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Perturbation Methods and Semilinear Elliptic Problems on R^n

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Foreword

Several important problems arising in Physics, Differential Geometry and other topics lead to consider semilinear variational elliptic equations on $\mathbb{R}^n$ and a great deal of work has been devoted to their study. From the mathematical point of view, the main interest relies on the fact that the tools of Nonlinear Functional Analysis, based on compactness arguments, in general cannot be used, at least in a straightforward way, and some new techniques have to be developed.

On the other hand, there are several elliptic problems on $\mathbb{R}^n$ which are perturbative in nature. In some cases there is a natural perturbation parameter, like in the bifurcation from the essential spectrum or in singularly perturbed equations or in the study of semiclassical standing waves for NLS. In some other circumstances, one studies perturbations either because this is the first step to obtain global results or else because it often provides a correct perspective for further global studies.

For these perturbation problems a specific approach, that takes advantage of such a perturbative setting, seems the most appropriate. These abstract tools are provided by perturbation methods in critical point theory. Actually, it turns out that such a framework can be used to handle a large variety of equations, usually considered different in nature.

The aim of this monograph is to discuss these abstract methods together with their applications to several perturbation problems, whose common feature is to involve semilinear Elliptic Partial Differential Equations on $\mathbb{R}^n$ with a variational structure.

The results presented here are based on papers of the Authors carried out in the last years. Many of them are works in collaboration with other people like D. Arcoya, M. Badiale, M. Berti, S. Cingolani, V. Coti Zelati, J.L. Gamez, J. Garcia Azorero, V. Felli, Y.Y. Li, W.M. Ni, I. Peral, S. Secchi. We would like to express our warm gratitude to all of them.

Notation

- $\mathbb{R}^n$ is the Euclidean n-dimensional space with points $x = (x_1, \ldots, x_n)$.
- $\langle x, y\rangle$ denote the Euclidean scalar product of $x, y \in \mathbb{R}^n$; we also set $|x|^2 = \langle x, x\rangle$.
- $B_r(y)$ is the ball $\{x \in \mathbb{R}^n : |x - y| < r\}$. We will write B_r to shorten $B_r(0)$.
- S^n denotes the unit n-dimensional sphere: $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$.
- If Ω is an open subset of $\mathbb{R}^n$ and $u : \Omega \mapsto \mathbb{R}$ is smooth, we denote by $D_i u$, $D^2_{ij} u$ the partial derivatives of u with respect to x_i, $x_i\, x_j$, etc.; we will also use the notation $\frac{\partial}{\partial x_i}$ or ∂_{x_i} instead of D_i, and $\frac{\partial^2}{\partial x_i \partial x_j}$ or $\partial^2_{x_i x_j}$ instead of D^2_{ij}.
- ∇u denotes the gradient of real-valued function u: $\nabla u = (D_1 u, \ldots, D_n u)$; sometime, for a real-valued function K, the notation K' will also be used instead of ∇K.
- $\nabla u \cdot \nabla v$ will be also used to denote $\langle \nabla u, \nabla v\rangle$.
- Δ denotes the Laplacian: $\Delta u = \sum_1^n \frac{\partial^2}{\partial x_i^2}$.
- If $u, v \in \mathcal{H}$, a (real) Hilbert space, the scalar product will be denoted by $(u|v)$ and the norm $\|u\|^2 = (u|u)$.
- Id denotes the identity map in $\mathbb{R}^n$ or $\mathcal{H}$.
- $L^p(\mathbb{R}^n)$, $L^p_{loc}(\mathbb{R}^n)$, $L^p(\Omega)$, etc. denote the usual Lebesgue spaces.
- $W^{m,p}(\mathbb{R}^n)$, $W^{m.p}(\Omega)$, etc. denote the usual Sobolev spaces. If M is a smooth manifold, $H^m(M)$ denotes the Sobolev space $H^{m,2}(M)$.
- 2^* stands for $\frac{2n}{n-2}$ if $n \geq 3$, and $2^* = +\infty$ if $n = 1, 2$.
- $\mathcal{D}^{1,2}(\mathbb{R}^n)$, $n \geq 3$, denotes the space $\{u \in L^{2^*}(\mathbb{R}^n) : \nabla u \in L^2(\mathbb{R}^n)\}$.
- If X, Y are Banach spaces, $L(X, Y)$ denotes the space of bounded linear maps from X to Y.
- If $f \in C^k(X, Y)$, $k \geq 1$, $df(u)$, $d^2 f(u)$, denote the Fréchet derivatives of f at $u \in X$. They are, respectively, a linear bounded map from X to Y, and a bilinear continuous map fro $X \times X$ to Y.
- If $I \in C^k(\mathcal{H}, \mathbb{R})$, $k \geq 1$, is a functional, $I'(u)$ denotes the gradient of I at $u \in \mathcal{H}$, defined by means of the Riesz representation Theorem setting $(I'(u)|v) = dI(u)[v]$, $\forall\, v \in \mathcal{H}$. Similarly, $I''(u)$ is the linear operator defined by setting $(I''(u)v|w) = d^2 I(u)[v, w]$, $\forall\, v, w \in \mathcal{H}$
- If $I \in C^1(\mathcal{H}, \mathbb{R})$, $\mathrm{Cr}[I]$ denotes the set of critical points of I.
- $u = o(\varepsilon^k)$ means that $u\, \varepsilon^{-k}$ tends to zero as $\varepsilon \to 0$.
- $u = O(\varepsilon^k)$ means that $|u\varepsilon^{-k}| \leq c$ as $\varepsilon \to 0$.
- $o_\varepsilon(1)$ denotes a function depending on ε that tends to 0 as $\varepsilon \to 0$. Similarly, $o_R(1)$ denotes a function depending on R that tends to 0 as $R \to +\infty$.
- The notation $\sim$ denotes quantities which, in the limit are of the same order.

Chapter 1

Examples and Motivations

In this initial chapter we will give an account of the main nonlinear variational problems that will be studied in more details in the rest of the monograph. A short outline of the abstract setting will be also given.

1.1 Elliptic equations on $\mathbb{R}^n$

To prove existence of solutions of elliptic problems on $\mathbb{R}^n$ one of the main difficulties is the lack of compactness. For example, let us take $n \geq 3$, $1 < p \leq \frac{n+2}{n-2}$ and consider an equation of the form

$$\begin{cases} -\Delta u + u = b(x) u^p, \\ u \in W^{1,2}(\mathbb{R}^n),\ u > 0, \end{cases} \tag{1.1}$$

whose solutions are the critical points in $W^{1,2}(\mathbb{R}^n)$ of the corresponding Euler functional

$$I_b(u) := \int_{\mathbb{R}^n} \tfrac{1}{2} \left[|\nabla u|^2 + u^2 \right] dx - \tfrac{1}{p+1} \int_{\mathbb{R}^n} b(x) |u|^{p+1} dx.$$

Since the embedding of $W^{1,2}(\mathbb{R}^n)$ into $L^{p+1}(\mathbb{R}^n)$ is not compact, even if $p+1 < 2^*$, then I_b does not satisfy, in general, the Palais-Smale (PS, to be short) compactness condition. For example, this is the case when b is constant. To overcome this difficulty a usual strategy is to apply the P.L. Lions Concentration-Compactness principle. Roughly, suppose that $1 < p < \frac{n+2}{n-2}$ and that $\lim_{|x| \to \infty} b(x) = b_\infty$. Let us consider the *limit functional*

$$I_\infty(u) := \int_{\mathbb{R}^n} \tfrac{1}{2} \left[|\nabla u|^2 + u^2 \right] dx - \tfrac{1}{p+1}\, b_\infty \int_{\mathbb{R}^n} |u|^{p+1} dx,$$

which has a mountain-pass critical level c_∞, the lowest nontrivial critical level of I_∞. In general, using the Concentration-Compactness principle, existence results are found by imposing conditions that permit to compare the critical levels (very often the mountain-pass critical level) of I_b with those of I_∞. For example, if $b(x) > b_\infty$ for all $x \in \mathbb{R}^n$, then it readily follows that the mountain-pass critical level of I_b is lower than the corresponding level c_∞ of I_∞. Since it is possible to prove that the PS condition holds at levels lower than c_∞, this yields the existence of a solution to (1.1). See Chapter 2, Section 2.1.

It is natural to ask the question whether there are other approaches that give rise to existence results for non-compact elliptic equations, which do not require the preceding comparison procedure.

Motivated by this question we will deal in Chapters 4 and 5 with elliptic problems on $\mathbb{R}^n$ whose model is the following one:

$$\begin{cases} -\Delta u + u = (1 + \varepsilon h(x))\, u^p, \\ u \in W^{1,2}(\mathbb{R}^n),\ u > 0, \end{cases}$$

where $n \geq 3$ and $1 < p \leq \frac{n+2}{n-2}$. Let us point out that in the sequel we will always take $n \geq 3$. If $n = 1, 2$ no restriction on $p > 1$ is required and most of the results we will discuss can be extended to this case as well.

Our approach will provide, for the class of perturbation problems like the preceding one, existence results which are, in some sense, complementary to those that can be found using the Concentration-Compactness principle.

It is convenient to distinguish between the subcritical case $1 < p < \frac{n+2}{n-2}$ and the critical one, $p = \frac{n+2}{n-2}$.

1.1.1 The subcritical case

Let us consider the problem

$$\begin{cases} -\Delta u + u = (1 + \varepsilon h(x)) u^p, \\ u \in W^{1,2}(\mathbb{R}^n),\ u > 0, \end{cases} \tag{1.2}$$

where $h(x)$ is a bounded function and the exponent $p > 1$ is subcritical. The preceding equation is just (1.1) with $b = 1 + \varepsilon h$ and $b_\infty = 1$. Solutions of (1.2) are critical points $u \in W^{1,2}(\mathbb{R}^n)$ of the functional

$$I_\varepsilon(u) := \int_{\mathbb{R}^n} \tfrac{1}{2} \left[|\nabla u|^2 + u^2 \right] dx - \tfrac{1}{p+1} \int_{\mathbb{R}^n} (1 + \varepsilon h(x)) |u|^{p+1} dx. \tag{1.3}$$

Remark that $W^{1,2}(\mathbb{R}^n) \hookrightarrow L^{2^*}(\mathbb{R}^n)$, where $2^* = 2n/(n-2)$ and thus I_ε is well defined on $W^{1,2}(\mathbb{R}^n)$ and is smooth. When $\varepsilon = 0$ we have the *unperturbed functional*

$$I_0(u) := \int_{\mathbb{R}^n} \tfrac{1}{2} \left[|\nabla u|^2 + u^2 \right] dx - \tfrac{1}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} dx.$$

We remark that I_0 is nothing but the *limit functional* I_∞ with $b_\infty = 1$. Plainly, $u = 0$ is a local strict minimum of I_0 and there exists $e \in W^{1,2}(\mathbb{R}^n) \setminus \{0\}$ such that

$I_\varepsilon(e) < 0$. Moreover, since the subspace $W_r^{1,2}(\mathbb{R}^n) = \{u \in W^{1,2}(\mathbb{R}^n) : u \text{ is radial}\}$ is compactly embedded in $L^q(\mathbb{R}^n)$ when $1 < q < 2^*$, see [135], then I_0 has a mountain-pass critical point $U > 0$ which is a solution of

$$-\Delta u + u = u^p, \quad u \in W_r^{1,2}(\mathbb{R}^n), \quad u > 0.$$

More precisely, setting $Z = \{U(\cdot - \xi) : \xi \in \mathbb{R}^n\}$, one has that every $z \in Z$ is a critical point of the unperturbed functional I_0 and the question becomes whether there exists $\bar{z} \in Z$ such that (1.2) has a solution $u_\varepsilon \sim \bar{z}$ for ε small enough.

In Chapter 4, where perturbation problems with subcritical growth like (1.2) will be discussed, we will show, e.g., that a solution exists, provided

$$\lim_{|x|\to\infty} h(x) = 0 \quad (\text{namely when } b_\infty = \lim_{|x|\to\infty} b(x) = 1).$$

It is worth pointing out that, in order to use the Concentration-Compactness principle as sketched before, we should assume that, roughly, h is greater or equal than 0, or h should tend to 0 in a suitable exponential way, see [34, 35] and Theorem 2.7 later on. Moreover, in some cases, like, e.g., when $h(x) < 0 \; \forall \, x \in \mathbb{R}^n$, our solutions are not mountain-pass critical points of I_ε and this would be another difficulty to be overcome in order to use the Concentration-Compactness principle.

1.1.2 The critical case: the Scalar Curvature Problem

Elliptic equations on $\mathbb{R}^n$ with critical exponent will be discussed in Chapter 5. We will be mainly concerned with problems as

$$-\Delta u = \left(1 + \varepsilon \widetilde{k}(x)\right) u^{\frac{n+2}{n-2}}, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^n), \quad u > 0, \tag{1.4}$$

which are critical points of the functional

$$I_\varepsilon(u) = \tfrac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 dx - \tfrac{1}{2^*} \int_{\mathbb{R}^n} (1 + \varepsilon \widetilde{k}(x))|u|^{2^*} dx, \qquad u \in \mathcal{D}^{1,2}(\mathbb{R}^n).$$

The new feature of the equation (1.4) is that the unperturbed problem

$$-\Delta u = u^{\frac{n+2}{n-2}}, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^n), \quad u > 0 \tag{1.5}$$

is invariant not only by translation (like in the subcritical case) but it is also invariant by dilations. Precisely, letting (up to a constant)

$$U(x) = \left(\frac{1}{1 + |x|^2}\right)^{\frac{n-2}{2}},$$

every function

$$z_{\mu,\xi}(x) = \mu^{-\frac{n-2}{2}} \, U\left(\frac{x - \xi}{\mu}\right)$$

is a solution of (1.5) and their union forms an $(n+1)$-dimensional critical manifold (with boundary) $Z \simeq \mathbb{R}^+ \times \mathbb{R}^n$. However, it is still possible to give conditions on k such that (1.4) has a solution for ε small enough. These topics will be discussed in Chapter 5

The class of problems above arises in differential geometry. Let (M, g) be a smooth compact Riemannian manifold. The *Scalar Curvature Problem* amounts to finding a metric $\widetilde{g}$ conformal to g such that the scalar curvature of $(M, \widetilde{g})$ is a prescribed function K. If $\widetilde{g} = u^{4/(n-2)} g$ ($n \geq 3$ [1]), $u > 0$, then one has to solve (omitting some multiplicative constants)

$$-\Delta_g u + R_g u = K u^{\frac{n+2}{n-2}}, \quad u \in H^1(M), \quad u > 0 \tag{1.6}$$

where Δ_g denotes the Laplace-Beltrami operator and R_g is the scalar curvature of (M, g). When $K \equiv$ const., this is called the *Yamabe problem.*

The most delicate case is when $(M, g) = (S^n, \bar{g}_0)$, the standard sphere. In this case, using the stereographic projection $\pi : S^n \to \mathbb{R}^n$, equation (1.6) becomes

$$-\Delta u = \widetilde{K} u^{\frac{n+2}{n-2}}, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^n), \quad u > 0, \tag{1.7}$$

where Δ is the standard Laplacian and $\widetilde{K} = K \circ \pi^{-1}$. If $\widetilde{K}$ is close to a positive constant, (1.7) is exactly of the form (1.4). Finding a solution of this latter perturbation problem can be used as a first ingredient to solve the (global) Scalar Curvature Problem with any $K > 0$. The argument is, roughly, the following. Let us consider the family of problems

$$-\Delta_{\bar{g}_0} u + R_{\bar{g}_0} u = K_t u^{\frac{n+2}{n-2}}, \quad u \in H^1(S^n), \quad u > 0, \tag{1.8}$$

where $K_t = (1-t) + tK$. When $t > 0$ is sufficiently small, problem (1.8) is equivalent, up to the stereographic projection, to a problem like (1.4). Once one is able to solve the latter (with an appropriate counting degree formula), a solution of the Scalar Curvature Problem can be found by a homotopy between K_t, t small and $K_1 = K$. This procedure relies on a compactness result [55, 100] stating that, under appropriate conditions on K, the set of solutions of (1.8) is bounded in the C^2 topology, uniformly with respect to $t \in [0, 1]$.

A perturbation technique can also be used to find multiple solutions of the Yamabe problem. In particular if $n \geq 4k + 3$ and $k \geq 2$, one can show that there exist C^k metrics g_ε on S^n, which converge to the standard one as $\varepsilon \to 0$, such that the Yamabe equation

$$-\Delta_{g_\varepsilon} u + R_{g_\varepsilon} u = u^{\frac{n+2}{n-2}}, \quad u \in H^1(S^n), \quad u > 0,$$

has, for every ε small, infinitely many solutions u_ε^i, $i \in \mathbb{N}$ and moreover

$$\|u_\varepsilon^i\|_{L^\infty(S^n)} \to +\infty \quad \text{as} \quad i \to \infty.$$

[1] Let us remark that the case $n = 2$ would require a different approach involving an exponential conformal factor, see (6.6) in Section 6.1

This should be compared with a well-known result by R. Schoen [131], see also [104], which establishes that if g is any C^∞ metric on M such that (M,g) is not conformally flat, then the solutions of the Yamabe problem

$$-\Delta_g u + R_g u = u^{\frac{n+2}{n-2}}, \quad u \in H^1(M), \quad u > 0,$$

are bounded in the C^2 norm.

Multiplicity results for the Yamabe problem will be discussed in Chapter 6, while the Scalar Curvature Problem as well as other problems arising in Conformal Geometry will be studied in Chapter 7.

1.2 Bifurcation from the essential spectrum

Let $\mathcal{H}$ be a Hilbert space, let $F : \mathbb{R} \times \mathcal{H} \to \mathcal{H}$ be a smooth function and suppose that $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. If there exists λ_0 with the property that the equation $F(\lambda, u) = 0$ has a sequence of solutions (λ_n, u_n), with $u_n \neq 0$ and such that $(\lambda_n, u_n) \to (\lambda_0, 0)$ as $n \to \infty$, we say that λ_0 is a *bifurcation point* (for $F = 0$). If $F(\lambda, u) = \lambda u - K'(u)$ and $K'(u)$ is a compact operator, a theorem by Krasnoselski [97] ensures that every eigenvalue of $K''(0)$ is a bifurcation point. Unlike the compact case, in the presence of the essential spectrum one tries to show that the infimum of such a spectrum is still a bifurcation point. A typical example is given by the problem

$$\psi'' + \lambda\psi + h(x)|\psi|^{p-1}\psi = 0, \qquad \lim_{|x|\to\infty} \psi(x) = 0, \tag{1.9}$$

where $p > 1$. If h is constant, say $h(x) \equiv 1$, (1.9) can be studied in a straightforward way by a phase plane analysis.

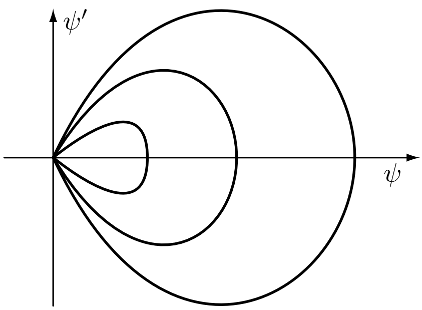

Figure 1.1. Phase plane portrait of $\psi'' + \lambda\psi + |\psi|^{p-1}\psi = 0$

It follows that from $\lambda = 0$, the bottom of the essential spectrum of $\psi'' + \lambda\psi = 0$, $\psi \in W^{1,2}(\mathbb{R})$, bifurcates a family of solutions (λ, ψ_λ), $\lambda < 0$, of $\psi'' + \lambda\psi + |\psi|^{p-1}\psi = 0$, with $(\lambda, \psi_\lambda) \to (0, 0)$ as $\lambda \uparrow 0$.

When h is not constant, an elementary approach as before cannot be carried out anymore and one needs to use a functional approach. Let us show that a suitable transformation brings (1.9) into a perturbation problem similar in nature to the examples in Subsection 1.1.1. Setting

$$\begin{cases} u(x) & = & \varepsilon^{-\frac{2}{p-1}}\psi\left(\frac{x}{\varepsilon}\right), \\ \lambda & = & -\varepsilon^2, \end{cases}$$

equation (1.9) becomes

$$-u'' + u = h\left(\frac{x}{\varepsilon}\right)|u|^{p-1}u, \quad u \in W^{1,2}(\mathbb{R}). \tag{1.10}$$

Suppose that $h(x) \to 1$ as $|x| \to \infty$, and write (1.10) as

$$-u'' + u = |u|^{p-1}u + \left[h\left(\frac{x}{\varepsilon}\right) - 1\right]|u|^{p-1}u, \quad u \in W^{1,2}(\mathbb{R}).$$

This form highlights that (1.10) can be viewed as a perturbation problem since $h\left(\frac{x}{\varepsilon}\right) - 1$ tends to zero (in an appropriate sense to be made precise) as $\varepsilon \to 0$. Here the unperturbed problem is

$$-u'' + u = |u|^{p-1}u, \quad u \in W^{1,2}(\mathbb{R}),$$

and, like in the problems of Section 1.1, the corresponding Euler functional has a one-dimensional critical manifold. This bifurcation problem will be discussed in Chapter 3. For example, we will show that if $h - 1 \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}}(h(x)-1)dx \neq 0$ then $\lambda = 0$ is a bifurcation point for (1.9), with solutions branching off on the left of $\lambda = 0$, like in the bifurcation diagram in Figure 1.2 below.

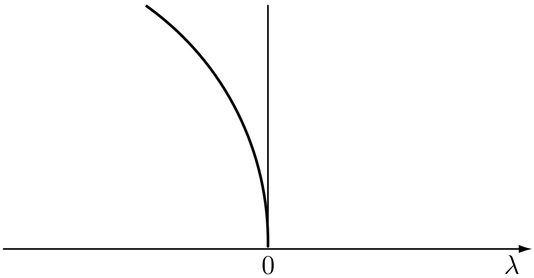

Figure 1.2. Bifurcation diagram of $\psi'' + \lambda\psi + |\psi|^{p-1}\psi = 0$

1.3 Semiclassical standing waves of NLS

In Quantum Mechanics the behavior of a single particle is governed by the linear Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\hbar^2\Delta\psi + Q(x)\psi,$$

where i is the imaginary unit, $\hbar$ is the Planck constant, $(t,x) \in \mathbb{R} \times \mathbb{R}^n$, Δ denotes the Laplace operator and $\psi = \psi(t,x)$ is a complex-valued function. Differently, in the presence of many particles, one can try to simulate the mutual interaction effect by introducing a nonlinear term. Expanding this nonlinearity in odd power series

$$a_0\psi + a_1|\psi|^{p-1}\psi + \cdots, \quad (p \geq 3)$$

and keeping only the first nonlinear term, one is led to a nonlinear equation of the form

$$i\hbar\frac{\partial\psi}{\partial t} = -\hbar^2\Delta\psi + (a_0 + Q(x))\psi + a_1|\psi|^{p-1}\psi. \tag{1.11}$$

We will consider the case in which $a_1 < 0$, say $a_1 = -1$. Nonlinear Schrödinger equations (in short NLS) of this sort are commonly used, for example, in Plasma Physics but they also arise, via Maxwell's equations, in Nonlinear Optics in the presence of a self-focusing material. Let us recall that in other cases, like in the Ginzburg-Landau theory, a nonlinearity of the form $|\psi|^2\psi - |\psi|^4\psi$ is introduced and this gives rise to problems quite different in nature, see, e.g., [46].
A *stationary wave* of (1.11) is a solution of (1.11) of the form

$$\psi(t,x) = \exp\left(i\,\alpha\,\hbar^{-1}t\right)u(x) \quad u(x) \in \mathbb{R}, \quad u > 0.$$

Thus, looking for solitary waves of (1.11) is equivalent to finding a $u > 0$ satisfying

$$-\hbar^2\Delta u + (\alpha + a_0 + Q(x))u = u^p. \tag{1.12}$$

Such a u will be called a *standing wave.* A particular interest is given to the so-called *semiclassical states* that are standing waves existing for $\hbar \to 0$. Setting $\hbar = \varepsilon$ and $V(x) = \alpha + a_0 + Q(x)$, we are finally led to

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u = u^p, \\ u \in W^{1,2}(\mathbb{R}^n),\ u > 0, \end{cases} \tag{1.13}$$

where the condition $u \in W^{1,2}(\mathbb{R}^n)$ is added in order to obtain *bound states*, namely solutions with finite energy.

To obtain a perturbation problem like the preceding ones, it is convenient to make the change of variables $x \mapsto \varepsilon x + x_0$, where $x_0 \in \mathbb{R}^n$ will be chosen in an appropriate way, that leads to

$$\begin{cases} -\Delta u + V(\varepsilon x + x_0)u = u^p, \\ u \in W^{1,2}(\mathbb{R}^n), \quad u > 0. \end{cases} \tag{1.14}$$

Above we assume that p is subcritical: $1 < p < \frac{n+2}{n-2}$ (if $n \geq 3$). The solutions of (1.14) are the critical points $u > 0$ of the functional

$$I_\varepsilon(u) = \int_{\mathbb{R}^n} \tfrac{1}{2}\left[|\nabla u|^2 + V(\varepsilon x + x_0)u^2\right]dx - \tfrac{1}{p+1}\int_{\mathbb{R}^n} |u|^{p+1}dx,\ u \in W^{1,2}(\mathbb{R}^n).$$

This functional is perturbative in nature: the unperturbed functional is

$$I_0(u) = \int_{\mathbb{R}^n} \tfrac{1}{2}\left[|\nabla u|^2 + V(x_0)u^2\right]dx - \tfrac{1}{p+1}\int_{\mathbb{R}^n} |u|^{p+1}dx$$

while the perturbation term is given by

$$\tfrac{1}{2}\int_{\mathbb{R}^n} \left[V(\varepsilon x + x_0) - V(x_0)\right]u^2 dx.$$

The unperturbed equation $I_0'(u) = 0$ becomes:

$$\begin{cases} -\Delta u + V(x_0)u = u^p, \\ u \in W^{1,2}(\mathbb{R}^n), \quad u > 0. \end{cases} \tag{1.15}$$

If $V(x_0) > 0$, it is known that (1.15) possesses a unique radial solution $U_0 > 0$, depending on x_0, such that $\nabla U_0(0) = 0$. Since (1.15) is an autonomous equation, then any $U_0(\cdot - \xi)$, $\xi \in \mathbb{R}^n$, is also a solution of (1.15). In other words, the unperturbed problem $I_0' = 0$ has an n-dimensional manifold of critical point $Z = \{U_0(\cdot - \xi) : \xi \in \mathbb{R}^n\}$. It will be shown that if x_0 is stationary point of the potential V which is *stable* (in a suitable sense specified later on), then (NLS) has for $\varepsilon \neq 0$ small a solution of the form

$$u_\varepsilon(x) \sim U_0\left(\frac{x - x_0}{\varepsilon}\right),$$

hence a solution that concentrates at x_0. This kind of solutions are called *spike layers* or simply *spikes*. From the physical point of view, spikes are important because they show that (focusing) NLS of the type (1.15) are not dispersive but the energy is localized in packets. These topics will be discussed in Chapter 8 together with more general results dealing with the case in which V has a manifold of stationary points.

We anticipate that for radial NLS it is possible to show that there exist solutions concentrating at higher-dimensional manifolds. This latter problem will studied in Chapter 10.

1.4 Other problems with concentration

There are several further problems whose main feature is that they possess solutions concentrating at points or at manifolds.

1.4.1 Neumann singularly perturbed problems

An important example is given by elliptic singularly perturbed problems with Neumann boundary conditions like

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^p, \quad \text{in } \Omega \\ u > 0, \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{1.16}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and ν denotes the unit outer normal at $\partial\Omega$. As before, we take $1 < p < \frac{n+2}{n-2}$. Problems like (1.16) arise in the study of some reaction-diffusion systems with chemical of biological motivation. A basic example is the following system, due to Gierer and Meinhardt, see [84], which models the densities of a chemical activator $\mathcal{U}$ and an inhibitor $\mathcal{V}$, and is used to describe experiments of regeneration of *hydra*

$$\begin{cases} \mathcal{U}_t = d_1 \Delta \mathcal{U} - \mathcal{U} + \frac{\mathcal{U}^p}{\mathcal{V}^q} & \text{in } \Omega \times (0, +\infty), \\ \mathcal{V}_t = d_2 \Delta \mathcal{V} - \mathcal{V} + \frac{\mathcal{U}^r}{\mathcal{V}^s} & \text{in } \Omega \times (0, +\infty), \\ \frac{\partial \mathcal{U}}{\partial \nu} = \frac{\partial \mathcal{V}}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, +\infty), \end{cases} \tag{GM}$$

Here $d_1, d_2, p, q, r, s > 0$, with the constraints

$$0 < \frac{p-1}{q} < \frac{r}{s+1}.$$

According to *Turing's instability*, [142], systems with different diffusivities may produce stable non-trivial patterns. If one considers steady states of (GM) in the limit $d_1 \ll 1 \ll d_2$, see the survey [116], it turns out that $\mathcal{V}$ is almost constant in Ω, and hence the significant equation in (GM) is the one for $\mathcal{U}$, which is of the form (1.16).

There is a great similarity between singular perturbation problems like (1.16) and NLS. Again, the specific feature of (1.16) is to possess spike layer solutions: in fact, dealing with spikes at $\partial\Omega$, the role of the potential V in the NLS is played here by the curvature of the boundary, in the sense that there exist solutions concentrating at *stable* stationary points of the mean curvature H.

The abstract setting appropriate to handle (1.16) is slightly different than the one used in the preceding problems, although it is similar in nature. Solutions of (1.16) are still critical points of a functional as I_ε, but unlike the preceding cases, there is not an unperturbed critical manifold. Rather, there is a manifold $\mathcal{Z}$ of points where I'_ε is sufficiently small. However, the same ideas used for the previous problems can be still carried out leading to show that spikes exist concentrating at stable stationary points of the mean curvature H of the boundary $\partial\Omega$. These topics will be discussed in Chapter 9.

1.4.2 Concentration on spheres for radial problems

Recently, see [111, 112], it has been proved that there exist solutions of (1.16) concentrating on all the boundary $\partial\Omega$, a fact conjectured long ago, see [116]. It is also natural to look for solutions concentrating on internal manifolds. Though in such a generality this remains an open problem, in the radial case solutions of this sort have been proved to exist, see [21, 22]. Similarly, one can also show that radial NLS like

$$\begin{cases} -\varepsilon^2 \Delta u + V(|x|) u = u^p, \\ u \in W^{1,2}(\mathbb{R}^n),\ u > 0, \end{cases} \tag{1.17}$$

possess solutions concentrating on a sphere. A new feature of this case is that there is an auxiliary weighted potential M that substitutes V. Roughly, one proves that if (1.17) has a radial solution concentrating at the sphere $\{|x| = r\}$, $r > 0$, then $M'(r) = 0$; conversely, if r satisfies $M'(r) = 0$ and is *stable*, then such a solution exists. Here the exponent p in the nonlinearity can be any number greater than 1.

The results dealing with concentration on sphere for radial NLS and for radial Neumann problems will be discussed in Chapter 10.

1.5 The abstract setting

The problems discussed above can be studied by means of a common abstract setting. Letting $\mathcal{H}$ be a Hilbert space, we look for critical points of a smooth functional $I_\varepsilon : \mathcal{H} \to \mathbb{R}$ depending a on a small parameter $\varepsilon \in \mathbb{R}$, namely solutions of equations in the form

$$I'_\varepsilon(u) = 0, \quad u \in \mathcal{H}. \tag{1.18}$$

Motivated by the preceding discussions, we will consider in Chapter 2 a class of functionals like $I_\varepsilon(u) = I_0(u) + \varepsilon G(u)$ or, more in general, $I_\varepsilon(u) = I_0(u) + G(\varepsilon, u)$ (see, e.g., the bifurcation problem discussed in Section 1.2), where $G(0, u) \equiv 0$. As in the applications, we will suppose to know some specific features of the *unperturbed functional* I_0. Precisely, we assume that I_0 possesses non-isolated critical points which form a *manifold* Z, usually referred to as *critical manifold*:

$$Z = \{z \in \mathcal{H} : I'_0(z) = 0\}.$$

In this case the problem of finding solutions of (1.18) becomes a kind of bifurcation problem in which $z \in Z$ is the bifurcation parameter and the set $\{0\} \times Z \subset \mathbb{R} \times \mathcal{H}$ is the set of the *trivial solutions*: one looks for conditions on the perturbation G that generate *non-trivial solutions* of (1.18) branching off from some $z \in Z$. Here by non-trivial solutions we mean a pair $(\varepsilon, u) \in \mathbb{R} \times \mathcal{H}$, with $\varepsilon \neq 0$, such that $I'_\varepsilon(u) = 0$.

More precisely, we will deal in the sequel with the case in which the critical manifold Z is not compact, although the abstract setting applies to the compact case as well (for some results in the compact case, see, e.g., [11]). The fact that Z is not compact usually depends on the invariance of the unperturbed problem $I'_0(u) = 0$ under the action of a non-compact group of transformations. In our setting, this is the counterpart of the fact that in the problems we will deal with, the Palais-Smale condition may not hold. From this point of view, our abstract results can be seen as an alternative way to overcome the lack of compactness in critical point theory, in the specific case of problems perturbative in nature.

In order to solve (1.18) we use a finite-dimensional reduction procedure. This is nothing but the classical Lyapunov-Schmidt method, with appropriate modifications which allow us to take advantage of the variational nature of our equations. To have an idea of the sort of results we will prove, let us consider the case in which $I_\varepsilon = I_0 + \varepsilon G$. Roughly, under an appropriate *non-degeneracy condition* on Z, always verified in our applications, we will show that the stable critical points of the perturbation G constrained on Z give rise to critical points of I_ε.

As anticipated before, in some applications we have to deal with the case in which Z is substituted by a manifold $\mathcal{Z}$ which does not consists of critical points of I_0 but is such that $I'_\varepsilon(z)$ is sufficiently small for every $z \in \mathcal{Z}$ and every $\varepsilon \ll 1$. This more general situation is not substantially different in nature to the preceding one. Actually, it turns out that the same finite-dimensional reduction method can be used to obtain, as before, quite similar results on the existence of critical points of I_ε.

We conclude this chapter pointing out that the abstract approach we will carry over, applies to several other equations as well, such as Hamiltonian Systems with chaotic dynamics, Arnold diffusion, periodic solutions of the nonlinear wave equations, surfaces with prescribed mean curvature (or related issues), and the list could continue. The interested reader can see, e.g., the papers [7, 40, 41, 42, 43, 45, 51, 79, 130] where these problems are studied essentially by the same methods. However, for the sake of brevity, we will not deal with these topics here but we will focus on elliptic problems.

Remarks on the exposition

In order to limit the monograph to a reasonable length, we will only give the outline of the proofs which are based on arguments already employed. This will be mainly the case in the last chapters.

Chapter 2

Pertubation in Critical Point Theory

In this chapter we will prove some abstract results on the existence of critical points of perturbed functionals I_ε on a Hilbert space $\mathcal{H}$ [1], whose norm and scalar product will be denoted, respectively, by $\|\cdot\|$ and $(\cdot|\cdot)$.

2.1 A review on critical point theory

In this section we will outline some topics in critical point theory. We will be sketchy, referring to [52, 136, 147] for proofs and more complete results.

A critical point of a functional $I \in C^1(\mathcal{H}, \mathbb{R})$ is an element $u \in \mathcal{H}$ such that $I'(u) = 0$. Hereafter I' denotes the gradient of I, defined through the relationship $dI(u)[v] = (I'(u)|v)$, $\forall\, v \in \mathcal{H}$. Critical points give rise to solutions of differential equation of variational type. For example, if $\mathcal{H} = W^{1,2}(\mathbb{R}^n)$ and

$$I_0(u) = \tfrac{1}{2}\|u\|^2 - \tfrac{1}{p+1}\int_{\mathbb{R}^n} |u|^{p+1} dx; \qquad \|u\|^2 = \int_{\mathbb{R}^n} \left(|\nabla u|^2 + u^2\right) dx, \quad (2.1)$$

where $0 < p \le \frac{n+2}{n-2}$ if $n \ge 3$ (otherwise any p is allowed), a critical point is a weak solution of the elliptic equation

$$-\Delta u + u = |u|^{p-1} u, \qquad u \in W^{1,2}(\mathbb{R}^n). \quad (2.2)$$

By elliptic regularity, u turns out to be indeed a classical solution. Moreover, it is easy to check that $u(x) \to 0$ as $|x| \to \infty$. Let us point out that in view of the embedding $W^{1,2}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ for every $q \in [2, 2^*]$, the functional I is well defined whenever $p + 1 \le 2^*$, namely $p \le \frac{n+2}{n-2}$.

[1] Most of the results we will discuss could be carried out in a Banach space.

Definition. *A number $c \in \mathbb{R}$ is called a* critical level *of I if there exists a critical point u of I such that $I(u) = c$.*

In general, critical levels can be found by min-max procedures. This is the case of the Mountain-Pass Theorem which applies to functionals which verify the following geometric condition: $\exists\, u_0,\, u_1 \in \mathcal{H}$ and $\alpha, r > 0$ such that

(MP.1) $\inf_{\|u-u_0\|=r} I(u) \geq \alpha > I(u_0)$;
(MP.2) $\|u_1\| > r$ and $I(u_1) \leq I(u_0)$.

If the above conditions hold, we can define a min-max level as follows. Letting $\Gamma = \{\gamma \in C([0,1], \mathcal{H}) : \gamma(0) = u_0,\, \gamma(1) = u_1\}$, we set

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)). \tag{2.3}$$

Let us remark that conditions (MP.1–2) imply that c is finite and different from $I(u_0)$. Actually, for every $\gamma \in \Gamma$ the path $\gamma(t)$ meets the sphere $\|u = u_0\| = r$. Then $\max_{t\in[0,1]} I(\gamma(t)) \geq \alpha$.

In order to prove that c is a critical level of I a compactness condition is in order. The following one is called *Palais-Smale* condition.

Definition. *A sequence $\{u_j\}$, $u_j \in \mathcal{H}$, is a $(PS)_c$ sequence if*

$$I(u_j) \to c, \quad \text{and} \quad I'(u_j) \to 0.$$

We say that the $(PS)_c$ condition holds if every $(PS)_c$ sequence has a converging sub-sequence.

The following result has been proved in [25].

Theorem 2.1. (Mountain-Pass) *Let $I \in C^1(\mathcal{H}, \mathbb{R})$ satisfy* (MP.1–2) *and suppose that $(PS)_c$ holds, where c is defined in* (2.3)*. Then c is a critical level of I.*

Remark 2.2. It is possible to show that a M-P critical point of a C^2 functional has Morse index at most equal to one. We recall that the Morse index if a critical point u is the maximal dimension of a subspace on which $I''(u)$ is negative definite. □

As an application of the M-P Theorem, we can find a radial solution of (2.2), following [135], see also [38]. Let let $\mathcal{H} = W^{1,2}_r(\mathbb{R}^n)$ be the space of the functions $u \in W^{1,2}(\mathbb{R}^n)$ which are radial. The critical points of the functional I_0 defined in (2.1), restricted to $W^{1,2}_r(\mathbb{R}^n)$, are the radial solutions of (2.2). It is easy to check that (MP.1–2) hold provided we assume $p > 1$. Moreover, since the embedding of $W^{1,2}_r(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ is compact whenever $q < 2^*$, it is possible to show that $(PS)_c$ is satisfied provided $p < 2^* - 1 = (n+2)/(n-2)$. In conclusion, we can infer that for every $1 < p < (n+2)/(n-2)$ equation (2.2) has a *radial* solution U. One can also easily show that U is positive. Finally one can also prove that U has an exponential decay as $|x| \to \infty$.

Remark 2.3. Of course, since the nonlinearity is homogeneous, the existence of U can also be found by looking for the minimum of $\|u\|^2$ constrained on the manifold $\{u \in W_r^{1,2}(\mathbb{R}^n) : \int |u|^{p+1} dx = 1\}$. One then finds u^* such that $-\Delta u^* + u^* = \lambda(u^*)^p$ for some Lagrange multiplier $\lambda \in \mathbb{R}$. Setting $U = \lambda^{1/(p-1)} u^*$ one obtains that $-\Delta U + U = U^p$. □

Let us explicitly point out that I does not satisfy the $(PS)_c$ condition if we work in $W^{1,2}(\mathbb{R}^n)$. Actually, for any $\xi \in \mathbb{R}^n$ the set of functions $U_\xi(x) = U(x - \xi)$ satisfy $I(U_\xi) \equiv c$ and $I'(U_\xi) \equiv 0$.

Below we will show that for the Euler functionals corresponding to problems like (2.2), we can recover the $(PS)_c$ condition under appropriate comparison assumptions. We will focus on the functional $I_b : W^{1,2}(\mathbb{R}^n) \to \mathbb{R}$,

$$I_b(u) = \tfrac{1}{2}\|u\|^2 - \tfrac{1}{p+1} \int_{\mathbb{R}^n} b(x)|u|^{p+1} dx,$$

where $p + 1 < 2^*$, $b \in L^\infty(\mathbb{R}^n)$ and $\|u\|$ denotes the standard norm in $W^{1,2}(\mathbb{R}^n)$. We will follow closely the arguments carried out in Sections 1.6, 1.7 and 1.8 of [147], to which we also refer for more details.

We shall assume:

$$\lim_{|x| \to \infty} b(x) = b_\infty > 0. \tag{2.4}$$

To simplify the notation we will take $b_\infty = 1$. It is natural to associate to I_b its *limit at infinity*, obtained substituting b with $b_\infty = 1$, namely

$$I_0(u) = \tfrac{1}{2}\|u\|^2 - \tfrac{1}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} dx.$$

Let c_0 denote the M-P critical level of I_0 (one has $c_0 = I_0(U)$) and let us set

$$S_{p+1} = \inf\{\|u\|^2 \,:\, u \in W^{1,2}(\mathbb{R}^n), \int_{\mathbb{R}^n} |u|^{p+1} dx = 1\}.$$

It is well known that $S_{p+1} > 0$ and it is achieved at some u^* such that $\|u^*\|^2 = S_{p+1}$. The reader should notice that S_{p+1} is the best Sobolev constant for the embedding $W^{1,2}(\mathbb{R}^n) \hookrightarrow L^{p+1}(\mathbb{R}^n)$ and hence

$$\|u\|^2_{L^{p+1}} \le S^{-1}_{p+1} \|u\|^2, \quad \forall u \in W^{1,2}(\mathbb{R}^n). \tag{2.5}$$

Moreover, according to Remark 2.3, we have that $U = S_{p+1}^{1/(p-1)} u^*$ satisfies $-\Delta U + U = U^p$ and hence

$$c_0 = I_0(U) = (\tfrac{1}{2} - \tfrac{1}{p+1})\|U\|^2 = (\tfrac{1}{2} - \tfrac{1}{p+1}) S_{p+1}^{\frac{p+1}{p-1}}.$$

The key lemma is the following:

Lemma 2.4. *Suppose that b satisfies (2.4), with $b_\infty = 1$. Then I_b satisfies $(PS)_c$ for any $c < c_0$.*

Proof. Let u_j be a $(PS)_c$ sequence for I_b. From

$$I_b(u_j) = \tfrac{1}{2}\|u_j\|^2 - \tfrac{1}{p+1}\int_{\mathbb{R}^n} b|u_j|^{p+1}dx = c + o(1),$$

jointly with

$$(I_b'(u_j), u_j) = \|u_j\|^2 - \int_{\mathbb{R}^n} b|u_j|^{p+1}dx = o(1)\|u_j\|$$

we infer

$$(\tfrac{1}{2} - \tfrac{1}{p+1})\|u_j\|^2 = c + o(1)\|u_j\| + o(1).$$

Thus there exists $a_1 > 0$ such that $\|u_j\| \le a_1$. Passing if necessary to a subsequence, we can assume that $u_j \to v$, weakly in $W^{1,2}(\mathbb{R}^n)$, strongly in $L^{p+1}_{loc}(\mathbb{R}^n)$ and a.e. in $\mathbb{R}^n$. It is clear that $(I_b'(v), \phi) = 0$ for every $\phi \in W^{1,2}(\mathbb{R}^n)$ and hence v is a critical point of I and satisfies

$$I_b(v) = (\tfrac{1}{2} - \tfrac{1}{p+1})\|v\|^2 \ge 0.$$

Let us now recall the following result due to Brezis and Lieb, [48]:

Let $h_j \in L^q(\mathbb{R}^n)$ $(1 \le q < \infty)$ be bounded in L^q and such that $h_j \to h$ a.e. in $\mathbb{R}^n$. Then one has

$$\int_{\mathbb{R}^n} |h_j|^q dx - \int_{\mathbb{R}^n} |h_j - h|^q dx = \int_{\mathbb{R}^n} |h|^q dx + o(1). \tag{2.6}$$

Applying (2.6) with $q = p+1$, $h_j = b^{\frac{1}{q}}\, u_j$ and $h = b^{\frac{1}{q}}\, v$ we get

$$\int_{\mathbb{R}^n} b|u_j|^{p+1}dx - \int_{\mathbb{R}^n} b|u_j - v|^{p+1}dx = \int_{\mathbb{R}^n} b|v|^{p+1}dx + o(1).$$

Using this equation and the fact that $(u_j - v, v) = o(1)$, it follows that

$$\begin{aligned} I_b(u_j) &= \tfrac{1}{2}\|(u_j - v) + v\|^2 - \tfrac{1}{p+1}\int_{\mathbb{R}^n} b|u_j|^{p+1}dx \\ &= \tfrac{1}{2}\|u_j - v\|^2 + \tfrac{1}{2}\|v\|^2 - \tfrac{1}{p+1}\int_{\mathbb{R}^n} b|u_j - v|^{p+1}dx - \tfrac{1}{p+1}\int_{\mathbb{R}^n} b|v|^{p+1}dx + o(1) \\ &= I_b(v) + I_b(u_j - v) + o(1). \end{aligned}$$

Since $I_b(u_j) \to c$ and $I_b(v) \ge 0$, we deduce

$$I_b(u_j - v) \le c + o(1). \tag{2.7}$$

By a similar calculation we get

$$\|u_j - v\|^2 - \int_{\mathbb{R}^n} b|u_j - v|^{p+1}dx = \|u_j\|^2 + \|v\|^2 - \int_{\mathbb{R}^n} b|u_j|^{p+1}dx$$
$$- \int_{\mathbb{R}^n} b|v|^{p+1}dx + o(1) = (I_b'(u_j)|u_j) + (I_b'(v)|v) + o(1).$$

Since $(I_b'(u_j)|u_j) \to 0$ and $(I_b'(v)|v) = 0$, we deduce that there is $\beta \geq 0$ satisfying

$$\lim \|u_j - v\|^2 = \lim \int_{\mathbb{R}^n} b|u_j - v|^{p+1}dx = \beta.$$

Let us point out that in view of the assumption (2.4) we also have

$$\int_{\mathbb{R}^n} |u_j - v|^{p+1}dx = \beta + o(1).$$

This and (2.5) imply

$$\beta \geq S_{p+1}\beta^{2/(p+1)}.$$

If $\beta = 0$ then $\|u_j - v\|^2 \to 0$ and we are done. Otherwise we get $\beta \geq S_{p+1}^{\frac{p+1}{p-1}}$. But in such a case we find

$$c_0 = (\tfrac{1}{2} - \tfrac{1}{p+1})S_{p+1}^{\frac{p+1}{p-1}} \leq (\tfrac{1}{2} - \tfrac{1}{p+1})\beta. \tag{2.8}$$

From (2.7) we infer

$$(\tfrac{1}{2} - \tfrac{1}{p+1})\|u_j - v\|^2 = I_b(u_j - v) \leq c + o(1)$$

and hence $(\frac{1}{2} - \frac{1}{p+1})\beta \leq c$. Finally, this and (2.8) imply $c_0 \leq c$, in contradiction with the assumption that $c < c_0$. □

It is now easy to check that the assumption

$$b(x) \geq b_\infty \ (= 1) \qquad \forall x \in \mathbb{R}^n \tag{2.9}$$

implies that the M-P level c_b of I_b satisfies $c_b \leq c_0$, with strict inequality provided $b \not\equiv b_\infty$ $(= 1)$ (if $b \equiv 1$ one has that $I_b \equiv I_0$). Then Lemma 2.4 implies that I_b satisfies $(PS)_c$ at $c = c_b$ and hence I_b has a M-P critical point. Thus we can state the following existence result

Theorem 2.5. *If* (2.4) *and* (2.9) *hold,* I_b *has a Mountain Pass critical point and hence the problem*

$$\begin{cases} -\Delta u + u = b(x)u^p, \\ u \in W^{1,2}(\mathbb{R}^n), \end{cases} \tag{2.10}$$

has a (nontrivial) solution.

Actually, it is possible to show that the Mountain Pass critical point gives rise to a positive solution of (2.10).

This result can be seen as a particular case of the *Concentration-Compactness* principle introduced by P.L. Lions [105, 106]. Limiting ourselves to a short discussion of the so-called *locally compact case*, let us state the main ingredient of this method, namely the following lemma

Lemma 2.6. [Concentration-Compactness Lemma] *Let $\rho_j \in L^1(\mathbb{R}^n)$ be such that $\rho_j \geq 0$ and $\int_{\mathbb{R}^n} \rho_j dx = \lambda$, where $\lambda > 0$ is fixed.*

Then there exists a subsequence, still denoted by ρ_j, satisfying one of the following three alternatives:

(i) (*compactness*) $\exists\, y_j \in \mathbb{R}^n$ *such that*

$$\forall\, \varepsilon > 0,\ \exists\, R > 0 \quad \textit{such that} \quad \int_{B_R(y_j)} \rho_j dx \geq \lambda - \varepsilon;$$

(ii) (*vanishing*) $\lim_{j\to\infty} \sup_{y\in\mathbb{R}^n} \int_{B_R(y)} \rho_j dx = 0,\ \forall\, R > 0$;

(iii) (*dichotomy*) $\exists\, \alpha \in]0, \lambda[$ *such that* $\forall\, \varepsilon > 0$ *there exist* $\rho_{1,j}, \rho_{2,j} > 0$ *such that for* $j \gg 1$ *one has:*

$$\int_{\mathbb{R}^n} |\rho_j - (\rho_{1,j} + \rho_{2,j})| dx \leq \varepsilon, \qquad \left| \int_{\mathbb{R}^n} \rho_{1,j} dx - \alpha \right| \leq \varepsilon,$$
$$\left| \int_{\mathbb{R}^n} \rho_{2,j} dx - (\lambda - \alpha) \right| \leq \varepsilon, \qquad \lim_{j\to\infty} \operatorname{dist}(\operatorname{supp}\rho_{1,j}, \operatorname{supp}\rho_{2,j}) = +\infty.$$

This Lemma can be used to find minima of some classes of functionals J constrained on a manifold $\mathcal{M}$. Roughly, if $u_j \in \mathcal{M}$ is a minimizing sequence, one rules out vanishing and dichotomy. For example, dealing with solutions of (2.10), one takes $J(u) = \|u\|^2$ and $\mathcal{M} = \{u \in W^{1,2}(\mathbb{R}^n) : \int |u|^{p+1} dx = 1\}$. *Vanishing* is readily excluded because $u_j \in \mathcal{M}$, while *dichotomy* is ruled out by the assumption (2.9). Then *compactness* holds and this implies that u_j converges strongly in $W^{1,2}(\mathbb{R}^n)$ up to translations.

We conclude this short review by stating the following existence result which is proved by using the Concentration-Compactness method, proved in [35], see also [34].

Theorem 2.7. *Let $1 < p < \frac{n+2}{n-2}$ and suppose that $b \in L^\infty(\mathbb{R}^n)$ satisfies*

(a) $b > 0$ *and* $\lim_{|x|\to\infty} b(x) = b_\infty > 0$;

(b) *there exist $R, C, \delta > 0$ such that*

$$b(x) \geq b_\infty - C \exp(-\delta x), \qquad \textit{for } |x| \geq R.$$

Then (2.10) *has a positive solution.*

Let us point out that in the present case the critical level of I_b can be greater than c_0. For this reason, more delicate topological arguments are required to prove existence.

2.2 Critical points for a class of perturbed functionals, I

In this and in the subsequent section we will discuss the existence of critical points for a class of functionals that do not satisfy the (PS) condition. The specific feature of these functionals is that they are perturbative in nature. For this specific class of functionals we will provide results that, in general, could not be obtained by means of the Concentration-Compactness method.

In this section we deal with functionals of the form

$$I_\varepsilon(u) = I_0(u) + \varepsilon G(u). \tag{2.11}$$

where $I_0 \in C^2(\mathcal{H}, \mathbb{R})$ plays the role of the unperturbed functional and $G \in C^2(\mathcal{H}, \mathbb{R})$ is the perturbation.

We will always suppose that there exists a d-dimensional smooth, say C^2, manifold Z, $0 < d = \dim(Z) < \infty$, such that all $z \in Z$ is a critical point of I_0. The set Z will be called a *critical manifold*(of I_0).

Remark 2.8. In our discussion Z will always be non-compact. Roughly, this is why I_ε does not satisfy, in general, the (PS) condition. We will investigate in which circumstances the perturbation G makes it possible to recover the compactness and allows us to find critical points of I_ε. □

Let T_zZ denote the tangent space to Z at z. If Z is a critical manifold then for every $z \in Z$ one has that $I_0'(z) = 0$. Differentiating this identity, we get

$$(I_0''(z)[v]|\phi) = 0, \quad \forall\, v \in T_zZ,\ \forall\, \phi \in \mathcal{H},$$

and this shows that every $v \in T_zZ$ is a solution of the *linearized equation* $I_0''(z)[v] = 0$, namely that $v \in \mathrm{Ker}[I_0''(z)]$: $T_zZ \subseteq \mathrm{Ker}[I_0''(z)]$. In particular, $I_0''(z)$ has a non-trivial Kernel (whose dimension is at least d) and hence all the $z \in Z$ are degenerate critical points of I_0. We shall require that this degeneracy is minimal. Precisely we will suppose that

(ND) $\quad T_zZ = \mathrm{Ker}[I_0''(z)], \quad \forall\, z \in Z.$

Remark 2.9. If, instead of a manifold, we consider an isolated critical point u_0, the condition (ND) corresponds to require that $I_0''(u_0)$ is invertible, namely that u_0 is non-degenerate critical point of I_0. Obviously, in such a case, a straight application of the Implicit Function Theorem allows us to find, for $|\varepsilon|$ small, a solution of (1.18). Differently, dealing with a critical manifold, proving that Z satisfies (ND) is equivalent to show that $\mathrm{Ker}[I_0''(z)] \subseteq T_zZ$, namely that every solution of the linearized equation $I_0''(z)[v] = 0$ belongs to T_zZ. □

In addition to (ND) we will assume that

(Fr) $\quad$ for all $z \in Z$, $I_0''(z)$ is an index 0 Fredholm map. [2]

[2] A linear map $T \in L(\mathcal{H}, \mathcal{H})$ is *Fredholm* if the kernel is finite-dimensional and the image is closed and has finite codimension. The *index* of T is $\dim(\mathrm{Ker}[T]) - \mathrm{codim}(\mathrm{Im}[T])$

Definition. *A critical manifold* Z *will be called non-degenerate,* ND *in short, if* (ND) *and* (Fr) *hold.*

2.2.1 A finite-dimensional reduction: the Lyapunov-Schmidt method revisited

As anticipated in Chapter 1, Section 1.5, the equation $I'_\varepsilon(u) = 0$ can be seen as a bifurcation problem and the method we will use is borrowed from the Theory of Bifurcation. Actually, the finite-dimensional reduction we are going to discuss, is nothing but the Lyapunov-Schmidt procedure, adapted to take advantage of the variational setting.

First some notation is in order. Let us set $W = (T_z Z)^\perp$ and let $\{q_i\}_{1\le i\le d}$ be an orthonormal basis such that $T_z Z = \text{span}\{q_1, \dots, q_d\}$. In the sequel we always assume (and understand) that Z has a (local) C^2 parametric representation $z = z_\xi$, $\xi \in \mathbb{R}^d$. Furthermore, we also suppose that $q_i = \partial_{\xi_i} z_\xi / \|\partial_{\xi_i} z_\xi\|$. This will be verified in all our applications.

We look for critical points of I_ε in the form $u = z + w$ with $z \in Z$ and $w \in W$. If $P : \mathcal{H} \to W$ denotes the orthogonal projection onto W, the equation $I'_\varepsilon(z + w) = 0$ is equivalent to the following system

$$\begin{cases} PI'_\varepsilon(z + w) = 0, & \text{(the \textit{auxiliary equation})} \\ (Id - P)I'_\varepsilon(z + w) = 0, & \text{(the \textit{bifurcation equation})}. \end{cases} \tag{2.12}$$

Let first solve the auxiliary equation, namely

$$PI'_0(z + w) + \varepsilon PG'(z + w) = 0, \tag{2.13}$$

by means of the Implicit Function Theorem, see, e.g., [24, Theorem 2.3]. Let $F : \mathbb{R} \times Z \times W \to W$ be defined by setting

$$F(\varepsilon, z, w) = PI'_0(z + w) + \varepsilon PG'(z + w).$$

F is of class C^1 and one has $F(0, z, 0) = 0$, for every $z \in Z$. Moreover, letting $D_w F(0, z, 0)$ denote the partial derivative with respect to w evaluated at $(0, z, 0)$, one has:

Lemma 2.10. *If* (ND) *and* (Fr) *hold, then* $D_w F(0, z, 0)$ *is invertible as a map from* W *into itself.*

Proof. The map $D_w F(0, z, 0)$ is given by

$$D_w F(0, z, 0) : v \mapsto PI''_0(z)[v].$$

Remark that, for any $i = 1, 2, \dots, d$, there holds:

$$(I''_0(z)[v] \mid q_i) = (I''_0(z)[q_i] \mid v) = 0,$$

because $q_i \in T_z Z$. Hence $PI_0''(z)[v] = I_0''(z)[v]$ and the equation $D_w F(0, z, 0)[v] = 0$ becomes $I_0''(z)[v] = 0$. Thus $v \in \mathrm{Ker}[I_0''(z)] \cap W$ and from (ND) it follows that $v = 0$, namely that $D_w F(0, z, 0)$ is injective. Using (Fr) we then deduce that $D_w F(0, z, 0) : W \to W$ is invertible. $\square$

Lemma 2.11. *Let* (ND) *and* (Fr) *hold. Given any compact subset Z_c of Z there exists $\varepsilon_0 > 0$ with the following property: for all $|\varepsilon| < \varepsilon_0$, for all $z \in Z_c$, the auxiliary equation* (2.13) *has a unique solution $w = w_\varepsilon(z)$ such that:*

(i) *$w_\varepsilon(z) \in W = (T_z Z)^\perp$ and is of class C^1 with respect to $z \in Z_c$ and $w_\varepsilon(z) \to 0$ as $|\varepsilon| \to 0$, uniformly with respect to $z \in Z_c$, together with its derivative with respect to z, w_ε';*

(ii) *more precisely one has that $\|w_\varepsilon(z)\| = O(\varepsilon)$ as $\varepsilon \to 0$, for all $z \in Z_c$.*

Proof. Lemma 2.10 allows us to apply the Implicit Function Theorem to $F(\varepsilon, z, w) = 0$ yielding a solution $w_\varepsilon = w_\varepsilon(z) \in W$, for all $z \in Z_c$, satisfying (i) (for brevity, in the sequel the dependence on z will be understood). Let us point out explicitly that w_ε' for $\varepsilon = 0$ is zero. Actually w_ε' satisfies

$$PI_0''(z + w_\varepsilon)[q + w_\varepsilon'] + \varepsilon PG''(z + w_\varepsilon)[q + w_\varepsilon'] = 0,$$

where $q = \sum_{i=1}^d \alpha_i q_i \in T_z Z$. Then for $\varepsilon = 0$ we get $PI_0''(z)[q + w_0'] = 0$. Since $q \in T_z Z \subseteq \mathrm{Ker}[I_0''(z)]$, then $PI_0''(z)[q] = 0$, and this implies $w_0' = 0$.

Let us now prove (ii). Setting $\widetilde{w}_\varepsilon = \varepsilon^{-1} w_\varepsilon(z)$ we have to prove that $\|\widetilde{w}_\varepsilon\| \le$ const. for $|\varepsilon|$ small. Recall that w_ε satisfies $PI_\varepsilon'(z + w_\varepsilon) = 0$; using a Taylor expansion we find

$$\begin{aligned} I_\varepsilon'(z + w_\varepsilon) &= I_0'(z + w_\varepsilon) + \varepsilon G'(z + w_\varepsilon) \\ &= I_0'(z) + I_0''(z)[w_\varepsilon] + \varepsilon G'(z) + \varepsilon G''(z)[w_\varepsilon] + o(\|w_\varepsilon\|). \end{aligned}$$

Since $I_0'(z) = 0$ we get

$$I_\varepsilon'(z + w_\varepsilon) = I_0''(z)[w_\varepsilon] + \varepsilon G'(z) + \varepsilon G''(z)[w_\varepsilon] + o(\|w_\varepsilon\|),$$

and the equation $PI_\varepsilon'(z + w_\varepsilon) = 0$ becomes

$$PI_0''(z)[w_\varepsilon] + \varepsilon PG'(z) + \varepsilon PG''(z)[w_\varepsilon] + o(\|w_\varepsilon\|) = 0. \tag{2.14}$$

Dividing by ε we infer that $\widetilde{w}_\varepsilon$ verifies

$$PI_0''(z)[\widetilde{w}_\varepsilon] + PG'(z) + PG''(z)[w_\varepsilon] + \varepsilon^{-1} o(\|w_\varepsilon\|) = 0.$$

Since $\varepsilon^{-1} o(\|w_\varepsilon\|) = o(\|\widetilde{w}_\varepsilon\|)$ we deduce

$$PI_0''(z)[\widetilde{w}_\varepsilon] = -PG'(z) - PG''(z)[w_\varepsilon] + o(\|\widetilde{w}_\varepsilon\|).$$

Recalling that $w_\varepsilon \to 0$ as $|\varepsilon| \to 0$, we get

$$PI_0''(z)[\widetilde{w}_\varepsilon] \to -PG'(z), \qquad \text{as } \varepsilon \to 0,$$

and this implies that (ii) holds. $\square$

2.2.2 Existence of critical points

We shall now solve the bifurcation equation. In order to do this, let us define the *reduced functional* $\Phi_\varepsilon : Z \to \mathbb{R}$ by setting

$$\Phi_\varepsilon(z) = I_\varepsilon(z + w_\varepsilon(z)). \tag{2.15}$$

Theorem 2.12. *Let $I_0, G \in C^2(\mathcal{H}, \mathbb{R})$ and suppose that I_0 has a smooth critical manifold Z which is non-degenerate, in the sense that* (ND) *and* (Fr) *hold. Given a compact subset Z_c of Z, let us assume that Φ_ε has, for $|\varepsilon|$ sufficiently small, a critical point $z_\varepsilon \in Z_c$. Then $u_\varepsilon = z_\varepsilon + w_\varepsilon(z_\varepsilon)$ is a critical point of $I_\varepsilon = I_0 + \varepsilon G$.*

Proof. We use the preceding notation and, to be short, we write below D_i for D_{ξ_i}, etc. Let ξ_ε be such that $z_\varepsilon = z_{\xi_\varepsilon}$, and set $q_i^\varepsilon = \partial z / \partial \xi_i|_{\xi_\varepsilon}$. Without loss of generality we can assume that $z_\varepsilon \to z^* \in Z_c$ as $\varepsilon \to 0$. From Lemma 2.11 we infer that there exists $\varepsilon_0 > 0$ such that the auxiliary equation (2.13) has a solution $w_\varepsilon(z_\varepsilon)$, defined for $|\varepsilon| < \varepsilon_0$. In particular, from (i) of that lemma and by continuity, one has that

$$\lim_{|\varepsilon| \to 0} (D_i w_\varepsilon(z_\varepsilon) \mid q_j^\varepsilon) = 0, \quad i, j = 1, \ldots, d.$$

Let us consider the matrix $B^\varepsilon = (b_{ij}^\varepsilon)_{ij}$, where

$$b_{ij}^\varepsilon = \left(D_i w_\varepsilon(z_\varepsilon) \mid q_j^\varepsilon \right).$$

From the above arguments we can choose $0 < \varepsilon_1 < \varepsilon_0$, such that

$$|\det(B^\varepsilon)| < 1, \quad \forall\, |\varepsilon| < \varepsilon_1. \tag{2.16}$$

Fix $\varepsilon > 0$ such that $|\varepsilon| < \min\{\varepsilon_0, \varepsilon_1\}$. Since z_ε is a critical point of Φ_ε we get

$$(I'_\varepsilon(z_\varepsilon + w_\varepsilon(z_\varepsilon)) \mid q_i^\varepsilon + D_i w_\varepsilon(z_\varepsilon)) = 0, \quad i = 1, \ldots, d.$$

From (2.13), namely $PI'_\varepsilon(z + w_\varepsilon(z_\varepsilon)) = 0$, we deduce that $I'_\varepsilon(z_\varepsilon + w_\varepsilon(z_\varepsilon)) = \sum A_{i,\varepsilon} q_i^\varepsilon$, where

$$A_{i,\varepsilon} = (I'_\varepsilon(z_\varepsilon + w_\varepsilon(z_\varepsilon)) \mid q_i^\varepsilon).$$

Then we find

$$\left(\sum_j A_{j,\varepsilon}\, q_j^\varepsilon \mid q_i^\varepsilon + D_i w_\varepsilon(z_\varepsilon) \right) = 0, \quad i = 1, \ldots, d,$$

namely

$$A_{i,\varepsilon} + \sum_j A_{j,\varepsilon} \left(q_j^\varepsilon \mid D_i w_\varepsilon(z_\varepsilon) \right) = A_{i,\varepsilon} + \sum_j A_{j,\varepsilon} b_{ij}^\varepsilon = 0, \quad i = 1, \ldots, d. \tag{2.17}$$

Equation (2.17) is a $(d \times d)$ linear system whose matrix $Id_{\mathbb{R}^d} + B^\varepsilon$ has entries $\delta_{ij} + b_{ij}^\varepsilon$, where δ_{ij} is the Kronecker symbol and b_{ij}^ε are defined above and satisfy (2.16). Then, for $|\varepsilon| < \varepsilon_1$, the matrix $Id_{\mathbb{R}^d} + B^\varepsilon$ is invertible. Thus (2.17) has the trivial solution only: $A_{i,\varepsilon} = 0$ for all $i = 1, \ldots, d$. Since the $A_{i,\varepsilon}$'s are the components of $\Phi_\varepsilon(z_\varepsilon)$, the conclusion follows. $\square$

Let us point out explicitly that when Z is compact the preceding result immediately implies

Corollary 2.13. *If, in addition to the assumptions of Theorem* 2.12, *the critical manifold Z is compact, then for $|\varepsilon|$ small enough, I_ε has at least* $\mathrm{Cat}(Z)$ [3] *critical points.*

Proof. It suffices to apply the usual Lusternik-Schnierelman theory (see [136]) to the functional $\Phi_\varepsilon : Z \to \mathbb{R}$. $\square$

Remarks 2.14. (i) From the geometric point of view the preceding arguments can be outlined as follows. Consider the manifold $Z_\varepsilon = \{z + w_\varepsilon(z)\}$. Since z_ε is a critical point of Φ_ε, it follows that $u_\varepsilon \in Z_\varepsilon$ is a critical point of I_ε constrained on Z_ε and thus u_ε satisfies $I'_\varepsilon(u_\varepsilon) \perp T_{u_\varepsilon} Z_\varepsilon$. Moreover the definition of w_ε, see (2.13), implies that $I'_\varepsilon(z + w_\varepsilon(z)) \in T_z Z$. In particular, $I'_\varepsilon(u_\varepsilon) \in T_{z_\varepsilon} Z$. Since, for $|\varepsilon|$ small, $T_{u_\varepsilon} Z_\varepsilon$ and $T_{z_\varepsilon} Z$ are close, see (i) in Lemma 2.11, it follows that $I'_\varepsilon(u_\varepsilon) = 0$. A manifold with these properties is called a *natural constraint* for I_ε.

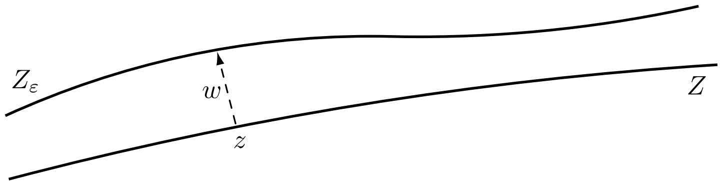

Figure 2.1. The manifold Z and the natural constraint Z_ε

(ii) In the proof of Theorem 2.12 we do not need to use that $w'_\varepsilon(z_\varepsilon) \to 0$, but only that $w'_\varepsilon(z_\varepsilon) \to 0$. Actually, from $(w_\varepsilon(z_\varepsilon) \,|\, q_j^\varepsilon) = 0, \quad j = 1, \dots, d$, we get

$$(D_i w_\varepsilon(z_\varepsilon) \,|\, q_j^\varepsilon) + (w_\varepsilon(z_\varepsilon) \,|\, D_i q_j^\varepsilon) = 0, \quad i, j = 1, \dots, d.$$

Since $w_\varepsilon(z_\varepsilon) \to 0$ as $|\varepsilon| \to 0$, we infer that $(D_i w_\varepsilon(z_\varepsilon) \,|\, q_j^\varepsilon) \to 0$ as $\varepsilon \to 0$ and this suffices to show that (2.16) holds. The rest of the proof is unaffected.

