + (125)(40) = 9800$$

$$[80 \quad 125]\begin{bmatrix} 60 & 85 & 70 \\ 40 & 100 & 20 \end{bmatrix}$$

At the Miami outlet, the total cost is

$$(80)(85) + (125)(100) = 19{,}300$$

$$[80 \quad 125]\begin{bmatrix} 60 & 85 & 70 \\ 40 & 100 & 20 \end{bmatrix}$$

At the Chicago outlet, the total cost is

$$(80)(70) + (125)(20) = 8100$$

$$[80 \quad 125]\begin{bmatrix} 60 & 85 & 70 \\ 40 & 100 & 20 \end{bmatrix}$$

The total cost at each outlet can then be displayed by the 1×3 matrix

$$[9800 \quad 19{,}300 \quad 8100]$$

which is the product of C and S. Thus,

$$\begin{aligned} CS &= [80 \quad 125]\begin{bmatrix} 60 & 85 & 70 \\ 40 & 100 & 20 \end{bmatrix} \\ &= [(80)(60) + (125)(40) \quad (80)(85) + (125)(100) \quad (80)(70) + (125)(20)] \\ &= [9800 \quad 19{,}300 \quad 8100] \end{aligned}$$

Our example illustrates the process for multiplying a matrix by a row matrix. If the matrix C had more than one row, we would repeat the process using each row of C. Here is an example.

EXAMPLE 5

Find the product AB if

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -6 & -2 & 4 \\ 2 & 0 & 1 & -5 \end{bmatrix}$$

SOLUTION

$$AB = \begin{bmatrix} (2)(4)+(1)(2) & (2)(-6)+(1)(0) & (2)(-2)+(1)(1) & (2)(4)+(1)(-5) \\ (3)(4)+(5)(2) & (3)(-6)+(5)(0) & (3)(-2)+(5)(1) & (3)(4)+(5)(-5) \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -12 & -3 & 3 \\ 22 & -18 & -1 & -13 \end{bmatrix}$$

PROGRESS CHECK

Find the product AB if

$$A = \begin{bmatrix} -2 & -1 & 2 \\ 4 & 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & -4 \\ 3 & 1 \\ -1 & 0 \end{bmatrix}$$

ANSWER

$$AB = \begin{bmatrix} -15 & 7 \\ 28 & -13 \end{bmatrix}$$

It is important to note that the product AB of an $m \times n$ matrix A and an $n \times r$ matrix B exists only when the number of columns of A equals the number of rows of B (see Figure 1). The product AB will then be of dimension $m \times r$.

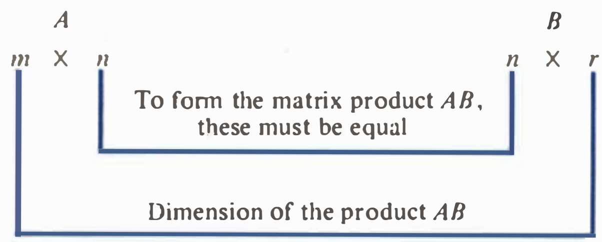


FIGURE 1

EXAMPLE 6

Given the matrices

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & -3 \\ -2 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & -1 & -2 \\ 1 & 0 & 4 \end{bmatrix} \qquad D = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(a) show that $AB \neq BA$;

(b) determine the dimension of AC.

SOLUTION

(a) $$AB = \begin{bmatrix} (1)(5) + (-1)(-2) & (1)(-3) + (-1)(2) \\ (2)(5) + (3)(-2) & (2)(-3) + (3)(2) \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ 4 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} (5)(1) + (-3)(2) & (5)(-1) + (-3)(3) \\ (-2)(1) + (2)(2) & (-2)(-1) + (2)(3) \end{bmatrix} = \begin{bmatrix} -1 & -14 \\ 2 & 8 \end{bmatrix}$$

Since the corresponding elements of AB and BA are not equal, $AB \neq BA$.

(b) The product of a 2×2 matrix and a 2×3 matrix is a 2×3 matrix.

PROGRESS CHECK

If possible, using the matrices of Example 6, find the dimension of the given product.

(a) CD (b) CB

ANSWERS

(a) 2×1 (b) not defined

We saw in Example 6 that $AB \neq BA$; that is, the commutative law does not hold for matrix multiplication. However, the associative law $A(BC) = (AB)C$ does hold when the dimensions of A, B, and C permit us to find the necessary products.

PROGRESS CHECK

Verify that $A(BC) = (AB)C$ for the matrices A, B, and C of Example 6.

MATRICES AND LINEAR SYSTEMS

Matrix multiplication provides a convenient shorthand means of writing a linear system. For example, the linear system

$$\begin{aligned} 2x - y - 2z &= 3 \\ 3x + 2y + z &= -1 \\ x + y - 3z &= 14 \end{aligned}$$

can be expressed as

$$AX = B$$

where

$$A = \begin{bmatrix} 2 & -1 & -2 \\ 3 & 2 & 1 \\ 1 & 1 & -3 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ -1 \\ 14 \end{bmatrix}$$

To verify this, simply form the matrix product AX and then apply the definition of matrix equality to the matrix equation $AX = B$.

EXAMPLE 7

Write out the linear system $AX = B$ if

$$A = \begin{bmatrix} -2 & 3 \\ 1 & 4 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \quad B = \begin{bmatrix} 16 \\ -3 \end{bmatrix}$$

SOLUTION

Equating corresponding elements of the matrix equation $AX = B$ yields

$$\begin{aligned} -2x + 3y &= 16 \\ x + 4y &= -3 \end{aligned}$$

EXERCISE SET 10.2

1. For what values of a, b, c, and d are the matrices A and B equal?

$$A = \begin{bmatrix} a & b \\ 6 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 3 & -4 \\ c & d \end{bmatrix}$$

2. For what values of a, b, c, and d are the matrices A and B equal?

$$A = \begin{bmatrix} a+b & 2c \\ a & c-d \end{bmatrix} \quad B = \begin{bmatrix} -1 & 6 \\ 5 & 10 \end{bmatrix}$$

In Exercises 3–18 the following matrices are given.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ -3 & 4 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 \\ 3 & 2 \\ 4 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -1 & 2 \\ 3 & 2 & 5 \end{bmatrix} \quad D = \begin{bmatrix} -3 & 2 \\ 4 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & -3 & 2 \\ 3 & 2 & 4 \\ 1 & 1 & 2 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix} \quad G = \begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 3 \end{bmatrix}$$

If possible, compute the indicated matrix.

3. $C + E$
4. $C - E$
5. $2A + 3G$
6. $3G - 4A$
7. $A + F$
8. $2B - D$
9. AB
10. BA
11. $CB + D$
12. $EB - FA$
13. $DF + AB$
14. $AC + 2DG$
15. $DA + EB$
16. $FG + B$
17. $2GE - 3A$
18. $AB + FG$

19. If $A = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} -4 & -3 \\ 0 & -4 \end{bmatrix}$, show that $AB = AC$.

20. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$, show that $AB \neq BA$.

21. If $A = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$, show that $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

22. If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, show that $A \cdot A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

23. If $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, show that $AI = A$ and $IA = A$.

24. Pesticides are sprayed on plants to eliminate harmful insects. However, some of the pesticide is absorbed by the plant, and the pesticide is then absorbed by herbivores (plant-eating animals such as cows) when they eat the plants that have been sprayed. Suppose that we have three pesticides and four plants and that the amounts of pesticide absorbed by the different plants are given by the matrix

$$A = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc} \text{Plant} & \text{Plant} & \text{Plant} & \text{Plant} \\ 1 & 2 & 3 & 4 \end{array}$$
$$A = \begin{bmatrix} 3 & 2 & 4 & 3 \\ 6 & 5 & 2 & 4 \\ 4 & 3 & 1 & 5 \end{bmatrix} \begin{array}{l} \text{Pesticide 1} \\ \text{Pesticide 2} \\ \text{Pesticide 3} \end{array}$$

where a_{ij} denotes the amount of pesticide i in milligrams that has been absorbed by plant j. Thus, plant 4 has absorbed 5 mg of pesticide 3. Now suppose that we have three herbivores and that the numbers of plants eaten by these animals are given by the matrix

$$\begin{array}{ccc} \text{Herbivore} & \text{Herbivore} & \text{Herbivore} \\ 1 & 2 & 3 \end{array}$$
$$B = \begin{bmatrix} 18 & 30 & 20 \\ 12 & 15 & 10 \\ 16 & 12 & 8 \\ 6 & 4 & 12 \end{bmatrix} \begin{array}{l} \text{Plant 1} \\ \text{Plant 2} \\ \text{Plant 3} \\ \text{Plant 4} \end{array}$$

How much of pesticide 2 has been absorbed by herbivore 3?

25. What does the entry in row 2, column 3, of the matrix product AB of Exercise 24 represent?

In Exercises 26–29 indicate the matrices A, X, and B so that the matrix equation $AX = B$ is equivalent to the given linear system.

26. $\begin{aligned} 7x - 2y &= 6 \\ -2x + 3y &= -2 \end{aligned}$

27. $\begin{aligned} 3x + 4y &= -3 \\ 3x - y &= 5 \end{aligned}$

28. $\begin{aligned} 5x + 2y - 3z &= 4 \\ 2x - \tfrac{1}{2}y + z &= 10 \\ x + y - 5z &= -3 \end{aligned}$

29. $\begin{aligned} 3x - y + 4z &= 5 \\ 2x + 2y + \tfrac{3}{4}z &= -1 \\ x - \tfrac{1}{4}y + z &= \tfrac{1}{2} \end{aligned}$

In Exercises 30–33 write out the linear system that is represented by the matrix equation $AX = B$.

30. $A = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \end{bmatrix}$ $B = \begin{bmatrix} -2 \\ 10 \end{bmatrix}$

31. $A = \begin{bmatrix} 1 & -5 \\ 4 & 3 \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

32. $A = \begin{bmatrix} 1 & 7 & -2 \\ 3 & 6 & 1 \\ -4 & 2 & 0 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix}$

33. $A = \begin{bmatrix} 4 & 5 & -2 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $B = \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix}$

34. The $m \times n$ matrix all of whose elements are zero is called the **zero matrix** and is denoted by O. Show that $A + O = A$ for every $m \times n$ matrix A.

35. The square matrix of order n such that $a_{ii} = 1$ and $a_{ij} = 0$ when $i \neq j$ is called the **identity matrix** of order n and is denoted by I_n. (Note: The definition indicates that the diagonal elements are equal to 1 and all elements off the diagonal are 0.) Show that $AI_n = I_nA$ for every square matrix A of order n.

36. The matrix B, each of whose entries is the negative of the corresponding entry of matrix A, is called the **additive inverse** of the matrix A. Show that $A + B = O$ where O is the zero matrix (see Exercise 34).

10.3 INVERSES OF MATRICES (Optional)

If $a \neq 0$, then the linear equation $ax = b$ can be solved easily by multiplying both sides by the reciprocal of a. Thus, we obtain $x = (1/a)b$. It would be nice if we could multiply both sides of the matrix equation $AX = B$ by the "reciprocal of A." Unfortunately, a matrix has *no* reciprocal. However, we shall discuss a notion that, for a square matrix, provides an analogue of the reciprocal of a real number and will enable us to solve the linear system in a manner distinct from the Gauss–Jordan method discussed earlier in this chapter.

In this section we confine our attention to square matrices. The $n \times n$ matrix

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

which has 1s on the main diagonal and 0s elsewhere, is called the **identity matrix.** Examples of identity matrices are

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If A is any $n \times n$ matrix, we can show that

$$\boxed{AI_n = I_nA = A}$$

(see Exercise 35, Section 7.2). Thus, I_n is the matrix analogue of the real number 1.

An $n \times n$ matrix A is called **invertible** or **nonsingular** if we can find an $n \times n$ matrix B such that

$$\boxed{AB = BA = I_n}$$

The matrix B is called an **inverse** of A.

EXAMPLE 1

Let

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

Since

$$AB = BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{(verify this)}$$

we conclude that A is an invertible matrix and that B is an inverse of A. Of course, if B is an inverse of A, then A is an inverse of B.

It can be shown that if an $n \times n$ matrix A has an inverse, it can have only one inverse. We denote the inverse of A by A^{-1}. Thus, we have

$$\boxed{AA^{-1} = I_n \quad \text{and} \quad A^{-1}A = I_n}$$

Note that the products AA^{-1} and $A^{-1}A$ yield the *identity matrix*, and that the products $a(1/a)$ and $(1/a)a$ yield the *identity element*. For this reason, A^{-1} may be thought of as the matrix analogue of the reciprocal $1/a$ of the real number a.

PROGRESS CHECK

Verify that the matrices

$$A = \begin{bmatrix} 4 & 5 \\ 2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & \frac{5}{2} \\ 1 & -2 \end{bmatrix}$$

are inverses of each other.

WARNING If $a \neq 0$ is a real number, then a^{-1} has the property that $aa^{-1} = a^{-1}a = 1$. Since $a^{-1} = 1/a$, we may refer to a^{-1} as the inverse *or* reciprocal of a. However, the matrix A^{-1} is the inverse of the $n \times n$ matrix A, since $AA^{-1} = A^{-1}A = I_n$, but cannot be referred to as the reciprocal of A, since *matrix division is not defined*.

We now develop a practical method for finding the inverse of an invertible matrix. Suppose we want to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

Let the inverse be denoted by

$$B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$$

Then we must have

$$AB = I_2 \tag{1}$$

and

$$BA = I_2 \tag{2}$$

Equation (1) now becomes

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

or

$$\begin{bmatrix} b_1 + 3b_3 & b_2 + 3b_4 \\ 2b_1 + 5b_3 & 2b_2 + 5b_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since two matrices are equal if and only if their corresponding entries are equal, we have

$$\begin{aligned} b_1 + 3b_3 &= 1 \\ 2b_1 + 5b_3 &= 0 \end{aligned} \tag{3}$$

and

$$\begin{aligned} b_2 + 3b_4 &= 0 \\ 2b_2 + 5b_4 &= 1 \end{aligned} \tag{4}$$

We solve the linear systems (3) and (4) by Gauss–Jordan elimination. We begin with the augmented matrices of the linear systems and perform a sequence of elementary row operations as follows:

(3)	(4)	
$\left[\begin{array}{cc:c} 1 & 3 & 1 \\ 2 & 5 & 0 \end{array}\right]$	$\left[\begin{array}{cc:c} 1 & 3 & 0 \\ 2 & 5 & 1 \end{array}\right]$	Augmented matrices of (3) and (4).
$\left[\begin{array}{cc:c} 1 & 3 & 1 \\ 0 & -1 & -2 \end{array}\right]$	$\left[\begin{array}{cc:c} 1 & 3 & 0 \\ 0 & -1 & 1 \end{array}\right]$	To make $a_{21} = 0$, replaced row 2 with the sum of itself and -2 times row 1.
$\left[\begin{array}{cc:c} 1 & 3 & 1 \\ 0 & 1 & 2 \end{array}\right]$	$\left[\begin{array}{cc:c} 1 & 3 & 0 \\ 0 & 1 & -1 \end{array}\right]$	Multiplied row 2 by -1 to obtain $a_{22} = 1$.
$\left[\begin{array}{cc:c} 1 & 0 & -5 \\ 0 & 1 & 2 \end{array}\right]$	$\left[\begin{array}{cc:c} 1 & 0 & 3 \\ 0 & 1 & -1 \end{array}\right]$	To make $a_{12} = 0$, replaced row 1 with the sum of itself and -3 times row 2.

Thus, $b_1 = -5$ and $b_3 = 2$ is the solution of (3), and $b_2 = 3$ and $b_4 = -1$ is the solution of (4). We can check that

$$B = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$$

also satisfies the requirement $BA = I_2$ of Equation (2).

Observe that the linear systems (3) and (4) have the same coefficient matrix (which is also the same as the original matrix A) and that an identical sequence of elementary row operations was performed in the Gauss–Jordan elimination. This suggests that we can solve the systems *at the same time*. We simply write the coefficient matrix A and next to it list the right-hand sides of (3) and (4) to obtain the matrix

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right] \tag{5}$$

Note that the columns to the right of the dashed line in (5) form the identity matrix I_2. Performing the same sequence of elementary row operations on matrix (5) as we did on matrices (3) and (4) yields

$$\left[\begin{array}{cc|cc} 1 & 0 & -5 & 3 \\ 0 & 1 & 2 & -1 \end{array}\right] \tag{6}$$

Then A^{-1} is the matrix to the right of the dashed line in (6).

The procedure outlined for the 2×2 matrix A applies in general. Thus, we have the following method for finding the inverse of an invertible $n \times n$ matrix A.

Computing A^{-1}

Step 1. Form the $n \times 2n$ matrix $[A \mid I_n]$ by adjoining the identity matrix I_n to the given matrix A.

Step 2. Apply elementary row operations to the matrix $[A \mid I_n]$ to transform the matrix A to I_n.

Step 3. The final matrix is of the form $[I_n \mid B]$ where B is A^{-1}.

EXAMPLE 2

Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 1 & 1 & 1 \end{bmatrix}$$

SOLUTION

We form the 3×6 matrix $[A \mid I_3]$ and transform it by elementary row operations to the form $[I_3 \mid A^{-1}]$. The pivot element at each stage is shown in color.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 7 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$ Matrix A augmented by I_3.

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right]$$

To make $a_{21} = 0$, replaced row 2 with the sum of itself and -2 times row 1. To make $a_{31} = 0$, replaced row 3 with the sum of itself and -1 times row 1.

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 1 & 5 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -3 & 1 & 1 \end{array}\right]$$

To make $a_{12} = 0$, replaced row 1 with the sum of itself and -2 times row 2. To make $a_{32} = 0$, replaced row 3 with the sum of itself and row 2.

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 1 & 5 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3 & -1 & -1 \end{array}\right]$$

Multiplied row 3 by -1.

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & -5 & 2 & 1 \\ 0 & 0 & 1 & 3 & -1 & -1 \end{array}\right]$$

To make $a_{13} = 0$, replaced row 1 with the sum of itself and -1 times row 3. To make $a_{23} = 0$, replaced row 2 with the sum of itself and -1 times row 3.

The final matrix is of the form $[I_3 \mid A^{-1}]$; that is,

$$A^{-1} = \begin{bmatrix} 2 & -1 & 1 \\ -5 & 2 & 1 \\ 3 & -1 & -1 \end{bmatrix}$$

We now have a practical method for finding the inverse of an invertible matrix, but we don't know whether a given square matrix *has* an inverse. It can be shown that if the preceding procedure is carried out with the matrix $[A \mid I_n]$ and we arrive at a point at which all possible candidates for the next pivot element are zero, then the matrix is not invertible and we may stop our calculations.

EXAMPLE 3

Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 0 & 2 \\ -3 & -6 & -9 \end{bmatrix}$$

SOLUTION

We begin with $[A \mid I_3]$.

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 6 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ -3 & -6 & -9 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 6 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 9 & 3 & 0 & 1 \end{array}\right]$$

To make $a_{31} = 0$, replaced row 3 by the sum of itself and 3 times row 1.

Note that $a_{22} = a_{32} = 0$ in the last matrix. We cannot perform any elementary row operations upon rows 2 and 3 that will produce a nonzero pivot element for a_{22}. We conclude that the matrix A does not have an inverse.

PROGRESS CHECK

Show that the matrix A is not invertible.

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 2 & 1 \\ 5 & 6 & -5 \end{bmatrix}$$

SOLVING LINEAR SYSTEMS

Consider a linear system of n equations in n unknowns.

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{12}x_2 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{array} \tag{7}$$

As has already been pointed out in Section 2 of this chapter, we can write the linear system (7) in matrix form as

$$AX = B \tag{8}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Suppose now that the coefficient matrix A is invertible so that we can compute A^{-1}. Multiplying both sides of (8) by A^{-1}, we have

$$\begin{aligned} A^{-1}(AX) &= A^{-1}B \\ (A^{-1}A)X &= A^{-1}B && \text{Associative law} \\ I_nX &= A^{-1}B && A^{-1}A = I_n \\ X &= A^{-1}B && I_nX = X \end{aligned}$$

Thus, we have the following result.

CODED MESSAGES

A	B	C	D	E	F	G
↕	↕	↕	↕	↕	↕	↕
1	2	3	4	5	6	7
H	I	J	K	L	M	N
↕	↕	↕	↕	↕	↕	↕
8	9	10	11	12	13	14
O	P	Q	R	S	T	
↕	↕	↕	↕	↕	↕	
15	16	17	18	19	20	
U	V	W	X	Y	Z	
↕	↕	↕	↕	↕	↕	
21	22	23	24	25	26	

Cryptography is the study of methods for encoding and decoding messages. One of the very simplest techniques for doing this involves the use of the inverse of a matrix.

First, attach a different number to every letter of the alphabet. For example, we can let A be 1, B be 2, and so on, as shown in the accompanying table. Suppose that we then want to send the message

ALGEBRA WORKS

Substituting for each letter, we send the message

$$1,\ 12,\ 7,\ 5,\ 2,\ 18,\ 1,\ 23,\ 15,\ 18,\ 11,\ 19 \tag{1}$$

Unfortunately, this simple code can be easily cracked. A better method involves the use of matrices.

Break the message (1) into four 3×1 matrices:

$$X_1 = \begin{bmatrix} 1 \\ 12 \\ 7 \end{bmatrix} \quad X_2 = \begin{bmatrix} 5 \\ 2 \\ 18 \end{bmatrix} \quad X_3 = \begin{bmatrix} 1 \\ 23 \\ 15 \end{bmatrix} \quad X_4 = \begin{bmatrix} 18 \\ 11 \\ 19 \end{bmatrix}$$

The sender and receiver jointly select an invertible 3×3 matrix such as

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

The sender forms the 3×1 matrices

$$AX_1 = \begin{bmatrix} 27 \\ 20 \\ 8 \end{bmatrix} \quad AX_2 = \begin{bmatrix} 43 \\ 25 \\ 23 \end{bmatrix} \quad AX_3 = \begin{bmatrix} 54 \\ 39 \\ 16 \end{bmatrix} \quad AX_4 = \begin{bmatrix} 67 \\ 48 \\ 37 \end{bmatrix}$$

and sends the message

$$27,\ 20,\ 8,\ 43,\ 25,\ 23,\ 54,\ 39,\ 16,\ 67,\ 48,\ 37 \tag{2}$$

To decode the message, the receiver uses the inverse of matrix A,

$$A^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

and forms

$$A^{-1}\begin{bmatrix} 27 \\ 20 \\ 8 \end{bmatrix} = X_1 \quad A^{-1}\begin{bmatrix} 43 \\ 25 \\ 23 \end{bmatrix} = X_2 \quad A^{-1}\begin{bmatrix} 54 \\ 39 \\ 16 \end{bmatrix} = X_3 \quad A^{-1}\begin{bmatrix} 67 \\ 48 \\ 37 \end{bmatrix} = X_4$$

which, of course, is the original message (1) and which can be understood by using the accompanying table.

If the receiver sends back the message

46, 37, 29, 50, 39, 30, 75, 52, 37

what is the response?

If $AX = B$ is a linear system of n equations in n unknowns and if the coefficient matrix A is invertible, then the system has exactly one solution, given by

$$X = A^{-1}B$$

WARNING Since matrix multiplication is not commutative, you must be careful to write the solution to the system $AX = B$ as $X = A^{-1}B$ and *not* $X = BA^{-1}$.

EXAMPLE 4

Solve the linear system by finding the inverse of the coefficient matrix.

$$\begin{aligned} x + 2y + 3z &= -3 \\ 2x + 5y + 7z &= 4 \\ x + y + z &= 5 \end{aligned}$$

SOLUTION

The coefficient matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 1 & 1 & 1 \end{bmatrix}$$

is the matrix whose inverse was obtained in Example 2 as

$$A^{-1} = \begin{bmatrix} 2 & -1 & 1 \\ -5 & 2 & 1 \\ 3 & -1 & -1 \end{bmatrix}$$

Since

$$B = \begin{bmatrix} -3 \\ 4 \\ 5 \end{bmatrix}$$

we obtain the solution of the given system as

$$X = A^{-1}B = \begin{bmatrix} 2 & -1 & 1 \\ -5 & 2 & 1 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -5 \\ 28 \\ -18 \end{bmatrix}$$

Thus $x = -5$, $y = 28$, $z = -18$.

PROGRESS CHECK

Solve the linear system by finding the inverse of the coefficient matrix.

$$\begin{aligned} x - 2y + z &= 1 \\ x + 3y + 2z &= 2 \\ -x \quad\quad + z &= -11 \end{aligned}$$

ANSWER

$x = 7$, $y = 1$, $z = -4$

The inverse of the coefficient matrix is especially useful when we need to solve a number of linear systems

$$AX = B_1, AX = B_2, \ldots, AX = B_k$$

where the coefficient matrix is the same and the right-hand side changes.

EXAMPLE 5

A steel producer makes two types of steel, regular and special. A ton of regular steel requires 2 hours in the open-hearth furnace and 5 hours in the soaking pit; a ton of special steel requires 2 hours in the open-hearth furnace and 3 hours in the soaking pit. How many tons of each type of steel can be manufactured daily if

(a) the open-hearth furnace is available 8 hours per day and the soaking pit is available 15 hours per day?

(b) the open-hearth furnace is available 9 hours per day and the soaking pit is available 15 hours per day?

SOLUTION

Let

x = the number of tons of regular steel to be made

y = the number of tons of special steel to be made

Then the total amount of time required in the open-hearth furnace is

$$2x + 2y$$

Similarly, the total amount of time required in the soaking pit is

$$5x + 3y$$

If we let b_1 and b_2 denote the number of hours that the open-hearth furnace and the soaking pit, respectively, are available per day, then we have

$$\begin{aligned} 2x + 2y &= b_1 \\ 5x + 3y &= b_2 \end{aligned}$$

or

$$\begin{bmatrix} 2 & 2 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Then

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 5 & 3 \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

We find the inverse of the coefficient matrix to be

$$\begin{bmatrix} 2 & 2 \\ 5 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{3}{4} & \frac{1}{2} \\ \frac{5}{4} & -\frac{1}{2} \end{bmatrix}$$

(verify this).

(a) We are given $b_1 = 8$ and $b_2 = 15$. Then

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} & \frac{1}{2} \\ \frac{5}{4} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 8 \\ 15 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{5}{2} \end{bmatrix}$$

That is, $\frac{3}{2}$ tons of regular steel and $\frac{5}{2}$ tons of special steel can be manufactured daily.

(b) We are given $b_1 = 9$ and $b_2 = 15$. Then

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} & \frac{1}{2} \\ \frac{5}{4} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 9 \\ 15 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ \frac{15}{4} \end{bmatrix}$$

That is, $\frac{3}{4}$ tons of regular steel and $\frac{15}{4}$ tons of special steel can be manufactured daily.

EXERCISE SET 10.3

In Exercises 1–4 determine whether the matrix B is the inverse of the matrix A.

1. $A = \begin{bmatrix} 2 & \frac{1}{2} \\ -1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -1 \\ -2 & 4 \end{bmatrix}$

2. $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$ $B = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 3 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 2 & -6 \\ 1 & 1 & -2 \\ -2 & -2 & 5 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 3 \\ -4 & 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & -2 \\ -2 & -4 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

In Exercises 5–10 find the inverse of the given matrix.

5. $\begin{bmatrix} -1 & 5 \\ 2 & -4 \end{bmatrix}$

6. $\begin{bmatrix} 2 & 0 \\ -1 & -2 \end{bmatrix}$

7. $\begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$

8. $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

9. $\begin{bmatrix} 1 & -2 & 3 \\ -1 & 3 & -4 \\ 0 & 5 & -4 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 2 \end{bmatrix}$

In Exercises 11–18 find the inverse, if possible.

11. $\begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$

12. $\begin{bmatrix} 6 & -4 \\ 9 & -6 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 1 & 3 \\ 2 & -8 & -4 \\ -1 & 2 & 0 \end{bmatrix}$

14. $\begin{bmatrix} 8 & 7 & -1 \\ -5 & -5 & 1 \\ -4 & -4 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$

16. $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 4 & 0 \\ 2 & 0 & -6 & 1 \end{bmatrix}$

In Exercises 19–24 solve the given linear system by finding the inverse of the coefficient matrix.

19. $\begin{aligned} 2x + y &= 5 \\ x - 3y &= 6 \end{aligned}$

20. $\begin{aligned} 2x - 3y &= -5 \\ 3x + y &= -13 \end{aligned}$

21. $\begin{aligned} 3x + y - z &= 2 \\ x - 2y &= 8 \\ 3y + z &= -8 \end{aligned}$

22. $\begin{aligned} 3x + 2y - z &= 10 \\ 2x - y + z &= -1 \\ -x + y - 2z &= 5 \end{aligned}$

23. $\begin{aligned} 2x - y + 3z &= -11 \\ 3x - y + z &= -5 \\ x + y + z &= -1 \end{aligned}$

24. $\begin{aligned} 2x + 3y - 2z &= 13 \\ 4x + 2y + z &= 3 \\ y - z &= 5 \end{aligned}$

25–34. Solve the linear systems of Section 1 of this chapter, Exercises 21–30, by finding the inverse of the coefficient matrix.

35. Solve the linear systems $AX = B_1$ and $AX = B_2$, given

$$A^{-1} = \begin{bmatrix} 3 & -2 & 4 \\ 2 & -1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \quad B_1 = \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} \quad B_2 = \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix}$$

36. Solve the linear systems $AX = B_1$ and $AX = B_2$, given

$$A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ -1 & -1 & 3 \end{bmatrix} \quad B_1 = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} \quad B_2 = \begin{bmatrix} 4 \\ -3 \\ -5 \end{bmatrix}$$

37. Show that the matrix

$$\begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ d & e & f \end{bmatrix}$$

is not invertible.

38. A trustee decides to invest \$30,000 in two mortgages, which yield 10% and 15% per year, respectively. How should the \$30,000 be invested in the two mortgages if the total annual interest is to be
(a) \$3600? (b) \$4000? (c) \$5000?
(*Hint:* Some of these investment objectives cannot be attained.)

10.4 DETERMINANTS AND CRAMER'S RULE

In this section we will define a determinant and will develop manipulative skills for evaluating determinants. We will then show that determinants have important applications and can be used to solve linear systems.

Associated with every square matrix A is a number called the **determinant** of A, denoted by $|A|$. If A is the 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

then $|A|$ is said to be a **determinant of second order** and is defined by the rule

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

EXAMPLE 1

Compute the real number represented by

$$\begin{vmatrix} 4 & -5 \\ 3 & -1 \end{vmatrix}$$

SOLUTION

We apply the rule for a determinant of second order:

$$\begin{vmatrix} 4 & -5 \\ 3 & -1 \end{vmatrix} = (4)(-1) - (3)(-5) = 11$$

PROGRESS CHECK

Compute the real number represented by the following.

(a) $\begin{vmatrix} -6 & 2 \\ -1 & -2 \end{vmatrix}$ (b) $\begin{vmatrix} \frac{1}{2} & \frac{1}{4} \\ -4 & -2 \end{vmatrix}$

ANSWERS

(a) 14 (b) 0

To simplify matters, when we want to compute the determinant of a matrix we will say "evaluate the determinant." This wording is not technically correct, however, since a determinant *is* a real number.

MINORS AND COFACTORS

The rule for evaluating a determinant of order 3 is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{array}{l} a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} \\ + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} \end{array}$$

The situation becomes even more cumbersome for determinants of higher order! Fortunately, we don't have to memorize this rule; instead, we shall see that it is possible to evaluate a determinant of order 3 by reducing the problem to that of evaluating three determinants of order 2.

The **minor of an element** a_{ij} is the determinant of the matrix remaining after deleting the row and column in which the element a_{ij} appears. Given the matrix

$$\begin{bmatrix} 4 & 0 & -2 \\ 1 & -6 & 7 \\ -3 & 2 & 5 \end{bmatrix}$$

the minor of the element in row 2, column 3, is

$$\begin{vmatrix} 4 & 0 & \cancel{-2} \\ \cancel{1} & \cancel{-6} & \cancel{7} \\ -3 & 2 & \cancel{5} \end{vmatrix} = \begin{vmatrix} 4 & 0 \\ -3 & 2 \end{vmatrix} = 8 - 0 = 8$$

The **cofactor** of the element a_{ij} is the minor of the element a_{ij} multiplied by $(-1)^{i+j}$. Since $(-1)^{i+j}$ is $+1$ if $i + j$ is even and -1 if $i + j$ is odd, we see that the cofactor is the minor with a sign attached. The cofactor attaches the sign to the minor according to this pattern:

$$\begin{matrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \end{matrix}$$

EXAMPLE 2

Find the cofactor of each element in the first row of the matrix

$$\begin{bmatrix} -2 & 0 & 12 \\ -4 & 5 & 3 \\ 7 & 8 & -6 \end{bmatrix}$$

SOLUTION

The cofactors are

$$(-1)^{1+1}\begin{vmatrix}-2 & 0 & 12\\ -4 & 5 & 3\\ 7 & 8 & -6\end{vmatrix} = \begin{vmatrix}5 & 3\\ 8 & -6\end{vmatrix} = -30 - 24 = -54$$

$$(-1)^{1+2}\begin{vmatrix}-2 & 0 & 12\\ -4 & 5 & 3\\ 7 & 8 & -6\end{vmatrix} = -\begin{vmatrix}-4 & 3\\ 7 & -6\end{vmatrix} = -(24 - 21) = -3$$

$$(-1)^{1+3}\begin{vmatrix}-2 & 0 & 12\\ -4 & 5 & 3\\ 7 & 8 & -6\end{vmatrix} = \begin{vmatrix}-4 & 5\\ 7 & 8\end{vmatrix} = -32 - 35 = -67$$

PROGRESS CHECK

Find the cofactor of each entry in the second column of the matrix

$$\begin{bmatrix}16 & -9 & 3\\ -5 & 2 & 0\\ -3 & 4 & -1\end{bmatrix}$$

ANSWER

Cofactor of -9 is -5; cofactor of 2 is -7; cofactor of 4 is -15.

The cofactor is the key to the process of evaluating determinants of order 3 or higher.

Expansion by Cofactors

To evaluate a determinant, form the sum of the products obtained by multiplying each entry of any row or any column by its cofactor. This process is called **expansion by cofactors.**

Let's illustrate the process with an example.

EXAMPLE 3

Evaluate the determinant by cofactors.

$$\begin{vmatrix}-2 & 7 & 2\\ 6 & -6 & 0\\ 4 & 10 & -3\end{vmatrix}$$

SOLUTION

Expansion by Cofactors	
Step 1. Choose a row or column about which to expand. In general, a row or column containing zeros will simplify the work.	*Step 1.* We will expand about column 3.
Step 2. Expand about the cofactors of the chosen row or column by multiplying each entry of the row or column by its cofactor. Repeat the procedure until all determinants are of order 2.	*Step 2.* The expansion about column 3 is $(2)(-1)^{1+3}\begin{vmatrix} 6 & -6 \\ 4 & 10 \end{vmatrix} + (0)(-1)^{2+3}\begin{vmatrix} -2 & 7 \\ 4 & 10 \end{vmatrix} + (-3)(-1)^{3+3}\begin{vmatrix} -2 & 7 \\ 6 & -6 \end{vmatrix}$
Step 3. Evaluate the determinants of order 2 and form the sum resulting from Step 2.	*Step 3.* Using the rule for evaluating a determinant of order 2, we have $(2)(1)[(6)(10) - (4)(-6)] + 0 + (-3)(1)[(-2)(-6) - (6)(7)] = 2(60 + 24) - 3(12 - 42) = 258$

Note that expansion by cofactors about *any row or any column* will produce the same result. This important property of determinants can be used to simplify the arithmetic. The best choice of a row or column about which to expand is the one that has the most zero entries. The reason is that if an entry is zero, the entry times its cofactor will be zero, so we don't have to evaluate that cofactor.

PROGRESS CHECK

Evaluate the determinant of Example 3 by expanding about the second row.

ANSWER

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EXAMPLE 4

Verify the rule for evaluating a determinant of order 3:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}$$

SOLUTION

Expanding about the first row, we have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}$$

PROGRESS CHECK

Show that the following determinant is equal to zero:

$$\begin{vmatrix} a & b & c \\ a & b & c \\ d & e & f \end{vmatrix}$$

The process of expanding by cofactors works for determinants of any order. If we apply the method to a determinant of order 4, we will produce determinants of order 3; applying the method again will result in determinants of order 2.

EXAMPLE 5

Evaluate the determinant

$$\begin{vmatrix} -3 & 5 & 0 & -1 \\ 1 & 2 & 3 & -3 \\ 0 & 4 & -6 & 0 \\ 0 & -2 & 1 & 2 \end{vmatrix}$$

SOLUTION

Expanding about the cofactors of the first column, we have

$$\begin{vmatrix} -3 & 5 & 0 & -1 \\ 1 & 2 & 3 & -3 \\ 0 & 4 & -6 & 0 \\ 0 & -2 & 1 & 2 \end{vmatrix} = -3\begin{vmatrix} 2 & 3 & -3 \\ 4 & -6 & 0 \\ -2 & 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 5 & 0 & -1 \\ 4 & -6 & 0 \\ -2 & 1 & 2 \end{vmatrix}$$

Each determinant of order 3 can then be evaluated.

$$-3\begin{vmatrix} 2 & 3 & -3 \\ 4 & -6 & 0 \\ -2 & 1 & 2 \end{vmatrix} = (-3)(-24) = 72 \qquad -1\begin{vmatrix} 5 & 0 & -1 \\ 4 & -6 & 0 \\ -2 & 1 & 2 \end{vmatrix} = (-1)(-52) = 52$$

The original determinant has the value $72 + 52 = 124$.

PROGRESS CHECK

Evaluate

$$\begin{vmatrix} 0 & -1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 5 & 0 & -3 \\ 1 & 0 & 1 & 0 \end{vmatrix}$$

ANSWER

7

CRAMER'S RULE

Determinants provide a convenient way of expressing formulas in many areas of mathematics, particularly in geometry. One of the best-known uses of determinants is in solving systems of linear equations, a procedure known as **Cramer's rule.**

In an earlier chapter we solved systems of linear equations by the method of elimination. Let's apply this method to the general system of two equations in two unknowns,

$$a_{11}x + a_{12}y = c_1 \qquad (1)$$

$$a_{21}x + a_{22}y = c_2 \qquad (2)$$

If we multiply Equation (1) by a_{22} and Equation (2) by $-a_{12}$ and add, we will eliminate y.

$$\begin{array}{r} a_{11}a_{22}x + a_{12}a_{22}y = c_1a_{22} \\ -a_{21}a_{12}x - a_{12}a_{22}y = -c_2a_{12} \\ \hline a_{11}a_{22}x - a_{21}a_{12}x = c_1a_{22} - c_2a_{12} \end{array}$$

Thus,

$$x(a_{11}a_{22} - a_{21}a_{12}) = c_1a_{22} - c_2a_{12}$$

or

$$x = \frac{c_1a_{22} - c_2a_{12}}{a_{11}a_{22} - a_{21}a_{12}}$$

Similarly, multiplying Equation (1) by a_{21} and Equation (2) by $-a_{11}$ and adding, we can eliminate x and solve for y:

$$y = \frac{c_2a_{11} - c_1a_{21}}{a_{11}a_{22} - a_{21}a_{12}}$$

The denominators in the expressions for x and y are identical and can be written as the determinant of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

If we apply this same idea to the numerators, we have

$$x = \frac{\begin{vmatrix} c_1 & a_{12} \\ c_2 & a_{22} \end{vmatrix}}{|A|} \qquad y = \frac{\begin{vmatrix} a_{11} & c_1 \\ a_{21} & c_2 \end{vmatrix}}{|A|} \qquad |A| \neq 0$$

What we have arrived at is **Cramer's rule,** which is a means of expressing the solution of a system of linear equations in determinant form.

The following example outlines the steps for using Cramer's rule.

EXAMPLE 6

Solve by Cramer's rule.

$$\begin{aligned} 3x - y &= 9 \\ x + 2y &= -4 \end{aligned}$$

SOLUTION

Cramer's Rule	
Step 1. Write the determinant of the coefficient matrix A.	*Step 1.* $|A| = \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix}$
Step 2. The numerator for x is the determinant of the matrix obtained from A by replacing the column of coefficients of x with the column of right-hand sides of the equations.	*Step 2.* $x = \dfrac{\begin{vmatrix} 9 & -1 \\ -4 & 2 \end{vmatrix}}{|A|}$
Step 3. The numerator for y is the determinant of the matrix obtained from A by replacing the column of coefficients of y with the column of right-hand sides of the equations.	*Step 3.* $y = \dfrac{\begin{vmatrix} 3 & 9 \\ 1 & -4 \end{vmatrix}}{|A|}$
Step 4. Evaluate the determinants to obtain the solution. If $|A| = 0$, Cramer's rule cannot be used.	*Step 4.* $|A| = 6 + 1 = 7$ $x = \dfrac{18 - 4}{7} = \dfrac{14}{7} = 2$ $y = \dfrac{-12 - 9}{7} = -\dfrac{21}{7} = -3$

PROGRESS CHECK

Solve by Cramer's rule.

$$\begin{aligned} 2x + 3y &= -4 \\ 3x + 4y &= -7 \end{aligned}$$

ANSWER

$x = -5,\ y = 2$

The steps outlined in Example 6 can be applied to solve any system of linear equations in which the number of equations is the same as the number of variables and in which $|A| \neq 0$. Here is an example with three equations and three unknowns.

EXAMPLE 7

Solve by Cramer's rule.

$$\begin{aligned} 3x \qquad\;\; + 2z &= -2 \\ 2x - \;y \qquad\;\; &= \;\;\,0 \\ 2y + 6z &= -1 \end{aligned}$$

SOLUTION

We form the determinant of coefficients:

$$|A| = \begin{vmatrix} 3 & 0 & 2 \\ 2 & -1 & 0 \\ 0 & 2 & 6 \end{vmatrix}$$

Then

$$x = \frac{|A_1|}{|A|} \qquad y = \frac{|A_2|}{|A|} \qquad z = \frac{|A_3|}{|A|}$$

where A_1 is obtained from A by replacing the first column of A with the column of right-hand sides, A_2 is obtained by replacing the second column of A with the column of right-hand sides, and A_3 is obtained from A by replacing the third column with the column of right-hand sides. Thus

$$x = \frac{\begin{vmatrix} -2 & 0 & 2 \\ 0 & -1 & 0 \\ -1 & 2 & 6 \end{vmatrix}}{|A|} \qquad y = \frac{\begin{vmatrix} 3 & -2 & 2 \\ 2 & 0 & 0 \\ 0 & -1 & 6 \end{vmatrix}}{|A|} \qquad z = \frac{\begin{vmatrix} 3 & 0 & -2 \\ 2 & -1 & 0 \\ 0 & 2 & -1 \end{vmatrix}}{|A|}$$

Expanding by cofactors, we calculate $|A| = -10$, $|A_1| = 10$, $|A_2| = 20$, and $|A_3| = -5$, obtaining

$$x = \frac{10}{-10} = -1 \qquad y = \frac{20}{-10} = -2 \qquad z = \frac{-5}{-10} = \frac{1}{2}$$

PROGRESS CHECK

Solve by Cramer's rule.

$$\begin{aligned} 3x \qquad\quad - z &= 1 \\ -6x + 2y \qquad\quad &= -5 \\ -4y + 3z &= 5 \end{aligned}$$

ANSWER

$x = \frac{2}{3}$, $y = -\frac{1}{2}$, $z = 1$

WARNING

(a) Each equation of the linear system must be written in the form

$$ax + by + cz = k$$

before using Cramer's rule.

(b) If $|A| = 0$, Cramer's rule cannot be used.

Determinants have significant theoretical importance but are not of much use for computational purposes. The matrix methods discussed in this chapter provide the basis for techniques better suited for computer implementation.

EXERCISE SET 10.4

In Exercises 1–6 evaluate the given determinant.

1. $\begin{vmatrix} 2 & -3 \\ 4 & 5 \end{vmatrix}$

2. $\begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix}$

3. $\begin{vmatrix} -4 & 1 \\ 0 & 2 \end{vmatrix}$

4. $\begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix}$

5. $\begin{vmatrix} 0 & 0 \\ 1 & 3 \end{vmatrix}$

6. $\begin{vmatrix} -4 & -1 \\ -2 & 3 \end{vmatrix}$

In Exercises 7–10 let

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 1 & -3 \\ 5 & -2 & 0 \end{bmatrix}$$

7. Compute the minor of each of the following elements.
(a) a_{11} (b) a_{23} (c) a_{31} (d) a_{33}

8. Compute the minor of each of the following elements.
(a) a_{12} (b) a_{22} (c) a_{23} (d) a_{32}

9. Compute the cofactor of each of the following elements.
(a) a_{11} (b) a_{23} (c) a_{31} (d) a_{33}

10. Compute the cofactor of each of the following elements.
(a) a_{12} (b) a_{22} (c) a_{23} (d) a_{32}

In Exercises 11–20 evaluate the given determinant.

11. $\begin{vmatrix} 4 & -2 & 5 \\ 5 & 2 & 0 \\ 2 & 0 & 4 \end{vmatrix}$

12. $\begin{vmatrix} 4 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & -4 \end{vmatrix}$

13. $\begin{vmatrix} -1 & 2 & 0 \\ 3 & 4 & 1 \\ 6 & 5 & 2 \end{vmatrix}$

14. $\begin{vmatrix} -1 & 3 & 2 \\ 0 & 7 & 7 \\ 2 & 1 & 3 \end{vmatrix}$

15. $\begin{vmatrix} 2 & 2 & 4 \\ 3 & 8 & 1 \\ 1 & 1 & 2 \end{vmatrix}$

16. $\begin{vmatrix} 0 & 1 & 3 \\ 2 & 5 & -1 \\ 4 & 2 & -2 \end{vmatrix}$

17. $\begin{vmatrix} 3 & 2 & 1 & 0 \\ -1 & -3 & -1 & 0 \\ 0 & 0 & 2 & 2 \\ 4 & 1 & 3 & 3 \end{vmatrix}$

18. $\begin{vmatrix} -1 & 2 & 4 & 0 \\ 3 & -2 & -3 & 0 \\ 0 & 4 & 2 & 5 \\ 0 & -3 & 1 & 4 \end{vmatrix}$

19. $\begin{vmatrix} 2 & -3 & 2 & -4 \\ 0 & 4 & -1 & 9 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 3 & -1 \end{vmatrix}$

20. $\begin{vmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 4 & -1 \\ -2 & 3 & 1 & -4 \\ 0 & 2 & 0 & 2 \end{vmatrix}$

In Exercises 21–28 solve the given linear system by use of Cramer's rule.

21. $\begin{aligned} 2x + y + z &= -1 \\ 2x - y + 2z &= 2 \\ x + 2y + z &= -4 \end{aligned}$

22. $\begin{aligned} x - y + z &= -5 \\ 3x + y + 2z &= -5 \\ 2x - y - z &= -2 \end{aligned}$

23. $\begin{aligned} 2x + y - z &= 9 \\ x - 2y + 2z &= -3 \\ 3x + 3y + 4z &= 11 \end{aligned}$

24. $\begin{aligned} 2x + y - z &= -2 \\ -2x - 2y + 3z &= 2 \\ 3x + y - z &= -4 \end{aligned}$

25. $\begin{aligned} -x - y + 2z &= 7 \\ x + 2y - 2z &= -7 \\ 2x - y + z &= -4 \end{aligned}$

26. $\begin{aligned} 4x + y - z &= -1 \\ x - y + 2z &= 3 \\ -x + 2y - z &= 0 \end{aligned}$

27. $\begin{aligned} x + y - z + 2w &= 0 \\ 2x + y - w &= -2 \\ 3x + 2z &= -3 \\ -x + 2y + 3w &= 1 \end{aligned}$

28. $\begin{aligned} 2x + y - 3w &= -7 \\ 3x + 2z + w &= -1 \\ -x + 2y + 3w &= 0 \\ -2x - 3y + 2z - w &= 8 \end{aligned}$

29. Show that

$$\begin{vmatrix} a_1 + b_1 & a_2 + b_2 \\ c & d \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ c & d \end{vmatrix} + \begin{vmatrix} b_1 & b_2 \\ c & d \end{vmatrix}$$

30. Show that

$$\begin{vmatrix} ka_{11} & ka_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ ka_{21} & ka_{22} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

31. Prove that if a row or column of a square matrix consists entirely of zeros, the determinant of the matrix is zero. (*Hint:* Expand by cofactors.)

32. Prove that if matrix B is obtained by multiplying each element of a row of a square matrix A by a constant k, then $|B| = k|A|$.

33. Prove that if A is an $n \times n$ matrix and $B = kA$, where k is a constant, then $|B| = k^n|A|$.

34. Prove that if matrix B is obtained from a square matrix A by interchanging the rows and columns of A, then $|B| = |A|$.

TERMS AND SYMBOLS

matrix (p. 415)
entries, elements (p. 415)
dimension (p. 416)
square matrix (p. 416)
order (p. 416)
row matrix (p. 416)
column matrix (p. 416)
$[a_{ij}]$ (p. 417)
coefficient matrix (p. 417)
augmented matrix (p. 417)
elementary row operations (p. 418)
pivot element (p. 419)
pivot row (p. 419)
Gauss–Jordan elimination (p. 422)
equality of matrices (p. 424)
scalar (p. 426)
scalar multiplication (p. 426)
zero matrix (p. 432)
identity matrix, I_n (p. 432)
additive inverse (p. 432)
invertible or nonsingular matrix (p. 433)
inverse (p. 433)
A^{-1} (p. 434)
determinant (p. 443)
minor (p. 445)
cofactor (p. 445)
expansion by cofactors (p. 446)
Cramer's rule (p. 449)

KEY IDEAS FOR REVIEW

- ☐ A matrix is simply a rectangular array of numbers.
- ☐ Systems of linear equations can be conveniently handled in matrix notation. By dropping the names of the variables, matrix notation focuses on the coefficients and the right-hand side of the system. The elementary row operations are then seen to be an abstraction of the operations that produce equivalent systems of equations.
- ☐ Gaussian elimination and Gauss–Jordan elimination both involve the use of elementary row operations on the augmented matrix corresponding to a linear system. In the case of a system of three equations with three unknowns, the final matrices will be of this form:

$$\left[\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{array}\right]$$

Gaussian elimination Gauss–Jordan elimination

If Gauss–Jordan elimination is used, the solution can be read from the final matrix; if Gaussian elimination is used, back-substitution is then performed with the final matrix.

- ☐ A linear system can be written in the form $AX = B$, where A is the coefficient matrix, X is a column matrix of the unknowns, and B is the column matrix of the right-hand sides.
- ☐ The sum and difference of two matrices A and B can be formed only if A and B are of the same dimension. The product AB can be formed only if the number of columns of A is the same as the number of rows of B.
- ☐ The $n \times n$ matrix B is said to be the inverse of the $n \times n$ matrix A if $AB = I_n$ and $BA = I_n$. We denote the inverse of A by A^{-1}. The inverse can be computed by using elementary row operations to transform the matrix $[A \mid I_n]$ to the form $[I_n \mid B]$. Then $B = A^{-1}$.
- ☐ If the linear system $AX = B$ has a unique solution, then $X = A^{-1}B$.
- ☐ Associated with every square matrix is a number called a determinant. The rule for evaluating a determinant of order 2 is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- ☐ For determinants of order greater than 2, the method of expansion by cofactors may be used to reduce the problem to one of evaluating determinants of order 2.
- ☐ When expanding by cofactors, choose the row or column that contains the most zeros. This will ease the arithmetic burden.
- ☐ Cramer's rule provides a means for solving a linear system by expressing the value of each unknown as a quotient of determinants.

REVIEW EXERCISES

Solutions to exercises whose numbers are in color are in the Solutions section in the back of the book.

10.1 Exercises 1–4 refer to the matrix

$$A = \begin{bmatrix} -1 & 4 & 2 & 0 & 8 \\ 2 & 0 & -3 & -1 & 5 \\ 4 & -6 & 9 & 1 & -2 \end{bmatrix}$$

1. Determine the dimension of the matrix A.

2. Find a_{24}. 3. Find a_{31}. 4. Find a_{15}.

Exercises 5 and 6 refer to the linear system

$$\begin{aligned} 3x - 7y &= 14 \\ x + 4y &= 6 \end{aligned}$$

5. Write the coefficient matrix of the linear system.

6. Write the augmented matrix of the linear system.

In Exercises 7 and 8 write a linear system corresponding to the augmented matrix.

7. $\left[\begin{array}{cc|c} 4 & -1 & 3 \\ 2 & 5 & 0 \end{array}\right]$ 8. $\left[\begin{array}{ccc|c} -2 & 4 & 5 & 0 \\ 6 & -9 & 4 & 0 \\ 3 & 2 & -1 & 0 \end{array}\right]$

In Exercises 9–12 use back-substitution to solve the linear system corresponding to the given augmented matrix.

9. $\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 1 & -4 \end{array}\right]$ 10. $\left[\begin{array}{cc|c} 1 & 2 & \frac{21}{2} \\ 0 & 1 & 5 \end{array}\right]$

11. $\left[\begin{array}{rrr|r} 1 & -4 & 2 & -18 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & -1 \end{array}\right]$

12. $\left[\begin{array}{rrr|r} 1 & -2 & 2 & -9 \\ 0 & 1 & 3 & -8 \\ 0 & 0 & 1 & -3 \end{array}\right]$

In Exercises 13–16 use matrix methods to solve the given linear system.

13. $\begin{aligned} x + y &= 2 \\ 2x - 4y &= -5 \end{aligned}$

14. $\begin{aligned} 3x - y &= -17 \\ 2x + 3y &= -4 \end{aligned}$

15. $\begin{aligned} x + 3y + 2z &= 0 \\ -2x + 3z &= -12 \\ 2x - 6y - z &= 6 \end{aligned}$

16. $\begin{aligned} 2x - y - 2z &= 3 \\ -2x + 3y + z &= 3 \\ 2y - z &= 6 \end{aligned}$

10.2 In Exercises 17 and 18 solve for x.

17. $\begin{bmatrix} 5 & -1 \\ 3 & 2x \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 3 & -6 \end{bmatrix}$

18. $\begin{bmatrix} 6 & x^2 \\ 4x & -2 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ -12 & -2 \end{bmatrix}$

Exercises 19–28 refer to the following matrices.

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 0 \\ 0 & 4 \\ 2 & -2 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 3 & 4 \\ -1 & 0 & -6 \end{bmatrix}$$

If possible, find the following.

19. $A + B$ 20. $B - A$ 21. $A + C$
22. $5D$ 23. CD 24. DC
25. BC 26. CB 27. $A + 2B$
28. $-AB$

10.3 In Exercises 29 and 30 find the inverse of the given matrix.

29. $\begin{bmatrix} -2 & 3 \\ 1 & 4 \end{bmatrix}$

30. $\begin{bmatrix} 1 & 1 & -4 \\ -5 & -2 & 0 \\ 4 & 2 & -1 \end{bmatrix}$

In Exercises 31 and 32 solve the given system by finding the inverse of the coefficient matrix.

31. $\begin{aligned} 2x - y &= 1 \\ x + y &= 5 \end{aligned}$

32. $\begin{aligned} x + 2y - 2z &= -4 \\ 3x - y &= -2 \\ y + 4z &= 1 \end{aligned}$

10.4 In Exercises 33–38 evaluate the given determinant.

33. $\begin{vmatrix} 3 & 1 \\ -4 & 2 \end{vmatrix}$ 34. $\begin{vmatrix} -1 & 2 \\ 0 & 6 \end{vmatrix}$ 35. $\begin{vmatrix} 2 & -1 \\ 6 & -3 \end{vmatrix}$

36. $\begin{vmatrix} 1 & 0 & -1 \\ 2 & 3 & -5 \\ 0 & 4 & 0 \end{vmatrix}$ 37. $\begin{vmatrix} 1 & -1 & 2 \\ 0 & 5 & 4 \\ 2 & 3 & 8 \end{vmatrix}$

38. $\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 4 \\ 0 & 0 & -1 \end{vmatrix}$

In Exercises 39–44 use Cramer's rule to solve the given linear system.

39. $\begin{aligned} 2x - y &= -3 \\ -2x + 3y &= 11 \end{aligned}$

40. $\begin{aligned} 3x - y &= 7 \\ 2x + 5y &= -18 \end{aligned}$

41. $\begin{aligned} x + 2y &= 2 \\ 2x - 7y &= 48 \end{aligned}$

42. $\begin{aligned} 2x + 3y - z &= -3 \\ -3x + 4z &= 16 \\ 2y + 5z &= 9 \end{aligned}$

43. $\begin{aligned} 3x + z &= 0 \\ x + y + z &= 0 \\ -3y + 2z &= -4 \end{aligned}$

44. $\begin{aligned} 2x + 3y + z &= -5 \\ 2y + 2z &= -3 \\ 4x + y - 2z &= -2 \end{aligned}$

PROGRESS TEST 10A

Problems 1 and 2 refer to the matrix

$$A = \begin{bmatrix} -1 & 2 \\ -2 & 4 \\ 0 & 7 \end{bmatrix}$$

1. Find the dimension of the matrix A.
2. Find a_{31}.
3. Write the augmented matrix of the linear system

$$\begin{aligned} -7x + 6z &= 3 \\ 2y - z &= 10 \\ x - y + z &= 5 \end{aligned}$$

4. Write a linear system corresponding to the augmented matrix

$$\left[\begin{array}{rr|r} -5 & 2 & 4 \\ 3 & -4 & 4 \end{array}\right]$$

5. Use back-substitution to solve the linear system corresponding to the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} \end{array}\right]$$

6. Solve the linear system

$$\begin{aligned} -x + 2y &= 2 \\ \frac{1}{2}x + 2y &= -7 \end{aligned}$$

by applying Gaussian elimination to the augmented matrix.

7. Solve the linear system

$$\begin{aligned} 2x - y + 3z &= 2 \\ x + 2y - z &= 1 \\ -x + y + 4z &= 2 \end{aligned}$$

by applying Gauss–Jordan elimination to the augmented matrix.

8. Solve for x.

$$\begin{bmatrix} 2x - 1 & 0 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 1 & -3 \end{bmatrix}$$

Problems 9–12 refer to the matrices

$$A = \begin{bmatrix} -4 & 0 & 3 \\ 6 & 2 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

$$C = \begin{bmatrix} 4 & 2 \\ -2 & 0 \\ 3 & -1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & -6 \\ 0 & 2 \\ 4 & -1 \end{bmatrix}$$

If possible, find the following.

9. $C - 2D$ 10. AC 11. CB 12. BA

13. Find the inverse of the matrix

$$\begin{bmatrix} -1 & 0 & 4 \\ 2 & 1 & -1 \\ 1 & -3 & 2 \end{bmatrix}$$

14. Solve the given linear system by finding the inverse of the coefficient matrix.

$$\begin{aligned} 3x - 2y &= -8 \\ 2x + 3y &= -1 \end{aligned}$$

In Problems 15 and 16 evaluate the given determinant.

15. $\begin{vmatrix} -6 & -2 \\ 2 & 1 \end{vmatrix}$ 16. $\begin{vmatrix} 0 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & 4 & 5 \end{vmatrix}$

17. Use Cramer's rule to solve the linear system

$$\begin{aligned} x + 2y &= -2 \\ -2x - 3y &= 1 \end{aligned}$$

PROGRESS TEST 10B

Problems 1 and 2 refer to the matrix

$$B = \begin{bmatrix} -1 & 5 & 0 & 6 \\ 4 & -2 & 1 & -2 \end{bmatrix}$$

1. Determine the dimension of the matrix B.
2. Find b_{23}.
3. Write the coefficient matrix of the linear system

$$\begin{aligned} 2x - 6y &= 5 \\ x + 3y &= -2 \end{aligned}$$

4. Write a linear system corresponding to the augmented matrix

$$\left[\begin{array}{ccc|c} 16 & 0 & 6 & 10 \\ -4 & -2 & 5 & 8 \\ 2 & 3 & -1 & -6 \end{array}\right]$$

5. Use back-substitution to solve the linear system corresponding to the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

6. Solve the linear system

$$\begin{aligned} 2x + 3y &= -11 \\ 3x - 2y &= 3 \end{aligned}$$

by applying Gaussian elimination to the augmented matrix.

7. Solve the linear system

$$\begin{aligned} 2x + y - 2z &= 7 \\ -3x - 5y + 4z &= -3 \\ 5x + 4y &= 17 \end{aligned}$$

by applying Gauss–Jordan elimination to the augmented matrix.

8. Solve for y.

$$\begin{bmatrix} 2 & -5 & 1 \\ -3 & 1-y & 2 \end{bmatrix} = \begin{bmatrix} 2 & -5 & 1 \\ -3 & 6 & 2 \end{bmatrix}$$

Problems 9–12 refer to the matrices

$$A = \begin{bmatrix} 2 & -3 \\ -1 & 0 \\ 2 & 1 \end{bmatrix} \qquad B = [1 \quad -2 \quad 3]$$

$$C = \begin{bmatrix} 0 & -4 \\ 3 & 1 \\ -2 & 5 \end{bmatrix}$$

If possible, find the following.

9. BA 10. $2C + 3A$ 11. CB 12. $BC - A$

13. Find the inverse of the matrix

$$\begin{bmatrix} 2 & -5 & -2 \\ -1 & 3 & 0 \\ -2 & 0 & 4 \end{bmatrix}$$

14. Solve the given linear system by finding the inverse of the coefficient matrix.

$$\begin{aligned} 5x - 2y &= -6 \\ -2x + 2y &= 3 \end{aligned}$$

In Problems 15 and 16 evaluate the given determinant.

15. $\begin{vmatrix} -2 & 4 \\ 3 & 5 \end{vmatrix}$ 16. $\begin{vmatrix} -2 & 0 & 1 \\ 1 & 2 & 0 \\ 2 & -1 & -1 \end{vmatrix}$

17. Use Cramer's rule to solve the linear system

$$\begin{aligned} x + y &= -1 \\ 2x - 4y &= -5 \end{aligned}$$

11

THEORY OF POLYNOMIALS

In Section 3.3 we observed that the polynomial function

$$f(x) = ax + b \tag{1}$$

is called a linear function and that the polynomial function

$$g(x) = ax^2 + bx + c, \qquad a \neq 0 \tag{2}$$

is called a quadratic function. To facilitate the study of polynomial functions in general, we will use the notation

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0, \qquad a_n \neq 0 \tag{3}$$

to represent a **polynomial function of degree *n*.** Note that the subscript k of the coefficient a_k is the same as the exponent of x in x^k. In general, the coefficients a_k may be real or complex numbers; our work in this chapter will focus on real values for a_k, but we will indicate which results hold true when the coefficients a_k are complex numbers.

If $a \neq 0$ in Equation (1), we set the polynomial function equal to zero and obtain the linear equation

$$ax + b = 0$$

which has precisely one solution, $-b/a$. If we set the polynomial function in Equation (2) equal to zero, we have the quadratic equation

$$ax^2 + bx + c = 0$$

which has the two solutions given by the quadratic formula. If we set the polynomial function in Equation (3) equal to zero, we have the **polynomial equation of degree *n***

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0 \tag{4}$$

Any complex number satisfying Equation (4) is called a **solution** or **root** of the polynomial equation $P(x) = 0$. Such values are also called the **zeros** of the polynomial $P(x)$. In short, a complex number satisfying $P(x) = 0$ is a *root of the equation* or a *zero of the polynomial* $P(x)$.

Here are some of the fundamental questions concerning zeros of a polynomial that have attracted the attention of mathematicians for centuries.

- Does a polynomial always have a zero?
- Including complex zeros, what is the total number of zeros of a polynomial of degree n?
- How many of the zeros of a polynomial are real numbers?
- If the coefficients of a polynomial are integers, how many of the zeros are rational numbers?
- Is there a relationship between the zeros and the factors of a polynomial?
- Can we find a formula for expressing the zeros of a polynomial in terms of the coefficients of the polynomial?

Some of these questions are tough mathematical problems. We will explore them and provide the answers in the course of this chapter.

11.1 POLYNOMIAL DIVISION AND SYNTHETIC DIVISION

POLYNOMIAL DIVISION

To find the zeros of a polynomial, it will be necessary to divide the polynomial by a second polynomial. There is a procedure for polynomial division that parallels the long division process of arithmetic. In arithmetic, if we divide an integer p by an integer $d \neq 0$, we obtain a quotient q and a remainder r, so we can write

$$\frac{p}{d} = q + \frac{r}{d} \tag{1}$$

where

$$0 \leq r < d \tag{2}$$

This result can also be written in the form

$$p = qd + r, \qquad 0 \leq r < d \tag{3}$$

For example,

$$\frac{7284}{13} = 560 + \frac{4}{13}$$

or

$$7284 = (560)(13) + 4$$

In the long division process for polynomials, we divide the dividend $P(x)$ by the divisor $D(x) \neq 0$ to obtain a quotient $Q(x)$ and a remainder $R(x)$. We then have

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)} \tag{4}$$

where $R(x) = 0$ or where

$$\text{degree of } R(x) < \text{degree of } D(x) \tag{5}$$

This result can also be written as

$$P(x) = Q(x)D(x) + R(x) \tag{6}$$

Note that Equations (1) and (4) have the same form and that Equation (6) has the same form as Equation (3). Equation (2) requires that the remainder be less than the divisor, and the parallel requirement for polynomials in Equation (5) is that the *degree* of the remainder be less than that of the divisor.

We illustrate the long division process for polynomials by an example.

EXAMPLE 1
Divide $3x^3 - 7x^2 + 1$ by $x - 2$.

SOLUTION

Polynomial Division	
Step 1. Arrange the terms of both polynomials by descending powers of x. If a power is missing, write the term with a zero coefficient.	*Step 1.* $x - 2\overline{)3x^3 - 7x^2 + 0x + 1}$
Step 2. Divide the first term of the dividend by the first term of the divisor. The answer is written above the first term of the dividend.	*Step 2.* $\begin{array}{r} 3x^2 \\ x - 2\overline{)3x^3 - 7x^2 + 0x + 1} \end{array}$
Step 3. Multiply the divisor by the quotient obtained in Step 2, and then subtract the product.	Step 3. $\begin{array}{r} 3x^2 \\ x - 2\overline{)3x^3 - 7x^2 + 0x + 1} \\ \underline{3x^3 - 6x^2} \\ -x^2 + 0x + 1 \end{array}$
Step 4. Repeat Steps 2 and 3 until the remainder is zero or the degree of the remainder is less than the degree of the divisor.	*Step 4.* $\begin{array}{rl} 3x^2 - x - 2 & = Q(x) \\ x - 2\overline{)3x^3 - 7x^2 + 0x + 1} & \\ \underline{3x^3 - 6x^2} & \\ -x^2 + 0x + 1 & \\ \underline{-x^2 + 2x} & \\ -2x + 1 & \\ \underline{-2x + 4} & \\ -3 & = R(x) \end{array}$
Step 5. Write the answer in the form of Equation (4) or Equation (6).	*Step 5.* $\begin{aligned} P(x) &= 3x^3 - 7x^2 + 1 \\ &= \underbrace{(3x^2 - x - 2)}_{Q(x)} \cdot \underbrace{(x - 2)}_{D(x)} + \underbrace{-3}_{R(x)} \end{aligned}$

PROGRESS CHECK

Divide $4x^2 - 3x + 6$ by $x + 2$.

ANSWER

$4x - 11 + \dfrac{28}{x+2}$

SYNTHETIC DIVISION

Our work in this chapter will frequently require division of a polynomial by a first-degree polynomial $x - r$, where r is a constant. Fortunately, there is a shortcut called **synthetic division** that simplifies this task. To demonstrate synthetic division we will do Example 1 again, writing only the coefficients.

$$
\begin{array}{r|rrrr}
 & \mathbf{3} & \mathbf{-1} & \mathbf{-2} & \\
\cline{2-5}
-2 & 3 & -7 & 0 & 1 \\
 & \mathbf{3} & -6 & & \\
\cline{2-5}
 & & \mathbf{-1} & 0 & 1 \\
 & & -1 & 2 & \\
\cline{2-5}
 & & & \mathbf{-2} & 1 \\
 & & & -2 & 4 \\
\cline{2-5}
 & & & & -3
\end{array}
$$

Note that the boldface numerals are duplicated. We can use this to our advantage and simplify the process as follows.

$$
\begin{array}{r|rrr|r}
-2 & 3 & -7 & 0 & 1 \\
 & & -6 & 2 & 4 \\
\hline
 & \multicolumn{3}{l|}{\underbrace{3 \quad -1 \quad -2}_{\text{coefficients of the quotient}}} & -3 \; \text{remainder}
\end{array}
$$

In the third row we copied the leading coefficient (3) of the dividend, multiplied it by the divisor (-2), and wrote the result (-6) in the second row under the next coefficient. The numbers in the second column were subtracted to obtain $-7 - (-6) = -1$. The procedure was repeated until the third row was of the same length as the first row.

