ALGEBRAIC STRUCTURES IN AUTOMATA AND DATABASES THEORY

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ALGEBRAIC STRUCTURES IN AUTOMATA AND DATABASES THEORY

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PREFACE

This book is devoted to the investigation of algebraic structures whispered by the concept of automaton. The emphasis is made on algebraic framework of real automata and databases. Thus, these notions appear as many-sorted algebraic structures, allowing developed algebraic theory. Since treating of algebraic structure paves way to the structure and behavior of real automata and databases, we hope that the constructed theory will find its applications. On the other hand, we have pursued quite a lot of algebraic purposes, searching for ways of enriching algebra itself.

This book consists of five chapters. First four of them are related directly to the algebraic theory of automata while the last one, describing an algebraic model of database, takes a special part. Nevertheless, this chapter is closely connected with previous ones and we think that this book owns a certain unity. In both cases of automata and databases, we concentrate on mathematical models of the corresponding systems.

This book is addressed primarily for mathematicians graduates, postgraduates and conceivably for programmers, engineers and scientists who wish to apply algebraic methods in their research. We have tried to minimize the algebraic background of the potential reader, therefore the definitions in the "Preliminary notions" and in the sequel attempt to make the book self-contained. However, touching the advanced topics, we assume a certain familiarity with universal algebra, theory of groups and semigroups. Chapter 5 will be much easier for those who are acquainted with set-theoretical and logical methods.

The concept of automaton arises in various problems associated with computer science, network systems, control theory etc. Its mathematical structure is based on intuitive arguments, reflecting the entity of real (not necessarily physical) automata. Now, automata theory is a developed mathematical field. On our opinion, two aspects could be pointed out in automata research. They are the combinatorial approach and algebraic theory. The first one ([38], [54], [99] etc.) is in a greater extent connected with behavior, analysis and synthesis of automata. Certainly, both these directions are not mutually independent: algebraic methods are used in combinatorial problems. For example, algebraic automata theory plays a significant role in the theory of algorithms and languages ([37], [38]). However, speaking on algebraic aspect of automata theory, first of all we bear in mind an automaton as an algebraic structure (this point of view has been reflected in [29], [34], [2] etc.). A consequent analysis of this algebraic structure is one of the main goals of this book. Moreover, the algebraic structure of automaton provides an important information about the structure of real automata. Krohn-Rhodes decomposition Theorem turns out to be the most impressive witness of this. Another important direction is presented by application of algebraic methods for the classification of automata, description of its behavior by identities, study of varieties of automata. That is why the standard algebraic notions: automorphisms, identities and varieties are widely used in this book. Finally, we have tried to follow the categorical point of view on automaton as on algebraic systems.

As it was mentioned, chapter 5 is meant for the construction of adequate algebraic model of databases. This theme is studied in detail in the book of B. Plotkin ([86]) and we frequently refer to it. In this chapter, we describe the sketch of the theory. A number of problems in the theory of relational databases can be reformulated in terms of an algebraic model. Among them are the problems of informational equivalence and isomorphism of databases, its composition and decomposition, classification on the base of symmetries and so on.

An *automaton* $\mathscr{A}=(A,X,B)$ is an algebraic system with three basic sets, A, X, B called the sets of *states*, *input signals* and *output signals* respectively, and two binary operations:

$$\circ : A\times X \longrightarrow A,$$

$$* : A\times X \longrightarrow B.$$

The operation $\circ$ is a function of two variables $a\in A$, $x\in X$, whose values lie in the set of states A: $a\circ x = a' \in A$. It is often called a *transition function* and shows how the input signal x transforms each state a into the new a'. The operation * assigns to a pair $a \in A$, $x \in X$ the element $b \in B$ and is called *input function* of the automaton $\mathcal{A}$.

In chapter 1 we consider pure automata, i.e. automata with the sets of states and outputs being arbitrary sets, without additional structure. Let the automaton (A,X,B) be transformed by the element $x_1 \in X$ from the state a_1 into the state $a_2 = a_1 \circ x_1$. Considering an element x_2 as a new input signal, we get a new state $a_2 \circ x_2 = (a_1 \circ x_1) \circ x_2$. Simultaneously, the element $a_2 * x_2 = (a_1 \circ x_1) * x_2$ arises as an output signal. This means that we can consider a new input signal $x_1 x_2$, which takes the state a_1 into a new one $(a_2 \circ x_1) \circ x_2$ and gives the signal $(a_1 \circ x_1) * x_2$ in output. It is natural to consider $x_1 x_2$ as a product of input elements x_1 and x_2, and the multiplication thus defined has to be associative. It is assumed that the set of input signals is a semigroup. Denote it by Γ. An automaton (A,Γ,B) is called a semigroup automaton if the set of inputs is a semigroup, and the axioms

$$a \circ \gamma_1 \gamma_2 = (a \circ \gamma_1) \circ \gamma_2,$$

$$a * \gamma_1 \gamma_2 = (a \circ \gamma_1) * \gamma_2, \qquad a \in A, \qquad \gamma_i \in \Gamma$$

are satisfied. In this book we deal mainly with semigroup automata.

In the first chapter we study also Moore automata, i.e. automata determined by the operation ∘ and the defining mapping $\mu: A \longrightarrow B$. There are various criteria for an arbitrary automaton to be a Moore one. One of them points out the class of semigroups being the semigroup of input signals of Moore automata.

Automata can have extra inputs, outputs and states. The construction of three types of universal automata is connected, in particular, with this problem.

More often, the problem of constructing more complicated automata from the simple ones leads us to search for suitable constructions. The most important of them is cascade connection and its universal variant, the wreath product of automata. The inverse task of how to decompose an arbitrary pure automaton has a decision. It is the classic Krohn-Rhodes decomposition theorem. These and the related questions are treated in chapter 2.

There are various generalizations of the notion of pure automaton. We can consider automata in varieties and in arbitrary categories. A particular case of latter ones present affine automata, stochastic and fuzzy automata.

Chapter 3 deals with well organized generalizations of pure automata: linear automata and biautomata. In linear automata, sets if states and output signals are linear spaces over field or modules over a commutative ring. All corresponding mappings are also linear.

Biautomata appear when not only states, but also output signals are subjected to transformation. In this case, relation between multiplying in semigroup and the rest operations carries character of differentiation. This concept is quite warranted physically and implies meaningful algebraic theory.

Several problems considered in chapter 1 for pure automata obtain new features in the linear case. Defining an important construction of triangular product, we establish Krohn-Rhodes type decomposition theorem for linear automata. We also study symmetries of pure and linear automata, that is, we treat their automorphisms.

Identities and varieties of automata and biautomata are examined in chapter 4. The main result here is the theorem that the semigroup of proper varieties of biautomata over a field is a free semigroup. This means that each proper variety of biautomata admits unique decomposition in product of indecomposable multiples. Moreover, there is a canonical homomorphism of this semigroup onto the semigroup of varieties of automata, which is already not free. This theorem is close by its nature to the theorem of Newmann-Shmelkin relating to group varieties theory and the theorem of Plotkin for varieties of representations of groups. In this chapter, the quasiidentities and quasivarieties of automata are considered too.

As it was mentioned, chapter 5 deals with algebraic model of database. In the first instance, a database is a triplet (*-automaton) of the form $\mathcal{A}=(F,Q,R)$, where F is the set of states and Q,R are the algebras of queries and replies respectively. We denote by $f*q$ the reply to a query q in the given state f. An automaton (F,Q,R) has to be associated with data algebra $\mathcal{D}$; algebra Q and R should be chosen in some variety of algebras. Already these arguments show that the construction of algebraic model needs significant efforts. It is accomplished in the first part of the chapter. The remainder is devoted to various problems: homomorphisms of databases, constructivization of databases, defining of the constructions.

We have tried to make not only the direct references but also include in bibliography the papers close in some aspects to the theme of the book. However, we would like to painfully put forward for not being able to provide the complete reference list.

The book is based on a series of lecture courses on algebraic systems delivered by B.Plotkin at Latvian University and on various talks given at some universities in the former USSR, former Yugoslavia, Hungary, Israel and Bulgaria. The book reflects not all the aspects of algebraic theory of automata but only those close to the results obtained by the authors and the other participants of Riga Algebraic Seminar during its past long years.

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CONTENTS

PRELIMINARIES

1. Sets

As usual, $a\in A$ means that a is an element of a set A and $A\subset B$ means that A is a subset of B. $A\cap B$ and $A\cup B$ are the intersection an the union of the sets A and B. Finally, $A\setminus B$ is the complement of B in A. A mapping of sets is denoted by $f\colon A \longrightarrow B$ or $A \xrightarrow{f} B$. For each $a\in A$ the corresponding $b\in B$ is denoted by $b=af$ or $b=f(a)$ or $b=a^f$. A mapping f is called *surjective* if for each $b\in B$ there exists $a\in A$ such that $b=af$; f is *injective* if $a_1\neq a_2$ implies $a_1f\neq a_2f$. If the mapping f is surjective and injective then it is called *bijective* one.

The product of mappings $f\colon A \longrightarrow B$ and $\varphi\colon B \longrightarrow C$ is defined by the equality $a(f\varphi)=(af)\varphi$. Identity mapping is a mapping of the form $\varepsilon_A\colon A \longrightarrow A$, defined by the rule $\varepsilon_A(a)=a$ for each $a\in A$. Mapping $f^{-1}\colon B \longrightarrow A$ is called an inverse one with respect to $f\colon A \longrightarrow B$ if $ff^{-1}=\varepsilon_A$ and $f^{-1}f=\varepsilon_B$. Fun(A,B) or B^A denotes the set of all mappings from A to B.

Any mapping $A \longrightarrow A$ is called a *transformation* of the set A. All these mappings are denoted by S_A. Bijective transformation of A is called *substitution* on A.

Cartesian product $A_1\times\ldots\times A_n$ consists of the sequences (n-tuples) of the form $(a_1,a_2,\ldots,a_n)$, $a_i\in A_i$, $i=1,\ldots,n$. If J is an arbitrary set and A_α corresponds to each $\alpha\in J$, then the Cartesian product $A=\prod_{\alpha\in J} A_\alpha$ is a set of all functions a defined on J, such that for any $\alpha\in J$ holds $a(\alpha)=a_\alpha\in A_\alpha$. In particular, if all A_α are copies of A then $\prod_{\alpha\in J} A_\alpha=A^J$.

Binary relation ρ on a set A can be treated as a subset in the Cartesian product $A\times A$. This subset consists of all pairs (a_1,a_2) such that elements a_1,a_2 are in the relation ρ. Henceforth we will use notations $a_1\rho a_2$ or $(a_1,a_2)\in\rho$. Since binary relation is a set, one can consider intersection, union and inclusion of binary relations. Relation ρ is called *reflexive* if $a\rho a$ holds for each $a\in A$. Relation ρ is called *transitive* if $a\rho b$ and $b\rho c$ implies $a\rho c$. Finally, ρ is the relation of *symmetry* if $a\rho b$ implies $b\rho a$. An *equivalence* is a relation satisfying these three properties. If ρ is an equivalence on the set A then the set of all $a'\in A$ such that $a\rho a'$ is called a *coset* by ρ with the representative a. It is denoted by [a] or $[a]_\rho$. An equivalence uniquely defines the decomposition of a set into cosets. The set of all cosets $[a]_\rho$, $a\in A$, is called *quotient set* of A by ρ and is denoted by A/ρ. Let the mapping $f: A \longrightarrow B$ be given. Consider relation $\rho=\rho(f)$ on A defined by the rule: $a\rho a'$ if $f(a)=f(a')$. It is evident that ρ is an equivalence, called the ***kernel equivalence*** of f or simply the kernel of f, and denoted by Ker$f$. Let $\rho_1,\rho_2,\ldots\rho_n$ be binary relations on A. There exists at least one equivalence containing all ρ_i, $i=1,2,\ldots,n$. Denote by $\bar{\rho}$ the intersection of all such equivalences. Then $\bar{\rho}$ is also an equivalence and it is the minimal equivalence containing all these ρ_i In this case $\bar{\rho}$ is said to be generated by binary relations $\rho_1,\rho_2,\ldots\rho_n$.

One can define a relation of partial order on the set of all equivalences on the set A, namely $\rho_1<\rho_2$ if for each $a\in A$ the inclusion $[a]_{\rho_1}\subset[a]_{\rho_2}$ takes place. In minimal equivalence each class consists of one element while in a maximal one there is only one class, coinciding with A.

Let M be a nonempty set, and ρ a subset of Cartesian product $M\times M$, i.e. binary relation on M. A pair (M,ρ) is called an *oriented graph* with the set of vertexes M and the set ρ of oriented edges. It can be presented by the diagram, where vertexes are displayed by circles and edges by arrows.

2. Ω-algebras

n-ary algebraic operation on the set A is a function of the form $\omega: A\times\ldots\times A \longrightarrow A$. The result of operation ω on the element $(a_1,\ldots,a_n)$ of the Cartesian product is denoted by $a_1a_2\ldots a_n\omega$. If n=2 then the operation is called binary. In various specific cases we use infix notation a+b, ab, etc. *Ω-algebra* is a set with the certain system of operations Ω. Let A be Ω-algebra; equivalence ρ on the set A is called a *congruence* of Ω-algebra A if $a_i \rho a'_i$, $i=1,2,\dots,n$ implies $(a_1 \dots a_n \omega)\rho(a'_1 \dots a'_n \omega)$ for all $\omega \in \Omega$. For a congruence ρ of A all the operations from Ω can be naturally transferred to quotient set A/ρ by the rule: if ω is n-ary operation, then $[a_1]\dots[a_n]\omega=[a_1 \dots a_n \omega]$. This definition does not depend on the choice of representatives in the cosets. Thus we obtain Ω-algebra A/ρ which is called the *quotient algebra* of the algebra A by the congruence ρ. Let A and B be two Ω-algebras with the same set of symbols of operations Ω. A mapping $\mu: A \rightarrow B$ is called the *homomorphism of Ω-algebras* if for all $\omega \in \Omega$, $a \in A$, the equality $(a_1 \dots a_n \omega)^{\mu}=a_1^{\mu} \dots a_n^{\mu}\omega$ takes place. This means that μ preserves all operations from Ω. The set of all homomorphisms from A to B is denoted by Hom(A,B). *Endomorphism* of Ω-algebra A is a homomorphism of A into itself. The set of all endomorphisms of Ω-algebra A is denoted by EndA. Injective homomorphism is called *monomorphism*; bijective one is *isomorphism* and bijective endomorphism is *automorphism*.

Let Ω be a fixed set of operations and A_{α}, $\alpha \in I$ some set of Ω-algebras. Consider $A=\prod_{\alpha \in I} A_{\alpha}$ All operations from Ω can be defined on the set A: if $a^1, a^2, \dots, a^n$ belong to A then $a^1 a^2 \dots a^n \omega$ is such a function that $(a^1 \dots a^n \omega)(\alpha)=a^1(\alpha)\dots a^n(\alpha)\omega$. This Ω-algebra A is a Cartesian product of Ω-algebras A_{α}. If A is some Ω-algebra and $B \subset A$ is a closed subset with respect to operations from Ω (i.e. if $\omega \in \Omega$, then for all $b_1 \in B$ holds $b_1 \dots b_n \omega \in B$), then B is *$\Omega$-subalgebra* in A. If X is some subset in Ω-algebra A then intersection of all subalgebras containing the set X is called *Ω-subalgebra generated by the set* X.

Let X be a set and Ω a set of symbols of operations. Define Ω-word over X. All elements of X and all symbols of null-ary operations are considered to be words. If $w_1, \dots, w_n$ are Ω-words and ω is a symbol of n-ary operation then the formal expression $w_1 w_2 \dots w_n \omega$ is also a word. Inductively we get a set of such words denoted by $F=F(X,\Omega)$. This set is an Ω-algebra, which is called *free algebra* with the set of generators X. Ω-algebra $F(X,\Omega)$ satisfies the following universal property: for any Ω-algebra A each mapping $X \rightarrow A$ is uniquely extended up to the homomorphism $F(X,\Omega) \rightarrow A$.

Each Ω-word $w=w(x_1,\dots,x_n)$ defines a set of elements

$w(a_1,\dots,a_n)$, $a_i \in A$, in A. Identity $w_1 \equiv w_2$ is said to be satisfied in Ω-algebra A if for all $a_1,\dots,a_n$ of A holds

$$w_1(a_1,a_2,\dots,a_n)=w_2(a_1,a_2,\dots,a_n).$$

A class of Ω-algebras is called a *variety of Ω-algebras* if it consists of all Ω-algebras satisfying a fixed set of identities.

Birkhoff's theorem [13]. *A class of Ω-algebras is a variety if and only if it is closed with respect to subalgebras, homomorphic images and Cartesian products.*

Ω-algebra is defined on one basic set and it is so-called *one-sorted algebraic system*. One can consider algebraic systems with several basic sets, automata for example. The above stated theorem is true in this case too [86]. We move now to the definition of main algebraic structures: semigroups, groups, linear spaces, modules, etc. Each of them is an Ω-algebra with the given set of operations Ω. Therefore, all general definitions (of homomorphism, congruence, Cartesian product, etc) hold true.

3. Main structures

Semigroup is a set S with one binary operation (often called multiplication) which satisfies the *associative identity*: $(xy)z=x(yz)$. The above set S_A of all transformations of A is a semigroup. A semigroup with unit (*monoid*), is a semigroup with designated element $e \in S$, called *unit element*, satisfying the identity $xe=ex=x$. Consider the construction of semigroup with external unit. Take a semigroup S with unit or without it. Let ε be a symbol not belonging to S and S^1 a union of two sets: S and {ε}. One can introduce a multiplication in S^1 by the rule: if a,b lie in S then their product is already defined in S; if $a \in S^1$, $b=\varepsilon$, then assume $a\varepsilon=\varepsilon a=a$. Evidently S^1 is a semigroup and S is a subsemigroup in S^1.

Right (left) unit of S is an element t of S such that for all $a \in S$ the equality $at=a$ ($ta=a$) holds. *Right (left) zero* of S is such element t, that for all $a \in S$ the quality $at=t$ ($ta=t$) takes place. A semigroup S is said to be a *semigroup with two sided cancellation* if for every

$x,y,z\in S$ each of the equalities $xz=yz$ and $zx=zy$ imply $x=y$. If for any $x,y\in S$ there exists $z=z(x,y)$ such that $zx=y$ then S is called a *semigroup with left division*. A subset $T\subset S$ is called *left (right) ideal* if for all $t\in T$ and $s\in S$ the inclusion $ts\in T$ $(st\in T)$ holds. If both inclusions hold in T, then T is called an *ideal* of S. A semigroup is *monogenic (cyclic)*, if it is generated by one element. Along with congruence one can consider *left (right) congruence* in S, i.e. an equivalence which preserves left (right) multiplication: if $s,s_1,s_2\in S$ and $s_1\rho s_2$ then $ss_1\rho ss_2$ $(s_1s\rho s_2s)$.

Element a of the semigroup S with unit e is called *invertible*, if there exists $b\in S$ such that $ab=ba=e$. Given invertible a, there is only one such b, denoted by a^{-1}. In the semigroup S_A only bijections are invertible. A *group* is a semigroup G with unit such that all $g\in G$ are invertible. A subgroup H of G is called *invariant (normal) subgroup* (or *divisor*) if any $h\in H$ and $a\in G$ satisfy $a^{-1}ha\in H$. If ρ is a congruence of a group G, and $H=[e]_\rho$, where e is a unit, then H is an invariant subgroup. Cosets on this congruence have the form $[a]_\rho=aH=\{ah \mid h\in H\}$. Quotient group G/ρ is denoted by G/H. A group G is called *simple* if it does not contain invariant subgroups except $\{e\}$ and G. A group G is called *commutative (Abelian)* if for all elements $a,b\in G$ holds $ab=ba$.

A series (chain) of subgroups

$$1=G_0\subset G_1\subset \dots \subset G_n=G$$

of a group G is a *normal series*, if G_i is normal subgroup in G_{i+1}, for each $i=0,\dots,n-1$.

A *ring* is a set K with two binary operations, addition (+) and multiplication (•), satisfying the conditions:

1. K is an Abelian group with respect to addition,
2. K is a semigroup with respect to multiplication,
3. A distributive law connects the above operations:

$$x(y+z)=xy+xz; \quad (x+y)z=xz+yz.$$

Thus, a ring joins the structure of Abelian group and the structure of semigroup. If the multiplication is commutative then a ring is called *commutative*. For example the set of integers $\mathbb{Z}$ is a commutative ring.

A *field* is a commutative ring in which all nonzero elements form a group on multiplication. The set of rational numbers $\mathbb{Q}$ and real ones $\mathbb{R}$ are the examples of fields.

A *linear space* A over field K is an Abelian group on addition in which the multiplication of elements $a\in A$ by scalars $\alpha\in K$ is defined. The following axioms have to be satisfied:

1. $\alpha(x+y)=\alpha x+\alpha y$
2. $1x=x$
3. $(\alpha+\beta)x=\alpha x+\beta x$
4. $(\alpha\beta)x=\alpha(\beta x)$

Replacing a field K by a ring in the definition of linear space we obtain the notion of a *module over ring* (or K-*module*). Homomorphisms of modules are also called *linear mappings* and endomorphisms - *linear operators*. Every linear space is freely generated by its basis. However, not all modules over rings are free. A *free module over ring* K with the basis X consists of all formal sums

$$\alpha_1x_1+\alpha_2x_2+\ \dots\ +\alpha_nx_n,\quad \alpha_i\in K,\quad x_i\in X.$$

It is assumed in this case that K has a unit and 1x is identified with x. Operations are defined componentwise. The K-module thus defined is called *left* K-*module*, since we consider left-hand multiplication. *Right* K-*module* is defined in a similar way. Let K, L be commutative rings. If A is a right K-module and left L-module and a condition $(\alpha a)\beta=\alpha(a\beta)$ takes place then A is called a *bimodule*.

Linear space (module) A is a direct sum of its subspaces (submodules) A_1, A_2, if $A_1\cap A_2=0$ and A_1, A_2 generate A. In this case A_2 is a direct complement of A_1 in A.

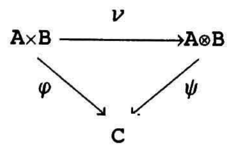
Define a *tensor product of modules* A, B over a commutative ring K. Consider a Cartesian product $A\times B$ and generate a free K-module M over $A\times B$. Denote by N its submodule generated by elements of the form

$$(a_1+a_2,b)-(a_1,b)-(a_2,b);\quad (a,b_1+b_2)-(a,b_1)-(a,b_2);$$

$$(\alpha a,b)-\alpha(a,b);\quad (a,\alpha b)-\alpha(a,b);\quad a,a_1,a_2\in A;\quad b,b_1,b_2\in B,\quad \alpha\in K.$$

Quotient module M/N is called a tensor product of modules A and B over ring K. It is denoted by $A\otimes_K B$ or simply $A\otimes B$. Tensor product satisfies the universal property: for each K-module C and bilinear mapping $\varphi: A\times B \rightarrow C$

there exists the uniquely defined homomorphism $\psi: A \otimes_K B \to C$ such that the following diagram is commutative



(ν is a canonic homomorphism).

Let H be a K-module over a commutative ring K and, moreover, H has a structure of semigroup relative to multiplication. H is an *associative algebra* if

1. H is a ring,
2. $\lambda(xy)=\lambda x \cdot y = x \cdot \lambda y$; $\lambda \in K$, $x, y, \in H$.

The following examples are of special interest. Let A be a K-module. One can consider EndA as an associative algebra over K defining operations by the rule: if $\varphi, \psi \in \mathrm{End}A$, $a \in A$, $\alpha \in K$, then $(\varphi+\psi)(a)=\varphi(a)+\psi(a)$; $(\varphi\psi)(a)=\varphi(\psi(a))$; $(\alpha\varphi)(a)=\alpha\varphi(a)$. Let now K be a commutative ring with unit and S be a semigroup. Construct a *semigroup algebra* KS. Its elements are formal sums of the kind $\alpha_1 s_1 + \ldots + \alpha_n s_n$, $\alpha_i \in K$, $s_i \in S$, $i=1,\ldots,n$. Addition and multiplication by scalars of K are defined componentwise, and multiplication in KS inherits one in S:

$$(\sum_i \alpha_i s_i)(\sum_j \beta_j s_j) = \sum_{i,j} \alpha_i \beta_j s_i s_j.$$

Replacing semigroup S from the definition by group G we come to *group algebra* KG.

4. Representations

A semigroup S_A of transformations and a group G_A of substitutions can be associated with each set A. If S is an abstract group then any homomorphism $\nu: S \to S_A$ defines a *representation of* S *by transformations* of A. Similarly, a *representation of* G *by substitutions* of A are defined by homomorphism $\nu: G \to G_A$. A representation can be also treated as an operation $\circ: A \times S \to A$. Taking this into account we speak about action of S on A and denote it by $(A, S, \circ)$ or simply (A, S). If S is a semigroup, then the definition of representation implies the condition: $a \circ y_1 y_2 =$

$(a\circ y_1)\circ y_2$, $a\in A$, $y_1,y_2\in S$; for groups we must add $a\circ e=a$, where e is a unit of S. If a semigroup S acts on A then every element of S can be considered as an unary operation on A and thus A is S-algebra. Such algebra is also called S-*polygon*. A representation (A,S) of semigroup S can be extended to representation (A,S^1) of the semigroup S^1 with adjoint unit ε, assuming $a\circ\varepsilon=a$ for all $a\in A$.

Thus defined representation (A,S) of the semigroup S is also called a *right representation*. Along with it, sometimes it is natural to consider left representation of semigroup S by the transformations of the set A, defined by antihomomorphism $\nu:S \to S_A$. Here $a\circ s_1s_2=(a\circ s_2)\circ s_1$, $a\in A$, $s_1,s_2\in S$. It is more convenient in this case to use left notation of the action: $s_1s_2\circ a=s_1\circ(s_2\circ a)$ and write (S,A) instead of (A,S).

Let S be a semigroup. To each element $s\in S$ corresponds a transformation λ_s of this very semigroup: $\lambda_s:x \to sx$ for all $x\in S$. A mapping $s \to \lambda_s$ defines an antihomomorphism of the semigroup S into semigroup of all transformations of S. So it defines left representation (S,S). In this case, if $s\in S$, then $s\circ x=sx$, $x\in S$. This representation is called *left regular* one. The *right regular representation* is defined in a similar way.

Let (A,S) be a representation. A subset $B\subset A$ is called invariant with respect to S (S-invariant) if the set $B\circ S=\{b\circ s\mid b\in B, s\in S\}$ lies in B. A representation (A,S) is called *irreducible* if there are no proper S-invariant subsets in A. For each subset $B\subset A$ one can consider its *normalizer* in S, i.e. the set of all elements $s\in S$ such that $B\circ s\subset B$. A *kernel of representation* (A,S) is the kernel congruence of the homomorphism $\nu:S \to S_A$. A representation is called *exact* one if this kernel is trivial, i.e. all its classes consist of one element. Along with S-invariant subset in A we consider S-invariant equivalence ρ in A such that $a_1\rho a_2$ implies $(a_1\circ s)\rho(a_2\circ s)$, for all $s\in S$, $a_i\in A$, $i=1,2$.

Let A,B be two semigroups, (C,B) right representation, A^C a Cartesian power of A. Right representation (C,B) induces left representation (B,A^C): if $\bar{a}\in A^C$, $b\in B$, then $b\circ\bar{a}$ is such element of A^C that $(b\circ\bar{a})(c)=\bar{a}(c\circ b)$; $c\in C$. Take the Cartesian product $A^C\times B$ and define multiplication on it by the rule: $(\bar{a}_1,b_1)(\bar{a}_2,b_2)=(\bar{a}_1(b_1\circ\bar{a}_2),b_1b_2)$. Given semigroup is called *wreath product* (*right wreath product*) of semigroups A and B over the set C and is denoted by $A\overset{C}{wr}B$. Left representation (B,C) of semigroup B induces right representation (A^C,B): if $\bar{a}\in A^C$, $b\in B$, $c\in C$, then $(\bar{a}\circ b)(c)=\bar{a}(b\circ c)$, in a dual way. Defining multiplication on $B\times A^C$ by $(b_1,\bar{a})(b_2,\bar{a}_2)=(b_1b_2,(\bar{a}_1\circ b_2)\bar{a}_2$ we come to semigroup $B\times A^C$, which is *left wreath product* of A and B over the set C.

5. Categories

A category $\mathfrak{K}$ consists of objects and morphisms. A class of all objects of the given $\mathfrak{K}$ is denoted by $Ob\mathfrak{K}$ and a class of all morphisms by $Mor\mathfrak{K}$. A set of morphisms Mor(A,B) corresponds to each pair A,B. Elements of this set are denoted by $f: A \rightarrow B$. The class $Mor\mathfrak{K}$ is a union of pair-wise nonintersected sets Mor(A,B). For each triplet of objects A,B,C there is a mapping

$$Mor(A,B)\times Mor(B,C) \rightarrow Mor(A,C),$$

which allows to speak of composition or product of morphisms. It is assumed that this product has the following properties:

a) the product of morphisms is associative

b) for each object A there exists such unit morphism $\varepsilon_A\in Mor(A,A)$, that for any $f\in Mor(A,B)$ and $\varphi\in Mor(C,A)$ holds $\varepsilon_A f=f$, $\varphi\varepsilon_A=\varphi$.

The concept of category is one of the most important joining notions of mathematics. The category of sets gives the example of category. If A and B are two sets then Mor(A,B) is the set of all mappings from A to B, where ε_A is an identical mapping of the set A into itself. The category of Ω-algebras is considered for fixed set of operations Ω. If A and B are two Ω-algebras then Mor(A,B) is Hom(A,B). In particular we have a category of linear spaces with linear mappings as morphisms, a category of semigroups with homomorphisms as morphisms, etc.

An object A of category $\mathfrak{K}$ is called *initial object* of $\mathfrak{K}$ if for any object B of $\mathfrak{K}$ there is a unique morphism $f: A \rightarrow B$. Dually, A is a *terminal object* if for any object B of $\mathfrak{K}$ there is a unique morphism $f: B \rightarrow A$.

Functors are homomorphisms of categories. Let $\mathfrak{K}_1,\mathfrak{K}_2$ be two categories. *Covariant functor* $\mathfrak{F}:\mathfrak{K}_1 \rightarrow \mathfrak{K}_2$ is a mapping assigning to each $A\in Ob\mathfrak{K}_1$ some $\mathfrak{F}(A)$ of $Ob\mathfrak{K}_2$, and to each $f\in Mor\mathfrak{K}_1$ some $\mathfrak{F}(f)\in Mor(\mathfrak{K}_2)$. The following conditions have to be satisfied:

a) if $f \in \mathrm{Mor}(A,B)$ then $\mathfrak{F}(f) \in \mathrm{Mor}(\mathfrak{F}(A),\mathfrak{F}(B))$,

b) $\mathfrak{F}(\varepsilon_A)=\varepsilon_{\mathfrak{F}(A)}$ for each $A \in \mathrm{Ob}\mathfrak{K}_1$,

c) $\mathfrak{F}(f\varphi)=\mathfrak{F}(f)\mathfrak{F}(\varphi)$ for each pair of morphisms of $\mathrm{Mor}\mathfrak{K}_1$.

The notion of *contravariant functor* arises by replacing of the conditions a) and c) by the following ones:

a) if $f \in \mathrm{Mor}(A,B)$ then $\mathfrak{F}(f) \in \mathrm{Mor}(\mathfrak{F}(B),\mathfrak{F}(A))$,

c) $\mathfrak{F}(f\varphi)=\mathfrak{F}(\varphi)\mathfrak{F}(f)$.

Contravariant functor is an antihomomorphism of categories: the order of multiplication changes to contrary one.

CHAPTER 1

PURE AUTOMATA

1.1. Basic concepts

1.1.1. Definitions and examples

Definitions of an automaton and of a semigroup automaton were given in the preface. Recall (see Preface), that we consider *automaton* $\mathfrak{A}=(A,X,B)$ as an algebraic system with three basic sets A, X, B called the sets of *states*, *input signals* and *output signals* respectively, and two binary operations:

$$\circ : A\times X \rightarrow A,$$

$$* : A\times X \rightarrow B.$$

An automaton (A,Γ,B) is called a semigroup automaton if the set of inputs is a semigroup, and the axioms

$$\begin{aligned} a\circ\gamma_1\gamma_2&=(a\circ\gamma_1)\circ\gamma_2, \\ a*\gamma_1\gamma_2&=(a\circ\gamma_1)*\gamma_2, \quad a\in A, \ \gamma_i\in\Gamma. \end{aligned} \tag{1.1}$$

are satisfied.

An automaton $\mathfrak{A}=(A,X,B)$ is called *finite*, if the sets A, X, B are finite. In a number of cases the sets A, B have to be provided with some algebraic structures, for example those of linear space. However, in this chapter we study only pure automata, i.e. automata whose sets of states and outputs do not have algebraic structure. Unlike the case of a semigroup automaton, an automaton (A, X, B) will be called an *absolutely pure* one if the set of inputs also does not have any algebraic structure.

In order to define an automaton, the basic sets and operations $\circ$ and $*$ should be defined.

Example. An RS flip-flop is used in electrical and radio engineering. It is a device with two input lines, on each of them 0 or 1 signals can be fed; the device could be in one of two states 0 or 1, whose values coincide with the values of the output signal. Inputs are two-dimensional vectors with coordinates taken from the set {0,1}. Though four various input vectors are possible, in a RS flip-flop three different vectors $x_1=(0,0)$, $x_2=(0,1)$, $x_3=(1,0)$ can be fed into its input. These input vectors act on the states of the device in the following way: if RS flip-flop was in the 0 state, then with x_1 and x_2 inputs, the state does not change, while with x_3 input it changes its state to 1; if the RS flip-flop is in the state 1, the state does not change with inputs x_1 and x_3, and changes to 0 with input x_2.

Thus, a RS flip-flop is an automaton (A,X,B) with the set of inputs $X=\{x_1,x_2,x_3\}$, the set of states $A=\{a_0,a_1;\ a_0=0,a_1=1\}$ and with outputs which are identical to the states. The functions of transitions and outputs of the automaton coincide and are defined by the following rule:

$$a_0\circ x_1=a_0\ ,\quad a_0\circ x_2=a_0\ ,\quad a_0\circ x_3=a_1$$
$$a_1\circ x_1=a_1\ ,\quad a_1\circ x_2=a_0\ ,\quad a_1\circ x_3=a_1$$

The example is of interest since the automaton thus defined is one of blocks with which other automata can be constructed, (see Chapter 2).

Example. Let $\mathfrak{A}=(A,X,B)$ be an automaton with two states $A=\{0,1\}$, with two inputs, $X=\{0,1\}$, and with two outputs $B=\{0,1\}$. The $\circ$ and $*$ operations are defined according to the rule:

$$a\circ x=a+x(\mathrm{mod}2)$$
$$a*x=a\cdot x(\mathrm{mod}2)$$

There exist different ways of setting of automata: the analytical one, defining with tables , with plots etc. In the latter example, the operations $\circ$ and $*$ are defined in an analytical way. The corresponding tables are as follows:

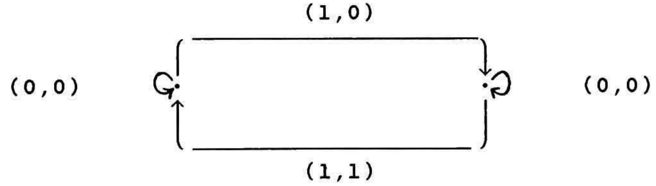
Table of transitions to new states

Initial state	Input 0	Input 1
0	0	1
1	1	0

Table of outputs

Initial state	Input 0	Input 1
0	0	0
1	0	1

In the case of description with a plot, the automaton is defined by an oriented graph, with vertices being states of the automaton, while the edge connecting the vertex a with the vertex a' is denoted by the pair of symbols (x,y), where x is the input signal effecting transition of the automaton from the state a into the state a', and y is the output signal of the automaton which is equal to a*x. For instance, the graph of the automaton of last example is of the form:



Example. Let A and B be arbitrary sets, S_A the semigroup of all transformations of the set A, Fun(A,B) the set of all mappings on A to B. Consider the Cartesian product $S(A,B)=S_A\times Fun(A,B)$ and define the multiplication operation on the set S(A,B), assuming

$$(\varphi_1,\psi_1)(\varphi_2,\psi_2)=(\varphi_1\varphi_2,\varphi_1\psi_2),\varphi_i\in S_A,\psi_i\in Fun(A,B),i=1,2$$

A direct check shows the associativity of this operation; hence

S(A,B) is a semigroup. Define the automaton (A,S(A,B),B); the operations $\circ$ and $*$ are defined by:

$$a\circ(\sigma,\varphi)=a^{\sigma},$$
$$a*(\sigma,\gamma)=a^{\varphi},$$

where $a\in A$, $(\sigma,\varphi)\in S(A,B)$, $\sigma\in S_A$, $\varphi\in Fun(A,B)$.

This is a semigroup automaton. Indeed,

$$a\circ((\sigma_1,\varphi_1)(\sigma_2,\varphi_2))=a\circ(\sigma_1\sigma_2,\sigma_1\varphi_2)=a^{\sigma_1\sigma_2}=(a^{\sigma_1})^{\sigma_2}=$$
$$=((a\circ(\sigma_1,\varphi_1))\circ(\sigma_2,\varphi_2);$$

$$a*((\sigma_1,\varphi_1)(\sigma_2,\varphi_2)=a*(\sigma_1\sigma_2,\sigma_1\varphi_2)=(a^{\sigma_1})^{\varphi_2}=$$
$$=(a\circ(\sigma_1,\varphi_2))*(\sigma_2,\varphi_2).$$

So the axioms (1.1) of a semigroup automaton are fulfilled. This automaton is denoted by $Atm^1(A,B)$ and plays an important part in the sequel.

Along with the introduced automata in real situations one can often find the so-called partially defined automata, whose operations $\circ$ and $*$ are defined not for all elements. Automata for which only sets A,X are essential and only the operation $\circ$ is given, are called *semiautomata* or the *automata of the input-state* type. In fact such automata are representations $\mathfrak{A}$=(A,X). There are also automata (A,X,B), in which only the $*$ operations is defined. Such automata will be called *automata of the input-output type*, or **-automata*. It is quite natural to ask, whether arbitrary semiautomaton and *-automaton could be joined to form an automaton? How it could be done?, and whether such an union would be unique. The answer to this question will be given in Section 1.1.2.

1.1.2. The automaton representation of a set and a semigroup

Let A be a set and S_A be the semigroup of all its transformations. If X is some other set, then each mapping $f:X \to S_A$ produces a representation of elements from X by transformations of A. We have simultaneously a binary operation $\circ: A\times X \to A$ defined by the equality

$a\circ x=af(x)$. On the other hand, given the operation $\circ$, each x can be considered as a transformation of A, and so the representation $f:X \to S_A$ arises. This is one-to-one correspondence. In the case when $X=\Gamma$ is a semigroup, it can be directly shown that the relation $a\circ\gamma_1\gamma_2=(a\circ\gamma_1)\circ\gamma_2$; $\gamma_1\in\Gamma$ holds if and only if the representation $f:\Gamma \to S_A$ is a homomorphism. Basing on these well known arguments, we define the notion of a automaton representation.

Take an automaton $\mathfrak{A}=(A,X,B)$ and a semigroup $S(A,B)$. The input elements $x\in X$ act, on the one hand, as transformations on the set A, i.e. as elements of S_A, and on the other hand, as elements of Fun(A,B). Thus, we define two mappings: $\alpha:X \to S_A$ and $\beta:X \to \mathrm{Fun}(A,B)$. For each $x\in X$ we define the transformation x^α of S_A and the mapping x^β of Fun(A,B) in the following way: if $a\in A$, then $ax^\alpha=a\circ x$, $ax^\beta=a*x$.

Define the representation $f:X \to S(A,B)$, setting $x^f=(x^\alpha,x^\beta)$. This representation is associated with the automaton $\mathfrak{A}$. If $X=\Gamma$ is a semigroup, it can be easily seen that $f:X \to S(A,B)$ is a homomorphism of the semigroup Γ to $S(A,B)$.

On the other hand, let us consider a mapping $f:X \to S(A,B)$, and the element $x^f=(\varphi,\psi)\in S(A,B)=S_A\times\mathrm{Fun}(A,B)$ being the image of the element x. The automaton $\mathfrak{A}=(A,X,B,\circ,*)$ with $a\circ x=a^\varphi$, $a*x=a^\psi$ corresponds to this mapping. If $X=\Gamma$ is a semigroup, the homomorphism $f:\Gamma \to S(A,B)$ defines the semigroup automaton (A,Γ,B) (to prove this, it suffices to verify fulfillment of the semigroup automaton axioms.)

Thus, defining of the automaton (A,X,B) is equivalent to that of the representation $f:X \to S(A,B)$, while defining of the semigroup automaton (A,Γ,B) is equivalent to that of the homomorphism $f:\Gamma \to S(A,B)$. We will call this homomorphism the *automaton representation* of the semigroup Γ.

An absolutely pure automaton (A,X,B) is called an *exact automaton* if the associated mapping $X \to S(A,B)$ is injective. Respectively, a semigroup automaton (A,Γ,B) is an exact one, if the homomorphism $f:\Gamma \to S(A,B)$ is a monomorphism of semigroups, i.e. different elements of Γ correspond to different elements of $S(A,B)$. The kernel congruence $\rho=\mathrm{Ker}f$ of the semigroup Γ is called the kernel of the automaton representation, or the *kernel of the automaton* $\mathfrak{A}$. An exact automaton $(A,\Gamma/\rho,B)$ can be assigned to each automaton $\mathfrak{A}=(A,\Gamma,B)$.

Together with the kernel $\rho=\mathrm{Ker}f$ consider the kernels of the mappings $\alpha:\Gamma\rightarrow S_A$, $\beta:\Gamma\rightarrow \mathrm{Fun}(A,B)$, which will be denoted by $\rho_\alpha=\mathrm{Ker}\alpha$, $\rho_\beta=\mathrm{Ker}\beta$ respectively. Since α is a homomorphism, ρ_α is the congruence of the semigroup Γ. The equivalence ρ_β could be not a congruence, but it endures the left multiplication: if $\gamma_1\rho_\beta\gamma_2$ then $\gamma\gamma_1\rho_\beta\gamma\gamma_2$ (it follows from the equality $(\gamma_1\gamma_2)^\beta=\gamma_1^\alpha\gamma_2^\beta$, $\gamma_1,\gamma_2\in\Gamma$). Besides, $\rho=\rho_\alpha\cap\rho_\beta$.

Using the automaton representation, an absolutely pure automaton $\mathfrak{A}=(A,X,B)$ can be extended to a semigroup automaton $\mathcal{F}(\mathfrak{A})=(A,F,B)$, where $F=F(X)$ is a free semigroup generated by the set X. Indeed, as it was mentioned above, defining of the automaton $\mathfrak{A}$ is equivalent to defining of the representation $f:X\rightarrow S(A,B)$. By the universal property of a free semigroup, the mapping $f:X\rightarrow S(A,B)$ is uniquely extended up to the homomorphism $f:F(X)\rightarrow S(A,B)$, while defining of latter is equivalent to defining of the semigroup automaton $(A,F(X),B)$.

Example. $\mathfrak{A}=(A,X,B)$ is an automaton with the set of states $A=\{a_0,a_1,\dots,a_{n-1}\}$, the set of input signals X consisting of one element x, and the set of output signals $B=\{0,1\}$. The operations $\circ$ and $*$ are defined by the equalities:

$$a_i\circ x=a_k \text{ where } k=i+1(mod n);$$

$$a_i*x=\begin{cases}0, & i\equiv 0(mod 2),\\ 1, & i\equiv 1(mod 2).\end{cases}$$

Recall that, if t, m, and n are integer numbers then $t=m(mod n)$ means that t is the remainder of division of m by n, and $t\equiv m(mod n)$ that t and m have the same remainder of division by n.

In the corresponding semigroup automaton $\mathcal{F}(\mathfrak{A})=(A,F(X),B)$ the semigroup of input sequences F(X) is the infinite cyclic semigroup with generator x. The elements of the semigroup are of the form x^m, $m=1,2,\dots$, and act in the following way:

$$a_i \circ x^m = a_k \ , \text{ where } k=i+m(mod\, n);$$

$$a_i * x^m = \begin{cases} 0, & i+m-1\equiv 0(mod 2); \\ 1, & i+m-1\equiv 1(mod 2). \end{cases}$$

1.1.3. Homomorphisms of automata

Functioning of an automaton can be described by the functioning of another one. Homomorphism is a mathematical notion which reflects the physical concept of modeling. A triplet of mappings $\mu=(\mu_1,\mu_2,\mu_3)$, $\mu_1:A\rightarrow A'$, $\mu_2:X \rightarrow X'$, $\mu_3:B \rightarrow B'$ is defined to be the *homomorphism* $\mu:\mathfrak{A} \rightarrow \mathfrak{A}'$ *of the automaton* $\mathfrak{A}=(A,X,B)$ *to the automaton* $\mathfrak{A}'=(A',X',B')$, if the following conditions are satisfied:

$$\begin{aligned} (a\circ x)^{\mu_1} &= a^{\mu_1}\circ x^{\mu_2}, \\ (a*x)^{\mu_3} &= a^{\mu_1} * x^{\mu_2}; \quad a\in A,\ x\in X \end{aligned} \qquad (1.2)$$

In order to define a homomorphism of semigroup automata $\mu:(A,\Gamma,B) \rightarrow (A',\Gamma',B')$ we should add the condition:

μ_2 is a homomorphism of the semigroup Γ to semigroup Γ' (1.3)

If the mappings μ_1, μ_2, μ_3 of the latter definition are one-to-one, μ is called an *isomorphism of automata*. A homomorphism (isomorphism) of an automaton $\mathfrak{A}$ into itself is called an endomorphism (automorphism) of an automaton.

Absolutely pure automata with their homomorphisms form a category. Semigroup automata together with their homomorphisms also form a category. Consider the mapping $\mathcal{F}$ which assigns to each pure automaton $\mathfrak{A}$ a semigroup automaton $\mathcal{F}(\mathfrak{A})$. Further, we take an arbitrary homomorphism of absolutely pure automata $\mu=(\mu_1,\mu_2,\mu_3)$: $\mathfrak{A}=(A,X,B) \rightarrow \mathfrak{A}'=(A',X',B')$ and define the corresponding homomorphism of semigroup automata $(\tilde{\mu}_1,\tilde{\mu}_2,\tilde{\mu}_3)=\mathcal{F}(\mu):\mathcal{F}(\mathfrak{A}) \rightarrow \mathcal{F}(\mathfrak{A}')$ by: $\tilde{\mu}_1=\mu_1$, $\tilde{\mu}_3=\mu_3$. As $\tilde{\mu}_2$ we take the unique extension of the mapping $\mu_2:X \rightarrow X'\subset F(X')$ to the homomorphism $\tilde{\mu}_2:F(X) \rightarrow F(X')$. It is easy to understand that $\mathcal{F}(\mu)$ is a homomorphism of semigroup automata and that $\mathcal{F}$ is a functor on the category of absolutely pure automata into the category of semigroup automata.

Consider some special homomorphisms. Let automata $\mathfrak{A}=(A,X,B)$ and $\mathfrak{A}'=(A',X,B)$ have identical sets of input and output signals. A homomorphism $\mathfrak{A} \rightarrow \mathfrak{A}'$ of the form $\tilde{\mu}_1=(\mu_1,\varepsilon_X,\varepsilon_B)$, where ε_X and ε_B are identity mappings of sets X and B respectively is called a *homomorphism in states*. If the mapping μ_1 is surjective (injective), $\tilde{\mu}_1$ is called an epimorphism (monomorphism) in states. In this case the work of the automaton $\mathfrak{A}$ is modeled by functioning of the automaton $\mathfrak{A}'$ with the same sets of input and output signals, but different set of states. It is natural to restrict the number of states of an automaton without limiting the number of its options. For finite automata, given automaton $\mathfrak{A}$, there exists an algorithm of constructing of its epimorphic in states image $\mathfrak{A}'$ with the the smallest number of states (see e.g. [62]). A homomorphism in input signals and a homomorphism in output signals are defined in a similar way as the homomorphism in states. For example, a homomorphism in input signals of an automaton $\mathfrak{A}=(A,X,B)$ into automaton $\mathfrak{A}'=(A,X',B)$ is a homomorphism of the type $\tilde{\mu}_2=(\varepsilon_A,\mu_2,\varepsilon_B)$ with ε_A and ε_B being identity mappings of sets A and B respectively. The condition (1.2) implies $a\circ x=(a\circ x)^{\varepsilon_A}=a^{\varepsilon_A}\circ x^{\mu_2}=a\circ x^{\mu_2}$, $a\in A$, $x\in X$; similarly, $a*x=a*x^{\mu_2}$. The homomorphism in input signals means making an automaton more exact; if $\mathfrak{A}$ is an exact automaton, the automaton $\mathfrak{A}'$ is again exact and the homomorphism $\tilde{\mu}_2:\mathfrak{A} \rightarrow \mathfrak{A}'$ is an isomorphism.

One can consider also homomorphisms of automata which are identical only on one of the sets A,X,B. Note that homomorphisms in states define the category of automata having a variable set A and fixed X and B, homomorphisms in output signals define the category of automata with fixed representation (A,X), and finally, homomorphisms in input signals yield the category of automata with given A and B.

If μ is a homomorphism of an automaton $\mathfrak{A}$ into $\mathfrak{A}'$, and ν a homomorphism $\mathfrak{A}'$ into $\mathfrak{A}''$, the multiplication of mappings $\mu\nu=(\mu_1\nu_1,\mu_2\nu_2,\mu_3\nu_3):\mathfrak{A} \rightarrow \mathfrak{A}''$ is defined in a natural way. Fulfillment of (1.2) for this mapping is obvious:

$$(a\circ x)^{\mu_1\nu_1}=((a\circ x)^{\mu_1})^{\nu_1}=(a^{\mu_1}\circ x^{\mu_2})^{\nu_1}=a^{\mu_1\nu_1}\circ x^{\mu_2\nu_2}.$$

The second condition of (1.2) is verified in the similar way. Hence, the mapping $\mu\nu:\mathfrak{A} \rightarrow \mathfrak{A}''$ is a homomorphism of automata.

Proposition 1.1. *Any homomorphism $\mu=(\mu_1,\mu_2,\mu_3)$ of an automaton can be represented as a product $\mu=\tilde{\mu}_3\tilde{\mu}_1\tilde{\mu}_2$ of homomorphisms in outputs $\tilde{\mu}_3$, in states $\tilde{\mu}_1$ and in inputs $\tilde{\mu}_2$.*

Proof. Let $\mu=(\mu_1,\mu_2,\mu_3)$ be a homomorphism of the semigroup automata, $\mu:\mathfrak{A}=(A,\Gamma,B,\circ,*) \rightarrow \mathfrak{A}'=(A',\Gamma',B',\circ',*')$. Consider the automaton $\mathfrak{A}=(A,\Gamma,B')$ with the operations $\circ_1$ and $*_1$, defined according to the rule: $a\circ_1\gamma=a\circ\gamma$, $a*_1\gamma=(a*\gamma)^{\mu_3}$, $a\in A$, $\gamma\in\Gamma$. This automaton is a semigroup one:

$$a\circ_1\gamma_1\gamma_2=(a\circ_1\gamma_1)\circ_1\gamma_2,\ a*_1\gamma_1\gamma_2=(a*\gamma_1\gamma_2)^{\mu_3}=((a\circ\gamma_1)*\gamma_2)^{\mu_3}=(a\circ_1\gamma_1)*_1\gamma_2;$$

$\tilde{\mu}_3=(\varepsilon_A,\varepsilon_\Gamma,\mu_3)$ is a homomorphism of (A,Γ,B) into (A,Γ,B'). Now, by the homomorphism $\mu_2:\Gamma \rightarrow \Gamma'$ and the automaton (A',Γ',B') define the automaton $\mathfrak{A}_2=(A',\Gamma,B')$ with the operations $\circ_2$ and $*_2$: if $a\in A',\gamma\in\Gamma$, then $a\circ_2\gamma=a\circ'\gamma^{\mu_2}$, $a*_2\gamma=a*'\gamma^{\mu_2}$. It is also a semigroup automaton and $\tilde{\mu}_2=(\varepsilon_{A'},\mu_2,\varepsilon_{B'})$ is a homomorphism of (A',Γ,B') into (A',Γ',B'). Finally, it is immediately verified, that $\tilde{\mu}_1=(\mu_1,\varepsilon_\Gamma,\varepsilon_{B'})$ is a homomorphism of (A,Γ,B') into (A',Γ,B'). We derived the sequence of homomorphisms

$$(A,\Gamma,B) \xrightarrow{\tilde{\mu}_3} (A,\Gamma,B') \xrightarrow{\tilde{\mu}_1} (A',\Gamma',B') \xrightarrow{\tilde{\mu}_2} (A',\Gamma',B').$$

It is clear that $\mu=\tilde{\mu}_3\tilde{\mu}_1\tilde{\mu}_2$

Note, that the given order of multiples is important. Such representation of the automaton homomorphism is called a *canonical decomposition of a homomorphism*. The proof is given for the case of the homomorphism of the semigroup automaton. If we consider a homomorphism of an absolutely pure automaton, the arguments will be even simpler.

It is interesting to remark, that if $\mu=(\mu_1,\mu_2,\mu_3)$ is an endomorphism of the automaton $\mathfrak{A}$ into itself, and $\mu=\tilde{\mu}_3\tilde{\mu}_1\tilde{\mu}_2$ is its canonical decomposition, then $\tilde{\mu}_1,\tilde{\mu}_2,\tilde{\mu}_3$ are homomorphisms and not endomorphisms. For example, $\tilde{\mu}_3$ maps the automaton (A,Γ,B) into the automaton $(A,\Gamma,B,\circ_1,*_1)$. These automata have common basic sets, but different operations.

An important role in the theory of automata can play the notion of a *homomorphism with substitution.* A homomorphism of the automaton $\mathfrak{A}=(A,X,B)$ into the automaton $\mathfrak{A}'=(A',X',B')$ with the *substitution of output signals* is a triplet of mappings $\mu_1:A \rightarrow A'$, $\mu_2:X \rightarrow X'$, $\mu_3:B' \rightarrow B$, $\mu=(\mu_1,\mu_2,\mu_3)$, which agrees with the basic operations $\circ$ and $*$

$$
\begin{aligned}
&a^{\mu_1}\circ x^{\mu_2}=(a\circ x)^{\mu_1}\\
&(a^{\mu_1}*x^{\mu_2})^{\mu_3}=a*x\ ;\ a\in A,\ x\in X.
\end{aligned}
\tag{1.4}
$$

(In the given example the mapping μ_3 acts in the opposite direction in comparison with μ_1 and μ_2). The homomorphism with the substitution of the output signal can be naturally explained: input signals of the automaton $\mathfrak{A}$ are coded by the signals of the automaton $\mathfrak{A}'$ and operation of the second automaton imitates the operation of the first one and decoding is performed in the output.

Homomorphism of the automaton $\mathfrak{A}=(A,X,B)$ into the automaton $\mathfrak{A}'=(A',X',B')$ with the *substitution of input signals* is a triplet of mappings $\mu_1:A \rightarrow A'$, $\mu_2:X' \rightarrow X$, $\mu_3:B \rightarrow B'$, $\mu=(\mu_1,\mu_2,\mu_3)$, satisfying the conditions

$$
\begin{aligned}
&a^{\mu_1}\circ x'=(a\circ x'^{\mu_2})^{\mu_1}\\
&a^{\mu_1}*x'=(a*x'^{\mu_2})^{\mu_3},\quad a\in A,\quad x'\in X'.
\end{aligned}
$$

The homomorphism with the substitution of input and output signals constitute a triplet of mappings $\mu_1:A \rightarrow A'$, $\mu_2:X' \rightarrow X$, $\mu_3:B' \rightarrow B$ $\mu=(\mu_1,\mu_2,\mu_3)$, with

$$
\begin{aligned}
&a^{\mu_1}\circ x'=(a\circ x'^{\mu_2})^{\mu_1}\\
&(a^{\mu_1}*x')^{\mu_3}=a*x'^{\mu_2},\quad a\in A,\quad x'\in X'.
\end{aligned}
$$

Consider the definitions of subautomata and quotient automata. Let $\mathfrak{A}=(A,X,B)$ be a certain automaton, $A_1\subset A$, $X_1\subset X$, $B_1\subset B$ and for any $a\in A_1$, $x\in X_1$ hold $a\circ x\in A_1$, $a*x\in B_1$. Then the sets A_1,X_1,B_1 define the automaton $(A_1,X_1,B_1)=\mathfrak{A}_1$ with respect to the same operations $\circ$ and $*$. It is called a *subautomaton* of the given automaton $\mathfrak{A}$. The subautomaton (A_1,X_1,B_1) is called a *subautomaton in states*, if $X_1=X$ and $B_1=B$. Similarly, we can define *subautomata in inputs* and *subautomata in outputs*. If (A,Γ,B) is a semigroup automaton, then a subautomaton of the form (A_1,Γ,B_1) is called a *Γ-subautomaton*.

Congruence of an automaton $\mathfrak{A}=(A,X,B)$ is a triplet of equivalences $\rho=(\rho_1,\rho_2,\rho_3)$: ρ_1 on the set A , ρ_2 on the set X, ρ_3 on the set B, which satisfies the following condition:

$$a\rho_1 a'\wedge x\rho_2 x' \Rightarrow (a\circ x)\rho_1(a'\circ x')\wedge(a*x)\rho_3(a'*x') \tag{1.5}$$

Let $\rho=(\rho_1,\rho_2,\rho_3)$ be a congruence of the automaton $\mathfrak{A}$. An automaton with basic sets $A/\rho_1, X/\rho_2, B/\rho_3$ and operations ∘ and *, defined according to the rule

$$[a]_{\rho_1}\circ[x]_{\rho_2}=[a\circ x]_{\rho_1};\quad [a]_{\rho_1}*[x]_{\rho_2}=[a*x]_{\rho_3}$$

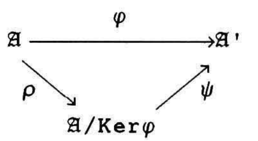
where $[a]_{\rho_1}$, $[x]_{\rho_2}$, $[b]_{\rho_3}$ are classes by corresponding equivalences ρ_1,ρ_2,ρ_3, is called a *quotient automaton* of the automaton $\mathfrak{A}$ by congruence ρ and denoted as $\mathfrak{A}/\rho$. According to (1.5) operations ∘ and * in the automaton are correctly defined.

If (A,Γ,B) is a semigroup automaton, then the definition of congruence ρ has to be complemented by the requirement: ρ_2 is a congruence of the semigroup Γ. In this case the quotient automaton $\mathfrak{A}/\rho=(A/\rho_1,\Gamma/\rho_2,B/\rho_3)$ also be a semigroup automaton.

Example. Let $\mathfrak{A}=(A,\Gamma,B)$ be a semigroup automaton, (A_0,Γ,B_0) Γ-subautomaton in $\mathfrak{A}$, ρ_1 an equivalence on the set A, classes of which are the set A_0 and individual elements not belonging to A_0; ρ_3 is a similar equivalence on the set B. Let $\rho_2=\rho_\varepsilon$ be a trivial equivalence on the set Γ, i.e. the equivalence, whose classes coincide with the elements of Γ. It is obvious, that $\rho=(\rho_1,\rho_2,\rho_3)$ is a congruence of the automaton $\mathfrak{A}$. In this case quotient sets A/ρ_1 and B/ρ_3 are denoted by A/A_0 and B/B_0 respectively. Factor automaton $\mathfrak{A}/\rho=(A/A_0,\Gamma,B/B_0)$ corresponds to the congruence ρ. Take in Γ a set J of all σ, such that $a\circ\sigma\in A_0$, $a*\sigma\in B_0$ for every $a\in A$. J is a two-sided ideal in Γ, associated with Rees congruence τ of the semigroup Γ. The ideal J and the elements not belonging to J are the classes of this congruence. Factor-semigroup Γ/τ in the theory of semigroups is denoted by Γ/J. It is easy to verify that (ρ_1,τ,ρ_3) is a congruence of the automaton $\mathfrak{A}$. If I is an arbitrary two sided ideal of the semigroup Γ, belonging to J, then the corresponding Rees congruence ρ_2 belongs to τ and (ρ_1,ρ_2,ρ_3) is also a congruence of the automaton $\mathfrak{A}$.

Now we can introduce a notion of the kernel of homomorphism. Let $\mu=(\mu_1,\mu_2,\mu_3)$ is a homomorphism of the semigroup automata $\mathfrak{A}=(A,\Gamma,B) \to \mathfrak{A}'=(A',\Gamma',B')$. Denote by τ_1, τ_2, τ_3 kernels of mappings μ_1, μ_2, μ_3, respectively. Recall that $a_1\tau_1a_2$ for $a_1^{\mu_1}=a_2^{\mu_1}$ with $a_1,a_2\in A$; τ_2 and τ_3 are defined similarly. In this case τ_2 is a congruence of Γ. For the triplet (τ_1,τ_2,τ_3) conditions of (1.5) are satisfied. Hence, $\tau=(\tau_1,\tau_2,\tau_3)$ is a congruence of the automaton $\mathfrak{A}$. This congruence is called a *kernel of the homomorphism* μ and is denoted by $\tau=\mathrm{Ker}\mu$. The homomorphism τ of the automaton $\mathfrak{A}$ on $\mathfrak{A}/\mathrm{Ker}\mu$ is called natural.

Theorem of homomorphisms 1.2. *Let φ be a homomorphism of the automaton $\mathfrak{A}$ on the automaton $\mathfrak{A}'$ and ρ a natural homomorphism of $\mathfrak{A}$ on $\mathfrak{A}/\mathrm{Ker}\varphi$. Then the automaton $\mathfrak{A}'$ is isomorphic to the automaton $\mathfrak{A}/\mathrm{Ker}\varphi$, and there exists a unique isomorphism ψ such that $\rho\psi=\varphi$.*



Denote by τ_Γ the kernel congruence of the automaton representation of the semigroup Γ: $\gamma_1\tau_\Gamma\gamma_2$, when for all $a\in A$ hold $a\circ\gamma_1=a\circ\gamma_2$ and $a*\gamma_1=a*\gamma_2$. Let τ_A be the equivalence on the set A defined by: $a_1\tau_Aa_2$, when $a_1*\gamma=a_2*\gamma$ for all $\gamma\in\Gamma$. Thus, input elements γ_1 and γ_2 are τ_Γ-equivalent, if they act identically as operators on the set of states A and from A to B; states a_1 and a_2 are τ_A-equivalent, if a_1 and a_2 act identically as functions from the input set Γ to the output signal set B.

The automaton (A,Γ,B) is exact, if classes of the congruence τ_Γ consist of the separate elements, i.e. $a\circ\gamma_1=a\circ\gamma_2$, $a*\gamma_1=a*\gamma_2$ for all $a\in A$

imply $\gamma_1=\gamma_2$; $\gamma_1,\gamma_2\in\Gamma$.

Let us call an automaton $\mathfrak{A}$ *left-reduced*, if classes of the congruence τ_A consist of separate elements, i.e. if different elements from the set A act as different operators from Γ to B, and right-reduced, if B coincides with the set $A*\Gamma=\{a*\gamma \mid a\in A,\gamma\in\Gamma\}$. We will call left reduced automaton simply a reduced automaton.

Consider the following congruences of the automaton $\mathfrak{A}=(A,\Gamma,B)$: $\tau^*_\Gamma=(\varepsilon_A,\tau_\Gamma,\varepsilon_B)$ and $\tau^*_A=(\tau_A,\varepsilon_A,\varepsilon_B)$. The automaton $\mathfrak{A}/\tau^*_\Gamma$ is exact, $\mathfrak{A}/\tau^*_A$ is a left-reduced automaton. As it has been mentioned, in finite case there is an algorithm restricting the number of essential states of $\mathfrak{A}$ or, in other words, realizing $\mathfrak{A}/\tau^*_A$. The problem of construction of the exact automaton $\mathfrak{A}/\tau^*_\Gamma$ also has practical argumentation and for finite automata there exists the corresponding simple algorithm. The automaton (A,Γ,B) and automaton (A',Γ,B) are called *equivalent in states*, if reduced automata are isomorphic in states. The automata (A,Γ,B) and (A,Γ',B) are called *equivalent in inputs*, if the corresponding exact automata are isomorphic in inputs.

Proposition 1.3. *If there exists an epimorphism in states* $\varphi=(\varphi_A,\varepsilon_\Gamma,\varepsilon_B):\mathfrak{A}=(A,\Gamma,B) \rightarrow \mathfrak{A}'=(A',\Gamma,B)$, *then the automata* $\mathfrak{A}$ *and* $\mathfrak{A}'$ *are equivalent in states.*

Indeed, since in the given case $a*\gamma=a^{\varphi_A}*\gamma$, then the equality $a_1*\gamma=a_2*\gamma$ is equivalent to $a_1^{\varphi_A}*\gamma=a_2^{\varphi_A}*\gamma$; $a_1,a_2\in A$, $\gamma\in\Gamma$. It means that $a_1\tau_A a_2$ is equivalent to $a_1^\varphi\tau_A a_2^\varphi$ and that the automata $\mathfrak{A}/\tau^*_A$ and $\mathfrak{A}'/\tau^*_{A'}$ are isomorphic in states.

A similar property holds for the equivalence in inputs.

1.1.4. Cyclic automata

We shall consider subautomata $\mathfrak{A}_\alpha$, $\alpha\in I$, in the fixed semigroup automaton $\mathfrak{A}=(A,\Gamma,B)$ under the relation of inclusion: $\mathfrak{A}_1\subset\mathfrak{A}_2$, if $\mathfrak{A}_1$ is a subautomaton in $\mathfrak{A}_2$. For any set of subautomata $\mathfrak{A}_\alpha$, $\alpha\in I$, it is possible to consider the least upper bound and the greatest lower bound of this set. The greatest lower bound $\tilde{\mathfrak{A}}$ is an intersection of the subautomata $\mathfrak{A}_\alpha=(A_\alpha,\Gamma_\alpha,B_\alpha)$, $\tilde{\mathfrak{A}}=\cap\mathfrak{A}_\alpha=(\cap A_\alpha,\cap\Gamma_\alpha,\cap B_\alpha)$, $\alpha\in I$. It is assumed that the corresponding intersections are not empty.

The least upper bound or union of the automata is defined in the following way: this is the subautomaton $\cup\mathfrak{A}_\alpha = (\bar{A},\Sigma,\bar{B})$, $\alpha\in I$, where Σ is a subsemigroup from Γ generated by all Γ_α; $\bar{A}$ is the least invariant with respect to Σ subset from A containing all A_α; $\bar{B}$ is a union of the set of all elements from B of the form $a*\sigma$, where $a\in\bar{A}$, $\sigma\in\Sigma$, and of all sets B_α, $\alpha\in I$.

Let the automaton $\mathfrak{A}=(A,\Gamma,B)$ and triplet of sets (Z,X,Y), $Z\subset A$, $X\subset\Gamma$, $Y\subset B$ be given. The least subautomaton $\mathfrak{A}'=(A',\Gamma',B')$ from $\mathfrak{A}$ with the property $Z\subset A'$, $X\subset\Gamma'$, $Y\subset B'$ will be called the subautomaton generated by this triplet In its turn, the triplet of sets (Z,X,Y) is called a *generator system of the automaton* $\mathfrak{A}'$ It is clear, that $\mathfrak{A}'$ is equal to the intersection of all such subautomata $\mathfrak{A}_\alpha=(A_\alpha,\Gamma_\alpha,B_\alpha)$ from $\mathfrak{A}$, that $Z\subset A_\alpha$, $X\subset\Gamma_\alpha$, $Y\subset B_\alpha$.

The following proposition describes an automata induced by the system (Z,X,Y).

Proposition 1.4. *If the subautomaton* $\mathfrak{A}'=(A',\Gamma',B')$ *from* (A,Γ,B) *is induced by the generator system* (Z,X,Y), *then* Γ' *is a subsemigroup from* Γ *generated by the set* X; *the set* A' *is a union of the set* Z *with the set* $Z\circ\Gamma'$ *of all* $a\circ\gamma$, $a\in Z$, $\gamma\in\Gamma'$; *the set* B' *is a union of the set* Y *and the set of all elements* $a*\gamma$, $a\in A'$, $\gamma\in\Gamma'$.

The proof is evident.

Remark. We regard mainly semigroup automata. Consider the automaton $\mathfrak{A}=(A,X,B)$ with an arbitrary set of input signals X. Take in $\mathfrak{A}$ a triplet of sets (Z,X',Y) and generate by it a subautomaton in $\mathfrak{A}$. This can be done in the following way. First generate the semigroup subautomaton (A',Σ',B') in $\mathcal{F}(\mathfrak{A})$ by the given triplet and then take its part (A',X',B') "forgetting" about the semigroup Σ. In particular, the given system (Z,X',Y) generates the automaton $\mathfrak{A}$ if and only if $X'=X$ and the same triplet generates $\mathcal{F}(\mathfrak{A})$.

Together with stated in 1.1.1 these simple arguments constitute one of the reasons why we suggest to associate the semigroup automaton $\mathcal{F}(\mathfrak{A})$ with every $\mathfrak{A}$. Further we, as a rule, consider semigroup automata.

As particular cases of the generator systems (Z,X,Y) it is possible to consider the systems with an empty set Y; with $X=\Gamma$; and finally, with an empty Y and $X=\Gamma$. If in the latter case Z consists of one element we come to the concept of cyclic automaton. In view of proposition 1.4 the definition of the cyclic automaton can be formulated in the following way.

An automaton $\mathfrak{A}=(A,\Gamma,B)$ is said to be a *cyclic automaton* with a generator element a, if $A=\{a\}\cup a\circ\Gamma$, where $a\circ\Gamma=\{a\circ\gamma,\gamma\in\Gamma\}$ and $B=\{a*\gamma,\gamma\in\Gamma\}=A*\Gamma$. (Henceforth we shall denote $\{a\}\cup a\circ\Gamma$ by $a\circ\Gamma^1$).

Now we describe all the cyclic automata with the given semigroup Γ.

Consider the automaton $\mathrm{Atm}(\Gamma)=(\Gamma^1,\Gamma,\Gamma)$ with operations $\circ$ and $*$ defined by the rules: $x\circ\gamma=x\gamma$, $x*\gamma=x\gamma$, $x\in\Gamma^1$, $\gamma\in\Gamma$. The axioms of the semigroup automaton are evident.

This automaton is cyclic with the unit of the semigroup Γ^1 as a generator element. $\mathrm{Atm}(\Gamma)$ is called a *regular cyclic automaton* of the semigroup Γ. It is clear, that a homomorphic image of the cyclic automaton is also cyclic and therefore all quotient automata of the automaton $\mathrm{Atm}(\Gamma)$ are cyclic.

Proposition 1.5. *Every cyclic automaton with the semigroup Γ is isomorphic to a certain quotient automaton of the automaton* $\mathrm{Atm}(\Gamma)$.

Proof. Let $\mathfrak{A}=(A,\Gamma,B)$ be a cyclic automaton with the generating element a. Define mappings $\mu_1:\Gamma^1\to A$ and $\mu_3:\Gamma\to B$ by:

$$\gamma^{\mu_1}=a\circ\gamma,\ a\circ1=a,\ \gamma\in\Gamma^1;\ \gamma^{\mu_3}=a*\gamma,\ \gamma\in\Gamma.$$

Then the triplet of mappings $(\mu_1,\varepsilon_\Gamma,\mu_3)$ is a homomorphism of the automaton $\mathrm{Atm}(\Gamma)$ into $\mathfrak{A}$. Really,

$$(x\circ\gamma)^{\mu_1}=(x\gamma)^{\mu_1}=a\circ x\gamma=(a\circ x)\circ\gamma=x^{\mu_1}\circ\gamma^{\varepsilon_\Gamma};$$

$$(x*\gamma)^{\mu_3}=(x\gamma)^{\mu_3}=a*x\gamma=(a\circ x)*\gamma=x^{\mu_1}*\gamma^{\varepsilon_\Gamma};\ x\in\Gamma^1,\ \gamma\in\Gamma.$$

Image Γ^1 under the mapping μ_1 is the set of all elements of the form $\gamma^{\mu_1}=a\circ\gamma$, $\gamma\in\Gamma^1$. Since the automaton $\mathfrak{A}$ is cyclic with the generating element a, then this set coincides with the set A; similarly, image Γ

under the mapping μ_3 is B. Hence, $(\mu_1, \varepsilon_\Gamma, \mu_3)$ is an epimorphism of Atm(Γ) onto $\mathfrak{A}$, which by the Theorem 1.2 proves the Proposition.

Let the mapping ψ of the semigroup Γ to the set B be given. A natural automaton realizing such mapping is the automaton (Γ^1, Γ, B) with the operations ∘ and * introduced as follows: $x \circ \gamma = x\gamma$; $x * \gamma = (x\gamma)^\psi$; $x \in \Gamma^1$, $\gamma \in \Gamma$. Denote this automaton by $\mathrm{Atm}^*(\psi : \Gamma \to B)$. If the mapping $\psi : \Gamma \to B$ is surjective, i.e. is a mapping onto all B, then $\mathrm{Atm}^*(\psi : \Gamma \to B)$ is a cyclic automaton with generator element 1 (unit element) from Γ^1. In this case we obtain the reduced automaton $\overline{\mathrm{Atm}}(\psi : \Gamma \to B) = (\Gamma^1/\rho, \Gamma, B)$ where ρ is the kernel of the corresponding mapping $\tau : \Gamma^1 \to \mathrm{Fun}(\Gamma, B)$.

Remark. $\gamma_1 \rho \gamma_2$ means that $\gamma_1 * x = \gamma_2 * x$ for all $x \in \Gamma$. It is equivalent (by the definition of the operation *) to that for every $x \in \Gamma$ holds $(\gamma_1 x)^\psi = (\gamma_2 x)^\psi$. Thus defined equivalence ρ is called *Nerode equivalence.*

Proposition 1.6. *Every reduced cyclic automaton* $\mathfrak{A}(A, \Gamma, B)$ *is isomorphic to a certain automaton* $\overline{\mathrm{Atm}}(\psi : \Gamma \to B)$.

Proof. By Proposition 1.5 the cyclic automaton $\mathfrak{A} = (A, \Gamma, B)$ is an epimorphic image of the regular automaton Atm(Γ). Let $\mu = (\mu_1, \varepsilon_\Gamma, \mu_3)$ be a corresponding epimorphism and $\mu = \tilde{\mu}_3 \tilde{\mu}_1 \tilde{\mu}_2$ its canonical decomposition (see 1.3.):

$$\mathrm{Atm}(\Gamma) = (\Gamma^1, \Gamma, \Gamma) \xrightarrow{\tilde{\mu}_3} (\Gamma^1, \Gamma, B) \xrightarrow{\tilde{\mu}_1} (A, \Gamma, B)$$

where $\tilde{\mu}_3$ is an epimorphism in outputs, $\tilde{\mu}_2$ is an identity mapping and $\tilde{\mu}_1$ is an epimorphism in states. Show that (Γ^1, Γ, B) is the automaton $\mathrm{Atm}^*(\psi : \Gamma \to B)$ for the mapping $\psi : \Gamma \to B$ defined by the rule: $\gamma^\psi = 1 * \gamma$, $1 \in \Gamma^1$, $\gamma \in \Gamma$. Really, if $x \in \Gamma^1$, $\gamma \in \Gamma$, then $x \circ \gamma = x\gamma$. (As (Γ^1, Γ, B) is epimorphic in outputs image of the automaton Atm(Γ)). Thus $x * \gamma = (1 \circ x) * \gamma = 1 * x\gamma = (x\gamma)^\psi$. So we have $x \circ \gamma = x\gamma$, $x * \gamma = (x\gamma)^\psi$. Hence, $(\Gamma^1, \Gamma, B) = \mathrm{Atm}^*(\psi : \Gamma \to B)$. Therefore, if (A, Γ, B) is a reduced automaton, then it is isomorphic to the reduced automaton $\mathrm{Atm}^*(\psi : \Gamma \to B)/\mathrm{Ker}\tilde{\mu}_1 = \overline{\mathrm{Atm}}(\psi : \Gamma \to B)$.

An automaton $\mathfrak{A} = (A, \Gamma, B)$ is called *Γ-irreducible*, if it does not contain Γ-subautomata. It is clear that $\mathfrak{A}$ is Γ-irreducible if and only if this automaton is cyclic and every element of A is its generator. An automaton $\mathfrak{A}=(A,\Gamma,B)$ is called *completely reducible*, if it is generated by its Γ-irreducible subautomata.

Proposition 1.7. *An automaton* $\mathfrak{A}=(A,\Gamma,B)$ *is completely reducible if and only if* $A=\biguplus A_\alpha$, $\alpha\in I$, *where* $A_\alpha\cap A_\beta=\varnothing$, $\alpha\neq\beta$, $A_\alpha\circ\Gamma=A_\alpha$, and $B=A*\Gamma=\{a*\gamma,\ a\in A,\ \gamma\in\Gamma\}$. *The irreducible subautomata in* $\mathfrak{A}$ *have the form* $\mathfrak{A}_\alpha=(A_\alpha,\Gamma,B_\alpha)$ *where* $B_\alpha=A_\alpha*\Gamma=\{a*\gamma,\ a\in A_\alpha,\ \gamma\in\Gamma\}$.

The proof of this Proposition is rather simple and is omitted.

1.2. Universal automata

1.2.1. Universal automata definition. Universal property

The automaton $Atm^1(A,B)$ discussed in section 1.1 has the universal property, namely:

Proposition 2.1. *For any semigroup automaton* $\mathfrak{A}=(A,\Gamma,B)$ *there exists a unique homomorphism in input signals* $\mu:\mathfrak{A}\to Atm^1(A,B)$.

Proof. Let $f:\Gamma\to S(A,B)$ be an automaton representation of the semigroup Γ. Homomorphism of semigroups f defines the homomorphism in inputs $\mu=(\varepsilon_A,f,\varepsilon_B)$ of the automaton $\mathfrak{A}$ into $Atm(A,B)$. Indeed, if $a\in A$, $\gamma\in\Gamma$, $\gamma^f=(\sigma,\varphi)\in S(A,B)=S_A\times Fun(A,B)$, then

$$(a\circ\gamma)^{\varepsilon_A}=a\circ\gamma=a^\sigma=a\circ(\sigma,\varphi)=a^{\varepsilon_A}\circ\gamma^f$$

$$(a*\gamma)^{\varepsilon_B}=a*\gamma=a^\varphi=a*(\sigma,\varphi)=a^{\varepsilon_A}*\gamma^f$$

Show the uniqueness of the mapping μ. Let $(\varepsilon_A,\psi,\varepsilon_B)$ be another homomorphism $\mathfrak{A}$ into $Atm^1(A,B)$ and let $\gamma^\psi=(\sigma',\varphi')$. By axioms (1.2), for all $a\in A$ we have $a\circ\gamma=(a\circ\gamma)^{\varepsilon_A}=a^{\varepsilon_A}\circ\gamma^\psi=a^{\sigma'}$; $a\circ\gamma=(a,\gamma)^{\varepsilon_A}=a^{\varepsilon_A}\circ\gamma^f=a^\sigma$. Therefore, $a^{\sigma'}=a^\sigma$, and $\sigma'=\sigma$. Similarly, $a^{\varphi'}=a^\varphi$ and $\varphi'=\varphi$.

Proposition 2.1 means that the automaton $Atm^1(A,B)$ is a terminal object in the category of automata with fixed sets A and B.

The kernel of homomorphism $\mu:\mathfrak{A}\to Atm^1(A,B)$ coincides with the congruence τ_Γ^* (see 1.1.3) and corresponding exact automaton $\mathfrak{A}/\tau_\Gamma^*$ is monomorphically embedded into $Atm^1(A,B)$. Because of uniqueness of this embedding we can say that any exact automaton with the set of states A and the set of outputs B lies in $Atm^1(A,B)$.

Each input signal x transforms the state a of the automaton into the output signal b: $a*x=b$. On the other hand, it is possible to say that the state a transforms the input signal x into the output signal, i.e. each fixed state a acts as mapping from X into B, that is, as an element from Fun(X,B). The following construction corresponds to this point of view.

Let the semigroup Γ and the set B be given. Define the automaton (A,Γ,B) where A=Fun(Γ,B) is a set of all mappings from Γ into B. Operations ∘ and * are introduced in the following way: if $a\in A=Fun(\Gamma,B)$, $\gamma\in\Gamma$, then $a\circ\gamma$ is such function from Fun(Γ,B) that $(a\circ\gamma)(x)=a(\gamma x)$ for all $x\in\Gamma$; $a*\gamma=a(\gamma)$. This automaton is a semigroup one:

$$(a\circ\gamma_1\gamma_2)(x)=a(\gamma_1\gamma_2x)=a(\gamma_1(\gamma_2x))=(a\circ\gamma_1)(\gamma_2x)=((a\circ\gamma_1)\circ\gamma_2)(x)$$
$$a*\gamma_1\gamma_2=a(\gamma_1\gamma_2)=(a\circ\gamma_1)(\gamma_2)=(a\circ\gamma_1)*\gamma_2$$

Denote it by $Atm^2(\Gamma,B)$. The role of this automaton is shown by the following property:

Proposition 2.2. *For any automaton* $\mathfrak{A}=(A,\Gamma,B)$ *there exists an unique homomorphism in states* $\mu:\mathfrak{A}\to Atm^2(\Gamma,B)$.

Proof. Define the mapping $\nu:A\to Fun(\Gamma,B)$ setting $a^\nu(x)=a*x\in B$ for each $a\in A$, $x\in\Gamma$. Then $\mu=(\nu,\varepsilon_\Gamma,\varepsilon_B)$ is a homomorphism (in states) of the automaton $\mathfrak{A}$ into $Atm^2(\Gamma,B)$: $(a\circ\gamma)^\nu(x)=(a\circ\gamma)*x=a*\gamma x=a^\nu(\gamma x)=(a^\nu\circ\gamma)(x)$,

that is, $(a\circ\gamma)^\nu=a^\nu\circ\gamma^{\varepsilon_\Gamma}$; $a*\gamma=(a*\gamma)^{\varepsilon_B}=a^\nu(\gamma)=a^\nu*\gamma=a^\nu*\gamma^{\varepsilon_\Gamma}$

This homomorphism is unique. Indeed, for another homomorphism $(h,\varepsilon_\Gamma,\varepsilon_B):\mathfrak{A}\to Atm(A,B)$ holds $a^h*\gamma=a^h(\gamma)=(a*\gamma)^{\varepsilon_B}=a^\nu(\gamma)$ and as it is true for all $a\in A$, $\gamma\in\Gamma$, then $h=\nu$.

Proposition 2.2 implies that the automaton $Atm^2(\Gamma,B)$ is an *terminal object* in the category of the pure automata with fixed sets of inputs and outputs.

The kernel of the homomorphism $\mu=(\nu,\varepsilon_\Gamma,\varepsilon_B):\mathfrak{A}\to Atm^2(\Gamma,B)$ coincides with the congruence τ_A^* (see 1.1.3). This means that the automaton $\mathfrak{A}$ is reduced (left) if the corresponding $\nu:A\to Fun(\Gamma,B)$ is a monomorphism, that is, different functions in Fun(Γ,B) correspond to different states in A. It also means that the left reduced automaton $\mathfrak{A}/\tau_A^*$ is monomorphically embedded into $Atm^2(\Gamma,B)$. In other words, each left reduced automaton "lies" in $Atm^2(\Gamma,B)$.

Observe the following fact: let two representations (A,Γ) and (A',Γ), an automaton $\mathfrak{A}'=(A',\Gamma,B)$ and mapping $\nu:A \to A'$ which preserves the action of Γ in A and in A' (that is $(a\circ\gamma)^\nu=a^\nu\circ\gamma$) be given. Setting $a*\gamma=a^\nu*\gamma$ we define the automaton $\mathfrak{A}=(A,\Gamma,B)$ and $(\nu,\varepsilon_\Gamma,\varepsilon_B)$ is a homomorphism in states of $\mathfrak{A}$ into $\mathfrak{A}'$.

In particular, to the given (A,Γ), (Fun(Γ,B),Γ) and $\nu:A \to$ Fun(Γ,B) corresponds the automaton (A,Γ,B) with the homomorphism $(\nu,\varepsilon_\Gamma,\varepsilon_B):(A,\Gamma,B) \to Atm^2(\Gamma,B)$. On the other hand it follows that any automaton (A,Γ,B) may be defined in this way.

The automata $Atm^1(A,B)$ and $Atm^2(\Gamma,B)$ are associated with the exact and left reduced automata respectively. Next we are going to introduce a universal automaton $Atm^3(A,\Gamma)$ associated with right reduction, that is with the elimination of extraneous output signals.

Let for a set A and a semigroup Γ an action $a\circ\gamma$ of the elements $\gamma\in\Gamma$ on the elements $a\in A$ be defined. That is, the representation (A,Γ) be given. Take a Cartesian product $A\times\Gamma$ and generate the equivalence ρ on it by the binary relation $(a,\gamma_1\gamma_2)\tilde{\rho}(a\circ\gamma_1,\gamma_2)$. Let $A\otimes\Gamma$ denotes the quotient set $A\times\Gamma/\rho$ and the bar denotes the mapping $\bar{\ }:A\times\Gamma \to A\otimes\Gamma$. Consider the automaton $(A,\Gamma,A\otimes\Gamma)$. The operations $\circ$ and $*$ in it are defined by the representation (A,Γ) and the relation $a*\gamma=\overline{(a,\gamma)}$, respectively. Thus defined semigroup automaton is denoted by $Atm^3(A,\Gamma)$.

Proposition 2.3. *For any automaton* $\mathfrak{A}=(A,\Gamma,B)$ *there exists an unique homomorphism in output signals from* $Atm^3(A,\Gamma)$ *into* $\mathfrak{A}$ *(i.e.* $Atm^3(A,\Gamma)$ *is an initial object in the category of the automata with fixed representation* (A,Γ)*).*

Proof. Let us define the mapping $\nu:A\otimes\Gamma \to B$ according to the rule: $\overline{(a,\gamma)}^\nu=a*\gamma$. Then $(\varepsilon_A,\varepsilon_\Gamma,\nu)$ is a homomorphism in output signals of the corresponding semigroup automata. Its unicity is verified immediately in a similar way, as it was done in the previous propositions.

The given proposition implies that the operation of any automaton (A,Γ,B) can be modeled by the operation of $Atm^3(A,\Gamma)$, the automaton without extraneous output signals.

If the automaton (A,Γ,B) is right reduced then the homomorphism $(\varepsilon_A,\varepsilon_\Gamma,\nu):Atm^3(A,\Gamma) \rightarrow \mathfrak{A}$ is an epimorphism.

Remark. From the uniqueness of the homomorphisms given by the Propositions 2.1-2.3 follows the uniqueness (up to isomorphism) of the corresponding universal objects i.e automata having the given universal properties. For example, if the automaton $\mathfrak{B}=(A,\Gamma,C)$ is such that for any automaton $\mathfrak{A}=(A,\Gamma,B)$ with the same as in $\mathfrak{B}$ operation $\circ$ there exists the unique homomorphism from $\mathfrak{B}$ into $\mathfrak{A}$, then $\mathfrak{B}$ is isomorphic to $Atm^3(A,\Gamma)$.

1.2.2. Exactness of the universal automata; left and right reducibility

It is clear that $Atm^1(A,B)$ and $Atm^2(\Gamma,B)$ are exact and left and right reduced, and $Atm^3(A,\Gamma)$ is right reduced. If (A,Γ) is an exact representation, then it is obvious that the automaton $Atm^3(A,\Gamma)$ is also exact. The following statements are easily verified:

Proposition 2.4. a) *The automaton* $Atm^3(A,\Gamma)$ *is exact if and only if for given* (A,Γ) *there exists at least one exact automaton* (A,Γ,B).
b) *The automaton* $Atm^3(A,\Gamma)$ *is left reduced if and only if for given* (A,Γ) *there exists at least one left reduced automaton* (A,Γ,B).

It is clear that only sufficiency conditions have to be proved. Let there exists an exact automaton $\mathfrak{A}=(A,\Gamma,B)$ and let $\mu=(\varepsilon_A,\varepsilon_\Gamma,\nu)$ be the unique homomorphism from $Atm^3(A,\Gamma)$ to $\mathfrak{A}$ (see Proposition 2.3). Let us denote by $\circ$ and $\bar{*}$ operations of the automaton $Atm^3(A,\Gamma)$ and by $\circ$ and $*$, as we usually do, operations of $\mathfrak{A}$. Then $(a\bar{*}\gamma)^\nu=(\overline{a,\gamma})^\nu=a*\gamma$. If in the automaton $Atm^3(A,\Gamma)$ for all $a\in A$ the equalities $a\circ\gamma_1=a\circ\gamma_2$, $a\bar{*}\gamma_1=a\bar{*}\gamma_2$, $\gamma_1,\gamma_2\in\Gamma$ are satisfied, then similar equalities $a\circ\gamma_1=a\circ\gamma_2$ and $a*\gamma_1=a*\gamma_2$ hold for $\mathfrak{A}$ (as $(a\bar{*}\gamma)^\nu=a*\gamma$). Since the automaton $\mathfrak{A}$ is exact it implies that $\gamma_1=\gamma_2$ and consequently the automaton $Atm^3(A,\Gamma)$ is exact.

The second statement is verified in a similar way.

It is a natural question, whether there exists a representation (A,Γ) such that either

a) the automaton $Atm^3(A,\Gamma)$ (and thus, any automaton (A,Γ,B)) is not exact, or

b) the automaton $Atm^3(A,\Gamma)$ and also any automaton (A,Γ,B) is not left reduced.

A positive answer to these questions is given in the following examples.

Examples. a) Let the set A and semigroup Γ be given. Suppose that the semigroup Γ is not a semigroup of the right zeros. Define the representation (A,Γ) by the rule: $a\circ\gamma=a$ for all $a\in A$, $\gamma\in\Gamma$. Let this representation be arbitrarily extended to the automaton (A,Γ,B). This automaton cannot be exact. Indeed, let $f:\Gamma \to S(A,B)$ be the mapping of Γ to $S(A,B)$ defined by this automaton, and $\gamma^f=(\nu,\varphi)$. As $a\circ\gamma=a$, then $\gamma^f=(\varepsilon,\varphi)$, where ε is an identical transformation of the set A. Since $(\varepsilon,\varphi)(\varepsilon,\psi)=(\varepsilon,\psi)$, then the image Γ^f of the semigroup Γ in $S(A,B)$ is a semigroup of the right zeros. But by the condition Γ is not a semigroup of the right zeros. Hence, f cannot be a monomorphism and the automaton (A,Γ,B) is not an exact one. In particular, the automaton $Atm^3(A,\Gamma)$ is also not exact.

b) Let Γ be a semigroup with a unit and (A,Γ) be such a representation that the unit does not act in A identically. Then any automaton (A,Γ,B) extending this representation cannot be left reduced. Indeed, take the element $a\in A$ for which $a\circ 1\neq a$. For all $\gamma\in\Gamma$, $a*\gamma=a*1\cdot\gamma=(a\circ 1)*\gamma$. This means that different elements a and $a\circ 1$ from A act on Γ in the same way, that is that the automaton (A,Γ,B) is not left reduced. Consequently, $Atm^3(A,\Gamma)$ is not left reduced also.

Define the automaton $(A,S(A,B),A\times B)$ assuming $a\circ(\tau,\varphi)=a\tau$, $a*(\tau,\varphi)=(a\tau,a\varphi)$ for all $a\in A$, $(\tau,\varphi)\in S(A,B)$.

Proposition 2.5. *The automaton* $Atm^3(A,S(A,B))$ *is isomorphic to the automaton* $(A,S(A,B),A\times B)$.

Proof. In virtue of the remark to the propositions 2.1-2.3 it suffices to show that for any automaton $\mathfrak{A}=(A,S(A,B),C)$ there exists unique homomorphism in output signals from $(A,S(A,B),A\times B)$ to $\mathfrak{A}$.

Let us define the mapping $\mu: A\times B \to C$ for an arbitrary automaton

$(A,S(A,B),C)$. Each element $(a,b)\in A\times B$ can be represented in the form $(a,b)=(\alpha\sigma,\alpha\varphi)$, where $\alpha\in A$, $(\sigma,\varphi)\in S(A,B)$. Suppose that $(a,b)^{\mu}=(\alpha\sigma,\alpha\varphi)^{\mu}=\alpha*(\sigma,\varphi)$. This definition is correct, i.e. it does not depend on the manner of the representation of the element (a,b) in the form $(\alpha\sigma,\alpha\varphi)$.

Indeed, let $(a,b)=(\alpha_1\sigma_1,\alpha_1\varphi_1)=(\alpha_2\sigma_2,\alpha_2\varphi_2)$ be such two different representations. Hence,

$$\alpha_1\sigma_1=\alpha_2\sigma_2, \tag{2.1}$$

$$\alpha_1\varphi_1=\alpha_2\varphi_2 \tag{2.2}$$

It is necessary to show that $(\alpha_1\sigma_1,\alpha_1\varphi_1)^{\mu}=(\alpha_2\sigma_2,\alpha_2\varphi_2)^{\mu}$. Denote by c_α the transformation of the set A which sends each element of A to $\alpha\in A$, and by d_β the mapping of A to B which carries each element of A to the element β of B. The following equalities are immediate:

$$\begin{aligned} &a\circ(c_\alpha,\varphi)=ac_\alpha=\alpha \ , \ a\in A \ , \ \alpha\in A \\ &c_\alpha\sigma=c_{\alpha\sigma} \ , \ \text{if } \sigma\in S_A \end{aligned} \tag{2.3}$$

$$c_\alpha\varphi=d_{\alpha\varphi} \ , \ \text{if } \varphi\in Fun(A,B) \tag{2.4}$$

The first of these equalities, in particular, means that $\alpha_1=\alpha_1\circ(c_{\alpha_1},\varphi_1)$. So,

$\alpha_1*(\sigma_1,\varphi_1)=(\alpha_1\circ(c_{\alpha_1},\varphi_1))*(\sigma_1,\varphi_1)=\alpha_1*((c_{\alpha_1},\varphi_1)\cdot(\sigma_1,\varphi_1))=$

$\alpha_1*(c_{\alpha_1}\sigma_1,c_{\alpha_1}\varphi_1)=\alpha_1*(c_{\alpha_1\sigma_1},d_{\alpha_1\varphi_1})$.

Similarly,

$$\alpha_2*(\sigma_2,\varphi_2)=\alpha_2*(c_{\alpha_2\sigma_2},d_{\alpha_2\varphi_2}). \tag{2.5}$$

Since according to (2.3) $\alpha_1=\alpha_2\circ(c_{\alpha_1},\varphi_1)$, then $\alpha_1*(\sigma_1,\varphi_1)=$

$\alpha_1*(c_{\alpha_1\sigma_1},d_{\alpha_1\varphi_1})=(\alpha_2\circ(c_{\alpha_1},\varphi_1))*(c_{\alpha_1\sigma_1},d_{\alpha_1\varphi_1})=\alpha_2*((c_{\alpha_1},\varphi_1)\cdot$

$(c_{\alpha_1\sigma_1},d_{\alpha_1\varphi_1}))=\alpha_2*(c_{\alpha_1}c_{\alpha_1\sigma_1},c_{\alpha_1}d_{\alpha_1\varphi_1})=\alpha_2*(c_{\alpha_1\sigma_1},d_{\alpha_1\varphi_1})=$ (according to (2.1) and (2.2)) $=\alpha_2*(c_{\alpha_2\sigma_2},d_{\alpha_2\varphi_2})=$ (according to (2.5)) $=\alpha_2*(\sigma_2,\varphi_2)$.

Thus, $\alpha_1 * (\sigma_1, \varphi_1) = \alpha_2 * (\sigma_2, \varphi_2)$ or $(\alpha_1\sigma_1, \alpha_1\varphi_1)^{\mu} = (\alpha_2\sigma_2, \alpha_2\varphi_2)^{\mu}$, that is μ is correctly defined. The homomorphism in output signals of the automaton $(A, S(A,B), A\times B)$ into automaton $(A, S(A,B), C)$ corresponds to this $\mu: A\times B \to C$. Simple arguments show that such homomorphism is unique.

1.2.3. Universal connection of the semiautomaton and input-output type automaton

Let $\mathfrak{A}_1 = (A, X_1)$ be a semiautomaton with an arbitrary set of input signals X_1 and $\mathfrak{A}_2 = (A, X_2, B)$ the input-output type automaton (*-automaton). Consider the triplet (X, α, β) with X being a certain set with the power not less than the powers of the sets X_1 and X_2 and α, β are the mappings of X on the sets X_1 and X_2, respectively.

The automaton (A, X, B) with the operations $\circ$ and $*$

$$a \circ x = a \circ x^{\alpha}, \quad a * x = a * x^{\beta},$$

is said to be the *connection of the semiautomaton* $\mathfrak{A}_1 = (A, X_1)$ *and input-output type automaton* $\mathfrak{A}_2$ by the triplet (X, α, β).

This automaton is denoted by $\mathfrak{A}(X, \alpha, \beta)$. To each triplet (X, α, β) corresponds its connection $\mathfrak{A}(X, \alpha, \beta)$.

Let us fix the semiautomaton $\mathfrak{A}_1$ and the *-automaton $\mathfrak{A}_2$. Consider the category $\mathcal{T}$ whose objects are the given triplets (X, α, β) and morphisms are the mappings $\mu: X \to X'$ such that for triplets (X, α, β) and (X', α', β') the following diagrams are commutative

This μ we shall call a *homomorphism of triplets* $\mu: (X, \alpha, \beta) \to (X', \alpha', \beta')$.

Now consider the category $\mathcal{C}$ whose objects are all possible connections $\mathfrak{A}(X, \alpha, \beta)$ of the semiautomaton $\mathfrak{A}_1$ and the *-automaton $\mathfrak{A}_2$ and whose morphisms are homomorphisms of the automata. The mapping which assigns to each triplet (X, α, β) the automaton $\mathfrak{A}(X, \alpha, \beta)$ is a functor from the category of triplets to the category $\mathfrak{C}$ of connections of the automata $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Consider the triplet (X,π_1,π_2) where $X=X_1\times X_2$ is a Cartesian product of the sets X_1 and X_2; π_1 and π_2 are projections of the set X on X_1 and X_2, respectively. It is a terminal object of the category of triplets, and the automaton $\mathfrak{A}(X_1\times X_2,\pi_1,\pi_2)$ is a terminal object in the category $\mathfrak{C}$ of the given connections of the automata $\mathfrak{A}_1$ and $\mathfrak{A}_2$.

Proposition 2.6. *For each connection $\mathfrak{A}(X,\alpha,\beta)$ of the semiautomaton (A,X_1) and input-output type automaton (A,X_2,B) there exists an unique homomorphism in input signals of this automaton into the automaton $\mathfrak{A}(X_1\times X_2,\pi_1,\pi_2)$.*

Proof. It is necessary to define the homomorphism $(\varphi_1,\varphi_2,\varphi_3)$ of the automaton $\mathfrak{A}(X,\alpha,\beta)=(A,X,B)$ to the automaton $\mathfrak{A}(X_1\times X_2,\pi_1,\pi_2)=$ $A,X_1\times X_2,B)$. As φ_1 and φ_3 we take identity mappings of the sets A and B. The mapping $\varphi_2:X \to X_1\times X_2$ is defined as follows: $x^{\varphi_2}=(x^{\alpha},x^{\beta})$. The triplet of mappings $(\varphi_1,\varphi_2,\varphi_3)$ is a homomorphism of automata. Indeed, if $a\in A$, $x\in X$, then

$$(a\circ x)^{\varphi_1}=a\circ x=a\circ x^{\alpha}=a\circ(x^{\alpha},x^{\beta})^{\pi_1}=a^{\varphi_1}\circ(x^{\varphi_2})^{\pi_1}=a^{\varphi_1}\circ x^{\varphi_2};$$

$$(a*x)^{\varphi_3}=a*x=a*x^{\beta}=a*(x^{\alpha},x^{\beta})^{\pi_2}=a^{\varphi_1}*(x^{\varphi_2})^{\pi_2}=a^{\varphi_1}*x^{\varphi_2}$$

From these equations also follows the uniqueness of the given homomorphism.