(iii) In general, one can show the following perturbation result: *suppose that $f \in C^2(\mathcal{H}, \mathbb{R})$ has a compact non-degenerate critical manifold Z of critical points and satisfies* (Fr). *Let $\mathcal{N}$ be a neighborhood of Z and let $g \in C^2(\mathcal{N}, \mathbb{R})$. If $\|f - g\|_{C^2}$ is sufficiently small, then g has at least* $\mathrm{Cat}(Z)$ *critical points in $\mathcal{N}$*. See [11]. The result can be improved to cover the case in which g is close to f in the C^1 norm, provided $\mathrm{Cat}(Z)$ is substituted by the cup-long of Z [4], see [54]. $\square$

[3] $\mathrm{Cat}(Z)$ denotes the Lusternik-Schnierelman category of Z, namely the smallest integer k such that $Z \subset \bigcup_{1 \le i \le k} \mathcal{C}_i$, where the sets $\mathcal{C}_i$ are closed and contractible in Z.

[4] The *cup long* $l(Z)$ of Z is defined by $l(Z) = 1 + \sup\{k \in \mathbb{N} : \exists\, \alpha_1, \dots, \alpha_k \in \check{H}^*(Z) \setminus 1,\ \alpha_1 \cup \dots \cup \alpha_k \neq 0\}$. If no such class exists, we set $l(Z) = 1$. Here $\check{H}^*(Z)$ is the Alexander cohomology of Z with real coefficients and $\cup$ denotes the cup product. In many cases $\mathrm{Cat}(Z) = l(Z)$ but in general one has that $l(Z) < \mathrm{Cat}(Z)$.

2.2.3 Other existence results

In order to use Theorem 2.12 it is convenient to expand Φ_ε in powers of ε.

Lemma 2.15. *One has:*

$$\Phi_\varepsilon(z) = c_0 + \varepsilon G(z) + o(\varepsilon), \quad \textit{where } c_0 = I_0(z).$$

Proof. Recall that

$$\Phi_\varepsilon(z) = I_0(z + w_\varepsilon(z)) + \varepsilon G(z + w_\varepsilon(z)).$$

Let us evaluate separately the two terms above. First we have

$$I_0(z + w_\varepsilon(z)) = I_0(z) + (I_0'(z) \,|\, w_\varepsilon(z)) + o(\|w_\varepsilon(z)\|).$$

Since $I_0'(z) = 0$ we get

$$I_0(z + w_\varepsilon(z)) = c_0 + o(\|w_\varepsilon(z)\|). \tag{2.18}$$

Similarly, one has

$$\begin{aligned} G(z + w_\varepsilon(z)) &= G(z) + (G'(z) \,|\, w_\varepsilon(z)) + o(\|w_\varepsilon(z)\|) \\ &= G(z) + O(\|w_\varepsilon(z)\|). \end{aligned} \tag{2.19}$$

Putting together (2.18) and (2.19) we infer that

$$\Phi_\varepsilon(z) = c_0 + \varepsilon \left[G(z) + O(\|w_\varepsilon(z)\|)\right] + o(\|w_\varepsilon(z)\|). \tag{2.20}$$

Since $\|w_\varepsilon(z)\| = O(\varepsilon)$, see Lemma 2.11-(ii), the result follows. □

The preceding lemma, jointly with Theorem 2.12 yields

Theorem 2.16. *Let $I_0, G \in C^2(\mathcal{H}, \mathbb{R})$ and suppose that I_0 has a smooth critical manifold Z which is non-degenerate. Let $\bar{z} \in Z$ be a strict local maximum or minimum of $\Gamma := G_{|Z}$.*

Then for $|\varepsilon|$ small the functional I_ε has a critical point u_ε and if $\bar{z}$ is isolated, then $u_\varepsilon \to \bar{z}$ as $\varepsilon \to 0$.

Proof. We will prove the theorem when $\bar{z}$ is a minimum of Γ: the other case is quite similar. Let $\gamma > 0$ and let $\mathcal{U}_\delta$ be a δ-neighborhood of $\bar{z}$ such that

$$\Gamma(z) \geq \Gamma(\bar{z}) + \gamma, \quad \forall\, z \in \partial\mathcal{U}_\delta.$$

Using Lemma 2.15 we find, for $|\varepsilon|$ small

$$\Phi_\varepsilon(z) - \Phi_\varepsilon(\bar{z}) = \varepsilon\ (\Gamma(z) - \Gamma(\bar{z})) + o(\varepsilon).$$

Then, there exists $\varepsilon_1 > 0$ small such that for every $z \in \partial \mathcal{U}_\delta$ one has

$$\begin{cases} \Phi_\varepsilon(z) - \Phi_\varepsilon(\bar z) > 0 & \text{if} \quad 0 < \varepsilon < \varepsilon_1, \\ \Phi_\varepsilon(z) - \Phi_\varepsilon(\bar z) < 0 & \text{if} \quad -\varepsilon_1 < \varepsilon < 0. \end{cases}$$

In the former case Φ_ε has a local minimum in $\mathcal{U}_\delta$, while in the latter it has a local maximum. In any case, Φ_ε has a critical point $z_\varepsilon \in \mathcal{U}_\delta$ and hence, by Theorem 2.12, $u_\varepsilon = z_\varepsilon + w_\varepsilon(z_\varepsilon)$ is a critical point of I_ε. If $\bar z$ is an isolated minimum or maximum of Γ we can take δ arbitrarily small and hence $z_\varepsilon \to \bar z$ as well as $u_\varepsilon \to \bar z$. □

Theorem 2.16 is a particular case of the following general result in which $\bar z \in Z$ is a critical point of $\Gamma = G_{|Z}$ satisfying

(G′) $\quad \exists\, \mathcal{N} \subset \mathbb{R}^d$ open bounded such that the topological degree $d(\Gamma', \mathcal{N}, 0) \neq 0$.

$\bar z \in Z$ is called *stable critical point* if $\exists B_r(\bar z)$ such that (G') holds with $\mathcal{N} = B_r(\bar z)$. For the definition of the topological degree and its properties see for example [81]. Let us point out that if (G′) holds then Γ has a critical point in $\mathcal{N}$. Moreover, if Γ has either a strict local maximum (or minimum), or any non-degenerate critical point $\bar z$, we can take as $\mathcal{N}$ the ball $B_r(\bar z)$ with $r \ll 1$, and (G′) holds true.

Theorem 2.17. *Let $I_0, G \in C^2(\mathcal{H}, \mathbb{R})$. Suppose that I_0 has a smooth critical manifold Z which is non-degenerate and that* (G′) *holds.*

Then for $|\varepsilon|$ small the functional I_ε has a critical point u_ε and there exists $\hat z \in \mathcal{N}$, $\Gamma'(\hat z) = 0$, such that $u_\varepsilon \to \hat z$ as $\varepsilon \to 0$. Therefore if, in addition, $\mathcal{N}$ contains only an isolated critical point $\bar z$ of Γ', then $u_\varepsilon \to \bar z$ as $\varepsilon \to 0$.

Proof. From the definition of Φ_ε we infer that, for all $v \in T_z Z$,

$$(\Phi'_\varepsilon(z) \,|\, v) = (I'_0(z + w_\varepsilon) \,|\, v + w'_\varepsilon) + \varepsilon (G(z + w_\varepsilon) \,|\, v + w'_\varepsilon). \tag{2.21}$$

Moreover, as $\varepsilon \to 0$, one has

$$I'_0(z + w_\varepsilon) = I'_0(z) + I''_0(z)[w_\varepsilon] + o(\|w_\varepsilon\|) = I''_0(z)[w_\varepsilon] + o(\|w_\varepsilon\|), \tag{2.22}$$

$$G'(z + w_\varepsilon) = G'(z) + G''(z)[w_\varepsilon] + o(\|w_\varepsilon\|). \tag{2.23}$$

From (2.22) it follows (as $\varepsilon \to 0$)

$$\begin{aligned} (I'_0(z + w_\varepsilon) \,|\, v + w'_\varepsilon) &= (I''_0(z)[w_\varepsilon] \,|\, v + w'_\varepsilon) + o(\|w_\varepsilon\|) \\ &= (I''_0(z)[v] \,|\, w_\varepsilon) + (I''_0(z)[w_\varepsilon] \,|\, w'_\varepsilon) + o(\|w_\varepsilon\|) \\ &= (I''_0(z)[w_\varepsilon] \,|\, w'_\varepsilon) + o(\|w_\varepsilon\|). \end{aligned}$$

Since $\|w_\varepsilon\| = O(\varepsilon)$ and $w'_\varepsilon \to 0$ as $\varepsilon \to 0$, we deduce:

$$(I'_0(z + w_\varepsilon) \,|\, v + w'_\varepsilon) = o(\varepsilon), \quad (\varepsilon \to 0). \tag{2.24}$$

Similarly, from (2.23) we get:

$$\begin{aligned} (G'(z + w_\varepsilon) \,|\, v + w'_\varepsilon) &= (G'(z) \,|\, v + w'_\varepsilon) + (G''(z)[w_\varepsilon] \,|\, v + w'_\varepsilon) + o(\|w_\varepsilon\|) \\ &= (G'(z) \,|\, v) + (G'(z) \,|\, w'_\varepsilon) + (G''(z)[w_\varepsilon] \,|\, v) \\ &\quad + (G''(z)[w_\varepsilon] \,|\, w'_\varepsilon) + o(\|w_\varepsilon\|). \end{aligned}$$

Using again the fact that $\|w_\varepsilon\| = O(\varepsilon)$ and $w'_\varepsilon \to 0$ as $\varepsilon \to 0$, we find:

$$\lim_{\varepsilon\to 0} [(G'(z) \,|\, w'_\varepsilon) + (G''(z)[w_\varepsilon] \,|\, v) + (G''(z)[w_\varepsilon] \,|\, w'_\varepsilon)] = 0,$$

and this yields

$$(G'(z + w_\varepsilon) \,|\, v + w'_\varepsilon) = (G'(z) \,|\, v) + o(1), \quad (\varepsilon \to 0). \tag{2.25}$$

Inserting (2.24) and (2.25) into (2.21) it follows that, for all $v \in T_z Z$,

$$(\Phi'_\varepsilon(z) \,|v\,) = \varepsilon\,(G'(z) \,|\, v) + o(\varepsilon), \quad (\varepsilon \to 0),$$

namely

$$\Phi'_\varepsilon(z) = \varepsilon\,\Gamma'(z) + o(\varepsilon), \quad (\varepsilon \to 0).$$

Then the continuity property of the topological degree and (G$'$) yield, for $|\varepsilon|$ small,

$$d(\Phi'_\varepsilon, \mathcal{N}, 0) = d(\Gamma', \mathcal{N}, 0) \neq 0.$$

This implies that, for $|\varepsilon|$ small, the equation $\Phi'_\varepsilon(z) = 0$ has a solution in $\mathcal{N}$, proving the theorem. □

2.2.4 A degenerate case

If $G(z) \equiv 0$, Theorem 2.17 is useless and we need to evaluate further terms in the expansion of Φ_ε.

For $z \in Z$ we set $L_z = (PI''_0(z))^{-1}$.

Lemma 2.18. *If $G(z) = 0$ for every $z \in Z$, then for the solution $w_\varepsilon(z)$ of the auxiliary equation $PI'_\varepsilon(z + w) = 0$ one has:*

$$w_\varepsilon(z) = \varepsilon\bar{w} + o(\varepsilon), \quad \textit{where} \quad \bar{w} = \bar{w}(z) = -L_z G'(z).$$

Proof. From (2.14) and the fact that $w_\varepsilon \to 0$ it follows that

$$PI''_0(z)[w_\varepsilon] = -\varepsilon\, PG'(z) + o(\varepsilon).$$

Moreover, $G(z) \equiv 0$ implies $G'(z) \perp T_z Z$. Therefore, $PG'(z) = G'(z)$ and we find $w_\varepsilon = -\varepsilon\, L_z G'(z) + o(\varepsilon)$. □

Let us now expand Φ_ε. One has:

$$\Phi_\varepsilon(z) = \tfrac{1}{2}(I''_0(z)[w_\varepsilon] \,|\, w_\varepsilon) + \varepsilon G(z) + \varepsilon(G'(z) \,|\, w_\varepsilon) + o(\varepsilon^2).$$

Since $G(z) \equiv 0$, using the preceding lemma, we infer

$$\Phi_\varepsilon(z) = \tfrac{1}{2}\,\varepsilon^2 (I''_0(z)[\bar{w}] \,|\, \bar{w}) + \varepsilon^2 (G'(z) \,|\, \bar{w}) + o(\varepsilon^2).$$

Since $\bar{w} = -L_z G'(z)$, we get

$$\Phi_\varepsilon(z) = -\tfrac{1}{2}\,\varepsilon^2 (G'(z) \,|\, L_z G'(z)) + o(\varepsilon^2). \tag{2.26}$$

In the sequel, e.g., in the applications to the Yamabe problem, we will deal with a C^2 functional of the form

$$I_\varepsilon(u) = I_0(u) + \varepsilon G_1(u) + \varepsilon^2 G_2(u) + o(\varepsilon), \quad (\varepsilon \to 0).$$

In such a case the preceding arguments yield:

Lemma 2.19. *If $G_1(z) = 0$, then for every $z \in Z$ then one has*

$$\Phi_\varepsilon(z) = \varepsilon^2 \, \left[G_2(z) - \tfrac{1}{2}\,(G_1'(z) \,|\, L_z G_1'(z))\right] + o(\varepsilon^2).$$

At this point we can repeat the arguments carried out in the proofs of Theorems 2.16 and 2.17 with Γ replaced by

$$\widetilde{\Gamma}(z) = G_2(z) - \tfrac{1}{2}\,(G_1'(z) \,|\, L_z G_1'(z)), \tag{2.27}$$

yielding

Theorem 2.20. *Let $I_0 \in C^2(\mathcal{H}, \mathbb{R})$ and suppose that I_0 has a smooth critical manifold Z which is non-degenerate. Furthermore, let $G_1, G_2 \in C^2(\mathcal{H}, \mathbb{R})$, with $G_1(z) = 0$ for all $z \in Z$. Let $\tilde{z} \in Z$ be a stationary point of $\widetilde{\Gamma}$ and let $\mathcal{N}$ be a neighborhood of $\tilde{z}$ such that $d(\widetilde{\Gamma}', \mathcal{N}, 0) \neq 0$.*

Then for $|\varepsilon|$ small the functional $I_\varepsilon = I_0 + \varepsilon G_1 + \varepsilon^2 G_2 + o(\varepsilon)$, $(\varepsilon \to 0)$ has a critical point u_ε and if $\tilde{z}$ is isolated, then $u_\varepsilon \to \tilde{z}$ as $\varepsilon \to 0$.

2.2.5 A further existence result

Another way to use Theorem 2.12 is to investigate the asymptotic behavior of $\Phi_\varepsilon(z)$. For example, if

$$\lim_{|\xi| \to \infty} \Phi_\varepsilon(z_\xi) = \text{const.},$$

uniformly with respect to ε, then either $\Phi_\varepsilon(z_\xi) \equiv \text{const.}$, or it has a global maximum or minimum. In any case Φ_ε possesses a critical point z_ε which will give rise, through Theorem 2.12, to a solution u_ε of $I_\varepsilon'(u) = 0$. To carry over this procedure, we need first of all a global version of Lemma 2.11 which, on the contrary, is local in nature. The following lemma provides such a global tool.

Lemma 2.21. *Suppose that:*

(i) *the operator $PI_0''(z_\xi)$ is invertible on $W = (T_{z_\xi}(Z))^\perp$ uniformly with respect to $\xi \in \mathbb{R}^d$, in the sense that there exists $C > 0$ such that*

$$\|(PI_0''(z_\xi))^{-1}\|_{L(W,W)} \leq C \quad \forall\, \xi \in \mathbb{R}^d. \tag{2.28}$$

(ii) *the remainder* $R_\xi(w) = I_0'(z_\xi + w) - I_0''(z_\xi)[w]$ *is such that* $R_\xi(w) = o(\|w\|)$ *as* $\|w\| \to 0$, *uniformly with respect to* $\xi \in \mathbb{R}^d$.

(iii) *There exists* $C_1 > 0$ *such that* $\|PG'(z_\xi + w)\| \le C_1 \quad \forall \xi \in \mathbb{R}^2$, $\forall w \in W$, $\|w\| \le 1$.

Then there exists $\bar{\varepsilon} > 0$ *such that for every* $|\varepsilon| < \bar{\varepsilon}$, *for every* $\xi \in \mathbb{R}^d$, *the auxiliary equation* (2.13) *has a unique solution* $w = w_\varepsilon(z_\xi)$ *and* $w_\varepsilon(z_\xi) \to 0$ *as* $\varepsilon \to 0$, *uniformly with respect to* $\xi \in \mathbb{R}^d$.

Proof. Since $I_0'(z_\xi+w)+\varepsilon G'(z_\xi+w) = I_0''(z_\xi)[w]+R_\xi(w)+\varepsilon G'(z_\xi+w)$, equation (2.13), namely $PI_0'(z_\xi+w)+\varepsilon PG'(z_\xi+w) = 0$, becomes $PI_0''(z_\xi)[w]+PR_\xi(w)+\varepsilon PG'(z_\xi+w) = 0$. Since $PI_0''(z_\xi)$ is invertible, then (2.13) is equivalent to

$$w = N_{\varepsilon,\xi}(w) := (PI_0''(z_\xi))^{-1}\left[\varepsilon PG'(z_\xi+w) - PR_\xi(w)\right]. \tag{2.29}$$

If (i)–(iii) hold there exists $\hat{\varepsilon} > 0$ such that for every $|\varepsilon| \le \hat{\varepsilon}$ and every $\xi \in \mathbb{R}^d$ the nonlinear operator $N_{\varepsilon,\xi} : W \to W$ is a contraction. Furthermore, for $|\varepsilon|$ possibly smaller, there exists $\rho(\varepsilon) > 0$, $\lim_{\varepsilon\to 0}\rho(\varepsilon) = 0$, such that $N_{\varepsilon,\xi}$ maps the ball $B_{\rho(\varepsilon)} \subset W$ into itself. Thus, for such ε, the auxiliary equation (2.13) has a unique solution $w_\varepsilon(z_\xi) \in W$, for all $\xi \in \mathbb{R}^d$ such that $\|w_\varepsilon(z_\xi)\| \le \rho(\varepsilon)$. □

Remark 2.22. Since we can still apply the Implicit Function Theorem, the local properties proved in Lemma 2.11 continue to hold: for each $|\varepsilon|$ small, $w_\varepsilon(z_\xi)$ is of class C^1 with respect to ξ. □

From Lemma 2.21 and Remark 2.22 we readily infer

Theorem 2.23. *Let* $I_0, G \in C^2(\mathcal{H}, \mathbb{R})$ *and assume that* I_0 *has a smooth critical manifold* Z *which is non-degenerate. Suppose also that the assumptions of Lemma* 2.21 *hold, and that there exists* C_0 *such that*

$$\lim_{|\xi|\to\infty} \Phi_\varepsilon(z_\xi) = C_0,$$

uniformly with respect to $|\varepsilon|$ *small. Then, for* $|\varepsilon|$ *small,* $I_\varepsilon = I_0 + \varepsilon G$ *has a critical point.*

Proof. If Φ_ε is identically equal to C_0, then any $z \in Z$ is a critical point of Φ_ε, for all $|\varepsilon|$ small, and $z + w_\varepsilon(z)$ is a critical point of I_ε. Otherwise, Φ_ε achieves the global maximum (or minimum) at $z_\varepsilon = z_{\xi_\varepsilon}$. Moreover, there exists $R > 0$ such that $|\xi_\varepsilon| \le R$ for all $|\varepsilon|$ small. At this point, taking also into account Lemma 2.21 and Remark 2.22, we can repeat the arguments carried out in the proof of Theorem 2.12. In particular, as pointed out in Remark 2.14-(ii), in the proof of that Theorem we only need that $w_\varepsilon(z_\varepsilon) \to 0$ as $\varepsilon \to 0$, and this has been established in Lemma 2.21. It follows that $u_\varepsilon = z_\varepsilon + w_\varepsilon(z_\varepsilon)$ is a critical point of I_ε, as required. □

2.2.6 Morse index of the critical points of I_ε

Under some further regularity assumptions, it is possible to evaluate the Morse index of the critical points of I_ε found above. As before, we will suppose that $Z = \{z_\xi : \xi \in \mathbb{R}^d\}$ is a non-degenerate critical manifold of I_0, with tangent space spanned by $q_i = \partial_{\xi_i} z_\xi / \|\partial_{\xi_i} z_\xi\|$. Moreover, we will assume that

$$(D_k q_i \,|\, q_j) = 0, \quad \forall\, i,j,k = 1,\dots,d. \tag{2.30}$$

Let ξ_ε be a sequence of critical points of $\Gamma = G_{|Z}$ and suppose that $\xi_\varepsilon \to \xi^*$ as $\varepsilon \to 0$.

Theorem 2.24. *Suppose that I_0 and G are of class C^3 and that* (2.30) *holds. Furthermore, let ξ^* be a non-degenerate maximum (resp. minimum) of Γ and let m_0 denote the Morse index of $z^* = \lim_{\xi_\varepsilon \to \xi^*} z_{\xi_\varepsilon}$ as critical point of the restriction of I_0 to $T_{z^*}Z^\perp$. Then, for $|\varepsilon|$ small, $u_\varepsilon = z_\varepsilon + w_\varepsilon(z_\varepsilon)$ is a non-degenerate critical point of I_ε and its Morse index is given by $m_0 + d$, resp. m_0.*

For the proof we refer to Section 5 of [32].

2.3 Critical points for a class of perturbed functionals, II

Motivated by the bifurcation problem discussed in Chapter 3, see also Section 1.2, we will consider in this section the case in which I_ε has the form

$$I_\varepsilon(u) = I_0(u) + G(\varepsilon, u),$$

where $G : \mathbb{R} \times \mathcal{H} \to \mathcal{H}$ satisfies

(G.0) $G \in C(\mathbb{R} \times \mathcal{H}, \mathcal{H})$ and is such that $G(0,u) = 0$, for all $u \in \mathcal{H}$. Moreover the map $u \mapsto G(\varepsilon,u)$ is of class C^2, $\forall\, \varepsilon \in \mathbb{R}$ and

$$\begin{array}{ccc} \mathbb{R} \times \mathcal{H} & \to & \mathcal{H} \\ (\varepsilon,u) & \mapsto & D_u G(\varepsilon,u) \end{array} \quad \text{as well as} \quad \begin{array}{ccc} \mathbb{R} \times \mathcal{H} & \to & L(\mathcal{H},\mathcal{H}) \\ (\varepsilon,u) & \mapsto & D^2_{uu} G(\varepsilon,u) \end{array}$$

are continuous.

Let us point out explicitly that one has $D_u G(0,u) = 0$ as well as $D^2_{uu} G(0,u) = 0$.

We shall still suppose that I_ε has a ND critical manifold Z. Using (G.0), in particular the regularity assumptions of the maps $(\varepsilon,u) \mapsto D_u G(\varepsilon,u)$ and $(\varepsilon,u) \mapsto D^2_{uu} G(\varepsilon,u)$, we can again solve the auxiliary equation $PI'_\varepsilon(z+w) = 0$ by means of the Implicit Function Theorem getting, for $|\varepsilon|$ small, a solution $w_\varepsilon(z)$ satisfying the properties stated in Lemma 2.11-(i). Indeed, as pointed out in Remark 2.14-(ii), in the proof of Theorem 2.12 we have merely used the first statement of Lemma 2.11. Hence we can conclude as before that the following result holds:

Theorem 2.25. *Suppose that $I_0 \in C^2(\mathcal{H}, \mathbb{R})$ has a smooth critical manifold Z which is non-degenerate and let* (G.0) *hold. Then any critical point $z_\varepsilon \in Z$ of Φ_ε gives rise to a critical point $u_\varepsilon = z_\varepsilon + w_\varepsilon(z_\varepsilon)$ of $I_\varepsilon = I_0 + G(\varepsilon, u)$.*

In order to prove the counterpart of Theorem 2.17 some lemmas are in order. The first one provides the information contained in Lemma 2.11-(ii).

Lemma 2.26. *Suppose that, in addition to* (G.0)*, there exists $\beta > 0$ such that*

(G.1) $\quad \|D_u G(\varepsilon, z)\| = o(\varepsilon^\beta), \quad$ *as* $\varepsilon \to 0$.

Then $\|w_\varepsilon(z)\| = o(\varepsilon^\beta)$ as $\varepsilon \to 0$, uniformly in any compact subset Z_c of Z.

Proof. The proof follows the same lines of the proof of (ii) of Lemma 2.11. Let us set $\widetilde{w}_\varepsilon = \varepsilon^{-\beta} w_\varepsilon(z)$ (again, for brevity, in the sequel the dependence on z will be understood). We first prove that $\|\widetilde{w}_\varepsilon\| \le$ const. for $|\varepsilon|$ small. Precisely let us start by showing

$$\|D_u G(\varepsilon, z)\| = O(\varepsilon^\beta), \text{ as } \varepsilon \to 0, \quad \Longrightarrow \quad \|w_\varepsilon(z)\| = O(\varepsilon^\beta), \text{ as } \varepsilon \to 0. \tag{2.31}$$

By contradiction, assume

$$\lim_{|\varepsilon| \to 0} \|\widetilde{w}_\varepsilon\| = +\infty.$$

From $PI'_\varepsilon(z + w_\varepsilon) = 0$, namely

$$I'_\varepsilon(z + w_\varepsilon) = \sum A_{i,\varepsilon} q_i, \quad \text{where } A_{i,\varepsilon} = (I'_\varepsilon(z + w_\varepsilon(z)) \,|\, q_i),$$

using the Taylor expansion

$$\begin{aligned} I'_\varepsilon(z + w) &= I'_0(z + w) + D_u G(\varepsilon, z + w) \\ &= I''_0(z)[w] + D_u G(\varepsilon, z) + D^2_{uu} G(\varepsilon, z)[w] + o(\|w\|), \end{aligned} \tag{2.32}$$

and dividing by $\varepsilon^\beta \|\widetilde{w}_\varepsilon\|$, we find

$$I''_0(z)\left[\frac{\widetilde{w}_\varepsilon}{\|\widetilde{w}_\varepsilon\|}\right] = -\frac{D_u G(\varepsilon, z)}{\varepsilon^\beta \|\widetilde{w}_\varepsilon\|} - D^2_{uu} G(\varepsilon, z)\left[\frac{\widetilde{w}_\varepsilon}{\|\widetilde{w}_\varepsilon\|}\right] + \frac{o(\|w_\varepsilon\|)}{\varepsilon^\beta \|\widetilde{w}_\varepsilon\|} + \sum \frac{A_{i,\varepsilon}}{\varepsilon^\beta \|\widetilde{w}_\varepsilon\|} q_i. \tag{2.33}$$

Let us evaluate separately the above terms in the right-hand side.

First, from $\|D_u G(\varepsilon, z)\| = O(\varepsilon^\beta)$ we infer

$$\frac{\|D_u G(\varepsilon, z)\|}{\varepsilon^\beta} \le c_1$$

and hence

$$\lim_{\varepsilon \to 0} \frac{D_u G(\varepsilon, z)}{\varepsilon^\beta \|\widetilde{w}_\varepsilon\|} = 0.$$

By continuity, one has that $\lim_{\varepsilon \to 0} \|D^2_{uu} G(\varepsilon, z)\|_{L(\mathcal{H}, \mathcal{H})} = 0$ and then we get

$$\lim_{\varepsilon \to 0} D^2_{uu} G(\varepsilon, z)\left[\frac{\widetilde{w}_\varepsilon}{\|\widetilde{w}_\varepsilon\|}\right] = 0.$$

Moreover,

$$\frac{o(\|w\|)}{\varepsilon^\beta \|\widetilde{w}_\varepsilon\|} = \frac{o(\|w_\varepsilon\|)}{\|w_\varepsilon\|} = o(1).$$

Finally, let us show that

$$\lim_{\varepsilon \to 0} \frac{A_{i,\varepsilon}}{\varepsilon^\beta \|\widetilde{w}_\varepsilon\|} = 0. \tag{2.34}$$

Using (2.32) one has

$$\begin{aligned} A_i(\varepsilon, z) = (I_0''(z)[w_\varepsilon(z)] \mid q_i) + (D_u G(\varepsilon, z) \mid q_i) \\ + (D^2_{uu} G(\varepsilon, z)[w_\varepsilon(z)] \mid q_i) + o(\|w_\varepsilon(z)\|). \end{aligned}$$

Since $(I_0''(z)[w_\varepsilon(z)] \mid q_i) = (I_0''(z)[q_i] \mid w_\varepsilon(z)) = 0$, $\|D_u G(\varepsilon, z)\| = O(\varepsilon^\beta)$ yields

$$A_i(\varepsilon, z) = O(\varepsilon^\beta) + (D^2_{uu} G(\varepsilon, z)[w_\varepsilon(z)] \mid q_i) + o(\|w_\varepsilon(z)\|). \tag{2.35}$$

Moreover, $\|D^2_{uu} G(\varepsilon, z)\|_{L(\mathcal{H},\mathcal{H})} \to 0$ as $\varepsilon \to 0$ implies that $(D^2_{uu} G(\varepsilon, z)[w_\varepsilon(z)] \mid q_i) = o(\|w_\varepsilon(z)\|)$ and hence (2.32) becomes

$$A_i(\varepsilon, z) = O(\varepsilon^\beta) + o(\|w_\varepsilon(z)\|), \quad \text{as } |\varepsilon| \to 0, \tag{2.36}$$

proving (2.34).

Inserting the above equations into (2.33) we deduce

$$\lim_{\varepsilon \to 0} I_0''(z) \left[\frac{\widetilde{w}_\varepsilon}{\|\widetilde{w}_\varepsilon\|} \right] = 0.$$

Since $I_0''(z)$ is an index zero Fredholm map, $\widetilde{w}_\varepsilon \|\widetilde{w}_\varepsilon\|^{-1}$ converges strongly in $\mathcal{H}$ to some w^* satisfying $\|w^*\| = 1$ and $I_0''(z)[w^*] = 0$. This means that $w^* \in \mathrm{Ker}[I_0''(z)] = T_z Z$. On the other hand we have

$$(w_\varepsilon \mid q_i) = 0 \quad \Longrightarrow \quad (w^* \mid q_i) = 0,$$

namely $w^* \perp T_z Z$. Thus $w^* = 0$, a contradiction that proves (2.31).

To complete the proof of the lemma let us show that $\|\widetilde{w}_\varepsilon\| \to 0$. We will follow arguments similar to the preceding ones. Instead of (2.33) we consider

$$I_0''(z)[\widetilde{w}_\varepsilon] = -\frac{D_u G(\varepsilon, z)}{\varepsilon^\beta} - D^2_{uu} G(\varepsilon, z)[\widetilde{w}_\varepsilon] + \frac{o(\|w_\varepsilon\|)}{\varepsilon^\beta} + \sum \frac{A_{i,\varepsilon}}{\varepsilon^\beta} q_i$$

and claim that $I_0''(z)[\widetilde{w}_\varepsilon] \to 0$ as $\varepsilon \to 0$, provided that (G.1) holds. Actually, if $D_u G(\varepsilon, z) = o(\varepsilon^\beta)$, then instead of (2.36) one now gets

$$A_i(\varepsilon, z) = o(\varepsilon^\beta) + o(\|w_\varepsilon(z)\|), \quad \text{as } |\varepsilon| \to 0.$$

Since $\|w_\varepsilon(z)\| = O(\varepsilon^\beta)$ we infer that $A_i(\varepsilon, z) = o(\varepsilon^\beta)$ as $|\varepsilon| \to 0$. This together with (G.1) immediately implies that $I_0''(z)[\widetilde{w}_\varepsilon] \to 0$ as $\varepsilon \to 0$. As before we deduce that $\widetilde{w}_\varepsilon$ converges to some $w^* \in \mathcal{H}$ which belongs both to W as well as to $T_z Z = W^\perp$. Hence $\|\widetilde{w}_\varepsilon\| \to 0$, completing the proof. $\square$

The next lemma is the counterpart of Lemma 2.15.

Lemma 2.27. *Let* (G.0) *hold and suppose that there exist* $\alpha > 0$ *and* $\mathcal{G} : Z \to \mathbb{R}$ *such that*

(G.2) $\lim_{\varepsilon\to 0} \frac{G(\varepsilon,z)}{\varepsilon^\alpha} = \mathcal{G}(z)$.

Moreover, let us assume that (G.1) *holds with* $\beta = \frac{1}{2}\alpha$. *Then one has:*

$$\Phi_\varepsilon(z) = c_0 + \varepsilon^\alpha \mathcal{G}(z) + o(\varepsilon^\alpha), \quad \textit{as } \varepsilon \to 0.$$

Proof. As in Lemma 2.15 we get

$$\begin{aligned}\Phi_\varepsilon(z) &= I_0'(z + w_\varepsilon) + G(\varepsilon, z + w_\varepsilon)\\ &= c_0 + \tfrac{1}{2}(I_0''(z)[w_\varepsilon] \,|\, w_\varepsilon) + G(\varepsilon, z)\\ &\quad + (D_u G(\varepsilon, z) \,|\, w_\varepsilon) + \tfrac{1}{2}(D^2_{uu} G(\varepsilon, z)[w_\varepsilon] \,|\, w_\varepsilon) + o(\|w_\varepsilon\|^2).\end{aligned} \tag{2.37}$$

Applying Lemma 2.26 with $\beta = \frac{1}{2}\alpha$ we find

$$(I_0''(z)[w_\varepsilon] \,|\, w_\varepsilon) = O(\|w_\varepsilon\|^2) = o(\varepsilon^\alpha), \quad \text{as } \varepsilon \to 0. \tag{2.38}$$

One also has

$$(D^2_{uu} G(\varepsilon, z)[w_\varepsilon] \,|\, w_\varepsilon) = o(\|w_\varepsilon)\|^2) = o(\varepsilon^\alpha), \quad \text{as } \varepsilon \to 0. \tag{2.39}$$

Moreover, since $\|D_u G(\varepsilon, z)\| = o(\varepsilon^{\alpha/2})$ and $\|w_\varepsilon\| = o(\varepsilon^{\alpha/2})$, we get

$$(D_u G(\varepsilon, z) \,|\, w_\varepsilon) = o(\varepsilon^\alpha), \quad \text{as } \varepsilon \to 0. \tag{2.40}$$

Finally, from (G.2) we deduce

$$G(\varepsilon, z) = \varepsilon^\alpha \, \mathcal{G}(z) + o(\varepsilon^\alpha), \quad \text{as } \varepsilon \to 0. \tag{2.41}$$

Inserting (2.38)–(2.41) into (2.37) we find that $\Phi_\varepsilon(z) = c_0 + \varepsilon^\alpha \, \mathcal{G}(z) + o(\varepsilon^\alpha)$. □

At this point, we can repeat the arguments carried out in the preceding section to prove the following result, which is the counterpart of Theorem 2.16.

Theorem 2.28. *Suppose that* $I_0 \in C^2(\mathcal{H}, \mathbb{R})$ *has a smooth critical manifold* Z *which is non-degenerate. Let* G *satisfy* (G.0), (G.1) *and* (G.2) *and let* $\bar{z} \in Z$ *be a strict local maximum or minimum of* $\mathcal{G}$.

Then for $|\varepsilon|$ *small the functional* $I_\varepsilon = I_0 + G(\varepsilon, \cdot)$ *has a critical point* u_ε *and if* $\bar{z}$ *is isolated, then* $u_\varepsilon \to \bar{z}$ *as* $\varepsilon \to 0$.

Remark 2.29. (i) Clearly, the case $I_\varepsilon = I_0 + \varepsilon G$, discussed in Section 2.2, fits in the preceding frame, with $\alpha = 1$ and $\mathcal{G} = \Gamma$.

(ii) It is possible to extend to the present case also the result of Theorem 2.24, dealing with the Morse index of u_ε. □

2.4 A more general case

Dealing with NLS and with singular perturbation problems, it is convenient to modify the abstract setting. We will give here only an idea of these tools, referring for more details to Chapters 8, 9 and 10.

Unlike the preceding cases when there was a critical unperturbed manifold, one needs to consider functionals I_ε which possess a manifold $\mathcal{Z}^\varepsilon$ of *pseudo-critical points*. By this we mean that the norm of $I_\varepsilon(z)$ is small for all $z \in \mathcal{Z}^\varepsilon$, in an appropriate uniform way. In the applications, the manifold $\mathcal{Z}^\varepsilon$ satisfies a sort of non-degeneracy condition in the sense that, again, $PI_\varepsilon''(z)$ is uniformly invertible on $W = (T_z\mathcal{Z}^\varepsilon)^\perp$ (for ε small). Furthermore, an inspection to the proof of Lemma 2.21 highlights that we can still solve the auxiliary equation $PI_\varepsilon'(z+w) = 0$. Actually, writing

$$I_\varepsilon'(z+w) = I_\varepsilon'(z) + I_\varepsilon''(z)[w] + R(z,w),$$

the auxiliary equation can be transformed into an equation which is the counterpart of (2.29):

$$w = N_\varepsilon(z,w) = -(PI_\varepsilon''(z))^{-1}\left[PI_\varepsilon'(z) + PR(z,w)\right].$$

Using the fact that $\|I_\varepsilon'(z)\| \ll 1$, one shows that N_ε is still a contraction, which maps a ball in W into itself. Thus, as in Lemma 2.21, one proves that there exists $w = w_\varepsilon(z)$ solving $PI_\varepsilon'(z+w) = 0$. At this point one can repeat the arguments carried out in the preceding sections to find, in analogy with Theorem 2.12, that any critical point of the reduced functional $\Phi_\varepsilon(z) = I_\varepsilon(z + w_\varepsilon(z))$ gives rise to a critical point of I_ε, namely that the manifold $\widetilde{\mathcal{Z}}^\varepsilon = \{z + w_\varepsilon(z)\}$ is a natural constraint for I_ε. Once that this general result is proved, one can obtain the other existence theorems as well.

Bibliographical remarks

Existence of critical points for perturbed functionals in the presence of a compact critical manifold, and applications to forced oscillations of Hamiltonian systems, has been studied, e.g., in [62, 90, 127, 143] and in [11]. The latter contains, as particular case, our Corollary 2.13.

The case of non-compact manifolds is handled in [7] and [8]. The topics discussed in Sections 2.2 and 2.3 follow closely these two papers, where we also refer for more details and further results.

Chapter 3

Bifurcation from the Essential Spectrum

In this chapter we will apply the perturbation techniques, in particular those discussed in Section 2.3, to study some problems concerning the bifurcation from the infimum of the essential spectrum.

3.1 A first bifurcation result

Here we deal with the following equation on the whole real line $\mathbb{R}$

$$-u''(x) + u(x) = h(x/\varepsilon)|u(x)|^{p-1}u(x), \quad u \in W^{1,2}(\mathbb{R}), \tag{3.1}$$

where $p > 1$ and h satisfies

(h.1) $\quad \exists\, \ell > 0 \,:\, h - \ell \in L^1(\mathbb{R})$, and $\int_{\mathbb{R}}(h-\ell)dx \neq 0$.

As anticipated in Section 1.2 the change of variable

$$\begin{cases} \psi(x) = \varepsilon^{\frac{2}{p-1}} u(\varepsilon\, x), \\ \quad \lambda = -\varepsilon^2, \end{cases}$$

transforms (3.1) into

$$\psi'' + \lambda\psi + h(x)|\psi|^{p-1}\psi = 0, \qquad \lim_{|x|\to\infty} \psi(x) = 0, \tag{3.2}$$

and if (3.1) has for all $|\varepsilon|$ small, a family of solutions $u_\varepsilon \neq 0$ then the corresponding ψ_λ is a family of non-trivial solutions of (3.2) branching off $\lambda = 0$. Let us point out that the spectrum of the linearized equation

$$\psi'' + \lambda\psi = 0, \quad \psi \in W^{1,2}(\mathbb{R}).$$

is the half real line $[0, +\infty)$. Hence $\lambda = 0$ is the infimum of the essential spectrum[1] of the linearized equation. Actually, since $\lambda = -\varepsilon^2$, the bifurcation arises on the left of the essential spectrum.

In order to fit (3.1) into the abstract frame, we set $\mathcal{H} = W^{1,2}(\mathbb{R})$ endowed with the norm $\|u\|^2 = \int_{\mathbb{R}}(|u'|^2 + u^2)dx$ and $I_\varepsilon(u) = I_0(u) + G(\varepsilon, u)$ where

$$I_0(u) = \tfrac{1}{2}\|u\|^2 - \frac{\ell}{p+1}\int_{\mathbb{R}} |u|^{p+1}dx,$$

and

$$G(\varepsilon, u) = \begin{cases} -\frac{1}{p+1}\int_{\mathbb{R}} \left[h\left(\frac{x}{\varepsilon}\right) - \ell\right] |u|^{p+1}dx & \text{if } \varepsilon \neq 0; \\ 0 & \text{if } \varepsilon = 0. \end{cases} \tag{3.3}$$

Clearly, if u is a critical point of I_ε for $\varepsilon \neq 0$, then u is a solution of (3.1).

3.1.1 The unperturbed problem

The unperturbed problem $I_0'(u) = 0$ is the equation

$$-u''(x) + u(x) = \ell|u(x)|^{p-1}u(x), \quad u \in W^{1,2}(\mathbb{R}) \tag{3.4}$$

which has a unique even positive solution $z_0(x)$ such that

$$z_0'(0) = 0, \qquad \lim_{|x|\to\infty} z_0(x) = 0.$$

Then I_0 has a one-dimensional critical manifold given by

$$Z = \{z_\xi(x) := z_0(x + \xi) : \xi \in \mathbb{R}\}.$$

Moreover, every z_ξ is a Mountain-Pass critical point of I_0. In order to show that Z is non-degenerate we will make use of the following elementary result, see, e.g., [39, Theorem 3.3]:

Lemma 3.1. *Let $y(x)$ be a solution of*

$$-y''(x) + Q(x)y(x) = 0,$$

where $Q(x)$ is continuous and there exist $a, R > 0$ such that $Q(x) \geq a > 0$, for all $|x| > R$. Then either $\lim_{|x|\to\infty} y(x) = 0$ or $\lim_{|x|\to\infty} y(x) = \infty$. Moreover, the solutions y satisfying the first alternative are unique, up to a constant.

Lemma 3.2. *Z is non-degenerate.*

[1] The essential spectrum is the set of all points of the spectrum that are not isolated jointly with the eigenvalues of infinite multiplicity.

Proof. Let $v \in \mathrm{Ker}[I_0''(z_\xi)]$, namely a solution of the linearized equation $I_0''(z_\xi)[v] = 0$,

$$-v''(x) + v(x) = \ell\, p\, z_\xi^{p-1}(x) v(x), \quad v \in W^{1,2}(\mathbb{R}). \tag{3.5}$$

A solution of (3.5) is given by $z_\xi'(x) = z_0'(x+\xi)$, spanning the tangent space $T_{z_\xi} Z$. Set $Q = 1 - \ell\, p z_\xi^{p-1}$. Since $\lim_{|x|\to\infty} z_\xi(x) = 0$ then $\lim_{|x|\to\infty} Q(x) = 1$ and we can apply Lemma 3.1 yielding that all the solutions $v \in W^{1,2}(\mathbb{R})$ of (3.5) are given by $c\, z_\xi'$, for some constant $c \in \mathbb{R}$. This shows that $\mathrm{Ker}[I_0''(z_\xi)] \subseteq T_{z_\xi} Z$ and implies that Z is ND. □

3.1.2 Study of G

First we prove the continuity of G and its derivatives.

Lemma 3.3. *If $h - \ell \in L^1(\mathbb{R})$ then G satisfies* (G.0).

Proof. For brevity, we will only prove the continuity of $(\varepsilon, u) \mapsto G(\varepsilon, u)$ and $(\varepsilon, u) \mapsto D_u G(\varepsilon, u)$ when $(\varepsilon, u) \to (0, u_0)$. The other properties require some more technicalities, but they follow from similar arguments. For details we refer to [8].

By the change of variable $y = x/\varepsilon$ we find

$$|G(\varepsilon, u)| \le \frac{|\varepsilon|}{p+1} \int_{\mathbb{R}} \left| [h(y) - \ell] |u(\varepsilon y)|^{p+1} \right| dy.$$

Since $W^{1,2}(\mathbb{R}) \subset C(\mathbb{R})$ we infer that

$$|G(\varepsilon, u)| \le \tfrac{1}{p+1} |\varepsilon|\, \|u\|_{L^\infty}^{p+1} \int_{\mathbb{R}} |h(y) - \ell| dy,$$

and this shows that $G(\varepsilon, u) \to 0$ as $(\varepsilon, u) \to (0, u_0)$.

As for $D_u G(\varepsilon, u)$ we find, for any $\phi \in W^{1,2}(\mathbb{R})$,

$$|(D_u G(\varepsilon, u) \,|\, \phi)| = \left| \int_{\mathbb{R}} \varepsilon [h(y) - \ell] |u(\varepsilon y)|^{p-1} u(\varepsilon y)\, \phi(\varepsilon y) dy \right|.$$

As before, it follows that

$$|(D_u G(\varepsilon, u) \,|\, \phi)| \le |\varepsilon|\, \|u\|_{L^\infty}^{p}\, \|\phi\|_{L^\infty} \int_{\mathbb{R}} |h(y) - \ell| dy$$

and hence

$$\|D_u G(\varepsilon, u)\| \le c_1\, |\varepsilon| \int_{\mathbb{R}} |h(y) - \ell| dy, \tag{3.6}$$

for some $c_1 > 0$, proving that $\|D_u G(\varepsilon, u)\| \to 0$ as $(\varepsilon, u) \to (0, u_0)$. □

Next, we set $\gamma = \int_{\mathbb{R}}[h(x) - \ell]dx$ and

$$\mathcal{G}(\xi) = -\frac{1}{p+1}\,\gamma\, z_0^{p+1}(\xi).$$

Lemma 3.4. *If* (h.1) *holds then* G *satisfies* (G.1) *with* $\beta = \frac{1}{2}$ *and* (G.2) *with* $\alpha = 1$. *Precisely, one has*

(i) $\|D_u G(\varepsilon, u)\| = O(\varepsilon)$ *as* $|\varepsilon| \to 0$*;*
(ii) $G(\varepsilon, z_\xi) = \varepsilon \mathcal{G}(\xi) + o(\varepsilon)$ *as* $|\varepsilon| \to 0$*, uniformly for* $|\xi|$ *bounded.*

Proof. Property (i) follows immediately from (3.6). Moreover we have

$$G(\varepsilon, z_\xi) = -\frac{\varepsilon}{p+1}\int_{\mathbb{R}}[h(y) - 1] z_0^{p+1}(\varepsilon y + \xi)dy.$$

By the Dominated Convergence Theorem we infer

$$\lim_{|\varepsilon|\to 0}\frac{G(\varepsilon, z_\xi)}{\varepsilon} = -\tfrac{1}{p+1}\left(\int_{\mathbb{R}}[h(y) - 1]dy\right) z_0^{p+1}(\xi) = \mathcal{G}(\xi),$$

and this shows that (ii) holds. □

The preceding lemmas allow us to show:

Theorem 3.5. *Let* (h.1) *hold. Then* (3.2) *has a family of solutions* (λ, ψ_λ) *such that* $\lambda \to 0^-$ *and* $\psi_\lambda \to 0$ *as* $\lambda \to 0^-$ *in the* $C(\mathbb{R})$ *topology. Moreover, one has:*

$$\lim_{\lambda\to 0^-}\|\psi_\lambda\|^2_{L^2(\mathbb{R})} = \begin{cases} 0 & \text{if} \quad 1 < p < 5 \\ \text{const.} > 0 & \text{if} \quad p = 5 \\ +\infty & \text{if} \quad p > 5 \end{cases} \tag{3.7}$$

Finally, if $p \geq 2$*, the family* (λ, ψ_λ) *is a curve.*

Proof. From Lemmas 2.27 and 3.4 we deduce that

$$\Phi_\varepsilon(\xi) = I_\varepsilon(z_\xi + w_\varepsilon(z_\xi)) = c_0 + \varepsilon\mathcal{G}(\xi) + o(\varepsilon), \quad \text{as } \varepsilon \to 0.$$

The function $\mathcal{G}$ equals, up to a (negative) constant, the function z_0^{p+1} and hence it has a strict global minimum at $\xi = 0$. Then the abstract existence Theorem 2.28 applies yielding, for all $|\varepsilon| > 0$ small, a solution to the equation (3.1) of the form $u_\varepsilon = z_{\xi_\varepsilon} + w_\varepsilon(\xi_\varepsilon)$, with $\xi_\varepsilon \to 0$. These u_ε correspond to a family (λ, ψ_λ) of solutions to (3.2) given by

$$\lambda = -\varepsilon^2, \qquad \psi_\lambda(x) = (-\lambda)^{1/(p-1)} u_\varepsilon(\varepsilon\, x).$$

Moreover, one has

$$\|\psi_\lambda\|^2_{L^2(\mathbb{R})} = \varepsilon^{4/(p-1)}\int_{\mathbb{R}} u_\varepsilon^2(\varepsilon x)dx = (-\lambda)^{(5-p)/2(p-1)}\|u_\varepsilon\|^2_{L^2(\mathbb{R})},$$

which proves (3.7).

If $p \geq 2$ it is possible to use the Morse theoretic results stated in Theorem 2.24, see also Remark 2.29-(ii). Actually, one has:

1) any z_ξ is a Mountain-Pass critical point of I_0 which is non-degenerate for the restriction of I_0 to $T_{z_\xi} Z^\perp$;
2) G is of class C^3;
3) $\xi = 0$ is a strict global non-degenerate minimum of $\mathcal{G}$.

It follows that u_ε is a non-degenerate critical point of I_ε and this implies that the family (λ, ψ_λ) is a curve. □

3.2 A second bifurcation result

Here we deal with (3.1) in the case in which $\int_{\mathbb{R}}(h(x) - \ell)dx = 0$. As before we shall look for critical points of $I_\varepsilon = I_0 + G(\varepsilon, \cdot)$, where G is defined in (3.3). Let

$$h^*(x) = \int_0^x (h(s) - \ell)ds.$$

Remark that $h^* \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Moreover, let $\ell^* \in \mathbb{R}$ be defined by setting

$$\ell^* = \lim_{x \to +\infty} h^*(x) = \int_0^{+\infty} (h(s) - \ell)ds.$$

From $\int_{\mathbb{R}}(h(x) - \ell)dx = 0$ it follows that

$$\ell^* = -\int_{-\infty}^0 (h(s) - \ell)ds = \lim_{x \to -\infty} h^*(x),$$

namely that

$$\lim_{|x| \to +\infty} h^*(x) = \ell^*.$$

We will suppose

(h.2) $h - \ell \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}}(h(x) - \ell)dx = 0$;
(h.3) $h^* - \ell^* \in L^1(\mathbb{R})$ and $\gamma^* := \int_{\mathbb{R}}(h^*(x) - \ell^*)dx \neq 0$.

Of course, since Lemma 3.3 relies only on the fact that $h - \ell \in L^1(\mathbb{R})$, we still have that G satisfies (G.0). On the other hand, the definition of $\mathcal{G}$ and Lemma 3.4 need to be modified. Let

$$\mathcal{G}^*(\xi) = \gamma^* \, z_0^p(\xi) \, z_0'(\xi).$$

Lemma 3.6. *If* (h.2) *and* (h.3) *hold then G satisfies* (G.1) *with $\beta = 1$ and* (G.2) *with $\alpha = 2$. Precisely one has that*

(i) $\|D_u G(\varepsilon, u)\| = O(\varepsilon^{3/2})$ *as* $|\varepsilon| \to 0$;
(ii) $G(\varepsilon, z_\xi) = \varepsilon^2 \mathcal{G}^*(\xi) + o(\varepsilon^2)$ *as* $|\varepsilon| \to 0$, *uniformly for* $|\xi|$ *bounded.*

Proof. For all $\phi \in W^{1,2}(\mathbb{R})$ integrating by parts we find

$$
\begin{aligned}
(D_u G(\varepsilon, z_\xi) \,|\, \phi) &= -\int_{\mathbb{R}} \left[h(\frac{x}{\varepsilon}) - \ell\right] z_\xi^p(x)\phi(x)dx \\
&= \varepsilon \int_{\mathbb{R}} h^*(\frac{x}{\varepsilon})(z_\xi^p(x)\phi(x))' dx \\
&= \varepsilon \int_{\mathbb{R}} \left[h^*(\frac{x}{\varepsilon}) - \ell^*\right] (z_\xi^p(x)\phi(x))' dx \\
&= \varepsilon \int_{\mathbb{R}} \left[h^*(\frac{x}{\varepsilon}) - \ell^*\right] z_\xi^p(x)\phi'(x) dx \\
&\quad + \varepsilon \int_{\mathbb{R}} \left[h^*(\frac{x}{\varepsilon}) - \ell^*\right] (z_\xi^p(x))'\phi(x) dx
\end{aligned} \tag{3.8}
$$

The first integral above can be estimated by means of the Hölder inequality:

$$
\begin{aligned}
\left| \int_{\mathbb{R}} \left[h^*(\frac{x}{\varepsilon}) - \ell^*\right] z_\xi^p(x)\phi'(x)dx \right| &\le \left[\int_{\mathbb{R}} |h^*(\frac{x}{\varepsilon}) - \ell^*|^2 z_\xi^{2p}(x)dx\right]^{\frac{1}{2}} \cdot \left[\int_{\mathbb{R}} |\phi'(x)|^2\right]^{\frac{1}{2}} \\
&\le c_0\|\phi\| \cdot \left[\int_{\mathbb{R}} |h^*(\frac{x}{\varepsilon}) - \ell^*|^2 z_\xi^{2p}(x)dx\right]^{\frac{1}{2}} \\
&\le c_1\, \varepsilon^{\frac{1}{2}} \|\phi\| \cdot \left[\int_{\mathbb{R}} |h^*(y) - \ell^*|^2 dy\right]^{\frac{1}{2}}.
\end{aligned} \tag{3.9}
$$

Remark that $h^* - \ell^* \in L^2(\mathbb{R})$ because $h^* - \ell^* \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Let us now estimate the last integral in (3.8). Since $\|(z_\xi^p(x))'\|_{L^\infty(\mathbb{R})} \le c_2$, one infers

$$
\left| \int_{\mathbb{R}} \left[h^*(\frac{x}{\varepsilon}) - \ell^*\right] (z_\xi^p(x))'\phi(x)dx \right| \le c_2\varepsilon\|\phi\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |h^*(y) - \ell^*| dy.
$$

From this and (3.9) we get

$$
|(D_u G(\varepsilon, z_\xi) \,|\, \phi)| \le \varepsilon^2 c_3 \|\phi\|_{L^\infty(\mathbb{R})} + \varepsilon^{3/2} c_4 \|\phi\| \le (c_4 \varepsilon^{\frac{3}{2}} + c_5\varepsilon^2)\|\phi\|,
$$

and this implies that (i) holds.

Similarly, one finds

$$
\begin{aligned}
G(\varepsilon, z_\xi) &= \tfrac{1}{(p+1)}\, \varepsilon \int_{\mathbb{R}} \left[h^*(\frac{x}{\varepsilon}) - \ell^*\right] (z_\xi^{p+1}(x))' dx \\
&= \tfrac{1}{(p+1)}\, \varepsilon^2 \int_{\mathbb{R}} [h^*(y) - \ell^*](z_0^{p+1}(\varepsilon y + \xi))' dy.
\end{aligned}
$$

Using the Dominated Convergence Theorem we get

$$
\begin{aligned}
\lim_{\varepsilon \to 0} \frac{G(\varepsilon, z_\xi)}{\varepsilon^2} &= \tfrac{1}{(p+1)}\, (z_0^{p+1}(\xi))' \int_{\mathbb{R}} [h^*(y) - \ell^*] dy \\
&= \gamma^*\, z_0^p(\xi)\, z_0'(\xi)) = \mathcal{G}^*(\xi).
\end{aligned}
$$

This proves (ii). □

From the preceding lemma we infer that

$$\Phi_\varepsilon(\xi) = c_0 + \varepsilon^2 \mathcal{G}^*(\xi) + o(\varepsilon^2).$$

Since $\mathcal{G}^*$ has a maximum and a minimum, an application of Theorem 2.28 yields

Theorem 3.7. *Let* (h.2) *and* (h.3) *hold. Then* (3.2) *has two distinct families of solutions* (λ, ψ_λ) *bifurcating from the left of* $\lambda = 0$*, with the same properties listed in Theorem* 3.5.

Remark 3.8. Since the critical points of $\mathcal{G}^*$ are different from 0, the solutions found in the preceding theorem are non-symmetric in x. □

3.3 A problem arising in nonlinear optics

In this section we will shortly show how the abstract setting can be used for a bifurcation problem arising in nonlinear optics, dealing with the propagation of light in a medium with dielectric function $f(x,u)$. We will be sketchy, referring to [6] for more details.

We consider a layered medium, such that the internal layer with thickness $\varepsilon > 0$ has a *linear* response, while the eternal layer has a *non-linear* self-focusing response. This model leads to study the following differential equation, see [5],

$$-u''(x) + \omega^2 u(x) = f_\varepsilon(x,u)u(x), \quad u \in W^{1,2}(\mathbb{R}), \tag{3.10}$$

where ω is the bifurcation parameter and

$$f_\varepsilon(x,u) = \begin{cases} 1 & \text{if } |x| < \varepsilon \\ u^2 & \text{if } |x| > \varepsilon. \end{cases}$$

It is convenient to introduce the characteristic function χ of the interval $[-1,1]$. With this notation we can write

$$f_\varepsilon(x,u) = \chi(\frac{x}{\varepsilon}) + \left(1 - \chi(\frac{x}{\varepsilon})\right) u^2$$

and (3.10) becomes

$$-u''(x) + \omega^2 u(x) = u^3 + \chi(\frac{x}{\varepsilon})(u - u^3), \quad u \in W^{1,2}(\mathbb{R}).$$

Then the solutions of (3.10) are the critical points on $\mathcal{H} = W^{1,2}(\mathbb{R})$ of

$$I_{\varepsilon,\omega}(u) = \tfrac{1}{2}\int_{\mathbb{R}} |u'(x)|^2 dx + \tfrac{1}{2}\omega^2 \int_{\mathbb{R}} |u(x)|^2 dx - \tfrac{1}{4}\int_{\mathbb{R}} u^4(x) dx + G(\varepsilon, u),$$

where

$$G(\varepsilon,u) = \begin{cases} -\int_{\mathbb{R}} \left[\frac{1}{2} u^2(x) - \frac{1}{4} u^4(x)\right] \chi(\frac{x}{\varepsilon}) dx & \text{if } \varepsilon > 0 \\ 0 & \text{if } \varepsilon = 0. \end{cases}$$

It is easy to check that $I_{\varepsilon,\omega}$ can be studied by means of the abstract set-up discussed in Section 2.3. Specifically, one has that the critical manifold is given by $Z_\omega = \{z_\omega(x+\xi) : \xi \in \mathbb{R}\}$ with

$$z_\omega(x) = \frac{\sqrt{2}\,\omega}{\cosh(\omega\, x)},$$

and the function G satisfies (G.0) as well as $\|D_u G(\varepsilon, z)\| = O(\varepsilon)$. Moreover, one has (for $\varepsilon > 0$)

$$\begin{aligned} G(\varepsilon, z_\omega(\cdot+\xi)) &= -\int_{\mathbb{R}} \chi\big(\frac{x}{\varepsilon}\big) \left[\tfrac{1}{2}\, z_\omega^2(x+\xi) - \tfrac{1}{4}\, z_\omega^4(x+\xi)\right] dx \\ &= -\int_{-\varepsilon}^{\varepsilon} \left[\tfrac{1}{2}\, z_\omega^2(x+\xi) - \tfrac{1}{4}\, z_\omega^4(x+\xi)\right] dx \\ &= -\varepsilon \int_{-1}^{1} \left[\tfrac{1}{2}\, z_\omega^2(\varepsilon x+\xi) - \tfrac{1}{4}\, z_\omega^4(\varepsilon x+\xi)\right] dx\,. \end{aligned}$$

Thus we find that (G.2) holds with $\alpha = 1$ and

$$\mathcal{G}_\omega(\xi) = -z_\omega^2(\xi) + \tfrac{1}{2}\, z_\omega^4(\xi).$$

The behavior of $\mathcal{G}_\omega$ depends on the value of ω, see the figures below.

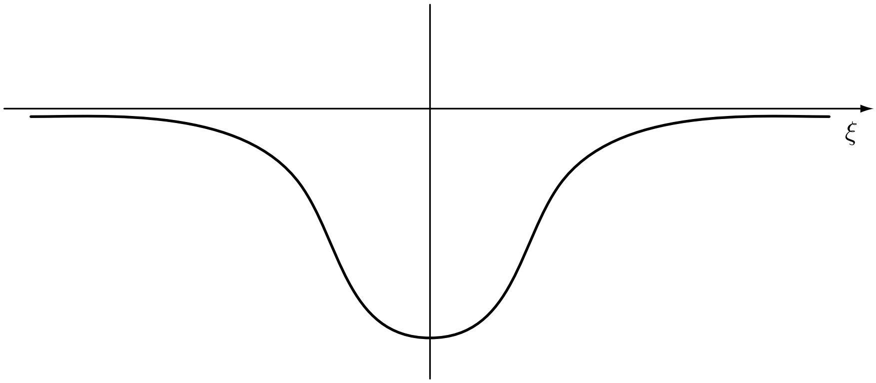

Figure 3.1. Graph of $\mathcal{G}_\omega$ for $\omega < \omega_0$

In particular, there exists $\omega_0 > 0$ such that, for ε small,

(i) for $0 < \omega < \omega_0$, $\mathcal{G}_\omega$ has a unique global minimum at $\xi = 0$;
(ii) for $\omega > \omega_0$, $\mathcal{G}_\omega$ has a (local or global) maximum at $\xi = 0$, while the global minimum is achieved at some $\pm\xi_\omega \neq 0$.

As a consequence (3.10) has, for ε small, a solution $u_{\varepsilon,\omega}$ for all $\omega > 0$ branching from the trivial solution at $\omega = 0$. In addition, at $\omega = \omega_0$ there is a secondary bifurcation of solutions $\tilde{u}_{\varepsilon,\omega}$ of (3.10), corresponding to ξ_ω. See the bifurcation

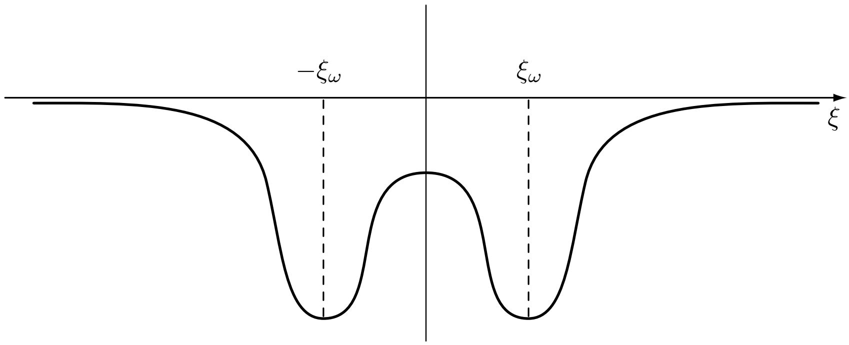

Figure 3.2. Graph of $\mathcal{G}_\omega$ for $\omega > \omega_0$

diagram below. The solutions $\tilde{u}_{\varepsilon,\omega}$ are not symmetric because $\xi_\omega \neq 0$. On the other hand, it is possible to show that $u_{\varepsilon,\omega}$ are even functions. Actually one can prove that in this case, fixing ω, the solution $w(\varepsilon, \xi)$ of the auxiliary equation satisfies $w(\varepsilon, \xi)(x) = w(\varepsilon, -\xi)(-x)$ for every x and ξ, and that the reduced functional Φ_ε is even in ξ. These two properties imply that $\xi = 0$ is a critical point of Φ_ε, and that $u_{\varepsilon,\omega}$ is symmetric.

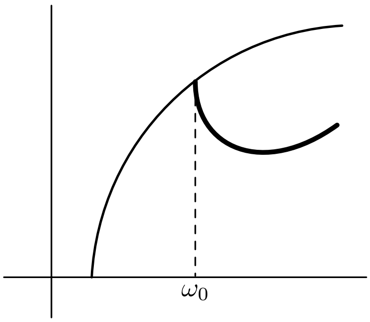

Figure 3.3. Bifurcation diagram for (3.10).The curve in bold represents the asymmetric solutions

Arguing as in the last part of the proof of Theorem 3.5, we can evaluate the Morse index of the solutions $u_{\varepsilon,\omega}$ and $\tilde{u}_{\varepsilon,\omega}$. For $\omega < \omega_0$ (resp. for $\omega > \omega_0$), $u_{\varepsilon,\omega}$ corresponds to a minimum (resp. a maximum) of $\mathcal{G}_\omega$. Moreover $\tilde{u}_{\varepsilon,\omega}$ ($\omega > \omega_0$) correspond to a minimum of $\mathcal{G}_\omega$. It follows that the Morse index of $u_{\varepsilon,\omega}$ is 1 or 2 provided that, respectively, $\omega < \omega_0$ and $\omega > \omega_0$. Similarly, the Morse index of $\tilde{u}_{\varepsilon,\omega}$ is 1. Using the stability results of [86], see also Remark 8.4, one infers that the stationary wave corresponding to the symmetric solution is (orbitally) stable if $\omega < \omega_0$. When ω crosses ω_0 there is a change of stability: the symmetric solution becomes unstable while the a-symmetric one is stable.

Bibliographical remarks

Bifurcation results for equations like (3.2) in the case that (h.1) holds, have been given in [108] and in [138, 139]. The topics discussed in Sections 3.1 and 3.2, that include the case in which (h.2) holds, are taken from [8] where we refer for other bifurcation results under different assumptions on h. The extension to the PDE analogue of (3.2) is addressed in [32]. In [8] and [32] the Morse index of the critical points of I_ε is also discussed. The case in which h is periodic is studied in [9]. The problem arising in nonlinear optic discussed in Section 3.3 is taken from [6]. Further results on such a problem have been obtained in [26, 65], where the general non-perturbative case (namely equation (3.10) with $\varepsilon = 1$) is handled. The physical backgrounds can be found in [5].

Chapter 4

Elliptic Problems on $\mathbb{R}^n$ with Subcritical Growth

In this chapter we will deal with the equation (1.2), in the case of a subcritical growth. We will closely follow the work [13].

4.1 The abstract setting

We will consider the elliptic problem

$$\begin{cases} -\Delta u + u = (1 + \varepsilon h(x))u^p, \\ u \in W^{1,2}(\mathbb{R}^n),\ u > 0, \end{cases} \tag{P_ε}$$

where $n \geq 3$ and p is a subcritical exponent, namely

$$1 < p < \frac{n+2}{n-2}. \tag{4.1}$$

Let $\mathcal{H} = W^{1,2}(\mathbb{R}^n)$ be the usual Sobolev space, endowed with the standard scalar product, resp. norm,

$$(u|v) = \int_{\mathbb{R}^n} (\nabla u \cdot \nabla v + uv)dx, \qquad \|u\|^2 = \int_{\mathbb{R}^n} (|\nabla u|^2 + u^2)dx.$$

Solutions of (P_ε), or even of a more general equation like

$$\begin{cases} -\Delta u + u = b(x)u^p, \\ u \in W^{1,2}(\mathbb{R}^n),\ u > 0, \end{cases} \tag{4.2}$$

with $b \in L^\infty(\mathbb{R}^n)$, are the critical points of the Euler functional $I_b : \mathcal{H} \mapsto \mathbb{R}$

$$I_b(u) = \tfrac{1}{2}\|u\|^2 - \tfrac{1}{p+1} \int_{\mathbb{R}^n} b(x)u_+^{p+1}dx,$$

where u_+ denotes the positive part of u (the fact that critical points correspond to *positive solutions* can be readily deduced in the following way: testing the equation on the negative part of u, u_-, one easily finds that $u_- \equiv 0$, hence $u \geq 0$, and by the strong maximum principle it follows that $u > 0$). Let us also remark that we use the same notation introduced in Section 2.1 because we are dealing with positive solutions. As seen in Section 2.1, even if $I_b(u)$ has the Mountain-Pass geometry, the M-P theorem cannot be directly applied because the lack of compactness of the embedding of $\mathcal{H}$ in $L^{p+1}(\mathbb{R}^n)$. We have also seen that to overcome this difficulty one can use the P.L. Lions *Concentration-Compactness* method which leads to the existence result stated in Theorem 2.7, Section 2.1.