Since subtraction is more apt to produce errors than is addition, we can modify this process slightly. If the divisor is $x - r$, we will write r instead of $-r$ in the box and use addition in each step instead of subtraction. Repeating our example, we have

$$
\begin{array}{r|rrr|r}
2 & 3 & -7 & 0 & 1 \\
 & & 6 & -2 & -4 \\
\hline
 & 3 & -1 & -2 & -3
\end{array}
$$

EXAMPLE 2

Divide $4x^3 - 2x + 5$ by $x + 2$ using synthetic division.

SOLUTION

Synthetic Division

Step 1. If the divisor is $x - r$, write r in the box. Arrange the coefficients of the dividend by descending powers of x, supplying a zero coefficient for every missing power.

Step 1.

$$\begin{array}{r|rrrr} -2 & 4 & 0 & -2 & 5 \\ \end{array}$$

Step 2. Copy the leading coefficient in the third row.

Step 2.

$$\begin{array}{r|rrrr} -2 & 4 & 0 & -2 & 5 \\ & & & & \\ \hline & 4 & & & \end{array}$$

Step 3. Multiply the last entry in the third row by the number in the box and write the result in the second row under the next coefficient. Add the numbers in that column.

Step 3.

$$\begin{array}{r|rrrr} -2 & 4 & 0 & -2 & 5 \\ & & -8 & & \\ \hline & 4 & -8 & & \end{array}$$

Step 4. Repeat Step 3 until there is an entry in the third row for each entry in the first row. The last number in the third row is the remainder; the other numbers are the coefficients of the quotient in descending order.

Step 4.

$$\begin{array}{r|rrr|r} -2 & 4 & 0 & -2 & 5 \\ & & -8 & 16 & -28 \\ \hline & 4 & -8 & 14 & -23 \end{array}$$

$$\frac{4x^3 - 2x + 5}{x + 2} = 4x^2 - 8x + 14 - \frac{23}{x + 2}$$

PROGRESS CHECK

Use synthetic division to obtain the quotient $Q(x)$ and the constant remainder R when $2x^4 - 10x^2 - 23x + 6$ is divided by $x - 3$.

ANSWER

$Q(x) = 2x^3 + 6x^2 + 8x + 1$; $R = 9$

WARNING

(a) Synthetic division can be used only when the divisor is a linear factor. Don't forget to write a zero for the coefficient of each missing term.

(b) When dividing by $x - r$, place r in the box. For example, when the divisor is $x + 3$, place -3 in the box, since $x + 3 = x - (-3)$. Similarly, when the divisor is $x - 3$, place $+3$ in the box, since $x - 3 = x - (+3)$.

EXERCISE SET 11.1

In Exercises 1–10 use polynomial division to find the quotient $Q(x)$ and the remainder $R(x)$ when the first polynomial is divided by the second polynomial.

1. $x^2 - 7x + 12, \quad x - 5$
2. $x^2 + 3x + 3, \quad x + 2$
3. $2x^3 - 2x, \quad x^2 + 2x - 1$
4. $3x^3 - 2x^2 + 4, \quad x^2 - 2$
5. $3x^4 - 2x^2 + 1, \quad x + 3$
6. $x^5 - 1, \quad x^2 - 1$
7. $2x^3 - 3x^2, \quad x^2 + 2$
8. $3x^3 - 2x - 1, \quad x^2 - x$
9. $x^4 - x^3 + 2x^2 - x + 1, \quad x^2 + 1$
10. $2x^4 - 3x^3 - x^2 - x - 2, \quad x - 2$

In Exercises 11–20 use synthetic division to find the quotient $Q(x)$ and the constant remainder R when the first polynomial is divided by the second polynomial.

11. $x^3 - x^2 - 6x + 5, \quad x + 2$
12. $2x^3 - 3x^2 - 4, \quad x - 2$
13. $x^4 - 81, \quad x - 3$
14. $x^4 - 81, \quad x + 3$
15. $3x^3 - x^2 + 8, \quad x + 1$
16. $2x^4 - 3x^3 - 4x - 2, \quad x - 1$
17. $x^5 + 32, \quad x + 2$
18. $x^5 + 32, \quad x - 2$
19. $6x^4 - x^2 + 4, \quad x - 3$
20. $8x^3 + 4x^2 - x - 5, \quad x + 3$

11.2 THE REMAINDER AND FACTOR THEOREMS

THE REMAINDER THEOREM

From our work with the division process we may surmise that division of a polynomial $P(x)$ by $x - r$ results in a quotient $Q(x)$ and a constant remainder R such that

$$P(x) = (x - r) \cdot Q(x) + R$$

Since this identity holds for all real values of x, it must hold when $x = r$. Consequently,

$$P(r) = (r - r) \cdot Q(r) + R$$
$$P(r) = 0 \cdot Q(r) + R$$

or

$$P(r) = R$$

We have proved the Remainder Theorem.

Remainder Theorem	If a polynomial $P(x)$ is divided by $x - r$, the remainder is $P(r)$.

EXAMPLE 1

Determine the remainder when $P(x) = 2x^3 - 3x^2 - 2x + 1$ is divided by $x - 3$.

SOLUTION

By the Remainder Theorem, the remainder is $R = P(3)$. We then have

$$R = P(3) = 2(3)^3 - 3(3)^2 - 2(3) + 1 = 22$$

We may verify this result by using synthetic division:

FERMAT'S LAST THEOREM

a^n + b^n = c^n

If you were asked to find natural numbers a, b, and c that satisfy the equation

$$a^2 + b^2 = c^2$$

you would have no trouble coming up with such "triplets" as (3, 4, 5) and (5, 12, 13). In fact, there are an infinite number of solutions, since any multiple of (3, 4, 5), such as (6, 8, 10), is also a solution.

Generalizing the above problem, suppose we seek natural numbers a, b, and c that satisfy the equation

$$a^n + b^n = c^n$$

for integer values of $n > 2$. Pierre Fermat, a great French mathematician of the seventeenth century, stated that there are no natural numbers a, b, and c that satisfy these conditions. This seductively simple conjecture is known as Fermat's Last Theorem. Fermat wrote in his notebook that he had a proof but that it was too long to include in the margin.

A proof of this theorem or a counterexample has eluded mathematicians for three hundred years. In 1983 the German mathematician Gerd Faltings proved that the equation

$$x^n + y^n = 1$$

has only a finite number of rational solutions. This proof may well be a significant step in establishing Fermat's Last Theorem.

$$\begin{array}{r|rrrr} 3 & 2 & -3 & -2 & 1 \\ & & 6 & 9 & 21 \\ \hline & 2 & 3 & 7 & \mathbf{22} \end{array}$$

The number in boldface is the remainder, so we have verified that $R = 22$.

PROGRESS CHECK

Determine the remainder when $3x^2 - 2x - 6$ is divided by $x + 2$.

ANSWER

10

Graphing $P(x)$

We can use the Remainder Theorem to tabulate values of a polynomial function from which we can sketch the graph. Recall that if a polynomial $P(x)$ is divided by $x - r$, the remainder is $P(r)$. Thus, the point $(r, P(r))$ lies on the graph of the function $P(x)$.

An efficient scheme for evaluating $P(r)$ is a streamlined form of synthetic division in which the addition is performed without writing the middle row. Given the polynomial $P(x)$ of Example 1,

$$P(x) = 2x^3 - 3x^2 - 2x + 1$$

we can find $P(3)$ by synthetic division without writing the middle line:

$$\begin{array}{r|rrrl} & 2 & -3 & -2 & 1 \\ 3 & 2 & 3 & 7 & 22 = P(3) \end{array}$$

so the point (3, 22) lies on the graph of $P(x)$. Repeating this procedure for a number of values of r will provide a table of values for plotting.

EXAMPLE 2
Sketch the graph of $P(x) = 2x^3 - 3x^2 - 2x + 1$.

SOLUTION
For each value r of x, the point $(r, P(r))$ lies on the graph of $y = P(x)$. We will allow x to assume integer values from -3 to $+3$ and will find $P(x)$ by using synthetic division.

	2	−3	−2	1	$(x, y) = (x, P(x))$
−3	2	−9	25	−74	(−3, −74)
−2	2	−7	12	−23	(−2, −23)
−1	2	−5	3	−2	(−1, −2)
0	2	−3	−2	1	(0, 1)
1	2	−1	−3	−2	(1, −2)
2	2	1	0	1	(2, 1)
3	2	3	7	22	(3, 22)

The ordered pairs shown at the right of each row are the coordinates of points on the graph shown in Figure 1.

FACTOR THEOREM

Let's assume that a polynomial $P(x)$ can be written as a product of polynomials, that is,

$$P(x) = D_1(x)D_2(x) \cdots D_n(x)$$

where $D_1(x), D_2(x), \ldots, D_n(x)$ are polynomials each of degree greater than zero. Then $D_1(x), D_2(x), \ldots, D_n(x)$ are called **factors** of $P(x)$. If we focus on $D_1(x)$ and let

$$Q(x) = D_2(x)D_3(x) \cdots D_n(x)$$

then

$$P(x) = D_1(x)Q(x)$$

This equation suggests the following formal definition:

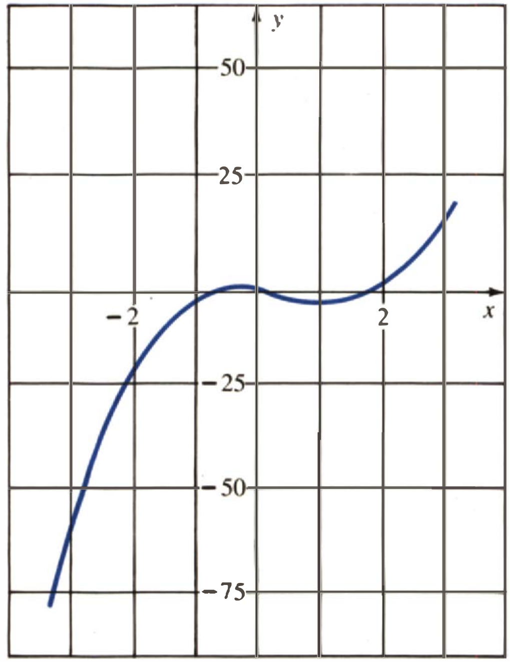


FIGURE 1

The polynomial $D(x)$ is a factor of a polynomial $P(x)$ if division of $P(x)$ by $D(x)$ results in a remainder of zero.

We can now combine this rule and the Remainder Theorem to prove the Factor Theorem.

Factor Theorem A polynomial $P(x)$ has a factor $x - r$ if and only if $P(r) = 0$.

If $x - r$ is a factor of $P(x)$, then division of $P(x)$ by $x - r$ must result in a remainder of 0. By the Remainder Theorem, the remainder is $P(r)$, and hence $P(r) = 0$. Conversely, if $P(r) = 0$, then the remainder is 0 and $P(x) = (x - r)\,Q(x)$ for some polynomial $Q(x)$ of degree one less than that of $P(x)$. By definition, $x - r$ is then a factor of $P(x)$.

EXAMPLE 3
Show that $x + 2$ is a factor of

$$P(x) = x^3 - x^2 - 2x + 8$$

SOLUTION

By the Factor Theorem, $x + 2$ is a factor if $P(-2) = 0$. Using synthetic division to evaluate $P(-2)$,

$$\begin{array}{r|rrrr} -2 & 1 & -1 & -2 & 8 \\ & & -2 & 6 & -8 \\ \hline & 1 & -3 & 4 & 0 \end{array}$$

we see that $P(-2) = 0$. Alternatively, we can evaluate

$$P(-2) = (-2)^3 - (-2)^2 - 2(-2) + 8 = 0$$

We conclude that $x + 2$ is a factor of $P(x)$.

PROGRESS CHECK

Show that $x - 1$ is a factor of $P(x) = 3x^6 - 3x^5 - 4x^4 + 6x^3 - 2x^2 - x + 1$.

SUMMARY

We can summarize our work thus far in this neat way.

The following are equivalent statements:

- r is a zero of $P(x)$.
- r is a root of the equation $P(x) = 0$.
- $P(r) = 0$. **(Remainder Theorem)**
- $x - r$ is a factor of $P(x)$. **(Factor Theorem)**
- Dividing $P(x)$ by synthetic division with r as the divisor will result in a remainder of zero.

EXERCISE SET 11.2

In Exercises 1–6 use the Remainder Theorem and synthetic division to find $P(r)$.

1. $P(x) = x^3 - 4x^2 + 1, \quad r = 2$
2. $P(x) = x^4 - 3x^2 - 5x, \quad r = -1$
3. $P(x) = x^5 - 2, \quad r = -2$
4. $P(x) = 2x^4 - 3x^3 + 6, \quad r = 2$
5. $P(x) = x^6 - 3x^4 + 2x^3 + 4, \quad r = -1$
6. $P(x) = x^6 - 2, \quad r = 1$

In Exercises 7–12 use the Remainder Theorem to determine the remainder when $P(x)$ is divided by $x - r$.

7. $P(x) = x^3 - 2x^2 + x - 3, \quad x - 2$
8. $P(x) = 2x^3 + x^2 - 5, \quad x + 2$
9. $P(x) = -4x^3 + 6x - 2, \quad x - 1$
10. $P(x) = 6x^5 - 3x^4 + 2x^2 + 7, \quad x + 1$
11. $P(x) = x^5 - 30, \quad x + 2$
12. $P(x) = x^4 - 16, \quad x - 2$

In Exercises 13–18 use the Remainder Theorem and synthetic division to sketch the graph of the given polynomial for $-3 \leq x \leq 3$.

13. $P(x) = x^3 + x^2 + x + 1$
14. $P(x) = 3x^4 + 5x^3 + x^2 + 5x - 2$
15. $P(x) = 2x^3 + 3x^2 - 5x - 6$
16. $P(x) = x^3 + 3x^2 - 4x - 12$
17. $P(x) = x^4 - 3x^3 + 1$
18. $P(x) = 4x^4 + 4x^3 - 9x^2 - x + 2$

In Exercises 19–26 use the Factor Theorem to decide whether or not the first polynomial is a factor of the second polynomial.

19. $x - 2, \quad x^3 - x^2 - 5x + 6$
20. $x - 1, \quad x^3 + 4x^2 - 3x + 1$
21. $x + 2, \quad x^4 - 3x - 5$
22. $x + 1, \quad 2x^3 - 3x^2 + x + 6$
23. $x + 3, \quad x^3 + 27$
24. $x + 2, \quad x^4 + 16$
25. $x + 2, \quad x^4 - 16$
26. $x - 3, \quad x^3 + 27$

In Exercises 27–30 use synthetic division to determine the value of k or r as requested.

27. Determine the values of r for which division of $x^2 - 2x - 1$ by $x - r$ has a remainder of 2.
28. Determine the values of r for which

$$\frac{x^2 - 6x - 1}{x - r}$$

has a remainder of -9.

29. Determine the values of k for which $x - 2$ is a factor of $x^3 - 3x^2 + kx - 1$.
30. Determine the values of k for which $2k^2x^3 + 3kx^2 - 2$ is divisible by $x - 1$.

31. Use the Factor Theorem to show that $x - 2$ is a factor of $P(x) = x^8 - 256$.
32. Use the Factor Theorem to show that $P(x) = 2x^4 + 3x^2 + 2$ has no factor of the form $x - r$, where r is a real number.
33. Use the Factor Theorem to show that $x - y$ is a factor of $x^n - y^n$, where n is a natural number.

11.3 FACTORS AND ZEROS

COMPLEX NUMBERS AND THEIR PROPERTIES

We introduced the complex number system in Section 1.8 and then used this number system in Section 2.5 to provide solutions to quadratic equations. Recall that $a + bi$ is said to be a complex number where a and b are real numbers and the imaginary unit $i = \sqrt{-1}$ has the property that $i^2 = -1$. We defin᾿ɹ fundamental operations with complex numbers in the following way.

Equality:	$a + bi = c + di$ if $a = c$ and $b = d$
Addition:	$(a + bi) + (c + di) = (a + c) + (b + d)i$
Multiplication:	$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$

With this background, we can now explore further properties of the complex number system.

The complex number $a - bi$ is called the **complex conjugate** (or simply the **conjugate**) of the complex number $a + bi$. For example, $3 - 2i$ is the conjugate of $3 + 2i$, $4i$ is the conjugate of $-4i$, and 2 is the conjugate of 2. Forming the product $(a + bi)(a - bi)$, we have

$$\begin{aligned}(a + bi)(a - bi) &= a^2 - abi + abi - b^2i^2 \\ &= a^2 + b^2 \qquad \text{Since } i^2 = -1\end{aligned}$$

Because a and b are real numbers, $a^2 + b^2$ is also a real number. We can summarize this result as follows.

> The product of a complex number and its conjugate is a real number:
> $$(a + bi)(a - bi) = a^2 + b^2$$

We can now demonstrate that the quotient of two complex numbers is also a complex number. The quotient

$$\frac{q + ri}{s + ti}$$

can be written in the form $a + bi$ by multiplying both numerator and denominator by $s - ti$, the conjugate of the denominator. We then have

$$\frac{q + ri}{s + ti} = \frac{q + ri}{s + ti} \cdot \frac{s - ti}{s - ti} = \frac{(qs + rt) + (rs - qt)i}{s^2 + t^2}$$
$$= \frac{qs + rt}{s^2 + t^2} + \frac{(rs - qt)}{s^2 + t^2}i$$

which is a complex number of the form $a + bi$. Of course, the reciprocal of the complex number $s + ti$ is the quotient $1/(s + ti)$, which can also be written as a complex number by using the same technique. In summary:

> - The quotient of two complex numbers is a complex number.
> - The reciprocal of a nonzero complex number is a complex number.

EXAMPLE 1

(a) Write the quotient $\dfrac{-2 + 3i}{3 - 2i}$ in the form $a + bi$.

(b) Write the reciprocal of $2 - 5i$ in the form $a + bi$.

SOLUTION

(a) Multiplying numerator and denominator by the conjugate $3 + 2i$ of the denominator, we have

$$\frac{-2 + 3i}{3 - 2i} = \frac{-2 + 3i}{3 - 2i} \cdot \frac{3 + 2i}{3 + 2i} = \frac{-6 - 4i + 9i + 6i^2}{3^2 + 2^2} = \frac{-6 + 5i + 6(-1)}{9 + 4}$$
$$= \frac{-12 + 5i}{13} = -\frac{12}{13} + \frac{5}{13}i$$

(b) The reciprocal is $1/(2 - 5i)$. Multiplying both numerator and denominator by the conjugate $2 + 5i$, we have

$$\frac{1}{2 - 5i} \cdot \frac{2 + 5i}{2 + 5i} = \frac{2 + 5i}{2^2 + 5^2} = \frac{2 + 5i}{29} = \frac{2}{29} + \frac{5}{29}i$$

Verify that $(2 - 5i)\left(\dfrac{2}{29} + \dfrac{5}{29}i\right) = 1$.

PROGRESS CHECK

Write the following in the form $a + bi$.

(a) $\dfrac{4-2i}{5+2i}$ (b) $\dfrac{1}{2-3i}$ (c) $\dfrac{-3i}{3+5i}$

ANSWERS

(a) $\dfrac{16}{29} - \dfrac{18}{29}i$ (b) $\dfrac{2}{13} + \dfrac{3}{13}i$ (c) $-\dfrac{15}{34} - \dfrac{9}{34}i$

If we let $z = a + bi$, it is customary to write the conjugate $a - bi$ as $\bar{z}$. We will have need to use the following properties of complex numbers and their conjugates.

Properties of Complex Numbers

If z and w are complex numbers, then

1. $\bar{z} = \bar{w}$ if and only if $z = w$;
2. $\bar{z} = z$ if and only if z is a real number;
3. $\overline{z + w} = \bar{z} + \bar{w}$;
4. $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$;
5. $\overline{z^n} = \bar{z}^n$, n a positive integer.

To prove Properties 1–5, let $z = a + bi$ and $w = c + di$. Properties 1 and 2 follow directly from the definition of equality of complex numbers. To prove Property 3, we note that $z + w = (a + c) + (b + d)i$. Then, by the definition of a complex conjugate,

$$\begin{aligned}\overline{z + w} &= (a + c) - (b + d)i \\ &= (a - bi) + (c - di) \\ &= \bar{z} + \bar{w}\end{aligned}$$

Properties 4 and 5 can be proved in a similar manner, although a rigorous proof of Property 5 requires the use of mathematical induction, a method we will discuss in a later chapter.

EXAMPLE 2

If $z = 1 + 2i$ and $w = 3 - i$, verify the following statements.

(a) $\overline{z + w} = \bar{z} + \bar{w}$ (b) $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$ (c) $\overline{z^2} = \bar{z}^2$

SOLUTION

(a) Adding, we get $z + w = 4 + i$. Therefore $\overline{z + w} = 4 - i$. Also,

$$\bar{z} + \bar{w} = (1 - 2i) + (3 + i) = 4 - i$$

Thus, $\overline{z + w} = \bar{z} + \bar{w}$.

(b) Multiplying, we get $z \cdot w = (1 + 2i)(3 - i) = 5 + 5i$. Therefore

$$\overline{z \cdot w} = 5 - 5i$$

Also,

$$\bar{z} \cdot \bar{w} = (1 - 2i)(3 + i) = 5 - 5i$$

Thus, $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$.
(c) Squaring, we get

$$z^2 = (1 + 2i)(1 + 2i) = -3 + 4i$$

Therefore $\overline{z^2} = -3 - 4i$. Also,

$$\bar{z}^2 = (1 - 2i)(1 - 2i) = -3 - 4i$$

Thus, $\overline{z^2} = \bar{z}^2$.

PROGRESS CHECK
If $z = 2 + 3i$ and $w = \frac{1}{2} - 2i$, verify the following statements.

(a) $\overline{z + w} = \bar{z} + \bar{w}$ (b) $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$

(c) $\overline{z^2} = \bar{z}^2$ (d) $\overline{w^3} = \bar{w}^3$

FACTOR THEOREM

We are now in a position to answer some of the questions posed in the introduction to this chapter. By using the Factor Theorem we can show that there is a close relationship between the factors and the zeros of the polynomial $P(x)$. By definition, r is a zero of $P(x)$ if and only if $P(r) = 0$. But the Factor Theorem tells us that $P(r) = 0$ if and only if $x - r$ is a factor of $P(x)$. This leads to the following alternative statement of the Factor Theorem.

Factor Theorem	A polynomial $P(x)$ has a zero r if and only if $x - r$ is a factor of $P(x)$.

EXAMPLE 3
Find a polynomial $P(x)$ of degree 3 whose zeros are -1, 1, and -2.

SOLUTION
By the Factor Theorem, $x + 1$, $x - 1$, and $x + 2$ are factors of $P(x)$. The product

$$P(x) = (x + 1)(x - 1)(x + 2) = x^3 + 2x^2 - x - 2$$

is a polynomial of degree 3 with the desired zeros. Note that multiplying $P(x)$ by any nonzero real number results in another polynomial that has the same zeros. For example, the polynomial

$$5 \cdot P(x) = 5x^3 + 10x^2 - 5x - 10$$

also has -1, 1, and -2 as its zeros. Thus, the answer is not unique.

PROGRESS CHECK
Find a polynomial $P(x)$ of degree 3 whose zeros are 2, 4, and -3.

ANSWER

$x^3 - 3x^2 - 10x + 24$

We began this chapter with the question, Does a polynomial always have a zero? The answer was supplied by Carl Friedrich Gauss in his doctoral dissertation in 1799. Unfortunately, the proof of this theorem is beyond the scope of this book.

The Fundamental Theorem of Algebra—Part 1

Every polynomial $P(x)$ of degree $n \geq 1$ has at least one zero among the complex numbers.

Note that the zero guaranteed by this theorem may be a real number since the real numbers are a subset of the complex number system.

Gauss, who is considered by many to have been the greatest mathematician of all time, supplied the proof at age 22. The importance of the theorem is reflected in its title. We now see why it was necessary to create the complex numbers and that we need not create any other number system beyond the complex numbers in order to solve polynomial equations.

How many zeros does a polynomial of degree n have? The next theorem will bring us closer to an answer.

Linear Factor Theorem

A polynomial $P(x)$ of degree $n \geq 1$ can be written as the product of n linear factors:

$$P(x) = a(x - r_1)(x - r_2) \cdots (x - r_n)$$

Note that a is the leading coefficient of $P(x)$ and that $r_1, r_2, \ldots, r_n$ are, in general, complex numbers.

To prove this theorem, we first note that the Fundamental Theorem of Algebra guarantees us the existence of a zero r_1. By the Factor Theorem, $x - r_1$ is a factor, and, consequently,

$$P(x) = (x - r_1)Q_1(x) \tag{1}$$

where $Q_1(x)$ is a polynomial of degree $n - 1$. If $n - 1 \geq 1$, then $Q_1(x)$ must have a zero r_2. Thus

$$Q_1(x) = (x - r_2)Q_2(x) \tag{2}$$

where $Q_2(x)$ is of degree $n - 2$. Substituting in Equation (1) for $Q_1(x)$ we have

$$P(x) = (x - r_1)(x - r_2)Q_2(x) \tag{3}$$

This process is repeated n times until $Q_n(x) = a$ is of degree 0. Hence,

$$P(x) = a(x - r_1)(x - r_2) \cdots (x - r_n) \tag{4}$$

Since a is the leading coefficient of the polynomial on the right side of Equation (4), it must also be the leading coefficient of $P(x)$.

EXAMPLE 4

Find the polynomial $P(x)$ of degree 3 that has the zeros -2, i, and $-i$ and that satisfies $P(1) = -3$.

SOLUTION

Since -2, i, and $-i$ are zeros of $P(x)$, we may write

$$P(x) = a(x + 2)(x - i)(x + i)$$

To find the constant a, we use the condition $P(1) = -3$:

$$P(1) = -3 = a(1 + 2)(1 - i)(1 + i) = 6a$$

$$a = -\frac{1}{2}$$

So

$$P(x) = -\frac{1}{2}(x + 2)(x - i)(x + i)$$

Recall that the zeros of a polynomial need not be distinct from each other. The polynomial

$$P(x) = x^2 - 2x + 1$$

can be written in the factored form

$$P(x) = (x - 1)(x - 1)$$

which shows that the zeros of $P(x)$ are 1 and 1. Since a zero is associated with a factor and a factor may be repeated, we may have repeated zeros. If the factor $x - r$ appears k times, we say that r is a **zero of multiplicity k.**

It is now easy to establish the following, which may be thought of as an alternative form of the Fundamental Theorem of Algebra.

The Fundamental Theorem of Algebra—Part II	If $P(x)$ is a polynomial of degree $n \geq 1$, then $P(x)$ has precisely n zeros among the complex numbers when a zero of multiplicity k is counted k times.

We may prove this theorem as follows: If we write $P(x)$ in the form of Equation (4), we see that $r_1, r_2, \ldots, r_n$ are zeros of the polynomial $P(x)$ and that hence there exist n zeros. If there is an additional zero r that is distinct from the zeros $r_1, r_2, \ldots, r_n$, then $r - r_1, r - r_2, \ldots, r - r_n$ are all different from 0. Substituting r for x in Equation (4) yields

$$P(r) = a(r - r_1)(r - r_2) \ . \ . \ . \ (r - r_n) \tag{5}$$

which cannot equal 0, since the product of nonzero numbers cannot equal 0. Thus, $r_1, r_2, \ldots, r_n$ are zeros of $P(x)$ and there are no other zeros. We conclude that $P(x)$ has precisely n zeros.

EXAMPLE 5
Find all zeros of the polynomial

$$P(x) = (x - \tfrac{1}{2})^3(x + i)(x - 5)^4$$

SOLUTION
The distinct zeros are $\frac{1}{2}$, $-i$, and 5. Further, $\frac{1}{2}$ is a zero of multiplicity 3; $-i$ is a zero of multiplicity 1; 5 is a zero of multiplicity 4.

If we know that r is a zero of $P(x)$, we may write

$$P(x) = (x - r)Q(x)$$

If r_1 is a zero of $Q(x)$, then $Q(r_1) = 0$ and

$$P(r_1) = (r_1 - r)Q(r_1) = (r_1 - r) \cdot 0 = 0$$

which shows that r_1 is also a zero of $P(x)$. We call $Q(x) = 0$ the **depressed equation,** since $Q(x)$ is of lower degree than $P(x)$. In the next example we illustrate the use of the depressed equation in finding the zeros of a polynomial.

EXAMPLE 6
If 4 is a zero of the polynomial $P(x) = x^3 - 8x^2 + 21x - 20$, find the other zeros.

SOLUTION
Since 4 is a zero of $P(x)$, $x - 4$ is a factor of $P(x)$. Therefore,

$$P(x) = (x - 4)Q(x)$$

To find the depressed equation, we compute $Q(x) = P(x)/(x - 4)$ by synthetic division.

$$\begin{array}{r|rrr|r}
4 & 1 & -8 & 21 & -20 \\
 & & 4 & -16 & 20 \\
\hline
 & \underbrace{1 \quad -4 \quad 5}_{\text{coefficients of } Q(x)} & & & \underset{\text{remainder}}{0}
\end{array}$$

The depressed equation is

$$x^2 - 4x + 5 = 0$$

Using the quadratic formula, we find the roots of the depressed equation to be $2 + i$ and $2 - i$. The zeros of $P(x)$ are then seen to be 4, $2 + i$, and $2 - i$.

PROGRESS CHECK

If -2 is a zero of the polynomial $P(x) = x^3 - 7x - 6$, find the remaining zeros.

ANSWER

$-1, 3$

EXAMPLE 7

If -1 is a zero of multiplicity 2 of $P(x) = x^4 + 4x^3 + 2x^2 - 4x - 3$, find the remaining zeros and write $P(x)$ as a product of linear factors.

SOLUTION

Since -1 is a double zero of $P(x)$, then $(x + 1)^2$ is a factor of $P(x)$. Therefore,

$$P(x) = (x + 1)^2 Q(x)$$

or

$$P(x) = (x^2 + 2x + 1)Q(x)$$

Using polynomial division, we can divide both sides of the last equation by $x^2 + 2x + 1$ to obtain

$$\begin{aligned} Q(x) &= \frac{x^4 + 4x^3 + 2x^2 - 4x - 3}{x^2 + 2x + 1} \\ &= x^2 + 2x - 3 \\ &= (x - 1)(x + 3) \end{aligned}$$

The roots of the depressed equation $Q(x) = 0$ are 1 and -3, and these are the remaining zeros of $P(x)$. By the Linear Factor Theorem,

$$P(x) = (x + 1)^2(x - 1)(x + 3)$$

PROGRESS CHECK

If -2 is a zero of multiplicity 2 of $P(x) = x^4 + 4x^3 + 5x^2 + 4x + 4$, write $P(x)$ as a product of linear factors.

ANSWER

$P(x) = (x + 2)(x + 2)(x + i)(x - i)$

We know from the quadratic formula that if a quadratic equation with real coefficients has a complex root $a + bi$, then the conjugate $a - bi$ is the other root. The following theorem extends this result to a polynomial of degree n with real coefficients.

Conjugate Zeros Theorem	If $P(x)$ is a polynomial of degree $n \geq 1$ with real coefficients, and if $a + bi$, $b \neq 0$, is a zero of $P(x)$, then the complex conjugate $a - bi$ is also a zero of $P(x)$.

PROOF OF CONJUGATE ZEROS THEOREM (Optional)

To prove the Conjugate Zeros Theorem, we let $z = a + bi$ and make use of the properties of complex conjugates developed earlier in this section. We may write

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \tag{6}$$

and, since z is a zero of $P(x)$,

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_n z + a_0 = 0 \tag{7}$$

But if $z = w$, then $\bar{z} = \bar{w}$. Applying this property of complex numbers to both sides of Equation (7), we have

$$\overline{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0} = \bar{0} = 0 \tag{8}$$

We also know that $\overline{z + w} = \bar{z} + \bar{w}$. Applying this property to the left side of Equation (8) we see that

$$\overline{a_n z^n} + \overline{a_{n-1} z^{n-1}} + \cdots + \overline{a_1 z} + \overline{a_0} = 0 \tag{9}$$

Further, $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$, so we may rewrite Equation (9) as

$$\overline{a_n}\,\overline{z^n} + \overline{a_{n-1}}\,\overline{z^{n-1}} + \cdots + \overline{a_1}\,\bar{z} + \overline{a_0} = 0 \tag{10}$$

Since $a_0, a_1, \ldots, a_n$ are all real numbers, we know that $\overline{a_0} = a_0$, $\overline{a_1} = a_1$, $\ldots$, $\overline{a_n} = a_n$. Finally, we use the property $\overline{z^n} = \bar{z}^n$ to rewrite Equation (10) as

$$a_n \bar{z}^n + a_{n-1} \bar{z}^{n-1} + \cdots + a_1 \bar{z} + a_0 = 0$$

which establishes that $\bar{z}$ is a zero of $P(x)$.

EXAMPLE 8

Find a polynomial $P(x)$ with real coefficients whose degree is 3 and whose zeros include -2 and $1 - i$.

SOLUTION

Since $1 - i$ is a zero, it follows from the Conjugate Zeros Theorem that $1 + i$ is also a zero of $P(x)$. By the Factor Theorem, $(x + 2)$, $[x - (1 - i)]$, and $[x - (1 + i)]$ are factors of $P(x)$. Therefore,

$$\begin{aligned} P(x) &= (x + 2)[x - (1 - i)][x - (1 + i)] \\ &= (x + 2)(x^2 - 2x + 2) \\ &= x^3 - 2x + 4 \end{aligned}$$

PROGRESS CHECK

Find a polynomial $P(x)$ with real coefficients whose degree is 4 and whose zeros include i and $-3 + i$.

ANSWER

$P(x) = x^4 + 6x^3 + 11x^2 + 6x + 10$

The following is a corollary of the Conjugate Zeros Theorem.

> A polynomial $P(x)$ of degree $n \geq 1$ with real coefficients can be written as a product of linear and quadratic factors with real coefficients so that the quadratic factors have no real zeros.

By the Linear Factor Theorem, we may write

$$P(x) = a(x - r_1)(x - r_2) \cdots (x - r_n)$$

where $r_1, r_2, \ldots, r_n$ are the n zeros of $P(x)$. Of course, some of these zeros may be complex numbers. A complex zero $a + bi$, $b \neq 0$, may be paired with its conjugate $a - bi$ to provide the quadratic factor

$$[x - (a + bi)][x - (a - bi)] = x^2 - 2ax + a^2 + b^2 \tag{11}$$

which has real coefficients. Thus, a *quadratic* factor with real coefficients results from each pair of complex conjugate zeros; a *linear* factor with real coefficients results from each real zero. Further, the discriminant of the quadratic factor in Equation (11) is $-4b^2$ and is therefore always negative, which shows that the quadratic factor has no real zeros.

POLYNOMIALS WITH COMPLEX COEFFICIENTS

Although the definition of a polynomial given in Section 3.1 permits the coefficients to be complex numbers, we have limited our examples to polynomials with real coefficients. To round out our work, we point out that both the Linear Factor Theorem and the Fundamental Theorem of Algebra hold for polynomials with complex coefficients.

On the other hand, the Conjugate Zeros Theorem may not hold if the polynomial $P(x)$ has complex coefficients. To see this, consider the polynomial

$$P(x) = x - (2 + i)$$

which has a complex coefficient and has the zero $2 + i$. Note that the complex conjugate $2 - i$ is *not* a zero of $P(x)$ and that, therefore, the Conjugate Roots Theorem fails to apply to $P(x)$.

EXAMPLE 9

Find a polynomial $P(x)$ of degree 2 that has the zeros -1 and $1 - i$.

SOLUTION

Since -1 is a zero of $P(x)$, $x + 1$ is a factor. Similarly, $[x - (1 - i)]$ is also a factor of $P(x)$. We can then write

$$\begin{aligned} P(x) &= (x + 1)[x - (1 - i)] \\ &= x^2 + ix - 1 + i \end{aligned}$$

which is a polynomial of degree 2 (with complex coefficients) that has the desired zeros.

EXERCISE SET 11.3

In Exercises 1–6 multiply by the conjugate and simplify.

1. $2 - i$
2. $3 + i$
3. $3 + 4i$
4. $2 - 3i$
5. $-4 - 2i$
6. $5 + 2i$

In Exercises 7–15 perform the indicated operations and write the answer in the form $a + bi$.

7. $\dfrac{2 + 5i}{1 - 3i}$
8. $\dfrac{1 + 3i}{2 - 5i}$
9. $\dfrac{3 - 4i}{3 + 4i}$
10. $\dfrac{4 - 3i}{4 + 3i}$
11. $\dfrac{3 - 2i}{2 - i}$
12. $\dfrac{2 - 3i}{3 - i}$
13. $\dfrac{2 + 5i}{3i}$
14. $\dfrac{5 - 2i}{-3i}$
15. $\dfrac{4i}{2 + i}$

In Exercises 16–21 find the reciprocal and write the answer in the form $a + bi$.

16. $3 + 2i$
17. $4 + 3i$
18. $\frac{1}{2} - i$
19. $1 - \frac{1}{3}i$
20. $-7i$
21. $-5i$
22. Prove that the multiplicative inverse of the complex number $a + bi$ (a and b not both 0) is
$$\frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$$
23. If z and w are complex numbers, prove that
$$\overline{z \cdot w} = \bar{z} \cdot \bar{w}$$
24. If z is a complex number, verify that $\overline{z^2} = \bar{z}^2$ and $\overline{z^3} = \bar{z}^3$.

In Exercises 25–30 find a polynomial $P(x)$ of lowest degree that has the indicated zeros.

25. $2, -4, 4$
26. $5, -5, 1, -1$
27. $-1, -2, -3$
28. $-3, \sqrt{2}, -\sqrt{2}$
29. $4, 1 \pm \sqrt{3}$
30. $1, 2, 2 \pm \sqrt{2}$

In Exercises 31–34 find the polynomial $P(x)$ of lowest degree that has the indicated zeros and satisfies the given condition.

31. $\frac{1}{2}, \frac{1}{2}, -2; P(2) = 3$
32. $3, 3, -2, 2; P(4) = 12$
33. $\sqrt{2}, -\sqrt{2}, 4; P(-1) = 5$
34. $\frac{1}{2}, -2, 5; P(0) = 5$

In Exercises 35–42 find the roots of the given equation.

35. $(x - 3)(x + 1)(x - 2) = 0$
36. $(x - 3)(x^2 - 3x - 4) = 0$
37. $(x + 2)(x^2 - 16) = 0$
38. $(x^2 - x)(x^2 - 2x + 5) = 0$

39. $(x^2 + 3x + 2)(2x^2 + x) = 0$

40. $(x^2 + x + 4)(x - 3)^2 = 0$

41. $(x - 5)^3(x + 5)^2 = 0$

42. $(x + 1)^2(x + 3)^4(x - 2) = 0$

In Exercises 43–46 find a polynomial that has the indicated zeros and no others.

43. -2 of multiplicity 3

44. 1 of multiplicity 2, -4 of multiplicity 1

45. $\frac{1}{2}$ of multiplicity 2, -1 of multiplicity 2

46. -1 of multiplicity 2, 0 and 2 each of multiplicity 1

In Exercises 47–52 use the given root(s) to help in finding the remaining roots of the equation.

47. $x^3 - 3x - 2 = 0$; -1

48. $x^3 - 7x^2 + 4x + 24 = 0$; 3

49. $x^3 - 8x^2 + 18x - 15 = 0$; 5

50. $x^3 - 2x^2 - 7x - 4 = 0$; -1

51. $x^4 + x^3 - 12x^2 - 28x - 16 = 0$; -2

52. $x^4 - 2x^2 + 1 = 0$; 1 (double root)

In Exercises 53–58 find a polynomial that has the indicated zeros and no others.

53. $1 + 3i, -2$

54. $1, -1, 2 - i$

55. $1 + i, 2 - i$

56. $-2, 3, 1 + 2i$

57. -2 is a root of multiplicity 2, $3 - 2i$

58. 3 is a triple root, $-i$

In Exercises 59–64 use the given root(s) to help in writing the given equation as a product of linear and quadratic factors with real coefficients.

59. $x^3 - 7x^2 + 16x - 10 = 0$; $3 - i$

60. $x^3 + x^2 - 7x + 65 = 0$; $2 + 3i$

61. $x^4 + 4x^3 + 13x^2 + 18x + 20 = 0$; $-1 - 2i$

62. $x^4 + 3x^3 - 5x^2 - 29x - 30 = 0$; $-2 + i$

63. $x^5 + 3x^4 - 12x^3 - 42x^2 + 32x + 120 = 0$; $-3 - i, -2$

64. $x^5 - 8x^4 + 29x^3 - 54x^2 + 48x - 16 = 0$; $2 + 2i$, 2

65. Write a polynomial $P(x)$ with complex coefficients that has the zero $a + bi$, $b \neq 0$, and that does not have $a - bi$ as a zero.

66. Prove that a polynomial equation of degree 4 with real coefficients has 4 real roots, 2 real roots, or no real roots

67. Prove that a polynomial equation of odd degree with real coefficients has at least one real root.

11.4 REAL AND RATIONAL ZEROS

In this section we will restrict our investigation to polynomials with real coefficients. Our objective is to obtain some information concerning the number of positive real zeros and the number of negative real zeros of such polynomials.

If the terms of a polynomial with real coefficients are written in descending order, then a **variation in sign** occurs whenever two successive terms have opposite signs. In determining the number of variations in sign, we ignore terms with zero coefficients. The polynomial

$$4x^5 - 3x^4 - 2x^2 + 1$$

has two variations in sign. The French mathematician René Descartes (1596–1650), who provided us with the foundations of analytic geometry, also gave us a theorem that relates the nature of the real zeros of polynomials to the variations in sign. The proof of Descartes's theorem is outlined in Exercises 39–44.

Descartes's Rule of Signs	If $P(x)$ is a polynomial with real coefficients, then • the number of positive zeros either is equal to the number of variations in sign of $P(x)$ or is less than the number of variations in sign by an even number, and • the number of negative zeros either is equal to the number of variations in sign of $P(-x)$ or is less than the number of variations in sign by an even number.

If it is determined that a polynomial of degree n has r real zeros, then the remaining $n - r$ zeros must be complex numbers.

To apply Descartes's Rule of Signs to the polynomial

$$P(x) = 3x^5 + 2x^4 - x^3 + 2x - 3$$

we first note that there are 3 variations in sign as indicated. Thus, either there are 3 positive zeros or there is 1 positive zero. Next, we form $P(-x)$,

$$\begin{aligned} P(-x) &= 3(-x)^5 + 2(-x)^4 - (-x)^3 + 2(-x) - 3 \\ &= -3x^5 + 2x^4 + x^3 - 2x - 3 \end{aligned}$$

which can be obtained by changing the signs of the coefficients of the odd-power terms. We see that $P(-x)$ has two variations in sign and conclude that $P(x)$ has either 2 negative zeros or no negative zeros.

EXAMPLE 1

Use Descartes's Rule of Signs to analyze the roots of the equation

$$2x^5 + 7x^4 + 3x^2 - 2 = 0$$

SOLUTION

Since

$$P(x) = 2x^5 + 7x^4 + 3x^2 - 2$$

has 1 variation in sign, there is precisely 1 positive zero. The polynomial $P(-x)$ is formed—

$$P(-x) = -2x^5 + 7x^4 + 3x^2 - 2$$

and is seen to have 2 variations in sign, so $P(-x)$ has either 2 negative zeros or no negative zeros. Since $P(x)$ has 5 zeros, the possibilities are

1 positive zero, 2 negative zeros, 2 complex zeros

1 positive zero, 0 negative zeros, 4 complex zeros

PROGRESS CHECK

Use Descartes's Rule of Signs to analyze the nature of the roots of the equation

$$x^6 + 5x^4 - 4x^2 - 3 = 0$$

ANSWER

1 positive root, 1 negative root, 4 complex roots

When the coefficients of a polynomial are all integers, a systematic search for the *rational* zeros is possible by using the following theorem.

Rational Zero Theorem If the coefficients of the polynomial

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0, \qquad a_n \neq 0$$

are all integers and p/q is a rational zero in lowest terms, then
(a) p is a factor of the constant term a_0, and
(b) q is a factor of the leading coefficient a_n.

PROOF OF RATIONAL ZERO THEOREM (Optional)

Since p/q is a zero of $P(x)$, then $P(p/q) = 0$. Thus,

$$a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_1\left(\frac{p}{q}\right) + a_0 = 0 \tag{1}$$

Multiplying Equation (1) by q^n, we have

$$a_np^n + a_{n-1}p^{n-1}q + \cdots + a_1pq^{n-1} + a_0q^n = 0 \tag{2}$$

or

$$a_np^n + a_{n-1}p^{n-1}q + \cdots + a_1pq^{n-1} = -a_0q^n \tag{3}$$

Taking the common factor p out of the left-hand side of Equation (3) yields

$$p(a_np^{n-1} + a_{n-1}p^{n-2}q + \cdots + a_1q^{n-1}) = -a_0q^n \tag{4}$$

Since $a_1, a_2, \ldots, a_n, p,$ and q are all integers, the quantity in parentheses in the left-hand side of Equation (4) is an integer. Division of the left-hand side by p results in an integer, and we conclude that p must also be a factor of the right-hand side, $-a_0q^n$. But p and q have no common factors since, by hypothesis, p/q is in lowest terms. Hence, p must be a factor of a_0; thus we have proved part a of the Rational Zero Theorem.

We may also rewrite Equation (2) in the form

$$q(a_{n-1}p^{n-1} + a_{n-2}p^{n-2}q + \cdots + a_1pq^{n-2} + a_0q^{n-1}) = -a_np^n \tag{5}$$

An argument similar to the preceding one then establishes part b of the theorem.

SOLVING POLYNOMIAL EQUATIONS

Cardan's Formula

Cardano provided this formula for one root of the cubic equation $x^3 + bx + c = 0$:

$$x = \sqrt[3]{\sqrt{\frac{b^3}{27} + \frac{c^2}{4}} - \frac{c}{2}} - \sqrt[3]{\sqrt{\frac{b^3}{27} + \frac{c^2}{4}} + \frac{c}{2}}$$

Try it for the cubics

$$x^3 - x = 0$$
$$x^3 - 1 = 0$$
$$x^3 - 3x + 2 = 0$$

The quadratic formula provides us with the solutions of a polynomial equation of second degree. How about polynomial equations of third degree? of fourth degree? of fifth degree?

The search for formulas expressing the roots of polynomial equations in terms of the coefficients of the equations intrigued mathematicians for hundreds of years. A method for finding the roots of polynomial equations of degree three was published around 1535 and is known as Cardan's formula despite the possibility that Girolamo Cardano stole the result from his friend Nicolo Tartaglia. Shortly afterward a method that is attributed to Ferrari was published for solving polynomial equations of degree four.

The next 250 years were spent in seeking formulas for the roots of polynomial equations of degree five or higher—without success. Finally, early in the nineteenth century, the Norwegian mathematician N. H. Abel and the French mathematician Evariste Galois proved that *no such formulas exist*. Galois's work on this problem was completed a year before his death in a duel at age 20. His proof, using the new concepts of group theory, was so advanced that his teachers wrote it off as being unintelligible gibberish.

EXAMPLE 2

Find the rational roots of the equation

$$8x^4 - 2x^3 + 7x^2 - 2x - 1 = 0$$

SOLUTION

If p/q is a rational root in lowest terms, then p is a factor of 1 and q is a factor of 8. We can now list the possibilities:

possible numerators: ± 1 (the factors of 1)
possible denominators: ± 1, ± 2, ± 4, ± 8 (the factors of 8)
possible rational roots: ± 1, $\pm\frac{1}{2}$, $\pm\frac{1}{4}$, $\pm\frac{1}{8}$

Synthetic division can be used to test whether these numbers are roots. Trying $x = 1$ and $x = -1$, we find that the remainder is not zero and they are therefore not roots. Trying $\frac{1}{2}$, we have

$$\begin{array}{r|rrrrr} \frac{1}{2} & 8 & -2 & 7 & -2 & -1 \\ & & 4 & 1 & 4 & 1 \\ \hline & 8 & 2 & 8 & 2 & 0 \end{array}$$

which demonstrates that $\frac{1}{2}$ is a root. Similarly,

$$\begin{array}{r|rrrrr} -\frac{1}{4} & 8 & -2 & 7 & -2 & -1 \\ & & -2 & 1 & -2 & 1 \\ \hline & 8 & -4 & 8 & -4 & 0 \end{array}$$

which shows that $-\frac{1}{3}$ is also a root. The student may verify that none of the other possible rational roots will result in a zero remainder when synthetic division is employed.

PROGRESS CHECK

Find the rational roots of the equation

$$9x^4 - 12x^3 + 13x^2 - 12x + 4 = 0$$

ANSWER

$\frac{2}{3}, \frac{2}{3}$

We can combine the Rational Zero Theorem and the depressed equation to give us even greater power in seeking the zeros of a polynomial.

EXAMPLE 3

Find the rational roots of the polynomial equation

$$8x^5 + 12x^4 + 14x^3 + 13x^2 + 6x + 1 = 0$$

SOLUTION

Since the coefficients of the polynomial are all integers, we may use the Rational Zero Theorem to list the possible rational roots:

possible numerators: ± 1 (factors of 1)
possible denominators: ± 1, ± 2, ± 4, ± 8 (factors of 8)
possible rational roots: ± 1, $\pm\frac{1}{2}$, $\pm\frac{1}{4}$, $\pm\frac{1}{8}$

Trying $+1$, -1, and $+\frac{1}{2}$, we find that they are not roots. Testing $-\frac{1}{2}$ by synthetic division results in a remainder of zero.

$$\begin{array}{r|rrrrrr}
-\frac{1}{2} & 8 & 12 & 14 & 13 & 6 & 1 \\
 & & -4 & -4 & -5 & -4 & -1 \\
\hline
 & \underbrace{8 & 8 & 10 & 8 & 2}_{} & \vert\ 0
\end{array}$$

coefficients of depressed equation

Rather than return to the original equation to continue the search, we will use the depressed equation

$$8x^4 + 8x^3 + 10x^2 + 8x + 2 = 0$$

The values $+1$, -1, and $+\frac{1}{2}$ have been eliminated, but the value $-\frac{1}{2}$ must be tried again:

$$\begin{array}{r|rrrrr} -\frac{1}{2} & 8 & 8 & 10 & 8 & 2 \\ & & -4 & -2 & -4 & -2 \\ \hline & \underbrace{8 \quad 4 \quad 8 \quad 4}_{\text{coefficients of depressed equation}} & & & & 0 \end{array}$$

Since the remainder is zero, $-\frac{1}{2}$ is again a root. This illustrates an important point: A rational root may be a multiple root! Applying the same technique to the resulting depressed equation

$$8x^3 + 4x^2 + 8x + 4 = 0$$

we see that $-\frac{1}{2}$ is once again a root:

$$\begin{array}{r|rrrr} -\frac{1}{2} & 8 & 4 & 8 & 4 \\ & & -4 & 0 & -4 \\ \hline & \underbrace{8 \quad 0 \quad 8}_{\text{coefficients of depressed equation}} & & & 0 \end{array}$$

The final depressed equation

$$8x^2 + 8 = 0 \quad \text{or} \quad x^2 + 1 = 0$$

has the roots $\pm i$. Thus, the original equation has the rational roots

$$-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}$$

PROGRESS CHECK

Find all zeros of the polynomial

$$P(x) = 9x^4 - 3x^3 + 16x^2 - 6x - 4$$

ANSWER

$\frac{2}{3}, -\frac{1}{3}, \pm\sqrt{2}i$

EXAMPLE 4

Use Descartes's Rule of Signs, the Rational Zero Theorem, and the depressed equation to write

$$P(x) = 3x^4 + 2x^3 + 2x^2 + 2x - 1$$

as a product of linear and quadratic factors with real coefficients such that the quadratic factors have no real zeros.

SOLUTION

We first use the Rational Zero Theorem to list the possible rational zeros.

possible numerators: ± 1 (factors of 1)
possible denominators: ± 1, ± 3 (factors of 3)
possible rational roots: ± 1, $\pm \frac{1}{3}$

Next, we note that $P(x)$ has real coefficients and that Descartes's Rule of Signs may therefore be employed. Since $P(x)$ has 1 variation in sign, there is precisely 1 positive real zero. If this real zero is a rational number, it must be either $+1$ or $+\frac{1}{3}$. Trying $+1$, we quickly see that $P(1) = 8$ and that $+1$ is not a zero. Using synthetic division,

$$\begin{array}{r|rrrrr} \frac{1}{3} & 3 & 2 & 2 & 2 & -1 \\ & & 1 & 1 & 1 & 1 \\ \hline & \underbrace{3 \quad 3 \quad 3 \quad 3}_{\text{coefficients of depressed equation}} & & & & \multicolumn{1}{|r}{0} \end{array}$$

we see that $\frac{1}{3}$ is a zero and the depressed equation is

$$Q_1(x) = 3x^3 + 3x^2 + 3x + 3 = 0$$

which has the same roots as

$$Q_2(x) = x^3 + x^2 + x + 1 = 0$$

Since any root of $Q_2(x)$ is also a zero of $P(x)$ and since we have removed the only positive zero, we know that $Q_2(x)$ cannot have any positive zeros. (Verify that $Q_2(x)$ has no variations in sign!) However, forming

$$Q_2(-x) = -x^3 + x^2 - x + 1$$

we see that $Q_2(x)$ has at least 1 negative zero. By the Rational Zero Theorem, the only possible rational zeros of $Q_2(x)$ are ± 1. Using synthetic division,

$$\begin{array}{r|rrrr} -1 & 1 & 1 & 1 & 1 \\ & & -1 & 0 & -1 \\ \hline & \underbrace{1 \quad 0 \quad 1}_{\text{coefficients of depressed equation}} & & & \multicolumn{1}{|r}{0} \end{array}$$

we verify that -1 is indeed a zero. Finally, we note that the depressed equation $x^2 + 1 = 0$ has no real roots, since the discriminant is negative. Thus,

$$P(x) = 3x^4 + 2x^3 + 2x^2 + 2x - 1 = 3\left(x - \frac{1}{3}\right)(x + 1)(x^2 + 1)$$

TRANSCENDENTAL NUMBERS

Theorem: Every rational number p/q is algebraic.
Proof: The number p/q is a root of the equation

$$qx - p = 0$$

since

$$q\left(\frac{p}{q}\right) - p = p - p = 0$$

Further, by definition of a rational number, q and p are integers and $q \neq 0$. So p/q is a root of a polynomial equation with integer coefficients and is therefore algebraic.

A real number that is a root of some polynomial equation with integer coefficients is said to be **algebraic.** We see that $\frac{2}{3}$ is algebraic since it is the root of the equation $3x - 2 = 0$; $\sqrt{2}$ is also algebraic, since it satisfies the equation $x^2 - 2 = 0$.

Note that every real number a satisfies the equation $x - a = 0$; that is, it satisfies a polynomial equation with *real* coefficients. But to be algebraic the number a must satisfy a polynomial equation with *integer* coefficients. To show that a real number a is *not* algebraic we must demonstrate that there is *no* polynomial equation with integer coefficients that has a as one of its roots. Although this appears to be an impossible task, it was performed in 1844 when Joseph Liouville exhibited specific examples of such numbers, called **transcendental** numbers. Subsequently, Georg Cantor (1845–1918), in his brilliant work on infinite sets, provided a more general proof of the existence of transcendental numbers.

You are already familiar with a transcendental number: the number π is not a root of any polynomial equation with integer coefficients.

In Chapter 1 we discussed number systems and said that numbers such as $\sqrt{2}$ and $\sqrt{3}$ were irrational. The Rational Zero Theorem provides a direct means of verifying that this is indeed so.

EXAMPLE 5

Prove that $\sqrt{3}$ is not a rational number.

SOLUTION

If we let $x = \sqrt{3}$, then $x^2 = 3$ or $x^2 - 3 = 0$. By the Rational Zero Theorem, the only possible rational roots are ± 1, ± 3. Synthetic division can be used to show that none of these are roots. However, $\sqrt{3}$ is a root of $x^2 - 3 = 0$. Hence, $\sqrt{3}$ is not a rational number.

EXERCISE SET 11.4

In Exercises 1–12 use Descartes's Rule of Signs to analyze the nature of the roots of the given equation. List all possibilities.

1. $3x^4 - 2x^3 + 6x^2 + 5x - 2 = 0$
2. $2x^6 + 5x^5 + x^3 - 6 = 0$
3. $x^6 + 2x^4 + 4x^2 + 1 = 0$
4. $3x^3 - 2x + 2 = 0$
5. $x^5 - 4x^3 + 7x - 4 = 0$
6. $2x^3 - 5x^2 + 8x - 2 = 0$
7. $5x^3 + 2x^2 + 7x - 1 = 0$
8. $x^5 + 6x^4 - x^3 - 2x - 3 = 0$

9. $x^4 - 2x^3 + 5x^2 + 2 = 0$

10. $3x^4 - 2x^3 - 1 = 0$

11. $x^8 + 7x^3 + 3x - 5 = 0$

12. $x^7 + 3x^5 - x^3 - x + 2 = 0$

In Exercises 13–22 use the Rational Zero Theorem to find all rational roots of the given equation.

13. $x^3 - 2x^2 - 5x + 6 = 0$

14. $3x^3 - x^2 - 3x + 1 = 0$

15. $6x^4 - 7x^3 - 13x^2 + 4x + 4 = 0$

16. $36x^4 - 15x^3 - 26x^2 + 3x + 2 = 0$

17. $5x^6 - x^5 - 5x^4 + 6x^3 - x^2 - 5x + 1 = 0$

18. $16x^4 - 16x^3 - 29x^2 + 32x - 6 = 0$

19. $4x^4 - x^3 + 5x^2 - 2x - 6 = 0$

20. $6x^4 + 2x^3 + 7x^2 + x + 2 = 0$

21. $2x^5 - 13x^4 + 26x^3 - 22x^2 + 24x - 9 = 0$

22. $8x^5 - 4x^4 + 6x^3 - 3x^2 - 2x + 1 = 0$

In Exercises 23–30 use the Rational Zero Theorem and the depressed equation to find all roots of the given equation.

23. $4x^4 + x^3 + x^2 + x - 3 = 0$

24. $x^4 + x^3 + x^2 + 3x - 6 = 0$

25. $5x^5 - 3x^4 - 10x^3 + 6x^2 - 40x + 24 = 0$

26. $12x^4 - 52x^3 + 75x^2 - 16x - 5 = 0$

27. $6x^4 - x^3 - 5x^2 + 2x = 0$

28. $2x^4 - \frac{3}{2}x^3 + \frac{11}{2}x^2 + \frac{23}{2}x + \frac{5}{2} = 0$

29. $2x^4 - x^3 - 28x^2 + 30x - 8 = 0$

30. $12x^4 + 4x^3 - 17x^2 + 6x = 0$

In Exercises 31–34 find the integer value(s) of k for which the given equation has rational roots, and find the roots. (*Hint:* Use synthetic division.)

31. $x^3 + kx^2 + kx + 2 = 0$

32. $x^4 - 4x^3 - kx^2 + 6kx + 9 = 0$

33. $x^4 - 3x^3 + kx^2 - 4x - 1 = 0$

34. $x^3 - 3kx^2 + k^2x + 4 = 0$

35. If $P(x)$ is a polynomial with real coefficients that has one variation in sign, prove that $P(x)$ has exactly one positive zero.

36. If $P(x)$ is a polynomial with integer coefficients and the leading coefficient is $+1$ or -1, prove that the rational zeros of $P(x)$ are all integers and are factors of the constant term.

37. Prove that $\sqrt{5}$ is not a rational number.

38. If p is a prime, prove that $\sqrt{p}$ is not a rational number.

39. Prove that if $P(x)$ is a polynomial with real coefficients and r is a positive zero of $P(x)$, then the depressed equation

$$Q(x) = \frac{P(x)}{(x - r)}$$

has at least one fewer variations in sign than $P(x)$. (*Hint:* Assume the leading coefficient of $P(x)$ to be positive, and use synthetic division to obtain $Q(x)$. Note that the coefficients of $Q(x)$ remain positive at least until there is a variation in sign in $P(x)$.)

40. Prove that if $P(x)$ is a polynomial with real coefficients, the number of positive zeros is not greater than the number of variations in sign in $P(x)$. (*Hint:* Let r_1, $r_2, \cdots, r_k$ be the positive zeros of $P(x)$, and let

$$P(x) = (x - r_1)(x - r_2) \cdots (x - r_k)\, Q(x)$$

Use the result of Exercise 39 to show that $Q(x)$ has at least k fewer variations in sign than does $P(x)$.)

41. Prove that if $r_1, r_2, \ldots, r_k$ are positive numbers, then

$$P(x) = (x - r_1)(x - r_2) \cdots (x - r_k)$$

has alternating signs. (*Hint:* Use the result of Exercise 40.)

42. Prove that the number of variations in sign of a polynomial with real coefficients is even if the first and last coefficients have the same sign and is odd if they are of opposite sign.

43. Prove that if the number of positive zeros of the polynomial $P(x)$ with real coefficients is less than the number of variations in sign, it is less by an even number. (*Hint:* Write $P(x)$ as a product of linear factors corresponding to the positive and negative zeros, and of quadratic factors corresponding to complex zeros. Apply the results of Exercises 41 and 42.)

44. Prove that the positive zeros of $P(-x)$ correspond to the negative zeros of $P(x)$; that is, prove that if $a > 0$ is a zero of $P(-x)$, then $-a$ is a zero of $P(x)$.

11.5 RATIONAL FUNCTIONS AND THEIR GRAPHS

A function f of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials and $Q(x) \neq 0$, is called a **rational function.** We will study the behavior of rational functions with the objective of sketching their graphs.

We first note that the polynomials $P(x)$ and $Q(x)$ are defined for all real values of x. Since we must avoid division by zero, the domain of the function f will consist of all real numbers except those for which $Q(x) = 0$.

EXAMPLE 1

Determine the domain of each function.

(a) $f(x) = \dfrac{x+1}{x-1}$ (b) $g(x) = \dfrac{x^2+9}{x^2-4}$ (c) $h(x) = \dfrac{x^2}{x^2+1}$

SOLUTION

(a) We must exclude all real values for which the denominator $x - 1 = 0$. Thus, the domain of f consists of all real numbers except $x = 1$.

(b) Since $x^2 - 4 = 0$ when $x = \pm 2$, the domain of g consists of all real numbers except $x = \pm 2$.

(c) Since $x^2 + 1 = 0$ has no real solutions, the domain of h is the set of all real numbers.

PROGRESS CHECK

Determine the domain of each function.

(a) $S(x) = \dfrac{x}{2x^2 - 3x - 2}$ (b) $T(x) = \dfrac{-1}{x^4 + x^2 + 2}$

ANSWERS

(a) $x \neq -\frac{1}{2}, 2$ (b) all real numbers

ASYMPTOTES

Let's first consider rational functions for which the numerator is a constant, for example,

$$f(x) = \frac{1}{x} \quad \text{and} \quad g(x) = \frac{1}{x^2}$$

The domain of both f and g is the set of all nonzero real numbers. Furthermore, the graph of f is symmetric with respect to the origin, since the equation $y = 1/x$ remains unchanged when x and y are replaced by $-x$ and $-y$, respectively. Similarly, the graph of g is symmetric with respect to the y-axis, since the equation $y = 1/x^2$ is unchanged when x is replaced by $-x$. We therefore need plot only those points corresponding to positive values of x (see Table 1) and can utilize symmetry to obtain the graphs of Figure 2.

TABLE 1

x	$\frac{1}{x}$	$\frac{1}{x^2}$
0.001	1000	1,000,000
0.01	100	10,000
0.1	10	100
1	1	1
2	0.5	0.25
4	0.25	0.06

When a graph gets closer and closer to a line, we say that the line is an **asymptote** of the graph. Note the behavior of the graphs of f and g (Figure 2) as x gets closer and closer to 0. We say that the line $x = 0$ is a **vertical asymptote** for each of these graphs. Similarly, we note that the line $y = 0$ is a **horizontal asymptote** in both cases. We will later show that the x-axis is a horizontal asymptote for any rational function for which the numerator is a constant and the denominator is a polynomial of degree one or higher.

The determination of asymptotes is extremely helpful in the graphing of rational functions. The following theorem provides the means for finding all vertical asymptotes.

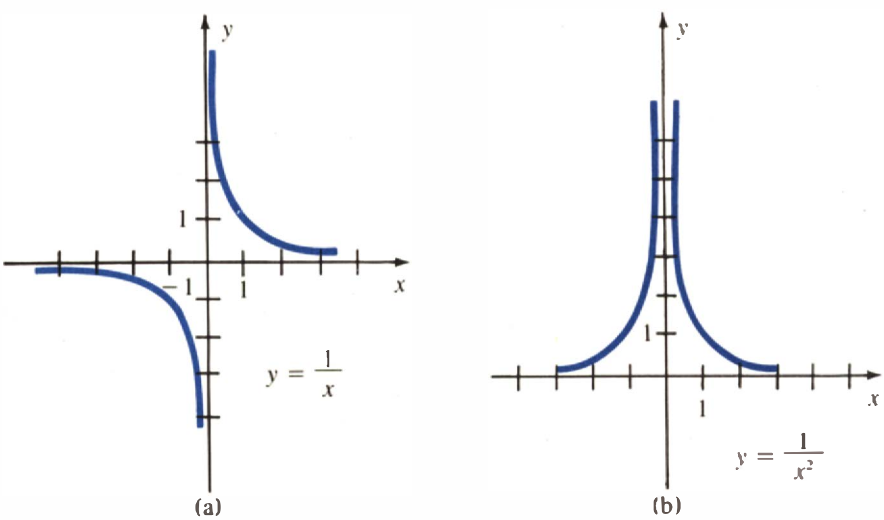


FIGURE 2

Vertical Asymptote Theorem

The graph of the rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

has a vertical asymptote at $x = r$ if r is a real root of $Q(x)$ but not of $P(x)$.

EXAMPLE 2

Determine the vertical asymptotes of the graph of the function

$$T(x) = \frac{2}{x^3 - 2x^2 - 3x}$$

SOLUTION

Factoring the denominator, we have

$$T(x) = \frac{2}{x(x + 1)(x - 3)}$$

and we conclude that $x = 0$, $x = -1$, and $x = 3$ are vertical asymptotes of the graph of T.

Let's examine the behavior of the function $T(x)$ of Example 2 when x is in the neighborhood of $+3$. When x is slightly more than $+3$, $x - 3$ is positive, as are x and $x + 1$; therefore, $T(x)$ is positive and growing larger and larger as x gets closer and closer to $+3$. When x is slightly less than $+3$, $x - 3$ is negative, but both x and $x + 1$ are positive; therefore, $T(x)$ is negative and growing smaller and smaller as x gets closer and closer to $+3$. This reasoning leads to the portion of the graph of $T(x)$ shown in Figure 3c. Similarly, when x is slightly more than 0, $T(x)$ is negative, and when x is slightly less than 0, $T(x)$ is positive (Figure 3b). The behavior of $T(x)$ when x is close to -1 is shown in Figure 3a. Since the numerator of $T(x)$ is constant, $T(x) \neq 0$ for any value of x and the graph of $T(x)$ does not cross the x-axis. Moreover, since $T(x)$ is of the form $k/Q(x)$, where k is a constant, the x-axis is a horizontal asymptote. Combining these observations with the portions of the graph of $T(x)$ sketched in Figure 3 leads to the graph of $T(x)$ sketched in Figure 4.

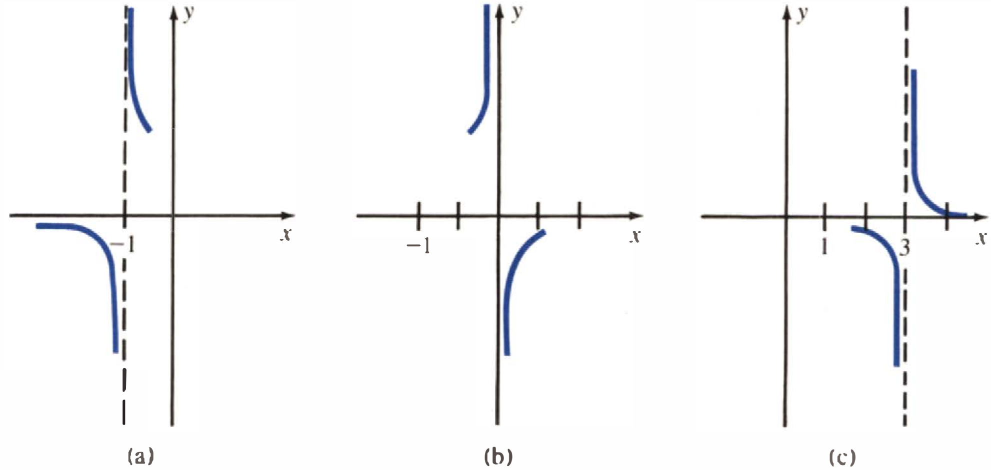


FIGURE 3

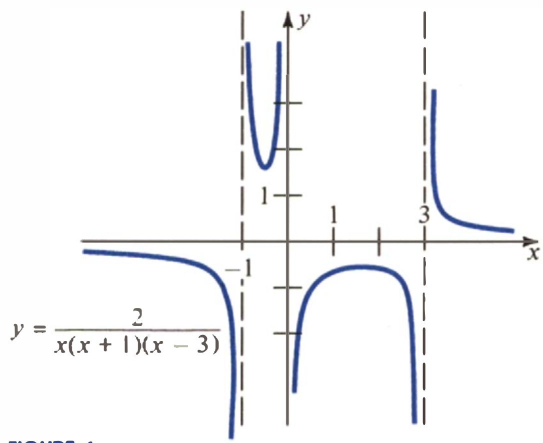


FIGURE 4

EXAMPLE 3

Sketch the graphs of the rational functions.