A similar problem of connection can be considered for semigroup automata as well. In this case $\mathfrak{A}_1=(A,X_1)$ is a semigroup automaton and $\mathfrak{A}_2=(A,X_2,B)$ is an arbitrary $*$-automaton. We have to assume that the semigroup X_1 acts left-hand on the set X_2, i.e. X_2 is the left X_1-polygon. Such situation, for example, occurs if X_1 is a subsemigroup from S_A and X_2 is a subset from Fun(A,B) closed under the natural left-hand action (multiplication) of the elements from X_1. In this case the triplet (X,α,β) where X is a semigroup, α is a homomorphism from X to X_1, β is a mapping from X to X_2, defines (similarly to the case of absolutely pure automata) the automaton $\mathfrak{A}(X,\alpha,\beta)=(A,X,B)$ satisfying the conditions (1.1) of a semigroup automaton. Let's call it a *semigroup* *connection of the semiautomaton* $\mathfrak{A}_1$ *with the input-output type automaton* $\mathfrak{A}_2$. For the fixed $\mathfrak{A}_1$ and $\mathfrak{A}_2$ a class of the semigroup connection form a category with homomorphisms of the semigroup automata as morphisms. Let us construct a terminal object in this category. Define a multiplication on the set $X=X_1\times X_2$ by the rule: $(x_1,x_2)(x'_1,x'_2)=(x_1x'_1,x_1x'_2)$. Then, X is a semigroup with respect to this multiplication. As before, denote the projections of the set X on X_1,X_2 by π_1,π_2 respectively. The semigroup automaton $\mathfrak{A}(X_1\times X_2,\pi_1,\pi_2)$ which is a terminal object in the category of the semigroup connections of the automata $\mathfrak{A}_1$ and $\mathfrak{A}_2$ corresponds to the triplet $(X_1\times X_2,\pi_1,\pi_2)$.

Proceed to the case when (A,X_1) is a semigroup semiautomaton, but the action of X_1 on X_2 is not defined, or X_2 in not closed under the action of X_1. In this situation the above definition of the connection does not lead to the semigroup automaton. Consider an example: let $X_1\subset S_A$, $X_2\subset \mathrm{Fun}(A,B)$ and an action of the elements of S_A on the elements from Fun(A,B) is defined by the rule: if $x_1\in S_A$, $x_2\in$ Fun(A,B), then x_1x_2 is such element from Fun(A,B) that $ax_1x_2=(ax_1)x_2$, $a\in A$. Let X_2 be not closed under action of the elements from X_1 and x_1,x_2 are such elements from X_1,X_2 respectively, that $x_1x_2\notin X_2$. Let now X be a semigroup, $\alpha: X \to X_1$ be a homomorphism of the semigroups, $\beta: X \to X_2$ be a mapping and (A,X,B) be a union of the semiautomaton (A,X_1) and $*$-automaton (A,X_2,B). Take such elements x and x' from X, that $x^\alpha=x_1$ and $(x')^\beta=x_2$. The automaton (A,X,B) is not a semigroup one; indeed, if for all $a\in A$ the condition of the semigroup automaton $a*xx'=(a\circ x)*x'$ is satisfied, then $a*xx'=a(xx')^\beta=(a\circ x)*x'=ax^\alpha(x')^\beta$, $(xx')^\beta=x^\alpha(x')^\beta$. Since $x^\alpha(x')^\beta=x_1x_2\notin X_2$ and $(xx')^\beta\in X_2$, the latter equality contradicts the choice of x and x'.

In order the considered connection to be a semigroup automaton we must extend the $*$-automaton (A,X_2,B) to an automaton $(A,\tilde{X}_2,B)$ whose set $\tilde{X}_2$ is a closure of X_2 under the action of X_1. Note that $\tilde{X}_2$ is X_1-polygon with the set of generators X_2 and if $a\in A$, $x_1x_2\in\tilde{X}_2$, then $a*x_1x_2=(a\circ x_1)*x_2$. In this way we come to the following definition: the automaton $\mathfrak{A}$ is called a *semigroup connection of the semiautomaton* $\mathfrak{A}_1=(A,X_1)$ *and* $*$*-automaton* $\mathfrak{A}_2=(A,X_2,B)$ if it is a connection of the semiautomaton (A,X_1) and some $*$-automaton (A,Z,B) whose Z is a X_1-polygon with the set of generators X_2, and the actions $\circ$ in (A,X_1) and $*$ in (A,Z,B) satisfy the condition: $a*x_1x_2=(a\circ x_1)*x_2$ for all $a\in A$, $x_1\in X_1$, $x_2\in X_2$, $z=x_1x_2\in Z$.

According to this definition the semigroup connection is defined not only by the triplet (X,α,β) but also by the choice of the $*$-automaton (A,Z,B). If the automaton (A,Z,B) is fixed, then, as it has been already mentioned, the automaton $\mathfrak{A}(X_1\times Z,\pi_1,\pi_2)$ appears to be a terminal object in the category of all connections of (A,X_1) and (A,Z,B). Let us call the automaton $\mathfrak{A}(X_1\times Z,\pi_1,\pi_2)$ a *p-universal connection* of (A,X_1) and (A,X_2,B). If we deduce from all possible (A,Z,B) then we obtain the category of p-universal connections of the semiautomaton $\mathfrak{A}_1$ with the $*$-automaton $\mathfrak{A}_2$. It is natural to define morphisms in this category as such homomorphisms in states $(\varepsilon_1,\varphi_2,\varepsilon_3)$ in which mapping $\varphi_2:X_1\times Z \rightarrow X_1\times Z'$ is identical on the component X_1 of the Cartesian product. Construct a initial object in this category. Let Y be a free X_1-polygon over X_2. It is a set of all possible formal expressions x_1x_2, $x_1\in X_1^1$, $x_2\in X_2$ (Here X_1^1 is the semigroup X_1 with the adjoined external unit). Action of X_1 in Y is defined by the rule: $x_1'(x_1x_2)=(x_1'x_1)x_2$. Furthermore, consider the automaton (A,Y,B) with the following operation $*$: $a*x_1x_2=(a\circ x_1)*x_2$; $a\in A$, $x_1\in X_1^1$, $x_2\in X_2$. The semigroup connection $\mathfrak{A}(X_1\times Y,\pi_1,\pi_2)$ of the semiautomaton (A,X_1) and the $*$-automaton (A,Y,B) is an initial object in the given category, i.e, for any p-universal connection $\mathfrak{A}(X_1\times Z,\pi_1,\pi_2)$ there exists a homomorphism in inputs $(\varepsilon_1,\varphi_2,\varepsilon_3):\mathfrak{A}(X_1\times Y,\pi_1,\pi_2) \rightarrow \mathfrak{A}(X_1\times Z,\pi_1,\pi_2)$ whose mapping $\varphi_2:X_1\times Y \rightarrow X_1\times Z$ is identical on X_1. The proof of this statement is easy. It is based on the fact that the free X_1-polygon Y over X_2 is an initial object in the category of X_1-polygons.

1.3. Moore automata

1.3.1. Definition and some properties

The automaton defined in the item 1.1.1 is called a *Mealy automaton*. An automaton (A,X,B) is called a *Moore automaton* if there exists the mapping $\psi:A \rightarrow B$, such that $a*x=(a\circ x)^{\psi}$ The mapping ψ is called a *determining mapping* of the Moore automaton. The condition $a*x=(a\circ x)^{\psi}$

means that in the Moore automaton operation $*$ is modeled by the operation $\circ$ and the mapping ψ. Therefore, Moore automata are simpler for investigation.

Proposition 3.1. *The automaton* $\mathfrak{A}=(A,X,B)$ *is a Moore automaton if and only if the equality* $a_1\circ x_1=a_2\circ x_2$ *implies* $a_1*x_1=a_2*x_2$.

Proof. The necessity of this condition is obvious. Let $a_1\circ x_1=a_2\circ x_2$ implies $a_1*x_1=a_2*x_2$. Consider the subset $A\circ X=\{a\circ x,\ a\in A,\ x\in X\}$ in the set of states A. Define the mapping $\psi:A\to B$ in the following way: if $a\in A\circ X$, i.e. $a=a_1\circ x$ for some $a_1\in A$ and $x\in X$, then $a^{\psi}=a_1*x$; if $a\notin A\circ X$, define the mapping ψ arbitrarily. Since $a_1\circ x_1=a_2\circ x_2$ implies $a_1*x_1=a_2*x_2$, this definition is correct - it does not depend on the representation of the element a in the form $a=a_1\circ x$. By the definition of ψ, $a*x=(a\circ x)^{\psi}$. Hence, $\mathfrak{A}$ is a Moore automaton.

Let us denote by Γ^1 the semigroup, which is a result of adjoining the external unit to the semigroup Γ.

Proposition 3.2. *The semigroup automaton* $\mathfrak{A}=(A,\Gamma,B)$ *is a Moore automaton if and only if it can be extended to the automaton* (A,Γ^1,B).

Proof. Since the semigroup Γ^1 contains a unit, the automaton (A,Γ^1,B) is a Moore automaton. Indeed, denote by ψ the mapping $a^{\psi}=a*1$. Then $a*\gamma=a*\gamma 1=(a\circ\gamma)*1=(a\circ\gamma)^{\psi}$, that is (A,Γ^1,B) is a Moore automaton. So, (A,Γ,B) is also a Moore automaton as a subautomaton of (A,Γ^1,B). On the other hand, let $\mathfrak{A}=(A,\Gamma,B)$ be a Moore automaton with the determining mapping ψ. Then the automaton $\mathfrak{A}$ can be extended to the automaton $\mathfrak{A}^1=(A,\Gamma^1,B)$ assuming $a\circ 1=a$, $a*1=a^{\psi}$. Axioms (1.1) of the semigroup automaton for $\mathfrak{A}^1$ are immediately verified.

Note that from the definitions of the Moore automaton and the automaton $\mathcal{F}(\mathfrak{A})$ it follows: if $\mathfrak{A}=(A,X,B)$ is a Moore automaton, then $\mathcal{F}(\mathfrak{A})=(A,\mathcal{F}(X),B)$ is also a Moore automaton with the same determining mapping.

Remarks. 1) *On the exactness of the Moore automaton.* Recall that the automaton $\mathfrak{A}=(A,\Gamma,B)$ is exact, if the kernel of the automaton representation $f:\Gamma\to S(A,B)$ is trivial. Generally speaking, the exactness of the automaton $\mathfrak{A}$ does not mean the exactness of the associated representation $\alpha:\Gamma \to S_A$. Really, $\mathrm{Ker}f=\mathrm{Ker}\alpha\cap\mathrm{Ker}\beta$ where β is a mapping of Γ into $\mathrm{Fun}(A,B)$ corresponding to the given automaton. However, if $\mathrm{Ker}\alpha\subset\mathrm{Ker}\beta$, then $\mathrm{Ker}f=\mathrm{Ker}\alpha$ and the exactness of the automaton $\mathfrak{A}$ is equivalent to the exactness of the representation (A,Γ). By the definition of Moore automata the inclusion $\mathrm{Ker}\alpha\subset\mathrm{Ker}\beta$ holds. Therefore, the exactness of the Moore automaton (A,Γ,B) is equivalent to the exactness of the representation (A,Γ).

2) *On the uniqueness of the determining mapping.*

If $\mathfrak{A}=(A,\Gamma,B)$ is a Moore automaton, then on the elements of the form $a\circ\gamma$ the determining mapping ψ is uniquely defined by the condition $(a\circ\gamma)^{\psi}=a*\gamma$. Beyond the set $A\circ\Gamma$ the mapping ψ can be taken arbitrarily. Hence, if $A\circ\Gamma\neq A$, then the Moore automaton may have many determining mappings but they differ only on the set $A\setminus A\circ\Gamma$.

1.3.2. Moore automata and universal automata

Proposition 3.3. *If the set* B *is not one-element, then for any* A *the automaton* $\mathrm{Atm}^1(A,B)$ *is not a Moore automaton.*

Proof. For a semigroup automaton $\mathfrak{A}=(A,\Gamma,B)$ and an arbitrary mapping $\psi: A \to B$, denote by $\Delta=\Delta(\psi)$ the set of all elements $\delta\in\Gamma$ for which $a*\delta=(a\circ\delta)^{\psi}$, $a\in A$ is satisfied. Δ is a left ideal in Γ, i.e $\gamma\delta\in\Delta$ for any $\gamma\in\Gamma$ and $\delta\in\Delta$. Indeed, $a*\gamma\delta= (a\circ\gamma)*\delta=((a\circ\gamma)\circ\delta)^{\psi}=(a\circ\gamma\delta)^{\psi}$. The subautomaton (A,Δ,B) is a Moore part of the initial automaton under the given $\psi: A\to B$. Consider the Moore part of the automaton $\mathrm{Atm}^1(A,B)=(A,S(A,B),B)$. For any $a\in A$ holds $a*\gamma=a^{\varphi}=(a\circ\gamma)^{\psi}=a^{\sigma\psi}$, where $\psi: A \to B$, $\gamma=(\sigma,\varphi)\in S(A,B)$ and $\gamma\in\Delta(\psi)$. It means that $\varphi=\sigma\psi$ and the semigroup $\Delta(\psi)$ consists of all elements $\gamma\in S(A,B)$ of the form $\gamma=(\sigma,\sigma\psi)$ where $\sigma\in S_A$ From this follows that $\Delta(\psi)$ is less than $S(A,B)$ and that the automaton $\mathrm{Atm}^1(A,B)$ is not a Moore automaton for any ψ . Moreover, it is possible to pick the elements of $S(A,B)$ not belonging to any $\Delta(\psi)$.

The second universal automaton $\mathrm{Atm}^2(\Gamma,B)$ may be a Moore automaton (for example, if Γ contains a unit) or may be not.

Let us consider an example of the automaton $\mathrm{Atm}^2(\Gamma,B)$ which is not a Moore automaton. Let Γ be a semigroup containing two different elements γ_1 and γ_2, such that $\gamma_1 x=\gamma_2 x$ for all $x\in\Gamma$. (This condition means that the regular left-hand action of Γ on itself is not exact). Such semigroups exist. Show that if the set B contains more than one element, then for the given semigroup Γ the automaton $Atm^2(\Gamma,B)$ is not a Moore one. Take the function $a\in Fun(\Gamma,B)$ with the condition $a(\gamma_1)\neq a(\gamma_2)$. Then $a*\gamma_1=a(\gamma_1)\neq a(\gamma_2)=a*\gamma_2$. At the same time $a\circ\gamma_1=a\circ\gamma_2$, since $(a\circ\gamma_1)(x)=a(\gamma_1x)=a(\gamma_2x)=(a\circ\gamma_2)(x)$ is fulfilled under any $x\in\Gamma$. By the Proposition 3.1. it follows that the given automaton $Atm^2(\Gamma,B)$ is not a Moore one.

An unexpected at first sight fact follows from the existence of the non-Moore automaton $Atm^2(\Gamma,B)$: the automaton $Atm^2(\Gamma,B)$ may not be embedded into the automaton $Atm^2(\Gamma^1,B)$. Indeed, if $Atm^2(\Gamma,B)$ is not a Moore automaton, then it cannot be a subautomaton of the Moore automaton.

The following Lemma gives a construction suitable to produce of the examples.

Lemma 3.4. *Given set* Z *and semigroup* Γ, *let* $H=Z\times\Gamma^1$. *Define the representation* (H,Γ): *if* $h=(z,\sigma)\in H$, $z\in Z$, $\sigma\in\Gamma^1$ *and* $\gamma\in\Gamma$, *then* $h\circ\gamma=(z,\sigma\gamma)$. *Then*

a) *The representation* (H,Γ) *is freely generated by the set* Z, *i.e. for any representation* (A,Γ) *and the mapping* $\nu:Z\rightarrow A$ *there is an unique extension to the mapping* $\nu:H\rightarrow A$ *which commutates with the action of* Γ *in* H *and* A;

b) *If the representation* (H,Γ) *is extended to the automaton* (H,Γ,B), *then this automaton is a Moore one.*

Proof. a) Define the mapping $\nu:H\rightarrow A$ by the rule: if $h=(z,\gamma)\in H$, then $h^\nu=(z,\gamma)^\nu=z^\nu\circ\gamma\in A$. Then for any element $\gamma_1\in\Gamma$ holds $(h\circ\gamma_1)^\nu=(z,\gamma\gamma_1)^\nu=z^\nu\circ\gamma\gamma_1=(z^\nu\circ\gamma)\circ\gamma_1=h^\nu\circ\gamma_1$.

We can assume that $Z\subset H$, identifying elements $z\in Z$ with the elements $(z,1)\in H$. Since the action of Γ in H and A commutes with the mapping ν, the extended mapping ν is unique. Indeed, if $(z,\gamma)\in Z\times\Gamma^1$, then $(z,\gamma)^\nu=((z,1)\circ\gamma)^\nu=(z,1)^\nu\circ\gamma=z^\nu\circ\gamma$

b) Define the mapping $\psi:H\rightarrow B$ in the following way: if $h=(z,\gamma)$, $\gamma\in\Gamma$, then $h^\psi=(z,1)*\gamma$; if $h=(z,1)$ then h^ψ is supposed to be arbitrary. In this case for $h=(z,\gamma)\in H$, and for $\gamma_1\in\Gamma$ holds $(h\circ\gamma_1)^\psi=(z,\gamma\gamma_1)^\psi=(z,1)*\gamma\gamma_1$. On the other hand, $h*\gamma_1=(z,\gamma)*\gamma_1=(z,1\gamma)*\gamma_1=((z,1)\circ\gamma)*\gamma_1=$

$(z,1)*\gamma\gamma_1$ Hence, $(h\circ\gamma_1)^{\psi}=h*\gamma_1$ and therefore (H,Γ,B) is a Moore automaton.

1.3.3. Homomorphisms of Moore automata

Proposition 3.5. *Each automaton is a homomorphic (in states) image of the Moore automaton.*

Proof. Let us construct a new automaton $\tilde{\mathfrak{A}}=(A\times B,X,B)$ by the automaton $\mathfrak{A}=(A,X,B)$ setting: $(a,b)\circ x=(a\circ x,a*x)$; $(a,b)*x= a*x$. Define the mapping $\psi: A\times B \to B$ as $(a,b)^{\psi}=b$. Then $((a,b)\circ x)^{\psi}=(a\circ x,a*x)^{\psi}=a*x=(a,b)*x$. Hence, $\tilde{\mathfrak{A}}$ is a Moore automaton. The triplet of mappings $(\mu,\varepsilon_X,\varepsilon_B)$ with the mapping $\mu: A\times B \to A$ defined by the rule $(a,b)^{\mu}=a$, is a homomorphism of the automaton $\tilde{\mathfrak{A}}$ on $\mathfrak{A}$:

$$((a,b)\circ x)^{\mu}=(a\circ x,a*x)^{\mu}=a\circ x=(a,b)^{\mu}\circ x^{\varepsilon_x}$$

$$(a,b)*x)^{\varepsilon_B}=(a,b)*x=a*x=(a,b)^{\mu}*x^{\varepsilon_x}$$

Proposition 3.5 means that each automaton is equivalent in states to a Moore automaton and therefore, any automaton can be modeled by a Moore automaton. It is essential that in this case the number of states of the automaton increases.

Since not every automaton is a Moore automaton (Proposition 3.3), then from Proposition 3.5 follows:

Corollary. *A homomorphic image of the Moore automaton may not be a Moore automaton.*

In view of the Corollary the question arises, when a homomorphic image of the Moore automaton is again a Moore automaton.

Proposition 3.6. *Let $\mathfrak{A}$ be a Moore automaton with the determining mapping ψ and $\rho=(\rho_1,\rho_2,\rho_3)$ be a congruence of $\mathfrak{A}$, such that $\rho_1\subset\mathrm{Ker}\psi$. Then the quotient automaton $\mathfrak{A}/\rho$ is also a Moore automaton.*

Proof. Let $\mu=(\mu_1,\mu_2,\mu_3)$ be a natural homomorphism of the automaton $\mathfrak{A}$ on $\mathfrak{A}/\rho$. Since $\rho_1\subset\mathrm{Ker}\psi$, then the mapping $\psi: A \to B$ induces a mapping $\psi': A/\rho_1 \to B$ and $\psi=\mu_1\psi'$ Take the mapping $\varphi=\psi'\mu_3: A/\rho_1 \to B/\rho_3$

$$\begin{array}{ccc} A & \xrightarrow{\ \psi\ } & B \\ \mu_1\big\downarrow & \overset{\psi'}{\nearrow} & \big\downarrow\mu_3 \\ A/\rho_1 & \xrightarrow{\ \varphi\ } & B/\rho_3 \end{array} \qquad \varphi=\psi'\mu_3 \qquad (3.1)$$

and show that $\mathfrak{A}/\rho=(A/\rho_1,\Gamma/\rho_2,B/\rho_3)$ is a Moore automaton with the determining mapping φ.

Indeed,

$$[a]*[\gamma]=[a*\gamma]=(a*\gamma)^{\mu_3}=((a\circ\gamma)^{\psi})^{\mu_3}=(a\circ\gamma)^{\psi\mu_3}=(a\circ\gamma)^{\mu_1\psi'\mu_3}=$$
$$=[a\circ\gamma]^{\psi'\mu_3}=[a\circ\gamma]^{\varphi}=([a]\circ[\gamma])^{\varphi},$$

where $[a]\in A/\rho_1$, $[\gamma]\in\Gamma/\rho_2$, $a\in A$, $\gamma\in\Gamma$.

We have shown that $\mathfrak{A}/\rho$ is a Moore automaton and that the determining mapping ψ of $\mathfrak{A}/\rho$ can be "passed" through B.

Corollary. *If there exists a homomorphism* $\mu=(\varepsilon_A,\mu_2,\mu_3)$ *of the Moore automaton* $\mathfrak{A}=(A,\Gamma,B)$ *on the automaton* $\mathfrak{A}'=(A,\Gamma',B')$ *such that* ε_A *is an identity mapping, in particular, if* μ *is an epimorphism in inputs or outputs, then* $\mathfrak{A}'$ *is a Moore automaton.*

Let $\rho=(\rho_1,\rho_2,\rho_3)$ be a congruence of the Moore automaton $\mathfrak{A}$ and let $\mu=(\mu_1,\mu_2,\mu_3)$ be a natural homomorphism of $\mathfrak{A}$ on $\mathfrak{A}/\rho$. The congruence ρ is called a *Moore congruence* (corresponding to it homomorphism μ is called a *Moore homomorphism*) if for a certain determining mapping ψ holds $\rho_1\subset\mathrm{Ker}\,\psi\mu_3$.

Proposition 3.7. *If* ρ *is a Moore congruence of the Moore automaton* $\mathfrak{A}$, *then* $\mathfrak{A}/\rho$ *is also a Moore automaton.*

Proof. According to Proposition 1.1 the homomorphism μ of the automaton $\mathfrak{A}$ on $\mathfrak{A}'=(A',\Gamma',B')=\mathfrak{A}/\rho$ admits the decomposition of the form $\mu=\tilde{\mu}_3\tilde{\mu}_1\tilde{\mu}_2$:

$$(A,\Gamma,B)\xrightarrow{\tilde{\mu}_3}(A,\Gamma,B')\xrightarrow{\tilde{\mu}_1}(A',\Gamma,B')\xrightarrow{\tilde{\mu}_2}(A',\Gamma',B')$$

The automaton (A,Γ,B') is an image of the Moore automaton under the epimorphism in outputs $\tilde{\mu}_3$. Hence, by the Corollary of Proposition 3.6 it is also a Moore automaton with the determining mapping $\psi_1=\psi\mu_3$:

$$\begin{array}{ccc} A & \xrightarrow{\ \psi\ } & B \\ \varepsilon_A\downarrow & & \downarrow\mu_3 \\ A & \xrightarrow{\ \psi_1\ } & B' \end{array} \qquad \psi_1=\psi\mu_3 \qquad (3.2)$$

The automaton $\mathfrak{A}_2=(A',\Gamma,B')=(A/\rho_1,\Gamma,B/\rho_3)$ is a homomorphic image of the automaton $\mathfrak{A}_1=(A,\Gamma,B')$. Since $\rho_1\subset\mathrm{Ker}\psi\mu_3=\mathrm{Ker}\psi_1$, by Proposition 3.6 $\mathfrak{A}_2$ is a Moore automaton

$$\begin{array}{ccc} A & \xrightarrow{\ \psi_1\ } & B' \\ \mu_1\downarrow & \nearrow\psi' & \downarrow\varepsilon_{B'} \\ A'=A/\rho_1 & \xrightarrow{\ \psi_2\ } & B' \end{array} \qquad \begin{array}{l} \psi_1=\mu_1\psi' \\ \psi_2=\psi'\varepsilon_{B'}=\psi' \end{array} \qquad (3.3)$$

with the determining mapping ψ_2

Finally, the automaton $\mathfrak{A}'=\mathfrak{A}/\rho$ is an epimorphic in inputs image of the Moore automaton $\mathfrak{A}_2$ and consequently it is Moore automaton.

Proposition 3.8. *The homomorphism* μ *of the Moore automaton* $\mathfrak{A}=(A,\Gamma,B)$ *with the determining mapping* ψ *on the automaton* $\mathfrak{A}'=(A',\Gamma',B')$ *is a Moore homomorphism, if and only if there exists a mapping* $\tilde{\psi}: A' \longrightarrow B'$ *with the commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{\ \psi\ } & B \\ \mu_1\downarrow & & \downarrow\mu_3 \\ A' & \xrightarrow{\ \tilde{\psi}\ } & B' \end{array} \qquad \mu_1\tilde{\psi}=\psi\mu_3 \qquad (3.4)$$

Proof. Let μ be a Moore homomorphism, i.e. $\rho_1=\mathrm{Ker}\mu_1$ satisfies the condition $\rho_1\subset\mathrm{Ker}\psi\mu_3$. Arguing as above, we represent μ in the form of $\tilde{\mu}_3\tilde{\mu}_1\tilde{\mu}_2$. By the equalities (3.2) and (3.3) we have $\mu_1\psi_2=\mu_1\psi'=\psi_1=\psi\mu_3$. Thus $\mu_1\psi_2=\psi\mu_3$, that is, taking the mapping ψ_2 as $\tilde{\psi}$, we get the required equality $\mu_1\tilde{\psi}=\psi\mu_3$.

Conversely, if for some mapping $\tilde{\psi}: A' \to B'$ the equality $\psi\mu_3=\mu_1\tilde{\psi}$ holds, then $\rho_1=\mathrm{Ker}\mu_1 \subset \mathrm{Ker}\mu_1\tilde{\psi}=\mathrm{Ker}\psi\mu_3$, that is μ is a Moore homomorphism.

Corollary. *If the homomorphism μ of the Moore automaton $\mathfrak{A}$ on $\mathfrak{A}'$ satisfies conditions (3.4), then $\mathfrak{A}'$ is a Moore automaton with the determining mapping $\tilde{\psi}$.*

The inverse statement is not true: from the fact that $\mu:\mathfrak{A}\to\mathfrak{A}'$ is a homomorphism of the Moore automata does not follow that μ is a Moore homomorphism.

Consider the case when the homomorphism $\mu:\mathfrak{A}\to\mathfrak{A}'$ of the Moore automata is a Moore homomorphism.

1) If the automaton $\mathfrak{A}=(A,\Gamma,B)$ satisfies the condition $A\circ\Gamma=A$ and $\mu=(\mu_1,\mu_2,\mu_3):\mathfrak{A}\to\mathfrak{A}'$ is a homomorphism of the Moore automata, then μ is a Moore homomorphism.

Indeed, let $\mathfrak{A}'=(A',\Gamma',B')$, ψ and ψ' be the determining mappings of the automata $\mathfrak{A}$ and $\mathfrak{A}'$, respectively. Take an arbitrary element $a_1\in A$. Since $A\circ\Gamma=A$, then for some $a\in A$, $\gamma\in\Gamma$, $a_1=a\circ\gamma$. Then $a_1^{\psi\mu_3}=(a\circ\gamma)^{\psi\mu_3}=(a*\gamma)^{\mu_3}=a^{\mu_1}*\gamma^{\mu_2}$; on the other hand, $a_1^{\mu_1\psi'}=(a\circ\gamma)^{\mu_1\psi'}=(a^{\mu_1}\circ\gamma^{\mu_2})^{\psi'}=a^{\mu_1}*\gamma^{\mu_2}$. Hence, $\psi\mu_3=\mu_1\psi'$ and μ is a Moore homomorphism.

2) Assume that $A\circ\Gamma$ is less than A. Take $a\in A\setminus A\circ\Gamma$ and suppose that $a^{\mu_1\psi'}$ belongs to the image of the set B under the mapping μ_3 (in particular, it takes place when μ is the homomorphism of the automaton $\mathfrak{A}$ on the automaton $\mathfrak{A}'$). Let us denote by $\tilde{b}$ an arbitrary fixed element from B for which $\tilde{b}^{\mu_3}=a^{\mu_1\psi'}$. Since the determining mapping $\psi: A\to B$ beyond the set $A\circ\Gamma$ can be defined arbitrarily, let $a^{\psi}=\tilde{b}$ Thus, $a^{\mu_1\psi'}=a^{\psi\mu_3}$.

If $a^{\mu_1}\notin A'\circ\Gamma'$, then by the arbitrariness of the determining mapping ψ' beyond the set $A'\circ\Gamma'$ one can assume that $a^{\mu_1\psi'}\in B^{\mu_3}$. If $a^{\mu_1}\in A^{\mu_1}\circ\Gamma^{\mu_2}$, then for some $a_1\in A$, $\gamma_1\in\Gamma$, $a^{\mu_1}=a_1^{\mu_1}\circ\gamma_1^{\mu_2}$ and $a^{\mu_1\psi'}=a_1^{\mu_1}*\gamma_1^{\mu_2}=(a_1*\gamma_1)^{\mu_3}\in B^{\mu_3}$. It is left to consider the occasion, when for some element $a\in A\setminus A\circ\Gamma$ its image a^{μ_1} lies in $A'\circ\Gamma'\setminus A^{\mu_1}\circ\Gamma^{\mu_2}$. In this case it is possible to construct an example of the homomorphism of the Moore automata which is not a Moore homomorphism.

Example. Let $A'=\{a_1,a_2,\dots,a_n\}$, $n>1$. Consider the representation (A',Γ') where Γ' is a semigroup $S_{A'}$ of all transformations of the set A'. Take the set $B'=\{b_1,b_2,\dots,b_n\}$ and extend the representation (A',Γ') to the automaton $\mathfrak{A}'=(A',\Gamma',B')$ by setting $a_i*\gamma=b_j$, if $a_i\circ\gamma=a_j$. The axioms of the semigroup automata are evident. $\mathfrak{A}'$ is a Moore automaton with the determining mapping $\psi'\cdot$ $a_i^{\psi'}=b_i$ Take the subautomaton $(A,\Gamma,B)\subset\mathfrak{A}'$ as the automaton $\mathfrak{A}$. In this subautomaton $A=A'$, Γ is a subsemigroup of Γ' consisting of all mappings of A to the set $\{a_2,\dots,a_n\}$; $B=\{b_2,\dots,b_n\}$. $\mathfrak{A}$ is also a Moore automaton with the following determining mapping: $a_i^{\psi}=b_i$, if $i>1$; a_1^{ψ} can be defined arbitrarily. The identity mapping of the automaton $\mathfrak{A}$ on itself is a homomorphism $\mathfrak{A}\to\mathfrak{A}'$, but it is not a Moore homomorphism, since the element $a_1^{\mu_1\psi'}=b_1$ does not belong to the set B^{μ_3} and, therefore, it cannot be equal to $a^{\psi\mu_3}$.

1.3.4. Moore semigroups

The example of the automaton $\mathrm{Atm}^2(\Gamma,B)$ shows that the property of being a Moore automaton depends on the properties of the semigroup Γ. The next theorem 3.10 gives the necessary and sufficient conditions for the semigroup Γ under which the automaton (A,Γ,B) is a Moore automaton.

Let Γ be an arbitrary semigroup, $\gamma\in\Gamma$. Consider the mapping $\hat{\gamma}:\Gamma\to\Gamma$, defined by the rule: for every $x\in\Gamma$, $\hat{\gamma}(x)=\gamma x$. Let $\rho=\rho(\gamma)=\mathrm{Ker}\hat{\gamma}$. By the definition of $\hat{\gamma}$, $x\rho(\gamma)y$, $x,y\in\Gamma$, is equivalent to $\gamma x=\gamma y$. For each pair of elements $\gamma_1,\gamma_2\in\Gamma$ denote by $\rho(\gamma_1,\gamma_2)$ the equivalence generated by the equivalences $\rho(\gamma_1)$ and $\rho(\gamma_2)$. Let us call the semigroup Γ a *Moore semigroup*, if each pair of its elements γ_1 and γ_2 has the right-hand units x and y (i.e. $\gamma_1x=\gamma_1$, $\gamma_2y=\gamma_2$) which are equivalent by $\rho(\gamma_1,\gamma_2)$. For example, each semigroup with a unit is a Moore semigroup. Given the automaton $\mathrm{Atm}^2(\Gamma,B)=(\mathrm{Fun}(\Gamma,B),\Gamma,B)$, the element $g\in\mathrm{Fun}(\Gamma,B)$ is called *divisible* by $\gamma\in\Gamma$, if there is such element $\varphi\in\mathrm{Fun}(\Gamma,B)$, that $g=\varphi\circ\gamma$.

Lemma 3.9. *The element* $g\in\mathrm{Fun}(\Gamma,B)$ *is divisible by* $\gamma\in\Gamma$ *if and only if* $x\rho(\gamma)y$ *implies* $g(x)=g(y)$.

Indeed, let $g=\varphi\circ\gamma$ and $x\rho(\gamma)y$. Then $g(x)=(\varphi\circ\gamma)(x)=\varphi(\gamma x)=(\varphi\circ\gamma)(y)=$

$g(y)$. Inversely, let function g satisfies the assertion of the Lemma. Let us find φ. For each element $z\in\Gamma$ of the form $z=\gamma x$ (for some x) take $\varphi(z)=g(x)$. Here, $\varphi(z)$ does not depend on the choice of x. Beyond the set of all γx define φ arbitrarily. Then $(\varphi\circ\gamma)(x)=\varphi(\gamma x)=g(x)$, that is $\varphi\circ\gamma=g$.

Theorem 3.10. [89]

a) *If in the semigroup automaton* $\mathfrak{A}=(A,\Gamma,B)$ *the semigroup* Γ *is a Moore semigroup, then* $\mathfrak{A}$ *is a Moore automaton.*

b) *If* $Atm^2(\Gamma,B)$ *is a Moore automaton, then the semigroup* Γ *is a Moore semigroup.*

Proof. a) It suffices to show that the equality $a_1\circ\gamma_1=a_2\circ\gamma_2$ implies $a_1*\gamma_1=a_2*\gamma_2$. Let $a_1\circ\gamma_1=a_2\circ\gamma_2=a$. We associate with this a the mapping $\hat{a}:\Gamma\longrightarrow B$ defined by $\hat{a}(x)=a*x$, and let $\tau=\mathrm{Ker}\hat{a}$. Then $x\tau y$ is equivalent to that $a*x=a*y$. Show that for the given γ_1 and γ_2, $\rho(\gamma_1)$ and $\rho(\gamma_2)$ belong to τ. Let $x\rho(\gamma_1)y$, $x,y\in\Gamma$ (i.e. $\gamma_1x=\gamma_1y$). Then

$$a*x=(a_1\circ\gamma_1)*x=a_1*\gamma_1x=a_1*\gamma_1y=(a_1\circ\gamma_1)*y=a*y.$$

Hence, $x\tau y$ and $\rho(\gamma_1)\subset\tau$. Similarly, $\rho(\gamma_2)\subset\tau$. Thus, $\rho(\gamma_1,\gamma_2)\subset\tau$.

Now let x and y be right-hand units for γ_1 and γ_2 respectively, which are equivalent by $\rho(\gamma_1,\gamma_2)$ (they exist, as Γ is a Moore semigroup). Since $\rho(\gamma_1,\gamma_2)\subset\tau$, then $x\tau y$, i.e. $a*x=a*y$. Then

$$a_1*\gamma_1=a_1*\gamma_1x=(a_1\circ\gamma_1)*x=a*y=(a_2\circ\gamma_2)*y=a_2*\gamma_2y=a_2*\gamma_2.$$

So, $a_1*\gamma_1=a_2*\gamma_2$ and $\mathfrak{A}$ is a Moore automaton.

b) Let $Atm^2(\Gamma,B)$ be a Moore automaton and the semigroup Γ be a non-Moore semigroup. Then there are elements γ_1 and γ_2 of Γ, such that the corresponding property is not satisfied. This means that either for one of the elements, say for γ_1, there is no right-hand unit, or any pair x,y of the right-hand units for γ_1 and γ_2 respectively, is not equivalent by $\rho(\gamma_1,\gamma_2)$.

At first assume that γ_1 does not have a right-hand unit. Take an arbitrary function $g\in Fun(\Gamma,B)$, divisible by both γ_1 and γ_2. It is obvious that these functions exist. Let $g=\varphi\circ\gamma_1$ and $g=\psi\circ\gamma_2$ The function φ may be chosen in such a way that $\varphi(\gamma_1)\neq\psi(\gamma_2)$. Indeed, (see Lemma 3.9) its value beyond the set of the elements of the form γ_1x was defined arbitrarily. Since γ_1 does not have the right unit, that is, for all $x\in\Gamma$ holds $\gamma_1\neq\gamma_1 x$, the value of $\varphi(\gamma_1)$ can be chosen arbitrarily. In particular, we can take $\varphi(\gamma_1)\neq\psi(\gamma_2)$. Then $\varphi*\gamma_1=\varphi(\gamma_1)\neq\psi(\gamma_2)=\psi*\gamma_2$. Since $g=\varphi\circ\gamma_1=\psi\circ\gamma_2$, the inequality $\varphi*\gamma_1\neq\psi*\gamma_2$ contradicts that $Atm^2(\Gamma,B)$ is a Moore automaton.

Now let x, y be the right units for γ_1 and γ_2 respectively, and x and y are not equivalent by $\rho=\rho(\gamma_1,\gamma_2)$. Then the classes $[x]_\rho$ and $[y]_\rho$ are different. Take the function $g\in Fun(\Gamma,B)$ satisfying the following conditions:

1) if $u,v\in\Gamma$ and $u\rho v$, then $g(u)=g(v)$,

2) $g(x)\neq g(y)$.

Such function exists, since $[x]_\rho\neq[y]_\rho$ and it is divisible by γ_1 and by γ_2. Indeed, if $u\rho_1 v$ (or $u\rho_2 v$), then $u\rho v$ and $g(u)=g(v)$. By Lemma 3.9 this implies, that g is divisible by γ_1 (and by γ_2). Let $g=\varphi\circ\gamma_1$ and $g=\psi\circ\gamma_2$. Then

$$\varphi*\gamma_1=\varphi(\gamma_1)=\varphi(\gamma_1 x)=(\varphi\circ\gamma_1)(x)=g(x);$$

$$\psi*\gamma_2=\psi(\gamma_2)=\psi(\gamma_2 y)=(\psi\circ\gamma_2)(y)=g(y).$$

We get $\varphi*\gamma_1\neq\psi*\gamma_2$, contradicting to the fact that $Atm^2(\Gamma,B)$ is a Moore automaton. The theorem is proved.

Corollary. *All semigroup automata* $\mathfrak{A}=(A,\Gamma,B)$ *are Moore automata if and only if* Γ *is a Moore semigroup.*

1.3.5. Cyclic Moore automata

Let us observe the case when the cyclic automaton is a Moore automaton. It is clear that $Atm(\Gamma)=(\Gamma',\Gamma,\Gamma)$ is a Moore automaton with the determining mapping $\psi:\Gamma^1\to\Gamma$ identical on Γ and arbitrary on the external unit. Not every cyclic automaton is a Moore automaton. For example, the universal automaton $Atm^1(A,B)$ is cyclic, but it is not a Moore automaton. Each cyclic automaton is a homomorphic image of the automaton $Atm(\Gamma)$. Hence, not every quotient automaton $Atm(\Gamma)/\rho$ is a Moore automaton.

Proposition 3.11. *Let* $\rho=(\rho_1,\rho_2,\rho_3)$ *be a congruence of the automaton* $Atm(\Gamma)$. *The cyclic automaton* $Atm(\Gamma)/\rho$ *is a Moore automaton if and* *only if ρ is a Moore congruence.*

Proof. Atm(Γ) is a Moore automaton. By Proposition 3.7, if ρ is a Moore congruence, then Atm(Γ)/ρ is also a Moore automaton.

On the other hand, let Atm(Γ)/ρ be a Moore automaton with the determining mapping ψ'; $\mu=(\mu_1,\mu_2,\mu_3)$ be a natural homomorphism corresponding to the congruence ρ.

Since $(\Gamma^1)^{\mu_1\psi'}\subset\Gamma/\rho_3=\Gamma^{\mu_3}$, then the homomorphism μ is a Moore homomorphism (and ρ is a Moore congruence).

Proposition 3.11 gives necessary and sufficient conditions for an arbitrary cyclic automaton to be a Moore automaton. The following Proposition gives such conditions for a reduced cyclic automaton.

Proposition 3.12. $\overline{\text{Atm}}(\psi:\Gamma\to B)$ *is a Moore automaton, if and only if for* $\gamma_1,\gamma_2\in\Gamma$ *the equality* $(\gamma_1x)^{\psi}=(\gamma_2x)^{\psi}$ *for all* $x\in\Gamma$ *implies the equality* $\gamma_1^{\psi}=\gamma_2^{\psi}$.

Proof. Let $\overline{\text{Atm}}(\psi:\Gamma\to B)=(\Gamma^1/\rho,\Gamma,B)$ be a Moore automaton with the determining mapping $\varphi:\Gamma^1/\rho\to B$. Denote by $\bar{\tau}=(\tau,\varepsilon_\Gamma,\varepsilon_B)$ the homomorphism of the automaton $\text{Atm}^*(\psi:\Gamma\to B)=(\Gamma^1,\Gamma,B)$ on $\overline{\text{Atm}}(\psi:\Gamma\to B)$ and consider the mapping $\tau\varphi:\Gamma^1\to\Gamma^1/\rho\to B$. Then,

$$\gamma^{\tau\varphi}=(\gamma^{\tau})^{\varphi}=((1\circ\gamma)^{\tau})^{\varphi}=(1^{\tau}\circ\gamma^{\varepsilon_\Gamma})^{\varphi}=(1^{\tau}\circ\gamma)^{\varphi}=1^{\tau}*\gamma=1^{\tau}*\gamma^{\varepsilon_\Gamma}=(1*\gamma)^{\varepsilon_B}=1*\gamma=(1\circ\gamma)^{\psi}=\gamma^{\psi}.$$

So, $\gamma^{\psi}=\gamma^{\tau\varphi}$ for any $\gamma\in\Gamma$.

Now let $(\gamma_1x)^{\psi}=(\gamma_2x)^{\psi}$ (i.e. $\gamma_1*x=\gamma_2*x$) for all $x\in\Gamma$. Since $\overline{\text{Atm}}(\psi:\Gamma\to B)$ is a reduced automaton then $\gamma_1^{\tau}=\gamma_2^{\tau}$. From the equalities $\gamma^{\psi}=\gamma^{\tau\varphi}$ and $\gamma_1^{\tau}=\gamma_2^{\tau}$ follows that $\gamma_1^{\psi}=\gamma_2^{\psi}$. Really, $\gamma_1^{\psi}=\gamma_1^{\tau\varphi}=(\gamma_1^{\tau})^{\varphi}=(\gamma_2^{\tau})^{\varphi}=\gamma_2^{\tau\varphi}=\gamma_2^{\psi}$.

Conversely, let for any $\gamma_1,\gamma_2\in\Gamma$ from the equality $(\gamma_1x)^{\psi}=(\gamma_2x)^{\psi}$, $x\in\Gamma$ (and hence, from $\gamma_1\rho\gamma_2$) follows the equality $\gamma_1^{\psi}=\gamma_2^{\psi}$. It is necessary to show that $\overline{\text{Atm}}(\psi:\Gamma\to B)$ is a Moore automaton.

First let us consider the automaton $\text{Atm}^*(\psi:\Gamma\to B)$. It is a Moore automaton. As a determining mapping of this automaton one can take any extension of the mapping $\psi:\Gamma\to B$ to the mapping $\Gamma^1\to B$. Denote it also by ψ. The automaton $\overline{\text{Atm}}(\psi:\Gamma\to B)=\text{Atm}^*(\psi:\Gamma\to B)/\bar{\rho}$ is a homomorphic image of the Moore automaton $Atm^*(\psi:\Gamma \to B)$; here $\bar{\rho}=(\rho_1,\rho_2,\rho_3)=Ker\bar{\tau}$. By the Proposition 3.6 it suffices to show that $\rho_1 \subset Ker\psi$.

As it was mentioned in the Remark before Proposition 1.6 (Section 1) the condition $\gamma_1\rho_1\gamma_2$ and the equality $(\gamma_1x)^\psi=(\gamma_2x)^\psi$ for all $x\in\Gamma$ are equivalent. This implies that $\gamma_1^\psi=\gamma_2^\psi$ for $\gamma_1,\gamma_2\in\Gamma$. But if one of the elements, say γ_1 is equal to 1 and the class $[1]_\rho$ contains elements from Γ, then using the arbitrariness of the extension of the mapping $\psi:\Gamma \to B$ to $\psi:\Gamma^1 \to B$, set $1^\psi=\gamma^\psi$, where $\gamma\in[1]_\rho$ This definition of 1^ψ does not depend on the choice of $\gamma\in\Gamma$, since, if $\gamma_1,\gamma_2\in[1]_\rho$, then $\gamma_1\rho\gamma_2$ and thus $\gamma_1^\psi=\gamma_2^\psi$ Now the condition $\gamma_1\rho\gamma_2$ implies the equality $\gamma_1^\psi=\gamma_2^\psi$ for all γ_1,γ_2 from Γ^1 This means that $\rho\subset Ker\psi$, and $\overline{Atm}(\psi:\Gamma \to B)$ is a Moore automaton.

If the semigroup Γ has a unit, then $\overline{Atm}(\psi:\Gamma \to B)$ is a Moore automaton (see Proposition 3.2). Using Proposition 3.12 construct an example of non-Moore automaton $\overline{Atm}(\psi:\Gamma \to B)$.

Example. Let the semigroup $\Gamma=\{\gamma_1,\gamma_2,\gamma_3,\gamma_4\}$ has the Cayley table

1 \ 2	γ_1	γ_2	γ_3	γ_4
γ_1	γ_1	γ_2	γ_3	γ_4
γ_2	γ_4	γ_3	γ_2	γ_1
γ_3	γ_1	γ_2	γ_3	γ_4
γ_4	γ_4	γ_3	γ_2	γ_1

with the natural order of multiplication (it is a semigroup of inputs of the semigroup automaton $\mathcal{F}(\mathfrak{A})$ generated by the automaton $\mathfrak{A}$ from the example in 1.1.1). Take the set $B=\{b_1,b_2,b_3,b_4\}$ and assume $\gamma_i^\psi=b_i$; $i=1,2,3,4$. For Γ,B and ψ thus defined consider the automaton $\overline{Atm}(\psi:\Gamma\to B)$. From the Cayley table we can see that $\gamma_1x=\gamma_3x$ for all $x\in\Gamma$, then of course $(\gamma_1 x)^{\psi}=(\gamma_3 x)^{\psi}$. At the same time $b_1=\gamma_1^{\psi}\neq\gamma_3^{\psi}=b_3$. By Proposition 3.12 this means that $\overline{Atm}(\psi:\Gamma \to B)$ is not a Moore automaton.

1.4. Free pure automata

1.4.1. Definition. Implementation

Let us consider the category of pure Γ-automata, whose objects are pure automata with the fixed semigroup of inputs Γ and morphisms are the homomorphisms of the form $(\mu_1, \varepsilon_\Gamma, \mu_3)$, where ε_Γ is an identity mapping of the semigroup Γ. An automaton (A,Γ,B) is called a *free Γ-automaton* with a free system of generators (Z,Y), Z⊂A, Y⊂B if for any Γ-automaton $\mathfrak{A}'=(A',\Gamma',B')$ and every mappings $\mu_1:Z \to A'$, $\mu_3:Y\to B'$ there exists a unique extension of these mappings to the homomorphism $\mu=(\mu_1,\varepsilon_\Gamma,\mu_3):\mathfrak{A} \to \mathfrak{A}'$. From the uniqueness of the extension follows the uniqueness (up to the isomorphism) of the free Γ-automaton $\mathfrak{A}$ with the system of free generators (Z,Y).

For each semigroup Γ and each pair of sets (Z,Y) it is possible to construct a free over (Z,Y) automaton with the semigroup of input signals Γ. For this purpose take a Cartesian product $H=Z\times\Gamma^1$ and define (as in Lemma 3.4) the action of the elements of Γ in H by the rule: if $h=(z,\sigma)\in H, \gamma\in\Gamma$, then $h\circ\gamma=(z,\sigma\gamma)$. By Lemma 3.4 the representation (H,Γ) is freely generated by the set Z, that is, for any representation (A,Γ) and mapping $\nu:Z \to A$ there is a unique extension to the mapping $\nu:H\to A$ permutable with the action of Γ in H and A. Let us assume that the set Z is included into H by identification of the elements z∈Z with the elements (z,1)∈H. Then elements $(z,\gamma)=(z,1)\circ\gamma$ are identified with $z\circ\gamma$. Correspondingly, the set $H=Z\times\Gamma^1$ can be identified with the set $Z\circ\Gamma^1$. Take $Y\cup(Z\times\Gamma)$ as the set Φ of output signals and define the operation * according to the rule: if $h=(z,\sigma)\in H$, $\gamma\in\Gamma$, then $h*\gamma=(z,\sigma\gamma)\in\Phi$. Then it is possible to identify elements of the form $(z,\gamma)\in\Phi$ with the elements $z*\gamma$. Denote the set $Z\times\Gamma\subset\Phi$ by $Z*\Gamma$. We obtain the automaton (H,Γ,Φ).

Let us verify that this automaton is free and has the free system of Γ-generators (Z,Y). Take an arbitrary automaton (A,Γ,B) and mappings $\mu_1:Z \to A$, $\mu_3:Y \to B$. The representation (H,Γ) is freely generated by the set Z, therefore the mapping $\mu_1: Z \rightarrow A$ is uniquely extended to the mapping $\mu_1: H \rightarrow A$ in such a way that for all $h\in H$ and $\gamma\in\Gamma$ holds $(h\circ\gamma)^{\mu_1}=h^{\mu_1}\circ\gamma$. For the element $(z,\gamma)\in Z\times\Gamma$ set $(z,\gamma)^{\mu_3}=z^{\mu_1}*\gamma$. Together with $\mu_3: Y \rightarrow B$ this gives the mapping $\mu_3: \Phi \rightarrow B$; and if $h=(z,\sigma)\in H$, $\gamma\in\Gamma$, then $(h*\gamma)^{\mu_3}=(z,\sigma\gamma)^{\mu_3}=z^{\mu_1}*\sigma\gamma=(z^{\mu_1}\circ\sigma)*\gamma=(z,\sigma)^{\mu_1}*\gamma=h^{\mu_1}*\gamma$. Thus, the pair of mappings (μ_1,μ_3) is extended to the homomorphism of the automata $(H,\Gamma,\Phi) \rightarrow (A,\Gamma,B)$. The uniqueness of this extension follows from the freeness of the representation (H,Γ) and from the axioms of the homomorphism of automata. So the automaton (H,Γ,Φ) is freely generated by the pair of sets (Z,Y). Denote this automaton by $Atm_\Gamma(Z,Y)$.

Our next aim is to consider the category of the automata with a variable semigroup of input signals. Objects of this category are arbitrary pure semigroup automata while morphisms are the homomorphisms of these automata. In the given case the system of free generators of a free automaton consists of the three sets Z, X, and Y. The automaton $\mathfrak{A}$ is free with the system of generators (Z,X,Y), if for any automaton $\mathfrak{A}'=(A',\Gamma',B')$ of the given category and for any triplet of the mappings $\mu_1: Z \rightarrow A'$, $\mu_2: X \rightarrow \Gamma'$, $\mu_3: Y \rightarrow B'$ there exists a unique extension of these mappings to the homomorphism $\mu: \mathfrak{A} \rightarrow \mathfrak{A}'$. Such automaton is denoted by $Atm(Z,X,Y)$. To construct such automaton it is necessary to take the free semigroup $F=F(X)$ with the free system of generators X and then the automaton $Atm_F(Z,Y)$. It is easy to see that the constructed automaton is actually free.

1.4.2. Criterion of freeness

The following proposition gives the criterion of freeness of an arbitrary automaton. Let us consider the automaton $\mathfrak{A}=(A,\Gamma,B)$. The element $a\in A$ is called *the divisor of the element* $b\in A$ if for a certain element $\gamma\in\Gamma$ holds $b=a\circ\gamma$.

Theorem 4.1. *The automaton* $\mathfrak{A}=(A,\Gamma,B)$ *is free, that is,* $\mathfrak{A}=Atm(Z,X,Y)$, *if and only if the following conditions are satisfied*:
1) *for any* $\gamma\in\Gamma$ *and* $a\in A$ *the element* $a\circ\gamma$ *is not a divisor of* a;
2) *for any* $a,b\in A$, $\gamma_1,\gamma_2,\gamma\in\Gamma$ *the equality* $a\circ\gamma=b\circ\gamma$ *implies* $a=b$ *and the*

equality $a\circ\gamma_1=a\circ\gamma_2$ *implies* $\gamma_1=\gamma_2$;

3) *from the equality* $a\circ\gamma_1=b\circ\gamma_2$ *it follows that either* $a=b$ *or* a *is a divisor of* b; *or* b *is a divisor of* a;

4) *each element* $a\in A$ *may have only a finite number of divisors*;

5) *the equality* $a*\gamma_1=b*\gamma_2$ *implies* $a\circ\gamma_1=b\circ\gamma_2$.

Proof. Recall first the well-known theorem on the structure of a free semigroup (see, for example, [53], chapter IX). The semigroup F is free if and only if the following conditions are satisfied:

(α) F does not have a unit,

(β) F is a semigroup with two-sided cancellation,

(γ) the equality $f_1f_2=f_3f_4$, where $f_1, f_2, f_3, f_4\in F$ implies that either $f_1=f_3$ or f_1 is left-divided by f_3, or f_3 is left divided by f_1,

(δ) each element of the semigroup F may have only a finite number of different left divisors.

Now let $\mathfrak{A}=\mathrm{Atm}(Z,X,Y)=(H,F,\Phi)$. Show that conditions 1-5 are satisfied.

1) Let $a=(z,f)\in H=\mathbf{Z}\times F^1$ and $\gamma_1\in F$. Assume that $a\circ\gamma_1=(z,f)\circ\gamma_1=(z,f\gamma_1)$ is a divisor of a. This means that $a=(a\circ\gamma_1)\circ\gamma_2$ for a certain $\gamma_2\in F$, that is, $(z,f)=(z,f\gamma_1\gamma_2)$. Therefore, $f=f\gamma_1\gamma_2$, which contradicts the condition that the semigroup F is free.

2) Let $a=(z_1,f_1)$ and $b=(z_2,f_2)$ belong to H, $\gamma_1,\gamma_2,\gamma\in F$ and $a\circ\gamma=b\circ\gamma$. Then $(z_1,f_1\gamma)=(z_2,f_2\gamma)$. This means that $z_1=z_2$ and $f_1\gamma=f_2\gamma$. Since the free semigroup is a semigroup with the cancellation, then $f_1=f_2$. Thus, $(z_1,f_1)=(z_2,f_2)$, that is, $a_1=a_2$.

If $a\circ\gamma_1=a\circ\gamma_2$, then $(z_1,f_1\gamma_1)=(z_1,f_1\gamma_2)$. Therefore, $f_1\gamma_1=f_1\gamma_2$. Cancelling by f_1 we get $\gamma_1=\gamma_2$.

3) Let $a\circ\gamma_1=b\circ\gamma_2$, that is, $(z_1,f_1\gamma_1)=(z_2,f_2\gamma_2)$. Then $z_1=z_2$, $f_1\gamma_1=f_2\gamma_2$. Since the semigroup F is free, then from the latter equality it follows that either $f_1=f_2$ or $f_1=f_2x$, or $f_2=f_1y$ for certain $x,y\in F$. If $f_1=f_2$, then $a=b$; if $f_1=f_2x$, then $a=(z_1,f_1)=(z_1,f_2x)=(z_2,f_2x)=(z_2,f_2)\circ x=b\circ x$, that is, b is a divisor of a; if $f_2=f_1y$, then $b=a\circ y$, i.e., a is a divisor of b.

4) If $b=(z_2,f_2)$ is a divisor of $a=(z_1,f_1)$, that is, $a=b\circ x$ for a certain $x\in F$, then $z_1=z_2$ and $f_1=f_2x$; f_2 is a left divisor of f_1. Since in

the free semigroup each element may have only a finite number of different left divisors, then for the given f_1 there exist only a finite number of different f_2 such that $f_1=f_2x$ and therefore, only a finite number of the elements $b=(z_2,f_2)$ which are divisors of the element a.

5) Satisfaction of the given condition immediately follows from the definition of the operations $\circ$ and $*$ for the automaton Atm(Z,X,Y).

Conversely, let the automaton $\mathfrak{A}=(A,\Gamma,B)$ satisfies conditions 1-5. First verify that Γ is a free semigroup. It suffices to check the conditions (α)-(δ).

(α) From condition 1 it follows, in particular, that for all $a\in A$ and $\gamma_1,\gamma_2\in\Gamma$, $(a\circ\gamma_1)\circ\gamma_2\neq a\circ\gamma_1$, that is $a\circ\gamma_1\gamma_2\neq a\circ\gamma_1$. Therefore $\gamma_1\gamma_2\neq\gamma_1$ and there is no unit in Γ.

(β) Let $\gamma_1\gamma=\gamma_2\gamma$. Take an arbitrary $a\in A$. Then $(a\circ\gamma_1)\circ\gamma=(a\circ\gamma_2)\circ\gamma$. By the condition 2 it follows that $a\circ\gamma_1=a\circ\gamma_2$ and $\gamma_1=\gamma_2$. If $\gamma\gamma_1=\gamma\gamma_2$, then $(a\circ\gamma)\circ\gamma_1=(a\circ\gamma)\circ\gamma_2$ and again $\gamma_1=\gamma_2$. Thus, Γ satisfies cancellation law.

(γ) Let $\gamma_1\gamma_2=\gamma_3\gamma_4$ is satisfied in Γ. For $a\in A$ holds $(a\circ\gamma_1)\circ\gamma_2=(a\circ\gamma_3)\circ\gamma_4$. By the condition 3 either $a\circ\gamma_1=a\circ\gamma_3$ or $a\circ\gamma_1$ is a divisor of $a\circ\gamma_3$, or $a\circ\gamma_3$ is a divisor of $a\circ\gamma_1$. If $a\circ\gamma_1=a\circ\gamma_3$, then $\gamma_1=\gamma_3$; if $a\circ\gamma_1$ is a divisor of $a\circ\gamma_3$, then $a\circ\gamma_3=(a\circ\gamma_1)\circ x=a\circ\gamma_1x$ and $\gamma_3=\gamma_1x$, i.e. γ_1 is a divisor of γ_3; if $a\circ\gamma_3$ is a divisor of $a\circ\gamma_1$, then γ_3 is a divisor of γ_1.

(δ) Verify that each $\gamma\in\Gamma$ has only a finite number of left divisors in Γ. Let $\gamma=\sigma\psi$; $\sigma,\psi\in\Gamma$. Then $a\circ\gamma=(a\circ\sigma)\circ\psi$ and $a\circ\sigma$ is a divisor of $a\circ\gamma$. By the condition 4 there is a finite number of different $a\circ\sigma$. However, in virtue of the condition 2, $\sigma_1\neq\sigma_2$ implies $a\circ\sigma_1\neq a\circ\sigma_2$. Therefore there is also a finite number of different σ.

Show now that the representation (A,Γ) is freely generated by a certain set Z, that is, $A=Z\times\Gamma^1$ and the operation $\circ$ is defined by the rule: if $a=(z,\gamma)\in A$ and $x\in\Gamma$, then $a\circ x=(z,\gamma x)$. An element $a\in A$ is defined to be *prime* if it does not have proper divisors. Denote the set of all prime elements of A by Z. This set is not void in virtue of the condition 4. Let z be a fixed element of Z and $z\circ\Gamma=\{z\circ\gamma,\gamma\in\Gamma\}$ be a set of all $z\circ\gamma$, $\gamma\in\Gamma$. By the condition 2 all $z\circ\gamma$ are different. Besides, show that if z_1 and z_2 are different elements of Z, then the sets $z_1\circ\Gamma$ and $z_2\circ\Gamma$

are disjoint. Indeed, let $z_1\circ\gamma_1=z_2\circ\gamma_2$. Since $z_1\neq z_2$, by the condition 3 either $z_1=z_2\circ x$ or $z_2=z_1\circ y$ for certain $x,y\in\Gamma$. This contradicts the fact that the elements z_1 and z_2 are prime. Thus, if $z_1\circ\gamma_1\neq z_2\circ\gamma_2$, $z_1,z_2\in\Gamma$, then $z_1=z_2$ and $\gamma_1=\gamma_2$. Moreover, A is a union of the set Z and sets $z\circ\Gamma$, $z\in Z$. Hence, $A=Z\times\Gamma^1$. So we can write now $z\circ\gamma$ also in the form (z,γ) as for Cartesian products. Then $(z,\gamma)\circ x=(z\circ\gamma)\circ x=z\circ\gamma x=(z,\gamma x)$.

Denote the set of "actually observed" output signals by Φ_0 (Φ_0 being the set of such output signals b, that $b=a*\gamma$ under certain $a\in A$ and $\gamma\in\Gamma$), and the set of all the remaining output signals of B by Y. To complete the proof it is necessary to show that $\Phi_0=Z\times\Gamma$ holds and that the operation $*$ is defined by the rule: if $a=(z,\gamma)\in A=Z\times\Gamma^1$ and $x\in\Gamma$, then $a*x=(z,\gamma x)$. Let $z*\Gamma=\{z*\gamma, \gamma\in\Gamma\}$, $z\in Z$. By the condition 5 and inasmuch as all $z\circ\gamma$ differ, it follows that all $z*\gamma$ are also different. Arguing as above, we conclude that $z_1*\Gamma$ and $z_2*\Gamma$, $z_1\neq z_2$, are disjoint and Φ_0 is a union of the sets $z*\Gamma$ on all $z\in\Gamma$. It means that $\Phi_0=Z\times\Gamma$. So, along with $z*\gamma$ we can write (z,γ), and for any $a=(z,\gamma)\in A$ and $x\in\Gamma$ we have $a*x=(z,\gamma)*x=(z\circ\gamma)*x=z*\gamma x=(z,\gamma x)$.

Thus, it is shown that if the automaton $\mathfrak{A}=(A,\Gamma,B)$ satisfies conditions 1-5, then the semigroup Γ is free, the representation (A,Γ) is freely generated by certain set Z, $B=(Z\times\Gamma)\cup Y$ for a certain set Y and the operations $\circ$ and $*$ are defined according to the given rules, i.e. the automaton $\mathfrak{A}$ is an automaton of the Atm(Z,X,Y) type.

If the set Z consists of one element and the set Y is empty, then $\mathrm{Atm}_\Gamma(Z,Y)$ is a free cyclic automaton and it can be identified with the automaton $\mathrm{Atm}(\Gamma)=(\Gamma^1,\Gamma,\Gamma)$.

1.4.3. Some properties

By Lemma 3.4 each free automaton is a Moore one. It is clear that each free automaton $\mathrm{Atm}_\Gamma(Z,Y)$ is an exact automaton. However, the next Proposition 4.2 shows that not every free automaton $\mathrm{Atm}_\Gamma(Z,Y)$ is left-reduced.

Consider left regular action of the semigroup Γ^1 on the semigroup Γ given by the rule: if $\gamma\in\Gamma^1, x\in\Gamma$, then $\gamma\circ x=\gamma x$. Denote its kernel by ρ, i.e. $\gamma_1\rho\gamma_2$, $\gamma_1,\gamma_2\in\Gamma^1$ means that $\gamma_1x=\gamma_2x$ for all $x\in\Gamma$. A semigroup Γ is called *exact* if the kernel ρ is trivial. Exactness of the semigroup Γ implies that for any elements $\gamma_1, \gamma_2 \in \Gamma^1$, $\gamma_1 \neq \gamma_2$ there exists such $x \in \Gamma$ that $\gamma_1 x \neq \gamma_2 x$.

Proposition 4.2. *Let τ be the kernel of reduction of the automaton* $Atm_\Gamma(Z,Y)=(H,\Gamma,\Phi)$ *and* $h_1=(z_1,\gamma_1)$, $h_2=(z_2,\gamma_2)$ *be the elements of* H. *Then* $h_1 \tau h_2$ *if and only if* $z_1=z_2$ *and* $\gamma_1 \rho \gamma_2$

Proof. $h_1 \tau h_2$ implies that $(z_1,\gamma_1)*x=(z_2,\gamma_2)*x$ is satisfied for all $x \in \Gamma$, that is $(z_1,\gamma_1 x)=(z_2,\gamma_2 x)$. The latter is possible only if $z_1=z_2$ and $\gamma_1 x=\gamma_2 x$, i.e., if $z_1=z_2$ and $\gamma_1 \rho \gamma_2$. The inverse statement is verified in a similar way.

Corollary. *The automaton* $Atm_\Gamma(Z,Y)$ *is (left) reduced if and only if the semigroup* Γ *is exact. In particular,* Atm(Z,X,Y) *is a reduced automaton.*

We have considered free semigroup automata. It is possible to define the automaton, which is free in the class of the automata with the arbitrary set X of the input signals. In order to construct such free automaton it is necessary to take $Atm(Z,X,Y)=(H,F,\Phi)$ and reduce it to the automaton (H,X,Φ).

Proposition 4.3. *If the automaton* (H,X,Φ) *is free in the class of automata with an arbitrary set of input signals, then each subautomaton in* (H,X,Φ) *is also free.*

Proof. Let (H_1,X_1,Φ_1) be a subautomaton in (H,X,Φ) and (Z,X,Y) be a free generator system for (H,X,Φ). Consider the automata $\mathfrak{A}=(H_1,F(X_1),\Phi_1)$ and $(H,F(X),\Phi)=Atm(Z,X,Y)$. Since (H_1,X_1,Φ_1) is a restriction of the automaton $\mathfrak{A}$ then it suffices to show that the semigroup automaton $\mathfrak{A}$ is free. Conditions 1,2,4,5 of the Theorem 4.1 are satisfied in $\mathfrak{A}$ since they are satisfied in Atm(Z,X,Y). It is left to verify the fulfillment of the condition 3 in $\mathfrak{A}$. Let $a \circ \gamma_1 = b \circ \gamma_2$ be given in $\mathfrak{A}$ and let a be a divisor of b in Atm(Z,X,Y), that is $b=a \circ \gamma$ for a certain $\gamma \in F(X)$. It is necessary to show that $\gamma \in F(X_1)$. From $a \circ \gamma_1 = b \circ \gamma_2$ and $b=a \circ \gamma$ follows $a \circ \gamma_1 = a \circ \gamma\gamma_2$. By the condition 2 we have $\gamma_1=\gamma\gamma_2$. Since $\gamma_1,\gamma_2 \in F(X_1)$, then $\gamma \in F(X_1)$. Therefore, condition 3 is satisfied and $\mathfrak{A}$ turns to be a free semigroup automaton.

Appendix. Since a subsemigroup of a free semigroup is not necessarily free, the statement similar to statement 4.3 is not true for the semigroup automata.

In the appendix let us consider automata of the type (H,Γ,B) which are partially free: it is assumed that the action of the semigroup Γ in H (the representation (H,Γ)) is freely generated by a certain set Z and the operation $*$ is an arbitrary one. Show that such automata can be defined by the system of mappings of the type $\psi_z:\Gamma \to B$ by all $z\in Z$. At the beginning of Section 1 it was noted that if the representation (H,Γ) is freely generated by the set Z, then H can be written in the form $Z\times\Gamma^1$ and the action $\circ$ is defined by the rule: if $h=(z,\tau)\in Z\times\Gamma^1$, $\gamma\in\Gamma$, then $h\circ\gamma=(z,\tau\gamma)$.

By Lemma 3.4 (H,Γ,B) is a Moore automaton and the determining mapping $\psi:H \to B$ is uniquely defined on the set $Z\times\Gamma$: $(z,\gamma)^{\psi}=z*\gamma$. In its turn, given the free representation (H,Γ) and the mapping $\psi:H \to B$, the operation $*$ in the automaton (H,Γ,B) is uniquely defined. Let us fix a mapping $\psi:H \to B$. To each $z\in Z$ corresponds the mapping ψ_z defined by the rule: $\gamma^{\psi_z}=(z,\gamma)^{\psi}=z*\gamma$. Using all these ψ_z it is possible to reconstruct ψ. Therefore the automaton of the type (H,Γ,B) can be defined by the sets of the mappings $\psi_z:\Gamma \to B$ for all $z\in Z$. The notation $\mathrm{Atm}(\psi_z:\Gamma \to B, z\in Z)$ is accepted for such automaton $\mathfrak{A}=(H,\Gamma,B)$.

Each automaton (A,Γ,B) is a homomorphic in states image of the automaton of the type $\mathrm{Atm}(\psi_z:\Gamma \to B, z\in Z)=(H,\Gamma,B)$.

1.5. Generalizations

1.5.1. Automata in an arbitrary variety

In the previous Sections pure automata, i. e. automata whose sets of states and outputs do not have an additional structure, have been considered. Now, let us define automata with sets of states and outputs being the elements of a certain variety Θ of Ω-algebras. We shall study an important particular case of such automata, namely, linear automata.

Let Θ be a certain variety of Ω-algebras. A triplet (A,X,B) with the operations $\circ:A\times X \to A$ and $*:A\times X \to B$ is said to be an *automaton in* *the variety* θ if A,B are algebras of θ and for each $x\in X$ the mappings $a \to a\circ x$ and $a \to a*x$, $a\in A$ are homomorphism of algebras of θ.

Semigroup automata in the variety θ is obtained in a similar way. If $\mu=(\mu_1,\mu_2,\mu_3)$ is a homomorphism of the automata in the variety θ, then μ_1,μ_3 are supposed to be homomorphisms of the corresponding algebras of θ. As in Section 1, we can introduce special kinds of homomorphisms of automata (in inputs, in outputs, in states), establish for them a canonical decomposition, and construct the corresponding universal automata. To the pair A,B of the Ω-algebras corresponds the automaton $\mathrm{Atm}^1(A,B)=(A,\mathrm{End}(A,B),B)$ which differs from the similar pure one by $\mathrm{End}(A,B)=\mathrm{End}A\times\mathrm{Hom}(A,B)$, where EndA is a set of all endomorphisms of the algebra A and Hom(A,B) is a set of all homomorphisms of A into B. Multiplication in End(A,B) is defined as in S(A,B) (see 1.1.2). The operations $\circ$ and $*$ are defined as in the automaton $\mathrm{Atm}^1(A,B)$ for the pure case. For each automaton $\mathfrak{A}=(A,\Gamma,B)$ in the variety θ there is only one homomorphism in input signals $\mathfrak{A} \to \mathrm{Atm}^1(A,B)$, that is $\mathrm{Atm}^1(A,B)$ is a terminal object in the category of automata in θ with given A and B and with homomorphisms in input signals as morphisms.

Define now the universal automata $\mathrm{Atm}^2(\Gamma,B)$ and $\mathrm{Atm}^3(A,\Gamma)$. Let the semigroup Γ and Ω-algebra $B\in\theta$ be given. The set $\mathrm{Fun}(\Gamma,B)$ of all mappings from Γ into B is a Cartesian power B^Γ; since θ is a variety of algebras, then $B^\Gamma\in\theta$. If ω is an n-ary operation of Ω, then this operation in $\mathrm{Fun}(\Gamma,B)$ is defined as follows: if $\psi_1,\psi_2,\dots,\psi_n\in\mathrm{Fun}(\Gamma,B)$, $x\in\Gamma$, then

$$(\psi_1,\psi_2,\dots,\psi_n\omega)(x)=\psi_1(x)\dots\psi_n(x)\omega.$$

Take the algebra $\mathrm{Fun}(\Gamma,B)\in\theta$ as an algebra of states of the automaton $\mathrm{Atm}^2(\Gamma,B)=(\mathrm{Fun}(\Gamma,B),\Gamma,B)$ in which operations $\circ$ and $*$ are defined by the following way: if $\psi\in\mathrm{Fun}(\Gamma,B)$, $\gamma,x\in\Gamma$, then $(\psi\circ\gamma)(x)=\psi(\gamma x)$; $\psi*\gamma=\psi(\gamma)$. It is easy to verify that

$$(\psi_1\dots\psi_n\omega)\circ\gamma=(\psi_1\circ\gamma)\dots(\psi_n\circ\gamma)\omega,\quad (\psi_1\dots\psi_n\omega)*\gamma=(\psi_1*\gamma)\dots(\psi_n*\gamma)\omega,$$

that is, the mappings $\psi \to \psi\circ\gamma$, $\psi \to \psi*\gamma$ are homomorphisms of algebras, and that axioms of the semigroup automata are satisfied:

$$\psi\circ\gamma_1\gamma_2=(\psi\circ\gamma_1)\circ\gamma_2;\quad \psi*\gamma_1\gamma_2=(\psi\circ\gamma_1)*\gamma_2.$$

The automaton $Atm^2(\Gamma,B)$ is a terminal object in the category of automata (in the variety θ) with fixed Γ and B and with homomorphisms in states as morphisms.

If the homomorphism of the automaton (A,Γ,B) to $Atm^2(\Gamma,B)$ is an isomorphism, then (A,Γ,B) is called a *reduced automaton*.

The homomorphism of an arbitrary automaton (A,Γ,B) to $Atm^1(A,B)$ means a transition to the corresponding exact automaton while its homomorphism to $Atm^2(\Gamma,B)$ means a transition to the corresponding reduced automaton.

Both these transitions provide a compression of the information about input signals or states without loss of the information in output.

Let us introduce the universal automaton $Atm^3(A,\Gamma)$ related to the elimination of the "extra" output signals. Let Ω-algebra $A\in\theta$, the semigroup Γ and the representation $\Gamma\to EndA$, defining the action of Γ in A, be given. Denote by H a θ-free algebra with a set of generators $A\times\Gamma$. Consider the binary relation ρ on the set H: $(a,\gamma_1\gamma_2)\rho(a\circ\gamma_1,\gamma_2)$ on all $a\in A$ and $\gamma_1,\gamma_2\in\Gamma$, and $((a_1,\gamma)\dots(a_n,\gamma))\omega\rho(a_1,\dots,a_n\omega,\gamma)$ on all sets of the elements $a_1,a_2,\dots,a_n$ of A, $(n=0,1,\dots)$ and on the n-ary operations $\omega\in\Omega$; $\gamma\in\Gamma$. By this relation generate the congruence $\tilde{\rho}$ of the algebra H. Denote the quotient algebra $H/\tilde{\rho}$ by $A\otimes\Gamma$. If $h\in H$, then the corresponding element of $A\otimes\Gamma$ denote by $[h]$. Now define $Atm^3(A,\Gamma)$ as the automaton $(A,\Gamma,A\otimes\Gamma)$ with the operation $\circ$ defined by the given representation (A,Γ) and the operation $*$ defined by the rule: $a*\gamma=[(a,\gamma)]$. These conditions provide that $(A,\Gamma,A\otimes\Gamma)$ is really an automaton in the given variety of algebras. It is easy to see that the automaton $Atm^3(A,\Gamma)$ is an initial object in the category of automata (in the variety θ) with the given representation (A,Γ) and with homomorphisms in output signals as morphisms.

1.5.2. Automata in categories

In the previous item we have discussed the automata in arbitrary varieties. Generalizing further, it is possible to consider automata whose sets of states and outputs are replaced by the objects of an arbitrary category $\mathcal{K}$. For a fixed pair of the objects A, B of $\mathcal{K}$ consider the set End(A, B)= EndA×Hom(A, B) where EndA is a set of all the morphisms from A into A and Hom(A, B) is a set of all the morphisms from A to B. Multiplication in End(A, B) is defined in a usual way: if (φ_1, ψ_1) and (φ_2, ψ_2) are the elements of End(A, B), then $(\varphi_1, \psi_1)(\varphi_2, \psi_2) = (\varphi_1\varphi_2, \varphi_1\psi_2)$. End(A, B) is a semigroup with respect to this multiplication. The *automaton over* $\mathcal{K}$ is defined as a triplet (A, X, B) in which A, B are the objects of $\mathcal{K}$ and X is a set with the mapping $f: X \to \mathrm{End}(A, B)$. If $X=\Gamma$ is a semigroup and the mapping $f: \Gamma \to \mathrm{End}(A, B)$ is a homomorphism of semigroups, then (A, Γ, B) is a semigroup automaton in the category $\mathcal{K}$. The homomorphism f is called an automaton representation of the semigroup Γ. The mapping $f: X \to \mathrm{End}(A, B)$ defines two mappings $\alpha: X \to \mathrm{End}A$ and $\beta: X \to \mathrm{Hom}(A, B)$; if $x \in X$ and $x^f = (\varphi, \psi)$, then $x^\alpha = \varphi$, $x^\beta = \psi$. For the semigroup automaton (A, Γ, B), α is homomorphism and β satisfies the condition: $(\gamma_1\gamma_2)^\beta = \gamma_1^\alpha\gamma_2^\beta$; $\gamma_1\gamma_2 \in \Gamma$.

The operations $\circ$ and $*$ in this general case are not defined. However, e.g. in the case if the category $\mathcal{K}$ is a variety of Ω-algebras, these operations can be defined by the rule $a \circ x = a\varphi$, $a * x = a\psi$.

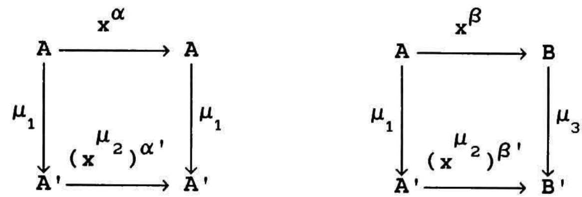
Let $\mathfrak{A} = (A, X, B)$ and $\mathfrak{A}' = (A', X', B')$ be automata over $\mathcal{K}$. A triplet $\mu = (\mu_1, \mu_2, \mu_3)$ where $\mu_1: A \to A'$, $\mu_3: B \to B'$ are morphisms in the category $\mathcal{K}$ and $\mu_2: X \to X'$ is a mapping of sets, such that for automata representations

$$f: X \to \mathrm{End}(A, B) \text{ and } f': X' \to \mathrm{End}(A', B')$$

holds

$$\begin{aligned} x^\alpha \mu_1 &= \mu_1 (x^{\mu_2})^{\alpha'}, \\ x^\beta \mu_3 &= \mu_1 (x^{\mu_2})^{\beta'}, \end{aligned} \tag{5.1}$$

is called a *homomorphism of the automata* $\mathfrak{A} \to \mathfrak{A}'$. Here $x \in X$, and the pairs (α, β) and (α', β') are defined by the representations f and f'. Conditions (5.1) imply the commutativity of the following diagrams



This generalizes the conditions (1.2) defined before.

1.5.3. Linear automata

We have already introduced automata in the arbitrary variety Θ. Taking a variety of the linear spaces over a certain field K or a variety of modules over a certain commutative ring as Θ, we get a *linear automaton*. In other words, an automaton (A,X,B) is a linear one if A and B are linear spaces over the field K (or modules over some ring) and the mappings A into A and A into B defined by the operations $\circ$ and $*$ are linear mappings. As a rule, we shall consider the automata in the category of the linear spaces.

An automaton is called *finite-dimensional* if the spaces A and B are finite-dimensional. If, moreover, the field K is finite, then the sets A and B are also finite and the automaton is called *finite*.

Examples. 1. Let A and B be the spaces of n-dimensional and m-dimensional rows over a certain field K, X be a set whose elements are pairs of matrices $x=(M_x,N_x)$, with M_x being n×n matrix and N_x being n×m matrix. The operations $\circ$ and $*$ are defined in the following way: if $a\in A, x\in X$, then $a*x=aM_x$, $a*x=aN_x$. The linear automaton (A,X,B) obtained in such a way is called a *matrix automaton*. It is clear, that each exact linear finite-dimensional automaton is isomorphic to a certain matrix automaton.

If $\mathfrak{A}=(A,X,B)$ is an exact linear automaton, then the elements of X can be represented as generalized matrices of the form $\begin{pmatrix} \alpha & \varphi \\ 0 & 0 \end{pmatrix}$, where $\alpha\in End\ A$, $\varphi\in Hom(A,B)$; in this case $a\circ\gamma=a\alpha$, $a*\gamma=a\varphi$.

2. Let L=K[x] be the ring of polynomials of one variable x over

the field K, U and V ideals in L, such that $U\subset V$. The quotient rings L/U and L/V one can consider as vector spaces over K; denote these spaces by A and B. Let, further, X be a certain set of the polynomials of L. Define the operations $\circ$ and $*$: if $\varphi\in L$, $a=[\varphi]=(\varphi+U)\in A$ and $g\in X$, then assume that $a\circ g=\varphi g+U$, $a*g=\varphi g+V$. The condition $U\subset V$ provides the independence of the operation $*$ from the choice of the representative φ for the given a. The obtained linear automaton (A,X,B) is called a *polynomial automaton*.

Let $\mathfrak{A}=(A,\Gamma,B)$ be a linear semigroup automaton. Its congruence is a triplet $\rho=(\rho_1,\rho_2,\rho_3)$ in which ρ_1,ρ_3 are congruences of the linear spaces A, B respectively and ρ_2 is a congruence of the semigroup Γ. The conditions (1.5) should be satisfied: if $a_1\rho_1 a_2$, $\gamma_1\rho_2\gamma_2$, $a_i\in A$, $\gamma_i\in\Gamma$, then $(a_1\circ\gamma_1)\rho_1(a_2\circ\gamma_2)$ and $(a_1*\gamma_1)\rho_3(a_2*\gamma_2)$.

Since the congruence of the linear space is determined by its subspace containing zero, the *congruence of the linear automaton* $\mathfrak{A}$ can be defined as the triplet $\rho=(A_0,\rho_2,B_0)$, where A_0,B_0 are the subspaces in A and B respectively, ρ_2 is a congruence of the semigroup Γ such that: $A_0\circ\Gamma\subset A_0$, $A_0*\Gamma\subset B_0$ and if $a\in A$, $\gamma_1\rho_2\gamma_2$, $\gamma_i\in\Gamma$, then $a\circ\gamma_1-a\circ\gamma_2\in A_0$, $a*\gamma_1-a*\gamma_2\in B_0$.

As for pure automata, the quotient automaton $\mathfrak{A}/\rho=(A/\rho_1,\Gamma/\rho_2,B/\rho_3)=(A/A_0,\Gamma/\rho_2,B/B_0)$ can be constructed.

1.5.4. Affine automata

In practice one has to consider automata, which are more general than linear ones. These are the automata whose input signals act on the states as compositions of the linear transformations and transformations of the special form, which are called translations. In order to consider this case define the affine automata.

Let A be a vector space, $a\in A$. A mapping $\hat{a}: A \rightarrow A$ defined by the rule: if $x\in A$, then $x\hat{a}=x+a$, is called a *translation* $\hat{a}$ of the space A corresponding to the element a. The relation $\hat{a}_1\hat{a}_2=a_1+a_2$ is obvious; thus the translations constitute a group isomorphic to the additive group of the space A. Denote this group by $\hat{A}$. It is easy to see that the translation is not a linear transformation.

Consider vector spaces A and B. A mapping of the form $\sigma\hat{b}$, where σ is a linear mapping from A into B and $\hat{b}$ is a translation of the space B corresponding to a certain element $b\in B$ is called *an affine mapping* from A into B. The set of all affine mappings from A into B is denoted by Aff(A,B). Elements of Aff(A,A) are called affine transformations of the space A. If $\sigma\hat{b}$ is an affine mapping from A into B and $\varphi\hat{c}$ from B into C, then their product is also an affine mapping from A into C, and holds

$$(\sigma\hat{b})(\varphi\hat{c})=\sigma\varphi(\widehat{b\varphi+c}). \tag{5.2}$$

Associativity of this multiplication is easily verified.

Linear spaces with affine mappings as morphisms form a category. Automata in this category are called *affine automata*. In other words, an affine automaton is a system (A,X,B,f) where A,B are vector spaces, X is a set, f is a representation: $X \to \mathrm{Aff}(A,A)\times\mathrm{Aff}(A,B)$. As before, defining of f is equivalent to that of two representations $\alpha: X\to\mathrm{Aff}(A,A)$ and $\beta: X \to \mathrm{Aff}(A,B)$. These representations define the corresponding operations $\circ$ and $*$. From now on we shall omit the symbol f in the automaton notation. The semigroup affine automaton $\mathfrak{A}=(A,\Gamma,B)$ is defined in a similar way, but the representations $\alpha:\Gamma \to \mathrm{Aff}(A,A)$ and $\beta:\Gamma \to \mathrm{Aff}(A,B)$ have to satisfy the following additional conditions

1) α is a homomorphism of the semigroups;
2) if $\gamma_1,\gamma_2\in\Gamma$, then $(\gamma_1\gamma_2)^{\beta}=\gamma_1^{\alpha}\ \gamma_2^{\beta}$; (5.3)

In contrast to linear automata, affine ones cannot be considered as the automata in a certain variety, since affine mappings are not homomorphisms of the linear spaces.

A linear automaton can be associated with each affine automaton. Consider this correspondence. Let the semigroup affine automaton $\mathfrak{A}=(A,\Gamma,B)$ be defined by the representations $\alpha:\Gamma \to \mathrm{Aff}(A,A)$, $\beta:\Gamma \to \mathrm{Aff}(A,B)$. Aff(A,A) is a semidirect product (see [60]) of the semigroups EndA and $\hat{A}$; Aff(A,B) is Cartesian product of Hom(A,B) and $\hat{B}$. In accordance with this the following four mappings $\alpha_1,\alpha_2,\beta_1,\beta_2$ correspond to the mappings α and β: if $\gamma\in\Gamma$, $\gamma^{\alpha}=\sigma\hat{a}$, $\gamma^{\beta}=\varphi\hat{b}$, $\sigma\in\mathrm{End}A$, $a\in A$, $\varphi\in\mathrm{Hom}(A,B)$, $b\in B$, then $\alpha_1:\Gamma \to \mathrm{End}A$ and $\alpha_2:\Gamma \to A$ are defined by

$$\gamma^{\alpha_1}=\sigma, \quad \gamma^{\alpha_2}=a, \tag{5.4a}$$

and $\beta_1:\Gamma \to \mathrm{Hom}(A,B)$ and $\beta_2:\Gamma \to B$ are defined by

$$\gamma^{\beta_1}=\varphi, \quad \gamma^{\beta_2}=b. \tag{5.4b}$$

By the definition of these mappings we have:

$$(\gamma_1\gamma_2)^{\alpha_1}=\gamma_1^{\alpha_1}\gamma_2^{\alpha_1} \quad ; \quad (\gamma_1\gamma_2)^{\beta_1}=\gamma_1^{\alpha_1}\gamma_2^{\beta_1}; \tag{5.5}$$

$$(\gamma_1\gamma_2)^{\alpha_2}=\gamma_1^{\alpha_2}\gamma_2^{\alpha_1}+\gamma_2^{\alpha_2}; \quad (\gamma_1\gamma_2)^{\beta_2}=\gamma_1^{\alpha_2}\gamma_2^{\beta_1}+\gamma_2^{\beta_2}. \tag{5.6}$$

Indeed,

$$(\gamma_1\gamma_2)^{\alpha}=\gamma_1^{\alpha}\gamma_2^{\alpha}=(\gamma_1\gamma_2)^{\alpha_1}(\gamma_1\gamma_2)^{\alpha_2}=(\gamma_1^{\alpha_1}{}^\wedge\gamma_1^{\alpha_2})(\gamma_2^{\alpha_1}{}^\wedge\gamma_2^{\alpha_2})=\gamma_1^{\alpha_1}\gamma_2^{\alpha_1}(\gamma_1^{\alpha_2}\gamma_2^{\alpha_1}+\gamma_2^{\alpha_2}).$$

(The latter equality follows from the equality (5.2)). Similarly for β too.

Equalities (5.5) mean that the mappings $\alpha_1:\Gamma \to \mathrm{End}A$ and $\beta_1:\Gamma \to \mathrm{Hom}(A,B)$ define the linear automaton $\mathfrak{A}^{\ell}=(A,\Gamma,B)$ with the same basic sets as in the automaton $\mathfrak{A}$ and with new operations $\circ'$ and $*'$. Using these notations we can write

$$a\circ\gamma=a\circ'\gamma+\gamma^{\alpha_2}; \quad a*\gamma=a*'\gamma+\gamma^{\beta_2};$$

Conversely, a linear automaton and two mappings of the form (5.4) with conditions (5.6) define the affine automaton.

If $\mu=(\mu_1,\mu_2,\mu_3)$ is a homomorphism of affine automata: $(A_1,\Gamma_1,B_1) \to (A_2,\Gamma_2,B_2)$, then the same triplet of mappings (μ_1,μ_2,μ_2) is a homomorphism of the corresponding linear automata. Indeed, let $a\in A$, $\gamma\in\Gamma$. Then

$$(a\circ\gamma)^{\mu_1}=(a\circ'\gamma+\gamma^{\alpha_2})^{\mu_1}=(a\circ'\gamma)^{\mu_1}+(\gamma^{\alpha_2})^{\mu_1}.$$

On the other hand, $(a\circ\gamma)^{\mu_1}=a^{\mu_1}\circ\gamma^{\mu_2}=a^{\mu_1}\circ'\gamma^{\mu_2}+(\gamma^{\mu_2})^{\alpha_2}$. Thus,

$$(a\circ'\gamma)^{\mu_1}+(\gamma^{\alpha_2})^{\mu_1}=a^{\mu_1}\circ'\gamma^{\mu_2}+(\gamma^{\mu_2})^{\alpha_2}.$$

Assuming in this equality that a=0, we get $(\gamma^{\alpha_2})^{\mu_1}=(\gamma^{\mu_2})^{\alpha_2}$. Therefore, $(a\circ'\gamma)^{\mu_1}=a^{\mu_1}\circ'\gamma^{\mu_2}$. Similarly, $(a*'\gamma)^{\mu_1}=a^{\mu_1}*'\gamma^{\mu_2}$.

1.5.5. Stochastic and fuzzy automata

We shall define such automata using the category approach outlined in the second item of the current Section.

Let us start with the stochastic automata and consider the category of sets with random mappings. Objects of this category are the sets A, B, C, ... and for the simplicity confine ourselves to the finite sets. The morphisms $\mu: A \to B$ are random mappings. They can be interpreted as stochastic matrices. Rows of the matrix μ are enumerated by the elements of the set A while columns by the elements of the set B. Each element $\mu(a,b)$ of the matrix μ is a real number from the segment I=[0,1]. This number is interpreted as a probability of taking $a\in A$ to $b\in B$ by the mapping μ. Clearly, for each $a\in A$ the natural condition

$$\sum_b \mu(a,b)=1$$

has to be satisfied. It means that a is transformed to a certain b.

A simple verification shows that if $\mu: A \to B$ and $\nu: B \to C$ are two stochastic matrices, then their ordinary matrix product $\mu\nu$ is again a stochastic matrix corresponding to the random mapping $\mu\nu$. Thus we arrive at the category. Units $\varepsilon_A: A \to A$ in this category are unit matrices. Correspondingly, we have the sets of morphisms Hom(A,B) and the semigroups of random transformations EndA.

If, further, Γ is a semigroup, then the automaton $\mathfrak{A}=(A,\Gamma,B)$ is defined by the representation

$$f: \Gamma \to \mathrm{End}A\times\mathrm{Hom}(A,B)=\mathrm{End}(A,B).$$

Here f is a pair (α,β), $\alpha: \Gamma \to \mathrm{End}A$ and $\beta: \Gamma \to \mathrm{Hom}(A,B)$. In this case the representation α realizes each element γ from Γ as random transformation of the set A and the assignment β implements each γ as the random representation from A into B.

Note now, that stochastic automaton defined here is only a special type of the stochastic automata. It would be possible to proceed from the situation when the representation f is also random in some sense; there are some other possibilities too (see, for example, [20]). One more approach to stochastic automata is presented below.

Now let us proceed to fuzzy automata and consider the category of fuzzy sets.

Fuzzy sets are usually considered as the subsets of ordinary sets. If M is an ordinary set and A its subset, then we have a characteristic function $\rho_A : M \to \{0,1\}$, which determines A as an ordinary subset in M. On the other hand, it is possible to consider the function of the form $\rho_A : M \to [0,1]$ and in this case such ρ_A is identified with the *fuzzy subset* A in M. For each $x \in M$ the value $\rho_A(x)$ is a measure of membership of the element x in the fuzzy subset A.

For example, if M is a set of components of a certain device, then we can speak of the fuzzy subset A of serviceable components. Each $x \in M$ belongs to A to a certain degree, which is somehow estimated.

Along with fuzzy subsets of ordinary sets it is quite natural to consider fuzzy quotient sets. If M is a set and ρ is a usual equivalence on M, then in the quotient set M/ρ elements of M are considered up to the equivalence ρ, i.e the equivalence goes into equality. It is also possible to consider a fuzzy equivalence. The *fuzzy equivalence* ρ on the set M is the mapping of the type $\rho : M \times M \to I=[0,1]$ satisfying the following conditions:

$$1)\ \rho(x,y) \le \rho(y,x) \text{ and } 2)\ \rho(x,y) \wedge \rho(y,z) \le \rho(x,z).$$

Here we write $\wedge$ instead of min and $\rho(x,y)$ is a measure of the equivalence of the elements x and y. We say that fuzzy quotient set M/ρ is a pair (M,ρ), M is a common set, ρ is a fuzzy equivalence on it. The elements of the fuzzy quotient set M/ρ are identified by the measure ρ.

Let, for example, M be a set of the decimal fractions of the form $0,\alpha_1\alpha_2...\alpha_n$, where n is fixed. Given $x=0,\alpha_1\alpha_2...\alpha_n$ and $y=0,\beta_1\beta_2...\beta_n$, assume $\rho(x,y)=k/n$, if x and y coincide on the first k digits and differ further. It is clear that 1) and 2) are satisfied and we can pass to the fuzzy quotient set M/ρ. In this example we have $\rho(x,x)=1$ and $\rho(x,y)=0$, if x and y differ already in the first digit.

Fuzzy equalities in the fuzzy quotient sets allow us to consider fuzzy mappings of the latter. Therefore it is expedient to proceed from the idea of the fuzzy quotient set. Fuzzy subsets in this case are naturally realized as subobjects of the corresponding objects. (The concept of the subobject of an object of category see, for example, in [42]). Henceforth, speaking about fuzzy sets, we shall mean fuzzy quotient sets of the ordinary sets.

Now we consider the category with fuzzy sets as its objects and fuzzy mappings as its morphisms.

Let $\mathbb{A}=A/\rho_1$ and $\mathbb{B}=B/\rho_2$ be fuzzy sets. The morphism $f:\mathbb{A} \to B$ is a mapping $f: A\times B \to I$ (a fuzzy subset in the Cartesian product) satisfying the following conditions:

1) $\rho_1(x,x')\wedge f(x,y)\le f(x',y)$
2) $f(x,y)\wedge\rho_2(y,y')\le f(x,y')$
3) $f(x,y)\wedge f(x,y')\le\rho_2(y,y')$
4) $\rho_1(x,x)=\cup\{f(x,y);y\in B\}$; $x,x'\in A$; $y,y'\in B$.

The first two conditions mean compatibility of the mapping with the equality of the elements and the third imitates the property of single-valuedness of the mapping. In the fourth condition the sign $\cup$ denotes the least upper bound of the set of elements in I and this condition substitutes the requirement that each x should have a certain f-image y. The notation $f(x,y)=\rho_2(f(x),y)$ is also used and by this the degree of equality between the potential $f(x)$ and y is shown.

Assume that the two morphisms $f:\mathbb{A} \to \mathbb{B}$ and $g:\mathbb{A} \to \mathbb{B}$ coincide if their fuzzy plots $f: A\times B \to I$ and $g: A\times B \to I$ also coincide.

It is easy to understand that if $\mathbb{A}$ and $\mathbb{B}$ are ordinary sets with trivial ρ_1 and ρ_2, then their fuzzy mappings actually turn to be ordinary mappings.

Let now $f:\mathbb{A} \to \mathbb{B}$ and $g:\mathbb{B} \to \mathbb{C}$ be morphisms. The morphism $gf:\mathbb{A} \to \mathbb{C}$ is defined as the function $gf: A\times C \to I$ specified by the following equality:

$$gf(x,z)=\bigcup_{y\in B}(f(x,y)\wedge g(y,z)) \quad x\in A,\ z\in C.$$

This equality, as it is easy to see, substitutes the condition: there exists such y, that $f(x)=y$ and $g(y)=z$.

Associativity of the multiplication is easily verified.

From the definition of the fuzzy mapping immediately follows that the assignment $(x,y) \to \rho(x,y)$ using the fuzzy equality ρ in A is at the same time a fuzzy mapping $\mathbb{A} \to \mathbb{A}$ which we denote by $\varepsilon_{\mathbb{A}}$. It can be verified that $\varepsilon_{\mathbb{A}}:\mathbb{A} \to \mathbb{A}$ plays the role of a unit: if $f:\mathbb{A} \to \mathbb{B}$ is a morphism, then we have: $f\varepsilon_{\mathbb{A}}=f$ and $\varepsilon_{\mathbb{B}}f=f$.

In this way we come to the category of fuzzy sets. It is proved that fuzzy subsets of the ordinary sets are implemented in the given category as subobjects of the suitable objects.

The category of fuzzy sets constructed here enables us to consider fuzzy automata. As before, for any fuzzy sets $\mathbb{A}$ and $\mathbb{B}$ and the semigroup Γ the corresponding automaton is defined by the representation:

$$f=(\alpha,\beta):\Gamma \to \mathrm{End}\mathbb{A}\times\mathrm{Hom}(\mathbb{A},\mathbb{B})$$

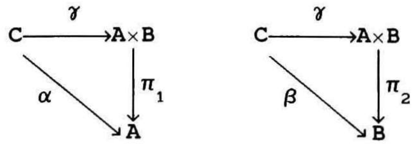
The representation $\alpha:\Gamma \to \mathrm{End}\mathbb{A}$ associates the elements of Γ with fuzzy transformations of the fuzzy set $\mathbb{A}$ and β implementing each $\gamma\in\Gamma$ as the fuzzy mapping from $\mathbb{A}$ into $\mathbb{B}$.

General arguments of the item 1.5.2 allow us to consider the categories of fuzzy and stochastic automata and one can speak of certain constructions in these categories. In particular, it is natural to bring forth a problem of the universal fuzzy automata and stochastic automata, to consider problems of decomposition, etc.

1.5.6. Another view on stochastic and fuzzy automata

Now we abandon consideration of the semigroup automata and consider the automata in which all three domains belong to the defined category. In this case we deal again with the operations $\circ$ and $*$, which were not present in the above consideration.

Define the notion of the product of two objects of the category. Let A and B be the objects of the category $\mathcal{K}$. Their product $A\times B$ is an object of the category $\mathcal{K}$ considered together with the projections $\pi_1: A\times B \to A$ and $\pi_2: A\times B \to B$. The following conditions have to be satisfied: if $\alpha: C \to A$ and $\beta: C \to B$ are defined, then there exists an unique morphism $\gamma: C \to A\times B$ such that the diagrams



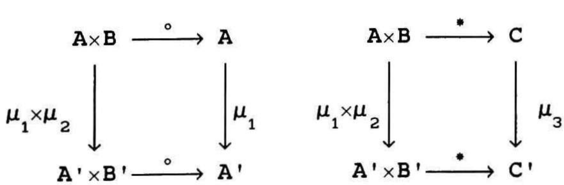
are commutative.

It is possible to define arbitrary finite and non-finite products of the objects of the category. Cartesian product in any variety of algebras is a product in this sense. Not every category allows a construction of the objects product and we can distinguish, for example, categories with finite products. The category of sets and the category of fuzzy sets are such categories, but the category of sets with random mappings does not possess the necessary property.

Now, if $\mathcal{K}$ is a category with finite products, then an *automaton in* $\mathcal{K}$ can be understood as a triplet of objects $\mathfrak{A}=(A,B,C)$ with two morphisms $\circ: A\times B \rightarrow A$ and $*: A\times B \rightarrow C$. This generalizes the initial definition of the pure automaton. Let us generalize also the notion of homomorphism of an automaton.

First, let us make a remark. Let $\alpha: A \rightarrow A'$ and $\beta: B \rightarrow B'$ be two arrows. There are also projections $\pi_1: A\times B \rightarrow A$ and $\pi_2: A\times B \rightarrow B$. Further take compositions $\alpha'=\pi_1\alpha: A\times B \rightarrow A'$ and $\beta'=\pi_2\beta: A\times B \rightarrow B'$. These α' and β' uniquely define $\gamma: A\times B \rightarrow A'\times B'$. This γ is denoted by $\alpha\times\beta$.