Below we will show that the methods discussed in Chapter 2, Section 2.2 allow us to obtain existence results different from Theorem 2.7. Roughly, the idea is the following. The lack of compactness in the Sobolev embedding is due to the presence of the non-compact group of translations in $\mathbb{R}^n$, $x \mapsto x + \xi$. In some cases the function $h(x)$ breaks this invariance and allows to recover the (PS) condition. The drawback of this approach is that we must restrict ourselves to the perturbative problem (P_ε). On the other hand, we will be able to prove the existence of solutions of (P_ε) for a class of coefficients $b = 1 + \varepsilon h$ which cannot be handled by Theorem 2.7.

In order to use the techniques discussed in Chapter 2, Section 2.2, we set,

$$I_\varepsilon(u) = \tfrac{1}{2}\|u\|^2 - \tfrac{1}{p+1}\int_{\mathbb{R}^n} u_+^{p+1} dx - \varepsilon \cdot \tfrac{1}{p+1}\int_{\mathbb{R}^n} h(x)\, u_+^{p+1} dx.$$

Above it is understood that $h|u|^{p+1} \in L^1(\mathbb{R}^n)$ provided $u \in \mathcal{H}$. This is the case if

$$h \in L^s(\mathbb{R}^n), \qquad s = \frac{2^*}{2^* - (p+1)}. \tag{4.3}$$

Plainly, $I_\varepsilon \in C^2(\mathcal{H}, \mathbb{R})$ and solutions of (P_ε) are critical points of I_ε. For $\varepsilon = 0$ the *unperturbed functional* I_0 is given by

$$I_0(u) = \tfrac{1}{2}\|u\|^2 - \tfrac{1}{p+1}\int_{\mathbb{R}^n} u_+^{p+1} dx,$$

which is nothing but the limit functional considered in Section 2.1 [1]. The perturbation is given here by

$$G(u) = -\tfrac{1}{p+1}\int_{\mathbb{R}^n} h(x)\, u_+^{p+1} dx.$$

With this notation, we have that

$$I_\varepsilon(u) = I_0(u) + \varepsilon G(u),$$

[1] The fact that we have u_+^{p+1} instead of $|u|^{p+1}$ is not relevant to our purposes.

The unperturbed problem $I_0'(u) = 0$ is equivalent to the elliptic equation

$$-\Delta u + u = u^p, \quad u \in \mathcal{H}, \quad u > 0 \tag{4.4}$$

which has a (positive) radial solution U, see the arguments following Theorem 2.1 in Section 2.1. It has been shown in [98] that such a solution is unique. Let us recall that U and its radial derivative satisfy the following decay properties, see [38]

$$U(r) \sim e^{-|r|}|r|^{-\frac{n-1}{2}}; \qquad \lim_{r\to\infty} \frac{U'(r)}{U(r)} = 1, \qquad\qquad r = |x|. \tag{4.5}$$

Since (4.4) is translation invariant, it follows that any

$$z_\xi(x) := U(x - \xi)$$

is also a solution of (4.4). In other words, I_0 has a (non-compact) critical manifold given by

$$Z = \{z_\xi(x) : \xi \in \mathbb{R}^n\} \simeq \mathbb{R}^n.$$

4.2 Study of the $\mathrm{Ker}[I_0''(z_\xi)]$

The purpose of this section is to show:

Lemma 4.1. *Z is non-degenerate, namely the following properties are true:*

(ND) $\quad T_{z_\xi} = \mathrm{Ker}[I_0''(z_\xi)], \quad \forall\, \xi \in \mathbb{R}^n$;

(Fr) $\quad I_0''(z_\xi)$ *is an index* 0 *Fredholm map, for all* $\xi \in \mathbb{R}^n$.

Proof. We will prove the lemma by taking $\xi = 0$, hence $z_0 = U$. The case of a general ξ will follow immediately. The proof will be carried out in several steps.

Step 1. In order to characterize $\mathrm{Ker}[I_0''(U)]$, let us introduce some notation. We set

$$r = |x|, \qquad \vartheta = \frac{x}{|x|} \in S^{n-1}$$

and let Δ_r, resp. $\Delta_{S^{n-1}}$ denote the Laplace operator in radial coordinates, resp. the Laplace-Beltrami operator:

$$\Delta_r = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}$$

$$\Delta_{S^{n-1}} = \frac{1}{\sqrt{g}} \sum \frac{\partial}{\partial y_j}\left(\sqrt{g}\, g^{ij} \frac{\partial}{\partial y_i}\right).$$

In the latter formula standard notation is used: $ds^2 = g_{ij}dy^i dy^j$ denotes the standard metric on S^{n-1}, $g = \det(g_{ij})$ and $[g^{ij}] = [g_{ij}]^{-1}$. Consider the spherical harmonics $Y_k(\vartheta)$ satisfying

$$-\Delta_{S^{n-1}} Y_k = \lambda_k Y_k, \tag{4.6}$$

and recall that this equation has a sequence of eigenvalues

$$\lambda_k = k(k+n-2), \qquad k = 0, 1, 2, \ldots$$

whose multiplicity is given by $N_k - N_{k-2}$, where

$$N_k = \frac{(n+k-1)!}{(n-1)!\,k!}, \quad (k \geq 0), \quad N_k = 0, \ \forall\, k < 0,$$

see [39]. In particular, one has that

$$\lambda_0 = 0 \quad \text{has multiplicity } 1,$$

and

$$\lambda_1 = n-1 \quad \text{has multiplicity } n.$$

Every $v \in \mathcal{H}$ can be written in the form

$$v(x) = \sum_{k=0}^{\infty} \psi_k(r) Y_k(\vartheta), \quad \text{where} \quad \psi_k(r) = \int_{S^{n-1}} v(r\vartheta) Y_k(\vartheta) d\vartheta \in W^{1,2}(\mathbb{R}).$$

One has that

$$\Delta(\psi_k Y_k) = Y_k(\vartheta) \Delta_r \psi_k(r) + \frac{1}{r^2} \psi_k(r) \Delta_{S^{n-1}} Y_k(\vartheta). \tag{4.7}$$

Recall that $v \in \mathcal{H}$ belongs to $\mathrm{Ker}[I_0''(U)]$ iff

$$-\Delta v + v = pU^{p-1}(x) v, \quad v \in \mathcal{H}. \tag{4.8}$$

Substituting (4.7) and (4.6) into (4.8) we get the following equations for ψ_k:

$$A_k(\psi_k) := -\psi_k'' - \frac{n-1}{r}\psi_k' + \psi_k + \frac{\lambda_k}{r^2}\psi_k - pU^{p-1}\psi_k = 0, \quad k = 0, 1, 2, \ldots$$

Step 2. Let us first consider the case $k = 0$. Since $\lambda_0 = 0$ we infer that ψ_0 satisfies

$$A_0(\psi_0) = -\psi_0'' - \frac{n-1}{r}\psi_0' + \psi_0 - pU^{p-1}\psi_0 = 0.$$

It has been shown in [98] that all the solutions of $A_0(u) = 0$ are unbounded. Since we are looking for solutions $\psi_0 \in W^{1,2}(\mathbb{R})$, it follows that $\psi_0 = 0$.

Step 3. For $k = 1$, one has that $\lambda_1 = n-1$ and we find

$$A_1(\psi_1) = -\psi_1'' - \frac{n-1}{r}\psi_1' + \psi_1 + \frac{n-1}{r^2}\psi_1 - pU^{p-1}\psi_1 = 0.$$

Let $\widehat{U}(r)$ denote the function such that $U(x) = \widehat{U}(|x|)$. Since $U(x)$ satisfies $-\Delta U + U = U^p$, then $\widehat{U}$ solves

$$-\widehat{U}'' - \frac{n-1}{r}\widehat{U}' + \widehat{U} = \widehat{U}^p.$$

Differentiating, we get

$$-(\widehat{U}')'' - \frac{n-1}{r}(\widehat{U}')' + \frac{n-1}{r^2}\widehat{U}' + \widehat{U}' = p\widehat{U}^{p-1}\widehat{U}'. \tag{4.9}$$

In other words, $\widehat{U}'(r)$ satisfies $A_1(\widehat{U}') = 0$, and $\widehat{U}' \in W^{1,2}(\mathbb{R})$. Let us look for a second solution of $A_1(\psi_1) = 0$ in the form $\psi_1(r) = c(r)\widehat{U}'(r)$. By a straight calculation, we find that $c(r)$ solves

$$-c''\widehat{U}' - 2c' \cdot (\widehat{U}')' - \frac{n-1}{r}c'\widehat{U}' = 0.$$

If $c(r)$ is not constant, it follows that

$$-\frac{c''}{c'} = 2\frac{\widehat{U}''}{\widehat{U}'} + \frac{n-1}{r},$$

and hence

$$c'(r) \sim \frac{1}{r^{n-1}\widehat{U}'^2}, \qquad (r \to +\infty).$$

This and (4.5) imply that $c(r) \sim e^{2r}$ and therefore $c(r)\widehat{U}'(r) \sim -e^r r^{(1-n)/2}$ as $r \to +\infty$. From this we infer that $c(r)\widehat{U}'(r)$ does not belong to $W^{1,2}(\mathbb{R})$ unless $c(r)$ is constant. In conclusion, the family of solutions of $A_1(\psi_1) = 0$, with $\psi_1 \in W^{1,2}(\mathbb{R})$, is given by $\psi_1(r) = \overline{c}\widehat{U}'(r)$, for some $\overline{c} \in \mathbb{R}$.

Step 4. Let us show that the equation $A_k(\psi_k) = 0$ has only the trivial solution in $W^{1,2}(\mathbb{R})$, provided that $k \geq 2$. To prove this fact, let us first remark that the operator A_1 has the solution $\widehat{U}'$ which does not change sign in $(0,\infty)$ and therefore is a non-negative operator. Actually, if ω denotes its smallest eigenvalue, any corresponding eigenfunction, φ_ω does not change sign. If $\omega < 0$, then φ_ω should be be orthogonal to $\widehat{U}'$, a contradiciton. Thus $\omega \geq 0$ and A_1 is non-negative. Next, from

$$\lambda_k = (n+k-2)k = \lambda_1 + \delta_k, \qquad \delta_k = k(n+k-2) - (n-1),$$

we infer that

$$A_k = A_1 + \frac{\delta_k}{r^2}.$$

Since $\delta_k > 0$ whenever $k \geq 2$, it follows that A_k is a *positive* operator for any $k \geq 2$. Thus $A_k(\psi_k) = 0$ implies that $\psi_k = 0$.

Conclusion. Putting together all the previous information, we deduce that any $v \in \mathrm{Ker}[I_0''(U)]$ has to be a constant multiple of $\widehat{U}'(r)Y_1(\vartheta)$. Here Y_1 is such that

$$-\Delta_{S^{n-1}}Y_1 = \lambda_1 Y_1 = (n-1)Y_1,$$

namely it belongs to the kernel of the operator $-\Delta_{S^{n-1}} - \lambda_1 Id$. Recalling that such a kernel is n-dimensional and letting $Y_{1,1}, \ldots, Y_{1,n}$ denote a basis on it, we finally find that

$$v \in \mathrm{span}\{\widehat{U}'Y_{1,i} : 1 \leq i \leq n\} = \mathrm{span}\{U_{x_i} : 1 \leq i \leq n\} = T_U Z.$$

This proves that (ND) holds. It is also easy to check that the operator $I_0''(U)$ is a compact perturbation of the identity, showing that (Fr) holds true, too. This completes the proof of Lemma 4.1. $\square$

Remark 4.2. Since U is a Mountain-Pass solution satisfying $-\Delta U + U = U^p$, the spectrum of $PI_0''(U)$ has exactly one negative simple eigenvalue, $p-1$, with eigenspace spanned by U itself. Moreover, we have shown in the preceding Lemma, that $\lambda = 0$ is an eigenvalue with multiplicity n and eigenspace spanned by D_iU, $i = 1, 2, \dots, n$. Moreover, there exists $\kappa > 0$ such that

$$(PI_0''(U)v|v) \geq \kappa \|v\|^2, \qquad \forall\, v \perp \langle U \rangle \oplus T_U Z, \tag{4.10}$$

and hence the rest of the spectrum is positive. $\square$

4.3 A first existence result

Here we will prove a first existence result by showing that (P_ε) has a solution provided that h satisfies some integrability conditions.

According to the general procedure, Lemma 4.1 allows us to say that, for $|\varepsilon|$ small, one has that

$$\Phi_\varepsilon(z_\xi) := I_\varepsilon(z_\xi + w_\varepsilon(\xi)) = c_0 + \varepsilon G(z_\xi) + o(\varepsilon), \quad c_0 = I_0(z_\xi) \equiv I_0(U).$$

Let $\Gamma : \mathbb{R}^n \mapsto \mathbb{R}$ be defined by setting

$$\Gamma(\xi) = G(z_\xi) = -\tfrac{1}{p+1} \int_{\mathbb{R}^n} h(x) U^{p+1}(x-\xi)dx, \qquad \xi \in \mathbb{R}^n.$$

Lemma 4.3. *Suppose that* (4.3) *holds. Then*

$$\lim_{|\xi|\to\infty} \Gamma(\xi) = 0.$$

Proof. Taken $\rho > 0$ we set

$$\Gamma_\rho(\xi) := \int_{|x|<\rho} h(x)U^{p+1}(x-\xi)dx, \quad \Gamma_\rho^*(\xi) = \int_{|x|>\rho} h(x)U^{p+1}(x-\xi)dx,$$

in such a way that $\Gamma(\xi)$ splits as

$$\Gamma(\xi) = -\tfrac{1}{p+1}\left[\Gamma_\rho(\xi) + \Gamma_\rho^*(\xi)\right].$$

Let s' denote the conjugate exponent of s (> 1). Using the Hölder inequality, we get

$$\begin{aligned}|\Gamma_\rho(\xi)| &\le \left(\int_{|x|<\rho} |h(x)|^s dx\right)^{1/s} \left(\int_{|x|<\rho} U^{s'(p+1)}(x-\xi)dx\right)^{1/s'} \\ &= \left(\int_{|x|<\rho} |h(x)|^s dx\right)^{1/s} \left(\int_{|x+\xi|<\rho} U^{s'(p+1)}(x)dx\right)^{1/s'} \\ &\le c_1 \left(\int_{|x+\xi|<\rho} U^{s'(p+1)}(x)dx\right)^{1/s'}.\end{aligned}$$

Since U decays exponentially to zero as $|x| \to \infty$, the last integral tends to zero as $|\xi| \to \infty$ and hence

$$\lim_{|\xi|\to\infty} \Gamma_\rho(\xi) = 0, \qquad \forall\, \rho > 0. \tag{4.11}$$

On the other hand we also have

$$\begin{aligned}|\Gamma^*_\rho(\xi)| &\le \left(\int_{|x|>\rho} |h(x)|^s dx\right)^{1/s} \left(\int_{|x+\xi|>\rho} U^{s'(p+1)}(x)dx\right)^{1/s'} \\ &\le \left(\int_{|x|>\rho} |h(x)|^s dx\right)^{1/s} \left(\int_{\mathbb{R}^n} U^{s'(p+1)}(x)dx\right)^{1/s'} \\ &\le c_2 \left(\int_{|x|>\rho} |h(x)|^s dx\right)^{1/s}.\end{aligned}$$

Thus, given any $\eta > 0$ there exists $\rho > 0$ so large that $|\Gamma^*_\rho(\xi)| \le \eta$. This, together with (4.11), proves the Lemma. □

The previous lemma allows us to prove the existence of solutions of (P_ε), provided $\Gamma(\xi) \not\equiv 0$. Actually, we can show

Theorem 4.4. *Let* (4.1) *hold and let* h *satisfy* (4.3). *Moreover, suppose that either*

(h$_1$) $\int_{\mathbb{R}^n} h(x)U^{p+1}(x) \neq 0$;

or

(h$_2$) $h \not\equiv 0$ *and* $\exists\, r \in [1,2]$ *such that* $h \in L^r(\mathbb{R}^n)$.

Then (P_ε) *has a solution provided* $|\varepsilon|$ *is small enough.*

Proof. Since h satisfies (4.3), then Lemma 4.3 applies and hence $\Gamma(\xi)$ tends to zero as $|\xi| \to \infty$.

If (h$_1$) holds then $\Gamma(0) = -\frac{1}{p+1}\int_{\mathbb{R}^n} h(x)U^{p+1}(x) \neq 0$. Then Γ is not identically zero and it follows that Γ has a maximum or a minimum on $\mathbb{R}^n$, and the existence of a solution follows from Theorem 2.16.

If (h_2) holds we need to use a different argument to show that $\Gamma(\xi) \not\equiv 0$. We will be sketchy. Setting $b(x) = U^{p+1}(x)$ we can write $\Gamma(\xi) = -\frac{1}{p+1}(h * b)$, where $*$ denotes the convolution. Taking the Fourier transform we get $\widehat{\Gamma} = -\frac{1}{p+1}(\widehat{h} \cdot \widehat{b})$. Using the Morera Theorem, see [4], it is easy to check that $\widehat{b}$ is analytic in the strip $\{\zeta \in C^n : |\mathbf{Im}\zeta| < \alpha\}$ for some $\alpha > 0$ and hence it has at most a countable number of zeroes there. Moreover, since $h \in L^r(\mathbb{R}^n)$ for some $r \in [1, 2]$ we deduce, by the Hausdorff-Young inequality , that $\widehat{h} \in L^{r'}(\mathbb{R}^n)$, where r' denotes the conjugate exponent of r (if $r = 1$, $\widehat{h}$ is continuous), and $\widehat{h} \not\equiv 0$, since $h \not\equiv 0$. Then $\widehat{\Gamma} = -\frac{1}{p+1}(\widehat{h} \cdot \widehat{b}) \not\equiv 0$, which implies that $\Gamma(\xi) \not\equiv 0$, and the conclusion follows as before. □

Remarks 4.5. (i) A condition which implies $\int_{\mathbb{R}^n} h(x)U^{p+1}(x) \neq 0$ is that h has constant sign in $\mathbb{R}^n$.

(ii) If h does not satisfy (h_2) we do not know if, in general, Γ is not identically zero. The argument sketched before does not work because $\widehat{h}$ could be merely a tempered distribution which could have no L^1_{loc} representation.

(iii) There are situations in which we can prove that (P_ε) has multiple solutions. For example, if

$$\int_{\mathbb{R}^n} h(x)U^{p+1}(x) = 0, \quad \int_{\mathbb{R}^n} D_i h(x)U^{p+1}(x) \neq 0, \quad \text{for some } i = 1, 2, \ldots, n,$$

then $\Gamma(0) = 0$ while $D_i\Gamma(0) \neq 0$. Thus Γ possesses a positive maximum and a negative minimum, which give rise to a pair of distinct solutions of (P_ε), for $|\varepsilon|$ small enough.

(iv) If Γ has a maximum (e.g., when $\int_{\mathbb{R}^n} h(x)U^{p+1}(x) < 0$), the Morse index of the corresponding solution is greater or equal than $1 + n$, see Theorem 2.24. In particular, in such a case the solution cannot be found by means of the Mountain-Pass Theorem. □

4.4 Another existence result

The main purpose of this section is to prove

Theorem 4.6. *Let* (4.1) *hold and suppose that* h *satisfies*

(h_3) $h \in L^\infty(\mathbb{R}^n)$ *and* $\lim_{|x|\to\infty} h(x) = 0$.

Then for all $|\varepsilon|$ *small, problem* (P_ε) *has a solution.*

The new feature of this result is that, unlike Theorem 4.4, we do not assume any integrability condition on h, nor any hypothesis like (h_1).

Although a simple modification of the arguments carried out in the proof of Lemma 4.3 would lead to show that if (h_3) holds then $\Gamma(\xi) \to 0$ as $|\xi| \to \infty$, we cannot use this information because we do not know whether $\Gamma \equiv 0$ or not, see Remark 4.5-(ii). We will overcome this problem by studying directly $\lim_{|\xi|\to\infty} \Phi_\varepsilon(\xi)$

and using Theorem 2.23. Having this goal in mind, we first show that Lemma 2.21 holds. Following the notation introduced in Chapter 2, $P = P_\xi : \mathcal{H} \mapsto W_\xi$ denotes the orthogonal projection onto $W_\xi = (T_{z_\xi} Z)^\perp$, where $z_\xi = U(\cdot - \xi)$. Moreover,

$$R_\xi(w) = I_0'(z_\xi + w) - I_0''(z_\xi)[w].$$

According to the statement of Lemma 2.21, we shall show

Lemma 4.7.

(i) *there is* $C > 0$ *such that* $\|(PI_0''(z_\xi))^{-1}\|_{L(W_\xi, W_\xi)} \le C, \qquad \forall \xi \in \mathbb{R}^n$,
(ii) $R_\xi(w) = o(\|w\|)$, *uniformly with respect to* $\xi \in \mathbb{R}^n$.

Proof. Since z_ξ is a Mountain-Pass solution satisfying $-\Delta z_\xi + z_\xi = z_\xi^p$, the arguments of Remark 4.2 readily imply that it suffices to prove that there is $\kappa > 0$ such that

$$(PI_0''(z_\xi)[v]|v) \ge \kappa\|v\|^2, \quad \forall \xi \in \mathbb{R}^n, \quad \forall v \perp \widetilde{W}_\xi := \langle z_\xi \rangle \oplus (T_{z_\xi} Z). \tag{4.12}$$

We already pointed out that for any fixed $\xi \in \mathbb{R}^n$, say $\xi = 0$, the operator $PI_0''(z_0) = PI_0''(U)$ is invertible and, see (4.10), there exists $\kappa > 0$ such that

$$(PI_0''(U)[v]|v) \ge \kappa\|v\|^2, \qquad \forall v \perp \widetilde{W} := \langle U \rangle \oplus (T_U Z).$$

Setting $v^\xi(x) = v(x + \xi)$ we get by a straight calculation that

$$(PI_0''(z_\xi)[v]|v) = (PI_0''(U)[v^\xi]|v^\xi).$$

Moreover, $v^\xi \perp \widetilde{W}$ whenever $v \perp \widetilde{W}_\xi$. Thus we deduce:

$$(PI_0''(z_\xi)[v]|v) = (PI_0''(U)[v^\xi]|v^\xi) \ge \kappa\|v^\xi\|^2 = \kappa\|v\|^2, \quad \forall \xi \in \mathbb{R}^n, \quad \forall v \perp \widetilde{W}_\xi,$$

proving (4.12), and (i) follows.

To prove (ii) it suffices to remark that, in the present case, one has that $R_\xi(w) = (z_\xi + w)^p - z_\xi^p - p z_\xi^{p-1} w$. $\square$

The preceding statements (i)–(ii) allow us to use Lemma 2.21 yielding that there exists $\varepsilon_0 > 0$ such that for all $|\varepsilon| \le \varepsilon_0$ and all $\xi \in \mathbb{R}^n$ the auxiliary equation $PI_\varepsilon'(z_\xi + w) = 0$ has a unique solution $w_{\varepsilon,\xi} := w_\varepsilon(z_\xi)$ with

$$\lim_{\varepsilon \to 0} \|w_{\varepsilon,\xi}\| = 0, \tag{4.13}$$

uniformly with respect to $\xi \in \mathbb{R}^n$. In the sequel ε is fixed, with $|\varepsilon| \ll 1$, and for brevity we will write w_ξ instead of $w_{\varepsilon.\xi}$.

We now prove

Lemma 4.8. *There exists $\varepsilon_1 > 0$ such that for all $|\varepsilon| \le \varepsilon_1$, the following result holds:*

$$\lim_{|\xi|\to\infty} w_\xi = 0, \qquad \textit{strongly in } \mathcal{H}. \tag{4.14}$$

Proof. We begin with the following preliminary results:

(a) w_ξ weakly converges in $\mathcal{H}$ to some $w_\infty = w_{\varepsilon,\infty} \in \mathcal{H}$, as $|\xi| \to \infty$. Moreover, the weak limit w_∞ is a weak solution of

$$-\Delta w_\infty + w_\infty = (1+\varepsilon h(x))w_\infty^p.$$

(b) One has that $w_\infty = 0$.

Proof of (a). First, let us remark that, as a byproduct of (4.13), w_ξ weakly converges in $\mathcal{H}$ to some $w_\infty = w_{\varepsilon,\infty} \in \mathcal{H}$, as $|\xi| \to \infty$.

Next, recall that the function w_ξ is a solution of the auxiliary equation $PI'_\varepsilon(z_\xi + w_\xi) = 0$, namely

$$-\Delta w_\xi + w_\xi = (1+\varepsilon h(x))(z_\xi + w_\xi)^p - z_\xi^p - \sum_{i=1}^n a_i D_i z_\xi,$$

where

$$a_i = \int_{\mathbb{R}^n} \left[(1+\varepsilon h(x))(z_\xi + w_\xi)^p - z_\xi^p\right] D_i z_\xi dx.$$

Let ϕ denote any test function. Then one finds

$$\begin{aligned}(w_\xi|\phi) = &\int_{\mathbb{R}^n} (1+\varepsilon h(x))(z_\xi(x) + w_\xi(x))^p \phi(x) dx \\ &- \int_{\mathbb{R}^n} z_\xi^p(x)\phi(x)dx + \sum a_i \int_{\mathbb{R}^n} D_i z_\xi(x)\phi(x)dx.\end{aligned} \tag{4.15}$$

In order to pass to limit in the above integrals, let us first show that

$$\lim_{|\xi|\to\infty} \int_{\mathbb{R}^n} z_\xi^{p-k} w_\xi^k \phi dx = 0, \qquad \forall\, k \in [0,p). \tag{4.16}$$

The argument is similar to the one used in the proof of Lemma 4.3 and so we will be sketchy. We split the integral in (4.16) as

$$\int_{\mathbb{R}^n} A dx = \int_{|x|<\rho} A dx + \int_{|x|>\rho} A dx, \qquad (A = z_\xi^{p-k} w_\xi^k \phi),$$

where $\rho > 0$ will be chosen later on. Using the Hölder inequality with $\alpha = 2^*/(2^* - k - 1)$, we get

$$\left|\int_{|x|<\rho} A dx\right| \le \left(\int_{|x|<\rho} z_\xi^{(p-k)\alpha} dx\right)^{1/\alpha} \left(\int_{|x|<\rho} |w_\xi|^{2^*} dx\right)^{k/2^*} \left(\int_{|x|<\rho} |\phi|^{2^*} dx\right)^{1/2^*}$$
$$\le c_1 \left(\int_{|x|<\rho} z_\xi^{(p-k)\alpha} dx\right)^{1/\alpha} = c_1 \left(\int_{|x+\xi|<\rho} U^{(p-k)\alpha}(x) dx\right)^{1/\alpha}.$$

The last integral tends to zero as $|\xi| \to \infty$ and thus $|\int_{|x|<\rho} A dx| \to 0$, too. Similarly, one finds that

$$\left|\int_{|x|>\rho} A dx\right| \le c_2 \left(\int_{|x|>\rho} |\phi|^{2^*} dx\right)^{1/2^*}$$

and deduces that $|\int_{|x|>\rho} A dx| \to 0$ as $\rho \to \infty$, whence (4.16) follows.

Furthermore, since $h \in L^\infty(\mathbb{R}^n)$, the same arguments yield

$$\lim_{|\xi|\to\infty} \int_{\mathbb{R}^n} h(x) z_\xi^{p-k} w_\xi^k \phi dx = 0, \qquad \forall\, k \in [0, p). \tag{4.17}$$

Finally one trivially finds, as $|\xi| \to \infty$,

$$\int_{\mathbb{R}^n} w_\xi^p(x)\phi(x) dx \to \int_{\mathbb{R}^n} w_\infty^p \phi dx,$$
$$\int_{\mathbb{R}^n} h(x) w_\xi^p(x)\phi(x) dx \to \int_{\mathbb{R}^n} h(x) w_\infty^p \phi dx,$$
$$\int_{\mathbb{R}^n} a_i D_i z_\xi \phi dx \to 0, \quad (i = 1, 2, \ldots, n).$$

By this, jointly with (4.16), (4.17) we can pass to the limit in (4.15) proving

$$(w_\infty|\phi) = \int_{\mathbb{R}^n} (1 + \varepsilon h(x)) w_\infty^p \phi dx,$$

namely that (a) holds.

Proof of (b). As a consequence of (4.13) one has that $\lim_{|\varepsilon|\to 0} w_{\varepsilon,\infty} = 0$. Since the unique solution $w \in \mathcal{H}$ of $-\Delta w + w = (1 + \varepsilon h) w^p$ with small norm is $w = 0$ we infer that $w_{\varepsilon,\infty} = 0$, provided $|\varepsilon| \ll 1$.

Proof of the Lemma completed. Let us recall that w_ξ satisfies equation (2.29), which in the present case becomes

$$w_\xi = (PI_0''(z_\xi))^{-1}[\varepsilon PG'(z_\xi + w_\xi) - PR_\xi(w_\xi)], \tag{4.18}$$

where

$$G'(z_\xi + w) = h\ (z_\xi + w)^p,$$
$$R_\xi(w) = (z_\xi + w)^p - z_\xi^p - p z_\xi^{p-1} w.$$

From (4.18) and Lemma 4.7-(i) it follows that

$$\|w_\xi\|^2 \le C \cdot \left[|\varepsilon||(G'(z_\xi + w_\xi)|w_\xi)| + |(R_\xi(w_\xi)|w_\xi)|\right]. \tag{4.19}$$

We claim that

$$\lim_{|\xi|\to\infty} |(G'(z_\xi + w_\xi)|w_\xi)| = 0. \tag{4.20}$$

Actually, let us set

$$g_r(\xi) = \int_{|x|>r} |h(x)(z_\xi + w_\xi)^p w_\xi| dx.$$

Since (h_3) holds, then, fixed any $\eta > 0$, there exists $\rho > 0$ such that $|h(x)| \le \eta$ for all $|x| > \rho$ and hence there exists $c_1 > 0$ such that

$$g_\rho(\xi) \le c_1 \eta, \qquad \forall\, \xi \in \mathbb{R}^n.$$

Moreover, we find

$$\begin{aligned}\int_{|x|<\rho} |h(x)(z_\xi + w_\xi)^p w_\xi| dx &\le \|h\|_\infty \int_{|x|<\rho} |(z_\xi + w_\xi)^p w_\xi| dx \\ &\le c_2 \left[\int_{|x|<\rho} z_\xi^p |w_\xi| dx + \int_{|x|<\rho} |w_\xi|^{p+1} dx\right].\end{aligned}$$

In the ball $B_\rho = \{x \in \mathbb{R}^n : |x| < \rho\}$, we have that $W^{1,2}(B_\rho)$ is compactly embedded into $L^q(B_\rho)$ for all $q \in [1, 2^*)$. Hence we have that $w_\xi \to 0$ strongly in $L^q(B_\rho)$ for all $q \in [1, 2^*)$. Thus we infer that

$$\int_{|x|<\rho} |h(x)(z_\xi + w_\xi)^p w_\xi| dx \to 0, \qquad \text{as } |\xi| \to \infty.$$

Since

$$|(G'(z_\xi + w_\xi)|w_\xi)| \le \int_{|x|<\rho} |h(x)(z_\xi + w_\xi)^p w_\xi| dx + g_\rho(\xi),$$

then the claim (4.20) follows.

Next, we estimate $|(R_\xi(w_\xi)|w_\xi)|$. For this, let us remark that, for $a > 0$ and $0 < b \ll 1$ the elementary inequalities hold

$$\left|(a+b)^p - a^p - p a^{p-1} b\right| \le c_3(a^{p-2} + b^p), \quad \text{if } p \ge 2,$$
$$\left|(a+b)^p - a^p - p a^{p-1} b\right| \le c_4 b^p, \quad \text{if } 1 < p < 2.$$

Then we readily find

$$|(R_\xi(w_\xi)|w_\xi)| \le \int_{\mathbb{R}^n} |(z_\xi + w_\xi)^p - z_\xi^p - p z_\xi^{p-1} w_\xi|\, |w_\xi| dx \le c_5 \|w_\xi\|^{2+\beta},$$

for some $\beta > 0$. Inserting the above inequality into (4.19) and using (4.20) we get

$$\|w_\xi\|^2 \le c_6 \|w_\xi\|^{2+\beta} + o(\varepsilon), \qquad \text{as } |\xi| \to \infty.$$

Passing to the limit as $|\xi| \to \infty$ we find

$$\lim_{|\xi|\to\infty} \|w_\xi\|^2 \le c_6 \lim_{|\xi|\to\infty} \|w_\xi\|^{2+\beta}.$$

Finally, since $w_\xi = w_{\varepsilon,\xi}$ is small (in $\mathcal{H}$) as $|\varepsilon| \to 0$, we conclude that

$$\lim_{|\xi|\to\infty} \|w_\xi\| = 0 \quad \text{provided} \quad |\varepsilon| \ll 1.$$

This completes the proof. □

We are now in the position to prove Theorem 4.6.

Proof of Theorem 4.6. Consider the functional $\Phi_\varepsilon(\xi) = I_\varepsilon(z_\xi + w_\xi)$. One has

$$\Phi_\varepsilon(\xi) = \tfrac{1}{2}\|z_\xi + w_\xi\|^2 - \tfrac{1}{(p+1)} \int_{\mathbb{R}^n} (1 + \varepsilon h(x))(z_\xi(x) + w_\xi(x))^{p+1} dx. \tag{4.21}$$

From $I_0(z_\xi) = \frac{1}{2}\|z_\xi\|^2 - \frac{1}{(p+1)} \int_{\mathbb{R}^n} z_\xi^{p+1} dx$ and setting $c_0 = I_0(z_\xi) \equiv I_0(U)$, we have that

$$\tfrac{1}{2}\|z_\xi\|^2 = c_0 + \tfrac{1}{(p+1)} \int_{\mathbb{R}^n} z_\xi^{p+1} dx.$$

Moreover, $-\Delta z_\xi + z_\xi = z_\xi^p$ implies

$$(z_\xi|w_\xi) = \int_{\mathbb{R}^n} z_\xi^p w_\xi dx.$$

Substituting these equations into (4.21) we infer

$$\Phi_\varepsilon(\xi) = c_0 + \tfrac{1}{2}\|w_\xi\|^2 - \tfrac{1}{(p+1)} \int_{\mathbb{R}^n} (z_\xi + w_\xi)^{p+1} dx + \tfrac{1}{(p+1)} \int_{\mathbb{R}^n} z_\xi^{p+1} dx$$
$$+ \int_{\mathbb{R}^n} z_\xi^p w_\xi dx - \tfrac{1}{(p+1)}\, \varepsilon \int_{\mathbb{R}^n} h(x)(z_\xi + w_\xi)^{p+1} dx.$$

Now, we estimate

$$\int_{\mathbb{R}^n} \left|(z_\xi + w_\xi)^{p+1} - z_\xi^{p+1} - (p+1) z_\xi^p w_\xi\right| dx \le c_1 \int \left|z_\xi^{p-1} w_\xi^2 + w_\xi^{p+1}\right| dx.$$

Repeating the arguments employed in Lemma 4.8 and using (4.14), we infer that the latter integral in the right-hand side tends to zero and hence

$$\frac{1}{(p+1)}\int_{\mathbb{R}^n}(z_\xi+w_\xi)^{p+1}dx-\frac{1}{(p+1)}\int_{\mathbb{R}^n}z_\xi^{p+1}dx-\int_{\mathbb{R}^n}z_\xi^{p}w_\xi dx\to 0,\qquad(|\xi|\to\infty). \tag{4.22}$$

Similarly, taking also into account (h_3), we get

$$\int_{\mathbb{R}^n}h(x)(z_\xi+w_\xi)^{p+1}dx\to 0,\qquad(|\xi|\to\infty). \tag{4.23}$$

Finally, using (4.14), (4.22) and (4.23), we deduce that, for all $|\varepsilon|\le\varepsilon_1$,

$$\lim_{|\xi|\to\infty}\Phi_\varepsilon(\xi)=c_0.$$

As a consequence, Φ_ε has at least a maximum or a minimum (unless $\Phi_\varepsilon\equiv c_0$). In any case Φ_ε has a critical point which, according to Theorem 2.23, gives rise to a solution of (P_ε), proving Theorem 4.6. □

The same arguments, with obvious changes, can be used to find solutions of

$$-\Delta u+(1+\varepsilon a_0(x))u=u^p,\quad u>0,\quad u\in W^{1,2}(\mathbb{R}^n).$$

For example, one can show that if $a_0\in L^\infty(\mathbb{R}^n)$ and $a_0(x)\to 0$ as $|x|\to\infty$ then the preceding equation has a solution for any $|\varepsilon|\ll 1$.

Bibliographical remarks

There is an extensive bibliography dealing with elliptic equations on $\mathbb{R}^n$, like

$$-\Delta u+a(x)u=f(x,u),\qquad u\in W^{1,2}(\mathbb{R}^n),$$

in the case that $f(\cdot,u)\sim|u|^{p-1}u$ as $|u|\to\infty$, with $1<p<(n+2)/(n-2)$, and under various assumptions on $a(x)\ge a_0>0$. As anticipated in Section 4.1, the problem is usually studied by using Critical Point Theory, the main difficulty being the failure of the (PS) compactness condition. It has been shown in [126] that when the potential $a(x)$ diverges at infinity, the (PS) condition can be recovered. On the other hand, when a is bounded, a general tool which has been used is the Concentration-Compactness method. Various results dealing with these problems are discussed in the books [52, 147], where we also refer for a more complete bibliography.

Recently some result dealing with nonlinear elliptic subcritical problems on $\mathbb{R}^n$ with potentials $a(x)$ that decay to zero at infinity has been also obtained, see [12].

Chapter 5

Elliptic Problems with Critical Exponent

In this chapter we will deal with the equation in $\mathbb{R}^n$

$$-\Delta u = u^{(n+2)/(n-2)} + \varepsilon k(x)u^q, \tag{5.1}$$

where $1 \leq q \leq (n+2)/(n-2)$. We mainly follow [14] where we refer for more details and other results.

After a first section devoted to studying the unperturbed problem $-\Delta u = u^{\frac{n+2}{n-2}}$, we consider, in Section 5.2, the case in which q is also critical, $q = (n+2)/(n-2)$. As seen in the introduction, the corresponding equation is particularly relevant for its relation with problems arising in differential geometry and will be called *Yamabe-like* equation. The rest of the chapter deals with the case $1 \leq q < (n+2)/(n-2)$.

5.1 The unperturbed problem

We will work in $\mathcal{H} := \mathcal{D}^{1,2}(\mathbb{R}^n)$, the space of $u \in L^{2^*}(\mathbb{R}^n)$ such that $\nabla u \in L^2(\mathbb{R}^n)$, endowed with scalar product and norm, respectively

$$(u|v) = \int_{\mathbb{R}^n} \nabla u \cdot \nabla v dx, \qquad \|u\|^2 = \int_{\mathbb{R}^n} |\nabla u|^2 dx.$$

The choice of this space is due to the specific form of the linear part of (5.1), which is $-\Delta u$. Indeed this is the natural choice for the geometric applications. Hereafter, k and q are such that $k|u|^{q+1} \in L^1(\mathbb{R}^n) \forall u \in \mathcal{H}$. Positive solutions of

$$-\Delta u = u^{\frac{n+2}{n-2}} + \varepsilon k(x)u^q, \qquad u \in \mathcal{H},$$

are the critical points of $I_\varepsilon : \mathcal{H} \to \mathbb{R}$,

$$I_\varepsilon(u) = \tfrac{1}{2}\|u\|^2 - \tfrac{1}{2^*}\int_{\mathbb{R}^n} u_+^{2^*}\,dx - \varepsilon\,\tfrac{1}{q+1}\int_{\mathbb{R}^n} k(x)u_+^{q+1}dx,$$

where u_+ denotes the positive part of u.

Remark 5.1. The arguments sketched in Section 4.1 can be repeated here to show that critical points of I_ε are positive solutions of (5.1). Moreover, let us recall that in the presence of the critical exponent, the regularity follows from a result by Brezis and Kato, see [49]. Unfortunately, when $q = 1$ the functional I_ε is not C^2 but merely $C^{1,1}$. For this reason, in such a case, it is convenient to define I_ε by setting

$$I_\varepsilon(u) = \tfrac{1}{2}\|u\|^2 - \tfrac{1}{2^*}\int_{\mathbb{R}^n} u_+^{2^*}\,dx - \tfrac{1}{2}\varepsilon\int_{\mathbb{R}^n} k(x)u^2dx.$$

The fact that critical points of I_ε give rise to positive solutions will require an *ad hoc* argument, see the proof of Theorem 5.10 in Section 5.3. □

Setting

$$I_0(u) = \tfrac{1}{2}\|u\|^2 - \tfrac{1}{2^*}\int_{\mathbb{R}^n} u_+^{2^*}\,dx, \tag{5.2}$$

and

$$G(u) = \begin{cases} \int_{\mathbb{R}^n} k(x)u_+^{q+1}dx, & \text{if } 1 < q \le \frac{n+2}{n-2}; \\ \int_{\mathbb{R}^n} k(x)u^2dx, & \text{if } q = 1, \end{cases} \tag{5.3}$$

we can write $I_\varepsilon(u) = I_0(u) - \frac{1}{q+1}\varepsilon G(u)$.

In the rest of the section we will study the unperturbed problem

$$-\Delta u = u^{(n+2)/(n-2)}, \quad u > 0, \quad u \in \mathcal{H}.$$

It is well known that this problem possesses the following family of solutions, depending on $(n+1)$ parameters $\xi \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^+$,

$$z_{\mu,\xi}(x) = \mu^{-(n-2)/2}U\left(\frac{x-\xi}{\mu}\right),$$

where

$$U(x) = [n(n-2)]^{(n-2)/4}\left(\frac{1}{1+|x|^2}\right)^{(n-2)/2}.$$

Correspondingly, we have an $(n+1)$-dimensional manifold of solutions given by

$$Z = \{z = z_{\mu,\xi} : \mu > 0,\ \xi \in \mathbb{R}^n\}.$$

With respect to the subcritical equations discussed in the preceding section, the new feature here is that the unperturbed problem is invariant not only by translation but also by the dilation $x \mapsto x/\mu$, $\mu > 0$.

It is easy to see that $I_0''(z)$ is Fredholm index zero for all $z \in Z$. Next we prove that Z is non-degenerate.

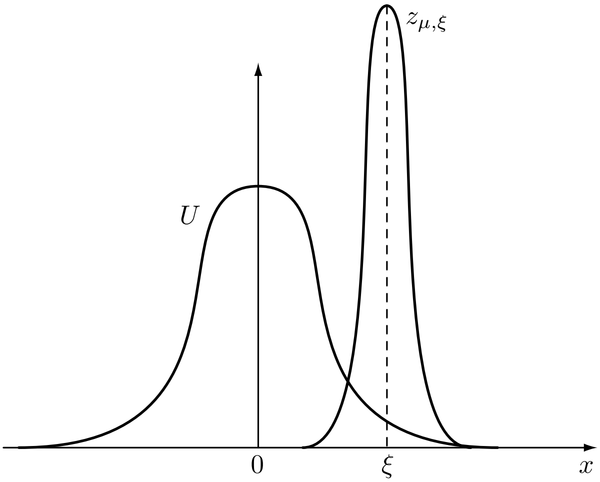

Figure 5.1

Lemma 5.2. *For all $z = z_{\mu,\xi} \in Z$, every solution of the linearized problem*

$$\Delta v = \tfrac{n+2}{n-2}\, z^{4/(n-2)} v, \qquad v \in \mathcal{H}, \tag{5.4}$$

has the form

$$v = aD_\mu z + \mathbf{b} \cdot \nabla z, \qquad a \in \mathbb{R},\ \mathbf{b} = (b_1, \dots, b_n) \in \mathbb{R}^n.$$

Thus $\mathrm{Ker}[I_0''(z)] = T_z Z$*, for all $z \in Z$ and Z satisfies the non-degeneracy condition* (ND)*.*

Proof. The proof is similar to that carried out for the subcritical case, see Lemma 4.1, and thus we will indicate the new features, only. To simplify notation, we carry over the arguments with $z = U$ instead of a generic $z_{\mu,\xi}$. Looking again for solutions of (5.4) in the form $v = \sum_{k \geq 0} \psi_k(r) Y_k(\vartheta)$, we find the following equations for ψ_k:

$$A_k(\psi_k) = -\psi_k'' - \frac{n-1}{r}\psi_k' + \frac{k(n+k-2)}{r^2}\psi_k - \tfrac{n+2}{n-2}\, U^{4/(n-2)} \psi_k = 0, \quad k = 0, 1, 2, \dots$$

For $k = 0$ this equation becomes

$$-\psi_0'' - \frac{n-1}{r}\psi_0' = \tfrac{n+2}{n-2}\, U^{4/(n-2)} \psi_0.$$

A first solution is given by $\varphi = D_\mu z_{\mu,0}|_{\mu=1} \in \mathcal{H}$. A second linearly independent solution of the form $\psi_0 = c(r)\varphi(r)$ satisfies $A_0(\psi_0) = 0$ provided

$$-c''\varphi - c'(2\varphi' + \tfrac{n-1}{r}\varphi) = 0.$$

It follows that

$$c'(r) = \frac{\text{const.}}{r^{n-1}\varphi^2(r)} \sim \text{const.}\ \frac{(1+r^2)^{n-2}}{r^{n-1}} \sim r^{n-3}.$$

Hence

$$c(r)\varphi(r) \sim \text{const.}\ \frac{r^{n-2}}{(1+r^2)^{(n-2)/2}}, \qquad (r \to \infty)$$

and thus $c\varphi \in \mathcal{H}$ implies that $c(r) \equiv 0$. This shows that the solutions of the equation $A_0(\psi_0) = 0$, $\psi_0 \in \mathcal{H}$, are of the form $a\psi_0(r) \equiv aD_\mu U(r)$ with $a \in \mathbb{R}$. Next, for $k = 1$ the equation $A_1(\psi_1) = 0$ becomes

$$A_1(\psi_1) = -\psi_1'' - \frac{n-1}{r}\psi_1' + \frac{n-1}{r^2}\psi_1 = \tfrac{n+2}{n-2}\, U^{4/(n-2)}\psi_1.$$

As in Lemma 4.1, one has that $\widehat{U}'(r)$ is a solution of $A_1(\psi_1) = 0$. Moreover, one shows that any linearly independent solution $v(r)$ behaves at infinity like

$$v(r) \sim r^n\, \frac{r^{n-2}}{(1+r^2)^{n/2}} \sim r, \qquad (r \to \infty).$$

This shows that the solutions of $A_1(\psi_1) = 0$, $\psi_1 \in \mathcal{H}$, are those spanned by $\psi_1(r)\widehat{U}'(r)$, which correspond to solution of (5.4) like $\mathbf{b}\cdot\nabla U$ with $\mathbf{b} \in \mathbb{R}^n$. Finally, the equation $A_k(\psi_k) = 0$, $\psi_k \in \mathcal{H}$, has the trivial solution only. $\square$

According to the general theory, we find

$$\Phi_\varepsilon(z_{\mu,\xi}) = c_0 - \varepsilon\,\frac{1}{q+1}\int_{\mathbb{R}^n} k(x) z_{\mu,\xi}^{q+1}(x)dx + o(\varepsilon), \quad c_0 = I_0(U),$$

and hence we are led to study the finite-dimensional functional

$$\Gamma(\mu,\xi) := \int_{\mathbb{R}^n} k(x) z_{\mu,\xi}^{q+1}(x)dx.$$

This will be done hereafter, distinguishing various ranges of q.

5.2 On the Yamabe-like equation

Here we deal with the case $q = (n+2)/(n-2)$, namely with the Yamabe-like problem

$$-\Delta u = (1+\varepsilon k(x))u^{(n+2)/(n-2)}, \quad u > 0, \quad u \in \mathcal{H} = \mathcal{D}^{1,2}(\mathbb{R}^n). \tag{5.5}$$

Problems of this sort has been studied, e.g., in [47, 73, 115, 128] (actually these papers deal with more general equations with critical exponent like $-\Delta u = K(x)u^{(n+2)/(n-2)}$). Moreover, up to the stereographic projection, (5.5) is the equation arising in the scalar curvature problem for the sphere S^n on which there is a broad literature, see also Section 7.1.

In the main result of this section we will make the following assumptions on $k(x)$. Let $\mathrm{Cr}[k]$, denote the set of critical points of k.

(k.0) $k \in L^\infty(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$;
(k.1) $\mathrm{Cr}[k]$ is finite and $\Delta k(x) \neq 0$, $\forall\, x \in \mathrm{Cr}[k]$;
(k.2) $\exists\, \rho > 0$ such that $\langle k'(x), x\rangle < 0$, $\forall\, |x| \geq \rho$;
(k.3) $\langle k'(x), x\rangle \in L^1(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \langle k'(x), x\rangle dx < 0$.

From (k.1) it follows that for every $x \in \mathrm{Cr}[k]$ the index $i(k', x)$ (namely the local degree) of k' at x is well defined. The next theorem is essentially taken from [14], Section 3, where one can find other results of the same sort.

Theorem 5.3. *Let* (k.1–3) *hold and suppose that*

$$\sum_{x \in \mathrm{Cr}[k],\, \Delta k(x) < 0} i(k', x) \neq (-1)^n. \tag{5.6}$$

Then (5.5) *has at least a solution, provided* $|\varepsilon| \ll 1$.

The proof of Theorem 5.3 will be carried out by showing that the finite-dimensional functional Γ defined in the previous section has a "stable" critical point, in the sense that Theorem 2.17 proved in Section 2.2, Chapter 2, applies. This will require some topological theoretic arguments carried out below.

5.2.1 Some auxiliary lemmas

First, let us point out that dealing with (5.5), the finite-dimensional functional Γ takes the form

$$\begin{aligned}\Gamma(\mu, \xi) := \int_{\mathbb{R}^n} k(x) z_{\mu,\xi}^{2^*}(x) dx &= \mu^{-n} \int_{\mathbb{R}^n} k(x) U^{2^*}\left(\frac{x-\xi}{\mu}\right) dx \\ &= \int_{\mathbb{R}^n} k(\mu y + \xi) U^{2^*}(y) dy.\end{aligned}$$

By a straight calculation we find

$$\lim_{\mu \downarrow 0} \Gamma(\mu, \xi) = a_0 k(\xi), \qquad a_0 = \int_{\mathbb{R}^n} U^{2^*}(y) dy.$$

Moreover, from

$$D_\mu \Gamma(\mu, \xi) = \int_{\mathbb{R}^n} \langle k'(\mu y + \xi), y\rangle U^{2^*}(y) dy$$

and since

$$\int_{\mathbb{R}^n} y_i U^{2^*}(y) dy = 0,$$

it follows

$$\lim_{\mu \downarrow 0} D_\mu \Gamma(\mu, \xi) = 0.$$

As a consequence, we can extend Γ to all of $\mathbb{R}^n$ by setting $\widetilde{\Gamma}(0,\xi) = a_0 k(\xi)$ and $\widetilde{\Gamma}(\mu,\xi) = \Gamma(-\mu,\xi)$ if $\mu < 0$. The extended function is of class C^1 and satisfies

$$D_\mu \widetilde{\Gamma}(0,\xi) = 0, \qquad \forall\, \xi \in \mathbb{R}^n. \tag{5.7}$$

From (5.7) we infer

$$\xi \in \mathrm{Cr}[k] \quad \Longleftrightarrow \quad (0,\xi) \in \mathrm{Cr}[\widetilde{\Gamma}], \tag{5.8}$$

where $\mathrm{Cr}[\widetilde{\Gamma}]$ denotes the set of critical points of $\widetilde{\Gamma}$ (on $\mathbb{R}^{n+1}$). Next, we evaluate the second derivatives of $\widetilde{\Gamma}$. We find

$$D^2_{\mu\mu}\widetilde{\Gamma}(\mu,\xi) = \int_{\mathbb{R}^n} \sum D^2_{ij} k(\mu y + \xi) y_i y_j U^{2^*}(y) dy.$$

Since $\int_{\mathbb{R}^n} y_i y_j U^{2^*}(y) dy = 0 \iff i \neq j$, we infer

$$D^2_{\mu\mu}\widetilde{\Gamma}(0,\xi) = a_1 \Delta k(\xi), \qquad a_1 = \int_{\mathbb{R}^n} |y|^2 U^{2^*}(y) dy. \tag{5.9}$$

Furthermore, differentiating (5.7) with respect to ξ_i we deduce

$$D^2_{\mu\xi_i}\widetilde{\Gamma}(0,\xi) = 0, \qquad i = 1,\dots,n. \tag{5.10}$$

Putting together (5.9) and (5.10) one finds that the Hessian matrix $\widetilde{\Gamma}''(0,\xi)$ at any $\xi \in \mathbb{R}^n$ has the form

$$\widetilde{\Gamma}''(0,\xi) \quad = \quad \begin{pmatrix} a_0 k''(\xi) & 0 \\ & \\ 0 & a_1\,\Delta k(\xi) \end{pmatrix}. \tag{5.11}$$

In particular, $(0,\xi)$ is an isolated critical point of $\tilde{\Gamma}$ and, by the multiplicative property of the degree, we have $i(\tilde{\Gamma}', (0,\xi)) = \mathrm{sgn}(\Delta K(\xi)) i(k',\xi)$. Let us collect the above results in the following Lemma.

Lemma 5.4. *Let* (k.0) *and* (k.1) *hold. Then* $(0,\xi)$ *is an isolated critical point of* $\widetilde{\Gamma}$ *if and only if* $\xi \in \mathrm{Cr}[k]$. *Moreover one has*

$$i(\widetilde{\Gamma}', (0,\xi)) = \begin{cases} i(k',\xi) & \text{if} \quad \Delta k(\xi) > 0, \\ -i(k',\xi) & \text{if} \quad \Delta k(\xi) < 0. \end{cases}$$

Our next lemma takes into account the consequences of assumptions (k.2) and (k.3). Let $B^d_R = \{x \in \mathbb{R}^d : |x| < R\}$.

Lemma 5.5. *Let* (k.2) *and* (k.3) *hold. Then* $\exists\, R > 0$ *such that*

$$\langle \widetilde{\Gamma}'(\mu,\xi), (\mu,\xi)\rangle < 0, \qquad \forall\, (\mu,\xi) \in \mathbb{R}^{n+1},\ \mu^2 + |\xi|^2 \widetilde{\Gamma} \geq R^2.$$

Therefore, $\deg(\widetilde{\Gamma}', B^{n+1}_R, 0) = (-1)^{n+1}$.

Proof. We set $g(x) = \langle k'(x), x\rangle$. From

$$\widetilde{\Gamma}(\mu,\xi) = \int_{\mathbb{R}^n} k(\mu y + \xi) U^{2^*}(y) dy$$

we infer

$$\langle \widetilde{\Gamma}'(\mu,\xi), (\mu,\xi)\rangle = \int_{\mathbb{R}^n} g(\mu y + \xi) U^{2^*}(y) dy = \mu^{-n} \int_{\mathbb{R}^n} g(x) U^{2^*}((x-\xi)/\mu) dx.$$

Setting

$$J_{1,R}(\mu,\xi) = \int_{|x|<R} g(x) U^{2^*}((x-\xi)/\mu) dx,$$
$$J_{2,R}(\mu,\xi) = \int_{|x|>R} g(x) U^{2^*}((x-\xi)/\mu) dx,$$

we find

$$\langle \widetilde{\Gamma}'(\mu,\xi), (\mu,\xi)\rangle = J_{1,R}(\mu,\xi) + J_{2,R}(\mu,\xi).$$

Assumption (k.2), namely $g(x) < 0 \; \forall\, |x| \geq \rho$, implies that

$$J_{2,R}(\mu,\xi) < 0, \qquad \forall\, (\mu,\xi) \in \mathbb{R}^{n+1}, \quad \forall\, R \geq \rho. \tag{5.12}$$

We claim that, taking R possibly larger, one has that $J_{1,R}(\mu,\xi) < 0$ provided $\mu^2 + |\xi|^2 \geq R^2$. Actually, for $x \in B_R^n$ one has

$$g(x) U^{2^*}\left(\frac{x-\xi}{\mu}\right) \leq \max_{x\in B_R^n} U^{2^*}\left(\frac{x-\xi}{\mu}\right) g_+(x) - \min_{x\in B_R^n} U^{2^*}\left(\frac{x-\xi}{\mu}\right) g_-(x)$$

where g_+, resp g_-, denotes the positive, resp. negative, part of g.

As $\mu + |\xi| \to \infty$, we get

$$\max_{x\in B_R^n} U^{2^*}\left(\frac{x-\xi}{\mu}\right) \sim \frac{\mu^{2n}}{(\mu^2 + (R-|\xi|)^2)^n};$$
$$\min_{x\in B_R^n} U^{2^*}\left(\frac{x-\xi}{\mu}\right) \sim \frac{\mu^{2n}}{(\mu^2 + (R+|\xi|)^2)^n}.$$

This implies that for $\mu + |\xi| \to \infty$,

$$J_{1,R}(\mu,\xi) \sim \max_{x\in B_R^n} U^{2^*}\left(\frac{x-\xi}{\mu}\right) \int_{B_R^n} g(x) dx.$$

Then, using (k.3), there exists $R' > 0$ such that $J_{1,R}(\mu,\xi) < 0$ provided that $R \geq R'$ and $\mu + |\xi| \geq R'$. This, jointly with (5.12), proves the lemma. $\square$

We are now in the position to prove Theorem 5.3.

5.2.2 Proof of Theorem 5.3

Let C_+ denote the set of points of $\mathrm{Cr}[\widetilde{\Gamma}]$ with $\mu > 0$. Using (5.8) and the fact that $\tilde{\Gamma}$ is even in μ, it follows that $\mathrm{Cr}[\widetilde{\Gamma}] = C_+ \cup C_0 \cup C_-$, where $C_- := \{(-\mu,\xi) : (\mu,\xi) \in C_+\}$ and $C_0 = \{(0,\xi) : \xi \in \mathrm{Cr}[k]\}$. Remark that as a consequence of (k.2), resp. Lemma 5.5, C_0 and $C_\pm$ are compact.

In order to apply Theorem 2.17, discussed in the abstract setting, we will show that for any open bounded set $\mathcal{N} \subset]0,\infty) \times \mathbb{R}^n$ with $C_+ \subset \mathcal{N}$ one has that $\deg(\Gamma', \mathcal{N}, 0) \neq 0$. As usual, $\deg(\phi, \Omega, 0)$ denotes the topological degree of a map ϕ with respect to Ω and 0 and it is always understood that it is well defined, in particular that $0 \notin \phi(\partial\Omega)$.

Let us argue by contradiction. Let $\mathcal{O} \subset]0,\infty) \times \mathbb{R}^n$ be an open bounded set with $C_+ \subset \mathcal{O}$ and such that $\deg(\Gamma', \mathcal{O}, 0) = 0$. Let us introduce the following notation:

$$\mathcal{O}_- = \{(-\mu,\xi) : (\mu,\xi) \in \mathcal{O}\}, \quad \mathcal{O}' = \mathcal{O} \cup \mathcal{O}_- \, .$$

Since $\Gamma = \tilde{\Gamma}$ in $]0,\infty) \times \mathbb{R}^n$, using Lemma 5.5 we deduce

$$\deg(\widetilde{\Gamma}', B_R^{n+1} \setminus \mathcal{O}', 0) = (-1)^{n+1}. \tag{5.13}$$

Since the only critical points of $\widetilde{\Gamma}'$ in $B_R^{n+1} \setminus \mathcal{O}'$ are those in C_0 and taking into account that C_0 consists of isolated points, we get

$$\begin{aligned}\deg(\widetilde{\Gamma}', B_R^{n+1} \setminus \mathcal{O}', 0) &= \sum_{\xi \in \mathrm{Cr}[k]} i(\widetilde{\Gamma}', (0,\xi)) \\ &= \sum_{\xi \in \mathrm{Cr}[k], \Delta k(\xi) > 0} i(\widetilde{\Gamma}', (0,\xi)) + \sum_{\xi \in \mathrm{Cr}[k], \Delta k(\xi) < 0} i(\widetilde{\Gamma}', (0,\xi)).\end{aligned}$$

Using Lemma 5.4 we infer

$$\deg(\widetilde{\Gamma}', B_R^{n+1} \setminus \mathcal{O}', 0) = \sum_{\xi \in \mathrm{Cr}[k], \Delta k(\xi) > 0} i(k',\xi) - \sum_{\xi \in \mathrm{Cr}[k], \Delta k(\xi) < 0} i(k',\xi).$$

This and (5.13) yield

$$\sum_{\xi \in \mathrm{Cr}[k], \Delta k(\xi) > 0} i(k',\xi) - \sum_{\xi \in \mathrm{Cr}[k], \Delta k(\xi) < 0} i(k',\xi) = (-1)^{n+1}. \tag{5.14}$$

On the other hand, from (k.2) it immediately follows that $\deg(k', B_R^n, 0) = (-1)^n$ and hence

$$\sum_{\xi \in \mathrm{Cr}[k]} i(k',\xi) = \sum_{\xi \in \mathrm{Cr}[k], \Delta k(\xi) > 0} i(k',\xi) + \sum_{\xi \in \mathrm{Cr}[k], \Delta k(\xi) < 0} i(k',\xi) = (-1)^n.$$

This and (5.14) imply

$$\sum_{\xi \in \mathrm{Cr}[k], \Delta k(\xi) < 0} i(k',\xi) = (-1)^n,$$

a contradiction to (5.13). This proves that, for any open bounded set $\mathcal{N} \subset]0,\infty) \times \mathbb{R}^n$ such that $C_+ \subset \mathcal{N}$, one has

$$\deg(\Gamma', \mathcal{N}, 0) \neq 0.$$

Now we can apply Theorem 2.17 yielding a critical point of I_ε and hence a solution of (5.5). This completes the proof of Theorem 5.3. □

Remarks 5.6. (i) If $u \in \mathcal{H}$ is any (positive) solution of (5.5), then the Pohozaev identity yields that $\int_{\mathbb{R}^n} \langle k'(x), x\rangle u^{(n+2)/(n-2)} dx = 0$ (a similar result indeed holds for the more general non-perturbative equation $\Delta u + K(x) u^{(n+2)/(n-2)} = 0$). Thus $\langle k'(x), x\rangle$ has to change sign. For example, if k is radial, then k cannot be monotone on $\mathbb{R}^+$. Notice also that, if the only critical point of k is a maximum, say $0 \in \mathbb{R}^n$, with $\Delta k(0) < 0$, then $\sum_{x\in \mathrm{Cr}[k], \Delta k(x)<0} i(k', x) = i(k', 0) = (-1)^n$, in contrast to assumption (5.6).

(ii) For future references (see Section 7.1 later on) let us point out that Theorem 5.3 can also be proved when (k.1) is substituted by the following conditions:

(k.1′) $\forall\, x \in \mathrm{Cr}[k]\ \exists\, \beta \in]1, N[$ and $a_j \in C(\mathbb{R}^n)$, with $\sum_j a_j(y) \neq 0$ and such that $k(y) = k(x) + \sum a_j |y_j - x_j|^\beta + o(|y-x|^\beta)$ as $y \to x$; and

(k.1″) there holds

$$\sum_{x\in \mathrm{Cr}[k],\, \sum a_j(x)<0} i(k', x) \neq (-1)^n.$$

The proof is similar to the previous one. Actually, one shows that

$$\deg(\Gamma', \mathcal{N}, 0) = \sum_{x\in \mathrm{Cr}[k],\, \sum a_j(x)<0} i(k', x) - (-1)^n \neq 0.$$

Notice that, when $\beta = 2$, we have $\sum_j a_j = \frac{1}{2}\Delta k(x)$ and we recover Theorem 5.3. □

5.2.3 The radial case

Here we will briefly discuss the case in which $k(x)$ is radial: $k(x) = \widehat{k}(|x|)$, for some $\widehat{k} : \mathbb{R}^+ \to \mathbb{R}$. In this case it is possible to prove some different result, as the following one, in which no assumption like (5.6) is made.

Theorem 5.7. *Let $\widehat{k} \in L^\infty(\mathbb{R}^+)$ and there exists $\alpha < n$ such that $\widehat{k}(r) r^{n-1} \in L^1([1,+\infty))$. Moreover, suppose that either*

(a) *$\widehat{k} \in C^2(\mathbb{R}^+)$ and $\widehat{k}(0)\widehat{k}''(0) > 0$;*

or, letting $\gamma := \int_0^\infty \widehat{k}(r)(1+r^2)^{-n} r^{n-1} dr$, that

(b) *$\gamma \neq 0$ and $\gamma\, \widehat{k}(0) \leq 0$.*

Then (5.5) has a radial solution, provided $|\varepsilon| \ll 1$.

Proof. We work in $\mathcal{H}_r = \mathcal{D}^{1,2}_r$, the space of radial $\mathcal{D}^{1,2}(\mathbb{R}^n)$ functions. Now the critical manifold is $Z_r = \{\mu^{-(n-2)/2}\widehat{U}(\cdot/\mu) : \mu > 0\} \simeq \mathbb{R}^+$, which is still non-degenerate in $\mathcal{H}_r$. The finite-dimensional functional Γ here becomes

$$\Gamma_r(\mu) = \mu^{-n}\int_0^\infty \widehat{k}(r)\widehat{U}^{2^*}(r/\mu)r^{n-1}dr = \int_0^\infty \widehat{k}(\mu\, r)\widehat{U}^{2^*}(r)r^{n-1}dr.$$

There holds

$$\begin{aligned}\Gamma_r(\mu) &= \int_0^1 \widehat{k}(r)\widehat{U}^{2^*}(r/\mu)r^{n-1}dr + \int_1^\infty \widehat{k}(r)\widehat{U}^{2^*}(r/\mu)r^{n-1}dr \\ &\le c_1\mu^{-n}\int_0^1 \widehat{k}(r)r^{n-1}dr + c_2\mu^{\alpha-n}\int_1^\infty \frac{\widehat{k}(r)}{r^\alpha}r^{n-1}dr.\end{aligned}$$

Since $\alpha < n$ and $\widehat{k}(r)r^{n-1} \in L^1([1,+\infty)$, it follows that

$$\lim_{\mu\to\infty}\Gamma_r(\mu) = 0.$$

Moreover, as before, Γ_r can be extended to $\mu = 0$ by continuity setting $\Gamma_r(0) = a_0\widehat{k}(0)$, with $a_0 > 0$.

Now, let (a) hold. Then one has

$$\Gamma_r'(0) = 0 \qquad \Gamma_r''(0) = a_1\widehat{k}''(0), \quad a_1 > 0,$$

and the condition $\widehat{k}(0)\widehat{k}''(0) > 0$ implies that Γ_r has a maximum (if $\widehat{k}(0) > 0$), or a minimum (if $\widehat{k}(0) < 0$), at some $\bar{\mu} > 0$. This allows us to use the abstract results, yielding a radial solution of (5.5), for $|\varepsilon| \ll 1$.

As for the case (b), it suffices to remark that $\Gamma_r(1) = [n(n-2)]^{\frac{n}{2}}\gamma$. If $\gamma > 0$ (resp. $\gamma < 0$) then $\widehat{k}(0) \le 0$ (resp. $\widehat{k}(0) \ge 0$) and, once more, Γ_r has a maximum (resp. a minimum) at some $\bar{\mu} > 0$. $\square$

5.3 Further existence results

In this section we will study the problem

$$-\Delta u = u^{\frac{n+2}{n-2}} + \varepsilon k(x)u^q, \qquad u > 0, \quad u \in \mathcal{H}, \tag{5.15}$$

where $1 \le q < \frac{n+2}{n-2}$.

Throughout the section we will assume that $k \not\equiv 0$ and satisfies

(k.4) $\quad k \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

It is worth mentioning that here we will not make the sharpest assumptions in order to avoid technicalities, the main purpose being to highlight the ideas of the approach we use. For (further and) more general results as well as for more details, we still refer to [14].

If (k.4) holds then $k \in L^s(\mathbb{R}^n)$, where s denotes the conjugate exponent of $\frac{2n}{(n-2)(q+1)}$, and hence $k\,|u|^{q+1} \in L^1(\mathbb{R}^n)$ so that the perturbation $G(u)$ is well defined on $\mathcal{H}$, see (5.3). Let us recall that the Euler functional $I_\varepsilon = I_0 - \frac{1}{q+1}\varepsilon G$, see (5.2), is of class C^2 on $\mathcal{H}$, see Remark 5.1.