(a) $F(x) = \dfrac{1}{x-1}$ (b) $G(x) = \dfrac{1}{(x+2)^2}$

SOLUTION

The graphs are shown in Figure 5. Note that the graphs are identical to those of Figure 2 with all points moved right one unit in the case of F and moved left two units in the case of G. In both cases we say that the y-axis has been **translated**.

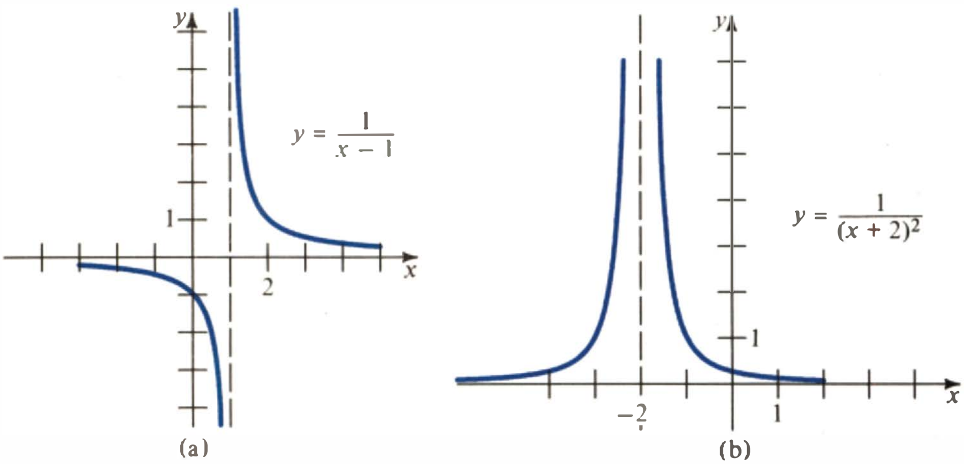


FIGURE 5

The x-axis is a horizontal asymptote for each of the rational functions sketched in Figures 2, 4, and 5. In general, we can determine the existence of a horizontal asymptote by studying the behavior of a rational function as x approaches $+\infty$ and $-\infty$, that is, as $|x|$ becomes very large. Recall that the expression

$$\frac{k}{x^n}$$

where n is a positive integer and k is a constant, will become very small as $|x|$ becomes very large; that is, k/x^n approaches 0 as $|x|$ approaches $+\infty$. The procedure for determining horizontal asymptotes employs the technique, used earlier, of factoring out the highest power of x to determine the behavior of the function as $|x|$ becomes large.

EXAMPLE 4

Determine the horizontal asymptote of the function

$$f(x) = \frac{2x^2 - 5}{3x^2 + 2x - 4}$$

SOLUTION

We illustrate the steps of the procedure.

Horizontal Asymptotes	
Step 1. Factor out the highest power of x found in the numerator; factor out the highest power of x found in the denominator.	*Step 1*. $f(x) = \dfrac{x^2\left(2 - \dfrac{5}{x^2}\right)}{x^2\left(3 + \dfrac{2}{x} - \dfrac{4}{x^2}\right)}$
Step 2. Since we are interested in large values of $\lvert x \rvert$, we may cancel common factors in the numerator and denominator.	*Step 2*. $f(x) = \dfrac{2 - \dfrac{5}{x^2}}{3 + \dfrac{2}{x} - \dfrac{4}{x^2}}, \qquad x \neq 0$
Step 3. Let $\lvert x \rvert$ increase. Then all terms of the form k/x^n approach 0 and may be discarded.	*Step 3*. The terms $-5/x^2$, $2/x$, and $-4/x^2$ approach 0 as $\lvert x \rvert$ approaches $+\infty$.
Step 4. If what remains is a real number c, then $y = c$ is the horizontal asymptote. Otherwise there is no horizontal asymptote.	*Step 4*. Discarding these terms, we have $y = \frac{2}{3}$ as the horizontal asymptote.

EXAMPLE 5

Determine the horizontal asymptote of the function

$$f(x) = \frac{2x^3 + 3x - 2}{x^2 + 5}$$

if there is one.

SOLUTION

Factoring, we have

$$f(x) = \frac{x^3\left(2 + \dfrac{3}{x^2} - \dfrac{2}{x^3}\right)}{x^2\left(1 + \dfrac{5}{x^2}\right)}$$

$$= \frac{x\left(2 + \dfrac{3}{x^2} - \dfrac{2}{x^3}\right)}{1 + \dfrac{5}{x^2}}, \qquad x \neq 0$$

As $|x|$ increases, the terms $3/x^2$, $-2/x^3$, and $5/x^2$ approach zero and can be discarded. What remains is $2x$, which becomes larger and larger as $|x|$ increases. Thus, there is no horizontal asymptote, and $|y|$ becomes larger and larger as $|x|$ approaches infinity.

The following theorem can be proved by utilizing the procedure of Example 4.

Horizontal Asymptote Theorem

The graph of the rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

has a horizontal asymptote if the degree of $P(x)$ is less than or equal to the degree of $Q(x)$.

Note that the graph of a rational function may have many vertical asymptotes but at most one horizontal asymptote.

PROGRESS CHECK

Determine the horizontal asymptote of the graph of each function.

(a) $f(x) = \dfrac{x-1}{2x^2+1}$ (b) $g(x) = \dfrac{4x^2-3x+1}{-3x^2+1}$

(c) $h(x) = \dfrac{3x^3-x+1}{2x^2-1}$

ANSWERS

(a) $y = 0$ (b) $y = -\dfrac{4}{3}$ (c) no horizontal asymptote

SKETCHING GRAPHS

We now summarize the information that can be gathered in preparation for sketching the graph of a rational function:

- symmetry with respect to the axes and the origin
- x-intercepts
- vertical asymptotes
- horizontal asymptotes
- brief table of values including points near the vertical asymptotes

EXAMPLE 6

Sketch the graph of

$$f(x) = \frac{x^2}{x^2-1}$$

SOLUTION

Symmetry. Replacing x with $-x$ results in the same equation, establishing symmetry with respect to the y-axis.

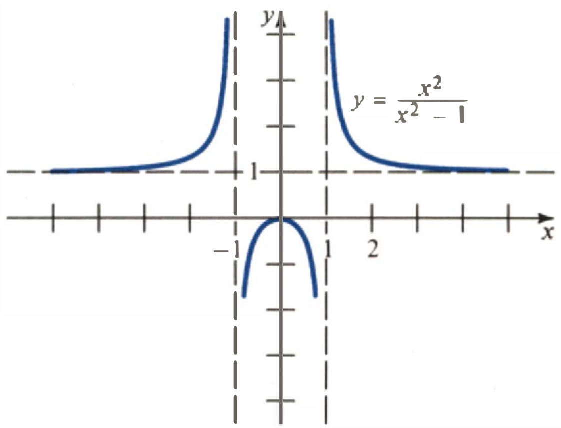
Intercepts. Setting the numerator equal to zero, we see that the graph of f crosses the x-axis at the point (0, 0).

Vertical asymptotes. Setting the denominator equal to zero, we find that $x = 1$ and $x = -1$ are vertical asymptotes of the graph of f.

Horizontal asymptotes. We note that

$$f(x) = \frac{x^2}{x^2\left(1 - \frac{1}{x^2}\right)} = \frac{1}{1 - \frac{1}{x^2}}, \qquad x \neq 0$$

As $|x|$ gets larger and larger, $1/x^2$ approaches 0 and the values of $f(x)$ approach 1. Thus, $y = 1$ is the horizontal asymptote. Plotting a few points, we sketch the graph of Figure 6.



x	y
1/2	−0.33
3/4	−1.29
5/4	2.78
3/2	1.80
2	1.33

FIGURE 6

PROGRESS CHECK

Sketch the graph of the function

$$f(x) = \frac{x^2 - x - 6}{x^2 - 2x}$$

ANSWER

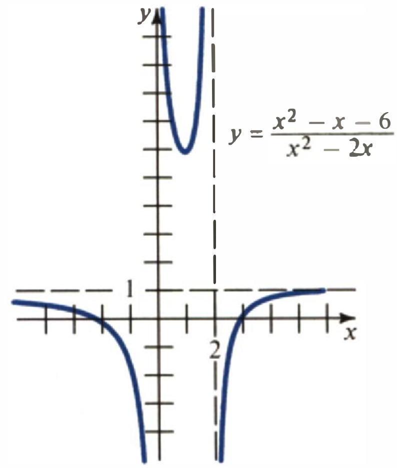


FIGURE 7

We conclude this section with an example of a rational function that is not in reduced form, that is, one in which the numerator and denominator have a common factor.

EXAMPLE 7

Sketch the graph of the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

SOLUTION

We observe that

$$f(x) = \frac{x^2 - 1}{x - 1} = \frac{(x + 1)(x - 1)}{x - 1} = x + 1, \qquad x \neq 1$$

Thus, the graph of the function f coincides with the straight line $y = x + 1$, with the exception that f is undefined when $x = 1$ (Figure 8).

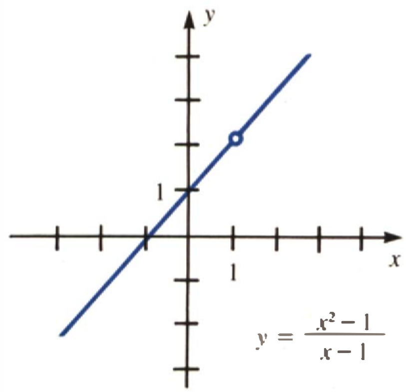


FIGURE 8

PROGRESS CHECK

Sketch the graph of the function

$$f(x) = \frac{8 - 2x^2}{x + 2}$$

ANSWER

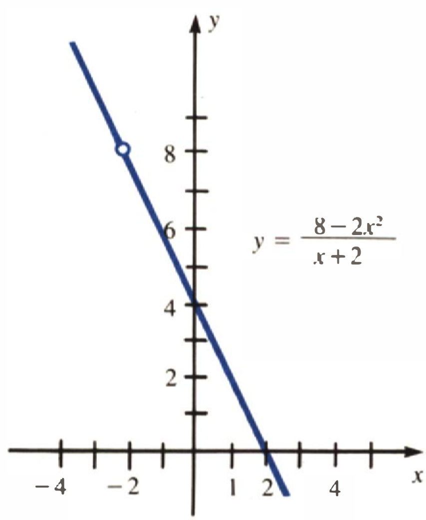


FIGURE 9

EXERCISE SET 11.5

In Exercises 1–6 determine the domain of the given function.

1. $f(x) = \frac{x^2}{x - 1}$

2. $f(x) = \frac{x - 1}{x^2 + x - 2}$

3. $g(x) = \frac{x^2 + 1}{x^2 - 2x}$

4. $g(x) = \frac{x^2 + 2}{x^2 - 2}$

5. $F(x) = \dfrac{x^2 - 3}{x^2 + 3}$

6. $T(x) = \dfrac{3x + 2}{2x^3 - x^2 - x}$

In Exercises 7–21 determine the vertical and horizontal asymptotes of the graph of the given function. Sketch the graph.

7. $f(x) = \dfrac{1}{x - 4}$

8. $f(x) = \dfrac{-2}{x - 3}$

9. $f(x) = \dfrac{3}{x + 2}$

10. $f(x) = \dfrac{-1}{(x - 1)^2}$

11. $f(x) = \dfrac{1}{(x + 1)^2}$

12. $f(x) = \dfrac{-1}{x^2 + 1}$

13. $f(x) = \dfrac{x + 2}{x - 2}$

14. $f(x) = \dfrac{x}{x + 2}$

15. $f(x) = \dfrac{2x^2 + 1}{x^2 - 4}$

16. $f(x) = \dfrac{x^2 + 1}{x^2 + 2x - 3}$

17. $f(x) = \dfrac{x^2 + 2}{2x^2 - x - 6}$

18. $f(x) = \dfrac{x^2 - 1}{x + 2}$

19. $f(x) = \dfrac{x^2}{4x - 4}$

20. $f(x) = \dfrac{x - 1}{2x^3 - 2x}$

21. $f(x) = \dfrac{x^3 + 4x^2 + 3x}{x^2 - 25}$

In Exercises 22–27 determine the domain and sketch the graph of the reducible function.

22. $f(x) = \dfrac{x^2 - 25}{2x - 10}$

23. $f(x) = \dfrac{2x^2 - 8}{x + 2}$

24. $f(x) = \dfrac{2x^2 + 2x - 12}{3x - 6}$

25. $f(x) = \dfrac{x^2 + 2x - 8}{2x^2 - 8x + 8}$

26. $f(x) = \dfrac{x + 2}{x^2 - x - 6}$

27. $f(x) = \dfrac{2x}{x^2 + x}$

TERMS AND SYMBOLS

polynomial function of degree n (p. 459)
polynomial equation of degree n (p. 459)
zeros of a polynomial (p. 460)
synthetic division (p. 462)
complex conjugate (p. 469)
$\bar{z}$ (p. 471)
zero of multiplicity k (p. 474)
depressed equation (p. 475)
variation in sign (p. 480)
rational function (p. 489)
vertical asymptote (p. 490)
horizontal asymptote (p. 490)

KEY IDEAS FOR REVIEW

- ☐ Polynomial division results in a quotient and a remainder, both of which are polynomials. Either the remainder is zero or its degree is less than the degree of the divisor.
- ☐ Synthetic division is a quick way to divide a polynomial by a first-degree polynomial $x - r$, where r is a real constant.
- ☐ The zeros of the polynomial $P(x)$ are the roots of the equation $P(x) = 0$.
- ☐ The following are the primary theorems concerning polynomials and their roots:

Remainder Theorem

If a polynomial $P(x)$ is divided by $x - r$, the remainder is $P(r)$.

Factor Theorem

A polynomial $P(x)$ has a zero r if and only if $x - r$ is a factor of $P(x)$.

Linear Factor Theorem

A polynomial $P(x)$ of degree $n \geq 1$ can be written as the product of n linear factors,

$$P(x) = a(x - r_1)(x - r_2) \cdots (x - r_n)$$

where $r_1, r_2, \ldots, r_n$ are the complex zeros of $P(x)$ and a is the leading coefficient of $P(x)$.

Fundamental Theorem Of Algebra

If $P(x)$ is a polynomial of degree $n \geq 1$, then $P(x)$ has precisely n zeros among the complex numbers when a zero of multiplicity k is counted k times.

Conjugate Zeros Theorem

If $a + bi$, $b \neq 0$, is a zero of the polynomial $P(x)$ with real coefficients, then $a - bi$ is also a zero of $P(x)$.

Rational Zero Theorem

If p/q is a rational zero (in lowest terms) of the polynomial $P(x)$ with integer coefficients, then p is a factor of the constant term a_0 of $P(x)$, and q is a factor of the leading coefficient a_n of $P(x)$.

☐ If r is a real zero of the polynomial $P(x)$, the roots of the depressed equation are the other zeros of $P(x)$. The depressed equation can be found by using synthetic division.

☐ Descartes's Rule of Signs tells us the maximum number of positive zeros and the maximum number of negative zeros of a polynomial $P(x)$ with real coefficients.

☐ If $P(x)$ has integer coefficients, the Rational Zero Theorem enables us to list all possible rational zeros of $P(x)$. Synthetic division can then be used to test these potential rational zeros, since r is a zero if and only if the remainder is zero, that is, if and only if $P(r) = 0$.

☐ Always determine the vertical and horizontal asymptotes of a rational function before attempting to sketch its graph.

REVIEW EXERCISES

Solutions to exercises whose numbers are in color are in the Solutions section in the back of the book.

11.1 In Exercises 1 and 2 use synthetic division to find the quotient $Q(x)$ and the constant remainder R when the first polynomial is divided by the second polynomial.

1. $2x^3 + 6x - 4, \quad x - 1$
2. $x^4 - 3x^3 + 2x - 5, \quad x + 2$

In Exercises 3 and 4 use synthetic division to find $P(2)$ and $P(-1)$.

3. $7x^3 - 3x^2 + 2$ 4. $x^5 - 4x^3 + 2x$

11.2 In Exercises 5 and 6 use the Factor Theorem to show that the second polynomial is a factor of the first polynomial.

5. $2x^4 + 4x^3 + 3x^2 + 5x - 2, \quad x + 2$
6. $2x^3 - 5x^2 + 6x - 2, \quad x - \dfrac{1}{2}$

11.3 In Exercises 7–9 write the given quotient in the form $a + bi$.

7. $\dfrac{3 - 2i}{4 + 3i}$ 8. $\dfrac{2 + i}{-5i}$ 9. $\dfrac{-5}{1 + i}$

In Exercises 10–12 write the reciprocal of the given complex number in the form $a + bi$.

10. $1 + 3i$ 11. $-4i$ 12. $2 - 5i$

In Exercises 13–15 find a polynomial of lowest degree that has the indicated zeros.

13. $-3, -2, -1$ 14. $3, \pm\sqrt{-3}$
15. $-2, \pm\sqrt{3}, 1$

In Exercises 16–18 find a polynomial that has the indicated zeros and no others.

16. $\frac{1}{2}$ of multiplicity 2, -1 of multiplicity 2
17. $i, -i$, each of multiplicity 2
18. -1 of multiplicity 3, 3 of multiplicity 1

In Exercises 19–21 use the given root to assist in finding the remaining roots of the equation.

19. $2x^3 - x^2 - 13x - 6 = 0; \quad -2$
20. $x^3 - 2x^2 - 9x + 4 = 0; \quad 4$
21. $2x^4 - 15x^3 + 34x^2 - 19x - 20 = 0; \quad -\frac{1}{2}$

11.4 In Exercises 22–25 use Descartes's Rule of Signs to determine the maximum number of positive and negative real roots of the given equation.

22. $x^4 - 2x - 1 = 0$
23. $x^5 - x^4 + 3x^3 - 4x^2 + x - 5 = 0$
24. $x^3 - 5 = 0$ 25. $3x^4 - 2x^2 + 1 = 0$

In Exercises 26–28 find all the rational roots of the given equation.

26. $6x^3 - 5x^2 - 33x - 18 = 0$
27. $6x^4 - 7x^3 - 19x^2 + 32x - 12 = 0$
28. $x^4 + 3x^3 + 2x^2 + x - 1 = 0$

In Exercises 29 and 30 find all roots of the given equation.

29. $6x^3 + 15x^2 - x - 10 = 0$
30. $2x^4 - 3x^3 - 10x^2 + 19x - 6 = 0$

11.5 In Exercises 31 and 32 sketch the graph of the given function.

31. $f(x) = \dfrac{x}{x + 1}$ 32. $f(x) = \dfrac{x^2}{x + 1}$

PROGRESS TEST 11A

1. Find the quotient and remainder when $2x^4 - x^2 + 1$ is divided by $x^2 + 2$.
2. Use synthetic division to find the quotient and remainder when $3x^4 - x^3 - 2$ is divided by $x + 2$.
3. If $P(x) = x^3 - 2x^2 + 7x + 5$, use synthetic division to find $P(-2)$.
4. Determine the remainder when $4x^5 - 2x^4 - 5$ is divided by $x + 2$.
5. Use the Factor Theorem to show that $x - 3$ is a factor of $2x^4 - 9x^3 + 9x^2 + x - 3$.

In Problems 6 and 7 find a polynomial of lowest degree that has the indicated zeros.

6. $-2, 1, 3$
7. $-1, 1, 3 \pm \sqrt{2}$

In Problems 8 and 9 find the roots of the given equation.

8. $(x^2 + 1)(x - 2) = 0$
9. $(x + 1)^2(x^2 - 3x - 2) = 0$

In Problems 10–12 find a polynomial that has the indicated zeros and no others.

10. -3 of multiplicity 2, 1 of multiplicity 3
11. $-\frac{1}{4}$ of multiplicity 2, i, $-i$, 1
12. i, $1 + i$

In Problems 13 and 14 use the given root to help in finding the remaining roots of the equation.

13. $4x^3 - 3x + 1 = 0$; -1
14. $x^4 - x^2 - 2x + 2 = 0$; 1
15. If $2 + i$ is a root of $x^3 - 6x^2 + 13x - 10 = 0$, write the equation as a product of linear and quadratic factors with real coefficients.

In Problems 16 and 17 determine the maximum number of roots, of the type indicated, of the given equation.

16. $2x^5 - 3x^4 + 1 = 0$; positive real roots
17. $3x^4 + 2x^3 - 2x^2 - 1 = 0$; negative real roots

In Problems 18 and 19 find all rational roots of the given equation.

18. $6x^3 - 17x^2 + 14x + 3 = 0$
19. $2x^5 - x^4 - 4x^3 + 2x^2 + 2x - 1 = 0$
20. Find all roots of the equation
$$3x^4 + 7x^3 - 3x^2 + 7x - 6 = 0$$
21. Sketch the graph of the function $f(x) = \dfrac{x^2 + 2}{x^2 - 1}$

PROGRESS TEST 11B

1. Find the quotient and remainder when $3x^5 - x^4 - 5x^3 - x + 1$ is divided by $x^2 - x - 1$.
2. Use synthetic division to find the quotient and remainder when $-2x^3 + 3x^2 - 1$ is divided by $x - 1$.
3. If $P(x) = 2x^4 - 2x^3 + x - 4$, use synthetic division to find $P(-1)$.
4. Determine the remainder when $3x^4 - 5x^3 + 3x^2 + 4$ is divided by $x - 2$.
5. Use the Factor Theorem to show that $x + 2$ is a factor of $x^3 - 4x^2 - 9x + 6$.

In Problems 6 and 7 find a polynomial of lowest degree that has the indicated zeros.

6. $-\frac{1}{2}, 1, 1, -1$
7. $2, 1 \pm \sqrt{3}$

In Problems 8 and 9 find the roots of the given equation.

8. $(x^2 - 3x + 2)(x - 2)^2 = 0$
9. $(x^2 + 3x - 1)(x - 2)(x + 3)^2 = 0$

In Problems 10–12 find a polynomial that has the indicated zeros and no others.

10. $\frac{1}{2}$ of multiplicity 3, -2
11. -3 of multiplicity 2, $1 + i$, $1 - i$
12. $3 \pm \sqrt{-1}$, -1 of multiplicity 2

In Problems 13 and 14 use the given root(s) to help in finding the remaining roots of the equation.

13. $x^3 - x^2 - 8x - 4 = 0$; -2
14. $x^4 - 3x^3 - 22x^2 + 68x - 40 = 0$; 2, 5
15. If $1 - i$ is a root of $2x^4 - x^3 - 4x^2 + 10x - 4 = 0$, write the equation as a product of linear and quadratic factors with real coefficients.

In Problems 16 and 17 determine the maximum number of roots, of the type indicated, of the given equation.

16. $3x^4 + 3x - 1 = 0$; positive real roots
17. $2x^4 + x^3 - 3x^2 + 2x + 1 = 0$; negative real roots

In Problems 18 and 19 find all rational roots of the given equation.

18. $3x^3 + 7x^2 - 4 = 0$

19. $4x^4 - 4x^3 + x^2 - 4x - 3 = 0$

20. Find all roots of the equation

$$2x^4 - x^3 - 4x^2 + 2x = 0$$

21. Sketch the graph of the function $f(x) = \dfrac{2x}{x^2 - 1}$.

12

TOPICS IN ALGEBRA

The topics in this chapter are related in that they all involve the set of natural numbers. As you might expect, despite our return to a simpler number system, the approach and results will be more advanced than in earlier chapters. For example, in discussing sequences, we will be dealing with functions whose domain is the set of natural numbers. Yet, sequences lead to considerations of series, and the underlying concepts of infinite series can be used as an introduction to calculus.

Another of the topics, mathematical induction, provides a means of proving certain theorems involving the natural numbers that appear to resist other means of proof. As an example, we will use mathematical induction to prove that the sum of the first n consecutive positive integers is $n(n + 1)/2$.

Yet another topic is the binomial theorem, which gives us a way to expand the expression $(a + b)^n$ where n is a natural number. One of the earliest results obtained in a calculus course requires the binomial theorem in its derivation.

Probability theory, a very useful topic in algebra, enables us to state the likelihood of occurrence of a given event and has obvious applications to games of chance. The theory of permutations and combinations, which enables us to count the ways in which we can arrange a set of objects or select a subset of the original set, is necessary background to a study of probability theory.

12.1 SEQUENCES AND SIGMA NOTATION

INFINITE SEQUENCES

Can you see a pattern or relationship that describes this string of numbers?

$$1, 4, 9, 16, 25, \ldots$$

If we rewrite this string as

$$1^2, 2^2, 3^2, 4^2, 5^2, \ldots$$

it is clear that these are the squares of successive natural numbers. Each number in the string is called a **term.** We could write the nth term of the list as a function a defined by

$$a(n) = n^2$$

where n is a natural number. Such a string of numbers is called an infinite sequence, since the list is infinitely long.

> An **infinite sequence** (often called simply a **sequence**) is a function whose domain is the set of all natural numbers.

The range of the function a is

$$a(1), a(2), a(3), \ldots, a(n), \ldots$$

which we write as

$$a_1, a_2, a_3, \ldots a_n, \ldots$$

That is, we indicate a sequence by using subscript notation rather than function notation. We say that a_1 is the **first term** of the sequence, a_2 is the **second term,** and so on, and we write the ***n*th term** as a_n where $a_n = a(n)$.

EXAMPLE 1

Write the first three terms and the tenth term of each of the sequences whose nth term is given.

(a) $a_n = n^2 + 1$ (b) $a_n = \dfrac{n}{n+1}$ (c) $a_n = 2^n - 1$

SOLUTION

The first three terms are found by substituting $n = 1, 2$, and 3 in the formula for a_n. The tenth term is found by substituting $n = 10$.

(a) $a_1 = 1^2 + 1 = 2$ $a_2 = 2^2 + 1 = 5$ $a_3 = 3^2 + 1 = 10$
$a_{10} = 10^2 + 1 = 101$

(b) $a_1 = \dfrac{1}{1+1} = \dfrac{1}{2}$ $a_2 = \dfrac{2}{2+1} = \dfrac{2}{3}$ $a_3 = \dfrac{3}{3+1} = \dfrac{3}{4}$

$a_{10} = \dfrac{10}{10+1} = \dfrac{10}{11}$

(c) $a_1 = 2^1 - 1 = 1 \quad a_2 = 2^2 - 1 = 3 \quad a_3 = 2^3 - 1 = 7$
$a_{10} = 2^{10} - 1 = 1023$

PROGRESS CHECK

Write the first three terms and the twelfth term of each of the sequences whose nth term is given.

(a) $a_n = 3(1 - n)$ (b) $a_n = n^2 + n + 1$ (c) $a_n = 5$

ANSWERS

(a) $a_1 = 0,\ a_2 = -3,\ a_3 = -6,\ a_{12} = -33$
(b) $a_1 = 3,\ a_2 = 7,\ a_3 = 13,\ a_{12} = 157$
(c) $a_1 = a_2 = a_3 = a_{12} = 5$

An infinite sequence is often defined by a formula expressing the nth term by reference to preceding terms. Such a sequence is said to be defined by a **recursive** formula.

EXAMPLE 2

Find the first four terms of the sequence defined by

$$a_n = a_{n-1} + 3 \quad \text{with} \quad a_1 = 2 \quad \text{and} \quad n \geq 2$$

SOLUTION

Any term of the sequence can be obtained if the preceding term is known. Of course, this recursive formulation requires a starting point, and we are indeed given a_1. Then

$$\begin{aligned} a_1 &= 2 \\ a_2 &= a_1 + 3 = 2 + 3 = 5 \\ a_3 &= a_2 + 3 = 5 + 3 = 8 \\ a_4 &= a_3 + 3 = 8 + 3 = 11 \end{aligned}$$

PROGRESS CHECK

Find the first four terms of the infinite sequence

$$a_n = 2a_{n-1} - 1 \quad \text{with} \quad a_1 = -1 \quad \text{and} \quad n \geq 2$$

ANSWER

$-1,\ -3,\ -7,\ -15$

SUMMATION NOTATION

In the following sections of this chapter, we will seek the sum of terms of a sequence such as

$$a_1 + a_2 + a_3 + \cdots + a_m$$

Since sums occur frequently in mathematics, a special notation has been developed that is defined in the following way.

Summation Notation	$\displaystyle\sum_{k=1}^{m} a_k = a_1 + a_2 + a_3 + \cdots + a_m$

This is often referred to as **sigma notation,** since the Greek letter Σ indicates a sum of terms of the form a_k. The letter k is called the **index of summation** and always assumes successive integer values, starting with the value written under the Σ sign and ending with the value written above the Σ sign.

EXAMPLE 3

Evaluate (a) $\displaystyle\sum_{k=1}^{3} 2^k(k+1)$ (b) $\displaystyle\sum_{i=2}^{4} (i^2+2)$.

SOLUTION

(a) The terms are of the form

$$a_k = 2^k(k+1)$$

and the sigma notation indicates that we want the sum of the terms a_1 through a_3. Forming the terms and adding,

$$\sum_{k=1}^{3} 2^k(k+1) = 2^1(1+1) + 2^2(2+1) + 2^3(3+1)$$

$$= 4 + 12 + 32 = 48$$

(b) Any letter may be used for the index of summation. Here, the letter i is used, and

$$\sum_{i=2}^{4} (i^2+2) = (2^2+2) + (3^2+2) + (4^2+2)$$

$$= 6 + 11 + 18 = 35$$

Note that the index of summation can begin with an integer value other than 1.

EXAMPLE 4

Write each sum using summation notation.

(a) $\dfrac{1}{2} + \dfrac{1}{2\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{2\cdot 4} + \dfrac{1}{2\cdot 5}$ (b) $\dfrac{2}{3} + \dfrac{3}{4} + \dfrac{4}{5} + \dfrac{5}{6}$

SOLUTION

(a) The denominator of each term is of the form $2 \cdot k$, where k assumes integer values from 1 to 5. Then

$$\sum_{k=1}^{5} \frac{1}{2 \cdot k}$$

expresses the desired sum.

(b) If the value of the numerator of a term is k, then the denominator is $k + 1$. Letting k range from 2 to 5,

$$\sum_{k=2}^{5} \frac{k}{k+1}$$

expresses the desired sum.

PROGRESS CHECK

Write each sum using summation notation.

(a) $x_1^2 + x_2^2 + x_3^2 + \cdots + x_{20}^2$ (b) $2^3 + 3^4 + 4^5 + 5^6$

ANSWERS

(a) $\displaystyle\sum_{k=1}^{20} x_k^2$ (b) $\displaystyle\sum_{k=2}^{5} k^{k+1}$

If a sequence is defined by $a_n = c$, where c is a real constant, then

$$\begin{aligned}\sum_{k=1}^{r} a_k &= a_1 + a_2 + \cdots + a_r \\ &= c + c + \cdots + c \\ &= rc\end{aligned}$$

This leads to the rule:

> For any real constant c,
>
> $$\sum_{k=1}^{n} c = nc$$

EXAMPLE 5

Evaluate (a) $\displaystyle\sum_{j=1}^{20} 5$ (b) $\displaystyle\sum_{k=1}^{4} (k^2 - 2)$.

SOLUTION

(a) $\displaystyle\sum_{j=1}^{20} 5 = 20 \cdot 5 = 100$

(b) $$\sum_{k=1}^{4} (k^2 - 2) = (1^2 - 2) + (2^2 - 2) + (3^2 - 2) + (4^2 - 2)$$
$$= -1 + 2 + 7 + 14 = 22$$

The following are properties of sums expressed in sigma notation.

Properties of Sums

For the sequences $a_1, a_2, \ldots,$ and $b_1, b_2, \ldots,$

(i) $\displaystyle\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

(ii) $\displaystyle\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

(iii) $\displaystyle\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k, \quad c$ a constant

EXAMPLE 6

Use the properties of sums to evaluate $\displaystyle\sum_{k=1}^{4} (k^2 - 2)$.

SOLUTION

Rather than write out the terms as was done in Example 4b, we may write

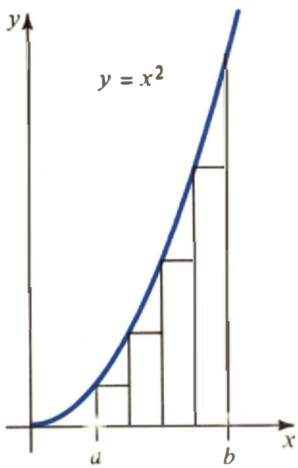
$$\sum_{k=1}^{4} (k^2 - 2) = \sum_{k=1}^{4} k^2 - \sum_{k=1}^{4} 2 = \sum_{k=1}^{4} k^2 - 8$$
$$= 1^2 + 2^2 + 3^2 + 4^2 - 8 = 30 - 8 = 22$$

SERIES

A sum of terms of a sequence is called a **series.** We denote by S_n the sum of the first n terms of an infinite sequence where n is a natural number. Summation notation is very useful in handling a series. For example, given the sequence

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

AREAS BY RECTANGLES



Many textbooks introduce the integral calculus by the use of rectangles to approximate area. In the accompanying figure, we are interested in calculating the area under the curve of the function $f(x) = x^2$ that is bounded by the x-axis and the lines $x = a$ and $x = b$. The interval $[a, b]$ is divided into n subintervals of equal width, and a rectangle is erected in each interval as shown. We then seek to use the sum of the areas of the rectangles as an approximation to the area under the curve.

To calculate the area of a rectangle, we need to know the height and the width. Since the interval $[a, b]$ has been divided into n parts of equal width, we see that

$$\text{width of rectangle} = \frac{b-a}{n}$$

Next, note that the height of the rectangle whose left endpoint is at x_k is determined by the value of the function at that point; that is,

$$\text{height of rectangle} = f(x_k) = x_k^2$$

The area of a "typical" rectangle is then

$$\left(\frac{b-a}{n}\right)f(x_k) = \left(\frac{b-a}{n}\right)x_k^2$$

and the sum of the areas of the rectangles is neatly expressed in summation notation by

$$\sum_{k=1}^{n}\left(\frac{b-a}{n}\right)x_k^2 = \left(\frac{b-a}{n}\right)\sum_{k=1}^{n} x_k^2$$

Intuitively, we see that the greater the number of rectangles, the better our approximation will be, and this concept is pursued in calculus. The student is urged to let $a = 1$ and $b = 3$ in the accompanying figure and to use the method of approximating rectangles with $n = 2$, $n = 4$, and $n = 8$. The *exact* answer is 26/3 square units, and the approximations improve as n grows larger.

we have

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$

and, in general,

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

The number S_n is called the ***n*****th partial sum,** and the numbers

$$S_1, S_2, S_3, \ldots, S_n, \ldots$$

form a sequence called the **sequence of partial sums.** This type of sequence is studied in calculus courses, where methods are developed for analyzing infinite series.

EXAMPLE 7

Given the infinite sequence

$$a_n = n^2 - 1$$

find S_4.

SOLUTION

The first four terms of the sequence are

$$a_1 = 0 \quad a_2 = 3 \quad a_3 = 8 \quad a_4 = 15$$

Then the sum S_4 is given by

$$S_4 = \sum_{k=1}^{4} a_k = 0 + 3 + 8 + 15 = 26$$

If a series alternates in sign, then a multiplicative factor of $(-1)^k$ or $(-1)^{k+1}$ can be used to obtain the proper sign. For example, the series

$$-1^2 + 2^2 - 3^2 + 4^2$$

can be written in sigma notation as

$$\sum_{k=1}^{4} (-1)^k k^2$$

while the series

$$1^2 - 2^2 + 3^2 - 4^2$$

can be written as

$$\sum_{k=1}^{4} (-1)^{k+1} k^2$$

EXAMPLE 8

The terms of a sequence are of the form $a_k = \sqrt{k}$, and the terms are negated when k is even. Write an expression for the general term a_n and for the sum S_n in summation notation.

SOLUTION

If we multiply each term by $(-1)^{k+1}$, then the odd terms will be positive and the even terms will be negative. The general term a_n is then

$$a_n = (-1)^{n+1}\sqrt{n}$$

and the sequence is

$$\sqrt{1}, -\sqrt{2}, \sqrt{3}, -\sqrt{4}, \ldots, (-1)^{n+1}\sqrt{n}, \ldots$$

Finally, the sum S_n is given by

$$S_n = \sum_{k=1}^{n} (-1)^{k+1}\sqrt{k}$$

EXERCISE SET 12.1

In Exercises 1–12 find the first four terms and the twentieth term of the sequence whose nth term is given.

1. $a_n = 2n$
2. $a_n = 2n + 1$
3. $a_n = 4n - 3$
4. $a_n = 3n - 1$
5. $a_n = 5$
6. $a_n = 1 - \dfrac{1}{n}$
7. $a_n = \dfrac{n}{n+1}$
8. $a_n = \sqrt{n}$
9. $a_n = 2 + 0.1^n$
10. $a_n = \dfrac{n^2 - 1}{n^2 + 1}$
11. $a_n = \dfrac{n^2}{2n + 1}$
12. $a_n = \dfrac{2n + 1}{n^2}$

In Exercises 13–18 a sequence is defined recursively. Find the indicated term of the sequence.

13. $a_n = 2a_{n-1} - 1$, $a_1 = 2$; find a_4
14. $a_n = 3 - 3a_{n-1}$, $a_1 = -1$; find a_3
15. $a_n = \dfrac{1}{a_{n-1} + 1}$, $a_3 = 2$; find a_6
16. $a_n = \dfrac{n}{a_{n-1}}$, $a_2 = 1$; find a_5
17. $a_n = (a_{n-1})^2$, $a_1 = 2$; find a_4
18. $a_n = (a_{n-1})^{n-1}$, $a_1 = 2$; find a_4

In Exercises 19–26 find the indicated sum.

19. $\displaystyle\sum_{k=1}^{5} (3k - 1)$
20. $\displaystyle\sum_{k=1}^{5} (3 - 2k)$
21. $\displaystyle\sum_{k=1}^{6} (k^2 + 1)$
22. $\displaystyle\sum_{k=0}^{4} \frac{k}{k^2 + 1}$
23. $\displaystyle\sum_{k=3}^{5} \frac{k}{k - 1}$
24. $\displaystyle\sum_{k=2}^{4} 4(2^k)$
25. $\displaystyle\sum_{j=1}^{4} 20$
26. $\displaystyle\sum_{i=1}^{10} 50$

In Exercises 27–36 use summation notation to express the sum. (The answer is not unique.)

27. $1 + 3 + 5 + 7 + 9$
28. $2 + 5 + 8 + 11 + 14$
29. $1 + 4 + 9 + 16 + 25$
30. $1 - 4 + 9 - 16 + 25$
31. $-1 + \dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{4}}$
32. $\dfrac{1}{2\cdot 4} + \dfrac{1}{2\cdot 5} + \dfrac{1}{2\cdot 6} + \dfrac{1}{2\cdot 7}$
33. $\dfrac{1}{1^2 + 1} - \dfrac{2}{2^2 + 1} + \dfrac{3}{3^2 + 1} - \dfrac{4}{4^2 + 1}$
34. $2 - 4 + 8 - 16$
35. $1 + \dfrac{1}{x} + \dfrac{1}{x^2} + \dfrac{1}{x^3} + \cdots + \dfrac{1}{x^n}$
36. $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \dfrac{1}{4\cdot 5} + \cdots + \dfrac{1}{49\cdot 50}$

12.2 ARITHMETIC SEQUENCES

The sequence

$$2, 5, 8, 11, 14, 17, \ldots$$

is an example of a special type of sequence in which each successive term is obtained by adding a fixed number to the previous term.

Arithmetic Sequence

In an **arithmetic sequence** there is a real number d such that

$$a_n = a_{n-1} + d$$

for all $n > 1$. The number d is called the **common difference.**

An arithmetic sequence is also called an **arithmetic progression.** Returning to the sequence

$$2, 5, 8, 11, 14, 17, \ldots$$

the nth term can be defined recursively by

$$a_n = a_{n-1} + 3, \quad a_1 = 2$$

This is an arithmetic progression with the first term equal to 2 and a common difference of 3.

EXAMPLE 1

Write the first four terms of an arithmetic sequence whose first term is -4 and whose common difference is -3.

SOLUTION

Beginning with -4, we add the common difference -3 to obtain

$$-4 + (-3) = -7 \qquad -7 + (-3) = -10 \qquad -10 + (-3) = -13$$

Alternatively, we note that the sequence is defined by

$$a_n = a_{n-1} - 3, \quad a_1 = -4$$

which leads to the terms

$$a_1 = -4, \quad a_2 = -7, \quad a_3 = -10, \quad a_4 = -13$$

PROGRESS CHECK

Write the first four terms of an arithmetic sequence whose first term is 4 and whose common difference is $-\frac{1}{3}$.

ANSWER

$$4, \frac{11}{3}, \frac{10}{3}, 3, \ldots$$

EXAMPLE 2
Show that the sequence

$$a_n = 2n - 1$$

is an arithmetic sequence, and find the common difference.

SOLUTION
We must show that the sequence satisfies

$$a_n - a_{n-1} = d$$

for some real number d. We have

$$a_n = 2n - 1$$
$$a_{n-1} = 2(n - 1) - 1 = 2n - 3$$

so

$$a_n - a_{n-1} = 2n - 1 - (2n - 3) = 2$$

This demonstrates that we are dealing with an arithmetic sequence whose common difference is 2.

For a given arithmetic sequence, it's easy to find a formula for the nth term a_n in terms of n and the first term a_1. Since

$$a_2 = a_1 + d$$

and

$$a_3 = a_2 + d$$

we see that

$$a_3 = (a_1 + d) + d = a_1 + 2d$$

Similarly, we can show that

$$a_4 = a_3 + d = (a_1 + 2d) + d = a_1 + 3d$$
$$a_5 = a_4 + d = (a_1 + 3d) + d = a_1 + 4d$$

In general,

The nth term a_n of an arithmetic sequence is given by

$$a_n = a_1 + (n - 1)d$$

EXAMPLE 3

Find the seventh term of the arithmetic progression whose first term is 2 and whose common difference is 4.

SOLUTION

We substitute $n = 7$, $a_1 = 2$, $d = 4$ in the formula

$$a_n = a_1 + (n - 1)d$$

obtaining

$$a_7 = 2 + (7 - 1)4 = 2 + 24 = 26$$

PROGRESS CHECK

Find the 16th term of the arithmetic progression whose first term is -5 and whose common difference is $\frac{1}{2}$.

ANSWER

$$\frac{5}{2}$$

EXAMPLE 4

Find the 25th term of the arithmetic sequence whose first and 20th terms are -7 and 31, respectively.

SOLUTION

We can apply the given information to find d.

$$a_n = a_1 + (n - 1)d$$
$$a_{20} = a_1 + (20 - 1)d$$
$$31 = -7 + 19d$$
$$d = 2$$

Now we use the formula for a_n to find a_{25}.

$$a_n = a_1 + (n - 1)d$$
$$a_{25} = -7 + (25 - 1)2$$
$$a_{25} = 41$$

PROGRESS CHECK

Find the 60th term of the arithmetic sequence whose first and 10th terms are 3 and $-\frac{3}{2}$, respectively.

ANSWER

$$-\frac{53}{2}$$

ARITHMETIC SERIES

The series associated with an arithmetic sequence is called an **arithmetic series.** Since an arithmetic sequence has a common difference d, we can write the nth partial sum S_n as

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_n - 2d) + (a_n - d) + a_n \quad (1)$$

where we write $a_2, a_3, \ldots$ in terms of a_1 and we write $a_{n-1}, a_{n-2}, \ldots$ in terms of a_n. Rewriting the right-hand side of Equation (1) in reverse order, we have

$$S_n = a_n + (a_n - d) + (a_n - 2d) + \cdots + (a_1 + 2d) + (a_1 + d) + a_1 \quad (2)$$

Summing the corresponding sides of Equations (1) and (2),

$$\begin{aligned} 2S_n &= (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots \qquad \text{Repeated } n \text{ times} \\ &= n(a_1 + a_n) \end{aligned}$$

Thus,

$$S_n = \frac{n}{2}(a_1 + a_n)$$

Since $a_n = a_1 + (n - 1)d$, we see that

$$\begin{aligned} S_n &= \frac{n}{2}[a_1 + a_1 + (n - 1)d] \qquad \text{Substituting for } a_n \\ &= \frac{n}{2}[2a_1 + (n - 1)d] \end{aligned}$$

We now have two useful formulas.

Arithmetic Series

For an arithmetic series,

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_n = \frac{n}{2}[2a_1 + (n - 1)d]$$

The choice of which formula to use depends on the available information. The following examples illustrate the use of the formulas.

EXAMPLE 5

Find the sum of the first 30 terms of an arithmetic sequence whose first term is -20 and whose common difference is 3.

SOLUTION

We know that $n = 30$, $a_1 = -20$, and $d = 3$. Substituting in

$$S_n = \frac{n}{2}[2a_1 + (n-1)d]$$

we obtain

$$\begin{aligned} S_{30} &= \frac{30}{2}[2(-20) + (30-1)3] \\ &= 15(-40 + 87) \\ &= 705 \end{aligned}$$

PROGRESS CHECK

Find the sum of the first 10 terms of the arithmetic sequence whose first term is 2 and whose common difference is $-\frac{1}{2}$.

ANSWER

$-\frac{5}{2}$

EXAMPLE 6

The first term of an arithmetic series is 2, the last term is 58, and the sum is 450. Find the number of terms and the common difference.

SOLUTION

We have $a_1 = 2$, $a_n = 58$, and $S_n = 450$. Substituting in

$$S_n = \frac{n}{2}(a_1 + a_n)$$

we have

$$\begin{aligned} 450 &= \frac{n}{2}(2 + 58) \\ 900 &= 60n \\ n &= 15 \end{aligned}$$

Now we substitute in

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ 58 &= 2 + (14)d \\ 56 &= 14d \\ d &= 4 \end{aligned}$$

PROGRESS CHECK

The first term of an arithmetic series is 6, the last term is 1, and the sum is 77/2. Find the number of terms and the common difference.

ANSWER

$n = 11, \quad d = -\frac{1}{2}$

EXERCISE SET 12.2

Write the next two terms of each of the following arithmetic sequences.

1. $3, 6, 9, 12, \ldots$
2. $2, -2, -6, -10, \ldots$
3. $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \ldots$
4. $y - 4, y, y + 4, y + 8, \ldots$
5. $0, \log 10, \log 100, \log 1000, \ldots$
6. $4, \frac{11}{2}, 7, \frac{17}{2}, \ldots$
7. $\sqrt{5} - 2, \sqrt{5}, \sqrt{5} + 2, \sqrt{5} + 4, \ldots$
8. $12, 8, 4, 0, \ldots$

Write the first four terms of the arithmetic sequence whose first term is a_1 and whose common difference is d.

9. $a_1 = 2, d = 4$
10. $a_1 = -2, d = -5$
11. $a_1 = 3, d = -\frac{1}{2}$
12. $a_1 = \frac{1}{2}, d = 2$
13. $a_1 = \frac{1}{3}, d = -\frac{1}{3}$
14. $a_1 = 6, d = \frac{5}{2}$

Find the specified term of the arithmetic sequence whose first term is a_1 and whose common difference is d.

15. $a_1 = 4, d = 3$; 8th term
16. $a_1 = -3, d = \frac{1}{4}$; 14th term
17. $a_1 = 14, d = -2$; 12th term
18. $a_1 = 6, d = -\frac{1}{3}$; 9th term

Given two terms of an arithmetic sequence, find the specified term.

19. $a_1 = -2, a_{20} = -2$; 24th term
20. $a_1 = \frac{1}{2}, a_{12} = 6$; 30th term
21. $a_1 = 0, a_{61} = 20$; 20th term
22. $a_1 = 23, a_{15} = -19$; 6th term
23. $a_1 = -\frac{1}{4}, a_{41} = 10$; 22nd term
24. $a_1 = -3, a_{18} = 65$; 30th term

Find the sum of the specified number of terms of the arithmetic sequence whose first term is a_1 and whose common difference is d.

25. $a_1 = 3, d = 2$; 20 terms
26. $a_1 = -4, d = \frac{1}{2}$; 24 terms
27. $a_1 = \frac{1}{2}, d = -2$; 12 terms
28. $a_1 = -3, d = -\frac{1}{3}$; 18 terms
29. $a_1 = 82, d = -2$; 40 terms
30. $a_1 = 6, d = 4$; 16 terms

31. How many terms of the arithmetic progression 2, 4, 6, 8, . . . add up to 930?
32. How many terms of the arithmetic progression 44, 41, 38, 35, . . . add up to 340?
33. The first term of an arithmetic series is 3, the last term is 90, and the sum is 1395. Find the number of terms and the common difference.
34. The first term of an arithmetic series is -3, the last term is $\frac{5}{2}$, and the sum is -3. Find the number of terms and the common difference.
35. The first term of an arithmetic series is $\frac{1}{2}$, the last term is $\frac{7}{4}$, and the sum is $\frac{27}{4}$. Find the number of terms and the common difference.
36. The first term of an arithmetic series is 20, the last term is -14, and the sum is 54. Find the number of terms and the common difference.
37. Find the sum of the first 16 terms of an arithmetic progression whose 4th and 10th terms are $-\frac{5}{4}$ and $\frac{1}{4}$, respectively.
38. Find the sum of the first 12 terms of an arithmetic progression whose 3rd and 6th terms are 9 and 18, respectively.
39. Show that the sum of the first n natural numbers is $n(n + 1)/2$.
40. Show that
$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

12.3 GEOMETRIC SEQUENCES

The sequence

$$3, 6, 12, 24, 48, \ldots$$

in which each term after the first is obtained by multiplying the preceding one by 2, is an example of a geometric sequence.

Geometric Sequence In a **geometric sequence** there is a real number r such that

$$a_n = ra_{n-1}$$

for all $n > 1$. The number r is called the **common ratio.**

A geometric sequence is also called a **geometric progression.** The common ratio r can be found by dividing any term a_k by the preceding term, a_{k-1}.

In a geometric sequence, the common ratio r is given by

$$r = \frac{a_k}{a_{k-1}}$$

Let's look at successive terms of a geometric sequence whose first term is a_1 and whose common ratio is r. We have

$$a_2 = ra_1$$
$$a_3 = ra_2 = r(ra_1) = r^2a_1$$
$$a_4 = ra_3 = r(r^2a_1) = r^3a_1$$

FIBONACCI COUNTS THE RABBITS

Here is a problem that was first published in the year 1202.

> A pair of newborn rabbits begins breeding at age one month and thereafter produces one pair of offspring per month. If we start with a newly born pair of rabbits, how many rabbits will there be at the beginning of each month?

The problem was posed by Leonardo Fibonacci of Pisa, and the resulting sequence is known as a **Fibonacci sequence.**

Month	Pairs of Rabbits
0	P_1
1	P_1
2	$P_1 \rightarrow P_2$
3	$P_1 \rightarrow P_3$ $\quad P_2$
4	$P_1 \rightarrow P_4$ $\quad P_2 \rightarrow P_5$ $\quad P_3$
5	$P_1 \rightarrow P_6$ $\quad P_2 \rightarrow P_7$ $\quad P_3 \rightarrow P_8$ $\quad P_4$ $\quad P_5$

The accompanying figure helps in analyzing the problem. At the beginning of month zero, we have the pair of newborn rabbits P_1. At the beginning of month 1, we still have the pair P_1, since the rabbits do not breed until age 1 month. At the beginning of month 2, the pair P_1 has the pair of offspring P_2. At the beginning of month 3, P_1 again has offspring, P_3, but P_2 does not breed during its first month. At the beginning of month 4, P_1 has offspring P_4, P_2 has offspring P_5, and P_3 does not breed during its first month.

If we let a_n denote the number of pairs of rabbits at the beginning of month n, we see that

$$a_0 = 1,\ a_1 = 1,\ a_2 = 2,\ a_3 = 3,\ a_4 = 5,\ a_5 = 8,\ \ldots$$

The sequence has the interesting property that each term is the sum of the two preceding terms; that is,

$$a_n = a_{n-1} + a_{n-2}$$

Strange as it seems, nature appears to be aware of the Fibonacci sequence. For example, arrangements of seeds on sunflowers and leaves on some trees are related to Fibonacci numbers. Stranger still, some researchers believe that cycle analysis, such as analysis of stock market prices, is also related in some way to Fibonacci numbers.

The pattern suggests that the exponent of r is one less than the subscript of a in the left-hand side.

> The nth term of a geometric sequence is given by
>
> $$a_n = a_1 r^{n-1}$$

Once again, mathematical induction is required to prove that the formula holds for all natural numbers.

EXAMPLE 1

Find the seventh term of the geometric sequence $-4, -2, -1, \ldots$.

SOLUTION

Since

$$r = \frac{a_k}{a_{k-1}}$$

we see that

$$r = \frac{a_3}{a_2} = \frac{-1}{-2} = \frac{1}{2}$$

Substituting $a_1 = -4$, $r = \frac{1}{2}$, and $n = 7$, we have

$$\begin{aligned} a_n &= a_1 r^{n-1} \\ a_7 &= (-4)\left(\frac{1}{2}\right)^{7-1} = (-4)\left(\frac{1}{2}\right)^6 \\ &= (-4)\left(\frac{1}{64}\right) = -\frac{1}{16} \end{aligned}$$

PROGRESS CHECK

Find the sixth term of the geometric sequence 2, −6, 18, ...

ANSWER

−486

GEOMETRIC MEAN

In a geometric sequence, the terms between the first and last terms are called **geometric means.** We will illustrate the method of calculating such means.

EXAMPLE 2

Insert three geometric means between 3 and 48.

SOLUTION

The geometric sequence must look like this.

$$3, a_2, a_3, a_4, 48, \ldots$$

Thus, $a_1 = 3$, $a_5 = 48$, and $n = 5$. Substituting in

$$\begin{aligned} a_n &= a_1 r^{n-1} \\ 48 &= 3r^4 \\ r^4 &= 16 \\ r &= \pm 2 \end{aligned}$$

Thus there are two geometric sequences with three geometric means between 3 and 48.

$$3, 6, 12, 24, 48, \ldots \quad r = 2$$
$$3, -16, 12, -24, 48, \ldots \quad r = -2$$

PROGRESS CHECK

Insert two geometric means between 5 and $\frac{8}{25}$.

ANSWER

$2, \frac{4}{5}$

GEOMETRIC SERIES

If $a_1, a_2, \ldots$ is a geometric sequence, then the nth partial sum

$$S_n = a_1 + a_2 + \cdots + a_n \tag{1}$$

is called a **geometric series.** Since each term of the series can be rewritten as $a_k = a_1 r^{k-1}$, we can rewrite Equation (1) as

$$S_n = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-2} + a_1 r^{n-1} \tag{2}$$

Multiplying each term in Equation (2) by r, we have

$$rS_n = a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1} + a_1 r^n \tag{3}$$

Subtracting Equation (2) from Equation (3) produces

$$\begin{aligned} rS_n - S_n &= a_1 r^n - a_1 \\ (r-1)S_n &= a_1(r^n - 1) && \text{Factoring} \\ S_n &= \frac{a_1(r^n - 1)}{r-1} && \text{Dividing by } r-1 \text{ (if } r \neq 1\text{)} \end{aligned}$$

Changing the signs in both the numerator and the denominator gives us the following formula for the nth partial sum.

Geometric Series	In a geometric series with first term a_1 and common ratio $r \neq 1$, $$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

EXAMPLE 3

Find the sum of the first six terms of the geometric sequence whose first three terms are 12, 6, 3.

SOLUTION

The common ratio can be found by dividing any term by the preceding term.

$$r = \frac{a_k}{a_{k-1}} = \frac{a_2}{a_1} = \frac{6}{12} = \frac{1}{2}$$

Substituting $a_1 = 12$, $r = \frac{1}{2}$, $n = 6$ in the formula for S_n, we have

$$S_n = \frac{a_1(1 - r^n)}{1 - r} = \frac{12\left[1 - \left(\frac{1}{2}\right)^6\right]}{1 - \frac{1}{2}} = \frac{189}{8}$$

PROGRESS CHECK

Find the sum of the first five terms of the geometric sequence whose first three terms are 2, $-\frac{4}{3}$, $\frac{8}{9}$.

ANSWER

$$\frac{110}{81}$$

EXAMPLE 4

A father promises to give each child 2 cents on the first day and 4 cents on the second day and to continue doubling the amount each day for a total of 8 days. How much will each child receive on the last day? How much will each child have received in total after 8 days?

SOLUTION

The daily payout to each child forms a geometric sequence 2, 4, 8, . . . with $a_1 = 2$ and $r = 2$. The term a_8 is given by substituting in

$$a_n = a_1 r^{n-1}$$
$$a_8 = a_1 r^{8-1} = 2 \cdot 2^7 = 256$$

Thus, each child will receive \$2.56 on the last day. The total received by each child is given by

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$
$$S_8 = \frac{a_1(1 - r^8)}{1 - r} = \frac{2(1 - 2^8)}{1 - 2}$$
$$= \frac{2(1 - 256)}{-1} = 510$$

Each child will have received a total of \$5.10 after 8 days.

PROGRESS CHECK

A ball is dropped from a height of 64 feet. On each bounce, it rebounds half the height it fell (Figure 1). How high is the ball at the top of the fifth bounce? What is the total distance the ball has traveled at the top of the fifth bounce?

ANSWER
2 feet; 186 feet

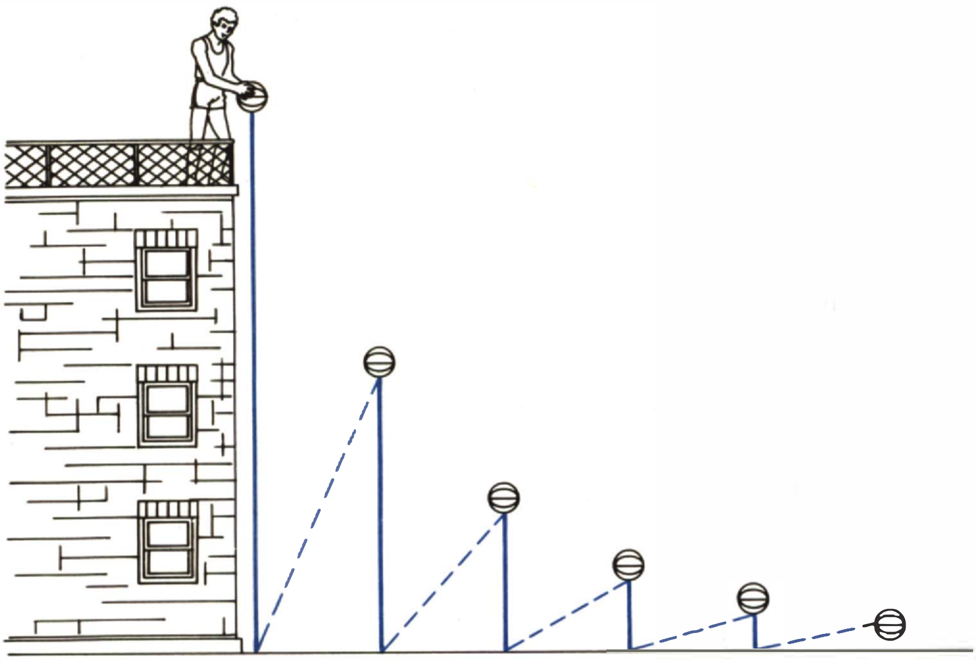


FIGURE 1

INFINITE GEOMETRIC SERIES

We now want to focus on a geometric series for which $|r| < 1$, say

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$$

To see how the sum increases as n increases, let's form a table of values of S_n.

n	1	2	3	4	5	6	7	8	9
S_n	0.500	0.750	0.875	0.938	0.969	0.984	0.992	0.996	0.998

We begin to suspect that S_n gets closer and closer to 1 as n increases. To see that this is really so, let's look at the formula

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

when $|r| < 1$. When a number r that is less than 1 in absolute value is raised to higher and higher positive integer powers, the absolute value of r^n gets smaller and smaller. Thus, the term r^n can be made as small as we like by choosing n sufficiently large. Since we are dealing with an infinite series, we say that "r^n approaches zero as n approaches infinity." We then replace r^n with 0 in the formula and denote the sum by S.

Sum of an Infinite Geometric Series

The sum S of the **infinite geometric series**

$$\sum_{k=0}^{\infty} a_1 r^k = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^n + \cdots$$

is given by

$$S = \frac{a_1}{1-r} \quad \text{when } |r| < 1$$

Applying this formula to the preceding series, we see that

$$S = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

which justifies the conjecture resulting from the examination of the above table. It is appropriate to remark that the ideas used in deriving the formula for an infinite geometric series have led us to the very border of the beginning concepts of calculus.

EXAMPLE 5

Find the sum of the infinite geometric series

$$\frac{3}{2} + 1 + \frac{2}{3} + \frac{4}{9} + \cdots$$

SOLUTION

The common ratio $r = \frac{2}{3}$. The sum of the infinite geometric series, with $|r| < 1$, is given by

$$S = \frac{a_1}{1-r} = \frac{\frac{3}{2}}{1 - \frac{2}{3}} = \frac{9}{2}$$

PROGRESS CHECK
Find the sum of the infinite geometric series

$$4 - 1 + \frac{1}{4} - \frac{1}{16} + \cdots$$

ANSWER
$\dfrac{16}{5}$

The notation

$$0.65252\overline{52}$$

indicates a repeating decimal with a pattern in which 52 is repeated indefinitely. Every repeating decimal can be written as a rational number. We will apply the formula for the sum of an infinite geometric series to find the rational number equal to a repeating decimal.

EXAMPLE 6
Find the rational number that is equal to $0.65252\overline{52}$.

SOLUTION
Note that

$$0.65252\overline{52} = 0.6 + 0.052 + 0.00052 + 0.0000052 + \cdots$$

We treat the sum

$$0.052 + 0.00052 + 0.0000052 + \cdots$$

as an infinite geometric series with $a = 0.052$ and $r = 0.01$. Then

$$S = \frac{a}{1-r} = \frac{0.052}{1 - 0.01} = \frac{0.052}{0.99} = \frac{52}{990}$$

and the repeating decimal is equal to

$$0.6 + \frac{52}{990} = \frac{6}{10} + \frac{52}{990} = \frac{646}{990} = \frac{323}{495}$$

PROGRESS CHECK
Write the repeating decimal $2.5454\overline{54}$ as a rational number.

ANSWER
$\dfrac{252}{99}$

EXERCISE SET 12.3

In Exercises 1–6 find the next term of the given geometric sequence.

1. $3, 6, 12, 24, \ldots$
2. $-4, 12, -36, 108, \ldots$
3. $-4, 3, -\frac{9}{4}, \frac{27}{16}, \ldots$
4. $2, -1, \frac{1}{2}, -\frac{1}{4}, \ldots$
5. $1.2, 0.24, 0.048, \ldots$
6. $\frac{1}{8}, \frac{1}{2}, 2, 8, \ldots$

In Exercises 7–12 write the first four terms of the geometric sequence whose first term is a_1 and whose common ratio is r.

7. $a_1 = 3,\ r = 3$
8. $a_1 = -4,\ r = 2$
9. $a_1 = 4,\ r = \frac{1}{2}$
10. $a_1 = 16,\ r = -\frac{3}{2}$
11. $a_1 = -3,\ r = 2$
12. $a_1 = 3,\ r = -\frac{2}{3}$

In Exercises 13–24 use the information given about a geometric sequence to find the requested item.

13. $a_1 = 3,\ r = -2$; find a_8
14. $a_1 = 18,\ r = -\frac{1}{2}$; find a_6
15. $a_1 = 16,\ a_2 = 8$; find a_7
16. $a_1 = 15,\ a_2 = -10$; find a_6
17. $a_1 = 3,\ a_5 = \frac{1}{27}$; find a_7
18. $a_1 = 2,\ a_6 = \frac{1}{16}$; find a_3
19. $a_1 = \frac{16}{81},\ a_6 = \frac{3}{2}$; find a_8
20. $a_4 = \frac{1}{4},\ a_7 = 1$; find r
21. $a_2 = 4,\ a_8 = 256$; find r
22. $a_3 = 3,\ a_6 = -81$; find a_8
23. $a_1 = \frac{1}{2},\ r = 2,\ a_n = 32$; find n
24. $a_1 = -2,\ r = 3,\ a_n = -162$; find n
25. Insert two geometric means between $\frac{1}{3}$ and 9.
26. Insert two geometric means between -3 and 192.
27. Insert two geometric means between 1 and $\frac{1}{64}$.
28. Insert three geometric means between $\frac{2}{3}$ and $\frac{32}{243}$.

In Exercises 29–32 find the requested partial sum for the geometric sequence whose first three terms are given.

29. $3, 1, \frac{1}{3}$; find S_7
30. $\frac{1}{3}, 1, 3$; find S_6
31. $-3, \frac{6}{5}, -\frac{12}{25}$; find S_5
32. $2, \frac{4}{3}, \frac{8}{9}$; find S_6

In Exercises 33–36 use the information given about a geometric sequence to find the requested partial sum.

33. $a_1 = 4,\ r = 2$; find S_8
34. $a_1 = -\frac{1}{2},\ r = -3$; find S_{10}
35. $a_1 = 2,\ a_4 = -\frac{54}{8}$; find S_5
36. $a_1 = 64,\ a_7 = 1$; find S_6

37. A Christmas Club calls for savings of \$5 in January, and twice as much in each successive month as in the previous month. How much money will have been saved by the end of November?

38. A city had 20,000 people in 1980. If the population increases 5% per year, how many people will the city have in 1990?

39. A city had 30,000 people in 1980. If the population increases 25% every 10 years, how many people will the city have in the year 2010?

40. For good behavior a child is offered a reward consisting of 1 cent on the first day, 2 cents on the second day, 4 cents on the third day, and so on. If the child behaves properly for two weeks, what is the total amount that the child will receive?

Evaluate the sum of each infinite geometric series.

41. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

42. $\frac{4}{5} + \frac{1}{5} + \frac{1}{20} + \frac{1}{80} + \cdots$

43. $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$

44. $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots$

45. $2 + \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots$

46. $1 + 0.1 + 0.01 + 0.001 + \cdots$

47. $0.5 + (0.5)^2 + (0.5)^3 + (0.5)^4 + \cdots$

48. $\frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \frac{16}{625} + \cdots$

49. $\frac{1}{3} - \frac{2}{9} + \frac{4}{27} - \frac{8}{81} + \cdots$

50. Find the rational number equal to $3.666\overline{6}$.

51. Find the rational number equal to $0.36767\overline{67}$.

52. Find the rational number equal to $4.1414\overline{14}$.

53. Find the rational number equal to $0.325\overline{325}$.

12.4 MATHEMATICAL INDUCTION

Mathematical induction is a method of proof that serves as one of the most powerful tools available to the mathematician. Viewed another way, mathematical induction is a property of the natural numbers that enables us to prove theorems that would otherwise appear unmanageable.

We begin by considering the sums of consecutive odd integers

$$1 = 1$$
$$1 + 3 = 4$$
$$1 + 3 + 5 = 9$$
$$1 + 3 + 5 + 7 = 16$$
$$1 + 3 + 5 + 7 + 9 = 25$$

We instantly recognize that the sequence

$$1, 4, 9, 16, 25$$

consists of the squares of the integers 1, 2, 3, 4, and 5. Is this coincidental or do we have a general rule? Is the sum of the first n consecutive odd integers always equal to n^2? Curiosity leads us to try yet one more case.

$$1 + 3 + 5 + 7 + 9 + 11 = 36 = 6^2$$

Indeed, the sum of the first six odd integers is 6^2. This strengthens our *suspicion* that the result may hold in general, but we cannot possibly verify a theorem for *all* positive integers by testing one integer at a time. At this point we need to turn to the principle of mathematical induction.

Principle of Mathematical Induction	If a statement involving a natural number n (I) is true when $n = 1$ and (II) whenever it is true for $n = k$, is also true for $n = k + 1$, then the statement is true for all positive integer values of n.

Let's examine the logic of the principle of mathematical induction. Part (I) says that we must verify the statement for $n = 1$. Then, by Part (II), the statement is also true for $n = 1 + 1 = 2$. But Part (II) then implies that the statement must also be true for $n = 2 + 1 = 3$, and so on. The effect is similar to an endless string of dominoes whereby each domino causes the next to fall. Thus, it is plausible that the principle has established the validity of the statement for *all* positive integer values of n.