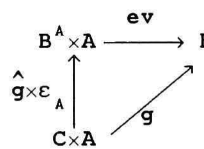
Given two automata $\mathfrak{A}=(A,B,C)$ and $\mathfrak{A}'=(A',B',C')$, a triplet of morphisms $\mu=(\mu_1,\mu_2,\mu_3)$, where $\mu_1: A \rightarrow A'$, $\mu_2=B \rightarrow B'$ and $\mu_3: C \rightarrow C'$, is a homomorphism $\mu=(\mu_1,\mu_2,\mu_3): \mathfrak{A} \rightarrow \mathfrak{A}'$ if the following two diagrams are commutative



It is easy to understand that this definition of the homomorphism actually generalizes the definition cited before.

In view of concept of fuzzy automata let us treat the category of fuzzy sets. Together with finite products there are also other constructions which make this category similar to the common category of sets. This likeness has an exact sense, meaning that the category of fuzzy sets is a topos (the definition of the topos see in [42]). We do not cite here this definition but note that in each topos there exist finite products and the operation of exponentiation.

Let A and B be the objects of the category. The exponent B^A is a new object considered together with the arrow $ev: B^A \times A \rightarrow B$. It is assumed that for any morphism $g: C\times A \rightarrow B$, where C is an object of the category $\mathcal{K}$ there is an unique morphism $\hat{g}: C \rightarrow B^A$ with the commutative diagram



In the category of sets the object B^A is, as usual, Fun(A,B) and if $(f,a)\in B^A\times A$, then $ev(f,a)=f(a)$.

If in the category $\mathcal{K}$ the exponent B^A exists for any pair of objects A and B, then it is said that $\mathcal{K}$ allows exponentiation. Each topos, in particular, the topos of fuzzy sets, allows exponentiation.

Consider further $*$-automata, i.e. the automata with the only operation $*$. Let $\mathcal{K}$ be a topos, B and C be its objects. Define the $*$-automaton $Atm^2(B,C)=(C^B,B,C)$, where morphism $*: C^B\times B \rightarrow C$ coincides with the corresponding mapping ev. It can easily be seen that if $\mathfrak{A}=(A,B,C)$ is also an $*$-automaton, then by the definition we have the unique morphism $A \rightarrow C^B$ defining the homomorphism $\mathfrak{A} \rightarrow Atm^2(B,C)$ identical on B and C. This means that $Atm^2(B,C)$ is a universal object in the corresponding category.

Given A and B, we have $Atm^3(A,B)=(A,B,A\times B)$ defined by the unit morphism $\varepsilon_{A\times B}$. If $\mathfrak{A}$=(A,B,C) is another *-automaton defined by $*: A\times B \to C$, then the latter morphism naturally produces the unique homomorphism $Atm^3(A,B) \to \mathfrak{A}$. Thus, $Atm^3(A,B)$ has the required universal property. Well-known duality [42] for exponentiation and multiplication defines the duality of the automata $Atm^2(B,C)$ and $Atm^3(A,B)$.

Let us make remarks on the category of sets with random mappings keeping in mind stochastic automata.

It is quite natural to construct a product in this category as a usual Cartesian product. Let A and B be two sets and $A\times B$ be their Cartesian product with the projections $\pi_1: A\times B \to A$ and $\pi_2: A\times B \to B$. π_1 and π_2 are represented by stochastic matrices by the rule:

$$\pi_1((a,b),a)=1 \text{ and } \pi_1((a,b),a')=0 \text{ if } a\neq a',$$

$$\pi_2((a,b),b)=1 \text{ and } \pi_2((a,b),b')=0 \text{ if } b\neq b'.$$

Given stochastic matrices $\alpha: C \to A$ and $\beta: C \to B$, let us construct $\gamma: C \to A\times B$ in such a way, that $\gamma\pi_1=\alpha$ and $\gamma\pi_2=\beta$ are satisfied.

We get:

$$\gamma\pi_1(c,a)= \sum_{(a',b)} \gamma(c,(a',b))\pi_1((a',b),a)=\sum_b \gamma(c,(a,b)),$$

$$\gamma\pi_2(c,b)= \sum_{(a,b')} \gamma(c,(a,b'))\pi_2((a,b'),b)=\sum_a \gamma(c,(a,b)).$$

Take $\gamma(c,(a,b))=\alpha(c,a)\beta(c,b)$, then

$$\gamma\pi_1(c,a)=\sum_b \alpha(c,a)\beta(c,b)=\alpha(c,a)\sum_b \beta(c,b)=\alpha(c,a),$$

$$\gamma\pi_2(c,b)=\sum_a \alpha(c,a)\beta(c,b)=\beta(c,b)\sum_a \alpha(c,a)=\beta(c,b).$$

(We use that $\sum_a \alpha(c,a)=1$, $\sum_b \beta(c,b)=1$).

Thus, the constructed γ satisfies the required condition. But, as we shall see, this γ is not unique.

Let the sets A, B and C consist of two elements: $A=\{a_1,a_2\}$, $B=\{b_1,b_2\}$ and $C=\{c_1,c_2\}$. The corresponding random mappings $\alpha: C \to A$, $\beta: C \to B$ and $\gamma: C \to A\times B$ are given by

$$\alpha=\begin{array}{|c|c|c|}\hline & a_1 & a_2 \\ \hline c_1 & \alpha_{11} & \alpha_{12} \\ \hline c_2 & \alpha_{21} & \alpha_{22} \\ \hline\end{array} \qquad \beta=\begin{array}{|c|c|c|}\hline & b_1 & b_2 \\ \hline c_1 & \beta_{11} & \beta_{12} \\ \hline c_2 & \beta_{21} & \beta_{22} \\ \hline\end{array}$$

$$\gamma=\begin{array}{|c|c|c|c|c|}\hline & (a_1,b_1) & (a_1,b_2) & (a_2,b_1) & (a_2,b_2) \\ \hline c_1 & \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} \\ \hline c_2 & \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} \\ \hline\end{array}$$

In this case, as before, we get:

$$\gamma\pi_1=\begin{pmatrix} \gamma_{11}+\gamma_{12} & \gamma_{13}+\gamma_{14} \\ \gamma_{21}+\gamma_{22} & \gamma_{23}+\gamma_{24} \end{pmatrix} \qquad \gamma\pi_2=\begin{pmatrix} \gamma_{11}+\gamma_{13} & \gamma_{12}+\gamma_{14} \\ \gamma_{21}+\gamma_{23} & \gamma_{22}+\gamma_{24} \end{pmatrix}$$

Take further

$$\begin{array}{ll} \gamma_{11}=\alpha_{11}\beta_{11}+\varepsilon & \gamma_{21}=\alpha_{21}\beta_{21}+\varepsilon \\ \gamma_{12}=\alpha_{11}\beta_{12}-\varepsilon & \gamma_{22}=\alpha_{21}\beta_{22}-\varepsilon \\ \gamma_{13}=\alpha_{12}\beta_{11}-\varepsilon & \gamma_{23}=\alpha_{22}\beta_{21}-\varepsilon \\ \gamma_{14}=\alpha_{12}\beta_{12}+\varepsilon & \gamma_{24}=\alpha_{22}\beta_{22}+\varepsilon \end{array}$$

Choose α,β and the number ε so that all γ_{ij} belong to the segment [0,1]. It is immediately verified that γ satisfies the conditions of the stochastic matrix and that the conditions $\gamma\pi_1=\alpha$ and $\gamma\pi_2=\beta$ are satisfied. Changing ε we obtain various suitable γ, and the condition of the uniqueness of γ is not satisfied.

This negative result implies that the definition of the stochastic automata is not subject to the common category scheme and one can act only by analogy.

Using this analogy define the stochastic automaton as a triplet of sets $\mathfrak{A}=(A,B,C)$ with the two random operations $\circ: A\times B \to A$ and $*: A\times B \to C$. Here $A\times B$ is an usual Cartesian product of sets.

Let stochastic matrices $\alpha: A \to A'$ and $\beta: B \to B'$ be given. Define $\alpha\times\beta: A\times B \to A'\times B'$, assuming $(\alpha\times\beta)((a,b),(a',b'))=\alpha(a,a')\beta(b,b')$. The triplet of random mappings $\mu(\mu_1,\mu_2,\mu_3): (A,B,C) \to (A',B',C')$ is a *homomorphism of stochastic automata*, if the following diagrams are commutative

$$\begin{array}{ccc} A\times B & \xrightarrow{\ \circ\ } & A \\ {\scriptstyle(\mu_1\times\mu_2)}\downarrow & & \downarrow{\scriptstyle\mu_1} \\ A'\times B' & \xrightarrow{\ \circ\ } & A' \end{array} \qquad \begin{array}{ccc} A\times B & \xrightarrow{\ \circ\ } & C \\ {\scriptstyle(\mu_1\times\mu_2)}\downarrow & & \downarrow{\scriptstyle\mu_3} \\ A'\times B' & \xrightarrow{\ *\ } & C' \end{array}$$

It is easy to understand that this definition agrees with the definition in the deterministic case. However, this algebraic approach to stochastic automata doesn't comprise the general situation, considered in [20].

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CHAPTER 2

CONSTRUCTIONS AND DECOMPOSITION OF PURE AUTOMATA

For the given automaton $\mathfrak{A}$ the problem of decomposition can be formulated as the problem of its representation by means of the automata (divisors), in some way simpler than $\mathfrak{A}$.

There are various types of decomposition, which differ by the choice of divisors, as well as by the used constructions. Decomposition theory for pure automata is based on the construction of cascade connection of automata and on its universal variant wreath product of automata.

This construction, in particular, has the property that information at the input of the decomposition component does not depend on the result, obtained at the output of this component at the previous step. In other words, it is a construction without feedback (without loops). It is known (see, for example, [5]) that if we allow arbitrary constructions with the feedback, then each automaton can be constructed only by modules - the automata with one state.

There are various types of relations between the components of decomposition and the initial automaton:

1. Each component is a homomorphic image of the subautomaton of the initial automaton (divides the initial automaton).
2. The semigroup of inputs of each component divides the semigroup of inputs of the initial automaton (i.e. is a homomorphic image of its subsemigroup).

Finally, we can consider the following case:

3. No conditions of connection with the initial automaton are imposed on the components.

The more free choice of the components implies in some sense more rational decomposition of the automaton (say from the point of view of the automaton complexity, its reliability etc.). On the other hand, decomposition being used for automaton analysis frequently requires more close connection of the components with the initial automaton. In the algebraic theory of automata the basic result on the decomposition of the automata without loops is well-known theorem of Krohn-Rhodes.

In this Chapter we introduce various automata constructions, consider the decomposition problem for automata (in particular, theorem of Krohn-Rhodes is proved) and treat indecomposable group automata.

2.1. Constructions

Some automata constructions have been already discussed in the previous Chapter. For example, in the Section 1.2 the constructions of the connection of semiautomaton and automaton of the input-output type have been discussed. In the given Section the main emphasis should be given to the constructions of the cascade connection and the wreath product of the automata as well as to the construction of the wreath product of the automaton and the semigroup.

2.1.1. Cascade connections of the absolutely pure automata

First define the Cartesian product of the automata. Let $\mathfrak{A}_\alpha=(A_\alpha,X_\alpha,B_\alpha)$ be an automata system, where α belongs to a certain set I. The automaton $\mathfrak{A}=(A,X,B)$ is called a *Cartesian product of the automata* $\mathfrak{A}_\alpha$, $\alpha\in I$, if A,B,X are Cartesian products of all $A_\alpha, B_\alpha, X_\alpha$ respectively, while the operations $\circ$ and $*$ are defined componentwise: if $\bar{a}\in A$, $\bar{x}\in X$, $\alpha\in I$, then

$$(\bar{a}\circ\bar{x})(\alpha)=\bar{a}(\alpha)\circ\bar{x}(\alpha),\quad (\bar{a}*\bar{x})(\alpha)=\bar{a}(\alpha)*\bar{x}(\alpha).$$

It is clear that if all $\mathfrak{A}_\alpha$ are semigroup automata, then their Cartesian product is also a semigroup automaton.

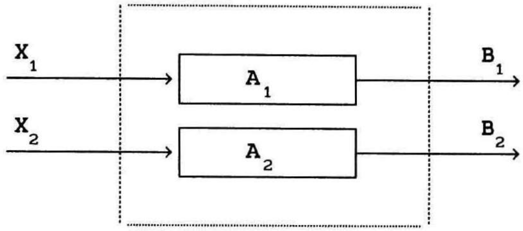
In particular, the *Cartesian product of two automata* is an automaton $\mathfrak{A}_1\times\mathfrak{A}_2=(A_1\times A_2, X_1\times X_2, B_1\times B_2)$ with the operations $\circ$ and $*$

$$(a_1,a_2)\circ(x_1,x_2)=(a_1\circ x_1, a_2\circ x_2),$$

$$(a_1,a_2)*(x_1,x_2)=(a_1*x_1, a_2*x_2),$$

where $(a_1, a_2) \in A_1 \times A_2$, $(x_1, x_2) \in X_1 \times X_2$.

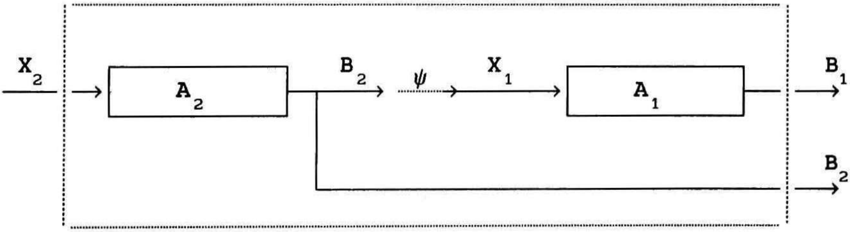
The Cartesian product of the automata realizes their *parallel connection*



The automaton $(A_1 \times A_2, X_2, B_1 \times B_2)$ with actions defined in the following way:

$$(a_1, a_2) \circ x_2 = (a_1 \circ (a_2 * x_2)^{\psi}, a_2 \circ x_2),$$
$$(a_1, a_2) * x_2 = (a_1 * (a_2 * x_2)^{\psi}, a_2 * x_2),$$

is called a *serial connection of the automata* $\mathfrak{A}_1$ and $\mathfrak{A}_2$ with the connection mapping $\psi: B_2 \rightarrow X_1$.



In the given construction an output signal of the automaton $\mathfrak{A}_2$ is transformed into an input signal of the automaton $\mathfrak{A}_1$ by means of the mapping ψ. The parallel (Cartesian product) and serial connections can be defined in this way also for the case of the semigroup automata.

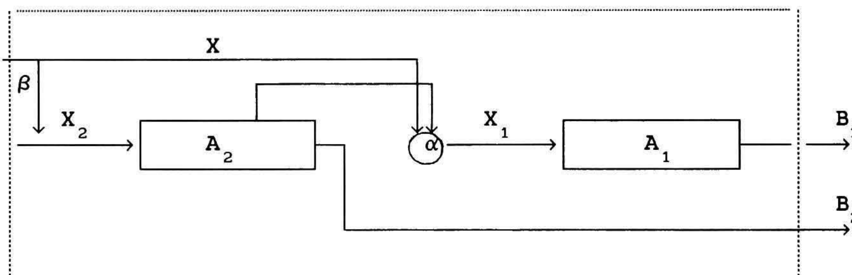
Serial and parallel connections of the automata are particular cases of the important construction of the cascade connection of the automata. Define this construction.

Let $\mathfrak{A}_1=(A_1,X_1,B_1)$ and $\mathfrak{A}_2=(A_2,X_2,B_2)$ be absolutely pure automata. Assume that a set X and two mappings $\alpha: X\times A_2 \rightarrow X_1$ and $\beta: X \rightarrow X_2$ are defined. The automaton $\mathfrak{A}_1 \times_{\alpha}^{\beta} \mathfrak{A}_2=(A_1\times A_2, X, B_1\times B_2)$ with operations $\circ$ and $*$ defined by the rule:

$$(a_1,a_2)\circ x=(a_1\circ\alpha(x,a_2), a_2\circ\beta(x)),$$
$$(a_1,a_2)*x=(a_1*\alpha(x,a_2), a_2*\beta(x)), \tag{1.1}$$

where $(a_1,a_2)\in A_1\times A_2$, $x\in X$, $\beta(x)\in X_2$, $\alpha(x,a_2)\in X_1$ is called a *cascade connection of the automata* $\mathfrak{A}_1$ and $\mathfrak{A}_2$ by the triplet (X,α,β).

It is possible to present a cascade connection of automata by the following design

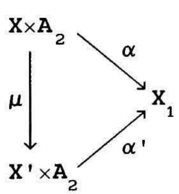


Different cascade connections of the automata $\mathfrak{A}_1$ and $\mathfrak{A}_2$ correspond to different triplets (X,α,β). For example, if we take a Cartesian product $X_1\times X_2$ as X and define the mappings α and β as corresponding projections $\alpha(x,a_2)=\alpha((x_1,x_2),a_2)=x_1$, $\beta(x)=\beta(x_1,x_2)=x_2$ where $x=(x_1,x_2)\in X$, then the cascade connection corresponding to such a triplet is a parallel connection of the automata.

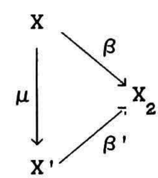
Let a mapping $\psi: B_2 \rightarrow X_1$ be defined. Setting $X=X_2$ and defining mappings $\alpha: X\times A_2 \to X_1$ and $\beta: X \to X_2$ by the rule $\alpha(x,a_2)=\psi(a_2*x)$, $\beta(x)=x$; $a_2\in A_2$, $x\in X=X_2$, we get that the corresponding cascade connection is a serial connection of the automata $\mathfrak{A}_1$ and $\mathfrak{A}_2$.

Remark. If the automaton $\mathfrak{A}=(A,X,B)$ is presented as the cascade connection $\mathfrak{A}_1\times^{\beta}_{\alpha}\mathfrak{A}_2$, then it is possible to say that the states and outputs of the automaton $\mathfrak{A}$ have two coordinates: $a=(a_1,a_2)$, $b=(b_1,b_2)$. The second coordinate of the new state (output) depends on the second (but not on both) coordinate of the previous state. The similar situation is in the case when an automaton is represented by cascade connection of several automata. This property can be called a weakened dependence. Presence of such dependence has a certain importance at the automata implementation (see [33]).

Let X_1, X_2, A_2 be the sets. Consider some special category $\mathcal{K}=\mathcal{K}(X_1,X_2,A_2)$. Objects of this category are triplets (X,α,β), where X is a set, α and β are mappings, $\alpha: X\times A_2 \to X_1$, $\beta: X \to X_2$. Morphisms of such triplets $\mu: (X,\alpha,\beta) \to (X',\alpha',\beta')$ are mappings of the sets $\mu: X \to X'$ for which the following two diagrams are commutative:



and



The mapping $\mu: X\times A_2 \to X'\times A_2$ in the first of these diagrams is defined by the rule: $(x,a)^{\mu}=(x^{\mu},a)$, $x\in X$, $a\in A_2$.

As it was mentioned above, a cascade connection of the automata $\mathfrak{A}_1=(A_1,X_1,B_1)$ and $\mathfrak{A}_2=(A_2,X_2,B_2)$ corresponds to each triplet (X,α,β).

Moreover, to each morphism of the triplets $\mu: (X,\alpha,\beta) \to (X',\alpha',\beta')$ corresponds a homomorphism in input signals of the respective cascade connections of the automata $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Really, if $(a_1,a_2)\in A_1\times A_2$, $x\in X$ then

$((a_1,a_2)\circ x)^{\mu}=(a_1,a_2)\circ x=(a_1\circ\alpha(x,a_2),a_2\circ\beta(x))=(a_1\circ\alpha'(x^{\mu},a_2),a_2\circ\beta'(x^{\mu}))=$
$(a_1,a_2)\circ x^{\mu}$.

The same for the operation *.

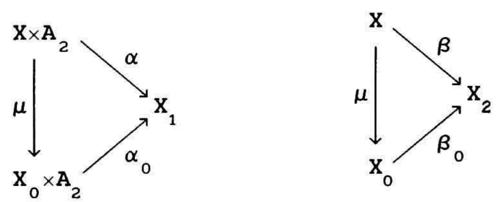
Consequently, the cascade connection is a functor on the category of triplets of the form (X,α,β) to the category of automata.

Further, consider a special triplet. Take a set $X_0=X_1^{A_2}\times X_2$ for X. Let us define mappings α_0 and β_0 by the rules:

$$\alpha_0(\bar{x}_1,x_2,a)=\bar{x}_1(a),$$
$$\beta_0(\bar{x}_1,x_2)=x_2;\ \bar{x}_1\in X_1^{A_2},\ x_2\in X_2,\ a\in A_2.$$

Proposition 1.1. (X_0,α_0,β_0) *is a terminal object in the category of triplets* $\mathcal{K}(X_1,X_2,A_2)$.

Proof. It is necessary to show that for an arbitrary triplet (X,α,β) there exists a unique morphism $\mu\colon(X,\alpha,\beta)\rightarrow(X_0,\alpha_0,\beta_0)$. Define $\mu\colon X\rightarrow X_0=X_1^{A_2}\times X_2$ by setting $\bar{x}_1(a_2)=\alpha(x,a_2)$, $x_2=\beta(x)$ for $x\in X$, $x^{\mu}=(\bar{x}_1,x_2)\in X_1^{A_2}\times X_2$. The diagrams



are commutative. Really,

$$\alpha_0(x,a_2)^{\mu})=\alpha_0(x^{\mu},a_2)=\alpha_0(\bar{x}_1,x_2,a_2)=\bar{x}_1(a_2)=\alpha(x,a_2);$$
$$\beta_0(x^{\mu})=\beta_0(\bar{x},x_2)=x_2=\beta(x).$$

Automaton corresponding to the universal triplet (X_0,α_0,β_0) we shall call a *wreath product of the automata* $\mathfrak{A}_1$ and $\mathfrak{A}_2$ and denote it by $\mathfrak{A}_1\mathrm{wr}\mathfrak{A}_2$. In accordance with the definition of the cascade connection by the triplet (X_0,α_0,β_0), the wreath product of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ is an automaton

$$\mathfrak{A}_1\mathrm{wr}\mathfrak{A}_2=(A_1\times A_2,X_1^{A_2}\times X_2,B_1\times B_2)$$

with the following operations $\circ$ and *: if $(a_1,a_2)\in A_1\times A_2$, $(\bar{x}_1,x_2)\in X_1^{A_2}\times X_2$, then

$$(a_1,a_2)\circ(\bar{x}_1,x_2)=(a_1\circ\bar{x}_1(a_2),a_2\circ x_2),$$
$$(a_1,a_2)*(\bar{x}_1,x_2)=(a_1*\bar{x}_1(a_2),a_2*x_2). \qquad (1.2)$$

Since a cascade connection is a functor on the category of triplets into the category of automata, from Proposition 1.1 it follows

Proposition 1.2. *For each cascade connection* $\mathfrak{A}=\mathfrak{A}_1\times^{\alpha}_{\beta}\mathfrak{A}_2$ *of the automata* $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ *there exists the homomorphism in input signals* $\mathfrak{A}\rightarrow\mathfrak{A}_1\mathrm{wr}\mathfrak{A}_2$

Since for exact automata (see Section 1.1) homomorphism in input signals is a monomorphism, then:

Corollary. *Each exact cascade connection of the automata* $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ *is isomorphic to a certain subautomaton of the wreath product* $\mathfrak{A}_1\mathrm{wr}\mathfrak{A}_2$.

2.1.2. Cascade connections and wreath products of pure semigroup automata

Consider two semigroup automata $\mathfrak{A}_1=(A_1,\Gamma_1,B_1)$ and $\mathfrak{A}_2=(A_2,\Gamma_2,B_2)$, a semigroup Γ and the mappings $\alpha:\Gamma\times A_2\rightarrow\Gamma_1$ and $\beta:\Gamma\rightarrow\Gamma_2$, satisfying supplementary conditions:

1. $\beta:\Gamma\rightarrow\Gamma_2$ is a semigroup homomorphism.
2. $\alpha(\gamma_1\gamma_2,a)=\alpha(\gamma_1,a)\cdot\alpha(\gamma_2,a\circ\beta(\gamma_1))$; $\gamma_1,\gamma_2\in\Gamma$, $a\in A_2$ (1.3)

A cascade connection of the semigroup automata by the given triplet (Γ,α,β) is defined in the same way as for absolute pure automata.

Proposition 1.3. *If the triplet* (Γ,α,β) *satisfies the conditions* (1.3), *then a cascade connection of the semigroup automata* $\mathfrak{A}=\mathfrak{A}_1\times^{\beta}_{\alpha}\mathfrak{A}_2$ *is a semigroup automata.*

Given semigroup automata $\mathfrak{A}_1$ and $\mathfrak{A}_2$, consider the category of triplets (Γ,α,β), satisfying the conditions (1.3). Morphisms in this category are defined as earlier but there is an additional requirement that the mapping $\mu:\Gamma\rightarrow\Gamma'$ must be a homomorphism of semigroups.

Construct the universal object $(\Gamma_0,\alpha_0,\beta_0)$ in this category. For Γ_0 take the wreath product $\Gamma_1\overset{A_2}{\mathrm{wr}}\Gamma_2=\Gamma_1^{A_2}\times\Gamma_2$ of the semigroups Γ_1 and Γ_2 by the set A_2. Mappings α_0 and β_0 are defined in the same way as in the previous item. Verify that this triplet satisfies the conditions (1.3). It is evident that $\beta_0:\Gamma_0 \to \Gamma_2$ is the homomorphism of the semigroups. Let $\gamma,\gamma'\in\Gamma_0$, $\gamma=(\bar{\gamma}_1,\gamma_2)$, $\gamma'=(\bar{\gamma}'_1,\gamma'_2)$, $a\in A_2$ Then

$\alpha_0(\gamma\gamma',a)=\alpha_0((\bar{\gamma}_1,\gamma_2)\cdot(\bar{\gamma}'_1,\gamma'_2),a)=\alpha_0(\bar{\gamma}_1\cdot(\gamma_2\circ\bar{\gamma}'_1),\gamma_2\gamma'_2,a)=(\bar{\gamma}_1\cdot(\gamma_2\circ\bar{\gamma}'_1))(a)=$

$\bar{\gamma}_1(a)\cdot(\gamma_2\circ\bar{\gamma}'_1)(a)=\bar{\gamma}_1(a)\cdot\bar{\gamma}'_1(a\circ\gamma_2)=\alpha_0(\bar{\gamma}_1,\gamma_2,a)\cdot\alpha_0(\bar{\gamma}'_1,\gamma'_2,a\circ\beta(\bar{\gamma}_1,\gamma_2))$.

Proposition 1.4. *The triplet* $(\Gamma_0,\alpha_0,\beta_0)$ *is a terminal object in the category of triplets satisfying the conditions* (1.3).

A cascade connection of the automata $\mathfrak{A}_1$ and $\mathfrak{A}_2$ corresponding to the universal triplet is denoted by $\mathfrak{A}_1\mathrm{wr}\mathfrak{A}_2$ and is called a *wreath product of the semigroup automata* $\mathfrak{A}_1$ and $\mathfrak{A}_2$:

$$\mathfrak{A}_1\mathrm{wr}\mathfrak{A}_2=(A_1\times A_2,\Gamma_1^{A_2}\times\Gamma_2,B_1\times B_2).$$

The operations $\circ$ and $*$ in $\mathfrak{A}_1\mathrm{wr}\mathfrak{A}_2$ are defined by the formulas (1.2). Statements, similar to Proposition 1.2 and its Corollary hold also for wreath product.

2.1.3. Properties of cascade connections

Proposition 1.5. *Let*

$$\varphi:\mathfrak{A}_1=(A_1,\Gamma_1,B_1)\to\mathfrak{B}_1=(A_1,\Sigma_1,B_1)$$

$$\psi:\mathfrak{A}_2=(A_2,\Gamma_2,B_2)\to\mathfrak{B}_2=(A_2,\Sigma_2,B_2)$$

be homomorphisms in input signals of the semigroup automata. Then, there is a corresponding homomorphism in input signals

$$\mu:\mathfrak{A}_1\mathrm{wr}\mathfrak{A}_2\to\mathfrak{B}_1\mathrm{wr}\mathfrak{B}_2$$

Proof. Dealing with homomorphisms in input signals, we denote homomorphisms of the input semigroups by the same letters as the homomorphisms of the corresponding automata. It is necessary to construct the following mapping

$$\mu: \Gamma_1^{A_2} \times \Gamma_2 \to \Sigma_1^{A_2} \times \Sigma_2.$$

This mapping must be a homomorphism of the semigroups, which defines the homomorphism of the wreath product of the automata.

First, define the mapping $\bar{\mu}: \Gamma_1^{A_2} \to \Sigma_1^{A_2}$, assuming $\bar{\gamma}_1^{\bar{\mu}}(a) = (\bar{\gamma}_1(a))^{\varphi}$ for $\bar{\gamma}_1 \in \Gamma_1^{A_2}$ and $a \in A_2$. If $(\bar{\gamma}_1, \gamma_2)$ is arbitrary element of $\Gamma_1^{A_2} \times \Gamma_2$, then define the required mapping $\mu: \Gamma_1^{A_2} \times \Gamma_2 \to \Sigma_1^{A_2} \times \Sigma_2$ in the following way: $(\bar{\gamma}_1, \gamma_2)^{\mu} = (\bar{\gamma}_1^{\bar{\mu}}, \gamma_2^{\psi})$. Show that μ is a semigroup homomorphism. Let $(\bar{\gamma}_1, \gamma_2)$, $(\bar{\gamma}'_1, \gamma'_2)$ be elements in $\Gamma_1^{A_2} \times \Gamma_2$. Then

$$((\bar{\gamma}_1, \gamma_2)(\bar{\gamma}'_1, \gamma'_2))^{\mu} = (\bar{\gamma}, \gamma_2\gamma'_2)^{\mu} = (\bar{\gamma}^{\bar{\mu}}, (\gamma_2\gamma'_2)^{\psi}), \tag{1.4}$$

where $\bar{\gamma} = \bar{\gamma}_1(\gamma_2 \circ \bar{\gamma}'_1)$,

$$\bar{\gamma}^{\bar{\mu}}(a) = (\bar{\gamma}(a))^{\varphi} = (\bar{\gamma}_1(a) \cdot \bar{\gamma}'_1(a \circ \gamma_2))^{\varphi} = (\bar{\gamma}_1(a))^{\varphi} \cdot (\bar{\gamma}'_1(a \circ \gamma_2))^{\varphi}.$$

On the other hand,

$$(\bar{\gamma}_1, \gamma_2)^{\mu} \cdot (\bar{\gamma}'_1, \gamma'_2)^{\mu} = (\bar{\gamma}_1^{\bar{\mu}}, \gamma_2^{\psi}) \cdot (\bar{\gamma}'^{\bar{\mu}}_1, \gamma'^{\psi}_2) = (\delta, (\gamma_2\gamma'_2)^{\psi}), \quad \delta \in \Sigma_1^{A_2} \tag{1.5}$$

where $\delta(a) = \bar{\gamma}_1^{\bar{\mu}}(a) \cdot \bar{\gamma}'^{\bar{\mu}}_1(a \circ \gamma_2^{\psi}) = (\bar{\gamma}_1(a))^{\varphi} \cdot (\bar{\gamma}'_1(a \circ \gamma_2^{\psi}))^{\varphi}$, $a \in A_2$.

As ψ is a homomorphism of the automata in input signals, then $a \circ \gamma_2^{\psi} = a \circ \gamma_2$, $a \in A_2$, $\gamma_2 \in \Gamma_2$. So, the equalities (1.4) and (1.5) imply that $((\bar{\gamma}_1, \gamma_2)(\bar{\gamma}'_1, \gamma'_2))^{\mu} = (\bar{\gamma}_1, \gamma_2)^{\mu}(\bar{\gamma}'_1, \gamma'_2)^{\mu}$. Therefore, μ is a homomorphism of the corresponding semigroups.

Besides, if $(a_1, a_2) \in A_1 \times A_2$, $(\bar{\gamma}_1, \gamma_2) \in \Gamma_1^{A_2} \times \Gamma_2$ then

$(a_1, a_2) \circ (\bar{\gamma}_1, \gamma_2)^{\mu} = (a_1, a_2) \circ (\bar{\gamma}_1^{\bar{\mu}}, \gamma_2^{\psi}) = (a_1 \circ \bar{\gamma}_1^{\bar{\mu}}(a_2), a_2 \circ \gamma_2^{\psi}) = (a_1 \circ \bar{\gamma}_1(a_2))^{\varphi}, a_2 \circ \gamma_2^{\psi}) =$
$(a_1 \circ \bar{\gamma}_1(a_2), a_2 \circ \gamma_2) = (a_1, a_2) \circ (\bar{\gamma}_1, \gamma_2)$.

Similarly, $(a_1, a_2) * (\bar{\gamma}_1, \gamma_2)^{\mu} = (a_1, a_2) * (\bar{\gamma}_1, \gamma_2)$. Hence, μ is a homomorphism in input signals of the automaton $\mathfrak{A}_1 \mathrm{wr} \mathfrak{A}_2$ to $\mathfrak{B}_1 \mathrm{wr} \mathfrak{B}_2$.

Corollary. *Let the exact automata $\mathfrak{A}'_1$, $\mathfrak{A}'_2$ correspond to the automata $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Then $\mathfrak{A}'_1 \mathrm{wr} \mathfrak{A}'_2$ is an exact automaton for $\mathfrak{A}_1 \mathrm{wr} \mathfrak{A}_2$.*

Really, if $\mu_1:\mathfrak{A}_1 \to \mathfrak{A}'_1$ and $\mu_2:\mathfrak{A}_2 \to \mathfrak{A}'_2$ are homomorphisms in input signals mapping the automata $\mathfrak{A}_1$ and $\mathfrak{A}_2$ into corresponding exact automata, then by Proposition 1.5 there is homomorphism in input signals $\mu:\mathfrak{A}_1\mathrm{wr}\mathfrak{A}_2 \to \mathfrak{A}'_1\mathrm{wr}\mathfrak{A}'_2$. Since the wreath product of the exact automata is again an exact automaton (this immediately follows from the definitions), then $\mathfrak{A}'_1\mathrm{wr}\mathfrak{A}'_2$ is an exact automaton corresponding to the automaton $\mathfrak{A}_1\mathrm{wr}\mathfrak{A}_2$.

It is quite natural to ask whether arbitrary cascade connection of the exact automata (and, respectively, of reduced ones) be exact (or reduced). The answer is given by the following statements.

Proposition 1.6. *In order that the cascade connection of the exact automata* $\mathfrak{A}_1=(A_1,\Gamma_1,B_1)$ *and* $\mathfrak{A}_2=(A_2,\Gamma_2,B_2)$ *by the triplet* (Γ,α,β) *be an exact automaton it is necessary and sufficient that the homomorphism* $\mu:\Gamma \to \Gamma_1^{A_2}\times\Gamma_2$ *corresponding to the mapping of the automaton* $\mathfrak{A}_1\times^{\beta}_{\alpha}\mathfrak{A}_2$ *into the wreath product* $\mathfrak{A}_1\mathrm{wr}\mathfrak{A}_2$ *(see Proposition* 1.1) *be a monomorphism.*

Proof. If the homomorphism $\mu:\Gamma \to \Gamma_1^{A_2}\times\Gamma_2$ given by the condition is a monomorphism, then the automaton $\mathfrak{A}_1\times^{\beta}_{\alpha}\mathfrak{A}_2$ is isomorphic to the subautomaton of the wreath product $\mathfrak{A}_1\mathrm{wr}\mathfrak{A}_2$ and, consequently, it is exact. On the other hand, if the homomorphism μ is not a monomorphism, the elements γ_1 and γ_2 from Γ having the same image in $\Gamma_1^{A_2}\times\Gamma_2$, act equally as input elements of the automaton $\mathfrak{A}_1\times^{\beta}_{\alpha}\mathfrak{A}_2$. Therefore this automaton is not exact.

Proposition 1.7. *If the automata* $\mathfrak{A}_1=(A_1,\Gamma_1,B_1)$ *and* $\mathfrak{A}_2=(A_2,\Gamma_2,B_2)$ *are reduced automata and the triplet* (Γ,α,β) *is such that for each* $a\in A$ *the mapping* $\alpha:\Gamma\times A_2 \to \Gamma_1$ *is a mapping onto* Γ_1 *and* $\beta:\Gamma \to \Gamma_2$ *is a mapping onto* Γ_2, *then the cascade connection* $\mathfrak{A}=(A_1\times A_2,\Gamma,B_1\times B_2)$ *by* (Γ,α,β) *is also a reduced automaton.*

Proof. Recall that the automaton (A,Γ,B) is called a reduced automaton if the equality $a*\gamma=a'*\gamma$ for all $\gamma\in\Gamma$ implies the equality $a=a'$. Let now $(a_1,a_2)*\gamma=(a'_1,a'_2)*\gamma$ be fulfilled under any $\gamma\in\Gamma$ By the definition of the cascade connection $(a_1,a_2)*\gamma=(a_1*\alpha(\gamma,a_2),a_2*\beta(\gamma))$. Therefore,

$$(a_1 * \alpha(\gamma, a_2), a_2 * \beta(\gamma)) = (a'_1 * \alpha(\gamma, a'_2), a'_2 * \beta(\gamma)) \text{ and}$$

$$a_1 * \alpha(\gamma, a_2) = a'_1 * \alpha(\gamma, a'_2); \quad a_2 * \beta(\gamma) = a'_2 * \beta(\gamma).$$

Since $\beta(\gamma)$ runs through the whole semigroup Γ_2 and the second automaton is reduced, then $a_2 = a'_2$. Similar arguments for $\alpha(\gamma, a_2)$ and Γ_1 imply $a_1 = a'_1$. Thus $(a_1, a_2) = (a'_1, a'_2)$, that is the automaton $\mathfrak{A}_1 \times^{\beta}_{\alpha} \mathfrak{A}_2$ is reduced.

Finally we would like to point out two rather simple but important facts:

1. Wreath product of the automata has the associative property:

$$\mathfrak{A}_1 \mathrm{wr}(\mathfrak{A}_2 \mathrm{wr} \mathfrak{A}_3) = (\mathfrak{A}_1 \mathrm{wr} \mathfrak{A}_2) \mathrm{wr} \mathfrak{A}_3.$$

2. Cascade connection of Moore automata is again a Moore automaton.

2.1.4. Cascade connection and transition to semigroup automata

We have seen (Section 1.1), that each automaton $\mathfrak{A} = (A, X, B)$ has an associate semigroup automaton $\mathcal{F}(\mathfrak{A}) = (A, F(X), B)$, where $F(X)$ is a free semigroup over the set X. Now we shall discuss the relation of this correspondence with the constructions of Cartesian product and cascade connection of automata.

Proposition 1.8. *Let the automaton $\mathfrak{A}$ be a Cartesian product of the automata $\mathfrak{A}_\alpha$, $\alpha \in I$. Then there exists an isomorphic embedding of the automata $\mathcal{F}(\mathfrak{A})$ into Cartesian product $\prod_\alpha \mathcal{F}_\alpha(\mathfrak{A})$, $\alpha \in I$. Besides, $\mathcal{F}(\mathfrak{A})$ is not isomorphic to $\prod_\alpha \mathcal{F}_\alpha(\mathfrak{A})$, $\alpha \in I$.*

Proof. Our aim is to construct a monomorphism of $\mathcal{F}(\mathfrak{A})$ into $\prod_\alpha \mathcal{F}(\mathfrak{A}_\alpha)$, $\alpha \in I$ and to show that this monomorphism is not an isomorphism. Let $\mathfrak{A} = (A, X, B)$, $\mathfrak{A}_\alpha = (A_\alpha, X_\alpha, B_\alpha)$, $\mathfrak{A} = \prod_\alpha \mathfrak{A}_\alpha$, $\alpha \in I$. Denote $F = F(X)$, $F_\alpha = F(X_\alpha)$. Let $\pi_\alpha : X \to X_\alpha$ be a projection: $x^{\pi_\alpha} = x(\alpha) \in X_\alpha$ for each $x \in X$. As $X_\alpha \subset F(X_\alpha)$, then π_α is a mapping of X into $\mathcal{F}(X_\alpha)$. Since $F = F(X)$ is a free semigroup generated by the set X, this mapping is uniquely extended to the homomorphism $\mu_\alpha : F \to F_\alpha$: if $u = x_1 \ldots x_n \in F(X)$, then $u^{\mu_\alpha} = x_1^{\pi_\alpha} \ldots x_n^{\pi_\alpha} = x_1(\alpha) \ldots x_n(\alpha)$. Homomorphisms μ_α define the homomorphism of the semigroup F into $\prod_\alpha \mathcal{F}(\mathfrak{A}_\alpha)$,

$\alpha\in I$: if $u\in F$, then $u^{\mu}(\alpha)=u^{\mu_\alpha}$. Show that μ is a monomorphism. It suffices to check that the kernel ρ of the homomorphism μ is the minimal congruence of the semigroup F. Denote $\rho_\alpha=\mathrm{Ker}\mu_\alpha$. It is clear that $\rho=\bigcap\limits_\alpha\rho_\alpha$. Let $u,v\in F$, $u=x_1\ldots x_n$, $v=y_1\ldots y_m$; $x_i,y_i\in X$ and $u\rho v$. Since $\rho=\bigcap\limits_\alpha\rho_\alpha$, then $u\rho_\alpha v$ for all $\alpha\in I$. This implies that $u^{\mu_\alpha}=v^{\mu_\alpha}$. Taking into account the definition of μ_α and the notation u,v we get: $x_1(\alpha)\ldots x_n(\alpha)=y_1(\alpha)\ldots y_m(\alpha)$. Both parts of this equality are elements of the free semigroup F_α. Therefore m=n and $x_i(\alpha)=y_i(\alpha)$, $i=1,\ldots,n$. This holds for any $\alpha\in I$, therefore, $x_i=y_i$ and u=v. Hence, ρ is a trivial congruence and $\mu:F\to\prod\limits_\alpha F_\alpha$, $\alpha\in I$, is a monomorphism.

It is obvious that $\varphi=(\varepsilon_A,\mu,\varepsilon_B)$, where $\varepsilon_A,\varepsilon_B$ are identity mappings of the sets A and B, is a monomorphism of $F(\mathfrak{A})$ into $\prod\limits_\alpha F(\mathfrak{A}_\alpha)$, $\alpha\in I$. Let us check that φ is not an isomorphism. Let $u=x_1\ldots x_n\in F$, $x_i\in X$. Number n is called the length of the element u. By the definition of mappings $\mu_\alpha:F\to F_\alpha$, $u^{\mu_\alpha}=x_1(\alpha)\ldots x_n(\alpha)\in F_\alpha$. Hence, the length of the element u^{μ_α} in the free semigroup F_α is also equal to n. This implies that the image of the semigroup F under the mapping μ (denote it by F^{μ}) in the Cartesian product $\prod\limits_\alpha F_\alpha$, $\alpha\in I$ consists of the functions with values having the same length for all α. It is clear that this image is less than $\prod\limits_\alpha F_\alpha$, $\alpha\in I$ and, moreover, it is not isomorphic to $\prod\limits_\alpha F_\alpha$, $\alpha\in I$. Therefore, the automaton $F(\mathfrak{A})$ is not isomorphic to $\prod\limits_\alpha F(\mathfrak{A}_\alpha)$, $\alpha\in I$.

From this Proposition it follows:

Proposition 1.9. *If $\mathfrak{B}=(A,\Gamma,B)$ is an exact automaton corresponding to the automaton $\mathcal{F}(\mathfrak{A})$, $\mathfrak{A}=\prod\limits_a\mathfrak{A}_\alpha$, $\alpha\in I$ and $\mathfrak{B}_\alpha=(A_\alpha,\Gamma_\alpha,B_\alpha)$ are exact automata corresponding to $\mathcal{F}(\mathfrak{A}_\alpha)$, then $\mathfrak{B}$ is isomorphically embedded into Cartesian product $\prod\limits_\alpha\mathfrak{B}_\alpha$, $\alpha\in I$.*

Proof. Let $\psi=(\varepsilon_A,\psi_2,\varepsilon_B)$ be a homomorphism taking the automaton $\mathcal{F}(\mathfrak{A})=(A,F,B)$ into the corresponding exact automaton (A,Γ,B). Denote by

ε_x an identity transformation of the sets X, and by ρ_ψ the kernel of the homomorphism ψ. Further, let $f_\alpha=(\varepsilon_{A_\alpha}, f^2_\alpha, \varepsilon_{B_\alpha})$ be a homomorphism, taking the automaton $(A_\alpha, F_\alpha, B_\alpha)$ into the corresponding exact automaton $(A_\alpha, \Gamma_\alpha, B_\alpha)$. The homomorphism

$$f=(\varepsilon_A, f_\alpha, \varepsilon_B): (\prod A_\alpha, \prod F_\alpha, \prod B_\alpha) \to (\prod A_\alpha, \prod \Gamma_\alpha, \prod B_\alpha)=(A, \prod \Gamma_\alpha, B),\ \alpha\in I$$

naturally corresponds to the homomorphisms f_α, $\alpha\in I$. As in Proposition 1.8, $\varphi=(\varepsilon_A, \mu, \varepsilon_B)$ is a monomorphism of the automaton (A,F,B) into the automaton $(A, \prod_\alpha F_\alpha, B)$, $\alpha\in I$.

Consider the homomorphism $\varphi f: (A,F,B) \to (A, \prod_\alpha \Gamma_\alpha, B)$ and denote its kernel by $\rho_{\varphi f}$. Show that $\rho_{\varphi f}=\rho_\psi$. From this follows the assertion of the Proposition.

Indeed, $\mathfrak{B}\simeq F(\mathfrak{A})/\rho_\psi$ while the image of $\mathcal{F}(\mathfrak{A})$ in $\prod_\alpha \mathfrak{B}_\alpha$ is $((\mathcal{F}(\mathfrak{A}))^{\varphi f} \simeq \mathcal{F}(\mathfrak{A})/\rho_{\varphi f}$. If $\rho_{\varphi f}=\rho_\psi$, then $\mathfrak{B}\simeq(\mathcal{F}(\mathfrak{A}))^{\varphi f} \subset \prod_\alpha \mathfrak{B}_\alpha$, that is $\mathfrak{B}$ is isomorphically embedded into Cartesian product of the automata $\mathfrak{B}_\alpha$. It is left to show that $\rho_{\varphi f}=\rho_\psi$. Let $\rho_{\varphi f}=(\rho^1_{\varphi f}, \rho^2_{\varphi f}, \rho^3_{\varphi f})$ and $\rho_\psi=(\rho^1_\psi, \rho^2_\psi, \rho^3_\psi)$. Here $\rho^1_{\varphi f}=\rho^1_\psi$ is a trivial equivalence of the set A, and $\rho^3_{\varphi f}=\rho^3_\psi$ - of the set B. It is necessary to show that $\rho^2_{\varphi f}=\rho^2_\psi$ on F.

Let $u, v\in F$ and $u\rho^2_{\varphi f}v$, that is $u^{\mu f_2}=v^{\mu f_2}$. This equality implies that u^μ and v^μ act equally in the automaton $(A, \prod_\alpha F_\alpha, B)$, $\alpha\in I$. Since $\varphi=(\varepsilon_A, \mu, \varepsilon_B)$ is a monomorphism of the automaton (A,F,B) into $(A, \prod_\alpha F_\alpha, B)$ which acts identically on A and B, then u^μ also acts as u and v^μ acts as v. Therefore, $u\rho^2_{\varphi f}v$ is equivalent to the fact that u and v act equally in the automaton $\mathcal{F}(\mathfrak{A})$. The latter implies that $u\rho^2_\psi v$. Therefore, $\rho^2_{\varphi f}=\rho^2_\psi$ and thus, $\rho_{\varphi f}=\rho_\psi$.

Let $\mathfrak{A}=\mathfrak{A}_1 \times^\beta_\alpha \mathfrak{A}_2$ be a cascade connection of the automata $\mathfrak{A}_1=(A_1, X_1, B_1)$ and $\mathfrak{A}_2=(A_2, X_2, B_2)$ by the triplet (X, α, β):

$$\mathfrak{A}=(A_1\times A_2, X, B_1\times B_2);\ \alpha: X\times A_2 \to X_1,\ \beta: X \to X_2.$$

By Proposition 1.2 $\mathfrak{A}$ is homomorphically embedded into the wreath product of the automata $\mathfrak{A}_1 wr \mathfrak{A}_2=(A_1\times A_2, X_1^{A_2}\times X_2, B_1\times B_2)$ by the homomorphism $\varphi=(\varepsilon_A,\varphi_2,\varepsilon_B)$, $A=A_1\times A_2$, $B=B_1\times B_2$, with the mapping φ_2 defined as follows: if $x\in X$, then $x^{\varphi_2}=(f,x_2)$, where $x_2=\beta(x)$ while f is such a function $f:A_2 \to X_1$, that $f(a_2)=\alpha(x,a_2)$ for all $a_2\in A_2$.

Let us proceed from this mapping φ_2. Since it is possible to consider $X_1^{A_2}\times X_2$ as a subset in $F(X_1)^{A_2}\times F(X_2)$, then φ_2 is a mapping $\varphi_2:X \to F(X_1)^{A_2}\times F(X_2)$. This mapping can be extended to the homomorphism $\psi:F(X) \to F(X_1)^{A_2}\times F(X_2)$. Homomorphism ψ defines two mappings $\psi_1:F(X) \to F(X_1)^{A_2}$ and $\psi_2:F(X) \to F(X_2)$: if $v\in F(X)$ and $u^{\psi}=(f,v)\in F(X_1)^{A_2}\times F(X_2)$, then $u^{\psi_1}=f$, $u^{\psi_2}=v$.

Proposition 1.10. *If* $\mathfrak{A}=\mathfrak{A}_1\times^{\beta}_{\alpha}\mathfrak{A}_2$ *is a cascade connection of the automata by the triplet* (X,α,β), *then* $\mathcal{F}(\mathfrak{A})$ *is a cascade connection of the automata* $\mathcal{F}(\mathfrak{A}_1)$ *and* $\mathcal{F}(\mathfrak{A}_2)$ *by the triplet* $(F(X),\bar{\alpha},\bar{\beta})$, *where the homomorphism* $\bar{\beta}:F(X) \to F(X_2)$ *is induced by the mapping* $\beta:X \to X_2$, *and the mapping* $\bar{\alpha}:F(X)\times A_2 \to F(X_2)$ *is defined by the rule: if* $u\in F(X)$, $a_2\in A_2$, *then* $\bar{\alpha}(u,u_2)=u^{\psi_1}(a_2)$.

Proof. First note that the mappings $\bar{\alpha},\bar{\beta}$ satisfy the conditions (1.3) and therefore, the cascade connection of the automata $\mathcal{F}(\mathfrak{A}_1)$ and $\mathcal{F}(\mathfrak{A}_2)$ corresponding to the triplet $(F(X),\alpha,\beta)$ is a semigroup automaton. Clearly, we have to check only the second condition. Since ψ is a homomorphism of the semigroups and

$$\bar{\beta}=\psi_2:\bar{\alpha}(u_1u_2,u_2)=(u_1u_2)^{\psi_1}(a_2)=u_1^{\psi_1}(a_2)u_2^{\psi_1}(a_2\circ u_1^{\psi_2})=$$

$$u_1^{\psi_1}(a_2)u_2^{\psi_1}(a_2\circ\bar{\beta}(u_1))=\bar{\alpha}(u_1,a_2)\bar{\alpha}(u_2,a_2\circ\bar{\beta}(u_1)),$$

the second condition is immediate.

The automaton $\mathcal{F}(\mathfrak{A})$ and corresponding cascade connection $\mathcal{F}(\mathfrak{A}_1)\times^{\bar{\beta}}_{\bar{\alpha}}\mathcal{F}(\mathfrak{A}_2)$ are defined on the same set $(A_1\times A_2, F(X), B_1\times B_2)$. We must verify that the corresponding actions $\circ$ and $*$ coincide. Moreover, it suffices to verify it only for elements $x\in X$. Since operations $\circ$ and $*$ in cascade connections are defined by the corresponding mappings α and β: $(a_1,a_2)\circ x=(a_1\circ\alpha(x,a_2),a_2\circ\beta(x))$, $(a_1,a_2)*x=(a_1*\alpha(x,a_2),a_2*\beta(x))$, the coincidence of the actions $\circ$ and $*$ in both automata follows from the equalities $\alpha(x,a)=\bar{\alpha}(x,a)$; $\beta(x)=\bar{\beta}(x)$. These equalities are either given by the definition $(\beta(x)=\bar{\beta}(x))$, or they immediately follow from them: $\bar{\alpha}(x,a_2)=x^{\psi_1}(a_2)=\alpha(x,a_2)$.

The proposition means that the automaton $\mathcal{F}(\mathfrak{A}_1\mathrm{wr}\mathfrak{A}_2)$ is a cascade connection of the automata $\mathcal{F}(\mathfrak{A}_1)$ and $\mathcal{F}(\mathfrak{A}_2)$. Therefore, there is a uniquely defined homomorphism in input signals $\mathcal{F}(\mathfrak{A}_1\mathrm{wr}\mathfrak{A}_2)\rightarrow\mathcal{F}(\mathfrak{A}_1)\mathrm{wr}\mathcal{F}(\mathfrak{A}_2)$.

2.1.5. Wreath product of automata and representations

Let us consider two constructions of wreath product of automaton and representation. Their particular cases will be constructions of the wreath product of automaton and semigroup.

Let (A,Σ,B) be a semigroup automaton and let (C,Φ) be a right representation of the semigroup Φ by transformations of the set C. The automaton (C,Φ,C) one-to-one corresponds to the representation (C,Φ). In this automaton the operation $*$ coincides with the operation $\circ$. The wreath product of the automata $(A,\Sigma,B)\mathrm{wr}(C,\Phi,C)=(A\times C,\Sigma^C\times\Phi,B\times C)$ is denoted by $(A,\Sigma,B)\mathrm{wr}(C,\Phi)$ and called a *right wreath product of the automaton and representation*. If, in particular, (Φ,Φ) is a right regular representation of the semigroup Φ, then the wreath product $(A,\Sigma,B)\mathrm{wr}(\Phi,\Phi)=(A\times\Phi,\Sigma^\Phi\times\Phi,B\times\Phi)$ is called a *right wreath product of the automaton* (A,Σ,B) *with the semigroup* Φ. It is denoted by $(A,\Sigma,B)\mathrm{wr}\Phi$.

Now, let (Φ,C) be a left representation of the semigroup Φ by transformations of the set C. It induces the right representation (A^C,Φ): if $\bar{a}\in A^C$, $\varphi\in\Phi$, then $\bar{a}\circ\varphi$ is such function from A^C that $(\bar{a}\circ\varphi)(c)=\bar{a}(\varphi\circ c)$. Define the *left wreath product of the representation* (Φ,C) *and the automaton* (A,Σ,B) as the automaton $(\Phi,C)\mathrm{wr}_\ell(A,\Sigma,B)=(A^C,\Phi\overset{C}{\mathrm{wr}}_\ell\Sigma,B^C)$, where $\Phi\overset{C}{\mathrm{wr}}_\ell\Sigma=\Phi\times\Sigma^C$ is the left wreath product of the corresponding semigroups and the operations $\circ$ and $*$ are defined in the following way: if $\bar{a}\in A^C$, $(\varphi,\bar{\tau})\in\Phi\times\Sigma^C$, then $\bar{a}\circ(\varphi,\bar{\tau})=(\bar{a}\circ\varphi)\circ\bar{\tau}$, $\bar{a}*(\varphi,\bar{\tau})=(\bar{a}\circ\varphi)*\bar{\tau}$.

Recall that $\bar{a}\circ\varphi\in A^C$, $\bar{a}\circ\bar{\tau}\in A^C$, $\bar{a}*\bar{\tau}\in B^C$ are the following functions: $(\bar{a}\circ\varphi)(c)=\bar{a}(\varphi c)$, $(\bar{a}\circ\bar{\tau})(c)=\bar{a}(c)\circ\bar{\tau}(c)$; $(\bar{a}*\bar{\tau})(c)=\bar{a}(c)*\bar{\tau}(c)$, $c\in C$. From the definition of $\bar{a}\circ\varphi$ and $\bar{a}\circ\bar{\tau}$ follows the equality $(\bar{a}\circ\bar{\tau})\circ\varphi=(\bar{a}\circ\varphi)\circ(\bar{\tau}\circ\varphi)$. Indeed, if $c\in C$, then

$((\bar{a}\circ\bar{\tau})\circ\varphi)(c)=(\bar{a}\circ\bar{\tau})(\varphi\circ c)=\bar{a}(\varphi\circ c)\circ\bar{\tau}(\varphi\circ c)=((\bar{a}\circ\varphi)(c))\circ(\bar{\tau}\circ\varphi)((c))=$

$((\bar{a}\circ\varphi)\circ(\bar{\tau}\circ\varphi))(c)$.

Now, direct calculations show that the introduced automaton satisfies the axioms of semigroup automaton:

$(\bar{a}\circ(\varphi_1,\bar{\tau}_1))\circ(\varphi_2,\bar{\tau}_2)=(((\bar{a}\circ\varphi_1)\circ\bar{\tau}_1)\circ\varphi_2)\circ\bar{\tau}_2=(((\bar{a}\circ\varphi_1)\circ\varphi_2)\circ(\bar{\tau}_1\circ\varphi_2))\circ\bar{\tau}_2=$

$(\bar{a}\circ\varphi_1\varphi_2)\circ(\bar{\tau}_1\circ\varphi_2)\bar{\tau}_2=\bar{a}\circ(\varphi_1\varphi_2,(\bar{\tau}_1\circ\varphi_2)\bar{\tau}_2)=\bar{a}\circ((\varphi_1,\bar{\tau}_1)(\varphi_2,\bar{\tau}_2))$.

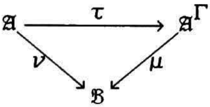
The latter equality follows from the definition of the multiplication in the left wreath product of the semigroups. Therefore, $\bar{a}\circ((\varphi_1,\bar{\tau}_1)(\varphi_2,\bar{\tau}_2))=(\bar{a}\circ(\varphi_1,\bar{\tau}_1))\circ(\varphi_2,\bar{\tau}_2)$. Validity of the second axiom is verified in a similar way: $\bar{a}*((\varphi_1,\bar{\tau}_1)(\varphi_2,\bar{\tau}_2))=(\bar{a}\circ(\varphi_1,\bar{\tau}_1))*(\varphi_2,\bar{\tau}_2)$.

If (Φ,Φ) is a left regular representation of the semigroup Φ, then the wreath product $(\Phi,\Phi)wr_\ell(A,\Sigma,B)=(A^\Phi,\Phi\times\Sigma^\Phi,B^\Phi)$ denoted $\Phi wr_\ell(A,\Sigma,B)$ is called a *left wreath product of the semigroup* Φ *with the automaton* (A,Σ,B).

2.1.6. Induced automata

In the Section 1.2, and in item 2.1.1 automata construction having universal properties were discussed. Now we shall introduce a construction of induced automaton which also has a certain universal property.

Let $\mathfrak{A}=(A,\Sigma,B)$ be an automaton and Σ be a subsemigroup in Γ. The automaton $\mathfrak{A}^\Gamma=(\tilde{A},\Gamma,\tilde{B})$ is called *induced* by $\mathfrak{A}$, if $\mathfrak{A}$ is isomorphic to a subautomaton from $\mathfrak{A}^\Gamma$; this isomorphism τ is identical on Σ, and for any automaton $\mathfrak{B}=(A',\Gamma,B')$ and monomorphism $\nu:\mathfrak{A}\to\mathfrak{B}$ identical on Σ, there exists a unique homomorphism $\mu:\mathfrak{A}^\Gamma\to\mathfrak{B}$ identical on Γ, such that the diagram



is commutative. It follows from the definition that the induced automaton is unique.

Construct such an automaton. As usual, denote by Γ^1 the semigroup obtained from Γ by adjoining of external unit element. First consider the automaton $\mathfrak{D}=(A\times\Gamma^1,\Gamma,B\cup(A\times\Gamma))$, whose actions of $\circ$ and $*$ are defined in the following way:

$$(a,\gamma)\circ\gamma'=(a,\gamma\gamma');\ (a,\gamma)*\gamma'=(a,\gamma\gamma');\ (a,\gamma)\in A\times\Gamma^1,\ \gamma'\in\Gamma.$$

Generate an equivalence α_1 on the set $A\times\Gamma^1$ and an equivalence α_3 on the set $B\cup(A\times\Gamma)$ by the relations σ_1 and σ_3: $(a,\tau)\sigma_1(a\circ\tau,1)$; $(a,\tau)\sigma_3(a*\tau)$, $a\in A$, $\tau\in\Sigma$. Denote by α_2 the trivial congruence on Γ. It is clear that the triplet $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ is a congruence of the automaton $\mathfrak{D}$. Quotient automaton $\tilde{\mathfrak{A}}=\mathfrak{D}/\alpha=(\tilde{A},\Gamma,\tilde{\tilde{B}})$ is a desired automaton.

2.2 Decomposition of finite pure automata

2.2.1. Krohn-Rhodes Decomposition Theorem

In this Section we confine ourselves to the discussion of the decomposition of the input-state type finite automata (semiautomata). To do this, we first develop the necessary definitions. Since in a Moore automaton the operation $*$ is expressed by the operation $\circ$ and a determining mapping, the decomposition of Moore automata is reduced to the decomposition of the input-state type automata.

The semigroup Γ_1 is called a divisor of the semigroup Γ_2, if Γ_1 is a homomorphic image of a certain subsemigroup from Γ_2. Similarly, the automaton $\mathfrak{A}_1=(A_1,\Gamma_1)$ is called a *divisor of the automaton* $\mathfrak{A}_2=(A_2,\Gamma_2)$, if $\mathfrak{A}_1$ is a homomorphic image of the subautomaton from $\mathfrak{A}_2$. If in this case the automaton $\mathfrak{A}_1$ is not isomorphic to $\mathfrak{A}_2$, it is called a proper divisor of the automaton $\mathfrak{A}_2$. We shall use the notations: $\Gamma_1|\Gamma_2$; $\mathfrak{A}_1|\mathfrak{A}_2$. Representation of the automaton $\mathfrak{A}$ as the divisor of the cascade connection (wreath product) of other automata is called *decomposition of the automaton* $\mathfrak{A}$.

Let a be a state. An input element (denote it by φ_a) acting by the rule: $x\circ\varphi_a=a$ for any state x of the given automaton, is called an *input constant* corresponding to the state a.

The automaton $\mathfrak{C}=(C,P)$ is called a *flip-flop* if the set of its states contains two elements: $C=\{c_0,c_1\}$, and the semigroup of inputs P consists of the identically acting element and input constants $\varphi_{c_0},\varphi_{c_1}$. Denote a flip-flop by $\mathfrak{C}$. The automaton (A,Γ) is called a *group automaton* if Γ is a group.

Theorem 2.1. (Krohn-Rhodes, [55], *see also* **[56] [29]).** *Each finite exact automaton* $\mathfrak{A}=(A,S)$ *admits decomposition of the form* $\mathfrak{A}|\mathfrak{A}_1\mathrm{wr}\mathfrak{A}_2\mathrm{wr}\ldots\mathrm{wr}\mathfrak{A}_n$, *whose components* $\mathfrak{A}_i$ *satisfy the conditions:*

1. *Each* $\mathfrak{A}_i$ *is either flip-flop or group automata.*
2. *If the automaton* $\mathfrak{A}_i=(A_i,S_i)$ *is a group one, then* $S_i|S$.

Krohn-Rhodes Theorem also describes indecomposable in a certain sense automata. We shall discuss this problem in the next item.

To prove the theorem we need a number of statements. Consider an automaton $\mathfrak{A}=(A,S)$. Denote by Φ_A the set of all its input constants: $\Phi_A=\{\varphi_a, a\in A\}$. It is a semigroup with respect to the multiplication $\varphi_{a_1}\varphi_{a_2}=\varphi_{a_2}$ Define multiplication of elements from S by elements from Φ_A by the rule: $s\varphi_a=\varphi_a$; $\varphi_a s=\varphi_{a\circ s}$, $a\in A$, $s\in S$. Then the set $\bar{S}=S\cup\Phi_A$ acquires the structure of a semigroup. Now the automaton $\bar{\mathfrak{A}}=(A,\bar{S})$ is defined in a natural way and called a *constant extension* of the automaton $\mathfrak{A}$.

Lemma 2.2. *Let* $\mathfrak{A}=(A,S)$, $A\circ S\neq A$ *be an exact automaton and* x *be such an element from* A *that* $A\circ S\subset A\setminus\{x\}=A_1$. *Then* $\mathfrak{A}|\bar{Y}\mathrm{wr}\mathfrak{C}_0$, *where* $Y=(A_1,S_1)$ *is an exact automaton corresponding to* (A_1,S) *and* $\mathfrak{C}_0=(C,\{\varphi_c\})$ *is a subautomaton of the flip-flop* $\mathfrak{C}$.

Proof. It is necessary to show that the automaton $\mathfrak{A}=(A,S)$ is a homomorphic image of a certain subautomaton $\mathfrak{B}$ from $\bar{Y}\mathrm{wr}\mathfrak{C}_0=(A_1\times C,\bar{S}_1^C\times\{\varphi_c\})$. To each element $s\in S$ we associate the element (f,φ_{c_0}) from $\bar{S}_1^C\times\{\varphi_{c_0}\}$. The component $f\in\bar{S}_1^c$ is defined by the rule $f(c_0)=s'$, where s' is the image of the element s under the natural homomorphism of S onto S_1, $f(c_1)=\varphi_{x\circ s}$. The set of such elements forms a subsemigroup in $\bar{S}_1^c\times\{\varphi_{c_0}\}$. Really, if (f_1,φ_{c_0}) and (f_2,φ_{c_0}) correspond to the elements s_1 and s_2 respectively, then $(f_1,\varphi_{c_0})\cdot(f_2,\varphi_{c_0})=(f_1(\varphi_{c_0}\circ f_2),\varphi_{c_0})=(f,\varphi_{c_0})$ where

$$
\begin{aligned}
f(c_0)&=f_1((c_0)\cdot f_2(c_0\circ\varphi_{c_0})=f_1(c_0)\cdot f_2(c_0)=s_1's_2'=(s_1s_2)'\\
f(c_1)&=f_1(c_1)\cdot f_2(c_1\circ\varphi_{c_0})=f_1(c_1)\cdot f_2(c_0)=\varphi_{x\circ s_1}s_2'=\varphi_{x\circ s_1s_2}
\end{aligned}
\qquad (2.1)
$$

Consequently, $(f_1,\varphi_{c_0})(f_2,\varphi_{c_0})=(f,\varphi_{c_0})$ is an element from $\bar{S}_1^c\times\{\varphi_{c_0}\}$ corresponding to the element s_1s_2 from S. Denote the obtained semigroup by V and the subautomaton $(A_1\times C,V)$ from $\bar{Y}\mathrm{wr}\mathfrak{C}_0$ by $\mathfrak{B}$. Define the mapping $\mu=(\mu_1,\mu_2):\mathfrak{B}\rightarrow\mathfrak{A}$ by the rule:

$$
(a,c_i)^{\mu_1}=\begin{cases} a, & i=0\\ x, & i=1\end{cases}\qquad (a,c_i)\in A_1\times C.
$$

if (f,φ_{c_0}) is an element from $\bar{S}_1^c\times\{\varphi_{c_0}\}$ associated with $s\in S$, then let $(f,\varphi_{c_0})^{\mu_2}=s$.

First of all, it is necessary to verify that the mappings μ_1,μ_2 are correctly defined. It is clear for μ_1 Let the element $(f,\varphi_{c_0})\in\bar{S}_1^c\times\{\varphi_{c_0}\}$ corresponds to $s_1,s_2\in S$. Then, on the one hand, $f(c_0)=s_1'$ and $f(c_1)=\varphi_{x\circ s_1}$; on the other hand, $f(c_0)=s_2'$ and $f(c_1)=\varphi_{x\circ s_2}$. It implies that, first, $s_1'=s_2'$, i.e. that elements s_1 and s_2 act equally on A_1, and that $\varphi_{x\circ s_1}=\varphi_{x\circ s_2}$ (i.e. that $x\circ s_1=x\circ s_2$). Thus, s_1 and s_2 act identically on the whole A. Since the automaton (A,S) is exact, then $s_1=s_2$. Verified feature of μ_2 and equalities (2.1) imply that μ_2 is a semigroup homomorphism. It remains to show that if $(a,c_i)\in A_1\times C$, $(f,\varphi_{c_0})\in V$, then $((a,c_i)\circ(f,\varphi_{c_0}))^{\mu_1}=(a,c_i)^{\mu_1}\circ(f,\varphi_{c_0})^{\mu_2}$, i=1,2. The validity of this equality means that $\mathfrak{A}$ is a homomorphic image of the automaton $\mathfrak{B}$.

Let (A,S) be an exact automaton and C be a subsemigroup in S. Denote by C^* the semigroup coinciding with C if C contains 1_A - identical transformation of the set A, and equal to $C^1=C\cup 1_A$ otherwise.

Lemma 2.3. *If the semigroup* S *of the exact automaton* $\mathfrak{A}=(A,S)$ *contains such left ideal* L *and subsemigroup* T, *that* $L\cup T=S$, *then the automaton* $\mathfrak{A}$ *admits the following decomposition* $\mathfrak{A}|\mathfrak{A}_1 wr\bar{\mathfrak{A}}_2$, *where* $\mathfrak{A}_1=(A,L^*)$, $\mathfrak{A}_2=(T^*,T)$.

Proof. By the definition, $\mathfrak{A}_1 wr\bar{\mathfrak{A}}_2=(A,L^*)wr(T^*,\bar{T})=(A\times T^*,(L^*)^{T^*}\times\bar{T})$. It is required to find such subautomaton $\mathfrak{B}=(Q,V)$ of this wreath product and such homomorphism $\mu=(\mu_1,\mu_2):\mathfrak{B}\rightarrow\mathfrak{A}$, that $\mathfrak{A}=\mathfrak{B}^{\mu}$. Assume $Q=A\times T^*$. As V take a union of the set of all the elements (ψ,t) from $(L^*)^{T^*}\times\bar{T}$ such that $t\in T$ and $\psi(x)=1_A$ for all $x\in T^*$, with the set of all the elements of the form $(f_s,\varphi_t)\in(L^*)^{T^*}\times\bar{T}$, $s\in L$, $t\in T$ for which $f_s(x)=xs$. Equalities

$$
\begin{aligned}
&(\psi,t)(\psi,t_1)=(\psi,tt_1); \qquad (\psi,t)(f_s,\varphi_{t_1})=(f_{ts},\varphi_{t_1}),\\
&(f_s,\varphi_t)(\psi,t_1)=(f_s,\varphi_{tt_1}); \quad (f_s,\varphi_t)(f_{s_1},\varphi_{t_1})=(f_{sts_1},\varphi_{t_1}),
\end{aligned}
\tag{2.2}
$$

following from the definition of the wreath product $(L^*)^{T^*}\times\bar{T}$, imply that V is a semigroup.

The homomorphism $\mu=(\mu_1,\mu_2):\mathfrak{B}\rightarrow\mathfrak{A}$ can be defined as follows:

μ_1: $A\times T^*\rightarrow A$; $\mu_1(a,t)=a\circ t$, $a\in A$, $t\in T^*$,

μ_2: $V\rightarrow S$; $\mu_2(\psi,t)=t$, $t\in T$; $\mu_2(f_s,\varphi_t)=st$, $s\in L$, $t\in T^*$.

From the definition of μ_2 and equalities (2.2) it follows that μ_2 is a homomorphism of the semigroups. It is clear that μ_2 is a surjective homomorphism of V onto S. Thus, it is left to verify that the conditions (1.2) (Chapter 1) are valid. Let $(a,t^*)\in A\times T^*$ Then $((a,t^*)\circ(\psi,t))^{\mu_1}=(a\circ\psi(t^*),t^*t)^{\mu_1}=(a,t^*t)^{\mu_1}=a\circ t^*t$. On the other hand, $(a,t^*)^{\mu_1}\circ(\psi,t)^{\mu_2}=(a\circ t^*)\circ t=a\circ t^*t$. Therefore, $((a,t^*)\circ(\psi,t))^{\mu_1}=(a,t^*)^{\mu_1}\circ(\psi,t)^{\mu_2}$. Similarly,

$$(a,t^*)\circ(f_s,\varphi_t))^{\mu_1}=(a\circ f_s(t^*),t^*\circ\varphi_t)^{\mu_1}=(a\circ t^*s,t)^{\mu_1}=a\circ t^*st,$$

$$(a,t^*)^{\mu_1}\circ(f_s,\varphi_t)^{\mu_2}=(a\circ t^*)\circ st=a\circ t^*st.$$

Therefore in this case $((a,t^*)\circ(f_s,\varphi_t))^{\mu_1}=(a,t^*)^{\mu_1}\circ(f_s,\varphi_t)^{\mu_2}$ too. Finally, note that from the definition of the homomorphism $\mu=(\mu_1,\mu_2)$ it follows immediately that μ is an epimorphism of $\mathfrak{B}$ on $\mathfrak{A}$.

The following lemma is proved similarly to Lemma 2.3.

Lemma 2.4. *Let* (A,S) *be an exact automaton with a monoid* S, T *be a subgroup of inverse elements of* S; L=S\T *be an ideal in* S. *Then* $(A,S)\,|\,(A,L^*)\mathrm{wr}(T,T)$.

Corollary. *For the group automaton* $\mathfrak{A}=(A,S)$ *holds*

$$\bar{\mathfrak{A}}\,|\,\overline{(A,1_A)}\mathrm{wr}(S,S). \qquad (2.3)$$

Really, $\bar{\mathfrak{A}}=(A,S\cup\Phi_A)$, where Φ_A is a set of all input constants of the automaton $\mathfrak{A}$; S is subgroup of inverse elements of the monoid $S\cup\Phi_A$. Applying Lemma 2.4 we get (2.3).

Lemma 2.5. *If* $\mathfrak{A}\,|\,\mathfrak{A}_1\mathrm{wr}\mathfrak{A}_2$, *then* $\bar{\mathfrak{A}}\,|\,\bar{\mathfrak{A}}_1\mathrm{wr}\bar{\mathfrak{A}}_2$.

Proof. Let $\mathfrak{A}=(A,S)$, $\mathfrak{A}_i=(A_i,S_i)$, i=1,2. Then $\mathfrak{A}_1\mathrm{wr}\mathfrak{A}_2=(A_1\times A_2,S_1^{A_2}\times S_2)$. Consider an epimorphism $\mu=(\mu_1,\mu_2):(Q,V)\rightarrow\mathfrak{A}$, where (Q,V) is a subautomaton from $\mathfrak{A}_1\mathrm{wr}\mathfrak{A}_2$. Let a be an arbitrary element from A and (a_1,a_2) its certain inverse image from $Q\subset(A_1,A_2)$: $(a_1,a_2)^{\mu_1}=a$; let further φ_a be an input constant from $\bar{S}$. Take the element $\omega_a=(\bar{f},\varphi_{a_2})\in\bar{S}_1^{A_2}\times\bar{S}_2$ with $\bar{f}(x)=\varphi_{a_1}$ for all $x\in A_2$. Let $\tilde{V}=V\cup\{\omega_a,a^{\mu_1}\in Q\}$. The homomorphism $\bar{\mu}_2:\tilde{V}\rightarrow\bar{S}$ is determined by the homomorphism $\mu_2:V\rightarrow S$ and by the mappings $(\omega_a)^{\bar{\mu}_2}=\varphi_a$. In this case the pair $\bar{\mu}=(\mu_1,\bar{\mu}_2):(Q,\tilde{V})\rightarrow\bar{\mathfrak{A}}$ satisfies the conditions of the homomorphism of automata.

Lemma 2.6. $\overline{(A,1_A)}\,|\,\mathfrak{C}\mathrm{wr}\ldots\mathrm{wr}\mathfrak{C}$.

Lemma 2.7. *If* (A,S) *is an exact semigroup automaton, then* $(S^*,S)\,|\,(A,S)\times\ldots\times(A,S)$.

The proofs of Lemmas 2.6 and 2.7 are rather simple. It is suggested to the reader as an exercise (see also [2]).

Proof of Theorem 2.1. To given automaton $\mathfrak{A}=(A,S)$ we assign a pair of integers (m,n) where $m=|A|$ and $n=|S|$. This pair is called a power of the automaton $\mathfrak{A}=(A,S)$. The set of all pairs is ordered by the rule: $(m_1,n_1)<(m_2,n_2)$ if $m_1<m_2$ or $m_1=m_2$, and $n_1<n_2$.

By induction on the introduced ordering show first that each exact automaton $\mathfrak{A}$ admits decomposition of the form

$$\mathfrak{A}|\bar{\mathfrak{A}}_1 \mathrm{wr} \bar{\mathfrak{A}}_2 \mathrm{wr} \dots \mathrm{wr} \bar{\mathfrak{A}}_k \tag{2.4}$$

where $\mathfrak{A}_i$ are either flip-flops or group automata dividing the automaton $\mathfrak{A}$.

Let the automaton $\mathfrak{A}=(A,S)$ has the power (m,n) and each automaton with the power less than (m,n) admits decomposition (2.4). It is known (see, for example, [53]) that for every finite semigroup S one of the three following conditions has to be satisfied:

1) the semigroup S is monogenic (cyclic);
2) the semigroup S does not contain proper left ideals;
3) in the semigroup S there are such proper left ideal L and subsemigroup T, that $L\cup T=S$.

Consider each case separately.

1) Let the semigroup S be monogenic with the generator element γ. Then either $A\circ\gamma=\{a\circ\gamma | a\in A\}$ is contained in A (we denote $A\circ\gamma<A$) or $A\circ\gamma=A$. Let first $A\circ\gamma<A$. Then for any natural ℓ holds $A\circ\gamma^{\ell}<A$ and thus $A\circ S<A$. Then Lemma 2.2 yields the decomposition $\mathfrak{A}|\bar{Y}\mathrm{wr}\mathfrak{C}$ in which the power of the automaton Y is less than the power of the automaton $\mathfrak{A}$ and the automaton $\mathfrak{C}$ is a flip-flop. By assumption of the induction, $Y|\bar{Y}_1\mathrm{wr}\dots\mathrm{wr}\bar{Y}_n$, where Y_i are either flip-flops or group automata dividing Y. Since Y is a divisor of $\mathfrak{A}$, then these group components also divide $\mathfrak{A}$. It is necessary to note that in this case the automaton Y is exact. Let now $A\circ\gamma=A$. Then γ is a permutation on the set A, and since S is a cyclic semigroup with the generator γ, it follows that S is a group.

2) Consider the case when the semigroup S does not have proper left ideals. If $A\circ S=\{a\circ s | a\in A, s\in S\}<A$, then as in the first case, we can

apply Lemma 2.2 and the induction.

Let $A\circ S=A$. Denote by $A_1, A_2, \ldots, A_k$ all such $A\circ s_i=\{a\circ s_i \mid a\in A\}$, $s_i\in S$, $i=1\ldots k$, that $A\circ s_i$ does not belong to $A\circ t$ for any $t\in S$ ($t\neq s_i$). If $k=1$ (i.e. exists unique A_1 with this property), then $A\circ s_1=A$. Indeed, if $A\circ s_1=A_1<A$, then there would be $A\circ S<A$ (otherwise A_2 with the given property would exist), but this contradicts the assumption $A\circ S=A$. Therefore $A\circ s_1=A$, where s_1 is a permutation on the set A. Then there exists a natural number n, such that s_1^n acts identically on the elements of A. The element s_1^n is a unit in S. It is known that semigroup without left ideals containing a left unit is a group [53].

If $k>1$, then $L_i=\{S \mid s\in S, A\circ s\subset A_i\}$ is a non-trivial left ideal in S. This contradicts the assumption that the semigroup S does not have proper left ideals.

3) Let now the semigroup S contain a non-trivial left ideal L and semigroup T satisfying the condition $L\cup T=S$. Then in virtue of Lemma 2.3 the automaton $\mathfrak{A}=(A,S)$ admits the decomposition: $\mathfrak{A}\mid\mathfrak{X}\mathrm{wr}\bar{Z}$ where $\mathfrak{X}=(A,L^*)$, $Z=(T^*,T)$.

By Lemma 2.7 the automaton (T^*,T) divides the automaton $(A,T)\times\ldots\times(A,T)$ embedded into $(A,T)\mathrm{wr}\ldots\mathrm{wr}(A,T)$. Powers of automata (A,L) and (A,T) are less than the power of the automaton $\mathfrak{A}$. We can again use the assumption of induction. (Here one should bear in mind that if $(A,L)\mid\mathfrak{X}_1\mathrm{wr}\ldots\mathrm{wr}\mathfrak{X}_n$ and the semigroup of inputs in all $\mathfrak{X}_i$ is a monoid, then $(A,L^*)\mid\mathfrak{X}_1\mathrm{wr}\ldots\mathrm{wr}\mathfrak{X}_n$). Thus, decomposition (2.4) is proved. If $\mathfrak{A}_i$ are flip-flops, then $\bar{\mathfrak{A}}_i=\mathfrak{A}_i$. Let $\mathfrak{B}=(B,Q)$ be a group automaton. Then by the corollary of Lemma 2.4 $\bar{\mathfrak{B}}\mid\overline{(B,1_B})\mathrm{wr}(Q,Q)$. In its turn by Lemma 2.6 holds $\overline{(B,1)}\mid\mathfrak{C}\mathrm{wr}\ldots\mathrm{wr}\mathfrak{C}$, where $\mathfrak{C}$ is a flip-flop, and by Lemma 2.7 holds $(Q,Q)\mid(B,Q)\mathrm{wr}\ldots\mathrm{wr}(B,Q)$. Using the decomposition (2.3), from the above it follows that

$$\mathfrak{A}\mid\mathfrak{X}_1\mathrm{wr}\ldots\mathrm{wr}\mathfrak{X}_n$$

where $\mathfrak{X}_i$ are either flip-flops or group automata dividing the initial automaton.

In fact, the result which has been proved can be formulated in the following way:

Theorem 2.1'. *Each finite exact automaton* $\mathfrak{A}$ *admits decomposition of the form* $\mathfrak{A}|\mathfrak{A}_1 \mathrm{wr}\ldots\mathrm{wr}\mathfrak{A}_n$. *Its components* $\mathfrak{A}_i$ *are either flip-flops or group automata dividing the initial automaton* $\mathfrak{A}$.

2.2.2. Indecomposable automata

Performing decomposition of the finite automaton several times we come to the automata which are, in a certain sense, indecomposable ones. Note that we deal only with indecomposable group automata. In the work by Krohn, Rhodes and Tilson [56] the decomposition theorem has been proved in semigroup terms with indecomposable automata being defined correspondingly. In accordance with this work, indecomposable automata are the automata with a simple acting group. In the present book we emphasize consideration of the automaton as many-sorted algebraic system; reformulation of theorem 2.1 in the form of theorem 2.1' as well as the following definition of indecomposable automata, correspond to this point of view.

Let us call the automaton (A,Γ) *non-trivial* if the set A is not a one-element set.

An automaton $\mathfrak{A}$ is called *indecomposable* if decomposition $\mathfrak{A}|\mathfrak{A}_1 \mathrm{wr}\ldots\mathrm{wr}\mathfrak{A}_n$ with $\mathfrak{A}_i$ being non-trivial automata and $\mathfrak{A}_i|\mathfrak{A}$, $i=1,\ldots n$, implies that $\mathfrak{A}$ divides certain $\mathfrak{A}_i$

An automaton $\mathfrak{A}$ is called s-*indecomposable or simple* if $\mathfrak{A}|\mathfrak{A}_1 \mathrm{wr}\mathfrak{A}_2$ implies, that $\mathfrak{A}$ divides $\mathfrak{A}_1$ or $\mathfrak{A}_2$. These two definitions correspond to different approaches pointed above, namely, whether components of decomposition are divisors of the initial automaton or not. Simple or s-indecomposable automata are considered in the book [60]. It has been proved that the automaton $\mathfrak{A}$ is simple if and only if it is a regular automaton of the form $\mathfrak{A}=(Z_p,Z_p)$ where Z_p is a cyclic group of the simple order p. In the present item indecomposable group automata will be considered.

Recall that for the input-state automaton (A,S) (i.e. for semiautomaton), congruence is a pair of equivalences (ρ_1,ρ_2) (ρ_1 is an equivalence on the set A and ρ_2 is a congruence of the semigroup S), satisfying the condition: if $a\rho_1 a'$, $s\rho_2 s'$ then $(a\circ s)\rho_1(a'\circ s')$. Congruence

$\rho=(\rho_1,\rho_2)$ of the automaton (A,S) is called *trivial* in one of the following cases:

1) if classes of the equivalences by ρ_1 and ρ_2 consist of individual elements of the sets A and S correspondingly (i.e. ρ_1 and ρ_2 are minimal congruences on A and S);

2) if there is the unique class of equivalence by ρ_1, equal to A, and the unique class of equivalence by ρ_2 equal to S (i.e. ρ_1 and ρ_2 are maximal congruences on A and S);

3) if ρ_1 is a maximal congruence on A and ρ_2 is a minimal congruence on S.

The automaton $\mathfrak{A}=(A,S)$ is called *transitive* if for any elements $a_1,a_2\in A$ there exists $s\in S$ such that $a_2=a_1\circ s$, in other words, if for any $a\in A$ holds $a\circ S=A$.

Proposition 2.8. *The group automaton* (A,Γ) *does not have non-trivial congruences if and only if it is isomorphic to the automaton of the form* $(\Gamma/\Sigma,\Gamma)$ *with* Γ *being a simple group,* Σ *a maximal subgroup in* Γ, Γ/S *a set of the right cosets by* Σ *and with the action defined as follows:* $\Sigma\gamma\circ\gamma'=\Sigma\gamma\gamma'$; $\gamma,\gamma'\in\Gamma$.

Proof. If the automaton $\mathfrak{A}=(A,\Gamma)$ is not transitive and $A_i=a_i\circ\Gamma$ are Γ-trajectories in A, then the equivalence $\rho=(\rho_1,\rho_2)$, for which A_i are classes of the equivalence by ρ_1 and ρ_2 is the minimal equivalence of the group Γ, is a non-trivial congruence of the automaton $\mathfrak{A}$. Therefore, the automaton without non-trivial congruences is a transitive one. Any transitive automaton (A,Γ) is isomorphic to the automaton $(\Gamma/\Sigma,\Gamma)$ where Γ/Σ is a set of the right cosets of the group Γ by a certain subgroup Σ and the action $\circ$ is defined in a way stated in the condition. Indeed, having fixed an arbitrary element $a\in A$ we have $a\circ\Gamma=A$, that is, for any element $a'\in A$ under certain $\gamma\in\Gamma$, holds $a\circ\gamma=a'$.

Let $\Sigma=\{\sigma\in\Gamma\,|\,a\circ\sigma=a\}$ be a centralizer of the element a; this is a subgroup of the group Γ. Assigning the right coset $\Sigma\gamma$ of Γ/Σ to the element $a'\in A$ we get the isomorphism of the automata $(\Gamma/\Sigma,\Gamma)$ and (A,Γ) identical on the group Γ. Each subgroup H of the group Γ defines the non-trivial congruence $\tau=(\tau_1,\tau_2)$ of the regular automaton (Γ,Γ) where τ_1 is an equivalence of the group Γ, classes of which are right cosets corresponding to the subsemigroup H, and τ_2 is the minimal equivalence of Γ. Then, if $\Sigma\subset H$, the congruence of the automaton (Γ,Γ) defined by H contains the congruence defined by the subgroup Σ. This implies that if the subgroup Σ is not a maximal one, then the automaton $(\Gamma/\Sigma,\Gamma)$ has a nontrivial congruence.

It is easy to prove that, conversely, the automaton $(\Gamma/\Sigma,\Gamma)$ has no nontrivial congruences.

Proposition 2.9. *If the finite simple group Γ divides the wreath product of the groups $\Gamma_1 wr \Gamma_2$, then it divides one of the components of the wreath product Γ_1 or Γ_2.*

Proof. Let us denote $\Gamma_1 wr \Gamma_2$ by $\tilde{\Gamma}$. Let Γ be a homomorphic image of the subgroup $\Delta\subset\tilde{\Gamma}$. Since $\Gamma_1^{\Gamma_2}$ is an invariant subgroup in $\tilde{\Gamma}$, then it is possible to construct in $\tilde{\Gamma}$ a normal series

$$1=\Sigma_0\subset\Sigma_1\subset \ldots \subset\Sigma_n=\tilde{\Gamma} \tag{2.5}$$

with factors Σ_i/Σ_{i-1} isomorphic either to Γ_1 or to Γ_2. Let $\Sigma'_i=\Delta\cap\Sigma_i$. Then the series

$$1=\Sigma'_0\subset\Sigma'_1\subset \ldots \subset\Sigma'_n=\Delta \tag{2.6}$$

is a normal series in Δ. From Zassenhaus's lemma ([58]) follows that the quotient group $(\Sigma_i\cap\Delta)/(\Sigma_{i-1}\cap\Delta)$ is isomorphic to the subgroup of the quotient group Σ_i/Σ_{i-1}. Therefore, each factor of the series (2.6) is isomorphic to a certain subgroup of one of the groups Γ_1 or Γ_2.

Let P be a kernel of the homomorphism of the group Δ on Γ, that is Δ/P is isomorphic to Γ. Consider one more normal series in Δ

$$1\subset P\subset\Delta. \tag{2.7}$$

By Schreier's theorem ([58]) normal series (2.6) and (2.7) have isomorphic refinements. Factors of the refined series are divisors of the factors of initial series. Since the factor Δ/P is isomorphic to the simple group Γ, it is not refinable. Hence, it is isomorphic to a certain factor of the refinement of series (2.6), which in its turn is a divisor of the corresponding factor of series (2.5). Thus, Γ is a divisor either of Γ_1 or of Γ_2

Recall that there is one-to-one correspondence between the congruences of groups and their invariant subgroups. Therefore each congruence (ρ_1,ρ_2) of the group automaton (A,Γ) can be written in the form of (ρ_1,ρ_H) where H is an invariant subgroup associate to the congruence ρ_2.

Proposition 2.10. (*Kaluzhnin-Krasner's theorem*). *Let* (A,Γ) *be a group automaton and* (ρ_1,ρ_H) *be such a congruence of this automaton that the quotient automaton* $(A/\rho,\Gamma/H)$ *is transitive. Denote by B an arbitrary coset of the congruence* ρ *and by* Σ *the normalizer of the set* B *in* Γ. *Then the automaton* (A,Γ) *divides the wreath product* $(B,\Sigma)\mathrm{wr}(A/\rho,\Gamma/H)$.

Proof. Denote the quotient group Γ/H by $\bar{\Gamma}$. The transitive automaton $(A/\rho,\Gamma/H)=(A/\rho,\bar{\Gamma})$ is isomorphic to the automaton $(\bar{\Gamma}/\bar{\Sigma},\bar{\Gamma})$. In its turn, this automaton is isomorphic to the automaton $(\Gamma/\Sigma,\Gamma/H)$ with the operation $\tilde{x}\circ\bar{\gamma}=\widetilde{x\gamma}$, $\tilde{x}\in\Gamma/\Sigma$, $\bar{\gamma}\in\Gamma/H$, $x\in\Gamma,\gamma\in\Gamma$. Therefore the statement of the theorem is equivalent to the following: $(A,\Gamma)\,|\,(B,\Sigma)\mathrm{wr}(\Gamma/\Sigma,\Gamma/H)$. It is necessary to find the subautomaton (Q,V) of the wreath product $(B\times\Gamma/\Sigma,\Sigma^{\Gamma/\Sigma}\times\bar{\Gamma})$, whose homomorphic image is (A,Γ).

Fix a certain complete system $T=\{t_1,\dots,t_n\}$ of the representatives of the right cosets Γ by Σ. Denote by ψ a mapping which takes each coset to its representative in T. Since $(A/\rho,\Gamma)$ is transitive we get $A=\bigcup_{\gamma\in\Gamma} B\circ\gamma=\bigcup_{t\in T} B\circ\Sigma t=\bigcup_{t\in T} B\circ t$. So each element $a\in A$ can be uniquely represented in the following form: $a=b\circ t$, $b\in B$, $t\in T$. If now we shall take $B\times\Gamma/\Sigma$ as Q, then the mapping $\mu\colon B\times\Gamma/\Sigma\to A$ defined by the rule $(b,\tilde{x})^{\mu_1}=b\circ\tilde{x}^{\psi}$, $b\in B$, $\tilde{x}\in\Gamma/\Sigma$ is a mapping of $B\times\Gamma/\Sigma$ on the set A.

Take as V the set of elements of $\Sigma^{\Gamma/\Sigma}\times\bar{\Gamma}$ having the form $(f_\gamma,\bar{\gamma})$, $\gamma\in\Gamma$ such that $\bar{\gamma}$ is an image of the element γ under the natural homomorphism of Γ on $\bar{\Gamma}$, while the function f_γ of $\Sigma^{\Gamma/\Sigma}$ is defined in the following way: if $\tilde{x}$ lies in Γ/Σ, then

$$f_\gamma(\tilde{x})=\tilde{x}^{\psi}\gamma((\tilde{x}\circ\bar{\gamma})^{\psi})^{-1}\in\Sigma.$$

From the definition it follows that

$$(f_{\gamma_1},\bar{\gamma}_1)(f_{\gamma_2},\bar{\gamma}_2)=(f_{\gamma_1\gamma_2},\bar{\gamma}_1\bar{\gamma}_2). \tag{2.8}$$

This, in particular, implies that V is a semigroup. Define $\mu_2: V \to \Gamma$ by the rule $(f_\gamma,\bar{\gamma})^{\mu_2}=\gamma$. This definition is correct: if $(f_{\gamma_1},\bar{\gamma}_1)=(f_{\gamma_2},\bar{\gamma}_2)$, then $\gamma_1=\gamma_2$. Indeed, if $(f_{\gamma_1},\bar{\gamma}_1)=(f_{\gamma_2},\bar{\gamma}_2)$ then for the arbitrary $\tilde{x}\in\Gamma/\Sigma$ the equality

$$\tilde{x}^\psi\gamma_1((\tilde{x}\circ\bar{\gamma}_1)^\psi)^{-1}=\tilde{x}^\psi\gamma_2((\tilde{x}\circ\bar{\gamma}_2)^\psi)^{-1})$$

is satisfied. Since $\bar{\gamma}_1=\bar{\gamma}_2$, then $\tilde{x}\circ\bar{\gamma}_1=\tilde{x}\circ\bar{\gamma}_2$ and $\tilde{x}^\psi\gamma_1=\tilde{x}^\psi\gamma_2$. Therefore $\gamma_1=\gamma_2$. From the equality (2.8) it follows that μ_2 is a homomorphism of the semigroup V. It remains to verify that $\mu=(\mu_1,\mu_2):(Q,V) \to (A,\Gamma)$ satisfies the condition (1.2) from Chapter 1 of the homomorphisms of automata.

Let $(b,\tilde{x})\in Q=B\times\Gamma/\Sigma$, $(f_\gamma,\bar{\gamma})\in V$. Then

$((b,\tilde{x})\circ(f_\gamma,\bar{\gamma}))^{\mu_1}=(b\circ f_\gamma(\tilde{x}),\tilde{x}\circ\bar{\gamma})^{\mu_1}=(b\circ\tilde{x}^\psi\gamma((\tilde{x}\circ\bar{\gamma})^\psi)^{-1},\tilde{x}\circ\bar{\gamma})^{\mu_1}=$

$b\circ\tilde{x}^\psi\gamma((\tilde{x}\circ\bar{\gamma})^\psi)^{-1}(\tilde{x}\circ\bar{\gamma})^\psi=b\circ\tilde{x}^\psi\gamma.$

On the other hand, $(b,\tilde{x})^{\mu_1}\circ(f_\gamma,\bar{\gamma})^{\mu_2}=(b\circ\tilde{x}^\psi)\circ\gamma=b\circ\tilde{x}^\psi\gamma$. Thus $(b,\tilde{x})\circ(f_\gamma,\bar{\gamma})^{\mu_1}=(b,\tilde{x})^{\mu_1}\circ(f_\gamma,\bar{\gamma})^{\mu_2}$, as required.

Corollary. *If Σ is an invariant subgroup of the group Γ, then* $(\Gamma,\Gamma)\,|\,(\Sigma,\Sigma)\mathrm{wr}(\Gamma/\Sigma,\Gamma/\Sigma)$.

Indeed, Σ defines the congruence ρ and one of the cosets by ρ is Σ itself. Σ also coincides with its normalizer in the representation $(\Gamma/\Sigma,\Gamma)$.

The key statement in this item is

Theorem 2.11. *In order the exact finite group automaton $\mathfrak{A}=(A,\Gamma)$ to be indecomposable it is necessary and sufficient that it would not have non-trivial congruences.*

Proof. Show first that if the automaton $\mathfrak{A}$ has a non-trivial congruence, then it is decomposable. This is equivalent to the fact that an indecomposable automaton does not have non-trivial congruences. Let $(\rho,\rho_2)=(\rho_1,\rho_H)$ be non-trivial congruence of the automaton $\mathfrak{A}$. Assume first that this automaton is not transitive and that $A_1,A_2,\dots,A_k$ are Γ-trajectories in A, $A_i=a_i\circ\Gamma$, $i=1,\dots,k$, $a_i\in A$. The automaton (A,Γ) divides the Cartesian product $(A,1)\times(\Gamma,\Gamma)=(A\times\Gamma,1\times\Gamma)$. The homomorphism $\mu=(\mu_1,\mu_2)$, $\mu_1:A\times\Gamma\to A$, $\mu_1(a,\gamma)=a\circ\gamma$; $\mu_2:1\times\Gamma\to\Gamma$, $\mu_2(1,\gamma)=\gamma$ maps the automaton $(A,1)\times(\Gamma,\Gamma)$ on (A,Γ).

Let further $\mathfrak{z}_i$ be a kernel of (A_i,Γ), $i=1,\dots,k$. By Lemma 2.7.

$$(\Gamma/\mathfrak{z}_i,\Gamma/\mathfrak{z}_i)\,|\,(A_i,\Gamma/\mathfrak{z}_i)\mathrm{wr}\ \dots\ \mathrm{wr}(A_i,\Gamma/\mathfrak{z}_i) \qquad (2.9)$$

Since $\bigcap_{i=1}^{k}\mathfrak{z}_i=1$, then by Remak's theorem the group Γ is isomorphically embedded into the direct product $\prod_{i=1}^{k}\Gamma/\mathfrak{z}_i$ and the automaton (Γ,Γ) into the automaton $(\prod_{i=1}^{k}\Gamma/\mathfrak{z}_i,\ \prod_{i=1}^{k}\Gamma/\mathfrak{z}_i)\cong\prod_{i=1}^{k}(\Gamma/\mathfrak{z}_i,\Gamma/\mathfrak{z}_i)$. Taking into account decomposition (2.9) this implies that the automaton $\mathfrak{A}$ divides the wreath product of certain automata $\mathfrak{X}_i$, $i=1,2,\dots,\ell$, such that each $\mathfrak{X}_i$ divides $\mathfrak{A}$ and is not isomorphic to it.

If the automaton $\mathfrak{A}$ is a transitive one and (ρ_1,ρ_H) is such its congruence that ρ_1 is a non-trivial equivalence of the set A, then by Kaluzhnin-Krasner's theorem $(A,\Gamma)\,|\,(B,\Sigma)\ \mathrm{wr}(A/\rho_1,\Gamma/H)$, where B is a certain class of the equivalence ρ_1 and Σ is a normalizer of this class in Γ. It is evident, that both components are proper divisors of (A,Γ), thus the automaton $\mathfrak{A}$ is decomposable.

Consider the following situation: the automaton (A,Γ) is transitive and allows only such non-trivial congruences (ρ_1,ρ_H) that ρ_1 is a trivial equivalence on the set A. From the definition of the automaton congruence follows that H acts identically on the set A/ρ_1. If the cosets by ρ_1 consist of the separate elements of the set A, this would imply that H belongs to the kernel of the representation (A,Γ), which contradicts the exactness of the automaton $\mathfrak{A}$. Note that $H\neq1$, otherwise the congruence (ρ_1,ρ_H) would be trivial. Thus, ρ_1 is a maximal equivalence with the unique class equal to A, and H a non-trivial invariant subgroup of the group Γ. Show that in the given case the automaton $\mathfrak{A}$ is also decomposable.