Using the finite-dimensional reduction, we have to study the functional

$$\Gamma(\mu,\xi) = \int_{\mathbb{R}^n} k(x) z_{\mu,\xi}^{q+1}(x)dx$$

which becomes

$$\Gamma(\mu,\xi) = \mu^{-\theta} \int_{\mathbb{R}^n} k(x) U^{q+1}\left(\frac{x-\xi}{\mu}\right) dx = \mu^{n-\theta} \int_{\mathbb{R}^n} k(\mu y + \xi) U^{q+1}(y) dy,$$

where $\theta = \frac{(n-2)(q+1)}{2}$. Let us remark that $n - \theta > 0$ iff $q+1 < 2^*$. This fact allows us to obtain results where, differently from the Yamabe-like equations discussed in the preceding section, no assumption involving Δk is made.

First, let us show a couple of lemmas.

Lemma 5.8. *One has that* $\lim_{\mu+|\xi|\to\infty} \Gamma(\mu,\xi) = 0$.

Proof. We distinguish between the case $\mu \to 0$ and $\mu \to \mu^* > 0$. In the former, we take t such that $\frac{n}{n-2} < t < 2^*$ and denote by τ the conjugate exponent of $t/(q+1)$. Since $t > \frac{n}{n-2}$, then $U^t \in L^1(\mathbb{R}^n)$, the Hölder inequality yields

$$\begin{aligned} |\Gamma(\mu,\xi)| &\le \mu^{-\theta} \left(\int_{\mathbb{R}^n} k^\tau(x) dx \right)^{1/\tau} \left(\int_{\mathbb{R}^n} U^t \left(\frac{x-\xi}{\mu}\right) dx \right)^{\frac{q+1}{t}} \\ &\le c_1 \mu^{\frac{n(q+1)}{t} - \theta}. \end{aligned}$$

Since $t < 2^*$ we have that $\frac{n(q+1)}{t} - \theta > 0$, and the conclusion follows.

Next, if $\mu \to \mu^* > 0$ (and hence $|\xi| \to \infty$), we use the dominated convergence theorem to infer that

$$\Gamma(\mu,\xi) = \mu^{-\theta} \int_{\mathbb{R}^n} k(x) U^{q+1}\left(\frac{x-\xi}{\mu}\right) dx \to 0.$$

Finally, if $\mu \to +\infty$ then we write

$$\begin{aligned} \Gamma(\mu,\xi) &= \mu^{n-\theta} \int_{\mathbb{R}^n} k(\mu y + \xi) U^{q+1}(y) dy \\ &\le \mu^{n-\theta} \|U\|_{L^\infty}^{q+1} \int_{\mathbb{R}^n} k(\mu y + \xi) dy \le \mu^{-\theta} \|U\|_{L^\infty}^{q+1} \|k\|_{L^1}. \end{aligned}$$

Thus $\Gamma(\mu,\xi) \to 0$ in this case, too. This completes the proof. □

Lemma 5.9. *Suppose that one of the two following conditions is satisfied:*

(k.5) $q > 1$ *or* $q = 1$ *and* $n > 4$*;*
(k.6) $\int_{\mathbb{R}^n} k(x) \neq 0$.

Then $\Gamma \not\equiv 0$.

Proof. If $n > 4$, taking advantage of the fact that $U^{q+1} \in L^1(\mathbb{R}^n)$ for $q \in [1, \frac{n+2}{n-2})$, we get

$$\lim_{\mu \to 0} \int_{\mathbb{R}^n} k(\mu y + \xi) U^{q+1}(y) dy = c_2 k(\xi), \quad c_2 = \int_{\mathbb{R}^n} U^{q+1}(y) dy.$$

This shows that $\mu^{\theta - n}\Gamma(\mu, \xi) \to c_2 k(\xi)$ as $\mu \to 0$ and implies that $\Gamma \not\equiv 0$ provided $k \not\equiv 0$. When $q > 1$ and $n = 2, 3$ we can use the Fourier analysis arguments employed in the second part of Theorem 4.4 in Section 4.3. This proves the lemma when (k.5) holds.

Next, we take $\xi = 0$ and evaluate

$$\lim_{\mu \to \infty} \int_{\mathbb{R}^n} k(x) U^{q+1}\left(\frac{x}{\mu}\right) dx = U^{q+1}(0) \int_{\mathbb{R}^n} k(x) dx.$$

This implies that $\mu^{\theta}\Gamma(\mu, \xi) \to U^{q+1}(0) \int_{\mathbb{R}^n} k(x) dx$ as $\mu \to \infty$ and shows that $\Gamma \not\equiv 0$, provided (k.6) holds. □

We are now in position to prove the main result of this section.

Theorem 5.10. *Let* (k.4) *holds and suppose that either* (k.5) *or* (k.6) *are satisfied. Then* (5.15) *has a solution, provided* $|\varepsilon| \ll 1$.

Proof. From $\Gamma(\mu, \xi) = \mu^{n-\theta} \int_{\mathbb{R}^n} k(\mu y + \xi) U^{q+1}(y) dy$ and since $n - \theta > 0$ it immediately follows that $\Gamma(0, \xi) \equiv 0$. This, jointly with Lemmas 5.8 and 5.9, implies that Γ has a maximum or a minimum at some $(\bar{\mu}, \bar{\xi})$, with $\bar{\mu} > 0$ and I_ε has a critical point u_ε close to $z_{\bar{\mu}, \bar{\xi}}$, hence a solution of (5.15). As anticipated in Remark 5.1 we need here to prove that when $q = 1$ we still have that $u_\varepsilon > 0$. We follow the arguments carried out in [63], pp. 1172-1173. From the equation we infer that

$$\|(u_\varepsilon)_\pm\|^2 = \int_{\mathbb{R}^n} |(u_\varepsilon)_\pm|^{2^*} dx + \varepsilon \int_{\mathbb{R}^n} k(x)(u_\varepsilon)_\pm^2 dx. \tag{5.16}$$

Let us set

$$S_\varepsilon = \inf_{u \in \mathcal{H},\, u \neq 0} \frac{\|u\|^2 - \varepsilon \int_{\mathbb{R}^n} k(x) u^2 dx}{(\int_{\mathbb{R}^n} u^{2^*} dx)^{2/2^*}}.$$

One has that $\lim_{\varepsilon \to 0} S_\varepsilon = S$, where S denotes the best Sobolev constant

$$S = \inf_{u \in \mathcal{H},\, u \neq 0} \frac{\|u\|^2}{(\int_{\mathbb{R}^n} u^{2^*} dx)^{2/2^*}},$$

and hence $S_\varepsilon > \frac{S}{2} > 0$ for ε small. From (5.16) we get

$$\int_{\mathbb{R}^n} |(u_\varepsilon)_\pm|^{2^*} dx = \|(u_\varepsilon)_\pm\|^2 - \varepsilon \int_{\mathbb{R}^n} k(x)(u_\varepsilon)_\pm^2 dx \geq S_\varepsilon \left(\int_{\mathbb{R}^n} |(u_\varepsilon)_\pm|^{2^*} dx \right)^{2/2^*}. \tag{5.17}$$

Notice that $(u_\varepsilon)_+ \not\equiv 0$ because $u_\varepsilon \sim z_{\overline{\mu},\overline{\xi}} > 0$. If, by contradiction, also $(u_\varepsilon)_- \not\equiv 0$ then (5.17) implies

$$\int_{\mathbb{R}^n} |(u_\varepsilon)_\pm|^{2^*} dx \geq S_\varepsilon^{n/2}.$$

It follows that

$$I_\varepsilon(u_\varepsilon) = (\tfrac{1}{2} - \tfrac{1}{2^*}) \left(\int_{\mathbb{R}^n} (u_\varepsilon)_+^{2^*} dx + \int_{\mathbb{R}^n} (u_\varepsilon)_-^{2^*} dx \right) \geq 2(\tfrac{1}{2} - \tfrac{1}{2^*}) S_\varepsilon^{n/2}. \tag{5.18}$$

On the other hand, we know that $u_\varepsilon \to z_{\bar{\mu},\bar{\xi}}$ as $\varepsilon \to 0$ and this implies

$$I_\varepsilon(u_\varepsilon) \to I_\varepsilon(z_{\bar{\mu},\bar{\xi}}) = (\tfrac{1}{2} - \tfrac{1}{2^*}) S_\varepsilon^{n/2},$$

a contradiction with (5.18). This shows that $u_\varepsilon \geq 0$ and, by the maximum principle, we get that $u_\varepsilon > 0$. The proof of Theorem 5.10 is now complete. □

Remarks 5.11.

(i) One can consider problems of the type $-\Delta u = u^{\frac{n+2}{n-2}} + \varepsilon k(x) u^{\frac{n+2}{n-2}} + \varepsilon h(x) u^q$. In such a case one can prove the existence of (positive) solutions assuming that h satisfies conditions like those made in this section, and assuming on k conditions like the ones made in the previous section.

(ii) Dealing with (5.15) we can prove multiplicity results. For example, if k satisfies (k.4) and (k.5) and if k changes sign, the preceding arguments show that Γ has a *positive* maximum and a *negative* minimum, yielding a pair of positive solutions of (5.15). □

Bibliographical remarks

As for the subcritical case, one can use the concentration-compactness principle to find positive solutions for equations like $-\Delta u = k u^{\frac{n+2}{n-2}} + h u^q$, see the references in the aforementioned books [52, 147], see also [1], [140]. Roughly, letting S denote the best Sobolev constant, one shows that I_ε satisfies $(PS)_c$ at any level $c < \frac{1}{n} S^{n/2}$. This method is also used in [37] where is proved that (using our notation) $-\Delta u = u^{\frac{n+2}{n-2}} + h(x) u$ has a positive solution in $\mathcal{H}$ provided h satisfies: (a) $h(x) \leq 0$, and $h(x) \leq -\nu < 0$ in some ball; (b) $h \in L^s$ for all $s \in (n/2 - \delta, n/2 + \delta)$, $\delta > 0$ if $n > 3$, $s \in (n/2 - \delta, 3)$ if $n = 3$; (c) $\|h\|_{L^{n/2}}$ is sufficiently small.

For a review on problems like those discussed in this chapter we also refer to the survey paper [16].

Chapter 6

The Yamabe Problem

This chapter is devoted to the study of the Yamabe problem. After recalling some basic notions and facts, we apply the perturbative method to find multiplicity results.

6.1 Basic notions and facts

In this section we recall some well-known concepts in Riemannian geometry. In the presentation we will be as concise as possible, in order to arrive soon to the Yamabe equation. We refer for example to [29, 93], for detailed derivations of the geometric quantities, their motivation and applications.

Given a Riemannian manifold (M, g) of dimension n, let (U, η), $U \subseteq M$, $\eta : U \to \mathbb{R}^n$, be a local coordinate system and let g_{ij} denote the components of the metric g. We also denote with g^{ij} the elements of the inverse matrix $(g^{-1})_{ij}$, and with dV_g the volume element, which is given by

$$dV_g = \sqrt{\det g}\, dx. \tag{6.1}$$

The *Christoffel symbols* are given by

$$\Gamma_{ij}^l = \frac{1}{2}[D_i g_{kj} + D_j g_{ki} - D_k g_{ij}] g^{kl}$$

while the *Riemann curvature tensor*, the *Ricci tensor* and the *scalar curvature* are given respectively by

$$\begin{aligned} R_{kij}^l &= D_i \Gamma_{jk}^l - D_j \Gamma_{ik}^l + \Gamma_{im}^l \Gamma_{jk}^m - \Gamma_{jm}^l \Gamma_{ik}^m; \\ R_{kj} &= R_{klj}^l; \qquad R_g = R_{kj} g^{kj}. \end{aligned} \tag{6.2}$$

Hereafter, we use the standard convention that repeated (upper and lower) indices are summed over all their range (usually between 1 and n). For $n \geq 3$, the

Weyl tensor W_{ijkl} is then defined as

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}\left(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il}\right) + \frac{R}{(n-1)(n-2)}(g_{jl}g_{ik} - g_{jk}g_{il}).$$

For a smooth function u the components of $\nabla_g u$ are

$$(\nabla_g u)^i = g^{ij}\partial_{x_j} u. \tag{6.3}$$

The Laplace-Beltrami operator, applied to a C^2 function $u : M \to \mathbb{R}$, is given by

$$\Delta_g u = g^{ij}\left(\partial^2_{x_i x_j} u - \Gamma^k_{ij}\,\partial_{x_k} u\right) = \frac{1}{|dV_g|}\,\partial_{x_m}\left(|dV_g| g^{mk} D_k u\right). \tag{6.4}$$

We say that the metrics g and $\tilde{g}$ are *conformally equivalent* if there is a smooth function $\rho(x) > 0$ such that $\tilde{g} = \rho\, g$. If $n \geq 3$, using the (convenient) notation $\tilde{g} = u^{\frac{4}{n-2}}\, g$, the scalar curvature $R_{\tilde{g}}$ of $(M, \tilde{g})$ is related to R_g by the following formula

$$-2c_n\Delta_g u + R_g u = R_{\tilde{g}} u^{\frac{n+2}{n-2}}; \qquad c_n = 2\frac{(n-1)}{(n-2)}. \tag{6.5}$$

The structure of equation (6.5) is variational, and the presence of the exponent $\frac{n+2}{n-2}$ makes the study of (6.5) a *non-compact variational problem.* This implies in particular that the associated Palais-Smale sequences do not converge in general, so the analytic study of (6.5) is rather difficult.

For the case $n = 2$, setting $\tilde{g} = e^{2u}\, g$, the corresponding equation is

$$-\Delta_g\, u + K_g = K_{\tilde{g}}\, e^{2u}, \tag{6.6}$$

where $K_g = R_g$ is the *Gauss curvature.* We note that the nonlinearity $u \mapsto e^{2u}$ can be seen as the two-dimensional analogue of the critical growth for the case $n \geq 3$.

6.1.1 The Yamabe problem

We recall the classical Uniformization Theorem, which asserts that every compact two-dimensional surface can be conformally deformed in such a way that its curvature becomes constant.

The prescription of the full curvature tensor in higher dimensions is not expectable, since for n large this has a number of components of order n^4. Hence, working in the same conformal class, one can try to obtain this result for the complete trace of the curvature tensor, namely the scalar curvature. Finding a conformal metric with constant scalar curvature $R_0 \in \mathbb{R}$ on a Riemannian manifold

(M, g), $n \geq 3$, is known as the *Yamabe problem.* Taking into account (6.5), this is equivalent to finding solutions to the equation

$$-2c_n \Delta_g u + R_g u = R_0\, u^{\frac{n+2}{n-2}}; \quad u > 0 \qquad \text{on } M. \tag{6.7}$$

Yamabe, [146], was the first to raise the question of finding such metrics and tried to solve problem (6.7) by using an approximation of the equation as

$$-2c_n \Delta_g u + R_g u = R_0\, u^q; \quad u > 0, \qquad \text{on } M, \tag{6.8}$$

for $q < \frac{n+2}{n-2}$. It is well known that equation (6.8) admits indeed a regular solution u_q for q subcritical, and Yamabe tried to prove that, when $q \to \frac{n+2}{n-2}$, u_q converge to some solution of (6.7). Unfortunately his proof was not correct, since he could not exclude that the limit of the u_q's is the trivial solution $u \equiv 0$.

A first rigorous answer to the problem was given by N. Trudinger, [141]. Setting

$$\mu_{M,g} = \inf_{u \in H^1(M), u \neq 0} \frac{\int_M \left(|\nabla_g u|^2 + R_g\, u^2\right)\, dV_g}{\left(\int_M |u|^{2^*} dV_g\right)^{\frac{2}{2^*}}}, \tag{6.9}$$

this number turns out to be a conformal invariant of g, and the manifold (M, g) is called of negative (resp. null and positive) type if $\mu_{M,g} < 0$ (resp. if $\mu_{M,g} = 0$ and $\mu_{M,g} > 0$). Trudinger proved the Yamabe conjecture in the negative and in the null case.

In the positive case, which is more difficult, a first improvement was obtained by T. Aubin, [28], who showed that for every manifold of positive type there holds $\mu_{M,g} \leq \mu_{S^n, \overline{g}_0}$, where $\overline{g}_0$ is the standard metric of S^n. Moreover, when $\mu_{M,g} < \mu_{S^n, \overline{g}_0}$, the infimum in (6.9) is achieved, so there exists a solution of (6.7). Through an accurate expansion, he proved also that when $n \geq 6$ and (M, g) is non-locally conformally flat (namely when the Weyl tensor is not identically zero), it is indeed $\mu_{M,g} < \mu_{S^n, \overline{g}_0}$. This is shown by using appropriate test functions which are highly concentrated at a point where the Weyl tensor does not vanish. The proof of the Yamabe conjecture in the remaining cases, namely for (M, g) locally conformally flat and for $n = 3, 4, 5$, is due to R. Schoen, [131]. In these cases the local geometry of the manifold does not give sufficient information, and to prove that $\mu_{M,g} < \mu_{S^n, \overline{g}_0}$ some global test functions is employed. These are similar to Aubin's functions near the concentration point, but away from it they are substituted with the Green's function of the conformal Laplacian (the linear operator in (6.7)). A crucial role in this proof is played by the so-called *Positive Mass Theorem*, see [134], arising in general relativity.

Being the existence part settled, one can ask for compactness or multiplicity results. Regarding the first question, in [132] R. Schoen stated the following result, giving the proof just for the locally conformally flat case.

Theorem 6.1. *Let $n \geq 3$ and let (M, g) be a smooth compact n-dimensional manifold. Then the set of solutions of* (6.7) *is bounded in $C^{2,\alpha}$ norm.*

The proof has been recently given in some other cases by O. Druet, Li-Zhang and Marques, and in particular they treat the cases of dimension less or equal to 7 and the case in which W_g never vanishes on M in higher dimensions.

Regarding multiplicity of solutions, some examples are given in [132], where the case of $S^1(T) \times S^n$ is considered. Here $S^1(T)$ is the one-dimensional circle of radius T. Using ODE analysis, it is proved that when $T \to +\infty$, then there is an increasing number of solutions with large energy and large Morse index. Other multiplicity results in the same spirit are given in [94] for the case of manifolds possessing some isometry group or some m-fold covering. More results were also obtained by D. Pollack in [123], where he showed that starting from any compact manifold of positive type, there are arbitrarily small perturbations of the metric for which the Yamabe problem possesses an arbitrarily large number of solutions.

We are going to obtain here the same result starting from the sphere S^n in high dimensions and then, by improving the technique, to obtain non-compactness of solutions in the case of some metrics of class C^k on S^n. The results we want to discuss here are the following.

Theorem 6.2. *Let $n \geq 6$ and $\ell \geq 2$. Then there exists a family of smooth metrics $\overline{g}_\varepsilon$ on S^n, converging (in $C^\infty(S^n)$) to $\overline{g}_0$ as $\varepsilon \to 0$ such that, for every ε small enough, problem (6.7) on $(S^n, \overline{g}_\varepsilon)$ possesses at least ℓ solutions.*

Theorem 6.3. *Let $k \geq 2$ and $n \geq 4k+3$. Then there exists a family of C^k metrics $\overline{g}_\varepsilon$ on S^n, with $\|\overline{g}_\varepsilon - \overline{g}_0\|_{C^k(S^n)} \to 0$ as $\varepsilon \to 0$, which has the following property. For every ε small enough, problem (6.7) on $(S^n, \overline{g}_\varepsilon)$ possesses a sequence of solutions v_ε^i with $\|v_\varepsilon^i\|_{L^\infty(S^n)} \to +\infty$ as $i \to \infty$.*

6.2 Some geometric preliminaries

In order to study problem (6.7), it is useful to understand how the Sobolev spaces are affected by a conformal change of the metric. Let $\tilde{g} = \varphi^{\frac{4}{n-2}} g$, $\varphi \geq 0$, and for $u \in H^1(M)$, define the function $\tilde{u} : M \to \mathbb{R}$ by

$$\tilde{u}(x) = \varphi(x)^{-1}\, u(x). \tag{6.10}$$

It is easy to check that the following relations hold

$$\int_M u^{2^*}\, dV_g = \int_M \tilde{u}^{2^*}\, dV_{\tilde{g}}, \qquad \forall u \in H^1(M); \tag{6.11}$$

$$\begin{aligned} &\int_M (2c_n \nabla_g u \cdot \nabla_g v + R_g\, uv)\, dV_g \\ &\quad = \int_M (2c_n \nabla_{\tilde{g}} u \cdot \nabla_{\tilde{g}} v + R_{\tilde{g}}\, uv)\, dV_{\tilde{g}}, \qquad \forall u, v \in H^1(M). \end{aligned} \tag{6.12}$$

The first equation is an easy consequence of the relation $|dV_{g'}| = \varphi^{2^*}\, |dV_g|$, while the second can be achieved using (6.5) and integrating by parts.

The map π will denote the stereographic projection

$$\pi : S^n = \left\{x \in \mathbb{R}^{n+1} : \ |x| = 1\right\} \to \mathbb{R}^n$$

through the north pole P_N of S^n, $P_N = (0, \dots, 0, 1)$, where we identify $\mathbb{R}^n$ with $\left\{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\right\}$. Letting $(x', x_{n+1}) \in S^n$, $x' = x_1, \dots, x_n$, the explicit expression of π is given by

$$\pi(x', x_{n+1}) = \left(x', \frac{1 + x_{n+1}}{|x'|^2}\right); \qquad (x', x_{n+1}) \in S^n,$$

while for the inverse map there holds

$$\pi^{-1}(x) = \left(\frac{2x}{1 + |x|^2}, \frac{|x|^2 - 1}{1 + |x|^2}\right); \qquad x \in \mathbb{R}^n.$$

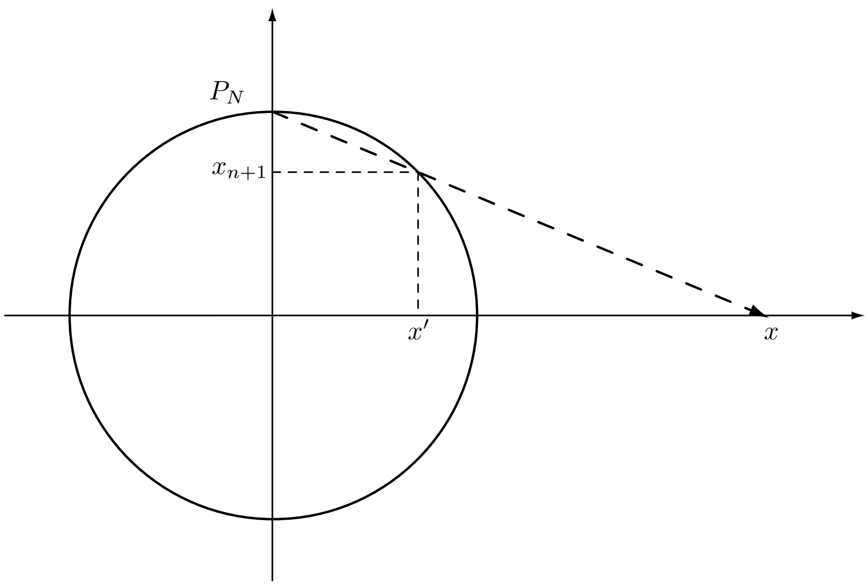

Figure 6.1. The stereographic projection $(x = \pi(x', x_{n+1}))$

The stereographic projection π is a *conformal map*, namely the pull-back $(\pi^{-1})^* \overline{g}_0$ of the standard metric on S^n is conformal to the standard metric dx^2 in $\mathbb{R}^n$. It follows that

$$(\pi^{-1})^* \overline{g}_0 = z_0(x)^{\frac{4}{n-2}} dx^2, \tag{6.13}$$

and one can check with straightforward computations that the explicit expression of z_0 is the following

$$z_0(x) = \kappa_n \frac{1}{(1 + |x|^2)^{\frac{n-2}{2}}}, \qquad \kappa_n = 4^{\frac{n-2}{4}}. \tag{6.14}$$

Since the scalar curvature of $(S^n, \overline{g}_0)$ is $n(n-1)$, which is also the scalar curvature of the pull-back $(\mathbb{R}^n, (\pi^{-1})^* \overline{g}_0)$, by equation (6.5) the function z_0 satisfies the equation

$$-2c_n \Delta z_0 = n(n-1) z_0^{\frac{n+2}{n-2}}; \qquad \text{in } \mathbb{R}^n. \tag{6.15}$$

Even if $(\mathbb{R}^n, g_0)$ is not compact, it is possible to reason as in (6.10), (6.11), and to prove that the stereographic projection π induces an isomorphism $\iota : H^1(S^n) \to \mathcal{D}^{1,2}(\mathbb{R}^n)$ defined by

$$(\iota u)(x) = z_0(x) u(\pi^{-1}(x)), \qquad u \in H^1(S^n), \quad x \in \mathbb{R}^n. \tag{6.16}$$

In particular the following relations hold for every $u, v \in H^1(S^n)$

$$\begin{cases} 2c_n \int_{\mathbb{R}^n} \nabla \iota u \cdot \nabla \iota v = \int_{S^n} \left(2c_n \nabla_{g_0} u \cdot \nabla_{g_0} v + n(n-1) uv\right) dV_{g_0}, \\ \int_{\mathbb{R}^n} (\iota u)^{2^*-1} \iota v = \int_{S^n} u^{2^*-1} v. \end{cases} \tag{6.17}$$

Let $\mathcal{R} : S^n \to S^n$ be the reflection through the hyperplane $\{x_{n+1} = 0\}$. Namely, given $(x', x_{n+1}) \in S^n$, one has $\mathcal{R}(x', x_{n+1}) = (x', -x_{n+1})$. In stereographic coordinates, this map corresponds to the Kelvin transform

$$x \to \frac{x}{|x|^2}, \qquad x \in \mathbb{R}^n. \tag{6.18}$$

Given a function $v : \mathbb{R}^n \to \mathbb{R}$, we define $v^\sharp : \mathbb{R}^n \to \mathbb{R}$ in the following way

$$v^\sharp(x) = v\left(\frac{x}{|x|^2}\right), \qquad x \in \mathbb{R}^n,$$

and for $u \in \mathcal{D}^{1,2}(\mathbb{R}^n)$, the function $u^* \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ is defined as

$$u^*(x) = \frac{1}{|x|^{n-2}} u\left(\frac{x}{|x|^2}\right), \qquad x \in \mathbb{R}^n.$$

One can check that the following relations hold

$$\iota(\mathcal{R}^* v) = (\iota v)^*, \qquad v \in H^1(S^n); \tag{6.19}$$

$$\int_{\mathbb{R}^n} K\, u^{2^*-1} v = \int_{\mathbb{R}^n} K^\sharp\, (u^*)^{2^*-1} v^*, \qquad K \in L^\infty(\mathbb{R}^n), u, v \in \mathcal{D}^{1,2}(\mathbb{R}^n). \tag{6.20}$$

For every $u \in \mathcal{D}^{1,2}(\mathbb{R}^n)$, $\mu \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$ we set $u_{\mu,\xi} = \mu^{-\frac{n-2}{2}} u\left(\frac{x-\xi}{\mu}\right)$. For the specific case of $u = z_0$ we use the notation

$$z_{\mu,\xi} = \mu^{-\frac{n-2}{2}} z_0\left(\frac{x-\xi}{\mu}\right), \qquad \mu, \in \mathbb{R}_+, \xi \in \mathbb{R}^n. \tag{6.21}$$

One can check with simple computations that

$$(z_{\mu,\xi})^* = z_{\overline{\mu},\overline{\xi}}, \qquad \text{with} \quad \overline{\mu} = \frac{\mu}{\mu^2+\xi^2}, \, \overline{\xi} = \frac{\xi}{\mu^2+\xi^2}. \tag{6.22}$$

Consider now the sphere S^n endowed with a Riemannian metric $\overline{g}$ (which is not necessarily the standard one). Next we describe how problem (6.7) (and also problem (7.1) below) can be reduced, with the stereographic projection, to a problems in $\mathbb{R}^n$. The Euler functional $J_{\overline{g}} : H^1(S^n) \to \mathbb{R}$ associated to (6.7) for the present case is

$$J_{\overline{g}}(v) = \int_{S^n} \left(c_n |\nabla_{\overline{g}} v|^2 + \frac{1}{2} R_{\overline{g}} v^2 - \frac{n(n-1)}{2^*} |v|^{2^*} \right) dV_{\overline{g}}, \qquad v \in H^1(S^n). \tag{6.23}$$

Using stereographic coordinates on S^n, we define the metric g on $\mathbb{R}^n$ as

$$g_{ij}(x) = z_0^{-\frac{4}{n-2}}(x) \cdot \overline{g}_{ij}(x) \tag{6.24}$$

and, associated to g, the following functional $I_g : \mathcal{D}^{1,2}(\mathbb{R}^n) \to \mathbb{R}$

$$I_g(u) = \int_{\mathbb{R}^n} \left(c_n |\nabla_g u|^2 + \frac{1}{2} R_g u^2 - \frac{n(n-1)}{2^*} |u|^{2^*} \right) dV_g, \qquad u \in \mathcal{D}^{1,2}(\mathbb{R}^n). \tag{6.25}$$

$J_{\overline{g}}$ is related to I_g by the equation

$$J_{\overline{g}}(u) = I_g(\iota u), \qquad u \in H^1(S^n). \tag{6.26}$$

Hence it is equivalent to study either the functional I_g or the functional $J_{\overline{g}}$. We also describe how the metric g in $\mathbb{R}^n$ given by (6.24) changes when $\overline{g}$ is transformed into $\mathcal{R}^*\overline{g}$. Letting $\overline{g}_{\mathcal{R}}$ denote the pull-back of $\overline{g}$ through $\mathcal{R}$, its transposition on $\mathbb{R}^n$ is given by

$$g^\sharp_{ij}(x) := z_0^{-\frac{4}{n-2}}(x) (\overline{g}_{\mathcal{R}})_{ij}(x), \qquad x \in \mathbb{R}^n, \tag{6.27}$$

where

$$\begin{aligned} \sum_{ij} g^\sharp_{ij}(x)\, dx_i\, dx_j = \delta_{ij} dx_i dx_j &+ \sum_{ij} \left(g_{ij}\left(\frac{1}{x}\right) - \delta_{ij} \right) \\ &\times \left(dx_i - \frac{2\, x_i \sum_k x_k\, dx_k}{|x|^2} \right) \left(dx_j - \frac{2\, x_j \sum_l x_l\, dx_l}{|x|^2} \right). \end{aligned} \tag{6.28}$$

6.3 First multiplicity results

In this section we prove Theorem 6.2. We consider a metric $g = g_\varepsilon = g_0 + \varepsilon h$ on $\mathbb{R}^n$ which is close to the standard one, where $h = (h_{ij})$ is some symmetric bilinear form with compact support. Working in stereographic coordinates and using (6.24), we obtain the corresponding metric $\overline{g}$ on S^n. Therefore we are reduced to find solutions of the following problem

$$-2c_n \Delta_{g_\varepsilon} u + R_{g_\varepsilon} u = n(n-1) u^{\frac{n+2}{n-2}} \qquad \text{in } \mathbb{R}^n. \tag{6.29}$$

Solutions of (6.29) can be found as critical points of the functional $I_\varepsilon = I_{g_\varepsilon}$ defined in (6.25). We show that this case requires the specialized setting of Theorem 2.20, since the first term in the expansion of I_{g_ε} in ε vanishes identically, see Proposition 6.6. Some computations here will be sketchy, hence we often refer to [19], or to [109].

6.3.1 Expansions of the functionals

In this subsection we perform the expansion in ε of the functional $I_\varepsilon : \mathcal{D}^{1,2}(\mathbb{R}^n) \to \mathbb{R}$ associated to the metric $g = g_\varepsilon = g_0 + \varepsilon h$. We recall that the bilinear form h has compact support in $\mathbb{R}^n$. We have first the following expansion in ε of the scalar curvature.

Lemma 6.4. *If $g_\varepsilon = g_0 + \varepsilon h$, and if R_{g_ε} denotes the scalar curvature of g_ε, then one has*

$$R_g(x) = \varepsilon\, R_1(x) + \varepsilon^2\, R_2(x) + o(\varepsilon^2),$$

where

$$R_1 = \sum_{i,j} D^2_{ij} h_{ij} - \Delta \operatorname{tr}\ h; \tag{6.30}$$

and

$$\begin{aligned} R_2 = &-2 \sum_{k,j,l} h_{kj} D^2_{lk} h_{lj} + \sum_{k,j,l} h_{kj} D^2_{ll} h_{kj} + \sum_{k,j,l} h_{kj} D^2_{jk} h_{ll} + \frac{3}{4} \sum_{k,j,l} D_k h_{jl} D_k h_{jl} \\ &- \sum_{k,j,l} D_l h_{jl} D_k h_{jk} + \sum_{k,j,l} D_l h_{jl} D_j h_{kk} - \frac{1}{4} \sum_{k,j,l} D_j h_{ll} D_j h_{kk} - \frac{1}{2} \sum_{k,j,l} D_j h_{lk} D_l h_{jk}. \end{aligned}$$

Proof. Writing $g^{-1} = I + \varepsilon A + \varepsilon^2 B$, from the relation

$$(I + \varepsilon h)(I + \varepsilon A + \varepsilon^2 B) = I + o(\varepsilon^2),$$

we obtain immediately

$$(g_\varepsilon)^{ij} = \delta_{ij} - \varepsilon\, h_{ij} + \varepsilon^2 \sum_s h_{is} h_{sj}. \tag{6.31}$$

Then the conclusion follows from the expression of the Christoffel symbols and (6.2). $\square$

In the sequel all the integrals are understood to be on $\mathbb{R}^n$ unless specified.

Lemma 6.5. *If $g_\varepsilon = g_0 + \varepsilon h$, then one has*

$$I_\varepsilon(u) = I_0(u) + \varepsilon\, G_1(u) + \varepsilon^2\, G_2(u) + o(\varepsilon^2), \tag{6.32}$$

where

$$I_0(u) = \frac{c_n}{2}\int |\nabla u|^2 dx - \frac{n(n-1)}{2^*}\int |u|^{2^*} dx; \tag{6.33}$$

$$\begin{aligned} G_1(u) = \int \Bigg(& - c_n \sum_{i,j} h_{ij} D_i u D_j u + \frac{1}{2} R_1 u^2 \\ & + \left(c_n |\nabla u|^2 - \frac{n(n-1)}{2^*}|u|^{2^*}\right) \frac{1}{2}\,\mathrm{tr}\, h \Bigg) dx, \end{aligned} \tag{6.34}$$

$$\begin{aligned} G_2(u) = \int \Bigg[& c_n \sum_{i,j,l} h_{il} h_{lj} D_i u D_j u + \frac{1}{2} R_2 u^2 \\ & + \left(c_n |\nabla u|^2 - \frac{n(n-1)}{2^*}|u|^{2^*}\right) \left(\frac{1}{8}(\mathrm{tr}\ h)^2 - \frac{1}{4}\,\mathrm{tr}(h^2)\right) \\ & + \frac{1}{2}\,\mathrm{tr}\, h \left(\frac{1}{2} R_1 u^2 - c_n \sum_{i,j} h_{ij} D_i u D_j u \right) \Bigg] dx. \end{aligned} \tag{6.35}$$

Proof. First we expand in powers of ε the term $|\nabla_{g_\varepsilon} u|^2$, which is given by $|\nabla_{g_\varepsilon} u|^2 = \sum_{i,j} (g_\varepsilon)^{ij} D_i u D_j u$. Using (6.31) we obtain

$$|\nabla_{g_\varepsilon} u|^2 = |\nabla u|^2 - \varepsilon \sum_{i,j} h_{ij} D_i u D_j u + \varepsilon^2 \sum_{i,j,l} h_{il} h_{lj} D_i u D_j u + o(\varepsilon^2). \tag{6.36}$$

In order to evaluate the volume element $dV_{g_\varepsilon} = |g_\varepsilon|^{1/2} dx$, let us expand first $|g_\varepsilon|$ in power series. Consider the determinant of the matrix

$$\begin{pmatrix} 1 + \varepsilon h_{11} & \varepsilon h_{12} & \cdots \\ \varepsilon h_{21} & 1 + \varepsilon h_{22} & \cdots \\ \cdots & \cdots & \ddots \end{pmatrix}.$$

Its linear part in ε is tr h, while its quadratic part is $\frac{1}{2}\left(\sum_{i \neq j} h_{ii} h_{jj} - \sum_{i\neq j} h_{ij} h_{ji}\right)$, which coincides with $\frac{1}{2}\left((\mathrm{tr}\ h)^2 - \mathrm{tr}(h^2)\right)$. Then we obtain

$$|g_\varepsilon|^{\frac{1}{2}} = 1 + \frac{\varepsilon}{2}\,\mathrm{tr}\, h + \varepsilon^2 \left(\frac{1}{8}(\mathrm{tr}\ h)^2 - \frac{1}{4}\,\mathrm{tr}(h^2)\right) + o(\varepsilon^2). \tag{6.37}$$

Now, using (6.36) and (6.37), we can write

$$\begin{aligned}
I_\varepsilon(u) = \int \Bigg(c_n \Big(|\nabla u|^2 - \varepsilon \sum_{i,j} h_{ij} D_i u D_j u + \varepsilon^2 \sum_{i,j,l} h_{il} h_{lj} D_i u D_j u \Big) \\
+ \frac{1}{2}(\varepsilon R_1 + \varepsilon^2 R_2) u^2 - \frac{n(n-1)}{2^*} |u|^{2^*} \Bigg) \\
\times \left(1 + \frac{\varepsilon}{2} \operatorname{tr} h + \varepsilon^2 \left(\frac{1}{8} (\operatorname{tr} h)^2 - \frac{1}{4} \operatorname{tr}(h^2) \right) \right) dx + o(\varepsilon^2).
\end{aligned}$$

Taking the coefficients of ε and ε^2 the conclusion follows. $\square$

6.3.2 The finite-dimensional functional

We start by studying the perturbation term G_1.

Proposition 6.6. *The functional G_1 given in* (6.34) *satisfies*

$$G_1(z) = 0, \qquad \text{for every } z \in Z.$$

Proof. From the expression of z_0 in (6.14) we deduce

$$D_i z_{\mu,\xi} = (2-n)\mu^{-\frac{n}{2}-1} \frac{\kappa_n}{\left(1 + \left|\frac{y-\xi}{\mu}\right|^2\right)^{\frac{n}{2}}} (x_i - \xi_i); \tag{6.38}$$

$$D_{ij} z_{\mu,\xi} = (2-n)\mu^{-\frac{n}{2}-1} \frac{\kappa_n \delta_{ij}}{\left(1 + \left|\frac{y-\xi}{\mu}\right|^2\right)^{\frac{n}{2}}} + n(n-2)\mu^{-\frac{n}{2}-3} \frac{\kappa_n (x_i - \xi_i)(x_j - \xi_j)}{\left(1 + \left|\frac{y-\xi}{\mu}\right|^2\right)^{\frac{n}{2}+1}}. \tag{6.39}$$

Therefore

$$z_{\mu,\xi} D_{ij} z_{\mu,\xi} = (2-n)\mu^{-n} \frac{\kappa_n^2 \delta_{ij}}{\left(1 + \left|\frac{y-\xi}{\mu}\right|^2\right)^{n-1}} + \frac{n}{n-2} D_i z_{\mu,\xi} D_j z_{\mu,\xi}. \tag{6.40}$$

Using (6.30) and integrating by parts, we obtain

$$\begin{aligned}
\int R_1(x) z_{\mu,\xi}^2(x) dx = \int \sum_{i,j} h_{ij}(x) \left(2 D_i z_{\mu,\xi} D_j z_{\mu,\xi} + 2 z_{\mu,\xi} D_{ij}^2 z_{\mu,\xi} \right) \\
+ \int \operatorname{tr} h(x) \left(2 z_{\mu,\xi} \Delta z_{\mu,\xi} - 2 |\nabla z_{\mu,\xi}|^2 \right) dx.
\end{aligned}$$

From the fact that z_0 solves (6.15), and from (6.40) we deduce the equality

$$\begin{aligned}&\int R_1(x) z^2_{\mu,\xi}(x)dx\\&= \int \sum_{i,j} h_{ij}(x)\left(2\left(1+\frac{n}{n-2}\right) D_i z_{\mu,\xi} D_j z_{\mu,\xi} + \frac{2(2-n)\mu^{-n}\kappa_n^2\delta_{ij}}{\left(1+\left|\frac{y-\xi}{\mu}\right|^2\right)^{n-1}}\right)dx\\&\quad+\int \operatorname{tr} h(x)\left(\frac{n(n-1)}{c_n} z^{2^*}_{\mu,\xi} - 2|\nabla z_{\mu,\xi}|^2\right)dx,\end{aligned}$$

which inserted in (6.34) yields

$$\begin{aligned}&G_1(z_{\mu,\xi})\\&=\frac{1}{2}\int \operatorname{tr} h\left(\frac{2}{n-2}|\nabla z_{\mu,\xi}|^2 + \frac{n-2}{2}|z_{\mu,\xi}|^{2^*} + \frac{2(2-n)\kappa_n^2\mu^{-n}}{\left(1+\left|\frac{y-\xi}{\mu}\right|^2\right)^{n-1}}\right)dx\\&=\frac{1}{2}\int \operatorname{tr} h \frac{(n-2)\kappa_n^2\mu^{-n}}{\left(1+\left|\frac{y-\xi}{\mu}\right|^2\right)^{n}}\left(2\left|\frac{y-\xi}{\mu}\right|^2 + \frac{4n(n-1)}{2n(n-1)} - 2\left(1+\left|\frac{y-\xi}{\mu}\right|^2\right)\right)dx = 0.\end{aligned}$$

This concludes the proof. □

According to Proposition 6.6 we need to apply Theorem 2.20 with the functional $\widetilde{\Gamma}$ given in (2.27). In this specific case we have

$$\widetilde{\Gamma}(\mu,\xi) = G_2(z_{\mu,\xi}) + \frac{1}{2}(G_1'(z_{\mu,\xi})|\overline{w}_{\mu,\xi}), \tag{6.41}$$

where $\overline{w}_{\mu,\xi} = \lim_{\varepsilon\to 0}\varepsilon^{-1}w_{\mu,\xi}$, see Lemma 2.18.

In order to find critical points of $\widetilde{\Gamma}$ it is convenient to study its behavior as $\mu \to 0$ and as $\mu + |\xi| \to \infty$.

Proposition 6.7. *$\widetilde{\Gamma}(\mu,\xi) \to 0$ as $\mu \to 0^+$. Hence $\widetilde{\Gamma}$ can be extended continuously to the hyperplane $\{(\mu,\xi)\,|\,\mu = 0\}$ by setting*

$$\widetilde{\Gamma}(0,\xi) = 0. \tag{6.42}$$

In the sequel, this extension will be still denoted by $\widetilde{\Gamma}$. Moreover there holds

$$\widetilde{\Gamma}(\mu,\xi) \to 0, \qquad \text{as } \mu + |\xi| \to +\infty. \tag{6.43}$$

Proof. We omit some of the details, for which we refer to [19]. First of all, by a change of variables and some direct computation, one finds the limit of G_2 as $\mu \to 0^+$ is given by

$$\lim_{\mu\to 0^+} G_2(z_{\mu,\xi}) = \kappa_n^2(n-1)^2(n-2)\left(\operatorname{tr}(h^2) - \frac{1}{2}(\operatorname{tr} h)^2\right)(\xi)\int_{\mathbb{R}^n}\frac{|x|^2}{(1+|x|^2)^n}dx. \tag{6.44}$$

For the second term in $\widetilde{\Gamma}$ we have $(G_1'(z_{\mu,\xi}), \bar{w}) = \alpha_1 + \alpha_2$, where

$$\alpha_1 = \int \frac{1}{2} \operatorname{tr} h \left(2c_n \langle \nabla z_{\mu,\xi}, \nabla \bar{w}_{\mu,\xi} \rangle - n(n-1)|z_{\mu,\xi}|^{2^*-1} \bar{w}_{\mu,\xi} \right) dx;$$

$$\alpha_2 = \int \left(-2c_n \sum_{ij} h_{ij} D_i z_{\mu,\xi} D_j \bar{w}_{\mu,\xi} + Rz\bar{w} \right) dx.$$

It is convenient to introduce $w^*(y) = w^*_{\mu,\xi}(y)$ by setting

$$w^*(y) = \mu^{\frac{n-2}{2}} \bar{w}_{\mu,\xi}(\mu y + \xi).$$

Then, a change of variable yields

$$\begin{aligned} \alpha_1 &= \int \frac{1}{2} \operatorname{tr} h(\mu y + \xi) \left(2c_n \langle \nabla z_0(y), \nabla w^*(y) \rangle - n(n-1)|z_0(y)|^{2^*-1} w^*(y) \right) dy \\ \alpha_2 &= \int \left(-2c_n \sum_{ij} h_{ij}(\mu y + \xi) D_i z_0(y) D_j w^*(y) \right) dy \\ &\qquad + \mu^2 \int R(\mu y + \xi) z_0(y) w^*(y) dy. \end{aligned} \tag{6.45}$$

Using the fact that $L_{z_{\mu,\xi}} \overline{w}_{\mu,\xi} = -G_1'(z_{\mu,\xi})$, we obtain a linear elliptic partial differential equation for $\overline{w}_{\mu,\xi}$, which is solved explicitly in [19], yielding

$$w^*_{\mu,\xi}(y) \to w_0(y) \quad \text{as } \mu \to 0^+, \tag{6.46}$$

where, setting $c_n' = c_n \kappa_n \frac{(n-2)^2}{4(n-1)}$,

$$w_0(y) = -\frac{c_n'}{(1+|y|^2)^{\frac{n}{2}}} \sum_{j,k} h_{jk} y_j y_k. \tag{6.47}$$

Then, from (6.46) and some elementary computations one finds

$$\lim_{\mu \to 0^+} \alpha_1 = \frac{1}{2} \operatorname{tr} h(\xi) \int \left(2c_n \langle \nabla z_0, \nabla w^* \rangle - n(n-1)|z_0|^{2^*-1} w^* \right) dy = 0;$$

$$\lim_{\mu \to 0^+} \alpha_2 = -2\kappa_n^2 (n-1)^2 (n-2) \left(\operatorname{tr}(h^2) - \frac{1}{2} (\operatorname{tr} h)^2 \right) \int_{\mathbb{R}^n} \frac{|x|^2}{(1+|x|^2)^n} dx.$$

The last two equations, together with (6.44), imply $\widetilde{\Gamma}(\mu, \xi) \to 0$ as $\mu \to 0^+$.

We now prove (6.43). Let $g_\varepsilon^\sharp$ be the metric given by (6.28), and consider the corresponding functional $I_{g_\varepsilon^\sharp}$. Similarly, let us consider $G_i^\sharp(u)$, $i = 1, 2$, etc. Letting

$$u^*(x) = |x|^{2-n} u \left(\frac{x}{|x|^2} \right),$$

it is easy to check from (6.26) that $I_{g^\sharp}(u) = I_g(u^*)$, $G_i^\sharp(u) = G_i(u^*)$, and $\widetilde{\Gamma}^\sharp(z) = \widetilde{\Gamma}(z^*)$. This in terms of coordinates (μ, ξ) becomes

$$\widetilde{\Gamma}(\mu, \xi) = \widetilde{\Gamma}^\sharp \left(\frac{\mu}{\mu^2 + |\xi|^2}, \frac{\xi}{\mu^2 + |\xi|^2} \right).$$

Finally one finds

$$\lim_{\mu + |\xi| \to \infty} \widetilde{\Gamma}(\mu, \xi) = \widetilde{\Gamma}^\sharp(0,0) = 0,$$

proving (6.43). □

Given a metric g of the form $g_e = g_0 + \varepsilon h$ (h with compact support), let W_ε denote the corresponding Weyl tensor. Expanding W_ε with respect to ε one finds

$$W_\varepsilon = \varepsilon \overline{W}_h + o(\varepsilon), \tag{6.48}$$

where $\overline{W}_h(x)$ is a tensor depending only on the second derivatives $D^2_{kl} h_{ij}(x)$. In [19], see also [109], it is proved the following result.

Proposition 6.8. *For $n > 6$, and for $g_\varepsilon = g_0 + \varepsilon h$ there holds*

$$\frac{\partial \widetilde{\Gamma}}{\partial \mu}(0, \xi) = 0, \quad \frac{\partial^2 \widetilde{\Gamma}}{\partial \mu^2}(0, \xi) = 0, \quad \frac{\partial^3 \widetilde{\Gamma}}{\partial \mu^3}(0, \xi) = 0, \qquad \forall \xi \in \mathbb{R}^n; \tag{6.49}$$

$$\frac{1}{4!} \frac{\partial^4 \widetilde{\Gamma}}{\partial \mu^4}(0, \xi) = - \sum_{i,j,k,l} c_{i,j,k,l} |\overline{W}_{ijkl}(\xi)|^2 \quad \forall \xi \in \mathbb{R}^n, \tag{6.50}$$

where $c_{i,j,k,l} > 0$. Furthermore, for $n = 6$ one has $\lim_{\mu \to 0^+} \frac{\widetilde{\Gamma}(\mu,\xi)}{\mu^4} = -\infty$ whenever $\overline{W}(\xi) \neq 0$.

It is worth mentioning that the above equations (6.49), (6.50) are obtained evaluating limits of the form $\lim_{\mu \to 0} \frac{\widetilde{\Gamma}(\mu,\xi)}{\mu^m}$, for $\mu = 1, \dots, 4$. These do not require to prove higher differentiability properties of $\overline{w}_{\mu,\xi}$ with respect to μ, but only the property (6.46).

Remarks 6.9. (i) The condition $\overline{W}_h \not\equiv 0$ is generic.

(ii) Suppose $n \geq 6$ and that $\overline{W}_h \not\equiv 0$. Then $\widetilde{\Gamma}$ achieves a minimum and hence we recover existence of the Yamabe problem for ε small.

(iii) The fact the $\widetilde{\Gamma}$ has a minimum when the Weyl tensor does not vanish can be related to the existence result of Aubin, which relies on minimizing the Sobolev quotient

$$Q(u) = \frac{\int_M c_n |\nabla_g u|^2 + \frac{1}{2} R_g u^2}{\|u\|^2_{2^*}}.$$

This can be done by testing the quotient on an appropriate function $u \in H^1(M)$ which is peaked near a point where the Weyl tensor does not vanish. □

6.3.3 Proof of Theorem 6.2

We consider in $\mathbb{R}^n$ a metric of the form

$$g_\varepsilon = g_0 + \varepsilon h(x) + \varepsilon h(x - x_0),$$

where, as before, h is a symmetric bilinear form with compact support, and $x_0 \in \mathbb{R}^n$ is a vector with large modulus.

We denote by $G_1^{x_0}, G_2^{x_0}$, $\widetilde{\Gamma}^{x_0}$, the functionals obtained from the translated perturbation $h(\cdot - x_0)$, and by G_1^*, etc., those obtained from the perturbation $h(\cdot) + h(\cdot - x_0)$. It is clear that

$$G_i^{x_0}(z_{\mu,\xi}) = G_i(z_{\mu,\xi - x_0}); \qquad i = 1, 2; \tag{6.51}$$

$$\widetilde{\Gamma}^{x_0}(\mu, \xi) = \widetilde{\Gamma}(\mu, \xi - x_0). \tag{6.52}$$

If $|x_0|$ is large enough, the supports of h and $h(\cdot - x_0)$ are disjoint, hence it follows that

$$G_i^*(z_{\mu,\xi}) = G_i(z_{\mu,\xi}) + G_i^{x_0}(z_{\mu,\xi - x_0}); \qquad i = 1, 2. \tag{6.53}$$

the same is true for ∇G_i. We need now the following result.

Lemma 6.10. *If G_1 and $G_1^{x_0}$ are as above, then there holds ($C_1, C_2 > 0$)*

$$\|\nabla G_1(z)\|, \|\nabla G_1^{x_0}(z)\| \to 0 \qquad \text{as } \mu \to +\infty, \ \text{uniformly in } \xi. \tag{6.54}$$

Proof. We denote by A the support of $h(\cdot)$. By (6.34) there holds

$$\begin{aligned} |(\nabla G_1(z), v)| &= \Bigg| -2c_n \int_A \sum_{i,j} h_{ij} D_i z D_j v + \int_A R_1 z v \\ &\qquad + \int_A \frac{1}{2} \operatorname{tr} h \left(2c_n \langle \nabla z, \nabla v \rangle - n(n-1)|z|^{2^*-1} v \right) \Bigg| \\ &\le C_1 \|h\|_\infty \|\nabla z\|_\infty \int_A |\nabla v| + \|R_1\|_\infty \|z\|_\infty \int_A |v| \\ &\qquad + C_2 \|h\|_\infty \left(\|\nabla z\|_\infty \int_A |\nabla v| + \|z\|_\infty^{2^*-1} \int_A |v| \right). \end{aligned}$$

Using the Hölder and the Sobolev inequalities we obtain

$$|(\nabla G_1(z), v)| \le C_3 \left(\|\nabla z\|_\infty + \|z\|_\infty + \|z\|_\infty^{2^*-1} \right) \|v\|,$$

for some $C_3 > 0$. Since $\|\nabla z\|_\infty, \|z\|_\infty \to 0$ when $\mu \to +\infty$, we find immediately $\|\nabla G_1(z)\| \to 0$. The same holds for $\nabla G_1^{x_0}(z)$. $\square$

In order to find a similar expression for $\widetilde{\Gamma}$, the following lemma is in order.

Lemma 6.11. *Given $M > 0$, there holds*

$$\|\nabla G_1(z)\| \, \|\nabla G_1^{x_0}(z)\| \to 0, \quad \text{as } |x_0| \to \infty, \tag{6.55}$$

uniformly in (μ, ξ), $\mu \le M$.

Proof. We have the estimate

$$|(\nabla G_1(z), v)| \le C_1 \|h\|_\infty \int_A |\nabla z||\nabla v| + \|R_1\|_\infty \int_A |v||z| + C_1 \|h\|_\infty \int_A |\nabla z||\nabla v| + C_1 \|h\|_\infty \int_A |z|^{2^*-1}|v|.$$

Using again the Hölder and the Sobolev inequalities, we find $|(\nabla G_1(z), v)| \le C_2 \|z\| \, \|v\|$ for some fixed $C_2 > 0$, so it is sufficient to show that

$$\min\{\|\nabla G_1(z)\|, \|\nabla G_1^{x_0}(z)\|\} \to 0 \text{ as } |x_0| \to \infty, \tag{6.56}$$

uniformly in (μ, ξ), $\mu \le M$. Looking at the expression of $z_{\mu,\xi}$ we deduce that for every $\eta > 0$ there exists $R > 0$ such that

$$|\nabla z_{\mu,0}(x)|, |z_{\mu,0}(x)| \le \eta, \qquad \text{for } |x| \ge R, \mu \le M. \tag{6.57}$$

Using the change of variables $y = x - \xi$, we find

$$\begin{aligned} \nabla G_1(z)[v] = & -2c_n \int_{A-\xi} \sum_{i,j} h_{ij}(y+\xi) D_i z_{\mu,0}(y) D_j v(y+\xi) dy \\ & + \int_{A-\xi} R_1(y+\xi) z_{\mu,0}(y) v(y+\xi) dy \\ & + \frac{1}{2} \int_{A-\xi} \operatorname{tr} h(y+\xi) \Big(2c_n \langle \nabla z_{\mu,0}(y), \nabla v(y) \rangle \\ & \qquad - n(n-1) |z_{\mu,0}(y)|^{2^*-2} z_{\mu,0}(y) v(y) \Big) dy. \end{aligned}$$

If $\operatorname{dist}(\xi, A) \ge R$ and if $\mu \le M$ then, using (6.57), the Hölder and the Sobolev inequalities we get

$$|(\nabla G_1(z), v)| \le C_3(\eta + \eta^{2^*-1}) \|v\|,$$

for some $C_3 > 0$. Since the above estimate is uniform in v, it follows that

$$\|\nabla G_1(z)\| \le C_3(\eta + \eta^{2^*-1}), \qquad \text{for } \operatorname{dist}(\xi, A) \ge R, \, \mu \le M,$$

as well as

$$\|\nabla G_1^{x_0}(z)\| \le C_3(\eta + \eta^{2^*-1}), \qquad \text{for } \operatorname{dist}(\xi - x_0, A) \ge R, \mu \le M.$$

When $|x_0|$ is large enough, it is always $\operatorname{dist}(\xi, A) \ge R$ or $\operatorname{dist}(\xi - x_0, A) \ge R$, and hence

$$\min\{\|\nabla G_1(z)\|, \|\nabla G_1^{x_0}(z)\|\} \le C_3(\eta + \eta^{2^*-1}).$$

By the arbitrarity of η, (6.56) follows. $\square$

Using the boundedness of L_z, Lemma 6.10 and Lemma 6.11 we finally deduce the decay

$$(L_z \nabla G_1(z), \nabla G_1^{x_0}(z)) \to 0 \quad \text{as } |x_0| \to \infty, \tag{6.58}$$

uniformly for $z \in Z$. Finally, from (6.51), (6.53) and (6.58) we infer this characterization of the finite-dimensional functional $\widetilde{\Gamma}^*$.

Lemma 6.12. *In the above notation there holds*

$$\widetilde{\Gamma}^*(\mu, \xi) = \widetilde{\Gamma}(\mu, \xi) + \widetilde{\Gamma}(\mu, \xi - x_0) + o(1),$$

where $o(1) \to 0$ *as* $|x_0| \to \infty$*, uniformly in* (μ, ξ).

Proof of Theorem 6.2. From Remark 6.9 it follows that $\widetilde{\Gamma}$ achieves a minimum at some point (μ_1, ξ_1). On the other hand from (6.52) we know that $\widetilde{\Gamma}^{x_0}$ achieves a minimum at $(\mu_1, \xi_1 + x_0)$. From Lemma 6.12 we infer that for $|x_0|$ sufficiently large there exists $\delta > 0$ such that the sublevel $\{\widetilde{\Gamma}^* < -\delta\}$ is disconnected, namely $\{\widetilde{\Gamma}^* < -\delta\} = A_1 \cup A_2$ with $A_1 \cap A_2 = \emptyset$. Applying the abstract result of Theorem 2.20, it follows that the two distinct minima of $\widetilde{\Gamma}^*$ give rise to two distinct solutions of (6.29). This concludes the proof. $\square$

6.4 Existence of infinitely-many solutions

This section is devoted to the proof of Theorem 6.3, which involves several technical lemmas. Therefore, for the reader's convenience, we will indicate the main steps of the arguments, postponing the technical details to an appendix.

We will consider metrics on $\mathbb{R}^n$ possessing *infinitely many bumps*. In order to describe precisely such metrics we introduce some notation. Let $\tau : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ symmetric bilinear form with compact support, satisfying $\overline{W}_\tau \not\equiv 0$, see formula (6.48). For $A > 0$, let $\mathcal{H}_A \subseteq \mathcal{S}_n$ be defined by

$$\mathcal{H}_A = \left\{ h(x) = \sum_{i \in \mathbb{N}} \sigma_i \tau(x - x_i), |x_i - x_j| \geq 4 \operatorname{diam}(\operatorname{supp} \tau), i \neq j, \sum_i |\sigma_i|^{\frac{n}{2}} \leq A \right\}. \tag{6.59}$$

We will consider the following class of metrics on $\mathbb{R}^n$ with components

$$g_{ij} = (g_\varepsilon)_{ij} = \delta_{ij} + \varepsilon h_{ij}, \tag{6.60}$$

where ε is a small parameter and $h = (h_{ij}) \in \mathcal{H}_A$.

As before, through the Lyapunov-Schmidt method, we will reduce problem (6.7) to a finite-dimensional one. As in Lemma 2.21, we need to find results that holds true uniformly for $h \in \mathcal{H}_A$. For the reader's convenience, we restate that lemma in the proposition below. For brevity, we denote by $\dot{z} \in (\mathcal{D}^{1,2}(\mathbb{R}^n))^{n+1}$ an orthonormal $(n+1)$-tuple in $T_z Z = \operatorname{span}\{D_\mu z, D_{\xi_1} z, \ldots, D_{\xi_n} z\}$. Precisely, we have

Proposition 6.13. *Let $n \geq 7$. Given $A > 0$, there exist $\varepsilon_0, C > 0$, such that for every $h \in \mathcal{H}_A$ there is a C^1-function $w(\varepsilon, z)$ which satisfies the following properties*

(i) *$w(\varepsilon, z)$ is orthogonal to $T_z Z \quad \forall z \in Z$, i.e. $(w, \dot{z}) = 0$;*
(ii) *$I'_\varepsilon(z + w(\varepsilon, z)) \in T_z Z \quad \forall z \in Z$;*
(iii) *$\|w(\varepsilon, z)\| \leq C|\varepsilon| \quad \forall z \in Z$.*

From (i)–(ii) it follows that

(iv) *the manifold $Z_\varepsilon = \{z + w(\varepsilon, z) \mid z \in Z\}$ is a natural constraint for I_ε.*

The proof of the above result can be found in Appendix 6.5. Although the idea is quite similar to the proof of Lemma 2.21, we carry out the details because the functional I_ε is not of the form $I_0 + \varepsilon G$, as in Section 2.2.5.

By Proposition 6.13-(iv) problem (6.29) is solved if one can find critical points of $I_\varepsilon|_{Z_\varepsilon}$. This is done by expanding the finite-dimensional functional in powers of ε as stated in (6.62) below.

First, it is possible to show (see the appendix) that

$$w(\varepsilon, z) = -\varepsilon L_z G'_1(z) + O\left(|\varepsilon|^{\frac{(n+2)}{(n-2)}}\right). \tag{6.61}$$

The preceding equation is in the spirit of Lemma 2.18, but with a quantitative estimate in ε on the error term. Using (6.61) one can prove, see the appendix, that

$$I_\varepsilon(z_{\mu,\xi} + w_\varepsilon(z_{\mu,\xi})) = b_0 + \varepsilon^2 \widetilde{\Gamma}(\mu, \xi) + o(\varepsilon^2), \tag{6.62}$$

where $\tilde{\Gamma} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is defined in (6.41). The new feature of this formula is that it holds uniformly in $z_{\mu,\xi} \in Z$ and in $h \in \mathcal{H}_A$.

We consider on $\mathbb{R}^n$ metrics g as in (6.60) with h of the form

$$h(x) = \sum_{i \in \mathbb{N}} \sigma_i \tau(x - x_i). \tag{6.63}$$

Since these metrics possess infinitely many *bumps*, from the analysis of the previous section we expect that the function $I_\varepsilon|_{Z_\varepsilon}$ inherits infinitely many local minima when the points x_i are sufficiently far away one from each other. On the other hand, we also need to choose the σ_i's appropriately in order that the metric g_ε, transposed on S^n, has the desired regularity. This will be shown at the end of the next subsection.

Let I^i_ε be the Euler functional corresponding to the metric $g^i(x) = g^i_\varepsilon(x) = \delta + \varepsilon\, \sigma_i\, \tau(x - x_i)$. Since $\sigma_i\, \tau(\cdot - x_i) \in \mathcal{H}_A$, the construction of Proposition 6.13 can be performed for I^i_ε as well. We denote by $Z^i = \{z + w^i_\varepsilon \mid z \in Z\}$ the corresponding natural constraint. We will often set for brevity

$$A_i := \text{supp } \tau(\cdot - x_i); \qquad z^i_\varepsilon := z + w^i_\varepsilon.$$

Let $\widetilde{\Gamma}^\tau$ be the function as in Lemma 2.19 associated to the metric $\delta(x) + \varepsilon\, \tau(x)$. By Proposition 6.7, $\widetilde{\Gamma}^\tau$ possesses some negative minimum and tends to zero at the boundary of $\mathbb{R}_+ \times \mathbb{R}^n$. Hence we can find a compact set $\mathcal{K}$ of $\mathbb{R}_+ \times \mathbb{R}^n$ such that

$$\left\{ y \in \mathbb{R}_+ \times \mathbb{R}^n : \widetilde{\Gamma}^\tau(y) \leq \frac{1}{2} \min \widetilde{\Gamma}^\tau \right\} \subseteq \mathcal{K}.$$

In the following this compact set $\mathcal{K}$ will be kept fixed.

Next, we need to estimate the difference between w_ε and w_ε^i; precisely one has

Lemma 6.14. *There exist $C > 0$, $\varepsilon_1 > 0$ such that for $|\varepsilon| \leq \varepsilon_1$ there holds*

$$\|w_\varepsilon - w_\varepsilon^i\| \leq C \|I'_\varepsilon(z + w_\varepsilon^i) - (I_\varepsilon^i)'(z + w_\varepsilon^i)\|. \tag{6.64}$$

Furthermore, the right-hand side of (6.64) can be estimated in the following way:

Lemma 6.15. *There exist $C > 0$, $L_1 > 0$ such that, if $|x_{i_0} - x_i| \geq L_1$ for all $i \neq i_0$, then*

$$\|I'_\varepsilon(z_{\mu,\xi} + w_\varepsilon^{i_0}) - (I_\varepsilon^{i_0})'(z_{\mu,\xi} + w_\varepsilon^{i_0})\| \leq C|\varepsilon| \sum_{i \neq i_0} \frac{\sigma_i}{|x_i - x_{i_0}|^{n-2}}, \tag{6.65}$$

for every $(\mu, \xi) \in (0, x_{i_0}) + \mathcal{K}$.

We finally need to compare $I_\varepsilon|_{Z_\varepsilon}$ with the reduced functional $I_\varepsilon^{i_0}|_{Z_{i_0}}$ corresponding to the one-bump metrics.

Proposition 6.16. *Define*

$$Q_{i_0} = I_\varepsilon(z_{\mu,\xi} + w_\varepsilon) - I_\varepsilon^{i_0}(z_{\mu,\xi} + w_\varepsilon^{i_0}).$$

Then, if $|x_{i_0} - x_i| \geq L_1$ for all $i \neq i_0$, for all $(\mu, \xi) \in (0, x_{i_0}) + \mathcal{K}$ and for all $|\varepsilon| < \varepsilon_1$ there holds

$$|Q_{i_0}| \leq C|\varepsilon| \Big(\sum_{i \neq i_0} \frac{1}{|x_i - x_{i_0}|^n} \Big)^{\frac{n-2}{n}}. \tag{6.66}$$

6.4.1 Proof of Theorem 6.3 completed

Fix $\mathbf{a} \in \mathbb{R}^n$ with $|\mathbf{a}| = 1$, and let h be of the form (6.63) with $\sigma_i = i^{-\beta}$ and $x_i = D i^\alpha \mathbf{a}$. We choose

$$D = \frac{C_0}{|\varepsilon|^{1/(n-2)}}; \qquad \alpha > 4k + 1; \qquad 2\alpha k < \beta < 2\alpha k + \frac{\alpha - (4k+1)}{2}, \tag{6.67}$$

where C_0 is a constant to be fixed later. With the above choice of $(\sigma_i)_i$ there holds $\sum_{i+1}^{+\infty} |\sigma_i|^{n/2} < +\infty$, since $\beta > 1 > \frac{2}{n}$. Since also $\alpha > 1$, we have $\inf_{i \neq j} |x_i - x_j| > 4 \operatorname{diam}(\operatorname{supp} \tau)$ for i, j large enough. Hence, if we take $\sigma_i = 0$ for i sufficiently small, then h belongs to $\mathcal{H}_A$.

From the expansion (6.62) we know that

$$I_\varepsilon^{i_0}(z_\varepsilon^{i_0}) = b_0 + \varepsilon^2\sigma_i^2\tilde{\Gamma}^{\tau(\cdot - x_{i_0})}(\mu,\xi) + o(\varepsilon^2\sigma_i^2), \qquad z_\varepsilon^{i_0} = z_{\mu,\xi} + w_\varepsilon^{i_0},$$

and so $I_\varepsilon^{i_0}|_{Z^{i_0}}$ attains an absolute minimum at a point $\widetilde{z}_\varepsilon^{i_0} = z_{\widetilde{\mu},\widetilde{\xi}} + w_\varepsilon^{i_0}$ with $(\widetilde{\mu},\widetilde{\xi}) \in (0, x_{i_0}) + \mathcal{K}$. Moreover there exists a smooth open set $U \subseteq \mathcal{K}$ such that for σ_{i_0} sufficiently small

$$\min_{(\mu,\xi)\in\partial U} I_\varepsilon^{i_0}(z_{\mu,\xi} + w_\varepsilon^{i_0}) - I_\varepsilon^{i_0}(\widetilde{z}_\varepsilon^{i_0}) \geq \frac{1}{4} d_\tau \sigma_{i_0}^2 \varepsilon^2; \qquad d_\tau = |\min \tilde{\Gamma}^\tau|. \tag{6.68}$$

We assume i_0 to be so large that $\min_{i\neq i_0}|x_{i_0} - x_i| \geq L_1$, so (6.66) holds. Hence we have that

$$|Q_{i_0}| \leq \frac{C|\varepsilon|}{D^{(n-2)}}\Big(\sum_{i\neq i_0}\frac{1}{|i^\alpha - {i_0}^\alpha|^n}\Big)^{\frac{n-2}{n}}.$$

By elementary arguments, see the appendix, one finds that

$$\sum_{i\neq i_0}\frac{1}{|i^\alpha - i_0^\alpha|^n} \sim \frac{1}{i_0^{(\alpha-1)n}}, \qquad i_0 \to +\infty. \tag{6.69}$$

Thus, for i_0 sufficiently large there holds

$$|Q_{i_0}| \leq \frac{C|\varepsilon|}{D^{(n-2)}}\frac{1}{i_0^{(\alpha-1)(n-2)}}. \tag{6.70}$$

By our choice of σ_i and by (6.68), in order to find for ε small a minimum of $I_\varepsilon|_{Z_\varepsilon}$ near $\widetilde{z}_\varepsilon^{i_0}$, it is sufficient that

$$|Q_{i_0}| \leq \frac{1}{8} d_\tau {i_0}^{-2\beta}|\varepsilon|^2. \tag{6.71}$$

Taking into account (6.70), inequality (6.71) is satisfied, for i_0 large enough, when $D = \frac{C_0}{|\varepsilon|^{1/(n-2)}}$, C_0 is sufficiently large, and

$$(\alpha - 1)(n-2) \geq 2\beta. \tag{6.72}$$

We have proved that if (6.72) holds, then for every i_0 large enough and every ε small enough $I_\varepsilon(z_{\mu,\xi} + w_\varepsilon)$ attains a minimum $(\widetilde{\mu}_{i_0}, \widetilde{\xi}_{i_0}) \in (0, x_{i_0}) + \mathcal{K}$. Hence there are infinitely many distinct solutions v_ε^i of (6.29). By the correspondence between $(\mathbb{R}^n, g_\varepsilon)$ and $(S^n, \overline{g}_\varepsilon)$, the existence of infinitely-many solutions of (6.7) follows.