We outline the steps involved in applying the principle of mathematical induction in the following example.

EXAMPLE 1

Prove that the sum of the first n consecutive integers is given by $n(n + 1)/2$.

SOLUTION

Mathematical Induction	
Step 1. Verify that the statement is true for $n = 1$.	*Step 1*. The "sum" of the first integer is 1. Evaluating the formula for $n = 1$ yields $$\frac{1(1 + 1)}{2} = \frac{2}{2} = 1$$ which verifies the formula for $n = 1$.

Step 2. Assume the statement is true for $n = k$. Show it is true for $n = k + 1$.	*Step 2.* For $n = k$ we have $$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$ Adding the next consecutive integer, $k + 1$, to both sides, we obtain $$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= (k+1)\left(\frac{k}{2} + 1\right) \\ &= (k+1)\left(\frac{k+2}{2}\right) \\ &= \frac{1}{2}(k+1)(k+2) \end{aligned}$$ Thus, the formula holds for $n = k + 1$. By the principle of mathematical induction, it is then true for all positive integer values of n.

EXAMPLE 2
Prove that the sum of the first n consecutive odd integers is given by n^2.

SOLUTION
To verify the formula for $n = 1$, we need only observe that $1 = 1^2$.

The following table shows the correspondence between the natural numbers and the odd integers. We see that when $n = k$, the value of the nth consecutive

n	1	2	3	4	$\cdots$	k
nth odd integer	1	3	5	7	$\cdots$	$2k - 1$

odd integer is $2k - 1$. Since the formula is assumed to be true for $n = k$, we have

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2$$

Adding the next consecutive odd integer, $2k + 1$, to both sides, we obtain

$$1 + 3 + \cdots + (2k - 1) + (2k + 1) = k^2 + (2k + 1)$$

or

$$1 + 3 + \cdots + (2k + 1) = (k + 1)^2$$

Thus, the sum of the first $k + 1$ consecutive odd integers is $(k + 1)^2$. By the principle of mathematical induction, the formula is true for all positive integer values of n.

The student should be aware that many of the theorems that were used in this book can be proved formally by using mathematical induction. Here is an example of a basic property of positive integer exponents that yields to this type of proof.

EXAMPLE 3

Prove that $(xy)^n = x^n y^n$ for all positive integer values of n.

SOLUTION

For $n = 1$, we have

$$(xy)^1 = xy = x^1 y^1$$

which verifies the validity of the statement for $n = 1$. Assuming the statement holds for $n = k$, we have

$$(xy)^k = x^k y^k$$

To show that the statement holds for $n = k + 1$, we write

$$\begin{aligned}
(xy)^{k+1} &= (xy)^k(xy) && \text{Definition of exponents} \\
&= (x^k y^k)(xy) && \text{Statement holds for } n = k \\
&= (x^k x)(y^k y) && \text{Associative and commutative laws} \\
&= x^{k+1} y^{k+1} && \text{Definition of exponents}
\end{aligned}$$

Thus, the statement holds for $n = k + 1$, and by the principle of mathematical induction the statement holds for all integer values of n.

EXERCISE SET 12.4

In Exercises 1–10 prove that the statement is true for all positive integer values of n by using the principle of mathematical induction.

1. $2 + 4 + 6 + \cdots + 2n = n(n + 1)$

2. $1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 = \dfrac{n(2n + 1)(2n - 1)}{3}$

3. $2 + 5 + 8 + \cdots + (3n - 1) = \dfrac{n(3n + 1)}{2}$

4. $4 + 8 + 12 + \cdots + 4n = 2n(n + 1)$

5. $5 + 10 + 15 + \cdots + 5n = \dfrac{5n(n + 1)}{2}$

6. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n + 1)(2n + 1)}{6}$

7. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n + 1) = \dfrac{n(n + 1)(n + 2)}{3}$

8. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n + 1)^2}{4}$

9. $1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)$

10. $\left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n}$

11. Prove that the nth term a_n of an arithmetic progression whose first term is a_1 and whose common difference is d is given by $a_n = a_1 + (n - 1)d$.

12. Prove that the nth term a_n of a geometric progression whose first term is a_1 and whose common ratio is r is given by $a_n = a_1 r^{n-1}$.

13. Prove that $2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$.

14. Prove that $a + ar + ar^2 + \cdots + ar^{n-1} = \dfrac{a(1 - r^n)}{1 - r}$.

15. Prove that $x^n - 1$ is divisible by $x - 1$, $x \neq 1$. [*Hint:* Recall that divisibility requires the existence of a polynomial $Q(x)$ such that $x^n - 1 = (x - 1)Q(x)$.]

16. Prove that $x^n - y^n$ is divisible by $x - y$, $x \neq y$. [*Hint:* Note that $x^{n+1} - y^{n+1} = (x^{n+1} - xy^n) + (xy^n - y^{n+1})$.]

12.5 THE BINOMIAL THEOREM

By sequential multiplication by $(a + b)$ you may verify that

$$(a + b)^1 = a + b$$
$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$
$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

The expression on the right-hand side of the equation is called the **expansion** of the left-hand side. If we were to predict the form of the expansion of $(a + b)^n$, where n is a natural number, the preceding example would lead us to conclude that it has the following properties.

(a) The expansion has $n + 1$ terms.

(b) The first term is a^n and the last term is b^n.

(c) The sum of the exponents of a and b in each term is n.

(d) In each successive term after the first, the exponent of a decreases by 1, and the exponent of b increases by 1.

(e) The coefficients may be obtained from the following array, which is known as **Pascal's triangle.** Each number, with the exception of those at the ends of the rows, is the sum of the two nearest numbers in the row above. The numbers at the ends of the rows are always 1.

$$\begin{array}{ccccccccccc}
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Pascal's triangle is not a convenient means for determining the coefficients of the expansion when n is large. Here is an alternative method.

(e′) The coefficient of any term (after the first) can be found by the following rule: In the preceding term, multiply the coefficient by the exponent of a and then divide by one more than the exponent of b.

EXAMPLE 1

Write the expansion of $(a + b)^6$.

SOLUTION

From Property (b) we know that the first term is a^6. Thus,

$$(a + b)^6 = a^6 + \cdots$$

From Property (e′) the next coefficient is

$$\frac{1 \cdot 6}{1} = 6$$

(since the exponent of b is 0). By Property (d) the exponents of a and b in this term are 5 and 1, respectively, so we have

$$(a + b)^6 = a^6 + 6a^5b + \cdots$$

Applying Property (e′) again, we find that the next coefficient is

$$\frac{6 \cdot 5}{2} = 15$$

and by Property (d) the exponents of a and b in this term are 4 and 2, respectively. Thus,

$$(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + \cdots$$

Continuing in this manner, we see that

$$(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

PROGRESS CHECK

Write the first five terms in the expansion of $(a + b)^{10}$.

ANSWER

$a^{10} + 10a^9b + 45a^8b^2 + 120a^7b^3 + 210a^6b^4$

The expansion of $(a + b)^n$ that we have described is called the **binomial theorem** or **binomial formula** and can be written

The Binomial Formula

$$(a + b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 + \cdots + \frac{n(n-1)(n-2)\cdots(n-r+1)}{1 \cdot 2 \cdot 3 \cdots r}a^{n-r}b^r + \cdots + b^n$$

The binomial formula can be proved by the method of mathematical induction discussed in the preceding section.

EXAMPLE 2

Find the expansion of $(2x - 1)^4$.

SOLUTION

Let $a = 2x$, $b = -1$, and apply the binomial formula.

$$\begin{aligned}(2x - 1)^4 &= (2x)^4 + \frac{4}{1}(2x)^3(-1) + \frac{4 \cdot 3}{1 \cdot 2}(2x)^2(-1)^2 \\ &\quad + \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3}(2x)(-1)^3 + (-1)^4 \\ &= 16x^4 - 32x^3 + 24x^2 - 8x + 1\end{aligned}$$

PROGRESS CHECK

Find the expansion of $(x^2 - 2)^4$.

ANSWER

$x^8 - 8x^6 + 24x^4 - 32x^2 + 16$

FACTORIAL NOTATION

Note that the denominator of the coefficient in the binomial formula is always the product of the first n natural numbers. We use the symbol $n!$, which is read as n **factorial,** to indicate this type of product. For example,

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$
$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

and, in general,

n Factorial

$$n! = n(n-1)(n-2)\cdots 4\cdot 3\cdot 2\cdot 1$$

Since

$$(n-1)! = (n-1)(n-2)(n-3)\cdots 4\cdot 3\cdot 2\cdot 1$$

we see that for $n > 1$

$$n! = n(n-1)!$$

For convenience, we define $0!$ by

$$0! = 1$$

EXAMPLE 3

Evaluate each of the following.

(a) $\dfrac{5!}{3!}$

Since $5! = 5\cdot 4\cdot 3!$ we may write

$$\frac{5!}{3!} = \frac{5\cdot 4\cdot 3!}{3!} = 5\cdot 4 = 20$$

(b) $\dfrac{9!}{8!} = \dfrac{9\cdot 8!}{8!} = 9$

(c) $\dfrac{10!4!}{12!} = \dfrac{10!4!}{12\cdot 11\cdot 10!} = \dfrac{4!}{12\cdot 11} = \dfrac{4\cdot 3\cdot 2\cdot 1}{12\cdot 11} = \dfrac{2}{11}$

(d) $\dfrac{n!}{(n-2)!} = \dfrac{n(n-1)(n-2)!}{(n-2)!} = n(n-1) = n^2 - n$

(e) $\dfrac{(2-2)!}{3!} = \dfrac{0!}{3\cdot 2} = \dfrac{1}{6}$

PROGRESS CHECK

Evaluate each of the following.

(a) $\dfrac{12!}{10!}$ (b) $\dfrac{6!}{4!2!}$ (c) $\dfrac{10!8!}{9!7!}$

(d) $\dfrac{n!(n-1)!}{(n+1)!(n-2)!}$ (e) $\dfrac{8!}{6!(3-3)!}$

ANSWERS

(a) 132 (b) 15 (c) 80 (d) $\dfrac{n-1}{n+1}$ (e) 56

Here is what the binomial formula looks like in factorial notation.

$$(a+b)^n = a^n + \frac{n!}{1!(n-1)!}a^{n-1}b + \frac{n!}{2!(n-2)!}a^{n-2}b^2 + \frac{n!}{3!(n-3)!}a^{n-3}b^3 + \cdots + \frac{n!}{r!(n-r)!}a^{n-r}b^r + \cdots + b^n$$

The symbol $\binom{n}{r}$, called the **binomial coefficient**, is defined in this way:

Binomial Coefficient

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

This symbol is useful in denoting the coefficients of the binomial expansion. Using this notation, the binomial formula can be written as

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + b^n$$

Sometimes we merely want to find a certain term in the expansion of $(a+b)^n$. We shall use the following observation to answer this question. In the binomial formula for the expansion of $(a+b)^n$, b occurs in the second term, b^2 occurs in the third term, b^3 occurs in the fourth term, and, in general, b^k occurs in the $(k+1)$th term. The exponents of a and b must add up to n in each term. Since the exponent of b in the $(k+1)$th term is k, we conclude that the exponent of a must be $n-k$.

EXAMPLE 4

Find the fourth term in the expansion of $(x - 1)^5$.

SOLUTION

The exponent of b in the fourth term is 3, and the exponent of a is then $5 - 3 = 2$. From the binomial formula we see that the coefficient of the term a^2b^3 is

$$\binom{n}{3} = \binom{5}{3} = \frac{5!}{3!2!}$$

Since $a = x$ and $b = -1$, the fourth term is

$$\frac{5!}{3!2!}x^2(-1)^3 = -10x^2$$

PROGRESS CHECK

Find the third term in the expansion of

$$\left(\frac{x}{2} - 1\right)^8$$

ANSWER

$\frac{7}{16}x^6$

EXAMPLE 5

Find the term in the expansion of $(x^2 - y^2)^6$ that involves y^8.

SOLUTION

Since $y^8 = (-y^2)^4$, we seek that term which involves b^4 in the expansion of $(a + b)^6$. Thus, $b^4 = (-y^2)^4 = y^8$ occurs in the fifth term. In this term the exponent of a is $6 - 4 = 2$. By the binomial formula the corresponding coefficient is

$$\binom{6}{4} = \frac{6!}{4!2!} = 15$$

Since $a = x^2$ and $b = -y^2$, the desired term is

$$15(x^2)^2(-y^2)^4 = 15x^4y^8$$

PROGRESS CHECK

Find the term in the expansion of $(x^3 - \sqrt{2})^5$ that involves x^6.

ANSWER

$-20\sqrt{2}x^6$

EXERCISE SET 12.5

Expand and simplify.

1. $(3x + 2y)^5$
2. $(2a - 3b)^6$
3. $(4x - y)^4$
4. $\left(3 + \frac{1}{2}x\right)^4$
5. $(2 - xy)^5$
6. $(3a^2 + b)^4$
7. $(a^2b + 3)^4$
8. $(x - y)^7$
9. $(a - 2b)^8$
10. $\left(\frac{x}{y} + y\right)^6$
11. $\left(\frac{1}{3}x + 2\right)^3$
12. $\left(\frac{x}{y} + \frac{y}{x}\right)^5$

Find the first four terms in the given expansion and simplify.

13. $(2 + x)^{10}$
14. $(x - 3)^{12}$
15. $(3 - 2a)^9$
16. $(a^2 + b^2)^{11}$
17. $(2x - 3y)^{14}$
18. $\left(a - \frac{1}{a^2}\right)^8$
19. $(2x - yz)^{13}$
20. $\left(x - \frac{1}{y}\right)^{15}$

Evaluate.

21. $5!$
22. $7!$
23. $\frac{12!}{11!}$
24. $\frac{13!}{12!}$
25. $\frac{11!}{8!}$
26. $\frac{7!}{9!}$
27. $\frac{10!}{6!}$
28. $\frac{9!}{6!}$
29. $\frac{6!}{3!}$
30. $\binom{8}{5}$
31. $\binom{10}{6}$
32. $\frac{(n+1)!}{(n-1)!}$

In each expansion find only the term specified.

33. The fourth term in $(2x - 4)^7$.
34. The third term in $(4a + 3b)^{11}$.
35. The fifth term in $\left(\frac{1}{2}x - y\right)^{12}$.
36. The sixth term in $(3x - 2y)^{10}$.
37. The fifth term in $\left(\frac{1}{x} - 2\right)^9$.
38. The next to last term in $(a + 4b)^5$.
39. The middle term in $(x - 3y)^6$.
40. The middle term in $\left(2a + \frac{1}{2}b\right)^6$.
41. The term involving x^4 in $(3x + 4y)^7$.
42. The term involving x^6 in $(2x^2 - 1)^9$.
43. The term involving x^6 in $(2x^3 - 1)^9$.
44. The term involving x^8 in $\left(x^2 + \frac{1}{y}\right)^8$.
45. The term involving x^{12} in $\left(x^3 + \frac{1}{2}\right)^7$.
46. The term involving x^{-4} in $\left(y + \frac{1}{x^2}\right)^8$.
47. Evaluate $(1.3)^6$ to four decimal places by writing it as $(1 + 0.3)^6$ and using the binomial formula.
48. Using the method of Example 47, evaluate
    (a) $(3.4)^4$ (b) $(48)^5$ (*Hint:* $48 = 50 - 2$.)

12.6 COUNTING: PERMUTATIONS AND COMBINATIONS

How many arrangements can be made using the letters *a*, *b*, *c*, and *d* three at a time? One way to solve this problem is to enumerate all the possible arrangements. The tree diagram shown in Figure 2 is a graphic device that yields precisely what we need. The letters *a*, *b*, *c*, and *d* are listed at the top and represent the candidates for the first letter. The three branches emanating from these lead to the possible choices for the second letter, and so on. For example, the portion of the tree shown in Figure 3 illustrates the arrangements *bda* and *bdc*. In this way we determine that there are a total of 24 arrangements.

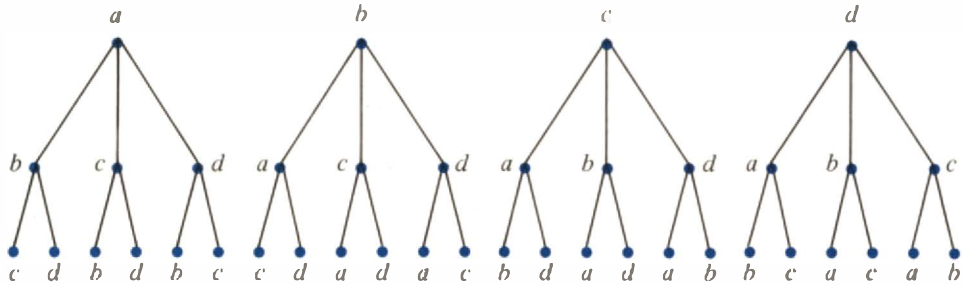


FIGURE 2

There is a more efficient way to solve this problem. Each arrangement consists of a choice of candidates to fill 3 positions in Figure 4. Any one of the 4 candidates *a, b, c,* or *d* can be assigned to the first position; once a candidate is

FIGURE 3

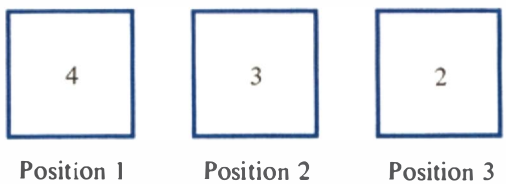


FIGURE 4

assigned to the first position, any one of the 3 remaining candidates can be assigned to the second position; finally, either one of the remaining 2 candidates can be assigned to the third position. Since each candidate for a position can be associated with any other candidate in the other positions, the *product*

$$4 \cdot 3 \cdot 2 = 24$$

yields the total number of arrangements. This simple example illustrates a very important principle.

Counting Principle	If one event can occur in m different ways and, after it has happened in one of these ways, a second event can occur in n different ways, then both events can occur in mn different ways.

Note that the order or sequence of events is significant since each arrangement is counted as one of the "mn different ways."

EXAMPLE 1
In how many ways can 5 students be seated in a row of 5 seats?

SOLUTION
We have 5 positions to be filled. Any one of the 5 students can occupy the first position, after which any one of the remaining 4 students can occupy the next position. Reapplying the counting principle to the other positions, we see that the number of arrangements is

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

PROGRESS CHECK
How many different 4-digit numbers can be formed using the digits 2, 4, 6, and 8? (Don't repeat any of the digits.)

ANSWER
24

EXAMPLE 2
How many different 3-letter arrangements can be made using the letters A, B, C, X, Y, and Z
(a) if no letter may be repeated in an arrangement, and
(b) if letters may be repeated?

SOLUTION
(a) We need to fill 3 positions. Any one of the 6 letters may occupy the first position; then, any one of the *remaining* 5 letters may occupy the second position (since repetitions are not allowed). Thus, the total number of arrangements is $6 \cdot 5 \cdot 4 = 120$.
(b) Any one of the 6 letters may fill any of the 3 positions (since repetitions are allowed). The total number of arrangements is $6 \cdot 6 \cdot 6 = 216$.

PROGRESS CHECK
The positions of president, secretary, and treasurer are to be filled from a class of 15 students. In how many ways can these positions be filled if no student may hold more than 1 position?

ANSWER
2730

PERMUTATIONS

Each arrangement that can be made by using all or some of the elements of a set of objects without repetition is called a **permutation.** The phrase "without repetition" means that no element of the set appears more than once. For example, the permutations of the letters a, b, and c taken 3 at a time include $b\ a\ c$ but exclude $a\ a\ b$.

We will use the notation $P(n, r)$ to indicate the number of permutations of n distinct objects taken r at a time. (There are a number of other notations in common use: nPr, P_r^n, ${}^nP_r$, $P_{n,r}$.) If $r = n$, then using the counting principle, we see that

$$P(n, n) = n(n-1)(n-2) \cdots 2 \cdot 1$$

since any one of the n objects may fill the first position, any one of the remaining $(n-1)$ objects may fill the second position, and so on. Using factorial notation,

$$\boxed{P(n, n) = n!}$$

Let's try to calculate $P(n, r)$, that is, the number of permutations of n distinct objects taken r at a time when r is less than n. We may think of this as the number of ways of filling r positions with n candidates. Once again, we may fill the first position with any one of the n candidates, the second position with any one of the remaining $(n-1)$ candidates, and so on, so

$$P(n, r) = \underbrace{n(n-1)(n-2) \cdots}_{r \text{ factors}}$$

We may write this as

$$P(n, r) = n(n-1)(n-2) \cdots (n-r+1) \tag{1}$$

since $(n-r+1)$ will be the rth factor. If we multiply the right-hand side of Equation (1) by

$$\frac{(n-r)!}{(n-r)!} = 1$$

we have

$$P(n, r) = \frac{n(n-1)(n-2) \cdots (n-r+1)(n-r)(n-r-1) \cdots 2 \cdot 1}{(n-r)!}$$

or

$$\boxed{P(n, r) = \frac{n!}{(n-r)!}}$$

EXAMPLE 3

Evaluate.

(a) $P(5, 5)$ (b) $P(5, 2)$ (c) $\dfrac{P(6, 2)}{3!}$

SOLUTION

(a) $P(5, 5) = \frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1} = 120$

(b) $P(5, 2) = \frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{5 \cdot 4 \cdot 3!}{3!} = 20$

(c) $\frac{P(6, 2)}{3!} = \frac{6!}{3!(6-2)!} = \frac{6!}{3!4!} = \frac{6 \cdot 5 \cdot 4!}{3 \cdot 2 \cdot 4!} = 5$

PROGRESS CHECK

Evaluate.

(a) $P(4, 4)$ (b) $P(6, 3)$ (c) $\frac{2\,P(6, 4)}{2!}$

ANSWERS

(a) 24 (b) 120 (c) 360

EXAMPLE 4

How many different arrangements can be made by taking 5 of the letters of the word *relation?*

SOLUTION

Since the word *relation* has 8 different letters, we are seeking the number of permutations of 8 objects taken 5 at a time or $P(8, 5)$. Thus,

$$P(n, r) = \frac{n!}{(n-r)!}$$

$$P(8, 5) = \frac{8!}{(8-5)!} = \frac{8!}{3!} = 6720$$

PROGRESS CHECK

There is space on a bookshelf for displaying 4 books. If there are 6 different novels available, how many arrangements can be made?

ANSWER

360

EXAMPLE 5

How many arrangements can be made using all the letters of the word *quartz* if the vowels are always to remain adjacent to each other?

SOLUTION

If we treat the vowel pair *ua* as a unit, then there are five "letters" (q, ua, r, t, z) that can be arranged in $P(5, 5)$ ways. But the vowels can themselves be arranged in $P(2, 2)$ ways. By the counting principle, the total number of arrangements is

$$P(5, 5) \cdot P(2, 2)$$

Since $P(5, 5) = 120$ and $P(2, 2) = 2$, the total number of arrangements is 240.

PROGRESS CHECK

A bookshelf is to be used to display 5 new textbooks. There are 7 mathematics textbooks and 4 biology textbooks available. If we wish to put 3 mathematics books and 2 biology books on display, how many arrangements can be made if the books in each discipline must be kept together?

ANSWER

5040

COMBINATIONS

Let's take another look at the arrangements of the letters a, b, and c taken 2 at a time:

$$ab \quad ba \quad ca$$

$$ac \quad bc \quad cb$$

Now let's ask a different question: In how many ways can we *select* 2 letters from the letters a, b, and c? In answering this question, we disregard the order in which the letters are chosen. The result is then

$$ab \quad ac \quad bc$$

In general, a set of r objects chosen from a set of n objects is called a **combination.** We denote the number of combinations of r objects chosen from n objects by $C(n, r)$. [Other notations in common use include nCr, C^n_r, ${}^nC_r$, $C_{n,r}$, and $\binom{n}{r}$.]

EXAMPLE 6

List the combinations of the letters a, b, c, and d taken 3 at a time.

SOLUTION

The combinations are seen to be

$$abc \quad abd \quad acd \quad bcd$$

Here is a rule that is helpful in determining whether a problem calls for the number of permutations or the number of combinations.

$P(n, r)$ or $C(n, r)$	If we are interested in calculating the number of arrangements, in which different orderings of the same objects are counted, we use permutations.
	If we are interested in calculating the number of ways of selecting objects, and the order of the selected objects does not matter, we use combinations.

For example, suppose we want to determine the number of different 4-card hands that can be dealt from a deck of 52 cards. Since a hand consisting of 4 cards is the same hand regardless of the order of the cards, we must use combinations.

Let's find a formula for $C(n, r)$. There are three combinations of the letters a, b, and c taken 2 at a time, namely

$$ab \qquad ac \qquad bc$$

so that $C(3, 2) = 3$. Now, each of these combinations can be arranged in $2!$ ways to yield the total list of permutations

$$ab \qquad ba \qquad ac \qquad ca \qquad bc \qquad cb$$

Thus, $P(3, 2) = 6 = 2!C(3, 2)$. In general, each of the $C(n, r)$ combinations can be permuted in $r!$ ways, so by the counting principle the total number of permutations is $P(n, r) = r!C(n, r)$ or

$$\boxed{C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}}$$

EXAMPLE 7

Evaluate.

(a) $C(5, 2)$ (b) $C(4, 4)$ (c) $\dfrac{P(6, 3)}{C(6, 3)}$

SOLUTION

(a) $C(5, 2) = \dfrac{5!}{2!(5-2)} = \dfrac{5!}{2!3!} = \dfrac{5 \cdot 4 \cdot 3!}{2 \cdot 3!} = 10$

(b) $C(4, 4) = \dfrac{4!}{4!(4-4)!} = \dfrac{4!}{4!0!} = 1$

(c) $P(6, 3) = \dfrac{6!}{(6-3)!} = \dfrac{6!}{3!} = 6 \cdot 5 \cdot 4 = 120$

$C(6, 3) = \dfrac{6!}{3!(6-3)!} = \dfrac{6!}{3!3!} = \dfrac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20$

$\dfrac{P(6, 3)}{C(6, 3)} = \dfrac{120}{20} = 6$

PROGRESS CHECK

Evaluate.

(a) $C(6, 2)$ (b) $C(10, 10)$ (c) $\dfrac{P(3, 2)}{3!C(5, 4)}$

ANSWERS

(a) 15 (b) 1 (c) $\dfrac{1}{5}$

EXAMPLE 8

In how many ways can a committee of 4 be selected from a group of 10 people?

SOLUTION

If A, B, C, and D constitute a committee, is the arrangement B, A, C, D a different committee? Of course not—the order does not matter. We are therefore interested in computing $C(10, 4)$:

$$C(10, 4) = \frac{10!}{4!6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210$$

PROGRESS CHECK

In how many ways can a 5-card hand be dealt from a deck of 52 cards?

ANSWER

2,598,960

EXAMPLE 9

In how many ways can a committee of 3 girls and 2 boys be selected from a class of 8 girls and 7 boys?

SOLUTION

The girls can be selected in $C(8, 3)$ ways, and the boys can be selected in $C(7, 2)$ ways. By the counting principle, each choice of boys can be associated with each choice of girls:

$$C(8, 3) \cdot C(7, 2) = \frac{8!}{3!5!} \cdot \frac{7!}{2!5!} = (56)(21) = 1176$$

PROGRESS CHECK

From 5 representatives of District A and 8 representatives of District B, in how many ways can 4 persons be chosen if only 1 representative from District A is to be included?

ANSWER

280

EXAMPLE 10

A bookstore has 12 French and 9 German books. In how many ways can a group of 6 books, consisting of 4 French and 2 German books, be placed on a shelf?

SOLUTION

The French books can be selected in $C(12, 4)$ ways and the German books in $C(9, 2)$ ways. The 6 books can then be selected in $C(12, 4) \cdot C(9, 2)$ ways. Each *selection* of 6 books can then be *arranged* on the shelf in $P(6, 6)$ ways, so the total number of arrangements is

$$C(12, 4) \cdot C(9, 2) \cdot P(6, 6) = \frac{12!}{4!8!} \cdot \frac{9!}{2!7!} \cdot \frac{6!}{(6-6)!} = 495 \cdot 36 \cdot 720 = 12{,}830{,}400$$

PROGRESS CHECK

From 6 different consonants and 4 different vowels, how many 5-letter words can be made consisting of 3 consonants and 2 vowels? (Assume every arrangement is a "word.")

ANSWER

14,400

EXERCISE SET 12.6

1. How many different 5-digit numbers can be formed using the digits 1, 3, 4, 6, and 8?
2. How many different ways are there to arrange the letters in the word *study*?
3. An employee identification number consists of 2 letters of the alphabet followed by a sequence of 3 digits selected from the digits 2, 3, 5, 6, 8, and 9. If repetitions are allowed, how many different identification numbers are possible?
4. In a psychological experiment, a subject has to arrange a cube, a square, a triangle, and a rhombus in a row. How many different arrangements are possible?
5. A coin is tossed 8 times and the result of each toss is recorded. How many different sequences of heads and tails are possible?
6. A die (from a pair of dice) is tossed 4 times, and the result of each toss is recorded. How many different sequences are possible?
7. A concert is to consist of 3 guitar pieces, 2 vocal numbers, and 2 jazz selections. In how many ways can the program be arranged?

In Exercises 8–19 evaluate the given expression.

8. $P(6, 6)$
9. $P(6, 5)$
10. $P(4, 2)$
11. $P(8, 3)$
12. $P(5, 2)$
13. $P(10, 2)$
14. $P(8, 4)$
15. $\dfrac{P(9, 3)}{3!}$
16. $\dfrac{4P(12, 3)}{2!}$
17. $P(3, 1)$
18. $\dfrac{P(7, 3)}{2!}$
19. $\dfrac{P(10, 4)}{4!}$
20. Find the number of ways in which 5 men and 5 women can be seated in a row
    (a) if any person may sit next to any other person;
    (b) if a man must always sit next to a woman.
21. Find the number of permutations of the letters in the word *money*.
22. Find the number of distinguishable permutations of the letters in the word *goose*. (*Hint:* Permutations in which the letters *o* and *o* exchange places are not distinguishable.)
23. Find the number of distinguishable permutations of the letters of the word *needed*. (See hint in Exercise 22.)
24. How many permutations of the letters *a*, *b*, *e*, *g*, *h*, *k*, and *m* are there when taken
    (a) 2 at a time?
    (b) 3 at a time?
25. How many 3-letter labels of new chemical products can be formed from the letters *a*, *b*, *c*, *d*, *f*, *g*, *l*, and *m*?

26. Find the number of distinguishable permutations that can be formed from the letters of the word *mississippi* taken 4 at a time.
27. A family consisting of a mother, a father, and 3 children is having a picture taken. If all 5 people are arranged in a row, how many different photographs can be taken?
28. List all the combinations of the numbers 4, 3, 5, 8, and 9 taken 3 at a time.

In Exercises 29–37 evaluate the given expression.

29. $C(9, 3)$
30. $C(7, 3)$
31. $C(10, 2)$
32. $C(7, 1)$
33. $C(7, 7)$
34. $C(5, 4)$
35. $C(n, n-1)$
36. $C(n, n-2)$
37. $C(n+1, n-1)$

38. In how many ways can a committee of 2 faculty members and 3 students be selected from 8 faculty members and 10 students?
39. In how many ways can a basketball team of 5 players be selected from among 15 candidates?
40. In how many ways can a 4-card hand be dealt from a deck of 52 cards?
41. How many three-letter moped plates can be formed for a local campus
    (a) if no letters can be repeated?
    (b) if letters can be repeated?
42. In a certain city each police car is staffed by 2 officers, 1 male and 1 female. A police captain, who needs to staff 8 cars, has 15 male officers and 12 female officers available. How many different teams can be formed?
43. How many different 10-card hands with 4 aces can be dealt from a deck of 52 cards?
44. A car manufacturer makes 3 different models, each of which is available in 5 different colors with 2 different engines. How many cars must a dealer stock in the showroom to display the full line?
45. A penny, a nickel, a dime, a quarter, a half-dollar, and a silver dollar are to be arranged in a row. How many different arrangements can be formed if the penny and dime must always be next to each other?
46. An automobile manufacturer that is planning an advertising campaign is considering 7 newspapers, 2 magazines, 3 radio stations, and 4 television stations. In how many ways can 5 advertisements be placed
    (a) if all 5 are to be in newspapers?
    (b) if 2 are to be in newspapers, 2 on radio, and 1 on television?
47. In a certain police station there are 12 prisoners and 10 police officers. How many possible lineups consisting of 4 prisoners and 3 officers can be formed?
48. The notation $\binom{n}{r}$ is often used in place of $C(n, r)$.

    Show that $\binom{n}{r} = \binom{n}{n-r}$.
49. How many different 10-card hands with 6 red cards and 4 black cards can be dealt from a deck of 52 cards?
50. A bin contains 12 transistors, 7 of which are defective. In how many ways can 4 transistors be chosen so that
    (a) all 4 are defective?
    (b) 2 are good and 2 are defective?
    (c) all 4 are good?
    (d) 3 are defective and 1 is good?

12.7 PROBABILITY

DEFINITION

There is a vast difference between the statements "It will probably rain today" and "It is equally probable that a tossed coin will come up heads or tails." The first statement conveys an expectation but only in a vague sense; the latter statement is much more useful because it quantifies the notion of probability.

Let's take a closer look at what happens when we toss a coin. The event has only 2 possible outcomes: heads and tails. Since heads represents 1 of 2 possible outcomes, we say that the probability of a head is 1/2. Thus, we can define probability in the following way: If an event can occur in a total of t ways and s of these are considered successful, the probability of success is s/t. In short,

$$\text{probability} = \frac{\text{number of successful outcomes}}{\text{total number of outcomes}}$$

EXAMPLE 1

A container holds 1 red ball, 2 white balls, and 2 blue balls. If 1 ball is drawn, what is the probability that it will be white?

SOLUTION

The selection of a ball represents a possible outcome, so there are a total of 5 possible outcomes. Since there are 2 ways of achieving a successful outcome (a white ball),

$$\begin{array}{c}\text{probability of}\\ \text{selecting a white ball}\end{array} = \frac{\text{number of successful outcomes}}{\text{total number of outcomes}} = \frac{2}{5}$$

PROGRESS CHECK

One card is drawn from an ordinary deck of 52 cards. What is the probability it is an ace?

ANSWER

$\frac{1}{13}$

EXAMPLE 2

A single die (whose faces contain the numbers 1, 2, 3, 4, 5, and 6) is tossed. What is the probability that the result is less than 5?

SOLUTION

There are 4 successful outcomes, occurring when the die shows a 1, 2, 3, or 4. Since there are 6 possible outcomes, we see that

$$\begin{aligned}\text{probability} &= \frac{\text{number of successful outcomes}}{\text{total number of outcomes}}\\ &= \frac{4}{6} = \frac{2}{3}\end{aligned}$$

PROGRESS CHECK

A bag of coins contains 4 nickels, 5 dimes, and 10 quarters. If 1 coin is withdrawn, what is the probability that it will be worth less than 25 cents?

ANSWER

$\frac{9}{19}$

PRINCIPLES OF PROBABILITY

Let's consider a bag containing 3 red marbles and 5 brown marbles. It is easy to verify that the probability of drawing a red marble in a single draw is 3/8 and that the probability of drawing a brown marble is $5/8 = 1 - 3/8$. What is the probability of drawing either a red marble or a brown marble? Since any of the 8 possible outcomes is considered a success, the probability is $8/8 = 1$. What is the probability of drawing a black marble? Since there are no successful outcomes, the probability is $0/8 = 0$. Generalizing these results, we can state the following principles:

- A probability of 1 indicates certainty.
- A probability of 0 indicates impossibility.
- If p is the probability that an event will happen, $1 - p$ is the probability that it will not happen.

EXAMPLE 3

While shuffling an ordinary deck of 52 cards, you drop 1 card. What is the probability that it is not a king?

SOLUTION

Since there are 4 kings in a deck, the probability of a king is $p = 4/52 = 1/13$. Then the probability that it is *not* a king is $1 - p = 12/13$.

PROGRESS CHECK

Two people throw a single die. If player A rolls a 4, what is the probability that player B will not also roll a 4?

ANSWER

$\frac{5}{6}$

APPLICATIONS

The rules for computing permutations and combinations are useful in solving probability problems.

EXAMPLE 4

A bag contains 3 green, 5 white, and 7 yellow balls. If 3 balls are drawn at random, what is the probability that they will all be white?

SOLUTION

We can select 3 white balls from 5 white balls in $C(5, 3)$ ways; we can select 3 balls from the bag of 15 balls in $C(15, 3)$ ways. Then

$$\text{probability of selecting three white balls} = \frac{C(5, 3)}{C(15, 3)} = \frac{10}{455} = \frac{2}{91}$$

PROGRESS CHECK

Three cards are drawn from an ordinary deck of 52 cards. What is the probability that they are all aces?

ANSWER

$\frac{1}{5525}$

Many problems in probability involve the tossing of a pair of dice. Since the faces of the dice contain the numbers 1, 2, 3, 4, 5, and 6, the sum of the numbers on the 2 dice can be any of the numbers 2 through 12. The outcomes, however, are not equally probable. In Table 1 we display the possible outcomes of tossing a pair of dice. In Table 2 we then summarize the number of ways in which each sum can be obtained. The probability of tossing a 3 with a pair of dice is therefore 2/36, or 1/18; the probability of tossing a 7 is 6/36, or 1/6.

TABLE 1

Die 1	Die 2					
	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

TABLE 2

Sum of 2 dice	2	3	4	5	6	7	8	9	10	11	12
Number of ways	1	2	3	4	5	6	5	4	3	2	1

EXAMPLE 5

What is the probability of throwing a 10 or higher with a single throw of a pair of dice?

SOLUTION

The favorable outcomes are 10, 11, and 12, which, by Table 2, can occur in a total of 6 ways. Then

$$\text{probability} = \frac{\text{number of successful outcomes}}{\text{total number of outcomes}}$$

$$= \frac{6}{36} = \frac{1}{6}$$

PROGRESS CHECK

What is the probability of throwing no higher than a 5 with a single throw of a pair of dice?

ANSWER

$\frac{5}{18}$

INDEPENDENT EVENTS

We conclude our introduction to probability by considering the probability of successive, independent events. For example, if a card is drawn from a deck of 52 cards, that card is replaced in the deck, and a second card is drawn, what is the probability that both cards will be aces? Note that these events are independent, since the second outcome in no way depends on the first outcome. Here is the principle that permits us to solve this type of problem:

> If p_1 is the probability that an event will occur and p_2 is the probability that a second, independent event will occur, then p_1p_2 is the probability that both events will occur.

In our example, the probability that the first card drawn will be an ace is $p_1 = 4/52 = 1/13$; the probability that the second card drawn will be an ace is also $1/13 = p_2$. Then the probability of drawing aces successively is $p_1p_2 = (1/13)(1/13) = 1/169$. Of course, we can extend this principle to more than two events by forming the product of the probabilities of the independent events.

EXAMPLE 6

What is the probability of throwing a 7 twice in succession with a pair of dice?

SOLUTION

From Table 2 we see that a 7 can occur in 6 ways, so the probability of throwing a

7 is $p_1 = 6/36 = 1/6$. The probability of throwing a 7 on the second roll is again $p_2 = 1/6$, so the probability of throwing a 7 on both rolls is $p_1p_2 = (1/6)(1/6) = 1/36$.

PROGRESS CHECK

What is the probability of throwing an 11 twice in succession with a pair of dice?

ANSWER

$\dfrac{1}{324}$

EXAMPLE 7

What is the probability of drawing an ace 3 times in succession from a deck of 52 cards if the drawn cards are not replaced?

SOLUTION

On the first draw, the probability of obtaining an ace is $p_1 = 4/52$. Since the ace is not replaced in the deck, there remain 3 aces and a total of 51 cards, so the probability of obtaining a second ace is $p_2 = 3/51$. Arguing the same way, we see that there now remain 2 aces and 50 cards, so the probability of drawing a third ace is $p_3 = 2/50$. Thus, the probability of drawing aces 3 times in succession without replacement is

$$p_1p_2p_3 = \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} = \frac{1}{5525}$$

PROGRESS CHECK

What is the probability of drawing a spade 3 times in succession from a deck of 52 cards if the drawn cards are not replaced?

ANSWER

$\dfrac{11}{850}$

EXAMPLE 8

What is the probability of throwing a 5 only on the first of 2 successive throws with a single die?

SOLUTION

The probability of throwing a 5 on the first toss is $p_1 = 1/6$. A success on the second toss consists of *not* throwing a 5 and has a probability $p_2 = 1 - p_1 = 5/6$. The probability of the desired result is

$$p_1p_2 = \frac{1}{6} \cdot \frac{5}{6} = \frac{5}{36}$$

EXAMPLE 9

A transistor manufacturer finds that 81 of every 1000 transistors made are defective. What is the probability that 2 transistors selected at random will both prove to be defective?

SOLUTION

This problem is an example of **empirical probability,** that is, probability obtained from experience or measurement rather than by theoretical means. The probability that a transistor is defective is $p_1 = 81/1000$. The probability that a second transistor is defective is $p_2 = 81/1000$. Thus, the probability that both transistors will be defective is

$$p_1p_2 = \frac{81}{1000} \cdot \frac{81}{1000} = \frac{6561}{1,000,000}$$

PROGRESS CHECK

The probability of rain in a certain town on any given day is 1/4. What is the probability of having a rainy Monday, a dry Tuesday, and a rainy Wednesday?

ANSWER

$$\frac{3}{64}$$

EXERCISE SET 12.7

1. If a single die is tossed, what is the probability that an odd number will appear?
2. If two dice are tossed, what is the probability of having at least one 5 showing on the top faces of the dice?
3. If a card is randomly selected from an ordinary deck of 52 cards, what is the probability that it will be
   (a) a red card? (b) a spade? (c) a king?
4. Suppose that 2 coins are tossed. What is the probability of having
   (a) both tails?
   (b) at least 1 heads?
   (c) neither tails?
   (d) 1 heads and 1 tails?
5. If 2 dice are tossed, what is the probability that
   (a) at least 1 of the dice will show a 4 on its top face?
   (b) the sum of the numbers on the dice will be 8?
   (c) neither a 3 nor a 4 will appear on the top face of a die?
6. If a card is picked at random from a standard deck of 52 cards, what is the probability that it will be
   (a) a club?
   (b) a 4?
   (c) not an ace?
   (d) a 4 of spades?
   (e) either an ace or a king?
   (f) neither an ace nor a king?
7. The quality control department of a calculator manufacturer determines that 1 percent of all calculators made are defective. What is the probability that a buyer of a calculator will get
   (a) a good calculator?
   (b) a defective calculator?
8. A photography club consisting of 18 women and 12 men wishes to elect a steering committee composed of 3 members. If every member is equally likely to be elected, find the probability that
   (a) all 3 members will be women?
   (b) none of the members will be women?

(c) exactly 1 member will be a woman?
(d) at least 1 member will be a woman?

9. A box contains 97 good bulbs and 5 defective bulbs. If 3 bulbs are chosen at random, what is the probability that
(a) all three bulbs will be defective?
(b) exactly one of the bulbs will be defective?
(c) none of the bulbs will be defective?

10. Suppose that 2 cards are to be drawn in succession from a deck of 52 cards. What is the probability that both cards will be aces if
(a) drawn cards are replaced?
(b) drawn cards are not replaced?

11. Suppose that 4 cards are selected, without replacement, from a deck of 52 cards. What is the probability that they are all hearts?

12. If the probability of getting an A in this course is 0.2, what is the probability of not getting an A?

13. The board of trustees of a university consists of 14 women and 12 men. Suppose that an executive committee of 6 persons is to be elected and that every trustee is equally likely to be elected. Find the probability that the committee will consist of 3 men and 3 women.

14. Suppose that the probability of a cloudy day in a certain town in England is 0.6.
(a) What is the probability of a clear day?
(b) What is the probability of 2 consecutive clear days?

15. A bag contains 6 blue marbles, 5 green marbles, and 7 yellow marbles. If 5 marbles are chosen without replacement, what is the probability that 2 will be blue, 2 will be green, and 1 will be yellow?

16. Suppose that 2 cards have been chosen at random from a deck of 52 cards. What is the probability that 1 card is an ace and the other card is not a king?

17. If 2 percent of cameras made on a production line are defective, what is the probability that 4 cameras chosen at random will all be
(a) good? (b) defective?

18. A fraternity consists of 12 seniors, 10 juniors, and 14 sophomores. A steering committee of 7 members is randomly chosen. What is the probability that it consists of 3 seniors, 2 juniors, and 2 sophomores?

TERMS AND SYMBOLS

infinite sequence (p. 504)
term of a sequence (p. 504)
recursive formula (p. 505)
sigma (Σ) notation (p. 506)
summation notation (p. 506)
index of summation (p. 506)
series (p. 508)
partial sum (p. 510)
arithmetic sequence (p. 512)
arithmetic progression (p. 512)
common difference (p. 512)
arithmetic series (p. 515)
geometric sequence (p. 518)
geometric progression (p. 518)
common ratio (p. 518)
geometric mean (p. 520)
geometric series (p. 521)
infinite geometric series (p. 523)
$0.537537\overline{537}$ (p. 525)
mathematical induction (p. 527)
expansion of $(a + b)^n$ (p. 531)
Pascal's triangle (p. 532)
binomial formula (p. 533)
$n!$ (p. 533)
factorial (p. 533)
$\binom{n}{r}$ (p. 535)
binomial coefficient (p. 535)
permutation (p. 539)
$P(n, r)$ (p. 540)
combination (p. 542)
$C(n, r)$ (p. 542)
probability (p. 547)
empirical probability (p. 552)

KEY IDEAS FOR REVIEW

☐ An infinite sequence is a function whose domain is restricted to the set of natural numbers. We generally write a sequence by using subscript notation; that is, a_n replaces $a(n)$.

☐ An infinite sequence is defined recursively if each term is defined by reference to preceding terms.

☐ Sigma (Σ) or summation notation is a handy means of indicating a sum of terms. The values written below and above the Σ indicate the starting and ending values, respectively, of the index of summation.

☐ An arithmetic sequence has a common difference d between terms. We can define an arithmetic sequence recursively by writing $a_n = a_{n-1} + d$ and specifying a_1.

☐ A geometric sequence has a common ratio r between terms. We can define a geometric sequence recursively by writing $a_n = ra_{n-1}$ and specifying a_1.

- ☐ The formulas for the nth terms of arithmetic and geometric sequences are

$$a_n = a_1 + (n-1)d \quad \text{Arithmetic}$$
$$a_n = a_1 r^{n-1} \quad \text{Geometric}$$

- ☐ A series is the sum of the terms of a sequence.
- ☐ The formulas for the sums S_n of the first n terms of arithmetic and geometric sequences are

$$S_n = \frac{n}{2}(a_1 + a_n) \quad \text{Arithmetic}$$

$$S_n = \frac{n}{2}[2a_1 + (n-1)d] \quad \text{Arithmetic}$$

$$S_n = \frac{a_1(1-r^n)}{1-r} \quad \text{Geometric}$$

- ☐ If the common ratio r satisfies $-1 < r < 1$, the infinite geometric series has the sum S given by the formula

$$S = \frac{a_1}{1-r}$$

where a_1 is the first term of the series.

- ☐ Mathematical induction is useful in proving certain types of theorems involving natural numbers.
- ☐ The notation $n!$ indicates the product of the natural numbers 1 through n:

$$n! = n(n-1)(n-2)\cdots 2\cdot 1 \quad \text{for } n \geq 1$$
$$0! = 1$$

- ☐ The binomial formula provides the terms of the expansion of $(a+b)^n$:

$$(a+b)^n = a^n + \frac{n!}{1!(n-1)!}a^{n-1}b + \frac{n!}{2!(n-2)!}a^{n-2}b^2 + \frac{n!}{3!(n-3)!}a^{n-3}b^3 + \cdots + \frac{n!}{r!(n-r)!}a^{n-r}b^r + \cdots + b^n$$

- ☐ The notation $\binom{n}{r}$ is defined by the formula

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

and is useful in writing out the binomial formula.

- ☐ Permutations involve arrangements or the order of objects; thus, *abc* and *bac* are distinct permutations of the letters a, b, and c.
- ☐ Combinations involve selection of objects; the order is not significant. If we are selecting 3 letters from a box containing the letters a, b, c, and d, then *abc* and *bac* are the same combination.
- ☐ The formulas for counting permutations and combinations of n things taken r at a time are

$$P(n, r) = \frac{n!}{(n-r)!} \quad \text{Permutations}$$

$$C(n, r) = \frac{n!}{r!(n-r)!} \quad \text{Combinations}$$

- ☐ Probability is a means of expressing the likelihood of the occurrence of an event. It is the ratio of the number of successful outcomes to the total number of outcomes.
- ☐ A probability of 1 indicates that an event is certain to occur, whereas a probability of 0 indicates that an event cannot possibly occur.
- ☐ If p_1 and p_2 are the probabilities of the occurrence of two independent events, then p_1p_2 is the probability that both events will occur.

REVIEW EXERCISES

Solutions to exercises whose numbers are in color are in the Solutions section in the back of the book.

12.1 In Exercises 1 and 2 write the first three terms and the tenth term of the sequence whose nth term is given.

1. $a_n = n^2 + n + 1$
2. $a_n = \dfrac{n^3 - 1}{n + 1}$

In Exercises 3 and 4 find the fifth term of the recursive sequence.

3. $a_n = n - a_{n-1}, \quad a_1 = 0$
4. $a_n = na_{n-1}, \quad a_1 = 1$

In Exercises 5–7 find the indicated sum.

5. $\sum_{k=1}^{4}(1-2k)$ 6. $\sum_{k=3}^{5}k(k+1)$

7. $\sum_{i=1}^{5}10$

In Exercises 8–10 express the sum in sigma notation.

8. $\frac{1}{3}+\frac{2}{4}+\frac{3}{5}+\frac{4}{6}$

9. $1-x+x^2-x^3+x^4$

10. $\log x+\log 2x+\log 3x+\cdots+\log nx$

12.2 In Exercises 11 and 12 find the specified term of the arithmetic sequence whose first term is a_1 and whose common difference is d.

11. $a_1=-2,\ d=2$; 21st term

12. $a_1=6,\ d=-1$; 16th term

In Exercises 13 and 14, given two terms of an arithmetic sequence, find the specified term.

13. $a_1=4,\ a_{16}=9$; 13th term

14. $a_1=-4,\ a_{23}=-15$; 26th term

In Exercises 15 and 16 find the sum of the first 25 terms of the arithmetic sequence whose first term is a_1 and whose common difference is d.

15. $a_1=-\frac{1}{3},\ d=\frac{1}{3}$ 16. $a_1=6,\ d=-2$

12.3 In Exercises 17 and 18 determine the common ratio of the given geometric sequence.

17. $2, -6, 18, -54, \ldots$

18. $-\frac{1}{2}, \frac{3}{4}, -\frac{9}{8}, \frac{27}{16}, \ldots$

In Exercises 19 and 20, write the first four terms of the geometric sequence whose first term is a_1 and whose common ratio is r.

19. $a_1=5,\quad r=\frac{1}{5}$ 20. $a_1=-2,\quad r=-1$

21. Find the sixth term of the geometric sequence $-4, 6, -9, \ldots$.

22. Find the eighth term of a geometric sequence for which $a_1=-2$ and $a_5=-32$.

23. Insert two geometric means between 3 and 1/72.

24. Find the sum of the first six terms of the geometric progression whose first three terms are $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{12}$.

25. Find the sum of the first six terms of the geometric progression for which $a_1=-2$ and $r=3$.

In Exercises 26 and 27 find the sum of the infinite geometric series.

26. $5+\frac{5}{2}+\frac{5}{4}+\cdots$ 27. $3-2+\frac{4}{3}-\cdots$

12.4 28. Use the principle of mathematical induction to show that

$$3+6+9+\cdots+3n=\frac{3n(n+1)}{2}$$

is true for all positive integer values of n.

12.5 In Exercises 29–31 expand and simplify.

29. $(2x-y)^4$ 30. $\left(\frac{x}{2}-2\right)^4$

31. $(x^2+1)^3$

In Exercises 32–37 evaluate the expression.

32. $6!$ 33. $\frac{13!}{11!2!}$

34. $\frac{(n-1)!(n+1)!}{n!n!}$ 35. $\binom{6}{4}$

36. $\binom{3}{0}$ 37. $\binom{10}{8}$

12.6 38. Four novels have been selected for display on a shelf. How many different arrangements are possible?

39. Find the number of distinguishable permutations of the letters in the word *soothe*.

40. In how many ways can a tennis team of 6 players be selected from 10 candidates?

41. In how many ways can a consonant and a vowel be chosen from the letters in the word *fouled*?

12.7 42. If 2 dice are tossed, what is the probability that the sum of the numbers on the dice will be 7 or 11?

43. A box contains 3 red pens and 4 white pens. If 2 pens are selected at random, what is the probability that they will both be white?

44. Two cards are drawn in succession from a deck of 52 cards. What is the probability that the cards

will be a king and an ace if the first card drawn is not replaced?

45. If 10 percent of the trees in a region are found to be diseased, what is the probability that 2 trees chosen at random are both free of disease?

46. Six husband-wife teams volunteer for an experiment in parapsychology. If 4 persons are selected at random to participate in the experiment, what is the probability that they will be two husband-wife teams?

PROGRESS TEST 12A

1. Write the first four terms of a sequence whose nth term is $a_n = n/(n+1)^2$.
2. Evaluate $\sum_{j=2}^{4} \frac{j}{j-1}$.
3. Write the first four terms of the arithmetic sequence whose first term is -1 and whose common difference is $\frac{3}{2}$.
4. Find the 25th term of the arithmetic sequence whose first term is -4 and whose common difference is $\frac{1}{2}$.
5. Find the 15th term of an arithmetic sequence whose first term is -1 and whose tenth term is 26.
6. Find the sum of the first 10 terms of an arithmetic sequence whose first term is -4 and whose ninth term is 8.
7. Find the common ratio of the geometric sequence 12, 4, $\frac{4}{3}$, $\frac{4}{9}$, ...
8. Write the first four terms of the geometric sequence whose first term is $-\frac{2}{3}$ and whose common ratio is 2.
9. Find the tenth term of the sequence 2, -2, 2, ...
10. Insert two geometric means between -4 and 32.
11. Find the sum of the first seven terms of the geometric sequence whose first three terms are -8, 4, -2.
12. Find the sum of the infinite geometric series $-4 - \frac{4}{3} - \frac{4}{9} - \dots$.
13. Use the principle of mathematical induction to show that $2 + 6 + 10 + \cdots + (4n - 2) = 2n^2$ is true for all positive integer values of n.
14. Find the first four terms in the expansion of $(a + 1/b)^{10}$.
15. Evaluate $12!/10!3!$.
16. Evaluate $P(6, 4)$.
17. Evaluate $C(n + 1, n)$.
18. Three buses arrive simultaneously at a terminal that has 4 parking stalls. In how many ways can the buses be parked?
19. The 1980 census staff divided a region into 3 districts that were to be canvassed by 10, 8, and 15 staff members, respectively. If 40 staff members were available, in how many ways could they have been assigned to the 3 districts?
20. The telephone company uses white, black, and green telephones, which are distributed to new customers at random. If an apartment dweller requests 2 telephones, what is the probability that they will be the same color?
21. Four marbles are removed at random from a box containing 4 purple, 3 blue, and 3 red marbles. What is the probability that these are 2 purple and 2 blue marbles?

PROGRESS TEST 12B

1. Write the first four terms of a sequence whose nth term is

$$a_n = n^2 + \frac{2n}{n+2}$$

2. Write the sum $2! + 3! + 4! + \cdots + n!$ in summation notation.
3. Write the first four terms of the arithmetic sequence whose first term is 6 and whose common difference is $-\frac{2}{3}$.
4. Find the sixth term of the arithmetic sequence whose first term is -5 and whose common difference is 3.
5. Find the thirtieth term of an arithmetic sequence

whose first and twentieth terms are 3 and -35, respectively.

6. The first term of an arithmetic series is -5, the last term is -2, and the sum is $-91/2$. Find the number of terms and the common difference.
7. Find the common ratio of the geometric sequence 20, 4, 0.8, 0.16.
8. Write the first four terms of the geometric sequence whose first term is -1 and whose common ratio is $-\frac{1}{4}$.
9. Find the sixth term of a geometric sequence for which $a_1 = 3$ and $a_4 = -\frac{1}{9}$.
10. Insert two geometric means between -6 and $-16/9$.
11. Find the sum of the first five terms of a geometric sequence if $a_1 = -8$ and $a_4 = -1$.
12. Find the sum of the infinite geometric progression $5 - 2 + \frac{4}{5} - \dots$.
13. Use the principle of mathematical induction to show that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

is true for all positive integer values of n.

14. Find the third term in the expansion of $(2x - 1)^{10}$.
15. Evaluate $n \cdot n!/(n + 1)!$.
16. Evaluate $P(7, 2)/6$.
17. Evaluate $P(5, 2)/C(5, 2)$.
18. Four plants are to be displayed on a window shelf. If 6 different plants are available, how many arrangements are possible?
19. A student-faculty committee of 6 members is to be set up, composed of 3 faculty members and 3 students. In how many ways can this be done if 5 faculty members and 6 students volunteer to serve on the committee?
20. A manufacturer finds that 2 percent of his products are defective. If 3 items are selected at random, what is the probability that they will all be defective?
21. What is the probability that a throw of two dice will result in a sum of 7 or greater?

TABLES APPENDIX

TABLE I Exponentials and Their Reciprocals

x	e^x	e^{-x}	x	e^x	e^{-x}
0.00	1.0000	1.0000	1.4	4.0552	0.2466
0.01	1.0101	0.9900	1.5	4.4817	0.2231
0.02	1.0202	0.9802	1.6	4.9530	0.2019
0.03	1.0305	0.9704	1.7	5.4739	0.1827
0.04	1.0408	0.9608	1.8	6.0496	0.1653
0.05	1.0513	0.9512	1.9	6.6859	0.1496
0.06	1.0618	0.9418	2.0	7.3891	0.1353
0.07	1.0725	0.9324	2.1	8.1662	0.1225
0.08	1.0833	0.9231	2.2	9.0250	0.1108
0.09	1.0942	0.9139	2.3	9.9742	0.1003
0.10	1.1052	0.9048	2.4	11.023	0.0907
0.11	1.1163	0.8958	2.5	12.182	0.0821
0.12	1.1275	0.8869	2.6	13.464	0.0743
0.13	1.1388	0.8781	2.7	14.880	0.0672
0.14	1.1503	0.8694	2.8	16.445	0.0608
0.15	1.1618	0.8607	2.9	18.174	0.0550
0.16	1.1735	0.8521	3.0	20.086	0.0498
0.17	1.1853	0.8437	3.1	22.198	0.0450
0.18	1.1972	0.8353	3.2	24.533	0.0408
0.19	1.2092	0.8270	3.3	27.113	0.0369
0.20	1.2214	0.8187	3.4	29.964	0.0334
0.21	1.2337	0.8106	3.5	33.115	0.0302
0.22	1.2461	0.8025	3.6	36.598	0.0273
0.23	1.2586	0.7945	3.7	40.447	0.0247
0.24	1.2712	0.7866	3.8	44.701	0.0224
0.25	1.2840	0.7788	3.9	49.402	0.0202
0.26	1.2969	0.7711	4.0	54.598	0.0183
0.27	1.3100	0.7634	4.1	60.340	0.0166
0.28	1.3231	0.7558	4.2	66.686	0.0150
0.29	1.3364	0.7483	4.3	73.700	0.0136
0.30	1.3499	0.7408	4.4	81.451	0.0123
0.35	1.4191	0.7047	4.5	90.017	0.0111
0.40	1.4918	0.6703	4.6	99.484	0.0101
0.45	1.5683	0.6376	4.7	109.95	0.0091
0.50	1.6487	0.6065	4.8	121.51	0.0082
0.55	1.7333	0.5769	4.9	134.29	0.0074
0.60	1.8221	0.5488	5	148.41	0.0067
0.65	1.9155	0.5220	6	403.43	0.0025
0.70	2.0138	0.4966	7	1,096.6	0.0009
0.75	2.1170	0.4724	8	2,981.0	0.0003
0.80	2.2255	0.4493	9	8,103.1	0.0001
0.85	2.3396	0.4274	10	22,026	0.00005
0.90	2.4596	0.4066	11	59,874	0.00002
0.95	2.5857	0.3867	12	162,754	0.000006
1.0	2.7183	0.3679	13	442,413	0.000002
1.1	3.0042	0.3329	14	1,202,604	0.0000008
1.2	3.3201	0.3012	15	3.269,017	0.0000003
1.3	3.6693	0.2725			

TABLE II Common Logarithms

N	0	1	2	3	4	5	6	7	8	9
1.0	.0000	.0043	.0086	.0128	.0170	.0212	.0253	.0294	.0334	.0374
1.1	.0414	.0453	.0492	.0531	.0569	.0607	.0645	.0682	.0719	.0755
1.2	.0792	.0828	.0864	.0899	.0934	.0969	.1004	.1038	.1072	.1106
1.3	.1139	.1173	.1206	.1239	.1271	.1303	.1335	.1367	.1399	.1430
1.4	.1461	.1492	.1523	.1553	.1584	.1614	.1644	.1673	.1703	.1732
1.5	.1761	.1790	.1818	.1847	.1875	.1903	.1931	.1959	.1987	.2014
1.6	.2041	.2068	.2095	.2122	.2148	.2175	.2201	.2227	.2253	.2279
1.7	.2304	.2330	.2355	.2380	.2405	.2430	.2455	.2480	.2504	.2529
1.8	.2553	.2577	.2601	.2625	.2648	.2672	.2695	.2718	.2742	.2765
1.9	.2788	.2810	.2833	.2856	.2878	.2900	.2923	.2945	.2967	.2989
2.0	.3010	.3032	.3054	.3075	.3096	.3118	.3139	.3160	.3181	.3201
2.1	.3222	.3243	.3263	.3284	.3304	.3324	.3345	.3365	.3385	.3404
2.2	.3424	.3444	.3464	.3483	.3502	.3522	.3541	.3560	.3579	.3598
2.3	.3617	.3636	.3655	.3674	.3692	.3711	.3729	.3747	.3766	.3784
2.4	.3802	.3820	.3838	.3856	.3874	.3892	.3909	.3927	.3945	.3692
2.5	.3979	.3997	.4014	.4031	.4048	.4065	.4082	.4099	.4116	.4133
2.6	.4150	.4166	.4183	.4200	.4216	.4232	.4249	.4265	.4281	.4298
2.7	.4314	.4330	.4346	.4362	.4378	.4393	.4409	.4425	.4440	.4456
2.8	.4472	.4487	.4502	.4518	.4533	.4548	.4564	.4579	.4594	.4609
2.9	.4624	.4639	.4654	.4669	.4683	.4698	.4713	.4728	.4742	.4757
3.0	.4771	.4786	.4800	.4814	.4829	.4843	.4857	.4871	.4886	.4900
3.1	.4914	.4928	.4942	.4955	.4969	.4983	.4997	.5011	.5024	.5038
3.2	.5051	.5065	.5079	.5092	.5105	.5119	.5132	.5145	.5159	.5172
3.3	.5185	.5198	.5211	.5224	.5237	.5250	.5263	.5276	.5289	.5302
3.4	.5315	.5328	.5340	.5353	.5366	.5378	.5391	.5403	.5416	.5428
3.5	.5441	.5453	.5465	.5478	.5490	.5502	.5514	.5527	.5539	.5551
3.6	.5563	.5575	.5587	.5599	.5611	.5623	.5635	.5647	.5658	.5670
3.7	.5682	.5694	.5705	.5717	.5729	.5740	.5752	.5763	.5775	.5786
3.8	.5798	.5809	.5821	.5832	.5843	.5855	.5866	.5877	.5888	.5899
3.9	.5911	.5922	.5933	.5944	.5955	.5966	.5977	.5988	.5999	.6010
4.0	.6021	.6031	.6042	.6053	.6064	.6075	.6085	.6096	.6107	.6117
4.1	.6128	.6138	.6149	.6160	.6170	.6180	.6191	.6201	.6212	.6222
4.2	.6232	.6243	.6253	.6263	.6274	.6284	.6294	.6304	.6314	.6325
4.3	.6335	.6345	.6355	.6365	.6375	.6385	.6395	.6405	.6415	.6425
4.4	.6435	.6444	.6454	.6464	.6474	.6484	.6493	.6503	.6513	.6522
4.5	.6532	.6542	.6551	.6561	.6571	.6580	.6590	.6599	.6609	.6618
4.6	.6628	.6637	.6646	.6656	.6665	.6675	.6684	.6693	.6702	.6712
4.7	.6721	.6730	.6739	.6749	.6758	.6767	.6776	.6785	.6794	.6803
4.8	.6812	.6821	.6830	.6839	.6848	.6857	.6866	.6875	.6884	.6893
4.9	.6902	.6911	.6920	.6928	.6937	.6946	.6955	.6964	.6972	.6981
5.0	.6990	.6998	.7007	.7016	.7024	.7033	.7042	.7050	.7059	.7067
5.1	.7076	.7084	.7093	.7101	.7110	.7118	.7126	.7135	.7143	.7152
5.2	.7160	.7168	.7177	.7185	.7193	.7202	.7210	.7218	.7226	.7235
5.3	.7243	.7251	.7259	.7267	.7275	.7284	.7292	.7300	.7308	.7316
5.4	.7324	.7332	.7340	.7348	.7356	.7364	.7372	.7380	.7388	.7396

TABLE II *(continued)*

N	0	1	2	3	4	5	6	7	8	9
5.5	.7404	.7412	.7419	.7427	.7435	.7443	.7451	.7459	.7466	.7474
5.6	.7482	.7490	.7497	.7505	.7513	.7520	.7528	.7536	.7543	.7551
5.7	.7559	.7566	.7574	.7582	.7589	.7597	.7604	.7612	.7619	.7627
5.8	.7634	.7642	.7649	.7657	.7664	.7672	.7679	.7686	.7694	.7701
5.9	.7709	.7716	.7723	.7731	.7738	.7745	.7752	.7760	.7767	.7774
6.0	.7782	.7789	.7796	.7803	.7810	.7818	.7825	.7832	.7839	.7846
6.1	.7853	.7860	.7868	.7875	.7882	.7889	.7896	.7903	.7910	.7917
6.2	.7924	.7931	.7938	.7945	.7952	.7959	.7966	.7973	.7980	.7987
6.3	.7993	.8000	.8007	.8014	.8021	.8028	.8035	.8041	.8048	.8055
6.4	.8062	.8069	.8075	.8082	.8089	.8096	.8102	.8109	.8116	.8122
6.5	.8129	.8136	.8142	.8149	.8156	.8162	.8169	.8176	.8182	.8189
6.6	.8195	.8202	.8209	.8215	.8222	.8228	.8235	.8241	.8248	.8254
6.7	.8261	.8267	.8274	.8280	.8287	.8293	.8299	.8306	.8312	.8319
6.8	.8325	.8331	.8338	.8344	.8351	.8357	.8363	.8370	.8376	.8382
6.9	.8388	.8395	.8401	.8407	.8414	.8420	.8426	.8432	.8439	.8445
7.0	.8451	.8457	.8463	.8470	.8476	.8482	.8488	.8494	.8500	.8506
7.1	.8513	.8519	.8525	.8531	.8537	.8543	.8549	.8555	.8561	.8567
7.2	.8573	.8579	.8585	.8591	.8597	.8603	.8609	.8615	.8621	.8627
7.3	.8633	.8639	.8645	.8651	.8657	.8663	.8669	.8675	.8681	.8686
7.4	.8692	.8698	.8704	.8710	.8716	.8722	.8727	.8733	.8739	.8745
7.5	.8751	.8756	.8762	.8768	.8774	.8779	.8785	.8791	.8797	.8802
7.6	.8808	.8814	.8820	.8825	.8831	.8837	.8842	.8848	.8854	.8859
7.7	.8865	.8871	.8876	.8882	.8887	.8893	.8899	.8904	.8910	.8915
7.8	.8921	.8927	.8932	.8938	.8943	.8949	.8954	.8960	.8965	.8971
7.9	.8976	.8982	.8987	.8993	.8998	.9004	.9009	.9015	.9020	.9025
8.0	.9031	.9036	.9042	.9047	.9053	.9058	.9063	.9069	.9074	.9079
8.1	.9085	.9090	.9096	.9101	.9106	.9112	.9117	.9122	.9128	.9133
8.2	.9138	.9143	.9149	.9154	.9159	.9165	.9170	.9175	.9180	.9186
8.3	.9191	.9196	.9201	.9206	.9212	.9217	.9222	.9227	.9232	.9238
8.4	.9243	.9248	.9253	.9258	.9263	.9269	.9274	.9279	.9284	.9289
8.5	.9294	.9299	.9304	.9309	.9315	.9320	.9325	.9330	.9335	.9340
8.6	.9345	.9350	.9355	.9360	.9365	.9370	.9375	.9380	.9385	.9390
8.7	.9395	.9400	.9405	.9410	.9415	.9420	.9425	.9430	.9435	.9440
8.8	.9445	.9450	.9455	.9460	.9465	.9469	.9474	.9479	.9484	.9489
8.9	.9494	.9499	.9504	.9509	.9513	.9518	.9523	.9528	.9533	.9538
9.0	.9542	.9547	.9552	.9557	.9562	.9566	.9571	.9576	.9581	.9586
9.1	.9590	.9595	.9600	.9605	.9609	.9614	.9619	.9624	.9628	.9633
9.2	.9638	.9643	.9647	.9652	.9657	.9661	.9666	.9671	.9675	.9680
9.3	.9685	.9689	.9694	.9699	.9703	.9708	.9713	.9717	.9722	.9727
9.4	.9731	.9736	.9741	.9745	.9750	.9754	.9759	.9763	.9768	.9773
9.5	.9777	.9782	.9786	.9791	.9795	.9800	.9805	.9809	.9814	.9818
9.6	.9823	.9827	.9832	.9836	.9841	.9845	.9850	.9854	.9859	.9863
9.7	.9868	.9872	.9877	.9881	.9886	.9890	.9894	.9899	.9903	.9908
9.8	.9912	.9917	.9921	.9926	.9930	.9934	.9939	.9943	.9948	.9952
9.9	.9956	.9961	.9965	.9969	.9974	.9978	.9983	.9987	.9991	.9996