Denote by Σ the centralizer of the arbitrary element $a_0 \in A$; $\Sigma=\{\delta\in\Gamma \mid a_0\circ\delta=a_0\}$. Since the automaton (A,Γ) is transitive, then $(A,\Gamma)\cong(\Gamma/\Sigma,\Gamma)$, where Γ/Σ is a set of the right cosets by the subgroup Σ. Subsets in A of the form $a_0\circ(\Delta\gamma)$, $\gamma\in\Gamma$ define a non-trivial Γ-congruence on A for the arbitrary subgroup Δ containing Σ. This contradicts the assumption on non-trivial congruences of the automaton (A,Γ). Therefore, Σ is a maximal subgroup in Γ.

If the invariant subgroup H belongs to Σ, then H lies in the kernel of the automaton $(\Gamma/\Sigma,\Gamma)$. However, $(\Gamma/\Sigma,\Gamma)\cong(A,\Gamma)$ and (A,Γ) is an exact automaton. Therefore, H does not belong to Σ and in virtue of maximality of the subgroup Σ the equality $H\Sigma=\Gamma$ takes place. Consequently, $\Gamma/H=H\Sigma/H\cong\Sigma/H\cap\Sigma$.

The automaton $(\Gamma/\Sigma,\Gamma)$ is a divisor of the automaton (Γ,Γ). Therefore, $(A,\Gamma)\mid(\Gamma,\Gamma)$. In its turn, by the corollary of the statement 2.10

$$(\Gamma,\Gamma)\mid(H,H)\mathrm{wr}(\Gamma/H,\Gamma/H)\cong(H,H)\mathrm{wr}(\Sigma/H\cap\Sigma,\Sigma/H\cap\Sigma).$$

The latter automaton is a homomorphic image of the automaton (Σ,Σ). Thus $(A,\Gamma)\mid(H,H)\mathrm{wr}(\Sigma,\Sigma)$. Since H,Σ are subgroups in Γ, then by Lemma 2.7 $(H,H)\mid(A,H)\mathrm{wr}\ldots\mathrm{wr}(A,H)$. Finally we get

$$(A,\Gamma)\mid(A,H)\mathrm{wr}\ldots\mathrm{wr}(A,H)\mathrm{wr}(A,\Sigma)\mathrm{wr}\ldots\mathrm{wr}(A,\Sigma).$$

This implies that the automaton (A,Γ) is decomposable.

Prove the converse statement. Let $\mathfrak{A}=(A,\Gamma)$ be the automaton without non-trivial congruences. In accordance with proposition 2.8 it is isomorphic to the automaton $(\Gamma/\Sigma,\Gamma)$, where the group Γ is simple and Σ is a maximal subgroup in Γ. Assume that this automaton divides the wreath product of the automata $(A_1,\Gamma_1)\mathrm{wr}\ldots\mathrm{wr}(A_n,\Gamma_n)$, where each (A_i,Γ_i) is exact and divides the initial automaton $\mathfrak{A}$. Then the group Γ divides the wreath product of the groups $\Gamma_1\mathrm{wr}\ldots\mathrm{wr}\Gamma_n$ By proposition 2.9 the simple group Γ divides certain component Γ_i of this wreath product. By the condition the automaton (A_i,Γ_i) divides the initial automaton $\mathfrak{A}$. Therefore, the group Γ_i is isomorphic to the group Γ. So, the automaton

(A_1,Γ_1) defines the isomorphic automaton (A_1,Γ). The initial automaton (A,Γ) is transitive. The automaton (A_1,Γ), dividing it and having the same group of inputs Γ, is also transitive. The automaton (A_1,Γ) is a homomorphic image of a certain subautomaton of (A,Γ). Since these automata are transitive and have one group of inputs, then actually (A_1,Γ) is a homomorphic image of the automaton (A,Γ). Since the latter does not have non-trivial congruences, this means that the automata (A,Γ) and (A_1,Γ), and consequently the automata (A,Γ) and (A_1,Γ_1) are also isomorphic. Thus, the automaton (A,Γ) is indecomposable.

Joining the results of Theorem 2.11 and Proposition 2.8 we complete the description of the indecomposable group automata.

Corollary. *The finite group automaton* (A,Γ) *is indecomposable if and only if it is isomorphic to the automaton of the form* $(\Gamma/\Sigma,\Gamma)$ *where* Γ *is a simple group,* Γ/Σ *is a set of the right cosets by the maximal subgroup* Σ *and with the action defined by:* $\Sigma\gamma\circ\gamma'=\Sigma\gamma\gamma'$; $\gamma,\gamma'\in\Gamma$.

2.2.3. Decomposition of Mealy automata

From Krohn-Rhodes theorem for Moore automata it follows that each Mealy automaton $\mathfrak{A}=(A,S,B)$ also allows decomposition of the form $\mathfrak{A}|\mathfrak{A}_1 wr\dots wr\mathfrak{A}_K$, where $\mathfrak{A}_i$ are either flip-flops or simple group automata whose groups of inputs divide the semigroup of inputs of the initial automaton.

Indeed, the automaton $\mathfrak{A}=(A,S,B)$ is homomorphic in states image of Moore automaton $\mathfrak{A}'=(A\times B,S,B)$.

By Krohn-Rhodes theorem the Moore automaton $\mathfrak{A}'$ admits a required decomposition and besides, the automata $\mathfrak{A}$ and $\mathfrak{A}'$ have the same semigroup of inputs.

Consider one more reduction to a Moore automaton which may prove to be more efficient for decomposition of the automata with great number of the output signals and small number of the input ones. Lemmas 2.11-2.14 are similar to the corresponding ones for Moore automata.

The automaton with two states $A_1=\{a_0,a_1\}$, with three outputs $A_1\cup\{b\}=\{a_0,a_1,b\}$, and the semigroup of inputs consisting of the input constants Φ_{A_1} and the element ε acting in the following way: $a_1\circ\varepsilon=a_1$, $a_1*\varepsilon=a_1$, is called a *flip-flop* (Mealy). The flip-flop will be denoted by $\mathfrak{F}=(A_1,\Phi_{A_1}\cup\varepsilon,A_1\cup\{b\})$.

Recall that a semigroup automaton (S,S,S) is called *regular* if the operations $\circ$ and $*$ are defined by the rules: $s_1\circ s=s_1s$; $s_1*s=s_1s$; $s_1\in S$, $s\in S$.

Lemma 2.12. *If the automaton* $\mathfrak{A}=(A,S,B)$ *satisfies the condition* $A\circ S=A$, *then* $\mathfrak{A}|\mathfrak{A}_1 wr \mathfrak{A}_2$, *where* $\mathfrak{A}_1=(A,\varepsilon,A\cup B)$ *is an automaton with the operations* $a\circ\varepsilon=a$, $a*\varepsilon=a$, $a\in A$, *and* $\mathfrak{A}_2$ *is a regular automaton* (S,S,S).

Lemma 2.13. *Let the automaton* (A,S,B) *be given and* x *be such an element of* A *that* $A\circ S\subset A_1=A\setminus x\neq\varnothing$. *Then* $\mathfrak{A}|\bar{\mathfrak{J}} wr \mathfrak{F}$, *where* $\mathfrak{J}$ *is an exact automaton corresponding to* (A_1,S,B) *and* $\mathfrak{F}$ *is a flip-flop.*

Lemma 2.14. *The automaton* $(A,\varepsilon,A\cup B)$ *of Lemma* 2.12 *divides the direct product of flip-flops.*

Applying Lemma 2.13 several times we obtain that $\mathfrak{A}|\bar{\mathfrak{X}}_1 wr\mathfrak{F}wr\ldots wr\mathfrak{F}$, where the automaton $\mathfrak{X}=(A',T,D)$ is a divisor of $\mathfrak{A}$ and satisfies $A'\circ T=A'$. By Lemma 2.12 $\mathfrak{X}|\mathfrak{X}_1 wr\mathfrak{X}_2$, where $\mathfrak{X}_1=(A',\varepsilon,A'\cup D)$ and $\mathfrak{X}_2=(T,T,T)$ and T divides S. By Lemma 2.14 $\mathfrak{X}_1$ divides the product of the flip-flops and besides, $\mathfrak{X}_2$ is a Moore automaton. Thus, we came to the decomposition of the Moore automaton.

Having a consistent approach to the automaton as to the three-sorted algebraic system it is quite natural to study a decomposition of Mealy automata, components of which divide the initial automaton and to describe automata indecomposable in this sense.

CHAPTER 3

LINEAR AUTOMATA

3.1. Basic concepts

In this Chapter we consider the general theory of linear automata, introduce various constructions (in particular, the construction of the triangular product), and apply them to decomposition of such automata.

3.1.1. Linear automata, linearization, universal linear automata

Linear automata (A,X,B,), i.e. automata whose state and output sets are linear spaces (or modules over commutative rings) with actions ∘ and * being linear operations, were introduced in 1.5.3. Linear semigroup automata were defined in the same item. In the similar way, as in the case of pure automata, to each linear automaton (A,X,B) can be assigned a semigroup linear automaton (A,F(X),B), where F(X) is a free semigroup over its set of generators X. The assigning is a functor on the category of all linear automata to the category of semigroup linear automata.

A linear automaton can be associated with to each pure one. Let (Z,X,Y) be an absolutely pure automaton. Fix a field K and consider vector spaces A and B defined over sets Z and Y respectively. Operations ∘: $Z \times X \to X$ and $*: Z \times X \to Y$ are extended by linearity to the corresponding operations $A \times X \to A$, $A \times X \to B$. The result is the linear automaton (A,X,B). Since homomorphisms of linear automata correspond to homomorphisms of pure ones, the linearization is a functor on the category of pure automata to the category of linear automata (over the given field K).

If (Z,X,Y) is an exact finite automaton, then linearizing it, we get an automaton, which is isomorphic to a matrix one. It turns to be convenient that matrices can be transtormed to different canonical forms. We show how matrix forms of an automaton can be changed.

Let $\mathfrak{A}=(A,X,B)$ be an exact linear automaton, $X\subset End(A,B)$, $x=(\sigma,\varphi)\in X$. Take a pair of automorphisms: μ_1 of the space A, μ_3 of the space B, and define a mapping μ_2 of the set X in the following way:

$$\text{if } x=(\sigma,\varphi)\in X, \quad \text{then set} \quad x^{\mu_2}=(\mu_1^{-1}\sigma\mu_1,\mu_1^{-1}\sigma\mu_3).$$

The image of X under μ_2 is denoted by X_1, $X_1\subset End(A,B)$. The mapping μ_2 is one-to-one, and the triplet $\mu=(\mu_1,\mu_2,\mu_3)$ satisfies the conditions (1.2), Ch. 1. This means that μ is an isomorphic mapping of the automaton (A,X,B) onto the automaton (A,X_1,B). By suitable choice of μ_1,μ_3 one can change the matrix form of the initial automaton.

Along with linear semigroup automata, it is advisable to consider also ring automata. In the case of linear automata, the set End(A,B)= EndA×Hom(A,B) is a ring, the multiplication in which is defined in the same way, as in the general case (see Section 1.5), while the addition is defined componentwise, i.e. if (φ_1,ψ_1) and (φ_2,ψ_2) are the elements of End(A,B), then

$$(\varphi_1,\psi_1)(\varphi_2,\psi_2)=(\varphi_1\varphi_2,\varphi_1\psi_2),\quad (\varphi_1,\psi_1)+(\varphi_2,\psi_2)=(\varphi_1+\varphi_2,\psi_1+\psi_2).$$

The linear automaton (A,R,B), such that R is a ring and operations $\circ$ and $*$ are determined by the ring homomorphism of R into End(A,B), is called *a ring automaton*.

If (Γ,Γ,Γ) is a regular pure semigroup automaton, then as the result of the linearization, we get a *regular linear semigroup automaton* $(K\Gamma,\Gamma,K\Gamma)$. The operations $\circ$ and $*$ in this automaton are defined as follows:

$$\text{if } \gamma\in\Gamma,\ u=\sum_{i=1}^{n}\alpha_i\gamma_i\in K\Gamma,\ \alpha_i\in K,\ \gamma_i\in\Gamma,\ \text{then}$$

$$u\circ\gamma=u*\gamma=u\gamma=\sum_{i=1}^{n}\alpha_i\gamma_i\gamma.$$

For the linear automaton (A,Γ,B) the operations $\circ$ and $*$ defined for elements of Γ can be extended by linearity to corresponding operations on elements of $K\Gamma$:

$$\text{if} \quad u=\sum_{i=1}^{n}\alpha_i\gamma_i\in K\Gamma, \quad \alpha_i\in K, \ \gamma_i\in\Gamma; \ a\in A \text{ then}$$

$$a\circ u=\sum_{i=1}^{n}\alpha_i(a\circ\gamma_i), \ a*u=\sum_{i=1}^{n}\alpha_i(a*\gamma_i).$$

In this way the automaton $(A,K\Gamma,B)$ whose set of inputs is a semigroup algebra is defined. This automaton is a particular case of the ring automaton defined above.

A linear automaton (A,Γ,B) is called *a cyclic* one, if there exists an element $a\in A$, such that $A=a\circ K\Gamma$, $B=a*K\Gamma$. In this case a is called a *generating element.*

Any cyclic automaton (A,Γ,B) is the homomorphic image of a regular one $(K\Gamma,\Gamma,K\Gamma)$. Indeed, let $u\in K\Gamma$ be a state, and $v\in K\Gamma$ be an output signal of a regular automaton. Let a be a generating element of a cyclic automaton. Define linear mappings $\mu_1:K\Gamma \to A$ and $\mu_3:K\Gamma \to B$ according to the rule: $u^{\mu_1}=a\circ u$, $v^{\mu_1}=a*u$, and let $\mu_2:\Gamma \to \Gamma$ be the identity mapping. Then $\mu=(\mu_1,\mu_2,\mu_3)$ is a homomorphism of the regular automaton onto a cyclic one.

Three types of the *universal* automata $Atm^1(A,B)$, $Atm^2(\Gamma,B)$, $Atm^3(A,\Gamma)$ were defined for automata in arbitrary varieties in Section 1.5. We preserve this notation in the case of linear automata. Let (A,Γ,B) be a linear automaton. Each element $a\in A$ defines a mapping f_a of the set of inputs into the set of outputs, $f_a:\Gamma \to B$, according to the rule $f_a(\gamma)=a*\gamma, \gamma\in\Gamma$. Assignment of the mapping f_a to each element $a\in A$ defines a homomorphism in states of the automaton (A,Γ,B) into $Atm^2(\Gamma,B)$. The automaton is a *reduced* one, if the homomorphism is a monomorphism, i.e. if the equality $f_{a_1}=f_{a_2}$ implies $a_1=a_2$. In the case of linear automata reduction is equivalent to the fact that the subspace $A_1=\{a\in A \mid a*\gamma=0, \gamma\in\Gamma\}$ of elements of A, mapped by each element of Γ to zero of the space B, equals zero.

Return to the definition of the universal automaton $Atm^3(A,\Gamma)$. By the definition of $Atm^3(A,\Gamma)$ in arbitrary variety, it follows that in the case of linear automata, H is a linear space with the free set of generators $A\times\Gamma$. Congruence ρ of this space is determined by the subspace H_0 generated by all elements of the form

$$(a,\gamma_1\gamma_2)-(a\circ\gamma_1,\gamma_2) \text{ and } \alpha(a_1,\gamma)+\beta(a_2,\gamma)-(\alpha a_1+\beta a_2,\gamma),$$

where $a_1,a_2,a_3\in A$; $\alpha,\beta\in K$; $\gamma\in\Gamma$. Finally $A\otimes\Gamma=H/H_0$. The quotient space H/H_0 can be regarded as the tensor product of the linear spaces A and $K\Gamma$ over the ring $K\Gamma$.

3.1.2. Linear Moore automata

A linear automaton (A,Γ,B) is called a *Moore automaton* if there exists a linear mapping $\psi: A \to B$ such that for any elements $a\in A$, and $\gamma\in\Gamma$ the equality $a*\gamma=(a\circ\gamma)^{\psi}$ holds.

In this case ψ is called the *determining mapping*. A number of statements formulated and proved in Section 1.3 for pure Moore automata, holds also for linear automata. For instance: a linear automaton (A,Γ,B) is a Moore automaton, if and only if it can be extended to the automaton (A,Γ^1,B). The same, as in the case of pure automata (statement 3.7, Chapter 1), is the proof of the following

Proposition 1.1. *If $\rho=(\rho_1,\rho_2,\rho_3)$ is a congruence of a linear Moore automaton $\mathfrak{A}=(A,\Gamma,B)$, with ψ being the determining mapping for which the condition $\rho_1\subset\mathrm{Ker}\psi$ is true then the quotient automaton $\mathfrak{A}/\rho$ is also a Moore automaton.*

Theorem 1.2. *In order that a linear automaton $\mathfrak{A}=(A,\Gamma,B)$ over the field K be a Moore automaton, it is necessary and sufficient that the equality $\sum_{i=1}^{n}\lambda_i(a_i\circ\gamma_i)=0$, $a_i\in A$, $\gamma_i\in\Gamma$, $\lambda_i\in K$, implies $\sum_{i=1}^{n}\lambda_i(a_i*\gamma_i)=0$.*

Proof. Let $\mathfrak{A}$ be a linear Moore automaton with the determining mapping $\psi: A \to B$ and let $\sum_{i=1}^{n}\lambda_i(a_i\circ\gamma_i)=0$. Then

$$\sum_{i=1}^{n}\lambda_i(a_i\circ\gamma_i)^{\psi} = \sum_{i=1}^{n}\lambda_i(a_i*\gamma_i)=0.$$

In order to prove the inverse statement, denote by $A\circ\Gamma$ the linear hull of all the elements of the form $a\circ\gamma$, $a\in A$, $\gamma\in\Gamma$, and by A_0 the direct complement of $A\circ\Gamma$ in A. Then $A=A_0\oplus A\circ\Gamma$. Each element h of $A\circ\Gamma$ can be written in the form $h=\sum_{i=1}^{n}\lambda_i(a_i\circ\gamma_i)$ (Clearly, this form is not unique). Define the mapping $\psi\colon A\to B$ by the rule: if $a_0\in A_0$ and $h=\sum^{n}\alpha_i(a_i\circ\gamma_i)\in A\circ\Gamma$, we set

$$a_0^{\psi}=0 \text{ and } h^{\psi}=\Big(\sum_{i=1}^{n}\alpha_i(a_i\circ\gamma_i)\Big)^{\psi}=\sum_{i=1}^{n}\alpha_i(a_i*\gamma_i).$$

By virtue of the condition, the definition does not depend on the form of presentation of the element h. Obviously, the mapping ψ is a linear one, and $(a\circ\gamma)^{\psi}=a*\gamma$ holds for each $a\in A$, $\gamma\in\Gamma$. Thus, (A,Γ,B) is a Moore automaton.

Theorem 1.3. *If K is a field and semigroup Γ is finite then the following conditions are equivalent:*

1. *Semigroup algebra $K\Gamma$ has a right unit ε.*
2. *Every linear automaton (A,Γ,B) is a Moore automaton*
3. *If $|B|>1$ then $Atm^2(\Gamma,B)$ is a Moore automaton.*

Proof. 1 ⇒ 2. Extending any automaton (A,Γ,B) to automaton $(A,K\Gamma,B)$ one can define the determining mapping by the rule $a^{\psi}=a*\varepsilon$, $a\in A$.

Clearly, 2 implies 3.

3 ⇒ 1. Let $Atm^2(\Gamma,B)=(B^{\Gamma},\Gamma,B)$ be a Moore automaton. We can consider the case of one-dimensional space B. One can identify B^{Γ} with the linear space $Hom(K\Gamma,B)$. Let $\psi\colon Hom(K\Gamma,B)\to B$ be the determining mapping of automaton $Atm^2(\Gamma,B)$. With each element $u\in K\Gamma$ we associate an element $\bar{u}\in Hom(Hom(K\Gamma,B),B)$ by the rule $\bar{u}(\varphi)=\varphi(u)$ for any $\varphi\in Hom(K\Gamma,B)$. The mapping $u\to\bar{u}$ is an injective one. Since B is one-dimensional linear space, $\dim(K\Gamma)=\dim(Hom(Hom(K\Gamma,B),B))$. It follows that mapping $u\to\bar{u}$ is an isomorphism. Take an element $v\in K\Gamma$ such that $\psi=\bar{v}$. Then

$$\varphi(\gamma)=\varphi*\gamma=\bar{v}(\varphi\circ\gamma)=(\varphi\circ\gamma)(v)=\varphi(\gamma v),\ \text{for all } \varphi\in Hom(K\Gamma,B),\ \gamma\in\Gamma.$$

If $\gamma v\neq\gamma$ then there is $\varphi\in Hom(K\Gamma,B)$ that $\varphi(\gamma v)\neq\varphi(\gamma)$. It follows that $\gamma v=\gamma$, hence v is a right unit in $K\Gamma$.

Similar to the case of pure automata one can show that for linear Moore automaton (A,Γ,B) the kernel of automaton representation of the semigroup Γ coincides with the kernel of the representation (A,Γ).

Linear group automata, i.e. the automata with their input sets being groups, are a particular case of linear Moore automata.

Let $\mathfrak{A}=(A,\Gamma,B)$ be a group automaton, and the mapping $\beta:\Gamma \to \mathrm{Hom}(A,B)$ be determined by the automaton representation of the group Γ. We denote by Φ the set of all the elements $\gamma\in\Gamma$ such that $a*\gamma=a*\varepsilon$ for each $a\in A$. Show that Φ is a subgroup of Γ, and that it does not have to be a normal subgroup of the group Γ.

Denote by A_0 the kernel of a linear mapping of A to B, defined by the unit ε of the group Γ: $A_0=\{a\in A \mid a*\varepsilon=0\}$, and by Σ the normalizer of the subspace A_0 in the representation (A,Γ): $\Sigma=\{\sigma\in\Gamma \mid A_0\circ\sigma=A_0\}$.

Let $\varphi\in\Phi$. Then for each $a\in A$ it is true that $a*\varphi=a*\varepsilon$ and $a*\varphi-a*\varepsilon=0$. Since $a*\varphi=a*\varphi\varepsilon=(a\circ\varphi)*\varepsilon$, then it follows from the above that $(a\circ\varphi)*\varepsilon-a*\varepsilon=0$ and $(a\circ\varphi-a)*\varepsilon=0$. It means that

$$a\circ\varphi-a\in A_0 \tag{1.1}$$

Now, if φ_1 and φ_2 are two elements of Φ, then

$$(a\circ\varphi_1\varphi_2-a)*\varepsilon=(a\circ\varphi_1\varphi_2-a\circ\varphi_1)+(a\circ\varphi_1-a))*\varepsilon=$$

$$=(a\circ\varphi_1\varphi_2-a\circ\varphi_1)*\varepsilon+(a\circ\varphi_1-a)*\varepsilon=(\bar{a}\circ\varphi_2-\bar{a})*\varepsilon+(a\circ\varphi_1-a)*\varepsilon.$$

The first summand of the sum equals zero, since $\bar{a}\in A$ and $\varphi_2\in\Phi$. In its turn, it follows from the inclusion $\varphi_1\in\Phi$ that the second summand equals zero. Thus $(a\circ\varphi_1\varphi_2-a)*\varepsilon=0$ and $\varphi_1\varphi_2\in\Phi$. In a similar way, if $\varphi\in\Phi$ then also $\varphi^{-1}\in\Phi$. So, Φ is a subgroup in Γ.

If $\rho=\mathrm{Ker}\beta$, then, by the definition of Φ, it is a coset of this relation containing ε: $\Phi=[\varepsilon]_\rho$. Inclusion (1.1) means, in particular, that Φ belongs to Σ, and that Φ is the kernel of the representation $(A/A_0,\Sigma)$. Thus, Φ is a normal subgroup in Σ. However, Φ can be non-invariant in Γ. Consider a corresponding

Example. Let A be an n-dimensional vector space over the field of real numbers, Γ is the group of all the automorphisms of the space, A_0 a subspace in A, and $B = A/A_0$ is the quotient space of A by A_0. If $a\in A, \gamma\in\Gamma$ then $a\circ\gamma$ is defined as the image of the automorphism γ. Further, assume $a*\varepsilon=\bar{a}$, where $\bar{a}$ is the image of a under the natural homomorphism A onto $B=A/A_0$. Finally, $a*\gamma$ is defined according to the rule $a*\gamma=(a\circ\gamma)*\varepsilon$. We get a group automaton (A,Γ,B). It is easy to show that in this automaton

Φ is not an invariant subgroup in Γ. To do that, it suffices to choose a basis of the space A passing through A_0, to consider the matrix forms of elements of Φ and to select elements $\gamma\in\Gamma$ and $\varphi\in\Phi$ such that $\gamma\varphi\neq\varphi'\gamma$ for no one φ' from Φ. The subspace A_0 can be taken one-dimensional.

3.1.3. Biautomata

The concept of biautomaton is another generalization of the concept of a semigroup linear automaton. We speak about the situation when input signals act not only on states of an automaton, but also on its outputs.

A *biautomaton* is a system consisting of three basic sets A,Γ,B where A,B are vector spaces over field K (modules over a ring), Γ is a semigroup, and of three representations $\alpha_1:\Gamma \to \mathrm{End}A$, $\beta:\Gamma \to \mathrm{Hom}(A,B)$, $\alpha_2:\Gamma \to \mathrm{End}B$. Operations defined by the representations α_1 and β will be denoted by the symbols $\circ$ and $*$. We denote by $\cdot$ the operation defined by the representation α_2. Upon this, the following conditions should be fulfilled: if $a\in A$, $b\in B$, $\gamma_1,\gamma_2\in\Gamma$, then

$$a\circ\gamma_1\gamma_2=(a\circ\gamma_1)\circ\gamma_2; \quad b\cdot\gamma_1\gamma_2=(b\cdot\gamma_1)\cdot\gamma_2;$$
$$a*\gamma_1\gamma_2=(a\circ\gamma_1)*\gamma_2+(a*\gamma_1)\cdot\gamma_2 \tag{1.2}$$

An automaton is a particular case of biautomaton: it is sufficient to suppose that elements of Γ act in B as zeros, i.e. for each $b\in B$, $\gamma\in\Gamma$, holds $b\cdot\gamma=0$.

A *coautomaton* is another special case of biautomata: elements of Γ act in a zero way in A.

As well as an automaton, a biautomaton (A,Γ,B) is called *finite-dimensional*, if A and B are finite-dimensional linear spaces.

A *homomorphism of biautomata*

$$\mu:\mathfrak{A}=(A,\Gamma,B) \to \mathfrak{A}'=(A',\Gamma',B')$$

is a collection of three mappings $\mu_1:A \to A'$, $\mu_2:\Gamma \to \Gamma'$, $\mu_3:B \to B'$, where μ_1,μ_3 are linear mappings, μ_2 is a homomorphism of semigroups. The following consistency conditions

$$(a\circ\gamma)^{\mu_1}=a^{\mu_1}\circ\gamma^{\mu_2},\quad (a*\gamma)^{\mu_3}=a^{\mu_1}*\gamma^{\mu_2},\quad (b\cdot\gamma)^{\mu_3}=b^{\mu_3}\cdot\gamma^{\mu_2}$$

must be satisfied. One can consider separately homomorphisms in states, in input and output signals. As in the case of automata, here the canonical decomposition also takes place.

Consider the *universal biautomaton* $\mathrm{Atm}^1(A,B)$. Let us take the Cartesian product

$$\mathrm{End}^b(A,B)=\mathrm{End}A\times\mathrm{Hom}(A,B)\times\mathrm{End}B$$

and define multiplication according to the rule

$$(\sigma'_A,\varphi',\sigma'_B)(\sigma''_A,\varphi'',\sigma''_B)=(\sigma'_A\sigma''_A,\sigma'_A\varphi''+\varphi'\sigma''_B,\sigma'_B\sigma''_B) \qquad (1.3)$$

with σ'_A,σ''_A from EndA; φ',φ'' from Hom(A,B) and σ'_B,σ''_B from EndB. Assosiativity of the multiplication is easily verified, hence $\mathrm{End}^b(A,B)$ is a semigroup. If, further, $\gamma=(\sigma_A,\varphi,\sigma_B)\in\mathrm{End}^b(A,B)$, $a\in A$, $b\in B$, we set $a\circ\gamma=a\sigma_A$, $a*\gamma=a\varphi$, $b\cdot\gamma=b\sigma_B$. It follows from the definitions of the operations that conditions (1.2) are satisfied. In this way, we get the *biautomaton*

$$\mathrm{Atm}^1(A,B)=(A,\mathrm{End}^b(A,B),B).$$

It is a *terminal object in the category of biautomata, with fixed* A,B *and homomorphisms in inputs as morphisms.*

It is easy to understand that to define an arbitrary biautomaton (A,Γ,B), it is sufficient to point out a homomorphism $\tau:\Gamma\rightarrow\mathrm{End}^b(A,B)$ determining all the actions of the semigroup Γ. This homomorphism is called the *biautomaton representation of the semigroup* Γ. It is interesting to notice that while the semigroup End(A,B) has no unit, the semigroup $\mathrm{End}^b(A,B)$ has one, $(\varepsilon_A,0,\varepsilon_B)$, and there are many inverse elements in it. It is precisely the set of elements of the form $\gamma=(\pi_A,\varphi,\pi_B)$ with the inverse π_A and π_B.

If in $\mathrm{End}^b(A,B)$ the addition and multiplication by a scalar of the field K are defined componentwise, then $\mathrm{End}^b(A,B)$ becomes a linear associative algebra over K. The homomorphism $\tau:\Gamma\rightarrow\mathrm{End}^b(A,B)$ can be extended up to the homomorphism $\tau:K\Gamma\rightarrow\mathrm{End}^b(A,B)$, and this extends the biautomaton (A,Γ,B) to the biautomaton $(A,K\Gamma,B)$ with the semigroup algebra of the input signals.

The *universal biautomaton* $\mathrm{Atm}^2(\Gamma,B)$ is being constructed on the same sets B^Γ,Γ,B, as the corresponding linear automaton (see item 3.1.1), but it is supposed that the representation (B,Γ) is defined. This fact changes the definition of the operation Γ in B^Γ. For any $\varphi\in B^\Gamma$; $\gamma,x\in\Gamma$, we set:

$$(\varphi\circ\gamma)(x)=\varphi(\gamma x)-\varphi(\gamma)\cdot x \qquad (1.4)$$

In this way, a representation (B^Γ,Γ) is defined. The operation $*$ is defined as follows: $\varphi*\gamma=\varphi(\gamma)$. We get the *biautomaton* $\mathrm{Atm}^2(\Gamma,B)$ which is the *terminal object in the category biautomata having the representation* (B,Γ) *and homomorphisms with respect to states as morphisms.*

Originally, the *universal biautomaton* $\mathrm{Atm}^3(A,\Gamma)$ with the given representation (A,Γ) is constructed similarly to an universal linear automaton of the third form (see 3.1.1). Take a free object over the Cartesian product $A\times\Gamma$, i.e. the vector space H with the basis $A\times\Gamma$ (when we deal with modules over a commutative ring, we take a free module over the ring, which is generated by the set $A\times\Gamma$), then take a subspace (submodule) H_0, generated by all the elements of the form

$$\alpha(a_1,\gamma)+\beta(a_2,\gamma)-(\alpha a_1+\beta a_2,\gamma),\ \alpha,\beta\in K,\ a_1,a_2\in A,\ \gamma\in\Gamma. \qquad (1.5)$$

It can be proved that the corresponding quotient space H/H_0 is isomorphic to the tensor product of linear K-spaces A and $K\Gamma$. Indeed, assign to each free generator (a,γ) the element $a\otimes\gamma$ in $A\otimes K\Gamma$. This gives the canonical epimorphism $\mu:H \longrightarrow A\otimes K\Gamma$. Let H_1 be the kernel of μ. Show, that $H_1=H_0$. It is clear, that $H_0\subset H_1$. Let $h_1=\sum\alpha_i(a_i,\gamma_i)\in H_1$. This element can be rewritten in the form $h_1=\sum(a_i,\gamma_i)+h_0$, where $h_0\in H_0$ and all γ_i are different. Then, $h_1^\mu=\sum a_i\otimes\gamma_i=0$. Since the elements γ_i form a basis of $K\Gamma$, it follows that always $a_i=0$. Hence, $h_1\in H_0$ and $H_1=H_0$. Denote $H/H_0=A\underset{K}{\otimes}K\Gamma$. Its elements can be written as $\sum a_i\otimes u_i$, $a_i\in A$, $u_i\in K\Gamma$. Define the operation $*$: $a*\gamma=a\otimes\gamma$. Finally, define the operation of Γ in $A\underset{K}{\otimes}K\Gamma$: if $a_i\in A$, $u_i\in K\Gamma$, $\gamma\in\Gamma$, $\sum_i a_i\otimes u_i\in A\underset{K}{\otimes}K\Gamma$ then

$$(a\otimes u)\cdot\gamma=a\otimes u\gamma-(a\circ u)\otimes\gamma. \qquad (1.6)$$

Conditions (1.2) are easily verified: we obtain the *biautomaton*

$Atm^3(A,\Gamma)=(A,\Gamma,A\underset{K}{\otimes}K\Gamma)$, which is an *initial object in the category of biautomata with the given representation* (A,Γ) *and homomorphisms in outputs as morphisms.* If, while constructing $A\underset{K}{\otimes}K\Gamma$, the subspace H_0 is generated not only by the elements of form (1.5), but by all the elements of the form $(a,\gamma_1\gamma_2)(a\circ\gamma_1,\gamma_2)$, as it was done in 3.1.1, then we get $A\underset{K\Gamma}{\otimes}K\Gamma$ instead of $A\underset{K}{\otimes}K\Gamma$. The operation $\cdot$ in it becomes the zero one, and the universal biautomaton $Atm^3(A,\Gamma)$ will be transformed into a corresponding universal linear automaton.

The definition of the congruence of the biautomaton $\mathfrak{A}=(A,\Gamma,B)$ is obtained from the corresponding definition of the linear automaton. Add the condition:

$$\text{if } \gamma_1\rho_2\gamma_2,\ b_1\rho_3 b_2,\ b_i\in B,\ \gamma_i\in\Gamma,\ i=1,2 \text{ then } (b_1\cdot\gamma_1)\rho_3(b_2\cdot\gamma_2).$$

A quotient automaton is being constructed on the basis of the congruence $\rho=(\rho_1,\rho_2,\rho_3)=(A_0,\rho_2,B_0)$. We use here the term "quotient automaton", not the "quotient biautomaton". The latter is less convenient. Just in the same way, we will speak about a subautomaton of a biautomaton, not about a subbiautomaton.

3.1.4. Automata, biautomata and representations

Each automaton $\mathfrak{A}=(A,\Gamma,B)$ comprises a representation (A,Γ). Here we mean to consider another, one-to-one relation. Let the initial category $\mathcal{K}$ be the variety of Ω-algebras, and $\mathfrak{A}$ an automaton over $\mathcal{K}$. The Cartesian product $A\times B$ is an algebra in $\mathcal{K}$. For each $\gamma\in\Gamma$ and $(a,b)\in A\times B$, we set: $(a,b)\circ'\gamma=(a\circ\gamma,a*\gamma)$. It is easy to verify that it defines a representation $(A\times B,\Gamma)$. Let, on the other hand, $(A\times B,\Gamma)$ be such a representation, that if $(a,b)\circ'\gamma=(a',b')$ then a' and b' depend only on a and γ. Denote $a'=a\circ\gamma$ and $b'=a*\gamma$. This defines the automaton (A,Γ,B). In this way we come to one-to-one correspondence between automata over $\mathcal{K}$ and representations of a special form.

Let $\mathcal{K}$ be the category of vector spaces (modules) and $\mathfrak{A}=(A,\Gamma,B)$ a biautomaton over $\mathcal{K}$. Define the action of Γ in the direct sum $A\oplus B$:

$$\text{if } a+b\in A\oplus B, \gamma\in\Gamma \text{ then } (a+b)\circ'\gamma=a\circ\gamma+a*\gamma+b\cdot\gamma.$$

This defines a representation $(A \oplus B, \Gamma)$ and obviously, B is invariant with respect to action of Γ. Inversely, if the action $\circ'$ of the semigroup Γ in $A \oplus B$ is specified, and the subspace B is invariant in this representation, we get the biautomaton (A, Γ, B). The operations in it are defined in a following way: if $a \circ' \gamma = a' + b'$, $a, a' \in A, b' \in B, \gamma \in \Gamma$ then $a \circ \gamma = a'$, $a * \gamma = b'$, $b \cdot \gamma = b \circ' \gamma$, $b \in B$.

For the universal biautomaton $Atm^1(A, B)$ the corresponding representation $(A \oplus B, End^1(A, B))$ can be illustrated by the following matrix picture:

$$\begin{pmatrix} EndA & Hom(A,B) \\ & \\ 0 & End\ B \end{pmatrix}$$

3.1.5. Moore biautomata

Let $(A, \Gamma, \circ)$ and $(B, \Gamma, \cdot)$ be two representations and $\psi : A \to B$ be a linear mapping. For arbitrary $a \in A$ and $\gamma \in \Gamma$ we set:

$$a * \gamma = (a \circ \gamma)^{\psi} - a^{\psi} \cdot \gamma$$

It can be shown directly that in such a way we always get a biautomaton , called *Moore biautomaton*. It is clear, that if ψ commutates with the action of Γ, the corresponding operation * becomes a zero one.

Let $\mathfrak{A} = (A, \Gamma, B)$ be an arbitrary biautomaton and $\psi : A \to B$ be a linear mapping. By M_{ψ} we denote the set of all $\gamma \in \Gamma$ for which $a * \gamma = (a \circ \gamma)^{\psi} - a^{\psi} \cdot \gamma$, $a \in A$ is satisfied. A straightforward calculation shows that M_{ψ} is a subsemigroup; by this, the Moore part (A, M_{ψ}, B) with the given ψ is selected. We apply this to $Atm^1(A, B)$. For given ψ the element $\gamma = (\sigma_1, \varphi, \sigma_2)$ belongs to M_{ψ}, if $a\varphi = a\sigma_1\psi - a\psi\sigma_2$; $\varphi = \sigma_1\psi - \psi\sigma_2$. Thus, neither $\psi : A \to B$ does make $Atm^1(A, B)$ a Moore biautomaton.

Proposition 1.4. *Any biautomaton* $\mathfrak{A} = (A, \Gamma, B)$ *is a homomorphic in states image of a Moore biautomaton.*

Proof. Take the corresponding triangular representation $(A \oplus B, \Gamma)$. Together with (B, Γ) and the projection $\psi : A \oplus B \to B$ compose the Moore biautomaton $(A\oplus B,\Gamma,B)$. Take further the projection $\nu:A\oplus B\to A$. We have:

$$((a+b)\circ\gamma)^{\nu}=(a\circ\gamma+a*\gamma+b\bullet\gamma)^{\nu}=a\circ\gamma=(a+b)^{\nu}\circ\gamma;$$

$$((a+b)*\gamma)=((a+b)\circ\gamma)^{\psi}-(a+b)^{\psi}\bullet\gamma=a*\gamma+b\bullet\gamma-b\bullet\gamma=a*\gamma=(a+b)^{\nu}*\gamma.$$

Corollary. *Homomorphisms in states do not preserve the Moore property of biautomata.*

However, it is easy to verify that homomorphisms in output signals do preserve the property. Indeed, if $\mathfrak{A}=(A,\Gamma,B)$ is a Moore biautomaton with the mapping $\psi:A\to B$, and $\nu:B\to B'$ defines a homomorphism in output signals: $\mathfrak{A}=(A,\Gamma,B)\to\mathfrak{A}'=(A,\Gamma,B')$, then the mapping $\psi\nu:A\to B'$ makes the second biautomaton a Moore one:

$$a\bar{*}\gamma=(a*\gamma)^{\nu}=((a\circ\gamma)^{\psi}-a^{\psi}\bullet\gamma)^{\nu}=(a\circ\gamma)^{\psi\nu}-a^{\psi\nu}\bullet\gamma.$$

Here $\bar{*}$ is the operation in $\mathfrak{A}'$

Proposition 1.5. *If* $\mathfrak{A}=(A,\Gamma,B)$ *is a biautomaton with free over the set Z representation* (A,Γ), *then* $\mathfrak{A}$ *is a Moore biautomaton.*

Proof. Take an arbitrary mapping $\psi:Z\to B$. For each $z\circ u$, $u\in K\Gamma^1$ set: $(z\circ u)^{\psi}=z*u+z^{\psi}\bullet u$. Here, if $u=1$ then assume $z*u=0$, $b\bullet u=b$, and the given ψ extends the initial mapping ψ. We got a linear mapping of each $zK\Gamma^1$ into B, which is extended up to the linear mapping $\psi:A\to B$. The definition of the Moore condition can be rewritten in the form $(a\circ\gamma)^{\psi}=a*\gamma+a^{\psi}\bullet\gamma$. Take now $a=z\circ u$; then

$$(a\circ\gamma)^{\psi}=(z\circ u\gamma)^{\psi}=z*u\gamma+z^{\psi}\bullet u\gamma=(z\circ u)*\gamma+(z*u)\bullet\gamma+z^{\psi}\bullet u\gamma=$$

$$=a*\gamma+((z\circ u)^{\psi}-z^{\psi}\bullet u)\bullet\gamma+z^{\psi}\bullet u\gamma=a*\gamma+a^{\psi}\bullet\gamma, \text{ as required.}$$

Now we will generalize this statement in the form of the following sign of a Moore biautomaton.

Proposition 1.6. *The biautomaton* $\mathfrak{A}=(A,\Gamma,B)$ *is a Moore one if and only if there exists a system Z of* $K\Gamma^1$*-generators of A and the mapping* $\psi:Z\to B$, *such that*

$$\sum_i z_i\circ u_i=0 \Rightarrow \sum_i(z_i*u_i+z_i^{\psi}\bullet u_i)=0$$

for any $z_i\in Z$ *and* $u_i\in K\Gamma^1$.

Proof. The if direction will be obtained, if we take Z=A and the corresponding ψ defining a Moore automaton. One should keep in mind that not only a $K\Gamma$, but also a $K\Gamma^1$-biautomaton corresponds to the biautomaton with the semigroup Γ. If (A,Γ,B) is a Moore biautomaton with the mapping $\psi: A \to B$, then $(A,K\Gamma^1,B)$ is a Moore biautomaton: for each $a\in A$ and $u\in K\Gamma^1$, $a*u=(a\circ u)^{\psi}-a^{\psi}\cdot u$ holds.

Let us verify the sufficiency. Let Z and $\psi: Z \to B$ be given. Each $a\in A$ can be written in the form $a=\sum_i z_i\circ u_i$, $u_i\in K\Gamma^1$ Set $a^{\psi}=\sum_i (z_i*u_i+z_i^{\psi}\cdot u_i)$. It follows from the conditions that mapping ψ is correctly defined, and that it is a linear one. A direct calculation yields

$$(a\circ\gamma)^{\psi}=\sum_i(z_i\circ u_i\gamma)^{\psi}=\sum_i(z_i*u_i\gamma+z_i^{\psi}\cdot u_i\gamma)=\sum_i(z_i\circ u_i)*\gamma+\sum_i(z_i*u_i)\cdot\gamma+$$

$$+\sum_i z_i^{\psi}\cdot u_i\gamma=a*\gamma+(\sum_i(z_i*u_i+z_i^{\psi}\cdot u_i))\gamma=a*\gamma+a^{\psi}\cdot\gamma.$$

Note the following problem.

Is it possible to find, in terms of semigroups or semigroup algebras, a class of semigroup Γ, for which all the biautomata over the given field K are the Moore biautomata?

A similar problem for automata is solved in Section 1.3 and in item 2 of the current Section.

Proposition 1.7. *Each biautomaton* (A,Γ,B) *can be isomorphically embedded as a subautomaton in output signals into Moore biautomaton.*

Proof. First of all, note that for each biautomaton (A,Γ,B) there exists the representation $(A\oplus B,\Gamma)$ defined by the formula

$$(a+b)\circ\gamma=a\circ\gamma-a*\gamma+b\cdot\gamma.$$

Now take a copy A' of the space A with the isomorphism $\psi: A \to A'$ and consider also a biautomaton (A',Γ,B) copying the initial one, (A,Γ,B). Here $a^{\psi}\circ\gamma=(a\circ\gamma)^{\psi}$ and $a^{\psi}*\gamma=a*\gamma$. Using the biautomaton (A',Γ,B) construct the representation $(A'\oplus B,\Gamma)$ according to the rule just mentioned. We combine the two representations (A,Γ) and $(A'\oplus B,\Gamma)$ into a biautomaton $(A,\Gamma,A'\oplus B)$, defining the operation * as in the case of the given (A,Γ,B). (A,Γ,B) is a subautomaton in $(A,\Gamma,A'\oplus B)$. We show now that the latter satisfies the Moore condition, with the mapping $\psi: A \rightarrow A' \subset A' \oplus B$. Indeed,

$$a^{\psi} \cdot \gamma = a^{\psi} \circ \gamma - a^{\psi} * \gamma = (a \circ \gamma)^{\psi} - a * \gamma;$$

$$a * \gamma = (a \circ \gamma)^{\psi} - a^{\psi} \cdot \gamma.$$

Corollary. *A subautomaton in output signals of a Moore biautomaton is not necessary a Moore one.*

On the other hand, it is obvious that a subautomaton in states keeps the Moore property.

3.1.6. Free linear automata and biautomata

The definitions of a free linear Γ-automaton and a free Γ-biautomaton with the system of generators (Z, Y) are submitted to the general definition of free algebra in a variety of many-sorted algebras (see, for instance, the corresponding definitions in the pure case). We will denote the free linear Γ-automaton and Γ-biautomaton by $Atm_{\Gamma}^{\ell}(Z,Y)$ and $Atm_{\Gamma}^{b}(Z,Y)$, respectively. In the places where it is clear what is the matter, indices ℓ and b can be omitted.

Construct a free linear Γ-automaton with the system of free generators (Z, Y). Denote by H the linear space over the field K generated by the Cartesian product $Z \times \Gamma^1$, by Φ the linear space generated by the set $(Z \times \Gamma) \cup Y$. The automaton (H, Γ, Φ) with operations $\circ$ and $*$, defined according to the rules

$$\Big(\sum_i \alpha_i (z_i, \gamma_i)\Big) \circ \gamma = \sum_i \alpha_i (z_i, \gamma_i \gamma)$$

$$\Big(\sum_i \alpha_i (z_i, \gamma_i)\Big) * \gamma = \sum_i \alpha_i (z_i, \gamma_i \gamma); \quad \alpha_i \in K, \ z_i \in Z, \ \gamma_i \in \Gamma^1, \ \gamma \in \Gamma,$$

is a free linear Γ-automaton with the system of free generators (Z, Y). Indeed, let $\mathfrak{A} = (A, \Gamma, B)$ be an arbitrary linear Γ-automaton and $\mu_1 : Z \rightarrow A$, $\mu_3 : Y \rightarrow B$ an arbitrary pair of mappings. Extensions of these mappings μ_1 and μ_3 to linear ones $\tilde{\mu}_1 : H \rightarrow A$ and $\tilde{\mu}_3 : \Phi \rightarrow B$ is defined as follows: if $h = \sum_i \alpha_i (z_i, \gamma_i)$, $\varphi = \sum_i \alpha_i (z_i, \gamma_i) + \sum_i \beta_j y_j$ then we set

$$h^{\tilde{\mu}_1}=\sum_i \alpha_i (z_i^{\mu_1} \circ \gamma_i), \quad \varphi^{\tilde{\mu}_3}=\sum_i \alpha_i (z_i^{\mu_1} * \gamma_i)+\sum_i \beta_j y_j^{\mu_3}.$$

The triplet $(\tilde{\mu}_1, \varepsilon_\Gamma, \tilde{\mu}_3)$ where ε_Γ is the identity mapping of the semigroup Γ, is homomorphism of the automaton (H,Γ,Φ) into the automaton $\mathfrak{A}$. The uniqueness of the homomorphism is obvious.

The linear space H can be regarded as a $K\Gamma^1$-module with the basis Z, i.e $H=ZK\Gamma^1$; the linear space Φ as the direct sum of a $K\Gamma$-module over the set of generators Z and of the linear space with the basis Y, $\Phi=ZK\Gamma \oplus KY$. In the latter case, the element $h \in H$ can be written in the form $h=\sum_i z_i u_i$, $z_i \in Z$, $u_i \in K\Gamma^1$, while the element $\varphi \in \Phi$ in the form

$$\varphi=(\sum_i z_i v_i + \alpha_i y_i)=\sum_i z_i v_i + \sum_i \alpha_i y_i, \quad z_i \in Z, \quad v_i \in K\Gamma, \quad \alpha_i \in K, \quad y_i \in Y.$$

Then the operations $\circ$ and $*$ have the form

$$(\sum_i z_i u_i) \circ \gamma = \sum_i z_i (u_i \gamma)$$

$$(\sum_i z_i u_i) * \gamma = \sum_i z_i (u_i \gamma), \quad u_i \in K\Gamma^1, \quad z_i \in Z, \quad \gamma \in \Gamma.$$

Now we construct a realization of a free Γ-biautomaton with the system of free generators (Z,Y). As in the previous case, we take first the linear space $H=K(Z \times \Gamma^1)=ZK\Gamma^1$. Then, consider the tensor product $\Phi_0=H \otimes K\Gamma$ of linear spaces H and $K\Gamma$, and the linear space $K(Y \times \Gamma^1)=YK\Gamma^1=\Phi_1$. By Φ we denote the direct sum

$$\Phi=\Phi_0 \oplus \Phi_1=(H \otimes K\Gamma) \oplus YK\Gamma^1.$$

In the triplet (H,Γ,Φ) we introduce the following operations $\circ$, $*$, $\cdot$:

$$\circ : \quad (\sum_i z_i u_i) \circ \gamma = \sum_i z_i (u_i \gamma),$$

$$* : \quad (\sum_i z_i u_i) * \gamma = (\sum_i z_i u_i) \otimes \gamma,$$

$$\cdot : \quad (\sum_i y_i u_i) \cdot \gamma = \sum_i y_i (u_i \gamma); \quad (\sum_i h_i \otimes u_i) \cdot \gamma = \sum_i (h_i \otimes u_i \gamma - h_i u_i \otimes \gamma),$$

where $u_i \in K\Gamma$, $z_i \in Z$, $\gamma \in \Gamma$, $y_i \in Y$.

(H,Γ,Φ) is a biautomaton, with respect to the introduced operations, and this biautomaton is free, with the free system of generators (Z,Y). If the set Y is empty, the constructed biautomaton, (H,Γ,Φ), is an universal biautomaton of the third type $Atm^3(H,\Gamma)$. A free linear automaton with variable semigroup of input signals and the system of free generators (Z,X,Y) is constructed in the following way: first we take a free semigroup $F=F(X)$ with the system of free generators X, and then the automaton $Atm_F^{\ell}(Z,Y)$. The required free automaton is thus obtained and it is denoted by $Atm^{\ell}(Z,X,Y)$. Correspondingly, a free biautomaton with the system of free generators (Z,X,Y) is the biautomaton $Atm_F^{b}(Z,Y)$, where $F=F(X)$. This biautomaton is denoted by $Atm^{b}(Z,X,Y)$.

3.2. Constructions. Decomposition of automata

3.2.1. Constructions. Triangular product

We consider the basic construction, called the triangular product of automata, playing the role in the theory of linear automata, which is similar to that of wreath product in the theory of pure automata.

First define the triangular product of exact linear semiautomata (representations). Let (A_1,Γ_1) and (A_2,Γ_2) be two such semiautomata. Denote by $\Phi=\mathrm{Hom}(A_2,A_1)$ the set of all linear mappings from A_2 to A_1 which is considered as an additive Abelian group, and by Γ the semigroup of generalized matrices of the form

$$\begin{pmatrix} \gamma_2 & \varphi \\ 0 & \gamma_1 \end{pmatrix}, \quad \gamma_1\in\Gamma_1, \gamma_2\in\Gamma_2, \quad \varphi\in\mathrm{Hom}(A_2,A_1) \qquad (2.1)$$

with the usual matrix product :

$$\begin{pmatrix} \gamma_2 & \varphi \\ 0 & \gamma_1 \end{pmatrix} \cdot \begin{pmatrix} \gamma'_2 & \varphi' \\ 0 & \gamma'_1 \end{pmatrix} = \begin{pmatrix} \gamma_2\gamma'_2 & \gamma_2\varphi'+\varphi\gamma'_1 \\ 0 & \gamma_1\gamma'_1 \end{pmatrix}$$

Then the semiautomaton $(A_1\oplus A_2,\Gamma)$ is defined by the natural actions of elements of Γ on elements of $A_1\oplus A_2$: if $(a_1,a_2)\in A_1\oplus A_2$, and $\gamma\in\Gamma$ is an element of the form (2.1), then $(a_1,a_2)\circ\gamma=(a_1\circ\gamma_1+a_2\varphi, a_2\circ\gamma_2)$. The semiautomaton thus obtained is called the *triangular product of the semiau-*

tomata (A_1,Γ_1) and (A_2,Γ_2). It is denoted as

$$(A_1,\Gamma_1)\nabla(A_2,\Gamma_2).$$

The semigroup Γ can be presented in the form of a Cartesian product $\Gamma_1\times\mathrm{Hom}(A_2,A_1)\times\Gamma_2$, with the multiplication:

$$(\gamma_1,\varphi,\gamma_2)(\gamma_1',\varphi',\gamma_2')=(\gamma_1\gamma_1',\gamma_2\varphi'+\varphi\gamma_1',\gamma_2\gamma_2');$$

$$\gamma_1,\gamma_1'\in\Gamma_1;\ i=1,2;\ \varphi,\varphi'\in\Phi=\mathrm{Hom}(A_2,A_1).$$

Correspondingly, the operation $\circ$ in the triangular product $(A_1\oplus A_2,\Gamma)=(A_1\oplus A_2,\Gamma_1\times\Phi\times\Gamma_2)$ can be written as

$$(a_1,a_2)\circ(\gamma_1,\varphi,\gamma_2)=(a_1\circ\gamma_1+a_2\varphi,a_2\circ\gamma_2). \tag{2.2}$$

We defined the triangular product of exact semiautomata. In order to define the triangular product of arbitrary semiautomata, one should indicate how the "products" $\gamma_2\circ\varphi$ and $\varphi\circ\gamma_1$ should be understood; $\gamma_1\in\Gamma_1$, $i=1,2$, $\varphi\in\Phi$. Define actions of Γ_1 on Φ from the right and Γ_2 on Φ from the left according to the rule: if $\varphi\in\Phi,\gamma_1\in\Gamma_1,\gamma_2\in\Gamma_2,a\in A_2$, then $\varphi\circ\gamma_1$ is an element of Φ such that

$$a(\varphi\circ\gamma_1)=(a\varphi)\circ\gamma_1, \tag{2.3}$$

and $\gamma_2\circ\varphi$ is an element of Φ such that

$$a(\gamma_2\circ\varphi)=(a\circ\gamma_2)\varphi. \tag{2.4}$$

Now the *triangular product of arbitrary semiautomata* is defined in a manner similar to that in the case of exact automata

$$(A_1,\Gamma_1)\nabla(A_2,\Gamma_2)=(A_1\oplus A_2,\Gamma_1\times\Phi\times\Gamma_2)=(A_1\oplus A_2,\Gamma),$$

where $\Phi=\mathrm{Hom}(A_2,A_1)$. $\Gamma=\Gamma_1\times\Phi\times\Gamma_2$ is a semigroup with the multiplication:

$$(\gamma_1,\varphi,\gamma_2)(\gamma_1',\varphi',\gamma_2')=(\gamma_1\gamma_1',\gamma_2\circ\varphi'+\varphi\circ\gamma_1',\gamma_2\gamma_2')$$

and the action of Γ on $A_1\oplus A_2$ is defined by (2.2).

Let us define triangular products of automata and biautomata. Consider linear automata $\mathfrak{A}_1=(A_1,\Gamma_1,B_1)$ and $\mathfrak{A}_2=(A_2,\Gamma_2,B_2)$; let $\Phi=\mathrm{Hom}(A_2,A_1)$ and $\Psi=\mathrm{Hom}(A_2,B_1)$. Action of semigroups Γ_1 and Γ_2 on Φ is defined by

(2.3) and (2.4), as in the case of semiautomata. Besides, define the action of Γ_2 on Ψ from the left:

$$a_2(\gamma_2\circ\psi)=(a_2\circ\gamma_2)\psi$$

and the action of Γ_1 on Φ to Ψ: if $\varphi\in\Phi,\gamma\in\Gamma_1$, then $\varphi*\gamma$ is an element of Ψ such that

$$a_2(\varphi*\gamma)=(a_2\varphi)*\gamma \quad \text{for each} \quad a_2\in A_2.$$

The Cartesian product $\Gamma_1\times\Phi\times\Psi\times\Gamma_2=\Gamma$ is a semigroup with respect to the multiplication operation:

$$(\gamma_1,\varphi,\psi,\gamma_2)(\gamma_1',\varphi',\psi',\gamma_2')=(\gamma_1\gamma_1',\gamma_2\circ\varphi'+\varphi\circ\gamma_1',\varphi*\gamma_1'+\gamma_2\circ\psi',\gamma_2\gamma_2'),$$

where γ_1,γ_1' are elements of Γ_1; γ_2,γ_2' of Γ_2; φ,φ' of Φ; ψ,ψ' of Ψ.

An automaton $\mathfrak{A}_1\nabla\mathfrak{A}_2=(A_1\oplus A_2,\Gamma,B_1\oplus B_2)$ in which the operations $\circ$ and $*$ are defined in the following way

$$(a_1,a_2)\circ\gamma=(a_1\circ\gamma_1+a_2\varphi,a_2\circ\gamma_2),$$

$$(a_1,a_2)*\gamma=(a_1*\gamma_1+a_2\psi,a_2*\gamma_2),$$

where $(a_1,a_2)\in A_1\oplus A_2,\gamma=(\gamma_1,\varphi,\psi,\gamma_2)\in\Gamma=\Gamma_1\times\Phi\times\Psi\times\Gamma_2$, is called the *triangular product of automata* $\mathfrak{A}_1$ and $\mathfrak{A}_2$.

If $\mathfrak{A}_1=(A_1,\Gamma_1,B_1)$ and $\mathfrak{A}_2=(A_2,\Gamma_2,B_2)$ are the exact linear automata, the triangular product $\mathfrak{A}_1\nabla\mathfrak{A}_2$ is the automaton $(A_1\oplus A_2,\Gamma,B_1\oplus B_2)$ for which Γ can be considered as the semigroup of generalized matrices of the form:

$$\begin{pmatrix} \alpha_{22} & \varphi_{22} & \alpha_{21} & \varphi_{21} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_{11} & \varphi_{11} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $(\alpha_{11},\varphi_{11})$ is the image of an element of Γ_1 under the automaton representation, $(\alpha_{22},\varphi_{22})$ is the image of an element of Γ_2 under the automaton representation, α_{21} and φ_{21} are any elements of $\mathrm{Hom}(A_2,A_1)$ and $\mathrm{Hom}(A_2,B_1)$ respectively, and the operations $\circ$ and $*$ are defined as follows:

$$(a_1, a_2)\circ\gamma=(a_1\alpha_{11}+a_2\alpha_{21}, a_2\alpha_{22}),$$

$$(a_1, a_2)*\gamma=(a_1\varphi_{11}+a_2\varphi_{21}, a_2\varphi_{22}).$$

As it was pointed out, the role of triangular products of linear automata is similar to that of wreath products in the category of pure automata. A wreath product $\mathfrak{A}_1 wr \mathfrak{A}_2$ of pure automata is a terminal object in the category of cascade connections of the automata $\mathfrak{A}_1$ and $\mathfrak{A}_2$ (see Section 2.1.). Let us define a cascade connection of linear automata $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Suppose that a semigroup Γ and the homomorphisms $\alpha_1:\Gamma \rightarrow \Gamma_1$, $\alpha_2:\Gamma \rightarrow \Gamma_2$ are given. Let there be mappings $\beta:\Gamma \rightarrow \mathrm{Hom}(A_2, A_1)$, $\delta:\Gamma \rightarrow \mathrm{Hom}(A_2, B_1)$ satisfying the relations: for $\gamma, \gamma' \in \Gamma$ holds

$$(\gamma\gamma')^{\beta}=\gamma^{\beta}(\gamma'^{\alpha_1})^{\mu_1}+(\gamma^{\alpha_2})^{\mu_2}\gamma'^{\beta}, \tag{2.5}$$

$$(\gamma\gamma')^{\delta}=\gamma^{\beta}(\gamma'^{\alpha_1})^{\nu_1}+(\gamma^{\alpha_2})^{\mu_2}\gamma'^{\delta};$$

here $(\mu_1, \nu_1):\Gamma_1 \rightarrow \mathrm{End}A\times\mathrm{Hom}(A_1, B_1)$ is the automaton representation of the semigroup Γ_1, and (μ_2, ν_2) is the automaton representation of the semigroup Γ_2.

An automaton $\mathfrak{A}=(A_1\oplus A_2, \Gamma, B_1\oplus B_2)$ with the following operations $\circ$ and $*$:

$$(a_1, a_2)\circ\gamma=(a_1\circ\gamma^{\alpha_1}+a_2\gamma^{\beta},\ a_2\circ\gamma^{\alpha_2});$$

$$(a_1, a_2)*\gamma=(a_1*\gamma^{\alpha_1}+a_2\gamma^{\delta},\ a_2*\gamma^{\alpha_2});\quad a_1\in A_1,\ a_2\in A_2$$

is called a *cascade connection of linear automata* $\mathfrak{A}_1$ and $\mathfrak{A}_2$ corresponding to a given semigroup Γ and to a set of mappings α_1, α_2, β, γ.

From the definition of the triangular product and of the cascade connection of automata $\mathfrak{A}_1$ and $\mathfrak{A}_2$ it follows that the triangular product of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ is the terminal object in the category of cascade connections of these automata. In other words, for any cascade connection of the automata $\mathfrak{A}_1$ and $\mathfrak{A}_2$ there exists the unique homomorphism of this connection to the triangular product $\mathfrak{A}_1\nabla\mathfrak{A}_2$.

Finally, let $\mathfrak{A}_1=(A_1,\Gamma_1,B_1)$, $\mathfrak{A}_2=(A_2,\Gamma_2,B_2)$ be biautomata; $\Phi_1=\mathrm{Hom}(A_2,A_1)$, $\Phi_2=\mathrm{Hom}(B_2,B_1)$, $\Psi=\mathrm{Hom}(A_2,B_1)$ are considered as additive Abelian groups. Let $\gamma_i\in\Gamma_i$, $\varphi_i\in\Phi_i$, $i=1,2$, $\psi\in\Psi$. As above, one can define actions of Γ_1 and Γ_2 on Φ_1: $\varphi_1\circ\gamma_1$, $\gamma_2\circ\varphi_1$; action of Γ_1 from Φ_1 to Ψ: $\varphi_1*\gamma_1\in\Psi$. Similarly, we define the elements $\varphi_2\circ\gamma_1$, $\varphi_2\circ\gamma_2$, $\psi*\gamma_1$, $\gamma_2*\varphi_2$, acting according to the rules: if $a_2\in A_2$, $b_2\in B_2$ then

$$b_2(\varphi_2\circ\gamma_1)=(b_2\varphi_2)\cdot\gamma_1; \qquad b_2(\gamma_2\circ\varphi_2)=(b_2\cdot\gamma_2)\varphi_2;$$
$$a_2(\psi*\gamma_1)=(a_2\psi)\cdot\gamma_1; \qquad a_2(\gamma_2*\varphi_2)=(a_2*\gamma_2)\varphi_2. \tag{2.6}$$

All these actions are compatible with the linear operations in Φ_1,Φ_2,Ψ. It is easy to verify that

$$\varphi_1*\gamma_1\gamma_1' =(\varphi_1\circ\gamma_1)*\gamma_1'+(\varphi_1*\gamma_1)\circ\gamma_1',$$

$$\gamma_2\gamma_2'*\varphi_2 =\gamma_2\circ(\gamma_2'*\varphi_2)+\gamma_2*(\gamma_2'\circ\varphi_2).$$

The Cartesian product $\Gamma=\Gamma_1\times\Phi_1\times\Psi\times\Phi_2\times\Gamma_2$ turns out to be a semigroup Γ with respect to the multiplication

$(\gamma_1,\varphi_1,\psi,\varphi_2,\gamma_2)(\gamma_1',\varphi_1',\psi',\varphi_2',\gamma_2')=(\gamma_1\gamma_1',\varphi_1\circ\gamma_1'+\gamma_2\circ\varphi_1',\varphi_1*\gamma_1'+\gamma_2*\varphi_2'+\gamma_2\circ\psi'+$
$\psi*\gamma_1',\varphi_2\circ\gamma_1'+\gamma_2\circ\varphi_2',\gamma_2\gamma_2')$.

The biautomaton $\mathfrak{A}_1\nabla\mathfrak{A}_2=(A_1\oplus A_2,\Gamma,B_1\oplus B_2)$, with the operations defined as follows: if $(a_1,a_2)\in A_1\oplus A_2$, $(b_1,b_2)\in B_1\oplus B_2$, $\gamma=(\gamma_1,\varphi_1,\psi,\varphi_2,\gamma_2)\in\Gamma$ then

$$(a_1,a_2)\circ\gamma=(a_1\circ\gamma_1+a_2\varphi_1,a_2\circ\gamma_2);$$

$$(a_1,a_2)*\gamma=(a_1*\gamma_1+a_2\psi,a_2*\gamma_2); \tag{2.7'}$$

$$(b_1,b_2)\cdot\gamma=(b_1\cdot\gamma_1+b_2\varphi_2,b_2\cdot\gamma_2),$$

is called *the triangular product of biautomata* $\mathfrak{A}_1$ and $\mathfrak{A}_2$. (The fulfillment of the biautomaton axioms is easily verified).

The comparatively voluminous definition of the triangular product of biautomata becomes more clear and visible, if one uses the matrix terms. Let $\mathfrak{A}_1=(A_1,\Gamma_1,B_1)$ and $\mathfrak{A}_2=(A_2,\Gamma_2,B_2)$ be exact biautomata. Their triangular product is the biautomaton $\mathfrak{A}_1\nabla\mathfrak{A}_2=(A_1\oplus A_2,\Gamma,B_1\oplus B_2)$, in which Γ is the semigroup of the generalized matrices of the form

$$\gamma \begin{pmatrix} \alpha_{22} & \varphi_{22} & \alpha_{21} & \varphi_{21} \\ 0 & \beta_{22} & 0 & \beta_{21} \\ 0 & 0 & \alpha_{11} & \varphi_{11} \\ 0 & 0 & 0 & \beta_{11} \end{pmatrix} \tag{2.7}$$

where $\begin{pmatrix} \alpha_{ii} & \varphi_{ii} \\ 0 & \beta_{ii} \end{pmatrix}$ are images of elements of Γ_i, i=1,2 under the biautomaton representation,

$$\alpha_{21}\in\Phi_1=\mathrm{Hom}(A_2,A_1),\ \beta_{21}\in\Phi_2=\mathrm{Hom}(B_2,B_1),\ \varphi_{21}\in\Psi=\mathrm{Hom}(A_2,B_1).$$

The operations $\circ$, $*$ and $\cdot$ are defined in the following way: if $(a_1,a_2)\in A_1\oplus A_2$, $(b_1,b_2)\in B_1\oplus B_2$ then

$$(a_1,a_2)\circ\gamma=(a_1\alpha_{11}+a_2\alpha_{21},\ a_2\alpha_{22});$$
$$(a_1,a_2)*\gamma=(a_1\varphi_{11}+a_2\varphi_{21},\ a_2\varphi_{22}); \tag{2.8}$$
$$(b_1,b_2)\cdot\gamma=(b_1\beta_{11}+b_2\beta_{21},\ b_2\beta_{22}).$$

This biautomaton is isomorphic to the biautomaton $(A_1\oplus A_2,\bar{\Gamma},B_1\oplus B_2)$ with $\bar{\Gamma}$ being the semigroup of generalized matrices of the form:

$$\gamma = \begin{pmatrix} \alpha_{22} & \alpha_{21} & \varphi_{22} & \varphi_{21} \\ 0 & \alpha_{11} & 0 & \varphi_{11} \\ 0 & 0 & \beta_{22} & \beta_{21} \\ 0 & 0 & 0 & \beta_{11} \end{pmatrix} \tag{2.8'}$$

while the operations $\circ$, $*$ and $\cdot$ are defined by the formula of (2.8).

As it was mentioned in the previous Section, the linear automaton (A,Γ,B) can be considered as biautomaton satisfying the condition: for any elements $b\in B$ and $\gamma\in\Gamma$ the equality $b\cdot\gamma=0$ is valid. The triangular product of such biautomata no longer satisfies the condition mentioned (since β_{21} is an arbitrary element of Φ_2 in the matrix of (2.7)). This fact means that the triangular product of automata cannot be considered as a particular case of the triangular product of biautomata. On the other hand, it is obvious that the triangular product of biautomata with zero outputs, i.e. semiautomata, is again a semiautomaton, hence the triangular product of semiautomata is a particular case of triangular multiplication of biautomata.

Without proof, note that

1) The triangular multiplication of automata (biautomata) is associative:

$$\mathfrak{A}_1\nabla(\mathfrak{A}_2\nabla\mathfrak{A}_3)=(\mathfrak{A}_1\nabla\mathfrak{A}_2)\nabla\mathfrak{A}_3.$$

2) If $\mathfrak{A}_1\cong\mathfrak{A}'_1$, $\mathfrak{A}_2\cong\mathfrak{A}'_2$, then $\mathfrak{A}_1\nabla\mathfrak{A}_2\cong\mathfrak{A}'_1\nabla\mathfrak{A}'_2$.

Note also that the representation corresponding to the universal biautomaton $Atm^1(A,B)$ is the triangular product of semigroup representations, namely $(B,EndB)\nabla(A,EndA)=(A\oplus B,End^b(A,B))$

Consider two constructions which are close to the triangular product.

Let $\mathfrak{A}=(A,\Gamma,B)$ be a biautomaton and $\mathfrak{C}=(X,\Sigma)$ the *right* representation of the semigroup Σ by transformations of the set X. The biautomaton

$$(A\otimes KX,\Gamma wr_r^X\Sigma,B\otimes KX)=(A',\Gamma',B')$$

where KX is the linear space over K with the basis X, $A\otimes KX$, $B\otimes KX$ are tensor products of corresponding linear spaces, and $\Gamma wr_r^X\Sigma$ is the right (corresponding to the right representation (X,Σ)) wreath product of semigroups Γ and Σ over the set X, is called the *tensor wreath product of the biautomaton* $\mathfrak{A}$ *with the right representation* $\mathfrak{C}$. To define $\circ$, $*$ and $\cdot$ operations of the semigroup Γ' on spaces A' and B' it is sufficient to define these operations on generating elements of the spaces, i.e., on elements of the form $a\otimes x$, $b\otimes x$, $a\in A$, $b\in B$, $x\in X$: if $(\bar{\gamma},\sigma)\in\Gamma wr_r^X\Sigma$, we set:

$$(a\otimes x)\circ(\bar{\gamma},\sigma)=(a\circ\bar{\gamma}(x))\otimes x\sigma;$$

$$(a\otimes x)*(\bar{\gamma},\sigma)=(a*\bar{\gamma}(x))\otimes x\sigma;$$

$$(b\otimes x)\bullet(\bar{\gamma},\sigma)=(b\bullet\bar{\gamma}(x))\otimes x\sigma.$$

The operations thus introduced satisfy axioms (1.2), i.e. $(A\otimes KX,\Gamma\text{wr}_r^X\Sigma,B\otimes KX)$ is indeed a biautomaton. Denote the *tensor wreath product of the biautomaton* $\mathfrak{A}$ *with the right representation* $\mathfrak{C}$ by $\mathfrak{A}\text{wr}_r\mathfrak{C}$. There is the following associative property:

$$\mathfrak{A}\text{wr}_r(\mathfrak{C}_1\text{wr}_r\mathfrak{C}_2)=(\mathfrak{A}\text{wr}_r\mathfrak{C})\text{wr}_r\mathfrak{C}_2$$

Define another construction of wreath product of biautomata with representations. Let $\mathfrak{A}=(A,\Gamma,B)$, as before, be a biautomaton, and $\mathfrak{C}=(\Sigma,X)$ be the *left* representation of the semigroup Σ by transformations of the set X. The *full wreath product of the biautomaton A and the left representation* $\mathfrak{C}$ is the biautomaton

$$(\text{Hom}(KX,A),\Gamma\text{wr}_\ell^X\Sigma,\text{Hom}(KX,B)),$$

where $\Gamma\text{wr}_\ell^X\Sigma$ is the left wreath product of the semigroups Γ and Σ, (i.e. corresponding to the left representation (Σ,X)).

The operations $\circ$, $*$ and $\bullet$ are defined as follows: if $\varphi\in\text{Hom}(KX,A)$, $\psi\in\text{Hom}(KX,B)$, $(\sigma,\bar{\gamma})\in\Gamma\text{wr}_\ell^X\Sigma$, then

$\varphi\circ(\sigma,\bar{\gamma})$ is an element of Hom(KX,A) such that for each $x\in X$

$$(\varphi\circ(\sigma,\bar{\gamma})(x)=\varphi(\sigma x)\circ\bar{\gamma}(x);$$

$\psi\bullet(\sigma,\bar{\gamma})$ is an element of Hom(KX,B) such that for each $x\in X$

$$(\psi\bullet(\sigma,\bar{\gamma}))(x)=\psi(\sigma x)\bullet\bar{\gamma}(x);$$

$\varphi*(\sigma,\bar{\gamma})$ is an element of Hom(KX,X) such that for each $x\in X$

$$(\varphi*(\sigma,\bar{\gamma}))(x)=\varphi(\sigma x)*\bar{\gamma}(x).$$

Define, finally, Cartesian and discrete direct product of automata. Let $\prod A_\alpha$, $\alpha \in J$, be a Cartesian product of linear spaces A_α (see Cartesian product of ω-algebras). A subspace in $\prod A_\alpha$, consisting of all functions taking nonzero values only on the finite sets of elements from J, is called a discrete direct product of linear spaces.

Let $\mathfrak{A}_\alpha=(A_\alpha,\Gamma_\alpha,B_\alpha)$, $\alpha \in J$ be a set of biautomata. Their Cartesian product $\mathfrak{A}=\prod\mathfrak{A}_\alpha=(A_\alpha,\Gamma_\alpha,B_\alpha)$, $\alpha \in J$, is defined componentwise. If instead of Cartesian product of linear spaces we take their discrete direct product, we obtain a biautomaton, called a discrete direct product of biautomata $\mathfrak{A}_\alpha$. If the set J is finite, the obtained biautomaton is called a finite direct product of biautomata $\mathfrak{A}_\alpha$

3.2.2. Decomposition of biautomata

A biautomaton $\mathfrak{A}=(A,\Gamma,B)$ is called a *simple biautomaton*, if it satisfies one of the following conditions:

1) A=0, while (B,Γ) is an irreducible representation.

2) B=0, while (A,Γ) is an irreducible representation.

A biautomaton $\mathfrak{A}$ is called a *divisor of the biautomaton* $\mathfrak{B}$, if the corresponding exact automaton $\tilde{\mathfrak{A}}$ is a homomorphic image of a subautomaton of $\mathfrak{B}$. A divisor $\mathfrak{A}$ is called a *prime divisor* of the automaton $\mathfrak{B}$ if $\mathfrak{A}$ is a simple biautomaton. The biautomaton decomposition is understood to be a representation of it in the form of a divisor of a construction made of other biautomata, which are called the components of a decomposition. In the given case, the triangular product is being taken as a construction and it is demanded that the components of the decomposition be divisors of the initial biautomaton. Now we shall prove a theorem that each biautomaton can be embedded into a triangular product of its subautomaton and the quotient automaton by it. Iterating the embedding process, we will get a decomposition whose components are simple biautomata.

Theorem 2.1. [80] (*on embedding*). *Let $\mathfrak{A}=(A,\Gamma,B)$ be an exact automaton, $\mathfrak{A}'_1=(A_1,\Gamma,B_1)$ its subautomaton, $\mathfrak{A}'_2=(A/A_1,\Gamma,B/B_1)$ the corresponding quotient automaton. Let $\mathfrak{A}_1=(A_1,\Gamma_1,B_1)$, $\mathfrak{A}_2=(A/A_1,\Gamma_2,B/B_1)$ be the exact automata associated with $\mathfrak{A}'_1$ and $\mathfrak{A}'_2$ respectively. Then the biautomaton $\mathfrak{A}$ is isomorphically embedded into triangular product $\mathfrak{A}_1\nabla\mathfrak{A}_2$*

The sketch of the proof is as follows. Denote by A_2 a complement of A_1 in A, by B_2 the complement of B_1 in B: $A=A_1\oplus A_2$, $B=B_1\oplus B_2$. First the biautomaton $\mathfrak{A}^*_2=(A_2,\Gamma_2,B_2)$ which is isomorphic to $\mathfrak{A}_2$ is constructed, and $\mathfrak{A}_1\nabla\mathfrak{A}_2\cong\mathfrak{A}_1\nabla\mathfrak{A}^*_2=(A,\tilde{\Gamma},B)$. After that, the biautomaton $(A,\Sigma,B)\cong(A,\tilde{\Gamma},B)$, such that $(A,\Gamma,B)\subset(A,\Sigma,B)$, is constructed. Then

$$(A,\Gamma,B)\subset(A,\Sigma,B)\cong(A,\tilde{\Gamma},B)=\mathfrak{A}_1\nabla\mathfrak{A}^*_2\cong\mathfrak{A}_1\nabla\mathfrak{A}_2,$$

as is required.

In order to construct the biautomaton $\mathfrak{A}^*_2$, one should suitably define operations $\circ,*,\cdot$ in the triplet of sets (A_2,Γ_2,B_2). Denote the natural homomorphism of the linear space $A=A_1\oplus A_2$ onto A/A_1 by ν, and by $\tilde{\nu}$ its restriction to A_2; $\tilde{\nu}:A_2\to A/A_1$, which is the isomorphism of linear spaces. Let $\tilde{\nu}^{-1}:A/A_1\to A_2$ be an inverse to $\tilde{\nu}$ mapping. It is clear that $\nu\nu^{-1}$ is a projection of A onto A_2. Basing on actions of elements of Γ_2 in A/A_1, we define actions of elements of Γ_2 in A_2, by the rule:

$$\text{if } a_2\in A_2,\ \gamma_2\in\Gamma_2 \text{ then } a_2\circ\gamma_2=(a_2^{\tilde{\nu}}\circ\gamma_2)^{\tilde{\nu}^{-1}}.$$

Similarly, let μ be a natural homomorphism $B=B_1\oplus B_2$ onto B/B_1, $\tilde{\mu}$ be its restriction to B_2: $\tilde{\mu}^{-1}:B/B_1\to B_2$ be the inverse to $\tilde{\mu}$ mapping. Define

$$a_2*\gamma_2=(a_2^{\tilde{\nu}}*\gamma_2)^{\tilde{\mu}^{-1}}$$

$$b_2\cdot\gamma_2=(b_2^{\tilde{\mu}}\cdot\gamma_2)^{\tilde{\mu}^{-1}};\quad a_2\in A_2,\ b_2\in B_2,\ \gamma_2\in\Gamma_2$$

The biautomaton thus constructed, $\mathfrak{A}^*_2=(A_2,\Gamma_2,B_2)$, is isomorphic to biautomaton $\mathfrak{A}_2=(A/A_1,\Gamma_2,B/B_1)$ and the triangular product $\mathfrak{A}_1\nabla\mathfrak{A}_2$ is isomorphic to

$$\mathfrak{A}_1\nabla\mathfrak{A}^*_2=(A_1\oplus A_2,\Gamma_1\times\mathrm{Hom}(A_2,A_1)\times\mathrm{Hom}(A_2,B_1)\times\mathrm{Hom}(B_2,B_1)\times\Gamma_2,B_1\oplus B_2)=(A,\tilde{\Gamma},B).$$

It remains to construct the group Σ and the biautomaton (A,Σ,B) containing (A,Γ,B) such that (A,Σ,B) is isomorphic to $(A,\tilde{\Gamma},B)$.

Since (A_1,Γ_1,B_1) and (A_2,Γ_2,B_2) are exact biautomata, one can assume that

$$\Gamma_1 \subset \mathrm{End}^b(A_1,B_1)=\mathrm{End}A_1\times \mathrm{Hom}(A_1,B_1)\times \mathrm{End}B_1,$$

$$\Gamma_2 \subset \mathrm{End}^b(A_2,B_2)=\mathrm{End}A_2\times \mathrm{Hom}(A_2,B_2)\times \mathrm{End}B_2$$

The semigroups Γ_1 and Γ_2 are embedded in $\mathrm{End}^b(A,B)$ in a natural way, according to the rule:
to any element $\gamma_1=(\sigma_1,\varphi_1,\tau_1)$ of $\Gamma_1\subset \mathrm{End}^b(A_1,B_1)$ is assigned the element $\tilde{\gamma}_1=(\bar{\sigma}_1,\bar{\varphi}_1,\bar{\tau}_1)$ of $\mathrm{End}^b(A,B)$, with components $\bar{\sigma}_1,\bar{\varphi}_1,\bar{\tau}_1$ acting in A_1 and B_1 as $\sigma_1,\varphi_1,\tau_1$ respectively. The element $\bar{\sigma}_1$ acts as the identity in A_2, the element $\bar{\varphi}_1$ maps A_2 to the zero of the space B_2, the element $\bar{\tau}_1$ acts as the identity in B_2. The set of such elements $\bar{\gamma}_1$, corresponding to elements γ_1 of Γ_1 forms a semigroup $\bar{\Gamma}_1$ included in $\mathrm{End}^b(A,B)$ and isomorphic to Γ_1.

The semigroup Γ_2 is embedded into $\mathrm{End}^b(A,B)$ in a similar way: to any element $\gamma_2=(\sigma_2,\varphi_2,\tau_2)$ of Γ_2 is assigned the element $\bar{\gamma}_2=(\bar{\sigma}_2,\bar{\varphi}_2,\bar{\tau}_2)$, with components $\bar{\sigma}_2,\bar{\varphi}_2,\bar{\tau}_2$ acting in A_2 and B_2 as $\sigma_2,\varphi_2,\tau_2$ respectively. Besides, $\bar{\sigma}_2$ in A_1 and $\bar{\tau}_2$ in B_2 act as the identities, and the element $\bar{\varphi}_2$ maps A_1 into zero of the space B_1. The elements $\bar{\gamma}_2$ corresponding to elements $\gamma_2\in\Gamma_2$ form a semigroup $\bar{\Gamma}_2$ included into $\mathrm{End}^b(A,B)$ and isomorphic to Γ_2.

Further, to each element ψ of $\mathrm{Hom}(A_2,A_1)$ is assigned the element $\bar{\psi}$ of $\mathrm{End}A$ acting by the rule: if $a=(a_1,a_2)\in A=A_1\oplus A_2$ then $a\bar{\psi}=(a_2\psi,0)$; to each element δ of $\mathrm{Hom}(B_2,B_1)$ is assigned the element $\bar{\delta}$ of $\mathrm{End}B$ such that if $b=(b_1,b_2)\in B=B_1\oplus B_2$, then $b\bar{\delta}=(b_2\delta,0)$ and, finally, to each η of $\mathrm{Hom}(A_2,B_1)$ the element $\bar{\eta}$ of $\mathrm{Hom}(A,B)$ such that $a\bar{\eta}=(a_2\eta,0)$. Consider the set Σ of elements of $\mathrm{End}^b(A,B)$ of the form

$$(\bar{\sigma}_1\bar{\sigma}_2+\bar{\psi},\ \bar{\varphi}_1+\bar{\eta}+\bar{\varphi}_2,\ \bar{\tau}_1\bar{\tau}_2+\bar{\delta}), \tag{2.9}$$

where $(\bar{\sigma}_1,\bar{\varphi}_1,\bar{\tau}_1)\in\bar{\Gamma}_1$, $(\bar{\sigma}_2,\bar{\varphi}_2,\bar{\tau}_2)\in\bar{\Gamma}_2$, and $\bar{\psi},\bar{\delta},\bar{\eta}$ are elements corresponding under the specified rule to the elements $\psi\in\mathrm{Hom}(A_2,A_1)$, $\delta\in\mathrm{Hom}(B_2,B_1)$ and $\eta\in\mathrm{Hom}(A_2,B_1)$ respectively. It is easy to verify that Σ is a semigroup, and, since it lies in $\mathrm{End}^b(A,B)$, (A,Σ,B) is a subautomaton in $(A,\mathrm{End}^b(A,B),B)$. If $\gamma=(\gamma_1,\psi,\eta,\delta,\gamma_2)$ is an element of

$$\tilde{\Gamma}=\Gamma_1\times\mathrm{Hom}(A_2,A_1)\times\mathrm{Hom}(A_2,B_1)\times\mathrm{Hom}(B_2,B_1)\times\Gamma_2;\ \ \gamma_i=(\sigma_i,\varphi_i,\tau_i);\ \ i=1,2$$

then the mapping assigning to it the element (2.9) of Σ is an isomorphism of semigroups $\tilde{\Gamma} \to \Sigma$, which, in its turn, defines an isomorphism in inputs of the biautomata $\mathfrak{A}_1\nabla\mathfrak{A}_2^*=(A,\tilde{\Gamma},B)\cong(A,\Sigma,B)$.

It remains only to prove that $\Gamma\subset\Sigma$. In order to do so, it is sufficient to show that any element of the semigroup Γ can be represented in the form (2.9).

Let $\gamma=(\sigma,\varphi,\tau)$ be an element of the semigroup Γ; $\gamma_1=(\sigma_1,\varphi_1,\tau_1)$, $\gamma_2=(\sigma_2,\varphi_2,\tau_2)$ be images of the element γ in Γ_1 and Γ_2 under mappings of Γ into Γ_1 and Γ_2, originated upon respective transitions from the biautomata (A_1,Γ,B_1) and $(A/A_1,\Gamma,B/B_1)$ to exact biautomata (A_1,Γ_1,B_1) and $(A/A_1,\Gamma_2,B/B_1)$. Let $\bar{\gamma}_1=(\bar{\sigma}_1,\bar{\varphi}_1,\bar{\tau}_1)$ and $\bar{\gamma}_2=(\bar{\sigma}_2,\bar{\varphi}_2,\bar{\tau}_2)$ be elements of $\mathrm{End}^b(A,B)$ corresponding to γ_1 and γ_2 under the embedding of Γ_1 and Γ_2 into $\mathrm{End}^b(A,B)$. Consider the difference $\sigma-\bar{\sigma}_1\bar{\sigma}_2$. It is an element of EndA acting as follows: if $a_1\in A_1$, $a_2\in A_2$ then

$$a_1(\sigma-\bar{\sigma}_1\bar{\sigma}_2)=a_1\sigma-a_1\bar{\sigma}_1\bar{\sigma}_2=a_1\sigma-a_1\bar{\sigma}_1=a_1\sigma-a_1\sigma_1=0;$$

$$a_2(\sigma-\bar{\sigma}_1\bar{\sigma}_2)=a_2\sigma-a_2\bar{\sigma}_1\bar{\sigma}_2=a_2\sigma-a_2\bar{\sigma}_2=a_2\sigma-a_2\sigma_2$$

Consider the last difference. Let $a_2\sigma=(a'_1,a'_2)$, $a'_1\in A_1$, $a'_2\in A_2$. Then, by the definition of the automaton (A_2,Γ_2,B_2) holds $a_2\sigma_2=a'_2$. Finally,

$$a_2(\sigma-\bar{\sigma}_1\bar{\sigma}_2)=a_2\sigma-a_2\sigma_2=(a'_1,a'_2-a'_2)=(a'_1,0) \text{ and}$$

$$(a_1,a_2)(\sigma-\bar{\sigma}_1\bar{\sigma}_2)=(a'_1,0).$$

Thus, $\sigma-\bar{\sigma}_1\bar{\sigma}_2$ is an element of the form $\bar{\psi}$, corresponding to some ψ of $\mathrm{Hom}(A_2,A_1)$, and $\sigma=\bar{\sigma}_1\bar{\sigma}_2+\bar{\psi}$. In a similar manner the possibility of representing of φ in the form $\bar{\varphi}_1+\eta+\bar{\varphi}_2$, and τ in the form $\bar{\tau}_1\bar{\tau}_2+\bar{\delta}$ is proved. By this, it is shown that $\Gamma\subset\Sigma$ and $(A,\Gamma,B)\subset(A,\Sigma,B)\cong\mathfrak{A}_1\nabla\mathfrak{A}_2$

Theorem 2.2. [80] (*on decomposition of biautomata*). *An exact finite dimensional biautomaton $\mathfrak{A}=(A,\Gamma,B)$ is isomorphically embedded into the triangular product of biautomata whose components are prime divisors of $\mathfrak{A}$.*

Proof. First we prove that $\mathfrak{A}$ can be embedded into the triangular product of biautomata with components (A_i,Γ_i,B_i) such that the representations (A_i,Γ_i) and (B_i,Γ_i) are irreducible ones. Denote by ℓ the length of Γ-compositional series in A, i.e. a series $0 \subset A_1 \subset \ldots A_\ell = A$ with each A_{i-1} being the maximal Γ-invariant subspace in A_i, $i \in \{1, 2, \ldots, \ell\}$; by m the length of Γ-compositional series in B. The compositional length of the biautomaton $\mathfrak{A}$ is $k=\ell+m$ and we shall use the induction by it. If k=0 or k=1, the validity of the statement is obvious. Assume that the statement is proved for k=n-1 and let k=n. Denote by A' the maximal Γ-invariant subspace in A, and by B' the maximal Γ-invariant subspace in B containing $A'*\Gamma = \{a*\gamma \mid a \in A', \gamma \in \Gamma\}$. Then (A',Γ,B') is a subautomaton in (A,Γ,B), its compositional length is less than n, and the quotient biautomaton (A/A',Γ,B/B') satisfies the condition of irreducibility of representations (A/A',Γ) and (B/B',Γ). The automaton (A,Γ,B) is embedded into the triangular product $(A',\Gamma_1,B')\nabla(A/A',\Gamma_2,B/B')$, where (A',Γ_1,B') and $(A/A',\Gamma_2,B/B')$ are exact biautomata corresponding respectively to biautomata (A',Γ,B') and (A/A',Γ,B/B'). Since the compositional length of (A',Γ_1,B') is less than n, the induction assumption is valid for it. Hence, the statement is valid also for k=n. Now the theorem follows from the proved fact and from the simple remark that any biautomaton (A,Γ,B) is embedded into the triangular product $(0,\Gamma,B)\nabla(A,\Gamma,0)$.

In view of the proved theorem it is natural to ask whether the decomposition above is in some sence unique. We say that the biautomaton $\mathfrak{A}$ uncancellably lies within the triangular product of simple biautomata $\mathfrak{A}_1\nabla\ldots\nabla\mathfrak{A}_n$, if it does not lie in the triangular product of these biautomata in the same order, in which at least one factor is absent. It can be proved ([80]) that if a biautomaton $\mathfrak{A}$ incancellably lies within the triangular products of its prime divisors $\mathfrak{A}_1\nabla\ldots\nabla\mathfrak{A}_n$ and $\mathfrak{A}'_1\nabla\ldots\nabla\mathfrak{A}'_m$, then m=n and triangular products differ only in orders.

3.2.3. Decomposition of the linear automata

In this item we consider decomposition of linear automata by means of the triangular product and its further reduction using the operation of the wreath product of linear automaton and pure one (see [91]).

Let identify the automaton (0,0,B) with the linear space B and automaton (A,Γ,0) with the representation (A,Γ). Consider the triangular product $(0,0,B)\nabla(A,\Gamma,0)$ and in this way define the multiplication of the space B by the automaton (A,Γ). We have the automaton

$$B\nabla(A,\Gamma)=(A,\mathrm{Hom}(A,B)\times\Gamma,B) \text{ with}$$

$$(\psi,\gamma)(\psi',\gamma')=(\gamma\circ\psi',\gamma\gamma'),$$

where $\gamma,\gamma'\in\Gamma$, $\psi,\psi'\in\mathrm{Hom}(A,B)$; $a\circ(\psi,\gamma)=a\circ\gamma$, $a*(\psi,\gamma)=a\psi$, $\gamma\circ\psi'\in\mathrm{Hom}(A,B)$ such that for $a\in A$ holds $a(\gamma\circ\psi')=(a\circ\gamma)\psi'$

Lemma 2.3. *Any linear automaton* $\mathfrak{A}=(A,\Gamma,B)$ *is embedded into the triangular product* $B\nabla(A,\Gamma)$.

Proof. The embedding is defined by the identity mappings on spaces A, B and by mapping $\mu:\Gamma\rightarrow\mathrm{Hom}(A,B)\times\Gamma$, $\mu:\gamma\rightarrow(\psi,\gamma)$, where $a\in A$, $\gamma\in\Gamma$, $\psi\in\mathrm{Hom}(A,B)$ and ψ is specified by the rule $a\psi=a*\gamma$. This mapping is the monomorphism of semigroups:

$$\gamma_1^{\mu}\gamma_2^{\mu}=(\psi_1,\gamma_1)(\psi_2,\gamma_2)=(\gamma_1\circ\psi_2,\gamma_1\gamma_2)=(\gamma_1\gamma_2)^{\mu}.$$

It is easy to see that we have also the automata homomorphism.

Lemma 2.4. *Let the automata* $\mathfrak{A}_1,\mathfrak{A}_2,\mathfrak{A}'_1,\mathfrak{A}'_2$ *be given and* $\mathfrak{A}_1|\mathfrak{A}'_1$, $\mathfrak{A}_2|\mathfrak{A}'_2$. *Then* $\mathfrak{A}_1\nabla\mathfrak{A}_2|\mathfrak{A}'_1\nabla\mathfrak{A}'_2$.

Proof. Let $\mathfrak{A}_1^0,\mathfrak{A}_2^0$ be subautomata in $\mathfrak{A}'_1,\mathfrak{A}'_2$ respectively, such that there are epimorphisms $\mu_1:\mathfrak{A}_1^0\rightarrow\mathfrak{A}_1$, $\mu_2:\mathfrak{A}_2^0\rightarrow\mathfrak{A}_2$. Then there is an epimorphism $\mu:\mathfrak{A}_1^0\nabla\mathfrak{A}'_2\rightarrow\mathfrak{A}_1\nabla\mathfrak{A}'_2$ Besides that there exists subautomaton $\mathfrak{B}\subset\mathfrak{A}_1\nabla\mathfrak{A}'_2$ such that there is an epimorphism $\nu:\mathfrak{B}\rightarrow\mathfrak{A}_1\nabla\mathfrak{A}_2$ ([82], proposition 7.1.3). The coimage of the automaton in respect to μ gives the necessary subautomaton in $\mathfrak{A}_1\nabla\mathfrak{A}_2$.

It follows from the theorem of biautomata decomposition that a similar statement for linear automata is valid. Indeed, any linear automaton (A,Γ,B) is embedded into triangular product $B\nabla(A,\Gamma)=(0,0,B)\nabla(A,\Gamma,0)$. Now, we can apply the theorem 2.2 to automaton (biautomaton) $(A,\Gamma,0)$. Hence, the biautomaton $(A,\Gamma,0)$ is embedded into triangular product $(A_1,\Gamma_1,0)\nabla\ldots\nabla(A_k,\Gamma_k,0)$, whose components $(A_i,\Gamma_i,0)$, $i=1,2,\ldots,k$ are irreducible representations. Thus every linear automaton can be embedded into the triangular product

$$(0,0,B_1)\nabla\ldots\nabla(0,0,B_m)\nabla(A_1,\Gamma_1,0)\nabla\ldots\nabla(A_k,\Gamma_k,0)$$

where the spaces B_i are one-dimensional, $B_1\oplus\ldots\oplus B_m=B$, and the representations (A_i,Γ_i) are irreducible. Therefore, taking into account the identifications of $(0,0,B)$ and $(A_i,\Gamma_i,0)$ with B and (A_i,Γ_i) respectively, we get

$$(A,\Gamma,B)\,|\,B_1\nabla\ldots\nabla B_m\nabla(A_1,\Gamma_1)\nabla\ldots\nabla(A_k,\Gamma_k) \qquad (2.10)$$

or

$$(A,\Gamma,B)\,|\,(\mathop{\nabla}_{i=1}^{m} B_i)\nabla(\mathop{\nabla}_{j=1}^{k}(A_j,\Gamma_j)). \qquad (2.10')$$

Let introduce now the operation of *wreath product of linear automaton with pure one* and consider the further reduction of the decomposition 2.10 in the case of finite completely 0-simple acting semigroup Γ.

Take linear automaton $\mathfrak{A}=(A,\Gamma_1,B)$ and pure automaton $\Lambda=(X,\Gamma_2,Y)$. The linear automaton

$$\mathfrak{A}\mathrm{wr}\Lambda=(A\otimes KX,\Gamma_1^X\times\Gamma_2,B\otimes KY), \text{ with}$$

$$(a\otimes x)\circ(\bar{\gamma}_1,\gamma_2)=(a\circ\bar{\gamma}_1(x))\otimes(x\circ\gamma_2),\quad (a\otimes x)*(\bar{\gamma}_1,\gamma_2)=(a*\bar{\gamma}_1(x))\otimes(x*\gamma_2),$$

where $a\in A$, $x\in X$, $\bar{\gamma}_1\in\Gamma_1^X$, $\gamma_2\in\Gamma_2$, KX is the linear space over X, is called the wreath product of $\mathfrak{A}$ and Λ. Thus defined actions $\circ$ and $*$ are the linear actions of the semigroup $\Gamma_1^X\times\Gamma_2$. Indeed, since X is the basis of the linear space KX then $A\otimes KX=\oplus\sum_{x\in X}(A\otimes x)$ and linear actions defined on the summands of the direct sum are extended to the linear actions on $A\otimes KX$.

For the wreath product of the linear automaton and pure one the following associative condition

$$(\mathfrak{A}_1\mathrm{wr}\Lambda_2)\mathrm{wr}\Lambda_3=\mathfrak{A}_1\mathrm{wr}(\Lambda_2\mathrm{wr}\Lambda_3)$$

is satisfied.

In order to verify this property, take a linear automaton $\mathfrak{A}_1=(A_1,\Gamma_1,B_1)$ and pure automata $\Lambda_2=(X_2,\Gamma_2,Y_2)$, $\Lambda_3=(X_3,\Gamma_3,Y_3)$. Then,

$$(\mathfrak{A}_1\mathrm{wr}\Lambda_2)\mathrm{wr}\Lambda_3=(A_1\otimes KX_2\otimes KX_3,(\Gamma_1^{X_2}\times\Gamma_2)^{X_3}\times\Gamma_3,B_1\otimes KY_2\otimes KY_3),$$

$$(\mathfrak{A}_1\mathrm{wr}(\Lambda_2\mathrm{wr}\Lambda_3))=(A_1\otimes K(X_2\times X_3),\Gamma_1^{X_2\times X_3}\times(\Gamma_2^{X_3}\times\Gamma_3),B_1\otimes K(Y_2\times Y_3)).$$

For linear spaces there is a canonical isomorphism $A_1 \otimes KX_2 \otimes KX_3 \simeq A_1 \otimes K(X_2 \times X_3)$, defined by the mapping: $a_1 \otimes (x_2, x_3) \rightarrow a_1 \otimes x_2 \otimes x_3$ (the same for B). Acting semigroups are also isomorphic. Really, setting $\mu: (\bar{\gamma}_1, \bar{\gamma}_2, \gamma_3) \rightarrow (\bar{\sigma}, \gamma_3)$, where $\bar{\gamma}_1 \in \Gamma_1^{X_2 \times X_3}$, $\bar{\gamma}_2 \in \Gamma_2^{X_3}$, $\gamma_3 \in \Gamma_3$, $\bar{\sigma} \in (\Gamma_1^{X_2} \times \Gamma_2)^{X_3}$ and $\bar{\sigma}(x_3)$ is a pair $(\bar{\sigma}_1, \bar{\gamma}_2(x_3))$, such that $\bar{\sigma}_1 \in \Gamma_1^{X_2}$ and $\bar{\sigma}_1(x_2) = \bar{\gamma}_1(x_2, x_3)$, we have the necessary isomorphism. Starting from these mappings we get the automata isomorphism:

$$(a_1 \otimes (x_2, x_3) \circ (\bar{\gamma}_1, (\bar{\gamma}_2, \gamma_3))^{\mu} = ((a_1 \circ \bar{\gamma}_1(x_2, x_3)) \otimes ((x_2, x_3) \circ (\bar{\gamma}_2, \gamma_3)))^{\mu} =$$
$$(a_1 \circ \bar{\gamma}_1(x_2, x_3)) \otimes (x_2 \circ \bar{\gamma}_2(x_3)) \otimes (x_3 \circ \gamma_3),$$
$$(a_1 \otimes (x_2, x_3))^{\mu} \circ (\bar{\gamma}_1, (\bar{\gamma}_2, \gamma_3))^{\mu} = (a_1 \otimes x_2 \otimes x_3) \circ (\bar{\sigma}, \gamma_3) = ((a_1 \otimes x_2) \circ \bar{\sigma}(x_3)) \otimes (x_3 \circ \gamma_3) =$$
$$((a_1 \otimes x_2) \circ (\bar{\sigma}_1, \bar{\gamma}_2(x_3))) \otimes (x_3 \circ \gamma_3) = (a_1 \circ \bar{\sigma}_1(x_3) \otimes (x_2 \circ \bar{\gamma}_2(x_3)) \otimes (x_3 \circ \gamma_3) =$$
$$(a_1 \circ \bar{\gamma}_1(x_2, x_3)) \otimes (x_2 \circ \bar{\gamma}_2(x_3)) \otimes (x_3 \circ \gamma_3).$$

The proved associative means, in other words, that the semigroup of pure automata acts on the class of linear automata.