Now we want to check the regularity of $\overline{g}_\varepsilon$ on S^n. Clearly $\overline{g}_\varepsilon$ is of class C^∞ on $S^n \setminus P_N$. Moreover, the regularity of $\overline{g}_\varepsilon$ at P_N is the same as that of $(\overline{g}_\varepsilon)_{\mathcal{R}}$ at the south pole P_S and so, recalling formula (6.27), it is the same of $g_\varepsilon^\sharp$ at $0 \in \mathbb{R}^n$. From equation (6.28), it follows that the functions $g_{ij}^\sharp(x)$ are of the form

$$g_{ij}^\sharp(x) = \delta_{ij} + \sum_{kj}\Lambda_{ijkl}\left(\frac{x}{|x|}\right)\left(g_{kl}\left(\frac{1}{x}\right) - \delta_{kl}\right), \tag{6.73}$$

where Λ_{ijkl} are smooth angular functions. Set $N_\varepsilon^i = \|(g_\varepsilon^i)^\sharp - \delta\|_{C^k}$. Since $(g_\varepsilon^i)^\sharp - \delta$ has support in $A^i := \left\{x \in \mathbb{R}^n : \frac{x}{|x|^2} \in A_i\right\}$, and since $\mathrm{diam}(A^i) \sim |x_i|^{-2}$, one can easily check from (6.73) that N_ε^i can be estimated by

$$N_\varepsilon^i \leq C\,|\varepsilon||\sigma_i||x_i|^{2k} \leq C|\varepsilon|^{1-\frac{2k}{n-2}} i^{2\alpha k-\beta}.$$

Let $g_{\varepsilon,j}^\sharp$ be the metric constituted by the first j bumps of $g_\varepsilon^\sharp$. Hence, since all the bumps of $g_\varepsilon^\sharp$ have disjoint support, there holds

$$\|g_{\varepsilon,j}^\sharp - g_{\varepsilon,l}^\sharp\|_{C^k(\mathbb{R}^n)} \leq \sup_{i=j+1,\dots,l} N_\varepsilon^i \leq C|\varepsilon|^{1-\frac{2k}{n-2}} \sup_{i=j+1,\dots,l} i^{2\alpha k-\beta}; \qquad j < l. \tag{6.74}$$

So, if $2\alpha k - \beta < 0$, the sequence $g_{\varepsilon,j}^\sharp$ is Cauchy in $C^k(B_1)$, and hence $\overline{g}_\varepsilon$ is also of class C^k. The two inequalities we are requiring, namely (6.72) and

$$\beta > 2\alpha k,$$

are satisfied provided $n \geq 4k+3$ by our choices in (6.67). This proves that $\overline{g}_\varepsilon$ is of class C^k on S^n. Moreover, from $n \geq 4k+3$ it also follows $1 - \frac{2k}{n-2} > 0$, and hence by (6.74) he have $\|\overline{g}_\varepsilon - \overline{g}_0\|_{C^k} \to 0$ when $\varepsilon \to 0$.

Since the solutions u_ε^i of (6.29) are close in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ to some $z_{\widetilde{\mu}_i,\widetilde{\xi}_i}$ with $(\widetilde{\mu}_i,\widetilde{\xi}_i) \in (0,x_i)+\mathcal{K}$, the solutions $v_\varepsilon^i = \iota^{-1}u_\varepsilon^i$ of (6.7) on S^n are close in $H^1(S^n)$ to $\iota^{-1}z_{\widetilde{\mu}_i,\widetilde{\xi}_i}$. From the fact that the functions $\iota^{-1}z_{\widetilde{\mu}_i,\widetilde{\xi}_i}$ blow-up at P_N as $i \to +\infty$, one can deduce that $\|v_\varepsilon^i\|_{L^\infty(S^n)} \to +\infty$ as $i \to +\infty$. Standard regularity arguments, see [49], imply that the weak solutions v_ε^i are indeed of class C^k on S^n. From the fact that $\|v_\varepsilon^i - \iota^{-1}z_{\widetilde{\mu}_i,\widetilde{\xi}_i}\|_{H^1(S^n)}$ is small and from the maximum principle, it is also easy to check that the solutions we find are positive (see the previous chapters). This concludes the proof.

Remark 6.17. It is an open problem to determine the sharpness of the condition $n \geq 4k+3$ to obtain non-compactness of solutions. $\square$

6.5 Appendix

In this section we collect the proofs of several technical results stated throughout the previous one. First we recall the following elementary inequalities.

Lemma 6.18. *Let $n \geq 3$ and $p > 0$. There exists $C > 0$ such that for all $a, b \in \mathbb{R}$*

$$|a+b|^p \leq C \cdot (|a|^p + |b|^p); \tag{6.75}$$

$$\left||a+b|^{2^*} - |a|^{2^*} - |b|^{2^*}\right| \leq C \cdot \left(|a|^{2^*-1} \cdot |b| + |a| \cdot |b|^{2^*-1}\right); \tag{6.76}$$

$$\left||a+b|^{2^*-2}(a+b) - |a|^{2^*-2}a - |b|^{2^*-2}b\right| \leq C \cdot (|a|^q \cdot |b|^r + |a|^r \cdot |b|^q), \tag{6.77}$$

where $q = \frac{(n+2)^2}{2n(n-2)}$, *and* $r = \frac{(n+2)}{2n}$. *Note that* $r + q = 2^* - 1$. *Moreover, for* $n \geq 6$

$$\left| |a+b|^{2^*-2} - |a|^{2^*-2} \right| \leq |b|^{2^*-2}, \qquad \forall\, a, b \in \mathbb{R}. \tag{6.78}$$

We also need the following estimates.

Lemma 6.19. *Let* $n \geq 7$, *let* $u, w \in \mathcal{D}^{1,2}(\mathbb{R}^n)$, *and let* $z \in Z$. *Then, in the above notation, there exists* $C > 0$ *such that the following inequalities hold*

$$I_\varepsilon(u) - I_0(u) - \varepsilon G_1(u) - \varepsilon^2 G_2(u) = o(\varepsilon^2)\left(\|u\|^2 + \|u\|^{2^*}\right); \tag{6.79}$$

$$\left\| I_\varepsilon'(u) - I_0'(u) - \varepsilon G_1'(u) \right\| \leq C\varepsilon^2 \left(\|u\| + \|u\|^{\frac{n+2}{n-2}}\right); \tag{6.80}$$

$$\|I_\varepsilon'(z)\| \leq C|\varepsilon|; \tag{6.81}$$

$$\left\| I_\varepsilon''(u) - I_0''(u) \right\| \leq C|\varepsilon| \left(1 + \|u\|^{\frac{4}{n-2}}\right); \tag{6.82}$$

$$\|I_\varepsilon(u+w) - I_\varepsilon(u)\| \leq C\|w\| \left(1 + \|u\|^{\frac{n+2}{n-2}} + \|w\|^{\frac{n+2}{n-2}}\right); \tag{6.83}$$

$$\left\| I_\varepsilon'(u+w) - E_\varepsilon'(u) \right\| \leq C\|w\| \left(1 + \|u\|^{\frac{4}{n-2}} + \|w\|^{\frac{4}{n-2}}\right); \tag{6.84}$$

$$\left\| G_1'(u+w) - G_1'(u) \right\| \leq C\|w\| \left(1 + \|u\|^{\frac{4}{n-2}} + \|w\|^{\frac{4}{n-2}}\right); \tag{6.85}$$

$$\left\| I_\varepsilon''(u+w) - I_\varepsilon''(u) \right\| \leq C\|w\|^{\frac{4}{n-2}}, \tag{6.86}$$

uniformly in u, w *and* z.

Proof. We start proving (6.86). Given two functions $v_1, v_2 \in \mathcal{D}^{1,2}(\mathbb{R}^n)$, there holds

$$\begin{aligned}
&\left|(I_\varepsilon''(u+w) - I_\varepsilon''(u))[v_1, v_2]\right| \\
&\quad = n(n-1)(2^*-1)\left| \int (|u+w|^{2^*-2} - |u|^{2^*-2}) v_1 v_2 dV_g \right| \\
&\quad \leq n(n-1)(2^*-1)(1+O(\varepsilon)) \left| \int \left| |u+w|^{2^*-2} - |u|^{2^*-2} \right| |v_1||v_2| dx \right|.
\end{aligned}$$

Using the Hölder and the Sobolev inequalities we get

$$\int \left| |u+w|^{2^*-2} - |u|^{2^*-2} \right| |v_1||v_2| dx \leq C \left(\int \left| |u+w|^{2^*-2} - |u|^{2^*-2} \right|^{\frac{n}{2}} \right)^{\frac{2}{n}} \|v_1\|\|v_2\|.$$

For $n \geq 6$, using the inequality (6.78) with $a = u(x)$, $b = w(x)$, we deduce that

$$\left| |u+w|^{2^*-2} - |u|^{2^*-2} \right|^{\frac{n}{2}} \leq C|w|^{2^*},$$

so (6.86) holds.

We now prove (6.82). Taking into account formulas (6.37) and (6.3), we have that

$$\begin{aligned}
&I_\varepsilon''(u)[v_1, v_2] \\
&= \int \left(\nabla v_1 \cdot \nabla v_2 (1 + O(\varepsilon)) + R_g v_1 v_2 - n(n-1)(2^*-1)|u|^{2^*-2} v_1 v_2 \right) dx (1 + O(\varepsilon)).
\end{aligned}$$

From the Hölder and the Sobolev inequalities, and using the fact that the support of R_g is compact, it follows that

$$(I''_\varepsilon(u) - I''_0(u))[v_1, v_2] = O(\varepsilon)\left(1 + O(\varepsilon) + \|u\|^{\frac{4}{n-2}}\right)\|v_1\|\|v_2\|,$$

and (6.82) is proved.

Let us turn to (6.84). For every $v \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ there holds

$$\begin{aligned}
&(I'_\varepsilon(u+w) - I'_\varepsilon(u), v) \\
&= \int \left(2c_n \nabla_g w \cdot \nabla_g v + R_g w v + |u+w|^{2^*-2}(u+w)v - n(n-1)|u|^{2^*-2}uv\right) dV_g.
\end{aligned} \tag{6.87}$$

This implies that

$$\begin{aligned}
\|I'_\varepsilon(u+w) - I'_\varepsilon(u)\| &\le O(1)\|w\|(1 + O(\varepsilon)) \\
&\quad + \left(\int \left||u+w|^{2^*-2}(u+w) - |u|^{2^*-2}u\right|^{\frac{2n}{n+2}}\right)^{\frac{n+2}{2n}} (1 + O(\varepsilon)).
\end{aligned}$$

Since

$$|u+w|^{2^*-2}(u+w) - |u|^{2^*-2}u = (2^*-1)\int_0^1 |u+sw|^{2^*-2} w \, ds,$$

setting $y(x) = (2^*-1)\int_0^1 |u+sw|^{2^*-2} ds$, we have $|u+w|^{2^*-2}(u+w) - |u|^{2^*-2}u = y(x)w(x)$. Hence there holds

$$\left(\int \left||u+w|^{2^*-2}(u+w) - |u|^{2^*-2}u\right|^{\frac{2n}{n+2}}\right)^{\frac{n+2}{2n}} \le C\|w\| \left(\int |y|^{\frac{n}{2}}\right)^{\frac{2}{n}}.$$

Using again the Hölder inequality, we have that $|y| \le \left(\int_0^1 |u+sw|^{2^*} ds\right)^{\frac{2}{n}}$. So from the Fubini Theorem we deduce

$$\int |y|^{\frac{n}{2}} dx \le \int \left|\int_0^1 |u+sw|^{2^*} ds\right| dx = \int_0^1 \left(\int |u+sw|^{2^*} dx\right) ds \le \sup_{s\in[0,1]} \|u+sw\|_{2^*}^{2^*}.$$

By (6.75) it turns out that

$$\left(\int |y|^{\frac{n}{2}}\right)^{\frac{2}{n}} \le \sup_{s\in[0,1]} \|u+sw\|^{\frac{4}{(n-2)}} \le C(\|u\|^{\frac{4}{(n-2)}} + \|w\|^{\frac{4}{(n-2)}}).$$

In conclusion we obtain (6.84).

We now prove (6.80). Given $v \in E$, we have

$$(I'_\varepsilon(u), v) = \int \left(2c_n \nabla_g u \cdot \nabla_g v + R_g uv - n(n-1)|u|^{2^*-2}uv\right) dV_g.$$

Taking into account formulas (6.3) and (6.37), we deduce

$$\begin{aligned}
(I'_\varepsilon(u), v) = \int \Bigg(&2c_n \nabla u \cdot \nabla v - \varepsilon \sum_{ij} h_{ij} D_i u D_j v + O(\varepsilon^2)|\nabla u||\nabla v| + \varepsilon R_1 uv \\
&+ O(\varepsilon^2)|u||v| - n(n-1)|u|^{2^*-2}uv\Bigg) \times \left(1 + \frac{1}{2}\varepsilon \operatorname{tr} h + O(\varepsilon^2)\right) dx.
\end{aligned}$$

Expanding the last expression in ε, up to order $O(\varepsilon^2)$, and using again the standard inequalities (we recall that the support of R_g is compact), we obtain (6.80). Formulas (6.79), (6.81), (6.83) and (6.85) can be obtained with similar procedures. $\square$

Next we give the proof of Proposition 6.13.

Proof. The function w can be found as a zero of the map

$$H : Z \times \mathcal{D}^{1,2}(\mathbb{R}^n) \times \mathbb{R}^{n+1} \times \mathbb{R} \to \mathcal{D}^{1,2}(\mathbb{R}^n) \times \mathbb{R}^{n+1}$$

defined by

$$H(z, w, \alpha, \varepsilon) = \begin{pmatrix} I'_\varepsilon(z + w) - \alpha \dot{z} \\ (w, \dot{z}) \end{pmatrix}.$$

Since $H(z, 0, 0, 0) = 0$ we have that

$$H(z, w, \alpha, \varepsilon) = 0 \quad \Leftrightarrow \quad \frac{\partial H}{\partial(w, \alpha)}|_{(z,0,0,0)}[w, \alpha] + R(z, w, \alpha, \varepsilon) = 0,$$

where $R(z, w, \alpha, \varepsilon) = H(z, w, \alpha, \varepsilon) - \frac{\partial H}{\partial(w,\alpha)}|_{(z,0,0,0)}[w, \alpha]$. Using Lemma 5.2, one can easily check that $\frac{\partial H}{\partial(w,\alpha)}|_{(z,0,0,0)}$ is uniformly invertible, and hence

$$H(z, w, \alpha, \varepsilon) = 0 \qquad \Leftrightarrow \qquad (w, \alpha) = F_{z,\varepsilon}(w, \alpha),$$

where

$$F_{z,\varepsilon}(w, \alpha) = -\left(\frac{\partial H}{\partial(w, \alpha)}(z, 0, 0, 0)\right)^{-1} R(z, w, \alpha, \varepsilon).$$

We will show that, for ρ and ε sufficiently small, $F_{z,\varepsilon}(w, \alpha)$ is a contraction in some set $B_\rho = \{(w, \alpha) \in \mathcal{D}^{1,2}(\mathbb{R}^n) \times \mathbb{R}^{n+1} \ : \ \|w\| + |\alpha| \leq \rho\}$. For this purpose, it is sufficient to show that there exists $C > 0$ such that for every $(w, \alpha), (w', \alpha')$ with $\|(w, \alpha)\|, \|(w', \alpha')\| \leq \rho$ small enough there holds

$$\begin{cases} \|F_{z,\varepsilon}(w, \alpha)\| \leq C\Big(|\varepsilon| + \rho^{\min\{2, \frac{n+2}{n-2}\}}\Big), \\ \|F_{z,\varepsilon}(w', \alpha') - F_{z,\varepsilon}(w, \alpha)\| \leq C\Big(|\varepsilon| + \rho^{\min\{1, \frac{4}{n-2}\}}\Big)\|(w, \alpha) - (w', \alpha')\|. \end{cases} \tag{6.88}$$

The system (6.88) is equivalent to the following two inequalities

$$\|I'_\varepsilon(z + w) - I''_0(z)[w]\| \leq C\Big(|\varepsilon| + \rho^{\min\{2, \frac{n+2}{n-2}\}}\Big); \tag{6.89}$$

$$\|(I'_\varepsilon(z+w) - I''_0(z)[w]) - (I'_\varepsilon(z+w') - I''_0(z)[w'])\| \leq C\Big(|\varepsilon| + \rho^{\min\{1, \frac{4}{n-2}\}}\Big)\|(w, \alpha) - (w', \alpha')\|. \tag{6.90}$$

We now prove (6.89). Using formulas (6.81) and (6.82) we have, since $\|z\|$ is bounded

$$\begin{aligned} I'_\varepsilon(z + w) - I''_0(z)[w] &= \Big(I'_\varepsilon(z + w) - I'_\varepsilon(z) - I''_\varepsilon(z)[w]\Big) + I'_\varepsilon(z) + \Big(I''_\varepsilon(z) - I''_0(z)\Big)[w] \\ &= \int_0^1 \Big(I''_\varepsilon(z + sw) - I''_\varepsilon(z)\Big)[w] ds + O(\varepsilon) + O(\varepsilon)\|w\|. \end{aligned}$$

Hence, using (6.86), since $\|z\|$ and $\|w\|$ are uniformly bounded, we deduce that

$$\|I'_\varepsilon(z + w) - I''_0(z)[w]\| \leq C(|\varepsilon| + \|w\|^{\min\{2, \frac{n+2}{n-2}\}} + |\varepsilon|\|w\|) \leq C\Big(|\varepsilon| + \rho^{\min\{2, \frac{n+2}{n-2}\}}\Big),$$

and (6.89) is proved. We turn now to (6.90). There holds

$$\begin{aligned}
&\|I'_\varepsilon(z+w) - I'_\varepsilon(z+w') - I''_0(z)[w-w']\| \\
&\qquad = \left| \int_0^1 \Big(I''_\varepsilon(z+w+s(w'-w)) - I''_0(z) \Big)[w'-w]ds \right| \\
&\qquad \leq \sup_{s\in[0,1]} \|I''_\varepsilon(z+w+s(w'-w)) - I''_0(z)\| \|w'-w\|.
\end{aligned}$$

Using again formulas (6.82), (6.86) and (6.86) we have that

$$\|I''_\varepsilon(z+w'+s(w-w')) - I''_0(z)\| \leq C(|\varepsilon| + \rho^{\min\{2,\frac{n+2}{n-2}\}}),$$

hence (6.90) is also satisfied. By (6.88), if $C(|\varepsilon|+\rho^{\min\{2,\frac{n+2}{n-2}\}}) < \rho$ and if $C(|\varepsilon|+\rho^{\frac{4}{n-2}}) < 1$, then $F_{z,\varepsilon}(w,\alpha)$ is a contraction in B_ρ. These inequalities hold true, for example, choosing $\rho = 2C|\varepsilon|$, for $|\varepsilon| \leq \varepsilon_0$ with ε_0 sufficiently small. Hence we find a unique solution $\|(w_\varepsilon, \alpha_\varepsilon)\| \leq 2C|\varepsilon|$. □

We now prove (6.61).

Proof. We can write $I'_\varepsilon(z+w_\varepsilon) = \beta_1 + \beta_2 + \beta_3 + (I''_0(z)[w_\varepsilon] + \varepsilon G'_1(z))$ where

$$\begin{aligned}
\beta_1 &= I'_\varepsilon(z+w_\varepsilon) - I'_0(z+w_\varepsilon) - \varepsilon G'_1(z+w_\varepsilon); \\
\beta_2 &= I'_0(z+w_\varepsilon) - I''_0(z)[w_\varepsilon]; \\
\beta_3 &= \varepsilon G'_1(z+w_\varepsilon) - \varepsilon G'_1(z).
\end{aligned}$$

From (6.80), since $\|z+w_\varepsilon\|$ is uniformly bounded, we have $\|\beta_1\| = O(\varepsilon^2)$. Moreover we can write

$$\beta_2 = \int_0^1 (I''_0(z+sw_\varepsilon) - I''_0(z))[w_\varepsilon]ds,$$

so (6.86) implies $\|\beta_2\| = O\left(|\varepsilon|^{\frac{(n+2)}{(n-2)}}\right)$. From (6.85), since $\|w_\varepsilon\| \leq C|\varepsilon|$, it follows that also $\|\beta_3\| = O(\varepsilon^2)$. Hence we deduce that $\beta_1+\beta_2+\beta_3 = O\left(|\varepsilon|^{\frac{(n+2)}{(n-2)}}\right)$. Thus the relation $I'_\varepsilon(z+w_\varepsilon) = \alpha_\varepsilon \dot z$ can be written as $I''_0(z)[w_\varepsilon] + \varepsilon G'_1(z) + O\left(|\varepsilon|^{\frac{(n+2)}{(n-2)}}\right) = \alpha_\varepsilon \dot z$. Projecting this equation onto $(T_z Z)^\perp$ and applying the operator L_z we obtain (6.61). □

The next one is the proof of (6.62).

Proof. We can write $I_\varepsilon(z+w_\varepsilon)$ as $I_\varepsilon(z+w_\varepsilon) = \gamma_1+\gamma_2+\gamma_3$, where

$$\gamma_1 = I_\varepsilon(z), \qquad \gamma_2 = I'_\varepsilon(z)[w_\varepsilon], \qquad \gamma_3 = I_\varepsilon(w_\varepsilon+z) - I_\varepsilon(z) - I'_\varepsilon(z)[w_\varepsilon].$$

By (6.79), since $G_1|_Z \equiv 0$, we deduce that

$$\gamma_1 = I_0(z) + \varepsilon G_1(z) + \varepsilon^2 G_2(z) + o(\varepsilon^2) = b_0 + \varepsilon^2 G_2(z) + o(\varepsilon^2).$$

Turning to γ_2, from (6.80), (6.61) and from $I'_0(z) = 0$ we obtain

$$\gamma_2 = (I'_0(z), w_\varepsilon) + \varepsilon(G'_1(z), w_\varepsilon) + o(\varepsilon^2) = -\varepsilon^2 \Big(L_z G'_1(z), G'_1(z) \Big) + o(\varepsilon^2).$$

We now estimate γ_3. We have

$$\gamma_3 = \int_0^1 (I_\varepsilon'(z + sw_\varepsilon) - I_\varepsilon'(z), w_\varepsilon) ds.$$

Using (6.80) we find

$$\gamma_3 = \int_0^1 \Big((I_0'(z + sw_\varepsilon) - I_0'(z)) + \varepsilon \left(G_1'(z + sw_\varepsilon) - G_1'(z) \right), w_\varepsilon \Big) ds + o(\varepsilon^2).$$

Using (6.85), (6.86) and $\|w_\varepsilon\| \leq C|\varepsilon|$, then it follows that

$$\begin{aligned}
\gamma_3 &= \int_0^1 (I_0'(z + sw_\varepsilon) - I_0'(z), w_\varepsilon) ds + o(\varepsilon^2) \\
&= \int_0^1 \left(\int_0^1 (I_0''(z + tsw_\varepsilon) - I_0''(z))[sw_\varepsilon] dt \right) [w_\varepsilon] ds \\
&\quad + \int_0^1 \left(\int_0^1 I_0''(z)[sw_\varepsilon] dt \right) [w_\varepsilon] ds + o(\varepsilon^2) = \frac{1}{2} I_0''(z)[w_\varepsilon, w_\varepsilon] + o(\varepsilon^2).
\end{aligned}$$

From the above estimates for $\gamma_1, \gamma_2, \gamma_3$ we deduce the claim. □

We are now in the position to prove Lemma 6.14.

Proof. Let us consider the function

$$\overline{H} \ : \ Z \times \mathcal{D}^{1,2}(\mathbb{R}^n) \times \mathbb{R}^{n+1} \to \mathcal{D}^{1,2}(\mathbb{R}^n) \times \mathbb{R}^{n+1} \times \mathbb{R}$$

with components $\overline{H}_1 \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ and $\overline{H}_2 \in \mathbb{R}^{n+1}$ given by

$$\begin{aligned}
\overline{H}_1(z, w, \alpha, \varepsilon) &= I_\varepsilon'(z + w_\varepsilon^i + w) - (\alpha_\varepsilon^i + \alpha)\, \dot{z}, \\
\overline{H}_2(z, w, \alpha, \varepsilon) &= (w, \dot{z}).
\end{aligned}$$

We have

$$\overline{H}(z, w, \alpha, \varepsilon) = 0 \quad \Leftrightarrow \quad \overline{H}(z, 0, 0, \varepsilon) + \frac{\partial \overline{H}}{\partial(w, \alpha)}|_{(z,0,0,\varepsilon)}[w, \alpha] + \overline{R}(z, w, \alpha, \varepsilon) = 0,$$

where $\overline{R}(z, w, \alpha, \varepsilon) = \overline{H}(z, w, \alpha, \varepsilon) - \overline{H}(z, 0, 0, \varepsilon) - \frac{\partial \overline{H}}{\partial(w,\alpha)}|_{(z,0,0,\varepsilon)}[w, \alpha]$.

It is easy to see that for $|\varepsilon|$ small enough there holds

$$\left| \left(\frac{\partial \overline{H}}{\partial(w, \alpha)}|_{(z,0,0,\varepsilon)} \right)^{-1} \right| \leq C, \qquad \forall z \in Z.$$

Moreover we have

$$\overline{H}(z, w, \alpha, \varepsilon) = 0 \qquad \Leftrightarrow \qquad (w, \alpha) = \overline{F}_{\varepsilon,z}(w, \alpha),$$

where

$$\overline{F}_{\varepsilon,z}(w, \alpha) := - \left(\frac{\partial \overline{H}}{\partial(w, \alpha)}|_{(z,0,0,\varepsilon)} \right)^{-1} \left(\overline{H}(z, 0, 0, \varepsilon) + \overline{R}(z, w, \alpha, \varepsilon) \right).$$

We claim that the following two estimates hold. For every (w,α) and (w',α') such that $\|(w,\alpha)\|$, $\|(w',\alpha')\| \le \rho$ small enough

$$\|\overline{F}_{\varepsilon,z}(w,\alpha)\| \le C\|I'_\varepsilon(z+w^i_\varepsilon) - (I^i_\varepsilon)'(z+w^i_\varepsilon)\| + C\rho^{\frac{n+2}{n-2}}, \tag{6.91}$$

$$\|\overline{F}_{\varepsilon,z}(w,\alpha) - \overline{F}_{\varepsilon,z}(w',\alpha')\| \le C\rho^{\frac{4}{n-2}}\|w'-w\|. \tag{6.92}$$

Let us prove (6.91). For every $(w,\alpha)\in B_\rho$ there holds

$$\|\overline{F}_{\varepsilon,z}(w,\alpha)\| \le C\|\overline{H}(z,0,0,\varepsilon)\| + C\|\overline{R}(z,w,\alpha,\varepsilon)\|. \tag{6.93}$$

We have, using the same arguments in the proof of Proposition 6.13

$$\begin{aligned}\|\overline{R}(\varepsilon,z,w,\alpha)\| &= \left\|\overline{H}(z,w,\alpha,\varepsilon) - \overline{H}(z,0,0,\varepsilon) - \frac{\partial\overline{H}}{\partial(w,\alpha)}|_{(z,0,0,\varepsilon)}[w,\alpha]\right\| \\ &= \|I'_\varepsilon(z+w^i_\varepsilon+w) - I'_\varepsilon(z+w^i_\varepsilon) - I''_\varepsilon(z+w^i_\varepsilon)[w]\| \le C\|w\|^{\frac{n+2}{n-2}}.\end{aligned}$$

Since $\overline{H}(z,0,0,\varepsilon) = I'_\varepsilon(z+w^i_\varepsilon) - (I^i_\varepsilon)'(z+w^i_\varepsilon)$, (6.91) follows from (6.93). Let us turn to (6.92). For all $(w,\alpha),(w',\alpha')\in B_\rho$ it is

$$\begin{aligned}\|\overline{F}_{\varepsilon,z}(w,\alpha) - \overline{F}_{\varepsilon,z}(w',\alpha')\| &= \left\|\left(\frac{\partial\overline{H}}{\partial(w,\alpha)}|_{(z,0,0,\varepsilon)}\right)^{-1}\left(\overline{R}(z,w,\alpha,\varepsilon) - \overline{R}(z,w',\alpha',\varepsilon)\right)\right\| \\ &\le C\left\|\int_0^1 I''_\varepsilon(z+w^i_\varepsilon+w'+s(w-w')) - I''_\varepsilon(z+w^i_\varepsilon)ds\right\| \\ &\quad\times\|w'-w\| \le C\rho^{2^*-2}\|w'-w\|,\end{aligned}$$

so (6.92) holds true. Now, arguing as before, we deduce that there exists a unique $(w^D_\varepsilon,\alpha^D_\varepsilon)$ such that

(j) $(w^D_\varepsilon,\dot z) = 0$;
(jj) $I'_\varepsilon(z+w^i_\varepsilon+w^D_\varepsilon) = (\alpha^i_\varepsilon+\alpha^D_\varepsilon)\dot z$;
(jjj) $\|w^D_\varepsilon\| \le C\|I'_\varepsilon(z+w^i_\varepsilon) - (I^i_\varepsilon)'(z+w^i_\varepsilon)\|$ for ε sufficiently small.

The couple $\left(w^i_\varepsilon + w^D_\varepsilon, \alpha^i_\varepsilon+\alpha^D_\varepsilon\right)$ satisfies (i)–(iii) in Proposition 6.13, hence by uniqueness it must be $w_\varepsilon = w^i_\varepsilon + w^D_\varepsilon$; by (jjj), inequality (6.64) follows. □

In order to prove Lemma 6.15 we need to show

$$|z^{i_0}_\varepsilon(x)| \le \frac{C}{|x-x_{i_0}|^{n-2}}, \qquad |\nabla z^{i_0}_\varepsilon(x)| \le \frac{C}{|x-x_{i_0}|^{n-1}} \qquad |x-x_{i_0}|\ge R, \tag{6.94}$$

where $(\mu,\xi)\in(0,\xi_0)+\mathcal{K}$ and $C>0$.

Proof of (6.94). We can suppose without loss of generality that the support of τ is contained in $B_1 = \{x\in\mathbb{R}^n : |x|\le 1\}$. The function $z^{i_0}_\varepsilon$, satisfies $(I^{i_0}_\varepsilon)'(z^{i_0}_\varepsilon) = \alpha^{i_0}_\varepsilon\dot z$, hence it solves the equation

$$-2c_n\Delta(z^{i_0}_\varepsilon) - n(n-1)|z^{i_0}_\varepsilon|^{2^*-2}z^{i_0}_\varepsilon = -\alpha^{i_0}_\varepsilon\Delta\dot z, \qquad \text{in } \mathbb{R}^n\setminus B_1.$$

Performing the transformation

$$z^{i_0}_\varepsilon(x) \to u^{i_0}_\varepsilon(x) := \mu^{\frac{n-2}{2}}(z^{i_0}_\varepsilon)^*(\mu x),$$

one easily verifies that the function $u_\varepsilon^{i_0}$ solves

$$-\Delta u_\varepsilon^{i_0}(x) = n(n-1)|u_\varepsilon^{i_0}|^{2^*-2}(x)u_\varepsilon^{i_0}(x) + \mu^{\frac{n+2}{2}} q_z(\mu x), \qquad \text{in } B_1, \tag{6.95}$$

where $q_z = -\alpha_\varepsilon^{i_0}(z)\Delta(\dot{z}^*)$. Since (μ_1, ξ_1) belongs to the fixed compact set $\mathcal{K}$, we have

$$\|q_z\|_{C^3(B_1)} \text{ is uniformly bounded for } (\mu_1, \xi_1) \in \mathcal{K}. \tag{6.96}$$

Moreover, since $w_\varepsilon^{i_0}$ is a continuous function of z, it turns out that

$$\zeta_\mu = \sup_{(\mu,\xi)\in\mathcal{K}} \int_{B_1} |\nabla u_\varepsilon^{i_0}|^2 \to 0, \quad \eta_\mu = \sup_{(\mu,\xi)\in\mathcal{K}} \int_{B_1} |u_\varepsilon^{i_0}|^{2^*} \to 0, \qquad \text{as } \mu \to 0. \tag{6.97}$$

Under conditions (6.95), (6.96) and (6.97), the arguments in the proof of Proposition 1.1 in [99] imply that for some $\mu = \mu_0$ sufficiently small it is $\|u_\varepsilon^{i_0}\|_{C^1(B_{1/2})} \leq C$ uniformly in for $(\mu_1, \xi_1) \in \mathcal{K}$. From this inequality one can easily deduce that

$$z_\varepsilon^{i_0}(x) \leq \frac{C}{\mu_0^{\frac{n-2}{2}}} \cdot \frac{1}{|x|^{n-2}}, \qquad \text{for } |x| \geq \frac{2}{\mu_0}; \qquad (\mu_1, \xi_1) \in \mathcal{K}.$$

The second inequality in (6.94) also follows from the boundedness of $\|u_\varepsilon^\tau\|_{C^1(B_{1/2})}$. □

We are now in the position to prove Lemma 6.15.

Proof. Given any $v \in \mathcal{D}^{1,2}(\mathbb{R}^n)$, there holds

$$\begin{aligned}
&|(I_\varepsilon'(z_\varepsilon^{i_0}) - (I_\varepsilon^{i_0})'(z_\varepsilon^{i_0}), v)| \\
&\quad = \Bigg| \sum_{i\neq i_0} \int_{A_i} 2c_n \nabla_g z_\varepsilon^{i_0} \cdot \nabla_g v + R_g z_\varepsilon^{i_0} v - n(n-1)|z_\varepsilon^{i_0}|^{2^*-2} z_\varepsilon^{i_0} v \, dV_g \\
&\qquad - \sum_{i\neq i_0} \int_{A_i} 2c_n \nabla z_\varepsilon^{i_0} \cdot \nabla v - n(n-1)|z_\varepsilon^{i_0}|^{2^*-2} z_\varepsilon^{i_0} v \, dx \Bigg| \\
&\quad \leq C|\varepsilon| \sum_{i\neq i_0} \sigma_i \int_{A_i} |\nabla z_\varepsilon^{i_0}||\nabla v| + |z_\varepsilon^{i_0}||v| + |z_\varepsilon^{i_0}|^{2^*-1}|v| \, dx.
\end{aligned}$$

Using (6.94), with the Hölder and the Sobolev inequalities we deduce that, if $|x_{i_0} - x_i| \geq L_1$, $i \neq i_0$, with $L_1 \geq R$, there holds

$$|(I_\varepsilon''(z_\varepsilon^{i_0}) - (I_\varepsilon^{i_0})'(z_\varepsilon^{i_0}), v)| \leq C|\varepsilon| \|v\| \sum_{i\neq i_0} \sigma_i \left(\frac{1}{|x_i - x_{i_0}|^{n-1}} + \frac{1}{|x_i - x_{i_0}|^{n-2}} + \frac{1}{|x_i - x_{i_0}|^{n+2}} \right).$$

This concludes the proof. □

We next prove Proposition 6.16.

Proof. We have by (6.83), (6.64) and (6.65)

$$\begin{aligned}|Q_{i_0}| &= |I_\varepsilon(z+w_\varepsilon) - I_\varepsilon^{i_0}(z_\varepsilon^{i_0})| \\ &\le |I_\varepsilon(z+w_\varepsilon) - I_\varepsilon(z_\varepsilon^{i_0})| + |I_\varepsilon(z_\varepsilon^{i_0}) - I_\varepsilon^{i_0}(z_\varepsilon^{i_0})| \\ &\le C\,\|w_\varepsilon - w_\varepsilon^{i_0}\| + |I_\varepsilon(z_\varepsilon^{i_0}) - I_\varepsilon^{i_0}(z_\varepsilon^{i_0})| \\ &\le C\,\|I'_\varepsilon(z_\varepsilon^{i_0}) - (I_\varepsilon^{i_0})'(z_\varepsilon^{i_0})\| + |I_\varepsilon(z_\varepsilon^{i_0}) - I_\varepsilon^{i_0}(z_\varepsilon^{i_0})| \\ &\le C|\varepsilon| \sum_{i\neq i_0} \frac{\sigma_i}{|x_i - x_{i_0}|^{n-2}} + |I_\varepsilon(z_\varepsilon^{i_0}) - I_\varepsilon^{i_0}(z_\varepsilon^{i_0})|. \end{aligned} \tag{6.98}$$

Arguing as in Lemma 6.15 we deduce

$$\begin{aligned}|I_\varepsilon(z_\varepsilon^{i_0}) - I_\varepsilon^{i_0}(z_\varepsilon^{i_0})| &= \sum_{i\neq i_0}\int_{A_i} c_n|\nabla_g(z_\varepsilon^{i_0})|^2 + R_g(z_\varepsilon^{i_0})^2 - \frac{n(n-1)}{2^*}|z_\varepsilon^{i_0}|^{2^*}\ dV_g \\ &\quad - \sum_{i\neq i_0}\int_{A_i} c_n|\nabla(z_\varepsilon^{i_0})|^2 - \frac{n(n-1)}{2^*}|z_\varepsilon^{i_0}|^{2^*}\ dx \\ &\le C|\varepsilon|\sum_{i\neq i_0}\sigma_i\int_{A_i}|\nabla(z_\varepsilon^{i_0})|^2 + |z_\varepsilon^{i_0}|^2 + |z_\varepsilon^{i_0}|^{2^*}\ dx. \end{aligned}$$

Then, using the fact that $|x_i - x_{i_0}| \ge L_1$, we obtain

$$|I_\varepsilon(z_\varepsilon^{i_0}) - I_\varepsilon^{i_0}(z_\varepsilon^{i_0})| \le C|\varepsilon|\sum_{i\neq i_0}\sigma_i\left(\frac{1}{|x_i - x_{i_0}|^{2(n-1)}} + \frac{1}{|x_i - x_{i_0}|^{2(n-2)}} + \frac{1}{|x_i - x_{i_0}|^{2n}}\right).$$

The last inequality and (6.98) imply that $|Q_{i_0}| \le C|\varepsilon|\sum_{i\neq i_0}\frac{\sigma_i}{|x_i - x_{i_0}|^{n-2}}$. Applying the Hölder inequality and taking into account that $\sum_i |\sigma_i|^{\frac{n}{2}} < A$, (6.66) follows. □

Finally we prove the estimate (6.69).

Proof. For i_0 large enough there holds

$$\sum_{i<i_0}\frac{1}{|i^\alpha - i_0^\alpha|^\gamma} \sim \int_0^{(i_0-1)}\frac{dx}{(i_0^\alpha - x^\alpha)^\gamma}, \qquad \sum_{i>i_0}\frac{1}{|i^\alpha - i_0^\alpha|^\gamma} \sim \int_{(i_0+1)}^\infty \frac{dx}{(x^\alpha - i_0^\alpha)^\gamma}.$$

Hence, we are reduced to estimate the above two integrals. Let us start with the first one. Using the change of variables $i_0 y = x$, we deduce that

$$\int_0^{(i_0-1)}\frac{dx}{(i_0^\alpha - x^\alpha)^\gamma} = i_0\int_0^{1-\frac{1}{i_0}}\frac{dy}{i_0^{\alpha\gamma}(1-y^\alpha)^\gamma} = \frac{1}{i_0^{\alpha\gamma-1}}\int_0^{1-\frac{1}{i_0}}\frac{dy}{(1-y^\alpha)^\gamma}.$$

Since $(1-y^\alpha)^\gamma \sim C(1-y)^\gamma$, for y close to 1 it follows that $\int_0^{1-\frac{1}{i_0}}\frac{dy}{(1-y^\alpha)^\gamma} \sim C i_0^{\gamma-1}$. Hence we have $\int_0^{(i_0-1)}\frac{dx}{(i_0^\alpha - x^\alpha)^\gamma} \sim C\frac{1}{i_0^{(\alpha-1)\gamma}}$. An analogous estimate holds for the other integral $\int_{(i_0+1)}^\infty\frac{dx}{(x^\alpha - i_0^\alpha)^\gamma}$. This concludes the proof. □

Chapter 7

Other Problems in Conformal Geometry

In this chapter we will survey some other problems arising in Conformal Geometry. First we will focus on the Scalar Curvature Problem for the standard sphere, see Sections 7.1 and 7.2. Next, in Section 7.3, we will deal with some problem on manifolds with boundary.

7.1 Prescribing the scalar curvature of the sphere

As a counterpart of the Yamabe problem one can ask whether, considering the standard sphere $(S^n, \overline{g}_0)$, $n \geq 3$, (for which $R_{\overline{g}_0}$ is constant), one can deform conformally the metric in such a way that the scalar curvature becomes a prescribed function on S^n. Denoting by $\widetilde{K}$ this function, the problem consists in solving the following equation, see (6.5)

$$-2c_n \Delta_{\overline{g}_0} u + R_{\overline{g}_0} u = \widetilde{K}\, u^{\frac{n+2}{n-2}}; \qquad u > 0 \text{ on } S^n. \tag{7.1}$$

In the case of $n = 2$, regarding the Gauss curvature, the problem was first raised by Nirenberg, and the corresponding equation is

$$-\Delta_{\overline{g}_0} u + R_{\overline{g}_0} = \widetilde{K}\, e^{2u}. \tag{7.2}$$

Unlike the Yamabe problem, (7.1) does not always admit a solution. A first necessary condition for the existence is that $\max_{S^n} \widetilde{K} > 0$, but there are also some obstructions, which are said *of topological type*. For example, Kazdan and Warner, [95], proved that every solution u of (7.1) must satisfy the condition

$$\int_{S^n} u^{2^*} \langle \widetilde{K}', a' \rangle = 0, \tag{7.3}$$

where a is the restriction to S^n of any affine function in $\mathbb{R}^{n+1}$. Hence, since u is positive, a necessary condition for the existence of solutions is that the function $\langle \widetilde{K}', a' \rangle$ changes sign. Other counterexamples to the existence are given in [137].

A first answer to the Nirenberg problem was given by J. Moser, [114], who proved that if $\widetilde{K}$ is an even function on S^2, then the problem is solvable. Further results in the presence of symmetries are discussed in the next section.

An existence result for the Nirenberg problem, without any symmetry assumption, was obtained in [56, 57]. Here the following two conditions are required:

(i) it is supposed that

$$x \in \mathrm{Cr}[\widetilde{K}] \qquad \Rightarrow \qquad \Delta_{\bar{g}_0} \widetilde{K}(x) \neq 0; \tag{7.4}$$

(ii) $\widetilde{K}$ possesses p local maxima and q saddle points with negative Laplacian, and that the following inequality holds

$$p \neq q + 1. \tag{7.5}$$

The Scalar Curvature Problem in dimension $n = 3$ was studied in [33] under the assumption that $\widetilde{K}$ is a Morse function (namely its critical points are non-degenerate) satisfying (7.4) and

$$\sum_{x \in \mathrm{Cr}[\widetilde{K}], \Delta_{\bar{g}_0} \widetilde{K}(x) < 0} (-1)^{m(\widetilde{K}, x)} \neq -1. \tag{7.6}$$

Here $m(\widetilde{K}, x)$ denotes the Morse index of $\widetilde{K}$ at x. The result of [33], which is based on a topological argument, has been extended in many directions.

An extension of condition (7.6), based on the Morse inequalities, was given in [133], again for the case $n = 3$. Therein they suppose that $\widetilde{K}$ is a Morse function satisfying (7.4) and, letting

$$D_q = \sharp\{x \in \mathrm{Cr}[\widetilde{K}] : m(\widetilde{K}, x) = 3 - q,\ \Delta_{\bar{g}_0} \widetilde{K}(x) < 0\},$$

it is required

$$D_0 - D_1 + D_2 \neq 1, \qquad \text{or} \qquad D_0 - D_1 > 1. \tag{7.7}$$

Note that the first condition in (7.7) is equivalent to (7.6), and for $n = 2$ it is analogous to (7.5).

The rest of the section is devoted to outline the main results dealing with the general case $n \geq 3$. First, let us show how Theorem 5.3 can be used to derive an existence result when $\widetilde{K}$ is close to a constant. For results of this sort, see also [58].

Let us suppose that $\widetilde{K}$ satisfies the following conditions:

(K1) $\widetilde{K} > 0$ is a C^2 Morse function and $\Delta_{\overline{g}_0}\widetilde{K}(x) \neq 0$ for any $x \in \mathrm{Cr}[\widetilde{K}]$;

(K2) there results

$$\sum_{x\in \mathrm{Cr}[\widetilde{K}],\Delta_{\overline{g}_0}\widetilde{K}(x)<0} (-1)^{m(\widetilde{K},x)} \neq (-1)^n. \tag{7.8}$$

Let us remark that the above condition is obviously a generalization of (7.6).

Theorem 7.1. *Let $n \geq 3$, and let* (K1) *and* (K2) *hold. Then* (7.1) *has a positive solution provided $\widetilde{K} = 1 + \varepsilon\widetilde{k}$ and ε is sufficiently small.*

Proof. Let $y_0 \in S^n$ denote the absolute minimum of $\widetilde{K}$. We use stereographic coordinates with north pole y_0 and, setting $K = \widetilde{K} \circ \pi^{-1}$ and $k = \widetilde{k} \circ \pi^{-1}$, we find that (7.1) is equivalent (up to an uninfluent constant) to the following equation on $\mathbb{R}^n$:

$$-\Delta u = K(x)\, u^{\frac{n+2}{n-2}}, \quad u > 0, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^n). \tag{7.9}$$

This is exactly the equation studied in Section 5.2 with $K = 1 + \varepsilon k$. It is easy to see that k satisfies the conditions (k.0)–(k.3) stated in Section 5.2. Furthermore, condition (K2) immediately implies that (5.6) holds. Then we are in position to apply Theorem 5.3 yielding a solution of (7.1) provided $\varepsilon \ll 1$. □

Theorem 7.1 can be used as a starting point to prove the following global result, see [100].

Theorem 7.2. *Let $n \geq 3$. In addition to* (K1) *let us suppose that $\widetilde{K} \in C^{2,\alpha}(S^n)$ and that the function $\widetilde{K}$ near any $x \in \mathrm{Cr}[\widetilde{K}]$ satisfies the flatness condition*

$$\widetilde{K}(y) = \widetilde{K}(x) + \sum_{i=1}^{n} a_i |y_i - x_i|^\beta; \qquad a_i \neq 0, \quad \sum_{i=1}^{n} a_i \neq 0, \quad \beta \in (n-2, n). \tag{7.10}$$

Then (7.1) *has a positive solution provided*

$$\sum_{x\in \mathrm{Cr}[\widetilde{K}],\sum a_i(x)<0} (-1)^{m(\widetilde{K},x)} \neq (-1)^n. \tag{7.11}$$

Proof. (Sketch) Roughly, the proof is based on three main steps.

Step 1. Let us consider the family $\widetilde{K}_t := t\widetilde{K} + (1-t)$ depending on the parameter $t \in [\varepsilon_0, 1]$ and the corresponding equations

$$\mathcal{L}u = \widetilde{K}_t u^{\frac{n+2}{n-2}}, \quad (\mathcal{L}u = -2c_n \Delta_{\overline{g}_0} u + R_{\overline{g}_0} u). \tag{7.12}$$

Let $X := \{u \in C^2(S^n)\ ;\ \ u > 0\}$ and consider the compact perturbation of the Identity F_t defined by setting $F_t(u) = u - \mathcal{L}^{-1}(\widetilde{K}_t u^{\frac{n+2}{n-2}})$, $u \in X$. According to Theorem 7.1, we know that for $t = \varepsilon$, $0 < \varepsilon \ll 1$, the scalar curvature problem for

$\widetilde{K}_\varepsilon$ has a solution. More precisely, using the assumption (7.10) with $\beta > n-2$, it is possible to show that for any $\delta > 0$ all the solutions of

$$-\Delta u = K_t u^{\frac{n+2}{n-2}}, \quad u \in D^{1,2}(\mathbb{R}^n), \quad u > 0, \quad (K_t = \widetilde{K}_t \circ \pi^{-1}),$$

with $t = \varepsilon_0$ are in a δ-neighborhood of the *critical manifold* Z (see Section 5.2) for sufficiently small ε. This fact and the degree arguments carried out in Section 5.2, see in particular Remark 5.6-(ii), readily imply that there exists a bounded open set $\mathcal{O}_{\varepsilon_0} \subset X$ such that

$$\deg(F_{\varepsilon_0}, \mathcal{O}_{\varepsilon_0}, 0) = \sum_{x \in \mathrm{Cr}[\widetilde{K}_{\varepsilon_0}], \sum a_i(x) < 0} (-1)^{m(\widetilde{K}_{\varepsilon_0}, x)} - (-1)^n.$$

Such an equation and the assumption (7.11) imply

$$\deg(F_{\varepsilon_0}, \mathcal{O}_{\varepsilon_0}, 0) \neq 0. \tag{7.13}$$

Step 2. Using the assumption (7.10) with $\beta < n$ jointly with a fine blow-up analysis, one proves that for all $\varepsilon_0 \le t \le 1$ the solutions of (7.12) stay in a compact subset (depending on t) of X. This compactness result is the counterpart of Theorem 6.1 dealing with the Yamabe problem.

Step 3. From Step 2 and using the homotopy invariance of the Leray-Schauder degree, it follows that there exists a bounded open set $\mathcal{O}_t \subset X$ such that $\deg(F_t, \mathcal{O}_t, 0)$ is constant. In particular, taking $t = 1$ and $t = \varepsilon_0$ and using (7.13) one infers that

$$\deg(F_1, \mathcal{O}_1, 0) = \deg(F_{\varepsilon_0}, \mathcal{O}_{\varepsilon_0}, 0) \neq 0.$$

Thus there exists $u \in X$ such that $F_1(u) = 0$, namely such that $\mathcal{L}u = \widetilde{K} u^{\frac{n+2}{n-2}}$. $\square$

Remark 7.3. It is worth pointing out that the flatness condition (7.10) is not necessary when $\widetilde{K}$ is close to a constant. On the other hand, counterexamples are given in [100] showing that, for the non-perturbative Scalar Curvature Problem, assumption (7.10) cannot be removed, in general. $\square$

We finally mention that in [36], [100], Part II and in [60], the Scalar Curvature Problem in dimension $n > 3$ without the flatness (7.10) has been discussed.

The non-perturbative problem for dimension greater than 3 requires different approaches, which we do not discuss here. About this topic, see [47], [73], [115].

7.2 Problems with symmetry

In this section we will shortly discuss the case in which $\widetilde{K}$ is invariant under a group of isometries $\Sigma \subset \mathbf{O}(n+1)$, namely $\widetilde{K}(\sigma x) = \widetilde{K}(x)$, for all $\sigma \in \Sigma$ and all $x \in S^n$, $n \geq 3$. We will denote by $F_\Sigma = \{x \in S^n : \sigma x = x, \forall \sigma \in \Sigma\}$ the fixed point set of Σ and by $O_\Sigma(x) = \{\sigma x : \sigma \in \Sigma\}$ the orbit of x through the action of Σ.

Extending Moser's work cited above, an existence result in the presence of symmetries was given in [78] for dimension $n = 3$ assuming that $\widetilde{K}$ is invariant under some group Σ such that $F_\Sigma = \emptyset$, and that $\widetilde{K}$ satisfies some suitable flatness assumptions, like (7.10), at its maximal points. Other sufficient conditions for the existence in the case of Σ-invariant functions were given in [93], removing the assumption that the action of Σ is fixed-point free.

Below we will first consider the perturbation case when $\widetilde{K}$ is close to a positive constant. We will always assume that $\widetilde{K}$ is positive and of class C^2 on S^n.

7.2.1 The perturbative case

When $\widetilde{K}$ is close to a positive constant, say $\widetilde{K} = 1 + \varepsilon \widetilde{k}$, it is possible to use the abstract perturbation method to find solutions of the symmetric Scalar Curvature Problem

$$-2c_n \Delta_{\overline{g}_0} u + R_{\overline{g}_0} u = (1 + \varepsilon \widetilde{k})\, u^{\frac{n+2}{n-2}}; \qquad u > 0 \text{ on } S^n. \tag{7.14}$$

We will outline below some of these results taken from [20] where we also refer for more details and further results.

Letting $k = \widetilde{k} \circ \pi^{-1}$, we are willing to find, for ε sufficiently small, a solution of a problem like

$$-\Delta u = (1 + \varepsilon \widetilde{k})\, u^{\frac{n+2}{n-2}}, \quad u > 0, \quad u \in D^{1,2}(\mathbb{R}^n). \tag{7.15}$$

According to the arguments carried out in Section 5.2, let us consider the reduced functional Φ_ε with its leading part given by

$$\Gamma(\mu, \xi) = \int_{\mathbb{R}^n} \widetilde{k}(\mu y + \xi) U^{2^*}(y) dy, \qquad (\mu, \xi) \in \mathbb{R}^+ \times \mathbb{R}^n.$$

The question that we have to address is which symmetry is induced to Γ and Φ_ε by the Σ-invariance of $\widetilde{k}$. For this, we first extend any $\sigma \in \Sigma$ to $\mathbb{R}^{n+1}$ by homogeneity and then consider the group $\widetilde{\Sigma}$ acting on S^{n+1} through the isometries $\widetilde{\sigma}$

$$\widetilde{\sigma}(x_1, x) = (x_1, \sigma(x)),$$

where the points of S^{n+1} are written in the form (x_1, x) with $x \in \mathbb{R}^{n+1}$. With this notation, we define the action σ^* on $\mathbb{R}^+ \times \mathbb{R}^n$ by the following diagram

$$\begin{array}{rccccccc} \sigma^*: & \mathbb{R}^+ \times \mathbb{R}^n & \xrightarrow{\pi^{-1}} & S^{n+1} & \xrightarrow{\widetilde{\sigma}} & S^{n+1} & \xrightarrow{\pi} & \mathbb{R}^+ \times \mathbb{R}^n \\ & (\mu, \xi) & \longrightarrow & (x_1, x) & \longrightarrow & (x_1, \sigma(x)) & \longrightarrow & \tau(\mu, \xi) \end{array}$$

Here we have used the same notation π to denote the stereographic projection from S^{n+1} to $\mathbb{R}^{n+2}$. Let Σ^* denote the group of all the σ^*'s.

To simplify the exposition, we will consider below three specific symmetry groups, which are the prototype of the general case. For points $x \in S^n$ we write $x = (x', x_{n+1})$ with $x' \in \mathbb{R}^n$.

- Σ_1, with elements Id and $\sigma : x = (x', x_{n+1}) \mapsto (-x', x_{n+1})$,
- Σ_2, with elements Id and $\sigma : x = (x', x_{n+1}) \mapsto (x', -x_{n+1})$,
- Σ_3, with elements Id and $\sigma : x = (x', x_{n+1}) \mapsto -x = (-x', -x_{n+1})$.

Using the definition of π,one finds that the groups Σ_i^* corresponding to Σ_i, $i = 1, 2, 3$, are the following ones:

- Σ_1^*, with elements Id and $\sigma^* : (\mu, \xi) \mapsto (\mu, -\xi)$,
- Σ_2^*, with elements Id and

$$\sigma^* : (\mu, \xi) \mapsto \left(\frac{\mu}{\mu^2 + |\xi|^2}, \frac{\xi}{\mu^2 + |\xi|^2} \right),$$

- Σ_3^*, with elements Id and

$$\sigma^* : (\mu, \xi) \mapsto \left(\frac{\mu}{\mu^2 + |\xi|^2}, \frac{-\xi}{\mu^2 + |\xi|^2} \right).$$

The role of Σ^* is made clear by the fact that it is possible to show:

Lemma 7.4. *If $\widetilde{k}$ is Σ-invariant then Γ and Φ_ε are invariant with respect to Σ^*. As a consequence, Γ' and Φ'_ε are tangent to the fixed point set F_{Σ^*}. In particular, at any isolated point of $(\bar{\mu}, \bar{\xi}) \in F_{\Sigma^*}$, one has that $\Gamma'(\bar{\mu}, \bar{\xi}) = 0$ and $\Phi'_\varepsilon(\bar{\mu}, \bar{\xi}) = 0$.*

To apply this lemma we need to find the fixed point set of each of the groups Σ_i^*. One immediately obtains:

- $F_{\Sigma_1^*} = \{(\mu, 0)\}_{\mu > 0}$,
- $F_{\Sigma_2^*} = \{(\mu, \xi) : \mu > 0,\ \mu^2 + |\xi|^2 = 1\}$,
- $F_{\Sigma_3^*} = \{(1, 0)\}$.

According to Lemma 7.4 it suffices to study Γ or Φ_ε constrained on $F_{\Sigma_i^*}$ $(i = 1, 2, 3)$ and this yields to find solutions of (7.14) by imposing conditions only on $\widetilde{k}$ restricted to F_{Σ_i}. For example, if $\widetilde{k}$ is invariant with respect to Σ_1, then $F_{\Sigma_1} = (P_N, -P_N)$, where $P_N = (0, \ldots, 0, 1)$ denotes the north pole on S^n. In this case we have:

Theorem 7.5. *Let $\widetilde{k}$ be Σ_1-invariant and suppose that one of the following conditions holds*

(a) $\widetilde{k}(P_N) \geq \widetilde{k}(-P_N)$ *and* $\Delta_{\bar{g}_0} \widetilde{k}(-P_N) < 0$,
(b) $\widetilde{k}(P_N) \leq \widetilde{k}(-P_N)$ *and* $\Delta_{\bar{g}_0} \widetilde{k}(-P_N) > 0$.

Then for ε sufficiently small, (7.14) *has a solution.*

Proof. Using Lemma 7.4 we can consider Γ restricted to $F_{\Sigma_1^*}$, namely

$$\Gamma(\mu,0)=\int_{\mathbb{R}^n} k(\mu y)U^{2^*}(y)dy.$$

Letting $y=\pi(x)$, we find

$$\Gamma(\mu,0)=\int_{S^n}\widetilde{k}(\mu x)dV_{\bar g_0}.$$

Then one gets

$$\lim_{\mu\to+\infty}\Gamma(\mu,0)=\omega_n\widetilde{k}(P_N),\qquad\qquad \lim_{\mu\to 0}\Gamma(\mu,0)=\omega_n\widetilde{k}(-P_N),$$

where $\omega_n=\int_{S^n}dV_{\bar g_0}$, and (a) implies

$$\Gamma(0,0):=\quad\lim_{\mu\to 0}\Gamma(\mu,0)\le\lim_{\mu\to+\infty}\Gamma(\mu,0).$$

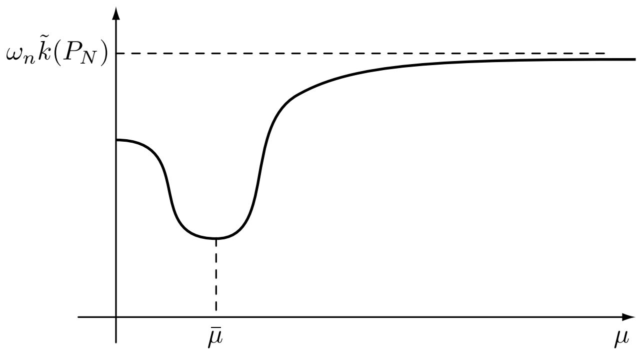

Figure 7.1. Graph of $\Gamma(\mu,0)$

Moreover, as in Section 5.2, we find that

$$D^2_{\mu\mu}\Gamma(0,0)=a_1\Delta k(0)=a_2\Delta_{\overline{g}_0}(-P_N)<0\quad(a_1,a_2>0).$$

Hence Γ achieves the absolute minimum at some $(\bar\mu,0)$ with $\bar\mu>0$ and the existence of a solution of (7.14) for ε sufficiently small, follows from Theorem 2.16. The proof in the case (b) is similar. □

Remark 7.6. If $\widetilde{k}(P_N)=\widetilde{k}(-P_N)$, the condition $\Delta_{\bar g_0}\widetilde{k}(-P_N)\neq 0$ is not necessary. Actually, it is possible to show that $\widetilde{k}(P_n)=\widetilde{k}(-P_N)$ implies

$$\lim_{\mu\to 0}\Phi_\varepsilon(\mu,0)=\lim_{\mu\to+\infty}\Phi_\varepsilon(\mu,0).$$

Hence $\Phi_\varepsilon(\mu,0)$ has a stationary point at some $\bar\mu>0$ and the result follows from Theorem 2.12. □

Arguments quite similar to those carried out in Theorem 7.5 can be used when $\widetilde{k}$ is invariant with respect Σ_2, yielding

Theorem 7.7. *Let $\widetilde{k}$ be Σ_2-invariant and suppose that there exists $\bar{x} \in F_{\Sigma_2}$ such that either $\widetilde{k}(\bar{x}) = \max\{\widetilde{k}(x) : x \in F_{\Sigma_2}\}$ and $\Delta_{\bar{g}_0}\widetilde{k}(\bar{x}) > 0$, or $\widetilde{k}(\bar{x}) = \min\{\widetilde{k}(x) : x \in F_{\Sigma_2}\}$ and $\Delta_{\bar{g}_0}\widetilde{k}(\bar{x}) < 0$. Then for ε sufficiently small,* (7.14) *has a solution.*

Finally, if $\widetilde{k}$ is invariant with respect Σ_3 then $F_{\Sigma_3} = \emptyset$ and $F_{\Sigma_3^*} = \{(1,0)\}$. Hence, using the last statement of Lemma 7.4 we immediately infer that the fixed point $(1,0)$ is stationary for Φ_ε and gives rise to a solution of (7.14). More in general, if the action of the group Σ is free, namely $F_\Sigma = \emptyset$, then one has that F_{Σ^*} is the single point $\{(1,0)\}$ which is a stationary point of Φ_ε. This shows

Theorem 7.8. *Let $\widetilde{k}$ be invariant with respect a group Σ such that $F_\Sigma = \emptyset$. Then for ε sufficiently small,* (7.14) *has a solution.*

We conclude this section dealing with a generalization of the Σ_1-invariance. Precisely, we consider the group $\Sigma_{1,\ell}$, $1 \le \ell < n$, acting on S^n through

$$x = (x_1, \dots, x_{n+1}) \quad \mapsto \quad (-x_1, \dots, -x_\ell, x_{\ell+1}, \dots, x_{n+1}).$$

For $\ell = n$ this is nothing but Σ_1. We introduce the notation

$$\mathcal{S}_\ell := F_{\Sigma_{1,\ell}} = \{x \in S^n : x_1 = \cdots = x_\ell = 0\}.$$

One readily finds that for the corresponding $\Sigma_{1,\ell}^*$ there results

$$F_{\Sigma_{1,\ell}^*} = \mathbb{R}^+ \times \{\xi \in \mathbb{R}^n : \xi_1 = \cdots = \xi_\ell = 0\}.$$

According to Lemma 7.4 we have to study Γ constrained on $F_{\Sigma_{1,\ell}^*}$. Repeating the arguments used in Section 5.2 and in Theorem 7.1, we find

Theorem 7.9. *Let $\widetilde{k}$ be invariant with respect to $\Sigma_{1,\ell}$, and suppose that the following conditions holds*

($\widetilde{\mathrm{k}}$1) *$X_\ell := \mathrm{Cr}[\widetilde{\ell}] \cap \mathcal{S}_\ell$ is finite, every $x \in X_\ell$ is non-degenerate for $\widetilde{k}$ on $\mathcal{S}_\ell$ and $\Delta_{\bar{g}_0}\widetilde{k}(x) \neq 0$ for any $x \in X_\ell$;*

($\widetilde{\mathrm{k}}$2) *there results*
$$\sum_{x \in X_\ell, \Delta_{\bar{g}_0}\widetilde{k}(x) < 0} (-1)^{m_\ell(\widetilde{k},x)} \neq (-1)^{n-\ell}, \tag{7.16}$$
where $m_\ell(\widetilde{k},x)$ denotes the Morse index of x as critical point of $\widetilde{k}$ on $\mathcal{S}_\ell$. Then for ε sufficiently small, (7.14) *has a solution.*

Remark 7.10. Adding a flatness condition like (7.10), it is possible to extend the preceding result proving the existence of a solution of the symmetric Scalar Curvature Problem in the non-perturbative case. In this way we obtain the symmetric version of Theorem 7.2. See [17]. □

7.3 Prescribing Scalar and Mean Curvature on manifolds with boundary

In this section we will deal with problems arising in conformal differential geometry on manifolds with boundary. We will focus on a specific but interesting case: when the manifold is the upper half-sphere $S^n_+ = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : |x| = 1, x_{n+1} > 0\}$, $n \geq 3$. More precisely, we consider the unit ball in $\mathbb{R}^n$, $B = \{x \in \mathbb{R}^n : |x| < 1\}$ endowed with a smooth metric g. Let ν_g and h_g denote, respectively, the outward unit normal to $\partial B = S^{n-1}$ with respect to g and the mean curvature of (S^{n-1}, g). Given two smooth functions K and h, we will look for positive solutions $u \in H^1(B)$ of

$$\begin{cases} -2c_n \Delta_g u + R_g u = K u^{\frac{n+2}{n-2}}, & \text{in } B \quad \left(c_n = 2\frac{(n-1)}{(n-2)}\right) \\ \frac{2}{(n-2)} \partial_{\nu_g} u + h_g u = h u^{\frac{n}{n-2}}, & \text{on } S^{n-1}. \end{cases} \tag{7.17}$$

If $u > 0$ is a smooth solution of (7.17) then $\widetilde{g} = u^{4/(n-2)} g$ is a metric, conformally equivalent to g, such that K is the scalar curvature of $(B, \widetilde{g})$ and h is the mean curvature of $(S^{n-1}, \widetilde{g})$. Up to a stereographic projection (through the south pole), this is equivalent to finding a conformal metric on the upper half-sphere S^n_+ such that the scalar curvature of S^n_+ and the mean curvature of $\partial S^n_+ = S^{n-1}$ are prescribed functions.

Following [18], we will discuss in the sequel the perturbative case. For the sake of brevity, we will state the main results but we will only outline the arguments, avoiding the technicalities.

7.3.1 The Yamabe-like problem

When K and h are constant functions, say $K \equiv 1$ and $h \equiv c$, (7.17) is the analogue of the Yamabe problem. In such a case, (7.17) becomes

$$\begin{cases} -2\, c_n \Delta_g u + R_g u = u^{\frac{n+2}{n-2}}, & \text{in } B \\ \frac{2}{(n-2)} \partial_{\nu_g} u + h_g u = c\, u^{\frac{n}{n-2}}, & \text{on } \partial B = S^{n-1}. \end{cases} \tag{7.18}$$

This problem has been first studied in [61], where the regularity of solutions is also proved. Further results can be found in, [76, 77]. More recently, some general results were proven in [91, 92]. It is shown that a solution to (7.18) exists provided that (B, g) is of positive type (for a definition see [91]) and satisfies one of the following assumptions:

(i) (B, g) is locally conformally flat and ∂B is umbilic (a point of ∂B is said *umbilic* if the differential of the Gauss map is diagonal, and ∂B is said *umbilic* if every point of ∂B is umbilic. In particular this is the case for the standard half-sphere S^n_+);

(ii) $n \geq 5$ and ∂B is not umbilic.

It is also proved that the set of solutions is compact in $C^{2,\alpha}(\overline{B})$.