TABLE III Natural Logarithms

N	ln N	N	ln N	N	ln N
		4.5	1.5041	9.0	2.1972
0.1	−2.3026	4.6	1.5261	9.1	2.2083
0.2	−1.6094	4.7	1.5476	9.2	2.2192
0.3	−1.2040	4.8	1.5686	9.3	2.2300
0.4	−0.9163	4.9	1.5892	9.4	2.2407
0.5	−0.6931	5.0	1.6094	9.5	2.2513
0.6	−0.5108	5.1	1.6292	9.6	2.2618
0.7	−0.3567	5.2	1.6487	9.7	2.2721
0.8	−0.2231	5.3	1.6677	9.8	2.2824
0.9	−0.1054	5.4	1.6864	9.9	2.2925
1.0	0.0000	5.5	1.7047	10	2.3026
1.1	0.0953	5.6	1.7228	11	2.3979
1.2	0.1823	5.7	1.7405	12	2.4849
1.3	0.2624	5.8	1.7579	13	2.5649
1.4	0.3365	5.9	1.7750	14	2.6391
1.5	0.4055	6.0	1.7918	15	2.7081
1.6	0.4700	6.1	1.8083	16	2.7726
1.7	0.5306	6.2	1.8245	17	2.8332
1.8	0.5878	6.3	1.8405	18	2.8904
1.9	0.6419	6.4	1.8563	19	2.9444
2.0	0.6931	6.5	1.8718	20	2.9957
2.1	0.7419	6.6	1.8871	25	3.2189
2.2	0.7885	6.7	1.9021	30	3.4012
2.3	0.8329	6.8	1.9169	35	3.5553
2.4	0.8755	6.9	1.9315	40	3.6889
2.5	0.9163	7.0	1.9459	45	3.8067
2.6	0.9555	7.1	1.9601	50	3.9120
2.7	0.9933	7.2	1.9741	55	4.0073
2.8	1.0296	7.3	1.9879	60	4.0943
2.9	1.0647	7.4	2.0015	65	4.1744
3.0	1.0986	7.5	2.0149	70	4.2485
3.1	1.1314	7.6	2.0281	75	4.3175
3.2	1.1632	7.7	2.0412	80	4.3820
3.3	1.1939	7.8	2.0541	85	4.4427
3.4	1.2238	7.9	2.0669	90	4.4998
3.5	1.2528	8.0	2.0794	95	4.5539
3.6	1.2809	8.1	2.0919	100	4.6052
3.7	1.3083	8.2	2.1041		
3.8	1.3350	8.3	2.1163		
3.9	1.3610	8.4	2.1282		
4.0	1.3863	8.5	2.1401		
4.1	1.4110	8.6	2.1518		
4.2	1.4351	8.7	2.1633		
4.3	1.4586	8.8	2.1748		
4.4	1.4816	8.9	2.1861		

$i = \frac{1}{2}\%$

n	$(1+i)^n$	n	$(1+i)^n$
1	1.0050 0000	51	1.2896 4194
2	1.0100 2500	52	1.2960 9015
3	1.0150 7513	53	1.3025 7060
4	1.0201 5050	54	1.3090 8346
5	1.0252 5125	55	1.3156 2887
6	1.0303 7751	56	1.3222 0702
7	1.0355 2940	57	1.3288 1805
8	1.0407 0704	58	1.3354 6214
9	1.0459 1058	59	1.3421 3946
10	1.0511 4013	60	1.3488 5015
11	1.0563 9583	61	1.3555 9440
12	1.0616 7781	62	1.3623 7238
13	1.0669 8620	63	1.3691 8424
14	1.0723 2113	64	1.3760 3016
15	1.0776 8274	65	1.3829 1031
16	1.0830 7115	66	1.3898 2486
17	1.0884 8651	67	1.3967 7399
18	1.0939 2894	68	1.4037 5785
19	1.0993 9858	69	1.4107 7664
20	1.1048 9558	70	1.4178 3053
21	1.1104 2006	71	1.4249 1968
22	1.1159 7216	72	1.4320 4428
23	1.1215 5202	73	1.4392 0450
24	1.1271 5978	74	1.4464 0052
25	1.1327 9558	75	1.4536 3252
26	1.1384 5955	76	1.4609 0069
27	1.1441 5185	77	1.4682 0519
28	1.1498 7261	78	1.4755 4622
29	1.1556 2197	79	1.4829 2395
30	1.1614 0008	80	1.4903 3857
31	1.1672 0708	81	1.4977 9026
32	1.1730 4312	82	1.5052 7921
33	1.1789 0833	83	1.5128 0561
34	1.1848 0288	84	1.5203 6964
35	1.1907 2689	85	1.5279 7148
36	1.1966 8052	86	1.5356 1134
37	1.2026 6393	87	1.5432 8940
38	1.2086 7725	88	1.5510 0585
39	1.2147 2063	89	1.5587 6087
40	1.2207 9424	90	1.5665 5468
41	1.2268 9821	91	1.5743 8745
42	1.2330 3270	92	1.5822 5939
43	1.2391 9786	93	1.5901 7069
44	1.2453 9385	94	1.5981 2154
45	1.2516 2082	95	1.6061 1215
46	1.2578 7892	96	1.6141 4271
47	1.2641 6832	97	1.6222 1342
48	1.2704 8916	98	1.6303 2449
49	1.2768 4161	99	1.6384 7611
50	1.2832 2581	100	1.6466 6849

$i = 1\%$

n	$(1+i)^n$	n	$(1+i)^n$
1	1.0100 0000	51	1.6610 7814
2	1.0201 0000	52	1.6776 8892
3	1.0303 0100	53	1.6944 6581
4	1.0406 0401	54	1.7114 1047
5	1.0510 1005	55	1.7285 2457
6	1.0615 2015	56	1.7458 0982
7	1.0721 3535	57	1.7632 6792
8	1.0828 5671	58	1.7809 0060
9	1.0936 8527	59	1.7987 0960
10	1.1046 2213	60	1.8166 9670
11	1.1156 6835	61	1.8348 6367
12	1.1268 2503	62	1.8532 1230
13	1.1380 9328	63	1.8717 4443
14	1.1494 7421	64	1.8904 6187
15	1.1609 6896	65	1.9093 6649
16	1.1725 7864	66	1.9284 6015
17	1.1843 0443	67	1.9477 4475
18	1.1961 4748	68	1.9672 2220
19	1.2081 0895	69	1.9868 9442
20	1.2201 9004	70	2.0067 6337
21	1.2323 9194	71	2.0268 3100
22	1.2447 1586	72	2.0470 9931
23	1.2571 6302	73	2.0675 7031
24	1.2697 3465	74	2.0882 4601
25	1.2824 3200	75	2.1091 2847
26	1.2952 5631	76	2.1302 1975
27	1.3082 0888	77	2.1515 2195
28	1.3212 9097	78	2.1730 3717
29	1.3345 0388	79	2.1947 6754
30	1.3478 4892	80	2.2167 1522
31	1.3613 2740	81	2.2388 8237
32	1.3749 4068	82	2.2612 7119
33	1.3886 9009	83	2.2838 8390
34	1.4025 7699	84	2.3067 2274
35	1.4166 0276	85	2.3297 8997
36	1.4307 6878	86	2.3530 8787
37	1.4450 7647	87	2.3766 1875
38	1.4595 2724	88	2.4003 8494
39	1.4741 2251	89	2.4243 8879
40	1.4888 6373	90	2.4486 3267
41	1.5037 5237	91	2.4731 1900
42	1.5187 8989	92	2.4978 5019
43	1.5339 7779	93	2.5228 2869
44	1.5493 1757	94	2.5480 5698
45	1.5648 1075	95	2.5735 3755
46	1.5804 5885	96	2.5992 7293
47	1.5962 6344	97	2.6252 6565
48	1.6122 2608	98	2.6515 1831
49	1.6283 4834	99	2.6780 3349
50	1.6446 3182	100	2.7048 1383

$i = 1\frac{1}{2}\%$

n	$(1+i)^n$	n	$(1+i)^n$
1	1.0150 0000	51	2.1368 2106
2	1.0302 2500	52	2.1688 7337
3	1.0456 7838	53	2.2014 0647
4	1.0613 6355	54	2.2344 2757
5	1.0772 8400	55	2.2679 4398
6	1.0934 4326	56	2.3019 6314
7	1.1098 4491	57	2.3364 9259
8	1.1264 9259	58	2.3715 3998
9	1.1433 8998	59	2.4071 1308
10	1.1605 4083	60	2.4432 1978
11	1.1779 4894	61	2.4798 6807
12	1.1956 1817	62	2.5170 6609
13	1.2135 5244	63	2.5548 2208
14	1.2317 5573	64	2.5931 4442
15	1.2502 3207	65	2.6320 4158
16	1.2689 8555	66	2.6715 2221
17	1.2880 2033	67	2.7115 9504
18	1.3073 4064	68	2.7522 6896
19	1.3269 5075	69	2.7935 5300
20	1.3468 5501	70	2.8354 5629
21	1.3670 5783	71	2.8779 8814
22	1.3875 6370	72	2.9211 5796
23	1.4083 7715	73	2.9649 7533
24	1.4295 0281	74	3.0094 4996
25	1.4509 4535	75	3.0545 9171
26	1.4727 0953	76	3.1004 1059
27	1.4948 0018	77	3.1469 1674
28	1.5172 2218	78	3.1941 2050
29	1.5399 8051	79	3.2420 3230
30	1.5630 8022	80	3.2906 6279
31	1.5865 2642	81	3.3400 2273
32	1.6103 2432	82	3.3901 2307
33	1.6344 7918	83	3.4409 7492
34	1.6589 9637	84	3.4925 8954
35	1.6838 8132	85	3.5449 7838
36	1.7091 3954	86	3.5981 5306
37	1.7347 7663	87	3.6521 2535
38	1.7607 9828	88	3.7069 0723
39	1.7872 1025	89	3.7625 1084
40	1.8140 1841	90	3.8189 4851
41	1.8412 2868	91	3.8762 3273
42	1.8688 4712	92	3.9343 7622
43	1.8968 7982	93	3.9933 9187
44	1.9253 3302	94	4.0532 9275
45	1.9542 1301	95	4.1140 9214
46	1.9835 2621	96	4.1758 0352
47	2.0132 7910	97	4.2384 4057
48	2.0434 7829	98	4.3020 1718
49	2.0741 3046	99	4.3665 4744
50	2.1052 4242	100	4.4320 4565

$i = 2\%$

n	$(1+i)^n$	n	$(1+i)^n$
1	1.0200 0000	51	2.7454 1979
2	1.0404 0000	52	2.8003 2819
3	1.0612 0800	53	2.8563 3475
4	1.0824 3216	54	2.9134 6144
5	1.1040 8080	55	2.9717 3067
6	1.1261 6242	56	3.0311 6529
7	1.1486 8567	57	3.0917 8859
8	1.1716 5938	58	3.1536 2436
9	1.1950 9257	59	3.2166 9685
10	1.2189 9442	60	3.2810 3079
11	1.2433 7431	61	3.3466 5140
12	1.2682 4179	62	3.4135 8443
13	1.2936 0663	63	3.4818 5612
14	1.3194 7876	64	3.5514 9324
15	1.3458 6834	65	3.6225 2311
16	1.3727 8571	66	3.6949 7357
17	1.4002 4142	67	3.7688 7304
18	1.4282 4625	68	3.8442 5050
19	1.4568 1117	69	3.9211 3551
20	1.4859 4740	70	3.9995 5822
21	1.5156 6634	71	4.0795 4939
22	1.5459 7967	72	4.1611 4038
23	1.5768 9926	73	4.2443 6318
24	1.6084 3725	74	4.3292 5045
25	1.6406 0599	75	4.4158 3546
26	1.6734 1811	76	4.5041 5216
27	1.7068 8648	77	4.5942 3521
28	1.7410 2421	78	4.6861 1991
29	1.7758 4469	79	4.7798 4231
30	1.8113 6158	80	4.8754 3916
31	1.8475 8882	81	4.9729 4794
32	1.8845 4059	82	5.0724 0690
33	1.9222 3140	83	5.1738 5504
34	1.9606 7603	84	5.2773 3214
35	1.9998 8955	85	5.3828 7878
36	2.0398 8734	86	5.4905 3636
37	2.0806 8509	87	5.6003 4708
38	2.1222 9879	88	5.7123 5402
39	2.1647 4477	89	5.8266 0110
40	2.2080 3966	90	5.9431 3313
41	2.2522 0046	91	6.0619 9579
42	2.2972 4447	92	6.1832 3570
43	2.3431 8936	93	6.3069 0042
44	2.3900 5314	94	6.4330 3843
45	2.4378 5421	95	6.5616 9920
46	2.4866 1129	96	6.6929 3318
47	2.5363 4352	97	6.8267 9184
48	2.5870 7039	98	6.9633 2768
49	2.6388 1179	99	7.1025 9423
50	2.6915 8803	100	7.2446 4612

$i = 2\frac{1}{2}\%$

n	$(1+i)^n$	n	$(1+i)^n$
1	1.0250 0000	51	3.5230 3644
2	1.0506 2500	52	3.6111 1235
3	1.0768 9063	53	3.7013 9016
4	1.1038 1289	54	3.7939 2491
5	1.1314 0821	55	3.8887 7303
6	1.1596 9342	56	3.9859 9236
7	1.1886 8575	57	4.0856 4217
8	1.2184 0290	58	4.1877 8322
9	1.2488 6297	59	4.2924 7780
10	1.2800 8454	60	4.3997 8975
11	1.3120 8666	61	4.5097 8449
12	1.3448 8882	62	4.6225 2910
13	1.3785 1104	63	4.7380 9233
14	1.4129 7382	64	4.8565 4464
15	1.4482 9817	65	4.9779 5826
16	1.4845 0562	66	5.1024 0721
17	1.5216 1826	67	5.2299 6739
18	1.5596 5872	68	5.3607 1658
19	1.5986 5019	69	5.4947 3449
20	1.6386 1644	70	5.6321 0286
21	1.6795 8185	71	5.7729 0543
22	1.7215 7140	72	5.9172 2806
23	1.7646 1068	73	6.0651 5876
24	1.8087 2595	74	6.2167 8773
25	1.8539 4410	75	6.3722 0743
26	1.9002 9270	76	6.5315 1261
27	1.9478 0002	77	6.6948 0043
28	1.9964 9502	78	6.8621 7044
29	2.0464 0739	79	7.0337 2470
30	2.0975 6758	80	7.2095 6782
31	2.1500 0677	81	7.3898 0701
32	2.2037 5694	82	7.5745 5219
33	2.2588 5086	83	7.7639 1599
34	2.3153 2213	84	7.9580 1389
35	2.3732 0519	85	8.1569 6424
36	2.4325 3532	86	8.3608 8834
37	2.4933 4870	87	8.5699 1055
38	2.5556 8242	88	8.7841 5832
39	2.6195 7448	89	9.0037 6228
40	2.6850 6384	90	9.2288 5633
41	2.7521 9043	91	9.4595 7774
42	2.8209 9520	92	9.6960 6718
43	2.8915 2008	93	9.9384 6886
44	2.9638 0808	94	10.1869 3058
45	3.0379 0328	95	10.4416 0385
46	3.1138 5086	96	10.7026 4395
47	3.1916 9713	97	10.9702 1004
48	3.2714 8956	98	11.2444 6530
49	3.3532 7680	99	11.5255 7693
50	3.4371 0872	100	11.8137 1635

$i = 3\%$

n	$(1+i)^n$
1	1.0300 0000
2	1.0609 0000
3	1.0927 2700
4	1.1255 0881
5	1.1592 7407
6	1.1940 5230
7	1.2298 7387
8	1.2667 7008
9	1.3047 7318
10	1.3439 1638
11	1.3842 3387
12	1.4257 6089
13	1.4685 3371
14	1.5125 8972
15	1.5579 6742
16	1.6047 0644
17	1.6528 4763
18	1.7024 3306
19	1.7535 0605
20	1.8061 1123
21	1.8602 9457
22	1.9161 0341
23	1.9735 8651
24	2.0327 9411
25	2.0937 7793
26	2.1565 9127
27	2.2212 8901
28	2.2879 2768
29	2.3565 6551
30	2.4272 6247
31	2.5000 8035
32	2.5750 8276
33	2.6523 3524
34	2.7319 0530
35	2.8138 6245
36	2.8982 7833
37	2.9852 2668
38	3.0747 8348
39	3.1670 2698
40	3.2620 3779
41	3.3598 9893
42	3.4606 9589
43	3.5645 1677
44	3.6714 5227
45	3.7815 9584
46	3.8950 4372
47	4.0118 9503
48	4.1322 5188
49	4.2562 1944
50	4.3839 0602

n	$i = 4\%$ $(1+i)^n$	$i = 5\%$ $(1+i)^n$	$i = 6\%$ $(1+i)^n$	$i = 7\%$ $(1+i)^n$	$i = 8\%$ $(1+i)^n$
1	1.0400 0000	1.0500 0000	1.0600 0000	1.0700 0000	1.0800 0000
2	1.0816 0000	1.1025 0000	1.1236 0000	1.1449 0000	1.1664 0000
3	1.1248 6400	1.1576 2500	1.1910 1600	1.2250 4300	1.2597 1200
4	1.1698 5856	1.2155 0625	1.2624 7696	1.3107 9601	1.3604 8896
5	1.2166 5290	1.2762 8156	1.3382 2558	1.4025 5173	1.4693 2808
6	1.2653 1902	1.3400 9564	1.4185 1911	1.5007 3035	1.5868 7432
7	1.3159 3178	1.4071 0042	1.5036 3026	1.6057 8148	1.7138 2427
8	1.3685 6905	1.4774 5544	1.5938 4807	1.7181 8618	1.8509 3021
9	1.4233 1181	1.5513 2822	1.6894 7896	1.8384 5921	1.9990 0463
10	1.4802 4428	1.6288 9463	1.7908 4770	1.9671 5136	2.1589 2500
11	1.5394 5406	1.7103 3936	1.8982 9856	2.1048 5195	2.3316 3900
12	1.6010 3222	1.7958 5633	2.0121 9647	2.2521 9159	2.5181 7012
13	1.6650 7351	1.8856 4914	2.1329 2826	2.4098 4500	2.7196 2373
14	1.7316 7645	1.9799 3160	2.2609 0396	2.5785 3415	2.9371 9362
15	1.8009 4351	2.0789 2818	2.3965 5819	2.7590 3154	3.1721 6911
16	1.8729 8125	2.1828 7459	2.5403 5168	2.9521 6375	3.4259 4264
17	1.9479 0050	2.2920 1832	2.6927 7279	3.1588 1521	3.7000 1805
18	2.0258 1652	2.4066 1923	2.8543 3915	3.3799 3228	3.9960 1950
19	2.1068 4918	2.5269 5020	3.0255 9950	3.6165 2754	4.3157 0106
20	2.1911 2314	2.6532 9771	3.2071 3547	3.8696 8446	4.6609 5714
21	2.2787 6807	2.7859 6259	3.3995 6360	4.1405 6237	5.0338 3372
22	2.3699 1879	2.9252 6072	3.6035 3742	4.4304 0174	5.4365 4041
23	2.4647 1554	3.0715 2376	3.8197 4966	4.7405 2986	5.8714 6365
24	2.5633 0416	3.2250 9994	4.0489 3464	5.0723 6695	6.3411 8074
25	2.6658 3633	3.3863 5494	4.2918 7072	5.4274 3264	6.8484 7520
26	2.7724 6978	3.5556 7269	4.5493 8296	5.8073 5292	7.3963 5321
27	2.8833 6858	3.7334 5632	4.8223 4594	6.2138 6763	7.9880 6147
28	2.9987 0332	3.9201 2914	5.1116 8670	6.6488 3836	8.6271 0639
29	3.1186 5145	4.1161 3560	5.4183 8790	7.1142 5705	9.3172 7490
30	3.2433 9751	4.3219 4238	5.7434 9117	7.6122 5504	10.0626 5689
31	3.3731 3341	4.5380 3949	6.0881 0064	8.1451 1290	10.8676 6944
32	3.5080 5875	4.7649 4147	6.4533 8668	8.7152 7080	11.7370 8300
33	3.6483 8110	5.0031 8854	6.8405 8988	9.3253 3975	12.6760 4964
34	3.7943 1634	5.2533 4797	7.2510 2528	9.9781 1354	13.6901 3361
35	3.9460 8899	5.5160 1537	7.6860 8679	10.6765 8148	14.7853 4429
36	4.1039 3255	5.7918 1614	8.1472 5200	11.4239 4219	15.9681 7184
37	4.2680 8986	6.0814 0694	8.6360 8712	12.2236 1814	17.2456 2558
38	4.4388 1345	6.3854 7729	9.1542 5235	13.0792 7141	18.6252 7563
39	4.6163 6599	6.7047 5115	9.7035 0749	13.9948 2041	20.1152 9768
40	4.8010 2063	7.0399 8871	10.2857 1794	14.9744 5784	21.7245 2150
41	4.9930 6145	7.3919 8815	10.9028 6101	16.0226 6989	23.4624 8322
42	5.1927 8391	7.7615 8756	11.5570 3267	17.1442 5678	25.3394 8187
43	5.4004 9527	8.1496 6693	12.2504 5463	18.3443 5475	27.3666 4042
44	5.6165 1508	8.5571 5028	12.9854 8191	19.6284 5959	29.5559 7166
45	5.8411 7568	8.9850 0779	13.7646 1083	21.0024 5176	31.9204 4939
46	6.0748 2271	9.4342 5818	14.5904 8748	22.4726 2338	34.4740 8534
47	6.3178 1562	9.9059 7109	15.4659 1673	24.0457 0702	37.2320 1217
48	6.5705 2824	10.4012 6965	16.3938 7173	25.7289 0651	40.2105 7314
49	6.8333 4937	10.9213 3313	17.3775 0403	27.5299 2997	43.4274 1899
50	7.1066 8335	11.4673 9979	18.4201 5427	29.4570 2506	46.9016 1251

TABLE V Trigonometric Functions of Radians and Real Numbers[a]

t	$\sin t$	$\cos t$	$\tan t$	$\cot t$	$\sec t$	$\csc t$
.00	.0000	1.0000	.0000	—	1.000	—
.01	.0100	1.0000	.0100	99.997	1.000	100.00
.02	.0200	.9998	.0200	49.993	1.000	50.00
.03	.0300	.9996	.0300	33.323	1.000	33.34
.04	.0400	.9992	.0400	24.987	1.001	25.01
.05	.0500	.9988	.0500	19.983	1.001	20.01
.06	.0600	.9982	.0601	16.647	1.002	16.68
.07	.0699	.9976	.0701	14.262	1.002	14.30
.08	.0799	.9968	.0802	12.473	1.003	12.51
.09	.0899	.9960	.0902	11.081	1.004	11.13
.10	.0998	.9950	.1003	9.967	1.005	10.02
.11	.1098	.9940	.1104	9.054	1.006	9.109
.12	.1197	.9928	.1206	8.293	1.007	8.353
.13	.1296	.9916	.1307	7.649	1.009	7.714
.14	.1395	.9902	.1409	7.096	1.010	7.166
.15	.1494	.9888	.1511	6.617	1.011	6.692
.16	.1593	.9872	.1614	6.197	1.013	6.277
.17	.1692	.9856	.1717	5.826	1.015	5.911
.18	.1790	.9838	.1820	5.495	1.016	5.586
.19	.1889	.9820	.1923	5.200	1.018	5.295
.20	.1987	.9801	.2027	4.933	1.020	5.033
.21	.2085	.9780	.2131	4.692	1.022	4.797
.22	.2182	.9759	.2236	4.472	1.025	4.582
.23	.2280	.9737	.2341	4.271	1.027	4.386
.24	.2377	.9713	.2447	4.086	1.030	4.207
.25	.2474	.9689	.2553	3.916	1.032	4.042
.26	.2571	.9664	.2660	3.759	1.035	3.890
.27	.2667	.9638	.2768	3.613	1.038	3.749
.28	.2764	.9611	.2876	3.478	1.041	3.619
.29	.2860	.9582	.2984	3.351	1.044	3.497
.30	.2955	.9553	.3093	3.233	1.047	3.384
.31	.3051	.9523	.3203	3.122	1.050	3.278
.32	.3146	.9492	.3314	3.018	1.053	3.179
.33	.3240	.9460	.3425	2.920	1.057	3.086
.34	.3335	.9428	.3537	2.827	1.061	2.999
.35	.3429	.9394	.3650	2.740	1.065	2.916
.36	.3523	.9359	.3764	2.657	1.068	2.839
.37	.3616	.9323	.3879	2.578	1.073	2.765
.38	.3709	.9287	.3994	2.504	1.077	2.696
.39	.3802	.9249	.4111	2.433	1.081	2.630

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t	$\sin t$	$\cos t$	$\tan t$	$\cot t$	$\sec t$	$\csc t$
.40	.3894	.9211	.4228	2.365	1.086	2.568
.41	.3986	.9171	.4346	2.301	1.090	2.509
.42	.4078	.9131	.4466	2.239	1.095	2.452
.43	.4169	.9090	.4586	2.180	1.100	2.399
.44	.4259	.9048	.4708	2.124	1.105	2.348
.45	.4350	.9004	.4831	2.070	1.111	2.299
.46	.4439	.8961	.4954	2.018	1.116	2.253
.47	.4529	.8916	.5080	1.969	1.122	2.208
.48	.4618	.8870	.5206	1.921	1.127	2.166
.49	.4706	.8823	.5334	1.875	1.133	2.125
.50	.4794	.8776	.5463	1.830	1.139	2.086
.51	.4882	.8727	.5594	1.788	1.146	2.048
.52	.4969	.8678	.5726	1.747	1.152	2.013
.53	.5055	.8628	.5859	1.707	1.159	1.987
.54	.5141	.8577	.5994	1.668	1.166	1.945
.55	.5227	.8525	.6131	1.631	1.173	1.913
.56	.5312	.8473	.6269	1.595	1.180	1.883
.57	.5396	.8419	.6410	1.560	1.188	1.853
.58	.5480	.8365	.6552	1.526	1.196	1.825
.59	.5564	.8309	.6696	1.494	1.203	1.797
.60	.5646	.8253	.6841	1.462	1.212	1.771
.61	.5729	.8196	.6989	1.431	1.220	1.746
.62	.5810	.8139	.7139	1.401	1.229	1.721
.63	.5891	.8080	.7291	1.372	1.238	1.697
.64	.5972	.8021	.7445	1.343	1.247	1.674
.65	.6052	.7961	.7602	1.315	1.256	1.652
.66	.6131	.7900	.7761	1.288	1.266	1.631
.67	.6210	.7838	.7923	1.262	1.276	1.610
.68	.6288	.7776	.8087	1.237	1.286	1.590
.69	.6365	.7712	.8253	1.212	1.297	1.571
.70	.6442	.7648	.8423	1.187	1.307	1.552
.71	.6518	.7584	.8595	1.163	1.319	1.534
.72	.6594	.7518	.8771	1.140	1.330	1.517
.73	.6669	.7452	.8949	1.117	1.342	1.500
.74	.6743	.7385	.9131	1.095	1.354	1.483
.75	.6816	.7317	.9316	1.073	1.367	1.467
.76	.6889	.7248	.9505	1.052	1.380	1.452
.77	.6961	.7179	.9697	1.031	1.393	1.437
.78	.7033	.7109	.9893	1.011	1.407	1.422
.79	.7104	.7038	1.009	.9908	1.421	1.408

t	sin t	cos t	tan t	cot t	sec t	csc t
.80	.7174	.6967	1.030	.9712	1.435	1.394
.81	.7243	.6895	1.050	.9520	1.450	1.381
.82	.7311	.6822	1.072	.9331	1.466	1.368
.83	.7379	.6749	1.093	.9146	1.482	1.355
.84	.7446	.6675	1.116	.8964	1.498	1.343
.85	.7513	.6600	1.138	.8785	1.515	1.331
.86	.7578	.6524	1.162	.8609	1.533	1.320
.87	.7643	.6448	1.185	.8437	1.551	1.308
.88	.7707	.6372	1.210	.8267	1.569	1.297
.89	.7771	.6294	1.235	.8100	1.589	1.287
.90	.7833	.6216	1.260	.7936	1.609	1.277
.91	.7895	.6137	1.286	.7774	1.629	1.267
.92	.7956	.6058	1.313	.7615	1.651	1.257
.93	.8016	.5978	1.341	.7458	1.673	1.247
.94	.8076	.5898	1.369	.7303	1.696	1.238
.95	.8134	.5817	1.398	.7151	1.719	1.229
.96	.8192	.5735	1.428	.7001	1.744	1.221
.97	.8249	.5653	1.459	.6853	1.769	1.212
.98	.8305	.5570	1.491	.6707	1.795	1.204
.99	.8360	.5487	1.524	.6563	1.823	1.196
1.00	.8415	.5403	1.557	.6421	1.851	1.188
1.01	.8468	.5319	1.592	.6281	1.880	1.181
1.02	.8521	.5234	1.628	.6142	1.911	1.174
1.03	.8573	.5148	1.665	.6005	1.942	1.166
1.04	.8624	.5062	1.704	.5870	1.975	1.160
1.05	.8674	.4976	1.743	.5736	2.010	1.153
1.06	.8724	.4889	1.784	.5604	2.046	1.146
1.07	.8772	.4801	1.827	.5473	2.083	1.140
1.08	.8820	.4713	1.871	.5344	2.122	1.134
1.09	.8866	.4625	1.917	.5216	2.162	1.128
1.10	.8912	.4536	1.965	.5090	2.205	1.122
1.11	.8957	.4447	2.014	.4964	2.249	1.116
1.12	.9001	.4357	2.066	.4840	2.295	1.111
1.13	.9044	.4267	2.120	.4718	2.344	1.106
1.14	.9086	.4176	2.176	.4596	2.395	1.101
1.15	.9128	.4085	2.234	.4475	2.448	1.096
1.16	.9168	.3993	2.296	.4356	2.504	1.091
1.17	.9208	.3902	2.360	.4237	2.563	1.086
1.18	.9246	.3809	2.427	.4120	2.625	1.082
1.19	.9284	.3717	2.498	.4003	2.691	1.077

t	sin t	cos t	tan t	cot t	sec t	csc t
1.20	.9320	.3624	2.572	.3888	2.760	1.073
1.21	.9356	.3530	2.650	.3773	2.833	1.069
1.22	.9391	.3436	2.733	.3659	2.910	1.065
1.23	.9425	.3342	2.820	.3546	2.992	1.061
1.24	.9458	.3248	2.912	.3434	3.079	1.057
1.25	.9490	.3153	3.010	.3323	3.171	1.054
1.26	.9521	.3058	3.113	.3212	3.270	1.050
1.27	.9551	.2963	3.224	.3102	3.375	1.047
1.28	.9580	.2867	3.341	.2993	3.488	1.044
1.29	.9608	.2771	3.467	.2884	3.609	1.041
1.30	.9636	.2675	3.602	.2776	3.738	1.038
1.31	.9662	.2579	3.747	.2669	3.878	1.035
1.32	.9687	.2482	3.903	.2562	4.029	1.032
1.33	.9711	.2385	4.072	.2456	4.193	1.030
1.34	.9735	.2288	4.256	.2350	4.372	1.027
1.35	.9757	.2190	4.455	.2245	4.566	1.025
1.36	.9779	.2092	4.673	.2140	4.779	1.023
1.37	.9799	.1994	4.913	.2035	5.014	1.021
1.38	.9819	.1896	5.177	.1913	5.273	1.018
1.39	.9837	.1798	5.471	.1828	5.561	1.017
1.40	.9854	.1700	5.798	.1725	5.883	1.015
1.41	.9871	.1601	6.165	.1622	6.246	1.013
1.42	.9887	.1502	6.581	.1519	6.657	1.011
1.43	.9901	.1403	7.055	.1417	7.126	1.010
1.44	.9915	.1304	7.602	.1315	7.667	1.009
1.45	.9927	.1205	8.238	.1214	8.299	1.007
1.46	.9939	.1106	8.989	.1113	9.044	1.006
1.47	.9949	.1006	9.887	.1011	9.938	1.005
1.48	.9959	.0907	10.983	.0910	11.029	1.004
1.49	.9967	.0807	12.350	.0810	12.390	1.003
1.50	.9975	.0707	14.101	.0709	14.137	1.003
1.51	.9982	.0608	16.428	.0609	16.458	1.002
1.52	.9987	.0508	19.670	.0508	19.695	1.001
1.53	.9992	.0408	24.498	.0408	24.519	1.001
1.54	.9995	.0308	32.461	.0308	32.476	1.000
1.55	.9998	.0208	48.078	.0208	48.089	1.000
1.56	.9999	.0108	92.620	.0108	92.626	1.000
1.57	1.0000	.0008	1255.8	.0008	1255.8	1.000

TABLE VI Trigonometric Functions of Angles in Degrees[a]

t degrees	sin t	cos t	tan t	cot t	sec t	csc t	
0°00′	.0000	1.0000	.0000	—	1.000	—	**90°00′**
10	.0029	1.0000	.0029	343.8	1.000	343.8	50
20	.0058	1.0000	.0058	171.9	1.000	171.9	40
30	.0087	1.0000	.0087	114.6	1.000	114.6	30
40	.0116	.9999	.0116	85.94	1.000	85.95	20
50	.0145	.9999	.0145	68.75	1.000	68.76	10
1°00′	.0175	.9998	.0175	57.29	1.000	57.30	**89°00′**
10	.0204	.9998	.0204	49.10	1.000	49.11	50
20	.0233	.9997	.0233	42.96	1.000	42.98	40
30	.0262	.9997	.0262	38.19	1.000	38.20	30
40	.0291	.9996	.0291	34.37	1.000	34.38	20
50	.0320	.9995	.0320	31.24	1.001	31.26	10
2°00′	.0349	.9994	.0349	28.64	1.001	28.65	**88°00′**
10	.0378	.9993	.0378	26.43	1.001	26.45	50
20	.0407	.9992	.0407	24.54	1.001	24.56	40
30	.0436	.9990	.0437	22.90	1.001	22.93	30
40	.0465	.9989	.0466	21.47	1.001	21.49	20
50	.0494	.9988	.0495	20.21	1.001	20.23	10
3°00′	.0523	.9986	.0524	19.08	1.001	19.11	**87°00′**
10	.0552	.9985	.0553	18.07	1.002	18.10	50
20	.0581	.9983	.0582	17.17	1.002	17.20	40
30	.0610	.9981	.0612	16.35	1.002	16.38	30
40	.0640	.9980	.0641	15.60	1.002	15.64	20
50	.0669	.9978	.0670	14.92	1.002	14.96	10
4°00′	.0698	.9976	.0699	14.30	1.002	14.34	**86°00′**
10	.0727	.9974	.0729	13.73	1.003	13.76	50
20	.0756	.9971	.0758	13.20	1.003	13.23	40
30	.0785	.9969	.0787	12.71	1.003	12.75	30
40	.0814	.9967	.0816	12.25	1.003	12.29	20
50	.0843	.9964	.0846	11.83	1.004	11.87	10
5°00′	.0872	.9962	.0875	11.43	1.004	11.47	**85°00′**
10	.0901	.9959	.0904	11.06	1.004	11.10	50
20	.0929	.9957	.0934	10.71	1.004	10.76	40
30	.0958	.9954	.0963	10.39	1.005	10.43	30
40	.0987	.9951	.0992	10.08	1.005	10.13	20
50	.1016	.9948	.1022	9.788	1.005	9.839	10
6°00′	.1045	.9945	.1051	9.514	1.006	9.567	**84°00′**
10	.1074	.9942	.1080	9.255	1.006	9.309	50
20	.1103	.9939	.1110	9.010	1.006	9.065	40
30	.1132	.9936	.1139	8.777	1.006	8.834	30
40	.1161	.9932	.1169	8.556	1.007	8.614	20
50	.1190	.9929	.1198	8.345	1.007	8.405	10
7°00′	.1219	.9925	.1228	8.144	1.008	8.206	**83°00′**
	cos t	sin t	cot t	tan t	csc t	sec t	t degrees

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t degrees	sin *t*	cos *t*	tan *t*	cot *t*	sec *t*	csc *t*	
7°00′	.1219	.9925	.1228	8.144	1.008	8.206	**83°00′**
10	.1248	.9922	.1257	7.953	1.008	8.016	50
20	.1276	.9918	.1287	7.770	1.008	7.834	40
30	.1305	.9914	.1317	7.596	1.009	7.661	30
40	.1334	.9911	.1346	7.429	1.009	7.496	20
50	.1363	.9907	.1376	7.269	1.009	7.337	10
8°00′	.1392	.9903	.1405	7.115	1.010	7.185	**82°00′**
10	.1421	.9899	.1435	6.968	1.010	7.040	50
20	.1449	.9894	.1465	6.827	1.011	6.900	40
30	.1478	.9890	.1495	6.691	1.011	6.765	30
40	.1507	.9886	.1524	6.561	1.012	6.636	20
50	.1536	.9881	.1554	6.435	1.012	6.512	10
9°00′	.1564	.9877	.1584	6.314	1.012	6.392	**81°00′**
10	.1593	.9872	.1614	6.197	1.013	6.277	50
20	.1622	.9868	.1644	6.084	1.013	6.166	40
30	.1650	.9863	.1673	5.976	1.014	6.059	30
40	.1679	.9858	.1703	5.871	1.014	5.955	20
50	.1708	.9853	.1733	5.769	1.015	5.855	10
10°00′	.1736	.9848	.1763	5.671	1.015	5.759	**80°00′**
10	.1765	.9843	.1793	5.576	1.016	5.665	50
20	.1794	.9838	.1823	5.485	1.016	5.575	40
30	.1822	.9833	.1853	5.396	1.017	5.487	30
40	.1851	.9827	.1883	5.309	1.018	5.403	20
50	.1880	.9822	.1914	5.226	1.018	5.320	10
11°00′	.1908	.9816	.1944	5.145	1.019	5.241	**79°00′**
10	.1937	.9811	.1974	5.066	1.019	5.164	50
20	.1965	.9805	.2004	4.989	1.020	5.089	40
30	.1994	.9799	.2035	4.915	1.020	5.016	30
40	.2022	.9793	.2065	4.843	1.021	4.945	20
50	.2051	.9787	.2095	4.773	1.022	4.876	10
12°00′	.2079	.9781	.2126	4.705	1.022	4.810	**78°00′**
10	.2108	.9775	.2156	4.638	1.023	4.745	50
20	.2136	.9769	.2186	4.574	1.024	4.682	40
30	.2164	.9763	.2217	4.511	1.024	4.620	30
40	.2193	.9757	.2247	4.449	1.025	4.560	20
50	.2221	.9750	.2278	4.390	1.026	4.502	10
13°00′	.2250	.9744	.2309	4.331	1.026	4.445	**77°00′**
10	.2278	.9737	.2339	4.275	1.027	4.390	50
20	.2306	.9730	.2370	4.219	1.028	4.336	40
30	.2334	.9724	.2401	4.165	1.028	4.284	30
40	.2363	.9717	.2432	4.113	1.029	4.232	20
50	.2391	.9710	.2462	4.061	1.030	4.182	10
14°00′	.2419	.9703	.2493	4.011	1.031	4.134	**76°00′**
	cos *t*	sin *t*	cot *t*	tan *t*	csc *t*	sec *t*	*t* degrees

TABLE VI (*continued*)

t degrees	sin *t*	cos *t*	tan *t*	cot *t*	sec *t*	csc *t*	
14°00′	.2419	.9703	.2493	4.011	1.031	4.134	76°00′
10	.2447	.9696	.2524	3.962	1.031	4.086	50
20	.2476	.9689	.2555	3.914	1.032	4.039	40
30	.2504	.9681	.2586	3.867	1.033	3.994	30
40	.2532	.9674	.2617	3.821	1.034	3.950	20
50	.2560	.9667	.2648	3.776	1.034	3.906	10
15°00′	.2588	.9659	.2679	3.732	1.035	3.864	75°00′
10	.2616	.9652	.2711	3.689	1.036	3.822	50
20	.2644	.9644	.2742	3.647	1.037	3.782	40
30	.2672	.9636	.2773	3.606	1.038	3.742	30
40	.2700	.9628	.2805	3.566	1.039	3.703	20
50	.2728	.9621	.2836	3.526	1.039	3.665	10
16°00′	.2756	.9613	.2867	3.487	1.040	3.628	74°00′
10	.2784	.9605	.2899	3.450	1.041	3.592	50
20	.2812	.9596	.2931	3.412	1.042	3.556	40
30	.2840	.9588	.2962	3.376	1.043	3.521	30
40	.2868	.9580	.2994	3.340	1.044	3.487	20
50	.2896	.9572	.3026	3.305	1.045	3.453	10
17°00′	.2924	.9563	.3057	3.271	1.046	3.420	73°00′
10	.2952	.9555	.3089	3.237	1.047	3.388	50
20	.2979	.9546	.3121	3.204	1.048	3.356	40
30	.3007	.9537	.3153	3.172	1.049	3.326	30
40	.3035	.9528	.3185	3.140	1.049	3.295	20
50	.3062	.9520	.3217	3.108	1.050	3.265	10
18°00′	.3090	.9511	.3249	3.078	1.051	3.236	72°00′
10	.3118	.9502	.3281	3.047	1.052	3.207	50
20	.3145	.9492	.3314	3.018	1.053	3.179	40
30	.3173	.9483	.3346	2.989	1.054	3.152	30
40	.3201	.9474	.3378	2.960	1.056	3.124	20
50	.3228	.9465	.3411	2.932	1.057	3.098	10
19°00′	.3256	.9455	.3443	2.904	1.058	3.072	71°00′
10	.3283	.9446	.3476	2.877	1.059	3.046	50
20	.3311	.9436	.3508	2.850	1.060	3.021	40
30	.3338	.9426	.3541	2.824	1.061	2.996	30
40	.3365	.9417	.3574	2.798	1.062	2.971	20
50	.3393	.9407	.3607	2.773	1.063	2.947	10
20°00′	.3420	.9397	.3640	2.747	1.064	2.924	70°00′
10	.3448	.9387	.3673	2.723	1.065	2.901	50
20	.3475	.9377	.3706	2.699	1.066	2.878	40
30	.3502	.9367	.3739	2.675	1.068	2.855	30
40	.3529	.9356	.3772	2.651	1.069	2.833	20
50	.3557	.9346	.3805	2.628	1.070	2.812	10
21°00′	.3584	.9336	.3839	2.605	1.071	2.790	69°00′
	cos *t*	sin *t*	cot *t*	tan *t*	csc *t*	sec *t*	*t* degrees

t degrees	sin t	cos t	tan t	cot t	sec t	csc t	
21°00′	.3584	.9336	.3839	2.605	1.071	2.790	69°00′
10	.3611	.9325	.3872	2.583	1.072	2.769	50
20	.3638	.9315	.3906	2.560	1.074	2.749	40
30	.3665	.9304	.3939	2.539	1.075	2.729	30
40	.3692	.9293	.3973	2.517	1.076	2.709	20
50	.3719	.9283	.4006	2.496	1.077	2.689	10
22°00′	.3746	.9272	.4040	2.475	1.079	2.669	68°00′
10	.3773	.9261	.4074	2.455	1.080	2.650	50
20	.3800	.9250	.4108	2.434	1.081	2.632	40
30	.3827	.9239	.4142	2.414	1.082	2.613	30
40	.3854	.9228	.4176	2.394	1.084	2.596	20
50	.3881	.9216	.4210	2.375	1.085	2.577	10
23°00′	.3907	.9205	.4245	2.356	1.086	2.559	67°00′
10	.3934	.9194	.4279	2.337	1.088	2.542	50
20	.3961	.9182	.4314	2.318	1.089	2.525	40
30	.3987	.9171	.4348	2.300	1.090	2.508	30
40	.4014	.9159	.4383	2.282	1.092	2.491	20
50	.4041	.9147	.4417	2.264	1.093	2.475	10
24°00′	.4067	.9135	.4452	2.246	1.095	2.459	66°00′
10	.4094	.9124	.4487	2.229	1.096	2.443	50
20	.4120	.9112	.4522	2.211	1.097	2.427	40
30	.4147	.9100	.4557	2.194	1.099	2.411	30
40	.4173	.9088	.4592	2.177	1.100	2.396	20
50	.4200	.9075	.4628	2.161	1.102	2.381	10
25°00′	.4226	.9063	.4663	2.145	1.103	2.366	65°00′
10	.4253	.9051	.4699	2.128	1.105	2.352	50
20	.4279	.9038	.4734	2.112	1.106	2.337	40
30	.4305	.9026	.4770	2.097	1.108	2.323	30
40	.4331	.9013	.4806	2.081	1.109	2.309	20
50	.4358	.9001	.4841	2.066	1.111	2.295	10
26°00′	.4384	.8988	.4877	2.050	1.113	2.281	64°00′
10	.4410	.8975	.4913	2.035	1.114	2.268	50
20	.4436	.8962	.4950	2.020	1.116	2.254	40
30	.4462	.8949	.4986	2.006	1.117	2.241	30
40	.4488	.8936	.5022	1.991	1.119	2.228	20
50	.4514	.8923	.5059	1.977	1.121	2.215	10
27°00′	.4540	.8910	.5095	1.963	1.122	2.203	63°00′
10	.4566	.8897	.5132	1.949	1.124	2.190	50
20	.4592	.8884	.5169	1.935	1.126	2.178	40
30	.4617	.8870	.5206	1.921	1.127	2.166	30
40	.4643	.8857	.5243	1.907	1.129	2.154	20
50	.4669	.8843	.5280	1.894	1.131	2.142	10
28°00′	.4695	.8829	.5317	1.881	1.133	2.130	62°00′
	cos t	sin t	cot t	tan t	csc t	sec t	t degrees

TABLE VI (*continued*)

t degrees	sin t	cos t	tan t	cot t	sec t	csc t	
28°00′	.4695	.8829	.5317	1.881	1.133	2.130	62°00′
10	.4720	.8816	.5354	1.868	1.134	2.118	50
20	.4746	.8802	.5392	1.855	1.136	2.107	40
30	.4772	.8788	.5430	1.842	1.138	2.096	30
40	.4797	.8774	.5467	1.829	1.140	2.085	20
50	.4823	.8760	.5505	1.816	1.142	2.074	10
29°00′	.4848	.8746	.5543	1.804	1.143	2.063	61°00′
10	.4874	.8732	.5581	1.792	1.145	2.052	50
20	.4899	.8718	.5619	1.780	1.147	2.041	40
30	.4924	.8704	.5658	1.767	1.149	2.031	30
40	.4950	.8689	.5696	1.756	1.151	2.020	20
50	.4975	.8675	.5735	1.744	1.153	2.010	10
30°00′	.5000	.8660	.5774	1.732	1.155	2.000	60°00′
10	.5025	.8646	.5812	1.720	1.157	1.990	50
20	.5050	.8631	.5851	1.709	1.159	1.980	40
30	.5075	.8616	.5890	1.698	1.161	1.970	30
40	.5100	.8601	.5930	1.686	1.163	1.961	20
50	.5125	.8587	.5969	1.675	1.165	1.951	10
31°00′	.5150	.8572	.6009	1.664	1.167	1.942	59°00′
10	.5175	.8557	.6048	1.653	1.169	1.932	50
20	.5200	.8542	.6088	1.643	1.171	1.923	40
30	.5225	.8526	.6128	1.632	1.173	1.914	30
40	.5250	.8511	.6168	1.621	1.175	1.905	20
50	.5275	.8496	.6208	1.611	1.177	1.896	10
32°00′	.5299	.8480	.6249	1.600	1.179	1.887	58°00′
10	.5324	.8465	.6289	1.590	1.181	1.878	50
20	.5348	.8450	.6330	1.580	1.184	1.870	40
30	.5373	.8434	.6371	1.570	1.186	1.861	30
40	.5398	.8418	.6412	1.560	1.188	1.853	20
50	.5422	.8403	.6453	1.550	1.190	1.844	10
33°00′	.5446	.8387	.6494	1.540	1.192	1.836	57°00′
10	.5471	.8371	.6536	1.530	1.195	1.828	50
20	.5495	.8355	.6577	1.520	1.197	1.820	40
30	.5519	.8339	.6619	1.511	1.199	1.812	30
40	.5544	.8323	.6661	1.501	1.202	1.804	20
50	.5568	.8307	.6703	1.492	1.204	1.796	10
34°00′	.5592	.8290	.6745	1.483	1.206	1.788	56°00′
10	.5616	.8274	.6787	1.473	1.209	1.781	50
20	.5640	.8258	.6830	1.464	1.211	1.773	40
30	.5664	.8241	.6873	1.455	1.213	1.766	30
40	.5688	.8225	.6916	1.446	1.216	1.758	20
50	.5712	.8208	.6959	1.437	1.218	1.751	10
35°00′	.5736	.8192	.7002	1.428	1.221	1.743	55°00′
	cos t	sin t	cot t	tan t	csc t	sec t	t degrees

TABLE VI (*continued*)

t degrees	$\sin t$	$\cos t$	$\tan t$	$\cot t$	$\sec t$	$\csc t$	
35°00′	.5736	.8192	.7002	1.428	1.221	1.743	**55°00′**
10	.5760	.8175	.7046	1.419	1.223	1.736	50
20	.5783	.8158	.7089	1.411	1.226	1.729	40
30	.5807	.8141	.7133	1.402	1.228	1.722	30
40	.5831	.8124	.7177	1.393	1.231	1.715	20
50	.5854	.8107	.7221	1.385	1.233	1.708	10
36°00′	.5878	.8090	.7265	1.376	1.236	1.701	**54°00′**
10	.5901	.8073	.7310	1.368	1.239	1.695	50
20	.5925	.8056	.7355	1.360	1.241	1.688	40
30	.5948	.8039	.7400	1.351	1.244	1.681	30
40	.5972	.8021	.7445	1.343	1.247	1.675	20
50	.5995	.8004	.7490	1.335	1.249	1.668	10
37°00′	.6018	.7986	.7536	1.327	1.252	1.662	**53°00′**
10	.6041	.7969	.7581	1.319	1.255	1.655	50
20	.6065	.7951	.7627	1.311	1.258	1.649	40
30	.6088	.7934	.7673	1.303	1.260	1.643	30
40	.6111	.7916	.7720	1.295	1.263	1.636	20
50	.6134	.7898	.7766	1.288	1.266	1.630	10
38°00′	.6157	.7880	.7813	1.280	1.269	1.624	**52°00′**
10	.6180	.7862	.7860	1.272	1.272	1.618	50
20	.6202	.7844	.7907	1.265	1.275	1.612	40
30	.6225	.7826	.7954	1.257	1.278	1.606	30
40	.6248	.7808	.8002	1.250	1.281	1.601	20
50	.6271	.7790	.8050	1.242	1.284	1.595	10
39°00′	.6293	.7771	.8098	1.235	1.287	1.589	**51°00′**
10	.6316	.7753	.8146	1.228	1.290	1.583	50
20	.6338	.7735	.8195	1.220	1.293	1.578	40
30	.6361	.7716	.8243	1.213	1.296	1.572	30
40	.6383	.7698	.8292	1.206	1.299	1.567	20
50	.6406	.7679	.8342	1.199	1.302	1.561	10
40°00′	.6428	.7660	.8391	1.192	1.305	1.556	**50°00′**
10	.6450	.7642	.8441	1.185	1.309	1.550	50
20	.6472	.7623	.8491	1.178	1.312	1.545	40
30	.6494	.7604	.8541	1.171	1.315	1.540	30
40	.6517	.7585	.8591	1.164	1.318	1.535	20
50	.6539	.7566	.8642	1.157	1.322	1.529	10
41°00′	.6561	.7547	.8693	1.150	1.325	1.524	**49°00′**
10	.6583	.7528	.8744	1.144	1.328	1.519	50
20	.6604	.7509	.8796	1.137	1.332	1.514	40
30	.6626	.7490	.8847	1.130	1.335	1.509	30
40	.6648	.7470	.8899	1.124	1.339	1.504	20
50	.6670	.7451	.8952	1.117	1.342	1.499	10
42°00′	.6691	.7431	.9004	1.111	1.346	1.494	**48°00′**
	$\cos t$	$\sin t$	$\cot t$	$\tan t$	$\csc t$	$\sec t$	t degrees

TABLE VI (*continued*)

t degrees	sin t	cos t	tan t	cot t	sec t	csc t	
42°00′	.6691	.7431	.9004	1.111	1.346	1.494	48°00′
10	.6713	.7412	.9057	1.104	1.349	1.490	50
20	.6734	.7392	.9110	1.098	1.353	1.485	40
30	.6756	.7373	.9163	1.091	1.356	1.480	30
40	.6777	.7353	.9217	1.085	1.360	1.476	20
50	.6799	.7333	.9271	1.079	1.364	1.471	10
43°00′	.6820	.7314	.9325	1.072	1.367	1.466	47°00′
10	.6841	.7294	.9380	1.066	1.371	1.462	50
20	.6862	.7274	.9435	1.060	1.375	1.457	40
30	.6884	.7254	.9490	1.054	1.379	1.453	30
40	.6905	.7234	.9545	1.048	1.382	1.448	20
50	.6926	.7214	.9601	1.042	1.386	1.444	10
44°00′	.6947	.7193	.9657	1.036	1.390	1.440	46°00′
10	.6967	.7173	.9713	1.030	1.394	1.435	50
20	.6988	.7153	.9770	1.024	1.398	1.431	40
30	.7009	.7133	.9827	1.018	1.402	1.427	30
40	.7030	.7112	.9884	1.012	1.406	1.423	20
50	.7050	.7092	.9942	1.006	1.410	1.418	10
45°00′	.7071	.7071	1.0000	1.0000	1.414	1.414	45°00′
	cos t	sin t	cot t	tan t	csc t	sec t	t degrees

ANSWERS TO ODD-NUMBERED EXERCISES, AND TO REVIEW EXERCISES AND PROGRESS TESTS

CHAPTER 1

EXERCISE SET 1.1, page 8

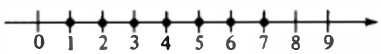
1. $\{3, 4, 5, 6, 7\}$ 3. $\{-9\}$ 5. $\{1, 2\}$
7. $\{1, 3, 7\}$ 9. F 11. F 13. T
15. T 17. T 19. F 21. F
23. commutative (addition) 25. distributive
27. associative (addition) 29. closure (multiplication)
31. commutative (multiplication) 33. commutative, associative (multiplication)
35. multiplicative inverse 41. symmetric 43. substitution

EXERCISE SET 1.2, page 14

1. 

3. $A{:}\,1,\ B{:}\,2.5,\ C{:}\,-2,\ D{:}\,4,\ O{:}\,0,\ E{:}\,-3.5$

5. 4 7. -2 9. -5

11. 

13. 

15. $10 > 9.99$

17. $a \geq 0$ 19. $x > 0$ 21. $b \leq -4$ 23. $b \geq 5$
25. multiplication by negative number 27. multiplication by negative number
29. multiplication by positive number
31. 2 33. 1.5 35. -2 37. 1
39. 4 41. 2 43. 1/5 45. 3
47. 2 49. 8/5

EXERCISE SET 1.3, page 21

1. 12   3. 8/3   5. 5/8   7. 13

9. (a) \$2160 (b) \$2080 (c) \$2106.67   11. 9.37

13. -9 15. 3/2   17. 0   19. b^7

21. $-20y^9$ 23. $-3x^4$ 25. c, d 27. 2; 3

29. 3/5; 4   31. 3   33. 4   35. 11

37. 176.20   39. $bh/2$ 41. cost of all purchases   43. $5x + 3$

45. $2s^2t^3 - 3s^2t^2 + 2s^2t + 3st^2 + st - s + 2t - 3$ 47. $-2a^2bc + ab^2c - 2ab^3 + 3$

49. $-(2x^4 - 4x^3 + x^2 - 4x + 4)$ 51. $6s^3 - s^2 - 11s + 6$

53. $-4y^5 - 2y^4 + 2y^3 - 5y^2 - 3y$ 55. $4a^5 - 16a^4 + 14a^3 - 3a^2 - 14a + 15$

57. $3b^4 + 3ab^3 + 2b^3 - 7ab^2 + 2b^2 - 4ab - 6a$

59. $-6x^2 + 22x - 12$ 61. $-260x + 13y + 17z$ 63. $x^2 + 2x - 3$

65. $4x^2 + 8x + 3$ 67. $3x^2 - 5x + 2$ 69. $x^2 + 2xy + y^2$

71. $9x^2 - 6x + 1$ 73. $4x^2 - 1$ 75. $x^4 + 2x^2y^2 + y^4$

EXERCISE SET 1.4, page 29

1. $5(x - 3)$ 3. $-2(x + 4y)$ 5. $5b(c + 5)$ 7. $-y^2(3 + 4y^3)$

9. $3x^2(1 + 2y - 3z)$ 11. $(x + 1)(x + 3)$ 13. $(y - 3)(y - 5)$ 15. $(a - 3b)(a - 4b)$

17. $(y - 1/3)(y + 1/3)$ 19. $(3 - x)(3 + x)$ 21. $(x - 7)(x + 2)$ 23. $(1/4 + y)(1/4 - y)$

25. $(x - 3)^2$ 27. $(x - 10)(x - 2)$ 29. $(x + 3)(x + 8)$ 31. $(2x + 1)(x - 2)$

33. $(3a - 2)(a - 3)$ 35. $(3x + 2)(2x + 3)$ 37. $(4m + 3)(2m - 3)$ 39. $(5x + 1)(2x - 3)$

41. $(3a + 2b)(2a - 3b)$ 43. $(5rs + 2t)(2rs + t)$ 45. $(4 + 3xy)(4 - 3xy)$ 47. $(4n + 1)(2n - 5)$

49. $2(x + 2)(x - 3)$ 51. $5(3x - 2)(2x - 1)$ 53. $3m(2x + 3)(3x + 1)$ 55. $2x^2(3 + 2x)(2 - 5x)$

57. $(x^2 + y^2)(x + y)(x - y)$ 59. $(b^2 + 4)(b^2 - 2)$ 61. $(x + 3y)(x^2 - 3xy + 9y^2)$

63. $(3x - y)(9x^2 + 3xy + y^2)$ 65. $(a + 2)(a^2 - 2a + 4)$ 67. $(\frac{1}{2}m - 2n)(\frac{1}{4}m^2 + mn + 4n^2)$

69. $(x + y - 2)(x^2 + 2xy + y^2 + 2x + 2y + 4)$ 71. $(2x^2 - 5y^2)(4x^4 + 10x^2y^2 + 25y^4)$

73. $4(y + 2)(x - 1)$ 75. $-(x + 2)^2(5x + 31)$

EXERCISE SET 1.5, page 37

1. $\dfrac{1}{x - 4}$ 3. $x - 4$ 5. $\dfrac{3x + 1}{x + 2}$

7. 4/9   9. $-2b(5 + a)$ 11. $\dfrac{5y}{x - 4}$

13. $\dfrac{(2x + 1)(x - 2)}{(x - 1)(x + 1)}$ 15. $\dfrac{(x + 2)(2x + 3)}{x + 4}$ 17. $\dfrac{(x + 3)(x^2 + 1)}{x - 2}$

19. $\dfrac{x + 4}{(x - 5)(x + 1)}$ 21. xy 23. $2a$

25. $(b - 1)^2$ 27. $(x - 2)(x + 3)$ 29. $x(x + 1)(x - 1)$

31. $\dfrac{4}{a - 2}$ 33. $\dfrac{x + 5}{3}$ 35. $\dfrac{4(a + 1)}{(a - 2)(a + 2)}$

37. $\dfrac{4y-15}{3xy}$ 39. $\dfrac{5-2x}{2(x+3)}$ 41. $\dfrac{23x+24}{6(x+3)(x-3)}$

43. $\dfrac{3x^2-4x-1}{(x-1)(x-2)(x+1)}$ 45. $\dfrac{5x-3}{(x+2)(x-1)}$

47. $\dfrac{x^2+x+3}{(x+1)(x+2)(x+3)}$ 49. $\dfrac{3x^2+10x+1}{(x+4)(x-1)(x+1)}$

51. $\dfrac{x+2}{x-3}$ 53. $\dfrac{x(x+1)}{x-1}$ 55. $4x(x+4)$

57. $\dfrac{a+2}{a+1}$ 59. $a-b$ 61. $\dfrac{x-2}{x}$

63. $\dfrac{(y-2)(y+1)}{(y+2)(y-1)}$ 65. $\dfrac{x+1}{2x+1}$

EXERCISE SET 1.6, page 41

1. x^6 3. b^4 5. $16x^4$ 7. $-1/128$
9. y^{8n} 11. $-x^3/y^3$ 13. x^{19} 15. $-32x^{10}$
17. x^{4n} 19. $1/x^2$ 21. $30x^8$ 23. 1
25. $(3/2)^n x^{2n} y^{3n}$ 27. $(2x+1)^{10}$ 29. $2^{2n}a^{4n}b^{6n}$ 31. 4/3
33. 3 35. 81 37. $-x^3$ 39. y^6
41. 25 43. $1/3^6$ 45. x^9 47. 32
49. $2x^2y$ 51. $a^4b^6/9$ 53. $-8y^{12}/x^9$ 55. $a^9/3b^4$
57. $4a^{10}c^6/b^8$ 59. $1/(a-2b^2)$ 61. $(a-b)^2/(a+b)$
63. $(b+a)/(b-a)$ 67. 0.074 69. 0.0113

EXERCISE SET 1.7, page 50

1. 8 3. 1/16 5. $2x^{13/12}$ 7. $x^{5/36}$
9. x^2y^{12} 11. x^9/y^6 13. $\sqrt[5]{1/16}$ 15. $\sqrt[4]{a^3}$
17. $\sqrt[3]{144x^6/y^4}$ 19. $8^{3/4}$ 21. $(-8)^{-2/5}$ 23. $(4a^3/9)^{-1/4}$
25. 2/3 27. not real 29. 5 31. 5/4
33. 54.82 35. 3, 4 37. $4\sqrt{3}$ 39. $3\sqrt[3]{2}$
41. $y^2\sqrt[3]{y}$ 43. $2x^2\sqrt[4]{6}\sqrt{x}$ 45. $x^2y\sqrt{xy}$ 47. $2x^2y\sqrt[4]{y}$
49. $\sqrt{5}/5$ 51. $\sqrt{3y}/3y$ 53. $2x\sqrt{2x}$ 55. $y^2\sqrt[3]{x^2y}$
57. $7\sqrt{3}$ 59. $7\sqrt{x}$ 61. $4\sqrt{3}$ 63. $11\sqrt{5}-\sqrt[3]{5}$
65. $-5\sqrt{5}$ 67. $3+4\sqrt{3}$ 69. $3xy$ 71. $5-2\sqrt{6}$
73. $3x-4y-\sqrt{6xy}$ 75. $3(3-\sqrt{2})/7$ 77. $2(4+\sqrt{3})/13$

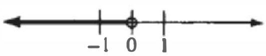
79. $\dfrac{-3(3\sqrt{a}-1)}{9a-1}$ 81. $\dfrac{-3(5-\sqrt{5y})}{5(5-y)}$ 83. $3+2\sqrt{2}$

85. $2+\sqrt{6}+3\sqrt{2}+2\sqrt{3}$ 87. 9, 16

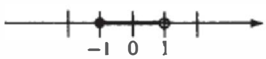
EXERCISE SET 1.8, page 55

1. 1
3. $-i$
5. $-i$
7. 1
9. i
11. $-1/2 + 0i$
13. $0 + 5i$
15. $0 - 6i$
17. $3 - 7i$
19. $0.3 - 7\sqrt{2}i$
21. $-2 - 4i$
23. $x = 2/3,\ y = -8$
25. $x = -1,\ y = -9/2$
27. $3 + i$
29. $5 + i$
31. $-5 - 4i$
33. $2 - 6i$
35. $-1 - i/2$
37. $5 + 0i$
39. $2 + 14i$
41. $4 - 7i$
43. 0
45. 3
53. $y \geq 5$

REVIEW EXERCISES, page 57

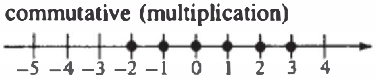
1. $\{1, 2, 3, 4\}$
2. $\{-3, -2, -1\}$
3. $\{2\}$
4. T
5. F
6. F
7. F
8. additive inverse
9. distributive
10. commutative (addition)
11. multiplicative identity
12. 

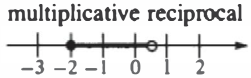
13. 

14. 

15. -1
16. 3/2
17. \$51
18. c
19. $-0.5, 7$
20. $-7, 5$
21. $a^2b^2 - 3a^2b + 4b$
22. $2x^3 + 3x^2 - 2x$
23. $12x^3 + 12x^2 + 3x$
24. $2(x + 1)(x - 1)$
25. $(x + 5y)(x - 5y)$
26. $(2a + 3b)(a + 3)$
27. $(4x - 1)(x + 5)$
28. $(x + 1)(x - 1)(x^2 + 1)(x^4 + 1)$
29. $(3r^2 + 2s^2)(9r^4 - 6r^2s^2 + 4s^4)$
30. $\dfrac{-6(y - 1)}{(x - y)xy^2}$
31. $\dfrac{-3(x + 2)}{2y}$
32. $\dfrac{a - 2b}{a - b}$
33. $\dfrac{3x(x + 1)}{2x - 1}$
34. $2x^2(x + 2)(x - 2)$
35. $x(x - 1)^2$
36. $5y^2(x - 1)^2$
37. $4x^2(y + 1)^2(y - 1)$
38. $\dfrac{2(a^2 - 2)}{(a + 2)(a - 2)}$
39. $\dfrac{-2x - 5}{(x + 4)(x - 4)}$
40. $\dfrac{x - 7}{(x - 1)^2(x + 2)}$
41. $\dfrac{x^3 - x^2 + 1}{x - 1}$
42. $b^9/8a^6$
43. 2
44. $1/x^4y^8$
45. x^3
46. $4\sqrt{5}$
47. $\sqrt{3}/3$
48. $x^3y^2\sqrt{xy}$
49. $2\sqrt[4]{2x^2y}\sqrt{y}$
50. $\dfrac{x - \sqrt{xy}}{x - y}$
51. $3\sqrt{|xy|}$
52. $8 + 2\sqrt{15}$
53. $x = -2,\ y = 4$
54. $-i$
55. $8 - i$
56. $3 + 4i$
57. $17 + 6i$

PROGRESS TEST 1A, page 59

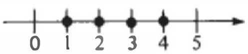
1. $\{2, 4, 6, 8, 10, 12\}$
2. $\{3\}$
3. F
4. F
5. commutative (multiplication)
6. multiplicative reciprocal
7. 

8. 

9. -1
10. 2
11. 25
12. $-7/3$

13. b 14. $-2.2, 5$ 15. $14, 6$ 16. $2xy + 3x + 4y + 1$
17. $3a^3 + 5a^2 + 3a + 10$ 18. $4a^2b(2ab^4 - 3a^3b + 4)$ 19. $(2 - 3x)(2 + 3x)$
20. $6m^5/n^2$ 21. $-(x - 1)/(x + 1)$ 22. $4x^2(x + 1)(x - 1)(x - 2)$
23. $(11x - 15)/3(x^2 - 9)$ 24. $2/(x + 1)$ 25. $1/x^{17}$ 26. y^{n+1}
27. -1 28. $4a^4/b^2$ 29. 0 30. $32 - 10\sqrt{7}$
31. $-11\sqrt{xy}/4$ 32. $x \leq 2$ 33. $-1 + 0i$ 34. $16 - 11i$

PROGRESS TEST 1B, page 60

1. $\{1, 3, 5, 7, 9\}$ 2. $\{0, 12, 15, 24\}$ 3. T 4. F
5. commutative (addition) 6. distributive
7.  8. 