Note without proving connection between triangular products and wreath products. The following inclusions take place

$$\mathfrak{A}_1 \nabla (\mathfrak{A}_2 \mathrm{wr} \Lambda) \subset (\mathfrak{A}_1 \nabla \mathfrak{A}_2) \mathrm{wr} \Lambda,$$

$$(\mathfrak{A}_1 \nabla \mathfrak{A}_2) \mathrm{wr} \Lambda \subset (\mathfrak{A}_1 \mathrm{wr} \Lambda) \nabla (\mathfrak{A}_2 \mathrm{wr} \Lambda).$$

Besides that, if $\mathfrak{A}_1 | \mathfrak{A}_2$ and $\Lambda_1 | \Lambda_2$, then $\mathfrak{A}_1 \mathrm{wr} \Lambda_1 | \mathfrak{A}_2 \mathrm{wr} \Lambda_2$.

Finally, one more remark. Wreath product of the representation $(A, \Gamma) = (A, \Gamma, 0)$ and certain Λ is again the representation. Wreath product of the space and pure automaton is not a space.

Semigroup Γ is called *0-simple* if it has a zero element, which is the unique proper two-sided ideal and besides, $\Gamma^2 \neq 0$. If the set of idempotents of 0-simple semigroup contains a primitive element then the semigroup is said to be *completely 0-simple semigroup*. (The set of idempotents is ordered according to the rule: $e_1 \leq e_2$ if and only if $e_1 e_2 = e_2 e_1 = e_1$). For finite semigroups both these notions coincide.

According to Rees theorem each completely 0-simple semigroup Γ is isomorphic to Rees matrix semigroup $\Gamma = (X, G, Y, [X, Y])$, where X, Y are the sets, G is the group with zero, and P=[X,Y] is the sandwich matrix with elements from G and without zero rows. P defines the multiplication in Γ. Elements from Γ are represented as triplets (x,g,y), x∈X, g∈G, y∈Y, with the multiplication:

$$(x_1,g_1,y_1)(x_2,g_2,y_2)=(x_1,g_1[y_1,x_2]g_2,y_2),$$

where $x_1,x_2\in X$, $g_1,g_2\in G$, $y_1,y_2\in Y$, $[y_1,x_2]\in G$.

Let M be a set. Define an associative operation on the set M assuming $m_1m_2=m_1$, $m_1,m_2\in M$. The obtained semigroup is called the *semigroup of left zeros* on the set M and is denoted by M^{ℓ}. Similar, if $m_1m_2=m_2$, $m_1,m_2\in M$, we get the *semigroup of right zeros* on the set M which is denoted by M^{r} Sometimes we shall identify elements of the set M and of the semigroups M^{ℓ} or M^{r}

For Rees semigroup Γ there is an inclusion

$$\mu:\Gamma \to X^{\ell}\times(\mathrm{Gwr}Y^{r})=X^{\ell}\times(\bar{G}\times Y^{r}),$$

where $\bar{G}=G^{Y^{r}}$ (see [29]). Recall it.

Lemma 2.5. *The mapping* $\mu:(x,g,y)\to(\bar{x},\bar{g},\bar{y})$, *where* $\bar{x}\in X^{\ell}$, $\bar{y}\in Y^{r}$, $\bar{g}\in\bar{G}=G^{Y^{r}}$ *and* $\bar{g}$ *is defined by the formula* $\bar{g}(y')=[y',x]g$, *is a monomorphism of semigroups* $\Gamma\to X^{\ell}\times(\mathrm{Gwr}Y^{r})$.

Proof. We have

$[(x_1,g_1,y_1)(x_2,g_2,y_2)]^{\mu}=(\bar{x}_1,\overline{g_1[y_1,x_2]g_2},\bar{y}_2)$;

$(x_1,g_1,y_1)^{\mu}(x_2,g_2,y_2)^{\mu}=(\bar{x}_1,\bar{g}_1,\bar{y}_1)(\bar{x}_2,\bar{g}_2,\bar{y}_2)=(\bar{x}_1,\bar{g}_1(\bar{y}_1\circ\bar{g}_2),\bar{y}_2)$;

But

$(\bar{g}_1(\bar{y}_1\circ\bar{g}_2))(y)=\bar{g}_1(y)(\bar{y}_1\circ\bar{g}_2)(y)=\bar{g}_1(y)\bar{g}_2(y_1)=[y,x_1]g_1[y_1,x_2]g_2=$

$\overline{g_1[y_1,x_2]g_2}(y)$. So, μ is a homomorphism.

Let $(\bar{x}_1,\bar{g}_1,\bar{y}_1)=(\bar{x}_2,\bar{g}_2,\bar{y}_2)$. Then, evidently, $x_1=x_2$, $y_1=y_2$ and $\bar{g}_1(y)=[y,x_1]g_1$, $\bar{g}_2(y)=[y,x_2]g_2$ Since sandwich matrix has no zero rows, there exists y such that $[y,x]\neq 0$. Taking into account that G is a group, we get $g_1=g_2$.

Let us consider the inclusion G → Γ defined by the rule g → (1,g,1) (with the units of sets X,Y respectively). Construct now an induced automaton from the automaton (A,G) in respect to inclusion

$G \to \Gamma$. Given automaton (A,Γ), one can define the automaton $\mathfrak{A}=(A\otimes KY^{r},\Gamma)$ by the rule: $(a\otimes y)\circ(x,g,y_1)= (a\circ[y,x]g)\otimes y_1$.

Lemma 2.6. *The automaton* $\mathfrak{A}=(A\otimes KY,\Gamma)$ *is the induced automaton for the automaton* $\mathfrak{A}_1=(A,G)$.

Proof. Let us introduce an automaton $\bar{\mathfrak{A}}=(\bar{A},\Gamma)$. Consider formal expressions of the form $A_y=A\circ(1,e,y)$, $y\in Y$, where e is the unit of group G. The automaton $\bar{\mathfrak{A}}=(\bar{A},\Gamma)$ is defined as follows:

$$(\bar{A},\Gamma)=(\otimes\sum_y A_y,\Gamma),\quad (a\circ(1,e,y))\circ(x,g,y_1)=(a\circ[y,x]g)\circ(1,e,y_1).$$

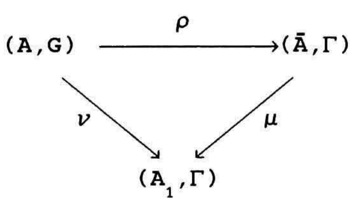
This automaton is a semigroup one. Indeed:

$$(a\circ(1,e,y))\circ(x,g_1,y_1)(x_1,g_2,y_2)=(a\circ(1,e,y))\circ(x_1,g_1[y_1,x_2]g_2,y_2)=$$
$$(a\circ([y,x]g_1[y_1,x_1]g_2))\circ(1,e,y_2);$$
$$((a\circ(1,e,y))\circ(x,g_1,y_1))\circ(x_1,g_2,y_2)=((a\circ[y,x]g_1)\circ(1,e,y_1))\circ(x_1,g_2,y_2)=$$
$$((a\circ[y,x]g_1)\circ([y_1,x_1]g_2))\circ(1,e,y_2)= (a\circ[y,x]g_1[y_1,x_1]g_2)(1,e,y_2).$$

(A,G) is naturally embedded into $(\bar{A},\Gamma)$ by the mapping $\rho=(\rho_1,\rho_2):a \to a\circ(1,e,1)$, $g \to (1,g,1)$. This mapping is the monomorphism of automata:

$$(a\circ g)^{\rho_1}=(a\circ g)\circ(1,e,1);\quad a^{\rho_1}\circ g^{\rho_2}=(a\circ(1,e,1))\circ(1,g,1)=(a\circ g)\circ(1,e,1).$$

Show, that $\bar{\mathfrak{A}}=(\bar{A},\Gamma)$ is the induced automaton for $\mathfrak{A}_1=(A,G)$, that is for any automaton (A_1,Γ) and homomorphism ν there is homomorphism μ, such that the following diagram is commutative:



Define μ by the rule: $(a\circ(1,e,y))^{\mu}=a^{\nu}\circ(1,e,y)$. It is clear that μ is the automata homomorphism. It remains to observe that automata $\bar{\mathfrak{A}}=(\bar{A},\Gamma)$ and $\mathfrak{A}=(A\otimes KY^{r},\Gamma)$ are isomorphic. The isomorphism is defined by the identity mapping on G and by $\varepsilon:\bar{A} \to A\otimes KY$ acting as follows:

$(a\circ(1,e,y))^{\varepsilon}=a\otimes y$.

Lemma 2.7. *Let* (A_1,Γ) *be irreducible automaton (representation) with the Rees semigroup* Γ *and* (A,G) *be its irreducible subautomaton. Then*

$$(A_1,\Gamma)\,|\,(A,G)\mathrm{wr}(Y,Y^{n})$$

Proof. Let consider the following mapping $\mu=(\mu_1,\mu_2)$ of the induced automaton $\bar{\mathfrak{A}}=(A\otimes KY,\Gamma)$ onto $\mathfrak{B}=(A,G)\mathrm{wr}(Y,Y^{n})=(A\otimes KY,G\mathrm{wr}Y^{n})$: μ_1 is the identity mapping on $A\otimes KY$; $\mu_2:\Gamma\to G\mathrm{wr}Y^{n}$ is the product of the mapping μ of Lemma 2.5 and of the projection $\nu:X^{\ell}\times(G\mathrm{wr}Y^{n})\to G\mathrm{wr}Y^{n}$. This μ is the automata homomorphism:

$$((a\otimes y)\circ(x,g,y_1))^{\mu}=((a\circ[y,x]g)\otimes y_1)^{\mu}=(a\circ[y,x]g)\otimes y_1;$$

$$(a\otimes y)^{\mu}\circ(x,g,y_1)^{\mu}=(a\otimes y)\circ(\bar{g},\bar{y}_1)=(a\otimes\bar{g}(y))\otimes y\bar{y}_1=(a\circ[y,x]g)\otimes y_1$$

Denote the image of the automaton $\bar{\mathfrak{A}}$ by $(A\otimes KY,\bar{\Gamma})$. Let $\rho=\mathrm{Ker}\mu_2$. Then, obviously, $(A\otimes KY,\Gamma/\rho)\cong(A\otimes KY,\bar{\Gamma})$. It is easy to verify that ρ coincides with the kernel of the automaton $(A\otimes KY,\Gamma)$. Therefore, $(A\otimes KY,\Gamma/\rho)$ belongs to $(A,G)\mathrm{wr}(Y,Y^{n})$.

Automaton $(A\otimes KY,\Gamma)$ is the induced one for $(A,G)\subset(A_1,\Gamma)$. So, there is a homomorphism $\mu:(A\otimes KY,\Gamma)\to(A_1,\Gamma)$. Since (A_1,Γ) is irreducible automaton, μ is the epimorphism. This implies that the kernel of the automaton $(A\otimes KY,\Gamma)$ lies in the kernel of the automaton (A_1,Γ). Therefore μ defines epimorphism $\mu':(A\otimes KY,\Gamma/\rho)\to(A_1,\Gamma/\rho)$. Thus, the automaton $(A_1,\Gamma/\rho)$ and every its homomorphic image is a homomorphic image of the automaton $(A\otimes KY,\Gamma/\rho)$. The latter lies in the wreath product and finally (A_1,Γ) is a divisor of wreath product.

From the proof of the last Lemma it follows that if a semigroup Γ admits an exact irreducible representation then the inclusion $(A\otimes KY,\Gamma)\subset(A,G)\mathrm{wr}(Y,Y^{n})$ takes place. In particular, Rees semigroup Γ is embedded into $G\mathrm{wr}Y^{n}$

Let us move to the main result.

Theorem 2.8. *Let* $\mathfrak{A}=(A,\Gamma,B)$ *be a linear semigroup automaton,* Γ *be* *a completely 0-simple semigroup. Then*

$$\mathfrak{A} \mid B\nabla\nabla_{i}(\mathfrak{A}_{i}\mathrm{wr}(Y_{i}, Y_{i}^{r})),$$

where each $\mathfrak{A}_i=(A_i, G_i)$ *is an irreducible group automaton being a divisor of the automaton* $\mathfrak{A}=(A,\Gamma,B)$, Y_i^{r} *are the semigroups of right zeros.*

Proof. Each linear automaton admits decomposition 2.10. Components (A_i,Γ_i) of the decomposition are irreducible divisors of (A,Γ). All semigroups Γ_i are also completely 0-simple semigroups (as homomorphic images of Γ).

Let us consider multiples $\mathfrak{A}_i=(A_i,\Gamma_i)$. We have (Lemma 2.7), that

$$(A_i,\Gamma_i) \mid (A'_i, G_i)\mathrm{wr}(Y_i, Y_i^{r}),$$

where (A'_i, G_i) is an irreducible group automaton. Therefore, by Lemma 2.3 it follows

$$(A,\Gamma,B) \mid B\nabla\nabla_{i}((A'_i, G_i)\mathrm{wr}(Y_i, Y_i^{r})).$$

All the automata (A'_i, G_i) are subautomata in (A_i,Γ_i) and, consequently, divisors of the initial $\mathfrak{A}=(A,\Gamma,B)$.

Theorem 2.8 reduces any semigroup automaton with completely 0-simple semigroup to group automata and pure automata. We show that there is further reduction to irreducible automata with simple acting group.

Lemma 2.9. *Let* (A,G) *be an irreducible group automaton and the space* A *is finite dimensional. Then*

$$(A,\Gamma) \mid (\nabla_{i}(A_i, H)\mathrm{wr}(X,\Phi),$$

where (A_i,H) *is irreducible simple group automaton,* (X,Φ) *is pure group automaton.*

Proof. Take in G a composition series

$$G\supset H_1\supset\ldots\supset H_{n-1}\supset H_n=1$$

Given H_1, we have [90]:

$$(A,G) \mid (A,H_1)\mathrm{wr}(X_1,G_1),$$

where $G_1=G/H_1$ is a simple group and X_1 is a group G/H_1, considered as a set. The same arguments for H_1 yeild $(A,H_1)|(A,H_2)wr(X_2,G_2)$, etc. As a result we obtain the decomposition

$$(A,G)|(A,H_{n-1})wr(X_{n-1},G_{n-1})wr...wr(X_1,G_1)=(A,H_{n-1})wr(X',G')$$

It is known that if a group is completely reducible then its normal subgroup is also completely reducible [81]. Therefore, (A,H_{n-1}) is completely reducible. Then $(A,H_{n-1})\subset\nabla_i(A_i,H_{n-1})$, where components (A_i,H_{n-1}) are irreducible. Denote $H=H_{n-1}$ This is a simple group. Finally,

$$(A,G)|(\nabla_i(A_i,H)wr(X',G').$$

Recall, that for pure automaton (X',G') there exists a decomposition into wreath product of simple group components:

$$(X',G')=(X_{n-1},G_{n-1})wr...wr(X_1,G_1)$$

where all G_i are simple groups.

Now, we can generalize the Theorem 2.8.

Theorem 2.10. *Let $\mathfrak{A}=(A,\Gamma,B)$ be a linear semigroup automaton, Γ be a completely 0-simple semigroup. Then*

$$\mathfrak{A}|B\nabla\nabla_i(((\nabla_j\mathfrak{A}_{ij})wr\Lambda_{is})wr\Lambda'_i),$$

where $\mathfrak{A}_{ij}$ - linear irreducible semigroup automata, Λ_{is} - pure group automata with simple acting group, Λ'_i pure semigroup automata in which acting semigroups are the semigroups of right zeros. Besides, all linear automata $\mathfrak{A}_{ij}$ are divisors of the initial automaton $\mathfrak{A}$.

At last, in order to find out the structure of decomposition it remains to describe the form of indecomposable automata.

3.2.4. Indecomposable linear automata

Lemma 2.11. *An automaton $\mathfrak{A}=(A,\Gamma,B)$ is indecomposable into triangular product, if and only if $\mathfrak{A}$ is an irreducible representation or one-dimensional linear space.*

Proof. Since $(A,\Gamma,B)|(0,0,B)\nabla(A,\Gamma,0)$, then in order $\mathfrak{A}$ to be indecomposable, either $(A,\Gamma,B)|(0,0,B)$ or $(A,\Gamma,B)|(A,\Gamma,0)$. In the first case (A,Γ,B) is obviously equal to $(0,0,B_1)$, $\dim B_1=1$; in the second one immediately B=0 and it follows from Theorem 2.1 that (A,Γ) has to be an irreducible representation.

Conversely. It is obvious that one-dimensional space B is indecomposable. Let irreducible representation (A,Γ) divide the triangular product of automata $\mathfrak{A}_1=(A_1,\Gamma_1)$ and $\mathfrak{A}_2=(A_2,\Gamma_2)$. This means that $\mathfrak{A}=(A,\Gamma)$ is a homomorphic image of subautomaton $\mathfrak{A}'=(A',\Gamma')$ of $\mathfrak{A}_1\nabla\mathfrak{A}_2=(A_1\oplus A_2,\Gamma_1\times\mathrm{Hom}(A_2,A_1)\times\Gamma_2)=(\tilde{A},\tilde{\Gamma})$. Denoting by (A_0',ρ) the kernel of this homomorphism we get $(A'/A_0,\Gamma'/\rho)\cong(A,\Gamma)$. Take in $\tilde{A}=A_1\oplus A_2$ two series $0\subset A_1\subset\tilde{A}$, $0\subset A_0\subset A'\subset\tilde{A}$ invariant with respect to Γ'. According to Jordan-Holder theorem [58], this series can be extended to Γ'-invariant compositional ones with isomorphic factors. Since the representation $(A'/A_0,\Gamma'/\rho)\cong(A,\Gamma)$ is irreducible then Γ'-factor A'/A_0 is a divisor of one of the factors of $0\subset A_1\subset\tilde{A}$. If it divides Γ'-factor A_1, then (A,Γ) is a divisor of (A_1,Γ_1). If it divides the second factor then (A,Γ) is a divisor of (A_2,Γ_2). This means that (A,Γ) is indecomposable.

Linear automaton (A,Γ) is called *decomposable in wreath product* if it can be represented in the form

$$(A,\Gamma)\,|\,(B,\Sigma)\mathrm{wr}(X,H),$$

where (A,Γ) is not a divisor of (B,Σ), and Γ is not a divisor of H. Otherwise an automaton is called *indecomposable in wreath product*. The definition of automaton indecomposable in triangular product has been given earlier. Joining these two notions we speak of *indecomposable linear automaton.*

Proposition 2.12. *Linear automaton* (A,Γ) *where* Γ *is a completely 0-simple semigroup is indecomposable if and only if* (A,Γ) *is an irreducible simple group automaton.*

Proof. Let (A,Γ) be an indecomposable automaton. Then by Lemma 2.11 (A,Γ) is an irreducible representation. According to Lemma 2.7 Γ is a group. If the group is not a simple one, then by Lemma 2.9 the automaton (A,Γ) is decomposable. So, Γ is a simple group.

Conversely, let (A,Γ) be an irreducible simple group automaton. Since (A,Γ) is an irreducible representation then by Lemma 2.11 (A,Γ) is indecomposable in triangular product.

Let now (A,Γ) is decomposable in wreath product, i.e.

$$(A,\Gamma)\,|\,(B,\Sigma)\mathrm{wr}(X,H),$$

and (A,Γ) is not a divisor of (B,Σ), Γ is not a divisor of H. Since $(A,\Gamma)|(B,\Sigma)\mathrm{wr}(X,H)$ then in $(B\otimes KX,\Sigma^X\times\Gamma)$ there exists a subautomaton (C,Φ) and an epimorphism μ such that $(C,\Phi)^\mu=(A,\Gamma)$. Consider $\Phi_1=\Phi\cap\Sigma^X$ Since Σ^X is an invariant subgroup in $(\Sigma^X\times H)$, then Φ_1 is invariant subgroup in Φ. Take Φ_1^μ It is invariant in $\Phi^\mu=\Gamma$ subgroup and since Γ is a simple group then either $\Phi_1^\mu=\Gamma$ or $\Phi_1^\mu=1$.

1. Let $\Phi_1^\mu=\Gamma$. Then $(C,\Phi_1)^\mu=(A,\Gamma)$ or in other words $(A,\Gamma)|(B\otimes KX,\Sigma^X)$. Show, that in this case $(A,\Gamma)|(B,\Sigma)$. Let X be a finite set (this assumption is quite natural for automata, but it is not necessary). Then

$$(B\otimes KX,\Sigma^X)=(\oplus\sum_{x\in X}(B\otimes x),\prod_i\Sigma),\ \ i=1,\dots,k;\ \ k=|X|.$$

The group $\prod_i\Sigma$ acts componentwise. Consider more general situation. Let

$$(D,G)=\prod_i(B_i,\Sigma_i)=(\oplus\sum_i B_i,\prod_i\Sigma_i),\ \ i=1,\dots,k;\ \ k=|X|$$

and $(A,\Gamma)|(D,G)$. Then if (A,Γ) is exact irreducible automaton then (A,Γ) divides one of the summands. Setting $B_i=B$, $\Sigma_i=\Sigma$ we get that $(A,\Gamma)|(B,\Sigma)$.

Take $(C,\Phi)\subset(D,G)$ such that $(C,\Phi)^\mu=(A,\Gamma)$. Then $(A,\Gamma)\cong(C/C_0,\Phi/\Phi_0)$, where (C_0,Φ_0) is the kernel of epimorphism μ. Denote

$$D_1=B_1,\ \ D_2=B_1\oplus B_2,\dots,D_n=(B_1\oplus B_2\oplus\dots\oplus B_n)=D.$$

We have a series of Φ-modules $0\subset D_1\subset D_2\subset\dots\subset D_n=D$ with the factors isomorphic to B_i

Let us consider some other series of Φ-invariant subspaces, containing $C_0\subset C$. Since Φ acts irreducibly in C/C_0 then according to Jordan-Holder theorem, C/C_0 is isomorphic to a factor of certain B_i (as Φ-module). Let $G_0=\prod_{j\neq i}\Sigma_j$. Since G_0 acts componentwise, it lies in the kernel (B_i,G) and, therefore, $G_0\cap\Phi$ lies in kernels (B_i,Φ) and $(C/C_0,\Phi)$. Furthermore, $(B_i,G/G_0)\cong(B_i,\Sigma_i)$. But

$$(B_i,\Sigma_i)\cong(B_i,G/G_0)\supset(B_i,\Phi G_0/G_0)\cong(B_i,\Phi/\Phi\cap G_0).$$

Since $(A,\Gamma)=(C/C_0,\Phi/\Phi_0)$ is an exact automaton, the kernel $(C/C_0,\Phi)$ is equal to Φ_0. Thus $G_0 \cap \Phi \subset \Phi_0$. Therefore, there exists a homomorphism

$$(C, \Phi/\Phi \cap G_0) \longrightarrow (C/C_0, \Phi/\Phi_0) \cong (A, \Gamma).$$

We get $(A,\Gamma) \mid (B,\Sigma)$ that contradicts indecomposability of the automaton (A,Γ).

2. $\Phi_1^{\mu}=1$. Then

$$H \cong H\Sigma^X/\Sigma^X \supset \Phi\Sigma^X/\Sigma^X; \quad (\Phi\Sigma^X/\Sigma^X)^{\mu} \cong (\Phi/\Sigma^X \cap \Phi)^{\mu} \cong (\Phi/\Phi_1)^{\mu} = \Gamma,$$

i.e. $\Gamma \mid H$, that also contradicts indecomposability of the automaton (A,Γ).

Thus, all linear components of the decomposition 2.10 are indecomposable automata.

3.2.5. Triangular products and homomorphisms of biautomata

The results described in this item ([41]) will be used in Chapter 4 in the proof of the Theorem of freeness of semigroup of biautomata varieties.

Proposition 2.13. *Let* $(A,\Gamma,B)=(A_1,\Sigma_1,B_1)\nabla(A_2,\Sigma_2,B_2)$ *be triangular product of biautomata and* (A',Γ,B') *be subautomata in* (A,Γ,B), $A_1 \subset A'$, $B_1 \subset B'$. *Then there is an epimorphism in input signals*:

$$(A',\Gamma,B') \longrightarrow (A_1,\Sigma_1,B_1)\nabla(A_2 \cap A',\Sigma_2,B_2 \cap B').$$

Proof. By the definition of triangular product, $\Gamma=\Sigma_1\times\Phi_1\times\Psi\times\Phi_2\times\Sigma_2$, $\Phi_1=\mathrm{Hom}(A_2,A_1)$, $\Psi=\mathrm{Hom}(A_2,B_1)$, $\Phi_2=\mathrm{Hom}(B_2,B_1)$. Denote $A_2 \cap A'=A_2'$, $B_2 \cap B'=B_2'$. Consider biautomaton $(A_1,\Sigma_1,B_1)\nabla(A_2',\Sigma_2,B_2')=(A',\Gamma',B')$, where $\Gamma'=\Sigma_1\times\Phi_1'\times\Psi'\times\Phi_2'\times\Sigma_2$, $\Phi_1'=\mathrm{Hom}(A_2',A_1)$, $\Psi'=\mathrm{Hom}(A_2',B_1)$, $\Phi_2'=\mathrm{Hom}(B_2',B_1)$.

Take the mapping $\mu_1:\Phi_1 \longrightarrow \Phi_1'$, assigning to the element $\varphi\in\mathrm{Hom}(A_2,A_1)$ the element of $\mathrm{Hom}(A_2',A_1)$ which is the restriction of φ on A_2'. The mappings $\mu_2:\Psi \longrightarrow \Psi'$ and $\mu_3:\Phi_2 \longrightarrow \Phi_2'$ are defined similarly. Identity mappings of Σ_1 and Σ_2 together with mentioned mappings μ_1,μ_2,μ_3 define epimorphism Γ on Γ'. The latter in its turn defines the epimorphism in input signals.

Proposition 2.14. *Let*

$$\mu=(\mu_1,\mu_2,\mu_3):(A_1,\Sigma_1,B_1) \longrightarrow (A_1',\Sigma_1',B_1')$$

be an automata homomorphism and (A_2,Σ_2,B_2) *be an arbitrary automaton.*

Then there exists identical on (A_2,Σ_2,B_2) *homomorphism*

$$\bar{\mu}:(A_1,\Sigma_1,B_1)\nabla(A_2,\Sigma_2,B_2)\rightarrow(A'_1,\Sigma'_1,B'_1)\nabla(A_2,\Sigma_2,B_2).$$

If μ *is monomorphism (epimorphism) then* $\bar{\mu}$ *is also monomorphism (epimorphism).*

Proof. Denote

$$(A,\Gamma,B)=(A_1,\Sigma_1,B_1)\nabla(A_2,\Sigma_2,B_2)=(A_1+A_2,\Sigma_1\times\Phi_1\times\Psi\times\Phi_2\times\Sigma_2,B_1+B_2),$$

$$(A',\Gamma',B')=(A'_1,\Sigma'_1,B'_1)\nabla(A_2,\Sigma_2,B_2)=(A'_1+A_2,\Sigma'_1\times\Phi'_1\times\Psi'\times\Phi'_2\times\Sigma_2,B'_1+B_2).$$

Define the homomorphism $\bar{\mu}=(\bar{\mu}_1,\bar{\mu}_2,\bar{\mu}_3)$ by the rule: if $a_1+a_2\in A$, where $a_i\in A_i$, $i=1,2$, then $(a_1+a_2)^{\bar{\mu}_1}=a_1^{\mu_1}+a_2$. The same for $\bar{\mu}_3$.

Now define the mapping $\mu_{21}:\Phi_1=\mathrm{Hom}(A_2,A_1)\rightarrow\Phi'_1=\mathrm{Hom}(A_2,A'_1)$ by the rule: if $\varphi\in\Phi_1$, $a_2\in A_2$, then $a_2\varphi^{\mu_{21}}=(a_2\varphi)^{\mu_1}$; the mapping $\mu_{22}:\Phi_2=\mathrm{Hom}(B_2,B_1)\rightarrow\Phi'_2=\mathrm{Hom}(B_2,B'_1)$ by the rule: if $\varphi\in\Phi_2$, $b_2\in B_2$, then $b_2\varphi^{\mu_{22}}=(b_2\varphi)^{\mu_3}$ and the mapping $\mu_{23}:\Psi=\mathrm{Hom}(A_2,B_1)\rightarrow\Psi'=\mathrm{Hom}(A_2,B'_1)$ by the rule: if $\psi\in\Psi$, $a_2\in A_2$, then $a_2\psi^{\mu_{23}}=(a_2\psi)^{\mu_1}$

The last three mappings together with the mapping $\mu_2:\Sigma_1\rightarrow\Sigma'_1$ and identity mapping $\varepsilon:\Sigma_2\rightarrow\Sigma_2$ define homomorphism $\bar{\mu}_2:\Gamma\rightarrow\Gamma'$. It is easy to verify that $\bar{\mu}=(\bar{\mu}_1,\bar{\mu}_2,\bar{\mu}_3)$ is the desired homomorphism of biautomata.

The kernel of this homomorphism $\rho=(\rho_1,\rho_2,\rho_3)$ has the form: $\rho_1=\mathrm{Ker}\mu_1$, $\rho_2=\mathrm{Ker}\mu_2\times\mathrm{Hom}(A_2,\mathrm{Ker}\mu_1)\times\mathrm{Hom}(B_2,\mathrm{Ker}\mu_3)\times\mathrm{Hom}(A_2,\ \mathrm{Ker}\mu_3)$, $\rho_3=\mathrm{Ker}\mu_3$. Therefore, if μ is a monomorphism (epimorphism) then $\bar{\mu}$ is also monomorphism (epimorphism).

Proposition 2.15. *Let* $(A,\Gamma,B)=\prod_{\alpha}(A_\alpha,\Gamma_\alpha,B_\alpha)$, $\alpha\in I$ *and let* (G,Γ',H) *be an arbitrary biautomaton. Then there exists an embedding* $\mu:(A,\Gamma,B)\nabla(G,\Gamma',H)\rightarrow\prod_{\alpha}((A_\alpha,\Gamma_\alpha,B_\alpha)\nabla(G,\Gamma',H))$.

Proof. Let

$$(A,\Gamma,B)\nabla(G,\Gamma',H)=(\sum_{\alpha}A_\alpha+G,\prod_{\alpha}\Gamma_\alpha\times\Phi_1\times\Psi\times\Phi_2\times\Gamma',\sum_{\alpha}B_\alpha+H),$$

$$\prod_{\alpha}((A_\alpha,\Gamma_\alpha,B_\alpha)\nabla(G,\Gamma',H))=(\sum_{\alpha}(A_\alpha+G),\prod_{\alpha}\Gamma_\alpha\times\prod_{\alpha}\Phi_{1\alpha}\times\prod_{\alpha}\Psi_\alpha\times\prod_{\alpha}\Phi_{2\alpha}\times\prod_{\alpha}\Gamma',\sum_{\alpha}(B_\alpha+H))$$ where

$\Phi_1=\mathrm{Hom}(G,\sum_{\alpha}A_\alpha)$, $\Psi=\mathrm{Hom}(G,\sum_{\alpha}B_\alpha)$, $\Phi_2=\mathrm{Hom}(H,\sum_{\alpha}B_\alpha)$, $\Phi_{1\alpha}=\mathrm{Hom}(G,A_\alpha)$, $\Psi_\alpha=\mathrm{Hom}(G,B_\alpha)$,

$\Phi_{2\alpha}=\mathrm{Hom}(H,B_\alpha)$.

Mapping $\mu=(\mu_1,\mu_2,\mu_3)$ is defined in the following way: if $\bar{a}\in\sum_\alpha A_\alpha$, $g\in G$, then $(\bar{a}+g)^{\mu_1}(\alpha)=\bar{a}(\alpha)+g$; if $\bar{b}\in B_\alpha$, $h\in H$, then $(\bar{b}+h)^{\mu_3}(\alpha)=\bar{b}(\alpha)+h$. Besides that let

$\mu_{22}:\Phi_1 \to \prod_\alpha \Phi_{1\alpha}$: if $\varphi_1\in\Phi_1$, $g\in G$, then $g(\varphi_1^{\mu_{22}}(\alpha))=(g\varphi_1)(\alpha)$;

$\mu_{23}:\Psi \to \prod_\alpha \Psi_\alpha$: if $\psi\in\Psi$, $g\in G$, then $g(\psi^{\mu_{23}}(\alpha))=(g\psi)(\alpha)$;

$\mu_{24}:\Phi_2 \to \prod_\alpha \Phi_{2\alpha}$: if $\varphi_2\in\Phi_2$, $h\in H$, then $h(\varphi_2^{\mu_{24}}(\alpha))=(h\varphi_2)(\alpha)$;

$\mu_{25}:\Gamma' \to \prod_\alpha \Gamma'$: if $\gamma\in\Gamma'$, then $\gamma^{\mu_{25}}(\alpha)=\gamma$; $\alpha\in I$.

These mappings together with identity mapping $\prod_\alpha \Gamma_\alpha \to \prod_\alpha \Gamma_\alpha$ define a homomorphism of the acting semigroups. It remains to verify that the constructed $\mu=(\mu_1,\mu_2,\mu_3)$ is a homomorphism of biautomata.

Corollary. *Given biautomata* (A_1,Γ_1,B_1), (A_2,Γ_2,B_2) *and the set* I, *there exists an embedding*:

$$\mu: (A_1,\Gamma_1,B_1)^I\nabla(A_2,\Gamma_2,B_2) \to ((A_1,\Gamma_1,B_1)\nabla(A_2,\Gamma_2,B_2))^I.$$

The following propositions can be established.

Proposition 2.16. *Given biautomata* (A_1,Γ_1,B_1), (A_2,Γ_2,B_2) *and the set* I, *there exists an embedding*:

$$\mu: (A_1,\Gamma_1,B_1)\nabla(A_2,\Gamma_2,B_2)^I \to ((A_1,\Gamma_1,B_1)\nabla(A_2,\Gamma_2,B_2))^I$$

Proposition 2.17. *Let* $(A,\Gamma,B)=(A_1,\Sigma_1,B_1)\nabla(A_2,\Sigma_2,B_2)$ *and let* (A'_2,Σ_2,B'_2) *be a subautomaton in* (A_2,Σ_2,B_2). *Then there exists an epimorphism in input signals*:

$$(A_1+A'_2,\Gamma,B_1+B'_2) \to (A_1,\Sigma_1,B_1)\nabla(A'_2,\Sigma_2,B'_2)$$

Proposition 2.18. *Let* $(A,\Gamma,B)=(A_1,\Sigma_1,B_1)\nabla(A_2,\Sigma_2,B_2)$ *and let* A'_1,B'_2 *be subspaces in* A_1,B_2 *respectively, such that* $A'_1\circ\Sigma_1\subset A'_1$, $B'_2\bullet\Sigma_2\subset B'_2$. *Then there exists an epimorphism in input signals*:

$$(A'_1,\Gamma,B_1+B'_2) \rightarrow (A'_1,\Sigma_1,B_1)\nabla(0,\Sigma_2,B'_2)$$

Note, finally, that in [80] is introduced the construction of triangular product of affine automata and the problem of their decomposition is considered. In [32] the same problem for ring automata is regarded.

3.3. Automorphisms of linear automata and biautomata

Automorphisms of a mathematical structure determine it in many aspects. It is by this, in particular, that the constant interest in the groups of automorphisms is explained. Our aim in this section is to consider automorphism of linear automata and biautomata, and to describe then for the corresponding universal objects and constructions (see [16], [17]).

3.3.1. Definitions and basic lemmas

An *automorphism of a linear representation* (A,Γ) is a pair of mappings (σ_A,α), where σ_A is an automorphism of the space A, α is an automorphism of the semigroup Γ and the condition

$$(a\circ\gamma)\sigma_A = a\sigma_A\circ\gamma\alpha \ , \quad a\in A \ , \quad \gamma\in\Gamma \tag{3.1}$$

is satisfied. An *automorphism of the linear automaton* $\mathfrak{A}=(A,\Gamma,B)$ is a triplet of mappings $(\sigma_A,\alpha,\sigma_B)$ satisfying the following conditions:

1) (σ_A,α) is an automorphism of the linear representation (A,Γ), σ_B is an automorphism of the space B ;

2) $(a*\gamma)\sigma_B = a\sigma_A*\gamma\alpha \ , \quad a\in A, \ \gamma\in\Gamma.$ (3.2)

In definition of an *automorphism of a biautomaton*, the third condition is added to the above ones:

3) (σ_B,α) is an automorphism of the linear representation (B,Γ). (3.3)

Since a linear automaton can be considered as a particular case of a biautomaton with $b\cdot\gamma=0$, $b\in B$, $\gamma\in\Gamma$, the definition of an automorphism of a linear automaton can be considered as a particular case of definition of an automorphism of a biautomaton. Due to this, we shall consider mainly the automorphisms of biautomata, if necessary, making corrections.

If $\mathfrak{A}=(A,\Gamma,B)$ is an exact biautomaton, the representation $\mu:\Gamma \rightarrow \Gamma' \subset \mathrm{End}^b(A,B)$ defines an isomorphism of the automata $\mathfrak{A}=(A,\Gamma,B)$ and $\mathfrak{A}'=(A,\Gamma',B)\subset \mathrm{Atm}^1(A,B)$. Since the symmetries of the isomorphic objects are identical, the automorphisms of one of the objects can be described by the automorphisms of the other. Thus, not violating the generality of considerations, one can assume, if necessary, that $\Gamma\subset \mathrm{End}^b(A,B)$.

Now we shall prove that in order to define automorphisms of an exact biautomaton, it is sufficient to point out automorphisms of the linear spaces A and B, satisfying an additional condition.

Lemma 3.1. 1) *If $(\sigma_A,\alpha,\sigma_B)$ is an automorphism of an exact linear automaton $\mathfrak{A}=(A,\Gamma,B)$, $\gamma=(\varphi_A,\psi)$ is an element of $\Gamma\subset \mathrm{End}^{\ell}(A,B)=\mathrm{End}\, A\times \mathrm{Hom}(A,B)$, then*

$$\gamma\alpha=(\sigma_A^{-1}\varphi_A\sigma_A,\ \sigma_A^{-1}\psi\sigma_B).$$

2) *If $(\sigma_A,\alpha,\sigma_B)$ is an automorphism of an exact biautomaton $\mathfrak{A}=(A,\Gamma,B)$, $\gamma=(\varphi_A,\psi,\varphi_B)$ is an element of $\Gamma\subset \mathrm{End}^b(A,B)=\mathrm{End}A\times \mathrm{Hom}(A,B)\times \mathrm{End}B$, then*

$$\gamma\alpha=(\sigma_A^{-1}\varphi_A\sigma_A,\ \sigma_A^{-1}\psi\sigma_B,\sigma_B^{-1}\varphi_B\sigma_B).$$

Proof. 1) Let $\gamma\alpha=(\varphi'_A,\psi')\in\Gamma$ and $a\in A$. Then, by the definition of an automorphism of an automaton $(a\circ\gamma)\sigma_A=a\sigma_A\circ\gamma\alpha$. Since $a\circ\gamma=a\varphi_A$, then $(a\circ\gamma)\sigma_A=a\varphi_A\sigma_A$, and since $\gamma\alpha=(\varphi'_A,\psi')$ then $a\sigma_A\circ\gamma\alpha=a\sigma_A\varphi'_A$. Hence, $a\varphi_A\sigma_A=a\sigma_A\varphi'_A$, $\varphi_A\sigma_A=\sigma_A\varphi'_A$, and finally $\varphi'_A=\sigma_A^{-1}\varphi_A\sigma_A$. Similarly, from the condition $(a*\gamma)\sigma_A=a\sigma_A*\gamma\alpha$ it follows $\psi'=\sigma_A^{-1}\psi\sigma_B$.

The validity of the second statement of the lemma is proved in the same way.

Denote by Σ_A,Σ_B the groups of all the automorphisms of the linear spaces A and B respectively. Let consider the Cartesian product $\Sigma=\Sigma_A\times\Sigma_B$ and define the action of Σ on $\mathrm{End}^b(A,B)$: if $(\varphi_A,\psi,\varphi_B)\in \mathrm{End}^b(A,B)$, $(\sigma_A,\sigma_B)\in\Sigma$, then

$$(\varphi_A,\psi,\varphi_B)\circ(\sigma_A,\sigma_B)=(\sigma_A^{-1}\varphi_A\sigma_A,\sigma_A^{-1}\psi\sigma_B,\sigma_B^{-1}\varphi_B\sigma_B).$$

Lemma 3.2. *Let $\mathfrak{A}=(A,\Gamma,B)$ be an exact biautomaton, $\Gamma\subset\mathrm{End}(A,B)$, and (σ_A,σ_B) be an element of $\Sigma_A\times\Sigma_B$ such that for all elements $\gamma=(\varphi_A,\psi,\varphi_B)\in\Gamma$*

the inclusion $(\varphi_A,\psi,\varphi_B)\circ(\sigma_A,\sigma_B)\in\Gamma$ *is valid. Then the mapping* $\alpha:\Gamma\to\Gamma$ *defined according to the rule:* $(\varphi_A,\psi,\varphi_B)\alpha=(\varphi_A,\psi,\varphi_B)\circ(\sigma_A,\sigma_B)$, *is an automorphism of the semigroup* Γ, *and the triplet* $(\sigma_A,\alpha,\sigma_B)$ *is an automorphism of the biautomaton* $\mathfrak{A}$.

This simple lemma means that the element (σ_A,σ_B) of Σ determines an automorphism of the biautomaton, if (σ_A,σ_B) satisfies the conditions of Lemma 3.2.

3.3.2. Automorphisms of universal biautomata

Let $\mathrm{Atm}^1(A,B)=(A,\mathrm{End}^b(A,B),B)$ be an universal biautomaton. It follows from the obvious inclusion $(\varphi_A,\psi,\varphi_B)\circ(\sigma_A,\sigma_B)\in\mathrm{End}^b(A,B)$ and from Lemma 3.2 that the following is true:

Proposition 3.3. *The group of automorphisms of an universal biautomaton* $\mathrm{Atm}^1(A,B)$ *is isomorphic to the Cartesian product* $\Sigma_A\times\Sigma_B$ *of the groups of automorphisms of linear spaces* A *and* B.

Consider automorphisms of the universal biautomaton $\mathrm{Atm}^2(\Gamma,B)$.

Proposition 3.4. *The group of automorphisms of an universal biautomaton* $\mathrm{Atm}^2(\Gamma,B)$ *is isomorphic to the group of automorphisms of the representation* (B,Γ).

Proof. Let (σ_B,α) be an automorphism of the representation (B,Γ) and $(\sigma_A,\alpha,\sigma_B)$ be an automorphism of the universal biautomaton $\mathrm{Atm}^2(\Gamma,B)=(B^\Gamma,\Gamma,B)$ having the same α and σ_B. By the definition of an automorphism of a biautomaton, if $\varphi\in B^\Gamma$, $\gamma\in\Gamma$, $b\in B$, then

$$(\varphi\circ\gamma)\sigma_A=\varphi\sigma_A\circ\gamma\alpha\ ,$$

$$(\varphi*\gamma)\sigma_B=\varphi\sigma_A*\gamma\alpha\ ,$$

$$(b\cdot\gamma)\sigma_B=b\sigma_B\cdot\gamma\alpha\ .$$

We write the second of equalities in a more involved form

$$(\varphi*\gamma)\sigma_B=\varphi(\gamma)\sigma_B;\qquad \varphi\sigma_A*\gamma\alpha=\varphi\sigma_A(\gamma\alpha),$$

Hence

$$\varphi(\gamma)\sigma_B=\varphi\sigma_A(\gamma\alpha).$$

Denote $\gamma\alpha=x$, then $\gamma=x\alpha^{-1}$; thus,

$$\varphi\sigma_A(x)=\varphi(x\alpha^{-1})\sigma_B. \qquad (3.4)$$

The last of equalities determines the automorphism σ_A of the linear space $A=B^{\Gamma}$. Thus if $(\sigma_A,\alpha,\sigma_B)$ is an automorphism of the biautomaton $Atm^2(\Gamma,B)$, then σ_A should be defined by the equality (3.4). Verify that a triplet $(\sigma_A,\alpha,\sigma_B)$, with σ_A thus defined, is indeed an automorphism of the biautomaton. The fact of σ_A being a linear transformation of the space $A=B^{\Gamma}$ is clear. The condition (3.3) follows from the initial condition. Equality (3.2) is fulfilled, since σ_A was chosen in an appropriate way. It remains to check the condition (3.1), namely one should show that $(\varphi\circ\gamma)\sigma_A=\varphi\sigma_A\circ\gamma\alpha$:

$((\varphi\circ\gamma)\sigma_A)(x)=((\varphi\circ\gamma)(x\alpha^{-1}))\sigma_B=(\varphi(\gamma(x\alpha^{-1}))-\varphi(\gamma)\cdot x\alpha^{-1})\sigma_B=$

$=(\varphi(\gamma(x\alpha^{-1}))\sigma_B-(\varphi(\gamma)\cdot x\alpha^{-1})\sigma_B=\varphi(\gamma(x\alpha^{-1}))\sigma_B-\varphi(\gamma)\sigma_B\cdot(x\alpha^{-1})\alpha=$

$=\varphi(\gamma(x\alpha^{-1}))\sigma_B-\varphi(\gamma)\sigma_B\cdot x.$

On the other hand,

$(\varphi\sigma_A\circ\gamma\alpha)(x))=\varphi\sigma_A(\gamma\alpha x)-\varphi\sigma_A(\gamma\alpha)\cdot x=\varphi((\gamma\alpha x)\alpha^{-1})\sigma_B-\varphi(\gamma\alpha\alpha^{-1})\sigma_B\cdot x=$

$=\varphi(\gamma(x\alpha^{-1}))\sigma_B-\varphi(\gamma)\sigma_B\cdot x.$

Hence

$$(\varphi\circ\gamma)\sigma_A=\varphi\sigma_A\circ\gamma\alpha.$$

Finally, it is easy to verify that the correspondence, uniquely assigning to each automorphism (σ_B,α) of the representation (B,Γ) an automorphism $(\sigma_A,\alpha,\sigma_B)$ of the automaton $Atm^2(\Gamma,B)$, preserves the multiplication, and hence defines an isomorphism of the groups of automorphisms mentioned in the conditions of the Proposition.

Proposition 3.5. *The group of automorphisms of an universal biautomaton* $Atm^3(A,\Gamma)$ *is isomorphic to the group of automorphisms of the representation* (A,Γ).

Proof. Let (σ_A,α) be an automorphism of the representation (A,Γ) and $(\sigma_A,\alpha,\sigma_B)$ be an automorphism of the biautomaton $Atm^3(A,\Gamma)=(A,\Gamma,A\otimes_K K\Gamma)$. We remind that the operations $*$ and $\cdot$ in the automaton are defined as follows: if $a\in A$, $\gamma\in\Gamma$, then $a*\gamma=a\otimes\gamma$; if $a\otimes u\in A\otimes_K K\Gamma$, $\gamma\in\Gamma$, then

$$(a\otimes u)\cdot\gamma=a\otimes u\gamma-(a\circ u)\otimes\gamma.$$

Since $(\sigma_A,\alpha,\sigma_B)$ is an automorphism of a biautomaton, the equality $(a*\gamma)\sigma_B=a\sigma_A*\gamma\alpha$ should be held, from which

$$(a\otimes\gamma)\sigma_B=a\sigma_A\otimes\gamma\alpha. \tag{3.5}$$

The latter equality defines the action of σ_B on generating elements of the space $A\otimes_K K\Gamma$, which by linearity is extended to the whole of the space $A\otimes_K K\Gamma$. We verify that the triplet $(\sigma_A,\alpha,\sigma_B)$ with the given σ_A, α and σ_B, defined by the equality (3.5), is indeed an automorphism of the biautomaton $Atm^3(A,\Gamma)$. By the condition, (σ_A,α) is an automorphism of the representation (A,Γ) and the validity of the condition (3.2) is determined by the construction of σ_B. It remains to show the validity of the condition (3.3): $(b\cdot\gamma)\sigma_B=b\sigma_B\cdot\gamma\alpha$, where $b=a\otimes u\in A\otimes_K K\Gamma$.

$(b\cdot\gamma)\sigma_B=((a\otimes u)\cdot\gamma)\sigma_B=(a\otimes u\gamma-(a\circ u)\otimes\gamma)\sigma_B=(a\otimes u\gamma)\sigma_B-((a\circ u)\otimes\gamma)\sigma_B=$

$=a\sigma_A\otimes(u\gamma)\alpha-(a\circ u)\sigma_A\otimes\gamma\alpha=a\sigma_A\otimes u\alpha\cdot\gamma\alpha-(a\sigma_A\circ u\alpha)\otimes\gamma\alpha.$

On the other hand,

$$b\sigma_B\cdot\gamma\alpha=(a\otimes u)\sigma_B\cdot\gamma\alpha=(a\sigma_A\otimes u\alpha)\cdot\gamma\alpha=a\sigma_A\otimes u\alpha\cdot\gamma\alpha-(a\sigma_A\circ u\alpha)\otimes\gamma\alpha.$$

Thus, $(b\cdot\gamma)\sigma_B=b\sigma_B\cdot\gamma\alpha$.

The one-to-one correspondence, assigning in such a way to each automorphism (σ_A,α) of the representation (A,Γ) an automorphism $(\sigma_A,\alpha,\sigma_B)$ of the universal biautomaton $Atm^3(A,\Gamma)$ is an isomorphism of the groups of automorphisms of the representation (A,Γ) and of the biautomaton $Atm^3(A,\Gamma)$.

Note that the groups of automorphisms of universal pure automata and universal linear automata are described in a similar way [17], and that groups of automorphisms of the universal biautomaton $Atm^1(A,B)$ and the corresponding universal linear automaton of the first type are isomorphic. The groups of automorphisms of an universal biautomaton $Atm^3(A,\Gamma)$ and the universal linear automaton of the third type are also isomorphic.

3.3.3. Automorphisms of the triangular product of biautomata

Lemma 3.6. *If* $(\sigma_A,\alpha,\sigma_B)$ *is an automorphism of the triangular pro-*

duct of biautomata $\mathfrak{A}_1\nabla\mathfrak{A}_2=(A_1\oplus A_2,\Gamma,B_1\oplus B_2)=(A,\Gamma,B)$, *then the elements* σ_A *and* σ_B *allow the matrix form*:

$$\sigma_A=\begin{pmatrix}\alpha_{22} & \alpha_{21}\\ 0 & \alpha_{11}\end{pmatrix}\qquad \sigma_A=\begin{pmatrix}\beta_{22} & \beta_{21}\\ 0 & \beta_{11}\end{pmatrix}$$

$\alpha_{ij}\in \mathrm{Hom}(A_i,A_j)$, $\beta_{ij}\in \mathrm{Hom}(B_i,B_j)$, $i,j=1,2$.

Proof. It follows from the definition of the triangular product that, if A' is Γ-invariant subspace of $A_1\oplus A_2$, then either $A'\supset A_1$, or $A'\subset A_1$.

Consider $(A_1^{\sigma_A},\Gamma,B_1^{\sigma_B})=(A_1^{\sigma_A},\Gamma,B_1^{\sigma_B})$, the image of the biautomaton $(A_1,\Gamma,B_1)\subset\mathfrak{A}$ under the automorphism $(\sigma_A,\alpha,\sigma_B)$ of the biautomaton $\mathfrak{A}$. Since $A_1^{\sigma_A}$ is a Γ-invariant subspace, either $A_1^{\sigma_A}\subset A_1$, or $A_1^{\sigma_A}\supset A_1$. It follows that $A_1^{\sigma_A}=A_1$ (since σ_A is automorphism of space A, and A is a finite-dimensional space). The proved fact means that the element σ_A has the required matrix form. Similarly the statement of the lemma for σ_B is proved.

Along with the matrix form of the elements σ_A and σ_B, it is convenient to use the following notations:

$$\sigma_A=(\alpha_{22},\alpha_{21},\alpha_{11}),\quad \sigma_B=(\beta_{22},\beta_{21},\beta_{11}).$$

Let $\mathfrak{A}_1=(A_1,\Gamma_1,B_1)$ and $\mathfrak{A}_2=(A_2,\Gamma_2,B_2)$ be the exact biautomata. Not restricting generality, one can assume that $\Gamma_i\subset \mathrm{End}(A_i,B_i)$, $i=1,2$. A description of automorphisms of the triangular product is given by following:

Theorem 3.7. *An element* $(\sigma_A,\sigma_B)=(\alpha_{22},\alpha_{21},\alpha_{11};\beta_{22},\beta_{21},\beta_{11})\in\Sigma_A\times\Sigma_B$ *defines an automorphism of the triangular product of exact biautomata* $\mathfrak{A}_1\nabla\mathfrak{A}_2$, *if and only if the element* (α_{22},β_{22}) *of* $\Sigma_{A_2}\times\Sigma_{B_2}$ *defines an automorphism of the biautomaton* $\mathfrak{A}_2=(A_2,\Gamma_2,B_2)$, *while the element* (α_{11},β_{11}) *of* $\Sigma_{A_1}\times\Sigma_{B_1}$ *defines an automorphism of the biautomaton* $\mathfrak{A}_1=(A_1,\Gamma_1,B_1)$.

Proof. The semigroup Γ of inputs of the triangular product of biautomata can be considered (item 3.2.1) as a semigroup of matrices of the form

$$\begin{pmatrix} \nu_{22} & \varphi_{22} & \nu_{21} & \varphi_{21} \\ 0 & \delta_{22} & 0 & \delta_{21} \\ 0 & 0 & \nu_{11} & \varphi_{11} \\ 0 & 0 & 0 & \delta_{11} \end{pmatrix} \qquad (3.6)$$

$\gamma_{ij} \in \mathrm{Hom}(A_i, A_j)$; $\varphi_{ij} \in \mathrm{Hom}(A_i, B_j)$; $\delta_{ij} \in \mathrm{Hom}(B_i, B_j)$;
satisfying the condition

$$(\nu_{22}, \varphi_{22}, \delta_{22}) \in \Gamma_2, \quad (\nu_{11}, \varphi_{11}, \delta_{11}) \in \Gamma_1 \qquad (3.7)$$

Consider the biautomaton representation $\mu: \Gamma \rightarrow \mathrm{End}(A_1 \oplus A_2, B_1 \oplus B_2)$. The image of Γ under this representation μ will be denoted by Γ'. If $x=(\nu, \varphi, \delta) \in \Gamma'$, then

$$\nu = \begin{pmatrix} \nu_{22} & \nu_{21} \\ 0 & \nu_{11} \end{pmatrix} = (\nu_{22}, \nu_{21}, \nu_{11}) \quad ; \qquad \varphi = \begin{pmatrix} \varphi_{22} & \varphi_{21} \\ 0 & \varphi_{11} \end{pmatrix} = (\varphi_{22}, \varphi_{21}, \varphi_{11})$$

$$\delta = \begin{pmatrix} \delta_{22} & \delta_{21} \\ 0 & \delta_{11} \end{pmatrix} = (\delta_{22}, \delta_{21}, \delta_{11})$$

Thus, the elements of the semigroup Γ' can be regarded as matrices of the form

$$\begin{pmatrix} \nu_{22} & \nu_{21} & \varphi_{22} & \varphi_{21} \\ 0 & \nu_{11} & 0 & \varphi_{11} \\ 0 & 0 & \delta_{22} & \delta_{21} \\ 0 & 0 & 0 & \delta_{11} \end{pmatrix} \qquad (3.8)$$

A matrix of form (3.8) belongs to Γ' if and only if the corresponding matrix of the form (3.6) belongs to Γ, i.e. when the conditions (3.7) are fulfilled.

If $x=(\nu,\varphi,\delta)=(\nu_{22},\nu_{21},\nu_{11};\varphi_{22},\varphi_{21},\varphi_{11};\delta_{22},\delta_{21},\delta_{11})$ is an arbitrary element of Γ' and

$$\sigma=(\sigma_A,\sigma_B)=(\alpha_{22},\alpha_{21},\alpha_{11};\beta_{22},\beta_{21},\beta_{11}) \text{ of } \Sigma_A\times\Sigma_B \text{ then,}$$

according to the definition, $x\circ\sigma=(\sigma_A^{-1}\nu\sigma_A,\sigma_A^{-1}\varphi\sigma_B,\sigma_B^{-1}\delta\sigma_B)$. Simple calculation shows that

$$(\sigma_A^{-1}\nu\sigma_A,\sigma_A^{-1}\varphi\sigma_B,\sigma_B^{-1}\delta\sigma_B)=(\nu'_{22},\nu'_{21},\nu'_{11},\varphi'_{22},\varphi'_{21},\varphi'_{11},\delta'_{22},\delta'_{21},\delta'_{11})$$

where, in particular,

$$\begin{aligned} &\nu'_{22}=\alpha_{22}^{-1}\nu_{22}\alpha_{22} && \nu'_{11}=\alpha_{11}^{-1}\nu_{11}\alpha_{11} \\ &\varphi'_{22}=\alpha_{22}^{-1}\varphi_{22}\beta_{22} && \varphi'_{11}=\alpha_{11}^{-1}\varphi_{11}\beta_{11} \\ &\delta'_{22}=\beta_{22}^{-1}\delta_{22}\beta_{22} && \delta'_{11}=\beta_{11}^{-1}\delta_{11}\beta_{11} \end{aligned} \qquad (3.9)$$

In order the element $x\circ\sigma$ belongs to Γ', it is necessary and sufficient that the condition (3.7) be fulfilled, i.e. that $(\nu'_{22},\varphi'_{22},\sigma'_{22})\in\Gamma_2$ and $(\nu'_{11},\varphi'_{11},\delta'_{11})\in\Gamma_1$. With accounting for the equalities (3.9), we have

$$(\nu'_{22},\varphi'_{22},\delta'_{22}) =(\alpha_{22}^{-1}\nu_{22}\alpha_{22},\alpha_{22}^{-1}\varphi_{22}\beta_{22},\beta_{22}^{-1}\delta_{22}\beta_{22})=$$

$$=\begin{pmatrix} \alpha_{22}^{-1} & 0 \\ 0 & \beta_{22}^{-1} \end{pmatrix} \begin{pmatrix} \nu_{22} & \varphi_{22} \\ 0 & \delta_{22} \end{pmatrix} \begin{pmatrix} \alpha_{22} & 0 \\ 0 & \beta_{22} \end{pmatrix}=$$

$$=(\alpha_{22},\beta_{22})^{-1}(\nu_{22},\varphi_{22},\delta_{22})(\alpha_{22},\beta_{22})=(\nu_{22},\varphi_{22},\delta_{22})\circ(\alpha_{22},\beta_{22}).$$

Similarly, $(\nu'_{11},\varphi'_{11},\delta'_{11})=(\nu_{11},\varphi_{11},\delta_{11})\circ(\alpha_{11},\beta_{11})$. Thus, in order that the element $x\circ\sigma$ lie in Γ', it is necessary and sufficient that

$$(\nu_{22},\varphi_{22},\delta_{22})\circ(\alpha_{22},\beta_{22})\in\Gamma_2 \text{ and } (\nu_{11},\varphi_{11},\delta_{11})\circ(\alpha_{11},\beta_{11})\in\Gamma_1.$$

The latter inclusions mean that (α_{22},β_{22}) determines an automorphism of the biautomaton $\mathfrak{A}_2$ and (α_{11},β_{11}) an automorphism of the biautomaton $\mathfrak{A}_1$, as required.

Note that there are papers on groups of automorphisms of other automata constructions. In particular, automorphisms of wreath products of pure automata are being studied in [16]. Some description of such wreath products has been given, while the problem of their complete description remains.

CHAPTER 4

VARIETIES OF AUTOMATA

Identities of automata give the important invariant, which describes automaton functioning. Using the language of varieties and identities we can classify various automata. Automata, free in certain variety, are of special interest from this point of view. We study the corresponding theory for pure automata, as well as for linear automata and biautomata.

4.1. Identities of pure automata

4.1.1. Defining and identical relations

Let us consider the automaton $\mathfrak{A}=(A,\Gamma,B)$ with a certain system of generators Z, X, Y. The given automaton is a homomorphic image of the free automaton Atm(Z, X, Y): natural embeddings $Z \to A$, $X \to \Gamma$ and $Y \to B$ are uniquely extended to the epimorphism μ: Atm(X, Y, Z) $\to \mathfrak{A}$. Let $\rho=(\rho_1,\rho_2,\rho_3)$ be a kernel of this epimorphism. Then $\mathfrak{A}\cong$Atm(Z, X, Y)$/\rho$.

The congruence ρ is called the *complete system of defining relations of the automaton* $\mathfrak{A}$ Three sets Z, X, Y of generators together with defining relations $\rho=(\rho_1,\rho_2,\rho_3)$ completely determine the automaton $\mathfrak{A}$. However, it is not necessary to proceed from the complete system of defining relations in order to define $\mathfrak{A}$. One may confine to a certain generating triplet of relations. Let $\mathfrak{A}=(A,\Gamma,B)$ be an automaton and $\tau=(\tau_1,\tau_2,\tau_3)$ be three binary relations on A, Γ, B respectively. This triplet is called the *relation on the automaton* $\mathfrak{A}$.

Given the relation $\tau=(\tau_1,\tau_2,\tau_3)$ on the automaton $\mathfrak{A}=(A,\Gamma,B)$, describe the congruence $\rho=(\rho_1,\rho_2,\rho_3)$ generated by τ. As ρ_2 take the congruence of the semigroup Γ generated by the relation τ_2. Define the relation τ'_1 on A by the rule: if $a\in A$; $\gamma_1,\gamma_2\in\Gamma$, then $(a\circ\gamma_1)\tau'_1(a\circ\gamma_2)$ if and only if $\gamma_1\rho_2\gamma_2$ Let us call the equivalence δ in A invariant with respect to Γ, if $a_1\delta a_2$ implies $(a_1\circ\gamma)\delta(a_2\circ\gamma)$. Now, for ρ_1, take equivalence on A containing $\tau_1\cup\tau'_1$, which is minimal invariant with respect to Γ. Let, further, the relation τ'_3 on the set B be defined in the following way: $(a_1*\gamma_1)\tau'_3(a_2*\gamma_2)$, if $a_1\rho_1a_2$ and $\gamma_1\rho_2\gamma_2$ where $a_1,a_2\in A$; $\gamma_1,\gamma_2\in\Gamma$. As ρ_3 take the equivalence generated by the relation $\tau_3\cup\tau'_3$ Then $\rho=(\rho_1,\rho_2,\rho_3)$ is a congruence. Indeed, let $a_1\rho_1a_2$ and $\gamma_1\rho_2\gamma_2$. Hence $(a_1\circ\gamma_1)\rho_1(a_1\circ\gamma_2)$, since $\rho_1\supset\tau'_1$ and $\gamma_1\rho_2\gamma_2$. In its turn, $(a_1\circ\gamma_2)\rho_1(a_2\circ\gamma_2)$, since $a_1\rho_1a_2$, and ρ_1 is invariant with respect to Γ. Finally, $(a_1\circ\gamma_1)\rho_1(a_2\circ\gamma_2)$. Since $\rho_3\supset\tau'_3$, then $((a_1*\gamma_1)\rho_3(a_2*\gamma_2)$. It is clear that ρ is the minimal congruence containing τ.

If the automaton $\mathfrak{A}=(A,\Gamma,B)$ is isomorphic to $Atm(Z,X,Y)/\rho$ and the congruence $\rho=(\rho_1,\rho_2,\rho_3,)$ on $Atm(Z,X,Y)$ is generated by a certain relation $\tau=(\tau_1,\tau_2,\tau_3)$, then $\mathfrak{A}$ is said to be defined by the *system of generators* (Z,X,Y) and by the *system of defining relations* $\tau=(\tau_1,\tau_2,\tau_3)$.

Now consider the identical relations of automata. Along with defining relations, they describe each given automaton by means of free automaton.

Let $Atm(Z,X,Y)=(H,F,\Phi)$ be a free automaton and $\mathfrak{A}=(A,\Gamma,B)$ be a certain automaton. Take a pair of elements $h_1,h_2\in H$. We say that the *identical relation*, or briefly, the *identity in the states* $h_1\equiv h_2$, is fulfilled in the automaton $\mathfrak{A}$, if for any homomorphism $\mu=(\mu_1,\mu_2,\mu_3)$: $Atm(Z,X,Y)\rightarrow\mathfrak{A}$, the equality $h_1^{\mu_1}=h_2^{\mu_1}$ holds. (Recall that defining relations are associated with a definite homomorphism). Identical relations in input signals (they have the form $f_1\equiv f_2$, $f_1,f_2\in F$) and in output signals ($\varphi_1\equiv\varphi_2$, $\varphi_1,\varphi_2\in\Phi$) are defined in a similar way. The system of all identical relations of the automaton can be understood as a union of identities in states, in inputs and in outputs.

Let us consider the identical relations of the given automaton $\mathfrak{A}$. Define relations ρ_1,ρ_2,ρ_3 on the sets H,F and Φ of the free automaton $Atm(Z,X,Y)=(H,F,\Phi)$ by the rules: $h_1\rho_1h_2$ if the identity in states $h_1\equiv h_2$ is fulfilled in $\mathfrak{A}$; $f_1\rho_2f_2$ if there is the identity in inputs $f_1\equiv f_2$ in $\mathfrak{A}$; $\varphi_1\rho_3\varphi_2$ if the identity in outputs $\varphi_1\equiv\varphi_2$ is satisfied.

It is natural to call $\rho=(\rho_1,\rho_2,\rho_3)$ the *system of identical rela-*

tions of the given automaton. The congruence $\theta=(\theta_1,\theta_2,\theta_3)$ of the automaton $\mathfrak{A}$ is called *completely characteristic* if $a_1\theta_1 a_2$ implies $a_1^{\nu_1}\theta_1 a_2^{\nu_1}$; $\gamma_1\theta_2\gamma_2$ implies $\gamma_1^{\nu_2}\theta_2\gamma^{\nu_2}$; $b_1\theta_3 b_2$ implies $b_1^{\nu_3}\theta_3 b_2^{\nu_3}$, for any endomorphism $\nu=(\nu_1,\nu_2,\nu_3)$: $\mathfrak{A}\to\mathfrak{A}$. In other words the congruence is said to be completely characteristic if it preserves all endomorphisms of the automaton $\mathfrak{A}$.

Proposition 1.1. *The system of all identical relations* $\rho=(\rho_1\rho_2,\rho_3)$ *of the given automaton* $\mathfrak{A}$ *is a completely characteristic congruence of the free automaton* Atm(Z,X,Y).

Proof. Let μ^{α} be a certain homomorphism of Atm(Z,X,Y) to $\mathfrak{A}$ and $\rho^{\alpha}=\mathrm{Ker}\mu^{\alpha}$. Then, evidently, ρ is the intersection of all ρ^{α} on all possible μ^{α}. Therefore, ρ is a congruence. It is left to verify that this congruence is completely characteristic. Take an arbitrary endomorphism $\nu=(\nu_1,\nu_2,\nu_3)$ of the free automaton Atm(Z,X,Y) and let the identities $h_1\equiv h_2$, $f_1\equiv f_2$, $\varphi_1\equiv\varphi_2$ be fulfilled in $\mathfrak{A}$. We must check that the identities $h_1^{\nu_1}\equiv h_2^{\nu_1}$, $f_1^{\nu_2}\equiv f_2^{\nu_2}$, $\varphi_1^{\nu_3}\equiv\varphi^{\nu_3}$ hold. Let $\mu=(\mu_1,\mu_2,\mu_3)$ be an arbitrary homomorphism Atm(Z,X,Y) $\to\mathfrak{A}$. Then $(h_1^{\nu_1})^{\mu_1}=h_1^{\nu_1\mu_1}=h_2^{\nu_1\mu_1}=(h_2^{\nu_1})^{\mu_1}$. Similarly, $(f_1^{\nu_2})^{\mu_2}\equiv(f_2^{\nu_2})^{\mu_2}$ and $(\varphi_1^{\nu\nu_3})^{\mu_3}=(\varphi_2^{\nu_3})^{\mu_3}$. The Proposition is proved.

On the other hand, it is clear that each completely characteristic congruence ρ is the system of all identical relations of the automaton Atm(Z,X,Y)/ρ. Together with the Proposition 1.1 it implies that the problem of description of identical relations of automata is equivalent to that of completely characteristic congruences of the automaton Atm(Z,X,Y).

It is necessary to remark, that speaking about all the identities of the automaton $\mathfrak{A}$, we consider them in the given free automaton Atm(Z,X,Y). This free automaton changes whenever the sets Z,X,Y change. In order to avoid this uncertainty proceed to the free automaton Atm(Z,X,Y) whose all three sets are countable. One can consider the identities of any automaton in this one. Indeed, every separate identity contains only a finite set of variables, while their number may increase.

Identities of the automaton $\mathfrak{A}=(A,\Gamma,B)$, considered here, are the identities from the point of view of the free automaton $Atm(Z,X,Y)$. Keeping in view the category of the automata with fixed semigroup of inputs Γ, let us introduce the concept of Γ-identity. *Γ-identity in states* $h_1 \equiv h_2$ is defined in a way similar to the identity in states, but in this case h_1 and h_2 are the elements of the set of states of the automaton $Atm_\Gamma(Z,Y)$, and $\mu=(\mu_1,\mu_2,\mu_3)$ is an arbitrary homomorphism of $Atm_\Gamma(Z,X)$ to $\mathfrak{A}$, identical on Γ. The definition of Γ-identities in outputs is analogous to that of Γ-identities in states. The system of all Γ-identities of the automaton $\mathfrak{A}$ is the union of Γ-identities in states and of Γ-identities in outputs. In order to emphasize the difference between identities and Γ-identities the first ones are sometimes called the *absolute identities*.

Γ-congruence of the automaton $\mathfrak{A}=(A,\Gamma,B)$ is a congruence of the form $(\rho_1,\delta_\Gamma,\rho_2)$, where δ_Γ is a trivial (minimal) congruence of the semigroup Γ. The next statement is similar to Proposition 1.1.

Proposition 1.2. *The system of all Γ-identities of the Γ-automaton $\mathfrak{A}$ is a completely characteristic* Γ*-congruence of the automaton* $Atm_\Gamma(Z,Y)$.

4.1.2. Compatible tuples and identities of Γ-automata

Our aim in this item is to examine completely characteristic Γ-congruences of the free automata of the type $Atm_\Gamma(Z,X)$ and to provide the description of Γ-identities.

Let us assign to each automaton $\mathfrak{A}=(A,\Gamma,B)$ the following four invariants. Denote by η the kernel of action of the semigroup Γ^1 in A: $\gamma_1\eta\gamma_2$, if for any $a\in A$ holds $a\circ\gamma_1=a\circ\gamma_2$; $\gamma_1,\gamma_2\in\Gamma^1$ It is a congruence of the semigroup Γ^1. Denote by τ the kernel of the external action (external kernel) of the semigroup Γ: $\gamma_1\tau\gamma_2$, if for any $a\in A$ holds $a*\gamma_1=a*\gamma_2$; $\gamma_1,\gamma_2\in\Gamma$. τ is the left congruence of the semigroup Γ. It is clear that the intersection $\eta\cap\tau$ is a kernel of the corresponding automaton representation of the given Γ. Denote by U the set of all elements of $\gamma\in\Gamma^1$ such that for any a_1 and a_2 from A holds $a_1\circ\gamma=a_2\circ\gamma$. U is a two-sided ideal of the semigroup Γ^1 called the *annihilator of the action* Γ in A. Denote by V the set of all elements $\gamma\in\Gamma$ such that for any a_1 and a_2 from A, $a_1*\gamma=a_2*\gamma$ takes place. V is the left ideal of the semigroup Γ called the *annihilator of the external action* of this semigroup. The sequence (η,U,τ,V) thus defined is called a *tuple of kernels and annihilators* corresponding to the automaton $\mathfrak{A}$.

Let Γ be a semigroup, η a certain congruence in Γ^1, τ a left congruence in Γ, U an ideal in Γ^1, V a left ideal in Γ. The sequence (tuple) (η,U,τ,V) is called a *compatible tuple* of the semigroup Γ if it satisfies the following conditions:

1) U is a union of certain cosets of the congruence η.
2) Each coset of the congruence η belonging to U is a left ideal in Γ^1.
3) V is a union of certain cosets of the equivalence τ.
4) Each coset of the equivalence τ belonging to V is a left ideal in Γ.
5) If $\gamma_1\eta\gamma_2$, $\gamma_1,\gamma_2\in\Gamma$, then for each $\gamma\in\Gamma$ holds $(\gamma_1\gamma)\tau(\gamma_2\gamma)$.
6) If $\sigma\in U$, then $\sigma\gamma\in V$ for any $\gamma\in\Gamma$.

Proposition 1.3. *The tuple* (η,U,τ,V) *consisting of kernels and annihilators, corresponding to the automaton* $\mathfrak{A}$, *is a compatible tuple.*

Proof. 1) Let a_1 and a_2 be arbitrary elements in A, $\delta\in U$ and $\delta'\eta\delta$. Then $a_1\circ\delta'=a_1\circ\delta=a_2\circ\delta=a\circ\delta'$ Therefore, δ' belongs to U and the whole coset $[\delta]$ of the congruence η containing δ, belongs to U

2) Let S be a coset of the congruence η lying in U; $\delta\in S$, $\gamma\in\Gamma$. Then for each element $a\in A$ the equality $(a\circ\gamma)\circ\delta=a\circ\delta$ takes place. Hence $a\circ\gamma\delta=(a\circ\gamma)\circ\delta=a\circ\delta$, i.e. $(\gamma\delta)\eta\delta$ and $\gamma\delta\in S$. Thus, S is the left ideal in Γ^1.

3) Now assume that $\delta\in V$, $\delta'\tau\delta$ and a_1,a_2 from A. Then $a_1*\delta'=a_1*\delta=a_2*\delta=a_2*\delta'$. Hence, $\delta'\in V$ and the whole coset $[\delta]$ of the equivalence τ belongs to V.

4) If S is a coset of the equivalence τ lying in V, $\delta\in S$ and $\gamma\in\Gamma$, then for any $a\in A$ holds $a*\gamma\delta=(a\circ\gamma)*\delta=a*\delta$. Hence, $(\gamma\delta)\tau\delta$ and $\gamma\delta\in S$. Therefore, S is the left ideal in Γ.

5) Let $\gamma_1\eta\gamma_2$, $\gamma\in\Gamma$. Then for any $a\in A$, $a*\gamma_1\gamma=(a\circ\gamma_1)*\gamma=(a\circ\gamma_2)*\gamma$ $=a*\gamma_2\gamma$, i.e. $(\gamma_1\gamma)\tau(\gamma_2\gamma)$.

6) Take $\delta\in U$, $\gamma\in\Gamma$ Then for any a_1, a_2 from A, $a_1*\delta\gamma=(a_1\circ\delta)*\gamma=(a_2\circ\delta)*\gamma=a_2*\delta\gamma$.

This implies that $\delta\gamma\in V$.

Note further that since in a Moore automaton from $a\circ\gamma_1=a\circ\gamma_2$ and $a_1\circ\gamma=a_2\circ\gamma$ follows $a*\gamma_1=a*\gamma_2$ and $a_1*\gamma=a_2*\gamma$ respectively, then tuples of kernels and annihilators corresponding to the Moore automata additionally satisfy the conditions: $\eta\subset\tau$, $U\subset V$.

The importance of compatible tuples is based on the fact that completely characteristic Γ-congruence of the free automaton $Atm_\Gamma(Z,Y)=(H,\Gamma,\Phi)$ corresponds to each of them. Moreover, there is one-to-one correspondence between these congruences which satisfy a certain additional condition of non-triviality (the condition will be formulated below) and compatible tuples. Let us construct this correspondence.

Given a compatible tuple (η,U,τ,V) of the semigroup Γ, define binary relations ρ_1 in H and ρ_3 in Φ. Let $h_1=(z_1,\gamma_1)=z_1\circ\gamma_1$, $h=(z_2,\gamma_2)=z_2\circ\gamma_2$ be the elements from $H=Z\times\Gamma^1$ Set $h_1\rho_1h_2$, if $z_1=z_2$ and $\gamma_1\eta\gamma_2$, or $z_1\neq z_2$, but γ_1,γ_2 from U and $\gamma_1\eta\gamma_2$. Remind that the set Φ is a free union of the sets Y and $Z\times\Gamma$. If now $\varphi_1=(z_1,\gamma_1)=z_1*\gamma_1$, $\varphi_2=(z_2,\gamma_2)=z_2*\gamma_2$, then assume $\varphi_1\rho_3\varphi_2$, if $z_1=z_2$ and $\gamma_1\tau\gamma_2$, or $z_1\neq z_2$, but γ_1,γ_2 from V and $\gamma_1\tau\gamma_2$. All elements from Y are considered as independent cosets by ρ_3.

Theorem 1.4. *The system $\rho=(\rho_1,\delta_\Gamma,\rho_3)$ where ρ_1,ρ_3 are constructed relations in* H *and* Φ *respectively, and δ_Γ is the minimal congruence of the semigroup* Γ, *is completely characteristic* Γ*-congruence of the automaton* $Atm_\Gamma(Z,Y)$.

Let us call the completely characteristic Γ-congruence $\rho=(\rho_1,\delta_\Gamma,\rho_3)$ of the automaton $Atm_\Gamma(Z,Y)$ trivial by ρ_3 if for certain elements $y\in Y$ and $\varphi\neq y$ $(\varphi\in\Phi)$, $y\rho_3\varphi$ is fulfilled. This means that all elements from Φ form one coset by the equivalence ρ_3. Really, for each element φ_1 from Φ one can take such endomorphism $(\mu_1,\varepsilon_\Gamma,\mu_3)$ of the automaton $Atm_\Gamma(Z,Y)$, that $y^{\mu_3}=\varphi_1$, $\varphi^{\mu_3}=\varphi$. As the congruence ρ is completely characteristic, then $y\rho_3\varphi$ implies $y^{\mu_3}\rho_3\varphi^{\mu_3}$, from which $\varphi_1\rho_3\varphi$. Since φ_1 is an arbitrary element from Φ, this means that the whole Φ forms one coset by the equivalence ρ_3.

Theorem 1.5. *If the set* Z *contains more than one element, then all non-trivial completely characteristic* Γ*-congruences of the free automaton* $\mathrm{Atm}_\Gamma(Z,Y)$ *are in one-to-one correspondence with the compatible tuples of the semigroup* Γ.

Not proving the Theorem, show the required correspondence. Let (η,U,τ,V) be a compatible tuple of the semigroup Γ and $\rho=(\rho_1,\delta_\Gamma,\rho_3)$ be the constructed above corresponding completely characteristic Γ-congruence of the automaton $\mathrm{Atm}_\Gamma(Z,Y)$. This congruence is non-trivial. On the other hand, if $\rho=(\rho_1,\delta_\Gamma,\rho_3)$ is a non-trivial completely characteristic congruence of the automaton $\mathrm{Atm}_\Gamma(Z,Y)$, then the tuple of the semigroup Γ composed of the kernels and annihilators of the automaton $\mathrm{Atm}_\Gamma(Z,Y)/\rho=(H/\rho,\Gamma,\Phi/\rho_3)$ corresponds to it. To prove the Theorem one must check that this assignment is one-to-one.

Now let us pass to Γ-identities of the automata.

Theorem 1.6. *Let* $\mathfrak{A}=(A,\Gamma,B)$ *be an automaton with the set* B *containing more than one element and let* (η,U,τ,V) *be the corresponding tuple of kernels and annihilators in* Γ. *Then all* Γ*-identities of the automaton* $\mathfrak{A}$ *have the form*

1. $z\circ\gamma_1\equiv z\circ\gamma_2$ *for elements* $\gamma_1,\gamma_2\in\Gamma^1$, *such that* $\gamma_1\eta\gamma_2$.
2. $z_1\circ\gamma=z_2\circ\gamma$ *for all* $\gamma\in U$.
3. $z*\gamma_1=z*\gamma_2$ *for elements* $\gamma,\gamma_2\in\Gamma$, *such that* $\gamma_1\tau\gamma_2$.
4. $z_1*\gamma=z_2*\gamma$ *for all* $\gamma\in V$.

Proof. All Γ-identities of the given automaton constitute a completely characteristic congruence $\rho=(\rho_1,\delta_\Gamma,\rho_2)$ in $\mathrm{Atm}_\Gamma(Z,Y)=(H,\Gamma,\Phi)$. This congruence is non-trivial. Really, let $\bar{\rho}$ be a complete system of the defining relations of the automaton $\mathfrak{A}$, i.e. $\mathfrak{A}\cong\mathrm{Atm}_\Gamma(Z,Y)/\bar{\rho}$. It is clear that $\rho\subset\bar{\rho}$. Therefore $\mathfrak{A}=(A,\Gamma,B)$ is a homomorphic image of the automaton $\mathrm{Atm}_\Gamma(Z,Y)/\rho=(H/\rho_1,\Gamma,\Phi/\rho_3)$. If the congruence ρ were trivial, then the set Φ/ρ_3 and together with it the set B would consist of one element, but this contradicts the condition of the Theorem. From the Theorem 1.5 follows that if (η',U',τ',V') is a tuple of kernels and annihilators corresponding to the automaton $(H/\rho_1,\Gamma,\Phi/\rho_3)$, and $\rho'=(\rho_1',\delta_\Gamma,\rho_3')$

is a congruence of the automaton $Atm_\Gamma(Z,Y)$ corresponding to this tuple, then $\rho=\rho'$. In accordance with the definition:

$(z\circ\gamma_1)\rho'_1(z_2\circ\gamma_2)$ if

1) $z_1=z_2=z$ and $\gamma_1\eta'\gamma_2$ or
2) $z_1\neq z_2$, but γ_1,γ_2 belong to U' and $\gamma_1\eta'\gamma_2$;

$(z_1*\gamma_1)\rho'_3(z_2*\gamma_2)$ if

3) $z_1=z_2=z$ and $\gamma_1\tau'\gamma_2$ or
4) $z_1\neq z_2$ but γ_1,γ_2 belong to V' and $\gamma_1\tau'\gamma_2$.

Since $\rho=\rho'$, then the identities of the automaton $\mathfrak{A}$ have the form:

1) $z\circ\gamma_1\equiv z\circ\gamma_2$; where $\gamma_1\eta'_1\gamma_2$, $\gamma_1,\gamma_2\in\Gamma^1$,
2) $z_1\circ\gamma\equiv z_2\circ\gamma$; where $\gamma\in U'$, (1.1)
3) $z*\gamma_1\equiv z*\gamma_2$; where $\gamma_1\tau'\gamma_2$, $\gamma_1,\gamma_2\in\Gamma$,
4) $z_1*\gamma\equiv z_2*\gamma$; where $\gamma\in V'$.

To complete the proof of the theorem it is necessary to show that the tuples (η',U',τ',V') and (η,U,τ,V) coincide. If $\gamma_1\eta\gamma_2$, then for any $\gamma\in\Gamma$, $\gamma\gamma_1\eta\gamma\gamma_2$. It means that for all $a\in A$ $a\circ\gamma\gamma_1=a\circ\gamma\gamma_2$. Thus the identity $z\circ\gamma\gamma_1\equiv z\circ\gamma\gamma_2$ is fulfilled in A, which is equivalent to $(z\circ\gamma\gamma_1)\rho_1(z\circ\gamma\gamma_2)$. Since $z\circ\gamma\gamma_1=(z\circ\gamma)\circ\gamma_1= h\circ\gamma_1$, then for arbitrary $h\in H$ holds $(h\circ\gamma_1)\rho_1(h\circ\gamma_2)$. Therefore, $\gamma_1\eta'\gamma_2$ and $\eta\subset\eta'$ On the other hand, let $\gamma_1\eta'\gamma_2$. Then $(z\circ\gamma_1)\rho_1(z\circ\gamma_2)$: the identity $z\circ\gamma_1\equiv z\circ\gamma_2$ is fulfilled in $\mathfrak{A}$. Hence, $a\circ\gamma_1=a\circ\gamma_2$ for all $a\in A$, that implies $\gamma_1\eta\gamma_2$. From this follows the inverse inclusion $\eta'\subset\eta$ and the equality $\eta=\eta'$.

Show that U=U' If $\gamma\in U$, then for arbitrary $\gamma_1,\gamma_2\in\Gamma$, $\gamma_1\gamma$ and $\gamma_2\gamma$ belong to U and $(\gamma_1\gamma)\eta(\gamma_2\gamma)$. Then for arbitrary elements $a_1,a_2\in A$, $a_1\circ\gamma_1\gamma=a_1\circ\gamma_2\gamma$ and $a_1\circ\gamma_2\gamma=a_2\circ\gamma_2\gamma$. From this follows $a_1\circ\gamma_1\gamma=a_2\circ\gamma_2\gamma$. Thus, for arbitrary fixed γ_1 and γ_2 in $\mathfrak{A}$ holds the identity $z_1\circ\gamma_1\gamma\equiv z_2\circ\gamma_2\gamma$, which is equivalent to $(z_1\circ\gamma_1\gamma)\rho_1(z_2\circ\gamma_2\gamma)$. Since $z_1\circ\gamma_1=h_1$ and $z_2\circ\gamma_2=h_2$, then for arbitrary h_1,h_2 from H holds $(h_1\circ\gamma)\rho_1(h_2\circ\gamma)$. This means that $\gamma\in U'$ and, consequently $U\subset U'$. Conversely, if $\gamma\in U'$, then $(z_1\circ\gamma)\rho_1(z_2\circ\gamma)$. Hence the identity $z_1\circ\gamma\equiv z_2\circ\gamma$ is fulfilled in $\mathfrak{A}$. Therefore for all $a_1,a_2\in A$, $a_1\circ\gamma=a_2\circ\gamma$ holds. So $\gamma\in U$ and $U'\subset U$. Thus, U=U' The equalities $\tau=\tau'$ and V=V' are verified in a similar way. The Theorem is proved.

Remark. Automaton operations are not explicitly included into the definition of the Γ-identity given at the beginning of this item. It is proved then that the identities of the automaton can be represented in the form (1.1). Here the identities of the automaton are the identities of the action.

It must be noted that to make a notation of the identities of an arbitrary Γ-automaton, it is sufficient that the set Z contains two elements.

4.1.3. Identities of arbitrary automata

Now, we consider completely characteristic congruences of absolutely free automata $Atm(Z,X,Y)=(H,F,\Phi)$.

Let $\mathfrak{A}=(A,\Gamma,B)$ be an automaton and let (η,U,τ,V) be the corresponding tuple of kernels and annihilators in Γ. Let $F=F(X)$ be a free semigroup over the set X. Define in F corresponding to the automaton $\mathfrak{A}$ tuple $(\eta_F,U_F,\xi_F,\tau_F,V_F)$:

η_F is the following binary relation in F^1: $f_1\eta_F f_2$ $(f_1,f_2\in F^1)$, if for any homomorphism $\mu:F^1 \to \Gamma^1$, transferring a unit into a unit, $f_1^\mu\eta f_2^\mu$ is fulfilled. It is easy to understand that η_F is a completely characteristic congruence of the semigroup F^1, namely a congruence of all identities of the semigroup Γ^1/η;

ξ_F is a completely characteristic congruence of all identities of the semigroup Γ in F.

τ_F is a binary relation in F defined by the rule: $f_1\tau_F f_2$ if for any homomorphism $\mu:F \to \Gamma$, $f_1^\mu\tau f_2^\mu$ is fulfilled. τ_F is a completely characteristic left congruence in F: for each endomorphism ν of the semigroup F from $f_1\tau_F f_2$ follows $f_1^\nu\tau_F f_2^\nu$. It can be shown that τ_F may not be a two-sided congruence.

U_F is a set of all $f\in F$ for which $f^\mu\in U$ for any homomorphism $\mu:F \to \Gamma$;

V_F is a set of all $f\in F$ for which $f^\mu\in V$ for any homomorphism $\mu:F \to \Gamma$.

U_F is a two-sided completely characteristic ideal in F^1 and V_F is a left completely characteristic ideal in F. It is obvious from the definition that (η_F,U_F,τ_F,V_F) is a compatible (in the sense defined earlier) tuple in F and that the inclusion $\xi_F \subset \eta_F \cap \tau_F$ holds. The constructed tuple $(\eta_F, U_F, \xi_F, \tau_F, V_F)$ is called an *external tuple* of the automaton $\mathfrak{A}$.

Remark. Show that the left completely characteristic congruence is not necessary a two-sided one. As an example take a free semigroup F=F(X) and denote by L the left ideal generated by the squares of all elements from F. This ideal is completely characteristic but not two-sided; for example, it does not contain the element $x_1^2 x_2$ where x_1, x_2 from X. Now take Rees congruence by the ideal L. This will be a left completely characteristic but not two-sided congruence.

Let us consider the tuple (η, U, ξ, τ, V) in which η and ξ are completely characteristic (two-sided) congruences of the semigroup F=F(X), τ is a left completely characteristic congruence of this semigroup, U is a two-sided and V is a left completely characteristic ideal in F. Let us call it a *completely characteristic tuple* in F if (η, U, τ, V) is a compatible tuple and $\xi \subset \eta \cap \tau$.

Theorem 1.7. *All non-trivial completely characteristic congruences of the free automaton* $\mathrm{Atm}(Z, X, Y) = (H, F, \Phi)$ *are in one-to-one correspondence with the completely characteristic tuples* (η, U, ξ, τ, V) *in* F.

Theorem 1.8. *Let* $\mathfrak{A} = (A, \Gamma, B)$ *be an automaton with the set* B *containing more than one element and let* $(\eta_F, U_F, \xi_F, \tau_F, V_F)$ *be its external tuple. Then all identities (absolute) of the automaton* $\mathfrak{A}$ *have the form:*

1) $z \circ f_1 \equiv z \circ f_2$ for the elements $f_1, f_2 \in F^1$ such that $f_1 \eta_F f_2$;
2) $z_1 \circ f \equiv z_2 \circ f$ for all $f \in U_F$;
3) $z * f_1 \equiv z * f_2$ for the elements $f_1, f_2 \in F$ such that $f_1 \tau_F f_2$;
4) $z_1 * f \equiv z_2 * f$ for all $f \in V_F$;
5) θ for all $\theta \in \xi_F$.

4.1.4. Identities of universal automata

a) Take first the universal automaton $\mathrm{Atm}^1(A, B) = (A, S(A, B), B)$, where $S(A, B) = S(A) \times \mathrm{Fun}(A, B)$. In order to describe identities of the automaton it is necessary to know its external tuple $(\eta_F, U_F, \xi_F, \tau_F, V_F)$. It is clear that the congruence η_F coincides with the identities of the semigroup S_A The congruence τ_F is the following: $f \tau_F \varphi$ (f, φ from F=F(X)) if and only if $f = f_1 x$, $\varphi = \varphi_1 x$ and $f_1 \eta_F \varphi_1$. Indeed, let $f \tau_F \varphi$, i.e. for any homomorphism $\mu:F \longrightarrow S(A,B)$ holds $a*f^{\mu}=a*\varphi^{\mu}$. Assume that $f=f_1x_1$, $\varphi=\varphi_1x_2$, $x_1,x_2\in X$. Then $a*f^{\mu}=(a\circ f_1^{\mu})*x_1^{\mu}=a*\varphi^{\mu}=(a\circ\varphi_1^{\mu})*x_2^{\mu}$. Since x_1^{μ},x_2^{μ} can be any elements from Fun(A,B), the latter equality holds only under the condition $a\circ f_1^{\mu}=a\circ\varphi_1^{\mu}$, $x_1=x_2=x$. For the same reason, if f or φ is equal to $x\in X$, then $f=\varphi$. Thus, $f=f_1x$, $\varphi=\varphi_1x$ and $f_1\eta_F\varphi_1$ The inverse statement is evident. From this follows that τ_F is a congruence (since η_F is a congruence) defined by η_F and it is less than η_F. It is clear that the sets U_F and V_F are empty. It remains to consider ξ_F There is the following fact: if the automaton (A,F,B) is exact, then $\xi_F=\eta_F\cap\tau_F$ Really, if $f_1\xi_Ff_2$, then $f_1^{\mu}=f_2^{\mu}$ for any homomorphism $\mu:F \longrightarrow \Gamma$; therefore $a\circ f_1^{\mu}=a\circ f_2^{\mu}$ and $a*f_1^{\mu}=a*f_2^{\mu}$ for all $a\in A$, that is $f_1\eta_Ff_2$ and $f_1\tau_Ff_2$ On the other hand, if $f_1(\eta_F\cap\tau_F)f_2$, then for all $a\in A$ and for any homomorphism $\mu:F \longrightarrow \Gamma$ hold $a\circ f_1^{\mu}=a\circ f_2^{\mu}$ and $a*f_1^{\mu}=a*f_2^{\mu}$ Since the automaton (A,Γ,B) is exact, then $f_1^{\mu}=f_2^{\mu}$, therefore, $f_1\xi_Ff_2$. The automaton $Atm^1(A,B)$ is exact, then as follows $\xi_F=\eta_F\cap\tau_F=\tau_F$. Thus, all the identities of the automaton $Atm^1(A,B)$ are defined by the identities of the semigroup S_A.

b) Identities of the universal automaton $Atm^2(\Gamma,B)=(B^{\Gamma},\Gamma,B)$. Show that in this case η_F coincides with the identities of the left regular action of the semigroup Γ in Γ. Let $f_1\eta_Ff_2$, and μ be an arbitrary homomorphism of F to Γ. Then for any element $\bar{a}\in\Gamma^B$ the equality $\bar{a}\circ f_1^{\mu}=\bar{a}\circ f_2^{\mu}$ holds. By the definition of the action $\circ$ in $Atm^2(\Gamma,B)$ this means that $\bar{a}(f_1^{\mu}\gamma)=\bar{a}(f_2^{\mu}\gamma)$ for all $\gamma\in\Gamma$. Since this is fulfilled for all functions $\bar{a}\in B^{\Gamma}$, then $f_1^{\mu}\gamma=f_2^{\mu}\gamma$. Therefore, $f_1\equiv f_2$ is an identity of the left regular representation (Γ,Γ). Inverse statement is proved in a similar way.

The congruence τ_F coincides with ξ_F, i.e. identities of Γ. Let $f_1\tau_Ff_2$. Then $\bar{a}*f_1^{\mu}=\bar{a}*f_2^{\mu}$ By the definition of the action $*$ in $Atm^2(\Gamma,B)$ hold $\bar{a}*f_1^{\mu}=\bar{a}(f_1^{\mu})$, $\bar{a}*f_2^{\mu}=\bar{a}(f_2^{\mu})$. Thus, $\bar{a}(f_1^{\mu})=\bar{a}(f_2^{\mu})$ for any $\bar{a}\in B^{\Gamma}$. Therefore, $f_1^{\mu}=f_2^{\mu}$, that is $f_1\equiv f_2$ is the identity of Γ. Inverse statement is proved in a similar way. The sets U_F and V_F are empty.

c) Identities of the universal automaton $Atm^3(A,\Gamma)$. In this case the representation (A,Γ) is given and we can proceed from the definite η_F, U_F and ξ_F. We must find τ_F,V_F. Show that $\tau_F\subset\eta_F$ and $V_F\subset U_F$. Construct the following Moore automaton (A,Γ,H) by the representation (A,Γ). Take the set $A\circ\Gamma=\{a\circ\gamma \mid a\in A, \gamma\in\Gamma\}$ and repeat it by a certain set H. Denote by ψ one-to-one correspondence between $A\circ\Gamma$ and H. Extend this ψ arbitrarily up to the mapping $\psi':A \rightarrow H$ and by this define the Moore automaton (A,Γ,H) with the determining mapping ψ'. Consider its external tuple $(\eta_F, U_F, \xi_F, \tau'_F, V'_F)$.

By the construction of this automaton $\eta_F=\tau'_F$, and $U_F=V'_F$. Really, for any Moore automaton $\eta_F\subset\tau'_F$ and $U_F\subset V'_F$. Let, on the other hand, $\gamma_1\tau'_F\gamma_2$, and $\mu:F^1 \rightarrow \Gamma^1$ be an arbitrary homomorphism of F^1 into Γ^1 conversing the unit of the semigroup F^1 into the unit of the semigroup Γ^1. Then $\gamma_1^{\mu}\tau\gamma_2^{\mu}$ and for any $a\in A$ holds $a*\gamma_1^{\mu}=a*\gamma_2^{\mu}$ From this by the construction of the automata follows $a\circ\gamma_1^{\mu}=a\circ\gamma_2^{\mu}$. Hence, $\gamma_1\eta_F\gamma_2$, $\tau'_F\subset\eta_F$ and $\tau'_F=\eta_F$. The equality $U_F=V'_F$ is verified in a similar way. Since (A,Γ,H) is an epimorhic in outputs image of the automaton $Atm^3(A,\Gamma)$, then $\tau_F\subset\tau'_F$ and $V_F\subset V'_F$. Thus, we have inclusions $\tau_F\subset\eta_F$, $V_F\subset U_F$ for the automaton $Atm^3(A,\Gamma)$. In particular, if Γ is a Moore semigroup, then $\tau_F=\eta_F$, $V_F\subset U_F$.

4.2. Varieties of pure automata

4.2.1. Definitions. Basic properties

A class of all the automata satisfying a certain set of identities is called a *variety of the automata*. Such varieties are sometimes called the varieties with variable semigroup of input signals in contrast to the variety of Γ-automata, which is a class of Γ-automata satisfying a given set of Γ-identities.

In the automata theory varieties can be applied for typical in algebra purposes. First of all, it is a classification of the automata by their identical relations. Each automaton is a homomorphic image of the suitable free automaton in the variety generated by the given automaton. Therefore free automata of different varieties are the matter of special interest.

To each variety correspond a special congruence in individual automaton, called a *verbal congruence*. Such congruences may be of important significance in the structural theorems, in the problem of automata decomposition, in particular. Perhaps all this may become useful also in the various technical applications. Let's consider several simple examples of the automata varieties.

Examples

1) The identity $z*\gamma_1\gamma_2=z*\gamma_2\gamma_1$, $\gamma_1,\gamma_2\in\Gamma$ defines the variety of Γ-automata in which the external action of the two given elements γ_1 and γ_2 is permutable.

2) The identity $z*x_1x_2=z*x_2x_1$, $x_1,x_2\in F=F(X)$ defines the variety of the automata with the variable Γ in which the external action of any two input elements is permutable.

3) The identity $z_1*x=z_2*x$ defines the variety of automata in which the result in output does not depend on the state of the automaton but depends only on the input signal.

4) Two identities $z*x_1=z*x_2$ and $z_1*x=z_2*x$ define the variety of automata, in which output signals do not depend neither on the state of the automaton nor on the signal on its input.

5) The identity $y_1=y_2$ assigns the automata with the unique output signal.

The following Theorem is a particular case of the classic theorem of Birkhoff which is true for the arbitrary many-sorted (heterogeneous) algebras and gives a description of varieties as closed classes of automata.

Theorem 2.1. *A class Θ of automata is a variety if and only if it is closed under taking subautomata, homomorphic images and Cartesian products of automata.*

Observe that Cartesian products can be naturally defined in the category of automata with the fixed semigroup Γ and for the varieties of Γ-automata there exists the theorem similar to Theorem 2.1.