We are going to show that in the perturbative case none of the above conditions is required. Precisely, we will deal with a metric g close to the standard one g_0. Our main result is:

Theorem 7.11. *Given $M > 0$ there exists $\varepsilon_0 > 0$ such that* (7.18) *has a positive solution provided $c > -M$, $0 < \varepsilon < \varepsilon_0$, and g satisfies*

$$\|g - g_0\|_{L^\infty(B)} < \varepsilon; \qquad \|\nabla g\|_{L^n(B)} < \varepsilon; \qquad \|\nabla g\|_{L^{n-1}(S^{n-1})} < \varepsilon. \tag{7.19}$$

The proof relies on the abstract perturbation results discussed in Chapter 2, see in particular Theorem 2.23. Here we take $\mathcal{H} = H^1(B)$, endowed with scalar product

$$(u|v) = 2c_n \int_B \nabla u \cdot \nabla v dx + 2(n-1) \int_{S^{n-1}} uv d\sigma$$

and norm $\|u\|^2 = (u|u)$, and set

$$I_g(u) = c_n \int_B |\nabla_g u|^2 dV_g + \tfrac{1}{2} \int_B R_g u^2 dV_g - \tfrac{1}{2^*} \int_B |u|^{2^*} dV_g$$
$$+ (n-1) \int_{S^{n-1}} h_g u^2 d\sigma_g - c(n-2) \int_{S^{n-1}} |u|^{2\frac{n-1}{n-2}} d\sigma_g.$$

Plainly, the critical points of I_g on $\mathcal{H}$ give rise to the solutions of (7.18). If $g = g_\varepsilon$ satisfies (7.19) the functional $I_\varepsilon := I_{g_\varepsilon}$ has the perturbative form

$$I_\varepsilon(u) = I_0(u) + O(\varepsilon),$$

where the unperturbed functional is given by

$$I_0^c(u) = \tfrac{1}{2}\|u\|^2 - \tfrac{1}{2^*} \int_B |u|^{2^*} dx - c(n-2) \int_{S^{n-1}} |u|^{2\frac{n-1}{n-2}} d\sigma.$$

Above, we have emphasized the dependence on the constant c because the result stated in Theorem 7.11 is not uniform with respect to c. The critical points of I_0^c in $\mathcal{H}$ are the solutions of

$$\begin{cases} -2\, c_n \Delta u = u^{\frac{n+2}{n-2}}, & \text{in } B \\ \frac{2}{(n-2)} \partial_\nu u + u = c u^{\frac{n}{n-2}}, & \text{on } \partial B = S^{n-1}. \end{cases} \tag{7.20}$$

In order to find the unperturbed critical manifold, we set

$$U(x) = \left(\frac{\kappa}{1 + |x|^2} \right)^{(n-2)/2}, \qquad \kappa = [4n(n-1)]^{\frac{1}{2}}$$

and

$$z_{\mu,\xi}(x) = \mu^{-(n-2)/2} U\left(\frac{x - \xi}{\mu} \right).$$

Clearly, $z = z_{\mu,\xi}$ solves the equation $-2\,c_n\Delta z = z^{\frac{n+2}{n-2}}$ in B. Moreover, a direct calculation shows that $z_{\mu,\xi}$ satisfies the boundary conditions whenever

$$\mu^2 + |\xi|^2 - c\kappa\mu = 1, \qquad \mu > 0.$$

Hence I_ε has an unperturbed critical manifold given by

$$Z = Z^c = \{z = z_{\mu,\xi} : (\mu, \xi) \in \mathbb{R}^+ \times \mathbb{R}^n,\ \mu^2 + |\xi|^2 - c\kappa\mu = 1\}.$$

Using arguments similar to those carried out in Lemma 5.2, it is possible to show that Z satisfies (ND), see [91], namely it is a non-degenerate critical manifold. Moreover, letting $\lambda_i(c)$ denote the non-zero eigenvalues of $D^2 I_0^c(z)[v] = \lambda v$, one can prove that

- the first eigenvalue $\lambda_1(c)$ is negative;
- Let $\lambda_2(c)$ denote the first *positive* eigenvalue of $D^2 I_0^c(z)[v] = \lambda v$. Then one has that $\forall\, M > 0$, $\exists\, C_M > 0$ such that

$$\frac{1}{C_M} \le |\lambda_i(c)| \le C_M, \quad \forall\, c \ge -M, \quad i = 1, 2.$$

This implies that the restriction of $D^2 I_0^c(z)$ to $(T_z Z)^\perp$ is invertible and the inverse $L_c(z)$ is uniformly bounded, in the sense that $\forall\, M > 0$, $\exists\, C > 0$ such that

$$\|L_c(z)\| \le C, \quad \forall\, z \in Z, \quad \forall\, c > -M.$$

Let us point out that there is a numerical evidence that $\lambda_2(c) \to 0$ as $c \to -\infty$ and hence it does not seem possible to obtain a bound on L_c *uniform* with respect to $c \in \mathbb{R}$.

The preceding results allow us to find a solution $w_{\mu,\xi}$ of the auxiliary equation $P(I_\varepsilon^c)'(z + w) = 0$ (for all $c > -M$ and $\varepsilon \ll 1$) is such a way that the stationary points of the reduced functional $\Phi_\varepsilon^c(\mu, \xi) = I_\varepsilon^c(z_{\mu,\xi} + w_{\mu,\xi})$ give rise to critical points of I_ε^c, according to Theorem 2.23. Finally, as for the Yamabe problem in Chapter 6, one proves that

$$\lim_{\mu \to 0} \Phi_\varepsilon^c(\mu, \xi) = \text{const. (depending on } c),$$

and hence Φ_ε^c can be continuously extended to ∂Z. Since the $Z \cup \partial Z$ is compact, it follows that either Φ_ε^c is identically constant, or it achieves the maximum or the minimum at some point in Z. In any case Φ_ε^c possesses a stationary point in Z, which yields a solution of (7.18), proving Theorem 7.11.

7.3.2 The Scalar Curvature Problem with boundary conditions

Here we consider the case in which g is the standard metric on B while $K = 1 + \varepsilon k(x)$ and $h = c + \varepsilon h_0(x)$. The corresponding equations become

$$\begin{cases} -2c_n \Delta u + u = (1 + \varepsilon k(x)) u^{\frac{n+2}{n-2}}, & \text{in } B \\ \frac{2}{(n-2)} \partial_\nu u + u = (c + \varepsilon h_0(x)) u^{\frac{n}{n-2}}, & \text{on } S^{n-1}. \end{cases} \tag{7.21}$$

In this case the functional $I_\varepsilon : \mathcal{H} \to \mathbb{R}$ (here c is fixed and so its dependence is omitted to simplify notation) has the form

$$I_\varepsilon(u) = I_0(u) + \varepsilon G(u),$$

where I_0 is as in the preceding subsection and

$$G(u) = \tfrac{1}{2^*} \int_B k(x)|u|^{2^*} dx + (n-2) \int_{S^{n-1}} h_0(x)|u|^{2\frac{n-1}{n-2}} d\sigma.$$

We point out that the exponent $\frac{2(n-1)}{n-2}$ is critical for the (trace) embedding $W^{1,2}(B) \hookrightarrow L^p(S^{n-1})$. Using the discussion made before, we are in position to apply here Theorem 2.16. In the present framework we have that

$$\Gamma(\mu,\xi) = \tfrac{1}{2^*} \int_B k(x) z_{\mu,\xi}^{2^*} dx + (n-2) \int_{S^{n-1}} h_0(\sigma) z_{\mu,\xi}^{2\frac{n-1}{n-2}} d\sigma.$$

As before, μ and ξ are related by the equation

$$\mu^2 + |\xi|^2 - c\kappa\mu = 1.$$

As for the problems with critical exponent discussed in Section 5.2, we need to study the behavior of Γ on the boundary of Z. Taking into account that

$$\partial Z = \{z_{\mu,\xi_0} : \mu = 0,\ |\xi_0| = 1\},$$

computations similar to those carried out in Section 5.2 yield

Lemma 7.12. *Let $|\xi_0| = 1$ and let ν_0 denote the outer normal direction to ∂Z at $(0,\xi_0)$. Then one has (a_i below denote positive constants depending explicitly on k, h_0):*

(i) $\Gamma(0,\xi_0) = a_1 k(\xi_0) + a_2 h_0(\xi_0)$,
(ii) $\Gamma'_{\nu_0}(0,\xi_0) = a_3 \langle k'(\xi_0), \xi_0\rangle$.

For $\xi \in S^{n-1} = \partial B$, let us put $\psi(\xi) = a_1 k(\xi) + a_2 h_0(\xi)$. With this notation one has that $\Gamma_{|\partial Z} = \psi$.

Theorem 7.13. *Suppose that one of the two following conditions holds:*

(ψ1) *ψ has an absolute maximum (or minimum) $\xi \in S^{n-1}$ such that $\langle k'(\xi), \xi\rangle < 0$ (resp. $\langle k'(\xi), \xi\rangle$);*

(ψ2) *ψ is a Morse function such that*

$$\langle k'(\xi), \xi\rangle \neq 0, \quad \forall\, \xi \in \mathrm{Cr}[\psi], \tag{7.22}$$

$$\sum_{\xi \in \mathrm{Cr}[\psi],\ \langle k'(\xi),\xi\rangle < 0} (-1)^{m(\psi,\xi)} \neq 1. \tag{7.23}$$

Then (7.21) has a positive solution provided ε is sufficiently small.

Proof. Let (ψ1) hold. Since $Z \cup \partial Z$ is compact, Γ achieves the absolute maximum (or minimum) at some $\bar{x} = (\bar{\mu}, \bar{\xi}) \in Z \cup \partial Z$. If such a point lies on ∂Z, we get that $\bar{\mu} = 0$ and $|\bar{\xi}| = 1$. By Lemma 7.12-(i) we know that $\Gamma(0, \bar{\xi}) = \psi(\bar{\xi})$ and thus $\bar{\xi}$ is an absolute maximum (or minimum) of ψ on S^{n-1}. Then assumption (ψ1) implies that $\langle k'(\bar{\xi}), \bar{\xi}\rangle < 0$ (resp. $\langle k'(\bar{\xi}), \bar{\xi}\rangle > 0$). According to Lemma 7.12-(ii) we infer that $\Gamma'_{\nu_0}(0, \bar{\xi}) < 0$ (resp. $\Gamma'_{\nu_0}(0, \bar{\xi}) > 0$), which is in contradiction with the fact that $(0, \bar{\xi})$ is the absolute maximum (or minimum) of Γ on $Z \cup \partial Z$. Therefore Γ achieves either the absolute maximum or the absolute minimum in the interior of $Z \cup \partial Z$. Then an application of Theorem 2.16 yields the result.

Let (ψ2) hold. We claim that

$$\deg(\Gamma', Z, 0) \neq 0, \tag{7.24}$$

where deg denotes, as usual, the topological degree. By Lemma 7.12-(ii) we have that $\Gamma'_{\nu_0}(x) \neq 0$ at any $x \in \mathrm{Cr}[\psi] = \mathrm{Cr}[\Gamma_{|\partial Z}]$. Moreover, the *negative boundary* of ∂Z, defined by $\partial Z^- = \{(0, \xi_0) \in \partial Z : \Gamma'_{\nu_0}(0, \xi_0) < 0\}$ is given by

$$\partial Z^- = \{(0, \xi_0) : |\xi_0| = 1,\ \langle k'(\xi_0), \xi\rangle_0 < 0\}.$$

In other words, the set $\{x \in \mathrm{Cr}[\psi] : \langle k'(x), x\rangle < 0\}$ coincides with the set $\mathrm{Cr}[\psi] \cap \partial Z^-$. Recalling a well-known result in the theory of the topological degree, see [85] and taking into account that $\Gamma_{|\partial Z} = \psi$, we get

$$\deg(\Gamma', Z, 0) = 1 - \sum_{x \in \mathrm{Cr}[\psi] \cap \partial Z^-} (-1)^{m(\psi, x)}.$$

Then the preceding arguments yield

$$\deg(\Gamma', Z, 0) = 1 - \sum_{x \in \mathrm{Cr}[\psi],\ \langle k'(x), x\rangle < 0} (-1)^{m(\psi, x)},$$

and the claim follows from the second assumption in (ψ2). Now, (7.24) allows us to use Theorem 2.17 from which we infer the existence of a stationary point of I_ε. This completes the proof of Theorem 7.13. $\square$

Remarks 7.14.

(i) It is possible to modify the preceding arguments to handle the case in which $K = \varepsilon k$, improving the results of [59]. It is also possible to prove some existence result for (7.21) when $\langle k'(x), x\rangle = 0$ at some $x \in \mathrm{Cr}[\psi]$, as well as when k, h_0 inherit some symmetry, like in Section 7.2. We do not carry over this material, referring to Theorems 7 and 8 of [18].

(ii) Condition (ψ2) is related to the assumptions (K1) and (K2) made in Theorem 7.1. On the contrary, the assumption (ψ1) has no counterpart in the Scalar Curvature Problem on S^n, but is a specific feature of problems dealing with manifolds with boundary.

(iii) Theorem 7.13 is the first step in proving the existence of solutions of (7.17) with K and h not necessarily close to constants. For this topics we refer to [74]. $\square$

Chapter 8

Nonlinear Schrödinger Equations

In this chapter we consider standing waves of the Nonlinear Schrödinger Equation, namely solutions to the following problem

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = u^p, & \text{in } \mathbb{R}^n \\ u > 0, \quad u \in W^{1,2}(\mathbb{R}^n), \end{cases} \tag{8.1}$$

where $p > 1$ is subcritical and V is a smooth bounded potential. We will be mainly interested in the behavior of the solutions as $\varepsilon \to 0^+$, the so-called *semiclassical limit*. Roughly we will show that there exist *spikes*, namely solutions concentrating at single points of $\mathbb{R}^n$ (the precise meaning of *concentration* is given in (8.2) below).

The chapter is organized as follows. First we show that concentration of spikes necessarily occurs at stationary points of V. In Section 8.2 we prove the existence of solutions concentrating at non-degenerate critical points of V. The remaining four sections of the chapter are devoted to deal with a more general situation, when V has a non-degenerate *manifold* of critical points and multiple spikes can possibly occur, see Theorem 8.5. This case requires a modification of the abstract setting. The interest of such a more general approach goes much beyond the proof of Theorem 8.5 because it is more flexible then the one discussed so far. Actually, this new tool can be used in other situations, in particular when one looks for solutions concentrating at spheres, see Chapter 10.

8.1 Necessary conditions for existence of spikes

Here and throughout in the sequel we assume that $1 < p < \frac{n+2}{n-2}$, and we make the following assumptions on the potential V

(V1) $V \in C^2(\mathbb{R}^n)$, and $\|V\|_{C^2(\mathbb{R}^n)} < +\infty$;
(V2) $\lambda_0^2 = \inf_{\mathbb{R}^n} V > 0$.

We say that a solution v_ε of (8.1) concentrates at x_0 (as $\varepsilon \to 0$) provided

$$\forall\, \delta > 0, \quad \exists\, \varepsilon_0 > 0,\ R > 0\ : v_\varepsilon(x) \le \delta,\ \forall\, |x - x_0| \ge \varepsilon\, R,\ \varepsilon < \varepsilon_0. \tag{8.2}$$

In this section we prove the following result:

Theorem 8.1. *Let* (V1) *and* (V2) *hold, and suppose that v_ε are solutions of* (8.1) *concentrating at x_0, in the sense of the definition* (8.2). *Then $V'(x_0) = 0$.*

Proof. We follow closely the arguments of [144]. First we prove that there exists $C > 0$ such that for all ε small one has

$$\|v_\varepsilon\|_{L^\infty} \le C. \tag{8.3}$$

Otherwise, there is a sequence $\varepsilon_k \to 0$ such that $v_k \equiv v_{\varepsilon_k}$ diverges in $L^\infty(\mathbb{R}^n)$. Let $m_k = \max v_k = v_k(x_k)$, $\mu_k = m_k^{-(p-1)/2}$ and

$$\phi_k(x) = \frac{1}{m_k} v_k(x_k + \varepsilon_k \mu_k x).$$

One has that ϕ_k verifies

$$\begin{cases} -\Delta \phi_k + \mu_k^2 V(x_k + \varepsilon_k \mu_k x)\phi_k = \phi_k^p, \\ \phi_k(0) = 1, \qquad 0 \le \phi_k(x) \le 1. \end{cases}$$

Since $\|\phi_k\|_{L^\infty} = 1$, up to subsequence $\phi_k \to \phi_0$ in $C^2_{loc}(\mathbb{R}^n)$. Moreover, since $\mu_k \to 0$ and V is bounded, it follows that ϕ_0 satisfies

$$-\Delta \phi_0 = \phi_0^p, \quad \text{in } \mathbb{R}^n, \quad \phi_0(0) = 1.$$

But, according to a well-known result by Gidas and Spruck, [82], the only entire non-negative solution of $-\Delta\phi = \phi^p$, with $1 < p < \frac{n+2}{n-2}$, is $\phi = 0$ and so we reach a contradiction, proving (8.3).

Next, let us set $\widetilde{v}_k(x) = v_k(x_0 + \varepsilon_k x)$. The function $\widetilde{v}_k$ satisfies

$$-\Delta \widetilde{v}_k + V(x_0 + \varepsilon_k x)\widetilde{v}_k = \widetilde{v}_k^p, \quad \text{in } \mathbb{R}^n.$$

As before, $\widetilde{v}_k$ converges to some $\widetilde{v}_0$ in $C^2_{loc}(\mathbb{R}^n)$ such that

$$-\Delta \widetilde{v}_0 + V(x_0)\widetilde{v}_0 = \widetilde{v}_0^p, \quad \text{in } \mathbb{R}^n.$$

Indeed, see [144], one can prove that $\widetilde{v}_k \to \widetilde{v}_0$ in $L^2(\mathbb{R}^n)$, as well. Furthermore, since v_k achieves its maximum at x_k, then $\Delta v_k(x_k) \le 0$. From $-\varepsilon_k^2 \Delta v_k(x_k) + V(x_k)v_k(x_k) = v_k^p(x_k)$ it follows that $m_k = v_k(x_k)$ satisfies $V(x_k)m_k \le m_k^p$. This, (V2) and $m_k > 0$ yield

$$m_k \ge \lambda_0^{1/(p-1)}.$$

In particular, this implies that $\widetilde{v}_0 \not\equiv 0$. Using a generalized Pohozaev identity, see [125], we have that

$$\begin{aligned}\tfrac{1}{2}\,\varepsilon_k \int_{B_R} & V'(x_0+\varepsilon_k x)\widetilde{v}_k^2 \\ &= \int_{\partial B_R} \left[(\tfrac{1}{2} V(x_0+\varepsilon_k x)\widetilde{v}_k^2 - \tfrac{1}{p+1}\widetilde{v}_k^{p+1} + \tfrac{1}{2}|\nabla \widetilde{v}_k|^2)\nu - \nabla \widetilde{v}_k \, \frac{\widetilde{v}_k}{\partial \nu} \right] d\sigma \qquad (8.4)\end{aligned}$$

where B_R is the ball centered in 0 with radius R and ν denotes the outer unit normal to ∂B_R. Let us denote by ℓ_R the integral on the right-hand side of (8.4). Since

$$|\ell_R| \le c_1 \int_{\partial B_R} \left[|\nabla \widetilde{v}_k|^2 + V(x_0+\varepsilon_k x)\widetilde{v}_k^2 + \widetilde{v}_k^{p+1} \right] d\sigma$$

we infer that, for each fixed k,

$$\int_0^\infty |\ell_R| dR \le c_1 \int_0^\infty dR \int_{\partial B_R} \left[|\nabla \widetilde{v}_k|^2 + V(x_0+\varepsilon_k x)\widetilde{v}_k^2 + \widetilde{v}_k^{p+1} \right] ds \le +\infty,$$

because $\widetilde{v}_k \in W^{1,2}(\mathbb{R}^n)$. Thus $\ell_R \to 0$ ar $R \to \infty$ (up to a subsequence) and, passing to the limit into (8.4), the Dominated Convergence Theorem (recall that V is bounded) yields, for each fixed k:

$$\int_{\mathbb{R}^n} V'(x_0+\varepsilon_k x)\widetilde{v}_k^2 = 0.$$

Therefore, letting $k \to \infty$ and recalling that $\widetilde{v}_k \to \widetilde{v}_0$ in $L^2(\mathbb{R}^n)$, we get

$$\int_{\mathbb{R}^n} V'(x_0)\widetilde{v}_0^2 = 0,$$

and this, since $\widetilde{v}_0 \not\equiv 0$, implies that $V'(x_0) = 0$. □

Remark 8.2. In [144] it is also proved that if v_ε is a solution of (8.1) with *minimal energy* concentrating at x_0, then x_0 is a global minimum of V. Moreover, any solution concentrating at some x_0 has a unique maximum which converges to x_0. This justifies the name *spikes* given to these solutions. □

8.2 Spikes at non-degenerate critical points of V

The main purpose of this section is to prove the following theorem.

Theorem 8.3. *Let* (V1) *and* (V2) *hold, and suppose x_0 is a non-degenerate critical point of V, namely for which $V''(x_0)$ is non-singular. Then there exists a solution $\bar{v}_\varepsilon$ of* (1.12) *which concentrates at x_0 as $\varepsilon \to 0$.*

Actually, this theorem is a particular case of a more general result, see Theorem 8.5 in Section 8.5 later on. For this reason, we will limit ourselves to outline the arguments, referring for more details to [10].

To simplify notation (and without loss of generality) we will suppose that $x_0 = 0$ and that $V(0) = 1$. To frame (8.1) in the abstract setting, we first make the change of variable $x \mapsto \varepsilon x$ and rewrite equation (8.1) as

$$\begin{cases} -\Delta u + V(\varepsilon x) u = u^p, & \text{in } \mathbb{R}^n, \\ u > 0, \quad u \in W^{1,2}(\mathbb{R}^n). \end{cases} \tag{8.5}$$

If $u_\varepsilon(x)$ is a solution of (8.5) then $v_\varepsilon(x) := u_\varepsilon(x/\varepsilon)$ solves (1.14). We set $\mathcal{H} = W^{1,2}(\mathbb{R}^n)$ and consider the functional $I_\varepsilon \in C^2(\mathcal{H}, \mathbb{R})$,

$$I_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^n} \left(|\nabla u|^2 + V(\varepsilon x) u^2 \right) - \frac{1}{p+1} \int_{\mathbb{R}^n} |u|^{p+1}. \tag{8.6}$$

Hereafter we endow $\mathcal{H}$ with the norm

$$\|u\|^2 = \int_{\mathbb{R}^n} \left(|\nabla u|^2 + V(\varepsilon x) u^2 \right) dx,$$

and we denote by $(\cdot|\cdot)$ the corresponding scalar product. With this notation, the functional I_ε takes, for $\varepsilon = 0$, the form

$$I_0(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p+1} \int_{\mathbb{R}^n} |u|^{p+1}.$$

Let us highlight that I_0 plays the role of the unperturbed functional by writing

$$I_\varepsilon(u) = I_0(u) + \tfrac{1}{2} \int_{\mathbb{R}^n} \left(V(\varepsilon x) - 1 \right) u^2 dx \equiv I_0(u) + G(\varepsilon, u).$$

Obviously, for any fixed $u \in \mathcal{H}$, we have $G(\varepsilon, u) \to 0$ as $\varepsilon \to 0$ and hence I_ε has the form discussed in Section 2.3. As in Chapter 4, letting U denote the radial positive solution of $-\Delta u + u = u^p$, $u \in W^{1,2}(\mathbb{R}^n)$, the unperturbed critical manifold is given by

$$Z = \{ z_\xi(x) := U(x - \xi) : \xi \in \mathbb{R}^n \}$$

and is non-degenerate. Unfortunately, as anticipated in Section 2.4, we cannot directly apply the results proven in Section 2.3 because, in general, $G''(\varepsilon, u)$ does not tend to zero $\varepsilon \to 0$. To see this, let us consider a sequence $v_j \in \mathcal{H}$ with compact support contained in $\{x \in \mathbb{R}^n : |x| > 1/j\}$. If, for example, the potential V is such that $V(x) - 1 \equiv c > 0$ for all $|x| \geq 1$, then evaluating $G''(\varepsilon, u)[v_j]^2$ for $\varepsilon = 1/j$ we find

$$G''(\varepsilon, u)[v_j]^2 = \int_{\mathbb{R}^n} \left(V(\varepsilon x) - 1 \right) v_j^2 dx = c\|v_j\|^2.$$

However, the first part of the abstract procedure can be still carried over. Denoted by P the orthogonal projection onto $W = (T_z Z)^{\perp}$, we look for solutions $u = z_\xi + w$, with $z_\xi \in Z$ and $w \in W$, of the system

$$\begin{cases} PI'_\varepsilon(z_\xi + w) & = 0, \\ (I - P)I'_\varepsilon(z_\xi + w) & = 0 \end{cases}$$

which is clearly equivalent to $I'_\varepsilon(z_\xi + w) = 0$. At this point the Implicit Function Theorem was used to find a solution $w_\varepsilon(z_\xi)$ of the auxiliary equation $PI'_\varepsilon(z_\xi + w) = 0$, for all $z_\xi \in Z$. Instead, we argue as in the proof of Lemma 2.21. First we write $PI'_\varepsilon(z_\xi + w) = PI'_\varepsilon(z_\xi) + PD^2 I_\varepsilon(z_\xi)[w] + R(z_\xi, w)$, where $R(z_\xi, w) = o(\|w\|)$, uniformly with respect to $z_\xi \in Z$ for bounded $|\xi|$. Next, using arguments similar to those carried out in Lemma 8.9 of the next section, one shows that there exists $C > 0$ such that for ε small enough one has

$$\|PI''_\varepsilon(z_\xi)\| \geq C, \qquad \forall\, z_\xi \in Z, \quad \text{for } |\xi| \text{ bounded.}$$

Setting $A_{\varepsilon,\xi} = -(PI''_\varepsilon(z_\xi))^{-1}$, the equation $PI'_\varepsilon(z_\xi + w) = 0$ can be written in the form

$$w = A_{\varepsilon,\xi}\left(PI'_\varepsilon(z_\xi) + R(z_\xi, w)\right) := N_{\varepsilon,\xi}(w).$$

It is also possible to show that $N_{\varepsilon,\xi}$ is a contraction in some ball of W provided ε is sufficiently small. This allows us to solve the auxiliary equation finding $w_\varepsilon(z_\xi)$ which is of class C^1 with respect to ξ. Furthermore, since $V'(0) = 0$, one finds that $w_\varepsilon(z_\xi) = O(\varepsilon^2)$, uniformly with respect to bounded ξ. At this point we can repeat the usual arguments that lead to look for stationary points of the (finite-dimensional) reduced functional $\Phi_\varepsilon(\xi) = I_\varepsilon(z_\xi + w_\varepsilon(z_\xi))$. One finds that

$$\Phi_\varepsilon(\xi) = c_0 + \varepsilon^2\, \Gamma(\xi) + o(\varepsilon^2),$$

where $c_0 = I_0(U)$ and

$$\Gamma(\xi) = \tfrac{1}{2} \int_{\mathbb{R}^n} \langle V''(0)x, x\rangle U^2(x - \xi)dx.$$

A straight calculation yields

$$\begin{aligned} \Gamma(\xi) &= \tfrac{1}{2} \int_{\mathbb{R}^n} \langle V''(0)(y + \xi), (y + \xi)\rangle U^2(y)dy \\ &= \tfrac{1}{2} \int_{\mathbb{R}^n} \langle V''(0)y, y\rangle U^2(y)dy + \tfrac{1}{2} \int_{\mathbb{R}^n} \langle V''(0)\xi, \xi\rangle U^2(y)dy \\ &= c_1 + c_2 \langle V''(0)\xi, \xi\rangle, \end{aligned}$$

where

$$c_1 = \tfrac{1}{2} \int_{\mathbb{R}^n} \langle V''(0)y, y\rangle U^2(y)dy, \qquad c_2 = \tfrac{1}{2} \int_{\mathbb{R}^n} U^2(x)dx.$$

Then $\xi = 0$ is a non-degenerate critical point of Γ and therefore, from the general theory it follows that for $\varepsilon \ll 1$, I_ε has a critical point $u_\varepsilon = z_{\xi_\varepsilon} + w_\varepsilon(z_{\xi_\varepsilon})$, with $\xi_\varepsilon \to 0$ as $\varepsilon \to 0$. In conclusion, coming back to the solutions v_ε of (8.1) , we find that this equation has a solution $\bar{v}_\varepsilon(x) \sim U(\frac{x-\xi_\varepsilon}{\varepsilon})$ that concentrates at $x = 0$, proving Theorem 8.3.

Remarks 8.4.

(i) According to Theorem 2.24 we infer that the solution $\bar{u}_\varepsilon$ has Morse index equal $1 + k$ where k is the index of $x_0 = 0$ as critical point of V on $\mathbb{R}^n$. In particular, the Morse index of $\bar{u}_\varepsilon$ is 1 whenever V has a minimum at $x_0 = 0$. This fact has an important consequence concerning the orbital stability of the standing waves found above. See below.

(ii) Simple modifications of the preceding arguments show that the same existence result holds if we suppose that $V(x) = 1 + a|x|^m + o(|x|^m)$ as $|x| \to 0$, where $a \neq 0$ and $m > 0$ is an even integer. □

We end this section with a brief discussion on the orbital stability of the standing wave $\bar{u}_\varepsilon$ found in Theorem 8.3. Let us consider the solitary wave corresponding to the solution $\bar{u}_\varepsilon$

$$\psi_\varepsilon(t, x) = \exp\left(i\,\alpha\,\hbar^{-1} t\right) \bar{u}_\varepsilon(x). \tag{8.7}$$

This function ψ is a solution of the evolutionary NLS introduced in Chapter 1, Section 1.3

$$i\hbar\,\frac{\partial \psi}{\partial t} = -\hbar^2 \Delta\psi + Q(x)\psi - |\psi|^{p-1}\psi, \tag{8.8}$$

where $V(x) = \alpha + a_0 + Q(x)$, see the notation used in Section 1.3.

We say that $\bar{u}_\varepsilon$ is *orbitally stable* if a solution $\psi(t, x)$ of the equation (8.8) exists for all $t \geq 0$ and remains $W^{1,2}$-close to the solitary wave (8.7) provided $\psi(0, x)$ is sufficiently close $\bar{u}_\varepsilon(x)$ in $W^{1,2}(\mathbb{R}^n)$. Since the orbital stability depends on the frequency α, we will write below $\bar{u}_{\varepsilon,\alpha}$ instead of $\bar{u}_\varepsilon$.

Let $m_{\varepsilon,\alpha}$ denote the Morse index of $\bar{u}_{\varepsilon,\alpha}$ as a critical point of I_ε and let

$$\mu(\varepsilon, \alpha) := \frac{\partial}{\partial \alpha} \int_{\mathbb{R}^n} |\bar{u}_{\varepsilon,\alpha}(x)|^2 dx.$$

According to Theorem 2 and Section 6.D of [86]-Part I, and to the Instability Theorem discussed in [86]-Part II, we know that $\bar{u}_{\varepsilon,\alpha}$ is orbitally stable provided $m_{\varepsilon,\alpha} = 1$ and $\mu(\varepsilon, \alpha) > 0$. Furthermore, if either $m_{\varepsilon,\alpha} > 1$ or $m_{\varepsilon,\alpha} = 1$ but $\mu(\varepsilon, \alpha) < 0$, we have instability.

Therefore, taking also into account the Remark 8.4-(i), a necessary condition for the standing wave $\bar{u}_{\varepsilon,\alpha}$ to be orbitally stable is that x_0 is a minimum of V. If this is the case, we do have orbital stability provided $\mu(\varepsilon, \alpha) > 0$. It has been shown in [86] that in the one-dimensional case $\mu(\varepsilon, \alpha) > 0$ provided Q is constant and $1 < p < 5$. Recently, this result has been extended to a class of potentials $Q(x)$, depending on x, see [107].

8.3 The general case: Preliminaries

The rest of the Chapter is devoted to consider the more general case in which V has a non-degenerate manifold of critical points (see the precise definition later on), which requires a different approach that is useful in other problems like those discussed in Chapters 9 and 10. As sketched in Section 2.4, the idea is to find an n-dimensional manifold $\mathcal{Z}^\varepsilon$ of *pseudo-critical points*, which can be perturbed to obtain a *natural constraint* $\tilde{\mathcal{Z}}^\varepsilon$ for I_ε. Namely, a critical point of I_ε restricted to $\tilde{\mathcal{Z}}^\varepsilon$ is also a critical point for I_ε. See Proposition 8.7 later on.

More precisely, we will suppose that V has a smooth compact manifold of critical points M, which is non-degenerate (for V) in the sense that for every $x \in M$ one has that $T_x M = \mathrm{Ker}[V''(x)]$. Obviously, this definition coincides with the non-degeneracy condition (ND) introduced in Chapter 2.

The main result of this second part is the following theorem:

Theorem 8.5. *Let* (V1) *and* (V2) *hold and suppose V has a non-degenerate smooth compact manifold of critical points M. Then for $\varepsilon > 0$ small,* (8.1) *has at least $l(M)$* [1] *solutions that concentrate near points of M.*

The proof of this theorem will be given at the end of Section 8.5.

First, in this section, we recall the variational structure of the problem and we collect some useful results. We consider again the equation(8.5) whose Euler functional is defined in (8.6). Throughout this section, we use again the space $\mathcal{H} = W^{1,2}$ with the norm

$$\|u\|^2 = \int_{\mathbb{R}^n} \left(|\nabla u|^2 + V(\varepsilon x) u^2\right) dx.$$

Let us recall that the radial solution U of

$$\begin{cases} -\Delta u + u = u^p & \text{in } \mathbb{R}^n_+; \\ u(x) \to 0 & \text{as } |x| \to +\infty; \\ u > 0 \end{cases} \tag{P_0}$$

satisfies

$$\lim_{r \to +\infty} r^{\frac{n-1}{2}} e^r U(r) = \alpha_{n,p}; \qquad \lim_{r \to +\infty} \frac{U'(r)}{U(r)} = -1; \tag{8.9}$$

for some positive constant $\alpha_{n,p}$ depending only on n and p.
We also need to consider the following variant of problem (P_0), namely

$$\begin{cases} -\Delta u + \lambda^2 u = u^p & \text{in } \mathbb{R}^n; \\ u(x) \to 0 & \text{as } |x| \to +\infty; \\ u > 0, \end{cases} \tag{P_λ}$$

[1] $l(M)$ denotes the cup long of M, defined in Section 2.2.

where $\lambda > 0$. It is immediate to check from the arguments of Chapter 4 and some scaling that the function U_λ and all its translates are solutions of (P_λ), where

$$U_\lambda(x) = \lambda^{\frac{2}{p-1}} U(\lambda x); \qquad\qquad x \in \mathbb{R}^n.$$

The function U_λ is a critical point of the functional $\overline{I}_\lambda : W^{1,2}(\mathbb{R}^n) \to \mathbb{R}$ defined as

$$\overline{I}_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^n} \left(|\nabla u|^2 + \lambda^2 u^2\right) - \frac{1}{p+1}\int_{\mathbb{R}^n} |u|^{p+1}; \qquad u \in W^{1,2}(\mathbb{R}^n), \tag{8.10}$$

and is natural here to endow the Sobolev space $W^{1,2}(\mathbb{R}^n)$ with the scalar product

$$(u,v)_\lambda = \int_{\mathbb{R}^n} \left(|\nabla u|^2 + \lambda^2 u^2\right); \qquad u, v \in W^{1,2}(\mathbb{R}^n). \tag{8.11}$$

The reason of considering the functional $\overline{I}_\lambda$ is that, freezing the argument of the potential V in I_ε as $x = \overline{x}$, and setting $\lambda = V(\varepsilon\overline{x})$, we obtain exactly $\overline{I}_\lambda$. We will find approximate solutions of the form U_λ, for suitable values of λ, and therefore it is fundamental to understand the properties of the linearization of (P_λ) at U_λ. This is the content of the next lemma, which proof is identical to that of Lemma 4.1.

Lemma 8.6. *For every $\xi \in \mathbb{R}^n$, $U_\lambda(\cdot - \xi)$ is a critical point of $\overline{I}_\lambda$. Moreover, the kernel of $\overline{I}''_\lambda(U_\lambda)$ is generated by $\frac{\partial U_\lambda}{\partial x_1}, \dots, \frac{\partial U_\lambda}{\partial x_n}$. The operator has only one negative eigenvalue, and therefore there exists $\delta_\lambda > 0$, depending continuously on λ such that*

$$\overline{I}''_\lambda(U_\lambda)[v,v] \geq \delta_\lambda \|v\|^2_\lambda \quad \forall\, v \in W^{1,2}(\mathbb{R}^n_+), v \perp_\lambda U_\lambda, v \perp_\lambda \frac{\partial U_\lambda}{\partial x_1}, \dots, v \perp_\lambda \frac{\partial U_\lambda}{\partial x_n}.$$

Here the symbol $\perp_\lambda$ means orthogonality with respect to the scalar product $(\cdot,\cdot)_\lambda$ defined in (8.11).

We also recall the following elementary inequalities, which hold true for all $a, b, b_1, b_2 \in \mathbb{R}$, with $|a| \leq 1$.

$$\left|(a+b)^p - a^p - pa^{p-1}b\right| \leq \begin{cases} C|b|^p & \text{for } p \leq 2, \\ C\left(|b|^2 + |b|^p\right) & \text{for } p > 2; \end{cases} \tag{8.12}$$

$$\begin{aligned} &\left|(a+b_1)^p - (a+b_2)^p - pa^{p-1}(b_1 - b_2)\right| \\ &\quad \leq \begin{cases} C\left(|b_1|^{p-1} + |b_2|^{p-1}\right)|b_1 - b_2| & \text{for } p \leq 2, \\ C\left(|b_1|^{p-1} + |b_2|^{p-1} + |b_1| + |b_2|\right)|b_1 - b_2| & \text{for } p > 2; \end{cases} \end{aligned} \tag{8.13}$$

$$\left|(a+b)^{p-1} - a^{p-1}\right| \leq \begin{cases} C|b|^{p-1} & \text{for } p \leq 2, \\ C\left(|b| + |b|^{p-1}\right) & \text{for } p > 2, \end{cases} \tag{8.14}$$

where the constant C depends only on p.

8.4 A modified abstract approach

We tackle the problem as follows. We find first a manifold $\mathcal{Z}^\varepsilon$ of *pseudo-critical points* for I_ε, namely a family of functions z_ξ, $\xi \in \mathbb{R}^n$, for which $\|I'_\varepsilon(z_\xi)\|$ is small. Then the Contraction Mapping Theorem allows us to perform a local inversion orthogonally to $T\mathcal{Z}^\varepsilon$, uniformly for $\xi \in \mathbb{R}^n$. This will provide a natural constraint $\tilde{\mathcal{Z}}^\varepsilon$ for I_ε, see Proposition 8.7, which is homeomorphic and close to $\mathcal{Z}^\varepsilon$.

We set

$$z^{\varepsilon\xi}(x) = U_\lambda(x) = \alpha(\varepsilon\xi)U(\beta(\varepsilon\xi)x); \qquad \xi \in \mathbb{R}^n, \tag{8.15}$$

where $\lambda^2 = V(\varepsilon\xi)$, and

$$\beta(\varepsilon\xi) = (V(\varepsilon\xi))^{\frac{1}{2}}; \qquad \alpha(\varepsilon\xi) = (\beta(\varepsilon\xi))^{\frac{2}{p-1}}.$$

Then we define

$$\mathcal{Z}^\varepsilon = \{z^{\varepsilon\xi}(x-\xi) : \xi \in \mathbb{R}^n\}.$$

When there is no possible misunderstanding we will write z, resp. $\mathcal{Z}$, instead of $z^{\varepsilon\xi}$, resp $\mathcal{Z}^\varepsilon$. We will also use the symbol z_ξ to denote the function $z_\xi(x) := z^{\varepsilon\xi}(x-\xi)$. All the functions in $z_\xi \in \mathcal{Z}$ are solutions of (P_λ), $\lambda^2 = V(\varepsilon\xi)$, or equivalently critical points of $\overline{I}_\lambda$. Basically, in order to find approximate solutions, we freeze the argument of V at the maximum of z_ξ. Since $V(\varepsilon\cdot)$ varies slowly for ε small and since z_ξ decays exponentially, z_ξ represents a good approximate solution to (8.5).

For future reference, let us point out some estimates. First of all, we evaluate

$$\begin{aligned}\partial_\xi z^{\varepsilon\xi}(x-\xi) &= \partial_\xi\left[\alpha(\varepsilon\xi)U(\beta(\varepsilon\xi)(x-\xi))\right]\\ &= \varepsilon\nabla\alpha(\varepsilon\xi)U(\beta(\varepsilon\xi)(x-\xi))\\ &\quad + \varepsilon\alpha(\varepsilon\xi)\nabla\beta(\varepsilon\xi)U(\beta(\varepsilon\xi)(x-\xi))\\ &\quad - \alpha(\varepsilon\xi)\beta(\varepsilon\xi)\nabla U(\beta(\varepsilon\xi)(x-\xi)).\end{aligned}$$

Recalling the definition of α, β and using the assumptions (V1) and (V2) one finds

$$\partial_\xi z^{\varepsilon\xi}(x-\xi) = -\nabla z^{\varepsilon\xi}(x-\xi) + O(\varepsilon|V'(\varepsilon\xi)|), \qquad \text{in } W^{1,2}(\mathbb{R}^n). \tag{8.16}$$

The main result of this section is the following proposition.

Proposition 8.7. *Let V satisfy the assumptions* (V1), (V2). *Then for $\varepsilon > 0$ small there exists a unique $w = w(\varepsilon,\xi) \in (T_{z_\xi}\mathcal{Z})^\perp$ such that $I'_\varepsilon(z_\xi + w) \in T_{z_\xi}\mathcal{Z}$. The function $w(\varepsilon,\xi)$ is of class C^1 with respect to ξ and there holds*

$$\|\partial_\xi w\| \le C\left[\left(\varepsilon|V'(\varepsilon\xi)| + \varepsilon^2\right) + \left(\varepsilon|V'(\varepsilon\xi)| + \varepsilon^2\right)^{p-1}\right]. \tag{8.17}$$

Moreover the functional $\Phi_\varepsilon(\xi) = I_\varepsilon(z_\xi + w(\varepsilon,\xi))$ is also of class C^1 in ξ and satisfies

$$\Phi'_\varepsilon(\xi_0) = 0 \quad \Longrightarrow \quad I'_\varepsilon\left(z_{\xi_0} + w(\varepsilon,\xi_0)\right) = 0.$$

In order to prove this proposition, we need to show that $\|I'_\varepsilon(z_\xi)\|$ is small, and that I''_ε is invertible on the orthogonal complement of $T_{z_\xi}Z^\varepsilon$. These two facts are proven respectively in Lemmas 8.8 and 8.9 below.

Lemma 8.8. *Assume* (V1), (V2) *hold. Then there exists* $C > 0$ *such that for all* $\xi \in \mathbb{R}^n$ *and all* $\varepsilon > 0$ *small, one has*

$$\|I'_\varepsilon(z_\xi)\| \le C\left(\varepsilon|V'(\varepsilon\xi)| + \varepsilon^2\right).$$

Proof. Since

$$I_\varepsilon(u) = \overline{I}_\lambda(u) + \frac{1}{2}\int_{\mathbb{R}^n}[V(\varepsilon x) - V(\varepsilon\xi)]\,u^2 dx; \qquad \lambda^2 = V(\varepsilon\xi),$$

and since z_ξ is a critical point of $\overline{I}_\lambda$, one has

$$I'_\varepsilon(z_\xi)[v] = \overline{I}'_\lambda(z_\xi)[v] + \int_{\mathbb{R}^n}[V(\varepsilon x) - V(\varepsilon\xi)]\,z_\xi v\,dx = \int_{\mathbb{R}^n}[V(\varepsilon x) - V(\varepsilon\xi)]\,z_\xi v\,dx.$$

Using the Hölder inequality, one finds

$$|I'_\varepsilon(z_\xi)[v]|^2 \le \|v\|^2_{L^2(R^n)}\int_{\mathbb{R}^n}|V(\varepsilon x) - V(\varepsilon\xi)|^2 z_\xi^2 dx. \tag{8.18}$$

From the assumption (V1), namely that $|V''(x)| \le C$, one infers

$$|V(\varepsilon x) - V(\varepsilon\xi)| \le C\varepsilon|V'(\varepsilon\xi)|\,|x-\xi| + C\varepsilon^2|x-\xi|^2, \qquad \forall x,\xi \in \mathbb{R}^n. \tag{8.19}$$

This implies

$$\begin{aligned}\int_{\mathbb{R}^n}&|V(\varepsilon x) - V(\varepsilon\xi)|^2 z_\xi^2 dx\\ &\le C\varepsilon^2|V'(\varepsilon\xi)|^2\int_{\mathbb{R}^n}|x-\xi|^2 z_\xi^2(x)dx + C\varepsilon^4\int_{\mathbb{R}^n}|x-\xi|^4 z_\xi^2(x)dx.\end{aligned} \tag{8.20}$$

Recalling the exponential decay of U and the definition of z_ξ, see (8.15), a direct calculation yields

$$\begin{aligned}\int_{\mathbb{R}^n}|x-\xi|^2 z^2(x-\xi)dx &= \alpha^2(\varepsilon\xi)\int_{\mathbb{R}^n}|y|^2U^2(\beta(\varepsilon\xi)y)dy\\ &= \alpha^2\beta^{-n-2}\int_{\mathbb{R}^n}|y'|^2U^2(y')dy' \le C.\end{aligned}$$

From this (and a similar calculation for the last integral in (8.20)) one derives

$$\int_{\mathbb{R}^n}|V(\varepsilon x) - V(\varepsilon\xi)|^2 z_\xi^2 dx \le C\varepsilon^2|V'(\varepsilon\xi)|^2 + C\varepsilon^4. \tag{8.21}$$

Putting together (8.18) and (8.21), the lemma follows. $\square$

Lemma 8.9. *Under the assumptions* (V1) *and* (V2) *there exists* $C > 0$ *such that for* ε *small enough one has*

$$I_\varepsilon''(z_\xi)[v,v] \geq C^{-1}\|v\|^2, \qquad \forall\, \xi \in \mathbb{R}^n,\ \forall\, v \in (z_\xi \oplus T_{z_\xi}\mathcal{Z}^\varepsilon)^\perp. \tag{8.22}$$

Proof. From (8.16) it follows that every element $\zeta \in T_{z_\xi}\mathcal{Z}$ can be written in the form $\zeta = -\nabla_x z^{\varepsilon\xi}(x-\xi)+O(\varepsilon)$. As a consequence it suffices to prove the following property

$$I_\varepsilon''(z_\xi)[v,v] \geq C^{-1}\|v\|^2, \qquad \forall\, \xi \in \mathbb{R}^n,\ \forall\, v \in \left(\mathrm{span}\,\{z_\xi, \partial_{x_1} z_\xi, \dots, \partial_{x_n} z_\xi\}\right)^\perp. \tag{8.23}$$

Let $R \gg 1$ and consider a radial smooth function $\chi_R : \mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{cases} \chi_R(x) = 1, & \text{in } B_R(0); \\ \chi_R(x) = 0 & \text{in } \mathbb{R}^n \setminus B_{2R}(0); \\ |\nabla \chi_R| \leq \frac{2}{R} & \text{in } B_{2R}(0) \setminus B_R(0), \end{cases} \tag{8.24}$$

and we set

$$v_1(x) = \chi_R(x-\xi)v(x); \qquad\qquad v_2 = (1-\chi_R)(x-\xi)v(x).$$

A straight computation yields

$$\|v\|^2 = \|v_1\|^2 + \|v_2\|^2 + 2\int_{\mathbb{R}^n} \left[\nabla v_1 \cdot \nabla v_2 + v_1\, v_2\right].$$

We write $\int_{\mathbb{R}^n} [\nabla v_1 \cdot \nabla v_2 + v_1\, v_2] = \tau_1 + \tau_2$, where

$$\tau_1 = \int_{\mathbb{R}^n} \chi_R(1-\chi_R)(v^2 + |\nabla v|^2); \quad \tau_2 = \int_{\mathbb{R}^n} v_2 \nabla v \cdot \nabla \chi_R - v_1 \nabla v \cdot \nabla \chi_R - v^2 |\nabla \chi_R|^2.$$

Since the integrand in τ_2 is supported in $\{R \leq |x| \leq 2R\}$, using the inequality in (8.24) and the Hölder's inequality we deduce that $|\tau_2| = o_R(1)\|v\|^2$. As a consequence we have

$$\|v\|^2 = \|v_1\|^2 + \|v_2\|^2 + 2\tau_1 + o_R(1)\|v\|^2. \tag{8.25}$$

After these preliminaries, let us evaluate $I_\varepsilon''(z_\xi)[v,v] = \sigma_1 + \sigma_2 + \sigma_3$, where

$$\sigma_1 = I_\varepsilon''(z_\xi)[v_1,v_1]; \qquad \sigma_2 = I_\varepsilon''(z_\xi)[v_2,v_2]; \qquad \sigma_3 = 2I_\varepsilon''(z_\xi)[v_1,v_2].$$

There holds

$$\sigma_1 = I_\varepsilon''(z_\xi)[v_1,v_1] = \overline{I}_\lambda''(U_\lambda)[v_1,v_1] + \int_{\mathbb{R}^n} \left[V(\varepsilon x) - V(\varepsilon\xi)\right] v_1^2. \tag{8.26}$$

We introduce now the function $\overline{v}_1 = v_1 - \psi$, where

$$\psi = \frac{1}{\|z_\lambda\|_\lambda^2}(v_1|z_\xi)_\lambda z_\xi + \sum_{i=1}^n \frac{1}{\|\partial_{x_i} z_\lambda\|_\lambda^2}(v_1|\partial_{x_i} z_\xi)_\lambda \partial_{x_i} z_\xi.$$

Then we have

$$\overline{I}_\lambda''(z_\xi)[v_1,v_1] = \overline{I}_\lambda''(z_\xi)[\overline{v}_1,\overline{v}_1] + \overline{I}_\lambda''(z_\xi)[\psi,\psi] + 2\overline{I}_\lambda''(z_\xi)[\overline{v}_1,\psi]. \tag{8.27}$$

Let us explicitly point out that $\overline{v}_1 \perp_\lambda \operatorname{span}\{z_\xi, \partial_{x_1} z_\xi, \dots, \partial_{x_n} z_\xi\}$ and hence Lemma 8.6 implies

$$\overline{I}_\lambda''(z_\xi)[\overline{v}_1,\overline{v}_1] \ge \delta_\lambda \|\overline{v}_1\|_\lambda^2. \tag{8.28}$$

On the other hand, since $(v|z_\xi) = 0$ it follows that

$$(v_1|z_\xi)_\lambda = (v|z_\xi)_\lambda - (v_2|z_\xi)_\lambda = -\int_{\mathbb{R}^n} v(V(\varepsilon x) - V(\varepsilon\xi))z_\xi - (v_2|z_\xi)_\lambda.$$

Since v_2 is supported in $|x-\xi| \ge R$ and since z_ξ tends exponentially to zero at infinity, we infer $(v_1|z_\xi)_\lambda = o_{R,\varepsilon}(1)\|v\|$. Similarly one shows $(v_1|\partial_{x_i} z_\xi)_\lambda = o_R(1)\|v\|$, and it follows that

$$\|\psi\| = o_{R,\varepsilon}(1)\|v\|. \tag{8.29}$$

Putting together (8.28) and (8.29) we infer

$$\overline{I}_\lambda''(z_\xi)[v_1,v_1] \ge \|\overline{v}_1\|_\lambda^2 + o_{R,\varepsilon}(1)\|v\|^2 = \|v_1\|_\lambda^2 + o_{R,\varepsilon}(1)\|v\|^2.$$

The last equation and (8.26) imply

$$\begin{aligned} \sigma_1 &\ge \delta_l \|v_1\|_\lambda^2 + \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon\xi)]v_1^2 + o_{R,\varepsilon}\|v\|^2 \\ &\ge \delta_\lambda \|v_1\|^2 - (1+\delta_\lambda)\int_{\mathbb{R}^n} |V(\varepsilon x) - V(\varepsilon\xi)|v_1^2. \end{aligned} \tag{8.30}$$

Using arguments already carried out before, the last integral can be estimated as

$$\begin{aligned} \int_{\mathbb{R}^n} |V(\varepsilon x) - V(\varepsilon\xi)|v_1^2 dx &\le \varepsilon C \int_{\mathbb{R}^n} |x-\xi|\chi_R^2(x-\xi)v^2(x)dx \\ &\le \varepsilon C \int_{\mathbb{R}^n} y\chi_R(y)v^2(y+\xi)dy \le \varepsilon C\|v\|^2. \end{aligned}$$

This and (8.30) yield

$$\begin{aligned} \sigma_1 = I_\varepsilon''(z_\xi)[v_1,v_1] &\ge C^{-1}\|v_1\|^2 - \varepsilon C\|v\|^2 + o_{R,\varepsilon}(1)\|v\|^2 \\ &\ge C^{-1}\|v_1\|^2 + o_{R,\varepsilon}(1)\|v\|^2. \end{aligned} \tag{8.31}$$

Let us now estimate σ_2. One finds

$$\begin{aligned} \sigma_2 = I_\varepsilon''(z_\xi)[v_2,v_2] &= \int_{\mathbb{R}^n} |\nabla v_2|^2 + \int_{\mathbb{R}^n} V(\varepsilon x)v_2^2 - p\int_{\mathbb{R}^n} z_\xi^{p-1}v_2^2 \\ &= \|v_2\|^2 - p\int_{\mathbb{R}^n} z_\xi^{p-1}v_2^2. \end{aligned}$$

As before, $v_2(x) = 0$ for all x with $|x - \xi| < R$ and the exponential decay of z at infinity imply

$$\sigma_2 \geq C^{-1}\|v_2\|^2 + o_R(1)\|v\|^2. \tag{8.32}$$

In a similar way one shows that

$$\sigma_3 \geq C^{-1}\tau_1 + o_R(1)\|v\|^2. \tag{8.33}$$

Finally, (8.31), (8.32), (8.33) yield

$$I''_\varepsilon(z_\xi)[v,v] = \sigma_1 + \sigma_2 + \sigma_3 \geq C^{-1}\left[\|v_1\|^2 + \|v_2\|^2 + \tau_1\right] + o_R(1)\|v\|^2.$$

Recalling (8.25) we infer that

$$I''_\varepsilon(z_\xi)[v,v] \geq C^{-1}\|v\|^2 + o_R(1)\|v\|^2.$$

Taking ε small and R large, equation (8.22) follows. This completes the proof of Lemma 8.9. □

Lemma 8.10. *Let P_ξ denote the projection onto $(T_{z_\xi}\mathcal{Z}^\varepsilon)^\perp$. Then for ε sufficiently small the operator $L_\xi = P_\xi \circ I''_\varepsilon(z_\xi) \circ P_\xi$ is invertible for every $\xi \in \mathbb{R}^n$ and there exists $C > 0$ such that*

$$\|L_\xi^{-1}\| \leq C; \qquad \xi \in \mathbb{R}^n.$$

Proof. We decompose $(T_{z_\xi}\mathcal{Z}^\varepsilon)^\perp$ as $(T_{z_\xi}\mathcal{Z}^\varepsilon)^\perp = V_1 \oplus V_2$, where

$$V_1 = \langle P_\xi z_\xi \rangle; \qquad V_2 = \left(z_\xi \oplus T_{z_\xi}\mathcal{Z}^\varepsilon\right)^\perp; \qquad V_1 \perp V_2.$$

We will prove the following two properties

$$\|z_\xi - P_\xi z_\xi\| = o_\varepsilon(1); \qquad\qquad I''_\varepsilon(z_\xi)[z_\xi] = -(p-1)z_\xi + o_\varepsilon(1). \tag{8.34}$$

These indeed imply

$$\begin{aligned} L_\xi(z_\xi) &= P_\xi I''_\varepsilon(z_\xi) P_\xi z_\xi = P_\xi(I''_\varepsilon(z_\xi)[z_\xi] + o_\varepsilon(1)) \\ &= P_\xi\left(-(p-1)z_\xi + o_\varepsilon(1)\right) = -(p-1)P_\xi z_\xi + o_\varepsilon(1). \end{aligned}$$

Hence the operator L_ξ, in matrix form with respect to the spaces V_1 and V_2, can be decomposed as

$$L_\xi = \begin{pmatrix} -(p-1)Id + o_\varepsilon(1) & o_\varepsilon(1) \\ o_\varepsilon(1) & A_\xi \end{pmatrix},$$

where A_ξ, according to (8.22), satisfies $A_\xi \geq C^{-1}Id$, and the Lemma would follow.

It remains to prove (8.34). Given $i \in \{1,\dots,n\}$, by (8.16), (8.19) and the exponential decay of z_ξ, there holds

$$\begin{aligned} (z_\xi|\partial_{\xi_i} z_\xi) &= (z_\xi|\partial_{x_i} z_\xi) + o(1) \\ &= (z_\xi|\partial_{x_i} z_\xi)_\lambda + \int_{\mathbb{R}^n} (V(\varepsilon x) - V(\varepsilon\xi)) z_\xi \partial_{x_i} z_\xi + o(1) = o(1). \end{aligned}$$

This proves the first estimate in (8.34). To prove the second one, we notice that for any $v \in W^{1,2}(\mathbb{R}^n)$ there holds

$$
\begin{aligned}
I''_\varepsilon(z_\xi)[v] &= \overline{I}''_\lambda(z_\xi)[v] + \int_{\mathbb{R}^n} (V(\varepsilon x) - V(\varepsilon\xi)) z_\xi v = -(p-1)(z_\xi|v)_\lambda + o(\|v\|) \\
&= -(p-1)(z_\xi|v) + o(\|v\|).
\end{aligned}
$$

Hence the proof is concluded. □

Proof of Proposition 8.7. Our aim is to find a solution $w \in (T_{z_\xi}\mathcal{Z}^\varepsilon)^\perp$ of $PI'_\varepsilon(z_\xi + w) = 0$. For every $w \in (T_{z_\xi}\mathcal{Z}^\varepsilon)^\perp$ we can write

$$
I'_\varepsilon(z_\xi + w) = I'_\varepsilon(z_\xi) + I''_\varepsilon(z_\xi)[w] + R(z_\xi, w),
$$

where $R(z_\xi, w)$ is given by

$$
R(z_\xi, w) = I'_\varepsilon(z_\xi + w) - I'_\varepsilon(z_\xi) - I''_\varepsilon(z_\xi)[w].
$$

Taking the projection P_ξ onto $(T_{z_\xi}\mathcal{Z}^\varepsilon)^\perp$, by the invertibility of $L_\xi = P_\xi \circ I''_\varepsilon(z_\xi) \circ P_\xi$, see Lemma 8.10, the function w solves $PI'_\varepsilon(z_\xi + w) = 0$ if and only if

$$
w = N_{\varepsilon,\xi}(w), \qquad \text{where} \qquad N_{\varepsilon,\xi}(w) = -L_\xi^{-1}\left(PI'_\varepsilon(z_\xi) + PR(z_\xi, w)\right).
$$

The norm of $I'_\varepsilon(z)$ has been estimated in Lemma 8.8, so we focus on $R(z_\xi, w)$. Given $v \in H^1(\mathbb{R}^n)$ there holds

$$
R(z_\xi, w)[v] = -\int_{R^n} \left[(z_\xi + w)^p - z_\xi^p - p z_\xi^{p-1} w\right] v.
$$

Using (8.12), the Hölder's inequality and the Sobolev embeddings we obtain

$$
\|R(z_\xi, w)[v]\| \le C \int_{\mathbb{R}^n} (|w|^2 + |w|^p)|v| \le C(\|w\|^2 + \|w\|^p)\, \|v\|. \tag{8.35}
$$

Similarly, from (8.13) we get

$$
\begin{aligned}
&\|R(z_\xi, w_1)[v] - R(z_\xi, w_2)[v]\| \\
&\qquad \le C \int_{\mathbb{R}^n} (|w_1|^2 + |w_1|^{p-1} + |w_2|^2 + |w_2|^{p-1}|)|v| \\
&\qquad \le C(\|w_1\|^2 + \|w_1\|^{p-1} + \|w_2\|^2 + \|w_2\|^{p-1})\|w_1 - w_2\|\, \|v\|. \qquad (8.36)
\end{aligned}
$$

Then from Lemma 8.8, (8.35) and (8.36) we obtain the two relations

$$
\|N_{\varepsilon,\xi}(w)\| \le C(\varepsilon|V'(\varepsilon\xi)| + \varepsilon^2) + C(\|w\| + \|w\|^{p-1})\|w\|; \tag{8.37}
$$

$$
\|N_{\varepsilon,\xi}(w_1) - N_{\varepsilon,\xi}(w_1)\| \le C(\|w_1\| + \|w_1\|^{p-1} + \|w_2\| + \|w_2\|^{p-1})\|w_1 - w_2\|. \tag{8.38}
$$

For $\overline{C} > 0$, we now define the set

$$W_{\overline{C}} = \left\{ w \in (T_{z_\xi}\mathcal{Z}^\varepsilon)^\perp \ : \ \|w\| \le \overline{C}\Lambda(\varepsilon,\xi) \right\},$$

where we have set

$$\Lambda(\varepsilon,\xi) = \varepsilon|V'(\varepsilon\xi)| + \varepsilon^2. \tag{8.39}$$

We show that $N_{\varepsilon,\xi}$ is a contraction in $W_{\overline{C}}$ for $\overline{C}$ sufficiently large and for ε small. Clearly, by (8.37), if $\overline{C} \ge 2C$ the set $W_{\overline{C}}$ is mapped into itself if ε is sufficiently small. Then, if $w_1, w_2 \in W_{\overline{C}}$, by (8.38) there holds

$$\|N_{\varepsilon,\xi}(w_1) - N_{\varepsilon,\xi}(w_1)\| \le C(\overline{C} + \overline{C}^{p-1}) \left[\Lambda(\varepsilon,\xi) + \Lambda(\varepsilon,\xi)^{p-1}\right] \|w_1 - w_2\|.$$

Therefore, again if ε is sufficiently small, the coefficient of $\|w_1 - w_2\|$ in the last formula is less than 1. Hence the Contraction Mapping Theorem applies, yielding the existence of a solution w satisfying the condition

$$\|w\| \le \overline{C}\left(\varepsilon|V'(\varepsilon\xi)| + \varepsilon^2\right). \tag{8.40}$$

This concludes the proof of the existence part.

We turn now to the C^1-dependence of w on ξ. This would follow from Remark 2.22, but in order to prove (8.17), we need to find quantitative estimates. Consider the map $H : \mathbb{R}^n \times W^{1,2}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R} \to W^{1,2} \times \mathbb{R}^n$ defined by

$$H(\xi, w, \alpha, \varepsilon) = \begin{pmatrix} I'_\varepsilon(z_\xi + w) - \sum_{i=1}^n \alpha_i \partial_{\xi_i} z_\xi \\ (w|\partial_{\xi_1} z_\xi), \dots, (w|\partial_{\xi_n} z_\xi) \end{pmatrix},$$

where $\alpha = \{\alpha_i\}_{i=1,\dots,n}$. Let us remark that H is nothing but the map introduced in the proof of Proposition 6.13.

Then $w \in (T_{z_\xi}\mathcal{Z}^\varepsilon)^\perp$ is a solution of $P_\xi I'_\varepsilon(z_\xi + w)$ if and only if $H(\xi, w, \alpha, \varepsilon) = 0$. Moreover, for $v \in W^{1,2}(\mathbb{R}^n)$ and $\beta = \{\beta_i\}_{i=1,\dots,n}$, there holds

$$\begin{aligned} \frac{\partial H}{\partial(w,\alpha)}(\xi, w, \alpha, \varepsilon)[v,\beta] &= \begin{pmatrix} I''_\varepsilon(z_\xi + w)[v] - \sum_{i=1}^n \beta_i \partial_{\xi_i} z_\xi \\ (v|\partial_{\xi_1} z_\xi), \dots, (v|\partial_{\xi_n} z_\xi) \end{pmatrix} \\ &= \begin{pmatrix} I''_\varepsilon(z_\xi)[v] - \sum_{i=1}^n \beta_i \partial_{\xi_i} z_\xi \\ (v|\partial_{\xi_1} z_\xi), \dots, (v|\partial_{\xi_n} z_\xi) \end{pmatrix} + O(\|w\| + \|w\|^{p-1}). \end{aligned} \tag{8.41}$$

To prove the last estimate it is sufficient to use (8.14) and to use the Sobolev embedding, similarly to the proof of (8.35) and (8.36).

By Lemma 8.10 it is easy to check that $\frac{\partial H}{\partial(w,\alpha)}(\xi, 0, 0, \varepsilon)$ is uniformly invertible in ξ for ε small. Hence, by (8.40) and (8.41), also $\frac{\partial H}{\partial(w,\alpha)}(\xi, w, \alpha, \varepsilon)$ is uniformly invertible in ξ for ε small. As a consequence, by the Implicit Function Theorem, the map $\xi \mapsto (w_\xi, \alpha_\xi)$ is of class C^1. Note that by the contraction mapping argument the vector α, similarly to w, satisfies the following estimate

$$|\alpha| \le C\left(\varepsilon|V'(\varepsilon\xi)| + \varepsilon^2\right). \tag{8.42}$$

Now we are in position to provide the norm estimate of $\partial_\xi w$. Differentiating the equation

$$H(\xi, w_\xi, \alpha_\xi, \varepsilon) = 0$$

with respect to ξ, we obtain

$$0 = \frac{\partial H}{\partial \xi}(\xi, w, \alpha, \varepsilon) + \frac{\partial H}{\partial(w,\alpha)}(\xi, w, \alpha, \varepsilon)\frac{\partial(w_\xi, \alpha_\xi)}{\partial \xi}.$$

Hence, by the uniform invertibility of $\frac{\partial H}{\partial(w,\alpha)}(\xi, w, \alpha, \varepsilon)$ it follows that

$$\begin{aligned}
\|\partial_\xi w\| &\le C \left\| \begin{pmatrix} I''_\varepsilon(z_\xi + w)[\partial_\xi z_\xi] - \sum_{i=1}^n \alpha_i \partial_\xi \partial_{\xi_i} z_\xi \\ (w|\partial_\xi \partial_{\xi_1} z_\xi), \dots, (w|\partial_\xi \partial_{\xi_n} z_\xi) \end{pmatrix} \right\| \\
&\le C \left(\|I''_\varepsilon(z_\xi + w)\partial_\xi z_\xi\| + |\alpha| + \|w\| \right).
\end{aligned}$$

By the estimate in (8.41), (8.16), and the fact that $\overline{I}''_\lambda(U_\lambda)[\nabla z_\xi] = 0$ we obtain

$$\begin{aligned}
&\|I''_\varepsilon(z_\xi + w)[\partial_\xi z_\xi]\| \\
&\qquad \le \|I''_\varepsilon(z_\xi)[\partial_\xi z_\xi]\| + C(\|w\| + \|w\|^{p-1}) \\
&\qquad \le \|I''_\varepsilon(z_\xi)[\nabla z_\xi]\| + C\varepsilon|\nabla V(\varepsilon\xi)| + C(\|w\| + \|w\|^{p-1}) \\
&\qquad \le \|(I''_\varepsilon(z_\xi)) - \overline{I}''_\lambda(U_\lambda))[\nabla z_\xi]\| + C\varepsilon|\nabla V(\varepsilon\xi)| + C(\|w\| + \|w\|^{p-1}).
\end{aligned}$$

For any $v \in W^{1,2}(\mathbb{R}^n)$, using (8.19) and reasoning as in the proof of Lemma 8.8, one finds

$$\begin{aligned}
\left|(I''_\varepsilon(z_\xi)) - \overline{I}''_\lambda(U_\lambda))[\nabla z_\xi, v]\right| &\le \int_{\mathbb{R}^n} |V(\varepsilon x) - V(\varepsilon\xi)||\nabla z_\xi||v| \\
&\le C\left(\varepsilon|V'(\varepsilon\xi)| + \varepsilon^2\right)\|v\|.
\end{aligned}$$

The last three formulas imply (8.17).

The final assertion in Proposition 8.7 is proved as for Theorem 2.12, see also Remark 2.14-(i). Roughly, from (8.17) it follows that

$$T_{z_\xi}\mathcal{Z}^\varepsilon \sim T_{z_\xi + w(\varepsilon,\xi)}\tilde{\mathcal{Z}}^\varepsilon \qquad \text{for } \varepsilon \text{ small},$$

where $\tilde{\mathcal{Z}}^\varepsilon = \{z_\xi + w(\varepsilon, \xi) \mid \xi \in \mathbb{R}^n\}$. Suppose $z_{\xi_0} + w(\varepsilon, \xi_0)$ is a critical point of $I_\varepsilon|_{\tilde{\mathcal{Z}}^\varepsilon}$. Then $I'_\varepsilon(z_{\xi_0} + w(\varepsilon, \xi_0))$ is perpendicular to $T_{z_{\xi_0}+w(\varepsilon,\xi_0)}\tilde{\mathcal{Z}}^\varepsilon$, and hence almost perpendicular to $T_{z_{\xi_0}}\mathcal{Z}^\varepsilon$. Since, by construction of $\tilde{\mathcal{Z}}^\varepsilon$, it is $I'_\varepsilon(z_{\xi_0} + w(\varepsilon, \xi_0)) \in T_{z_{\xi_0}}\mathcal{Z}^\varepsilon$, it must be $I'_\varepsilon(z_{\xi_0} + w(\varepsilon, \xi_0)) = 0$. This concludes the proof. $\square$

8.5 Study of the reduced functional

The main purpose of this section is to use the estimates on w established above to find an expansion of $\Phi_\varepsilon(\xi)$ and $\Phi'_\varepsilon(\xi)$, where Φ_ε was defined by $\Phi_\varepsilon(\xi) = I_\varepsilon(z_\xi + w(\varepsilon, \xi))$. In the sequel, to simplify the notation, we will often write z instead of z_ξ and w instead of $w(\varepsilon, \xi)$. It is always understood that ε is taken so small that all the results discussed in the preceding sections hold true.