9. 1 10. 7 11. 14 12. 2
13. a, d 14. 4, 5 15. 1.5, 10
16. $2s^2t^3 - 3s^2t^2 + 2s^2t + 3st^2 + st - s + 2t - 3$ 17. $-3b^3 - 7b^2 + 10b + 12$
18. $5r^3s^3(s - 8rt)$ 19. $(2x - 1)(x + 4)$ 20. $x^2/uv(y - 1)$ 21. $-1/x$
22. $4x^2(y + 1)^2(y - 1)$ 23. 0 24. $(-2x + 1)/x$ 25. $4/x$
26. b^{28} 27. x^6/y^9 28. -1 29. $x^7y^8\sqrt{y}$
30. $-2(\sqrt{x} + 1)/(x - 1)$ 31. $ab\sqrt[3]{b^2}$ 32. $x > 2$ 33. $2 - 3i/2$
34. $4 - 7i$

CHAPTER 2

EXERCISE SET 2.1, page 66

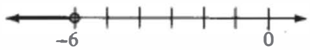
1. T 3. T 5. -2 7. $-2/3$
9. 6 11. $-4/3$ 13. 3/2 15. $-10/3$
17. -2 19. 5 21. $-7/2$ 23. 1
25. $8/(5 - k),\ k \neq 5$ 27. $(6 + k)/5$ 29. 10/3 31. 1
33. 4 35. 4 37. 12 39. 2
41. 12/7 43. none 45. 1 47. C
49. T 51. F 53. T

EXERCISE SET 2.2, page 74

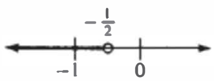
1. $2n + 3$ 3. $6n - 5 = 26$ 5. 16, 28 7. 6, 7, 8
9. 68° 11. 4 meters and 8 meters
13. 10 nickels, 25 dimes 15. 300 children, 400 adults
17. 61 three-dollar tickets, 40 five-dollar tickets, 20 six-dollar tickets
19. \$11,636.36 on 10-speeds, \$4363.64 on 3-speeds 21. \$7000
23. 20 hours 25. 50 and 54 mph 27. 40 kph, 80 kph
29. Ceylon: 2.4 ounces, Formosa: 5.6 ounces 31. 13.5 gal
33. 1/12, 1/4; $-1/4$, $-1/12$ 35. 12/5 hours 37. 9/2 days, 9 days
39. 8 hours 41. 140 mph 43. $C/2\pi$ 45. $5(F - 32)/9$
47. $-b + 2A/h$ 49. $ff_1/(f_1 - f)$ 51. $(a + Sr)/(r + S)$

EXERCISE SET 2.3, page 82

1. $[-5, 1)$
3. $(9, \infty)$
5. $[-12, -3]$
7. $(3, 7)$
9. $(-6, -4]$
11. $5 \le x \le 8$
13. $x > 3$
15. $x \le 5$
17. $x \ge 0$
19. $x < 4$
21. $x < -6$
23. $x \ge 5$



25. $a > -1$
27. $y < -1/2$
29. $x \ge 0$

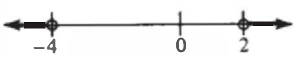


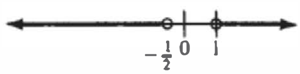
31. $r < 2$
33. $x \ge 1$
35. $x > 5/3$

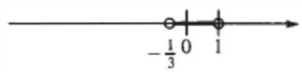
37. $(-\infty, 2]$
39. $[2, \infty)$
41. $(-\infty, 3/2)$
43. $(-12, \infty)$
45. $(-\infty, 9/2)$
47. $(-\infty, -6]$
49. $(-\infty, 3/2)$
51. $(-\infty, 7]$
53. $(-1/2, 5/4]$
55. $[-3, -2]$
57. $(-3, -1]$
59. $[-1, 2)$
61. 98
63. 5
65. 2924
67. $L \le 20$

EXERCISE SET 2.4, page 86

1. $1, -5$
3. $3, 1$
5. $2, -4/3$
7. $3/2, -3$
9. $4, -2$
11. $x < -4$ or $x > 2$
13. $x < -1/2$ or $x > 1$
15. $-1/3 < x < 1$







17. $(-\infty, -1], [7, \infty)$
19. $(-7/2, 9/2)$
21. no solution
23. $(-\infty, -7), (17, \infty)$
27. $|x - 100| \le 2$; $98 \le x \le 102$

EXERCISE SET 2.5, page 99

1. ± 3
3. $\pm\sqrt{5}$
5. $-5/2 \pm \sqrt{2}$
7. $5/3 \pm 2\sqrt{2}/3$
9. $\pm 8i/3$
11. $1, 2$
13. $1, -2$
15. $-2, -4$
17. $0, 4$
19. $1/2, 2$
21. ± 2
23. $1/3, 1/2$
25. $4, -2$
27. $-1/2, 4$
29. $1/3, -3$
31. $-1/2 \pm i/2$
33. $1, -3/4$
35. $-1/3 \pm \sqrt{2}i/3$
37. $0, -3/2$
39. $2/5 \pm \sqrt{11}i/5$
41. $2/5 \pm \sqrt{21}i/5$
43. $2/3, -1$
45. $\pm 2\sqrt{3}/3$
47. $0, -3/4$
49. $-1/2 \pm \sqrt{11}/2$
51. $-2, 2/3$
53. $-5/4 \pm \sqrt{7}i/4$
55. $\pm 1/2$
57. $-1/4 \pm \sqrt{11}i/4, 0$
59. $\pm\sqrt{c^2 - a^2}$
61. $\pm\sqrt{3V/\pi h}$
63. $(-v \pm \sqrt{v^2 + 2gs})/g$
65. two complex roots
67. double real root
69. two real roots
71. two real roots
73. two complex roots
75. two complex roots
77. two real roots
79. double real root

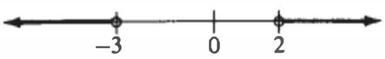
81. 4
83. 0, −8
85. 4
87. 3
89. 0, 4
91. 5
93. $u = x^2$; $\pm i\sqrt{2}$, $\pm \sqrt{3}/3$
95. $u = 1/x$; 2, −3/2
97. $u = x^{1/5}$; −32, −1/32
99. $u = 1 + 1/x$; 1/3, −2/7
103. −1/2
105. −6
107. ±9
109. 6

EXERCISE SET 2.6, page 103

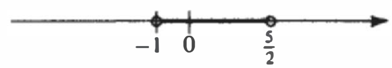
1. A: 3 hr; B: 6 hr
3. roofer: 6 hr; assistant: 12 hr
5. $L = 12$ ft, $W = 4$ ft
7. $L = 8$ cm, $W = 6$ cm
9. 10 ft
11. 5 or 1/5
13. 3, 7; −3, −7
15. 6, 8
17. 150 shares
19. 8 days

EXERCISE SET 2.7, page 107

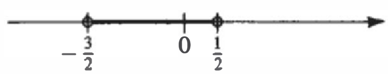
1. $x < -3$, $x > -2$
3. $-1/2 < x < 1$
5. $x < 0$, $x > 2$
7. $-5 \le x < -3$
9. $-1/2 \le r < 3$
11. $s \le -2/3$, $s > 1/2$
13. $-2 < x < 2/3$, $x > 1$
15. $x < -3$, $x > 2$



17. $-1 < x < 5/2$

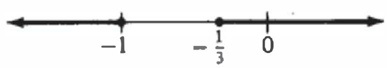


19. $-3/2 < r < 1/2$

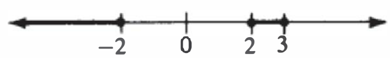


21. $x < -1$, $x \ge 1$

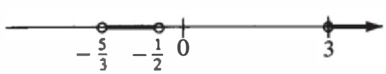
23. $x \le -1$, $x \ge -1/3$



25. $y \le -2$, $2 \le y \le 3$



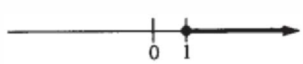
27. $-5/3 < x < -1/2$, $x > 3$



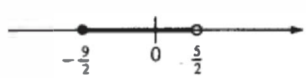
29. $x \le -1$, $x \ge 2$
31. $x \le -2$, $x \ge -3/2$
33. $0 \le x < 100$

REVIEW EXERCISES, page 109

1. 8/3
2. 0
3. 10/3
4. $k/2(2k + 1)$
5. 10/3 × 8/3
6. 5 quarters, 14 dimes
7. 240 mi
8. 6 hr
9. F
10. F
11. $x \ge 1$

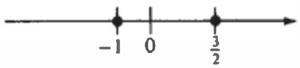


12. $-9/2 \le x < 5/2$

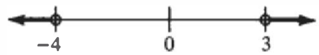


13. $(-\infty, 8)$
14. $(5/2, \infty)$
15. $[-9, \infty)$
16. 5/3, −3

17. $x = -1, x = 3/2$



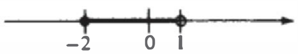
18. $x > 3$ or $x < -4$



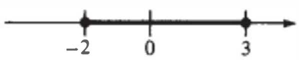
19. $(1/5, 3/5)$
20. $(-\infty, -4/3], [8/3, \infty)$
21. $5, -4$
22. $1/2, 4/3$
23. $1 \pm i\sqrt{5}$
24. $(2 \pm i\sqrt{2})/2$
25. $-1, 1/3$
26. $\pm 3/7$
27. $\pm\sqrt{3\pi k}/k$
28. $-4, 3$
29. two real roots
30. double root
31. two complex roots
32. 4
33. 6
34. $\pm 1, \pm 3$
35. $-1/2, -1$
36. 60
37. $x \le -3/2, x \ge 2$
38. $(-\infty, -5], [1, \infty)$
39. $(-\infty, -5), [-1/2, \infty)$
40. $(-2, -3/2), (3, \infty)$

PROGRESS TEST 2A, page 110

1. 3/4
2. 8/13
3. 18/5, 28/5, 29/5
4. \$6000 at 6.5%, \$6200 at 7.5%, \$12,300 at 9%
5. T
6. $-2 \le x < 1$



7. $(-\infty, 15/2]$
8. $[-4, 4]$
9. $5/2, -2$
10. $-2 \le x \le 3$



11. $(-4/3, 2)$
12. $-2, 7$
13. $(1 \pm i\sqrt{79})/10$
14. $-3/4, 1/3$
15. $(5 \pm 3i)/2$
16. $1, -3/2$
17. two real roots
18. two complex roots
19. -4
20. $\pm\sqrt{2}i, \pm\sqrt{3}/3$
21. 8×12 meters
22. $x \le 1/3, x \ge 1$
23. $(-\infty, 1/2], [1, \infty)$
24. $[-2, 2/3], [1, \infty)$

PROGRESS TEST 2B, page 111

1. -1
2. $k^2/(k + 3)$
3. 30 ounces 60%, 90 ounces 80%
4. $b(d - 1)/(a - cd)$
5. F
6. $1 \le x \le 2$

7. $(-\infty, 4/5]$
8. $(3, \infty)$
9. $8/3, -2$
10. $x \le 2, x \ge 6$

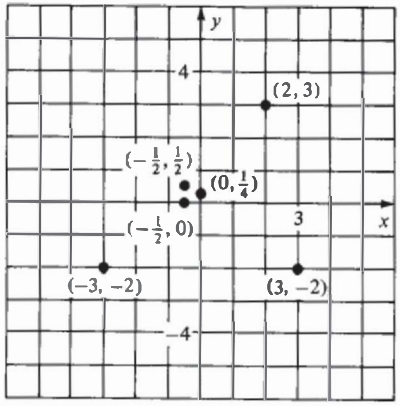
11. $(-\infty, -1), (1/5, \infty)$
12. $-5/2, 1/3$
13. $1/2, 2$
14. $(1 \pm i\sqrt{83})/6$
15. $3 \pm i\sqrt{2}$
16. $\pm\sqrt{k_2/(k_3 - k_1)}, k_3 \ne k_1$
17. two complex roots
18. double real root
19. 1

20. $\pm i$, $\pm(-8)^{3/2}$
21. 50¢
22. $x > 1/2$
23. $(-\infty, -5)$
24. $(-\infty, -4)$, $(2/3, 1)$

CHAPTER 3

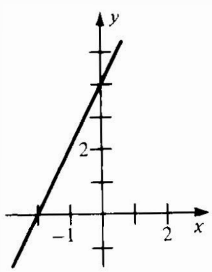
EXERCISE SET 3.1, page 123

1.

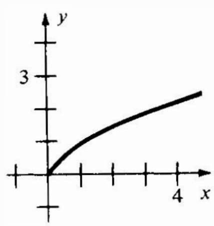


3. $3\sqrt{2}$
5. $4\sqrt{2}$
7. $\sqrt{1345/6}$
9. $\overline{BC} = \sqrt{37}$
11. $\overline{RS} = \sqrt{2/2}$
13. no
15. yes
21. $2\sqrt{10} + 7 + 5\sqrt{2} + \sqrt{37}$
25. any point satisfying $x^2 + y^2 - 10y - 6x + 29 = 0$

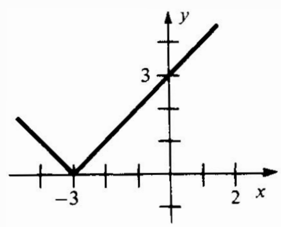
27. x-int.: -2
y-int.: 4



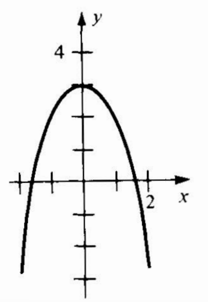
29. x-int.: 0
y-int.: 0



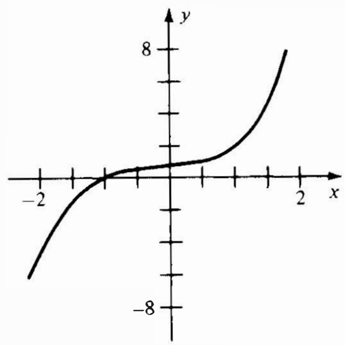
31. x-int.: -3
y-int.: 3



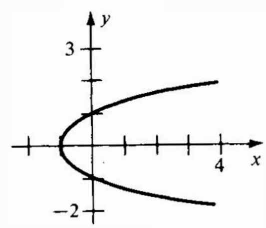
33. x-int.: $\pm\sqrt{3}$
y-int.: 3



35. x-int.: -1
y-int.: 1



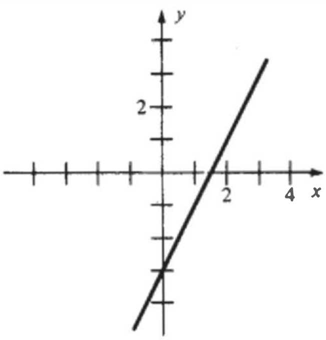
37. x-int.: -1
y-int.: ± 1



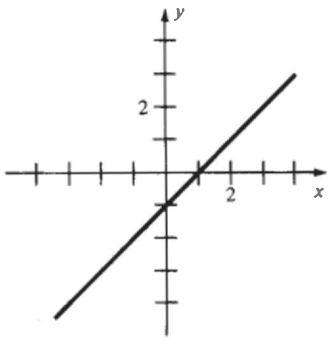
39. none 41. x-axis 43. x-axis 45. x-axis
47. none 49. y-axis 51. all 53. origin

EXERCISE SET 3.2, page 129

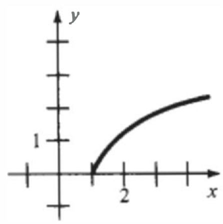
1. domain: all reals
   range: all reals



3. domain: all reals
   range: all reals



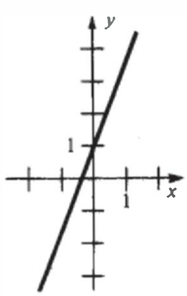
5. domain: $x \geq 1$
   range: $y \geq 0$



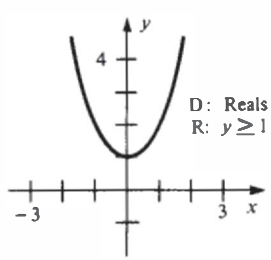
7. $x \geq 3/2$ 9. $x > 2$ 11. $x \geq 1, x \neq 2$
13. 7/2 15. 3/2 17. 5
19. $2a^2 + 5$ 21. $6x^2 + 15$ 23. 3
25. $1/(x^2 + 2x)$ 27. $a^2 + h^2 + 2ah + 2a + 2h$ 29. -0.92
31. $(3x - 1)/(x^2 + 1)$ 33. $2(4x^2 + 1)/6x - 1)$ 35. -0.21
37. $2(a - 1)/(4a^2 + 4a - 3)$ 39. $(a - 1)/a(a + 4)$ 41. $I(x) = 0.28x$
43. $d(C) = C/\pi$

EXERCISE SET 3.3, page 139

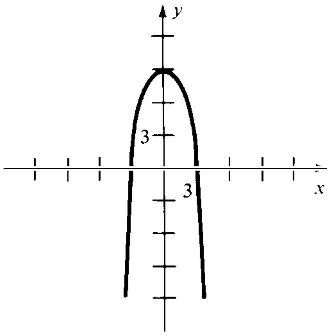
1. increasing: $(-\infty, \infty)$



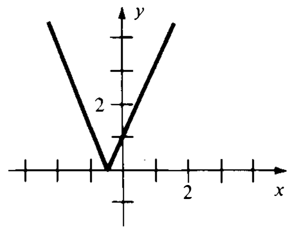
3. increasing: $x \geq 0$
   decreasing: $x \leq 0$



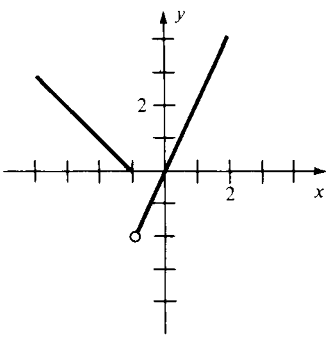
5. increasing: $x \leq 0$
decreasing: $x \geq 0$



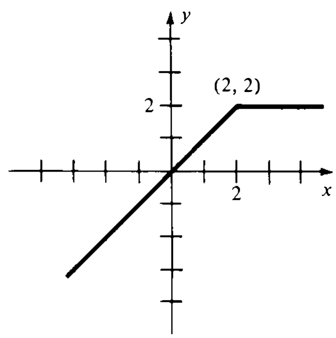
7. increasing: $x \geq -1/2$
decreasing: $x \leq -1/2$



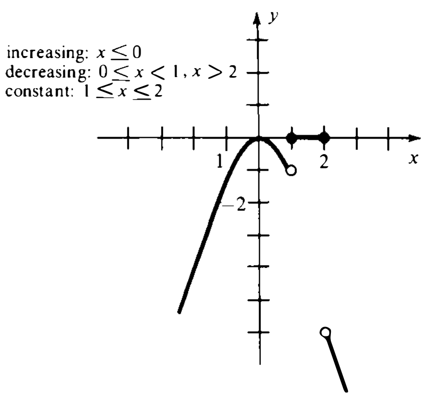
9. increasing: $x > -1$
decreasing: $x < -1$



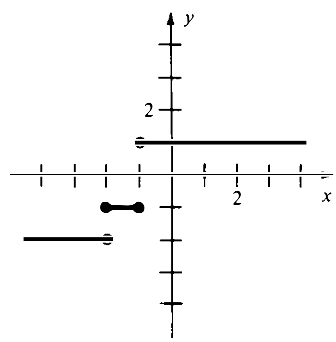
11. increasing: $x \leq 2$
constant: $x \geq 2$



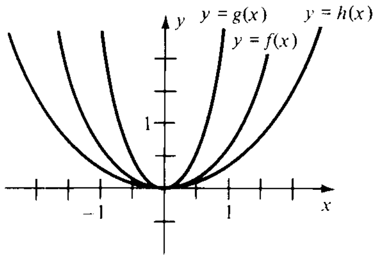
13. increasing: $x \leq 0$
decreasing: $0 \leq x < 1$, $x > 2$
constant: $1 \leq x \leq 2$



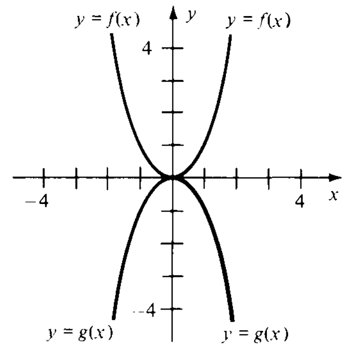
15. constant: $x < -2$, $-2 \leq x \leq -1$, $x > -1$



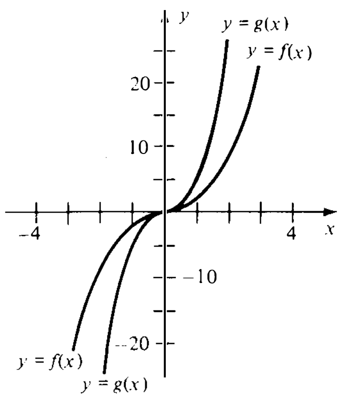
17.



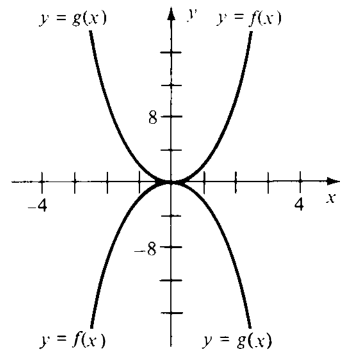
19.



21.



23.



25. $C(u) = \begin{cases} 6.50, & 0 \le u \le 100 \\ 6.50 + 0.06(u - 100), & 100 < u \le 200 \\ 12.50 + 0.05(u - 200), & u > 200 \end{cases}$

27. $R(x) = \begin{cases} 30{,}000, & 0 \le x \le 100 \\ 400x - x^2, & x > 100 \end{cases}$

29. (a) $C(m) = 14 + 0.08m$
(b) $m \ge 0$
(c) \$22

EXERCISE SET 3.4, page 149

1. 2; increasing
3. $-3/2$, decreasing
5. -1; decreasing
9. $2x - y + 5 = 0$
11. $3x - y = 0$
13. $2x - y = 0$
15. $2x - 3y = 0$
17. $2x - y = 0$
19. $3x - y + 2 = 0$
21. $y - 2 = 0$
23. $x - 3y - 15 = 0$
25. $m = -3/4$, $b = 5/4$
27. $m = 0$, $b = 4$
29. $m = -3/4$, $b = -1/2$
31. (a) $y = 3$ (b) $x = -6$
33. (a) $y = 0$ (b) $x = -7$
35. (a) $y = -9$ (b) $x = 9$
37. (a) -3 (b) $1/3$
39. (a) $4/3$ (b) $-3/4$
41. (a) $3x + y - 6 = 0$ (b) $x - 3y + 8 = 0$
43. (a) $3x + 5y - 1 = 0$ (b) $5x - 3y + 21 = 0$
45. (a) $F = \frac{9}{5}C + 32$ (b) 68° F
47. \$1,000,000
49. 5
51. $f(x) = 8x + 13$

EXERCISE SET 3.5, page 154

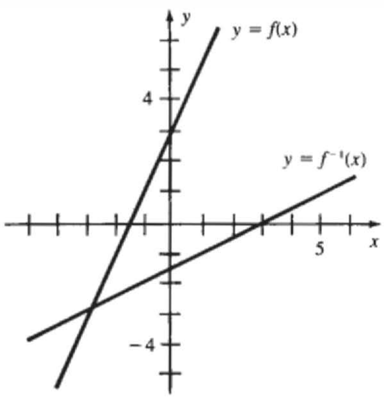
1. (a) 4 (b) $y = 4x$ (c)

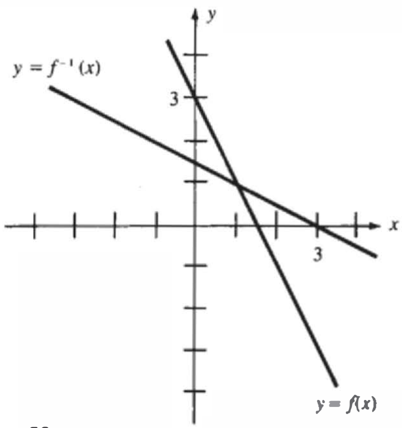
x	2	3	4	6	8	12	20	30
y	8	12	16	24	32	48	80	120

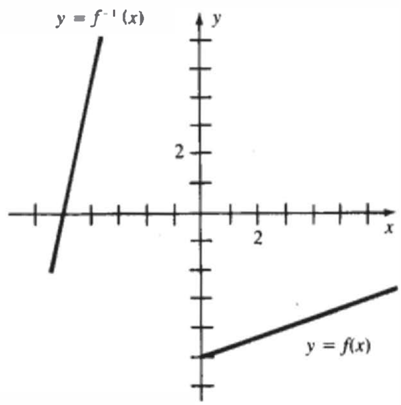
3. (a) $-1/32$ (b) $-3/8$
5. (a) $1/10$ (b) $5/2$
7. (a) -3 (b) $-1/4$
9. (a) 512 (b) 512/125
11. (a) $M = r^2/s^2$ (b) 36/25
13. (a) $T = 16pv^3/u^2$ (b) 2/3
15. (a) 400 feet (b) 5 seconds
17. 40/3 ohms
19. (a) 800/9 candlepower (b) 8 feet
21. 6
23. 120 candlepower/ft^2

EXERCISE SET 3.6, page 164

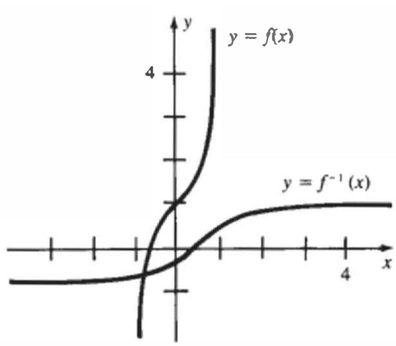
1. $x^2 + x - 1$
3. $x^2 - x + 3$
5. $x^3 - 2x^2 + x - 2$
7. $(x^2 + 1)/(x - 2)$
9. domain of f and of g: all reals
11. $4x^2 + 2x + 1$
13. 21
15. $4x^2 + 10x + 7$
17. $8x^2 - 6x + 1$
19. $x + 6,\ x \geq -2$
21. 29
23. all reals
25. $(f \circ g)(x) = x + 1$; $(g \circ f)(x) = x + 1$
27. $(f \circ g)(x) = (x - 1)/x,\ x \neq 1$; $(g \circ f)(x) = -(x + 1)/x,\ x \neq -1$
29. $f(x) = x + 3$; $g(x) = x^2$
31. $f(x) = x^8$; $g(x) = 3x + 2$
33. $f(x) = x^{1/3}$; $g(x) = x^3 - 2x^2$
35. $f(x) = |x|$; $g(x) = x^2 - 4$
37. $f(x) = \sqrt{x}$; $g(x) = 4 - x$
45. $f^{-1}(x) = (x - 3)/2$
47. $f^{-1}(x) = (3 - x)/2$
49. $f^{-1}(x) = 3x + 15$





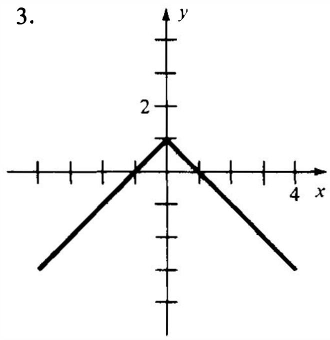


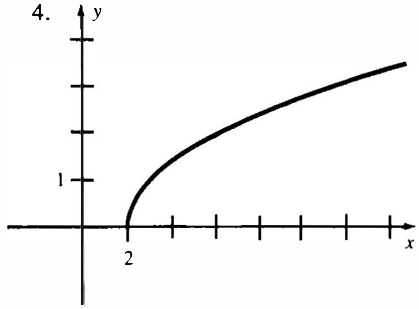
51. $f^{-1}(x) = (x - 1)^{1/3}$

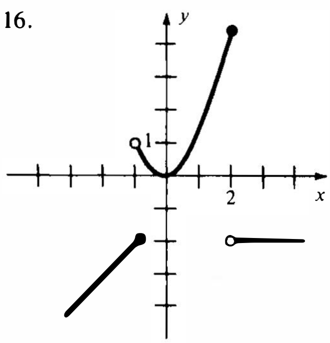


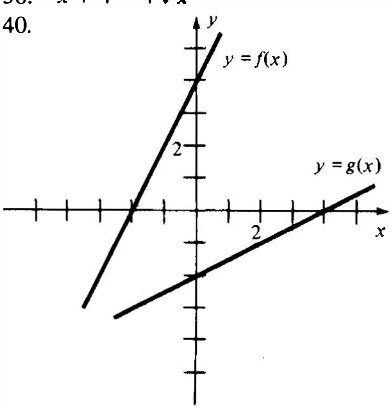
53. yes
55. no
57. yes
59. no
63. $(x - b)/a$

REVIEW EXERCISES, page 167

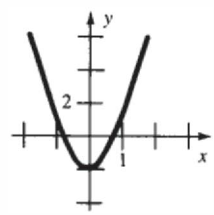
1. $\sqrt{61}$
2. $\sqrt{65}$
3. 

4. 

5. x-axis
6. all
7. yes
8. yes
9. $x \geq 5/3$
10. $x \neq -1$
11. 226
12. ± 3
13. 12
14. $y^2 - 3y + 2$
15. $3 + h$
16. 

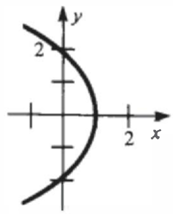
17. increasing: $x \leq -1$, $0 \leq x \leq 2$
decreasing: $-1 < x \leq 0$
constant: $x > 2$
18. -5
19. -2
20. 3
21. $y - 3x - 6 = 0$
22. $x = -4$
23. $y = 3$
24. $y - 2x - 2 = 0$
25. $y - 2x - 5 = 0$
26. 160
27. 1
28. $-1/4$
29. $x^2 + x$
30. 0
31. $(x + 1)/(x^2 - 1)$
32. $x \neq \pm 1$
33. $x^2 + 2x$
34. 4
35. $|x| - 2$
36. $x + 4 - 4\sqrt{x}$
37. 0
38. not defined
40. 

PROGRESS TEST 3A, page 168

1. $3 + \sqrt{26} + \sqrt{41}$
2. 

3. origin
4. $x \geq 0,\ x \neq 1$
5. 17
6. $8t^2 + 3$
7. increasing: $x \geq 0$
   decreasing: $-2 \leq x \leq 0$
   constant: $x < -2$
8. 0
9. 2
10. $2y - 3x - 19 = 0$
11. $x = -3$
12. $m = 1/2;\ b = 2$
13. $y = -1$
14. $3y + x - 7 = 0$
15. -1024
16. 65,536
17. -3
18. $x^2(x - 1)$
19. 1/4

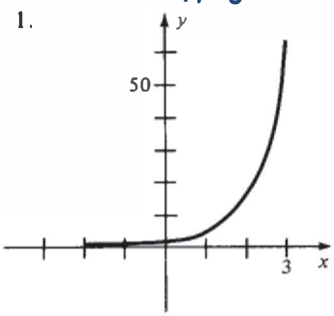
PROGRESS TEST 3B, page 168

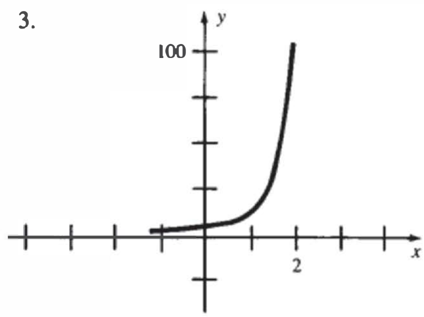
1. $6\sqrt{2}$
2. 

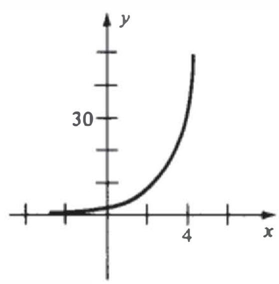
3. origin
4. $x \neq \pm 4$
5. 1
6. 1
7. increasing: $x > 3$
   decreasing: $x \leq -3$
   constant: $-3 < x \leq 3$
8. 10
9. 24
10. $-9/2$
11. $y = -5$
12. 1
13. -3
14. $x + 3y + 3 = 0$
15. $-32/9$
16. 1
17. 1
18. $\sqrt{2}/2$
19. $\sqrt{x}/x$

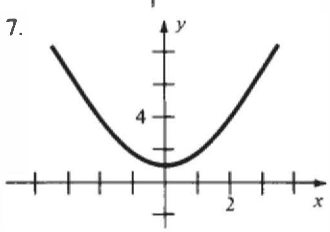
CHAPTER 4

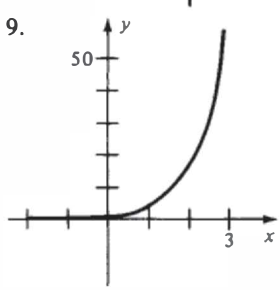
EXERCISE SET 4.1, page 181

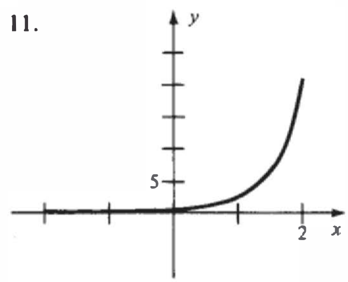
1. 

3. 

5. 

7. 

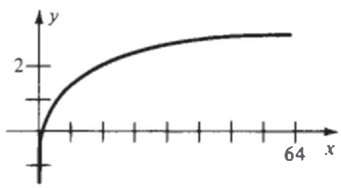
9. 

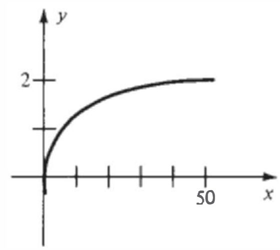
11. 

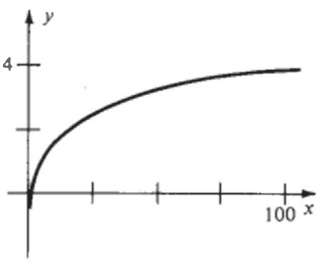
13. 3 15. 4 17. 2
19. 4 21. 2 23. 1
25. (a) 200 (b) 29,682 (c) 256.8, 543.7, 1478, 2436
27. 6.59 billion 29. 670.3 grams 31. $41,611
33. $45,417.50 35. $173.33 37. $2489
39. π^2

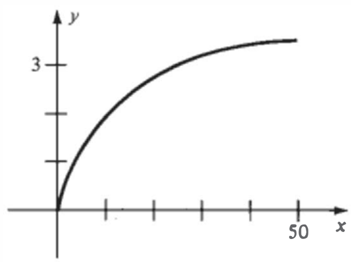
EXERCISE SET 4.2, page 190

1. $2^2 = 4$ 3. $9^{-2} = 1/81$ 5. $e^3 = 20.09$ 7. $10^3 = 1000$
9. $e^0 = 1$ 11. $3^{-3} = 1/27$ 13. $\log_5 25 = 2$ 15. $\log_{10} 10{,}000 = 4$
17. $\log_2 1/8 = -3$ 19. $\log_2 1 = 0$ 21. $\log_{36} 6 = 1/2$ 23. $\log_{16} 64 = 3/2$
25. $\log_{27} 1/3 = -1/3$ 27. 25 29. 1/5 31. $e^2 \approx 7.39$
33. $e^{-1/2} \approx 0.61$ 35. -2 37. 512 39. 124
41. 2 43. 3 45. 6 47. 2
49. 3 51. 1/2 53. 2 55. 1
57. 0 59. -2 61. 4 63. 2

65.


67.


69.


71.


EXERCISE SET 4.3, page 197

1. $\log_{10} 120 + \log_{10} 36$ 3. 4 5. $\log_a 2 + \log_a x + \log_a y$
7. $\log_a x - \log_a y - \log_a z$ 9. $5 \ln x$ 11. $2 \log_a x + 3 \log_a y$
13. $\frac{1}{2}(\log_a x + \log_a y)$ 15. $2 \ln x + 3 \ln y + 4 \ln z$ 17. $\frac{1}{2} \ln x + \frac{1}{3} \ln y$
19. $2 \log_a x + 3 \log_a y - 4 \log_a z$ 21. 0.77 23. 0.94
25. 1.07 27. 0.87 29. 0.435
31. $\log x^2\sqrt{y}$ 33. $\ln \sqrt[3]{xy}$ 35. $\log_a \dfrac{x^{1/3}y^2}{z^{3/2}}$

37. $\log_a \sqrt{xy}$
39. $\ln \dfrac{\sqrt[3]{x^2y^4}}{z^3}$
41. $\log_a \dfrac{\sqrt{x-1}}{(x+1)^2}$
43. $\log_a \dfrac{x^3(x+1)^{1/6}}{(x-1)^2}$
45. 1.2304
47. 4.5046
49. 2.3892

EXERCISE SET 4.4, page 203

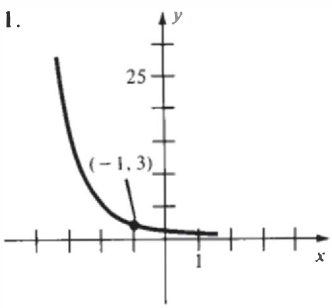
1. 2.725×10^3
3. 8.4×10^{-3}
5. 7.16×10^5
7. 2.962×10^2
9. 0.5514
11. 1.5740
13. 1.5476
15. 1.8692
17. 4.6830
19. −0.4660
21. 2.520
23. 2.9
25. 7.9
27. 0.257
29. 0.000607
31. 0.0219
33. 1.028
35. 2.115
37. 103.55
39. 0.002875
41. 1.93×10^{-5}
43. 2.59×10^{-8}
45. $10,453
47. $14,660.72
49. 8.75% compounded quarterly

EXERCISE SET 4.5, page 207

1. log 18/log 5
3. 1 + log 7/log 2
5. log 46/2 log 3
7. 5/2 + log 564/2 log 5
9. (log 2 + log 3)/(log 3 − 2 log 2)
11. −log 15/log 2
13. 1/2 − log 12/(2 log 4)
15. ln 18
17. (−3 + ln 20)/2
19. 500
21. 1/2
23. 5
25. 3
27. 8
29. $-1 + \sqrt{17}$
31. $\ln(y + \sqrt{y^2 + 1})$
33. 36.62 years
35. 12.6 hours
37. 8.8 years
39. 27.47 days
41. 1.386 days

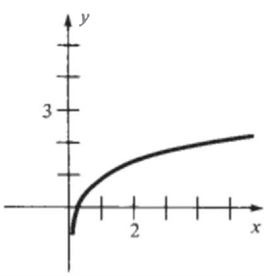
REVIEW EXERCISES, page 209

1.



2. 3
3. 2
4. $12,750.40
5. $\log_9 27 = 3/2$
6. $8 = 64^{1/2}$
7. $1/8 = 2^{-3}$
8. $\log_6 1 = 0$
9. 2
10. −2
11. e^{-4}
12. 26
13. 5
14. −1/3
15. −1
16. 3

17.

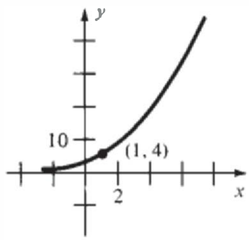


18. $\frac{1}{2}\log_a(x-1) - \log_a 2 - \log_a x$
19. $\log_a x + 2\log_a(2-x) - \frac{1}{2}\log_a(y+1)$
20. $4\ln(x+1) + 2\ln(y-1)$
21. $\frac{3}{5}\log y + \frac{1}{5}\log z - \frac{1}{5}\log(z+3)$
22. 1.15
23. 0.55
24. 0.4
25. −0.15
26. $\log_a \dfrac{\sqrt[3]{x}}{\sqrt{y}}$
27. $\log(x^2 - x)^{4/3}$

28. $\ln \frac{3xy^2}{z}$ 29. $\log_a \frac{(x+2)^2}{(x+1)^{3/2}}$ 30. 5/3 31. 15/7
32. 4.765×10^2 33. 9.8×10^{-2} 34. 2.6475×10^4 35. 7.767×10^1
36. 803 37. 7.9 38. 3.49×10^{-4} 39. 11.5 hours
40. $\frac{1}{3} + \frac{\log 14}{3 \log 2}$ 41. $\sqrt{5000}$ 42. $\frac{199}{98}$

PROGRESS TEST 4A, page 210

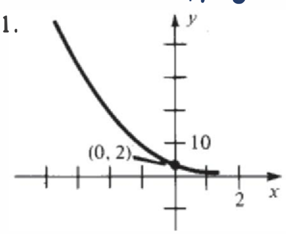
1.



2. $-2/3$ 3. $1/9 = 3^{-2}$ 4. $\log_{16} 64 = 3/2$
5. 3 6. -1 7. 5/2
8. 1/2 9. $3 \log_a x - 2 \log_a y - \log_a z$
10. $2 \log x + \frac{1}{2} \log(2y - 1) - 3 \log y$
11. 0.7 12. 0.45
13. $\log \frac{x^2}{(y+1)^3}$ 14. $\log_a\left(\frac{x+3}{x-3}\right)^{2/3}$
15. 2.73×10^{-4} 16. 5.972×10^0
17. 4.7×10^{-2} 18. 0.26
19. 34.6 hours 20. 200
21. 4

PROGRESS TEST 4B, page 210

1.



2. 8 3. $\log \frac{1}{1000} = -3$
4. $1 = 3^0$ 5. 3/2
6. 3/2 7. 10
8. 4 9. $\log_a(x - 1) + \frac{1}{4} \log_a(y + 3)$
10. $\frac{1}{2} \ln x + \frac{1}{2} \ln y + \frac{1}{2} \ln 2z$ 11. 0.85
12. 1.5 13. $\frac{1}{5}\ln\frac{(x-1)^3y^2}{z}$
14. $\log\frac{x^2}{y^2}$
15. 2.2684321×10^7 16. 2.97×10^{-1} 17. 9397 18. 6.2×10^{-4}
19. $530.76 20. 3 21. 10

CHAPTER 5

EXERCISE SET 5.1, page 223

1. IV 3. I 5. II 7. I
9. III 11. II 13. II 15. III
17. I 19. $\pi/6$ 21. $-5\pi/6$ 23. $5\pi/12$
25. $-5\pi/2$ 27. $3\pi/4$ 29. $2\pi/3$ 31. 0.251π
33. 45° 35. 270° 37. −90° 39. 240°
41. 450° 43. −300° 45. 98.55° 47. T

49. F
51. F
53. 4/7; 32.74°
55. $6/\pi$
57. 6.8 ft; ≈776.5 rotations
59. 10

EXERCISE SET 5.2, page 232

1. III
3. II
5. II
7. II

	$\sin t$	$\cos t$	$\tan t$
9.	4/5	$-3/5$	$-4/3$
11.	$-1/2$	$\sqrt{3}/2$	$-\sqrt{3}/3$
13.	$\sqrt{2}/2$	$-\sqrt{2}/2$	-1
15.	$-1/4$	$\sqrt{15}/4$	$-\sqrt{15}/15$

17. II
19. III
21. III
23. $-3/2$
25. -1
27. $-\sqrt{2}/2$
29. $-\sqrt{3}/2$
31. $-\sqrt{3}$
33. $-\sqrt{3}/2$
35. (a)$(-3/5, -4/5)$ (b) $(4/5, -3/5)$ (c) $(3/5, -4/5)$ (d) $(-3/5, 4/5)$
37. (a) a (b) $-b$
41. $-3/4$
43. $-12/13$
45. $-3/5$
47. 4/5

EXERCISE SET 5.3, page 243

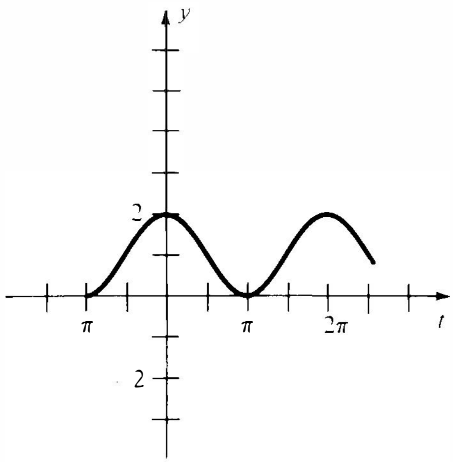
1. 0
3. $\pi/7$
5. $3\pi/2$
7. $5\pi/6$
9. π
11. $7\pi/5$
13. $\pi, -\pi$
15. $3\pi/4, -5\pi/4$
17. $5\pi/6, -7\pi/6$
19. $5\pi/3, -\pi/3$

		$\sin t$	$\cos t$	$\tan t$
21.	(a) $(-1, 0)$	(b) 0	-1	0
23.	(a) $(\sqrt{2}/2, -\sqrt{2}/2)$	(b) $-\sqrt{2}/2$	$\sqrt{2}/2$	-1
25.	(a) $(-\sqrt{2}/2, -\sqrt{2}/2)$	(b) $-\sqrt{2}/2$	$-\sqrt{2}/2$	1
27.	(a) $(-1/2, -\sqrt{3}/2)$	(b) $-\sqrt{3}/2$	$-1/2$	$\sqrt{3}$
29.	(a) $(-1/2, -\sqrt{3}/2)$	(b) $-\sqrt{3}/2$	$-1/2$	$\sqrt{3}$
31.	(a) $(-\sqrt{3}/2, -1/2)$	(b) $-1/2$	$-\sqrt{3}/2$	$\sqrt{3}/3$
33.	(a) $(-\sqrt{3}/2, -1/2)$	(b) $-1/2$	$-\sqrt{3}/2$	$\sqrt{3}/3$
35.	(a) $(1/2, \sqrt{3}/2)$	(b) $\sqrt{3}/2$	1/2	$\sqrt{3}$

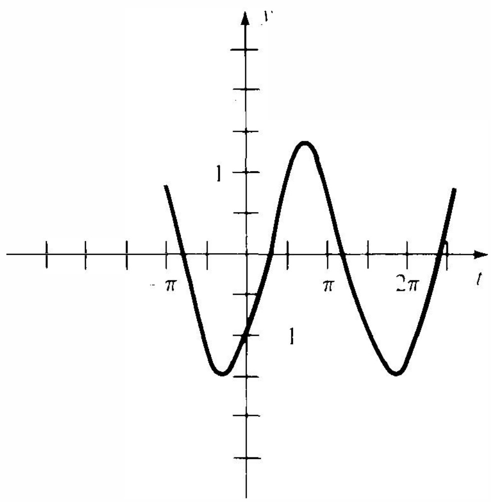
37. 0.4357
39. -5.471
41. 0.3093
43. 0.8415
45. -0.9737
47. -0.1307
61. -0.4663
63. -0.7880
65. -0.2250
67. -0.7193
69. 0.3476
71. 0.3007
85. 0.7174
87. -0.1987
89. 0.1003

EXERCISE SET 5.4, page 255

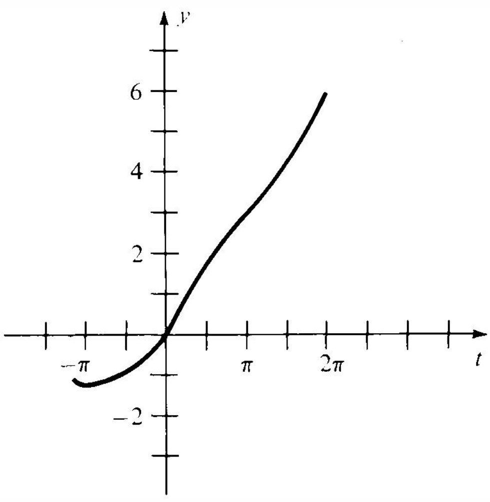
1.



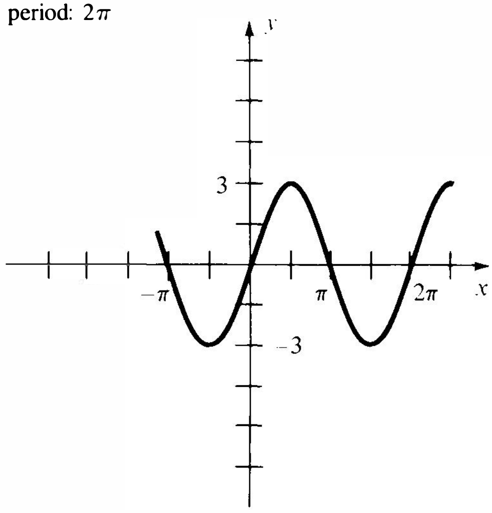
3.



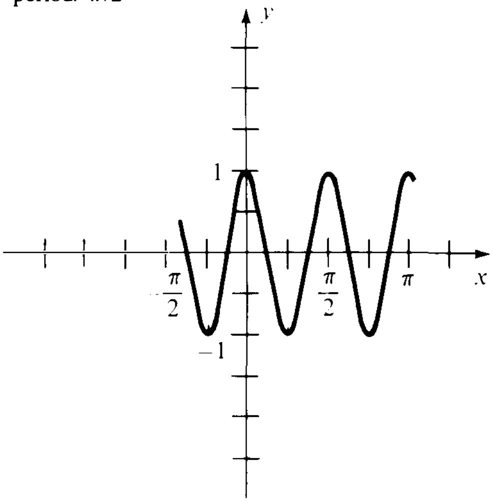
5.



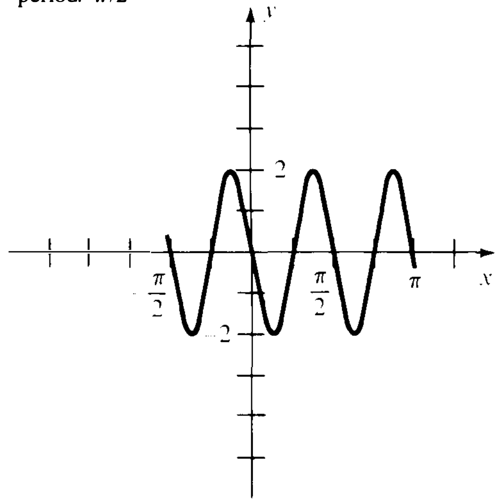
9. amplitude: 3
period: 2π



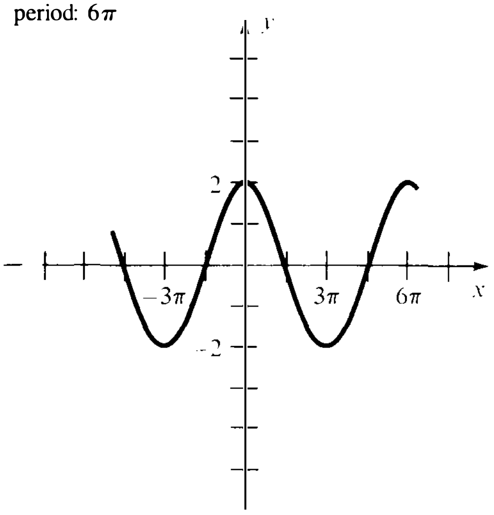
11. amplitude: 1
period: $\pi/2$



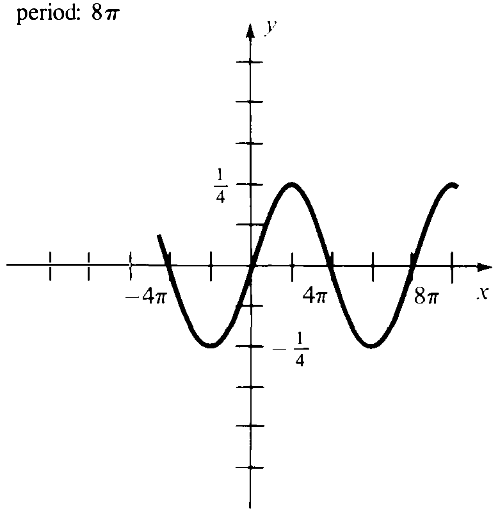
13. amplitude: 2
period: $\pi/2$



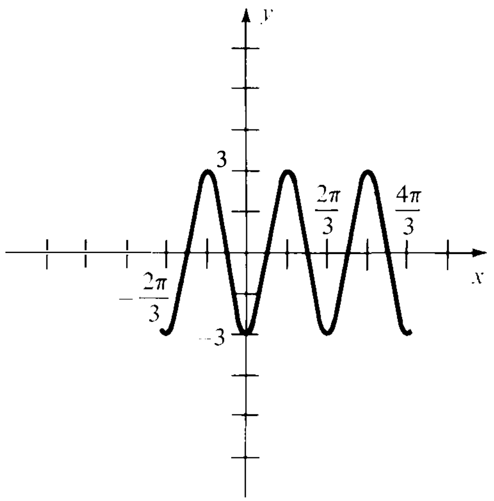
15. amplitude: 2
period: 6π



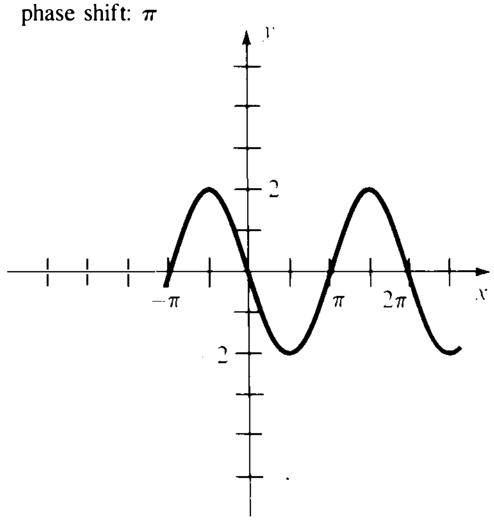
17. amplitude: 1/4
period: 8π



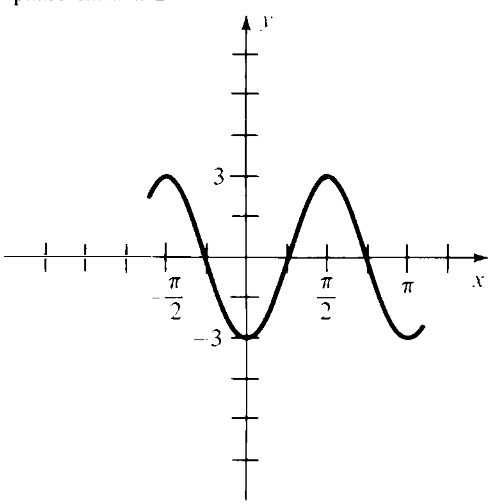
19. amplitude: 3
period: $2\pi/3$



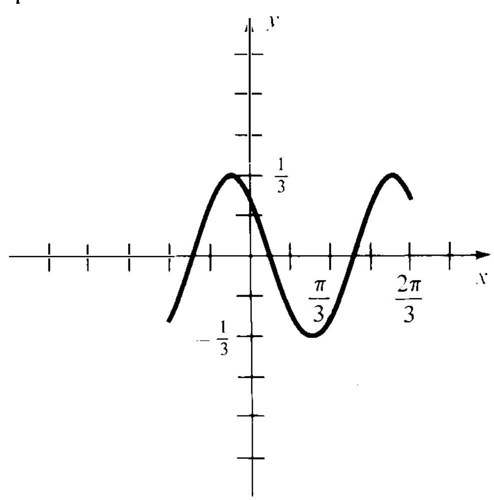
21. amplitude: 2
period: 2π
phase shift: π



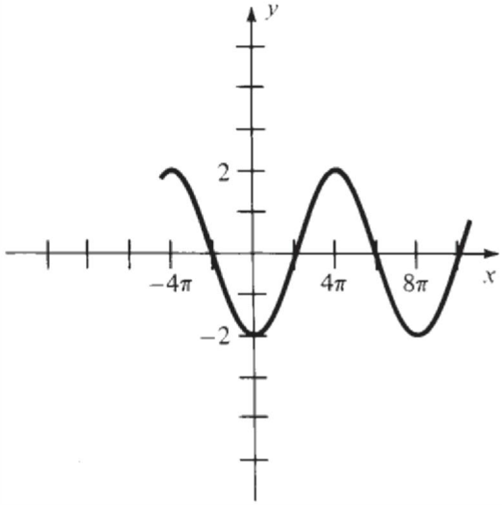
23. amplitude: 3
period: π
phase shift: $\pi/2$



25. amplitude: 1/3
period: $2\pi/3$
phase shift: $-\pi/4$



27. amplitude: 2
period: 8π
phase shift: 4π



29. $y = 2\sin(2x - \pi)$

31. $y = 3\cos(x/3 - 2\pi/3)$

EXERCISE SET 5.5, page 260

1. $\sec t = 2$, $\csc t = 2\sqrt{3}/3$, $\cot t = \sqrt{3}/3$
3. $\sec t = \sqrt{2}$, $\csc t = \sqrt{2}$, $\cot t = 1$
5. $\sec t = -2\sqrt{3}/3$, $\csc t = 2$, $\cot t = -\sqrt{3}$
7. $\sec t$ not defined, $\csc t = -1$, $\cot t = 0$
9. $\sec t = -\sqrt{2}$, $\csc t = \sqrt{2}$, $\cot t = -1$
11. $\sec t = -\sqrt{2}$, $\csc t = -\sqrt{2}$, $\cot t = 1$

13. $0, 2\pi$	15. $7\pi/6, 11\pi/6$	17. $\pi/4, 5\pi/4$	19. $3\pi/4, 7\pi/4$
21. $\pi/4, 7\pi/4$	23. $5\pi/6, 11\pi/6$	25. III	27. II
29. III	31. III	33. $5\pi/6$	35. $2\pi/3$
37. $5\pi/4$	39. $3\pi/4$	41. 4.271	43. 1.000
45. 3.270			

EXERCISE SET 5.6, page 268

1. $-\pi/6$	3. $\pi/3$	5. $-\pi/4$	7. $5\pi/6$
9. $-\pi/2$	11. $\pi/2$	13. 0	15. $-\pi/4$
17. $2\pi/3$	19. 0.38	21. 2.44	23. 1.30
31. $\sqrt{2}/2$	33. 0	35. $\pi/4$	37. $2\pi/3$

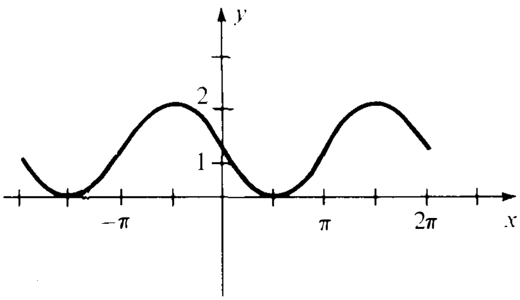
39. $\arcsin(\pm\sqrt{7}/7)$
41. $\cos^{-1}(1/3)$, $\cos^{-1}(-1/4)$
43. $\sin^{-1}(2/3)$
45. 0

REVIEW EXERCISES, page 270

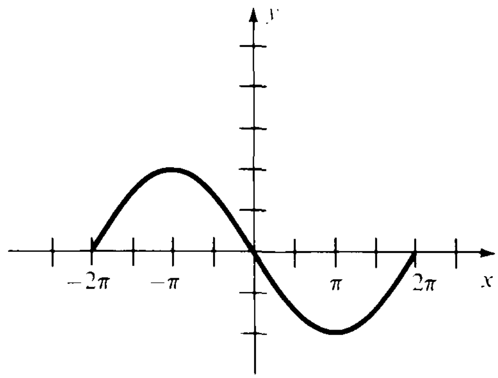
1. $-\pi/3$	2. 270°	3. −75°	4. $\pi/4$
5. yes	6. no	7. yes	8. 1.4
9. 15/4 cm			
10. IV	11. II	12. II	13. I
14. $\pi/2$	15. $\pi/2$	16. 0	17. $5\pi/3$

18. $(-4/5,\ 3/5)$ 19. $(3/5,\ 4/5)$ 20. $(4/5,\ 3/5)$ 21. $(-3/5,\ -4/5)$
22. $(-4/5,\ -3/5)$
23. IV 24. IV 25. III 26. I
27. $-3/4$ 28. $-5/3$ 29. $-12/5$ 30. $13/12$
33. $\sqrt{3}/2$ 34. $-\sqrt{2}$ 35. $-\sqrt{3}/3$ 36. -2
37. $5\pi/4$ 38. $11\pi/6$ 39. $\pi/3$ 40. $2\pi/3$
41. -1.8334 42. 0.4228

43.



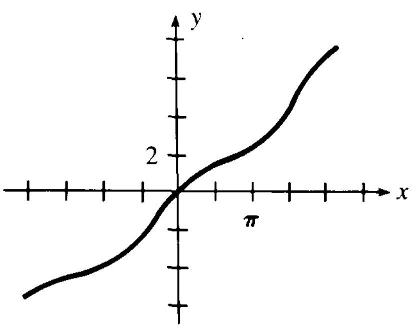
44.



45. amplitude: 1; period: π; phase shift: $\pi/2$
46. amplitude: 4; period: 2π; phase shift: $\pi/2$
47. amplitude: 2; period: 6π; phase shift: $-\pi$
48. none 49. $3\pi/4$
50. $-\pi/6$ 51. 0 52. 5 53. $\cos^{-1}(\pm 2\sqrt{5}/5)$

PROGRESS TEST 5A, page 272

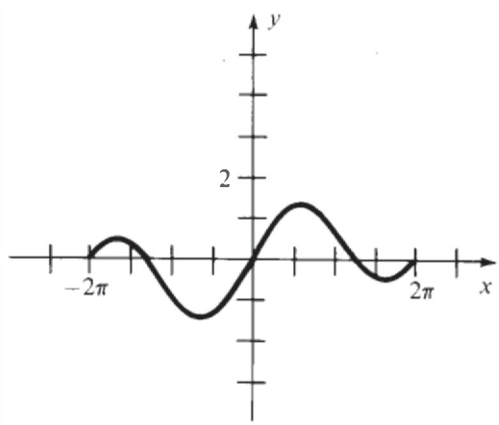
1. 300° 2. $-10\pi/9$ 3. $5\pi/12$ 4. 335°
5. 45° 6. $4/5$ 7. $\pi/3$ 8. 0
9. $(-\sqrt{3}/2,\ 1/2)$ 10. $(1/2,\ -\sqrt{3}/2)$ 11. $(5/13,\ -12/13)$ 12. $(12/13,\ 5/13)$
13. $(-5/13,\ -12/13)$ 14. 20° 15. $\pi/4$
16. $1/2$ 17. $-2\sqrt{3}/3$ 18. $5\pi/4$ 19. $7\pi/4$
20. $-5/13$ 21. $-5/4$ 23. -0.5973 24. -0.2509
25.



26. amplitude: 2; period: 2π; phase shift: π
27. amplitude: 2; period: 4π; phase shift: π
28. $-\pi/3$
29. $1/2$
30. $\arctan(2/3)$, $\arctan(3/2)$

PROGRESS TEST 5B, page 273

1. $-3\pi/4$
2. $135°$
3. $-150°$
4. $70°$
5. $240°$
6. $21/5$
7. 0
8. $\pi/5$
9. $(\sqrt{3}/2, -1/2)$
10. $(-\sqrt{2}/2, -\sqrt{2}/2)$
11. $(-4/5, 3/5)$
12. $(3/5, -4/5)$
13. $(4/5, -3/5)$
14. $7\pi/16$
15. $15°$
16. -1
17. 1
18. $\pi/3$
19. $2\pi/3$
20. $-5/12$
21. $-3/4$
23. 0.6378
24. -3.109
25.