As it has already been mentioned it is convenient to define the identities of the automata in the free automaton Atm(Z,X,Y) with countable sets Z, X and Y. Therefore in defining of the varieties of automata one proceeds from the automaton Atm(Z,X,Y) with countable sets Z,X,Y. If now Θ is a certain class of automata, then to each automaton $\mathfrak{A}_\alpha$ from Θ corresponds the completely characteristic congruence ρ_α in the given automaton Atm(Z,X,Y), namely, the congruence of all identities of the given $\mathfrak{A}_\alpha$ in Atm(Z, X, Y). The intersection $\rho=\bigcap\rho_\alpha$ on all $\mathfrak{A}_\alpha$ from θ is also a completely characteristic congruence, and it gives all identities of the class θ. In particular, to each variety of automata θ corresponds the completely characteristic congruence ρ of the identities which are fulfilled for all automata from θ. On the other hand, each completely characteristic congruence in Atm(Z, X, Y) can be regarded as the set of identities which defines the variety of automata. If we consider only completely characteristic congruences in Atm(Z, X, Y) then this correspondence between the varieties of automata and such congruences is a one-to-one correspondence, i.e.:

Theorem 2.2. *There is one-to-one correspondence between varieties of automata and completely characteristic congruences of a free automaton.*

Bearing in mind Theorem 1.7 it follows that the correspondence between varieties of automata and completely characteristic tuples of the free semigroup F is one-to-one.

Let θ be a certain class of automata, ρ be a completely characteristic congruence of all identities of this class, θ_ρ be a variety corresponding to this congruence. Then $\theta\subset\theta_\rho$ and it is clear that θ_ρ is the minimal variety containing θ. This variety is denoted by Varθ.

Introduce the following operators on the classes of the automata:

C is a Cartesian product operator: if θ is a class of automata, then Cθ is a class of all Cartesian products of the automata from θ;

S is an operator of the transition to subautomata: Sθ is a class of all subautomata of the automata from θ;

Q is a homomorphic image operator: Qθ is a class of all homomorphic images of the automata from θ;

V is a saturation operator: $(A,\Gamma,B)\in V\theta$ if there exist a homomorphism in input signals ψ such that $(A,\Gamma^\psi,B)\in\theta$.

Theorem 2.3. Varθ=QSCθ.

This theorem is proved in a similar way to those for varieties of groups and of representations of groups ([78], [90]). In this case theorem 2.1 and well known relations between introduced operators are used.

The variety of the automata $\mathfrak{X}$ is called *saturated* if it is closed under the saturation operator, that is, if $V\mathfrak{X}=\mathfrak{X}$.

The saturated variety generated by the class Θ coincides with $VQSC\Theta$. There are the automata with arbitrary Γ in the saturated varieties: considering such varieties we pay attention not so much on the construction of the semigroup of the input signals Γ but on its action. The identities of the semigroup Γ are not present in the set of identities of the saturated varieties.

Let $\mathfrak{X}$ be a variety of the automata, and $(\eta_F, U_F, \xi_F, \tau_F, V_F)$ be the corresponding external tuple. Associate with the given variety $\mathfrak{X}$ the saturated variety of the automata $\mathfrak{Y}=V\mathfrak{X}$, and the variety of the semigroups Θ satisfying the identities from ξ_F In this case the following condition is satisfied:

if (A,Γ,B) is an exact automaton from $\mathfrak{Y}$, then $\Gamma\in\Theta$. (*)

On the other hand, assign to each pair $(\mathfrak{Y},\Theta)$ with $\mathfrak{Y}$ and Θ being the saturated variety of automata and variety of semigroups respectively, satisfying the condition (*), the variety of the automata $\mathfrak{X}=\{(A,\Gamma,B)\mid(A,\Gamma,B)\in\mathfrak{Y},\ \Gamma\in\Theta\}$.

It is easy to show that this correspondence is a one-to-one. Thus, consideration of varieties of automata is reduced to that of saturated varieties of automata and of varieties of the semigroups.

Let us introduce the notion of a free union of the automata. Given the set of the automata $\mathfrak{A}_\alpha=(A_\alpha,\Gamma_\alpha,B_\alpha)$, $\alpha\in I$, an automaton $\mathfrak{A}=(A,\Gamma,B)$ is called a *free union* of these $\mathfrak{A}_\alpha$ if A is a free union of the sets A_α, B is a free union of the sets B_α, the semigroup Γ is a Cartesian product of the semigroups Γ_α and the operations $\circ$ and $*$ are defined by the rules: if $a\in A_\alpha$, then $a\circ\gamma=a\circ\gamma^{\pi_\alpha}$; $a*\gamma=a*\gamma^{\pi_\alpha}$, where π_α denotes projection of Γ on Γ_α It is clear, that $\mathfrak{A}$ is an automaton.

Remark. If $\mathfrak{A}=(A,\Gamma,B)$ is a free union of certain automata $\mathfrak{A}_1$ and $\mathfrak{A}_2$, then for such automaton $\mathfrak{A}$ annihilators U_F and V_F defined in 1.3 are empty. Really, if the element $\gamma\in\Gamma$ belongs to U_F, then $a_1\circ\gamma=a_2\circ\gamma$ takes place for any elements a_1,a_2 from A. However, for $a_1\in A_1$ and $a_2\in A_2$ hold $a_1\circ\gamma=a_1\circ\gamma^{\pi_1}\in A_1$, $a_2\circ\gamma\in A_2$. This contradicts the fact that the sets A_1 and A_2 are disjoint. It is also easy to see that the set V_F is empty.

Variety of automata is called *special* if it is closed with respect to free unions.

Proposition 2.4. *The variety* Θ *is special if and only if among its determining identities there are no identities of the types* $z_1\circ f\equiv z_2\circ f$ *and* $z_1*f\equiv z_2*f$ *(i.e. annihilator kind identities).*

Proof. Let Θ be a special variety and $\mathfrak{A}$ a free union of the automata $\mathfrak{A}_1,\mathfrak{A}_2$ from Θ. Then $\mathfrak{A}\in\Theta$. By the above Remark, U and V are empty sets for $\mathfrak{A}$. It means that among the identities of this automaton and consequently, also among the identities of the variety Θ there are no identities of the type $z_1\circ f\equiv z_2\circ f$, $f\in U_F$; $z_1*f\equiv z_2*f$, $f\in V_F$. On the other hand, if there are no identities of the given type among the identities of the variety Θ, then it is closed with respect to free unions. Indeed, it is clear that if the identities of the type $z\circ f_1\equiv z\circ f_2$ and $z*f_1\equiv z*f_2$ are fulfilled for the automata $\mathfrak{A}_\alpha$, $\alpha\in I$, then they are also fulfilled for the free union of these automata. Besides, it is known that if a certain identity is fulfilled for the semigroups Γ_α, $\alpha\in I$, then it is fulfilled for a Cartesian product of these semigroups.

Remark. Proposition 2.4 means that if (η,U,ξ,τ,V) is a completely characteristic tuple of the free semigroup F corresponding to the special variety Θ, then the ideals U, V are empty.

The following theorem is a version of the theorem of Remak for automata.

Theorem 2.5. Let ρ_α, $\alpha\in I$ *be a certain set of the congruences of the automaton* $\mathfrak{A}=(A,\Gamma,B)$ *and let* $\rho=\bigcap\rho_\alpha$. *Then the quotient automaton* $\mathfrak{A}/\rho$ *is isomorphically embedded into the Cartesian product* $\prod_\alpha\mathfrak{A}/\rho_\alpha$ *of the automata* $\mathfrak{A}/\rho_\alpha$.

Proof. Denote by μ_α a natural homomorphism of the automata $\mu_\alpha:\mathfrak{A}\to\mathfrak{A}/\rho_\alpha$. The homomorphism $\mu=(\mu_1,\mu_2,\mu_3):\mathfrak{A}\to\prod_\alpha\mathfrak{A}/\rho_\alpha$ defined by the rule: if $\mu_\alpha=(\mu_1^\alpha,\mu_2^\alpha,\mu_3^\alpha)$ and $a\in A$, $\gamma\in\Gamma$, $b\in B$, then $a^{\mu_1}(\alpha)=a^{\mu_1^\alpha}(\alpha)$; $\gamma^{\mu_2}(\alpha)=\gamma^{\mu_2^\alpha}(\alpha)$; $b^{\mu_3}(\alpha)=b^{\mu_3^\alpha}(\alpha)$, uniquely corresponds to the set of all these μ_α From the definition of μ it follows that $\mathrm{Ker}\mu=\bigcap_\alpha\mathrm{Ker}\mu_\alpha=\bigcap_\alpha\rho_\alpha=\rho$. It means that the automaton $\mathfrak{A}/\rho$ is isomorphically embedded into the automaton $\prod_{\alpha}\mathfrak{A}/\rho_{\alpha}$.

A congruence ρ equal to the intersection of all such congruences ρ_{α}, that $\mathfrak{A}/\rho_{\alpha}\in\Theta$, is called a *verbal congruence of the automaton* $\mathfrak{A}$ by the variety Θ. Denote the verbal congruence by $\Theta^{*}(\mathfrak{A})$. This congruence is a completely characteristic one.

Proposition 2.6. $\Theta^{*}(\mathfrak{A})$ *is minimal among the congruences with the property* $\mathfrak{A}/\rho_{\alpha}\in\Theta$.

Proof. By the definition, $\Theta^{*}(\mathfrak{A})=\bigcap_{\alpha}\rho_{\alpha}$ where ρ_{α} are all congruences with the property $\mathfrak{A}/\rho_{\alpha}\in\Theta$. Therefore the statement is equivalent to $\mathfrak{A}/\Theta^{*}(\mathfrak{A})\in\Theta$. By Remak's theorem $\mathfrak{A}/\Theta^{*}(\mathfrak{A})$ is isomorphically embedded into $\prod_{\alpha}\mathfrak{A}/\rho_{\alpha}$. Since all $\mathfrak{A}/\rho_{\alpha}\in\Theta$, then by Birkhoff's theorem, $\prod_{\alpha}\mathfrak{A}/\rho_{\alpha}$ is also contained in Θ. By the same theorem each subautomaton of the given product is also contained in Θ. Therefore, $\mathfrak{A}/\Theta^{*}(\mathfrak{A})\in\Theta$.

Corollary. *If* ρ *is a congruence of the automaton* $\mathfrak{A}$, *then* $\mathfrak{A}/\rho\in\Theta$ *is equivalent to* $\rho\supset\Theta^{*}(\mathfrak{A})$.

Along with the automaton which is free in the variety of all automata (or simply with a free automaton) consider the automaton which is free in the variety of the automata Θ. It is such automaton $\mathfrak{A}\in\Theta$, that for a certain system of its generators (Z,X,Y) any mapping of this system into an arbitrary automaton $\mathfrak{A}'$ from Θ is uniquely extended up to the homomorphism of $\mathfrak{A}$ to $\mathfrak{A}'$.

Denote such automaton by $\mathrm{Atm}_{\Theta}(Z,X,Y)$. By contrast to it the automaton $\mathrm{Atm}(Z,X,Y)$ is an absolutely free one, i.e. a free automaton in the variety of all automata.

Proposition 2.7. *If* ρ *is a verbal congruence of the absolutely free automaton* $\mathrm{Atm}(Z,X,Y)$ *by a variety* Θ, *then* $\mathrm{Atm}(Z,X,Y)/\rho$ *is the automaton, free in this variety* Θ.

Proof. Let $\mathfrak{A}=(A,\Gamma,B)$ be an arbitrary automaton in Θ and let the mappings $\mu_1:Z\to A$, $\mu_2:X\to\Gamma$, $\mu_3:Y\to B$ be given. It is necessary to verify that these mappings are uniquely extended up to the homomorphism

μ: Atm(Z, X, Y)/$\rho \to \mathfrak{A}$. First take the extension of the mappings μ_1, μ_2, μ_3 up to the homomorphism ν: Atm(Z, X, Y) $\to \mathfrak{A}$. Since $\mathfrak{A} \in \Theta$, then also Atm(Z, X, Y)/Ker$\nu \in \Theta$. By the Corollary of the previous proposition $\rho \subset$ Kerν. Therefore, there exists the homomorphism μ: Atm(Z, X, Y)/$\rho \to \mathfrak{A}$, extending the given mappings.

Proposition 2.8. *Let Θ be a class of automata,* VarΘ *be a variety generated by it. Consider the homomorphisms μ_α:* Atm(Z, X, Y) $\to \mathfrak{A}$ *for all $\mathfrak{A}$ from Θ. Let ρ_α=Kerμ_α, $\rho=\bigcap \rho_\alpha$. Then* Atm(Z, X, Y)/ρ=$\mathrm{Atm}_{\mathrm{Var}\Theta}$(Z, X, Y).

Indeed, from the theorems of Remak and Birkhoff follows that Atm(Z, X, Y)/$\rho \in$ VarΘ. From the definition of ρ it is clear that ρ is a completely characteristic congruence of all the identities of the class Θ which, in its turn, coincides with the congruence of all the identities of the variety VarΘ. This exactly means that Atm(Z, X, Y)/ρ= $\mathrm{Atm}_{\mathrm{Var}\Theta}$(Z, X, Y).

Recall that the automaton (A, Γ, B) is called a finite one, if A, Γ, B are finite sets.

Theorem 2.9. *If the class Θ contains a finite number of finite automata and sets Z, X, Y are finite, then the free automaton $\mathrm{Atm}_{\mathrm{Var}\Theta}$(Z, X, Y) is also finite.*

Proof. Let $\mathfrak{A}_i=(A_i, \Gamma_i, B_i)$, i=1,2,...,n be all the automata of the class Θ. Take a homomorphism μ^α: Atm(Z, X, Y) $\to \mathfrak{A}$, $\mathfrak{A} \in \Theta$ and let ρ^α=Kerμ^α. Then Atm(Z, X, Y)/ρ^α is a finite automaton. By virtue of finiteness of the sets Z, X, Y there is only a finite number of different mappings of the type $Z \to A_i$, $X \to \Gamma_i$, $Y \to B_i$, and therefore, a finite number of different μ^α and ρ^α If $\rho=\bigcap \rho_\alpha$, then by Proposition 2.8 the automaton $\mathrm{Atm}_{\mathrm{Var}\Theta}$(Z, X, Y) is equal to Atm(Z, X, Y)/ρ. By Remak's Theorem the latter is isomorphic to the subautomaton of the Cartesian product $\prod_\alpha$ Atm(Z, X, Y)/ρ_α, which is a finite one.

4.2.2. Varieties of group automata

Let (A, Γ, B) be a group automaton, that is, an automaton with semigroup Γ being a group. By Proposition 3.1 from Chapter 1, a group automaton is always a Moore automaton. As it was noted, the kernel (the kernel congruence) of the Moore automaton (A,Γ,B) coincides with the kernel of the representation (A,Γ). Any congruence of the group Γ is uniquely defined by the coset containing unit element ε. This coset is an invariant subgroup of the group Γ. Let us denote it by Σ and call a kernel of the automaton. The automaton is exact if its kernel is trivial, that is, if $\Sigma=\{\varepsilon\}$. Together with the kernel of the automaton (A,Γ,B) the *external kernel* τ of (A,Γ,B) is defined: $\gamma_1\tau\gamma_2$, $\gamma_1,\gamma_2\in\Gamma$, if and only if $a*\gamma_1=a*\gamma_2$ for each element $a\in A$. Denote by Σ_1 the class $[\varepsilon]_\tau$ of the equivalence τ which contains the unit ε of the group Γ; in other words, Σ_1 is a set of all $\gamma\in\Gamma$ satisfying the condition $a*\gamma=a*\varepsilon$ for all $a\in A$. Σ_1 is a subgroup in Γ. Indeed, if σ_1,σ_2 are the elements from Σ_1, then $a*\sigma_1\sigma_2=(a\circ\sigma_1)*\sigma_2=(a\circ\sigma_1)*\varepsilon=a*\sigma_1=a*\varepsilon$; if $\sigma\in\Sigma_1$, then $a*\varepsilon=a*\sigma^{-1}\sigma=(a\circ\sigma^{-1})*\sigma=(a\circ\sigma^{-1})*\varepsilon=a*\sigma^{-1}$. It is evident that $\Sigma\subset\Sigma_1$. Consider the set $\Phi=\bigcap\gamma^{-1}\Sigma_1\gamma$, $\gamma\in\Gamma$. This set is a maximal invariant subgroup from Γ contained in Σ_1. The external kernel τ uniquely defines this invariant subgroup. Therefore, let us call Φ also an external kernel of the automaton (A,Γ,B). It is clear that $\Sigma\subset\Phi$. The automaton is called *absolutely exact*, if $\Phi=\{\varepsilon\}$.

The following propositions [79] set the connection between the varieties of group automata and the triplets of the varieties of groups Θ_1, Θ_2, Θ_3, satisfying the condition $\Theta_1\supset\Theta_2\supset\Theta_3$.

Theorem 2.10. *Let $\mathfrak{X}$ be a variety of group automata, Θ_1 a class of groups which are the groups of input signals of the automata from $\mathfrak{X}$ (allowing a representation in $\mathfrak{X}$), Θ_2 a class of groups allowing an exact representation in $\mathfrak{X}$, Θ_3 a class of groups allowing an absolutely exact representation in $\mathfrak{X}$. Then Θ_1, Θ_2, Θ_3 are varieties of groups and $\Theta_1\supset\Theta_2\supset\Theta_3$.*

Proof. By Birkhoff's theorem, the class of groups is a variety of groups if it is closed with respect to taking of subgroups, homomorphic images and Cartesian products, i.e. it is closed under the operators S, Q and C.

Let us take the group Γ from Θ_1 and the subgroup Δ from Γ. Since $\Gamma\in\Theta_1$, then there exists the automaton $(A,\Gamma,B)\in\mathfrak{X}$. (A,Δ,B) is a subautomaton in (A,Γ,B) and therefore it also belongs to $\mathfrak{X}$. Hence, $\Delta\in\Theta_1$ and the class θ_1 is closed with respect to the operator S. Consider an arbitrary homomorphism of the groups $\nu:\Gamma^{\nu} \to \Delta$, $\Gamma\in\theta_1$, $(A,\Gamma,B)\in\mathfrak{X}$. Take the elements a_0,b_0 and define the automaton (A_0,Δ,B_0) with $A_0=\{a_0\}$, $B_0=\{b_0\}$ and $a_0\circ\delta=a_0$, $a_0*\delta=b_0$ for each $\delta\in\Delta$. It is easy to see that (A_0,Δ,B_0) is a homomorphic image of the automaton (A,Γ,B). Therefore, (A_0,Δ,B_0) also belongs to $\mathfrak{X}$, $\Delta\in\theta_1$ and the class θ_1 is closed with respect to the operator Q. Let now Γ_α be a set of groups from θ_1 (α runs through a certain set I) and let $\mathfrak{A}_\alpha=(A_\alpha,\Gamma_\alpha,B_\alpha)\in\mathfrak{X}$ be a set of the automata. The Cartesian product of these automata $\prod\mathfrak{A}_\alpha=(A,\prod\Gamma_\alpha,B)$, $\alpha\in I$, also belongs to $\mathfrak{X}$, and therefore, the Cartesian product of the groups Γ_α belongs to θ_1; this means that the class θ_1 is closed with respect to the operator C. Thus, the class θ_1 is a variety of groups.

Let $\Gamma\in\theta_2$. It means that there exists an exact automaton $(A,\Gamma,B)\in\mathfrak{X}$. Since the automaton (A,Γ,B) is a group one, then its exactness is equivalent to the exactness of the representation (A,Γ). From this immediately follows that the class θ_2 is closed under the operators S and C. Show that the class θ_2 is closed with respect to the operator Q. Take an arbitrary element b_0 and consider the automaton (A,Γ,B_0) in which $B_0=\{b_0\}$, the operation $\circ$ is defined as in the automaton (A,Γ,B) and $a*\gamma=b_0$ for any elements $a\in A$ and $\gamma\in\Gamma$. This automaton is a homomorphic image of the automaton (A,Γ,B) and therefore it belongs to $\mathfrak{X}$. Its subautomaton $(a\circ\Gamma,\Gamma,B_0)$ also lies in $\mathfrak{X}$ (here, as earlier, $a\circ\Gamma=\{a\circ\gamma,\gamma\in\Gamma\}$). The representation $(a\circ\Gamma,\Gamma)$ is isomorphic to the quotient representation $(\Gamma/\rho_a,\Gamma)$ of the regular representation (Γ,Γ). In this case $\gamma_1\rho_a\gamma_2$ means that $a\circ\gamma_1=a\circ\gamma_2$, and the automaton $(a\circ\Gamma,\Gamma,B_0)$ is isomorphic to the automaton $(\Gamma/\rho_a,\Gamma,B_0)$ with the following operation $*$: $\bar{\gamma}*x=b_0$, $\bar{\gamma}\in\Gamma/\rho_a$, $x\in\Gamma$. Since the representation (A,Γ) is exact, the intersection of all ρ_a, $a\in A$ is a congruence, classes of which are individual elements of the group Γ. By Remak's theorem the automaton $(\Gamma,\Gamma,B_0)\cong(\Gamma/\bigcap\rho_a,\Gamma,B_0)$, $a\in A$, is a subautomaton of the Cartesian product of the automata $(\Gamma/\rho_a,\Gamma,B_0)\in\mathfrak{X}$. Therefore, $(\Gamma,\Gamma,B_0)\in\mathfrak{X}$. If now Δ is a homomorphic image of the group Γ, then (Δ,Δ,B_0) is a homomorphic image of the automaton (Γ,Γ,B_0) and therefore it also belongs to $\mathfrak{X}$. Besides, the automaton (Δ,Δ,B_0) is exact. Hence, $\Delta\in\theta_2$.

In order to prove that the class θ_3 is a variety of groups let us introduce a class $\tilde{\theta}_3=\{\Gamma \mid (\Gamma,\Gamma,\Gamma)\in\mathfrak{X}\}$ and show that $\tilde{\theta}_3$ is a variety of groups which coincides with the class θ_3. We leave to the reader the verification of the first statement, i.e. that the class $\tilde{\theta}_3$ is closed under the operators S, Q, C, and check the second one. Let $\Gamma\in\tilde{\theta}_3$. Hence, an absolutely exact automaton (Γ,Γ,Γ) lies in $\mathfrak{X}$ and $\Gamma\in\theta_3$. Thus, $\tilde{\theta}_3\subset\theta_3$. Conversely, let $\Gamma\in\theta_3$. Then there exists the absolutely exact automaton $\mathfrak{A}=(A,\Gamma,B)\in\mathfrak{X}$. Take an arbitrary element $a\in A$; $(a\circ\Gamma,\Gamma,a*\Gamma)$ is a subautomaton in (A,Γ,B) and therefore it also belongs to $\mathfrak{X}$. Let us introduce the equivalences ρ_a and τ_a on Γ: $\gamma_1\rho_a\gamma_2$ if and only if $a\circ\gamma_1=a\circ\gamma_2$; $\gamma_1\tau_a\gamma_2$ if and only if $a*\gamma_1=a*\gamma_2$. Denote by δ_Γ the trivial equivalence on Γ, classes of which are individual elements. Then $(\rho_a,\delta_\Gamma,\tau_a)$ is a congruence of the regular automaton (Γ,Γ,Γ) and $(a\circ\Gamma,\Gamma,a*\Gamma)\cong(\Gamma/\rho_a,\Gamma,\Gamma/\tau_a)\in\mathfrak{X}$. The intersection $\cap\rho_a$, $a\in A$ is a kernel of the automaton (A,Γ,B). The intersection $\cap\tau_a$, $a\in A$ is an external kernel of this automaton. Since the automaton (A,Γ,B) is an absolutely exact one, these intersections are equal to δ_Γ. By Remak's theorem the automaton $(\Gamma,\Gamma,\Gamma)\cong(\Gamma/\bigcap\rho_a,\Gamma,\Gamma/\bigcap\tau_a)$ is isomorphically embedded into the Cartesian product of the automata $(\Gamma/\rho_a,\Gamma,\Gamma/\tau_a)$ and therefore it also lies in $\mathfrak{X}$; it means that $\Gamma\in\tilde{\theta}_3$, and $\theta_3=\tilde{\theta}_3$, that proves the Proposition.

On the other hand, note the following statement without being proved. Denote by $\theta^*(\Gamma)$ a *verbal subgroup* of the group Γ, i.e. a minimal invariant subgroup of Γ, such that $\Gamma/\theta^*(\Gamma)$ lies in θ.

Theorem 2.11. *Let $\theta_1\supset\theta_2\supset\theta_3$ be varieties of groups. Define the class of the automata $\mathfrak{X}$ by the rule: the automaton $\mathfrak{A}=(A,\Gamma,B)$ belongs to $\mathfrak{X}$ if the following three conditions are satisfied*

1) $\Gamma\in\theta_1$;

2) $\theta_2^*(\Gamma)\subset\Sigma$, *where Σ is the kernel of the automaton $\mathfrak{A}$;*

3) $\theta_3^*(\Gamma)\subset\Phi$, *where Φ is the external kernel of the automaton $\mathfrak{A}$.*

Then $\mathfrak{X}$ is a variety of automata.

Theorem 2.10 assigns to each variety of group automata three embedded into each other varieties of groups. Theorem 2.11 to each such triplet of the varieties of groups assigns a variety of group automata. The following statement shows that this assignment has the properties of closeness.

Theorem 2.12. *Let* $\theta_1\supset\theta_2\supset\theta_3$ *be varieties of groups,* $\mathfrak{X}$ *be a corresponding to them (by Theorem 2.11) variety of group automata,* $\theta'_1\supset\theta'_2\supset\theta'_3$ *be varieties of groups corresponding to the variety of the automata* $\mathfrak{X}$ *by Theorem 2.10. Then* $\theta'_1=\theta_1$, $\theta'_2=\theta_2$, $\theta'_3=\theta_3$.

Proof. Let $\Gamma\in\theta_1$ Denote $\Delta_2=\theta^*_2(\Gamma)$ and $\Delta_3=\theta^*_3(\Gamma)$ and consider the automaton $(\Gamma/\Delta_2,\Gamma,\Gamma/\Delta_3)$ with the following operations $\circ$ and $*$ · if $x\in\Gamma$, $[\gamma]_2$, $[\gamma]_3$ are cosets containing γ in Γ/Δ_2 and Γ/Δ_3 respectively, then $[\gamma]_2\circ x=[\gamma x]_2$, $[\gamma]_2*x=[\gamma x]_3$. Clearly, the operation $\circ$ is correct. The operation $*$ is correctly defined due to the condition $\theta_2\supset\theta_3$ and its consequence $\theta^*_3(\Gamma)\supset\theta^*_2(\Gamma)$. The kernel of this automaton contains $\Delta_2=\theta^*_2(\Gamma)$ and the external kernel contains $\theta^*_3(\Gamma)=\Delta_3$. Hence, this automaton belongs to the variety $\mathfrak{X}$. From the definition of θ'_1 it follows that $\Gamma\in\theta'_1$. Therefore, $\theta_1\subset\theta'_1$. On the other hand, let $\Gamma\in\theta'_1$. It means that there exists the automaton (A,Γ,B) with the given group of inputs Γ, belonging to $\mathfrak{X}$. Then by the definition of $\mathfrak{X}$ the group Γ belongs to θ_1 Hence, $\theta'_1\subset\theta_1$ and $\theta'_1=\theta_1$.

Take $\Gamma\in\theta'_2$. Then there exists an exact automaton (A,Γ,B) from $\mathfrak{X}$. By the definition of $\mathfrak{X}$, $\theta^*_2(\Gamma)$ belongs to Σ - the kernel of this automaton, but since the automaton is exact, then $\theta^*_2(\Gamma)=\{\varepsilon\}$. Hence, $\Gamma\in\theta_2$. On the other hand, let $\Gamma\in\theta_2$. Denote, as before, $\Delta_3=\theta^*_3(\Gamma)$ and define the automaton $\mathfrak{A}=(\Gamma,\Gamma,\Gamma/\Delta_3)$ with the operation $\circ$ being a multiplication in Γ and the operation $*$: $\gamma*x=[\gamma x]_3$, $\gamma,x\in\Gamma$. This automaton belongs to $\mathfrak{X}$, since $\Gamma\in\theta_2\subset\theta_1$, $\theta^*_2(\Gamma)=\{\varepsilon\}\subset\Sigma$ and $\theta^*_3(\Gamma)\subset\Phi$. (Recall that Σ, Φ are respectively the kernel and the external kernel of the automaton). Besides, the automaton $\mathfrak{A}$ is exact, hence, $\Gamma\in\theta'_2$ and $\theta'_2=\theta_2$

Show that $\theta_3=\theta'_3$. It has been proved (Theorem 2.10) that $\theta'_3=\tilde{\theta}_3$ where $\tilde{\theta}_3=\{\Gamma\,|\,(\Gamma,\Gamma,\Gamma)\in\mathfrak{X}\}$. Demonstrate that $\tilde{\theta}_3=\theta_3$. Let $\Gamma\in\tilde{\theta}_3$, then $(\Gamma,\Gamma,\Gamma)\in$ $\mathfrak{X}$. By the definition of $\mathfrak{X}$, $\theta^*_3(\Gamma)$ belongs to Φ - an external kernel of this automaton, equal to a unit. Hence, $\theta_3^*(\Gamma)=\{\varepsilon\}$ and $\Gamma\in\theta_3$. On the other hand, let $\Gamma\in\theta_3\subset\theta_2\subset\theta_1$. Then $\theta_2^*(\Gamma)=1$ and $\theta_3^*(\Gamma)=1$. Therefore, the automaton (Γ,Γ,Γ) lies in $\mathfrak{X}$. Thus, $\Gamma\in\tilde{\theta}_3$, as required.

If the variety of automata $\mathfrak{X}$ is generated by one automaton, then each group variety of the corresponding triplet $\theta_1\subset\theta_2\subset\theta_3$ is generated by one group. The following theorem specifies the generating groups of the varieties θ_i.

Theorem 2.13. *Let* $\mathfrak{X}=\mathrm{Var}\mathfrak{A}$ *be a variety generated by the automaton* $\mathfrak{A}=(A,\Gamma,B)$; $\theta_1\supset\theta_2\supset\theta_3$ *be varieties of groups corresponding to* $\mathfrak{X}$ *according to Theorem* 2.10. *Then* $\theta_1=\mathrm{Var}\Gamma$, $\theta_2=\mathrm{Var}\Gamma/\Sigma$, $\theta_3=\mathrm{Var}\Gamma/\Phi$, *where* Σ *and* Φ *are a kernel and an external kernel of the automaton* $\mathfrak{A}$, *respectively.*

Proof. a) By the definition $\Gamma\in\theta_1$, therefore $\mathrm{Var}\Gamma\subset\theta_1$ On the other hand, let the group G belong to the variety θ_1 It implies that G admits a representation in $\mathfrak{X}$, that is, there exists the automaton (A',G,B') belonging to $\mathfrak{X}=QSC\mathfrak{A}$. Therefore, there is such an automaton $(\tilde{A},\tilde{G},\tilde{B})$ lying in a certain Cartesian power $\mathfrak{A}^{\alpha}$ of the automaton $\mathfrak{A}=(A,\Gamma,B)$, that the automaton (A',G,B') is a homomorphic image of this automaton: $(A',G,B')=(\tilde{A},\tilde{G},\tilde{B})^{\varphi}$. Hence, $\tilde{G}\subset\Gamma^{\alpha}\in\mathrm{Var}\Gamma$, and $G=\tilde{G}^{\varphi}$ also belongs to $\mathrm{Var}\Gamma$. Thus, $\theta_1\subset\mathrm{Var}\Gamma$ and, finally, $\theta_1=\mathrm{Var}\Gamma$.

b) Since $(A,\Gamma/\Sigma,B)$ is an exact automaton of $\mathfrak{X}$, then $\Gamma/\Sigma\in\theta_2$, and $\mathrm{Var}\Gamma/\Sigma\subset\theta_2$. On the other hand, let $\varphi(z_1,\ldots,z_n)$ be a certain identity of the group Γ/Σ. Then $x\circ\varphi(z_1,\ldots,z_n)\equiv x$ is an identity of the automaton $(A,\Gamma/\Sigma,B)$ and therefore, of the automaton (A,Γ,B), and since $\mathfrak{X}=\mathrm{Var}(A,\Gamma,B)$, of the variety $\mathfrak{X}$. If $G\in\theta_2$, then there exists the exact automaton $(\tilde{A},G,\tilde{B})\in\mathfrak{X}$. Hence, the automaton $(\tilde{A},G,\tilde{B})$ and the representation $(\tilde{A},G)$ satisfy the identity $x\circ\varphi(z_1,\ldots,z_n)\equiv x$. The representation $(\tilde{A},G)$ is exact, hence, the identity $\varphi(z_1,\ldots,z_n)$ is also satisfied in the group G. Thus, any identity of the variety $\mathrm{Var}\Gamma/\Sigma$ is satisfied in the variety θ_2. Therefore, $\theta_2\subset\mathrm{Var}\Gamma/\Sigma$. Taking into consideration the inverse inclusion, we get $\theta_2=\mathrm{Var}\Gamma/\Sigma$.

c) First show that the group Γ/Φ, and therefore the variety $\mathrm{Var}\Gamma/\Phi$, belongs to θ_3. To do this it suffices to construct an absolutely exact automaton with the acting group Γ/Φ from $\mathfrak{X}$. Denote by ρ a splitting of the set A into Φ-orbits, that is, into the sets of the form $a\circ\Phi=\{a\circ\varphi|\varphi\in\Phi\}$, $a\in A$. Since Φ is an invariant subgroup in Γ, the equivalence ρ is invariant with respect to Γ. Really, $a\rho a'$ means that for a certain element $\varphi\in\Phi$, $a'=a\circ\varphi$. Then $a'\circ\gamma=a\circ\varphi\gamma=a\circ\gamma\varphi'$, $\varphi'\in\Phi$ (since Φ is a invariant subgroup in Γ). The equality $(a'\circ\gamma)=(a\circ\gamma)\circ\varphi'$ means that $(a'\circ\gamma)\rho(a\circ\gamma)$, i.e. ρ is invariant with respect to Γ. By virtue of this property it is possible to consider the representation $(A/\rho,\Gamma)$ in which the action $\circ$ is defined by the rule: $\bar{a}\circ\gamma=\overline{a\circ\gamma}$, where $\bar{a}$ is a coset of the equivalence ρ containing the element a. Φ belongs to the kernel of this representation. Now consider the automaton $(A/\rho,\Gamma,B)$ in which the action $\circ$ is defined by the representation $(A/\rho,\Gamma)$ while the operation $*$ is defined by the equality: $\bar{a}*\gamma=a*\gamma$. This definition is correct: if $\bar{a}_1=\bar{a}_2$, then $a_1=a_2\circ\varphi$, $\varphi\in\Phi$, and

$$\bar{a}_1*\gamma=a_1*\gamma=(a_2\circ\varphi)*\gamma=a_2*\varphi\gamma=a_2*\gamma\varphi'=(a_2\circ\gamma)*\varphi'=(a_2\circ\gamma)*\varepsilon=a_2*\gamma=\bar{a}_2*\gamma.$$

The group Φ belongs to the kernel of the constructed automata, since this kernel coincides with the kernel of the representation $(A/\rho,\Gamma)$. Therefore, it is possible to consider the automaton $(A/\rho,\Gamma/\Phi,B)=\bar{\mathfrak{A}}$. Show that it is a required automaton. It lies in $\mathfrak{X}$ as a homomorphic image of the automaton (A,Γ,B) from $\mathfrak{X}$. It remains to check that the automaton $(A/\rho,\Gamma/\Phi,B)$ is absolutely exact, that is, its external kernel is equal to a unit. This is equivalent to the fact that the external kernel of the automaton $(A/\rho,\Gamma,B)$ coincides with Φ. The latter is quite evident, since by the definition $\bar{a}*\gamma=a*\gamma$. Thus, Γ/Φ allows an absolutely exact representation in $\mathfrak{X}$, that is $\Gamma/\Phi\in\Theta_3$, and $\mathrm{Var}\Gamma/\Phi\subset\Theta_3$.

Conversely, let $\varphi(z_1,\dots,z_n)$ be an identity of the group Γ/Φ. Since Φ is an external kernel of the automaton (A,Γ,B), the $x*\varphi(z_1,\dots,z_n)=x*\varepsilon$ is an identity of this automaton and consequently, is an identity of the variety $\mathfrak{X}$. Let now $G\in\Theta_3$, and $(\tilde{A},G,\tilde{B})$ be an absolutely exact automaton from $\mathfrak{X}$. Since this automaton lies in $\mathfrak{X}$, it satisfies the identity $x*\varphi(z_1,\dots,z_n)=x*\varepsilon$; and since it is absolutely exact, $\varphi(z_1,\dots,z_n)$ is an identity of the group G. Thus, we have the inverse inclusion $\Theta_3\subset\mathrm{Var}\Gamma/\Phi$ and therefore the equality $\Theta_3=\mathrm{Var}\Gamma/\Phi$.

4.3. Identities of linear automata and biautomata

4.3.1. Identities of biautomata

Let Γ^1, as before, be a result of the external adjoining of the unit element to the semigroup Γ, K be a field. Consider the tensor product $\Psi=K\Gamma^1\otimes K\Gamma$ and define the structure of $K\Gamma$-bimodule on it: if $u\otimes v\in K\Gamma^1\otimes K\Gamma$, $u'\in K\Gamma$, then set $u'(u\otimes v)= u'u\otimes v$, $(u\otimes v)\cdot u'=u\otimes vu'-uv\otimes u'$.

The action, thus defined, (from the left and from the right) of the element $u'\in K\Gamma$ on the elements of the form $u\otimes v\in K\Gamma^1\otimes K\Gamma$ are extended by linearity to the action of the element u' on the arbitrary elements w of $K\Gamma^1\otimes K\Gamma$. All the axioms of bimodule are satisfied; in particular, if $u,u'\in K\Gamma$, $w\in K\Gamma^1\otimes K\Gamma$, then

$$u'(uw)=uu'w,$$
$$u(w\cdot u')=uw\cdot u'$$

An arbitrary biautomaton $\mathfrak{A}=(A,\Gamma, B)$ is uniquely extended up to the biautomaton $(A,K\Gamma,B)$. Proceeding from the operations $\circ$, $*$ of this biautomaton it is possible to define one more operation the action of the elements w of $K\Gamma^1\otimes K\Gamma$ on the elements $a\in A$. Namely, if $w=\sum_1^n\alpha_{ij}\gamma_i\otimes\gamma_j\in K\Gamma^1\otimes K\Gamma$, then by the definition: $aw=a\sum_1^n\alpha_{ij}\gamma_i\otimes\gamma_j=\sum_1^n\alpha_{ij}(a\circ\gamma_i)*\gamma_j$, $i,j=1,\dots,n$. This operation has a number of useful properties, in particular:

1) $a \longrightarrow aw$ is a linear mapping;
2) $a(w_1+w_2)=aw_1+aw_2$;
3) $a\alpha w=\alpha aw$;
4) if $w=\sum_{i=1}^n u_i\otimes v_i$, $u_i\in K\Gamma^1$, $v_i\in K\Gamma$, then $aw=\sum_{i=1}^n(a\circ u_i)*v_i$;
5) this operation agrees with the structure of the bimodule

$$a(uw)=(a\circ u)w;\ a(w\cdot u)=(aw)\cdot u.$$

Now, define the regular biautomaton $(K\Gamma^1,\Gamma,K\Gamma^1\otimes K\Gamma)$ in the following way:

$$\circ: x\circ\gamma=x\gamma,$$
$$*: x*\gamma=x\otimes\gamma,\ x\in K\Gamma^1,\ \gamma\in\Gamma.$$

The action $\cdot$ is determined by defining of $K\Gamma^1\otimes K\Gamma$ as a right $K\Gamma$-module. It is easy to verify that it is really a biautomaton.

In the Section 3.1, the free Γ-biautomaton $Atm_\Gamma(Z,Y)=(H,\Gamma,G)=(ZK\Gamma^1,\Gamma,(ZK\Gamma^1\otimes K\Gamma)\otimes YK\Gamma^1)$ was introduced. It is convenient to consider tensor product $ZK\Gamma^1\otimes K\Gamma=\Psi$ as a K-module of the formal sums of the form $\sum_{i=1}^n z_iw_i$, $z_i\in Z$, $w_i\in K\Gamma^1\otimes K\Gamma$ (in this case $\sum_{i=1}^n z_iw_i+\sum_{i=1}^n z_iw'_i$ is defined as $\sum_{i=1}^n z_i(w_i+w'_i)$) and denote it by $Z(K\Gamma^1\otimes K\Gamma)$. Then operations in the free Γ-biautomaton (H,Γ,G) are written in the following way: if $h=\sum_{i=1}^n z_i\circ u_i\in H$, $g_1=\sum_{i=1}^n z_iw_i\in Z(K\Gamma^1\otimes K\Gamma)$, $g_0=\sum_{i=1}^n y_i\cdot u_i\in YK\Gamma^1$, $\gamma\in\Gamma$, then

$$h\circ\gamma=(\sum_{i=1}^n z_i\circ u_i)\circ\gamma=\sum_{i=1}^n z_i\circ(u_i\gamma),$$

$$h*\gamma=(\sum_{i=1}^n z_i\circ u_i)*\gamma=\sum_{i=1}^n z_i(u_i\otimes\gamma),$$

$$g_1\cdot\gamma=(\sum_{i=1}^n z_iw_i)\cdot\gamma=\sum_{i=1}^n z_i(w_i\cdot\gamma),$$

$$g_0\cdot\gamma=(\sum_{i=1}^n y_i\cdot u_i)\cdot\gamma=\sum_{i=1}^n y_i\cdot(u_i\gamma).$$

Let $Atm_\Gamma(Z,Y)=(H,\Gamma,G)$ be a free Γ-biautomaton. As in the pure case, we say that in the arbitrary biautomaton $\mathfrak{A}=(A,\Gamma,B)$ Γ-identity in states $h_1\equiv h_2$, $h_i\in H$, is fulfilled if for each Γ-homomorphism $\mu: Atm_\Gamma(Z,Y)\to\mathfrak{A}$ the equality $h_1^\mu=h_2^\mu$ holds. Γ-identity in outputs is defined in a similar way. A completely characteristic Γ-congruence (i.e. trivial on Γ) $\rho=(\rho_1,\delta_\Gamma,\rho_3)$ of the free biautomaton $Atm_\Gamma(Z,Y)$ is in one-to-one correspondence with the system of all Γ-identities of the biautomaton. This congruence can be defined by the subspaces $M=[0]_{\rho_1}\subset H$ and $T=[0]_{\rho_3}\subset G$ which are the classes of the congruences ρ_1, ρ_3 respectively, containing zeros of the spaces H, G. Proceeding from this, an element h of H is called Γ-identity in states of the biautomaton $\mathfrak{A}=(A,\Gamma,B)$ if for any Γ-homomorphism $\mu: Atm_\Gamma(Z, Y)\to\mathfrak{A}$, h^μ is the zero of the space A. In this case we say that the identity in states $h\equiv 0$ is satisfied in $\mathfrak{A}$. A similar remark can be made with respect to the identities in outputs.

The element $h\in H$ of the form $h=\sum_{i=n}^{n} z_i\circ u_i$, $z_i\in Z$, $u_i\in K\Gamma^1$, is an identity in states if and only if each $z_i\circ u_i$, $i=1,\dots,n$ is an identity in states. Indeed, let $h=\sum_{i=1}^{n} z_i\circ u_i$ be an identity in states of the biautomaton $\mathfrak{A}$ and $\mu=(\mu_1,\varepsilon_\Gamma,\mu_3)$ be an arbitrary homomorphism from $Atm_\Gamma(Z,Y)$ to $\mathfrak{A}$. Let us fix i and consider the homomorphism $\mu=(\bar{\mu}_1,\varepsilon_\Gamma,\mu_3):Atm_\Gamma(Z, Y)\to\mathfrak{A}$ for which

$$z_j^{\bar{\mu}_1}=\begin{cases} 0 \ , & j\neq i \\ z_j^{\mu_1}, & j=i \end{cases} \tag{3.1}$$

The fact that $\bar{\mu}_1$ is really a homomorphism is evident. Since h is an identity in states, then

$$h^{\bar{\mu}_1}=\sum_{i=1}^{n} z_i^{\bar{\mu}_1}\circ u_i^{\mu_2}=z_i^{\bar{\mu}_1}\circ u_i^{\mu_2}=z_i^{\mu_1}\circ u_i^{\mu_2}=(z_i\circ u_i)^{\mu_1}=0.$$

Thus for any homomorphism μ, $(z_i\circ u_i)^{\mu_1}=0$, that is, for each $i\in\{1,2,\dots,n\}$, $z_i\circ u_i$ is the identity of the automaton $\mathfrak{A}$. Similarly, the element $g=\sum_{i=1}^{n} z_iw_i+\sum_{i=1}^{n} y_i\cdot u_i$ from G is an identity in outputs if this also true for each z_iw_i of $Z(K\Gamma^1\otimes K\Gamma)$ and for each $y_i\cdot w_i\in YK\Gamma^1$, $i=1,\dots,k$. Stated above implies that for the description of Γ-identities of the biautomata it is sufficient to take a free cyclic Γ-biautomaton (in the sense that the sets Z and Y contain one element: $Z=\{z\}$, $Y=\{y\}$), and the identities of Γ-biautomata in states and in outputs are reduced to the identities of the form

$$\begin{aligned} &z\circ u\equiv 0, \ u\in K\Gamma^1 \\ &zw\equiv 0, \ w\in K\Gamma^1\otimes K\Gamma \\ &y\cdot u\equiv 0, \ u\in K\Gamma^1 \end{aligned} \tag{3.2}$$

These identities are defined by the operations $\circ$, $*$, $\cdot$ of the automaton $\mathfrak{A}$. The current item is aimed to description of Γ-identities of the biautomaton $\mathfrak{A}$ in terms of the semigroup algebra $K\Gamma$.

Let $(\rho_1,\delta_\Gamma,\rho_3)$ be the system of all Γ-identities of the biautomaton $\mathfrak{A}$, $\tilde{U}_1=[0]_{\rho_1}\subset H$, $T=[0]_{\rho_3}\subset G$. Then $(\tilde{U}_1,\Gamma,T)$ is a completely characteristic subautomaton of the free biautomaton (H,Γ,G) holding all the endomorphisms of $Atm_\Gamma(Z,Y)$ identical on Γ (Such *subautomata* we shall call Γ-*completely characteristic*). On the other hand, each biautomaton of that type defines the system $(\rho_1,\delta_\Gamma,\rho_3)$ of Γ-identities of action of the automaton $(H/\tilde{U}_1,\Gamma,G/T)$. Thus, there is a one-to-one correspondence between Γ-identities of biautomata and Γ-completely characteristic subautomata of the type $(\tilde{U}_1,\Gamma,T)$ of the free biautomaton. In this case $\tilde{U}_1$ and T can be written in the form $\tilde{U}_1=zU_1$, $U_1\subset K\Gamma^1$, $T=zW\oplus yU_2$, $W\subset K\Gamma^1\otimes K\Gamma=\Psi$, $U_2\subset K\Gamma^1$, so that $(\tilde{U}_1,\Gamma,T)=(zU_1,\Gamma,zW\oplus yU_2)$.

In the given case U_1 is a set of all elements u_1 of $K\Gamma^1$ for which the identity $z\circ u_1\equiv 0$ is satisfied in $\mathfrak{A}$; W is a set of all elements $u\otimes v$ of $K\Gamma^1\otimes K\Gamma$ for which the identity $zw=(z\circ u)*v\equiv 0$ is satisfied in $\mathfrak{A}$ and U_2 is the set of all elements u_2 of $K\Gamma^1$ for which the identity $y\cdot u_2\equiv 0$ is satisfied in $\mathfrak{A}$. Thus, the tuple (U_1,W,U_2) of its Γ-identities is associated with the biautomaton $\mathfrak{A}$. The tuple for the class of biautomata is defined in a same way. It is easy to verify that this tuple satisfies the conditions:

1) U_1,U_2 are two-sided ideals in $K\Gamma^1$;
2) W is a submodule of the bimodule $\Psi=K\Gamma^1\otimes K\Gamma$; (3.3)
3) $U_1\otimes K\Gamma\subset W$;
4) $\Psi\cdot U_2\subset W$.

An arbitrary tuple (U_1,W,U_2) satisfying the given conditions (3.3) is called a *compatible Γ-tuple*.

The tuple of the biautomaton Γ-identities (or of the class of the biautomata Γ-identities), in particular, is a compatible one.

Theorem 3.1. *The subautomaton $\mathfrak{B}=(zU_1,\Gamma,zW\oplus yU_2)$ of the free cyclic Γ-biautomaton (H,Γ,G) is a Γ-completely characteristic one if and only if the tuple (U_1,W,U_2) is compatible.*

Proof. Let $\mathfrak{B}$ be a Γ-completely characteristic subautomaton. An arbitrary endomorphism of the free Γ-biautomaton is induced by the mappings $\nu_1:Z\to H$, $\nu_3:Y\to G$. Show that U_1 is a two-sided ideal in $K\Gamma^1$.

Really, if $u_1 \in U_1$ and $\gamma \in \Gamma$, then $u_1\gamma \in U_1$, since $(z\circ u_1)\circ\gamma = z\circ u_1\gamma \in zU_1$. In order to show that $\gamma u_1 \in U_1$ let us take such endomorphism $\nu=(\nu_1,\varepsilon_\Gamma,\nu_3)$ of the free Γ-biautomaton, that $z^{\nu_1}=z\circ\gamma$. Then $(z\circ u_1)^{\nu_1}=z^{\nu_1}\circ u_1=z\gamma u_1 \in z\,U_1$ and $\gamma u_1 \in U_1$. Thus, U_1 is a two-sided completely characteristic ideal in $K\Gamma^1$. A similar statement for U_2 also can be verified.

Show that W is a submodule in Ψ, that is, that $\tilde{u}w \in W$ and $w\cdot u \in W$ for all $w \in W$, $\tilde{u} \in K\Gamma^1$, $u \in K\Gamma^1$. Take such endomorphism $\nu=(\nu_1,\varepsilon_\Gamma,\nu_3)$ of the free biautomaton that $z^{\nu_1}=z\circ\tilde{u}_1$ and ν_3 is induced by the identity mapping of the set Y onto itself. Since the subautomaton B is a completely characteristic one, then $(zw)^{\nu_3}=((z\circ u)*v)^{\nu_3}=(z^{\nu_1}\circ u)*v=(z\tilde{u}_1\circ u)*v=z(\tilde{u}_1w)\in zW$ and $\tilde{u}_1w\in W$.

Inclusions $w\cdot u\in W$ and $U_1\otimes K\Gamma\subset W$ follow from the fact that $\mathfrak{B}$ is subautomaton.

Finally, show that $\Psi\cdot U_2\subset W$. Let $u\otimes v\in\Psi$, $u_2\in U_2$. Take such endomorphism $\nu=(\nu_2,\varepsilon_\Gamma,\nu_3)$ of the free automaton that $y^{\nu_3}=zu\otimes v$ and ν_1 is defined by the identity mapping of the set Z onto itself. Since the subautomaton $\mathfrak{B}$ is Γ-completely characteristic, then

$$(y\cdot u_2)^{\nu_3}=y^{\nu_3}\cdot u_2=(zu\otimes v)\cdot u_2\in zW, \text{ and } (u\otimes v)\cdot u_2\in W \text{ as required.}$$

Let now (U_1,W,U_2) be a compatible tuple. It is easy to verify that $(zU_1,\Gamma,zW\oplus yU_2)$ is a subautomaton in (H,Γ,G). Show that this subautomaton is a completely characteristic one. Take the endomorphism $\nu=(\nu_1,\varepsilon_\Gamma,\nu_3)$ of the free biautomaton (H,Γ,G) defined by the mappings $\nu_1:\{z\}\to H$, $\nu_3:\{y\}\to G$, and let $z^{\nu_1}=zu$, $u\in K\Gamma^1$, $y^{\nu_3}=z\varphi'+yv$, $\varphi'\in\Psi$, $v\in K\Gamma^1$. It is necessary to show that $\mathfrak{B}^\nu\subset\mathfrak{B}$. If $zu_1\in U_1$, then $(zu_1)^{\nu_1}=z^{\nu_1}\circ u_1=zu\circ u_1=z\circ uu_1$. Since U_1 is an ideal, then $uu_1\in U_1$, therefore, $(zu_1)^{\nu_1}\in zU_1$. Let $w=\tilde{u}\otimes\tilde{v}\in W$. Then

$$(zw)^{\nu_3}=((z\circ\tilde{u})*\tilde{v})^{\nu_3}=(z^{\nu_1}\circ\tilde{u})*\tilde{v}=z^{\nu_1}(\tilde{u}\otimes\tilde{v})=z^{\nu_1}(\tilde{u}\otimes\tilde{v})=zu(\tilde{u}\otimes\tilde{v})=(zu\circ\tilde{u})*\tilde{v}=$$

$$(z\circ u\tilde{u})*\tilde{v}=z(u\tilde{u}\otimes\tilde{v})=z(u(\tilde{u}\otimes\tilde{v})).$$

Since W is a subbimodule, then $u(\tilde{u}\otimes\tilde{v})\in W$ and $(zw)^{\nu_3}=z(u(\tilde{u}\otimes\tilde{v}))\in zW$. Take now $yu_2\in yU_2$. Then

$(yu_2)^{\nu_3}=y^{\nu_3}\cdot u_2=(z\varphi'+yv)\cdot u_2=(z\varphi')\cdot u_2+yv\cdot u_2=z(\varphi'\cdot u_2)+yvu_2$. Since (U_1,W,U_2) is a compatible tuple, then $\varphi'\cdot u_2\in W$ (statement 4 of the definition) and $vu_2\in U_2$ (statement 1 of the definition). Hence, $(yu_2)^{\nu_3}\in zW\oplus yU_2$. The theorem is proved.

Thus, Γ-identities of action of the automaton $\mathfrak{A}=(A,\Gamma,B)$ are the identities of the form (3.2) and tuples of the identities of action are compatible tuples of the form (U_1,W,U_2).

We have considered *Γ-identities of biautomata.* Now consider the *identities of arbitrary biautomata.*

Let $\mathcal{F}^b=(H,F,\Phi)$ be a free biautomaton (automaton). As earlier, the *identity in states* $h_1\equiv h_2$, $h_i\in H$, is said to be satisfied in the biautomaton (automaton) $\mathfrak{A}=(A,\Gamma,B)$ if for any homomorphism $\mu:\mathcal{F}^b\longrightarrow\mathfrak{A}$ the equality $h_1^\mu=h_2^\mu$ holds. *Identities in inputs and outputs* are defined in a similar way. Identities in states and in outputs are identities of the action; identities in inputs are identities of the semigroup Γ. In the sequel under the identities of the biautomaton (automaton) we shall understand only the identities of the action, i.e. identities in states and in outputs. Γ-identity in states of the biautomaton $\mathfrak{A}=(A,\Gamma,B)$ is an element of the free $K\Gamma^1$-module, while an identity in states of the same biautomaton is an element from ZKF^1 which is transformed into zero under any homomorphism $\mu:Atm(Z,X,Y)\longrightarrow\mathfrak{A}$ (recall, that F is a free semigroup with the set of free generators X). A similar remark is true for the identity in outputs of the biautomaton $\mathfrak{A}$. The same arguments as for Γ-identities yield:

for the description of the biautomata identities it is sufficient to proceed from the free biautomata cyclic in inputs and outputs;

identities of biautomata (of the class of biautomata) are reduced to the identities of the form

$$z\circ u\equiv 0,\ zw\equiv 0,\ y\cdot u\equiv 0,\ u\in KF^1,\ w\in KF^1\otimes KF;$$

a completely characteristic subautomaton in $Atm(\{z\},X,\{y\})$ of the type $(z\circ U_1,F,zW+y\cdot U_2)$, where $U_1,U_2\subset KF^1$ and $W\subset KF^1\otimes KF$, corresponds to the set of all biautomaton identities;

a biautomaton of the type $(z\circ U_1,F,zW+y\cdot U_2)$ from $\mathcal{F}^b$ is a completely characteristic one if and only if the tuple (U_1, W, U_2) satisfies the conditions:

1) U_1, U_2 are completely characteristic ideals in KF^1,
2) W is a completely characteristic submodule of bimodule $KF^1 \otimes KF$,
3) $U_1 \otimes KF \subset W$, (3.4)
4) $(KF^1 \otimes KF) \cdot U_2 \subset W$.

The tuple (U_1, W, U_2) satisfying these conditions is called compatible.

Let $\mathfrak{A}=(A, \Gamma, B)$ be a biautomaton and let $U_1=\{u \subset KF \mid z \circ u \equiv 0\}$, $W=\{w \in KF^1 \otimes KF \mid zw \equiv 0\}$, $U_2=\{u \in KF \mid y \cdot u \equiv 0\}$. Denote by V the set of elements $v \in KF$, such that for any element $a \in A$ and for any homomorphism $\mu : Atm(Z, X, Y) \longrightarrow \mathfrak{A}$ the equality $a * v^{\mu}=0$ takes place. Elements of U_1 are identities of the operation $\circ$, elements of V are identities of the operation $*$, elements of U_2 are identities of the operation $\cdot$. The tuple (U_1, W, U_2) of all identities of a biautomaton (class of biautomata) is a compatible one. By the definition of the compatible tuple $W \supset U_1 \otimes KF$ and $W \supset (KF^1 \otimes KF) \cdot U_2$. It is also verified that $W \supset KF^1 \otimes V$ and $W \supset (KF^1 \otimes V) \cdot KF$. Denote $\tilde{W}=U_1 \otimes KF + KF^1 \otimes V + (KF^1 \otimes V) \cdot KF + (KF^1 \otimes KF) \cdot U_2$. The tuple $(U_1, \tilde{W}, U_2)$ is also compatible. It is constructed according to the identities of the form $z \circ u \equiv 0$, $z * u \equiv 0$, $z \cdot u \equiv 0$. Show that this tuple may not describe all the identities of the original biautomaton (the class of biautomata). For this purpose let us construct an example of the biautomaton $\mathfrak{A}$ for which $\tilde{W} < W$.

Example. Let $F=F(X)$ be a free semigroup over countable set X. The regular biautomaton $(KF^1, F, KF^1 \otimes KF)$ is extended naturally up to the biautomaton $(KF^1, KF, KF^1 \otimes KF)$. Let U be equal to KF^2 where $F^2=FF$ (it is a completely characteristic ideal of the semigroup algebra KF), and $C=(U \otimes U)+(KF^1 \otimes U^2)$ be a subspace in $KF^1 \otimes KF$. This subspace is invariant with respect to the action $\cdot$ of the elements of U, i.e. if $c \in C$, $u \in U$, then $c \cdot u \in C$. Denote by B the quotient space

$$(KF^1 \otimes KF)/C=(KF^1 \otimes KF)/((U \otimes U)+(KF^1 \otimes U^2))$$

and define the action $\cdot$ of the elements $u \in U$ on the elements $b \in B$ by the rule: if $b=\overline{(v \otimes w)} \in B$ ($v \otimes w$ is an arbitrary representative of the coset

$\overline{(v\otimes w)})$, $u\in U$, then $b\cdot u=\overline{(v\otimes w)}\cdot u=\overline{(v\otimes w)\cdot u}$. Since C is invariant with respect to U, the action $b\cdot u$ is defined correctly. Denote now $KF^1=A$ and consider the biautomaton $\mathfrak{A}=(A,U,B)$ with the following operations $\circ$, $*$, $\cdot$: if $a\in KF^1=A$, $u\in U$, $b=\overline{(v\otimes w)}\in B$ then $a\circ u=au$, $a*u=\overline{a\otimes u}$, $b\cdot u=\overline{(v\otimes w)}\cdot u=\overline{(v\otimes w)\cdot u}$. We consider U in this biautomaton as a semigroup of inputs and it is easy to see that $\mathfrak{A}=(A,U,B)$ is really a semigroup biautomaton. Before we proceed to computation of the modules W and $\tilde{W}$ let us make one remark.

Let U be an associative algebra over a commutative ring. Consider U only as a semigroup and let KU be the semigroup algebra of this semigroup. Denote by α the homomorphism of the algebra KU into the algebra U, induced by the identity mapping of U into U. In this case $(\sum\limits_i^{\circ}\alpha_i u_i)^{\alpha}=\sum\limits_i\alpha_i u_i$; here $\sum\limits_i^{\circ}\alpha_i u_i$ is the sum in the algebra KU and $\sum\limits_{i=1}\alpha_i u_i$ is the sum in the algebra U.

If (A,U) is a representation of the algebra U (which is considered as a semigroup) and (A,KU) is a corresponding extension, then for $a\in A$, $x\in KU$ holds $a\circ x=a\circ x^{\alpha}$

The linear mapping $\mu: KF\otimes KF \rightarrow KU\otimes KU$ corresponds to the linear mapping $\mu: KF \rightarrow KU$. If $\sum\limits_{i=1}^{n}u_i\otimes v_i\in KF\otimes KF$ then $(\sum\limits_{i=1}^{n}u_i\otimes v_i)^{\mu}=\sum\limits_{i=1}^{n}u_i^{\mu}\otimes v_i^{\mu}$.

Denote by μ an arbitrary homomorphism of the free cyclic biautomaton in $\mathfrak{A}$.

By the definition $W=\{w\in KF^1\otimes KF \mid zw\equiv 0\}$. If $w=\sum\limits_{i=1}^{n}u_i\otimes v_i$ is an arbitrary element of $KF\otimes KF$ and $x\in A=KF^1$, then

$$xw^{\mu}=\sum_{i=1}^{n}(x\circ u_i^{\mu})*v_i^{\mu}=\sum_{i=1}^{n}(xu_i^{\mu\alpha})\otimes v_i^{\mu\alpha}\in U\otimes U.$$

Therefore, $W\supset KF\otimes KF$. Let us proceed now to the calculation of $\tilde{W}$.

$U_1=\{u\in KF \mid z\circ u\equiv 0\}$, that is the ideal U_1 consists of such elements $u\in KF$ that for any $x\in A=KF^1$, $x\circ u^{\mu}=x\circ u^{\mu\alpha}=xu^{\mu\alpha}=0$. If we take, in particular, $x=1$, we get that $u^{\mu\alpha}=0$, and $U_1^{\mu\alpha}=0$.

We have $V=\{u\in KF \mid z*v\equiv 0\}$. If $u\in V$, then for all $x\in A=KF^1$, $x*u^{\mu}=x*u^{\mu\alpha}=x\otimes u^{\mu\alpha}\in C=U\otimes U+KF^1\otimes U^2$. Since this inclusion is true for any element $x\in KF^1$, choosing $x=1$, we get $u^{\mu\alpha}\in U^2$. Hence, $V^{\mu\alpha}\subset U^2$.

$U_2=\{u\in KF \mid y\cdot u\equiv 0\}$. It is clear that $U_2\supset U^2$. Really, if $x\in U^2$, then $x^{\mu\alpha}\in U^2$ and $b\cdot x^{\mu\alpha}=\overline{v\otimes w}\cdot x^{\mu\alpha}=\overline{v\otimes wx}^{\mu\alpha}-\overline{vw\otimes x}^{\mu\alpha}=0$, that is $x\in U_2$.

If $u\in U_2$, then $b\cdot u^{\mu\alpha}=\overline{v\otimes w}\cdot u^{\mu\alpha}=\overline{v\otimes wu}^{\mu\alpha}-\overline{vw\otimes u}^{\mu\alpha}=0$, that is $v\otimes wu^{\mu\alpha}-vw\otimes u^{\mu\alpha}\in U\otimes U+KF^1\otimes U^2$. If, in particular, $v=1$, then $1\otimes wu^{\mu\alpha}-w\otimes u^{\mu\alpha}\in U\otimes U+KF^1\otimes U^2$ and $1\otimes wu^{\mu\alpha}\in KF^1\otimes U^2$. This means that $KFU_2^{\mu\alpha}\subset U_2$.

By the definition

$\tilde{W}=U_1\otimes KF+KF^1\otimes V+(KF^1\otimes V)\cdot KF+(KF^1\otimes KF)\cdot U_2=U_1\otimes KF+KF\otimes V+1\otimes V+(KF\otimes V)\cdot KF+(1\otimes V)\cdot KF+$ $+(KF\otimes KF)\cdot U_2+(1\otimes KF)\cdot U_2$.

In its turn, $(1\otimes KF)\cdot U_2\subset 1\otimes U_2+KF\otimes U_2$ and $(1\otimes V)\cdot KF\subset 1\otimes KF+V\otimes KF$. Then $\tilde{W}\cap(KF\otimes KF)\subset U_1\otimes KF+KF\otimes V+(KF\otimes V)\cdot KF+KF\otimes U_2+V\otimes KF$.

If it is possible to find such homomorphism μ that $(\tilde{W}\cap(KF\otimes KF))^{\mu\alpha}$ will be less than $(W\cap(KF\otimes KF))^{\mu\alpha}=(KF\otimes KF)^{\mu\alpha}$, this will imply that $\tilde{W}\cap(KF\otimes KF)<W\cap(KF\otimes KF)$ and $\tilde{W}<W$.

We have

$$(\tilde{W}\cap(KF\otimes KF))^{\mu\alpha}\subset U_1^{\mu\alpha}\otimes KF^{\mu\alpha}+KF^{\mu\alpha}\otimes V^{\mu\alpha}+(KF^{\mu\alpha}\otimes V^{\mu\alpha})\cdot KF^{\mu\alpha}+KF^{\mu\alpha}\otimes U_2^{\mu\alpha}+V^{\mu\alpha}\otimes KF^{\mu\alpha}.$$

Since $U_1^{\mu\alpha}=0$ and $V^{\mu\alpha}\subset U^2$, then

$$(\tilde{W}\cap(KF\otimes KF))^{\mu\alpha}\subset KF^{\mu\alpha}\otimes U^2+U^2\otimes KF^{\mu\alpha}+KF^{\mu\alpha}\otimes U_2^{\mu\alpha}.$$

Now denote by μ the mapping $X=\{x_i\}\longrightarrow U$ defined by $x_i^{\mu}=x_i^2$, as well as corresponding homomorphisms $\mu:F\longrightarrow U$ and $\mu:KF\longrightarrow KU$. Then $KF^{\mu\alpha}$ is the algebra of all the polynomials with the generators x_i^2, $i=1,2,\dots$ Since, according to the construction, $U=KF^2$ is an algebra of all polynomials over X with powers greater or equal to two, then U^2 is an algebra of all polynomials over X with powers greater or equal to four. It has been proved that $KFU_2^{\mu\alpha}\subset U^2$, therefore $U_2^{\mu\alpha}$ lies in the algebra P_3 of all polynomials with powers greater or equal to three. Thus

$$(\tilde{W}\cap(KF\otimes KF))^{\mu\alpha}\subset KF^{\mu\alpha}\otimes P_3+P_3\otimes KF^{\mu\alpha}.$$

It is clear that the elements of the form $x_i^2\otimes x_j^2$ from $KF^{\mu\alpha}\otimes KF^{\mu\alpha}$, $i,j=1,2,\dots$ do not lie in $KF^{\mu\alpha}\otimes P_3+P_3\otimes KF^{\mu\alpha}\supset(\tilde{W}\cap(KF\otimes KF))^{\mu\alpha}$. Thus, $(\tilde{W}\cap(KF\otimes KF))^{\mu\alpha}<(W\cap(KF\otimes KF))^{\mu\alpha}$ and $\tilde{W}<W$.

Certainly, it is possible to cite a number of examples when $W=\tilde{W}$.

Example. Let Z, X, Y be three sets. Denote, as usual, a free semigroup with the set of generators X by $F=F(X)$ and a linear space KZ by A. Take the biautomaton $\mathfrak{C}=(A,F,(A\otimes KF)\otimes YKF^1)$ with the following operations: if $a\in A$, $u\in KF^1$, $y\in Y$, $v\in KF$, $f\in F$, then $a\circ f=0$ (i.e. $\mathfrak{C}$ is a coautomaton), $a*f=a\otimes f$, $yu\cdot f=yuf$, $(a\otimes v)\cdot f=a\otimes vf$. (Biautomaton axioms are easily verified). Given biautomaton $\mathfrak{C}$, define sets $U_1, V, U_2, W, \tilde{W}$.

Since $a\circ f=0$ for any elements $a\in A$ and $f\in F$, then $U_1=KF$. If $u\in KF^1$, $v\in KF$, then $a(u\otimes v)=(a\circ u)*v=0$ if and only if $u\in KF$. Therefore, $W=KF\otimes KF$.

If now $a*f=a\otimes f=0$ holds for any element $a\in A$, then $f=0$. It means that $V=0$. It is also clear that $U_2=0$. Then

$$\tilde{W}=U_1\otimes KF+KF^1\otimes V+(KF^1\otimes V)\cdot KF+(KF^1\otimes KF)\cdot U_2=KF\otimes KF.$$

Thus, $W=\tilde{W}$.

It is useful to mention that any coautomaton (A,Γ,B) with the generator system (Z,X,Y) is a homomorphic image of the coautomaton $\mathfrak{C}$ considered in the given example.

4.3.2. Identities of linear automata

Let $\mathcal{F}^{\ell}=(H,F,\Phi)=(ZKF^1,F,ZKF\otimes KY)$ be a free linear automaton. By the definition, the identity in states of a linear automaton $\mathfrak{A}$ is an element from H transformed into zero under any homomorphism $\mathcal{F}^{\ell}\to\mathfrak{A}$; the identity in outputs is a corresponding element from Φ. Arguing as in the previous item one can see that:

- for the description of the identities of a linear automaton it is sufficient to proceed from the free linear automaton, cyclic in states and in outputs;

- identities of a linear automaton are reduced to the identities of the form $z\circ u\equiv 0$, $z*v\equiv 0$, $u\in KF^1$, $v\in KF$;

- a completely characteristic subautomaton of the type $(z\circ U,F,z*V)$ of $\mathcal{F}^{\ell}$ where $U=\{u\in KF^1 \mid z\circ u\equiv 0\}$, $V=\{v\in KF \mid z*v\equiv 0\}$ corresponds to the set of all identities (of action) of the linear automaton;

- an arbitrary subautomaton of the type $(z\circ U,F,\ z*V)$ of $\mathcal{F}^{\ell}$ is completely characteristic if and only if the tuple (U,V) satisfies the conditions:

1) U is a completely characteristic two-sided ideal in KF^1;

2) V is a completely characteristic left ideal in KF;

3) UKF⊂V.

Such tuple is also called *compatible*. The following theorems describe the identities of the universal linear automata.

Theorem 3.2. [79] *If* (U,V) *is a tuple of the identities of the universal linear automaton* $Atm^1(A,B)$, *then* V=UKF.

Proof. The inclusion UKF⊂V is evident. Show that V⊂UKF. Since V⊂KF, F=F(X), then the arbitrary element v∈V can be written in the form $v=\sum_i \lambda_i x_{i_1} x_{i_2} \ldots x_{i_{n_i}}$, $\lambda_i \in K$, $x_{i_j} \in X$. Let us group the summands by the last multiple and write v in the form $v=\sum_i u_i x_i$. Using the definitions of the ideals U, V show that $u_i \in U$, which implies the required inclusion V⊂UKΓ.

Show first that each u_i is not empty. Take such element φ∈Hom(A,B) that aφ≠0 under a certain a∈A and consider the following mapping α:X → End(A,B)

$$x_j^{\alpha}=\begin{cases}(0,\ 0), & \text{if } j\neq i\\ (0,\ \varphi), & \text{if } j=i, \quad j=1,2,\ldots,\ x_j\in X\end{cases}$$

Since the semigroup F is free, this mapping can be extended up to the homomorphism α:F → End(A,B) and further to the homomorphism of the algebras $\tilde{\alpha}$:KF → End(A,B). If u_i were empty, then under an appropriate a∈A, $a*v^{\alpha}=a*x_i^{\alpha}=a\varphi\neq 0$ would be satisfied. This contradicts the definition of V. Hence, each u_i is not empty.

Now show that $u_i x_i \in V$ for each i. It is necessary to show that for the extension of any homomorphism μ:F → End(A,B) to the homomorphism of the algebras $\tilde{\mu}$:KF → End(A,B) holds the equality $a*(u_i x_i)^{\tilde{\mu}}=0$ for any element a∈A. Assign the mappings μ_i:X → End(A,B) to the given μ by the rule: if $x_j \in X$ and $x_j^{\mu}=(\delta_j,\varphi_j)$ then

$$x_j^{\mu_i}=\begin{cases}(\delta_j,0)\ , & \text{if } j\neq i\\ (\delta_j,\varphi_j), & \text{if } j=i, \quad j=1,2,\ldots\end{cases}$$

These mappings are extended up to the homomorphism $\tilde{\mu}_i$: KF → End(A,B). By the definition of the mapping μ_i, for each a∈A, $a*x_j^{\mu_i}=0$ if j≠i,

$a*x_1^{\mu_1}=a*x_1^{\mu}$ and for any u∈KF $a\circ u^{\tilde{\mu}_1}=a\circ u^{\tilde{\mu}}$ Subject to this we have:

$$a*(u_1x_1)^{\tilde{\mu}}=a*u_1^{\tilde{\mu}}x_1^{\tilde{\mu}}=(a\circ u_1^{\tilde{\mu}})*x_1^{\tilde{\mu}}=(a\circ u_1^{\tilde{\mu}_1})*x_1^{\tilde{\mu}_1}=\sum_j(a\circ u_j^{\tilde{\mu}_1})*x_j^{\tilde{\mu}_1}=$$

$$\sum_j a*(u_jx_j)^{\tilde{\mu}_1}=a*(\sum_j u_jx_j)^{\tilde{\mu}_1}=a*v^{\tilde{\mu}_1}.$$

Since v∈V, then $a*v^{\tilde{\mu}_1}=0$. Thus, $a*(u_1x_1)^{\tilde{\mu}}=0$, that is, $u_1x_1\in V$.

As the final step show that $u_1\in U$. Suppose that $u_1\notin U$, that is there exists such homomorphism $\nu:F\rightarrow End(A,B)$ and such element a∈A, that $a\circ u_1^{\tilde{\nu}}\neq 0$. Denote $a\circ u_1^{\tilde{\nu}}$ by a'; a'≠0; take such mapping $\nu':X\rightarrow End(A,B)$ that for each x holds $a\circ x^{\nu'}=a\circ x^{\nu}$ and that $a'*x_1^{\nu'}\neq 0$. Then for the homomorphism $\tilde{\nu}':KF\rightarrow End(A,B)$ which is an extension of the given mapping ν' holds: $a*(u_1x_1)^{\tilde{\nu}'}=a*u_1^{\tilde{\nu}'}x_1^{\tilde{\nu}'}=(a\circ u_1^{\tilde{\nu}'})*x_1^{\tilde{\nu}'}=(a\circ u_1^{\tilde{\nu}'})*x_1^{\tilde{\nu}'}=a'*x_1^{\nu'}\neq 0$. This contradicts the fact that $u_1x_1\in V$. Hence, $u_1\in U$ and V⊂UKF. Thus, V=UKF.

Theorem 3.3. [79] *If* (U,V) *is a tuple of the identities of the universal linear over the field* K *automaton* $Atm^2(\Gamma,B)$, *then* U *coincides with the ideal* U_0 *of the identities of the left regular representation* (KΓ,KΓ), *and* V *coincides with the set of* KF*-identities of the semigroup algebra* KΓ. (*An element* v∈KF *is assumed to be a* KF*-identity if for any homomorphism* $\mu:F\rightarrow\Gamma$ *and its extension* $\tilde{\mu}:KF\rightarrow K\Gamma$ *holds the equality* $v^{\tilde{\mu}}=0$).

Proof. By the definition $Atm^2(\Gamma,B)=(A,\Gamma,B)=(Hom(K\Gamma,B),\Gamma,B)$; if $\bar{a}\in Hom(K\Gamma,B)$, $\gamma\in\Gamma$, then $(\bar{a}\circ\gamma)(x)=\bar{a}(\gamma x)$, $x\in K\Gamma$, $\bar{a}*\gamma=\bar{a}(\gamma)$. Take an arbitrary homomorphism $\mu:F\rightarrow\Gamma$. It can be extended up to the homomorphism of the semigroup algebras $\tilde{\mu}:KF\rightarrow K\Gamma$. Then $\bar{a}\circ u^{\tilde{\mu}}=0$ for any $\bar{a}\in A$, u∈U. This implies that for any x∈KΓ $(\bar{a}\circ u^{\tilde{\mu}})(x)=\bar{a}(u^{\tilde{\mu}}x)=0$. Since $\bar{a}$ is an arbitrary function of Hom(KΓ,B), then $u^{\tilde{\mu}}x=0$. It follows that if we consider the left regular representation (KΓ,KΓ), then for any element x∈KΓ and for the extension $\tilde{\mu}$ of an arbitrary homomorphism $\mu:F\rightarrow\Gamma$ holds $x\circ u^{\tilde{\mu}}=u^{\tilde{\mu}}x=0$, that is u is an identity of the left regular representation of the semigroup algebra $K\Gamma$, and hence, $U\subset U_0$. The inverse inclusion is verified in a similar way.

Let, as before, μ be an arbitrary homomorphism $F \to \Gamma$, $\tilde{\mu}$ its extension to the homomorphism $KF \to K\Gamma$. Take the element $v\in V$. Then for any $\bar{a}\in A=Hom(K\Gamma,B)$ holds $\bar{a}*v^{\tilde{\mu}}=\bar{a}(v^{\tilde{\mu}})=0$. Hence, $v^{\tilde{\mu}}=0$ and v is an identity of the semigroup algebra $K\Gamma$. On the other hand, if v is such identity, then for each $\tilde{\mu}$ of the given form, holds $v^{\tilde{\mu}}=0$ and $\bar{a}(v^{\tilde{\mu}})=\bar{a}*v^{\tilde{\mu}}=0$, that is $v\in V$.

In conclusion of the Section we shall make the following remark. Let (A,Γ,B) be a biautomaton. As earlier, assign to it the representation $(A\oplus B,\Gamma)$ with the action defined by the rule: if $a+b\in A\oplus B$, $\gamma\in\Gamma$, then $(a+b)\gamma=a\circ\gamma+a*\gamma+b\cdot\gamma$.

Proposition 3.4. *Let* U_1,V,U_2 *be the sets of identities of the biautomaton* (A,Γ,B). *Then* $U=U_1\cap V\cap U_2$ *is the set of all identities of the representation* $(A\oplus B,\Gamma)$.

Proof. Denote by U' all the identities of the given representation. It is clear that $U\subset U'$. On the other hand, let $u\in U'$. Then under any homomorphism $\mu:F \to \Gamma$ the equality $(a+b)u^{\mu}=0$ takes place for any $a\in A$ and $b\in B$. By the definition $(a+b)u^{\mu}=a\circ u^{\mu}+a*u^{\mu}+b\cdot u^{\mu}$. If $b=0$, then $a\circ u^{\mu}+a*u^{\mu}=0$. Since it is a direct sum, then $a\circ u^{\mu}=0$ and $a*u^{\mu}=0$, that is , $u\in U_1$ and $u\in V$. If $a=0$, then $b\cdot u^{\mu}=0$, i.e. $u\in U_2$. Thus, $u\in U_1\cap V\cap U_2=U$. Therefore $U'\subset U$, and $U'=U$.

4.4. Varieties of biautomata

4.4.1. Definitions and examples

Class $\mathfrak{X}$ of all the biautomata satisfying the given set of identities is called a *variety of biautomata*. Each variety is closed under Cartesian products, homomorphic images, and subautomata. As well as for the other algebraic systems, theorem of Birkhoff is valid, that is the class closed with respect to the given operations is a variety of biautomata. Moreover, it is possible to prove that if θ is a certain class of biautomata, then the variety generated by this class (denoted, as usual, by Varθ) is determined by the formula Varθ=QSCθ where Q,S,C are the operators of taking of homomorphic images, subautomata and Cartesian products respectively.

If the set of identities defining the variety of biautomata does not contain the identities of acting semigroup, then the variety of biautomata is called a *saturated* one. Saturation of the variety is equivalent to the fact that if the variety contains a biautomaton $\mathfrak{A}=(A,\Gamma,B)$, it contains also any biautomaton $\mathfrak{B}$ such that there is a homomorphism in inputs $\mu:\mathfrak{B} \rightarrow \mathfrak{A}$. We will consider only the saturated varieties. Therefore, a saturated variety generated by the class of biautomata θ we shall also denote by Varθ. Denote by V the saturation operator: if θ is a class of biautomata, then Vθ is a class of biautomata whose homomorphic in inputs images lie in θ. Then for an arbitrary class θ of biautomata holds

$$\mathrm{Var}\theta=\mathrm{VQSC}\theta$$

We do not present proofs of numerous facts since they are very close to analogous proofs for group representations (see the book [90]).

A class of biautomata closed under the operators V,Q,S,D (D is the operator of discrete direct products) is called a *radical class* of biautomata.

To each variety of biautomata $\mathfrak{X}$ can be assigned the following function on $\mathfrak{X}$: if $\mathfrak{A}$ is an arbitrary biautomaton, then $\mathfrak{X}^*(\mathfrak{A})$ is the intersection of all the biautomata $\mathfrak{B}\subset\mathfrak{A}$, such that $\mathfrak{A}/\mathfrak{B}\in\mathfrak{X}$. It is clear that $\mathfrak{A}/\mathfrak{X}^*(\mathfrak{A})\in\mathfrak{X}$. This function is called *verbal function* or simply *verbal*. To each radical class $\mathfrak{X}$ corresponds a function $\mathfrak{X}'$: if $\mathfrak{A}$ is an arbitrary biautomaton, then $\mathfrak{X}'(\mathfrak{A})$ is a subautomaton in $\mathfrak{A}$ generated by all the biautomata $\mathfrak{B}\subset\mathfrak{A}$, such that $\mathfrak{B}\in\mathfrak{X}$. This function is called *radical function* or simply *radical*. By the definition of radical class, $\mathfrak{X}'(\mathfrak{A})\in\mathfrak{X}$.

As usually, biautomaton $\mathcal{F}_{\mathfrak{X}}$ is called *free in the variety* $\mathfrak{X}$ if there exists a system of generators Z, X, Y of this biautomaton such that for any $\mathfrak{A}=(A,\Gamma,B)\in\mathfrak{X}$ a triplet of mappings $Z \rightarrow A$, $X \rightarrow \Gamma$, $Y \rightarrow B$ can be uniquely extended to the biautomata homomorphism $\mathcal{F}_{\mathfrak{X}} \rightarrow \mathfrak{A}$. If X={x}, Y={y} then $\mathcal{F}_{\mathfrak{X}}$ is called a free cyclic biautomaton.

Theorem 4.1. *Let $\mathcal{F}=(H,F,G)$ be an (absolutely) free biautomaton with the free system of generators (Z,X,Y), $\mathfrak{X}$ an arbitrary variety of biautomata, (U_1,W,U_2) - a tuple of identities corresponding to $\mathfrak{X}$. Then the biautomaton*

$$(H/ZU_1, F, G/ZW\oplus YU_2)$$

is free in the variety $\mathfrak{X}$.

Consider examples of biautomata varieties.

Each linear automaton (A,Γ,B) can be considered as a biautomaton with the following operation $\cdot$: if $b\in B$, $\gamma\in\Gamma$, then $b\cdot\gamma=0$. The class of all linear automata considered as biautomata forms a variety of biautomata. If (U_1,W,U_2) is a tuple of identities of this variety, then $U_1=0$, $U_2=KF$. Since $W\supset\tilde{W}$, then, in particular, $W\supset(KF^1\otimes KF)\cdot U_2=(KF^1\otimes KF)\cdot KF$. By Theorem 4.1 a free biautomaton with generators (Z,X,Y) of the given variety has the form $(H/ZU_1,F,G/(ZW\oplus YU_2))$, where $H=ZKF^1$, $G=(ZKF^1\otimes KF)\oplus YKF^1$. Then,

$$(H/ZU_1,F,G/(ZW\oplus YU_2))\cong(ZKF^1,F,(ZKF^1\otimes KF)/ZW\oplus YKF^1/YKF)\cong$$
$$(ZKF^1,F,(ZKF^1\otimes KF)/ZW\oplus KY)=\mathcal{F}_{\ell a},$$

where $\mathcal{F}_{\ell a}$ is a free biautomaton of the variety of all linear automata. Since $(KF^1\otimes KF)\cdot KF$ is a linear space generated by all elements of the form $u\otimes vw-uv\otimes w$, $u\in KF^1$, $v,w\in KF$ and since $W\supset(KF^1\otimes KF)\cdot KF$, then the quotient space $(ZKF^1\otimes KF)/ZW$ is a homomorphic image of the space ZKF, and the biautomaton $\mathcal{F}_{\ell a}$ is a homomorphic image of the biautomaton $(ZKF^1,F,ZKF\oplus KY)$ with the following operations: $zu\circ f=zuf$, $zu*f=zuf$, $zv\cdot f=0$, $b\cdot f=0$, where $u\in KF^1$, $f\in F$, $v\in KF$, $b\in KY$; that is $\mathcal{F}_{\ell a}$ is a homomorphic image of the automaton. Since a free object with the given system of generators is uniquely determined up to isomorphism, then $(ZKF^1,F,ZKF\oplus KY)$ is a free automaton in the variety of linear automata. Stated above implies that, in particular, $W=\tilde{W}=(ZKF^1\otimes KF)\cdot KF$.

The class of all coautomata considered as biautomata, is also a variety of biautomata. The biautomaton $(KZ,Y,A\otimes KF\oplus YKF^1)$ considered in the example at the end of the item 3.1 is a free biautomaton of this variety. The tuple of the identities of the variety coincides with the tuple of its free biautomaton. The tuple of the identities of the latter (as shows the mentioned example) is the following:

$$U_1=KF,\ W=KF\otimes KF,\ U_2=0.$$

4.4.2. Varieties and compatible tuples

To each compatible tuple corresponds the variety of biautomata for which the given tuple is a tuple of identities and each variety of biautomata defines the compatible tuple of identities of this variety. It can be proved that the given correspondence between the varieties of biautomata and the compatible tuples is one-to-one. Actually, the connection between the varieties and the compatible tuples is even more close. Now we shall introduce the concepts of the product of varieties and the product of compatible tuples. These products satisfy the associative law and the resulting semigroups of varieties and compatible tuples are antiisomorphic.

Let $\mathfrak{X}_1$ and $\mathfrak{X}_2$ be varieties of biautomata; their *product* $\mathfrak{X}_1\mathfrak{X}_2$ is defined in the following way: a biautomaton $\mathfrak{A}=(A,\Gamma,B)$ belongs to $\mathfrak{X}_1\mathfrak{X}_2$ if and only if there exists a subautomaton $\mathfrak{A}_1=(A_1,\Gamma,B_1)\subset\mathfrak{A}$ such that $\mathfrak{A}_1\in\mathfrak{X}_1$ and $\mathfrak{A}/\mathfrak{A}_1=(A/A_1,\Gamma,B/B_1)\in\mathfrak{X}_2$. Thus defined product of varieties is associative.

Let us introduce the concept of the product of compatible tuples. Let $\tau'=(U'_1,W',U'_2)$ and $\tau''=(U''_1,W'',U''_2)$ be compatible tuples. Define the *product of tuples* $\tau=\tau'\tau''$ by the rule:

$$\tau=\tau'\tau''=(U_1,W,U_2)=(U'_1U''_1,U'_1W''+W'\cdot U''_2,U'_2U''_2)$$

Verify that the tuple $\tau=\tau'\tau''$ is also compatible, that is the conditions (3.4) are satisfied. Since U'_1,U''_1,U'_2,U''_2 are completely characteristic ideals, the same property is valid for the products

$$U_1=U'_1U''_1,\quad U_2=U'_2U''_2$$

Show that $W=U'_1W''+W'\cdot U''_2$ is a submodule of the bimodule $KF^1\otimes KF$. Let $h\in KF^1$, $W\ni w=w_1+w_2$, where $w_1=u'_1w''$, $w_2=w'\cdot u''_2$, $u'_1\in U_1$, $w'\in W'$, $w''\in W''$, $u''_2\in U''_2$. Since U'_1 is an ideal, $hw_1=h(u'_1w'')=(hu'_1)w''\in U'_1W''$. Further, $hw_2=h(w'\cdot u''_2)=(hw')\cdot u''_2$. Since W' is a bimodule, then $hw'\in W'$. Thus,

$$hw_2=(hw')\cdot u''_2 \in W'\cdot U''_2.$$

Therefore,

$$hw=hw_1+hw_2 \in U'_1W''+W'\cdot U''_2=W.$$

This means that W is a left KF^1-module.

Let now $\ell \in KF$, and $w=w_1+w_2=u'_1w''+w'\cdot u''_2 \in W$. Then $w\cdot\ell=w_1\cdot\ell+w_2\cdot\ell$; $w_1\cdot\ell=(u'_1w'')\cdot\ell=u'_1(w''\cdot\ell)$. Since W'' is a right KF-module, then $w''\cdot\ell \in W''$. Therefore, $w_1\cdot\ell \in U'_1W''$. Similarly, $w_2\cdot\ell=(w'\cdot u''_2)\cdot\ell=w'\cdot u''_2\ell \in W'\cdot U''_2$. Thus, $w\cdot\ell \in U'_1W''+W'\cdot U''_2$. Hence, W is a right KF-module. Since $KF^1 \otimes KF$ is a bimodule and since it has been proved that W is a left KF^1-module and a right KF-module, then W is also bimodule. U'_1, W', U''_2, W'' are completely characteristic, therefore the bimodule $W=U'_1W''+W'\cdot U''_2$ is also completely characteristic.

If $u_1=u'_1u''_1 \in U'_1U''_1=U_1$, $v \in KF$, then $u_1 \otimes v=u'_1u''_1 \otimes v=u'_1(u''_1 \otimes v)$. Since $u''_1 \otimes v \in U''_1 \otimes KF \subset W''$, then $u_1 \otimes v=u'_1(u''_1 \otimes v) \in U'_1W'' \subset W$, that is, the third condition of (3.4) is satisfied.

Finally, if $w \in KF^1 \otimes KF$, $u_2=u'_2u''_2 \in U_2$, then $w\cdot u_2=w\cdot u'_2u''_2=(w\cdot u'_2)\cdot u''_2 \in W'\cdot U''_2 \subset W$. Thus, the tuple $\tau=\tau'\tau''$ is compatible.

The defined product of the compatible tuples is associative. Really, let $\tau^{(i)}=(U_1^{(i)},W^{(i)},U_2^{(i)})$, i=1,2,3 be compatible tuples. Then

$\tau^{(1)}\tau^{(2)}=(U_1^{(1)}U_1^{(2)},U_1^{(1)}W^{(2)}+W^{(1)}\cdot U_2^{(2)},U_2^{(1)}U_2^{(2)})$;

$(\tau^{(1)}\tau^{(2)})\tau^{(3)}=(U_1^{(1)}U_1^{(2)}U_1^{(3)},U_1^{(1)}U_1^{(2)}W^{(3)}+U_1^{(1)}W^{(2)}\cdot U_2^{(3)}+W^{(1)}\cdot U_2^{(2)}U_2^{(3)},$
$U_2^{(1)}U_2^{(2)}U_2^{(3)})$. On the other hand,

$\tau^{(2)}\tau^{(3)}=(U_1^{(2)}U_1^{(3)},U_1^{(2)}W^{(3)}+W^{(2)}\cdot U_2^{(3)},U_2^{(2)}U_2^{(3)})$;

$\tau^{(1)}(\tau^{(2)}\tau^{(3)})=(U_1^{(1)}U_1^{(2)}U_1^{(3)},U_1^{(1)}U_1^{(2)}W^{(3)}+U_1^{(1)}W^{(2)}\cdot U_2^{(3)}+W^{(1)}\cdot U_2^{(2)}U_2^{(3)},$
$U_2^{(1)}U_2^{(2)}U_2^{(3)})$.

Thus, $(\tau^{(1)}\tau^{(2)})\tau^{(3)}=\tau^{(1)}(\tau^{(2)}\tau^{(3)})$, as was to be shown.

Theorem 4.2. [41]. *The semigroup of varieties of biautomata is antiisomorphic to the semigroup of the compatible tuples.*

Proof. One-to-one correspondence of the biautomata varieties and compatible tuples was stated before. It is necessary to show that if

$\mathfrak{X}^{(i)}$ are varieties of automata, $\tau^{(i)}=(U_1^{(i)},W^{(i)},U_2^{(i)})$, i=1,2 are the corresponding tuples of identities, $\mathfrak{X}=\mathfrak{X}^{(1)}\mathfrak{X}^{(2)}$, $\tau=\tau^{(2)}\tau^{(1)}$ and $\mathfrak{X}_\tau$ is a variety of biautomata defined by the compatible tuple τ, then $\mathfrak{X}_\tau=\mathfrak{X}^{(1)}\mathfrak{X}^{(2)}$ By the definition $\tau=(U_1^{(2)}U_1^{(1)},U_1^{(2)}W^{(1)}+W^{(2)}\cdot U_2^{(1)},U_2^{(2)}U_2^{(1)})$. Denote $U_1^{(2)}W^{(1)}+W^{(2)}\cdot U_2^{(1)}=W$ and consider a free cyclic biautomaton of the variety $\mathfrak{X}_\tau$:

$$\mathfrak{A}=(KF^1/U_1^{(2)}U_1^{(1)},F,(KF^1\otimes KF\otimes KF^1)/(W\oplus U_2^{(2)}U_2^{(1)}))$$

and the subautomaton

$$\mathfrak{A}^{(1)}=(U_1^{(2)}/U_1^{(2)}U_1^{(1)},F,(W+W^{(2)}\oplus U_2^{(2)})/(W\oplus U_2^{(2)}U_2^{(1)}))$$

in it. If $u_i^{(j)}\in U_i^{(j)}$, $w^{(i)}\in W^{(i)}$, i,j=1,2 then $u_1^{(2)}u_1^{(1)}\in U_1^{(2)}U_1^{(1)}$ and $u_1^{(2)}w^{(1)}\in U_1^{(2)}W^{(1)}\subset W$. Furthermore,

$$(w+w^{(2)}+u_2^{(2)})\cdot u_2^{(1)}=w\cdot u_2^{(1)}+w^{(2)}\cdot u_2^{(1)}+u_2^{(2)}u_2^{(1)}\in W\oplus U_2^{(2)}U_2^{(1)}.$$

These inclusions mean that if $u_1^{(1)}\in U_1^{(1)}$, $w^{(1)}\in W^{(1)}$, $u_2^{(1)}\in U_2^{(1)}$, then identities

$$z\circ u_1^{(1)}\equiv 0,\quad zw^{(1)}\equiv 0,\quad y\cdot u_2^{(1)}\equiv 0$$

are satisfied in the biautomaton $\mathfrak{A}^{(1)}$ Therefore, $\mathfrak{A}^{(1)}\in\mathfrak{X}^{(1)}$

Take a quotient automaton

$$\mathfrak{A}/\mathfrak{A}^{(1)}=(KF^1/\mathfrak{A}_1^{(2)},F,((KF^1\otimes KF)\oplus KF^1)/(W+W^{(2)}\oplus U_2^{(2)})).$$

Arguing as above, we can check that $\mathfrak{A}/\mathfrak{A}^{(1)}\in\mathfrak{X}^{(2)}$ Therefore, $\mathfrak{A}\in\mathfrak{X}^{(1)}\mathfrak{X}^{(2)}$ and $\mathfrak{X}_\tau\subset\mathfrak{X}^{(1)}\mathfrak{X}^{(2)}$.

Let us verify the inverse inclusion. Take a biautomaton $\mathfrak{A}=(A,\Gamma,B)$ from $\mathfrak{X}^{(1)}\mathfrak{X}^{(2)}$ Hence, there is such subautomaton $\mathfrak{A}^{(1)}=(A_1,\Gamma,B_1)\in\mathfrak{X}_1$ in $\mathfrak{A}$ that $\mathfrak{A}/\mathfrak{A}^{(1)}=(A/A_1,\Gamma,B/B_1)\in\mathfrak{X}^{(2)}$. It is necessary to show that the identities of the tuple τ are satisfied in $\mathfrak{A}$, that is, that $\mathfrak{A}\in\mathfrak{X}_\tau$.

Let μ be an arbitrary homomorphism of the free cyclic biautomaton into $\mathfrak{A}$, $a\in A$, $b\in B$. Since $(A/A_1,\Gamma,B/B_1)\in\mathfrak{X}^{(2)}$, then $a\circ(u_1^{(2)})^\mu=a_1\in A_1$, $a(w^{(2)})^\mu=b_1'\in B_1$ and $b\cdot(u_2^{(2)})^\mu=b_1\in B_1$. In its turn, since the biautomaton (A_1,Γ,B_1) belongs to $\mathfrak{X}^{(1)}$, then $a_1\circ(u_1^{(1)})^\mu=0$, $a_1(w^{(1)})^\mu=0$, $b_1\cdot(u_2^{(1)})^\mu=0$ and $b_1'\cdot(u_2^{(1)})^\mu=0$. Consequently,

$$a\circ(u_1^{(2)}u_1^{(1)})^{\mu}=(a\circ(u_1^{(2)})^{\mu})\circ(u_1^{(1)})^{\mu}=a_1\circ(u_1^{(1)})^{\mu}=0;$$

$a(u_1^{(2)}w^{(1)}+w^{(2)}u_2^{(1)})^{\mu}=(a\circ(u_1^{(2)})^{\mu})(w^{(1)})^{\mu}+(a(w^{(2)})^{\mu})\cdot(u_2^{(1)})^{\mu}=$ $a_1(w^{(1)})^{\mu}+b_1'\cdot(u_2^{(1)})^{\mu}=0$ and, finally, $b\cdot(u_2^{(2)}u_2^{(1)})^{\mu}=(b\cdot(u_2^{(2)})^{\mu})\cdot(u_2^{(1)})^{\mu}=$ $b_1\cdot(u_2^{(1)})^{\mu}=0$.

Thus, the biautomaton $\mathfrak{A}$ satisfies the identities of the tuple τ, that is, it belongs to $\mathfrak{X}_\tau$.

This proves the inverse inclusion as well as the equality $\mathfrak{X}_\tau=\mathfrak{X}^{(1)}\mathfrak{X}^{(2)}$

4.4.3. Theorem of freeness of semigroup of biautomata varieties

In this item we prove two theorems [13] which play a significant role in biautomata varieties research.

Let θ_1,θ_2 be biautomata classes. Denote by $\theta_1\nabla\theta_2$ the class of biautomata consisting of all triangular products of biautomata from θ_1 by biautomata from θ_2. Then

Theorem 4.3. *The product of varieties of biautomata generated by the classes θ_1,θ_2 respectively, is equal to variety generated by the class $\theta_1\nabla\theta_2$, i.e.*

$$\mathrm{Var}\theta_1\cdot\mathrm{Var}\theta_2=\mathrm{Var}(\theta_1\nabla\theta_2).$$

Biautomata variety $\mathfrak{X}$ is called *left unproper* if the class of all representations (A,Γ) such that there exists a biautomaton $(A,\Gamma,B)\in\mathfrak{X}$, coincides with the class of all representations. *Right unproper* biautomata variety is defined in a similar way. A biautomata variety $\mathfrak{X}$ is called *unproper* if it is left or right unproper. Finally, biautomata variety $\mathfrak{X}$ is called *trivial* if it consists only of biautomata of the form $(0,\Gamma,0)$.

The theorem 4.3 lies in the base of the proof of the biautomata varieties semigroup freeness.

Theorem 4.4. *The semigroup of non-trivial proper varieties of biautomata is free.*

To prove the theorem we need several lemmas.

Lemma 4.5. *Let* $\mathfrak{X}_1, \mathfrak{X}_2$ *be the varieties of biautomata,* $\mathfrak{A}_1 \in \mathfrak{X}_1$, $\mathfrak{A}_2 \in \mathfrak{X}_2$. *Then*

$$\mathfrak{A}_1 \nabla \mathfrak{A}_2 \in \mathfrak{X}_1 \mathfrak{X}_2.$$

Proof. Let $\mathfrak{A}_i = (A_i, \Gamma_i, B_i)$ i=1,2 and $\mathfrak{A} = \mathfrak{A}_1 \nabla \mathfrak{A}_2 = (A_1 \oplus A_2, \Gamma, B_1 \oplus B_2)$. The automaton $\mathfrak{A}_1 \in \mathfrak{X}_1$ is a homomorphic on inputs image of the subautomaton $\mathfrak{A}'_1 = (A_1, \Gamma, B_1)$ from $\mathfrak{A}$. Since the variety $\mathfrak{X}_1$ is saturated then $\mathfrak{A}'_1$ also belongs to $\mathfrak{X}_1$.

Consider the quotient automaton $\mathfrak{A}/\mathfrak{A}'_1 = (A/A_1, \Gamma, B/B_1)$. It suffices to verify that it belongs to $\mathfrak{X}_2$. This implies that $\mathfrak{A} = \mathfrak{A}_1 \nabla \mathfrak{A}_2$ belongs to $\mathfrak{X}_1 \mathfrak{X}_2$. Take first an automaton $(A/A_1, \Gamma_2, B/B_1) \subset (A/A_1, \Gamma, B/B_1)$. One-to-one correspondence $\bar{a}_2 = a_2 + A_1 \rightarrow a_2$ and $\bar{b}_2 = b_2 + B_1 \rightarrow b_2$ defines the isomorphism of automaton $(A/A_1, \Gamma_2, B/B_1)$ on (A_2, Γ_2, B_2). Therefore it belongs to $\mathfrak{X}_2$. In its turn the automaton $(A/A_1, \Gamma_2, B/B_1)$ is an epimorphic in inputs image of the automaton $(A/A_1, \Gamma, B/B_1)$. Since the variety $\mathfrak{X}_2$ is saturated, this means that the latter automaton also belongs to $\mathfrak{X}_2$.