We have

$$\Phi_\varepsilon(\xi) = \frac{1}{2}\|z+w\|^2 + \frac{1}{2}\int_{\mathbb{R}^n} V(\varepsilon x)(z+w)^2 - \frac{1}{p+1}\int_{\mathbb{R}^n}(z+w)^{p+1}.$$

Since z satisfies $-\Delta z + V(\varepsilon\xi)z = z^p$ we infer that

$$\|z\|^2 = -V(\varepsilon\xi)\int_{\mathbb{R}^n} z^2 + \int_{\mathbb{R}^n} z^{p+1}; \qquad (z|w) = -V(\varepsilon\xi)\int_{\mathbb{R}^n} zw + \int_{\mathbb{R}^n} z^p w.$$

Then we find

$$\begin{aligned}\Phi_\varepsilon(\xi) = {} & \left(\tfrac{1}{2} - \tfrac{1}{p+1}\right)\int_{\mathbb{R}^n} z^{p+1} + \tfrac{1}{2}\int_{\mathbb{R}^n}\left[V(\varepsilon x) - V(\varepsilon\xi)\right] z^2 \\ & + \int_{\mathbb{R}^n}\left[V(\varepsilon x) - V(\varepsilon\xi)\right] zw + \tfrac{1}{2}\int_{\mathbb{R}^n} V(\varepsilon x) w^2 \\ & + \tfrac{1}{2}\|w\|^2 - \tfrac{1}{p+1}\int_{\mathbb{R}^n}\left[(z+w)^{p+1} - z^{p+1} - (p+1)z^p w\right].\end{aligned}$$

Since $z(x) = \alpha(\varepsilon\xi)U(\beta(\varepsilon\xi)x)$, where $\alpha = V^{1/(p-1)}$ and $\beta = V^{1/2}$, see (8.15), it follows that

$$\int_{\mathbb{R}^n} z^{p+1}dx = C_0(V(\varepsilon\xi))^\theta, \quad C_0 = \int_{\mathbb{R}^n} U^{p+1}; \qquad \theta = \frac{p+1}{p-1} - \frac{n}{2}.$$

Letting $C_1 = C_0[1/2 - 1/(p+1)]$ one has

$$\begin{aligned}\Phi_\varepsilon(\xi) = {} & C_1(V(\varepsilon\xi))^\theta + \tfrac{1}{2}\int_{\mathbb{R}^n}\left[V(\varepsilon x) - V(\varepsilon\xi)\right] z^2 \\ & + \int_{\mathbb{R}^n}\left[V(\varepsilon x) - V(\varepsilon\xi)\right] zw + \tfrac{1}{2}\int_{\mathbb{R}^n} V(\varepsilon x) w^2 \\ & + \tfrac{1}{2}\|w\|^2 - \tfrac{1}{p+1}\int_{\mathbb{R}^n}\left[(z+w)^{p+1} - z^{p+1} - (p+1)z^p w\right].\end{aligned} \tag{8.43}$$

We are now in the position to estimate the functions Φ_ε and Φ'_ε.

Lemma 8.11. *Let $a(\varepsilon\xi) = \theta C_1(V(\varepsilon\xi))^{\theta-1}$ and let $\gamma = \min\{1, p-1\}$. Then one has*

$$\Phi_\varepsilon(\xi) = C_1(V(\varepsilon\xi))^\theta + \rho_\varepsilon(\xi), \quad C_1 > 0, \quad \theta = \frac{p+1}{p-1} - \frac{n}{2}, \tag{8.44}$$

$$\Phi'_\varepsilon(\xi) = a(\varepsilon\xi)\varepsilon V'(\varepsilon\xi) + \varepsilon^{1+\gamma}R_\varepsilon(\xi), \tag{8.45}$$

where $|\rho_\varepsilon(\xi)| \le C\left(\varepsilon|V'(\varepsilon\xi)| + \varepsilon^2\right)$, and $|R_\varepsilon(\xi)| \le C$.

Proof. The first four error terms in (8.43) can be estimated as in Lemma 8.8, using the Hölder inequality and (8.40). Let us focus on the last term. Using the uniform boundedness of z and (8.12) one finds

$$\left|(z+w)^{p+1} - z^{p+1} - (p+1)z^p w\right| \le C\left(|w|^2 + |w|^{p+1}\right).$$

Hence, from the Sobolev inequality we deduce

$$\left|\int_{\mathbb{R}^n}\left[(z+w)^{p+1} - z^{p+1} - (p+1)z^p w\right]\right| \le C\left(\|w\|^2 + \|w\|^{p+1}\right).$$

Then, using (8.40), we obtain (8.44).

In order to prove (8.45) we compute first the expression $\partial_\xi I_\varepsilon(z_\xi)$. Using a Taylor's expansion for V and (8.16) we obtain

$$\begin{aligned}
\partial_\xi I_\varepsilon(z_\xi) &= C_1\partial_\xi V^\theta(\varepsilon\xi) + \frac{1}{2}\partial_\xi\int_{\mathbb{R}^n}(V(\varepsilon x) - V(\varepsilon\xi))z_\xi^2 \\
&= C_1\partial_\xi V^\theta(\varepsilon\xi) + \int_{\mathbb{R}^n}(V(\varepsilon x) - V(\varepsilon\xi))z_\xi\partial_\xi z_\xi - \tfrac{1}{2}\varepsilon V'(\varepsilon\xi)\int_{\mathbb{R}^n} z_\xi^2 \\
&= C_1\partial_\xi V^\theta(\varepsilon\xi) + \varepsilon V'(\varepsilon\xi)\int_{\mathbb{R}^n}(x-\xi)z_\xi\partial_\xi z_\xi + O(\varepsilon^2) - \tfrac{1}{2}\varepsilon V'(\varepsilon\xi)\int_{\mathbb{R}^n} z_\xi^2 \\
&= C_1\partial_\xi V^\theta(\varepsilon\xi) - \beta\varepsilon V'(\varepsilon\xi)\int_{\mathbb{R}^n}(x-\xi)z_\xi\partial_\xi z_\xi + O(\varepsilon^2) - \tfrac{1}{2}\varepsilon V'(\varepsilon\xi)\int_{\mathbb{R}^n} z_\xi^2.
\end{aligned}$$

Writing $z_\xi\partial_\xi z_\xi = \frac{1}{2}\partial_\xi z_\xi^2$ and integrating by parts we find

$$\partial_\xi I_\varepsilon(z_\xi) = C_1\partial_\xi V^\theta(\varepsilon\xi) + O(\varepsilon^2).$$

Then we write

$$\begin{aligned}
\partial_\xi\Phi_\varepsilon(\xi) &= I'_\varepsilon(z+w)[\partial_\xi z + \partial_\xi w] \\
&= \partial_\xi I_\varepsilon(z_\xi) + (I'_\varepsilon(z+w) - I'_\varepsilon(z))[\partial_\xi z] + I'_\varepsilon(z+w)[\partial_\xi w] \\
&= a(\varepsilon\xi)\varepsilon V'(\varepsilon\xi) + I''_\varepsilon(z)[w, \partial_\xi z_\xi] \\
&\qquad + R(z_\xi, w)[\partial_\xi z] + I'_\varepsilon(z)[\partial_\xi w] + R(z_\xi, w)[\partial_\xi w].
\end{aligned}$$

Using (8.35), (8.17) and arguing as in the proof of Proposition 8.7 we obtain the conclusion. □

We are finally ready to prove Theorem 8.5.

Proof of Theorem 8.5. We will use the result of [54] cited in Remark 2.14-(iii), that we report here for the reader convenience using the notation employed in this chapter:

> Let $f \in C^2(\mathbb{R}^n, \mathbb{R})$ and suppose that $\mathcal{M}$ is a non-degenerate compact manifold of critical points of f. Let $\mathcal{N}$ be a neighborhood of $\mathcal{M}$ and let $g_\varepsilon \in C^2(\mathcal{N}, \mathbb{R})$. If $\|f - g_\varepsilon\|_{C^1}$ is sufficiently small, then g_ε has at least $l(\mathcal{M})$ (cup long of $\mathcal{M}$) critical points in $\mathcal{N}$.

We take $f = C_1 V^\theta$ and $\mathcal{M} = M$. M is obviously a non-degenerate critical manifold of f. Fixed a neighborhood $\mathcal{N}$ of M we set $g_\varepsilon(\xi) = \Phi_\varepsilon(\xi/\varepsilon)$. From Lemma 8.11 it follows that $\|f - g_\varepsilon\|_{C^1} \ll 1$ provided $\varepsilon \ll 1$. Hence the result quoted above applies and we can infer the existence of at least $l(M)$ critical points of g_ε, provided $\varepsilon > 0$ is sufficiently small. Let $\xi_{\varepsilon,i} \in \mathcal{N}$ be any of those critical points. Then $\xi_{\varepsilon,i}/\varepsilon$ is a critical point of Φ_ε and Proposition 8.7 implies that $u_{\varepsilon,\xi_{\varepsilon,i}} = z^{\xi_{\varepsilon,i}}(x - \xi_{\varepsilon,i}/\varepsilon) + w(\varepsilon, \xi_{\varepsilon,i})$ is a critical point of I_ε. It follows that

$$v_{\varepsilon.i}(x) := u_{\varepsilon,\xi_{\varepsilon,i}}(x/\varepsilon) \simeq z^{\xi_{\varepsilon,i}}\left(\frac{x - \xi_{\varepsilon,i}}{\varepsilon}\right)$$

is a solution of (8.1). Any ξ_i converges to some $\xi_i^* \in \mathcal{N}$ as $\varepsilon \to 0$ and it is easy to see that ξ_i^* is a stationary point of V. Then, taking $\mathcal{N}$ possibly smaller, it follows that $\xi_i^* \in M$. This shows that $v_{\varepsilon,i}(x)$ concentrates near a point of M and completes the proof. □

Remarks 8.12.

(i) It is possible to handle the more general equation $-\varepsilon^2 \Delta u + V(x)u = K(x)u^p$. In this case, to determine the location of the concentration points, V must be replaced by the auxiliary function $V^\theta K^{-\frac{2}{p-1}}$.

(ii) If $p \geq 2$, then Φ_ε is of class C^2 and we can apply Corollary 2.13. In this case, $l(M)$ can be substituted by *cat*(M). The same holds if $p > 1$ and M is a compact set of local maxima or minima of V, without any smoothness assumption on M, see [23].

(iii) Expanding Φ_ε at higher order in ε, it would be possible to localize, generically, the concentration points on M, in the spirit of Theorem 8.1. □

Bibliographical remarks

The first rigorous proof of the existence of solutions in the semiclassical limit has been given in [80]. Since then, a lot of works have appeared, see, e.g., [10, 72, 87, 121, 145]. In particular, [72] deals with a nonlinearity $f(x, u) \sim K(x)u^p$, see also [10]. Solutions with many peaks have been found in, e.g., [67, 88, 122]. The case in which V has a critical manifold of critical points is discussed in [23], improving a preceding result of [66]. NLS with a magnetic potential have been studied in [27, 68]. NLS with (more general) linear part in divergence form has been considered in [31].

Chapter 9

Singularly Perturbed Neumann Problems

In this chapter we study the following singular perturbation problem on a bounded domain $\Omega \subset \mathbb{R}^n$ with Neumann boundary conditions:

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^p, & \text{in } \Omega; \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega; \\ u > 0 & \text{in } \Omega, \end{cases} \tag{N_ε}$$

where $p > 1$ is subcritical and ν denotes the outer unit normal at $\partial\Omega$. For motivations we refer to Section 1.4. We will see that the abstract tools carried over in the preceding chapter, see Sections 8.4 and following, can be also used to prove the existence of boundary spikes for (N_ε). Precisely, our aim is to prove the following result.

Theorem 9.1. *Suppose $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, is a smooth bounded domain, and that $1 < p < \frac{n+2}{n-2}$ ($1 < p < +\infty$ if $n = 2$). Suppose $X_0 \in \partial\Omega$ is a local strict maximum or minimum, or a non-degenerate critical point of the mean curvature H of $\partial\Omega$. Then for $\varepsilon > 0$ sufficiently small problem (N_ε) admits a solution concentrating at X_0.*

9.1 Preliminaries

In this section we introduce some preliminary material that will be used in the sequel. For $x \in \mathbb{R}^n$ we set $x = (x', x_n)$ with $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$. Let $\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$ and consider the problem

$$\begin{cases} -\Delta u + u = u^p & \text{in } \mathbb{R}^n_+; \\ u(x) \to 0 & \text{as } |x| \to +\infty; \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\mathbb{R}^n_+ & u > 0, \end{cases} \tag{P_0^+}$$

where $n \geq 2$ and $p > 1$.

If $p < \frac{n+2}{n-2}$ (in the case $n \geq 3$), and if $u \in W^{1,2}(\mathbb{R}^n_+)$, solutions of (P_0^+) can be found as critical points of the functional $\overline{I}_+ : W^{1,2}(\mathbb{R}^n_+) \to \mathbb{R}$ defined as

$$\overline{I}_+(u) = \frac{1}{2} \int_{\mathbb{R}^n_+} \left(|\nabla u|^2 + u^2 \right) - \frac{1}{p+1} \int_{\mathbb{R}^n_+} |u|^{p+1}. \tag{9.1}$$

Note that, by the Sobolev embedding theorem, $\overline{I}_+$ is well defined (and is actually of class C^2) on $W^{1,2}(\mathbb{R}^n_+)$.

Let us point out that, under the above restrictions on p, the function U introduced before is also a solution of problem (P_0^+).
It is essential to understand the spectral properties of the linearized equation at U, or equivalently of the operator $\overline{I}''_+(U)$, which is given by

$$\overline{I}''_+(U)[v_1, v_2] = (v_1|v_2)_+ - p \int_{\mathbb{R}^n_+} U^{p-1} v_1 v_2; \qquad v_1, v_2 \in W^{1,2}(\mathbb{R}^n_+), \tag{9.2}$$

where we have set

$$(v_1|v_2)_+ = \int_{\mathbb{R}^n_+} \left(\nabla v_1 \cdot \nabla v_2 + v_1 v_2 \right). \tag{9.3}$$

We have the following result, which is the counterpart of Lemma 4.1 for the half-space.

Proposition 9.2. *Let U be as above and consider the functional $\overline{I}_+$ given in* (9.1)*. Then for every $\xi \in \mathbb{R}^{n-1}$, $U(\cdot - (\xi, 0))$ is a critical point of $\overline{I}_+$. Moreover, the kernel of $\overline{I}''_+(U)$ is generated by $\frac{\partial U}{\partial x_1}, \dots, \frac{\partial U}{\partial x_{n-1}}$. The operator has only one negative eigenvalue, and therefore there exists $\delta > 0$ such that*

$$\overline{I}''_+(U)[v, v] \geq \delta \|v\|^2 \qquad \textit{for all} \quad v \in W^{1,2}(\mathbb{R}^n_+), v \perp_+ U, \frac{\partial U}{\partial x_1}, \dots, \frac{\partial U}{\partial x_{n-1}},$$

where we have used the symbol $\perp_+$ to denote orthogonality with respect to the scalar product $(\cdot, \cdot)_+$.

Proof. Given any $v \in W^{1,2}(\mathbb{R}^n_+)$, we define the function $\overline{v} \in W^{1,2}(\mathbb{R}^n)$ by an even extension across $\partial\mathbb{R}^n_+$, namely we set

$$\overline{v}(x', x_n) = \begin{cases} v(x', x_n), & \text{for } x_n > 0; \\ v(x', -x_n) & \text{for } x_n < 0. \end{cases}$$

We also recall the definition of the functional $I_0 : W^{1,2}(\mathbb{R}^n) \to \mathbb{R}$

$$I_0(u) = \frac{1}{2} \int_{\mathbb{R}^n} \left(|\nabla u|^2 + u^2 \right) - \frac{1}{p+1} \int_{\mathbb{R}^n} |u|^{p+1}; \qquad u \in W^{1,2}(\mathbb{R}^n),$$

see Chapter 4. We prove first the following claim.

Claim. Suppose $v \in W^{1,2}(\mathbb{R}^n_+)$ is an eigenfunction of $\overline{I}''_+(U)$ with eigenvalue λ. Then the function $\overline{v}$ is an eigenfunction of $I''_0(U)$ with eigenvalue λ.

In order to prove the claim, we notice that the function v satisfies the equation

$$\begin{cases} -\Delta v + v - pU^{p-1}v = \lambda(-\Delta v + v), & \text{in } \mathbb{R}^n_+; \\ \frac{\partial v}{\partial \nu} = 0, & \text{on } \partial\mathbb{R}^n_+. \end{cases}$$

Similarly, by symmetry, there holds

$$\begin{cases} -\Delta \overline{v} + \overline{v} - pU^{p-1}\overline{v} = \lambda(-\Delta \overline{v} + \overline{v}), & \text{in } \mathbb{R}^n_-; \\ \frac{\partial \overline{v}}{\partial \nu} = 0, & \text{on } \partial\mathbb{R}^n_-, \end{cases}$$

where we have set $\mathbb{R}^n_- = \{(x', x_n) \,:\, x' \in \mathbb{R}^{n-1}, x_n < 0\}$.

Then, considering any function $w \in W^{1,2}(\mathbb{R}^n)$, integrating by parts and using the Neumann boundary condition one finds

$$\begin{aligned} I''_0(U)[\overline{v}, w] &= \int_{\mathbb{R}^n} (\nabla \overline{v} \cdot \nabla w) + \overline{v}w - \frac{1}{p+1} \int_{\mathbb{R}^n} U^{p-1}\overline{v}w \\ &= \lambda \int_{\mathbb{R}^n_+} (-\Delta v + v)w + \lambda \int_{\mathbb{R}^n_-} (-\Delta \overline{v} + \overline{v}) \\ &= \lambda \int_{\mathbb{R}^n_+} (\nabla v \cdot \nabla w) + vw + \lambda \int_{\mathbb{R}^n_-} \nabla \overline{v} \nabla w + \overline{v}w = \lambda(\overline{v}|w)_{W^{1,2}(\mathbb{R}^n)}. \end{aligned}$$

This proves the above claim.

We know that the functions $\partial_{\xi_1} U, \dots, \partial_{\xi_{n-1}} U$ belong to the kernel of $\overline{I}''_+(U)$. Suppose by contradiction that there exists another element v in the kernel of $\overline{I}''_+(U)$, orthogonal to $\partial_{\xi_1} U, \dots, \partial_{\xi_{n-1}} U$. Then, by the above claim, its even extension $\overline{v}$ would belong to the kernel of $I''_0(U)$. But we know that the only element in the kernel of $I''_0(U)$ which is orthogonal to $\partial_{\xi_1} U, \dots, \partial_{\xi_{n-1}} U$ is $\partial_{\xi_n} U$. Since $\partial_{\xi_n} U$ is odd with respect to x_n, while $\overline{v}$ is even with respect to x_n, we get a contradiction. This concludes the proof. □

By a change of variables, problem (N_ε) can be transformed into

$$\begin{cases} -\Delta u + u = u^p & \text{in } \Omega_\varepsilon; \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon; \\ u > 0 & \text{in } \Omega_\varepsilon, \end{cases} \qquad (\tilde{N}_\varepsilon)$$

where $\Omega_\varepsilon = \frac{1}{\varepsilon}\Omega$.

Solutions of $(\tilde{N}_\varepsilon)$ can be found as critical points of the Euler functional

$$J_\varepsilon(u) = \frac{1}{2} \int_{\Omega_\varepsilon} \left(|\nabla u|^2 + u^2\right) - \frac{1}{p+1} \int_{\Omega_\varepsilon} |u|^{p+1}; \qquad u \in W^{1,2}(\Omega_\varepsilon). \tag{9.4}$$

Let us describe the $\partial\Omega_\varepsilon$ near a generic point $X \in \partial\Omega_\varepsilon$. Without loss of generality, we can assume that $X = 0 \in \mathbb{R}^n$, that $\{x_n = 0\}$ is the tangent plane of $\partial\Omega_\varepsilon$ (or $\partial\Omega$) at Q, and that $\nu(X) = (0, \dots, 0, -1)$. In a neighborhood of X, let $x_n = \psi(x')$ be a local parametrization of $\partial\Omega$. Then one has

$$x_n = \psi(x') := \tfrac{1}{2}\langle A_X x', x'\rangle + C_X(x') + O(|x'|^4); \qquad |x'| < \mu_0, \tag{9.5}$$

where A_X is the Hessian of ψ at 0 and C_X is a cubic polynomial, which is given precisely by

$$C_X(x') = \tfrac{1}{6} \sum_{i,j,k} \partial^3_{ijk}|_0 x'_i x'_j x'_k. \tag{9.6}$$

We have clearly $H(X) = \frac{1}{n-1}\operatorname{tr} A_X$. On the other hand, $\partial\Omega_\varepsilon$ is parameterized by $y_n = \psi_\varepsilon(x') := \frac{1}{\varepsilon}\psi(\varepsilon x')$, for which the following expansion holds

$$\begin{aligned} \psi_\varepsilon(x') &= \frac{\varepsilon}{2}\langle A_X x', x'\rangle + \varepsilon^2 C_X(x') + \varepsilon^3 O(|x'|^4); \\ \partial_i \psi_\varepsilon(x') &= \varepsilon(A_X x')_i + \varepsilon^2 Q^i_X(x') + \varepsilon^3 O(|x'|^3), \end{aligned} \tag{9.7}$$

where Q^i_X are quadratic forms in x' given by (see (9.6))

$$Q^i_X(x') = \frac{1}{2}\sum_{j,k} \partial^3_{ijk}|_0 x'_j x'_k.$$

In particular, from the Schwartz's Lemma, it follows that

$$(Q^i_X)_{jk} = (Q^i_X)_{kj} = (Q^j_X)_{ik} \qquad \text{for every } i, j, k. \tag{9.8}$$

Concerning the outer normal ν, we have also

$$\begin{aligned} \nu = \frac{\left(\frac{\partial\psi_\varepsilon}{\partial x_1}, \dots, \frac{\partial\psi_\varepsilon}{\partial x_{n-1}}, -1\right)}{\sqrt{1 + |\nabla\psi_\varepsilon|^2}} &= \left(\varepsilon(A_X x') + \varepsilon^2 Q_X(x'), -1 + \frac{1}{2}\varepsilon^2 |Ax'|^2\right) \\ &\quad + \varepsilon^3 O(|x'|^3). \end{aligned} \tag{9.9}$$

9.2 Construction of approximate solutions

We first prove the following technical lemma.

Lemma 9.3. *Let $T = (a_{ij})$ be an $(n-1)\times(n-1)$ symmetric matrix, and consider the following problem*

$$\begin{cases} Lw = -2\langle Tx', \nabla_{x'} U\rangle - 2\operatorname{tr} T\, \partial_{x_n} U, & \text{in } \mathbb{R}^n_+; \\ \frac{\partial}{\partial x_n} w = \langle Tx', \nabla_{x'} U\rangle, & \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{9.10}$$

where L is the operator

$$Lu = -\Delta u + u - pU^{p-1}u.$$

Then (9.10) *admits a solution* $\overline{w}_T$*, even in the variables* x'*, which satisfies the following decay estimates*

$$|\overline{w}_T(x)| + |\nabla \overline{w}_T(x)| + |\nabla^2 \overline{w}_T(x)| \leq \overline{C}|T|_\infty (1 + |x|^{\overline{C}}) e^{-|x|}, \tag{9.11}$$

where $\overline{C}$ *is a constant depending only on* n *and* p*, and* $|T|_\infty = \max_{ij} |a_{ij}|$.

Proof. Problem (9.10) can be reformulated as

$$\overline{I}''_+(U)[w] = v_T, \tag{9.12}$$

where v_T is an element of $W^{1,2}(\mathbb{R}^n_+)$ defined by duality as

$$(v_T|v)_{W^{1,2}(\mathbb{R}^n_+)} = \int_{\mathbb{R}^n_+} (-2\langle Tx', \nabla_{x'} U - \operatorname{tr} T \partial_{x_n} U\rangle)\, v - \int_{\partial \mathbb{R}^n_+} \langle Tx', \nabla_{x'} U\rangle v.$$

By Proposition 9.2, equation (9.12) is solvable if and only if v_T is orthogonal to $\frac{\partial U}{\partial x_1}, \ldots, \frac{\partial U}{\partial x_{n-1}}$. But this is the case since

$$\begin{aligned} \left(v_T, \frac{\partial U}{\partial x_i}\right)_{W^{1,2}(\mathbb{R}^n_+)} = & - \int_{\mathbb{R}^n_+} (2\langle Tx', \nabla_{x'} U\rangle + \operatorname{tr} T \partial_{x_n} U)\, \frac{\partial U}{\partial x_i} \\ & + \int_{\partial \mathbb{R}^n_+} \langle Tx', \nabla_{x'} U\rangle \frac{\partial U}{\partial x_i}, \qquad i = i, \ldots, n-1. \end{aligned}$$

Indeed, all the integrals in the last formula vanish because $\frac{\partial U}{\partial x_i}$ is odd in x' and the other functions are even, by the symmetry of T. The decay in (9.11) follows from (8.9) and standard elliptic estimates. $\square$

Given μ_0 as in (9.5), we introduce a new set of coordinates on $B_{\frac{\mu_0}{\varepsilon}}(X) \cap \Omega_\varepsilon$. Let

$$y' = x'; \qquad\qquad y_n = x_n - \psi_\varepsilon(x'). \tag{9.13}$$

The advantage of these coordinates is that $\partial \Omega_\varepsilon$ identifies with $\{y_n = 0\}$, but the corresponding metric will not be flat anymore. Its coefficients (g_{ij}) are given by

$$(g_{ij}) = \left(\langle \frac{\partial x}{\partial y_i}, \frac{\partial x}{\partial y_i}\rangle\right) = \begin{pmatrix} & & & \frac{\partial \psi_\varepsilon}{\partial y_1} \\ & \delta_{ij} + \frac{\partial \psi_\varepsilon}{\partial y_i}\frac{\partial \psi_\varepsilon}{\partial y_j} & & \vdots \\ & & & \frac{\partial \psi_\varepsilon}{\partial y_{n-1}} \\ \frac{\partial \psi_\varepsilon}{\partial y_1} & \cdots & \frac{\partial \psi_\varepsilon}{\partial y_{n-1}} & 1 \end{pmatrix}.$$

From the estimates in (9.7) it follows that

$$g_{ij} = Id + \varepsilon A + \varepsilon^2 B + O(\varepsilon^3 |y'|^3), \tag{9.14}$$

and

$$\partial_{y_k}(g_{ij}) = \varepsilon \partial_{y_k} A + \varepsilon^2 \partial_{y_k} B + O(\varepsilon^3 |y'|^2),$$

where

$$A = \begin{pmatrix} 0 & A_X y' \\ (A_X y')^t & 0 \end{pmatrix}; \qquad B = \begin{pmatrix} A_X y' \otimes A_X y' & Q_X(y') \\ (Q_X(y'))^t & 0 \end{pmatrix}.^1$$

It is also easy to check that the inverse matrix (g^{ij}) is of the form $g^{ij} = Id - \varepsilon A + \varepsilon^2 C + O(\varepsilon^3 |y'|^3)$, where

$$C = \begin{pmatrix} 0 & -Q_X(y') \\ -(Q_X(y'))^t & |A_X y'|^2 \end{pmatrix},$$

and

$$\partial_{y_k}(g^{ij}) = -\varepsilon \partial_{y_k} A + \varepsilon^2 \partial_{y_k} C + O(\varepsilon^3 |y'|^2).$$

Furthermore, since the transformation (9.13) preserves the volume, there holds

$$\det g \equiv 1.$$

We also recall that the Laplace operator in a general system of coordinates is given by the expression

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \partial_j \left(g^{ij} \sqrt{\det g} \right) \partial_i u + g^{ij} \partial^2_{ij} u,$$

so in our situation we get

$$\Delta_g u = g^{ij} u_{ij} + \partial_i (g^{ij}) \partial_j u.$$

In particular, by (9.14), for any smooth function u there holds

$$\begin{aligned} \Delta_g u = \Delta u &- \varepsilon \left(2\langle A_X y', \nabla_{y'} \partial_{y_n} u \rangle + \operatorname{tr} A_X \partial_{y_n} u \right) \\ &+ \varepsilon^2 \left(-2\langle Q_X, \nabla_{y'} \partial_{y_n} u \rangle + |A_X y'|^2 \partial^2_{y_n y_n} u - \operatorname{div} Q_X \partial_{y_n} u \right) \\ &+ O(\varepsilon^3 |y'|^2) |\nabla u| + O(\varepsilon^3 |y'|^3) |\nabla^2 u|. \end{aligned} \tag{9.15}$$

Here A_X is the Hessian of ψ at $x' = 0$, see Subsection 9.1. Now we choose a cut-off function ψ_{μ_0} with the following properties

$$\begin{cases} \psi_{\mu_0}(x) = 1 & \text{in } B_{\frac{\mu_0}{4}}; \\ \psi_{\mu_0}(x) = 0 & \text{in } B^n_{\frac{\mu_0}{2}} \setminus B_{\frac{\mu_0}{4}}; \\ |\nabla \psi_{\mu_0}| + |\nabla^2 \psi_{\mu_0}| \le C & \text{in } B_{\frac{\mu_0}{2}}(X) \setminus B_{\frac{\mu_0}{4}}, \end{cases}$$

and for any $X \in \partial\Omega$ we define the following function, in the coordinates (y', y_n)

$$z_{\varepsilon,X}(y) = \psi_{\mu_0}(\varepsilon y)(U(y) + \varepsilon \overline{w}_{A_X}(y)). \tag{9.16}$$

[1] If the vector v has components $(v_i)_i$, the notation $v \otimes v$ denotes the square matrix with entries $(v_i v_j)_{ij}$.

where $\overline{w}_{A_X}$ is given by Lemma 9.3 with $T = A_X$. We also give the expression of the unit outer normal to $\partial\Omega_\varepsilon$, $\tilde{\nu}$, in the new coordinates y. Letting ν_i, resp. $\tilde{\nu}_i$, be the components of ν, resp. $\tilde{\nu}$, from $\nu = \sum_{i=1}^n \nu^i \frac{\partial}{\partial x^i} = \sum_{i=1}^n \tilde{\nu}^i \frac{\partial}{\partial y^i}$, we have $\tilde{\nu}_k = \sum_{i=1}^n \nu^i \frac{\partial y^k}{\partial x^i}$. This implies

$$\tilde{\nu}^k = \nu^k, \quad k = 1, \dots, n-1; \qquad \tilde{\nu}^n = \sum_{i=1}^{n-1} \nu^i \frac{\partial \psi_\varepsilon}{\partial y^i} + \nu^n.$$

From (9.7) and the last formula in Subsection 9.1 we find

$$\tilde{\nu} = \left(\varepsilon A_X(y') + \varepsilon^2 Q_X(y'), -1 + \tfrac{3}{2}\varepsilon^2 |A_X(y')|^2\right) + \varepsilon^3 O(|y'|^3). \tag{9.17}$$

Finally the area-element of $\partial\Omega_\varepsilon$ can be estimated as

$$d\sigma = (1 + O(\varepsilon^2 |y'|^2)) dy'. \tag{9.18}$$

Next, we estimate the gradient of J_ε at $z_{\varepsilon,X}$ showing that $z_{\varepsilon,X}$ constitute, as X varies on $\partial\Omega_\varepsilon$, a manifold $\mathcal{Z}_\varepsilon$ of the pseudo-critical points of J_ε.

Lemma 9.4. *There exists $C > 0$ such that for ε small there holds*

$$\|J_\varepsilon'(z_{\varepsilon,X})\| \leq C\varepsilon^2; \qquad \qquad \textit{for all } X \in \partial\Omega_\varepsilon.$$

Proof. Let $v \in W^{1,2}(\Omega_\varepsilon)$. Since the function $z_{\varepsilon,X}$ is supported in $B_{\frac{\mu_0}{2\varepsilon}}(X)$, see (9.16), we can use the coordinates y in this set, and we obtain

$$J_\varepsilon'(z_{\varepsilon,X})[v] = \int_{\partial\Omega_\varepsilon} \frac{\partial z_{\varepsilon,X}}{\partial \tilde{\nu}} v d\sigma + \int_{\Omega_\varepsilon} \left(-\Delta_g z_{\varepsilon,X} + z_{\varepsilon,X} - z_{\varepsilon,X}^p\right) v dy. \tag{9.19}$$

Let us now evaluate $\frac{\partial z_{\varepsilon,X}}{\partial \tilde{\nu}}$. There holds

$$\frac{\partial z_{\varepsilon,X}}{\partial \tilde{\nu}} = (U + \varepsilon \overline{w}_{A_X}) \nabla \psi_{\mu_0}(\varepsilon y) \cdot \tilde{\nu} + \psi_{\mu_0}(\varepsilon y) \nabla (U + \varepsilon \overline{w}_{A_X}) \cdot \tilde{\nu}.$$

Since $\nabla \psi_{\mu_0}(\varepsilon \cdot)$ is supported in $\mathbb{R}^n \setminus B_{\frac{\mu_0}{4\varepsilon}}$, and both $U, \overline{w}_{A_X}$ have an exponential decay, we have

$$|(U + \varepsilon \overline{w}_{A_X}) \nabla \psi_{\mu_0}(\varepsilon y) \cdot \tilde{\nu}| \leq C(1 + |y|^C) e^{-\frac{1}{C\varepsilon}} e^{-|y|}.$$

On the other hand, from the boundary condition in (9.10) and from (9.17), the terms of order ε in $\psi_{\mu_0}(\varepsilon y) \nabla (U + \varepsilon \overline{w}_{A_X}) \cdot \tilde{\nu}$ cancel and we obtain

$$\frac{\partial z_{\varepsilon,X}}{\partial \tilde{\nu}} = O(\varepsilon^2 |y'||\nabla w|) + O(\varepsilon^2 |y'|^2 |\nabla U|); \qquad |y| \leq \frac{\mu_0}{4\varepsilon};$$
$$\left|\frac{\partial z_{\varepsilon,X}}{\partial \tilde{\nu}}\right| \leq C e^{-|y|} + \overline{C} \varepsilon (1 + |y|^C) e^{-|y|} \leq C \varepsilon^{-C} e^{-\frac{1}{C\varepsilon}}; \qquad \frac{\mu_0}{4\varepsilon} \leq |y| \leq \frac{\mu_0}{2\varepsilon}.$$

The last two estimates, (9.18), and the trace Sobolev inequalities readily imply

$$\left| \int_{\partial \Omega_\varepsilon} \frac{\partial z_{\varepsilon,X}}{\partial \tilde{\nu}} v d\sigma \right| \leq C\varepsilon^2 \|v\|. \tag{9.20}$$

On the other hand, using (9.11), (9.15) and the decay of U, the volume integrand can be estimated as

$$\begin{aligned} \left| -\Delta_g z_{\varepsilon,X} + z_{\varepsilon,X} - z_{\varepsilon,X}^p \right| &\leq C\varepsilon^2 \left(|y'||\nabla U| + |y'|^2 |\nabla^2 U| + |\nabla w| + |y'||\nabla^2 w| \right) \\ &\quad + \left| |U + \varepsilon w|^{p-1}(U + \varepsilon w) - U^p - p\varepsilon U^{p-1} w \right|, \end{aligned}$$

for $|y| \leq \left(\frac{1}{4\varepsilon \overline{C} \sup_X \|A_X\|} \right) \frac{1}{\overline{\overline{C}}}$, and

$$\begin{aligned} \left| -\Delta_g z_{\varepsilon,X} + z_{\varepsilon,X} - z_{\varepsilon,X}^p \right| &\leq C(1 + |y'|^{\overline{C}}) e^{-|y'|} \\ &\leq C\varepsilon^{-C} e^{-\frac{1}{C\varepsilon}}, \end{aligned}$$

for $\left(\frac{1}{4\varepsilon \overline{C} \sup_X \|A_X\|} \right) \frac{1}{\overline{\overline{C}}} \leq |y| \leq \frac{\mu_0}{2\varepsilon}$. We notice that the following inequality holds true

$$\left| (a+b)^p - a^p - p a^{p-1} b \right| \leq C b^2; \qquad a > 0, |b| \leq \frac{a}{2}.$$

In particular, by (9.11) we have

$$\varepsilon |\overline{w}(y)| \leq \frac{U(y)}{2}; \qquad \text{for } |y| \leq \left(\frac{1}{4\varepsilon \overline{C} \sup_X \|A_X\|} \right) \frac{1}{\overline{\overline{C}}}.$$

Hence it follows that

$$\left| -\Delta_g z_{\varepsilon,X} + z_{\varepsilon,X} - z_{\varepsilon,X}^p \right| \leq C\varepsilon^2 (1 + |y|^C) e^{-|y|}; \qquad |y| \leq \left(\frac{1}{4\varepsilon \overline{C} \sup_X \|A_X\|} \right) \frac{1}{\overline{\overline{C}}}.$$

Then, using the Hölder inequality we easily find

$$\left| \int_{\Omega_\varepsilon} \left(-\Delta_g z_{\varepsilon,X} + z_{\varepsilon,X} - z_{\varepsilon,X}^p \right) v dy \right| \leq C\varepsilon^2 \|v\|. \tag{9.21}$$

From (9.20) and (9.21) we obtain the conclusion. □

We also need to compute the expression of $\frac{\partial z_{\varepsilon,X}}{\partial X}$ in the coordinates y introduced in (9.13). We notice that in the definition of $z_{\varepsilon,X}$, see (9.16), not only the analytic expression of this function depends on X, but also the choice of the coordinates y. Therefore, when we differentiate in X, we have to take also this dependence into account. First we derive the variation in X of the coordinates x (introduced after (9.4)) of a fixed point in Ω. Using the dot to denote the differentiation with respect to X, one can prove that

$$\dot{x}' = \frac{\partial}{\partial X} x'_X = -\dot{X}; \qquad \dot{x}_n = \frac{\partial}{\partial X} (x_n)_X = -\langle x', \mathbf{H}_\varepsilon \dot{X} \rangle, \tag{9.22}$$

where $\mathbf{H}_\varepsilon = \varepsilon A_X$ is the second fundamental form of Ω_ε. The second equation in (9.22) is obtained by computing the variation of the distance of a fixed point in $\mathbb{R}^n$ from a moving tangent plane to Ω_ε. Similarly we get a dependence on X of the coordinates y. To emphasize the dependence of $z_{\varepsilon,X}$ on X we write

$$z_{\varepsilon,X} = U(y_X) + \varepsilon \overline{w}_{A_X}(y_X); \qquad\qquad y_X = (x'_X, (x_n)_X - \psi_\varepsilon(x'_X)). \tag{9.23}$$

Since the set Ω_ε is a dilation of Ω, the derivatives of A_X and ψ_ε with respect to X are of order ε (if $\dot{X}$ is of order 1). More precisely, if we set $\tilde{X} = \varepsilon X$, then we have

$$\frac{\partial A_X}{\partial X} = \varepsilon \frac{\partial A_{\tilde{X}}}{\partial \tilde{X}}; \qquad\qquad \frac{\partial \psi_\varepsilon}{\partial X} = \varepsilon \frac{\partial \psi}{\partial \tilde{X}},$$

where ψ is given in (9.5). Differentiating (9.23) with respect to X and using (9.22) it follows that, in the coordinates y

$$\dot{z}_{\varepsilon,X} = -\langle \dot{X}, \nabla_{y'} U \rangle + O(\varepsilon) \quad \text{ in } W^{1,2}(\mathbb{R}^n_+). \tag{9.24}$$

In this spirit, we also compute the variation of the matrix A_X, see (9.5), with respect to X. Differentiating the equation $x_n = \psi_\varepsilon(x')$ with respect to X and using (9.22) we find

$$-\langle x', \mathbf{H}_\varepsilon \dot{X} \rangle = \frac{1}{2} \varepsilon^2 \langle \frac{\partial A_{\tilde{X}}}{\partial \tilde{X}} x', x' \rangle - \varepsilon \langle A_X x', \dot{X} \rangle - \varepsilon^2 \sum_{i=1}^{n-1} Q_i \dot{X}_i.$$

If $e_1, \dots, e_{n-1}$ are an orthonormal system of tangent vectors to $\partial\Omega$ with $e_i = \frac{\partial \tilde{X}}{\partial x_i}$, the last equation implies

$$\langle \frac{\partial A_{\tilde{X}}}{\partial e_i} x', x' \rangle = 2 Q^i_X(x'), \qquad \text{namely} \qquad (Q^i_X)_{jk} = \left(\frac{\partial A_{\tilde{X}}}{\partial e_i} \right)_{jk}. \tag{9.25}$$

By the symmetries in (9.8), we have in particular

$$\left(\frac{\partial A_{\tilde{X}}}{\partial e_j} \right)_{ij} = \left(\frac{\partial A_{\tilde{X}}}{\partial e_i} \right)_{jj} \qquad \text{for every } i, j. \tag{9.26}$$

9.3 The abstract setting

The abstract method we use for studying problem (N_ε) is similar in spirit to the one introduced in Chapter 8. We find first a manifold of pseudo-critical points for I_ε, and then we prove the counterpart of Proposition 8.7.

Since $\partial\Omega_\varepsilon$ is almost flat for ε small and since the function U is radial, for $X \in \partial\Omega_\varepsilon$ we have $\frac{\partial}{\partial\nu} U(\cdot - X) \sim 0$. Thus $U(\cdot - X)$ is an approximate solution to $(\tilde{N}_\varepsilon)$. Hence, a natural choice of the manifold $\mathcal{Z}_\varepsilon$ could be the following

$$\{ U(\cdot - X) := U_X \ : \ X \in \partial\Omega_\varepsilon \}.$$

Actually one needs more accurate approximate solutions like $z_{\varepsilon,X}$, see Lemma 9.4. Hence we define

$$\mathcal{Z}_\varepsilon = \{z_{\varepsilon,X} \ : \ X \in \partial\Omega_\varepsilon\}\,. \tag{9.27}$$

We then have the following result, which allows us to perform a finite-dimensional reduction of problem $(\tilde{N}_\varepsilon)$ on the manifold $\mathcal{Z}_\varepsilon$.

Proposition 9.5. *Let J_ε be the functional defined in* (9.4). *Then for $\varepsilon > 0$ small there exists a unique $w = w(\varepsilon, X) \in (T_{z_{\varepsilon,X}}\mathcal{Z}_\varepsilon)^\perp$ such that $J'_\varepsilon(z_{\varepsilon,X} + w) \in T_{z_{\varepsilon,X}}\mathcal{Z}_\varepsilon$. The function $w(\varepsilon, X)$ is of class C^1 with respect to X. Moreover, the functional $\Psi_\varepsilon(\xi) = J_\varepsilon(z_{\varepsilon,X} + w(\varepsilon, X))$ is also of class C^1 in ξ and satisfies*

$$\Psi'_\varepsilon(X_0) = 0 \quad \Longrightarrow \quad J'_\varepsilon\left(z_{\varepsilon,X_0} + w(\varepsilon, X_0)\right) = 0.$$

In order to prove this proposition, we need as usual the following preliminary result.

Lemma 9.6. *There exists $\overline{\delta} > 0$ such that for ε small there holds*

$$J''_\varepsilon(z_{\varepsilon,X})[v, v] \geq \overline{\delta}\|v\|^2 \qquad \text{for every } v \perp z_{\varepsilon,X}, \frac{\partial z_{\varepsilon,X}}{\partial X}.$$

Proof. First of all we notice that, arguing as in (8.41) we have

$$J'_\varepsilon(z_{\varepsilon,X}) = J''_\varepsilon(U_X) + O(\varepsilon) + O(\varepsilon^p),$$

hence it is sufficient to prove the assertion for $J''_\varepsilon(U_X)$ instead of $J''_\varepsilon(z_{\varepsilon,X})$.

Let χ_R, v_1, v_2 and τ_1 be as in Chapter 8, with X replacing ξ. Then (8.25) holds true with no change. In the same spirit we also define

$$\tilde{\sigma}_1 = J''_\varepsilon(U_X)[v_1, v_1]; \qquad \tilde{\sigma}_2 = J''_\varepsilon(U_X)[v_2, v_2] \qquad \tilde{\sigma}_3 = 2J''_\varepsilon(U_X)[v_1, v_2],$$

and we can get immediately the counterparts of (8.32), (8.33), namely

$$\tilde{\sigma}_2 \geq C^{-1}\|v_2\|^2 + o_R(1)\|v\|^2; \qquad \tilde{\sigma}_3 \geq C^{-1}\tau_1 + o_R(1)\|v\|^2. \tag{9.28}$$

Hence it is sufficient to estimate the term $\tilde{\sigma}_1$.

By the exponential decay of $z_{\varepsilon,X}$, the fact that $(v|z_{\varepsilon,X}) = (v|\frac{\partial z_{\varepsilon,X}}{\partial X}) = 0$ and from (9.24) one easily finds

$$\begin{aligned} (v_1|z_{\varepsilon,X}) &= -(v_2|z_{\varepsilon,X}) = o_R(1)\|v\|; \\ \left(v_1|\frac{\partial z_{\varepsilon,X}}{\partial X}\right) &= -\left(v_2|\frac{\partial z_{\varepsilon,X}}{\partial X}\right) = o_R(1)\|v\|. \end{aligned} \tag{9.29}$$

Since both v_1, $z_{\varepsilon,X}$ and $\frac{\partial z_{\varepsilon,X}}{\partial X}$ are supported in $B_{\frac{\mu_0}{2\varepsilon}}(X)$, using the coordinates y we can identify them with their transposition on $\mathbb{R}^n_+$. Using (9.14), the decay of U, (9.16) and (9.29) one finds (recall the definition (9.3))

$$(v_1|U)_+ = (v_1|U) + o_\varepsilon(1)\|v_1\| = (v_1|z_{\varepsilon,X}) + o_\varepsilon(1)\|v_1\| = o_{\varepsilon,R}(1)\|v\|.$$

Similarly, from (9.24) we obtain

$$\left(v_1 | \frac{\partial U}{\partial x_i}\right)_+ = \left(v_1 | \frac{\partial U}{\partial x_i}\right) + o_\varepsilon(1)\|v\| = -\left(v_1 | \frac{\partial z_{\varepsilon,X}}{\partial X_i}\right) + o_\varepsilon(1)\|v_1\| = o_{\varepsilon,R}(1)\|v\|.$$

Hence using Proposition 9.2 and reasoning as in the proof of Lemma 8.9 we obtain

$$\overline{I}''_+(U)[v_1, v_1] \geq \delta\|v_1\|_+^2 + o_{\varepsilon,R}(1)\|v\|^2.$$

Using again (9.14) we finally find

$$\begin{aligned} \tilde{\sigma}_1 &= \overline{I}''_+(U)[v_1, v_1] + o_\varepsilon(1)\|v_1\|^2 \\ &\geq \delta\|v_1\|_+^2 + o_{\varepsilon,R}(1)\|v\|^2 \\ &\geq \delta\|v_1\|^2 + o_{\varepsilon,R}(1)\|v\|^2. \end{aligned} \tag{9.30}$$

In conclusion, from (9.28) and (9.30) we deduce

$$J''_\varepsilon(z_{\varepsilon,X})[v, v] \geq \delta\|v\|_1^2 + \|v_2\|^2 + I_v + o_{\varepsilon,R}(1)\|v\|^2 \geq \frac{\delta}{2}\|v\|^2,$$

provided R is taken large and ε is sufficiently small. This concludes the proof. □

Proof of Proposition 9.5. The argument is the same as Proposition 8.7. Letting $\overline{P}$ denote the projection onto $(T_{z_{\varepsilon,X}}\mathcal{Z}_\varepsilon)^\perp$, we want to find a solution of the two equations

$$\overline{P}w = 0; \qquad \overline{P}J'_\varepsilon(z + w) = 0.$$

As before we write

$$J'_\varepsilon(z_{\varepsilon,X} + w) = J'_\varepsilon(z_{\varepsilon,X}) + J''_\varepsilon(z_{\varepsilon,X})[w] + G_{\varepsilon,X}(w),$$

where

$$G_{\varepsilon,X}(w)[v] = \frac{1}{p+1}\int_{\Omega_\varepsilon}\left[|z_{\varepsilon,X} + w|^p - |z_{\varepsilon,X}|^p - pz_{\varepsilon,X}^{p-1}w\right] v.$$

From the inequalities in (8.12) and (8.13) we obtain the following estimates

$$\begin{cases} \|G_z(w)\| = o(\|w\|), & \|w\| \leq 1; \\ \|G_z(w_1 - w_2)\| = o(\|w_1\| + \|w_2\|)\|w_1 - w_2\|, & \|w_1\|, \|w_2\| \leq 1 \end{cases} \tag{9.31}$$

uniformly with respect to X. Then the function w is found as a fixed point in the set

$$\overline{W}_{\overline{C}} = \left\{w \in (T_{z_{\varepsilon,X}}\mathcal{Z}_\varepsilon)^\perp \ : \ \|w\| \leq \overline{C}\varepsilon^2\right\},$$

see Lemma 9.4. We omit the remaining details. □

9.4 Proof of Theorem 9.1

In view of Proposition 9.5, we can obtain existence of solutions to (N_ε) by finding critical points of the functional $\Psi_\varepsilon(X)$. The following lemma is devoted to the expansions of this functional with respect to X.

Lemma 9.7. *For ε small the following expansion holds*

$$J_\varepsilon(z_{\varepsilon,X}) = C_0 - C_1 \varepsilon H(X) + O(\varepsilon^2),$$

where

$$C_0 = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^n_+} U^{p+1}, \quad C_1 = \left(\int_0^\infty r^n U_r^2 dr\right) \int_{S^n_+} y_n |y'|^2 d\sigma.$$

Proof. To be short, we will often write z instead of $z_{\varepsilon,X}$ and w instead of $w(\varepsilon, X)$. Since z is supported in $B_{\frac{\mu_0}{2\varepsilon}}(X)$, we can use the coordinates y yielding

$$J_\varepsilon(z) = \tfrac{1}{2} \int_{\mathbb{R}^n_+} \left(|\nabla_g z|^2 + z^2\right) dy - \tfrac{1}{p+1} \int_{\mathbb{R}^n_+} z^{p+1} dy.$$

Integrating by parts, we get

$$J_\varepsilon(z) = \tfrac{1}{2} \int_{\partial \mathbb{R}^n_+} z \frac{\partial z}{\partial \tilde{\nu}} + \tfrac{1}{2} \int_{\mathbb{R}^n_+} z\left(-\Delta_g z + z\right) - \tfrac{1}{p+1} \int_{\mathbb{R}^n_+} |z|^{p+1}.$$

Using the definition of z given in (9.16) as well as the expression of the Laplace operator Δ_g given in (9.15) we find

$$\begin{aligned} &\tfrac{1}{2} \int_{\mathbb{R}^n_+} z\left(-\Delta_g z + z\right) - \tfrac{1}{p+1} \int_{\mathbb{R}^n_+} |z|^{p+1} \\ &\quad = \left(\tfrac{1}{2} - \tfrac{1}{p+1}\right) \int_{\mathbb{R}^n_+} U^{p+1} + \frac{\varepsilon}{2} \int_{\partial \mathbb{R}^n_+} U \langle A_X y', \nabla_{y'} U\rangle \\ &\qquad + \varepsilon \int_{\mathbb{R}^n_+} U \langle A_X y', \nabla_{y'} \partial_{y_n} U\rangle + \frac{\varepsilon}{2} \operatorname{tr} A_X \int_{\mathbb{R}^n_+} U \partial_{y_n} U + O(\varepsilon^2). \end{aligned}$$

Moreover, using (9.17), we get

$$\tfrac{1}{2} \int_{\partial \mathbb{R}^n_+} z \frac{\partial z}{\partial \tilde{\nu}} = \tfrac{\varepsilon}{2} \int_{\partial \mathbb{R}^n_+} U \langle A_X y', \nabla_{y'} U\rangle + O(\varepsilon^2).$$

Putting together the preceding formulas we have

$$\begin{aligned} J_\varepsilon(z) = &\left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^n_+} U^{p+1} + \frac{\varepsilon}{2} \int_{\partial \mathbb{R}^n_+} U \langle A_X y', \nabla_{y'} U\rangle \\ &+ \varepsilon \int_{\mathbb{R}^n_+} U \langle A_X y', \nabla_{y'} \partial_{y_n} U\rangle + \frac{\varepsilon}{2} \operatorname{tr} A_X \int_{\mathbb{R}^n_+} U \partial_{y_n} U + O(\varepsilon^2). \end{aligned}$$

Integrating by parts (more than once if needed), we find that the three terms of order ε are given by

$$\frac{1}{4}\int_{\partial\mathbb{R}^n_+}\langle A_X y', \nabla_{y'} U^2\rangle + \int_{\mathbb{R}^n_+} U\langle A_X y', \nabla_{y'}\partial_{y_n} U\rangle + \frac{1}{4}\operatorname{tr} A_X \int_{\mathbb{R}^n_+} \partial_{y_n} U^2$$
$$= -\frac{1}{2}\operatorname{tr} A_X \int_{\partial\mathbb{R}^n_+} U^2 - \int_{\partial\mathbb{R}^n_+} U\langle A_X y', \nabla_{y'} U\rangle - \int_{\mathbb{R}^n_+} \partial_{y_n} U\langle A_X y', \nabla_{y'} U\rangle$$
$$= -\int_{\mathbb{R}^n_+} \partial_{y_n} U\langle A_X y', \nabla_{y'} U\rangle.$$

Now we notice that, since U is radial, there holds

$$\partial_{y_n} U = \frac{y_n}{|y|} U_r; \qquad \nabla_{y'} U = \frac{y'}{|y|},$$

and hence

$$\int_{\mathbb{R}^n_+} \partial_{y_n} U\langle A_X(y'), \nabla_{y'} U\rangle = -\int_{\mathbb{R}^n_+} \frac{y_n\langle A_X(y'), y'\rangle}{|y|^2} dy.$$

At this point it is sufficient to express the last integral in radial coordinates. This concludes the proof. □

Proof of Theorem 9.1. First of all we have

$$\Psi_\varepsilon(X) = J_\varepsilon(z+w) = J_\varepsilon(z) + J'_\varepsilon(z)[w] + O(\|w\|^2).$$

Using Lemma 9.4 and the fact that $\|w\| \le C\,\varepsilon^2$ (see the end of the proof of Proposition 9.5) we infer

$$\Psi_\varepsilon(X) = J_\varepsilon(z) + O(\varepsilon^4).$$

Hence Lemma 9.7 yields

$$\Psi_\varepsilon(X) = C_0 - \varepsilon\, C_1\, H(X) + O(\varepsilon^2).$$

Therefore, if $X_0 \in \partial\Omega$ is a local strict maximum or minimum of the mean curvature H the result follows at once by usual arguments.

The general case in which X_0 is a non-degenerate critical point of H, requires a further estimate, contained in the following lemma.

Lemma 9.8. *For ε small the following expansion holds*

$$\frac{\partial}{\partial X} J_\varepsilon(z_{\varepsilon,X}) = -C_1\varepsilon^2 \frac{\partial H}{\partial X} + o(\varepsilon^2),$$

where C_1 is the constant given in the preceding Lemma.

Proof. There holds

$$J_\varepsilon'(z)[\partial_X z] = \int_{\mathbb{R}^n_+} (-\Delta_g z + z - |z|^p)\, \partial_X z + \int_{\partial\mathbb{R}^n_+} \partial_X z \frac{\partial}{\partial \tilde{\nu}} z d\sigma.$$

We notice that, by our construction, the terms $-\Delta_g z + z - |z|^p$ and $\frac{\partial}{\partial\tilde{\nu}} z$ are of order ε^2, hence it is sufficient to take the product only with the 0-th order term of $\partial_X z$, see (9.24). So we get $J_\varepsilon'(z)[\partial_X z] = (\alpha_1 + \alpha_2)\varepsilon^2 + o(\varepsilon^2)$, where

$$\alpha_1 = \int_{\mathbb{R}^n_+} \Big[2\langle Q, \nabla_{y'} \partial_{y_n} U\rangle - |A_p y'|^2 \partial^2_{y_n y_n} U + \operatorname{div} Q \partial_{y_n} U + 2\langle A_X y', \nabla_{y'} \partial_{y_n} \overline{w}\rangle$$
$$+ \operatorname{tr} A_X \partial_{y_n} \overline{w} - \frac{1}{2} p(p-1) U^{p-2} \overline{w}^2\Big]\, \partial_X U,$$

and

$$\alpha_2 = \int_{\partial\mathbb{R}^n_+} \langle Q, \nabla_{y'} U\rangle \partial_X U + \int_{\partial\mathbb{R}^n_+} \langle A_X y', \nabla_{y'} \overline{w}\rangle \partial_X U.$$

Since the function $\overline{w}$ is even in y' all the terms containing it vanish identically, and so does the term $|A_p y'|^2 \partial^2_{y_n y_n} U \partial_X U$. Hence we get

$$\alpha_1 = \int_{\mathbb{R}^n_+} [2\langle Q, \nabla_{y'} \partial_{y_n} U\rangle + \operatorname{div} Q \partial_{y_n} U]\, \partial_X U.$$

On the other hand, the boundary integral α_2 is given by

$$\alpha_2 = \int_{\partial\mathbb{R}^n_+} \langle Q, \nabla_{y'} U\rangle \partial_X U,$$

again by the oddness of $\overline{w}$.

In conclusion we have

$$\alpha_1 + \alpha_2 = \int_{\mathbb{R}^n_+} [2\langle Q, \nabla_{y'} \partial_{y_n} U\rangle + \operatorname{div} Q \partial_{y_n} U]\, \partial_X U + \int_{\partial\mathbb{R}^n_+} \langle Q, \nabla_{y'} U\rangle \partial_X U,$$

which we rewrite as

$$2\sum_j \int_{\mathbb{R}^n_+} Q_j(x') \partial_j \partial_{y_n} U \partial_i U + \sum_j \int_{\mathbb{R}^n_+} \partial_j Q_j(x') \partial_{y_n} U \partial_i U + \sum_j \int_{\partial\mathbb{R}^n_+} Q_j(x') \partial_j U \partial_i U.$$

If we integrate by parts in the variable y_j we find

$$\alpha_1 + \alpha_2 = \sum_j \int_{\mathbb{R}^n_+} Q_j(x') \partial_j \partial_{y_n} U \partial_i U - \sum_j \int_{\mathbb{R}^n_+} Q_j(x') \partial_{y_n} U \partial_j \partial_i U$$
$$+ \sum_j \int_{\partial\mathbb{R}^n_+} Q_j(x') \partial_j U \partial_i U.$$

Then, if we integrate by parts in the variable y_n and in the variable y_i we obtain

$$\sum_j \int_{\mathbb{R}^n_+} \partial_{y_i} \left(Q^j_X(y') \right) \partial_{y_j} U \partial_{y_n} U = \sum_j \left(\frac{\partial A_X}{\partial e_j} \right)_{ij} \int_{\mathbb{R}^n_+} y_j \partial_{y_j} U \partial_{y_n} U.$$

By the symmetry in (9.26) and using radial variables we finally get

$$\alpha_1 + \alpha_2 = \sum_j \left(\frac{\partial A_X}{\partial e_i} \right)_{jj} \int_{\mathbb{R}^n_+} y_j \partial_{y_j} U \partial_{y_n} U = \frac{\partial H}{\partial e_i} C_1.$$

which concludes the proof (recall that $\partial_i U = -\partial_{X_i} z$). □

Proof of Theorem 9.1 *completed.* Using a Taylor expansion for H, one can find a small positive number δ_0 such that

$$H' \neq 0 \text{ on } \partial B_{\delta_0}(X_0) \qquad \text{and} \qquad \deg(H', B_{\delta_0}(X_0), 0) = (-1)^{\operatorname{sgn} \det H''(X_0)}. \tag{9.32}$$

For $t \in [0,1]$, consider the homotopy $h_\varepsilon(t,X) = t\Psi_\varepsilon(X) + (1-t)H(X)$. From Lemma 9.8 and the first part of (9.32) one deduces that h is an admissible homotopy, namely that $h'_\varepsilon(t,X) \neq 0$ on $\partial B_{\delta_0}(X_0)$ for all $t \in [0,1]$. Then, by the homotopy property of the degree, it follows that

$$\deg(\Psi'_\varepsilon, B_{\delta_0}(X_0), 0) = \deg(H', B_{\delta_0}(X_0), 0) \neq 0.$$

As a consequence Ψ_ε possesses a critical point in $B_{\delta_0}(X_0)$ and hence, by Proposition 9.5, J_ε has a critical point of the form $z_{\varepsilon,X_0} + o(1)$. Scaling back in the variable x, we obtain the conclusion. □

Bibliographical remarks

There is a great deal of work on (N_ε) and it is not possible to make here an exhaustive list of papers. We limit ourselves to cite a few papers only, referring to their bibliography for further references. Boundary spikes have been found, e.g., in [69, 116, 118, 119] for subcritical nonlinearities. The critical exponent case has been studied, e.g., in [2, 117]. Solutions concentrating at interior points have been proved, e.g., in [120], see also [124]. There exist indeed solutions of (N_ε) which have multiple peaks both the boundary and at the interior of Ω, see, e.g., [89]. Spike-layers have also been found for singularly perturbed elliptic problems with Dirichlet boundary conditions, see, e.g., [103]. In [111, 112] it has been shown for the first time that there are solutions of (N_ε) concentrating at all the boundary $\partial\Omega$. It is worth pointing out that in such a case any power $p > 1$ is allowed. Solutions concentrating on a curve contained in the boundary also exist, see [110]. In the radial case, namely when Ω is a ball, one can show that (N_ε) possesses solutions concentrating on internal spheres, as proved in [22]. These latter results will be discussed in the next chapter.

Chapter 10

Concentration at Spheres for Radial Problems

In this chapter we will discuss the results of the two recent papers [21, 22] dealing with the existence of solutions of NLS and Singularly perturbed Neumann problems concentrating at spheres, in the radial case. For the sake of brevity we will mainly outline the main new features that arise in such a case. Many proofs will be omitted, especially when they are merely technical or based on arguments similar to those already carried out before. For complete arguments we refer to the aforementioned papers.

10.1 Concentration at spheres for radial NLS

In this section we consider radial NLS like

$$\begin{cases} -\varepsilon^2 \Delta u + V(|x|)u = u^p, & \text{in } \mathbb{R}^n \\ u > 0, \quad u \in W_r^{1,2}(\mathbb{R}^n), \end{cases} \tag{10.1}$$

where $W_r^{1,2}(\mathbb{R}^n)$ denotes the space of radial functions in $W^{1,2}(\mathbb{R}^n)$. We will denote by $\mathcal{H}_r$ such a space. We recall that the scalar product in $\mathcal{H}_r$ is given, up to a constant factor, by

$$(u|v) = \int_0^\infty (u'v' + uv) r^{n-1} dr.$$

We will use the same notation V to indicate both the function of one variable as well as the function on $\mathbb{R}^n$ induced by $V(r)$ As in Chapter 8, we will assume that V satisfies $(V1)$ and $(V2)$ namely

(V1) $V \in C^2(\mathbb{R}^n)$, and $\|V\|_{C^2(\mathbb{R}^n)} < +\infty$;
(V2) $\lambda_0^2 = \inf_{\mathbb{R}^n} V > 0$.

Moreover, we perform again the change of variable $x \mapsto \varepsilon x$ to get the perturbation problem

$$\begin{cases} -\Delta u + V(\varepsilon|x|)u = u^p, & \text{in } \mathbb{R}^n \\ u > 0, \quad u \in \mathcal{H}_r. \end{cases} \tag{10.2}$$

With this notation, the Euler functional of (10.2) I_ε has the form

$$I_\varepsilon(u) = \frac{1}{2}\int_0^\infty \left((u')^2 + V(\varepsilon r)u^2\right) r^{n-1} dr - \frac{1}{p+1}\int_0^\infty |u|^{p+1} r^{n-1} dr; \qquad u \in \mathcal{H}_r. \tag{10.3}$$

Here we are assuming that $1 < p \leq \frac{n+2}{n-2}$. It is easy to see that I_ε has a Mountain-Pass critical point which gives rise to a solution $u_\varepsilon \in \mathcal{H}_r$ of (10.2), provided $1 < p < \frac{n+2}{n-2}$. It suffices to remark that $\mathcal{H}_r$ is compactly embedded into $L^q(\mathbb{R}^n)$ provided $2 < q < 2^*$, see also the discussion in Section 2.1 after Remark 2.2. Scaling back we find a solution $v_\varepsilon(|x|/\varepsilon)$ of (10.1) and the arguments carried out in the proof of Theorem 8.1 readily imply that such a v_ε is a spike concentrating at the origin.

We now want to investigate whether (10.1) possesses a solution concentrating at a sphere $|x| = \bar{r}$.

To give an idea why (10.1) might possess solutions concentrating on a sphere, let us make the following heuristic considerations. A concentrated solution of (10.1) carries a *potential* energy due to V and a *volume energy*. The former would lead the region of concentration to approach the minima of V. On the other hand, unlike for the case of spike-layer solutions where the volume energy does not depend on the location, the volume energy of solutions concentrating on spheres tends to shrink the sphere. In the region where V is decreasing, there could possibly be a balance, that gives rise to solutions concentrating on a sphere. This phenomenon is quantitatively reflected by an *auxiliary weighted* potential M defined as follows. Let

$$\theta = \frac{p+1}{p-1} - \frac{1}{2},$$

and define M by setting

$$M(r) = r^{n-1}\, V^\theta(r), \qquad\qquad r > 0.$$

Our main result is the following.

Theorem 10.1. *Let* (V1) *and* (V2) *hold, let* $p > 1$ *and suppose that* M *has a point of local strict maximum or minimum at* $r = \overline{r}$. *Then, for* $\varepsilon > 0$ *small enough,* (10.1) *has a radial solution which concentrates near the sphere* $|x| = \overline{r}$.

Remarks 10.2.

(i) In the case $n = 1$, M and V have the same critical points. Otherwise, when $n > 1$ the stationary points of V do not determine the location of solutions concentrating at spheres.

Actually, one has

$$M'(r) = r^{n-2} V^{\theta-1}(r)\ [(n-1)V(r) + \theta r V'(r)]$$

and therefore critical points of M belong to the region $V' < 0$, as pointed out before.

(ii) Since $M(r) \sim r^{n-2}$ as $r \to 0$ and as $r \to \infty$, then stationary points of M arise generically in pairs.

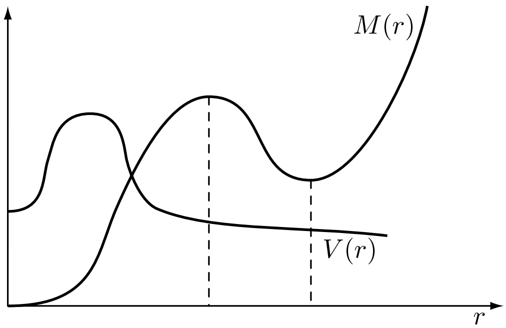

Figure 10.1. Graph of V versus M

(iii) Differently from the case of ordinary spikes, in Theorem 10.1 we do not require any upper bound on the exponent p, namely we can deal with the critical or supercritical case as well. This does not depend on the fact that we are dealing with radial problems but rather it is a consequence of the fact that the solutions concentrate on a sphere. Roughly, we will see that the asymptotic profile of a radial concentrating function is a solution of a one-dimensional problem, see (10.4) below, for which there is no restriction on p to get existence of a solution. Actually, looking for solutions concentrating on a k-dimensional sphere, with $1 \le k \le n-1$, one has to impose that $1 < p < \frac{n-k+2}{n-k-2}$ if $k < n-2$, see Theorem 10.11. □

The next three sections are mainly devoted to prove Theorem 10.1. Hereafter, until Subsection 10.3.1, we will assume that $1 < p \le \frac{n+2}{n-2}$, when the functional I_ε in (10.3) is well defined on $\mathcal{H}_r$. The general case will be handled by means of a truncation procedure.

10.2 The finite-dimensional reduction

As in Section 8.4, Chapter 8, we will perform a finite-dimensional reduction near a manifold of *pseudo-critical points*.