26. amplitude: 4; period: $2\pi/3$; phase shift: $\pi/3$
27. amplitude: $1/2$; period: π; phase shift: $-\pi/4$
28. $\pi/6$
29. 1
30. $\arcsin(-3/5)$, $\pi/2$

CHAPTER 6

EXERCISE SET 6.1, page 280

	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
1.	$3/5$	$4/5$	$3/4$	$5/3$	$5/4$	$4/3$
3.	$4/5$	$3/5$	$4/3$	$5/4$	$5/3$	$3/4$
5.	$2\sqrt{5}/5$	$\sqrt{5}/5$	2	$\sqrt{5}/2$	$\sqrt{5}$	$1/2$
7.	$\dfrac{\sqrt{x^2+1}}{x^2+1}$	$\dfrac{x\sqrt{x^2+1}}{x^2+1}$	$\dfrac{1}{x}$	$\sqrt{x^2+1}$	$\dfrac{\sqrt{x^2+1}}{x}$	x
9.	$12/13$	$-5/13$	$-12/5$	$13/12$	$-13/5$	$-5/12$
11.	$-\sqrt{2}/2$	$-\sqrt{2}/2$	1	$-\sqrt{2}$	$-\sqrt{2}$	1
13.	$3/5$	$-4/5$	$-3/4$	$5/3$	$-5/4$	$-4/3$
15.	$-5/13$	$12/13$	$-5/12$	$-13/5$	$13/12$	$-12/5$
17.	$-5/13$	$-12/13$	$5/12$	$-13/5$	$-13/12$	$12/5$
19.	$\sqrt{5}/5$	$-2\sqrt{5}/5$	$-1/2$	$\sqrt{5}$	$-\sqrt{5}/2$	-2

21. $5\sin\theta$
23. $6.5\cot\theta$
25. $3.7\csc\theta$
27. $36°50'$
29. 62.2
31. 30.3
33. 33.9
35. $-5/12$
37. $3/5$
39. $-4/3$

EXERCISE SET 6.2, page 286

1. 53°10′ 3. 61° 5. 7767 feet 7. 970 meters
9. 39°10′, 50°50′ 11. 18.7 cm 13. 53 feet 15. 24.19 miles

EXERCISE SET 6.3, page 291

1. 41°10′ 3. 14.4 5. 17.8 7. 62°30′
9. 90° 11. 82°10′ 13. 52.5 miles, S29°W 15. 32.8 miles
17. 68.5

EXERCISE SET 6.4, page 297

1. 29.1 3. 7.2 5. 15.7 7. none
9. 10.9 11. 8.3, 1.6 13. 98.9 meters 15. 682 meters
17. 40.5 miles 19. 18.8 meters 21. 115 cm

REVIEW EXERCISES, page 299

1. 5/13 2. 4/3 3. $\sqrt{65}/7$ 4. 4/3
5. $7\sqrt{33}/33$ 6. 2 7. -1 8. $-1/2$
9. $\sqrt{2}/2$ 10. 39°50′ 11. 14.6 12. 25.4
13. 16.6 14. 5.4 meters 15. 68°10′ 16. 36°
17. 51°50′ 18. 37°50′ 19. 14.1 20. 7.1

PROGRESS TEST 6A, page 299

1. 7/5 2. 3 3. 5/12 4. -1
5. -1 6. 2 7. $-2\sqrt{13}/13$ 8. 56°30′
9. 23.6 10. 53°10′ 11. 22°20′ 12. 14.5
13. 138 meters

PROGRESS TEST 6B, page 300

1. 5/4 2. 6/7 3. 3/5 4. undefined
5. $2\sqrt{3}/3$ 6. 1 7. $-\sqrt{5}/5$ 8. 10.3
9. 66°30′ 10. 12.2 11. 48°30′ 12. 15.8
13. 67.5 feet

CHAPTER 7

EXERCISE SET 7.1, page 306

47. $3\pi/2$ 49. $\pi/4$ 51. π

EXERCISE SET 7.2, page 313

1. $s = t = 0$ 3. $s = \pi$, $t = \pi/2$ 5. $s = t = \pi/4$ 7. $(\sqrt{6} - \sqrt{2})/4$
9. $(\sqrt{2} + \sqrt{6})/4$ 11. $-\sqrt{3}/2$ 13. $\sqrt{3}$ 15. $(\sqrt{6} - \sqrt{2})/4$

17. $(\sqrt{2}-\sqrt{6})/4$ 19. $-1/2$ 21. $2-\sqrt{3}$ 23. $\cos 43°$
25. $\cot \pi/3$ 27. $\sin \pi/6$ 29. $-4/5$ 31. -7
33. -2.29 35. $-16/65$ 37. $2/29$

EXERCISE SET 7.3, page 321

1. $7/25$ 3. $-\sqrt{3}/2$ 5. $-240/161$
7. $-24/25$ 9. $-161/289$ 11. 0.1022
13. $\sqrt{2-\sqrt{3}}/2$ 15. $\sqrt{(2-\sqrt{2})}/\sqrt{(2+\sqrt{2})}$ 17. $2/\sqrt{2-\sqrt{3}}$
19. $\sqrt{20}/5$ 21. $\sqrt{6}/3$ 23. -2
25. $-\sqrt{6}/3$

EXERCISE SET 7.4, page 325

1. $\sin 6\alpha + \sin 4\alpha$ 3. $(\cos 5x - \cos x)/2$ 5. $-(\cos 7\theta + \cos 3\theta)$
7. $(\cos 2\alpha + \cos 2\beta)/2$ 9. $-(2+\sqrt{2})/4$ 11. $\sqrt{3}/4$
13. $2 \sin 3x \cos 2x$ 15. $2 \cos 4\theta \cos 2\theta$ 17. $2 \sin \alpha \cos \beta$
19. $2 \cos 5x \cos 2x$ 21. $\sqrt{6}/2$ 23. $-\sqrt{2}$
35. $\dfrac{\sin(a+b)x + \sin(a-b)x}{2}$

EXERCISE SET 7.5, page 330

1. $\pi/6, 5\pi/6$; $30°, 150°$
3. π; $180°$
5. $\pi/6, 5\pi/6, 7\pi/6, 11\pi/6$; $30°, 150°, 210°, 330°$
7. $\pi/6, 5\pi/6, 7\pi/6, 11\pi/6$; $30°, 150°, 210°, 330°$
9. $0°, 30°, 150°, 180°$; $0, \pi/6, 5\pi/6, \pi$
11. $0, \pi/3, 5\pi/3$; $0°, 60°, 300°$
13. $\pi/10, \pi/2, 9\pi/10, 13\pi/10, 17\pi/10$; $18°, 90°, 162°, 234°, 306°$
15. $\pi/3, 5\pi/3$; $60°, 300°$
17. $\pi/6, 5\pi/6, 3\pi/2$; $30°, 150°, 270°$
19. 0; $0°$
21. $\pi/6 + \pi n$; $5\pi/6 + \pi n$
23. $\pi/3 + \pi n, 2\pi/3 + \pi n$
25. $\pi/6 + \pi n$; $5\pi/6 + \pi n$
27. $\pi n/4$
29. $\pi/12 + \pi n/2, 5\pi/12 + \pi n/2$
31. $\pi/2 + \pi n$
33. $\pi/2 + 2\pi n/3$
35. $\pi n, \pi/4 + \pi n$
37. $\pi/6 + 2\pi n, 5\pi/6 + 2\pi n$
39. 0.83, 2.31, 3.71, 5.71 radians
41. 6.05 radians, 5.32 radians

EXERCISE SET 7.6, page 338

1. $\sqrt{13}$ 3. $\sqrt{2}$ 5. $2\sqrt{10}$
7. $3\sqrt{2}(\cos 7\pi/4 + i \sin 7\pi/4)$ 9. $2(\cos 11\pi/6 + i \sin 11\pi/6)$ 11. $\sqrt{2}(\cos 3\pi/4 + i \sin 3\pi/4)$
13. $4(\cos \pi + i \sin \pi)$ 15. -4 17. $-1+i$
19. $-5i$ 21. 6 23. $\sqrt{3}-i$
25. $2(\cos 7\pi/4 + i \sin 7\pi/4)$, $2(\cos \pi/2 + i \sin \pi/2)$
27. $4(\cos 2\pi/3 + i \sin 2\pi/3)$, $3\sqrt{2}(\cos \pi/4 + i \sin \pi/4)$, $(-6-6\sqrt{3}) + (6\sqrt{3}-6)i$
29. $5(\cos 0 + i \sin 0)$, $2\sqrt{2}(\cos 5\pi/4 + i \sin 5\pi/4)$, $-10-10i$
31. $0+512i$ 33. $16-16i$ 35. $-8+8i$ 37. $\pm\sqrt{2} \pm \sqrt{2}i$
39. $\sqrt{2}(\cos 150° + i \sin 150°)$, $\sqrt{2}(\cos 330° + i \sin 330°)$
41. $-2, 1 \pm \sqrt{3}i$ 43. $\pm 2, \pm 2i$

REVIEW EXERCISES, page 340

4. $(\sqrt{2}+\sqrt{6})/4$
5. $-\sqrt{2}/2$
6. $-2-\sqrt{3}$
7. $(\sqrt{2}+\sqrt{6})/4$
8. sec 75°
9. sin 67°
10. $\cos 3\pi/8$
11. $\cot 3\pi/14$
12. 5/13
13. $10(3+4\sqrt{3})/39$
14. 3/4
15. 70
16. −16/65
17. −7/25
18. −24/25
19. 24/25
20. $-\sqrt{3}/2$
21. 120/169
22. $-\sqrt{10}/10$
23. −1/3
24. $-\sqrt{30}/6$
25. $\sqrt{2+\sqrt{3}}/2$
26. $\sqrt{2-\sqrt{2}}/2$
27. $-\sqrt{2+\sqrt{2}}/\sqrt{2-\sqrt{2}}$
31. $(\cos\alpha-\cos 2\alpha)/2$
32. $-2\sin 2x\sin x$
33. 1/4
34. $2(\cos\pi/2)(\cos\pi/4)$
35. $\pi/4,\ 3\pi/4,\ 5\pi/4,\ 7\pi/4$
36. $0,\ \pi/2,\ \pi,\ 3\pi/2$
37. $0,\ \pi/3,\ \pi,\ 5\pi/3$
38. $90°+360°n,\ 270°+360°n$
39. $45°+60°n$
40. $30°+90°n,\ 60°+90°n$
41. $\sqrt{5}$
42. $\sqrt{13}$
43. $\sqrt{41}$
44. $3\sqrt{2}(\cos 3\pi/4+i\sin 3\pi/4)$
45. $2i$
46. $1-i$
47. $2(\cos\pi+i\sin\pi)$
48. $24(\cos 37°+i\sin 37°)$
49. $5(\cos 21°+i\sin 21°)/3$
50. $2(\cos 90°+i\sin 90°)$
51. $-972+972i$
52. $0-8i$
53. $3(\cos 90°+i\sin 90°),\ 3(\cos 270°+i\sin 270°)$
54. $1,\ -1/2\pm\sqrt{3}i/2,$

PROGRESS TEST 7A, page 341

2. 1/2
3. $\sqrt{3}-2$
4. cos 43°
5. 3/5
6. −81/76
7. −119/169
8. 7/25
9. $-\sqrt{2-\sqrt{3}}/2$
10. $\sqrt{2-\sqrt{3}}/\sqrt{2+\sqrt{3}}$
12. $2(\sin 5x/2)(\cos x/2)$
13. $2(\cos 90°)(\cos 60°)=0$
14. $\pi/3,\ 2\pi/3,\ 4\pi/3,\ 5\pi/3$
15. $45°+90°n$
16. $5(\cos 86°+i\sin 86°)$
17. $(\cos 77°+i\sin 77°)/2$
18. $-\dfrac{1}{2\cdot 5^4}+\dfrac{\sqrt{3}}{2\cdot 5^4}i$
19. $3(\cos 60°+i\sin 60°);\ 3(\cos 180°+i\sin 180°);$ $3(\cos 300°+i\sin 300°)$

PROGRESS TEST 7B, page 342

2. 2
3. $(\sqrt{6}+\sqrt{2})/4$
4. cot 19°
5. $-13\sqrt{2}/7$
6. 0
7. 24/25
8. 336/625
9. $-3\sqrt{13}/13$
10. $\sqrt{2-\sqrt{2}}/2$
12. $(\sin 7\pi/12-\sin\pi/12)/2$
13. $(\cos 90°+\cos 60°)/2$
14. $0,\ 3\pi/4,\ \pi,\ 7\pi/4$
15. $90°+120°n$
16. $5(\cos 250°+i\sin 250°)$
17. $2(\cos 55°+i\sin 55°)$
18. $-64+0i$
19. $-1,\ 1/2\pm\sqrt{3}i/2$

CHAPTER 8

EXERCISE SET 8.1, page 346

1. (5/2, 5)
3. (1, 5/2)
5. (−7/2, −1)
7. (0, −1/2)
9. (−1, 9/2)
11. (0, 0)

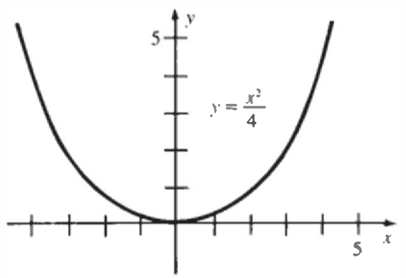
EXERCISE SET 8.2, page 351

1. $(x-2)^2+(y-3)^2=4$
3. $(x+2)^2+(y+3)^2=5$
5. $x^2+y^2=9$
7. $(x+1)^2+(y-4)^2=8$
9. $(h,k)=(2,3);\ r=4$
11. $(h,k)=(2,-2);\ r=2$
13. $(h,k)=(-4,-3/2);\ r=3\sqrt{2}$
15. $(h,k)=(1/3,0);\ r=1/3$
17. $(x+2)^2+(y-4)^2=16;\ (h,k)=(-2,4);\ r=4$
19. $(x-3/2)^2+(y-5/2)^2=11/2;\ (h,k)=(3/2,5/2);\ r=\sqrt{22}/2$

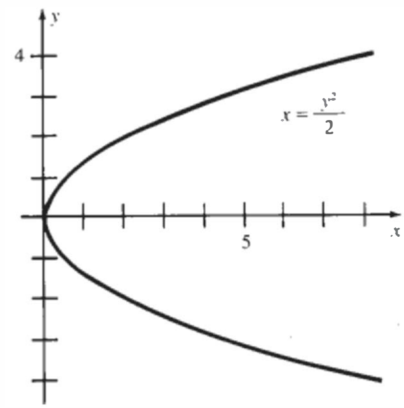
21. $(x-1)^2 + y^2 = 7/2$; $(h, k) = (1, 0)$; $r = \sqrt{14}/2$
23. $(x-2)^2 + (y+3)^2 = 8$; $(h, k) = (2, -3)$; $r = 2\sqrt{2}$
25. $(x-3)^2 + (y+4)^2 = 18$; $(h, k) = (3, -4)$; $r = 3\sqrt{2}$
27. $(x+3/2)^2 + (y-5/2)^2 = 3/2$; $(h, k) = (-3/2, 5/2)$; $r = \sqrt{6}/2$
29. $(x-3)^2 + y^2 = 11$; $(h, k) = (3, 0)$; $r = \sqrt{11}$
31. $(x-3/2)^2 + (y-1)^2 = 17/4$; $(h, k) = (3/2, 1)$; $r = \sqrt{17}/2$
33. $(x+2)^2 + (y-2/3)^2 = 100/9$; $(h, k) = (-2, 2/3)$; $r = 10/3$
35. neither
37. 9π
41. $(x+5)^2 + (y-2)^2 = 8$
43. $(x-5)^2 + (y-1)^2 = 20$

EXERCISE SET 8.3, page 360

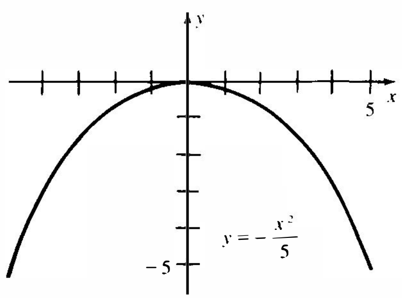
1. focus: $(0, 1)$; directrix: $y = -1$



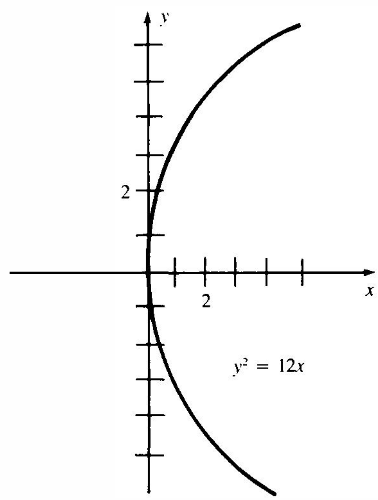
3. focus: $(1/2, 0)$; directrix: $x = -1/2$



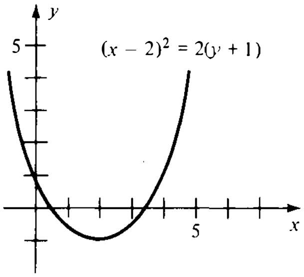
5. focus: $(0, -5/4)$; directrix: $y = 5/4$

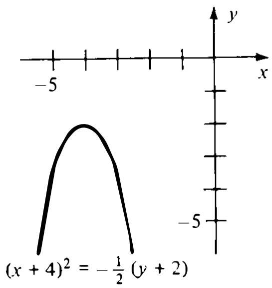


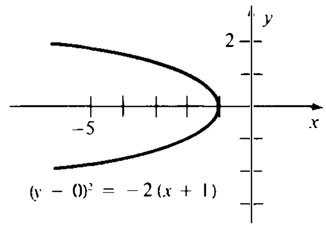
7. focus: $(3, 0)$; directrix: $x = -3$



9. $y^2 = 4x$
11. $y^2 = 6x$
13. $y^2 = \frac{1}{2}x$
15. $y^2 = -5x$
17. $y^2 = -4x$
19. $y^2 = x$
21. $(x - 1)^2 = 3(y - 2)$; vertex: $(1, 2)$; axis: $x = 1$; direction: up
23. $(y - 4)^2 = -2(x - 2)$; vertex: $(2, 4)$; axis: $y = 4$; direction: left
25. $(x - 1/2)^2 = -3(y + 1/4)$; vertex: $(1/2, -1/4)$; axis: $x = 1/2$; direction: down
27. $(y - 5)^2 = 3(x + 1/3)$; vertex: $(-1/3, 5)$; axis: $y = 5$; direction: right
29. $(x - 3/2)^2 = 3(y + 5/12)$; vertex: $(3/2, -5/12)$; axis: $x = 3/2$; direction: up
31. $(y + 3)^2 = -1/2(x - 4)$; vertex: $(4, -3)$; axis: $y = -3$; direction: left
33. $(x + 1)^2 = -2(y + 1)$; vertex: $(-1, -1)$; axis: $x = -1$; direction: down
35. vertex: $(2, -1)$; axis: $x = 2$; direction: up
37. vertex: $(-4, -2)$; axis: $x = -4$; direction: down
39. vertex: $(-1, 0)$; axis: $y = 0$; direction: left

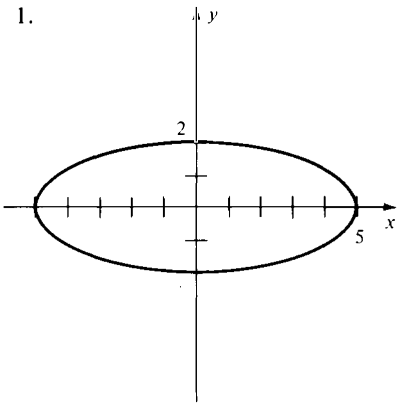




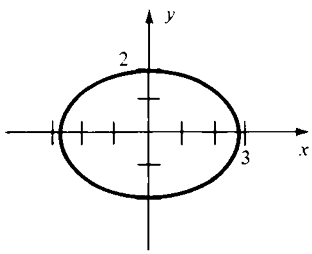


EXERCISE SET 8.4, page 367

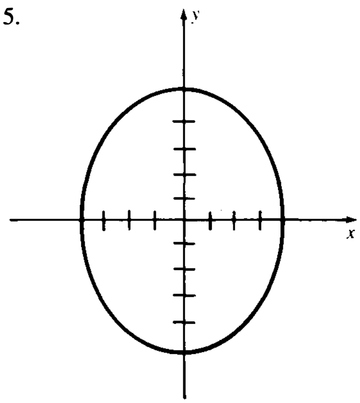
1.



3.

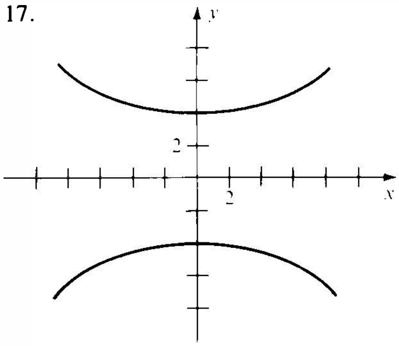


5.

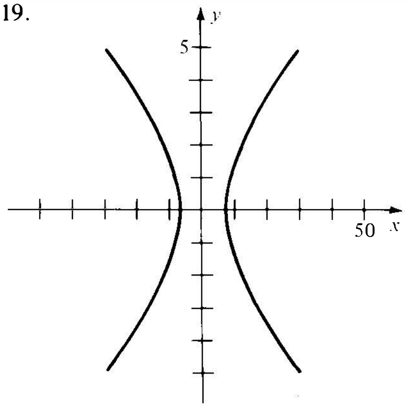


7. $\frac{x^2}{9} + \frac{y^2}{4} = 1$; $(0, \pm 2)$, $(\pm 3, 0)$
9. $\frac{x^2}{4} + \frac{y^2}{1} = 1$; $(0, \pm 1)$, $(\pm 2, 0)$
11. $\frac{x^2}{1} + \frac{y^2}{1/4} = 1$; $(0, \pm 1/2)$, $(\pm 1, 0)$
13. $\frac{x^2}{3} + \frac{y^2}{4} = 1$; $(0, \pm 2)$, $(\pm\sqrt{3}, 0)$
15. $\frac{x^2}{1/4} + \frac{y^2}{9/8} = 1$; $(0, \pm 3\sqrt{2}/4)$, $(\pm 1/2, 0)$

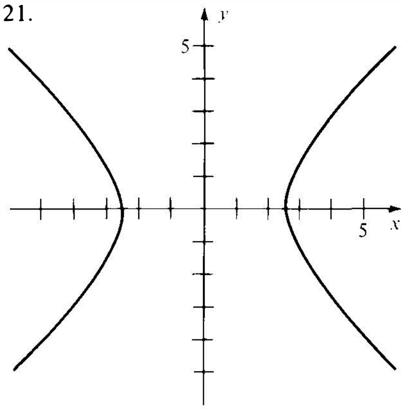
17.



19.



21.

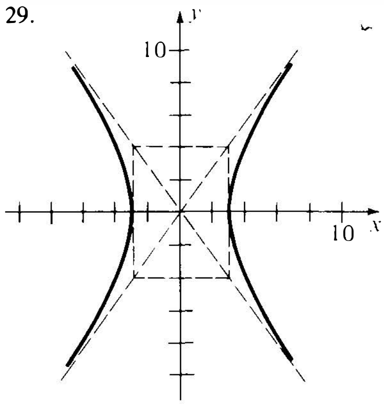


23. $\frac{x^2}{4} - \frac{y^2}{64} = 1$; $(\pm 2, 0)$

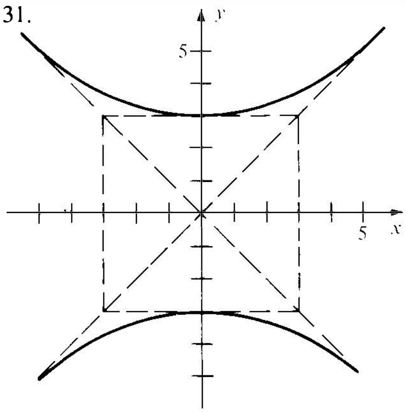
25. $\frac{y^2}{1/4} - \frac{x^2}{1/4} = 1$; $(0, \pm 1/2)$

27. $\frac{x^2}{5} - \frac{y^2}{4} = 1$; $(\pm\sqrt{5}, 0)$

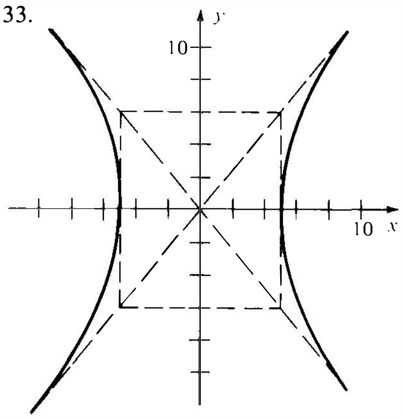
29.



31.



33.



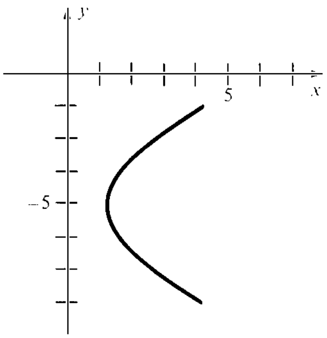
EXERCISE SET 8.5, page 370

1. parabola
3.
5. hyperbola
7. no graph
9. no graph
11.
13. hyperbola
15. point

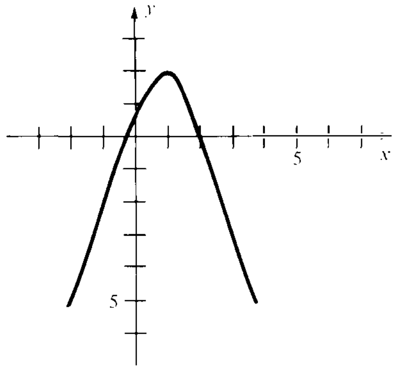
REVIEW EXERCISES, page 371

1. $(-1, -1)$
2. $(-5/2, 5/2)$
3. $(-1/2, -9/2)$
4. $(10, 7)$
5. $\overline{P_1P_2} = \overline{P_3P_4} = \sqrt{26}$, $\overline{P_1P_4} = \overline{P_2P_3} = 5$
6. $\overline{AB} = \sqrt{170}$, $\overline{AC} = \sqrt{136}$, $\overline{BC} = \sqrt{34}$, $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$
7. $10x + 12y + 15 = 0$
8. $(x + 5)^2 + (y - 2)^2 = 16$
9. $(x + 3)^2 + (y + 3)^2 = 4$
10. $(h, k) = (2, -3)$; $r = 3$
11. $(h, k) = (-1/2, 4)$; $r = 1/3$
12. $(h, k) = (-2, 3)$; $r = \sqrt{3}$
13. $(h, k) = (1, -1)$; $r = \sqrt{2}/2$
14. $(h, k) = (0, 3)$; $r = \sqrt{6}$
15. $(h, k) = (1, 1)$; $r = \sqrt{10}$

16. vertex: $(3/2, -5)$; axis: $y = -5$; direction: right



17. vertex: $(1, 2)$; axis: $x = 1$; direction: down



	Vertex	Axis	Direction
18.	$(-3, 0)$	$y = 0$	left
19.	$(2, -2)$	$y = -2$	left
20.	$(3, -2)$	$x = 3$	up
21.	$(-2, -1/2)$	$x = -2$	down
22.	$(0, 1)$	$y = 1$	right
23.	$(-3, 0)$	$x = -3$	down

24. $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$; $(\pm 2, 0)$

25. $\dfrac{x^2}{1} + \dfrac{y^2}{9} = 1$; $(\pm 1, 0)$, $(0, \pm 3)$

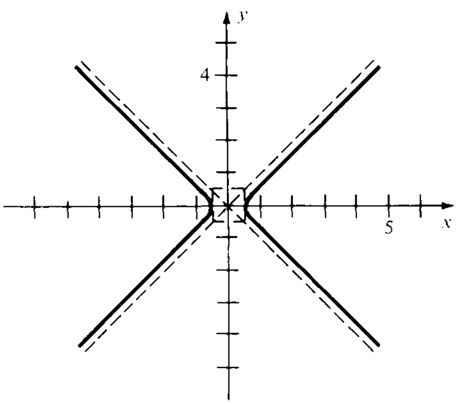
26. $\dfrac{x^2}{7} + \dfrac{y^2}{5} = 1$; $(\pm \sqrt{7}, 0)$, $(0, \pm \sqrt{5})$

27. $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$; $(\pm 4, 0)$

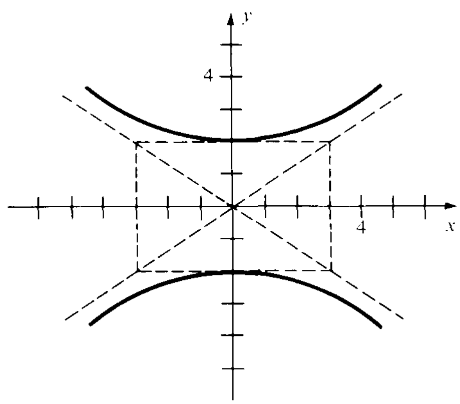
28. $\dfrac{x^2}{3} + \dfrac{y^2}{9/4} = 1$; $(\pm \sqrt{3}, 0)$, $(0, \pm 3/2)$

29. $\dfrac{y^2}{20/3} - \dfrac{x^2}{4} = 1$; $(0, \pm 2\sqrt{15}/3)$

30.



31.



32. parabola

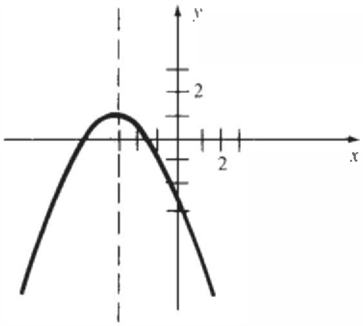
33. hyperbola

34. ellipse

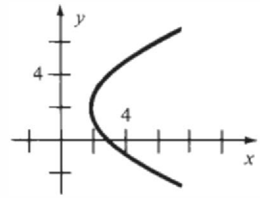
35. no graph

PROGRESS TEST 8A, page 372

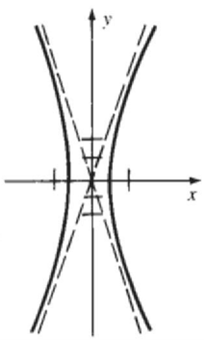
1. (0, 4)
2. (−1, 2)
3. slope AB = slope CD = −5/2; slope BC = slope AD = 2/7
4. $(x-2)^2 + (y+3)^2 = 36$
5. $(h, k) = (1, -2)$; $r = 2$
6. $(h, k) = (2, 0)$; $r = \sqrt{5}$
7. vertex: (−3, 1); axis: $x = -3$



8. vertex: (1, 2); axis: $y = 2$



9. vertex: (3, 2); axis: $x = 3$; direction: down
10. vertex: (−2, −4); axis: $y = -4$; direction: right
11. $\frac{x^2}{4} + \frac{y^2}{1} = 1$; (±2, 0), (0, ±1)
12. $\frac{y^2}{9} - \frac{x^2}{4} = 1$; (0, ±3)
13. $\frac{x^2}{1/4} - \frac{y^2}{1/4} = 1$; (±1/2, 0)
14.

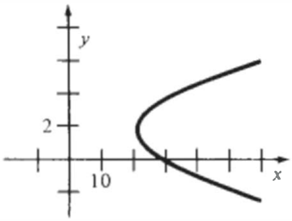


15. circle
16. ellipse

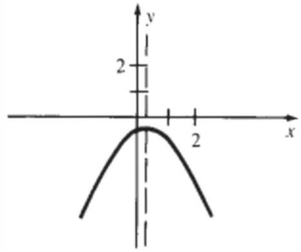
PROGRESS TEST 8B, page 372

1. (−1/2, −1)
2. (−7, 1)
3. $\overline{PR} = \overline{QS} = \sqrt{65}$
4. $(x+2)^2 + (y+5)^2 = 25$
5. $(h, k) = (-3, 2)$; $r = 3$
6. $(h, k) = (1/2, 1)$; $r = \sqrt{10}$

7. vertex: (22, 2); axis: $y = 2$; direction: right



8. vertex: (1/3, −1/3); axis: $x = 1/3$; direction: down



9. vertex: (−1, 2); axis: $y = 2$; direction: right

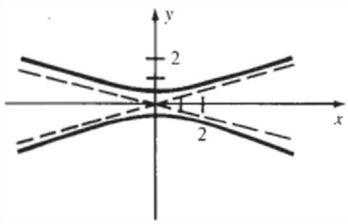
10. vertex: (1/2, −3); axis: $x = 1/2$; direction: up

11. $\frac{x^2}{5} + \frac{y^2}{25/9} = 1$; $(\pm\sqrt{5}, 0)$, $(0, \pm 5/3)$

12. $\frac{x^2}{3} + \frac{y^2}{7/2} = 1$; $(\pm\sqrt{3}, 0)$, $(0, \pm\sqrt{14}/2)$

13. $\frac{y^2}{9} - \frac{x^2}{3} = 1$; $(0, \pm 3)$

14.



15. hyperbola

16. parabola

CHAPTER 9

EXERCISE SET 9.1, page 380

1. $x = 2, y = -1$
3. none
5. $x = 1, y = -4$
7. $x = 3, y = 2$; $x = 3, y = -2$
9. no solution
11. $x = 2, y = -1$
13. $x = 3, y = 2$; $x = 1/5, y = -18/5$
15. $x = 1, y = 1$; $x = 9/16, y = -3/4$
17. $x = 1, y = 2$; $x = 13/5, y = -6/5$
19. $x = \frac{-1 + \sqrt{5}}{2}, y = \frac{1 + \sqrt{5}}{2}$; $x = \frac{-1 - \sqrt{5}}{2}, y = \frac{1 - \sqrt{5}}{2}$

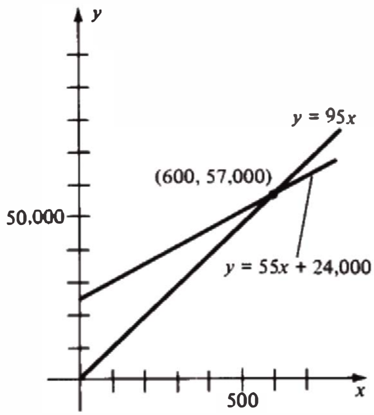
EXERCISE SET 9.2, page 383

1. $x = 3, y = -1$
3. no solution
5. $x = 3, y = 2$; $x = 3, y = -2$
7. none
9. $x = 3, y = 2$; $x = -3, y = 2$; $x = 3, y = -2$; $x = -3, y = -2$
11. I
13. I
15. I
17. C; all points on the line $3x - y = 18$
19. 22 nickels, 12 quarters
21. 6/5 pounds nuts, 4/5 pounds raisins
23. 6 and 8

EXERCISE SET 9.3, page 390

1. 25 nickels, 15 dimes
3. color: \$2.50; black and white: \$1.50
5. \$4000 in bond A, \$2000 in bond B
7. 10 rolls of 12″, 4 rolls of 15″
9. 8 pounds of \$1.20 coffee, 16 pounds of \$1.80 coffee
11. speed of bicycle: 105/8 mph; wind speed: 15/8 mph
13. 34
15. 30 pounds of nuts, 20 pounds of raisins
17. \$6000 in type A, \$12,000 in type B
19. 5 units Epiline I, 4 units Epiline II
21. (a) $R = 95x$

    (b)



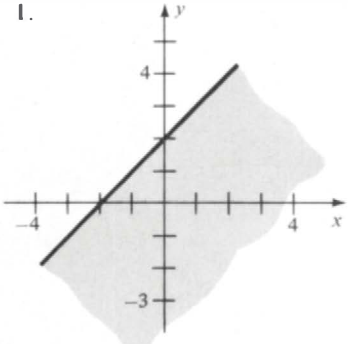
    (c) \$57,000
23. (a) $p = 4$ (b) 18
25. 4 and 5

EXERCISE SET 9.4, page 396

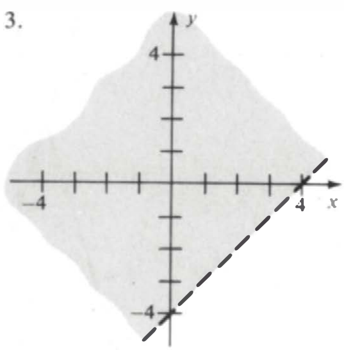
1. $x = 2, y = -1, z = -2$
3. $x = 1, y = 2/3, z = -2/3$
5. no solution
7. $x = 1, y = 2, z = 2$
9. $x = 1, y = 1, z = 0$
11. $x = 1, y = 27/2, z = -5/2$
13. no solution
15. no solution
17. $x = 5, y = -5, z = -20$
19. A:2; B:3; C:3
21. three 12″ sets, eight 16″ sets, five 19″ sets

EXERCISE SET 9.5, page 404

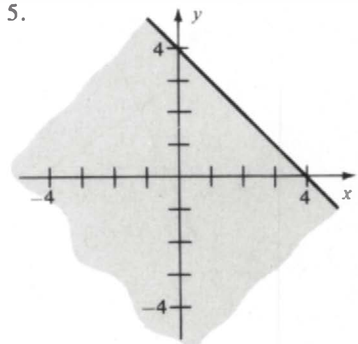
1.



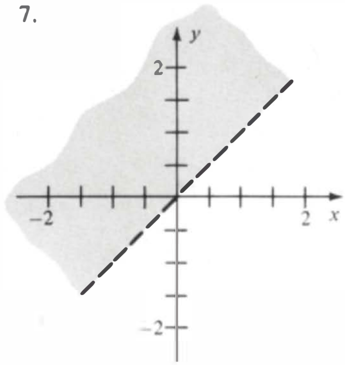
3.



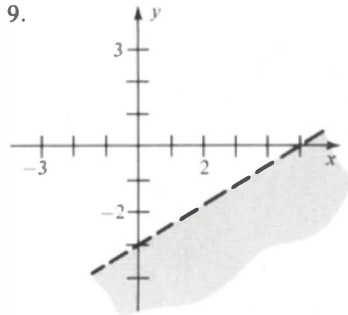
5.



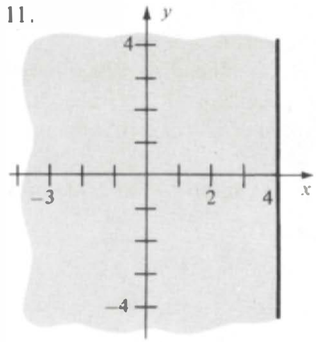
7.



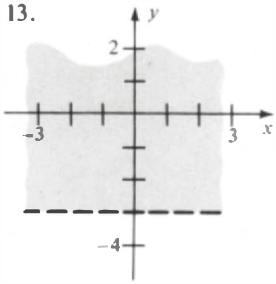
9.



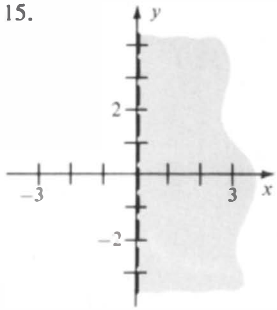
11.



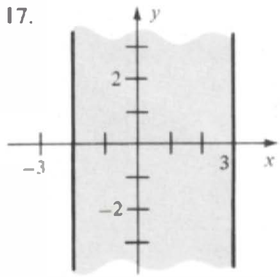
13.



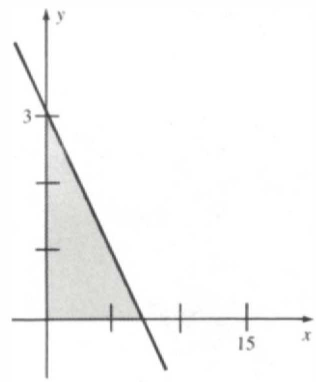
15.



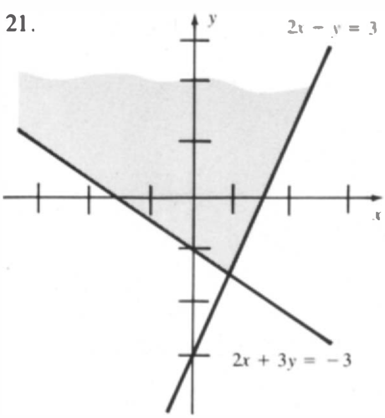
17.



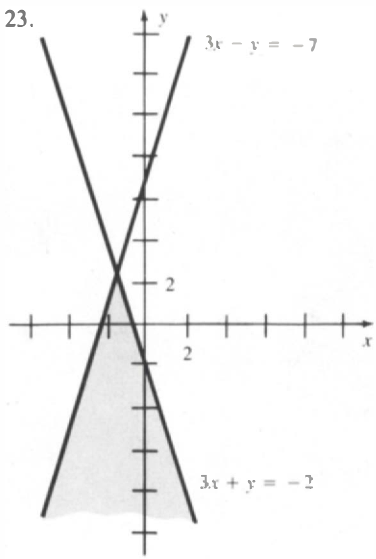
19. $2x + 5y \leq 15$; $x \geq 0$; $y \geq 0$



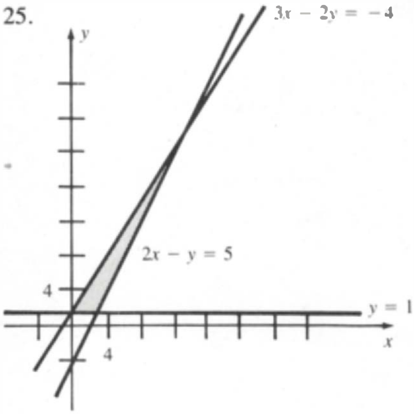
21.



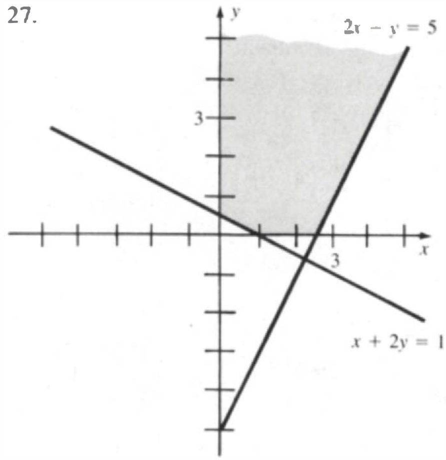
23.



25.

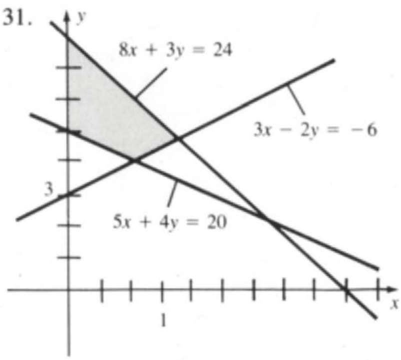


27.

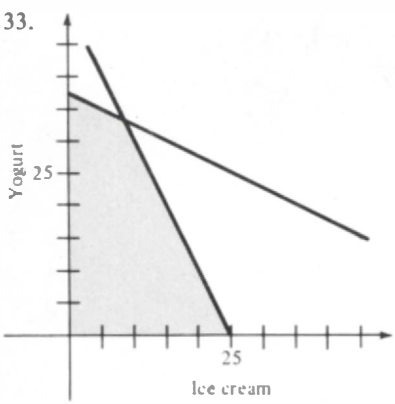


29. no solution

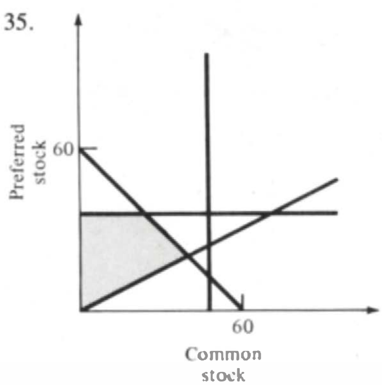
31.



33.



35.



EXERCISE SET 9.6, page 410

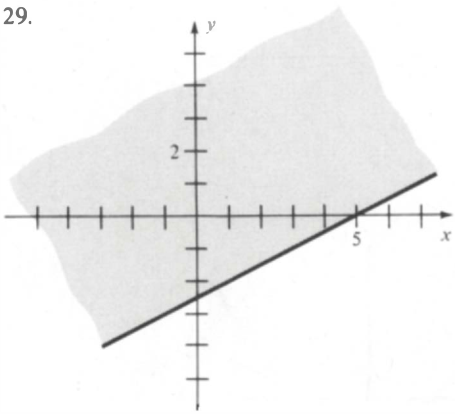
	Minimum	Maximum
1.	-2; (5, 14)	5; (5, 0)
3.	-3; (2, 2)	3; (6, 0)
5.	$\frac{6}{7}$; $\left(\frac{6}{7}, \frac{6}{7}\right)$	$\frac{19}{2}$; $\left(4, -\frac{3}{2}\right)$
7.	$\frac{1}{2}$; $\left(3, \frac{11}{2}\right)$	14; (8, 2)

9. preferred: 190/3 square feet
   regular: 30 square feet
11. large: 120
    small: 260
13. Java: 2000/11 pounds
    Colombian: 4000/11 pounds
15. crop A: 30 acres
    crop B: 70 acres
17. pack A: 6 pounds
    pack B: 12 pounds

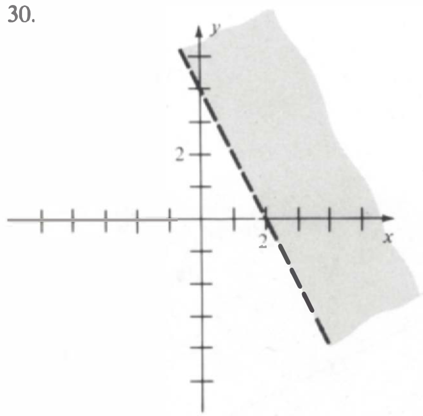
REVIEW EXERCISES, page 412

1. $x = -1/2,\ y = 1$
2. $x = 5,\ y = 2;\ x = 10,\ y = -3$
3. $x = 5,\ y = -1$
4. $x = -4,\ y = 3/2$
5. $x = 1/4,\ y = -1/2$
6. $x = 5,\ y = 0;\ x = -4,\ y = 3$
7. none
8. $x = 4,\ y = 4;\ x = 36/25,\ y = -12/5$
9. $x = -3,\ y = 5$
10. $x = 2,\ y = -2$
11. $x = 4,\ y = -1$
12. $x = -2,\ y = 3$
13. $x = 1,\ y = -1;\ x = 5,\ y = 3$
14. $x = 0,\ y = 3$
15. 45
16. 72
17. steak: \$3.25/lb; hamburger: \$1.80/lb
18. 600 kph
19. 3, 11
20. 575, \$9200
21. $x = -3,\ y = 1,\ z = 4$
22. $x = -2,\ y = 1/2,\ z = 3$
23. $x = 1,\ y = -1,\ z = 2$
24. $x = 3,\ y = 1/4,\ z = -1/3$
25. $x = -3,\ y = 4$
26. $x = -5/3,\ y = 5/6$
27. $x = -2,\ y = -1,\ z = -3$
28. $x = 1/2,\ y = -1,\ z = 1$

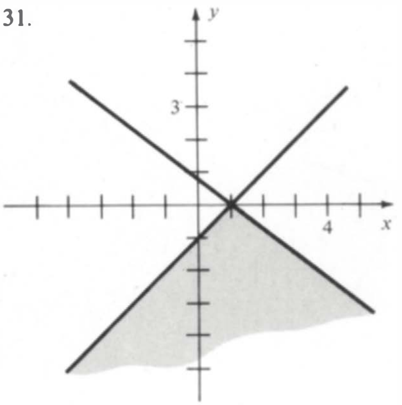
29.



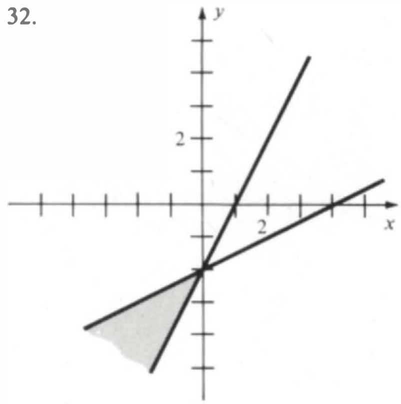
30.



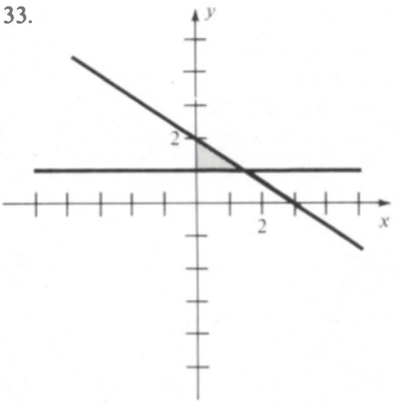
31.



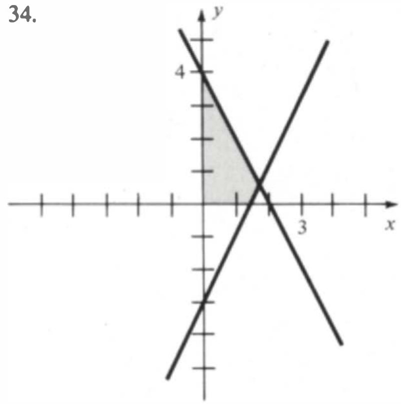
32.



33.



34.



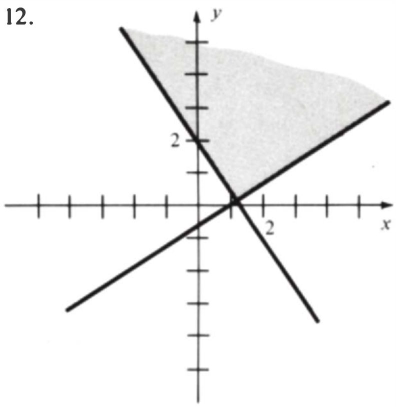
35. $x = 4$, $y = 6$, $z = 26$

36. $x = 11/2$, $y = 2$, $z = 27/2$

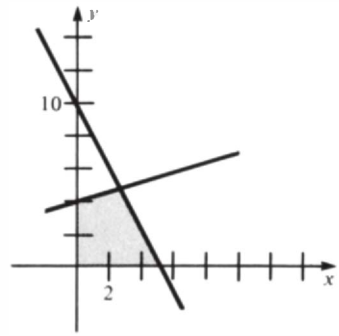
PROGRESS TEST 9A, page 414

1. $x = -5$, $y = 2$
2. $x = -1$, $y = 6$
3. $x = 2$, $y = \pm\sqrt{10}$; $x = 3$, $y = \pm\sqrt{15}$
4. $x = 1$, $y = -3$
5. $x = 3$, $y = \pm 4$; $x = -3$, $y = \pm 4$
6. 38
7. shirts: \$15; ties: \$10
8. 1100
9. $x = -2$, $y = 4$, $z = 6$
10. $x = -1/3$, $y = -1$
11. $x = 2/3$, $y = 2$, $z = -2$

12.



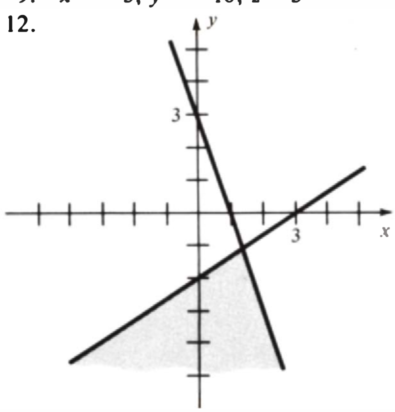
13.



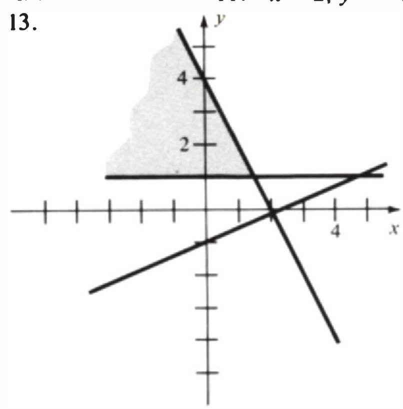
PROGRESS TEST 9B, page 414

1. $x = 0, y = 2; x = 3, y = 1$
2. $x = 2/3, y = -1$
3. $x = 1/5, y = -1/5$
4. $x = -7/2, y = -1/2$
5. $x = 5, y = \pm 4; x = -5, y = \pm 4$
6. 86
7. 5/2 kph
8. 6, 31
9. $x = -3, y = -10, z = 5$
10. $x = 1/2, y = -1/4$
11. $x = 2, y = -3, z = 1$

12.



13.



CHAPTER 10

EXERCISE SET 10.1, page 423

1. 2×2
3. 4×3
5. 3×3
7. (a) -4 (b) 7 (c) 6 (d) -3
9. $\begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix}$, $\left[\begin{array}{cc|c} 3 & -2 & 12 \\ 5 & 1 & -8 \end{array}\right]$
11. $\begin{bmatrix} 1/2 & 1 & 1 \\ 2 & -1 & -4 \\ 4 & 2 & -3 \end{bmatrix}$, $\left[\begin{array}{ccc|c} 1/2 & 1 & 1 & 4 \\ 2 & -1 & -4 & 6 \\ 4 & 2 & -3 & 8 \end{array}\right]$
13. $\begin{aligned} \tfrac{3}{2}x + 6y &= -1 \\ 4x + 5y &= 3 \end{aligned}$
15. $\begin{aligned} x + y + 3z &= -4 \\ -3x + 4y \quad &= 8 \\ 2x \quad + 7z &= 6 \end{aligned}$
17. $x = -13, y = 8, z = 2$
19. $x = 35, y = 14, z = -4$
21. $x = 2, y = 3$

23. $x = 2, y = -1, z = 3$
25. $x = 3, y = 2, z = -1$
27. $x = -5, y = 2, z = 3$
29. $x = -5/7, y = -2/7, z = -3/7, w = 2/7$

EXERCISE SET 10.2, page 431

1. $a = 3, b = -4, c = 6, d = -2$

3. $\begin{bmatrix} 2 & -1 & 5 \\ 7 & 1 & 6 \\ 4 & 3 & 7 \end{bmatrix}$

5. $\begin{bmatrix} -2 & 18 & 8 \\ -3 & 8 & 11 \end{bmatrix}$

7. not possible

9. $\begin{bmatrix} 17 & 5 \\ 10 & 12 \end{bmatrix}$

11. not possible

13. $\begin{bmatrix} 10 & 4 \\ 12 & 28 \end{bmatrix}$

15. not possible

17. $\begin{bmatrix} 18 & 23 & 29 \\ 17 & -12 & 13 \end{bmatrix}$

19. $AB = \begin{bmatrix} 8 & -6 \\ -8 & 6 \end{bmatrix}$ $AC = \begin{bmatrix} 8 & -6 \\ -8 & 6 \end{bmatrix}$

25. The amount of pesticide 2 eaten by herbivore 3.

27. $A = \begin{bmatrix} 3 & 4 \\ 3 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad B = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$

29. $A = \begin{bmatrix} 3 & -1 & 4 \\ 2 & 2 & 3/4 \\ 1 & -1/4 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 5 \\ -1 \\ 1/2 \end{bmatrix}$

31. $\begin{aligned} x_1 - 5x_2 &= 0 \\ 4x_1 + 3x_2 &= 2 \end{aligned}$

33. $\begin{aligned} 4x_1 + 5x_2 - 2x_3 &= 2 \\ 3x_2 - x_3 &= -5 \\ 2x_3 &= 4 \end{aligned}$

EXERCISE SET 10.3, page 442

1. no

3. yes

5. $\begin{bmatrix} 2/3 & 5/6 \\ 1/3 & 1/6 \end{bmatrix}$

7. $\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$

9. $\begin{bmatrix} 8 & 7 & -1 \\ -4 & -4 & 1 \\ -5 & -5 & 1 \end{bmatrix}$

11. $\begin{bmatrix} 4/7 & -3/7 \\ 1/7 & 1/7 \end{bmatrix}$

13. none

15. $\begin{bmatrix} 1/2 & 0 \\ 0 & -1/3 \end{bmatrix}$

17. $\begin{bmatrix} -1 & 1 & -1 \\ 2 & -1 & 2 \\ -2 & 1 & -1 \end{bmatrix}$

19. $x = 3, y = -1$

21. $x = 2, y = -3, z = 1$

23. $x = 0, y = 2, z = -3$

35. $x = \begin{bmatrix} 25 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 10 \end{bmatrix}$

EXERCISE SET 10.4, page 452

1. 22
3. -8
5. 0
7. (a) -6 (b) -1 (c) 1 (d) 7
9. (a) -6 (b) 1 (c) 1 (d) 7
11. 52
13. -3
15. 0
17. -12
19. 0
21. $x = 1, y = -2, z = -1$
23. $x = 3, y = 2, z = -1$
25. $x = -3, y = 0, z = 2$
27. $x = -5/7, y = -2/7, z = -3/7, w = 2/7$

REVIEW EXERCISES, page 454

1. 3×5

2. -1

3. 4

4. 8

5. $\begin{bmatrix} 3 & -7 \\ 1 & 4 \end{bmatrix}$

6. $\left[\begin{array}{cc|c} 3 & -7 & 14 \\ 1 & 4 & 6 \end{array}\right]$

7. $\begin{aligned} 4x - y &= 3 \\ 2x + 5y &= 0 \end{aligned}$

8. $\begin{aligned} -2x + 4y + 5z &= 0 \\ 6x - 9y + 4z &= 0 \\ 3x + 2y - z &= 0 \end{aligned}$

9. $x = -1, y = -4$

10. $x = 1/2, y = 5$

11. $x = -4, y = 3, z = -1$

12. $x = -1, y = 1, z = -3$

13. $x = 1/2, y = 3/2$

14. $x = -5, y = 2$

15. $x = 3, y = 1/3, z = -2$

16. $x = 3 + 5t/4, y = 3 + t/2, z = t$

17. -3

18. -3

19. $\begin{bmatrix} 1 & 4 \\ 7 & -1 \end{bmatrix}$

20. $\begin{bmatrix} -3 & 6 \\ 1 & -5 \end{bmatrix}$

21. not possible

22. $\begin{bmatrix} 5 & 15 & 20 \\ -5 & 0 & -30 \end{bmatrix}$

23. $\begin{bmatrix} -1 & -3 & -4 \\ -4 & 0 & -24 \\ 4 & 6 & 20 \end{bmatrix}$

24. $\begin{bmatrix} 7 & 4 \\ -11 & 12 \end{bmatrix}$

25. not possible

26. $\begin{bmatrix} 1 & -5 \\ 16 & -12 \\ -10 & 16 \end{bmatrix}$

27. $\begin{bmatrix} 0 & 9 \\ 11 & -4 \end{bmatrix}$

28. $\begin{bmatrix} 6 & -13 \\ -5 & -9 \end{bmatrix}$

29. $\begin{bmatrix} -4/11 & 3/11 \\ 1/11 & 2/11 \end{bmatrix}$

30. $\begin{bmatrix} 2/5 & -7/5 & -8/5 \\ -1 & 3 & 4 \\ -2/5 & 2/5 & 3/5 \end{bmatrix}$

31. $x = 2, y = 3$

32. $x = -1, y = -1, z = 1/2$

33. 10

34. -6

35. 0

36. 12

37. 0

38. -3

39. $x = 1/2, y = 4$

40. $x = 1, y = -4$

41. $x = 10, y = -4$

42. $x = -4, y = 2, z = 1$

43. $x = 1/3, y = 2/3, z = -1$

44. $x = 1/4, y = -2, z = 1/2$

PROGRESS TEST 10A, page 455

1. 3×2

2. 0

3. $\left[\begin{array}{ccc|c} -7 & 0 & 6 & 3 \\ 0 & 2 & -1 & 10 \\ 1 & -1 & 1 & 5 \end{array}\right]$

4. $\begin{aligned} -5x + 2y &= 4 \\ 3x - 4y &= 4 \end{aligned}$

5. $x = -1/2, y = 1/2$

6. $x = -6, y = -2$

7. $x = 1/2, y = 1/2, z = 1/2$

8. 3

9. $\begin{bmatrix} 2 & 14 \\ -2 & -4 \\ -5 & 1 \end{bmatrix}$

10. $\begin{bmatrix} -7 & -11 \\ 11 & 15 \end{bmatrix}$

11. $\begin{bmatrix} -10 \\ 2 \\ 0 \end{bmatrix}$

12. not possible

13. $\begin{bmatrix} 1/27 & 12/27 & 4/27 \\ 5/27 & 6/27 & -7/27 \\ 7/27 & 3/27 & 1/27 \end{bmatrix}$

14. $x = -2, y = 1$

15. -2

16. 27

17. $x = 4, y = -3$

PROGRESS TEST 10B, page 456

1. 2×4

2. 1

3. $\left[\begin{array}{rr|r} 2 & -6 & 5 \\ 1 & 3 & -2 \end{array}\right]$

4. $\begin{aligned} 16x \phantom{{}-2y} + 6z &= 10 \\ -4x - 2y + 5z &= 8 \\ 2x + 3y - z &= -6 \end{aligned}$

5. $x = 0, y = -1, z = -3$

6. $x = -1, y = -3$

7. $x = 5, y = -2, z = 1/2$

8. -5

9. $[10, 0]$

10. $\begin{bmatrix} 6 & -17 \\ 3 & 2 \\ 2 & 13 \end{bmatrix}$

11. not possible

12. not possible

13. $\begin{bmatrix} -3/2 & -5/2 & -3/4 \\ -1/2 & -1/2 & -1/4 \\ -3/4 & -5/4 & -1/8 \end{bmatrix}$

14. $x = -1, y = 1/2$

15. -22

16. -1

17. $x = -3/2, y = 1/2$

CHAPTER 11

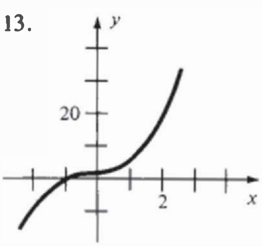
EXERCISE SET 11.1, page 464

1. $Q(x) = x - 2, R(x) = 2$
3. $Q(x) = 2x - 4, R(x) = 8x - 4$
5. $Q(x) = 3x^3 - 9x^2 + 25x - 75, R(x) = 226$
7. $Q(x) = 2x - 3, R(x) = -4x + 6$
9. $Q(x) = x^2 - x + 1, R(x) = 0$
11. $Q(x) = x^2 - 3x, R = 5$
13. $Q(x) = x^3 + 3x^2 + 9x + 27, R = 0$
15. $Q(x) = 3x^2 - 4x + 4, R = 4$
17. $Q(x) = x^4 - 2x^3 + 4x^2 - 8x + 16, R = 0$
19. $Q(x) = 6x^3 + 18x^2 + 53x + 159, R = 481$

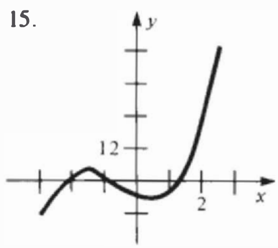
EXERCISE SET 11.2, page 468

1. -7
3. -34
5. 0
7. -1
9. 0
11. -62

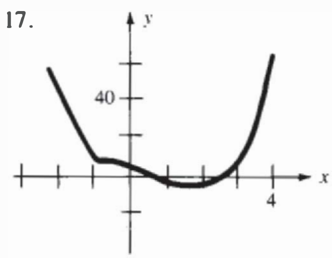
13.



15.



17.



19. yes
21. no
23. yes
25. yes
27. $r = 3, -1$
29. $5/2$

EXERCISE SET 11.3, page 479

1. 5
3. 25
5. 20
7. $-13/10 + 11i/10$
9. $-7/25 - 24i/25$
11. $8/5 - i/5$
13. $5/3 - 2i/3$
15. $4/5 + 8i/5$
17. $4/25 - 3i/25$
19. $9/10 + 3i/10$
21. $0 + i/5$
25. $x^3 - 2x^2 - 16x + 32$
27. $x^3 + 6x^2 + 11x + 6$
29. $x^3 - 6x^2 + 6x + 8$
31. $x^3/3 + x^2/3 - 7x/12 + 1/6$
33. $x^3 - 4x^2 - 2x + 8$
35. $3, -1, 2$
37. $-2, 4, -4$
39. $-2, -1, 0, -1/2$
41. $5, 5, 5, -5, -5$
43. $x^3 + 6x^2 + 12x + 8$
45. $4x^4 + 4x^3 - 3x^2 - 2x + 1$
47. $2, -1$
49. $(3 \pm i\sqrt{3})/2$
51. $-1, -2, 4$
53. $x^2 + (1 - 3i)x - (2 + 6i)$
55. $x^2 - 3x + (3 + i)$
57. $x^3 + (1 + 2i)x^2 + (-8 + 8i)x + (-12 + 8i)$
59. $(x^2 - 6x + 10)(x - 1)$
61. $(x^2 + 2x + 5)(x^2 + 2x + 4)$
63. $(x - 2)(x + 2)(x - 3)(x^2 + 6x + 10)$
65. $x - (a + bi)$

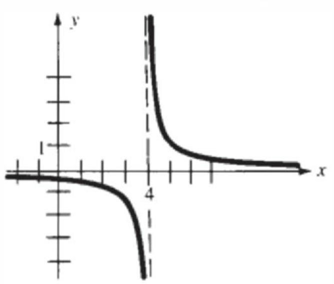
EXERCISE SET 11.4, page 487

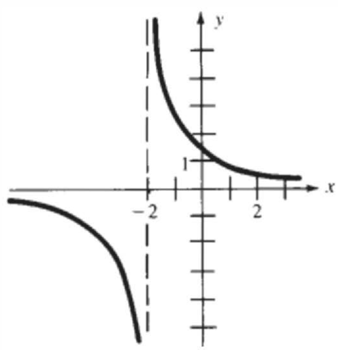
	Positive roots	Negative roots	Complex roots
1.	3	1	0
	1	1	2
3.	0	0	6
5.	3	2	0
	1	2	2
	3	0	2
	1	0	4
7.	1	2	0
	1	0	2
9.	2	0	2
	0	0	4
11.	1	1	6

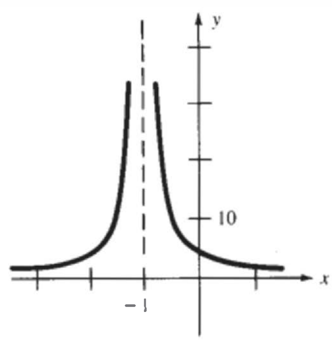
13. $1, -2, 3$
15. $2, -1, -1/2, 2/3$
17. $1, -1, -1, 1/5$
19. $1, -3/4$
21. $3, 3, 1/2$
23. $-1, 3/4, \pm i$
25. $3/5, \pm 2, \pm i\sqrt{2}$
27. $0, 1/2, 2/3, -1$
29. $1/2, -4, 2 \pm \sqrt{2}$
31. $k = 3, r = -2$
33. $k = 7, r = 1; k = -7, r = -1$

EXERCISE SET 11.5, page 498

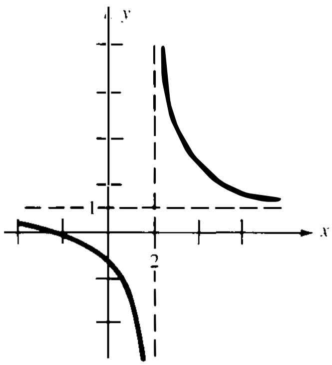
1. $x \neq 1$
3. $x \neq 0, 2$
5. all real numbers
7. $x = 4, y = 0$
9. $x = -2, y = 0$
11. $x = -1, y = 0$



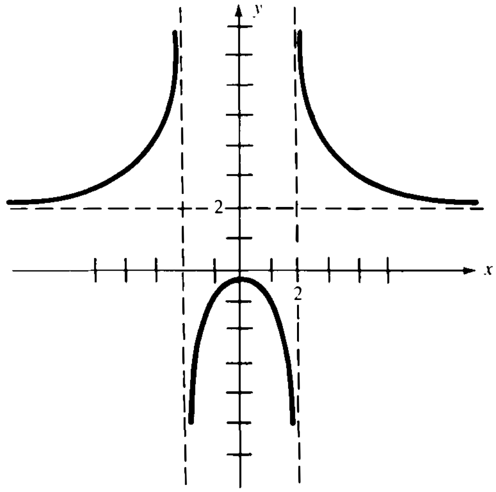




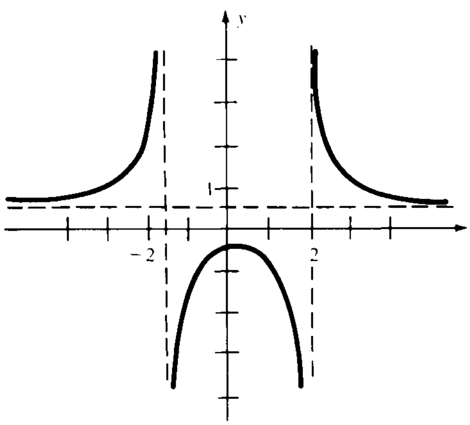
13. $x = 2, y = 1$



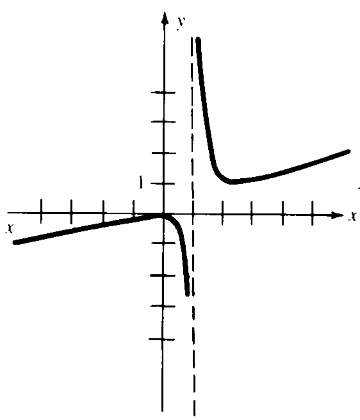
15. $x = 2, x = -2, y = 2$



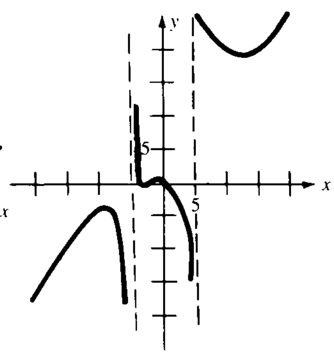
17. $x = 2, x = -3/2, y = 1/2$



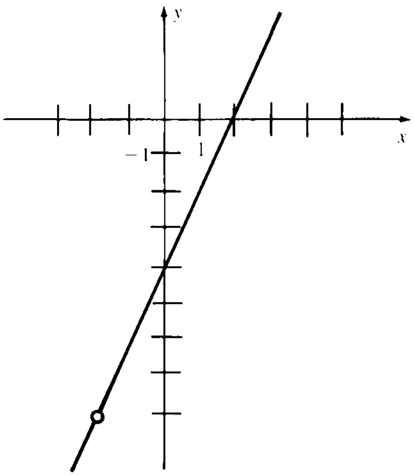
19. $x = 1$



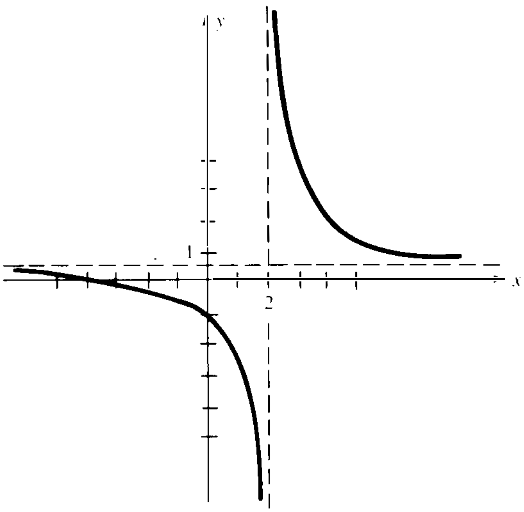
21. $x = 5, x = -5$



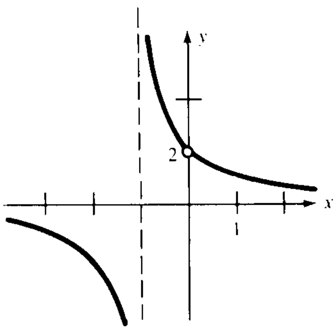
23. $x \neq -2$



25. $x \neq 2$



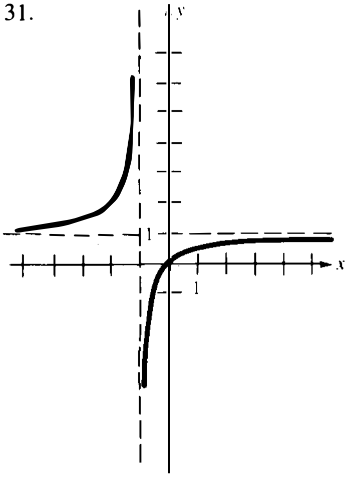
27. $x \neq 0, x \neq -1$



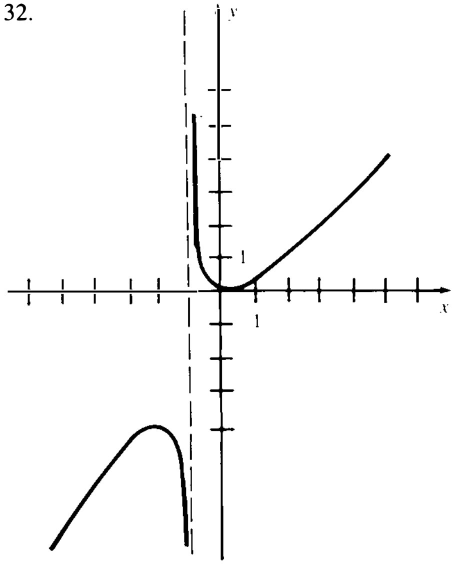
REVIEW EXERCISES, page 500

1. $Q(x) = 2x^2 + 2x + 8, R = 4$
2. $Q(x) = x^3 - 5x^2 + 10x - 18, R = 31$
3. 46, −8
4. 4, 1
7. $6/25 - 17i/25$
8. $-1/5 + 2i/5$
9. $-5/2 + 5i/2$
10. $1/10 - 3i/10$
11. $i/4$
12. $2/29 + 5i/29$
13. $x^3 + 6x^2 + 11x + 6$
14. $x^3 - 3x^2 + 3x - 9$
15. $x^4 + x^3 - 5x^2 - 3x + 6$
16. $4x^4 + 4x^3 - 3x^2 - 2x + 1$
17. $x^4 + 2x^2 + 1$
18. $x^4 - 6x^2 - 8x - 3$
19. −1/2, 3
20. $-1 \pm \sqrt{2}$
21. $4, 2 + i, 2 - i$
22. 1 positive, 1 negative
23. 5 positive, 0 negative
24. 1 positive, 0 negative
25. 2 positive, 2 negative
26. 3, −2/3, −3/2
27. 1, −2, 2/3, 3/2
28. none
29. $-1, (-9 \pm \sqrt{321})/12$
30. $2, 3/2, -1 \pm \sqrt{2}$

31.