Lemma 4.6. *Let* Θ *be a class of biautomata,* $\mathfrak{X} = \mathrm{Var}\Theta$. *Then all free biautomata from* $\mathfrak{X}$ *belong to* VSCQ.

Proof. Let $\mathfrak{A} = (A, \Gamma, B)$ be an arbitrary biautomaton from Θ, and $\mathcal{F} = (H, F, G)$ be an absolutely free biautomaton with the system of free generators (Z, X, Y). Mappings $\nu_1 : Z \rightarrow A$, $\nu_2 : X \rightarrow \Gamma$, $\nu_3 : Y \rightarrow B$ are uniquely extended to homomorphism $\nu = (\nu_1, \nu_2, \nu_3) : \mathcal{F} \rightarrow \mathfrak{A}$. Let $\rho_{\mathfrak{A}} = \mathrm{Ker}\nu$, and $\rho = \cap \rho_{\mathfrak{A}}$, $\mathfrak{A} \in \Theta$. Then $\mathcal{F}/\rho = (H/\rho_1, F/\rho_2, G/\rho_3)$ is a free biautomaton in $\mathfrak{X}$. Therefore $(H/\rho_1, F, G/\rho_3)$ is free in $\mathfrak{X}$. By Remak's theorem $\mathcal{F}/\rho$ is embedded into product $\prod \mathcal{F}/\rho_{\mathfrak{A}}$, $\mathfrak{A} \in \Theta$. Thus $\mathcal{F}/\rho \in SC\Theta$ and biautomaton $(H/\rho_1, F, G/\rho_3) \in VSC\Theta$.

Lemma 4.7. *Let* $\mathfrak{X}_i$, i=1,2 *be varieties of biautomata,* $\mathcal{F}_1$ *be a free in* $\mathfrak{X}_1$ *biautomaton with the countable set of generators,* $\mathfrak{A}_2$ *- free cyclic biautomaton of the variety* $\mathfrak{X}_2$. *Then*

$$\mathrm{Var}(\mathcal{F}_1 \nabla \mathfrak{A}_2) = \mathfrak{X}_1 \mathfrak{X}_2.$$

Proof. Let $\tau_i = (U_1^{(i)}, W^{(i)}, U_2^{(i)})$ be the tuples of identities of varieties $\mathfrak{X}_i$, i=1,2. By the theorem 4.2

$$\tau_2\tau_1=(U_1^{(2)}U_1^{(1)},U_1^{(2)}W^{(1)}+W^{(2)}U_2^{(1)},U_2^{(2)}U_2^{(1)})$$

is the tuple of the variety $\mathfrak{X}_1\mathfrak{X}_2$. Denote $U_1^{(2)}W^{(1)}+W^{(2)}U_2^{(1)}$ by W. Varieties $\mathfrak{X}_i$ are generated by the biautomata

$$\mathfrak{A}_i=(KF^1/U_i^{(i)},F,(KF^1\otimes KF\oplus KF^1)/(W^{(i)}\oplus U_2^{(i)})),\ i=1,2$$

and the variety $\mathfrak{X}_1\mathfrak{X}_2$ by the biautomaton

$$\mathfrak{A}=(KF^1/U_1^{(2)}U_1^{(1)},F,(KF^1\otimes KF\oplus KF^1)/(W\oplus U_2^{(2)}U_2^{(1)})).$$

Consider a subautomaton $\mathfrak{A}'$ in $\mathfrak{A}$

$$\mathfrak{A}'=(U_1^{(2)}/U_1^{(2)}U_1^{(1)},F,(W^{(2)}\oplus U_2^{(2)})/(W\oplus U_2^{(2)}U_2^{(1)})).$$

Quotient automaton $\mathfrak{A}/\mathfrak{A}'$ is isomorphic to the automaton $\mathfrak{A}_2$. Denote by $\bar{\mathfrak{A}}_i,\bar{\mathfrak{A}},\bar{\mathfrak{A}}'$ the exact biautomata corresponding to $\mathfrak{A}_i,\mathfrak{A},\mathfrak{A}'$ respectively. The automaton $\bar{\mathfrak{A}}$ is isomorphically embedded into triangular product $\bar{\mathfrak{A}}'\nabla\bar{\mathfrak{A}}/\bar{\mathfrak{A}}'$ belonging to $\mathfrak{X}_1\mathfrak{X}_2$ (by the theorem of embedding). Since $\mathrm{Var}\mathfrak{A}=\mathfrak{X}_1\mathfrak{X}_2$, then $\mathrm{Var}(\bar{\mathfrak{A}}'\nabla\bar{\mathfrak{A}}/\bar{\mathfrak{A}}')=\mathfrak{X}_1\mathfrak{X}_2$. By virtue of isomorphism $\mathfrak{A}/\mathfrak{A}'\cong\mathfrak{A}_2$ we get $\mathrm{Var}(\bar{\mathfrak{A}}'\nabla\bar{\mathfrak{A}}_2)=\mathfrak{X}_1\mathfrak{X}_2$. It is clear that $\mathrm{Var}(\mathfrak{A}'\nabla\mathfrak{A}_2)=\mathfrak{X}_1\mathfrak{X}_2$ also.

Since $\mathfrak{F}_1$ is a free in $\mathfrak{X}_1$ biautomaton with the countable set of generators then there is an epimorphism $\mu:\mathfrak{F}_1\to\mathfrak{A}_2$. By the proposition 2.14 of the previous chapter this epimorphism induces an epimorphism $\mathfrak{F}_1\nabla\mathfrak{A}_2\to\mathfrak{A}'\nabla\mathfrak{A}_2$. Therefore $\mathrm{Var}(\mathfrak{F}_1\nabla\mathfrak{A}_2)=\mathfrak{X}_1\mathfrak{X}_2$.

Lemma 4.8. *Given biautomaton $\mathfrak{B}$, let $\mathfrak{X}_1=\mathrm{Var}\mathfrak{B}$ and $\mathfrak{A}_2$ be a free cyclic biautomaton of the variety $\mathfrak{X}_2$. Then*

$$\mathrm{Var}(\mathfrak{B}\nabla\mathfrak{A}_2)=\mathfrak{X}_1\mathfrak{X}_2$$

Proof. Let $\mathfrak{F}_1$ be a free in $\mathfrak{X}_1$ biautomaton with the countable set of generators and $\bar{\mathfrak{F}}_1$ be the corresponding exact biautomaton. By Lemma 4.7, $\bar{\mathfrak{F}}_1\nabla\bar{\mathfrak{A}}_2$ (and also $\mathfrak{F}_1\nabla\mathfrak{A}_2$) generates the variety $\mathfrak{X}_1\mathfrak{X}_2$. Since $\bar{\mathfrak{F}}_1\in QSC\mathfrak{B}$ then for some set I there is a biautomaton $\mathfrak{B}'\subset\mathfrak{B}^I$ which is epimorphically mapped onto $\bar{\mathfrak{F}}$. Let $\mathfrak{X}=\mathrm{Var}(\mathfrak{B}\nabla\mathfrak{A}_2)$. By the proposition 2.15 (Chapter 3) there is the embedding $\mathfrak{B}^I\nabla\mathfrak{A}_2\to(\mathfrak{B}\nabla\mathfrak{A}_2)^I$. Therefore $\mathfrak{B}^I\nabla\mathfrak{A}_2\in\mathfrak{X}$. Since $\mathfrak{B}'\subset\mathfrak{B}^I$ then $\mathfrak{B}'\nabla\mathfrak{A}_2\in\mathfrak{X}$. By the proposition 2.14 (Chapter 3) there is the epimorphism $\mathfrak{B}'\nabla\mathfrak{A}_2\to\bar{\mathfrak{F}}\nabla\mathfrak{A}_2$. Thus $\bar{\mathfrak{F}}\nabla\mathfrak{A}_2\in\mathfrak{X}$. So $\mathfrak{X}_1\mathfrak{X}_2\subset\mathfrak{X}$. On the other hand, $\mathfrak{B}\nabla\mathfrak{A}_2\in\mathfrak{X}_1\mathfrak{X}_2$ by Lemma 4.5. Thus $\mathfrak{X}\subset\mathfrak{X}_1\mathfrak{X}_2$ and $\mathfrak{X}=\mathfrak{X}_1\mathfrak{X}_2$.

Lemma 4.9. *Let* θ *be a class of biautomata,* $\mathfrak{X}=\mathrm{Var}\theta$ *and* $\mathcal{F}_{\mathfrak{X}}=(A,F,B)$ *be a free biautomaton in* $\mathfrak{X}$. *Let* $a_1,\dots,a_n$ *and* $b_1,\dots,b_n$ *be systems of linear independent elements of* A *and* B *respectively. Then there exists such biautomaton* $\mathfrak{A}=(C,\Gamma,T)\in\mathcal{D}_0\theta$ *and homomorphism* $\nu:\mathcal{F}_{\mathfrak{X}}\to\mathfrak{A}$ *that* $a_1^{\nu},\dots,a_n^{\nu}$ *and* $b_1^{\nu},\dots,b_n^{\nu}$ *are the systems of linear independent elements also.*

Proof. Let I be a set and vector space H be a direct sum of spaces H_α, i.e. $H=\sum_{\alpha\in I}H_\alpha$. Take $M\subset I$ and consider projections $\varphi_M:H\to\sum_{\alpha\in M}H_\alpha$. If $h_1,\dots,h_k$ are linear independent elements in H then for some finite M the system of elements $h_1^{\varphi_M},\dots,h_k^{\varphi_M}$ is also linear independent. Indeed, let $H_0\subset H$ be a subspace generated by $h_1,\dots,h_k$ It is finite-dimensional, so there exists $M\subset I$ such that $H_0\cap\mathrm{Ker}\varphi_M=0$. Thus restriction φ_M on H_0 is a monomorphism for every such M and the elements $h_1^{\varphi_M},\dots,h_k^{\varphi_M}$ are linear independent.

$\mathcal{F}_{\mathfrak{X}}\subset\mathrm{VSC}\theta$ by Lemma 4.6. Let $\mathfrak{A}_\alpha=(A_\alpha,\Gamma_\alpha,B_\alpha)$ be such biautomata from θ that $\bar{\mathcal{F}}_{\mathfrak{X}}=(A,\bar{F},B)$ the exact biautomaton, isomorphically embedded into $\prod_{\alpha\in I}\mathfrak{A}_\alpha$. Let μ be an epimorphism in inputs, $\mu:\mathcal{F}_{\mathfrak{X}}\to\bar{\mathcal{F}}_{\mathfrak{X}}\subset\prod_{\alpha\in I}\mathfrak{A}_\alpha$. Consider finite subsets $M_1,M_2\subset I$ such that elements

$$a_1^{\varphi_{M_1}},\dots,a_n^{\varphi_{M_1}} \text{ and } b_1^{\varphi_{M_2}},\dots,b_n^{\varphi_{M_2}}$$

are linear independent in $\sum_{\alpha\in M_1}A_\alpha$, $\sum_{\alpha\in M_2}B_\alpha$ respectively. Take $M=M_1\cup M_2$ and $\mathfrak{A}=(C,\Gamma,T)=\prod_{\alpha\in M}\mathfrak{A}_\alpha$ It is clear that under projections $\varphi_M:\prod_{\alpha\in I}\mathfrak{A}_\alpha\to\mathfrak{A}=\prod_{\alpha\in M}\mathfrak{A}_\alpha$ elements $a_1^{\varphi_M},\dots,a_k^{\varphi_M}$ and $b_1^{\varphi_M},\dots,b_k^{\varphi_M}$ are linear independent in C and T respectively. Since the set M is finite then $\mathfrak{A}\in\mathcal{D}_0\theta$, and $\nu=\mu\varphi_M:\mathcal{F}_{\mathfrak{X}}\to\mathfrak{A}$ is a desired homomorphism.

Before the next lemma let us make some calculations. Let $(A,\Gamma,B)=(A_1,\Sigma_1,B_1)\nabla(H,\Sigma_2,G)$ where $\Gamma=\Sigma_1\times\Phi_1\times\Psi\times\Phi_2\times\Sigma_2$, $\Phi_1=\mathrm{Hom}(H,A_1)$, $\Psi=\mathrm{Hom}(H,B_1)$, $\Phi_2=\mathrm{Hom}(G,B_1)$. Consider an arbitrary element $u=u(x_1,\dots,x_n)$ from the semigroup algebra KF^1 over free semigroup $F=F(X)$. It may be written in the form

$$u=\sum_{i_1,\dots,i_k}\alpha_{i_1,\dots,i_k}x_{i_1}\dots x_{i_k},\quad \alpha_{i_1,\dots,i_k}\in K,$$

We need to calculate elements of the form $u(\gamma_1,\dots,\gamma_n)$ where $\gamma_i=(\sigma_{1i},\varphi_{1i},\psi_i,\varphi_{2i},\sigma_{2i})\in\Gamma$. Consider projections $\alpha:\Gamma\to\Sigma_1$, $d_1:\Gamma\to\Phi_1$, $\delta:\Gamma\to\Psi$, $d_2:\Gamma\to\Phi_2$, $\beta:\Gamma\to\Sigma_2$. It is clear that $\alpha(\gamma_1\dots\gamma_n)=\alpha(\gamma_1)\dots\alpha(\gamma_r)=\sigma_{11}\dots\sigma_{1n}$, and $\beta(\gamma_1\dots\gamma_r)=\beta(\gamma_1)\dots\beta(\gamma_n)=\sigma_{21}\dots\sigma_{2n}$. Inductively one can verify the following formulas:

$$d_1(\gamma_1\dots\gamma_n)=\sum_i\beta(\gamma_1\dots\gamma_{i-1})d_1(\gamma_i)\alpha(\gamma_{i+1}\dots\gamma_n)=\sum_i\sigma_{21}\dots\sigma_{2i}\varphi_{1i}\sigma_{1i+1}\dots\sigma_{1n},$$

$$d_2(\gamma_1\dots\gamma_n)=\sum_i\beta(\gamma_1\dots\gamma_{i-1})d_2(\gamma_i)\alpha(\gamma_{i+1}\dots\gamma_n)=\sum_i\sigma_{21}\dots\sigma_{2i}\varphi_{2i}\sigma_{1i+1}\dots\sigma_{1n}.$$

Further,

$$\gamma_{i_1}\dots\gamma_{i_k}=(\sigma_{1i_1}\dots\sigma_{1i_k},d_1(\gamma_{i_1}\dots\gamma_{i_k}),\delta(\gamma_{i_1}\dots\gamma_{i_k}),d_2(\gamma_{i_1}\dots\gamma_{i_k}),\sigma_{2i_1}\dots\sigma_{2i_k}),$$

$$u(\gamma_1,\dots,\gamma_r)=(u(\sigma_{11},\dots,\sigma_{1n}),\ \sum_{i_1,\dots,i_k}\alpha_{i_1,\dots,i_k}d_1(\gamma_{i_1}\dots\gamma_{i_k}),$$

$$\sum_{i_1,\dots,i_k}\alpha_{i_1,\dots,i_k}\delta(\gamma_{i_1}\dots\gamma_{i_k}),\ \sum_{i_1,\dots,i_k}\alpha_{i_1,\dots,i_k}d_2(\gamma_{i_1}\dots\gamma_{i_k}),$$

$$u(\sigma_{21}\dots\sigma_{2n})).$$

Lemma 4.10. *Let $\mathfrak{A}_1=(A_1,\Sigma_1,B_1)$ be a biautomaton and Θ be a class of biautomata, such that $\Theta=\mathcal{D}_0\Theta$. Let $\mathfrak{X}_1=\mathrm{Var}\mathfrak{A}_1$ and $\mathfrak{X}_2=\mathrm{Var}\Theta$. Then*

$$\mathrm{Var}(\mathfrak{A}_1\nabla\Theta)=\mathfrak{X}_1\mathfrak{X}_2$$

Proof. Let $\mathcal{F}_2=(H,F,G)$ be a cyclic free biautomaton in $\mathfrak{X}_2$. By Lemma 4.8 the triangular product $\mathfrak{A}_1\nabla\mathcal{F}_2$ generates $\mathfrak{X}_1\mathfrak{X}_2$. Then, any identity $z\circ u\equiv0$, $zw\equiv0$, $y\bullet u\equiv0$ which holds in $\mathfrak{A}_1\nabla\mathcal{F}_2$, holds also for biautomaton $\mathfrak{A}_1\nabla\mathfrak{A}_2$, for each $\mathfrak{A}_2\in\Theta$.

Conversely, let the identity $z\circ u\equiv0$ is not hold in $\mathfrak{A}_1\nabla\mathcal{F}_2=(A_1\oplus H,\Gamma,B_1\oplus G)$. This means that there exist the elements $a_1+h\in A_1+H$ and $\gamma_1,\dots,\gamma_n$ from Γ, such that $(a_1+h)\circ u(\gamma_1,\dots,\gamma_n)\neq0$. One can assume that $u\in U_1^{(1)}\cap U_1^{(2)}$, where $(U_1^{(i)},W^{(i)},U_2^{(i)})$ is the tuple of identities of the varieties $\mathfrak{X}_i$, i=1,2.

We have

$(a_1+h)\circ u(\gamma_1,\ldots,\gamma_n)=a_1\circ u(\sigma_1,\ldots,\sigma_n)+hd_1(\gamma_1,\ldots,\gamma_n)+h\circ u(\sigma_{21},\ldots,\sigma_{2n})$, $\sigma_{21}\in F$.

Since $u\in U_1^{(1)}\cap U_1^{(2)}$ then $a\circ u(\sigma_{11},\ldots,\sigma_{1n})=0$ and $h\circ u(\sigma_{21},\ldots,\sigma_{2n})=0$. Thus

$(a_1+h)\circ u(\gamma_1,\ldots,\gamma_n)=hd_1(\gamma_1,\ldots,\gamma_n)=$

$$\sum_{i_1,\ldots,i_k}\sum_j\alpha_{i_1,\ldots,i_k}((h\circ\sigma_{2i_1}\ldots\sigma_{2i_{j-1}})\varphi_{1i_j})\circ\sigma_{1i_{j+1}}\ldots\sigma_{1i_k}.$$

Denote by V the linear space in H generated by all the elements $h\circ\sigma_{2i_1}\ldots\sigma_{2i_{j-1}}$. Since the set of such elements is finite and (H,F,G) is a free biautomaton in the variety $\mathfrak{X}_2=\mathrm{Var}\theta$ then by Lemma 4.9 there are such biautomaton $\mathfrak{A}_2=(A_2,\Sigma_2,B_2)\in\theta=\mathcal{D}_0\theta$ and such homomorphism $\mu:\mathcal{F}_2\to(A_2,\Sigma_2,B_2)$ that the linear spaces V and V^μ have one and the same dimension. Consider a homomorphism $\nu:A_2\to H$, inverse to μ on the space V and arbitrary beyond it, and the homomorphism $\varphi'_{1i_j}=\nu\varphi_{1i_j}$ from A_2 to A_1. Then

$$(h\circ\sigma_{2i_1}\ldots\sigma_{2i_{j-1}})^\mu\varphi'_{1i_j}=(h\circ\sigma_{2i_1}\ldots\sigma_{2i_{j-1}})^{\mu\nu}\varphi_{1i_j}=(h\circ\sigma_{2i_1}\ldots\sigma_{2i_{j-1}})\varphi_{1i_j}.$$

Take the triangular product $\mathfrak{A}_1\nabla\mathfrak{A}_2=(A_1\oplus A_2,\tilde{\Gamma},B_1\oplus B_2)$, an arbitrary element $a_1\in A_1$, element $a_2=h_1^\mu\in A_2$ and elements $\tilde{\gamma}_i=(\sigma_{1i},\varphi'_{1i},0,0,\lambda_{2i})$ such that $\lambda_{2i}=\sigma_{2i}^\mu\in\Sigma_2$. Then

$$(a_1+a_2)\circ u(\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n)=h\circ u(\gamma_1,\ldots,\gamma_n)\neq0.$$

The identity $z\circ u\equiv0$ is not held in $\mathfrak{A}_1\nabla\mathfrak{A}_2$, where $\mathfrak{A}_2\in\theta$ by the construction. Thus the biautomaton $\mathfrak{A}_1\nabla\mathcal{F}_2$ and the class of biautomata $\mathfrak{A}_1\nabla\theta$ satisfy one and the same identities of the form $z\circ u\equiv0$. In a similar way one can verify that it is also true for identities of the form $zw\equiv0$ and $y\cdot u\equiv0$. So, $\mathfrak{A}_1\nabla\mathcal{F}_2$ and $\mathfrak{A}_1\nabla\theta$ generate the same variety.

The following lemma asserts that the condition $\theta=\mathcal{D}_0\theta$ in Lemma 4.10 is not essential.

Lemma 4.11. *Let $\mathfrak{A}_1=(A_1,\Sigma_1,B_1)$ be an arbitrary biautomaton and θ*

be a class of biautomata. Let $\mathfrak{X}_1=\mathrm{Var}\mathfrak{A}_1$, $\mathfrak{X}_2=\mathrm{Var}\theta$. *Then*

$$\mathfrak{X}_1\mathfrak{X}_2=\mathrm{Var}(\mathfrak{A}_1\nabla\theta).$$

Proof. Let $\theta'=\mathcal{D}_0\theta$. Denote $\mathfrak{X}=\mathrm{Var}(\mathfrak{A}_1\nabla\theta)$. Since $\mathrm{Var}\theta'=\mathrm{Var}\theta=\mathfrak{X}_2$ then according to the previous Lemma $\mathfrak{X}_1\mathfrak{X}_2=\mathrm{Var}(\mathfrak{A}_1\nabla\theta')$. It is clear that $\mathfrak{X}\subset\mathfrak{X}_1\mathfrak{X}_2$.

Verify the inverse inclusion. Let $\mathfrak{A}=(A,\Gamma,B)=\mathfrak{A}_1\nabla\mathfrak{A}_2$ where $\mathfrak{A}_2=(A_2,\Sigma_2,B_2)$ is a biautomaton from θ'. Since $\theta'=\mathcal{D}_0\theta$ then there exists such finite set of biautomata $\mathfrak{A}_{2i}=(A_{2i},\Sigma_{2i},B_{2i})\in\theta$, $i\in I$, that $\mathfrak{A}_2=\prod_{i\in I}\mathfrak{A}_{2i}$. It is easy to check that a triplet $(A_1+A_{2i},\Gamma,B_1+B_{2i})$ is biautomaton. Construct the epimorphism in input signals

$$\mu_i:(A_1+A_{2i},\Gamma,B_1+B_{2i})\to\mathfrak{A}_1\nabla\mathfrak{A}_{2i}=(A_1+A_{2i},\Gamma_i,B_1+B_{2i})$$

where $\Gamma_i=\Sigma_1\times\Phi_{1i}\times\Psi_i\times\Phi_{2i}\times\Sigma_{2i}$, $\Phi_{1i}=\mathrm{Hom}(A_{2i},A_1)$, $\Psi_i=\mathrm{Hom}(A_{2i},B_1)$, $\Phi_{2i}=\mathrm{Hom}(B_{2i},B_1)$.

Let $\sigma_1^{\mu_i}=\sigma_1$ if $\sigma_1\in\Sigma_1$; for $\varphi_1\in\Phi_1,\psi\in\Psi,a\in A_{2i}$ let $\varphi_1^{\mu_i}(a)=\varphi_1(a)$, $\psi^{\mu_i}(a)=\psi(a)$; for $\varphi_2\in\Phi_2,b\in B_{2i}$ let $\varphi_2^{\mu_i}(b)=\varphi_2(b)$; finally let $\mu_i:\Sigma_2\to\Sigma_{2i}$ be the projections. By this the epimorphism $\Gamma\to\Gamma_i$ is defined. This epimorphism together with the identity mappings on A_1+A_{2i} and B_1+B_{2i} defines in its turn the epimorphism of biautomata in input signals. Therefore, since $\mathfrak{A}_1\nabla\mathfrak{A}_{2i}\in\mathfrak{X}$, the biautomata $(A_1+A_{2i},\Gamma,B_1+B_{2i})$ belong to $\mathfrak{X}$, $i\in I$. The biautomaton $(A,\Gamma,B)=\mathfrak{A}_1\nabla\mathfrak{A}_2$ is generated by the biautomata $(A_1+A_{2i},\Gamma,B_1+B_{2i})$. Since the variety $\mathfrak{X}$ is simultaneously a radical class then $(A,\Gamma,B)\in\mathfrak{X}$. Therefore $\mathfrak{X}_1\mathfrak{X}_2\subset\mathfrak{X}$ and $\mathfrak{X}_1\mathfrak{X}_2=\mathfrak{X}$.

Proof of the theorem 4.3. Take some exact biautomaton $\mathfrak{A}_1$, generating variety $\mathfrak{X}_1$. By Lemma 4.10 $\mathrm{Var}(\mathfrak{A}_1\nabla\theta_2)=\mathfrak{X}_1\mathfrak{X}_2$. Since $\mathfrak{X}_1=\mathrm{Var}\theta_1$ then $\mathfrak{A}_1\in\mathrm{QSC}\theta_1$. This means that there exist a set of biautomata $\mathfrak{A}_{1i}\in\theta_1$, $i\in I$, and subautomaton $\mathfrak{A}'\subset\prod_{i\in I}\mathfrak{A}_{1i}$, such that $\mathfrak{A}_1$ is an epimorphic image of $\mathfrak{A}'$.

Denote $\mathrm{Var}(\theta_1\nabla\theta_2)$ by $\mathfrak{X}$ and let $\mathfrak{A}_2\in\theta_2$. Then there exists an inclusion $(\prod_{i\in I}\mathfrak{A}_{1i})\nabla\mathfrak{A}_2\to\prod_{i\in I}(\mathfrak{A}_{1i}\nabla\mathfrak{A}_2)\in\mathfrak{X}$. Therefore $(\prod_{i\in I}\mathfrak{A}_{1i})\nabla\mathfrak{A}_2\in\mathfrak{X}$. Since $\mathfrak{A}'\subset\prod_{i\in I}\mathfrak{A}_1$ then $\mathfrak{A}'\nabla\mathfrak{A}_2\in\mathfrak{X}$. According to the Proposition 2.14 from Chapter 3 there exists an epimorphism $\mathfrak{A}'\nabla\mathfrak{A}_2$ onto $\mathfrak{A}_1\nabla\mathfrak{A}_2$ So, $\mathfrak{A}_1\nabla\mathfrak{A}_2\in\mathfrak{X}$. Thus $\mathfrak{X}_1\mathfrak{X}_2\subset\mathfrak{X}$.

On the other hand, by Lemma 4.5 $\theta_1\nabla\theta_2\subset\mathfrak{X}_1\mathfrak{X}_2$ and $\mathfrak{X}\subset\mathfrak{X}_1\mathfrak{X}_2$. This proves the Theorem.

The similar Theorem for varieties of automata is not true. Let us show the corresponding example.

Take the automaton $(0,\Gamma_1,B)$. The tuple of its identities has the form (KF^1,KF). Therefore $\varepsilon=\mathrm{Var}(0,\Gamma_1,B)$ is the unit of the semigroup of automata varieties. Take this ε as $\mathfrak{X}_1$. As $\mathfrak{X}_2$ we take variety generated by the automaton (A,Γ_2,A) with the identities $z\circ x\equiv z$ and $z*x\equiv z$, $x\in X$. The tuple of identities of this automaton has the form (U,U), where $U=\Delta_F$ is the augmentation ideal in KF^1 The product $\mathfrak{X}_1\mathfrak{X}_2=\varepsilon\mathfrak{X}_2=\mathfrak{X}_2$ is associated with the tuple (U,U).

On the other hand, $(0,\Gamma_1,B)\nabla(A,\Gamma_2,A)=(A,\Gamma_1\times\mathrm{Hom}(A,B)\times\Gamma_2,A+B)$ $=(A,\Gamma,A+B)$. It is clear, that for each $a\in A$ and some $\gamma\in\Gamma$ holds $a*\gamma=a+a\psi\neq a$. Thus the tuple of identities of the variety $\mathrm{Var}(A,\Gamma,A+B)$ differs from (U,U) and

$$\mathfrak{X}_1\mathfrak{X}_2\neq\mathrm{Var}((0,\Gamma_1,B)\nabla(A,\Gamma_2,A)).$$

Proceed to the proof of the Theorem 4.4. At the beginning we prove some propositions which are also interesting by themselves.

Lemma 4.12. *Let* (A',Γ,B') *be a subautomaton of the triangular product of biautomata* $(A,\Gamma,B)=(A_1,\Sigma_1,B_1)\nabla(A_2,\Sigma_2,B_2)$. *Then either* $A_1\subset A'$ *and* $B_1\subset B'$, *or* $A'\subset A_1$ *and* $B'\subset B_1$ *or* $A'\subset A_1$ *and* $B_1\subset B'$.

Proof. Let A' does not belong to A_1 Take arbitrary elements $a_1\in A_1$ and $a_2\in A'\setminus A_1$. Write a_2 in the form $a_2'+a_2''$ where $a_2'\in A_1$, $a_2''\in A_2$. Denote by $\gamma=(\sigma_1,\varphi_1,\psi,\varphi_2,\sigma_2)$ such element from Γ that φ_1 satisfies the condition $a_2''\varphi_1=a_1$. Then $a_2\circ\gamma=a_2'\circ\sigma_1+a_2''\varphi_1+a_2''\circ\sigma_2=a_2'\circ\sigma_1+a_1+a_2''\circ\sigma_2$. Since A' is Γ-invariant subspace then $a_2\circ\gamma\in A'$ Consider an element $\gamma_0=(\sigma_1,0,\psi,\varphi_2,\sigma_2)$. It is clear that $\gamma_0\in\Gamma$, and $a_2\circ\gamma_0=a_2'\circ\sigma_1+a_2''\circ\sigma_2$ lies in A' Then $a_2\circ\gamma-a_2\gamma_0=a_1\in A'$ Since a_1 is an arbitrary element from A_1 then $A_1\subset A'$

Let now $A_1\subset A'$. Then $A'=A_1+A_2'$, where $A_2'\subset A_2$ Take an arbitrary $a_2\in A_2'$ For any $b_1\in B_1$ there exists $\psi\in\mathrm{Hom}(A_2,B_1)$ such that $a_2\psi=b_1$. Consider $\gamma=(\sigma_1,\varphi_1,\psi,\varphi_2,\sigma_2)\in\Gamma$ with arbitrary components $\sigma_1\in\Sigma_1$, $\varphi_1\in\mathrm{Hom}(A_2,A_1)$,

$\varphi_2 \in \mathrm{Hom}(B_2, B_1)$. Then $a_2 * \gamma = b_1 + a_2 * \sigma_2$. Consider also $\gamma_0 = (\sigma_1, \varphi_1, 0, \varphi_2, \sigma_2) \in \Gamma$. Elements $a*\gamma$ and $a*\gamma_0 = a*\sigma_2$ belong to B' Therefore $b_1 = a*\gamma - a*\gamma_0 \in B'$, and $B_1 \subset B'$.

The rest inclusions can be verified in a similar way.

Lemma 4.13. *Let* $(A,\Gamma,B)=(A_1,\Sigma_1,B_1)\nabla(A_2,\Sigma_2,B_2)$ *be a triangular product of biautomata and* $\mathfrak{X}$ *be a variety. Let*

$$\mathfrak{X}^*(A_2,\Sigma_2,B_2)=(H_1,\Sigma_2,H_2)\neq(0,\Sigma_2,0) \text{ and}$$

$$\mathfrak{X}'(A_1,\Sigma_1,B_1)=(G_1,\Sigma_1,G_2)\neq(A_1,\Sigma_1,B_1).$$

Then

$$\mathfrak{X}^*(A,\Gamma,B)=\begin{cases}(A_1+H_1,\Gamma,B_1+H_2), & \text{if } H_1\neq 0\\ (A',\Gamma,B_1+H_2), \text{ where } A'\subset A_1, & \text{if } H_1=0,\ H_2\neq 0\end{cases}$$

$$\mathfrak{X}'(A,\Gamma,B)=\begin{cases}(G_1,\Gamma,G_2), & \text{if } G_2\neq B_1\\ (G_1,\Gamma,G_2+B'_2), \text{ where } B'_2\subset B_2, & \text{if } G_2=B_1,\ G_1\neq A_1\end{cases}$$

Proof. 1) Let $\mathfrak{X}^*(A,\Gamma,B)=(A',\Gamma,B')$. Consider epimorphism of projection $\mu\colon (A,\Gamma,B) \rightarrow (A_2,\Sigma_2,B_2)$. Since verbal is permutable with epimorphism, we get

$$(A',\Gamma,B')^{\mu}=(\mathfrak{X}^*(A,\Gamma,B))^{\mu}=\mathfrak{X}^*(A,\Gamma,B)^{\mu}=\mathfrak{X}^*(A_2,\Sigma_2,B_2)=(H_1,\Sigma_2,H_2).$$

According to previous Lemma only the following cases are possible

a) $A'\subset A_1$, $B'\subset B_1$,

b) $A_1\subset A'$, $B_1\subset B'$,

c) $A'\subset A_1$, $B_1\subset B'$.

a) Let $A'\subset A_1$, $B'\subset B_1$ Then $(A',\Gamma,B')^{\mu}=(0,\Sigma_2,0)$. Therefore $\mathfrak{X}^*(A_2,\Sigma_2,B_2)=(0,\Sigma_2,0)$, which contradics the assertion of the Lemma.

b) Let $A_1\subset A'$, $B_1\subset B'$. Then (A',Γ,B') is the inverse image of (H_1,Σ_2,H_2) with respect to μ. Therefore $A'=A_1+H_1$, $B'=B_1+H_2$.

c) Let $A'\subset A_1$, $B_1\subset B'$. Then $(A',\Gamma,B')^{\mu}=(0,\Sigma_2,H_2)$ and B' is the inverse image of H_2 with respect to μ. Therefore $B'=B_1+H_2$ and the first statement is proved.

2) Let $(A_1,\Gamma,B_1)\subset(A,\Gamma,B)$. It is clear that if

$$\mathfrak{X}'(A_1,\Sigma_1,B_1)=(G_1,\Sigma_1,G_2) \text{ then } \mathfrak{X}'(A_1,\Gamma,B_1)=(G_1,\Gamma,G_2).$$

Let $\mathfrak{X}'(A,\Gamma,B)=(A',\Gamma,B')$. By Lemma 4.12 three cases are possible. Let consider them separately.

a) Let $A'\subset A_1$, $B'\subset B_1$ Then $\mathfrak{X}'(A_1,\Sigma_1,B_1)=(A',\Sigma_1,B')=(G_1,\Sigma_1,G_2)$ and $\mathfrak{X}'(A,\Gamma,B)=(G_1,\Gamma,G_2)$.

b) Let $A_1\subset A'$, $B_1\subset B'$. Then $\mathfrak{X}'(A_1,\Gamma,B_1)=(A_1,\Gamma,B_1)$ and $\mathfrak{X}'(A_1,\Sigma_1,B_1)=(A_1,\Sigma_1,B_1)$, that contradicts the assumption.

c) Let $A'\subset A_1$, $B_1\subset B'$. Then $(A',\Gamma,B_1)\in\mathfrak{X}$ as subautomaton of (A',Γ,B'). Therefore $\mathfrak{X}'(A_1,\Gamma,B_1)=(A',\Gamma,B_1)=(G_1,\Gamma,G_2)$. Since $B'\supset B_1$ then $B'=B_1+B_2'=G_2+B_2'$, where $B_2'\subset B_2$ Thus $\mathfrak{X}'(A,\Gamma,B)=(G_1,\Gamma,G_2+B_2')$.

Lemma 4.14. *Let* $\mathfrak{X}_1$, $\mathfrak{X}_2$, $\mathfrak{Y}_1$, $\mathfrak{Y}_2$ *be non-trivial varieties of biautomata such that* $\mathfrak{X}_1\mathfrak{X}_2=\mathfrak{Y}_1\mathfrak{Y}_2$ *and* $\mathfrak{X}_2$ *does not belong to* $\mathfrak{Y}_2$. *Then there exist non-trivial varieties* $\mathfrak{X}_3$ *and* $\mathfrak{X}_1'\subset\mathfrak{X}_1$ *such that*

$$\mathfrak{X}_1'\mathfrak{X}_3\mathfrak{Y}_2=\mathfrak{Y}_1\mathfrak{Y}_2=\mathfrak{X}_1\mathfrak{X}_2.$$

Proof. Take free biautomata (A_i,F,B_i) in $\mathfrak{X}_i$, $i=1,2$. By the assumption, $(A_2,F,B_2)\notin\mathfrak{Y}_2$. Let $(A,\Gamma,B)=(A_1,F,B_1)\nabla(A_2,F,B_2)$. Consider $\mathfrak{Y}_2^*(A_2,F,B_2)=(H,F,L)$. From the condition $(A_2,F,B_2)\notin\mathfrak{Y}_2$ it follows that $(H,F,L)\neq(0,F,0)$. According to Lemma 4.13

$$\mathfrak{Y}_2^*(A,\Gamma,B)=\begin{cases}(A_1+H,\Gamma,B_1+L), & \text{if } H\neq 0\\ (A',\Gamma,B_1+L), \text{ where } A'\subset A_1, & \text{if } H=0,\ L\neq 0\end{cases}$$

Assume that $H\neq 0$. Then $\mathfrak{Y}_2^*(A,\Gamma,B)=(A_1+H,\Gamma,B_1+L)$. Since the triangular product (A,Γ,B) belongs to $\mathfrak{X}_1\mathfrak{X}_2=\mathfrak{Y}_1\mathfrak{Y}_2$ then $\mathfrak{Y}_2^*(A,\Gamma,B)\in\mathfrak{Y}_1$.

By Proposition 2.17 (Chapter 3) there exists an epimorphism in input signals:

$$(A_1+H,\Gamma,B_1+L) \rightarrow (A_1,F,B_1)\nabla(H,F,L).$$

Let $\mathfrak{X}_3=\mathrm{Var}(H,F,L)$. By the Theorem 4.3,

$$\mathrm{Var}((A_1,F,B_1)\nabla(H,F,L))=\mathfrak{X}_1\mathfrak{X}_3.$$

Hence, $\mathrm{Var}(A_1+H,\Gamma,B_1+L)=\mathfrak{X}_1\mathfrak{X}_3$. Since $(A_1+H_1,\Gamma,B_1+L)\in\mathfrak{Y}_1$ then $\mathfrak{X}_1\mathfrak{X}_3\subset\mathfrak{Y}_1$. So, $\mathfrak{X}_1\mathfrak{X}_3\mathfrak{Y}_2\subset\mathfrak{Y}_1\mathfrak{Y}_2$.

On the other hand, since $\mathfrak{Y}_2^*(A,\Gamma,B)=(A_1+H,\Gamma,B_1+L)\in\mathfrak{X}_1\mathfrak{X}_3$, then $(A,\Gamma,B)\in\mathfrak{X}_1\mathfrak{X}_3\mathfrak{Y}_2$, and $\mathrm{Var}(A,\Gamma,B)=\mathfrak{X}_1\mathfrak{X}_2=\mathfrak{Y}_1\mathfrak{Y}_2\subset\mathfrak{X}_1\mathfrak{X}_3\mathfrak{Y}_2$. Thus we get $\mathfrak{X}_1\mathfrak{X}_3\mathfrak{Y}_2=\mathfrak{X}_1\mathfrak{X}_2=\mathfrak{Y}_1\mathfrak{Y}_2$.

Let now $H=0$ and $L\neq0$. Then $\mathfrak{Y}_2^*(A,\Gamma,B)=(A',\Gamma,B_1+L)$, where $A'\subset A_1$. Let $\mathfrak{X}'_1=\mathrm{Var}(A',\Gamma,B_1)$ and $\mathfrak{X}_3=\mathrm{Var}(0,F,L)$. By Proposition 2.18 (Chapter 3) there exists an epimorphism in input signals

$$(A',\Gamma,B_1+L)\to(A',F,B_1)\nabla(0,F,L).$$

The product of varieties $\mathfrak{X}'_1\mathfrak{X}_3$ is generated by this triangular product. Therefore $\mathfrak{X}'_1\mathfrak{X}_3=\mathrm{Var}(A',\Gamma,B_1+L)$. Since $(A',\Gamma,B_1+L)\in\mathfrak{Y}_1$, then $\mathfrak{X}'_1\mathfrak{X}_3\subset\mathfrak{Y}_1$ and $\mathfrak{X}'_1\mathfrak{X}_3\mathfrak{Y}_2\subset\mathfrak{Y}_1\mathfrak{Y}_2$. On the other hand, $(A,\Gamma,B)\in\mathfrak{X}'_1\mathfrak{X}_3\mathfrak{Y}_2$ and consequently $\mathfrak{Y}_1\mathfrak{Y}_2\subset\mathfrak{X}'_1\mathfrak{X}_3\mathfrak{Y}_2$. Finally we have

$$\mathfrak{X}'_1\mathfrak{X}_3\mathfrak{Y}_2=\mathfrak{Y}_1\mathfrak{Y}_2.$$

Lemma 4.15. *Let $\mathfrak{X}_1$, $\mathfrak{X}_2$, $\mathfrak{X}$ be varieties of biautomata and $\mathfrak{X}$ differs from the variety of all biautomata. Then*

1) *If $\mathfrak{X}\mathfrak{X}_1=\mathfrak{X}\mathfrak{X}_2$ then $\mathfrak{X}_1=\mathfrak{X}_2$;*

2) *If $\mathfrak{X}_1\mathfrak{X}=\mathfrak{X}_2\mathfrak{X}$ then $\mathfrak{X}_1=\mathfrak{X}_2$.*

Proof. Consider the first case: $\mathfrak{X}\mathfrak{X}_1=\mathfrak{X}\mathfrak{X}_2$. Assume that $\mathfrak{X}_1$ does not lie in $\mathfrak{X}_2$. Take free automata (A,F,B) from $\mathfrak{X}$ and (A_1,F,B_1) from $\mathfrak{X}_1$. Their triangular product $(\tilde{A},\Gamma,\tilde{B})=(A,F,B)\nabla(A_1,F,B_1)$ generates $\mathfrak{X}\mathfrak{X}_1=\mathfrak{X}\mathfrak{X}_2$. Let $\mathfrak{X}_2^*(A_1,F,B_1)=(A'_1,F,B'_1)$. If $A'_1=0$ and $B'_1=0$, then $(A_1,F,B_1)\in\mathfrak{X}_2$ and $\mathfrak{X}_1\subset\mathfrak{X}_2$, that contradicts the assumption. Thus either $A'_1\neq0$ or $B'_1\neq0$. By Lemma 4.13

$$\mathfrak{X}_2^*(\tilde{A},\Gamma,\tilde{B})=\begin{cases}(A+A'_1,\Gamma,B+B'_1), & \text{if } A'_1\neq0\\ (A',\Gamma,B+B'_1),\ A'\subset A, & \text{if } A'_1=0,\ B'_1\neq0\end{cases}$$

a) Let $A'_1\neq0$. Then $\mathfrak{X}_2^*(\tilde{A},\Gamma,\tilde{B})=(A+A'_1,\Gamma,B+B'_1)$. Since $(\tilde{A},\Gamma,\tilde{B})\in\mathfrak{X}\mathfrak{X}_2$, then $\mathfrak{X}_2^*(\tilde{A},\Gamma,\tilde{B})\in\mathfrak{X}$.

By Proposition 2.17 (Chapter 3) there exists an epimorphism in inputs: $(A+A'_1,\Gamma,B+B'_1) \to (A,F,B)\nabla(A'_1,F.B'_1)$. Thus,

$$(A,F,B)\nabla(A'_1,F,B'_1)\in\mathfrak{X}.$$

Let $\mathfrak{X}_0=\mathrm{Var}(A'_1,F,B'_1)$, $A'_1\neq 0$. Then $\mathrm{Var}((A,F,B)\nabla(A'_1,F,B'_1))=\mathfrak{X}\mathfrak{X}_0$. Thus $\mathfrak{X}\mathfrak{X}_0\subset\mathfrak{X}$ and $\mathfrak{X}\mathfrak{X}_0=\mathfrak{X}$. Denote the tuples associated with varieties $\mathfrak{X}$ and $\mathfrak{X}_0$ by (U_1,W,U_2) and (U_1^0,W^0,U_2^0) respectively. The product $\mathfrak{X}\mathfrak{X}_0$ is defined by the product of tuples $(U_1^0,W^0,U_2^0)(U_1,W,U_2)$. Hence $U_1^0U_1=U_1$, that is impossible. So we have the inclusion $\mathfrak{X}_1\subset\mathfrak{X}_2$. In a similar way one can show that $\mathfrak{X}_2\subset\mathfrak{X}_1$, and thus $\mathfrak{X}_1=\mathfrak{X}_2$.

b) Let now $A'_1=0$, $B'_1\neq 0$. Then $\mathfrak{X}_2^*(\tilde{A},\Gamma,\tilde{B})=(A',\Gamma,B+B'_1)$ where $A'\subset A$. It is clear that $\mathfrak{X}_2^*(\tilde{A},\Gamma,\tilde{B})\in\mathfrak{X}$. Since $(A,\Gamma,B)\in\mathfrak{X}$ and $(A',\Gamma,B+B'_1)\in\mathfrak{X}$, then $(A+A',\Gamma,B+B'_1)=(A,\Gamma,B+B'_1)\in\mathfrak{X}$. By the Proposition 2.18 from the Chapter 3 there is an epimorphism on inputs: $(A,\Gamma,B+B'_1) \to (A,F,B)\nabla(0,F,B'_1)$. Hence $(A,F,B)\nabla(0,F,B'_1)\in\mathfrak{X}$. Denote by $\mathfrak{X}_0$ variety $\mathrm{Var}(0,F,B'_1)$. Then $\mathfrak{X}\mathfrak{X}_0\subset\mathfrak{X}$ and $\mathfrak{X}\mathfrak{X}_0=\mathfrak{X}$. Considering tuples corresponding to the given varieties we get the equality $\mathfrak{X}_1=\mathfrak{X}_2$ as in the case a).

The proof of the second statement of the Lemma is similar to this one and uses the description of the radical of biautomaton given in Lemma 4.13.

A variety of automata is called *indecomposable* if it cannot be represented as a product of non-trivial biautomata.

Lemma 4.16. *Let varieties $\mathfrak{X}_1$, $\mathfrak{Y}_1$ be indecomposable ones and varieties $\mathfrak{X}_2$, $\mathfrak{Y}_2$ are different from the variety of all biautomata. Then $\mathfrak{X}_1\mathfrak{X}_1=\mathfrak{Y}_1\mathfrak{Y}_2$ implies $\mathfrak{X}_1=\mathfrak{Y}_1$, $\mathfrak{X}_2=\mathfrak{Y}_2$*

Proof. Suppose that $\mathfrak{X}_2$ does not belong to $\mathfrak{Y}_2$. By Lemma 4.14 there exist non-trivial varieties $\mathfrak{X}_3$ and $\mathfrak{X}'_1\subset\mathfrak{X}_1$, such that $\mathfrak{X}'_1\mathfrak{X}_3\mathfrak{Y}_2=\mathfrak{Y}_1\mathfrak{Y}_2$. This implies $\mathfrak{X}'_1\mathfrak{X}_3=\mathfrak{Y}_1$ that contradicts the indecomposability of $\mathfrak{Y}_1$. Thus $\mathfrak{X}_2\subset\mathfrak{Y}_2$ By similar arguments $\mathfrak{Y}_2\subset\mathfrak{X}_2$ and $\mathfrak{X}_2=\mathfrak{Y}_2$. So, $\mathfrak{X}_1=\mathfrak{Y}_1$

Proof of the theorem 4.4. The latter result implies that indecomposable non-trivial proper varieties generate a free semigroup. To complete the proof of the Theorem it remains to verify that each non-trivial variety of biautomata $\mathfrak{X}$ can be represented as a finite product of indecomposable ones.

Let us introduce the notion of the weight of an ideal. It is known [66] that $\bigcap_{n=1}^{\infty}(KF)^n=0$. Hence, for any nonzero ideal U of KF there exists such integer n that $U\subset(KF)^n$, while U does not belong to $(KF)^{n+1}$. This n is called the weight of the ideal U. If $U=KF^1$ then the weight of U is assumed to be zero. A sum of weights of ideals U_1 and U_2 is called a weight of a tuple (U_1,W,U_2) and of the corresponding variety $\mathfrak{X}$. Let $\mathfrak{X}$ be a non-trivial proper variety of biautomata, $\tau=(U_1,W,U_2)$ be a tuple of identities of the variety $\mathfrak{X}$. Let the weight of $\mathfrak{X}$ be equal to m. Since $U_1\neq0$, $U_2\neq0$ and one of them differs from KF^1, then $m>0$. If $\mathfrak{X}$ cannot be represented as a product of finite number of indecomposable varieties then it can be represented in the form

$$\mathfrak{X}=\mathfrak{X}_1\mathfrak{X}_2\ldots\mathfrak{X}_{m+1}$$

where $\mathfrak{X}_i$ are non-trivial proper varieties. Since the weight of the product is greater or equal to the sum of the weights of the factors, the weight of the given product is greater than m, that contradicts the assumption.

It must be mentioned that the semigroup of varieties of automata is not a free semigroup. Define the product of compatible tuples associated with linear automata by the rule

$$(U_1,V_1)(U_2,V_2)=(U_1U_2,U_1V_2).$$

It is easy to show that the compatible tuples form a semigroup with respect to this multiplication. This semigroup is antiisomorphic to the semigroup of varieties of linear automata. It is not a free one because, for example, holds the equality

$$(U_1,V_1)(U_2,V_2)=(U_1,\tilde{V}_1)(U_2,V_2)$$

which is not true in free semigroup.

The free semigroup of varieties of biautomata can be naturally homomorphically mapped onto the semigroup of varieties of linear automata. In particular, we can construct the canonical homomorphism, which associates a class of linear automata $\mathfrak{X}^0$ to each variety of biautomata $\mathfrak{X}$ by the rule: $(A,\Gamma,B)\in\mathfrak{X}^0$ if $(A,\Gamma,B)\in\mathfrak{X}$ and (A,Γ,B) is a linear automaton, that is if in (A,Γ,B) for each $u\in KF^1$ holds the identity $y\cdot u\equiv 0$. It can be verified that $\mathfrak{X}^0$ is the variety of automata and $(\mathfrak{X}_1\mathfrak{X}_2)^0=\mathfrak{X}_1^0\mathfrak{X}_2^0$. This means that the above mapping is homomorphism. It is clear that this homomorphism is actually epimorphism.

4.4.4. Indecomposable varieties

Since the semigroup of the biautomata varieties is free, the study of indecomposable varieties of biautomata is specially interesting. Now we shall consider several series of such varieties [14]. It is easy to prove the following

Proposition 4.17. *If $\mathfrak{X}^0$ is an indecomposable variety of representations, then the variety $\mathfrak{X}$ generated by all the biautomata of the type* (A,Γ,0) *where* $(A,\Gamma)\in\mathfrak{X}^0$ *is also indecomposable. The variety generated by all the biautomata of the type* (0,Γ,B) *where* $(B,\Gamma)\in\mathfrak{X}^0$ *is indecomposable in a similar way.*

From this Proposition and statement 19.2.1 of [90], in particular, follows,

Proposition 4.18 1) *If the class* θ *consists of the biautomata of the type* (A,Γ,0) *where* (A,Γ) *is an irreducible representation, then* Varθ *is an indecomposable variety of biautomata.*
2) *if* θ *consist of the biautomata* (0,Γ,B) *with the irreducible representation* (B,Γ), *then* Varθ *is also indecomposable.*

The varieties discussed in these Propositions in a certain sense are degenerate varieties. There are also non-degenerate indecomposable varieties of biautomata.

Theorem 4.19. *Let* $\mathfrak{A}$=(A,Γ,B) *be a Moore biautomaton with irreducible representations* (A,Γ) *and* (B,Γ) *and non-degenerate operation* ∗. *Then the variety* Var(A,Γ,B) *is indecomposable. (Non-degeneracy of the operation ∗ means that the identity* $z*x=0$ *is not satisfied in the automaton* (A,Γ,B)).

Proof. Assume that the variety $\mathfrak{X}$=Var(A,Γ,B) is decomposable, that is $\mathfrak{X}=\mathfrak{X}_1\mathfrak{X}_2$ where the varieties $\mathfrak{X}_1$, $\mathfrak{X}_2$ are non-trivial. Then in the biautomaton $\mathfrak{A}=(A,\Gamma,B)$ there exists the subautomaton $\mathfrak{A}_1$ from $\mathfrak{X}_1$, such that $\mathfrak{A}/\mathfrak{A}_1$ lies in $\mathfrak{X}_2$. Since the representations (A,Γ) and (B,Γ) are irreducible, the biautomaton $\mathfrak{A}_1$ coincides with one of the following automata: $(0,\Gamma,0)$, $(0,\Gamma,B)$, $(A,\Gamma,0)$, (A,Γ,B). The cases $\mathfrak{A}_1=(0,\Gamma,0)$ or $\mathfrak{A}_1=(A,\Gamma,B)$ fail away, since the varieties $\mathfrak{X}_1$ and $\mathfrak{X}_2$ must be nontrivial.

If $\mathfrak{A}_1=(A,\Gamma,0)$, then $\mathfrak{A}/\mathfrak{A}_1\cong(0,\Gamma,B)$. In the triangular product $(A,\Gamma,0)\nabla(0,\Gamma,B)$ the operation $*$ is a degenerate one. Since the biautomaton (A,Γ,B) is embedded into this product, then the operation $*$ is degenerate also in (A,Γ,B), that contradicts the assumption.

Let, finally, $\mathfrak{A}_1=(0,\Gamma,B)$. Then $\mathfrak{A}/\mathfrak{A}_1\cong(A,\Gamma,0)$. Consider the triangular product $(0,\Gamma,B)\nabla(A,\Gamma,0)=(A,\Sigma,B)\in\mathfrak{X}_1\mathfrak{X}_2$. If the element $u\in KF^1$ lies in the intersection of the ideals of identities of the representations (A,Γ) and (B,Γ), then the identity $z*u\equiv 0$ is satisfied in the biautomaton (A,Γ,B) (since (A,Γ,B) is a Moore biautomaton with a certain determining mapping ψ and by the definition of such automaton $a*\gamma=(a\circ\gamma)\psi-a\psi\cdot\gamma$). Hence, this identity is also satisfied in the variety $\mathfrak{X}=\mathrm{Var}(A,\Gamma,B)$. On the other hand, it is clear that it is not satisfied in the automaton (A,Σ,B) (it suffices to consider the matrix representation of the triangular product). The obtained contradiction implies that $\mathfrak{X}$ is indecomposable.

Proposition 4.20. *If $\mathfrak{A}=(A,\Gamma,B)$ is a biautomaton with (A,Γ) or (B,Γ) being an exact representation of the finite group of exponent n over the field of the characteristic zero, then the variety $\mathrm{Var}\mathfrak{A}$ is indecomposable.*

Proof. To prove the theorem we first develop one general idea.

Let $f_1\equiv f_2$ be an identity of the semigroup Γ, (A,Γ) a certain exact representation and $\mathfrak{A}=(A,\Gamma,B)$ a biautomaton with operation $\circ$ defined by the representation (A,Γ); naturally, it is also exact. Let the variety $\mathfrak{X}$ generated by this biautomaton be decomposable: $\mathfrak{X}=\mathfrak{X}_1\mathfrak{X}_2$. Then there is such subautomaton $\mathfrak{A}_1\subset\mathfrak{A}$, $\mathfrak{A}_1\in\mathfrak{X}$, that $\mathfrak{A}/\mathfrak{A}_1$ lies in $\mathfrak{X}_2$. Denote by (A_1,Γ_1,B_1) and (A_2,Γ_2,B_2) exact biautomata, corresponding to $\mathfrak{A}_1$ and $\mathfrak{A}/\mathfrak{A}_1$ respectively. (It is clear that in this case $A_1\oplus A_2\cong A$, $B_1\oplus B_2\cong B$). Consider, finally, the triangular product $(A_1,\Gamma_1,B_1)\nabla(A_2,\Gamma_2,B_2)=(A,\Sigma,B)\in\mathfrak{X}_1\mathfrak{X}_2$

where $\Sigma=\Gamma_1\times\mathrm{Hom}(A_2,A_1)\times \mathrm{Hom}(A_2,B_1)\times\mathrm{Hom}(B_2,B_1)\times\Gamma_2$.

If (C,Φ,D) is an exact biautomaton from $\mathrm{Var}(A,\Gamma,B)$, then the semigroup Φ should also satisfy the identity $f_1\equiv f_2$; this, in particular, relates to the considered biautomaton (A,Σ,B) and semigroup Σ. On the other hand, if the identity $f_1\equiv f_2$ is satisfied in Σ, then, since the biautomaton (A,Γ,B) is embedded as a subautomaton in (A,Σ,B) (Theorem 2.1, Chapter 3), it must be satisfied also in Γ, that is, Γ and Σ generate the same semigroup variety.

It follows that if Γ and Σ generate different varieties of the semigroups, then the variety of biautomata $\mathfrak{X}$ is indecomposable.

Return to the statement of the Proposition taking into account the above considerations.

The group Γ has the exponent n, hence, the identity $x^n\equiv1$ is satisfied in it. On the other hand, this identity is not satisfied in the group Σ, since Σ contains a subgroup consisting of all elements of the form $\{1,\varphi_1,\psi,\varphi_2,1\}$, where $\varphi_1\in\mathrm{Hom}(A_2,A_1)$, $\varphi_2\in\mathrm{Hom}(B_2,B_1)$, $\psi\in\mathrm{Hom}(A_2,B_1)$. This subgroup is isomorphic to a certain subgroup of the group of triangular matrices with the units on the main diagonal (over the field of characteristic zero). The latter is a nilpotent group without torsion.

Thus, the identity $x^n\equiv1$, satisfied in the group Γ, is not satisfied in Σ, hence, they generate different varieties. In accordance with the remark made above this implies that the variety $\mathfrak{X}$ is indecomposable.

From the last Proposition follows, in particular, that if (A,Γ) and (B,Γ) are the representations given in this Proposition, then $\mathrm{Var}(\mathrm{Atm}^3(A,\Gamma))$ and $\mathrm{Var}(\mathrm{Atm}^2(\Gamma,B))$ are indecomposable.

For comparison it is useful to note that since the first universal biautomaton $\mathrm{Atm}^1(A,B)=(A,\mathrm{End}(A,B),B)$ can be represented in the form of the triangular product

$$\mathrm{Atm}^1(A,B)=(0,\mathrm{End}B,B)\nabla(A,\mathrm{End}A,0),$$

the variety $\mathrm{Var}[\mathrm{Atm}^1(A,B)]$ is always decomposable:

$$\mathrm{Var}(\mathrm{Atm}^1(A,B)=\mathrm{Var}(0,\mathrm{End}B,B)\mathrm{Var}(A,\mathrm{End}A,0).$$

4.4.5. Example

Let us apply results of this item for the description of the tuple of identities of the universal biautomaton $Atm^1(A,B)$. Repeat once more that the biautomaton $Atm^1(A,B)=(A,EndA\oplus Hom(A,B)\oplus EndB,B)$ is a triangular product $\mathfrak{A}_2\nabla\mathfrak{A}_1$ of the biautomata $\mathfrak{A}_1=(A,EndA,0)$ and $\mathfrak{A}_2=(0,EndB,B)$.

It is evident that the tuple of identities of a biautomaton coincides with the tuple of identities of a biautomata variety generated by it. By Theorem 4.3 the variety of biautomata generated by the triangular product of arbitrary biautomata $\mathfrak{B}_1\nabla\mathfrak{B}_2$ is equal to the product of the varieties $\mathfrak{X}_1\mathfrak{X}_2$ generated by the biautomata $\mathfrak{B}_1$ and $\mathfrak{B}_2$, respectively. By Theorem 4.2 if $(U_1^{(1)},W^{(1)},U_2^{(1)})$ and $(U_1^{(2)},W^{(2)},U_2^{(2)})$ are tuples of identities of the biautomata $\mathfrak{B}_1$ and $\mathfrak{B}_2$ (or, what is equivalent, are the tuples of identities of the varieties $\mathfrak{X}_1$ and $\mathfrak{X}_2$), then the tuple of identities of the variety $\mathfrak{X}_2\mathfrak{X}_1$, and, hence, of the biautomaton $\mathfrak{B}_2\nabla\mathfrak{B}_1$ has the form

$$(U_1,W,U_2)=(U_1^{(1)}U_1^{(2)},U_1^{(1)}W^{(2)}+W^{(1)}\cdot U_2^{(2)},U_2^{(1)}U_2^{(2)}).$$

Now denote by U_1,U_2 the ideals of identities of the representations $(A,EndA)$, $(B,EndB)$ respectively. Then the tuples of identities of the biautomata $\mathfrak{A}_1$ and $\mathfrak{A}_2$ will have the form

$$(U_1^{(1)},W^{(1)},U_2^{(1)})=(U_1,KF^1\otimes KF,KF^1),$$

$$(U_1^{(2)},W^{(2)},U_2^{(1)})=(KF^1,KF^1\otimes KF,U_2).$$

Taking into account the previous passage we get that the tuple of identities of the biautomaton $Atm^1(A,B)=\mathfrak{A}_2\nabla\mathfrak{A}_1$ has the form

$$(U_1,U_1\otimes KF\oplus(KF^1\otimes KF)\cdot U_2,U_2).$$

In particular, for the biautomaton $Atm^1(A,B)$ the set W of the elements t for which there is an identity of the form $zt\equiv 0$, is equal to

$$W=(U_1\otimes KF)\oplus(KF^1\otimes KF)\cdot U_2. \qquad (4.1)$$

Let $\mathfrak{X}$ be a variety of biautomata. A completely characteristic subautomaton $(U_1, F, W+U_2)$ of the free cyclic biautomaton corresponds to the tuple (U_1, W, U_2) of identities of this variety. Denote, as earlier, by V the set of the elements $u \in KF$, for which the identity $z*u \equiv 0$ is satisfied, and by $\tilde{W}$ the following subspace of the tensor product $KF^1 \otimes KF$

$$\tilde{W}=U_1 \otimes KF+KF^1 \otimes V+(KF^1 \otimes V)\cdot F+(KF^1 \otimes KF)\cdot U_2. \tag{4.2}$$

It is clear, that $\tilde{W} \subset W$. The least completely characteristic F-subautomaton in the free cyclic biautomaton generated by the sets $U_1 \subset H$, $1 \otimes V \subset KF^1 \otimes KF$ and $U_2 \subset G$, is a biautomaton $(U_1, F, \tilde{W} \oplus U_2)$. Since this automaton is a completely characteristic one, then $(U_1, \tilde{W}, U_2)$ is a tuple of identities for a certain variety $\tilde{\mathfrak{X}}$. As we know (see Section 4.3), $\tilde{W}$ can be less than W. The equalities (4.1) and (4.2) show that for the biautomaton $Atm^1(A,B)$ the equality $\tilde{W}=W$ takes place. It means, in particular, that $(U_1, 1 \otimes V, U_2)$ is the basis of identities of the given biautomaton, that is, the set of identities defining the given variety. Consider in more detail the construction of V and prove that in the given case $V=U_1U_2$.

Since $V \subset KF$, $F=F(X)$, then arbitrary element $v \in V$ has the form

$$v=\sum_i \lambda_i x_{i_1} x_{i_2} \dots x_{i_{n_i}}, \lambda_i \in K, \ x_{i_k} \in X \tag{4.3}$$

Grouping the summands by the last factor it is possible to write v in the form $v=\sum_i u_i x_i$. At first show that each u_i is not empty. For this u_i take such element φ of Hom(A,B), that $a\varphi \neq 0$ under a certain $a \in A$, and consider the mapping $\alpha: X \rightarrow End(A,B)$

$$x_j^{\alpha}=\begin{cases}(0,0,0), & \text{if } j\neq i\\ (0,\varphi,0), & \text{if } j=i, \ x_j \in X\end{cases}$$

Since the semigroup F is free, this mapping can be extended up to the homomorphism $F \rightarrow End(A,B)$ and further to homomorphism of algebras $\tilde{\alpha}: KF \rightarrow End(A,B)$. If u_i were empty then under the appropriate $a \in A$, $a*v^{\tilde{\alpha}}=a*x_i^{\tilde{\alpha}}=a\varphi \neq 0$, but it contradicts the definition of V. Hence, each u_i is not empty. Now show that for any homomorphism $\mu: F \rightarrow End(A,B)$ and any

$a \in A$ the equality $(a \circ u_i^{\mu}) * x_i^{\mu} = 0$ takes place. Let $x_j^{\mu} = (\varphi_j, \delta_j, \psi_j)$. By μ and fixed i construct the mapping $\mu_i : X \rightarrow End(A,B)$:

$$x_j^{\mu_i} = \begin{cases} (\varphi_j, 0, 0) & \text{, if } j \neq i \\ (\varphi_j, \delta_j, 0) & \text{, if } j = i \end{cases}$$

Then

$$a * v^{\mu_i} = a * (\sum_j u_j x_j)^{\mu_i} = \sum_j a * (u_j x_j)^{\mu_i} = \sum_j ((a \circ u_j^{\mu_i}) * x_j^{\mu_i} + (a * u_j^{\mu_i}) \cdot x_j^{\mu_i}) = (a \circ u_i^{\mu_i}) * x_i^{\mu_i} =$$

$(a \circ u_i^{\mu}) * x_i^{\mu}$. Since $v \in V$, then $a * v^{\mu_i} = (a \circ u_i^{\mu}) * x_i^{\mu} = 0$.

Finally, show that $u_i \in U_1$. Suppose that $u_i \notin U_1$ and that $a \circ u_i^{\mu} \neq 0$ under certain a and μ. Denote $a \circ u_i^{\mu} = a'$. Let again $x_j^{\mu} = (\varphi_j, \delta_j, \psi_j)$. Take $\tilde{\mu} : X \rightarrow End(A,B)$, such that $x_i^{\tilde{\mu}} = (\varphi_i, \delta, 0)$, where $a'\delta \neq 0$. Then $(a \circ u_i^{\tilde{\mu}}) * x_i^{\tilde{\mu}} = (a \circ u_i^{\mu}) * x_i^{\tilde{\mu}} = a' * x_i^{\tilde{\mu}} = a'\delta \neq 0$ that contradicts to what has been proved in the previous passage. Thus, each $u_i \in U_1$, and since U_1 is a two-sided ideal, then also $v \in U_1$.

In order to prove the inclusion $v \in U_2$ group the summands of the element v in the notation (4.3) by the first factor: $v = \sum_i x_i v_i$. The same arguments as above imply that all v_i are not empty; using constructions dual relatively to the previous ones it is possible to show that for any homomorphism $\mu : F \rightarrow End(A,B)$ and for any $a \in A$ the equality $(a * x_i^{\mu}) \cdot v_i^{\mu} = 0$ is valid. Since $a * x_i^{\mu}$ can be arbitrary element of B, the latter equality implies that $v_i \in U_2$. Therefore, $v \in U_2$. It has been proved that $V \subset U_1 \cap U_2$. By Lemma 3.4 it means that the set of identities of the representation $(A \oplus B, End(A,B))$ is equal to V. Using this fact, show that $V = U_1 U_2$.

The representation $(A \oplus B, End(A,B))$ is a triangular product $\mathfrak{A}_2 \nabla \mathfrak{A}_1$ of the representations $\mathfrak{A}_1 = (A, End A)$ and $\mathfrak{A}_2 = (B, End B)$; U_1, U_2 are ideals of identities of the representations $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively. If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are representations of semigroups, then $Var(\mathfrak{A}_2 \nabla \mathfrak{A}_1) = Var\mathfrak{A}_2 Var\mathfrak{A}_1$. Since U_1, U_2 are ideals of identities of the representations $\mathfrak{A}_1, \mathfrak{A}_2$, and hence, of the varieties $Var\mathfrak{A}_1$, $Var\mathfrak{A}_2$, then ([90]) $U_1 U_2$ is an ideal of identities of the variety $Var\mathfrak{A}_2 Var\mathfrak{A}_1 = Var(\mathfrak{A}_2 \nabla \mathfrak{A}_1)$, and therefore, also of the representation $\mathfrak{A}_2 \nabla \mathfrak{A}_1 = (A \oplus B, End(A,B))$. Thus, $V = U_1 U_2$.

Finally, we have that the basis of identities of the universal biautomaton $Atm^1(A,B)$ has the form $(U_1, 1\otimes U_1U_2, U_2)$.

4.5. Quasivarieties of automata

4.5.1. Quasiidentities and quasivarieties

Along with varieties a great attention in algebra is paid to the quasivarieties. In particular, together with varieties of automata the quasivarieties of automata can be considered. For instance, such important class of automata as a class of Moore automata is a quasivariety of automata.

Let $\mathcal{F}=(H,F,\Phi)$ be a free pure or linear automaton. Elements u,v are called the elements of the same sort if they both lie in one and the same set of H, F or Φ. Let u_i, v_i be the elements of the same sort for each $i=1,\dots,n+1$. We say that a quasiidentity

$$u_1=v_1 \wedge u_2=v_2 \wedge \dots \wedge u_n=v_n \Rightarrow u_{n+1}=v_{n+1} \qquad (5.1)$$

is satisfied in the arbitrary automaton $\mathfrak{A}=(A,\Gamma,B)$, if for any homomorphism $\mu: F \to \mathfrak{A}$ the simultaneous fulfillment of the equalities $u_i^{\mu}=v_i^{\mu}$ for all $i=1,\dots,n$, implies the equality $u_{n+1}^{\mu}=v_{n+1}^{\mu}$ This definition relates both to pure and linear automata. A class of automata satisfying a certain set of quasiidentities is called *quasivariety of automata*.

Examples: 1) The quasiidentity $z_1\circ x_1=z_2\circ x_2 \Rightarrow z_1*x_1=z_2*x_2$ defines the quasivariety of Moore pure automata.

2) Quasiidentities of the form $\sum_i^n \alpha_i z_i \circ x_i=0 \Rightarrow \sum_i^n \alpha_i z_i * x_i=0$ define the quasivariety of Moore linear automata. It is clear that the number of such quasiidentities is infinitely great. The question of interest is whether the quasivariety of Moore linear automata is defined by a finite set of quasiidentities.

Note that Moore biautomata do not constitute a quasivariety.

Similar to varieties of automata, quasivarieties of automata allow the definition not connected with free objects. First cite the necessary notions. Let $\mathcal{K}$ be a certain category of automata. Its terminal object, that is, such automaton $\mathcal{E}\in\mathcal{K}$, that for any automaton $\mathfrak{A}\in\mathcal{K}$ there exists a unique homomorphism from $\mathfrak{A}$ into $\mathcal{E}$ is called a unit element of this category. It is evident that for $\mathcal{K}$ being a category of linear semigroup automata, $\mathcal{E}=(0,\{1\},0)$.

Let J be an arbitrary set, $\mathcal{D}$ be a set of non-empty subsets of J satisfying the conditions:

1) the intersection of two elements of $\mathcal{D}$ is again an element of $\mathcal{D}$,
2) if A is a subset of J, $A \supset B$, and B belongs to $\mathcal{D}$, then A also belongs to $\mathcal{D}$.

This $\mathcal{D}$ is called a *filter* on J. Let $\mathfrak{A}_i=(A_i,\Gamma_i,B_i)$, $i \in J$ be a set of automata. Define the filtered product of these automata by the filter $\mathcal{D}$. Take a Cartesian product of the automata $\mathfrak{A}=(A,\Gamma,B)=\prod_{i=J}\mathfrak{A}_i=(\prod_{i=J}A_i, \prod_{i\in J}\Gamma_i, \prod_{i\in J}B_i)$; associate to the filter $\mathcal{D}$ the congruence $\rho=(\rho_1,\rho_2,\rho_3)$ of the automata $\mathfrak{A}$: if $a_k \in A$, $\gamma_k \in \Gamma$, $b_k \in B$, $k=1,2$ then

$$
\begin{aligned}
a_1\rho_1 a_2 &\Leftrightarrow \{i \mid a_1(i)=a_2(i)\}\in\mathcal{D},\\
\gamma_1\rho_2\gamma_2 &\Leftrightarrow \{i \mid \gamma_1(i)=\gamma_2(i)\}\in\mathcal{D},\\
b_1\rho_3 b_3 &\Leftrightarrow \{i \mid b_1(i)=b_2(i)\}\in\mathcal{D}.
\end{aligned}
\tag{5.2}
$$

The quotient automaton $\mathfrak{A}/\rho=(A/\rho_1,\Gamma/\rho_2,B/\rho_3)$ is called a *filtered product of the automata* $\mathfrak{A}_i$, $i \in J$ by the filter $\mathcal{D}$; it is denoted by $\prod_{i\in J}\mathfrak{A}_i/\mathcal{D}$.

Recall that the class $\mathfrak{X}$ of automata is called hereditary if each subautomaton of the automaton of $\mathfrak{X}$ also lies in $\mathfrak{X}$. The following theorem presents an invariant characteristic of a quasivariety of automata.

Theorem 5.1. *A class of automata is a quasivariety if and only if it is closed with respect to filtered products, is a hereditary one and contains a unit automaton.*

The further considerations of this Section are devoted to quasivarieties of the automata saturated in input and output signals, as well as to some relations between quasivarieties of automata and quasivarieties of semigroups. Under the term "automaton" in this Section we shall understand a linear semigroup automaton, but it is necessary to note that most of the facts remain true also for pure automata.

4.5.2. Quasivarieties of automata saturated in input signals

The class $\mathfrak{X}$ of automata is called saturated in input signals if for each epimorphism in inputs $(A,\Gamma_1,B) \to (A,\Gamma_2,B)$ the inclusion $(A,\Gamma_1,B)\in\mathfrak{X}$ is equivalent to the inclusion $(A,\Gamma_2,B)\in\mathfrak{X}$.

Consider at first the category of Γ-automata. A unit automaton in this category is the automaton $(0,\Gamma,0)$, and a free automaton with the generators Z, Y has the form $\mathcal{F}_\Gamma=(Z\circ K\Gamma^1,\Gamma,Z*K\Gamma\otimes KY)$. Modifying the expression (5.1), we can say, that Γ-*quasiidentity* of the linear Γ-automaton $\mathfrak{A}$ is a formula

$$u_1=0\wedge u_2=0\wedge\ldots\wedge u_n=0 \Rightarrow v=0 \qquad (5.3)$$

where u_i, v belong either to $Z\circ K\Gamma^1$ or to $Z*K\Gamma\otimes KY$, and for any homomorphism $\mu:\mathcal{F}_\Gamma \to \mathfrak{A}$ identical on Γ (i.e. Γ-homomorphism) the simultaneous fulfillment of $u^\mu=0$ implies $v^\mu=0$. *Quasivariety of* Γ *automata* is a class of Γ-automata satisfying a certain set of Γ-quasiidentities. A filtered product of Γ-automata is defined similar to that of automata; as ρ_2 in $\rho=(\rho_1,\rho_2,\rho_3)$ we must take a trivial (minimal) congruence of the semigroup Γ. The theorem analogous to Theorem 5.1 holds for quasivarieties of Γ-automata: a class of Γ-automata is a quasivariety if and only if it is closed with respect to the filtered products, contains a unit Γ-automaton and is hereditary on Γ-subautomata.

For an arbitrary class of automata $\mathfrak{X}$ and arbitrary semigroup Γ denote by $\mathfrak{X}_\Gamma$ the class of all the Γ-automata from $\mathfrak{X}$.

Theorem 5.2. *If $\mathfrak{X}$ is a quasivariety of automata saturated in inputs, then for any semigroup Γ the class $\mathfrak{X}_\Gamma$ is a quasivariety of Γ-automata. Conversely, if each class $\mathfrak{X}_\Gamma$ of hereditary and saturated in inputs class of automata $\mathfrak{X}$ is a quasivariety, then $\mathfrak{X}$ is a quasivariety of automata.*

Proof. Let $\mathfrak{X}$ be a quasivariety of automata saturated in inputs. By Theorem 5.1 it contains a unit automaton $(0,1,0)$. For an arbitrary group Γ the mapping $(0,\Gamma,0) \to (0,1,0)$ is an epimorphism in inputs. Hence, $(0,\Gamma,0)$ belongs to $\mathfrak{X}$, and therefore, to $\mathfrak{X}_\Gamma$. Thus, $\mathfrak{X}_\Gamma$ contains a unit Γ-automaton. Closeness of the class $\mathfrak{X}_\Gamma$ in respect to Γ-automata immediately follows from the heredity of the class $\mathfrak{X}$. It remains to verify the closeness of $\mathfrak{X}_\Gamma$ with respect to filtered products of Γ-automata. Let J be a set, $\mathcal{D}$ a filter on J, $\mathfrak{A}_i=(A_i,\Gamma_i,B_i)$, $i\in J$, automata of $\mathfrak{X}_\Gamma$. The filtered product of the automata $(\prod_{i\in J} A_i/\rho_1,\bar{\Gamma}/\rho_2, \prod_{i\in J} B_i/\rho_3)$ belongs to $\mathfrak{X}$ since the quasivariety is closed with respect to filtered products. Here $\bar{\Gamma}=\prod_{i\in J}\Gamma_i$; $\Gamma_i=\Gamma$. By virtue of the saturation of the quasivariety $\mathfrak{X}$ in inputs, the automaton $(\prod_{i\in J} A_i/\rho_1,\bar{\Gamma}, \prod_{i\in J} B_i/\rho_3)$ also belongs to $\mathfrak{X}$. For each $\gamma\in\Gamma$ consider the function $\bar{\gamma}\in\bar{\Gamma}$ such that $\bar{\gamma}(i)=\gamma$ for any $i\in J$. Thus, the semigroup Γ is embedded into $\bar{\Gamma}$ as a semigroup of the constant functions. As a result the filtered product of Γ-automata $(\prod_{i\in J} A_i/\rho_1,\Gamma, \prod_{i\in J} B_i/\rho_3)$ belongs to $\mathfrak{X}$ as a subautomaton of the automaton $(\prod_{i\in J} A_i/\rho_1,\bar{\Gamma}, \prod_{i\in J} B_i/\rho_3)$ from $\mathfrak{X}$ and, therefore, it also belongs to $\mathfrak{X}_\Gamma$. Thus, the class $\mathfrak{X}_\Gamma$ is closed with respect to the filtered products of Γ-automata. Hence, $\mathfrak{X}_\Gamma$ is a quasivariety of Γ-automata.

Conversely. Since for each Γ the class $\mathfrak{X}_\Gamma$ is a quasivariety of Γ-automata, then $\mathfrak{X}_\Gamma$ contains the automaton $(0,\Gamma,0)$, and by virtue of saturation of $\mathfrak{X}$, it contains its epimorphic image $(0,1,0)$, that is, a unit automaton. Heredity of the class $\mathfrak{X}$ is given by the condition of the theorem. It remains to verify the closeness of $\mathfrak{X}$ with respect to filtered products. Let $(\prod_{i\in J} A_i/\rho_1, \prod_{i\in J}\Gamma_i/\rho_2, \prod_{i\in J} B_i/\rho_3)$, $i\in J$, be a filtered product of automata $\mathfrak{A}_i=(A_i,\Gamma_i,B_i)$ by a filter $\mathcal{D}$ and let $\prod_{i\in J}\Gamma_i=\bar{\Gamma}$. For each $i\in J$ the projection $\bar{\Gamma}\to\Gamma_i$ defines the automaton $(A_i,\bar{\Gamma},B_i)$ with the epimorphism in inputs $(A_i,\bar{\Gamma},B_i)\to(A_i,\Gamma_i,B_i)$. By virtue of saturation we get $(A_i,\bar{\Gamma},B_i)\in\mathfrak{X}$. Consider the class $\mathfrak{X}_{\bar{\Gamma}}$. Since $\mathfrak{X}_{\bar{\Gamma}}$ is a quasivariety of $\bar{\Gamma}$-automata, then the filtered product of $\bar{\Gamma}$-automata $(\prod_{i\in J} A_i/\rho_1,\bar{\Gamma}, \prod_{i\in J} B_i/\rho_3)$ belongs to $\mathfrak{X}_{\bar{\Gamma}}$ and thus also to $\mathfrak{X}$. The filtered product of the automata $\prod_{i\in J}\mathfrak{A}_i/\mathcal{D}=(\prod_{i\in J} A_i/\rho_1,\bar{\Gamma}/\rho_2, \prod_{i\in J} B_i/\rho_3)$ is an epimorphic in inputs image of the last automaton and due to saturation of the class $\mathfrak{X}$, belongs to it. The Theorem is proved.

Let us say that the quasiidentity of the form (5.1) does not contain a semigroup part if for each $i\in\{1,2,\dots,n+1\}$ elements u_i, v_i do not belong to F. It will be shown now that quasivarieties saturated in inputs are defined by the quasiidentities without the semigroup part.

Theorem 5.3. *A quasivariety of automata $\mathfrak{X}$ can be defined by the quasiidentities without the semigroup part if and only if it is saturated in inputs.*

Proof. Sufficiency. Let $\mathfrak{X}$ be a quasivariety of automata saturated in inputs and F=F(X) be a free semigroup over a countable set X. By Theorem 5.2 the class $\mathfrak{X}_F$ is F-quasivariety. Its quasiidentities are F-quasiidentities; they are associated with the free F-automaton $\mathcal{F}=(Z\circ KF^1, F, Z*KF\oplus KY)$: each of them has the form

$$(u_1=0)\wedge(u_2=0)\wedge\dots\wedge(u_n=0) \Rightarrow v=0 \qquad (5.4)$$

where u_i, v_i are elements of $Z\circ KF$ or $Z*KF\oplus KY$. Recall that the F-quasiidentity (5.4) is satisfied on the F-automaton if for any F-homomorphism $\mu:\mathcal{F} \to \mathfrak{A}$ from the equalities $u_i^{\mu}=0$, $i=1,\dots,n$ follows the equality $v^{\mu}=0$. Show that if we consider (5.4) as quasiidentities (not as F-quasiidentities) then the same set defines the quasivariety $\mathfrak{X}$. Preliminary make one remark.

Let the automaton (A,Γ,B) and arbitrary mappings $\mu_1:Z \to A$, $\mu_2:X \to \Gamma$, $\mu_3:Y\to B$ be given. Extend them to the homomorphism $\mu:\mathcal{F} \to (A,\Gamma,B)$. Denote $F^{\mu_2}=\Gamma_0$. Then we get the subautomaton (A,Γ_0,B) in (A,Γ,B). By μ_2 define the automaton (A,F,B): $a\circ f=a\circ f^{\mu_2}$; $a*f=a*f^{\mu_2}$ The mapping $(A,F,B) \to (A,\Gamma_0,B)$ is an epimorphism in inputs. The mapping $\mu'=(\mu_1,\varepsilon,\mu_3)$ of F-automaton $\mathcal{F}$ in (A,F,B) is a homomorphism. In this case the equality $u^{\mu}=u^{\mu'}$ is satisfied for each element u of $Z\circ KF'$ or of $Z*KF^1\oplus KY$.

Let us return to the proof of sufficiency. Let $(A,\Gamma,B)\in\mathfrak{X}$. Verify that each quasiidentity of the form (5.4) is satisfied in this automaton. We use the notations from the given remark. The subautomaton (A,Γ_0,B) is contained in the class $\mathfrak{X}$ together with (A,Γ,B). By virtue of saturation the corresponding F-automaton (A,F,B) belongs to $\mathfrak{X}$, to be more precise, to the class $\mathfrak{X}_F$ and, therefore, the F-quasiidentities of the form (5.4) are satisfied in this automaton. Let, as before, μ be an arbitrary homomorphism $\mathcal{F} \to (A,\Gamma,B)$ and μ' be a homomorphism $\mathcal{F} \to (A,F,B)$ corresponding to it. Assume that there are the equalities $(u_1^{\mu}=0) \wedge (u_2^{\mu}=0) \wedge \ldots \wedge (u_n^{\mu}=0)$. Then by virtue of the remark we have the equalities $(u_1^{\mu'}=0) \wedge (u_2^{\mu'}=0) \wedge \ldots \wedge (u_n^{\mu'}=0)$. Since the F-automaton (A,F,B) belongs to the F-quasivariety $\mathfrak{X}_F$, then from the latter equalities follows $v^{\mu'}=0$. But then $v^{\mu}=v^{\mu'}=0$. Thus, for an arbitrary homomorphism μ the equalities $u_i^{\mu}=0$, $i \in 1,\ldots,n$ imply the equality $v^{\mu}=0$, that is, quasiidentities of the form (5.4) are satisfied in the class $\mathfrak{X}$. Conversely, let quasiidentities of the form (5.4) be satisfied in (A,Γ,B). Show that $(A,\Gamma,B) \in \mathfrak{X}$. Take an epimorphism $\nu: F \to \Gamma$. Then as earlier, it can be shown that F-quasiidentities of the form (5.4) are satisfied in (A,F,B) and, therefore, $(A,F,B) \in \mathfrak{X}_F \subset \mathfrak{X}$.

The mapping $(\varepsilon_A, \nu, \varepsilon_B): (A,F,B) \to (A,\Gamma,B)$ is an epimorphism in inputs and by virtue of the saturation of $\mathfrak{X}$ we have $(A,\Gamma,B) \in \mathfrak{X}$.

Necessity. Let the quasivariety $\mathfrak{X}$ be defined by the quasiidentities of the form (5.4) without the semigroup part. Prove its saturation in inputs. Define an arbitrary epimorphism in inputs

$$\tilde{\nu}=(\varepsilon_A, \nu, \varepsilon_B): (A,\Gamma,B) \to (A,\Sigma,B).$$

Let $(A,\Gamma,B) \in \mathfrak{X}$. By the homomorphism $\mu=(\mu_1,\mu_2,\mu_3): \mathcal{F} \to (A,\Sigma,B)$ construct the homomorphism $\mu_2': F \to \Gamma$ with the property $\mu_2=\mu_2'\nu$. For arbitrary $x \in X$ choose such element $\gamma \in \Gamma$, that $x^{\mu_2}=\gamma^{\nu}$. Denote $x^{\mu_2'}=\gamma$. Thus we define the mapping $\mu_2': X \to \Gamma$ which can be uniquely extended up to the necessary homomorphism $\mu_2': F \to \Gamma$. From the definition of μ_2' follows that $\mu'=(\mu_1,\mu_2',\mu_3): \mathcal{F} \to (A,\Gamma,B)$ is a homomorphism and that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\quad\mu\quad} & (A,\Sigma,B) \\ & {}_{\mu'}\searrow \quad \swarrow_{\tilde{\nu}} & \\ & (A,\Gamma,B) & \end{array}$$

It is clear that for any element u from $Z\circ KF^1$ or $Z*KF\oplus KY$, the equality $u^{\mu}=u^{\mu'}$ takes place.

If now the assumptions of the implications (5.4) hold in (A,Σ,B):

$$(u_1^{\mu}=0)\wedge(u_2^{\mu}=0)\wedge\ldots\wedge(u_n^{\mu}=0)$$

then in (A,Γ,B) the equalities

$$(u_1^{\mu'}=0)\wedge(u_2^{\mu'}=0)\wedge\ldots\wedge(u_n^{\mu'}=0)$$

are satisfied. From these equalities follows $v^{\mu'}=0$ (since $(A,\Gamma,B)\in\mathfrak{X}$). Then $v^{\mu}=v^{\mu'}=0$ in (A,Σ,B); therefore, $(A,\Sigma,B)\in\mathfrak{X}$.

Let now $(A,\Sigma,B)\in\mathfrak{X}$. Defining the homomorphism $\mu':\mathcal{F}\to(A,\Gamma,B)$ by the rule $\mu'=\mu\tilde{\nu}$ we get the equality $u^{\mu}=u^{\mu'}$ for any u from $Z\circ KF^1$ or $Z*KF\oplus KY$. Arguing as above, we obtain $(A,\Gamma,B)\in\mathfrak{X}$. The Theorem is proved.

4.5.3. Quasivarieties of automata saturated in output signals

The class of automata $\mathfrak{X}$ is called *saturated in output signals* if it satisfies the condition: for the arbitrary automaton (A,Γ,B) and its subautomaton of the form (A,Γ,B_0) the inclusion $(A,\Gamma,B)\in\mathfrak{X}$ is equivalent to the inclusion $(A,\Gamma,B_0)\in\mathfrak{X}$.

The condition of saturation in output signals is connected with the exclusion of variables of the set Y from the quasiidentities (recall that (Z,X,Y) is a system of free generators of the free automaton $\mathcal{F}$).

Proposition 5.4. *If in quasiidentities defining the quasivariety of automata $\mathfrak{X}$ variables of* Y *are absent, then this quasivariety is saturated in output signals.*

Proof. Let $\mathfrak{X}$ be defined by the quasiidentities without variables of Y, (A,Γ,B) be a certain automaton in which there is a subautomaton (A,Γ,B_0) belonging to $\mathfrak{X}$. Show that $(A,\Gamma,B)\in\mathfrak{X}$. For this, it is necessary to verify that each quasiidentity

$$u_1=v_1\wedge u_2=v_2\wedge\ldots\wedge u_n=v_n\Rightarrow u_{n+1}=v_{n+1}$$

without variables of Y, defining the class $\mathfrak{X}$, is satisfied also in (A,Γ,B). Define the homomorphism $\mu:\mathcal{F}\to(A,\Gamma,B)$ and let in (A,Γ,B) the equalities $u_1^{\mu}=v_1^{\mu}$, $u_2^{\mu}=v_2^{\mu}$, ..., $u_n^{\mu}=v_n^{\mu}$ take place. Since variables of Y are not present in u_i, v_i, then the latter equalities are valid already in the subautomaton (A,Γ,B_0), and since the latter lies in $\mathfrak{X}$ then we also get $u^{\mu}_{n+1}=v^{\mu}_{n+1}$. The Proposition is proved.

Theorem 5.5. *If a quasivariety of automata $\mathfrak{X}$ contains such automaton (A,Γ,B) that $|B|>1$ (in particular, if $\mathfrak{X}$ is saturated in output signals), then this quasivariety can be defined by quasiidentities without variables of Y.*

Proof. In order to prove the theorem we consider the cases of quasivarieties of pure and of linear automata separately. Clearly, it suffices to consider only irreducible quasiidentities that is quasiidentities without iteration transitive equalities.

Proof for the case of pure automata. Recall that the free object with generators Z, X, Y in the category of pure automata has the form $\mathcal{F}=(Z\circ F^1, F, (Z*F)\cup Y)$. Show that each quasiidentity satisfied in the variety $\mathfrak{X}$ is equivalent to the quasiidentity without variables of Y. Consider the quasiidentity (5.1):

$$u_1=v_1\wedge u_2=v_2\wedge\ldots\wedge u_n=v_n \Rightarrow u=v$$

Denote the antecedent of this quasiidentity by $\mathcal{A}$ and divide it on three groups:

$\mathcal{A}_1$ is a conjunction of the equalities $u_i=v_i$ not containing variables of Y;

$\mathcal{A}_2$ is a conjunction of the equalities of the form $u_i=y_i$;

$\mathcal{A}_3$ is a conjunction of the equalities of the form $y_i=y_j$.

Then quasiidentity (5.1) can be represented in the form (5.1'):

$$\mathcal{A}_1\wedge\mathcal{A}_2\wedge\mathcal{A}_3 \Rightarrow u=v$$

The corollary u=v can also belong to one of these three groups. Consider the given cases separately.

1. The equality u=v does not contain variables of Y. Show that under this condition quasiidentity (5.1) is equivalent to the quasiidentity

$$\mathcal{B} \Rightarrow u=v \qquad (5.5)$$

where the antecedent $\mathcal{B}$ is a conjunction of $\mathcal{A}_1$ and of all the equalities $u_i=v_i$ not containing $y\in Y$ and satisfying the condition:

either the equalities $u_i=y_i$ and $v_i=y_i$ are involved in the group $\mathcal{A}_2$,

or the equalities $u_i=y_i$ and $v_i=y_k$ are involved in $\mathcal{A}_2$ while the equality $y_i=y_k$ is involved in $\mathcal{A}_3$.

Take an automaton $\mathfrak{A}=(A,\Gamma,B)$ satisfying the quasiidentity (5.5), an arbitrary homomorphism $\mu:\mathcal{F} \rightarrow \mathfrak{A}$ and assume that the conjunction $\mathcal{A}^\mu=\mathcal{A}^\mu_1\wedge\mathcal{A}^\mu_2\wedge\mathcal{A}^\mu_3$ is true. Then $\mathcal{B}^\mu$ is also true. Really, $\mathcal{A}^\mu_1$ is common for $\mathcal{A}^\mu$ and $\mathcal{B}^\mu$ Further, if $u_i=v_i$ is involved in $\mathcal{B}$, $u_i=y_i$ and $v_i=y_i$ are involved in $\mathcal{A}_2$, then $u^\mu_i=y^\mu_i$ and $v^\mu_i=y^\mu_i$, from which $u^\mu_i=v^\mu_i$. Finally, if $u_i=v_i$ is involved in $\mathcal{B}$, $u_i=y_i$ and $v_i=y_k$ are involved in $\mathcal{A}_2$, and $y_i=y_k$ are involved in $\mathcal{A}_3$, then $u^\mu_i=v^\mu_i$, $v^\mu_i=y^\mu_k$ and $y^\mu_i=y^\mu_k$ whence $u^\mu_i=v^\mu_i$. Thus, the truth of $\mathcal{A}^\mu$ in (A,Γ,B) implies the truth of $\mathcal{B}^\mu$ and therefore, $u^\mu=v^\mu$ is also true. Conversely, let quasiidentity (5.1) be satisfied in (A,Γ,B). It is necessary to show that quasiidentity (5.5) is valid. Let the antecedent $\mathcal{B}^\mu$ be true in (A,Γ,B). Denote by $y_1,y_2,\ldots,y_n$ all the variables of Y involved in the antecedent $\mathcal{A}$. Since variables of Y are not involved in (5.5), then the values of y^μ_i can be taken arbitrarily. Choose $y^\mu_1,\ldots,y^\mu_n$ such that the antecedent of quasiidentity (5.1) is satisfied.

It is clear, that $y_1,y_2,\ldots,y_n$ are involved in $\mathcal{A}_2$ and $\mathcal{A}_3$. If for a certain $y_i, i\in\{1,2,\ldots,n\}$ there exists v, such that $v=y_i$ is involved in $\mathcal{A}_2$, then assume $y^\mu_i=v^\mu$. If in $\mathcal{A}_2$ there is also the equality $y_i=v'$ then the equality $v=v'$ is involved in the antecedent $\mathcal{B}$ of quasiidentity (5.5) and since the antecedent $\mathcal{B}$ is true, then $v^\mu=v'^\mu$. This implies that the definition of y^μ_i does not depend on the choice of v.

If there is no such v, that $v=y_i$, but can be found y_j satisfying the conditions $(y_i=y_j)\in\mathcal{A}_3$, $(v=y_j)\in\mathcal{A}_2$, then assume $y^\mu_j=v^\mu$. By the same reasons as above the definition of y^μ_j in this case also does not depend on the choice of y_j and v. For all the rest y_i we assume that y^μ_i are arbitrary and coincide between themselves. Under such definition of the mapping μ on the set Y, antecedent (5.1) will be evidently satisfied and, therefore, also the corollary $u^\mu=v^\mu$ is also valid. Thus, quasiidentity (5.5) is satisfied in (A,Γ,B);, hence, quasiidentities (5.1) and (5.5) are equivalent.

2. The equality $u=v$ belongs to the second group, that is, it has the form $u=y$, $u\in Z*F$, $y\in Y$. In this case (5.1) takes the form

$$\mathcal{A}=\mathcal{A}_1\wedge\mathcal{A}_2\wedge\mathcal{A}_3 \Rightarrow u=y \qquad (5.6)$$

Consider all possible subcases.

2.1) $\mathcal{A}_2$ contains the equality $w=y$.
Replacing of the corollary (5.6) by $u=w$ we obtain the quasiidentity

$$\mathcal{A} \Rightarrow u=w \qquad (5.7)$$

which is equivalent to (5.6).

Really, let $(A,\Gamma,B)\in\mathfrak{X}$, μ be an arbitrary homomorphism $\mathcal{F}^{\mu} \rightarrow (A,\Gamma,B)$ and the antecedent $\mathcal{A}^{\mu}$ be valid. Then, (5.6) implies $u^{\mu}=y^{\mu}$, and since $\mathcal{A}$ contains the equality $w=y$, we get $u^{\mu}=y^{\mu}=w^{\mu}$, which means that quasiidentity (5.7) is satisfied.

Conversely, if in (A,Γ,B) is satisfied (5.7), then $u^{\mu}=w^{\mu}$, and $w^{\mu}=y^{\mu}$ that is, (5.6) is true. Replacing quasiidentity (5.6) by equivalent quasiidentity (5.7), we obtain the case 1.

2.2) $\mathcal{A}_3$ contains the equality $y=y_i$ while $\mathcal{A}_2$ contains $w=y_i$. As above, we can see that in this case (5.1) is equivalent to the quasiidentity $\mathcal{A} \Rightarrow u=w$ without variables of Y.

2.3) $\mathcal{A}_3$ contains $y=y_i$ while y_i does not belong to $\mathcal{A}_2$. Since (5.6) is irreducible in $\mathcal{A}_3$, there are no equalities $y=y_j$ for $j\neq i$. Delete the equality $y=y_i$ from $\mathcal{A}$ and show that thus obtained quasiidentity

$$\mathcal{A}' \Rightarrow u=y \qquad (5.8)$$

is equivalent to (5.6).

Let quasiidentity (5.6) be satisfied in (A,Γ,B) and under a certain homomorphism $\mu:\mathcal{F} \rightarrow (A,\Gamma,B)$ the antecedent $\mathcal{A}'$ of quasiidentity (5.8) is valid. The variable y_i is not involved in $\mathcal{A}'$ and, therefore, the value of y_i^{μ} can be chosen arbitrarily. Let $y_i^{\mu}=y^{\mu}$. Then the antecedent (5.6) will be satisfied and, therefore, also the corollary $u^{\mu}=y^{\mu}$, common with (5.8) is valid. The converse case is evident.

Thus, we come to the case when $\mathcal{A} \Rightarrow u=y$, and $\mathcal{A}$ does not contain y. Show that this is impossible. Take the automaton (A,Γ,B), $|B|>1$ and the homomorphism $\mu:\mathcal{F} \rightarrow (A,\Gamma,B)$ under which the antecedent $\mathcal{A}$ is true. Then the equality $u^{\mu}=y^{\mu}$ has to be satisfied. Let b be an arbitrary element of B. Along with μ define the homomorphism μ' coinciding with μ on all the elements except y; i.e. $y^{\mu'}=b\neq y^{\mu}$. Since $\mathcal{A}$ does not contain y, then the validity of $\mathcal{A}^{\mu}$ is equivalent to the validity of $\mathcal{A}^{\mu'}$; by the same reason $u^{\mu}=u^{\mu'}$ Since the quasiidentity $\mathcal{A} \Rightarrow u=y$ is satisfied in (A,Γ,B), then from the validity of $\mathcal{A}^{\mu'}$ the equalities $u^{\mu'}=y^{\mu'}=u^{\mu}=b$ follow. So, u^{μ} can be equal to any element of B. But it is impossible since B contains more than one element.

3. The equality u=v belongs to the third group, that is, it has the form $y_i=y_j$. Then quasiidentity (5.1) takes the form

$$\mathcal{A} \Rightarrow y_i=y_j \tag{5.9}$$

The following subcases are possible:

3.1) At least one of the variables y_i, y_j is contained in $\mathcal{A}_2$, say, the equality $v=y_j$ is contained in $\mathcal{A}_2$.

Then (5.9) is equivalent to the quasiidentity $\mathcal{A} \Rightarrow v=y_j$ and we have the case 2.

3.2) y_i, y_j are not involved in $\mathcal{A}_2$, while the equality $y_i=y_k$ is contained in $\mathcal{A}_3$

If $y_k=y_j$, then (5.9) is satisfied trivially: the corollary is contained in the antecedents. Therefore, it is necessary to consider the case $y_j\neq y_k$ when $y_k=v$ is not involved in $\mathcal{A}_2$, otherwise we have the sub-case 3.1. Then y_j may be also involved in $\mathcal{A}_3$, say, in the equality $(y_j=y_\ell)\in\mathcal{A}_3$ Assume that this is true. Then by removing the equality $y_j=y_\ell$ from $\mathcal{A}$ we get the quasiidentity

$$\mathcal{A}'' \Rightarrow y_i=y_j \tag{5.10}$$

where $\mathcal{A}''$ already does not contain the variable y_j Show that (5.10) is equivalent to (5.9).