First of all, we consider for any $\lambda > 0$ and any $p > 1$ the positive even solution $\overline{U}_\lambda$ of the one-dimensional equation

$$-\overline{U}_\lambda'' + \lambda^2 \overline{U}_\lambda = \overline{U}_\lambda^p, \qquad \overline{U}_\lambda \in W^{1,2}(\mathbb{R}). \tag{10.4}$$

Recall that one has

$$\overline{U}_\lambda(r) = \lambda^{\frac{2}{p-1}} \overline{U}(\lambda r),$$

where $\overline{U}$ stands for $\overline{U}_\lambda$ with $\lambda = 1$. The function $\overline{U}$ has an exponential decay to zero at infinity: indeed, there holds

$$\overline{U}(r) = \left(\frac{p+1}{2}\right)^{1/(p-1)} \left(\cosh\left[\frac{p-1}{2}\, r\right]\right)^{-2/(p-1)}.$$

Setting $\mathrm{Cr}[M] = \{r > 0 : M'(r) = 0\}$, let us fix $\rho_0 > 0$ with $8\rho_0 < \min \mathrm{Cr}[M]$ and let $\phi_\varepsilon(r)$ denote a smooth non-decreasing function such that

$$\phi_\varepsilon(r) = \begin{cases} 0, & \text{if} \quad r \le \frac{\rho_0}{2\varepsilon}, \\ 1, & \text{if} \quad r \ge \frac{\rho_0}{\varepsilon}. \end{cases} \qquad |\phi'_\varepsilon(r)| \le \frac{4\varepsilon}{\rho_0}, \quad |\phi''_\varepsilon(r)| \le \frac{16\varepsilon^2}{\rho_0^2}.$$

For $\rho \ge 4\rho_0/\varepsilon$, set

$$z_{\rho,\varepsilon}(r) = \phi_\varepsilon(r)\, \overline{U}_\lambda(r - \rho); \qquad \lambda^2 = V(\varepsilon\rho). \tag{10.5}$$

Fixed $\ell > \overline{r}$, see Theorem 10.1, consider the compact interval $\mathcal{T}_\varepsilon = \left[4\varepsilon^{-1}\rho_0, \varepsilon^{-1}\ell\right]$ and let

$$\mathcal{Z} = \mathcal{Z}_\varepsilon = \{z = z_{\rho,\varepsilon} : \rho \in \mathcal{T}_\varepsilon\}.$$

As usual, we set $W = (T_z\mathcal{Z})^\perp$ and let P denote the orthogonal projection on W. Given a positive constants $\gamma > 0$ (to be fixed later), we define

$$\mathcal{C}_\varepsilon = \{w \in W \;:\; \|w\|_{\mathcal{H}_r} \le \gamma\varepsilon \|z_{\rho,\varepsilon}\|_{\mathcal{H}_r},\ |w(r)| \le \gamma\varepsilon \text{ for } r > 0\}. \tag{10.6}$$ [1]

Remark 10.3. The reason for the introduction of the set $\mathcal{C}_\varepsilon$ is the following. The norms of the function $z_{\rho,\varepsilon}$ and of the gradient $I'_\varepsilon(z_{\rho,\varepsilon})$ diverge as ε goes to zero, see the estimates (E1) and (E2) in the next Subsection 10.2.1. For this reason it is not possible to perform the contraction argument using only norm estimates, as in the proof of Proposition 8.7. By means of the set $\mathcal{C}_\varepsilon$ we keep the function w small in L^∞ and the function $z + w$ concentrated near $|x| = \rho$. □

It is now convenient to collect some estimates we will need in the sequel.

10.2.1 Some preliminary estimates

For every $\rho \in \mathcal{T}_\varepsilon$, every $w \in \mathcal{C}_\varepsilon$ and $\varepsilon \ll 1$, the following estimates hold

(E1) $\|z_{\rho,\varepsilon}\|_{\mathcal{H}_r} \sim \varepsilon^{(1-n)/2}$;
(E2) $\|I'_\varepsilon(z_{\rho,\varepsilon})\| \sim \varepsilon\, \|z_{\rho,\varepsilon}\|$;
(E3) $\|I''_\varepsilon(z_{\rho,\varepsilon} + sw) - I''_\varepsilon(z_{\rho,\varepsilon})\| \sim \varepsilon^{1\wedge(p-1)} \quad (0 \le s \le 1)$;
(E4) $\|I'_\varepsilon(z_{\rho,\varepsilon} + w) - I'_\varepsilon(z_{\rho,\varepsilon}) - I''_\varepsilon(z_{\rho,\varepsilon})[w]\| \sim \varepsilon^{1\wedge(p-1)} \|w\|$.

[1] the reader should note that the set $\mathcal{C}_\varepsilon$ defined above is sligthly different from the one introduced in [21]. However, this suffices for the proof of Theorem 10.1.

Proof of (E1). By the definition of $z_{\rho,\varepsilon}$ we have

$$\|z_{\rho,\varepsilon}\|^2_{\mathcal{H}_r} = \int_0^{+\infty} r^{n-1}(|z'|^2 + V(\varepsilon r)z^2)dr \sim \rho^{n-1}.$$

Since $\rho \in \mathcal{T}_\varepsilon$, then $\rho \sim \varepsilon^{-1}$ and hence $\|z_{\rho,\varepsilon}\|_{\mathcal{H}_r} \sim \varepsilon^{(1-n)/2}$.

Proof of (E2). For all $v \in \mathcal{H}_r$ one has

$$\begin{aligned} I'_\varepsilon(z)[v] &= \int_0^{+\infty} r^{n-1}\left(z'v' + V(\varepsilon r)zv - z^p v\right)dr \\ &= -\int_0^{+\infty} v(r^{n-1}z')'dr + \int_0^{+\infty} r^{n-1}\left(V(\varepsilon r)zv - z^p v\right)dr \\ &= -(n-1)\underbrace{\int_0^{+\infty} r^{n-2}z'vdr}_{A_0(v)} \underbrace{-\int_0^{+\infty} r^{n-1}z''vdr + \int_0^{+\infty} r^{n-1}\left(V(\varepsilon r)zv - z^p v\right)dr}_{A_1(v)}. \end{aligned}$$

Using the Hölder inequality we get

$$|A_0(v)| \le C\,\|v\|_{\mathcal{H}_r}\left(\int_0^{+\infty}(r^{(n-3)/2}z')^2dr\right)^{1/2}.$$

Since z decays exponentially away from $r = \rho$ and since $\rho \in \mathcal{T}_\varepsilon$, it follows that

$$\int_0^{+\infty}(r^{(n-3)/2}z')^2dr = \int_0^{+\infty} r^{-2}\cdot r^{n-1}|z'|^2dr \sim \rho^{-2}\|z\|^2_{\mathcal{H}_r} \sim \varepsilon^2\|z\|^2_{\mathcal{H}_r}.$$

Then we find

$$\sup\{|A_0(v)| : \|v\|_{\mathcal{H}_r} \le 1\} \sim \varepsilon\,\|z\|. \tag{10.7}$$

To estimate $A_1(v)$ we write $A_1(v) = A_2(v) + A_3(v)$ where

$$A_2(v) = \int_0^{+\infty} r^{n-1}\left[\phi''\overline{U}_\lambda(r-\rho) + 2\phi'\overline{U}'_\lambda(r-\rho)\right]vdr$$

and

$$\begin{aligned} A_3(v) = \int_0^{+\infty} r^{n-1}\Big[&\phi\overline{U}_\lambda(r-\rho) + V(\varepsilon r)\phi\overline{U}_\lambda(r-\rho) \\ &- (\phi\overline{U}_\lambda(r-\rho))^p - \phi\overline{U}''_\lambda(r-\rho)\Big]vdr. \end{aligned}$$

Since the support of ϕ' is the interval $[\rho_0/2\varepsilon, \rho_0/\varepsilon]$ and $\overline{U}_\lambda$ decays exponentially to zero as $r \to \infty$ we get

$$\sup\{|A_2(v)| : \|v\|_{\mathcal{H}_r} \le 1\} \sim e^{-\frac{1}{C\varepsilon}}. \tag{10.8}$$

Finally, using the definition of $\overline{U}_\lambda$ we infer

$$
\begin{aligned}
A_3(v) &= \int_0^{+\infty} r^{n-1}\left(V(\varepsilon r) - V(\varepsilon\varrho)\right)\phi\overline{U}_\lambda(r-\rho)v dr\\
&= \int_0^{+\infty} r^{n-1}\left(V(\varepsilon r) - V(\varepsilon\varrho)\right) zv dr,
\end{aligned}
$$

hence

$$
|A_3| \le C\|v\|_{\mathcal{H}_r}\left(\int_0^{\infty}|V(\varepsilon r)-V(\varepsilon\rho)|^2 z^2 r^{n-1} dr\right)^{\frac{1}{2}}.
$$

By (V1) one has (see also (8.19))

$$
|V(\varepsilon r) - V(\varepsilon\rho)| \le C\varepsilon|r-\rho| + C\varepsilon^2|r-\rho|^2,
$$

hence, arguing as for (8.21) we infer

$$
\sup\{|A_3(v)| : \|v\|_{\mathcal{H}_r} \le 1\} \sim \varepsilon\ \|z\|. \tag{10.9}
$$

Putting together (10.7), (10.8) and (10.9), we find (E1).

The proofs of (E3) and (E4) are based on similar arguments and are omitted.

10.2.2 Solving $PI'_\varepsilon(z+w)=0$

We will look for critical points of I_ε of the form

$$
u = z + w, \quad z = z_{\rho,\varepsilon} \in Z, \quad w \in \mathcal{C}_\varepsilon .
$$

As usual we first solve the auxiliary equation $PI'_\varepsilon(z+w)=0$, which is equivalent to

$$
PI'_\varepsilon(z) + PR_w + PI''_\varepsilon(z)[w] = 0,
$$

where

$$
R_w = I'_\varepsilon(z+w) - I'_\varepsilon(z) - I''_\varepsilon(z)[w].
$$

As in the previous section, with only minor modifications, one can prove the following result.

Lemma 10.4. *There exists a positive constant C such that, for every $\rho \in \mathcal{T}_\varepsilon$ and for ε sufficiently small there holds*

$$
I''_\varepsilon(z_{\rho,\varepsilon})[v,v] \ge C^{-1}\|v\|^2, \qquad \text{for all } v \perp \{tz\} \oplus T_z Z.
$$

In particular the operator $L_\varepsilon = L_{\rho,\varepsilon} := P \circ I''_\varepsilon(z_{\rho,\varepsilon}) \circ P$ is invertible.

Setting

$$S_\varepsilon(w) := L_\varepsilon^{-1}\left(I'_\varepsilon(z) + R_w\right),$$

we deduce that

$$PI'_\varepsilon(z+w) = 0 \quad \Longleftrightarrow \quad w = S_\varepsilon(w).$$

In order to find the fixed points of S_ε we will prove that there is $\gamma > 0$ such that for ε sufficiently small S_ε is a contraction that maps $\mathcal{C}_\varepsilon$ into itself, namely:

(S1) $S_\varepsilon(\mathcal{C}_\varepsilon) \subset \mathcal{C}_\varepsilon$;
(S2) $\exists \kappa \in (0,1) : \|S_\varepsilon(w_1) - S_\varepsilon(w_2)\| \le \kappa \|w_1 - w_2\|, \ \forall\, w_1, w_2 \in \mathcal{C}_\varepsilon$.

Proof of (S1). First we show that there exists $C_1 > 0$ such that

$$\|S_\varepsilon(w)\| \le C_1\, \varepsilon\, \|z\|, \qquad \forall\, w \in \mathcal{C}_\varepsilon. \tag{10.10}$$

Actually, using (E4) we get

$$\|R_w\| = \|I'_\varepsilon(z+w) - I'_\varepsilon(z) - I''_\varepsilon(z)[w]\| \le c_1 \varepsilon^{1\wedge(p-1)} \|w\|.$$

Since $w \in \mathcal{C}_\varepsilon$ then $\|w\| \sim \varepsilon \|z\|$ and hence

$$\|R_w\| \le c_2\, \varepsilon \cdot \varepsilon^{1\wedge(p-1)} \|z\|.$$

From this, the definition of S_ε and (E2) we infer

$$\|S_\varepsilon(w)\| \le c_3\ \left[\|I'_\varepsilon(z) + \|R_w\|\right] \le c_4\, \varepsilon \left[1 + \varepsilon^{1\wedge(p-1)}\right] \|z\|$$

and (10.10) follows.

To complete the proof of (S1) it remains to show that, letting $\widetilde{w} = S_\varepsilon w$, there exists $C > 0$ such that

$$|\widetilde{w}(r)| \le C\, \varepsilon.$$

First of all, let us recall once more that for all $u \in \mathcal{H}_r$ there holds

$$|u(r)| \le c_1\, r^{(1-n)/2}\, \|u\|_{\mathcal{H}_r}, \quad (r \ge 1).$$

Using this estimate with $u = w$ and taking into account the equation (10.10), we get

$$|\widetilde{w}(r)| \le c_1\, r^{(1-n)/2}\, \|\widetilde{w}\|_{\mathcal{H}_r} \le c_2\, \varepsilon\, r^{(1-n)/2}\, \|z\|_{\mathcal{H}_r} \qquad (r \ge 1).$$

Then, recalling (E1) and taking $r \ge 4\rho_0/\varepsilon$ we find:

$$|\widetilde{w}(r)| \le c_3\, \varepsilon, \qquad (r \ge \rho_0/4\varepsilon). \tag{10.11}$$

To prove a similar inequality for $0 < r < 4\rho_0/\varepsilon$ we argue as follows.

The function $\tilde{w}$ satisfies the equation

$$-\Delta\tilde{w} + V(\varepsilon r)\tilde{w} - pz^{p-1}\tilde{w} = -\left((z+w)^p - z^p - pz^{p-1}w\right) + \beta\left(-\Delta\dot{z} + V(\varepsilon r)\dot{z}\right) + \left(-\Delta z + V(\varepsilon r)z - z^p\right), \qquad \text{in } \mathbb{R}^n,$$

where β is a real number with $|\beta| \sim \varepsilon$, and where $\dot{z} = \frac{\partial z}{\partial \varrho}$. Since for $0 < r < 4\rho_0/\varepsilon$ we have that $z \equiv 0$ and thus $\widetilde{w}$ satisfies

$$-\Delta\widetilde{w} + V(\varepsilon r)\widetilde{w} = -|w|^{p-1}w, \qquad \text{for } |x| < \rho_0/\varepsilon\ .$$

Since $|\widetilde{w}(x)| < c_3\varepsilon$ on the sphere $|x| = \rho_0/\varepsilon$, the maximum principle implies that $|\widetilde{w}(x)| < c_3\varepsilon$ in the ball $|x| < \rho_0/\varepsilon$. This, jointly with (10.11), proves that there exists $C > 0$ such that $w \in \mathcal{C}_\varepsilon \implies |\widetilde{w}(r)| < C\varepsilon$, completing the proof of (S1).

Proof of (S2). From

$$S_\varepsilon(w_1) - S_\varepsilon(w_2) = L_\varepsilon^{-1}\left[I'_\varepsilon(z+w_1) - I''_\varepsilon(z)[w_1] - I'_\varepsilon(z+w_2) + I''_\varepsilon(z)[w_2]\right]$$

we infer

$$\|S_\varepsilon(w_1) - S_\varepsilon(w_2)\| \le c_1\|I'_\varepsilon(z+w_1) - I''_\varepsilon(z)[w_1] - I'_\varepsilon(z+w_2) + I''_\varepsilon(z)[w_2]\|.$$

One also has:

$$\begin{aligned} &I'_\varepsilon(z+w_1) - I''_\varepsilon(z)[w_1] - I'_\varepsilon(z+w_2) + I''_\varepsilon(z)[w_2] \\ &\qquad = \int_0^1 \left(I''_\varepsilon(z+w_1+s(w_1-w_2) - I''_\varepsilon(z)\right)[w_1-w_2]ds. \end{aligned}$$

Putting together the preceding estimates and using (E3) we deduce

$$\|S_\varepsilon(w_1) - S_\varepsilon(w_2)\| \le c_2\varepsilon^{1\wedge(p-1)}\|w_1 - w_2\|,$$

and (S2) follows. $\square$

From (S1) and (S2) it follows that the equation $S_\varepsilon(w) = w$ has a solution in $\mathcal{C}_\varepsilon$. Repeating the arguments used in Section 8.4, we find the following result which is the counterpart of Proposition 8.7.

Proposition 10.5. *For ε sufficiently small there exists a positive constant γ such that for $\rho \in \mathcal{T}_\varepsilon$, there exists and a function $w = w(z_{\rho,\varepsilon}) \in W$ satisfying $PI'_\varepsilon(z + w) = 0$. Furthermore, setting*

$$\Phi_\varepsilon(\rho) = I_\varepsilon(z_{\rho,\varepsilon} + w_{\rho,\varepsilon}),$$

if, for some $\varepsilon \ll 1$, ρ_ε is stationary point of Φ_ε, then $\widetilde{u}_\varepsilon = z_{\rho_\varepsilon,\varepsilon} + w_{\rho_\varepsilon,\varepsilon}$ is a critical point of I_ε.

10.3 Proof of Theorem 10.1

Here we carry over the Proof of Theorem 10.1. First of all we expand the functional Φ_ε.

Lemma 10.6. *For $\varepsilon > 0$ small, there is a constant $C_0 > 0$ such that:*

$$\varepsilon^{n-1} I_\varepsilon(z_{\rho,\varepsilon} + w_{\rho,\varepsilon}) = C_0 M(\varepsilon\rho) + O(\varepsilon^2), \quad \rho \in \mathcal{T}_\varepsilon.$$

Proof. As usual, for brevity, we write z instead of $z_{\rho,\varepsilon}$ and w instead of $w_{\rho,\varepsilon}$. One has

$$I_\varepsilon(z+w) = I_\varepsilon(z) + I_\varepsilon'(z)[w] + \int_0^1 I_\varepsilon''(z+sw)[w]^2 ds.$$

Using (E1) and (E2) we infer that $I_\varepsilon'(z)[w] \sim \varepsilon^{(3-n)/2}\|w\|$. Moreover, from $\|w\| \le \varepsilon\|z\|$ and (E1) we get $\|w\| \sim \varepsilon^{(3-n)/2}$ and hence $I_\varepsilon'(z)[w] \sim \varepsilon^{3-n}$. Using arguments similar to those carried out in Subsection 10.2.1 we also find that $I_\varepsilon''(z+sw)[w]^2 \sim \varepsilon^{3-n}$ and thus we deduce

$$I_\varepsilon(z+w) = I_\varepsilon(z) + O(\varepsilon^{3-n}).$$

On the other hand, recall that by definition $z_{\rho,\varepsilon}(r) = \phi_\varepsilon(r)\overline{U}_\lambda(r-\rho)$. Then z concentrates near ρ and one finds

$$\begin{aligned} I_\varepsilon(z) &= \int_0^\infty r^{n-1}\left(\frac{|z'|^2 + V(\varepsilon r)z^2}{2} - \frac{z^{p+1}}{p+1}\right) dr \\ &= \rho^{n-1}\int_{\mathbb{R}}\left(\frac{|\overline{U}_\lambda'|^2 + V(\varepsilon\rho)\overline{U}_\lambda^2}{2} - \frac{\overline{U}_\lambda^{p+1}}{p+1}\right) dr\,(1+o(1)). \end{aligned}$$

We recall that

$$\overline{U}_\lambda(r) = \lambda^{2/(p-1)}\overline{U}(\lambda r), \qquad \lambda^2 = V(\varepsilon\rho).$$

It follows by a straightforward calculation that

$$\int_{\mathbb{R}}\left(\frac{|\overline{U}'|^2 + V(\varepsilon r)\overline{U}^2}{2} - \frac{\overline{U}^{p+1}}{p+1}\right) dr = C_0 V^\theta(\varepsilon\rho),$$

where $C_0 = \frac{1}{p+1}\int \overline{U}^{p+1}$. Substituting into the preceding equations we find

$$I_\varepsilon(z+w) = C_0\rho^{n-1}V^\theta(\varepsilon\rho) + O(\varepsilon^{3-n}).$$

Recalling the definition of M we get

$$I_\varepsilon(z+w) = \frac{C_0}{\varepsilon^{n-1}}(\varepsilon\rho)^{n-1}V^\theta(\varepsilon\rho) + O(\varepsilon^{3-n}) = \frac{C_0}{\varepsilon^{n-1}}M(\varepsilon\rho) + O(\varepsilon^{3-n}),$$

and the lemma follows. □

We are now in the position to prove Theorem 10.1 in the case $p \in \left(1, \frac{n+2}{n-2}\right]$. By Lemma 10.6, if $\overline{r}$ is a maximum (resp. minimum) of M then $\Phi_\varepsilon(\rho) = I_\varepsilon(z_{\rho_e,\varepsilon} + w_{\rho_e,\varepsilon})$ will possess a maximum (resp. minimum) at some $\rho_\varepsilon \sim \overline{r}/\varepsilon$, with $\rho_\varepsilon \in \mathcal{T}_\varepsilon$. Using Proposition 10.5, such a stationary point of Φ_ε gives rise to a critical point $\widetilde{u}_\varepsilon = z_{\rho_e,\varepsilon} + w_{\rho_\varepsilon,\varepsilon}$, which is a (radial) solution of (10.2). Since $\widetilde{U}_\varepsilon(r) \sim \overline{U}_\lambda(r-\rho_\varepsilon) \sim \overline{U}_\lambda(r - \overline{r}/\varepsilon)$, then the scaled $u_\varepsilon(r) = \widetilde{u}_\varepsilon(r/\varepsilon)$ is a solution of (10.1) such that $u_\varepsilon(r) \sim \overline{U}_\lambda\left((r-\overline{r})/\varepsilon\right)$, hence concentrating near the sphere $|x| = \overline{r}$.

10.3.1 Proof of Theorem 10.1 completed

Let us now consider the case $p > \frac{n+2}{n-2}$. The proof is done using some truncation for the nonlinear term, and then proving a priori L^∞ estimates on the solutions. We list the modifications which are necessary to handle this case.

For $K > 0$, we define a smooth positive function $F_K : \mathbb{R} \to \mathbb{R}$ such that

$$F_K(t) = |t|^{p+1} \quad \text{for } |t| \le K; \qquad F_K(t) = (K+1)^{p+1} \quad \text{for } |t| \ge K+1.$$

Let $I_{\varepsilon,K} : \mathcal{H}_r \to \mathbb{R}$ be the functional obtained substituting $|u|^{p+1}$ with $F_K(u)$ in I_ε, and let $K_0 = (\sup V)^{\frac{1}{p-1}}$. Since the non-linear term in $I_{\varepsilon,K}$ is sub-critical, this is a well-defined functional on $\mathcal{H}_r$.

We note that by the definition of $\overline{U}_\lambda$ and $z_{\rho,\varepsilon}$, it is $\|z_{\rho,\varepsilon}\| \le K_0$ for all $\rho \in \mathcal{T}_\varepsilon$ and ε sufficiently small.

In the above notation, if $K \ge K_0$, the operator $PI''_{\varepsilon,K}(z)$ remains invertible and its inverse A_ε has uniformly bounded norm, independent of K. In fact, the preceding arguments are based on local arguments and remain unchanged. Moreover, if $K \ge K_0 + \gamma$ (see the definition of $\mathcal{C}_\varepsilon$) and using the pointwise bounds on $|w(r)|$, one readily checks that the estimates (E2)–(E4) involving $I'_\varepsilon(z)$ and $I''_\varepsilon(z+w)$ are also independent of K. Hence the above method produces a solution u_ε of $I'_{\varepsilon,K} = 0$ for which $\|u_\varepsilon\|_\infty \le K$. Hence u_ε also solves (10.2). This completes the proof of Theorem 10.1. $\square$

10.4 Other results

In this section we collect some further results on the existence of solutions of (10.1) that concentrate at a sphere. We will not give the proofs, referring to [21].

First of all, let us state the following result which is the counterpart of Theorem 8.1 dealing with necessary conditions for concentration at points.

Theorem 10.7. *Suppose that, for all $\varepsilon > 0$ small,* (10.1) *has a radial solution u_ε concentrating on the sphere $|x| = \widehat{r}$, in the sense that $\forall\, \delta > 0$, $\exists\, \varepsilon_0 > 0$ and $R > 0$ such that*

$$u_\varepsilon(r) \le \delta, \qquad \textit{for } \varepsilon \le \varepsilon_0, \qquad \textit{and for } \ |r - \widehat{r}| \ge \varepsilon R.$$

Then u_ε has a unique maximum at $r = r_\varepsilon$, $r_\varepsilon \to \widehat{r}$ and $M'(\widehat{r}) = 0$.

Our second result is concerned with the bifurcation of non-radial solutions from a family of radial solutions concentrating on spheres. Let

$$\Lambda_{\overline{r},\overline{\varepsilon}} = \{(\varepsilon, u_\varepsilon) \ : \ 0 < \varepsilon < \overline{\varepsilon}\},$$

where u_ε denote the solutions of (10.1) obtained using Theorem 10.1 in correspondence of $r - \overline{r}$.

Theorem 10.8. *In addition to the assumption of Theorem* 10.1*, suppose that the potential V is smooth and that at a point $\overline{r} > 0$ of strict local maximum or minimum of M there holds*

$$M''(\overline{r}) \neq 0. \tag{10.12}$$

Then for $\overline{\varepsilon}$ sufficiently small $\Lambda_{\overline{r},\overline{\varepsilon}}$ is a smooth curve. Moreover, there exist a sequence $\varepsilon_j \downarrow 0$ such that from each $(\varepsilon_j, u_{\varepsilon_j}) \in \Lambda_{\overline{r},\overline{\varepsilon}}$ bifurcates a family of non-radial solutions of (10.1).

Roughly, the proof of Theorem 10.8 is based on the following two propositions which have an interest in itself.

Proposition 10.9. *Let $\widetilde{u}_\varepsilon$ be the family of solutions radial solutions of* (10.2) *having the form*

$$\widetilde{u}_\varepsilon = z_{\rho_\varepsilon,\varepsilon} + w_{\rho_\varepsilon,\varepsilon}, \qquad \textit{for some } \rho_\varepsilon \sim \frac{\overline{r}}{\varepsilon},$$

where $w_{\rho_\varepsilon,\varepsilon} \in \mathcal{C}_\varepsilon$. Then the Morse index of $\widetilde{u}_\varepsilon$ in $W^{1,2}(\mathbb{R}^n)$ tends to infinity as ε goes to zero.

Let us emphasize that it is the Morse index in $W^{1,2}(\mathbb{R}^n)$ which tends to infinity, while the Morse index of $\widetilde{u}_\varepsilon$ in $\mathcal{H}_r$ is 1 (resp. 2) if $\overline{r}$ is a local minimum (resp. maximum) of M.

Proposition 10.10. *Suppose $M''(\overline{r}) \neq 0$, and suppose $\widetilde{u}_\varepsilon$ is a solution of* (10.2) *as above. Then, for ε small, $\widetilde{u}_\varepsilon$ is non-degenerate in $\mathcal{H}_r$.*

By Proposition 10.10 the solution $\widetilde{u}_\varepsilon$ of (10.2) is non-degenerate and locally unique in the class of radial functions. This implies that the set Λ in Theorem 10.8 is a smooth curve. By Proposition 10.9 the Morse index of $I''_\varepsilon(\widetilde{u}_\varepsilon)$, in the space $W^{1,2}(\mathbb{R}^n)$, diverges as $\varepsilon \to 0$. To obtain the conclusion it is sufficient to apply a bifurcation result of Kielhofer [96].

We complete this section with a short discussion about concentration at k-dimensional spheres, $1 \leq k \leq n-1$. In such a case, the corresponding limit problem is of the form

$$\begin{cases} -\Delta U_{\lambda,k} + \lambda^2 U_{\lambda,k} = U_{\lambda,k}^p & \text{in } \mathbb{R}^{n-k}, \\ U_{\lambda,k} > 0, U_{\lambda,k} \in W^{1,2}(\mathbb{R}^{n-k}). \end{cases} \tag{10.13}$$

Here $\lambda^2 = V(\varepsilon \overline{r}_k)$ is the potential at the concentration radius (to be found) and the exponent p is subcritical with respect to $\mathbb{R}^{n-k}$, namely $1 < p < \frac{n-k+2}{n-k-2}$ if $n-k > 2$, $1 < p$ if $n-k \leq 2$.

From a simple scaling argument one finds $U_{\lambda,k}(x) = \lambda^{\frac{2}{p-1}}\overline{U}_k(\lambda x)$, $x \in \mathbb{R}^{n-k}$ as well as

$$\frac{1}{2}\int_{\mathbb{R}^{n-k}} |\nabla U_{\lambda,k}|^2 + \frac{1}{2}\lambda^2 \int_{\mathbb{R}^{n-k}} U_{\lambda,k}^2 - \frac{1}{p+1}\int_{\mathbb{R}^{n-k}} U_{\lambda,k}^{p+1} = \left(\tfrac{1}{2} - \tfrac{1}{p+1}\right)\lambda^{2\theta_k}\|\overline{U}_k\|_{W^{1,2}(\mathbb{R}^{n-k})},$$

where $\theta_k = \frac{p+1}{p-1} - \frac{1}{2}(n-k)$ and $\overline{U}_k$ is the solution of (10.13) with $\lambda = 1$. Hence the energy of an approximate solution z_ρ which is concentrated near a k-dimensional sphere of radius ρ can be estimated as

$$E(z_\rho) \sim \rho^k V^{\theta_k}(\varepsilon\rho).$$

As a consequence, solutions of (10.1) should concentrate at critical points of the auxiliary functional $M_k(r) := r^k V^{\theta_k}(r)$. When $k = n-1$, M_k coincides with M while for $k = 0$ the critical points of M_k coincide with those of V. Precisely, one can prove the following theorem:

Theorem 10.11. *Suppose that* $1 < p < \frac{n-k+2}{n-k-2}$ *if* $n-k > 2$, $1 < p$ *if* $n-k \leq 2$, *that* (V1) *and* (V2) *hold and let* $M_k(r) := r^k V^{\theta_k}(r)$, *where* $\theta_k = \frac{p+1}{p-1} - \frac{1}{2}(n-k)$. *If* (10.1) *has a (radial) solutions concentrating at a* k*-dimensional sphere of radius* $\widehat{r} > 0$, *then* $M_k'(\widehat{r}) = 0$. *Conversely, if* $\overline{r} > 0$ *is a local strict maximum or minimum of* M_k, *then there exists a radial solution of* (10.1) *concentrating at the* k*-dimensional sphere of radius* $\overline{r} > 0$.

As anticipated in Remark 10.2-(iii), in the singularly perturbed problems where solutions concentrate on a k-dimensional manifold, the number $\frac{n-k+2}{n-k-2}$ (if $n-k > 2$) which replaces the usual critical exponent $\frac{n+2}{n-2}$.

10.5 Concentration at spheres for (N_ε)

In this section we study concentration at spheres for (N_ε) in the case of the unit ball $\Omega = B_1 = \{x \in \mathbb{R}^n : |x| < 1\}$, $n \geq 2$, highlighting that new phenomena take place, due to the imposed boundary conditions. We give first some heuristic description of the situation.

As already mentioned at the beginning of the chapter, a solution concentrating at a sphere carries a volume energy which tends to shrink its radius. On the other hand, imposing Neumann conditions at the boundary of the domain correspond naively to add some *virtual* spherical spike outside the domain, at the same distance from $\partial\Omega$. It is standard to see from energy expansions (see Subsection 10.5.2 for precise estimates) that spikes attract each-other. Therefore, any spherical spike with interior profile is attracted by the boundary. As for NLS it turns out that the two competing forces balance each-other giving rise to a radial solution concentrating at a sphere close to $\partial\Omega$, preventing the collapsing to the origin. Our main result is the following theorem.

Theorem 10.12. *Given $n \geq 2$ and $p > 1$, consider the problem*

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^p, & \text{in } B_1, \\ \frac{\partial u}{\partial \nu} = 0 \ \text{on } \partial B_1, & u > 0. \end{cases} \tag{$\tilde{N}$}$$

Then there exists a family of radial solutions u_ε of $(\tilde{N})$ concentrating at $|x| = r_\varepsilon$, where r_ε is a local maximum point of u_ε satisfying $1 - r_\varepsilon \sim \varepsilon |\log \varepsilon|$.

As for the Schrödinger equation, it is convenient to scale $(\tilde{N})$ to the set $B_{\frac{1}{\varepsilon}}$, namely to consider the problem

$$\begin{cases} -\Delta u + u = u^p, & \text{in } \frac{1}{\varepsilon} B_{\frac{1}{\varepsilon}}, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial B_{\frac{1}{\varepsilon}}, & u > 0. \end{cases} \tag{10.14}$$

and to use the functional I_ε defined as

$$I_\varepsilon(u) = \tfrac{1}{2} \int_{\Omega_\varepsilon} \left(|\nabla u|^2 + V(\varepsilon |x|) u^2 \right) dx - \tfrac{1}{p+1} \int_{\Omega_\varepsilon} |u|^{p+1} dx, \qquad u \in H^1_r(B_{\frac{1}{\varepsilon}}).$$

In the sequel, it is understood that the norm $\| \cdot \| = \| \cdot \|_{H^1_r(B_{\frac{1}{\varepsilon}})}$.

Remark 10.13. As for (10.1), the phenomenon is peculiar of the higher-dimensional case since for $n = 1$ it is possible to prove that there are no interior spikes approaching the boundary of an interval. The other comments in Remarks 10.2 hold for this case with obvious changes. □

10.5.1 The finite-dimensional reduction

For any $r_0 < \frac{1}{2}$, let $\phi_\varepsilon(r)$ be a smooth cut-off function such that

$$\phi_\varepsilon(r) = \begin{cases} 0 & \text{for } r \in \left[0, \frac{r_0}{8\varepsilon}\right]; \\ 1 & \text{for } r \in \left[\frac{r_0}{4\varepsilon}, \frac{1}{\varepsilon}\right]; \\ |\phi'_\varepsilon(r)| \leq C\varepsilon & \text{for } r \in \left[\frac{r_0}{8\varepsilon}, \frac{r_0}{4\varepsilon}\right]; \\ |\phi''_\varepsilon(r)| \leq C\varepsilon^2 & \text{for } r \in \left[\frac{r_0}{8\varepsilon}, \frac{r_0}{4\varepsilon}\right]. \end{cases} \tag{10.15}$$

Let $\overline{\alpha} = \lim_{t \to +\infty} e^t \overline{U}(t)$, where the function $\overline{U}$ is given in (10.4), and $z_\rho(r) = \overline{U}(r - \rho)$. We define $\mathcal{Z}^N$ to be the following manifold

$$\mathcal{Z}^N = \left\{ \phi_\varepsilon \left(z_\rho + \overline{\alpha} e^{-\left(\frac{1}{\varepsilon} - \rho\right)} e^{-\left(\frac{1}{\varepsilon} - \cdot\right)} \right) \right\}_\rho := \left\{ z^N_\rho = \phi_\varepsilon \left(z_\rho + v_\rho \right) \right\}_\rho; \qquad \rho \geq \frac{3}{4\varepsilon}. \tag{10.16}$$

The range of ρ will be chosen appropriately later. The function z^N_ρ has been defined in such a way that it has a small normal derivative at $\partial \Omega_\varepsilon$. In fact, we have the following estimate

$$(z_\rho^N)'\left(\frac{1}{\varepsilon}\right) = z_\rho'\left(\frac{1}{\varepsilon}\right) - \overline{\alpha}e^{-\left(\frac{1}{\varepsilon}-\rho\right)} \tag{10.17}$$
$$= z_\rho\left(\frac{1}{\varepsilon}\right)\left(\frac{z_\rho'}{z_\rho}\left(\frac{1}{\varepsilon}\right) - \frac{\overline{\alpha}e^{-\left(\frac{1}{\varepsilon}-\rho\right)}}{z_\rho\left(\frac{1}{\varepsilon}\right)}\right) = o\left(e^{-\left(\frac{1}{\varepsilon}-\rho\right)}\right).$$

As already mentioned at the beginning of the section, the correction term v_ρ in the definition of z_ρ^N can be heuristically viewed as the contribution of a *virtual spike*outside Ω.

We collect first some preliminary estimates.

Lemma 10.14. *Let $\mathcal{Z}^N$ be as above, and let $w \in \mathcal{C}_\varepsilon$, where*

$$\widetilde{\mathcal{C}}_\varepsilon = \left\{ w \in H_r^1(B_{\frac{1}{\varepsilon}}) \ : \ \|w\|_{H_r^1(B_{\frac{1}{\varepsilon}})} \le \gamma\varepsilon\|z_\rho^N\|_{H_r^1(B_{\frac{1}{\varepsilon}})}, \ |w(r)| \le \gamma\varepsilon \text{ for } r > 0 \right\}.$$

Then there exists $C > 0$ such that the following properties hold true

($\widetilde{\mathrm{E}}1$) $\quad \|I_\varepsilon''(z_\rho^N + sw)\| \le C, \quad (0 \le s \le 1)$;

($\widetilde{\mathrm{E}}2$) $\quad \|I_\varepsilon''(z_\rho^N + sw) - I_\varepsilon(z_\rho^N)\| \le C \max\left\{\|w\|_\infty, \|w\|_\infty^{(p-1)}\right\}, \quad (0 \le s \le 1)$;

($\widetilde{\mathrm{E}}3$) $\quad \|I_\varepsilon'(z_\rho^N)\| \le C\varepsilon^{\frac{1-n}{2}}\left(\varepsilon + o\left(e^{-\left(\frac{1}{\varepsilon}-\rho\right)}\right)\right)$ *for every $z_\rho^N \in \mathcal{Z}^N$.*

Proof. We prove ($\widetilde{\mathrm{E}}3$) only, since ($\widetilde{\mathrm{E}}1$) and ($\widetilde{\mathrm{E}}2$) can be proved as in Subsection 10.2.1. Since $z_\rho = \overline{U}(\cdot - \rho)$ and v_ρ satisfy respectively the equations $-z_\rho'' + z_\rho = z_\rho^p$ and $-v_\rho'' + v_\rho = 0$, we have, for an arbitrary $u \in H_r^1(B_{\frac{1}{\varepsilon}})$

$$\begin{aligned} I_\varepsilon'(z^N)[u] = &\int_0^{\frac{1}{\varepsilon}} \left(-(z_\rho^N)'' - \frac{n-1}{r}(z_\rho^N)' + V(\varepsilon r)z_\rho^N - (z_\rho^N)^p\right) ur^{n-1}dr \\ &+ \varepsilon^{1-n}(z_\rho^N)'(1/\varepsilon)u(1/\varepsilon) \\ = &\ \varepsilon^{1-n}(z_\rho^N)'(1/\varepsilon)u(1/\varepsilon) - (n-1)\int_0^{\frac{1}{\varepsilon}} \frac{1}{r}(z_\rho^N)'ur^{n-1}dr \\ &- \int_0^{\frac{1}{\varepsilon}} \left(2\phi_\varepsilon'(z_\rho^N)' + \phi_\varepsilon''(z_\rho^N)\right) ur^{n-1}dr - \int_0^{\frac{1}{\varepsilon}} \left((z_\rho^N)^p - \phi_\varepsilon z_\rho^p\right) ur^{n-1}dr. \end{aligned}$$

In the sequel, for brevity, we will often omit the index ρ in z_ρ, Z_ρ^N and v_ρ and we will set

$$\int(\cdot) := \int_0^{\frac{1}{\varepsilon}} (\cdot)r^{n-1}dr. \tag{10.18}$$

From the Strauss Lemma, see [135], and (10.17) we find

$$\varepsilon^{1-n}|(z^N)'(1/\varepsilon)u(1/\varepsilon)| = \varepsilon^{\frac{1-n}{2}} o\left(e^{-\left(\frac{1}{\varepsilon}-\rho\right)}\right)\|u\|. \tag{10.19}$$

Moreover, as in (E1) of Subsection 10.2.1, one has that $\|(z^N)'\| \leq C\varepsilon^{\frac{1-n}{2}}$. On the other hand, since the function z^N is supported in $\left\{r \geq \frac{r_0}{8\varepsilon}\right\}$, one also has

$$\left| \int \frac{1}{r}(z^N)'u \right| \leq C\varepsilon \|(z^N)'\| \|u\| \leq C\varepsilon^{\frac{3-n}{2}} \|u\|. \tag{10.20}$$

From the exponential decay of $z = z_\rho$ and $v = v_\rho$, from the fact that $\phi_\varepsilon', \phi_\varepsilon''$ have support in $\left[\frac{r_0}{8\varepsilon}, \frac{r_0}{4\varepsilon}\right]$ and from $\rho \geq \frac{3}{4\varepsilon}$, one deduces the estimates

$$\begin{aligned} \left| \int \phi_\varepsilon'(z+v)'u \right| &\leq C\varepsilon^{1+\frac{1-n}{2}} e^{-\frac{r_0}{4\varepsilon}} \|u\|; \\ \left| \int \phi_\varepsilon''(z+v)u \right| &\leq C\varepsilon^{2+\frac{1-n}{2}} e^{-\frac{r_0}{4\varepsilon}} \|u\|. \end{aligned} \tag{10.21}$$

Let us consider now the term $\int \left((z^N)^p - \phi_\varepsilon z^p\right) u$. We can write

$$(z^N)^p - \phi_\varepsilon z^p = \phi_\varepsilon^p \left((z+v)^p - z^p\right) + \phi_\varepsilon^p \left(\phi_\varepsilon^p z^p - \phi_\varepsilon z^p\right).$$

Since z is uniformly bounded, we have

$$\left|(z+v)^p - z^p - pz^{p-1}v\right| \leq C \max\{|v|^2, |v|^p\}.$$

It follows that

$$\left| \int \left[(z+v)^p - z^p\right] u \right| \leq p \left| \int z^{p-1} v|u| \right| + C \left| \int |u| \max\{|v|^2, |v|^p\} \right|.$$

Again from the Hölder inequality we obtain

$$\left| \int |v|^{2\wedge p}|u| \right| \leq C e^{-(2\wedge p)\left(\frac{1}{\varepsilon}-\rho\right)} \int e^{-(2\wedge p)\left(\frac{1}{\varepsilon}-r\right)}|u| \leq C e^{-(2\wedge p)\left(\frac{1}{\varepsilon}-\rho\right)} \varepsilon^{\frac{1-n}{2}} \|u\|.$$

We have also $\left|\int z^{p-1}v|u|\right| \leq \left(\int z^{2(p-1)}v^2\right)^{\frac{1}{2}} \|u\|$. We divide the last integral in the two regions $r \leq \frac{\rho+\varepsilon^{-1}}{2}$ and $r \geq \frac{\rho+\varepsilon^{-1}}{2}$. When $r \leq \frac{\rho+\varepsilon^{-1}}{2}$, v satisfies $|v| \leq e^{-\frac{3}{2}\left(\frac{1}{\varepsilon}-\rho\right)}$ and hence

$$\begin{aligned} \left(\int_{r \leq \frac{\rho+\varepsilon}{2}} z^{2(p-1)} v^2 r^{n-1} dr \right)^{\frac{1}{2}} &\leq C e^{-\frac{3}{2}\left(\frac{1}{\varepsilon}-\rho\right)} \left(\int_{r \leq \frac{\rho+\varepsilon}{2}} z^{2(p-1)} r^{n-1} dr \right)^{\frac{1}{2}} \\ &\leq C e^{-\frac{3}{2}\left(\frac{1}{\varepsilon}-\rho\right)} \varepsilon^{\frac{1-n}{2}}. \end{aligned}$$

On the other hand when $r \geq \frac{\rho+\varepsilon^{-1}}{2}$, z satisfies $|z(r)| \leq e^{-\frac{1}{2}\left(\frac{1}{\varepsilon}-\rho\right)}$ so we obtain

$$\begin{aligned} \left(\int_{r \geq \frac{\rho+\varepsilon}{2}} z^{2(p-1)} v^2 r^{n-1} dr \right)^{\frac{1}{2}} &\leq C e^{-\frac{p-1}{2}\left(\frac{1}{\varepsilon}-\rho\right)} \left(\int_0^{\frac{1}{\varepsilon}} |v|^2 r^{n-1} dr \right)^{\frac{1}{2}} \\ &\leq C e^{-\left(1+\frac{p-1}{2}\right)\left(\frac{1}{\varepsilon}-\rho\right)} \varepsilon^{\frac{1-n}{2}}. \end{aligned}$$

We have also

$$\left|\int (\phi_\varepsilon^p z^p - \phi_\varepsilon z^p)\, u\right| \le C \left(\int (\phi_\varepsilon^p - \phi_\varepsilon)^2\, \overline{U}^{2p}\right)^{\frac{1}{2}} \|u\| \le C e^{-\frac{pr_0}{2\varepsilon}} \varepsilon^{\frac{1-n}{2}} \|u\|.$$

The above estimates yield

$$\left|\int \left((z^N)^p - \phi_\varepsilon \overline{u}^p\right) u\right| \le C\varepsilon^{\frac{1-n}{2}} \left(e^{-\left(\frac{3\wedge(p+1)}{2}\right)\left(\frac{1}{\varepsilon}-\rho\right)} + e^{-\frac{pr_0}{4\varepsilon}}\right) \|u\|. \tag{10.22}$$

Hence (10.19)–(10.22) imply

$$\|I_\varepsilon'(z_\rho^N)\| \le C\varepsilon^{\frac{1-n}{2}} \left(\varepsilon + o\left(e^{-\left(\frac{1}{\varepsilon}-\rho\right)}\right) + e^{-\frac{r_0}{4\varepsilon}}\right).$$

This concludes the proof of the $(\widetilde{\mathrm{E}}3)$. □

Similarly to Proposition 10.5 above, we obtain the following result, which reduces (10.14) to a finite-dimensional problem.

Proposition 10.15. *For ε sufficiently small there exists a positive constant μ such that for $\rho \in \left[\frac{r_0}{\varepsilon}, \frac{1}{\varepsilon} - \mu\right]$, there exists a function $w^N = w^N(z_{\rho,\varepsilon}) \in W = (T_{z^N} \mathcal{Z}_\varepsilon^N)^\perp$ satisfying $PI_\varepsilon'(z + w) = 0$, where P is the projection onto W, and $\|w^N\| \le C\|I_\varepsilon'(z_\rho^N)\|$. Furthermore, setting*

$$\Psi_\varepsilon(\rho) = I_\varepsilon(z_\rho^N + w_{\rho,\varepsilon}^N),$$

if, for some $\varepsilon \ll 1$, ρ_ε is stationary point of Ψ_ε, then $\widetilde{u}_\varepsilon = z_{\rho_\varepsilon}^N + w_{\rho_\varepsilon,\varepsilon}^N$ is a critical point of I_ε.

In order to use Proposition 10.15 we need a careful expansion of Ψ_ε, since we want to consider values of ρ which are *close* to the exterior boundary of Ω_ε.

10.5.2 Proof of Theorem 10.12

In this subsection we prove our main result finding a critical point of the reduced functional Ψ_ε. The first step is to expand $I_\varepsilon(z_\rho^N)$ as a function of ρ and ε. Integrating by parts and using the equations satisfied by z and v (as in the equation before (10.19)), we find (we use again the notation (10.18))

$$\begin{aligned}
I_\varepsilon(z^N) &= \tfrac{1}{2}\int \left(|(z^N)'|^2 + (z^N)^2\right) - \tfrac{1}{p+1}\int |z^N|^{p+1} \\
&= \tfrac{1}{2}\int \left(-(z^N)'' - \tfrac{n-1}{r}(z^N)' + z^N\right) z^N \\
&\quad + \tfrac{1}{2}\varepsilon^{1-n} z^N(1/\varepsilon)(z^N)'(1/\varepsilon) - \tfrac{1}{p+1}\int |z^N|^{p+1} \\
&= \tfrac{1}{2}\varepsilon^{1-n} z^N(1/\varepsilon)(z^N)'(1/\varepsilon) + \tfrac{1}{2}\int \phi_\varepsilon z^p z^N - \tfrac{1}{p+1}\int |z^N|^{p+1} \\
&\quad - \tfrac{n-1}{2}\int \frac{(z^N)' z^N}{r} - \int \phi_\varepsilon' z^N (z+v)' - \tfrac{1}{2}\int \phi_\varepsilon'' z^N (z+v).
\end{aligned} \tag{10.23}$$

Let us estimate each of the seven terms in the last expression. From equations (8.9) and (10.17) we deduce

$$\varepsilon^{1-n}\left|z^N(1/\varepsilon)(z^N)'(1/\varepsilon)\right| = \varepsilon^{1-n} o\left(e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}\right). \tag{10.24}$$

To estimate the second and the third term, we can write

$$\begin{aligned}&\tfrac{1}{2}\int \phi_\varepsilon z^p z^N - \tfrac{1}{p+1}\int |z^N|^{p+1} \qquad (10.25)\\ &= \left(\tfrac{1}{2}-\tfrac{1}{p+1}\right)\int \phi_\varepsilon^{p+1} z^{p+1} + \tfrac{1}{2}\int\left(\phi_\varepsilon^2-\phi_\varepsilon^{p+1}\right) z^p(z+v) - \tfrac{1}{2}\int \phi_\varepsilon^{p+1} z^p v\\ &\qquad - \tfrac{1}{p+1}\int \phi_\varepsilon^{p+1}\left(|z+v|^{p+1} - z^{p+1} - (p+1)z^p v\right).\end{aligned}$$

We have

$$\begin{aligned}\left|\int \phi_\varepsilon^{p+1} z^{p+1} - \rho^{n-1}\int_{\mathbb{R}} \overline{U}^{p+1} dr\right| &\le \rho^{n-1}\int_{r\ge 1/\varepsilon} \overline{U}^{p+1}(r-\rho)dr + \int (1-\phi_\varepsilon^{p+1}) z^{p+1}\\ &\quad + \left|\int_0^{\frac{1}{\varepsilon}} (r^{n-1}-\rho^{n-1})\overline{U}^{p+1}(r-\rho)dr\right|.\end{aligned}$$

Using a Taylor expansion for the function $r^{n-1}-\rho^{n-1}$ and the fact that $r \le C(r_0)\rho$ (since $\rho \ge r_0/\varepsilon$), we obtain

$$\begin{aligned}\left|\int_0^{\frac{1}{\varepsilon}} (r^{n-1}-\rho^{n-1})\overline{U}^{p+1}(r-\rho)dr\right| &\le C(n,r_0)\rho^{n-2}\int_0^{\frac{1}{\varepsilon}} |r-\rho|\overline{U}^{p+1}(r-\rho)dr\\ &\le C\rho^{n-2}.\end{aligned}$$

On the other hand, from the exponential decay of $\overline{U}$, see (8.9), we get

$$\begin{aligned}&\rho^{n-1}\int_{r\ge 1/\varepsilon} \overline{U}^{p+1}(r-\rho)dr \le C\varepsilon^{1-n}\left(e^{-(p+1)\left(\frac{1}{\varepsilon}-\rho\right)} + e^{-\frac{(p+1)r_0}{4\varepsilon}}\right);\\ &\int_0^{\frac{1}{\varepsilon}} r^{n-1}(1-\phi_\varepsilon^{p+1})\overline{U}^{p+1} \le C\varepsilon^{1-n} e^{-\frac{(p+1)r_0}{4\varepsilon}}.\end{aligned}$$

Hence from the last three equations we deduce

$$\left|\int \phi_\varepsilon^{p+1} z^{p+1} - \rho^{n-1}\int_{\mathbb{R}} \overline{U}^{p+1} dr\right| \le C\varepsilon^{1-n}\left(e^{-(p+1)\left(\frac{1}{\varepsilon}-\rho\right)} + \varepsilon\right). \tag{10.26}$$

The term $\int \phi_\varepsilon^{p+1}\left(|z+v|^{p+1} - z^{p+1} - (p+1)z^p v\right)$ in (10.25) can be estimated as follows. From the inequality

$$\left||z+v|^{p+1} - z^{p+1} - (p+1)z^p v - p(p+1)z^{p-1}v^2\right| \le C\max\{|v|^3, |v|^{p+1}\},$$

one finds

$$\int \left||z+v|^{p+1} - z^{p+1} - (p+1)z^p v\right| \le C\int z^{p-1}v^2 + C\int \max\{|v|^3, |v|^{p+1}\}.$$

The first integral in the last expression can be estimated dividing the domain into the two regions $r \le \frac{\rho+\varepsilon^{-1}}{2}$ and $r \ge \frac{\rho+\varepsilon^{-1}}{2}$, as before, while for the second it is sufficient to use the explicit expression of v. In this way we find

$$\begin{aligned}&\left|\int \phi_\varepsilon^{p+1}\left(|z+v|^{p+1} - z^{p+1} - (p+1)z^p v\right)\right| \\ &\qquad \le C\varepsilon^{1-n}\left(e^{-3\left(\frac{1}{\varepsilon}-\rho\right)} + e^{-\frac{(p+3)}{2}\left(\frac{1}{\varepsilon}-\rho\right)} + e^{-(3\wedge(p+1))\left(\frac{1}{\varepsilon}-\rho\right)}\right).\end{aligned} \tag{10.27}$$

The term $\int \phi_\varepsilon^{p+1} z^p v$ in (10.25) turns out to be of order $\varepsilon^{1-n}e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}$. We need to have a rather precise expansion of this term, so we treat it in some detail. There holds

$$\begin{aligned}\int \phi_\varepsilon^{p+1} z^p v = {}& \overline{\alpha}\rho^{n-1}e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}\int_{\mathbb{R}} \overline{U}^p e^r\,dr \\ & - \overline{\alpha}\rho^{n-1}e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}\int_{r\ge 1/\varepsilon} \overline{U}^p(r-\rho)e^{(r-\rho)}dr \\ & + \int_0^{\frac{1}{\varepsilon}} (r^{n-1}-\rho^{n-1})z^p v + \int (\phi_\varepsilon^{p+1}-1)z^p v.\end{aligned}$$

Reasoning as above, we obtain

$$\begin{aligned}\rho^{n-1}e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}\int_{r\ge 1/\varepsilon} \overline{U}^p(r-\rho)e^{(r-\rho)}dr &\le C\varepsilon^{1-n}e^{-(p+1)\left(\frac{1}{\varepsilon}-\rho\right)}; \\ \left|\int_0^{\frac{1}{\varepsilon}}(r^{n-1}-\rho^{n-1})z^p v\,dr\right| &\le C\varepsilon^{2-n}e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}; \\ \int (1-\phi_\varepsilon^{p+1})z^p v &\le C\varepsilon^{1-n}e^{-\frac{(p+1)r_0}{4\varepsilon}}.\end{aligned}$$

Hence the last three equations and the expression of $\overline{U}$ imply

$$\begin{aligned}\int \phi_\varepsilon^{p+1} z^p v &= \overline{\alpha}\rho^{n-1}e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}\int_{\mathbb{R}} \overline{U}^p e^r\,dr + \varepsilon^{1-n}e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}O\left(\varepsilon + e^{-(p-1)\left(\frac{1}{\varepsilon}-\rho\right)}\right) \\ &= \overline{\alpha}\varepsilon^{1-n}(\varepsilon\rho)^{n-1}e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}\int_{\mathbb{R}} \overline{U}^p e^r\,dr \\ &\qquad + \varepsilon^{1-n}e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}O\left(\varepsilon + e^{-(p-1)\left(\frac{1}{\varepsilon}-\rho\right)}\right),\end{aligned} \tag{10.28}$$

for ε small. The fourth term in (10.23) can be estimated as for (10.20), and gives

$$\left|\int \frac{(z^N)' z^N}{r}\right| \le C\varepsilon^{2-n}. \tag{10.29}$$

The fifth and the sixth terms in (10.23) can be estimated in the following way

$$\left|\int \phi_\varepsilon' z^N (z+v)'\right| \le C\varepsilon^{2-n} e^{-\frac{r_0}{2\varepsilon}} \qquad \left|\int \phi_\varepsilon'' z^N (z+v)\right| \le C\varepsilon^{3-n} e^{-\frac{r_0}{2\varepsilon}}. \tag{10.30}$$

From (10.24)–(10.30) we deduce the following result.

Lemma 10.16. *Let z_ρ^N be defined in* (10.16), *and set*

$$\alpha = \left(\tfrac{1}{2} - \tfrac{1}{p+1}\right) \int_{\mathbb{R}} \overline{U}^{p+1}; \qquad \beta = \tfrac{1}{2}\overline{\alpha} \int_{\mathbb{R}} \overline{U}^p e^r. \tag{10.31}$$

Then one has

$$I_\varepsilon(z_\rho^N) = \varepsilon^{1-n}(\varepsilon\rho)^{n-1}\left[\alpha - \beta e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}\right] + O(\varepsilon^{2-n}) + \varepsilon^{1-n} o\left(e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}\right)$$

for all $\rho \in \left[\frac{3}{4\varepsilon}, \frac{1}{\varepsilon}\right]$.

Proof of Theorem 10.12. For $s \in [0,1]$, using ($\widetilde{\mathrm{E}}1$) and ($\widetilde{\mathrm{E}}2$) in Lemma 10.14, we have

$$\begin{aligned}
&\left\|I_\varepsilon'(z^N + sw^N) - I_\varepsilon'(z^N)\right\| \\
&\quad \le \|I_\varepsilon''(z^N)[sw^N]\| + \left\|\int_0^1 \left(I_\varepsilon''(z^N + \zeta s w^N) - I_\varepsilon''(z^N)\right)[w]d\zeta\right\| \\
&\quad = O(\|w^N\|) + O\left(\max\left\{\|w^N\|^2, \|w^N\|^p\right\}\right).
\end{aligned}$$

Hence, using the estimate of $\|w^N\|$ in Proposition 10.15 and ($\widetilde{\mathrm{E}}1$), we deduce

$$\begin{aligned}
I_\varepsilon(z^N + w^N) &= I_\varepsilon(z^N) + I_\varepsilon'(z^N)[w^N] + \int_0^1 \left(I_\varepsilon'(z^N + sw^N) - I_\varepsilon'(z^N)\right)[w^N]ds \\
&= I_\varepsilon(z^N) + O\left(\|I_\varepsilon'(z^N)\|^2\right).
\end{aligned}$$

Using ($\widetilde{\mathrm{E}}3$) we infer that $O\left(\|I_\varepsilon'(z^N)\|^2\right) = O(\varepsilon^{3-n})$. Hence from Lemma 10.16 it turns out that

$$I_\varepsilon(z_\rho^N + w_\rho^N) = \rho^{n-1}\left[\alpha - \beta e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}\right] + O(\varepsilon^{2-n}) + \varepsilon^{1-n} o\left(e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}\right). \tag{10.32}$$

We are going to show that the function $\rho \mapsto I_\varepsilon(z_\rho^N + w_\rho^N)$ possesses a critical point ρ_ε with $\left|\frac{1}{\varepsilon} - \rho_\varepsilon\right| \sim |\log\varepsilon|$. We give first an heuristic argument, which justifies the choice of the numbers $\rho_{0,\varepsilon}, \rho_{1,\varepsilon}$ and $\rho_{2,\varepsilon}$ below. The main term in (10.32) is $\rho^{n-1}\left[\alpha - \beta e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}\right]$. Differentiating with respect to ρ we obtain

$$(n-1)\rho^{n-2}\left[\alpha - \beta e^{-2\lambda\left(\frac{1}{\varepsilon}-\rho\right)}\right] - 2\beta\rho^{n-1} e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}.$$

Since $\left|\frac{1}{\varepsilon} - \rho_\varepsilon\right| \sim |\log \varepsilon|$, the term $e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}$ converges to 0 as ε goes to 0, hence to get a critical point we must require, roughly

$$(n-1)\rho^{n-2} = 2\beta\rho^{n-1}e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}.$$

Taking the logarithm, and using the fact that all the terms except ε and $e^{-2\left(\frac{1}{\varepsilon}-\rho\right)}$ are uniformly bounded from above and from below by positive constants, we obtain the condition

$$|\log \varepsilon| \sim 2\left(\frac{1}{\varepsilon} - \rho\right) \qquad \Leftrightarrow \qquad \left(\frac{1}{\varepsilon} - \rho\right) \sim \frac{|\log \varepsilon|}{2}. \tag{10.33}$$

We now begin our justification of the above arguments. Given $C_0 > 0$ (to be fixed later sufficiently large), consider the three numbers

$$\rho_{0,\varepsilon} = \frac{1}{\varepsilon} - \frac{1}{2}|\log \varepsilon|; \qquad \rho_{1,\varepsilon} = \frac{1}{\varepsilon} - \frac{1}{C_0}|\log \varepsilon|; \qquad \rho_{2,\varepsilon} = \frac{1}{\varepsilon} - C_0|\log \varepsilon|. \tag{10.34}$$

By condition (10.33) we expect $\rho_{0,\varepsilon}$ to be almost critical for the function $\rho \mapsto \Psi_\varepsilon(\rho) = I_\varepsilon(z_\rho^N + w_\rho^N)$. Using Lemma 10.16 and some elementary computations, one finds

$$\begin{aligned}\Psi_\varepsilon(\rho_{0,\varepsilon}) = \varepsilon^{1-n}\left(1 + o(\varepsilon|\log \varepsilon|)\right)&\left[\alpha - \beta\varepsilon^{\left(1-\frac{\varepsilon|\log \varepsilon|}{2}\right)}\right] \\ &+ O(\varepsilon^{2-n}) + \varepsilon^{1-n}o\left(\varepsilon^{\left(1-\frac{\varepsilon|\log \varepsilon|}{2}\right)}\right).\end{aligned}$$

We have $\varepsilon^{\left(1-\frac{\varepsilon|\log \varepsilon|}{2}\right)} = \varepsilon^{1+O(\varepsilon|\log \varepsilon|)} = O(\varepsilon) \ll \varepsilon|\log \varepsilon|$, and hence

$$\Psi_\varepsilon(\rho_{0,\varepsilon}) = \varepsilon^{1-n}\alpha\left(1 + o(\varepsilon|\log \varepsilon|)\right).$$

On the other hand, there holds

$$\begin{aligned}\Psi_\varepsilon(\rho_{1,\varepsilon}) = \varepsilon^{1-n}\left(1 + o(\varepsilon|\log \varepsilon|)\right)&\left[\alpha - \beta\varepsilon^{2\left(1-\frac{\varepsilon|\log \varepsilon|}{C_0}\right)/C_0}\right] \\ &+ O(\varepsilon^{2-n}) + \varepsilon^{1-n}o\left(\varepsilon^{2\left(1-\frac{\varepsilon|\log \varepsilon|}{C_0}\right)/C_0}\right).\end{aligned}$$

If $C_0 > 2$, we use the estimate

$$\varepsilon^{2\left(1-\frac{\varepsilon|\log \varepsilon|}{C_0}\right)/C_0} = \varepsilon^{2/C_0+O(\varepsilon|\log \varepsilon|)} = \varepsilon^{2/C_0}(1 + o(1)) \gg \varepsilon|\log \varepsilon|,$$

to obtain

$$\Psi_\varepsilon(\rho_{1,\varepsilon}) = \varepsilon^{1-n}\left[\alpha - \beta\varepsilon^{\frac{2}{C_0}} + o\left(\varepsilon^{\frac{2}{C_0}}\right)\right].$$

For the third term, we can write

$$\begin{aligned}\Psi_\varepsilon(\rho_{2,\varepsilon}) = \varepsilon^{1-n}\left(1 + o(\varepsilon|\log \varepsilon|)\right)&\left[\alpha - \beta\varepsilon^{2C_0(1-C_0\varepsilon|\log \varepsilon|)}\right] \\ &+ O(\varepsilon^{2-n}) + \varepsilon^{1-n}o\left(\varepsilon^{2C_0(1-C_0\varepsilon|\log \varepsilon|)}\right).\end{aligned}$$

If $C_0 > \frac{1}{2}$, we obtain

$$\varepsilon^{2C_0(1-C_0\varepsilon|\log\varepsilon|)} = \varepsilon^{2C_0+O(\varepsilon|\log\varepsilon|)} = O(\varepsilon^{2C_0+O(\varepsilon|\log\varepsilon|)}) \ll \varepsilon|\log\varepsilon|,$$

and hence

$$\Psi_\varepsilon(\rho_{2,\varepsilon}) = \varepsilon^{1-n}\alpha\left(1 + o(\varepsilon|\log\varepsilon|)\right),$$

for ε sufficiently small. If C_0 is chosen sufficiently large, the last three equations imply

$$\sup_{[\rho_{2,\varepsilon},\rho_{1,\varepsilon}]} \Psi_\varepsilon \geq \Psi_\varepsilon(\rho_{0,\varepsilon}) > \max\left\{\Psi_\varepsilon(\rho_{1,\varepsilon}), \Psi_\varepsilon(\rho_{2,\varepsilon})\right\}.$$

Hence it follows that the reduced functional Ψ_ε possesses a critical point (maximum) ρ in the interval $(\rho_{1,\varepsilon}, \rho_{2,\varepsilon})$. By Proposition 10.15, we obtain a critical point of I_ε with the desired asymptotic profile. By construction, this solution is close in L^∞ to a positive function. Then from the maximum principle it is easy to conclude that u_ε is strictly positive. This concludes the proof of the theorem. □

10.5.3 Further results

As for the Schrödinger equation, we collect some related results without giving the proofs, since they are based on similar ideas. First of all, we can consider a generalization of $(\tilde{N})$, adding a radial potential V. Precisely, letting Ω denote either the unit ball B_1 or the annulus

$$A = \{x \in \mathbb{R}^n \,:\, a < |x| < 1\}, \qquad a \in (0,1),$$

we consider the problem

$$\begin{cases} -\varepsilon^2\Delta u + V(|x|)u = u^p & \text{in } \Omega; \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, & u > 0 \text{ in } \Omega. \end{cases} \tag{$\hat{N}$}$$

Theorem 10.12 admits the following extension.

Theorem 10.17. *Let $(V1)$ and (V2) hold, $p > 1$ and let $\Omega \subseteq \mathbb{R}^n$ be the unit ball B_1 (resp. the annulus A). Suppose that the function $M(r) = r^{n-1}V^\theta(r)$ satisfies the condition*

$$M'(1) > 0 \qquad\qquad (\textit{resp. } M'(a) < 0). \tag{10.35}$$

Then there exists a family of radial solutions u_ε of $(\hat{N})$ concentrating on $|x| = r_\varepsilon$, where r_ε is a local maximum for u_ε such that $1 - r_\varepsilon \sim \varepsilon|\log\varepsilon|$ (resp. $r_\varepsilon - a \sim \varepsilon|\log\varepsilon|$).

Similarly to Theorem 10.1, we can also prove concentration in the interior of Ω, in correspondence of suitable critical points of the auxiliary potential M.

Theorem 10.18. *Let* (V1) *and* (V2) *hold, $p > 1$ and suppose that M has a point of strict local maximum or minimum at $r = \overline{r}$. Then, for $\varepsilon > 0$ small enough, $(\hat{N})$ has a radial solution which concentrates near the sphere $|x| = \overline{r}$.*

If one is willing to sacrifice the information concerning the location of the concentration set $|x| = r_\varepsilon$, a more general existence result is in order.

Theorem 10.19. *Suppose that $\Omega = B_1$ (resp. $\Omega = A$), $p > 1$, and that $V : B_1 \to \mathbb{R}$ (resp. $V : A \to \mathbb{R}$) satisfies assumptions* (V1) *and* (V2). *Then problem $(\hat{N})$ admits a family of solutions concentrating on a sphere.*

Finally, we also consider the Dirichlet version of problem $(\hat{N})$, namely

$$\begin{cases} -\varepsilon^2 \Delta u + V(|x|)u = u^p & \text{in } \Omega, \\ u = 0 \text{ on } \partial\Omega, & u > 0 \text{ in } \Omega. \end{cases} \tag{D}$$

In this case, the effect of the boundary is the opposite with respect to the Neumann case, and this will *repel* the functions concentrated at a sphere. The result for this case is the following.

Theorem 10.20. *Let $\Omega \subseteq \mathbb{R}^n$ be the ball B_1 (resp. the annulus A). Suppose that the function M satisfies the condition*

$$M'(1) < 0 \qquad\qquad (\textit{resp. } M'(a) > 0). \tag{10.36}$$

Then there exists a family of radial solutions u_ε of (D) concentrating near $|x| = 1$ (resp. near $|x| = a$). More precisely, u_ε possesses a local maximum point $r_\varepsilon < 1$ (resp. $a < r_\varepsilon < 1$) for which $1 - r_\varepsilon \sim \varepsilon|\log\varepsilon|$ (resp. $r_\varepsilon - a \sim \varepsilon|\log\varepsilon|$).

Remarks 10.21.

(i) Theorem 10.18 holds also for (D).

(ii) The counterpart of Theorem 10.19 for problem (D) holds only for annulus. Indeed, in the case of problem (D) in the unit ball with $V \equiv 1$, the only solution is the spike at the origin (for p subcritical) by the results in [83] and [98]. □

Bibliographical remarks

There are at the moment only few papers dealing with concentration at spheres ir manifolds. In addition to the aforementioned [21, 22, 110, 111, 112] we can mention [70, 71, 113, 129].

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