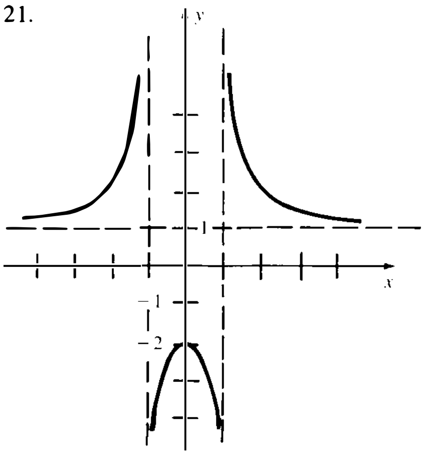


32.



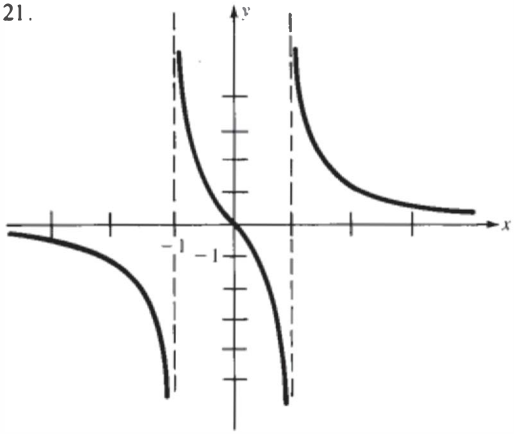
PROGRESS TEST 11A, page 501

1. $Q(x) = 2x^2 - 5$, $R(x) = 11$
2. $Q(x) = 3x^3 - 7x^2 + 14x - 28$, $R(x) = 54$
3. -25
4. -165
6. $x^3 - 2x^2 - 5x + 6$
7. $x^4 - 6x^3 + 6x^2 + 6x - 7$
8. $2, \pm i$
9. $-1, -1, (3 \pm \sqrt{17})/2$
10. $x^5 + 3x^4 - 6x^3 - 10x^2 + 21x - 9$
11. $16x^5 - 8x^4 + 9x^3 - 9x^2 - 7x - 1$
12. $x^2 - (1 + 2i)x + (-1 + i)$
13. $1/2, 1/2$
14. $1, -1 \pm i$
15. $(x^2 - 4x + 5)(x - 2)$
16. 2
17. 1
18. none
19. $1, 1, -1, -1, 1/2$
20. $2/3, -3, \pm i$
21.



PROGRESS TEST 11B, page 501

1. $Q(x) = 3x^3 + 2x^2 + 2$, $R(x) = x + 3$
2. $Q(x) = -2x^2 + x + 1$, $R(x) = 0$
3. -1
4. 24
6. $2x^4 - x^3 - 3x^2 + x + 1$
7. $x^3 - 4x^2 + 2x + 4$
8. 1, 2, 2, 2
9. $(-3 \pm \sqrt{13})/2$, -3, -3
10. $8x^4 + 4x^3 - 18x^2 + 11x - 2$
11. $x^4 + 4x^3 - x^2 - 6x + 18$
12. $x^4 - 4x^3 - x^2 + 14x + 10$
13. $(3 \pm \sqrt{17})/2$
14. $-2 \pm 2\sqrt{2}$
15. $(x^2 - 2x + 2)(2x - 1)(x + 2)$
16. 1
17. 2
18. -1, $2/3$, -2
19. $-1/2$, $3/2$, $\pm i$
20. 0, 1/2, $\pm\sqrt{2}$
21.



CHAPTER 12

EXERCISE SET 12.1, page 511

1. 2, 4, 6, 8; 40
3. 1, 5, 9, 13; 77
5. 5, 5, 5, 5; 5
7. 1/2, 2/3, 3/4, 4/5; 20/21
9. 2.1, 2.01, 2.001, 2.0001; $2 + 0.1^{20}$
11. 1/3, 4/5, 9/7, 16/9; 400/41
13. 9
15. 4/7
17. 256
19. 40
21. 97
23. 49/12
25. 80
27. $\sum_{k=1}^{5} (2k - 1)$
29. $\sum_{k=1}^{5} k^2$
31. $\sum_{k=1}^{4} \frac{(-1)^k}{\sqrt{k}}$
33. $\sum_{k=1}^{4} \frac{(-1)^{k+1}k}{k^2 + 1}$
35. $\sum_{k=0}^{n} \frac{1}{x^k}$

EXERCISE SET 12.2, page 517

1. 15, 18
3. 1, 5/4
5. 4, 5
7. $\sqrt{5} + 6$, $\sqrt{5} + 8$
9. 2, 6, 10, 14
11. 3, 5/2, 2, 3/2
13. 1/3, 0, $-1/3$, $-2/3$
15. 25
17. -8
19. -2
21. 19/3
23. 821/160
25. 440
27. -126
29. 1720
31. 30
33. $n = 30$, $d = 3$
35. $n = 6$, $d = 1/4$
37. -2

EXERCISE SET 12.3, page 526

1. 48
3. $-81/64$
5. 0.0096
7. 3, 9, 27, 81
9. 4, 2, 1, 1/2
11. $-3, -6, -12, -24$
13. -384
15. 1/4
17. 1/243
19. 27/8
21. 2
23. 7
25. 1, 3
27. 1/4, 1/16
29. 1093/243
31. $-1353/625$
33. 1020
35. 55/8
37. $10,235
39. 58,594
41. 2
43. 3/4
45. 8/3
47. 1
49. 1/5
51. 364/990
53. 325/999

EXERCISE SET 12.5, page 537

1. $243x^5 + 810x^4y + 1080x^3y^2 + 720x^2y^3 + 240xy^4 + 32y^5$
3. $256x^4 - 256x^3y + 96x^2y^2 - 16xy^3 + y^4$
5. $32 - 80xy + 80x^2y^2 - 40x^3y^3 + 10x^4y^4 - x^5y^5$
7. $a^8b^4 + 12a^6b^3 + 54a^4b^2 + 108a^2b + 81$
9. $a^8 - 16a^7b + 112a^6b^2 - 448a^5b^3 + 1120a^4b^4 - 1792a^3b^5 + 1792a^2b^6 - 1024ab^7 + 256b^8$
11. $\frac{1}{27}x^3 + \frac{2}{3}x^2 + 4x + 8$
13. $1024 + 5120x + 11{,}520x^2 + 15{,}360x^3$
15. $19{,}683 - 118{,}098a + 314{,}928a^2 - 489{,}888a^3$
17. $16{,}384x^{14} - 344{,}064x^{13}y + 3{,}354{,}624x^{12}y^2 - 20{,}127{,}744x^{11}y^3$
19. $8192x^{13} - 53{,}248x^{12}yz + 159{,}744x^{11}y^2z^2 - 292{,}864x^{10}y^3z^3$
21. 120
23. 12
25. 990
27. 5040
29. 120
31. 210
33. $-35{,}840x^4$
35. $\frac{495}{256}x^8y^4$
37. $2016x^{-5}$
39. $-540x^3y^3$
41. $181{,}440x^4y^3$
43. $-144x^6$
45. $\frac{35}{8}x^{12}$
47. 4.8268

EXERCISE SET 12.6, page 545

1. 120
3. 146,016
5. 256
7. 5040
9. 720
11. 336
13. 90
15. 84
17. 3
19. 210
21. 120
23. 60
25. 336
27. 120
29. 84
31. 45
33. 1
35. n
37. $(n^2 + n)/2$
39. 3003
41. (a) 15,600 (b) 17,576
43. 12,271,512
45. 240
47. 59,400
49. $(26!)^2/22!20!6!4!$

EXERCISE SET 12.7, page 552

1. 1/2
3. (a) 1/2 (b) 1/4 (c) 1/13
5. (a) 11/36 (b) 5/36 (c) 4/9
7. (a) 99/100 (b) 1/100
9. (a) 1/17170 (b) 1164/8585 (c) 7372/8585
11. 11/4165
13. 8008/23023
15. 75/612
17. (a) 0.922 (b) 1.6×10^{-7}

REVIEW EXERCISES, page 554

1. 3, 7, 13; 111
2. 0, 7/3, 13/2; 999/11
3. 2
4. 120
5. -16
6. 62
7. 50

8. $\sum_{k=1}^{4} \frac{k}{k+2}$

9. $\sum_{k=0}^{4} (-1)^k x^k$

10. $\sum_{k=1}^{n} \log kx$

11. 38
12. -9
13. 8
14. $-33/2$
15. 275/3
16. -450
17. -3
18. $-3/2$
19. 5, 1, 1/5, 1/25
20. $-2, 2, -2, 2$
21. 243/8
22. ± 256
23. 1/2, 1/12
24. 21/32
25. -728
26. 10
27. 9/5
29. $16x^4 - 32x^3y + 24x^2y^2 - 8xy^3 + y^4$
30. $x^4/16 - x^3 + 6x^2 - 16x + 16$
31. $x^6 + 3x^4 + 3x^2 + 1$
32. 720
33. 78
34. $(n+1)n$
35. 15
36. 1
37. 90
38. 24
39. 360
40. 210
41. 9
42. 2/9
43. 2/7
44. 8/663
45. 0.81
46. 1/33

PROGRESS TEST 12A, page 556

1. 1/4, 2/9, 3/16, 4/25
2. 29/6
3. -1, 1/2, 2, 7/2
4. 8
5. 41
6. 55/2
7. 1/3
8. $-2/3, -4/3, -8/3, -16/3$
9. -2
10. 8, -16
11. $-43/8$
12. -6
14. $a^{10} + 10a^9/b + 45a^8/b^2 + 120a^7/b^3$
15. 22
16. 360
17. $n + 1$
18. 24
19. 8.46×10^{20}
20. 1/3
21. 3/35

PROGRESS TEST 12B, page 556

1. 5/3, 5, 51/5, 52/3
2. $\sum_{k=2}^{n} k!$
3. 6, 16/3, 14/3, 4
4. 10
5. -55
6. 13, 1/4
7. 0.2
8. $-1, 1/4, -1/16, 1/64$
9. $-1/81$
10. $-4, -8/3$
11. $-31/2$
12. 25/7
14. $11520x^8$
15. $n/(n+1)$
16. 7
17. 2
18. 360
19. 200
20. 0.8×10^{-5}
21. 7/12

SOLUTIONS TO SELECTED REVIEW EXERCISES

CHAPTER 1

1. $\{1, 2, 3, 4\}$. (The negative integers and zero are not natural numbers.)
4. T. (Irrational numbers are a subset of the real numbers.)
6. F. (The negative integers and zero are a subset of the integers.)
15. $$|-3| - |1 - 5| = |-3| - |-4| = 3 - 4 = -1$$
16. $\overline{PQ} = |9/2 - 6| = |-3/2| = 3/2$
18. c. (Every exponent of a polynomial must be a nonnegative integer.)
22. $$x(2x - 1)(x + 2) = (2x^2 - x)(x + 2) = 2x^3 + 3x^2 - 2x$$
26. $2a^2 + 3ab + 6a + 9b$

$= (2a^2 + 6a) + (3ab + 9b)$	Grouping
$= 2a(a + 3) + 3b(a + 3)$	Common factors $2a$, $3b$
$= (a + 3)(2a + 3b)$	Common factor $a + 3$

28. $$\begin{aligned} x^8 - 1 &= (x^4)^2 - (1)^2 \\ &= (x^4 + 1)(x^4 - 1) \\ &= (x^4 + 1)(x^2 + 1)(x^2 - 1) \\ &= (x^4 + 1)(x^2 + 1)(x + 1)(x - 1) \end{aligned}$$
33. $$\begin{aligned} &\frac{x^2 - 2x - 3}{2x^2 - x} \div \frac{x^2 - 4x + 3}{3x^3 - 3x^2} \\ &= \frac{x^2 - 2x - 3}{2x^2 - x} \cdot \frac{3x^3 - 3x^2}{x^2 - 4x + 3} \\ &= \frac{(x + 1)(x - 3)}{x(2x - 1)} \cdot \frac{3x^2(x - 1)}{(x - 1)(x - 3)} \\ &= \frac{3x(x + 1)}{2x - 1}, \qquad x \neq 0, 1, 3 \end{aligned}$$
34. Factor each denominator:
$$\frac{-1}{2x^2} \qquad \frac{2}{(x + 2)(x - 2)} \qquad \frac{3}{x - 2}$$
Product of all factors each to its highest power:
$$2x^2(x + 2)(x - 2)$$
39. LCD $= (x + 4)(x - 4)$
$$\begin{aligned} \frac{3}{x^2 - 16} - \frac{2}{x - 4} &= \frac{3}{(x + 4)(x - 4)} - \frac{2(x + 4)}{(x + 4)(x - 4)} \\ &= \frac{3 - 2(x + 4)}{(x + 4)(x - 4)} = \frac{-5 - 2x}{(x + 4)(x - 4)} \end{aligned}$$
40. Multiply numerator and denominator by
$$\text{LCD} = (x + 2)(x - 1)$$
$$\frac{\dfrac{3}{x + 2} - \dfrac{2}{x - 1}}{x - 1} = \frac{3(x - 1) - 2(x + 2)}{(x + 2)(x - 1)^2} = \frac{x - 7}{(x + 2)(x - 1)^2}$$
42. $$(2a^2b^{-3})^{-3} = (2)^{-3}(a^2)^{-3}(b^{-3})^{-3} = \frac{1}{8}a^{-6}b^9 = \frac{b^9}{8a^6}$$
46. $\sqrt{80} = \sqrt{16 \cdot 5} = \sqrt{16} \cdot \sqrt{5} = 4\sqrt{5}$
48. $$\sqrt{x^7y^5} = (x^7y^5)^{1/2} = x^{7/2}y^{5/2} = x^3x^{1/2}y^2y^{1/2} = x^3y^2\sqrt{xy}$$
or
$$\sqrt{x^7y^5} = \sqrt{x^6xy^4y} = x^3y^2\sqrt{xy}$$
50. $$\frac{\sqrt{x}}{\sqrt{x} + \sqrt{y}} = \frac{\sqrt{x}}{\sqrt{x} + \sqrt{y}} \cdot \frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} - \sqrt{y}} = \frac{x - \sqrt{xy}}{x - y}$$
51. $$\sqrt[4]{x^2y^2} + 2\sqrt[4]{x^2y^2} = 3\sqrt[4]{x^2y^2} = 3|x|^{1/2}|y|^{1/2} = 3\sqrt{|xy|}$$
53. Equate the real and the imaginary parts.
$$x - 2 = -4 \qquad 2y - 1 = 7$$
$$x = -2 \qquad y = 4$$

54. $i^{47} = i^{44} \cdot i^3 = i^3 = -i$

56. $(2 + i)(2 + i) = 4 + 2i + 2i + i^2$
$$= 4 + 4i - 1$$
$$= 3 + 4i$$

CHAPTER 2

4. $k - 2x = 4kx$
$$k = 4kx + 2x$$
$$k = x(4k + 2)$$
$$x = \frac{k}{4k + 2} = \frac{k}{2(2k + 1)}$$

6. Let n be the number of quarters.

	Coins	× Cents per coin	= Value
Quarters	n	25	$25n$
Dimes	$2n + 4$	10	$10(2n + 4)$

$$\text{Total value} = \begin{pmatrix}\text{value of}\\ \text{quarters}\end{pmatrix} + \begin{pmatrix}\text{value of}\\ \text{dimes}\end{pmatrix}$$
$$265 = 25n + 10(2n + 4)$$
$$265 = 25n + 20n + 40$$
$$225 = 45n$$
$$n = 5 = \text{number of quarters}$$
$$2n + 4 = 14 = \text{number of dimes}$$

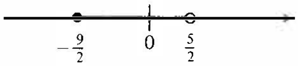
8. Let x be the number of hours for machine B.

	Rate	× Time	= Work
A	$\frac{1}{3}$	2	$\frac{2}{3}$
B	$\frac{1}{x}$	2	$\frac{2}{x}$

$$(\text{work done by A}) + (\text{work done by B}) = 1$$
$$\frac{2}{3} + \frac{2}{x} = 1$$
$$2x + 6 = 3x$$
$$x = 6$$

9. F. The equation does not hold for $x = 0$, and therefore it does not hold for all real values of x.

12. $-4 < -2x + 1 \leq 10$
$$-5 < -2x \leq 9$$
$$\frac{-5}{-2} > \frac{-2x}{-2} \geq \frac{9}{-2}$$
$$\frac{5}{2} > x \geq -\frac{9}{2}$$
or
$$-\frac{9}{2} \leq x < \frac{5}{2}$$



14. Since the numerator is negative, the denominator must be positive if the quotient is to be negative. Note also that the denominator cannot equal 0.
$$2x - 5 > 0$$
$$x > 5/2 \quad \text{or} \quad (5/2, \infty)$$

16. $|3x + 2| = 7$

$3x + 2 = 7$ $\qquad$ $-(3x + 2) = 7$

$3x = 5$ $\qquad$ $-3x = 9$

$x = 5/3$ $\qquad$ $x = -3$

19. $|2 - 5x| < 1$
$$-1 < 2 - 5x < 1$$
$$-3 < -5x < -1$$
$$\frac{-3}{-5} > \frac{-5x}{-5} > \frac{-1}{-5}$$
$$\frac{3}{5} > x > \frac{1}{5}$$
$$\frac{1}{5} < x < \frac{3}{5} \quad \text{or} \quad \left(\frac{1}{5}, \frac{3}{5}\right)$$

22. $6x^2 - 11x + 4 = (2x - 1)(3x - 4)$

$2x - 1 = 0$ $\qquad$ $3x - 4 = 0$

$x = \frac{1}{2}$ $\qquad$ $x = \frac{4}{3}$

23. $x^2 - 2x = -6$
$$x^2 - 2x + 1 = -6 + 1$$
$$(x - 1)^2 = -5$$
$$x - 1 = \pm\sqrt{-5}$$
$$x = 1 \pm i\sqrt{5}$$

27. $$\begin{aligned} kx^2 - 3\pi &= 0 \\ kx^2 &= 3\pi \\ x^2 &= \frac{3\pi}{k} \\ x &= \pm\sqrt{\frac{3\pi}{k}} = \pm\frac{\sqrt{3\pi k}}{k} \end{aligned}$$

29. $$\begin{aligned} 3r^2 - 2r - 5 &= 0 \\ a = 3,\ b = -2,\ c &= -5 \\ b^2 - 4ac &= 64 \end{aligned}$$

Since $b^2 - 4ac$ is positive and a square, the roots are real and rational.

33. $$\begin{aligned} &\sqrt{x+3} + \sqrt{2x-3} = 6 \\ &\sqrt{x+3} = 6 - \sqrt{2x-3} \\ &x + 3 = 36 - 12\sqrt{2x-3} + 2x - 3 \\ &12\sqrt{2x-3} = x + 30 \\ &144(2x-3) = x^2 + 60x + 900 \\ &x^2 - 228x + 1332 = 0 \\ &(x-6)(x-222) = 0 \\ &x = 6 \qquad x = 222 \end{aligned}$$

Check:

$$\begin{array}{cc} x = 6 & x = 222 \\ \sqrt{6+3} + \sqrt{12-3} \stackrel{?}{=} 6 & \sqrt{225} + \sqrt{441} \stackrel{?}{=} 6 \\ 3 + 3 \stackrel{?}{=} 6 & 15 + 21 \stackrel{?}{=} 6 \\ 6 \doteq 6 & 36 \neq 6 \end{array}$$

The solution is 6.

35. Let $u = 1 - \dfrac{2}{x}$.

$$\begin{aligned} u^2 - 8u + 15 &= 0 \\ (u-3)(u-5) &= 0 \\ u = 3 \qquad u &= 5 \end{aligned}$$

Substituting, we have

$$\begin{array}{cc} 3 = 1 - \dfrac{2}{x} & 5 = 1 - \dfrac{2}{x} \\ 2 = -\dfrac{2}{x} & 4 = -\dfrac{2}{x} \\ x = -1 & x = -\dfrac{1}{2} \end{array}$$

36. Let n = number of actual attendees.

	Number of attendees	Cost per attendee
Actual group	n	$\dfrac{420}{n}$
Enlarged group	$n + 10$	$\dfrac{420}{n+10}$

Since the larger group would have paid \$1 less per attendee,

$$\begin{aligned} \frac{420}{n+10} + 1 &= \frac{420}{n} \\ 420n + n(n+10) &= 420(n+10) \\ n^2 + 10n - 4200 &= 0 \\ (n-60)(n+70) &= 0 \\ n &= 60 \end{aligned}$$

37. The quantity under the radical sign must be nonnegative.

$$\begin{aligned} 2x^2 - x - 6 &\geq 0 \\ (2x+3)(x-2) &\geq 0 \end{aligned}$$

$2x + 3$	− − 0 + + + + + +
$x - 2$	− − − − − − − 0 + +
	$-\frac{3}{2}$ 0 2
$(2x + 3)(x - 2)$	+ + 0 − 0 − 0 + +

$$\{x | x \leq -3/2 \quad \text{or} \quad x \geq 2\}$$

or

$$(-\infty, -3/2], [2, \infty)$$

39.

$2x + 1$	− − − − − 0 + +
$x + 5$	− − 0 + + + + +
	−5 $-\frac{1}{2}$
$\dfrac{2x+1}{x+5}$	+ + − − 0 + +

$$\{x | x < -5 \quad \text{or} \quad x \geq -1/2\}$$

or

$$(-\infty, -5), [-1/2, \infty)$$

Exclude $x = -5$, since the denominator cannot equal 0.

CHAPTER 3

1. $$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(2+4)^2 + (-1+6)^2} \\ &= \sqrt{36 + 25} = \sqrt{61} \end{aligned}$$

5.

y-axis test	x-axis test
Replace x with $-x$:	Replace y with $-y$:
$y^2 = 1 - (-x)^3$	$(-y)^2 = 1 - x^3$
$y^2 = 1 + x^3$	$y^2 = 1 - x^3$
no	yes

origin test

Replace both:

$$(-y)^2 = 1 - (-x)^3$$
$$y^2 = 1 + x^3$$

no

7. Yes. No vertical line meets the graph in more than one point.

9. The quantity under the radical cannot be negative.

$$3x - 5 \geq 0$$
$$x \geq \frac{5}{3}$$

11. Solve the equation:

$$f(x) = 15 = \sqrt{x - 1}$$
$$225 = x - 1$$
$$x = 226$$

14. Replace x with $y - 1$:

$$f(x) = x^2 - x = (y - 1)^2 - (y - 1)$$
$$= y^2 - 2y + 1 - y + 1$$
$$= y^2 - 3y + 2$$

18. $f(x) = x - 1$ when $x \leq -1$

$f(-4) = -4 - 1 = -5$

19. $f(x) = -2$ when $x > 2$

$f(4) = -2$

20. $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{3 - (-6)}{-1 - (-4)} = \dfrac{9}{3} = 3$

21.

$$y - y_1 = m(x - x_1)$$
$$y - (-6) = 3[x - (-4)]$$
$$y + 6 = 3x + 12$$
$$y = 3x + 6$$

25.

$$2y + x - 5 = 0$$
$$y = -\frac{1}{2}x + \frac{5}{2}$$

The slope of the given line is $m_1 = -1/2$. The slope m of any line perpendicular to the given line is

$$m = -\frac{1}{m_1} = 2$$

Then, with $m = 2$ and $(x_1, y_1) = (-1, 3)$,

$$y - y_1 = m(x - x_1)$$
$$y - 3 = 2(x + 1)$$
$$y = 2x + 5$$

28. $P = k\dfrac{qr}{t^2}$

$$-3 = k\frac{(2)(-3)}{4^2}$$
$$k = 8$$
$$P = 8\frac{qr}{t^2}$$
$$P = -\frac{1}{4}$$

30.

$$(f \cdot g)(x) = (x + 1)(x^2 - 1)$$
$$= x^3 + x^2 - x - 1$$
$$(f \cdot g)(-1) = (-1)^3 + (-1)^2 - (-1) - 1 = 0$$

33. $(g \circ f)(x) = g(x + 1) = (x + 1)^2 - 1 = x^2 + 2x$

34. $g(x) = x^2 - 1$

$g(2) = 2^2 - 1 = 3$

$(f \circ g)(2) = f(3) = 3 + 1 = 4$

35. $(f \circ g)(x) = f(x^2) = \sqrt{x^2} - 2 = |x| - 2$

37. $(f \circ g)(-2) = |-2| - 2 = 0$

39. $(f \circ g)(x) = f\left(\dfrac{x}{2} - 2\right) = 2\left(\dfrac{x}{2} - 2\right) + 4 = x$

$(g \circ f)(x) = g(2x + 4) = \dfrac{2x + 4}{2} - 2 = x$

CHAPTER 4

2.

$2^{2x} = 8^{x-1} = (2^3)^{x-1}$	Write in terms of same base.
$2^{2x} = 2^{3x-3}$	$(a^m)^n = a^{mn}$
$2x = 3x - 3$	If $a^u = a^v$, then $u = v$.
$x = 3$	Solve for x.

4.

$S = P(1 + i)^n$	Compound interest formula.
$i = \dfrac{r}{k} = \dfrac{0.12}{2}$	Interest rate i per conversion period.
$i = 0.6$	
$n = 4 \times 2 = 8$	Number of conversion periods.
$S = 8000(1 + 0.6)^8$	Substitute for P, i, and n.
$= 8000(1.5938)$	Table IV in Tables Appendix or a calculator.
$= \$12{,}750.40$	

10.

$\log_5 \dfrac{1}{125} = x - 1$	
$5^{x-1} = \dfrac{1}{125}$	Equivalent exponential form.
$5^{x-1} = 5^{-3}$	Write in terms of same base.
$x - 1 = -3$	If $a^u = a^v$, then $u = v$.
$x = -2$	Solve for x.

12.

$\log_3(x + 1) = \log_3 27$	If $\log_a u = \log_a v$, then $u = v$.
$x + 1 = 27$	Solve for x.
$x = 26$	

13. $\log_3 3^5 = 5$ Since $\log_a a^x = x$.

or

$\log_3 3^5 = x$ Introduce unknown x.
$3^x = 3^5$ Equivalent exponential form.
$x = 5$ If $a^u = a^v$, then $u = v$.

16. $e^{\ln 3} = 3$ Since $a^{\log_a x} = x$.

or

$e^{\ln 3} = x$ Introduce unknown x.
$\ln x = \ln 3$ Equivalent logarithmic form.
$x = 3$ If $\log_a u = \log_a v$, then $u = v$.

18. $\log_a \dfrac{\sqrt{x-1}}{2x} = \log_a \dfrac{(x-1)^{1/2}}{2x}$
$= \log_a (x-1)^{1/2} - \log_a 2x$
$= \log_a (x-1)^{1/2} - [\log_a 2 + \log_a x]$
$= \frac{1}{2}\log_a (x-1) - \log_a 2 - \log_a x$

22. $\log 14 = \log (2 \cdot 7)$
$= \log 2 + \log 7$ Property 1.
$= 0.30 + 0.85 = 1.15$ Substitute given data.

25. $\log 0.7 = \log \dfrac{7}{10}$
$= \log 7 - \log 10$ Property 2.
$= 0.85 - 1$ $\log_a a = 1$.
$= -0.15$

26. $\frac{1}{3}\log_a x - \frac{1}{2}\log_a y$
$= \log_a x^{1/3} - \log_a y^{1/2}$ Property 3.
$= \log_a \dfrac{x^{1/3}}{y^{1/2}}$ Property 2.
$= \log_a \dfrac{\sqrt[3]{x}}{\sqrt{y}}$ Radical form.

27. $\frac{4}{3}[\log x + \log (x-1)]$
$= \frac{4}{3}\log (x)(x-1)$ Property 1.
$= \log (x^2 - x)^{4/3}$ Property 3.

30. $\log_b x = \dfrac{\log_a x}{\log_a b}$ Change of base formula.
$\log_8 32 = \dfrac{\log 32}{\log 8}$ $x = 32$, $b = 8$, $a = 10$.
$\log_8 32 = \dfrac{1.5}{0.9} = \dfrac{5}{3}$ Substitute given data.
Checking: $8^{5/3} \stackrel{?}{=} 32$
$32 = 32$

38. $x = \dfrac{2.1}{(32.5)^{5/2}}$ Introduce unknown x.

$\log x = \log 2.1 - \frac{5}{2}\log 32.5$ Properties of logarithms.
$= 0.3222 - \frac{5}{2}(1.5119)$ Table II in Tables Appendix.
$= -3.4575$
$= (4 - 3.4575) - 4$ Mantissa must be positive.
$= 0.5425 - 4$
$x \approx 3.49 \times 10^{-4}$ Table II in Tables Appendix.

39. $Q(t) = q_0 e^{-kt}$ Exponential decay model.
$\dfrac{q_0}{2} = q_0 e^{-0.06t}$ Substitute $k = 0.06$ and $Q(t) = q_0/2$
$\dfrac{1}{2} = e^{-0.06t}$
$\ln \dfrac{1}{2} = \ln e^{-0.06t}$ Take logs of both sides.
$\ln 0.5 = -0.06t(\ln e)$ Property 3.
$= -0.06t$ $\ln e = 1$.
$t = \dfrac{\ln 0.5}{-0.06}$
$= \dfrac{-0.6931}{-0.06}$ Table III in Tables Appendix.
$= 11.5$ hours

40. $2^{3x-1} = 14$
$(3x - 1)\log 2 = \log 14$ Take logs of both sides.
$x = \dfrac{1}{3} + \dfrac{\log 14}{3 \log 2}$ Solve for x.

41. $2 \log x - \log 5 = 3$
$\log x^2 - \log 5 = 3$ Property 3.
$\log \dfrac{x^2}{5} = 3$ Property 2.
$\dfrac{x^2}{5} = 10^3 = 1000$ Equivalent exponent form.
$x = \sqrt{5000}$ Solve for x.

CHAPTER 5

1. $1° = \dfrac{\pi}{180}$ radians

$-60° = -60\left(\dfrac{\pi}{180}\right) = -\dfrac{\pi}{3}$ radians

2. $1 \text{ radian} = \left(\dfrac{180}{\pi}\right)°$

$\dfrac{3\pi}{2}$ radians $= \dfrac{3\pi}{2}\left(\dfrac{180}{\pi}\right) = 270°$

6. $1 \text{ radian} = \left(\frac{180}{\pi}\right)^\circ$

$$\frac{4\pi}{3} \text{ radians} = \frac{4\pi}{3}\left(\frac{180}{\pi}\right) = 240^\circ$$

Since $480^\circ - 240^\circ = 240^\circ$ and 240° is not an integral multiple of 360°, they are not coterminal.

8. $\theta = \frac{s}{r} = \frac{14}{10} = 1.4$ radians

10. $\frac{11\pi}{6} \text{ radians} = \frac{11\pi}{6}\left(\frac{180}{\pi}\right) = 330^\circ$

Since

$$270^\circ < 330^\circ < 360^\circ$$

the angle is in quadrant IV.

11. $\theta = -220^\circ$ is coterminal with $\theta' = -220^\circ + 360^\circ = 140^\circ$, which is an angle in quadrant II.

14. The real numbers t and t' determine the same unit circle point if they differ by a multiple of 2π.

$$t' = \frac{9\pi}{2} - \frac{8\pi}{2} = \frac{\pi}{2}$$

15. The same unit circle point is determined by

$$t + 2\pi n$$

for all integer values of n. With $n = 4$,

$$t + 2\pi n = -\frac{15\pi}{2} + 8\pi = \frac{\pi}{2}$$

23. $\tan t < 0$ in quadrants II and IV
$\sin t < 0$ in quadrants III and IV
Both $\tan t$ and $\sin t$ are negative in quadrant IV.

25. From

$$\sin(-t) = -\sin t > 0$$

we conclude that

$$\sin t < 0$$

so t is in quadrant III or IV. Since $\tan t$ is positive in quadrants I and III, both conditions hold in quadrant III.

27. $\sin^2 t + \cos^2 t = 1$

$$\sin^2 t = 1 - \left(\frac{3}{5}\right)^2 = \frac{16}{25}$$

$\sin t = -\frac{4}{5}$ (Since t is in quadrant IV, sine is negative.)

$$\tan t = \frac{\sin t}{\cos t} = \frac{-4/5}{3/5} = -\frac{4}{3}$$

$$\cot t = \frac{1}{\tan t} = -\frac{3}{4}$$

28. $\sin^2 t + \cos^2 t = 1$

$$\cos^2 t = 1 - \left(-\frac{4}{5}\right)^2 = \frac{9}{25}$$

$\cos t = -\frac{3}{5}$ (Since $\sin t < 0$, $\tan t > 0$, t must be in quadrant III.)

$$\sec t = \frac{1}{\cos t} = -\frac{5}{3}$$

32. $\frac{\sin t}{\cos^2 t} = \frac{\sin t}{\cos t} \cdot \frac{1}{\cos t} = (\tan t)(\sec t)$

35. *Step 1*. The argument $t = \frac{5\pi}{6}$ is in quadrant II. Then

$$t' = \pi - t = \pi - \frac{5\pi}{6} = \frac{\pi}{6}$$

Step 2.

$$\tan \frac{\pi}{6} = \frac{\sqrt{3}}{3}$$

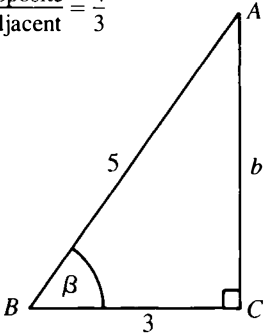
Step 3. Since tangent is negative in quadrant II,

$$\tan \frac{5\pi}{6} = -\tan \frac{\pi}{6} = -\frac{\sqrt{3}}{3}$$

CHAPTER 6

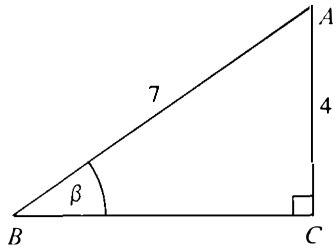
2. $b^2 = c^2 - a^2 = 25 - 9 = 16$
$b = 4$

$$\tan \beta = \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{3}$$



5. $a^2 = c^2 - b^2 = 49 - 16 = 33$
$a = \sqrt{33}$

$$\sec \beta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{7}{\sqrt{33}} = \frac{7\sqrt{33}}{33}$$

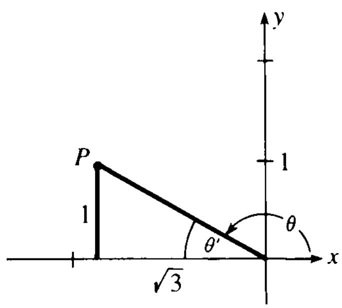


6. hypotenuse $\overline{OP} = \sqrt{3+1} = 2$

$$\csc \theta' = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{2}{1} = 2$$

Since θ is in quadrant II and cosecant is positive in quadrant II,

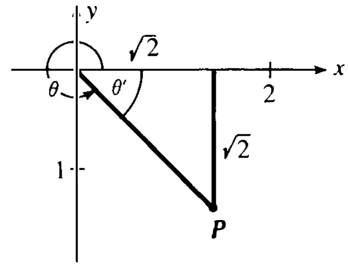
$$\csc \theta = \csc \theta' = 2$$



7. $\cot \theta' = \dfrac{\text{adjacent}}{\text{opposite}} = \dfrac{\sqrt{2}}{\sqrt{2}} = 1$

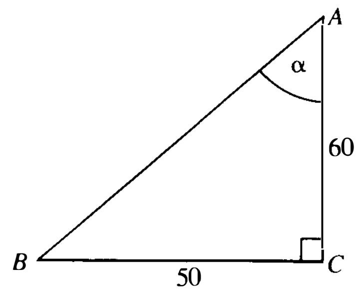
Since θ is in quadrant IV and cotangent is negative in quadrant IV,

$$\cot \theta = -\cot \theta' = -1$$



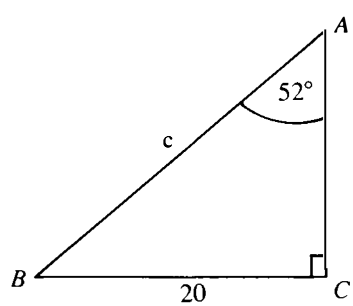
10. $\tan \alpha = \dfrac{50}{60} = 0.8333$

$\alpha = 39°50'$



12. $\sin 52° = \dfrac{20}{c}$

$c = \dfrac{20}{\sin 52°} = 25.4$



15. $\tan\theta = \dfrac{\text{tree height}}{\text{shadow length}} = \dfrac{25}{10} = 2.5$

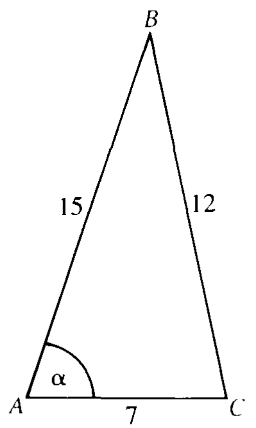
$\theta = 68°10'$

17. $a^2 = b^2 + c^2 - 2bc\cos\alpha$

$144 = 49 + 225 - 2(7)(15)\cos\alpha$

$\dfrac{-130}{-210} = 0.6190 = \cos\alpha$

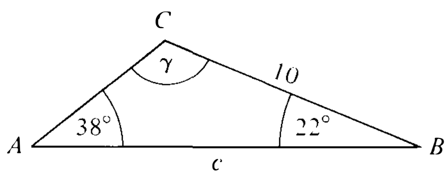
$\alpha = 51°50'$



19. $\gamma = 180° - 38° - 22° = 120°$

$\dfrac{c}{\sin 120°} = \dfrac{10}{\sin 38°}$

$c = 14.1$



CHAPTER 7

1. $\sin\sigma\sec\sigma + \tan\sigma$

$= \sin\sigma\dfrac{1}{\cos\sigma} + \tan\sigma$

$= \tan\sigma + \tan\sigma$

$= 2\tan\sigma$

3. $\sin\alpha + \sin\alpha\cot^2\alpha$

$= \sin\alpha(1 + \cot^2\alpha)$

$= \sin\alpha\csc^2\alpha$

$= \dfrac{\csc^2\alpha}{\csc\alpha} = \csc\alpha$

5. $\cos(45° + 90°)$

$= \cos 45°\cos 90° - \sin 45°\sin 90°$

$= \left(\dfrac{\sqrt{2}}{2}\right)(0) - \left(\dfrac{\sqrt{2}}{2}\right)(1) = -\dfrac{\sqrt{2}}{2}$

7. $\sin\dfrac{7\pi}{12} = \sin\left(\dfrac{3\pi}{12} + \dfrac{4\pi}{12}\right) = \sin\left(\dfrac{\pi}{4} + \dfrac{\pi}{3}\right)$

$= \sin\left(\dfrac{\pi}{4}\right)\cos\left(\dfrac{\pi}{3}\right) + \cos\left(\dfrac{\pi}{4}\right)\sin\left(\dfrac{\pi}{2}\right)$

$= \left(\dfrac{\sqrt{2}}{2}\right)\left(\dfrac{1}{2}\right) + \left(\dfrac{\sqrt{2}}{2}\right)\left(\dfrac{\sqrt{3}}{2}\right)$

$= \dfrac{\sqrt{2} + \sqrt{6}}{4}$

8. $\csc 15° = \sec(90° - 15°) = \sec 75°$

10. $\sin\dfrac{\pi}{8} = \cos\left(\dfrac{\pi}{2} - \dfrac{\pi}{8}\right) = \cos\dfrac{3\pi}{8}$

13. $\csc\left(\sigma + \dfrac{\pi}{3}\right) = \dfrac{1}{\sin(\sigma + \pi/3)}$

$\sin\left(\sigma + \dfrac{\pi}{3}\right) = \sin\sigma\cos\dfrac{\pi}{3} + \cos\sigma\sin\dfrac{\pi}{3}$

$\cos\sigma = \dfrac{1}{\sec\sigma} = \dfrac{8}{10}$

$\sin^2\sigma = 1 - \cos^2\sigma = \dfrac{36}{100}$

$\sin\sigma = -\dfrac{6}{10}$ Since σ is in quadrant IV.

Substituting, we have

$\sin\left(\sigma + \dfrac{\pi}{3}\right) = \left(-\dfrac{6}{10}\right)\left(\dfrac{1}{2}\right) + \left(\dfrac{8}{10}\right)\left(\dfrac{\sqrt{3}}{2}\right)$

$= \dfrac{4\sqrt{3} - 3}{10}$

$\csc\left(\sigma + \dfrac{\pi}{3}\right) = \dfrac{10}{4\sqrt{3} - 3} = \dfrac{10(4\sqrt{3} + 3)}{39}$

15. $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$, $\tan\beta = -\dfrac{5}{2}$

$\tan^2\alpha = \sec^2\alpha - 1 = \dfrac{1}{\cos^2\alpha} - 1$

$= \left(-\dfrac{13}{12}\right)^2 - 1 = \dfrac{25}{144}$

$\tan\alpha = -\dfrac{5}{12}$ Since α is in quadrant II.

Substituting, we have

$$\tan(\alpha + \beta) = 70$$

17. $\cos 2u = \cos^2 u - \sin^2 u$

$$\sin u = \frac{1}{\csc u} = -\frac{4}{5}$$

$$\cos^2 u = 1 - \sin^2 u = 1 - \left(-\frac{4}{5}\right)^2 = \frac{9}{25}$$

Substituting, we have

$$\cos 2u = \frac{9}{25} - \left(-\frac{4}{5}\right)^2 = -\frac{7}{25}$$

19. $\sin 4t = \sin 2u$, where $u = 2t$

$\sin 2u = 2 \sin u \cos u = 2 \sin 2t \cos 2t$

$$\sin 2t = \frac{3}{5}$$

$$\cos^2 2t = 1 - \sin^2 2t = \frac{16}{25}$$

$\cos 2t = \frac{4}{5}$ Since $2t$ is in quadrant I.

Substituting, we have

$$\sin 4t = 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = \frac{24}{25}$$

21. $\cos \frac{\sigma}{2} = \frac{12}{13} = \pm \sqrt{\frac{1 + \cos \sigma}{2}}$

$$\cos \sigma = 2\left(\frac{12}{13}\right)^2 - 1 = \frac{119}{169}$$

$$\sin^2 \sigma = 1 - \cos^2 \sigma = \frac{14400}{28561}$$

$\sin \sigma = \frac{120}{169}$ Since σ is acute (quadrant I).

23. $\tan \frac{t}{2} = \pm \sqrt{\frac{1 - \cos t}{1 + \cos t}}$

$$\csc^2 t = \cot^2 t + 1 = \left(-\frac{4}{3}\right)^2 + 1 = \frac{25}{9}$$

$$\sin^2 t = \frac{1}{\csc^2 t} = \frac{9}{25}$$

$$\cos^2 t = 1 - \sin^2 t = \frac{16}{25}$$

$\cos t = \frac{4}{5}$ Since $W(t)$ is in quadrant IV.

$$\tan \frac{t}{2} = \pm \sqrt{\frac{1 - \frac{4}{5}}{1 + \frac{4}{5}}} = \pm \sqrt{\frac{1}{9}} = \pm \frac{1}{3}$$

Since

$$270° < t < 360°$$

$135° < t/2 < 180°$
so $t/2$ is in quadrant II and
$\tan t/2 = -1/3$

25. $\cos 15° = \cos \frac{30°}{2} = \sqrt{\frac{1 + \cos 30°}{2}}$

$$\cos 15° = \frac{1}{2}\sqrt{2 + \sqrt{3}}$$

27. $\tan 112.5° = \tan \frac{225°}{2} = -\sqrt{\frac{1 - \cos 225°}{1 + \cos 225°}}$

Since $\cos 225° = -\cos 45° = -\frac{\sqrt{2}}{2}$,

$$\tan 112.5° = -\sqrt{\frac{2 + \sqrt{2}}{2 - \sqrt{2}}}$$

28. Let $u = 15x$.
Then $2u = 30x$.

$$\cos 2u = \cos^2 u - \sin^2 u$$
$$= 1 - 2 \sin^2 u$$

Substituting

$u = 15x$,

$$\cos 30x = 1 - 2 \sin^2 15x$$

30. $\tan \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} \cdot \sqrt{\frac{1 - \cos \alpha}{1 - \cos \alpha}}$

$$= \pm \frac{1 - \cos \alpha}{\sqrt{1 - \cos^2 \alpha}} = \pm \frac{1 - \cos \alpha}{\sqrt{\sin^2 \alpha}}$$

$$= \pm \frac{1 - \cos \alpha}{\sin \alpha}$$

Since $1 - \cos \alpha \geq 0$ for all α, the sign of $\tan \alpha/2$ is determined by the sign of $\sin \alpha$. Therefore,

$$\tan \frac{\alpha}{2} = \frac{1 - \cos \alpha}{\sin \alpha}$$

31. $\sin \frac{3\alpha}{2} \sin \frac{\alpha}{2} = \frac{1}{2}\left[\cos\left(\frac{3\alpha}{2} - \frac{\alpha}{2}\right) - \cos\left(\frac{3\alpha}{2} + \frac{\alpha}{2}\right)\right]$

$$= \frac{1}{2}[\cos \alpha - \cos 2\alpha]$$

32. $\cos 3x - \cos x = -2 \sin \dfrac{3x + x}{2} \sin \dfrac{3x - x}{2}$

$= -2 \sin 2x \sin x$

36. $2 \sin \sigma \cos \sigma = 0$

$\sin \sigma = 0$ or $\cos \sigma = 0$

$\sigma = 0, \pi \qquad \sigma = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

37. $\sin 2t - \sin t = 0$

$2 \sin t \cos t - \sin t = 0$

$\sin t(2 \cos t - 1) = 0$

$\sin t = 0$ or $2 \cos t - 1 = 0$

$t = 0, \pi \qquad \cos t = \dfrac{1}{2}$

$t = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

41. $|2 - i| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$

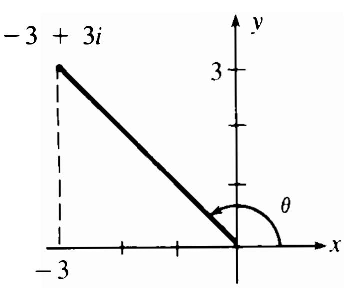
44. $r = \sqrt{a^2 + b^2} = \sqrt{(-3)^2 + (3)^2} = \sqrt{18} = 3\sqrt{2}$

$\tan \theta = \dfrac{3}{-3} = -1$

$\theta = 135°$

$-3 + 3i = r(\cos \theta + i \sin \theta)$

$= 3\sqrt{2}(\cos 135° + i \sin 135°)$



45. $2(\cos 90° + i \sin 90°) = 2(0 + i) = 0 + 2i$

49. $\dfrac{5(\cos 71° + i \sin 71°)}{3(\cos 50° + i \sin 50°)} = \dfrac{5}{3}[\cos(71° - 50°)$

$+ i \sin(71° - 50°)]$

$= \dfrac{5}{3} (\cos 21° + i \sin 21°)$

51. $r = \sqrt{a^2 + b^2} = \sqrt{3^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2}$

$\tan \theta = \dfrac{-3}{3} = -1$

$\theta = 315°$

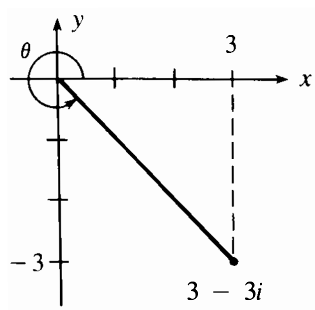
$(3 - 3i)^5 = [3\sqrt{2}(\cos 315° + i \sin 315°)]^5$

$= (3\sqrt{2})^5(\cos 1575° + i \sin 1575°)$

$= 972\sqrt{2}(\cos 135° + i \sin 135°)$

$= 972\sqrt{2}\left(-\dfrac{\sqrt{2}}{2} + i\dfrac{\sqrt{2}}{2}\right)$

$= -972 + 972i$



53. In trigonometric form,

$-9 = 9(\cos 180° + i \sin 180°)$

so $r = 9$, $\theta = 180°$, and $n = 2$. The square roots are

$$\sqrt{9}\left[\cos\left(\frac{180° + 360°k}{2}\right) + i \sin\left(\frac{180° + 360°k}{2}\right)\right]$$

for $k = 0, 1$. Substituting for k, we have

$3(\cos 90° + i \sin 90°) = 3i$

$3(\cos 270° + i \sin 270°) = -3i$

CHAPTER 8

1. $x = \dfrac{x_1 + x_2}{2} = \dfrac{-5 + 3}{2} = -1$

$y = \dfrac{y_1 + y_2}{2} = \dfrac{4 - 6}{2} = -1$

6. By the distance formula,

$\overline{AB} = \sqrt{170}, \qquad \overline{AC} = \sqrt{136}, \qquad \overline{BC} = \sqrt{34}$

Since $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$, triangle ABC satisfies the Pythagorean theorem and is a right triangle.

7. slope of AB:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 + 3}{1 + 4} = \frac{6}{5}$$

midpoint of AB:

$$x = \frac{x_1 + x_2}{2} = \frac{-4 + 1}{2} = -\frac{3}{2}$$

$$y = \frac{y_1 + y_2}{2} = \frac{-3 + 3}{2} = 0$$

The perpendicular bisector passes through $(-3/2, 0)$ and has slope $-5/6$. Then

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -\frac{5}{6}\left(x + \frac{3}{2}\right)$$

$$10x + 12y + 15 = 0$$

8. $h = -5$, $k = 2$, $r = 4$

$$(x - h)^2 + (y - k)^2 = r^2$$

$$(x + 5)^2 + (y - 2)^2 = 16$$

10. $x - h = x - 2 \qquad y - k = y + 3 \qquad r^2 = 9$

$h = 2 \qquad k = -3 \qquad r = 3$

center: $(2, -3)$; $r = 3$

12.

$$x^2 + 4x + y^2 - 6y = -10$$

$$(x^2 + 4x + 4) + (y^2 - 6y + 9) = -10 + 4 + 9$$

$$(x + 2)^2 + (y - 3)^2 = 3$$

center: $(-2, 3)$; $r = \sqrt{3}$

19.

$$y^2 + 4y = -x - 2$$

$$y^2 + 4y + 4 = -x - 2 + 4$$

$$(y + 2)^2 = -x + 2 = -(x - 2)$$

Since $(y - k)^2 = 4p(x - h)$,

vertex: $(h, k) = (2, -2)$;

axis: $y + 2 = 0$ or $y = -2$;

direction: opens left, since $p < 0$.

20.

$$2x^2 - 12x = y - 16$$

$$2(x^2 - 6x + 9) = y - 16 + 18$$

$$2(x - 3)^2 = y + 2$$

$$(x - 3)^2 = \frac{1}{2}(y + 2)$$

Since $(x - h)^2 = 4p(y - k)$,

vertex: $(3, -2)$;

axis: $x - 3 = 0$ or $x = 3$;

direction: opens up, since $p > 0$.

24. Dividing by 36, we have

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

Setting $y = 0$, we have

$$x^2 = 4 \quad \text{or} \quad x = \pm 2$$

Setting $x = 0$, we see that there are no y-intercepts.

28. Dividing by 9, we have

$$\frac{x^2}{3} + \frac{y^2}{\frac{9}{4}} = 1$$

With $x = 0$, $y = \pm 3/2$.
With $y = 0$, $x = \pm\sqrt{3}$.

34.

$$2x^2 + 12x + y^2 - 2y = -17$$

Completing the square, we have

$$2(x^2 + 6x + 9) + (y^2 - 2y + 1) = -17 + 18 + 1$$

$$2(x + 3)^2 + (y - 1)^2 = 2$$

Since the right-hand side is positive, $A \neq C$, and $AC > 0$, the graph is an ellipse.

CHAPTER 9

3. Substituting $x = 6y + 11$, we have

$$2(6y + 11) + 5y = 5$$

$$17y = -17$$

$$y = -1$$

$$x = 6y + 11 = 6(-1) + 11 = 5$$

Solution: $x = 5$, $y = -1$.

6. Substituting $x = 5 - 3y$, we have

$$(5 - 3y)^2 + y^2 = 25$$

$$25 - 30y + 9y^2 + y^2 = 25$$

$$10y^2 - 30y = 0$$

$$10y(y - 3) = 0$$

$$y = 0 \quad \text{or} \quad y = 3$$

$$x = 5 - 3y = 5 \qquad x = 5 - 3y = -4$$

9. To eliminate x, multiply the first equation by -2 and the second equation by 1. Then add the two equations:

$$\begin{array}{r} -2x - 8y = -34 \\ 2x - 3y = -21 \\ \hline -11y = -55 \\ y = 5 \\ x + 4(5) = 17 \\ x = -3 \end{array}$$

Solution: $x = -3$, $y = 5$.

14. Rewriting the equations and adding, we have

$$\begin{array}{r} x^2 + y^2 - 9 = 0 \\ -x^2 + y - 3 = 0 \\ \hline y^2 + y - 12 = 0 \\ (y - 3)(y + 4) = 0 \\ y = 3 \quad \text{or} \quad y = -4 \end{array}$$

$$x^2 = y - 3 = 0 \qquad x^2 = y - 3 = -7$$

$x = 0$ $\qquad$ no real solutions

The circle and parabola are tangent at $(0, 3)$.

15. t = tens digit, u = units digit

$10t + u$ = original number

Then

$$(10t + u) + t = 49$$

or

$$11t + u = 49 \qquad (1)$$

Also,

$10u + t$ = number with digits reversed

Then

$$10u + t = (10t + u) + 9$$

or

$$-9t + 9u = 9 \qquad (2)$$

Solving Equations (1) and (2) simultaneously, we find

$$t = 4, \qquad u = 5$$

The original number is 45.

17. x = cost per lb of hamburger

y = cost per lb of steak

Then

$$5x + 4y = 22.00$$
$$3x + 7y = 28.15$$

Solving, we find

$$x = \$1.80, \qquad y = \$3.25$$

21. Interchange equations 1 and 3:

$$\begin{aligned} -x + 4y + 2z &= 15 \\ 2x + 5y - 2z &= -9 \\ -3x - y + z &= 12 \end{aligned}$$

Add 2 times equation 1 to equation 2; add -3 times equation 1 to equation 3:

$$\begin{aligned} -x + 4y + 2z &= 15 \\ 13y + 2z &= 21 \\ -13y - 5z &= -33 \end{aligned}$$

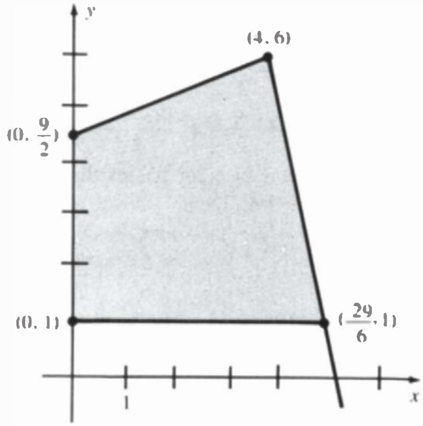
Add equation 2 to equation 3:

$$\begin{aligned} -x + 4y + 2z &= 15 \\ 13y + 2z &= 21 \\ -3z &= -12 \end{aligned}$$

Use back-substitution:

$$-3z = -12 \quad \text{or} \quad z = 4$$
$$13y + 2(4) = 21 \quad \text{or} \quad y = 1$$
$$-x + 4(1) + 2(4) = 15 \quad \text{or} \quad x = -3$$
$$x = -3 \qquad y = 1 \qquad z = 4$$

35. The figure shows the set of feasible solutions and the coordinates of the vertices.



Evaluating the objective function at these points gives us the following information:

x	y	$z = 5y - x$
0	1	5
0	$\frac{9}{2}$	$\frac{45}{2}$
4	6	26
$\frac{29}{6}$	1	$\frac{1}{6}$

The maximum value, $z = 26$, occurs at $x = 4$, $y = 6$.

CHAPTER 10

12. From the third row, $x_3 = -3$. Then, from row 2,

$$\begin{aligned} x_2 + 3x_3 &= -8 \\ x_2 + 3(-3) &= -8 \\ x_2 &= 1 \end{aligned}$$

From row 1,

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &= -9 \\ x_1 - 2(1) + 2(-3) &= -9 \end{aligned}$$
$$x_1 = -1 \qquad x_2 = 1 \qquad x_3 = -3$$

13. In matrix form,

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & -4 & -5 \end{array}\right]$$

Add -2 times row 1 to row 2:

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -6 & -9 \end{array}\right]$$

Multiply row 2 by $-1/6$:

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 3/2 \end{array}\right]$$

Add -1 times row 2 to row 1:

$$\left[\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & 3/2 \end{array}\right]$$

The solution is $x = 1/2$, $y = 3/2$.

18. Two matrices of the same dimension are equal if corresponding entries are equal. This requires that

$$x^2 = 9 \quad \text{and} \quad 4x = -12$$

The only value satisfying both equations is $x = -3$.

20. $$B - A = \begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1-2 & 5-(-1) \\ 4-3 & -3-2 \end{bmatrix} = \begin{bmatrix} -3 & 6 \\ 1 & -5 \end{bmatrix}$$

21. Addition of matrices is defined only when the matrices are of the same dimension.

29. Appending the identity matrix I_2 to the coefficient matrix, we have

$$\left[\begin{array}{cc|cc} -2 & 3 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ -2 & 3 & 1 & 0 \end{array}\right]$$ Interchanged rows.

$$\left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 0 & 11 & 1 & 2 \end{array}\right]$$ Added 2 times row 1 to row 2.

$$\left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 0 & 1 & 1/11 & 2/11 \end{array}\right]$$ Multiplied row 2 by 1/11.

$$\left[\begin{array}{cc|cc} 1 & 0 & -4/11 & 3/11 \\ 0 & 1 & 1/11 & 2/11 \end{array}\right]$$ Added -4 times row 2 to row 1.

$$\begin{bmatrix} -4/11 & 3/11 \\ 1/11 & 2/11 \end{bmatrix}$$ Inverse.

31. Appending the identity matrix I_2 to the coefficient matrix, we have

$$\left[\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 2 & -1 & 1 & 0 \end{array}\right]$$ Interchanged rows.

$$\left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & -3 & 1 & -2 \end{array}\right]$$ Added -2 times row 1 to row 2.

$$\left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 1 & -1/3 & 2/3 \end{array}\right]$$ Multiplied row 2 by $-1/3$.

$$\left[\begin{array}{cc|cc} 1 & 0 & 1/3 & 1/3 \\ 0 & 1 & -1/3 & 2/3 \end{array}\right]$$ Added -1 times row 2 to row 1.

Multiplying the coefficients of the right-hand side by the inverse, we have

$$\begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

so $x = 2$, $y = 3$ is the unique solution.

37. Expanding by the cofactors of the first column, we have

$$1\begin{vmatrix} 5 & 4 \\ 3 & 8 \end{vmatrix} + 2\begin{vmatrix} -1 & 2 \\ 5 & 4 \end{vmatrix}$$

$$= (40 - 12) + 2(-4 - 10) = 0$$

39. $$D = \begin{vmatrix} 2 & -1 \\ -2 & 3 \end{vmatrix} = 6 - 2 = 4$$

$$x = \frac{\begin{vmatrix} -3 & -1 \\ 11 & 3 \end{vmatrix}}{4} = \frac{2}{4} = \frac{1}{2}$$

$$y = \frac{\begin{vmatrix} 2 & -3 \\ -2 & 11 \end{vmatrix}}{4} = \frac{16}{4} = 4$$

CHAPTER 11

1. $$\begin{array}{r|rrrr} 1 & 2 & 0 & 6 & -4 \\ & & 2 & 2 & 8 \\ \hline & \underbrace{2 \quad 2 \quad 8}_{Q(x)} & & & \underset{R}{4} \end{array}$$

$Q(x) = 2x^2 + 2x + 8$; $R = 4$

3. $$\begin{array}{r|rrrr} -1 & 7 & -3 & 0 & 2 \\ & & -7 & 10 & -10 \\ \hline & 7 & -10 & 10 & -8 \end{array}$$

$P(-1) = -8$

$$\begin{array}{r|rrrr} 2 & 7 & -3 & 0 & 2 \\ & & 14 & 22 & 44 \\ \hline & 7 & 11 & 22 & 46 \end{array}$$

$P(2) = 46$

5. $$\begin{array}{r|rrrrr} -2 & 2 & 4 & 3 & 5 & -2 \\ & & -4 & 0 & -6 & 2 \\ \hline & 2 & 0 & 3 & -1 & 0 \end{array}$$

Since $P(-2) = 0$, $x + 2$ is a factor.

8. $$\frac{2+i}{0-5i}\left(\frac{0+5i}{0+5i}\right) = \frac{10i + 5i^2}{-25i^2}$$
$$= \frac{-5 + 10i}{25} = -\frac{1}{5} + \frac{2}{5}i$$

14. With $\sqrt{-3} = \sqrt{3}i$, form the product:
$$(x-3)(x-\sqrt{3}i)(x+\sqrt{3}i) = (x-3)(x^2+3)$$
$$= x^3 - 3x^2 + 3x - 9$$

16. The number 1/2 is a zero of the linear factor $(2x - 1)$, and -1 is a zero of the linear factor $(x + 1)$. Form the product:
$$(2x-1)^2(x+1)^2 = 4x^4 + 4x^3 - 3x^2 - 2x + 1$$

19. Divide by $x + 2$ to find the depressed equation:
$$\begin{array}{r|rrrr} -2 & 2 & -1 & -13 & -6 \\ & & -4 & 10 & 6 \\ \hline & 2 & -5 & -3 & 0 \end{array}$$

$\underbrace{2 \quad -5 \quad -3}_{\text{depressed equation}}$

Solving $2x^2 - 5x - 3 = 0$, we have
$$(2x+1)(x-3) = 0$$
$$x = -\frac{1}{2} \qquad x = 3$$

23. The polynomial
$$P(x) = x^5 - x^4 + 3x^3 - 4x^2 + x - 5$$
has 5 variations in sign and therefore has a maximum of 5 positive real zeros.

The polynomial
$$P(-x) = -x^5 - x^4 - 3x^3 - 4x^2 - x - 5$$
has no variations in sign, and therefore there are no negative real zeros.

25. The polynomial
$$P(x) = 3x^4 - 2x^2 + 1$$
has 2 variations in sign, so there can be at most 2 positive real roots. $P(-x) = P(x)$, so there can be at most 2 negative real roots.

28. The only possible rational roots are ± 1. Using condensed synthetic division, we find
$$\begin{array}{r|rrrrr} & 1 & 3 & 2 & 1 & -1 \\ 1 & 1 & 4 & 6 & 7 & \boxed{6} \\ -1 & 1 & 2 & 0 & 1 & \boxed{-2} \end{array}$$
Since neither remainder is zero, there are no rational roots.

29. Since the coefficients are all integers, the Rational Zero Theorem restricts the possible rational roots to
$$\pm 1, \quad \pm\frac{1}{2}, \quad \pm\frac{1}{3}, \quad \pm\frac{1}{6}, \quad \pm 2, \quad \pm\frac{2}{3},$$
$$\pm 5, \quad \pm\frac{5}{2}, \quad \pm\frac{5}{3}, \quad \pm\frac{5}{6}, \quad \pm 10, \quad \pm\frac{10}{3}$$
Testing by synthetic division,
$$\begin{array}{r|rrrr} -1 & 6 & 15 & -1 & -10 \\ & & -6 & -9 & 10 \\ \hline & 6 & 9 & -10 & 0 \end{array}$$
we show that -1 is a root. The remaining roots are those of the depressed equation
$$6x^2 + 9x - 10 = 0$$
and are found by the quadratic formula:
$$x = \frac{-9 \pm \sqrt{81 + 240}}{12} = \frac{-9 \pm \sqrt{321}}{12}$$

CHAPTER 12

3. $a_n = n - a_{n-1}$
$a_1 = 0$
$a_2 = 2 - 0 = 2$
$a_3 = 3 - 2 = 1$
$a_4 = 4 - 1 = 3$
$a_5 = 5 - 3 = 2$

5. $$\sum_{k=1}^{4} (1 - 2k) = (1 - 2) + (1 - 4) + (1 - 6) + (1 - 8) = -16$$

11. $a_n = a_1 + (n - 1)d$
$a_{21} = -2 + (21 - 1)(2) = 38$

13. Use the given information to determine d:

$$n = 16,\ a_{16} = 9,\ a_1 = 4$$
$$a_n = a_1 + (n - 1)d$$
$$9 = 4 + 15d$$
$$d = 1/3$$

Then find a_{13}:

$$a_{13} = a_1 + (n - 1)d = 4 + 12\left(\frac{1}{3}\right) = 8$$

15. $$S_n = \frac{n}{2}[2a_1 + (n - 1)d] = \frac{25}{2}\left[-\frac{2}{3} + 24\left(\frac{1}{3}\right)\right] = \frac{275}{3}$$

17. $$r = \frac{a_2}{a_1} = \frac{-6}{2} = -3$$

19. $$a_2 = a_1 r = 5\left(\frac{1}{5}\right) = 1$$
$$a_3 = a_2 r = 1\left(\frac{1}{5}\right) = \frac{1}{5}$$
$$a_4 = a_3 r = \left(\frac{1}{5}\right)\left(\frac{1}{5}\right) = \frac{1}{25}$$

21. $$r = \frac{a_2}{a_1} = \frac{6}{-4} = -\frac{3}{2}$$
$$a_n = a_1 r^{n-1}$$
$$a_6 = (-4)\left(-\frac{3}{2}\right)^5 = \frac{243}{8}$$

23. The sequence is

$$3,\ a_2,\ a_3,\ 1/72$$

With $a_1 = 3$, $a_4 = 1/72$, and $n = 4$,

$$a_n = a_1 r^{n-1}$$
$$a_4 = a_1 r^3$$
$$r^3 = \frac{1}{216}$$
$$r = \frac{1}{6}$$

Then

$$a_2 = a_1 r = 3\left(\frac{1}{6}\right) = \frac{1}{2}$$
$$a_3 = a_2 r = \left(\frac{1}{2}\right)\left(\frac{1}{6}\right) = \frac{1}{12}$$

24. $$r = \frac{a_2}{a_1} = \frac{1}{2}$$
$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$
$$S_6 = \frac{\frac{1}{3}\left[1 - \left(\frac{1}{2}\right)^6\right]}{1 - \frac{1}{2}} = \frac{21}{32}$$

27. $$r = \frac{a_2}{a_1} = -\frac{2}{3}$$
$$S = \frac{a_1}{1 - r} = \frac{3}{1 - \left(-\frac{2}{3}\right)} = \frac{9}{5}$$

30. By the binomial formula,

$$\left(\frac{x}{2} - 2\right)^4 = \left(\frac{x}{2}\right)^4 + \frac{4}{1}\left(\frac{x}{2}\right)^3(-2) + \frac{4 \cdot 3}{1 \cdot 2}\left(\frac{x}{2}\right)^2(-2)^2 + \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3}\left(\frac{x}{2}\right)(-2)^3 + (-2)^4$$
$$= \frac{x^4}{16} - x^3 + 6x^2 - 16x + 16$$

33. $$\frac{13!}{11!2!} = \frac{13 \cdot 12 \cdot 11!}{11!2!} = \frac{13 \cdot 12}{2} = 78$$

34. $$\frac{(n-1)!(n+1)!}{n!n!} = \frac{(n-1)!(n+1)n!}{n!n(n-1)!} = \frac{n+1}{n}$$

35. $$\binom{6}{4} = \frac{6!}{4!2!} = \frac{6 \cdot 5}{2} = 15$$

39. The six letters can be arranged in

$$P(6, 6) = 6! = 720$$

ways. However, the existence of two of the letter *o* will make half the arrangements indistinguishable. The answer is therefore

$$\frac{P(6, 6)}{P(2, 2)} = \frac{6!}{2!} = 360$$

40. $C(10, 6) = \dfrac{10!}{4!6!} = 210$

42. There are 8 successful outcomes:

1, 6	2, 5	3, 4
6, 1	5, 2	4, 3
5, 6	6, 5	

There are $6 \times 6 = 36$ total outcomes. So

$$\text{probability} = \frac{8}{36} = \frac{2}{9}$$

43. We can select 2 white pens in $C(4, 2)$ ways. We can select 2 pens in $C(7, 2)$ ways. Therefore,

$$\text{probability} = \frac{C(4, 2)}{C(7, 2)} = \frac{2}{7}$$

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