Let quasiidentity (5.9) be satisfied in the automaton $(A,\Gamma,B)\in\mathfrak{X}$ and under the homomorphism μ the antecedent $\mathcal{A}''$ of quasiidentity (5.10) is valid. Assuming $y_\ell^{\mu}=y_j^{\mu}$ we make true the antecedent $\mathcal{A}^{\mu}$ for (5.9). Thus its corollary $y_i^{\mu}=y_j^{\mu}$, which is also a corollary for (5.10), is true. The converse case is evident.

Since y_j is not contained in $\mathcal{A}''$ and does not coincide with y_i, then y_j^{μ} can be any element of B. Since $|B|>1$, we get a contradiction with the equality $y_i^{\mu}=y_j^{\mu}$. By this the proof of the theorem for the case of pure automata is completed.

Proof for the case of linear automata. As in the previous case we show that each quasiidentity valid in $\mathfrak{X}$ is equivalent to a quasiidentity without variables of Y. Each quasiidentity can be written in the form

$$\mathcal{A}_1 \wedge \mathcal{A}_2 \Rightarrow u=v \qquad (5.11)$$

where $\mathcal{A}_1$ does not contain variables of Y, while $\mathcal{A}_2$ contains such variables.

Recall that $\mathcal{F}=(Z\circ KF^1, F, Z*KF\oplus KY)$ is a free linear automaton and consider the following cases:

1. The equality u=v does not contain elements of Y. Write out all the components of the conjunction of $\mathcal{A}_2$:

$$\sum_{i=1}^{n} \lambda_{1i} y_i = v_1$$

$$\cdots\cdots\cdots$$

$$\sum_{i=1}^{n} \lambda_{si} y_i = v_s$$

$$\cdots\cdots\cdots$$

$$\sum_{i=1}^{n} \lambda_{mi} y_i = v_m$$

where $v_k=\sum_i z_i * v_{ki}$, $k=1,\dots,m$, $z_i \in Z$, $v_{ki} \in KF$, $\lambda_{jk} \in K$, $y_i \in Y$. Denote $\sum_i^n \lambda_{ji} y_i = y'_j$, $j=1,\dots,m$. Let the elements $y'_1,\dots,y'_s$ among the elements $y'_1,\dots,y'_s,\dots,y'_m$ of the vector space KY set up the maximal system of linearly independent vectors.

Thus we get a certain linear system

$$\begin{cases} y'_{s+1}=\alpha_{11}y'_1+\ldots+\alpha_{1s}y'_s \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y'_m\ \ =\alpha_{m-s1}y'_1+\ldots+\alpha_{m-ss}y'_s \end{cases}$$

Transforming this system on the elements $v_1,\ldots,v_s,\ldots,v_m$ we obtain

$$\begin{cases} v_{s+1}=\alpha_{11}v_1+\ldots+\alpha_{1s}v_s \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ v_m\ \ =\alpha_{m-s1}v_1+\ldots+\alpha_{m-ss}v_s \end{cases} \tag{5.12}$$

Conjunction of all the equalities of the system (5.12) denote by $\mathcal{A}'_2$ and show that the quasiidentity

$$\mathcal{A}_1\wedge\mathcal{A}'_2 \Rightarrow u=v \tag{5.13}$$

is equivalent to (5.11). Take an arbitrary automaton (A,Γ,B) for which the quasiidentity (5.13) is satisfied, and a homomorphism $\mu:\mathcal{F} \to (A,\Gamma,B)$ for which the antecedent of the quasiidentity (5.11) is true, that is $\mathcal{A}^{\mu}_1\wedge\mathcal{A}^{\mu}_2$ is valid. By virtue of the validity of $\mathcal{A}^{\mu}_2$ the following system of equalities is satisfied

$$\begin{cases} y'^{\mu}_1=\sum\limits_{i=1}^{n}\lambda_{1i}y^{\mu}_i=v^{\mu}_1 \\ \ldots\ldots\ldots\ldots \\ y'^{\mu}_s=\sum\limits_{i=1}^{n}\lambda_{si}y^{\mu}_i=v^{\mu}_s \\ \ldots\ldots\ldots\ldots \\ y'^{\mu}_m=\sum\limits_{i=1}^{n}\lambda_{mi}y^{\mu}_i=v^{\mu}_m \end{cases} \tag{5.14}$$

Besides, the equalities

$$\begin{cases} y'^{\mu}_{s+1}=\alpha_{11}y'^{\mu}_1+\ldots+\alpha_{1s}y'^{\mu}_s \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y'^{\mu}_m\ \ =\alpha_{m-s1}y'^{\mu}_1+\ldots+\alpha_{m-ss}y'^{\mu}_s \end{cases} \tag{5.15}$$

and, due to (5.14), the equalities

$$\begin{cases} v^{\mu}_{s+1}=\alpha_{11}v^{\mu}_{1}+\ldots+\alpha_{1s}v^{\mu}_{s} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ v^{\mu}_{m}=\alpha_{m-s1}v^{\mu}_{1}+\ldots+\alpha_{m-ss}v^{\mu}_{s} \end{cases} \tag{5.16}$$

are also satisfied. But in this case in (A,Γ,B) all the equalities involved in $\mathcal{A}'_2$ and the whole antecedent $\mathcal{A}_1\wedge\mathcal{A}'_2$ of the formula (5.13) are true, hence the equality $u^{\mu}=v^{\mu}$ which is a corollary for (5.11) is also true. Thus since quasiidentity (5.11) holds on (A,Γ,B) then (5.13) is satisfied.

Conversely, let (5.11) be satisfied in (A,Γ,B). The system of vectors $y'_1,\ldots,y'_s$ can be complemented arbitrarily to the basis Y' of the linear space KY. Take the arbitrary homomorphism $\mu:\mathcal{F}\to(A,\Gamma,B)$ for which antecedent (5.13) is true. Since in (5.13) variables of Y are not contained, then only the values of the homomorphism μ on the variables of X and Z are essential, while on the variables of Y it can be defined arbitrarily. Define the mapping $\nu:Y\to B$ by the rule: $y'^{\nu}_1=v^{\mu}_1,\ldots,y'^{\nu}_s=v^{\mu}_s$. Define ν on the remaining basic elements of Y' in an arbitrary way. Now let us proceed from the new homomorphism $\mu':\mathcal{F}\to(A,\Gamma,B)$ which differs from μ only on the variables of Y according to the given rule. Prove that in this case $\mathcal{A}^{\mu}_1\wedge\mathcal{A}^{\mu}_2$ is true, that is, the antecedent (5.11) is true. Really, $\mathcal{A}^{\mu'}_1$ is common for antecedents (5.11) and (5.13). Further, take from $\mathcal{A}_2$ an arbitrary component of the conjunction

$$v_j=\sum_{i=1}^{n}\lambda_{ji}y_i=y'_j,\ \ j=1,\ldots m$$

and show that the equality

$$v^{\mu'}_j=\sum_{i=1}^{n}\lambda_{ji}y^{\mu'}_i$$

takes place. Consider two cases.

1.1) $j\le s$. By the definition of μ', in this case $y'^{\mu'}_j$ coincides with $v^{\mu'}_j$. Since μ' is a homomorphism, then $y'^{\mu'}_j=v^{\mu'}_j=\sum_{i=1}^{n}\lambda_{ji}y^{\mu'}_i$.

1.2) $s<j\le m$. Let $j=s+k$. Then we have the equality

$$y'_j=\alpha_{k1}y'_1+\ldots+\alpha_{ks}y'_s.$$

At the same time the component of the conjunction $v_j=\alpha_{k1}v_1+\ldots+\alpha_{ks}v_s$ is involved in the antecedent $\mathcal{A}'_2$ and therefore, the equality

$$v^{\mu}_j=\alpha_{k1}v^{\mu}_1+\ldots+\alpha_{ks}v^{\mu}_s$$

is satisfied. The latter equality does not contain variables of Y, therefore, μ can be replaced by μ' in it:

$$v^{\mu'}_j=\alpha_{k1}v^{\mu'}_1+\ldots+\alpha_{ks}v^{\mu'}_s=\alpha_{k1}y'^{\mu'}_1+\ldots+\alpha_{ks}y'^{\mu'}_s.$$

Since $y'_j=\sum_{i=1}^{n}\lambda_{ij}y_i=\alpha_{k1}y'_1+\ldots+\alpha_{ks}y'_s$, then using μ' at the left and at the right we get the required equality.

Thus, in $\mathcal{A}^{\mu'}_2$ all the equalities are true and the antecedent $\mathcal{A}^{\mu'}_1\wedge\mathcal{A}^{\mu'}_2$ is satisfied. Therefore, $u^{\mu'}=v^{\mu'}$, and $u^{\mu}=v^{\mu}$, since u, v does not contain variables of Y. Thus, (5.13) is satisfied in (A,Γ,B).

2. The equality u=v contains variables of Y. In this case (5.11) can be written in the form

$$\mathcal{A}=\mathcal{A}_1\wedge\mathcal{A}_2 \Rightarrow u=\sum_{i=1}^{n}\lambda_i y_i=y' \qquad (5.17)$$

Consider the following two possibilities:

2.1) y' is involved in the linear hull of the vectors $y'_1,\ldots,y'_s$. Then $y'=\alpha_1y'_1+\ldots+\alpha_sy'_s$. As in the case of pure automata, it can be proved that the quasiidentity $\mathcal{A} \Rightarrow u=\alpha_1v_1+\ldots+\alpha_sv_s$ is equivalent to (5.17) and we come to the case 1.

2.2) y' is not involved in the linear hull of the vectors $y'_1,\ldots,y'_s$. Let (A,Γ,B), $|B|>1$ be an automaton and let $\mu:\mathcal{F} \rightarrow (A,\Gamma,B)$ be an arbitrary homomorphism for which the antecedent (5.17) is satisfied. Then, the equalities

$$v^{\mu}_j=y'^{\mu}_j \qquad (5.18)$$

take place. The corollary $u^{\mu}=y'^{\mu}$ also has to be satisfied. Since y' is not contained in the linear hull of the vectors $y'_1,\ldots,y'_s$, then it is possible to define the homomorphism μ' coinciding with μ on the variables of $X \cup Z \cup \{y'_1, \dots, y'_m\}$ and different from μ on the variable y'. (It can be done, since $|B|>1$). Then $\mathcal{A}_1^{\mu}$ coincides with $\mathcal{A}_1^{\mu'}$ and $\mathcal{A}_2^{\mu}$ with $\mathcal{A}_2^{\mu'}$, since (5.18) is satisfied under replacement of μ by μ'. Hence, under the homomorphism μ' the antecedent (5.17) and the corollary $u^{\mu'}=y'^{\mu'}$ are satisfied. Besides, $u^{\mu'}=u^{\mu}$ and $u^{\mu}=y'^{\mu}$. On the other hand, by the construction, $y'^{\mu} \neq y'^{\mu'}$. The obtained contradiction implies that case 2.2 cannot take place. The theorem is proved.

From the proof of the theorem follows that if the quasivariety $\mathfrak{X}$ considered in the theorem is defined by the finite set of quasiidentities, then it can be also defined by the finite set of quasiidentities, which do not contain variables of Y.

4.5.4. Quasivarieties of automata and quasivarieties of semigroups

First introduce the following operators on the classes of automata: if θ is a certain class of automata, then

- θ_{ℓ} denotes the result of the anointment to θ of a unit automaton;
- $S\theta$, as usual, is the class of all the subautomata of automata of θ;
- $C^{\varphi}\theta$ denotes the class of all the filtered products of automata of θ.

It is possible to prove ([47]) that the least quasivariety generated by the class of automata θ is $SC^{\varphi}\theta_{\ell}$. Along with mentioned above introduce the saturation operators V, V':

$V\theta$ is the class consisting of all the automata being the epimorphic in inputs images and coimages of the automata of θ;
- $V'\theta$ is the class consisting of all the automata (A,Γ,B) containing certain subautomaton (A,Γ,B_0) from θ.

It is immediately verified that if θ is a quasivariety of automata, then $V\theta$ is a quasivariety saturated in inputs and $V'\theta$ is saturated in outputs. Thus, the class $VSC^{\varphi}\theta_{\ell}$ is a quasivariety saturated in inputs generated by the class θ, and the class $V'SC^{\varphi}\theta_{\ell}$ is a quasivariety saturated in outputs generated by the class θ.

Similarly, as it was done for the linear representations ([90]), define the relation between the quasivarieties of automata and of semigroups. If $\mathfrak{X}$ is a class of semigroup automata, then denote by $\vec{\mathfrak{X}}$ a class of semigroups Γ for which there exists an exact automaton (A,Γ,B) from $\mathfrak{X}$.

Theorem 5.6. *If $\mathfrak{X}$ is a quasivariety of automata, then $\vec{\mathfrak{X}}$ is a quasivariety of semigroups.*

Proof. A unit automaton $(0,1,0)$ belongs to $\mathfrak{X}$. Therefore, a one-element semigroup belongs to the class $\vec{\mathfrak{X}}$.

Let $\Gamma\in\vec{\mathfrak{X}}$, and Σ be a subsemigroup in Γ. Then there exists an exact automaton $(A,\Gamma,B)\in\mathfrak{X}$. The automaton (A,Σ,B) belongs to $\mathfrak{X}$ as a subautomaton of the automaton from the quasivariety $\mathfrak{X}$. It is also exact, therefore, $\Sigma\subset\vec{\mathfrak{X}}$, hence, the class $\vec{\mathfrak{X}}$ is hereditary.

Prove the closeness of $\vec{\mathfrak{X}}$ by filtered products. Take a system of the semigroups $\Gamma_i\in\vec{\mathfrak{X}}$, $i\in I$, the filter $\mathcal{D}$ over I and set up the filtered product $\prod\limits_{i\in I}\Gamma_i/\mathcal{D}$. By the condition, for each $i\in I$ there exists the exact automaton $\mathfrak{A}_i=(A_i,\Gamma_i,B_i)\in\mathfrak{X}$. Then, $\bar{\mathfrak{A}}=\prod\limits_{i\in I}\mathfrak{A}_i/\mathcal{D}=(\prod\limits_{i\in I}A_i/\rho_1,\ \prod\limits_{i\in I}\Gamma_i/\rho_2,\ \prod\limits_{i\in I}B_i/\rho_3)=(\bar{A},\bar{\Gamma},\bar{B})$ belongs to $\mathfrak{X}$ since $\mathfrak{X}$ is a quasivariety. The semigroup $\prod\limits_{i\in I}\Gamma_i/\rho_2$ coincides with $\prod\limits_{i\in I}\Gamma_i/\mathcal{D}$ and it is a semigroup of the input signals for the automaton $\bar{\mathfrak{A}}$ of $\mathfrak{X}$. Verify that the automaton $\bar{\mathfrak{A}}$ is exact.

Together with the automaton $\bar{\mathfrak{A}}$ consider the automaton $\bar{\mathfrak{A}}'=(\prod\limits_{i\in I}A_i/\rho_1,\Gamma,\ \prod\limits_{i\in I}B_i/\rho_3)$ where $\Gamma=\prod\limits_{i\in I}\Gamma_i$ and the operations $\circ$ and $*$ are defined by the rule: if $\bar{a}\in\prod\limits_{i\in I}A_i/\rho_1$, γ is an element of Γ, $\bar{\gamma}$ is its image in Γ/ρ_2, then

$$\bar{a}\circ\gamma=\bar{a}\circ\bar{\gamma},\quad \bar{a}*\gamma=\bar{a}*\bar{\gamma}. \tag{5.19}$$

Introduce the congruence τ on Γ:

$$\gamma_1\tau\gamma_2 \Leftrightarrow I(\gamma_1,\gamma_2)\in\mathcal{D}, \tag{5.20}$$

where $I(\gamma_1,\gamma_2)=\{i\in I\,|\,(a\circ\gamma_1(i)=a\circ\gamma_2(i))\wedge(a*\gamma_1(i)=a*\gamma_2(i))$ for all $a\in A_i\}$.

Verify that τ satisfies the following two conditions:

1) τ coincides with the kernel of the automaton $\bar{\mathfrak{A}}'$;

2) $\rho_2 \subset \tau$; if all the automata $\mathfrak{A}_i$, $i \in I$ are exact, then $\tau = \rho_2$. This means that the automaton $\mathfrak{A}$ is an exact one.

1) Denote the kernel of the automaton $\bar{\mathfrak{A}}'$ by ρ' and first show that $\tau \subset \rho'$. Let $\gamma_1 \tau \gamma_2$. It is necessary to show that $\gamma_1 \rho' \gamma_2$, that is, for any element $\bar{a} \in \prod A_i/\rho_1 = \bar{A}$ the equalities $\bar{a} \circ \gamma_1 = \bar{a} \circ \gamma_2$ and $\bar{a} * \gamma_1 = \bar{a} * \gamma_2$ hold. By the definition of the automaton $\bar{\mathfrak{A}}'$, $\bar{a} \circ \gamma = \bar{a} \circ \bar{\gamma} = \overline{a \circ \gamma}$ and $\bar{a} * \gamma = \overline{a * \gamma}$. Thus, we must check that the equalities $\overline{a \circ \gamma_1} = \overline{a \circ \gamma_2}$ and $\overline{a * \gamma_1} = \overline{a * \gamma_2}$ hold for any $a \in A$.

By the definition of ρ_1 the equality $\overline{a \circ \gamma_1} = \overline{a \circ \gamma_2}$ implies that the set J_1 of the elements i, for which $(a \circ \gamma_1)(i) = (a \circ \gamma_2)(i)$, belongs to $\mathcal{D}$. Similarly, the equality $\overline{a * \gamma_1} = \overline{a * \gamma_2}$ implies that the set J_2 of the elements i, for which $(a * \gamma_1)(i) = (a * \gamma_2)(i)$, belongs to $\mathcal{D}$. Thus, it is necessary to show that for each a both J_1 and J_2 belong to $\mathcal{D}$. The condition $\gamma_1 \tau \gamma_2$ implies, by (5.20), that the set $I(\gamma_1, \gamma_2)$ belongs to $\mathcal{D}$. It is clear that $I(\gamma_1, \gamma_2) = J_1 \cap J_2$ Hence, $I(\gamma_1, \gamma_2) \subset J_1$ and $I(\gamma_1, \gamma_2) \subset J_2$ and since $\mathcal{D}$ is a filter, then $J_1 \in \mathcal{D}$ and $J_2 \in \mathcal{D}$ what was required.

Conversely, show that if γ_1 and γ_2 are not in the relation τ, then they are not in the relation ρ'. Let $I(\gamma_1, \gamma_2) \notin \mathcal{D}$ and $i \in I \setminus I(\gamma_1, \gamma_2)$. Then in A_i can be found the element a_i for which $a_i \circ \gamma_1(i) \neq a_i \circ \gamma_2(i)$ or $a_i * \gamma_1(i) \neq a_i * \gamma_2(i)$. Take such element $a \in \prod_{i \in I} A_i$ that $a(i) = a_i$ for all $i \in I \setminus I(\gamma_1, \gamma_2)$. For this element either $\bar{a} \circ \gamma_1 \neq \bar{a} \circ \gamma_2$ or $\bar{a} * \gamma_1 \neq \bar{a} * \gamma_2$ ($\bar{a}$ is an image of the element a in $\prod_{i \in I} A_i/\rho_1$). Indeed, by the construction of a all i for which the equalities

$$
\begin{aligned}
(a \circ \gamma_1)(i) &= (a \circ \gamma_2)(i) = a(i) \circ \gamma_1(i) = a(i) \circ \gamma_2(i) \\
(a * \gamma_1)(i) &= (a * \gamma_2)(i) = a(i) * \gamma_1(i) = a(i) * \gamma_2(i)
\end{aligned}
\tag{5.21}
$$

hold, lie in $I(\gamma_1, \gamma_2)$. If $\overline{a \circ \gamma_1} = \overline{a \circ \gamma_2}$ and $\overline{a * \gamma_1} = \overline{a * \gamma_2}$, then the set of all i, for which the equalities (5.21) are satisfied, belongs to $\mathcal{D}$. Then $I(\gamma_1, \gamma_2)$ also belongs to $\mathcal{D}$ that contradicts the assertion. Thus, either $\bar{a} \circ \gamma_1 \neq \bar{a} \circ \gamma_2$ or $\bar{a} * \gamma_1 \neq \bar{a} * \gamma_2$, that is, γ_1 and γ_2 are not in the relation ρ'.

2) Show that $\rho_2 \subset \tau$. Let $\gamma_1 \rho_2 \gamma_2$, that is $J_{\rho_2} = \{i \mid \gamma_1(i) = \gamma_2(i)\} \in \mathcal{D}$. It

is evident that $J_{\rho_2} \subset I(\gamma_1,\gamma_2)$. Since $\mathcal{D}$ is a filter, then $I(\gamma_1,\gamma_2)$ also belongs to $\mathcal{D}$. Therefore, $\gamma_1 \tau \gamma_2$.

Let now all the automata $\mathfrak{A}_i$ be exact and $\gamma_1 \tau \gamma_2$ The latter implies that $I(\gamma_1,\gamma_2)$ belongs to $\mathcal{D}$. If $i \in I(\gamma_1,\gamma_2)$, then for any element a of A_i the equalities $a \circ \gamma_1(i) = a \circ \gamma_2(i)$ and $a * \gamma_1(i) = a * \gamma_2(i)$ are satisfied. Since the automaton $\mathfrak{A}_i$ is exact, then $\gamma_1(i) = \gamma_2(i)$. Hence, $J_{\rho_2} \supset I(\gamma_1,\gamma_2) \in \mathcal{D}$. Therefore, J_{ρ_2} also belongs to $\mathcal{D}$, that is $\gamma_1 \rho_2 \gamma_2$. Thus, $\tau \subset \rho_2$ and $\tau = \rho_2$. This proves that the automaton $\bar{\mathfrak{A}}$ is exact and that $\vec{\mathfrak{X}}$ is closed with respect to the filtered products. The theorem is proved.

Theorem 5.6 assigns to each quasivariety of automata a quasivariety of semigroups. The following statement solves the converse problem.

Theorem 5.7. *Let $\mathfrak{Y}$ be a quasivariety of semigroups, $\mathfrak{X}_0$ be a class of such exact automata* (A,Γ,B) *that* $\Gamma \in \mathfrak{Y}$, *and let $\mathfrak{X}$ be a quasivariety of automata generated by the class $\mathfrak{X}_0$. Then*

$$\vec{\mathfrak{X}} = \mathfrak{Y}.$$

Proof. Let $\Gamma \in \mathfrak{Y}$. Then we have the exact regular automaton $(K\Gamma^1, \Gamma, K\Gamma)$. This automaton belongs to the class $\mathfrak{X}_0$ and hence, to the quasivariety $\mathfrak{X}$. By virtue of its exactness, $\Gamma \in \vec{\mathfrak{X}}$. Thus, $\mathfrak{Y} \subset \vec{\mathfrak{X}}$.

Conversely, let $\Gamma \in \vec{\mathfrak{X}}$, and (A,Γ,B) be an exact automaton belonging to the class $\mathfrak{X} = SC^{\varphi}\mathfrak{X}_0$; it is a subautomaton of certain automaton (A_1,Γ_1,B_1) in the class $C^{\varphi}\mathfrak{X}_0$ By the definition of the operator C^{φ} we have that (A_1,Γ_1,B_1) is a filtered product of the automata of $\mathfrak{X}_0$: $(A_1,\Gamma_1,B_1) = \prod_{\alpha \in I} (A_{1\alpha},\Gamma_{1\alpha},B_{1\alpha})/\mathcal{D}$, where $\mathcal{D}$ is a filter over I. Since $\Gamma_{1\alpha} \in \mathfrak{Y}$ for all $\alpha \in I$, then the filtered product $\Gamma_1 = \prod_{\alpha \in I} \Gamma_{1\alpha}/\mathcal{D}$ also belongs to the quasivariety $\mathfrak{Y}$. The subsemigroup Γ of the semigroup Γ_1 also belongs to $\mathfrak{Y}$ and, therefore, $\vec{\mathfrak{X}} \subset \mathfrak{Y}$. The theorem is proved.

Note that, for example, for linear automata there may not exist such a variety $\mathfrak{X}$ for which $\vec{\mathfrak{X}}$ coincides with the defined quasivariety of semigroups $\mathfrak{Y}$.

The system of quasivarieties of automata forms a semigroup with respect to the multiplication of the classes of automata. This semigroup has not been studied yet. The theory of quasivarieties of biautomata has not been considered too.

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CHAPTER 5

AUTOMATA MODEL OF DATABASE

The present chapter is in great extend of expository, survey character and some of the problems are only outlined. For detailed information we refer to the book [86].

5.1. *-automata

5.1.1. *-automata and databases

Our aim in this Chapter is to introduce a class of algebraic structures which may serve as adequate algebraic models of real databases. This will be done in several successive steps.

In the first instance, a database is treated as a *-automaton of the form

$$A=(F,Q,R),$$

where F is a set of states, Q,R are algebras of queries and replies respectively, both of the same similarity type. Moreover, Q and R are supposed to be algebras of certain variety $\mathcal{L}$. This variety which a priori is arbitrary, is associated with the fixed logical tools, allowing to determine a query and the reply to it. We shall specify the choice of $\mathcal{L}$ later, but at the moment it is assumed to be arbitrary. Let us denote by $f*q$ the reply to the query q in a given state f. It is supposed that the mapping $\hat{f}:Q \to R$ defined by the formula $\hat{f}(q)=f*q$ is a homomorphism of algebras. This natural assumption means merely that the structure of every reply reflects that of the corresponding query.

Since the concept of *-automaton is a framework of database model, the enriching of latter is withheld momentary in order to consider *-automata in more detail.

In each such automaton (F,Q,R) algebra of queries Q and and algebra of replies R lie in one and the same variety $\mathcal{L}$. In the *-automata considered earlier (Chapter 1) algebra of states and algebra of input signals belonged to one variety of algebras. We could combine both the approaches swapping the roles of F and Q. However, taking into account database semantics, it will be more convenient to use the approach, considered in the current Section. Let us consider some generalizations of it.

Let $\mathscr{L}$ be a category. $*$-automaton in $\mathscr{L}$ is a triplet (F,Q,R) where Q,R are objects of $\mathscr{L}$, and F is a set with the representation

$$\hat{}: F \to \mathrm{Hom}(Q,R).$$

F is treated as a set of states, Q and R as objects of queries and replies to queries. Morphism $\hat{f}: Q \to R$ corresponds to each $f \in F$. We speak about $*$-automaton, although there may be no operation $*$. The operation $*$ appears if $\mathscr{L}$ is a variety of one-sorted algebras $\mathscr{L}$. It is defined by the rule:

$$f*q=\hat{f}(q), f \in F, \quad q \in Q.$$

Further let R be an arbitrary algebra of $\mathscr{L}$, and F an arbitrary set. Construct the specific $*$-automaton Atm(F,R). Take as Q the Cartesian power R^F, which is the algebra of $\mathscr{L}$ and define the representation $\hat{}: F \to \mathrm{Hom}(Q,R)$ by the rule: for each $f \in F$ and $q \in Q = R^F$ homomorphism $\hat{f}: Q \to R$ is defined by $\hat{f}(q)=q(f)$. Verify that for each $f \in F$ element $\hat{f}: Q \to R$ is actually the homomorphism of algebras.

Let ω be n-ary operation in the variety $\mathscr{L}$ and $q_1, \dots, q_n$ be the elements of Q. The equality $\hat{f}(q_1 \dots q_n \omega)=\hat{f}(q_1) \dots \hat{f}(q_n)\omega$ follows directly from definitions. Indeed,

$$\hat{f}(q_1 \dots q_n \omega)=(q_1 \dots q_n \omega)(f)=q_1(f) \dots q_n(f)\omega=\hat{f}(q_1) \dots \hat{f}(q_n)\omega.$$

Show that any $*$-automaton (F,Q,R) can be naturally defined by the constructed Atm(F,R).

Proposition 1.1. *Let* $\mathfrak{A}$=(F,Q,R) *be a* $*$*-automaton. Define a mapping* $\mu: Q \to R^F$ *assuming*:

$$q^{\mu}(f)=f*q, \quad f \in F, \quad q \in Q.$$

*Then μ is a homomorphism in $\mathcal{L}$ and there is one-to-one correspondence between the operations of the kind * and homomorphisms of the kind μ.*

Proof. Verify first that μ is a homomorphism in $\mathcal{L}$. In notations above we must check that $(q_1 \ldots q_n \omega)^\mu = q_1^\mu \ldots q_n^\mu \omega$. Really, for each $f \in F$ holds

$$(q_1 \ldots q_n \omega)^\mu (f) = f * q_1 \ldots q_n \omega = \hat{f}(q_1 \ldots q_n \omega) = \hat{f}(q_1) \ldots \hat{f}(q_n)\omega,$$

$$q_1^\mu \ldots q_n^\mu \omega(f) = q_1^\mu(f) \ldots q_n^\mu(f)\omega = (f*q_1) \ldots f*q_n)\omega = \hat{f}(q_1) \ldots \hat{f}(q_n)\omega.$$

Conversely, given homomorphism $\mu: Q \rightarrow R^F$ define operation $*: F\times Q \rightarrow R$ assuming $f*q = q^\mu(f)$. Show that for each $f \in F$ the mapping $\hat{f}: Q \rightarrow R$ is a homomorphism in $\mathcal{L}$. We have:

$$\hat{f}(q_1 \ldots q_n \omega) = f*q_1 \ldots q_n \omega = (q_1 \ldots q_n \omega)^\mu (f) = q_1^\mu \ldots q_n^\mu \omega(f) =$$
$$q_1^\mu(f) \ldots q_n^\mu(f)\omega = (f*q_1) \ldots f*q_n)\omega = \hat{f}(q_1) \ldots \hat{f}(q_n)\omega.$$

The considered assignment is obviously bijective.

Thus, defining of *-automaton (F,Q,R) is equivalent to defining of the homomorphism $\mu: Q \rightarrow R^F$. This remark as well as the next one in fact was mentioned earlier: the set of states F can be treated as the set of input signals and the algebra Q can be regarded as the algebra of states.

Let $\mathfrak{A}$=(F,Q,R) be a *-automaton. For every $f \in F$ the kernel of the homomorphism $\hat{f}: Q \rightarrow R$ is denoted by Ker$f$. This is the kernel equivalence on Q. The kernel of *-automaton $\mathfrak{A}$ is the intersection of all Ker$f$ and it is denoted by Ker$\mathfrak{A}$. If Ker$\mathfrak{A}$ is trivial then $\mathfrak{A}$ is called an exact *-automaton. Therefore, *-automaton $\mathfrak{A}$ is exact if and only if for any pair of distinct queries q_1, q_2 of Q there exists some state $f \in F$ such that $f*q_1 \neq f*q_2$.

The following remark is easily verified. Let $\mathfrak{A}$=(F,Q,R) be a *-automaton and $\nu: Q \rightarrow Q'$ an epimorphism of algebras in $\mathcal{L}$, such that $\mathrm{Ker}\nu \subset \mathrm{Ker}\mathfrak{A}$. Then there is *-automaton $\mathfrak{A}'$=(F,Q',R) with the operation: $f*q' = f*q$, where $f \in F$, $q' = q^\nu \in Q'$, $q \in Q$. This definition of operation * in $\mathfrak{A}'$ is correct and $\mathfrak{A}'$ is really *-automaton. In particular, if $Q' = Q/\mathrm{Ker}\mathfrak{A}$ and $\nu: Q \rightarrow Q'$ is the natural homomorphism then $\mathfrak{A}'$=(F,Q',R) is the exact *-

automaton defined by $\mathfrak{A}$.

Proposition 1.2. *Let* $\mathfrak{A}=(F,Q,R)$ *be a* $*$*-automaton and* $\mu:Q\to R^F$ *the corresponding homomorphism. Then* $\mathrm{Ker}\mu=\mathrm{Ker}\mathfrak{A}$.

Proof. If $q_1\mathrm{Ker}\mu q_2$, then $q_1^{\mu}=q_2^{\mu}$ and for each $f\in F$ holds

$$f*q_1=q_1^{\mu}(f)=q_2^{\mu}(f)=f*q_2.$$

Thus $q_1\mathrm{Ker}\mathfrak{A}q_2$. Conversely, if q_1 and q_2 act in $\mathfrak{A}$ in the same way then for each $f\in F$ holds

$$q_1^{\mu}(f)=f*q_1=f*q_2=q_2^{\mu}(f);\ q_1^{\mu}=q_2^{\mu}.$$

5.1.2. Dynamic $*$-automata

A $*$-automaton $\mathfrak{A}=(F,Q,R)$ is called a *dynamic* $*$*-automaton* if there is a semigroup Σ acting on the set of states F and on the algebra of queries Q. We assume that the action is right-hand on F and left-hand on Q. The elements of Σ act on Q as endomorphisms and the equality

$$f*\sigma q=(f\circ\sigma)*q,\quad q\in Q,\quad \sigma\in\Sigma$$

has to be satisfied.

The latter equality means that the query σq in the state f produces the same reply as the query q in the state $f\circ\sigma$. In this case queries are of dynamic character, i.e. various changes of states can be reflected in queries. We shall see later that the concept of dynamic $*$-automaton is associated with the definition of dynamic database.

Consider the example of dynamic $*$-automaton. Take $\mathrm{Atm}(F,R)=(F,Q^F,R)$. Assume that Σ is an arbitrary semigroup with the given representation as a semigroup of transformations of F. Define the action of Σ on $Q=R^F$ For any $\sigma\in\Sigma$, $\xi\in R^F$ set

$$(\sigma\xi)(f)=\xi(f\circ\sigma).$$

Then Σ acts in R^F as a semigroup of endomorphisms. Indeed, for $f\in F$ and $\xi_1,\dots,\xi_n\in R^F$ holds:

$$\sigma(\xi_1\dots\xi_n\omega)(f)=(\xi_1\dots\xi_n\omega)(f\circ\sigma)=\xi_1(f\circ\sigma)\dots\xi_n(f\circ\sigma)\omega=$$
$$(\sigma\xi_1)(f)\dots(\sigma\xi_n)(f)\omega=\sigma\xi_1\dots\sigma\xi_n\omega(f).$$

It's left to note that $f*\sigma\xi=(f\circ\sigma)*\xi$. Really,

$$f*\sigma\xi=\sigma\xi(f)=\xi(f\circ\sigma)=(f\circ\sigma)*\xi.$$

Let $\mathfrak{A}=(F,Q,R)$ be a dynamic $*$-automaton with semigroup Σ. Then the $*$-automaton Atm(F,R) has the following universal property:

Proposition 1.3. *Homomorphism* $\mu: Q \to R^F$ *is coordinated with the action of* Σ *in* Q *and* R^F. *Each dynamic* $*$*-automaton is defined in such a way.*

Proof. We must check that $(\sigma q)^{\mu}=\sigma q^{\mu}$ For any $f\in F$ holds

$$(\sigma q)^{\mu}(f)=f*\sigma q=(f\circ\sigma)*q=q^{\mu}(f\circ\sigma)=\sigma q^{\mu}(f).$$

Let now semiautomaton (F,Σ), algebras Q and R of $\mathcal{L}$ and homomorphism $\mu: Q \to R^F$ be given. Let Σ acts on Q as a semigroup of automorphisms and μ is coordinated with the action of Σ. Assume $f*q=q^{\mu}(f)$. It is easy to verify that this defines dynamic $*$-automaton (F,Q,R).

Note finally that certain semigroup automaton corresponds to dynamic $*$-automaton (F,Q,R) with the semigroup Σ. Define the multiplication on $S=\Sigma\times Q$ by the rule $(\sigma,q)(\sigma',q')= (\sigma\sigma',\sigma q')$. Then it follows that S is a semigroup. For $s=(\sigma,q)$ and $f\in F$ let

$$f\circ s=f\circ\sigma;\quad f*s=f*\sigma.$$

These operations define on the triplet (F,S,R) the structure of semigroup automaton.

5.2. Database scheme, Halmos algebras

5.2.1. Database scheme

In the previous Section we had considered the first step of database model definition, having discussed its $*$-automaton framework. Henceforth, we should accomplish the following program. Actually, a database must be founded on some data algebra $\mathcal{D}$ that is many-sorted in general: $\mathcal{D}=(D_i, i\in\Gamma)$, where Γ is the set of sorts, and D_i are the domains of $\mathcal{D}$. Moreover, $\mathcal{D}$ is connected with certain variety θ which realizes the idea of data type. This leads to concept of database scheme, the emphasis of this item. Then one has to make the choice of a variety

$\mathcal{L}$ of algebras of replies and queries. Finally, with each $\mathcal{D}\in\theta$ will be associated definite universal *-automaton $\mathrm{Atm}\mathcal{D}=(\mathcal{F}_{\mathcal{D}}, U, V_{\mathcal{D}})$ and database model is defined via the representation of *-automata.

A database scheme consists of:

a) a set Γ of sorts (interpreted also as names of domains),

b) a set Ω of symbols of basic operations on data (the signature). Every $\omega\in\Omega$ has a definite type $\tau=(i_1,\dots,i_n;\ j)$, $i_k, j\in\Gamma$,

c) a variety θ of Ω-algebras $\mathcal{D}=(D_i, i\in\Gamma)$ (data algebras). Each ω of the type $\tau=(i_1,\dots,i_n;j)$ defines an operation

$$\omega: D_{i_1}\times\dots\times D_{i_n} \to D_j.$$

d) a set X of variables (or names of variables) together with a function $n: X \to \Gamma$. The splitting n gives rise to a family of sets, or *complex*, $X=(X_i, i\in\Gamma)$. We assume that no X_i is void,

e) a set Φ of symbols of relations, which are realized in algebras of θ. Every $\varphi\in\Phi$ has a definite type $\tau=(i_1,\dots,i_n)$, $i_k\in\Gamma$.

Realization (interpretation) is made by some function f defined on the set Φ, which for each $\varphi\in\Phi$ of the type $\tau=(i_1,\dots,i_n)$ chooses some subset in $D_{i_1}\times\dots\times D_{i_n}$. The function f will be also treated as a state of database. Any realization f allows to speak about a model $(\mathcal{D},\Phi,f)$ in the sense as this notion is used in mathematical logic.

The items a)-d) make up the main part of the scheme. In what follows, let $\mathcal{S}=(\Gamma,\Omega,\theta,X,n,\Phi,A)$ be a fixed database scheme.

5.2.2. Halmos algebras

In this item we specify the choice of the variety $\mathcal{L}$. The algebras Q and R are connected with a certain query language. In relational databases queries are usually written by means of language of first order logic and any query is represented as a class of equivalent formulas. Therefore, one must choose the appropriate algebraization of first order calculus. If we proceed from the pure first order calculus, both P and Q may be chosen to be *Halmos algebras* (they are usually called *polyadic*), though it is possible to start also from cylindric algebras or some other algebraic equivalent of first order logic. These algebras are in the same relation to first order logic as Boolean algebras to propositional calculus. However, it is very significant for applications to develop the extended notion of a specialized Halmos algebra, which is connected with an arbitrary but fixed variety of algebras Θ, where Θ plays the role of an abstract data type in programming. Hence, we will take the algebraic counterpart of that version of first order logic which is oriented to Θ (i.e. enriched by Θ-terms).

Specialized Halmos algebras are defined relatively to the main part of the scheme. Let $W=(W_i, i\in\Gamma)$ be a free algebra in Θ generated by the complex X.

Consider Boolean algebra H. A mapping $\exists: H \rightarrow H$ is called an *existential quantifier* of the Boolean algebra H if the following three conditions are fulfilled:

1) $\exists 0=0$,

2) $h<\exists h$,

3) $\exists(h_1\wedge\exists h_2)=\exists h_1\wedge\exists h_2$.

Here 0 denotes zero element in H, and $h_1, h_2, h\in H$.

Let H be a Boolean algebra and $Y\subset X$. H is called a *quantifier algebra* over X, if an existential quantifier $\exists(Y)$ corresponds to each subset $Y\subset X$ and holds

1) $\exists(\emptyset)h=h$,

2) $\exists(Y_1\cup Y_2)h=\exists(Y_1)\exists(Y_2)h$, $h\in H$.

Suppose further, that the semigroup $S=\mathrm{End}W$ acts on quantifier algebra H as a semigroup of Boolean endomorphisms. There are two more axioms controlling the interaction of quantifiers with these endomorphisms (here, $h\in H$, $\sigma, s\in\mathrm{End}W$):

α) $\sigma\exists(Y)=s\exists(Y)h$,

if σ and s are two elements of S, which act in the same way on those variables of X, which do not belong to the subset Y.

β) $\exists(Y)\sigma h=\sigma\exists(\sigma^{-1}Y)h$,

if σ is one-to-one on $\sigma^{-1}(Y)$, and, for all $x\in X$ with $\sigma(x)\notin Y$, no variable occurring in $\sigma(x)$ belongs to Y.

A quantifier algebra H over X with acting semigroup End$\mathcal{W}$ is called *Halmos (polyadic) algebra in the given scheme* if the conditions α) and β) are fulfilled.

Halmos algebras in the given scheme are also called *specialized* Halmos algebras. Since all the enumerated axioms are the identities, these algebras form a variety.

Consider the algebras of queries and replies as the examples of specialized Halmos algebras. Remember that we are working in the fixed scheme $\mathcal{S}$. It was already noted that queries in relational databases are written in terms of such version of first order calculus, which takes into account the variety Θ. Define elementary formulas of first order language as ones of the form $\varphi(w_1,\dots,w_n)$, where $\varphi\in\Phi$ has the type $\tau=(i_1,\dots,i_n)$ and $w_k\in W_{i_k}$.

Arbitrary formulas are constructed from elementary ones in a standard way, using Boolean operations $\vee,\wedge$, and quantifiers of the form $\exists x$, $x\in X$. Denote by $\tilde{\Phi}$ the set of all formulas. $\tilde{\Phi}$ can be regarded as a free algebra over the set of elementary formulas, in respect to Boolean operations and existential quantifiers of the form $\exists x$, $x\in X$. We should define interpretations of the formulas of $\tilde{\Phi}$.

To every algebra $\mathcal{D}\in\Theta$ there corresponds a certain Halmos algebra that is a derived structure of $\mathcal{D}$. It is defined as follows.

The semigroup End$\mathcal{W}$ naturally acts in Hom$(\mathcal{W},\mathcal{D})$ - the set of homomorphisms of $\mathcal{W}$ into $\mathcal{D}$. Its power set $\mathfrak{M}_{\mathcal{D}}$ is a Boolean algebra with respect to the set-theoretic operations. Let, further, A be a subset in Hom$(\mathcal{W},\mathcal{D})$. Setting $\mu\in\exists(Y)A$, if there exists $\nu\in A$ such that $\mu(x)=\nu(x)$ for all $x\notin Y$, we get the mapping $\exists(Y):\mathfrak{M}_{\mathcal{D}}\rightarrow\mathfrak{M}_{\mathcal{D}}$. It is easy to understand that these $\exists(Y)$ are the existential quantifiers in the sense of the definition above. In particular, in the Boolean algebra $\mathfrak{M}_{\mathcal{D}}$ act existential quantifiers of the form $\exists x$ (and dually, universal quantifiers $\forall x$).

For any $A\in\mathfrak{M}_{\mathcal{D}}$, $s\in$End$\mathcal{W}$ assume that $\mu\in sA$ if $\mu s\in A$. Then the semigroup S=End$\mathcal{W}$ acts in $\mathfrak{M}_{\mathcal{D}}$ as a semigroup of endomorphisms of Boolean algebra $\mathfrak{M}_{\mathcal{D}}$. Moreover, one can verify that $\mathfrak{M}_{\mathcal{D}}$ with S turns out to be a Halmos algebra.

Now, we might go back to interpretation of formulas of $\tilde{\Phi}$. If f is a state of symbols of relations of Φ in $\mathcal{D}$ then the interpretation, i.e. mapping

$$\tilde{f}:\tilde{\Phi} \rightarrow \mathfrak{M}_{\mathcal{D}},$$

is defined as follows.

Let $u=\varphi(w_1,\ldots,w_n)$ be an elementary formula. Suppose that $\mu\in\tilde{f}(u)$ if a sequence $w_1^{\mu},\ldots,w_n^{\mu}$ belongs to the set $f(\varphi)\subset D_{1_1}\times\ldots\times D_{1_n}$ The mapping $\tilde{f}$ is defined for elementary formulas and since $\tilde{\Phi}$ is a free algebra over the set of elementary formulas, it is extended up to the homomorphism of algebras.

In this case, if formula u is treated as a notation of a query then $\tilde{f}(u)$ is considered as the reply to the query u. One and the same reply can correspond to different notations of a query. Therefore, two formulas u and v are defined to be equivalent in $\tilde{\Phi}$ if for any state f holds $\tilde{f}(u)=\tilde{f}(v)$. Denote the given equivalence on $\tilde{\Phi}$ by ρ.

We say that by the definition a query is a class of its equivalent notations and thus the universal set of queries is the set $U=\tilde{\Phi}/\rho$.

It was shown that the algebra of replies $\mathfrak{M}_D$ have the structure of Halmos algebra. Our next goal is to provide U by the same structure.

Boolean operations and existential quantifiers of the form $\exists x$ are defined on the set of formulas $\tilde{\Phi}$ in the usual way. We ought to define action on $\tilde{\Phi}$ of the quantifiers of the form $\exists(Y)$ and of the elements $s\in\mathrm{End}W$. We define these actions as many-valued operations on $\tilde{\Phi}$, transformed to one-valued ones in $\tilde{\Phi}/\rho$.

First, introduce the notion of a support of a formula $u\in\tilde{\Phi}$. Roughly speaking, one can determine a finite subset of variables $Y_0\subset X$ to be a support of the formula $u\in\tilde{\Phi}$, if $u=u(x_1,\ldots,x_n)$, $x_1\in Y_0$. In other words, it is the set of variables, which are included in the notation of the formula u. One can verify, that if Y_0 is a support of an element u and f is a state of the model $(\mathcal{D},\Phi,f)$ then in $\mathfrak{M}_{\mathcal{D}}$ the equality $\exists(\bar{Y}_0)\tilde{f}(u)=\tilde{f}(u)$ is fulfilled.

For arbitrary $Y\subset X$ the set $\exists(Y)u$ is defined as follows. Let an element v belong to $\exists(Y)u$ if there exists a support Y_0 of u such that $v=\exists x_1 \ldots \exists x_n u$, where $x_1,\ldots,x_n$ are the elements of $Y \cap Y_0$ taken in the definite order.

Define inductively a set su, $s \in \mathrm{End}W$. Take $u=\varphi(w_1,\ldots,w_n)$. Then set

$$su=\varphi(sw_1,\ldots,sw_n).$$

Let su and sv be defined for u, v and all $s \in S$. Assume:

$$s(u \vee v)=\{u_0 \mid u_0=u' \vee v',\ u' \in su,\ v' \in sv\},$$

$$s(\bar{u})=\{u' \mid u'=\bar{u}'',\ u'' \in su\},$$

$$s(u \wedge v)=\{u_0 \mid u_0=u' \wedge v',\ u' \in su,\ v' \in sv\}.$$

It remains to define $s\exists xu$. Let Y_0 be a support of u. According to s, consider a set Σ of elements $\sigma \in \mathrm{End}W$ such that $\sigma y=sy$, for $y \neq x$, $y \in Y_0$; $\sigma x \in X$ and does not belong to supports of the elements σy, $y \neq x$, $y \in Y_0$. Let $v \in s\exists xu$ if there exists some $\sigma \in \Sigma$ and $v=\exists \sigma x u'$, $u' \in \sigma u$. Thus, all su for $u \in \tilde{\Phi}$ and $s \in \mathrm{End}W$ are defined.

Proposition 2.1. [81] *Let* $u \rho v$, $u' \in su$, $v' \in sv$ *hold for equivalence* ρ *of* $\tilde{\Phi}$. *Then* $u' \rho v'$. *Similarly, if* $u' \in \exists(Y)u$ *and* $v' \in \exists(Y)v$ *then* $u \rho v$ *implies* $u' \rho v'$.

Hence, all the operations $\exists(Y)$ and s are correctly defined and one-valued on the quotient set $U=\tilde{\Phi}/\rho$. Moreover, every $\tilde{f}$ induces a homomorphism $\hat{f}: U \to \mathfrak{M}_{\mathcal{D}}$, so it follows that U turns out to be a Halmos algebra.

The support of an element can be defined also for an arbitrary Halmos algebra H. Namely, we say that $Y \subset X$ is a support of $h \in H$ if $\exists(\bar{Y})h=h$ holds. The locally finite part of H is a subalgebra consisting of all the elements $h \in H$ with the finite support. U is always locally finite Halmos algebra while $\mathfrak{M}_{\mathcal{D}}$ is not. The locally finite part of $\mathfrak{M}_{\mathcal{D}}$ is denoted by $V_{\mathcal{D}}$. The restriction of the homomorphism $\hat{f}: U \to \mathfrak{M}_{\mathcal{D}}$ gives rise to a homomorphism $\hat{f}: U \to V_{\mathcal{D}}$.

Remark. A few words on kernels of homomorphisms of Halmos algebras. If $f: H \to H'$ is a homomorphism of Halmos algebras then its kernel can be considered as inverse image of unit or inverse image of zero. In the first case we get a filter and in the second one an ideal.

A subset T of a Halmos algebra H is called a *filter* if it is closed with respect to products of elements, action of a universal quantifiers and for $a \in T$, $b \in H$ holds $a \vee b \in T$. Dually, a subset R of H is called an *ideal* if it is closed with respect to sums of elements, action of an existential quantifiers and for $a \in R$, $b \in H$ holds $a \wedge b \in R$.

It is easy to show that the filter associated as a kernel with the homomorphism $\hat{f}: U \to V_{\mathcal{D}}$ is the elementary theory (see [23]) of the model $(\mathcal{D}, \Phi, f)$.

5.2.3. Equality in Halmos algebras

Equality in a Halmos algebra H in a given scheme is a function which assigns to each pair of elements w and w' of one and the same W_i, $i \in \Gamma$, an element $d(w,w')$ of H. This function has to satisfy some conditions, which imitate the axioms of usual equality. Namely,

1) $sd(w,w')=d(sw,sw')$, $s \in \mathrm{End}\mathcal{W}$;
2) $d(w,w)=1$ for each $w \in W_i$;
3) $d(w_1,w'_1) \wedge \ldots \wedge d(w_n,w'_n) \le d(w_1 \ldots w_n \omega, w'_1 \ldots w'_n \omega)$, ω is the operation from Ω of the corresponding type;
4) $s^x_{w_1} h \wedge d(w_1,w_2) \le s^x_{w_2} h$, where $x \in X$, $h \in H$, $w_1, w_2 \in W_i$, if $n(x)=i$ and s^x_w is the endomorphism of $\mathcal{W}$ which takes x into w and does not change $y \ne x$.

It is shown [86], [48] that the equality on H can be defined in the unique way. Moreover, it can be proved that each Halmos algebra in the given scheme is embedded into the Halmos algebra with the equality.

In the Halmos algebras of the kind $\mathfrak{M}_{\mathcal{D}}$ the equality d is defined by the rule: an element $\mu \in \mathrm{Hom}(\mathcal{W}, \mathcal{D})$ belongs to $d(w_1,w_2)$ if $w_1^{\mu}=w_2^{\mu}$. It is obvious that all such elements $d(w_1,w_2)$ lie in the subalgebra $V_{\mathcal{D}}$.

5.3. Database model

5.3.1. Universal database

At the next step of the definition of an abstract database we shall associate a universal *-automation

$$\mathrm{Atm}\mathcal{D}=(\mathcal{F}_{\mathcal{D}},\mathrm{U},\mathrm{V}_{\mathcal{D}})$$

with every $\mathcal{D}\in\Theta$.

We have already constructed the algebras U and $V_{\mathcal{D}}$. The set of states $\mathcal{F}_{\mathcal{D}}$ of Atm$\mathcal{D}$ might be defined in an invariant way to be the set of homomorphisms Hom$(U,V_{\mathcal{D}})$. However, we prefer to consider the set $\mathcal{F}_{\mathcal{D}}$ as the set of states f - interpretations of Φ in $\mathcal{D}$ i.e. functions which realize every $\varphi\in\Phi$ of type $\tau=(i_1,\dots,i_n)$ as a subset of $D_{i_1}\times\dots\times D_{i_n}$. Boolean operations on such f are defined componentwise that makes $\mathcal{F}_{\mathcal{D}}$ to be a Boolean algebra. To complete the construction of Atm$\mathcal{D}$, we define the operation * by the rule

$$f*u=\hat{f}(u).$$

The triplet Atm$\mathcal{D}=(\mathcal{F}_{\mathcal{D}},U,V_{\mathcal{D}})$ is obviously a *-automaton, which plays the role of universal database in the given scheme.

Consider a functor property of Atm$\mathcal{D}=(\mathcal{F}_{\mathcal{D}},U,V_{\mathcal{D}})$. Every homomorphism $\delta:\mathcal{D}\to\mathcal{D}'$ induces a mapping

$$\tilde{\delta}:\mathrm{Hom}(W,\mathcal{D})\to\mathrm{Hom}(W,\mathcal{D}')$$

defined by the rule: if $\mu\in\mathrm{Hom}(W,\mathcal{D})$ then $\mu^{\tilde{\delta}}=\mu\delta$. In its turn $\tilde{\delta}$ implies a homomorphism of Boolean algebras

$$\delta_*:\mathfrak{M}_{\mathcal{D}'}\to\mathfrak{M}_{\mathcal{D}}.$$

Proposition 3.1. [81] *If $\delta:\mathcal{D}\to\mathcal{D}'$ is a surjective homomorphism then $\tilde{\delta}$ is also a surjection and $\delta_*:\mathfrak{M}_{\mathcal{D}'}\to\mathfrak{M}_{\mathcal{D}}$ is a monomorphism of Halmos algebras.*

The homomorphism δ induces also a monomorphism of Boolean algebras $\delta_*:\mathcal{F}_{\mathcal{D}'}\to\mathcal{F}_{\mathcal{D}}$. It is easy to verify that

$$(f*u)^{\delta_*}=(\delta_*f)*u,\quad f\in\mathcal{F}_{\mathcal{D}'},\quad u\in U.$$

Thus, the surjection $\delta:\mathcal{D}\to\mathcal{D}'$ induces an automata monomorphism:

$$\delta_*:\mathrm{Atm}\mathcal{D}'\to\mathrm{Atm}\mathcal{D},$$

identical on U.

5.3.2. Automata model of database

Let $\mathfrak{A}=(F,Q,R)$ be an abstract $*$-automaton in the given scheme $\mathcal{S}$ and $Atm\mathcal{D}=(\mathcal{F}_{\mathcal{D}},U,V_{\mathcal{D}})$ be an universal automaton in the given scheme.

The final step in the algebraic definition of a database presents the latter as a representation

$$\rho: (F,Q,R) \rightarrow Atm\mathcal{D}$$

of an abstract $*$-automaton into the universal one. More specifically, a *database over the scheme* $\mathcal{S}$ is a triplet $\rho=(\alpha,\beta,\gamma)$, where

- α is a mapping $F \rightarrow \mathcal{F}_{\mathcal{D}}$ which transforms the abstract states into real ones from $\mathcal{F}_{\mathcal{D}}$,
- β is a homomorphism $U \rightarrow Q$ which connects the abstract queries with formulas,
- γ is a homomorphism $R \rightarrow V_{\mathcal{D}}$ which associates a relation from $V_{\mathcal{D}}$ with any abstract reply,

and the following axiom holds:

$$(f*u^{\beta})^{\gamma}=f^{\alpha}*u.$$

Consider the particular case of the definition. In the automaton $Atm\mathcal{D}=(\mathcal{F}_{\mathcal{D}},U,V_{\mathcal{D}})$ take a subautomaton (F,U,R) where F is a subset of $\mathcal{F}_{\mathcal{D}}$ and R is a subalgebra of $V_{\mathcal{D}}$. Define a congruence τ on U by the rule: $u_1\tau u_2$ if for every $f\in F$ holds $f*u_1= f*u_2$. So we obtain a database (F,Q,R) where $Q=U/\tau$, β is the natural homomorphism $U \rightarrow U/\tau$, and α,γ are the inclusions. Such databases with compressed set of queries will be called *specific databases*.

One more point f) is usually added to the database scheme a)-e):

f) a set of axioms A which are satisfied for all admittable states.

Then each state $f\in F$ has to satisfy the axioms of A, i.e. $f*u=1$, $f\in F$, $u\in A$ hold true.

Henceforth,

$$\mathfrak{A}=(F,Q,R;U,\mathcal{D},\rho)$$

will denote database in the scheme $\mathcal{S}$, associated with automaton $\mathfrak{A}=(F,Q,R)$, algebra $\mathcal{D}$ and representation ρ. Algebra U presents the scheme

$\mathcal{S}$ in the notation of U.

We have a precise definition of a database as an algebraic structure. Consequently, we may naturally define homomorphisms, isomorphisms and automorphisms of databases. This allows us to define accurately the important concept of informational equivalence of two databases. Moreover, various constructions for databases (a wreath product, for example) can be defined, and the problem of decomposition of a database may be considered. We provide also a classification of databases with a fixed data algebra on the ground of Galois theory of databases [85], [86]. Here a group G is considered to be the group of automorphisms of data algebra $\mathcal{D}$.

5.3.3 Dynamic databases

Dynamic databases are associated with dynamic $*$-automata. A semigroup Σ which regulates updating of states must be added to the $*$-automaton constituting the database. Variation of states is taken into account also in the notation of queries.

In order to pass from dynamic $*$-automaton to dynamic database the universal dynamic $*$-automaton has to be defined. So we must define a dynamic Halmos algebra in the given scheme.

Let a semigroup Σ act on the set Φ of symbols of relations preserving the type of a relation, i.e. $\tau(\sigma\varphi)=\tau(\varphi)$, $\sigma\in\Sigma$, $\varphi\in\Phi$. It is proved that the action of Σ on Φ is uniquely extended to the representation of the semigroup Σ as the semigroup of endomorphisms of the algebra U. This means that U becomes a *dynamic Halmos algebra* with respect to Σ.

In a general case it is not assumed that Σ acts on Φ and we expand the set of relations. Suppose that Σ is the semigroup with the unit and consider a set $\Sigma\Phi$ of the formal expressions of the kind $\sigma\varphi$, $\sigma\in\Sigma$, $\varphi\in\Phi$. Identifying φ with 1φ, we get $\Phi\subset\Sigma\Phi$. The set $\Sigma\Phi$ is considered as a new set of symbols of relations where $\tau(\sigma\varphi)=\tau(\varphi)$. Setting $\sigma\circ(\sigma'\varphi)=\sigma\sigma'\varphi$ we have the free left action of the semigroup Σ on the set $\Sigma\Phi$. The transition from Φ to $\Sigma\Phi$ changes the database scheme. Therefore, the universal algebra of queries is also transformed into a new one (denoted by U^{Σ}) which is a Σ-dynamic Halmos algebra, as well as arises the new set of states $\mathcal{F}^{\Sigma}_{\mathcal{D}}$. However, the algebra of replies $V_{\mathcal{D}}$ is not changed. An action of the semigroup Σ in $\mathcal{F}^{\Sigma}_{\mathcal{D}}$ is defined by the rule: $f\circ\sigma(\sigma'\varphi)=f(\sigma\sigma'\varphi)$. So we come to $*$-automaton

$$(\mathcal{F}^{\Sigma}_{\mathcal{D}}, U^{\Sigma}, V_{\mathcal{D}}),$$

but it is not yet quite what we are looking for. The desired universal dynamic $*$-automaton Atm$\mathcal{D}$ arises if we replace the semigroup Σ by a free semigroup $\mathfrak{S}$ over a set of generators Y and adjoin the unit element.

The representation $\rho:\mathfrak{A} \to \mathrm{Atm}\mathcal{D}$ of an abstract dynamic Σ-automaton $\mathfrak{A}=(F,Q,R)$ in the dynamic $\mathfrak{S}$-automaton $\mathrm{Atm}\mathcal{D}=(\mathcal{F}^{\mathfrak{S}}_{\mathcal{D}}, U^{\mathfrak{S}}, V_{\mathcal{D}})$ is the tuple $\rho=(\alpha,\beta,\gamma,\eta)$ where the first three components are defined as above and $\eta:\mathfrak{S} \to \Sigma$ is a homomorphism of monoids subject to following conditions:

$$(f\circ\sigma^{\eta})^{\alpha}=f^{\alpha}\circ\sigma;\quad (\sigma u)^{\beta}=\sigma^{\eta}u^{\beta},\quad f\in F,\ \sigma\in\mathfrak{S},\ u\in U.$$

A *dynamic database* is a dynamic $*$-automaton together with its representation in the universal dynamic $*$-automaton Atm$\mathcal{D}$, $\mathcal{D}\in\Theta$.

The transition to the free semigroup $\mathfrak{S}$ allows to specify the notation of a dynamic query.

Henceforth we confine the subject only by static databases.

5.4. Homomorphisms of databases

5.4.1. Homomorphisms in the fixed scheme

First of all recall that each homomorphism of algebras $\delta:\mathcal{D}' \to \mathcal{D}$ generates two homomorphisms of Boolean algebras $\delta_*:V_{\mathcal{D}} \to V_{\mathcal{D}'}$ and $\delta_*:\mathcal{F}_{\mathcal{D}} \to \mathcal{F}_{\mathcal{D}'}$. If the initial δ is a surjection, then $\delta_*:V_{\mathcal{D}} \to V_{\mathcal{D}'}$ is the injection of Halmos algebras. Both δ_* induce injection of automata:

$$\delta_*:\mathrm{Atm}\mathcal{D} \to \mathrm{Atm}\mathcal{D}',$$

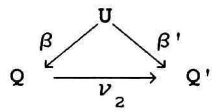
which does not change queries.

Let $\mathfrak{A}=(F,Q,R;U,\mathcal{D},\rho)$ and $\mathfrak{A}'=(F',Q',R';U,\mathcal{D}',\rho')$ be two databases with $\rho=(\alpha,\beta,\gamma)$ and $\rho'=(\alpha',\beta',\gamma')$.

A homomorphism of databases $\mu:\mathfrak{A} \to \mathfrak{A}'$ is a pair of homomorphisms $\mu=(\nu,\delta)$, where ν is a homomorphism of $*$-automata $\nu=(\nu_1,\nu_2,\nu_3)$: $(F,Q,R) \to (F',Q',R')$ and δ is a homomorphism of data algebras

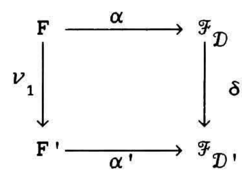
$\delta:\mathcal{D}' \to \mathcal{D}$, subject to two conditions:

1) the natural diagram



is commutative,

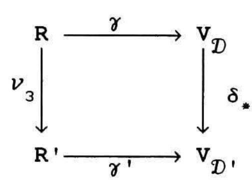
2) second condition is described by the diagram



with weakened commutativity: $f^{\nu_1\alpha'} \le f^{\alpha\delta_*}$ for any $f \in F$.

The second condition means merely that transition $f^{\nu_1\alpha'} \to f^{\alpha}$ together with $\delta:\mathcal{D}' \to \mathcal{D}$ is a homomorphism of the corresponding models, i.e. of algebras $\mathcal{D}'$ and $\mathcal{D}$ with relations of Φ realized there.

It can be checked, that if $\mu=(\nu,\delta):\mathfrak{A} \to \mathfrak{A}'$ is an isomorphism of databases, then the second diagram becomes commutative diagram in the usual sense, that is $f^{\nu_1\alpha'} = f^{\alpha\delta_*}$ Moreover, in this case it is possible to add the third commutative diagram to the definition:



Thus, in the definition of isomorphism we can obtain a natural symmetry, which is absent in the definition of an arbitrary database homomorphism.

We now examine connections between homomorphisms of models and homomorphisms of databases. For fixed Θ and Φ we consider models

$(\mathcal{D},\Phi,f)$, $\mathcal{D}\in\Theta$. A database $\mathfrak{A}=(F,Q,R)$ where F consists of a unique state f, R is the image of the homomorphism $\hat{f}:U \to V_{\mathcal{D}}$, i.e. $R=\mathrm{Im}\hat{f}$, is a subbase in Atm$\mathcal{D}$ and it can be assigned to every such model. Denote Ker$\hat{f}$ by T. It is a filter in U, which coincides with elementary theory of the model f. Then an algebra Q of $\mathfrak{A}$ is determined by Q=U/T. The operation in Atm$\mathcal{D}$ gives rise to the operation * in $\mathfrak{A}$. The question is to what extent this transition $(\mathcal{D},\Phi,f) \to (F,Q,R)$ from models to databases enjoys the functorial properties.

Let $(\mathcal{D}',\Phi',f')$ be another model and $\mathfrak{A}'=(F',Q',R')$ be its database. It follows directly from definitions, that every database homomorphism $\mu=(\nu,\delta):\mathfrak{A} \to \mathfrak{A}'$ induces a model homomorphism. On the other hand, assume, that the transition $f \to f'$ defines the homomorphism of models $\delta:(\mathcal{D}',\Phi',f') \to (\mathcal{D},\Phi,f)$ so as we could reconstruct the homomorphism of databases $\mu=(\nu,\delta)=((\nu_1,\nu_2,\nu_3),\delta)$. In fact δ is already determined by μ and ν_1 is defined by the transition $f \to f'$. It is natural to define $\nu_3:R \to R'$ by a homomorphism, induced by the injection $\delta_*:V_{\mathcal{D}} \to V_{\mathcal{D}'}$. ν_2 is determined according to the condition of commutativity of the corresponding first diagram.

However, we have not obtained a database homomorphism yet. The fact is that the homomorphism δ_* does not always inject algebra R into algebra R', there is no correlation with the operation, the second diagram is fulfilled trivially, but the commutativity of the first diagram does not hold because there is no connection between the elementary theories for f and f'. The situation significantly changes if we take f^{δ_*} as f'. Elementary theories for f and f^{δ_*} coincide, thus Q=Q', and ν is a trivial homomorphism. Besides, we have

$$(\mathrm{Im}\hat{f})^{\delta_*}=\mathrm{Im}f^{\delta_*}\ .$$

As a result we get a homomorphism of databases, which is simultaneously an isomorphism of corresponding *-automata $\mathfrak{A}$ and $\mathfrak{A}'$. This is true, in particular, in the case, when $\delta:\mathcal{D}' \to \mathcal{D}$ is an isomorphism of models. Thus isomorphisms of databases imply isomorphisms of models.

5.4.2 Homomorphisms with the modification of the scheme

Now we consider homomorphisms of databases with variable set of symbols of relations Φ. To the two given sets of symbols of relations Φ and Φ' correspond different Halmos algebras U and U'. We assume that there is the homomorphism $\xi: U' \to U$ which determines the scheme modification. Now, for every $\delta: \mathcal{D}' \to \mathcal{D}$ we shall give the new definition of the mapping $\delta_*: \mathcal{F}_{\mathcal{D}} \to \mathcal{F}_{\mathcal{D}'}$, which is associated with this ξ.

Let us note first that the mapping $\hat{}: \mathcal{F}_{\mathcal{D}} \to Hom(U, V_{\mathcal{D}})$ which connects states and homomorphisms, is a bijection [81]. We consider the composition of homomorphisms $\hat{f}: U \to V_{\mathcal{D}}$ and $\delta_*: V_{\mathcal{D}} \to V_{\mathcal{D}'}$. For a fixed Φ we have

$$\widehat{f^{\delta_*}} = \hat{f}\delta_*: U \to V_{\mathcal{D}'}$$

For variable scheme and given $\xi: U' \to U$ we define

$$\widehat{f^{\delta_*}} = \xi\hat{f}\delta_*: U' \to V_{\mathcal{D}'} .$$

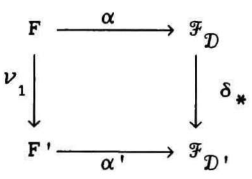
Now we can define homomorphisms of databases in the case of variable set Φ. Let $\mathfrak{A}=(F,Q,R;U,\mathcal{D},\rho)$ and $\mathfrak{A}'=(F',Q',R';U',\mathcal{D}',\rho')$ be two databases.

Homomorphism $\mu: \mathfrak{A} \to \mathfrak{A}'$ of databases with modification of the scheme is a triplet $\mu=(\nu,\xi,\delta)$ where $\nu=(\nu_1,\nu_2,\nu_3)$ is a homomorphism of automata, $\xi: U' \to U$ is a homomorphism of algebras of queries, $\delta: \mathcal{D}' \to \mathcal{D}$ is a homomorphism of data algebras. For symmetry, it is convenient to direct the mapping ν_2 opposite to the mappings ν_1 and ν_3, that is $\nu_2: Q' \to Q$. The mapping $\mu=(\nu,\xi,\delta)$ should satisfy the following diagrams:

1) Commutative diagram

$$\begin{array}{ccc} U' & \xrightarrow{\xi} & U \\ \beta' \downarrow & & \downarrow \beta \\ Q' & \xrightarrow[\nu_2]{} & Q \end{array}$$

2) Weakened commutative diagram



The second diagram guarantees that the correspondence of the states is a homomorphism of models with the variable scheme.

Now we deal with specific databases, i.e. the case when the mappings α and γ in the representation $\rho=(\alpha,\beta,\gamma)$ are trivial.

A homomorphism $\mu=(\nu,\xi,\delta)$ is called a *replacement of the scheme* if data algebras and replies algebras in databases $\mathfrak{A}$ and $\mathfrak{A}'$ coincide, and the mappings ν_3 and δ are trivial.

Theorem 4.1. *Every homomorphism* $\mu=(\nu,\xi,\delta):\mathfrak{A} \rightarrow \mathfrak{A}'$ *where* $\nu_2:Q' \rightarrow Q$ *is a surjection, admits a canonical decomposition in to a product of two homomorphisms* μ_1 *and* μ_2. *The first of them does not change the scheme, and the second one is connected only with the replacement of the scheme.*

This proposition is used in the problem of reconstruction of databases, i.e. replacement of a database by an equivalent one which is more convenient to use.

5.5. Constructive databases

5.5.1. General notes

Real database model has to be connected with the programming means. The intermediate step in this direction is the database constructivization i.e. study of existence of algorithms Databases with the finite $\mathcal{D}$ always admit constructivization, thus we face a problem of effective algorithms and programs. The situation is essentially different in the case of infinite $\mathcal{D}$.

The well-known definition of a constructive algebraic structure [30], [69] may be applied to models of databases as well. However, there arise some fundamental difficulties because of which the constructivization never can be carried out completely.

According to A. I. Malcev, algebraic system is constructive, if all its domains consist of constructive elements and are constructively defined as sets, and all main operations and relations are also constructive, i.e. the required algorithms do exist. This general definition is applied to the various components of databases, for example to data algebra $\mathcal{D}$ or $*$-automaton (F,Q,R). However, its application to a whole database faces conceptual difficulties and leads to very strong and rarely fulfilled conditions. Therefore, as far as this is possible under given conditions, we must strive for a reasonable constructivization, taking into account the main problem: how to calculate a reply to a given query.

Each reply to a query q in the state f looks like $f*q$ and one needs an algorithm which checks if $\mu\in f*q$ for arbitrary $\mu\in\mathrm{Hom}(W,\mathcal{D})$. There are a number of approaches here. For example, we can speak of algorithms for various fixed f and q or we can fix f and search for an algorithm, enclosing all q from a certain class. However we apply the strongest form of constructivization for specific databases.

Let $\mathfrak{A}=(F,Q,R;U,\mathcal{D},\rho)$ be such database. *Database* $\mathfrak{A}$ *is constructive* if there exists an algorithm, which checks up the inclusion $\mu\in f*q$ for all $\mu\in\mathrm{Hom}(W,\mathcal{D})$, $f\in F$, $q\in Q$.

The existence of such algorithm leads to the restrictions on database and we will discuss some of them.

We consider Halmos algebras with the equality. Hence, there arise the queries of a type $x_1\ldots x_n\omega=y$, where $\omega\in\Omega$ have the type $\tau=(i_1,\ldots,i_n;j)$ and $x_1,\ \ldots,x_n,y$ are the variables of corresponding sorts. The reply to such a query is independent from the state f. It is easy to understand that the existence of required algorithm implies that all operations of $\mathcal{D}$ are constructive. Thus, it makes sense to include the constructiveness of data algebra $\mathcal{D}$ into every definition of constructive database.

Let now, $\varphi\in\Phi$ be a relation of type $\tau=(i_1,\ldots,i_n)$. Let q be defined by the elementary formula $\varphi(x_1,\ldots,x_n)$. Given f, the existence of algorithm for calculation of reply to such query means that the state f defines the constructive realization of all relations, i.e. $(\mathcal{D},\Phi,f)$ is a constructive structure. Moreover, if algebra $\mathcal{D}$ is constructive, then for constructive f one can calculate the replies to the queries of the type $\varphi(w_1,\dots,w_n)$, where $w_i \in W_i$ are the θ-terms of corresponding sorts. So, it is quite reasonable to claim that all states $f \in F$ should be constructive.

A query q is called an *opened* query if it can be written without quantifiers. Hence, we can assert, that if the algebra $\mathcal{D}$ and the state f are constructive then there is an algorithm for calculating replies to any opened query in the state f.

Constructiveness of database, as defined above, actually implies the uniform constructiveness for all states $f \in F$. Moreover, the required possibility to get reply to any query means that every state $f \in F$ is not only constructive, but also has a decidable elementary theory.

Now, we consider one more natural definition of constructive specific database.

Database $\mathfrak{A}=(F,Q,R;U,\mathcal{D},\rho)$ is *constructive*, if

1) Data algebra $\mathcal{D}$ is constructive.
2) The automaton (F,Q,R) is a constructive one, in particular, Halmos algebras Q and R are constructive.
3) Every state $f \in F$ is constructive and every $r \in R$ is a constructive relation.

Some connections between these two definitions have been discussed, but it would be of great interest to study them in detail. In particular, the investigation of constructive Halmos algebras becomes very important, while constructive Boolean algebras have been studied before [43].

Let us note the following question: how should a model $(\mathcal{D},\Phi,f)$ look so that Halmos algebra $\mathrm{Im}\hat{f}=R$ is constructive. It is also interesting to study filters T in U, for which quotient algebra U/T allows constructivization.

5.5.2. Introduction of data into language

We would like to point out that it is often convenient to introduce the data algebras $\mathcal{D}=(D_i, i\in\Gamma)$ into the language and algebra of queries for the constructivization of databases. Description of $\mathcal{D}$ by generators and defining relations may be used here.

Let $M=(M_i, i\in\Gamma)$ be the set of generators of $\mathcal{D}$, and let τ be the set of defining relations of $\mathcal{D}$. We associate a variable y_a with each $a\in M_i$. Therefore, a set of variables Y_i corresponds to every set M_i, and we obtain a complex $Y=(Y_i, i\in\Gamma)$. Furthermore, let $W_{\mathcal{D}}$ be an algebra over Y free in θ. The transition $y_a \to a$ for all a gives rise to an epimorphism $\nu: W_{\mathcal{D}} \to \mathcal{D}$. Its kernel ρ is generated by the set τ, and we have an isomorphism $W_{\mathcal{D}}/\rho \to \mathcal{D}$.

We also associate a symbol of nullary operation ω_a with every $a\in M_i$, $i\in\Gamma$, and denote by Ω' the union of the set of all these symbols with Ω. Let θ' be the variety of Ω'-algebras defined by the identities of θ together with the defining relations of $\mathcal{D}$. Now, if

$$w(y_{a_1},\dots,y_{a_n})=w'(y_{a_1},\dots,y_{a_n})$$

is one of such relations, it must be rewritten as

$$w(\omega_{a_1},\dots,\omega_{a_n})=w'(\omega_{a_1},\dots,\omega_{a_n}).$$

There are no variables in the latter equality, therefore it may be considered as an identity.

Let W be the free algebra over the complex X in θ, and W' be such an algebra in θ' All ω_a are elements of W' and generate a subalgebra in it. We denote it by $\mathcal{D}'$.

Proposition 5.1. *The algebra $\mathcal{D}'$ is isomorphic to $\mathcal{D}$. Let $A=(A_i, i\in\Gamma)$ be any algebra from θ', and $B=(B_i, i\in\Gamma)$ be the subalgebra of A generated by the nullary operations. Then B is a homomorphic image of $\mathcal{D}$.*

The initial scheme $\mathcal{S}$ together with the algebra $\mathcal{D}$ determine the universal database

$$\mathrm{Atm}\mathcal{D}=(\mathcal{F}_{\mathcal{D}}, U, V_{\mathcal{D}}).$$

Let $\mathcal{S}'$ be the scheme corresponding to θ' $\mathcal{D}$ may be considered as an algebra from θ', too. So, the set of all states for the same set of symbols of relations Φ is not changed. Obviously, we may identify the set of homomorphisms Hom($\mathcal{W}'$, $\mathcal{D}$) with Hom($\mathcal{W}$, $\mathcal{D}$); hence the algebra $V_{\mathcal{D}}$ also is not changed. Only the algebra U must be replaced by a new algebra of queries U'. There exists a canonical mapping U $\rightarrow$ U', which allows us to consider elements from U as elements from U'. Therefore, we have obtained an automaton

$$\mathrm{Atm}'\mathcal{D}=(\mathcal{F}_{\mathcal{D}}, U', V_{\mathcal{D}}).$$

Now, we shall consider an application of such transition.

Let $u \in U$, $\mu \in \mathrm{Hom}(\mathcal{W},\mathcal{D})$, and let $Y=(y_1,\ldots,y_n)$ be the support of u. We denote $y_1^{\mu}=a_1$, ..., $y_n^{\mu}=a_n$ Let us extend the homomorphism $\mu:\mathcal{W} \rightarrow \mathcal{D}$ up to the homomorphism $\mu:\mathcal{W}' \rightarrow \mathcal{D}$ and then consider a restriction $\mu:\mathcal{D}' \rightarrow \mathcal{D}$. The mapping μ is an isomorphism, and by 0_i we denote the image of y_i^{μ} in $\mathcal{D}'$, i=1,...,n. Finally, let $v=v_{\mu}$ be the element of U determined by the formula

$$((y_1=0_1)\wedge\ldots\wedge(y_n=0_n) \Rightarrow u)=v.$$

Proposition 5.2. *Let f be a state. Then* $\mu \in f*u$ *if and only if the equality* $f*v_{\mu}=1$ *holds true.*

Let $(\mathcal{D},\Phi,f)$ be a model with decidable elementary theory in the expanded algebra of queries U', and R=ImU under the homomorphism $\hat{f}:U \rightarrow V_{\mathcal{D}}$. Then we have the corollary:

Proposition 5.3. *Each element from Halmos algebra* R *is a constructive subset in* Hom($\mathcal{W}$, $\mathcal{D}$).

Proposition 5.3 shows that the verification of the inclusion $\mu \in f*u$ is reduced, in the extended scheme, to the verification whether the definite proposition v is true in the model $(\mathcal{D},\Phi,f)$. In model theory, such questions of effective verification were considered long ago [72].

It is supposed that there is a good finite description of the entity domain, i.e. of the model $(\mathcal{D},\Phi,f)$, allowing us to reduce the problem of validity of sentences to a suitable problem of derivability of formulas in pure first order language. Namely, if a set of elements

$u_1, u_2, \ldots, u_n$ in U' serves as a description of this model , and v is a sentence the truth of which is verified on this model, then we construct a new proposition

$$u_1 \wedge \ldots \wedge u_n \Rightarrow v$$

and verify its absolute truth in U', i.e. whether it is equal to 1 in U' Furthermore, if the variety θ has a finite description, then one can pass to the first order calculus. Thus, there arises a possibility to apply methods of proof theory and mechanical theorem proving.

Generally speaking, a computer should act like a human. If there is some hypothesis, one has to act in two parallel directions: to search for its possible proofs on the one hand, and the possible counter-examples on the other hand. Both these parallel lines ought to be well formalized for the computer. Of particular importance is the case when one construct counter-examples in finite models. The ideas of McKinsey's well-known work [72] are based on these remarks.

In applications to databases all this is connected with the ideas of logical programming and the PROLOG language. This language is designed for the special type propositions, the so-called Horn clauses. There is some optimal strategy of proofs searching for them but there does not exist methods which would lead us to counter it.

The main idea of logical programming is that the possible corollaries are derived from the given description of entity domain. As to queries, their derivability from the description is verified. As a rule, the entity domain is finite and there is no problem of algorithm existence, the main efforts should be directed to the looking for effective algorithms.

There is no set of symbols of operations Ω in PROLOG. Such set is associated with functional programming and LISP language. The exploration of combination of logical and functional programming based on λ-calculus, do hold.

Now we consider the situation, when data algebras are approximated by finite algebras.

As it was mentioned, the computer can construct counter-examples, if they may be picked out among finite models. This remark inspires the following definition [72]: a closed element $u \in U$ is called *finite-reduced* in respect to class of models K, if falsehood of u on some model from K implies its falsehood on a finite model from K.

This definition can be applied also to open elements, using its enclosing on universal quantifiers. In particular, it may be applied to the formulas of the kind $v=v_{\mu}$ from proposition 5.2.

We consider below a similar notion, which can be applied to arbitrary queries u.

Given $\mathcal{D}$, let $\delta_{\alpha}:\mathcal{D} \to \mathcal{D}_{\alpha}$, $\alpha \in I$ be a system of homomorphisms. The set I is assumed to be ordered, and every α and β of I are covered by some γ. Suppose that if $\beta > \alpha$, then there is a surjective homomorphism $\delta_{\alpha\beta}:\mathcal{D}_{\beta} \to \mathcal{D}_{\alpha}$ with commutative diagram

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\delta_{\alpha}} & \mathcal{D}_{\alpha} \\ & {\scriptstyle \delta_{\beta}} \searrow \quad \nearrow {\scriptstyle \delta_{\alpha\beta}} & \\ & \mathcal{D}_{\beta} & \end{array}$$

Such a diagram defines the commutative diagram for automata

$$\begin{array}{ccc} \mathbf{Atm}\mathcal{D} & \xleftarrow{\delta_{\alpha *}} & \mathbf{Atm}\mathcal{D}_{\alpha} \\ & {\scriptstyle \delta_{\beta *}} \nwarrow \quad \swarrow {\scriptstyle \delta_{\alpha\beta *}} & \\ & \mathbf{Atm}\mathcal{D}_{\beta} & \end{array}$$

In particular, the following diagrams hold:

$$\begin{array}{ccc} \mathcal{F}_{\mathcal{D}} & \xleftarrow{\delta_{\alpha *}} & \mathcal{F}_{\mathcal{D}_{\alpha}} \\ & {\scriptstyle \delta_{\beta *}} \nwarrow \quad \swarrow {\scriptstyle \delta_{\alpha\beta *}} & \\ & \mathcal{F}_{\mathcal{D}_{\beta}} & \end{array} \qquad\qquad \begin{array}{ccc} V_{\mathcal{D}} & \xleftarrow{\delta_{\alpha *}} & V_{\mathcal{D}_{\alpha}} \\ & {\scriptstyle \delta_{\beta *}} \nwarrow \quad \swarrow {\scriptstyle \delta_{\alpha\beta *}} & \\ & V_{\mathcal{D}_{\beta}} & \end{array}$$

where all the homomorphisms are injections.

Denote by R_{α} the image of $\delta_{\alpha *}:V_{\mathcal{D}_{\alpha}} \to V_{\mathcal{D}}$. All the subalgebras R_{α} form a local system in $V_{\mathcal{D}}$, and therefore $R=\cup R_{\alpha}$ is a subalgebra in $V_{\mathcal{D}}$.

Proposition 5.4. *If algebra $\mathcal{D}$ is countable, all $\mathcal{D}_\alpha$ are finite and homomorphisms $\delta_\alpha:\mathcal{D} \to \mathcal{D}_\alpha$ are computable, then all elements from algebra* R *are constructive subsets in* $\mathrm{Hom}(W,\mathcal{D})$.

In database theory it is important to consider subalgebras $R\subset V_{\mathcal{D}}$ containing equalities. Unlike the proposition 5.4, in the proposition 5.5 the algebra R is, as a rule, without equality. In accordance with general theory of Halmos algebras the diagonal $D_{w,w'}\subset\mathrm{Hom}(W,\mathcal{D})$ corresponds to the equality $w\equiv w'$ in $V_{\mathcal{D}}$. It consists of all $\mu:W \to \mathcal{D}$ for which $w^\mu=w'^\mu$ holds.

The next proposition describes the structure of diagonal in $V_{\mathcal{D}}$. Let the set of homomorphisms $\delta_\alpha:\mathcal{D} \to \mathcal{D}_\alpha$, $\alpha\in I$ be complete, i.e. if a and b are two different elements of the same sort in $\mathcal{D}$, then there exists $\alpha\in I$ such that $\delta_\alpha(a)\neq\delta_\alpha(b)$.

Proposition 5.5. *The following equality takes place*

$$D_{w,w'}=\bigcap_\alpha (D^\alpha_{w,w'})^{\delta_{\alpha *}}$$

We generalize it in proposition 5.6.

Along with injections $\delta_{\alpha *}:\mathcal{F}_{\mathcal{D}_\alpha} \to \mathcal{F}_{\mathcal{D}}$ we consider mappings $\mathcal{F}_{\mathcal{D}} \to \mathcal{F}_{\mathcal{D}_\alpha}$ which are defined as follows: if $(\mathcal{D},\Phi,f)$ is a model, then $(\mathcal{D}_\alpha,\Phi,f^{\delta^*_\alpha})$ is a model defined by the natural transition to the corresponding quotient model. For simplicity we write f^{δ_α} instead of $f^{\delta^*_\alpha}$.

Let $f\in\mathcal{F}_{\mathcal{D}}$ and $\mu\in\mathrm{Hom}(W,\mathcal{D})$. The query u in state f is said to be *compatible* with the given system of homomorphisms, if for any $\mu\in\mathrm{Hom}(W,\mathcal{D})$, $\mu\notin f*u$ implies that $\mu\delta_\alpha\notin f^{\delta_\alpha}*u$ for some $\alpha\in I$. The query u is called *co-compatible* with the given system of homomorphisms if from $\mu\in f*u$ it follows that $\mu\delta_\alpha\in f^{\delta_\alpha}*u$, for some $\alpha\in I$.

It is easy to verify, that if u is compatible, then its negation $\bar{u}$ is co-compatible, and vice versa. Coordination of equations with the given set of homomorphisms means the completeness of this set.

Proposition 5.6. *If* u *is a positive query, compatible with the set of homomorphisms in state* f, *then* $f{*}u=\cap(f^{\delta_\alpha}{*}u)^{\delta_{\alpha*}}$. *If* u *is negative and co-compatible query, then* $f{*}u=\cup_\alpha(f^{\delta_\alpha}{*}u)^{\delta^\alpha_{\alpha*}}$.

Here positive query is a query which is constructed from primitive ones without negations. Negative query is a negation of the positive query.

This proposition may be useful for the calculation of reply to the query if the set of homomorphisms $\delta_\alpha:\mathcal{D}\to\mathcal{D}_\alpha$, $\alpha\in I$ defines the finite approximation of algebra $\mathcal{D}$.

Finally, we formulate the proposition, devoted to the conditions of compatibility. Fix a state f and consider elements of algebra u, compatible with the given set of homomorphisms of algebra $\mathcal{D}$.

Proposition 5.7.

1) *If* u *and* v *are compatible, then* $u\wedge v$ *is compatible too.*
2) *If* u *is compatible, then* $\forall(Y)u$ *is compatible for any* $Y\in X$.
3) *If* u *is compatible, then* su *is compatible for any* $s\in \mathrm{End}W$.
4) *If* u *and* v *are positive and compatible, then* $u\vee v$ *is also compatible.*
5) *If* u *is a positive element and* v *is compatible, then* $u\Rightarrow v$ *is compatible.*

The existential quantifiers can break the coordination property.

5.6. Constructions in database theory

We will consider only specific databases. In this case the algebra of replies is always a simple Halmos algebra, and therefore, all homomorphisms of the kind $\gamma:R\to R'$ turn out to be monomorphisms.

Let $\mathcal{D}=\mathcal{D}_1\times\mathcal{D}_2$ be a Cartesian product of algebras and $\pi_1:\mathcal{D}\to\mathcal{D}_1$, $\pi_2:\mathcal{D}\to\mathcal{D}_2$ the natural projections. Take the corresponding injections of Halmos algebras $\pi_{1*}:V_{\mathcal{D}_1}\to V_{\mathcal{D}}$ and $\pi_{2*}:V_{\mathcal{D}_2}\to V_{\mathcal{D}}$. The algebra, generated by Halmos algebras $\pi_1(V_{\mathcal{D}_1})$ and $\pi_2(V_{\mathcal{D}_2})$ is denoted by $V_{\mathcal{D}_1}\otimes V_{\mathcal{D}_2}$. We describe the structure of $V_{\mathcal{D}_1} \otimes V_{\mathcal{D}_2}$

First of all, the set Hom($\mathcal{W},\mathcal{D}$) may be canonically represented as a Cartesian product Hom($\mathcal{W},\mathcal{D}_1$)$\times$Hom($\mathcal{W},\mathcal{D}_2$). For $A \subset$ Hom($\mathcal{W},\mathcal{D}_1$), $B \subset$ Hom($\mathcal{W},\mathcal{D}_2$) their Cartesian product $A\times B$ may be represented as $A\times B=\pi_{1*}(A)\cap\pi_{2*}(B)$. If the supports of A and B are finite, then $A\times B$ also has a finite support.

Proposition 6.1. *The subalgebra* $V_{\mathcal{D}_1} \otimes V_{\mathcal{D}_2}$ *in* $V_{\mathcal{D}}$ *consists of the finite unions of elements* $A\times B$, $A\in V_{\mathcal{D}_1}$, $B\in V_{\mathcal{D}_2}$ *If* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *are finite, then* $V_{\mathcal{D}_1} \otimes V_{\mathcal{D}_2} = V_{\mathcal{D}}$.

It is easy to verify that if D, D^1 and D^2 are the diagonals in $\mathcal{D}$, $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively, then for any w,w' holds

$$D_{w,w'} = D^1_{w,w'} \times D^2_{w,w'}$$

and this means, that the algebra $V_{\mathcal{D}_1} \otimes V_{\mathcal{D}_2}$ contains all equalities. This is also true for $R_1 \otimes R_2 \subset V_{\mathcal{D}_1} \otimes V_{\mathcal{D}_2}$ where R_1 is a subalgebra in $V_{\mathcal{D}_1}$

Our next goal is to define the product of databases. Let the scheme of database $\mathcal{S}$ be fixed. This defines the universal algebra of queries U. Consider algebras $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}=\mathcal{D}_1\times\mathcal{D}_2$ of Θ. The automata Atm$\mathcal{D}=(\mathcal{F}_{\mathcal{D}}, U, V_{\mathcal{D}})$, Atm$\mathcal{D}_1=(\mathcal{F}_{\mathcal{D}_1}, U, V_{\mathcal{D}_1})$ and Atm$\mathcal{D}_2=(\mathcal{F}_{\mathcal{D}_2}, U, V_{\mathcal{D}_2})$ correspond to them. Projections $\pi_1:\mathcal{D} \to \mathcal{D}_1$ and $\pi_2:\mathcal{D} \to \mathcal{D}_2$ give rise to the injections of automata $\pi_1:$Atm$\mathcal{D}_1 \to$ Atm$\mathcal{D}$; $\pi_2:$Atm$\mathcal{D}_2 \to$ Atm$\mathcal{D}$. In particular, we have the homomorphisms of Boolean algebras

$$\pi_1:\mathcal{F}_{\mathcal{D}_1} \to \mathcal{F}_{\mathcal{D}}, \qquad \pi_2:\mathcal{F}_{\mathcal{D}_2} \to \mathcal{F}_{\mathcal{D}}.$$

Let $\mathcal{F}_{\mathcal{D}_1} \otimes \mathcal{F}_{\mathcal{D}_2}$ be a subalgebra of the Boolean algebra $\mathcal{F}_{\mathcal{D}}$ generated by the algebras $\pi_{1*}(\mathcal{F}_{\mathcal{D}_1})$ and $\pi_{2*}(\mathcal{F}_{\mathcal{D}_2})$.

Take $f_1\in\mathcal{F}_{\mathcal{D}_1}$, $f_2\in\mathcal{F}_{\mathcal{D}_2}$ and let $(\mathcal{D}_1,\Phi_1,f_1)$, $(\mathcal{D}_2,\Phi_2,f_2)$ be the relative models. We can define the product $f_1\times f_2$ in accordance with the definition of Cartesian product of the models: $(\mathcal{D}_1\times\mathcal{D}_2,\Phi,f_1\times f_2)$. It can be checked that if u is an element of U, defined by elementary formula, then

$$(f_1\times f_2)*u=(f_1\times u)*(f_2\times u)$$

Hence, $f_1\times f_2=\pi_{1*}(f_1)\cap\pi_{2*}(f_2)$ and it is proved that the Boolean algebra $\mathcal{F}_{\mathcal{D}_1}\otimes\mathcal{F}_{\mathcal{D}_2}$ consists of all sums of elements $f_1\times f_2$, with $f_1\in\mathcal{F}_{\mathcal{D}_1}$, $f_2\in\mathcal{F}_{\mathcal{D}_2}$. As a result, we have a subautomaton $\mathrm{Atm}\mathcal{D}_1\otimes\mathrm{Atm}\mathcal{D}_2$ in $\mathrm{Atm}\mathcal{D}$ generated by the copies of automata $\mathrm{Atm}\mathcal{D}_1$ and $\mathrm{Atm}\mathcal{D}_2$:

$$(\mathcal{F}_{\mathcal{D}_1}\otimes\mathcal{F}_{\mathcal{D}_2},U,V_{\mathcal{D}_1}\otimes V_{\mathcal{D}_2})=\mathrm{Atm}\mathcal{D}_1\otimes\mathrm{Atm}\mathcal{D}_2$$

Let now $\mathfrak{A}_1=(F_1,U,R_1;U,\mathcal{D}_1)$ and $\mathfrak{A}_2=(F_2,U,R_2;U,\mathcal{D}_2)$ be two data bases with one and the same algebra of queries U. Construct the product $\mathfrak{A}_1\otimes\mathfrak{A}_2$. First, let us take the product of automata

$$\mathrm{Atm}\mathcal{D}_1\otimes\mathrm{Atm}\mathcal{D}_2=(\mathcal{F}_{\mathcal{D}_1}\otimes\mathcal{F}_{\mathcal{D}_2},U,V_{\mathcal{D}_1}\otimes V_{\mathcal{D}_2})$$

and then consider the subset $F_1\times F_2$ in $\mathcal{F}_{\mathcal{D}_1}\otimes\mathcal{F}_{\mathcal{D}_2}$ consisting of all $f_1\times f_2$, $f_1\in F_1$, $f_2\in F_2$.

Let $R_1\otimes R_2$ be a subalgebra in $V_{\mathcal{D}_1}\otimes V_{\mathcal{D}_2}$. Then, for any $u\in U$ holds

$$(f_1\times f_2)*u=(f_1\times u)*(f_2\times u)\in R_1\otimes R_2$$

So, we obtain an automaton $(F_1\times F_2,U,R_1\otimes R_2)$. The associated with this automaton database

$$\mathfrak{A}=\mathfrak{A}_1\otimes\mathfrak{A}_2=(F_1\times F_2,U,R_1\otimes R_2;U,\mathcal{D}_1\times\mathcal{D}_2)$$

is called the *product of databases* $\mathfrak{A}_1$ and $\mathfrak{A}_2$.

Thus defined product of databases is coordinated with the product of models.

A union of databases is defined in a similar way, as a database

$$(F,Q,R_1\otimes R_2;U,\mathcal{D}_1\times\mathcal{D}_2),$$

associated with the subautomaton $(F,U,R_1\otimes R_2)\subset\mathrm{Atm}\mathcal{D}_1\otimes\mathrm{Atm}\mathcal{D}_2$, where F is the union of the copies of the sets F_1 and F_2.

Both these operations, multiplication and union of databases, can be defined also for variable set Φ. In this case we must take into account that in $\mathfrak{A}_1$ and $\mathfrak{A}_2$ algebras of queries U_1 and U_2 are different.

The definitions of cascade connections and wreath products in the database theory are complicated. These constructions are based on the cascade connections and wreath products of the models, which, in their turn, generalize similar notions of the automata theory (see Chapter 2). We refer to [86] for the general definitions and consider here only some remarks.

Let $M=M_1\times M_2$ be Cartesian product of sets, and $\mathfrak{M}$, $\mathfrak{M}_1$, $\mathfrak{M}_2$ be the corresponding power sets. Consider an arbitrary mapping $\chi: M_2 \to \mathfrak{M}_1$ and a set $B\subset M_2$. Then the set (χ, B) is defined by the rule: $(a,b)\in(\chi,B)$ if and only if $b\in B$ and $a\in\chi(b)$.

Take a set A of M and define $\chi=\tilde{A}: M_2 \to \mathfrak{M}_1$ by the rule: $a\in\chi(b)$, if $(a,b)\in A$, where $b\in M_2$, $a\in M_1$. Then $A=(\tilde{A}, A^{\pi_2})$, where $\pi_2: M \to M_2$ is the projection.

Consider two applications of this simple remark. Let $\mathcal{D}_1$, $\mathcal{D}_2$ be the algebras of Θ, $\mathcal{D}=\mathcal{D}_1\times\mathcal{D}_2$ and R_1, R_2 be subalgebras in $V_{\mathcal{D}_1}$ and $V_{\mathcal{D}_2}$. Evidently, $\mathrm{Hom}(W,\mathcal{D})=\mathrm{Hom}(W,\mathcal{D}_1)\times\mathrm{Hom}(W,\mathcal{D}_2)$. In accordance with the definition above, let $\chi: \mathrm{Hom}(W,\mathcal{D}_2) \to R_1$ The subset (χ, B), $B\in R_2$ is an element of $V_{\mathcal{D}_1\times\mathcal{D}_2}$. Then $\mu=(\mu_1,\mu_2)\in\mathrm{Hom}(W,\mathcal{D})$ belongs to (χ,B) if $\mu_2\in B$ and $\mu_1\in\chi(\mu_2)$. The wreath product of R_1 and R_2 is the subalgebra in $V_{\mathcal{D}_1\times\mathcal{D}_2}$, generated by all these (χ,B) . It is denoted by $R=R_1 \mathrm{wr} R_2$.

This wreath product is used in an arbitrary cascade connection of databases as an algebra of replies.

We study now the decomposition of symbols of relations. Let $\mathcal{D}=(D_i, i\in\Gamma)$ be an algebra, and φ a symbol of relation of the type $\tau=(\tau_1,\tau_2)=(i_1,\ldots i_n, j_1,\ldots j_m)$. Then $M_1=D_{i_1}\times\ldots\times D_{i_n}$; $M_2=D_{j_1}\times\ldots D_{j_m}$; $M=M_1\times M_2$.

Let φ_2 be a symbol of type $\tau_2=(j_1,\ldots j_m)$ and for each $b\in M_2$, φ^b be a set of symbols of relations of the type $\tau_1=(i_1,\ldots i_n)$. Thus, we have three sets $\Phi=\{\varphi\}$, $\Phi_1=\{\varphi^b, b\in M_2\}$, $\Phi_2=\{\varphi_2\}$. Take the states f, f_1, f_2 connected with these symbols of relations.

Let $f(\varphi)=A\subset M$ and $\chi=\tilde{A}: M_2 \to \mathfrak{M}_1$ be the corresponding mapping. Set $f_1(\varphi^b)=\tilde{A}(b)$, $f_2(\varphi_2)=A^{\pi_2}$. The equality $A=(\tilde{A}, f_2(\varphi_2))$ means that

$$(a,b)\in f(\varphi) \text{ if and only if } b\in f_2(\varphi_2)\wedge a\in f_1(\varphi^b),$$

and this decomposition of $f(\varphi)$ is associated with the general idea of cascade connections of data algebras.

As we have seen, constructions in databases are associated with the respective constructions for the corresponding models, i.e. databases states. The same for the problem of decomposition. In its turn, decomposition of models supposes, that there is some decomposition of data algebra $\mathcal{D}$. We founded on decomposition $\mathcal{D}$ in Cartesian product, but it is also possible to start from the approximation of $\mathcal{D}$ by some $\mathcal{D}_\alpha$, $\alpha\in I$. Then the above notion is useful. It can be proposed, that for some special θ there exist another "good" constructions. On the other hand, probably the main attention must be paid to the decomposition of relations (may be under the fixed data algebra $\mathcal{D}$).

Perhaps, instead of searching for a good universal theory of decomposition of databases, it is more useful to obtain the sufficient amount of constructions and in each specific case to apply the suitable construction.

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